Krittika Summer Projects

Eclipsing Binaries

Theoretical assignment solution

week-2

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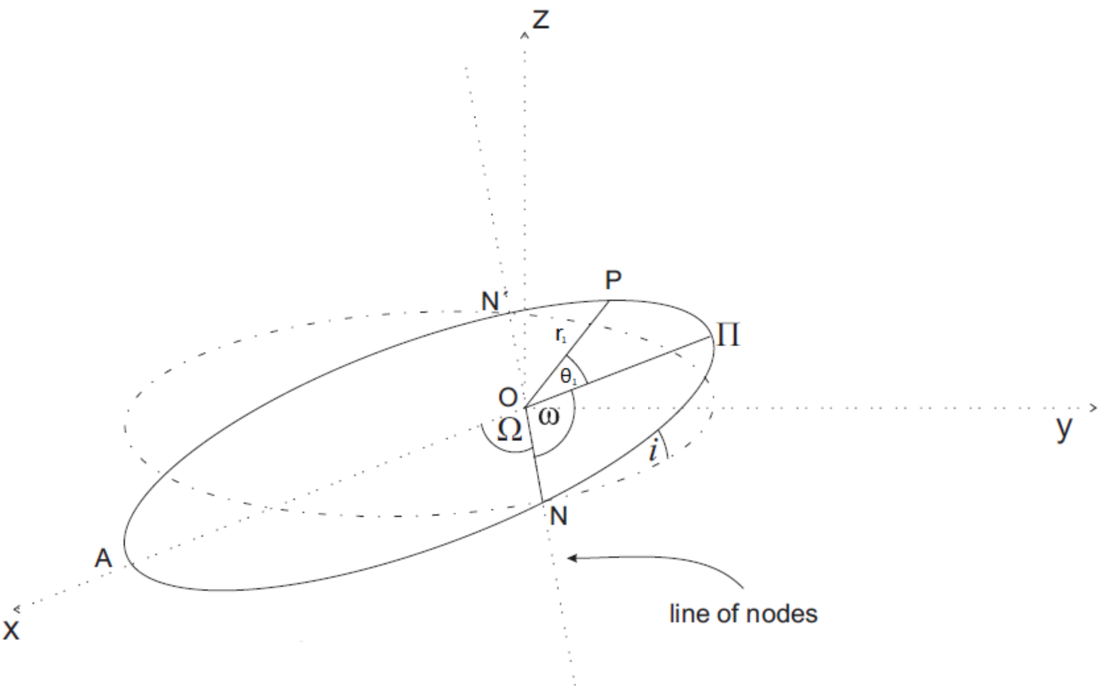
Projected semi major axis



In the above figure, the ellipse represents the orbit of the primary star with the barycentre of the binary star system as one of its foci.

The z axis is along the line of observation, or the line of vision. θ_1 is the true anomaly and r_1 is the radial distance of the primary star from the barycentre.

Solving the equations of motion of a two-body system we have obtained the following relation of r_1 and θ_1 :

$$r_1(\theta_1) = \frac{a_1(1-e^2)}{1+e\cos\theta_1} \tag{1}$$

where, a_1 is the semi major axis length of the orbit of the primary star

The component of r_1 along the direction of observation is (from above figure) :

$$z_1 = r_1 \sin(i)\sin(\theta_1+\omega) \tag{2}$$

Differentiating with respect to time, we obtain the form of the observed radial velocity of the star with respect to the barycentre,

$$V_{1,\mathrm{rad}} = \dot{z_1} = \dot{r_1}\sin(i)\sin(\theta_1+\omega) + r_1\sin(i)\cos(\theta_1+\omega)\dot{\theta_1} \tag{3}$$

Now we differentiate (1) wrt time,

$$\dot{r_1} = \frac{a_1(1-e^2)}{(1+e\cos\theta_1)^2}\cdot(-e\sin\theta_1)\cdot\dot{\theta_1} \tag{4}$$

While solving for the equations of motion of a two-body system, we also obtained the relation :

$$r_1^2\dot{\theta}_1 = \frac{2\pi(a_1)^2\sqrt{1-e^2}}{P}$$
$$\implies \dot{\theta}_1 = \frac{2\pi(a_1)^2\sqrt{1-e^2}}{(r_1)^2 P}$$

Replacing r_1 from (1) we get,

$$\dot{\theta}_1 = \frac{2\pi(a_1)^2\sqrt{1-e^2}}{P} \cdot \frac{(1+e\cos\theta_1)^2}{(a_1)^2(1-e^2)^2}$$

$$\implies \dot{\theta}_1 = \frac{2\pi}{P} \cdot (1-e^2)^{-\frac{3}{2}} \cdot (1+e\cos\theta_1)^2 \tag{5}$$

Hence replacing the forms of $\dot{r_1}$, $\dot{\theta}_1$ in (3), we get,

$$V_{1,\text{rad}} = \dot{z} = \frac{2\pi a_1 \sin(i)}{P\sqrt{1-e^2}}(\cos(\theta_1+\omega) + e\cos\omega) \tag{6}$$

From the radial velocity curve, we obtain K_1, which is the semiamplitude of the rv curve. From (6), we observe that,

$$K_1 = \frac{2\pi a_1 \sin(i)}{P\sqrt{1-e^2}} \tag{7}$$

We have carried out the derivation for the primary star, however the same process is applicable to the companion star as well, hence,

$$K_2 = \frac{2\pi a_2 \sin(i)}{P\sqrt{1-e^2}} \tag{8}$$

Now, the semi-major axis, a, of the binary star system is given by :

$$a = a_1 + a_2$$

Hence adding (7) and (8), we get,

$$K_1 + K_2 = \frac{2\pi \sin(i)}{P\sqrt{1-e^2}}(a_1+a_2) = \frac{2\pi a \sin(i)}{P\sqrt{1-e^2}}$$

$$\implies \boxed{a\sin(i) = \frac{P}{2\pi}(K_1+K_2)(1-e^2)^{\frac{1}{2}}} \tag{9}$$

Minimum Masses

From Kepler's Third Law, we get the following relation for binary star systems :

$$a^3 = \frac{GM}{(2\pi)^2}P^2 \tag{10}$$

where, a is the semi-major axis of the binary mass system, $M = m_1 + m_2$ is the total mass of the system, and P is the orbital period of each star.

Now, as r_1 and r_2 represent the radial distances of the primary and the secondary stars respectively from the centre of mass, hence the following holds true at all times :

$$m_1 r_1 = m_2 r_2 \tag{11}$$

Therefore, $m_1 a_1 = m_2 a_2$. Then, $m_2 = \frac{a_1}{a_2} m_1$. From (8),

$\frac{a_1}{a_2} = \frac{K_1}{K_2}$. Hence we can write,

$$M = m_1 + m_2 = m_1 \left(1 + \frac{K_1}{K_2}\right) \tag{12}$$

Replacing M in (10) we get,

$$a^3 = \frac{G}{4\pi^2}P^2 \cdot m_1 \left(1 + \frac{K_1}{K_2}\right) \tag{13}$$

$$\implies a^3 \sin^3(i) = \frac{G}{4\pi^2}P^2 \cdot m_1 \left(1 + \frac{K_1}{K_2}\right) \sin^3(i)$$

$$\implies (a \sin(i))^3 = \frac{G}{4\pi^2}P^2 \cdot \left(1 + \frac{K_1}{K_2}\right) \cdot m_1 \sin^3(i)$$

Replacing $a\sin(i)$ from (9) in above relation we get,

$$\left(\frac{P}{2\pi}\right)^3 \cdot (K_1 + K_2)^3 \cdot (1 - e^2)^{\frac{3}{2}} = \frac{G}{4\pi^2}P^2 \cdot \left(1 + \frac{K_1}{K_2}\right) \cdot m_1 \sin^3(i)$$

$$\implies \boxed{m_1 \sin^3(i) = \frac{P}{2\pi G} K_2 (K_1 + K_2)^2 (1 - e^2)^{\frac{3}{2}}} \tag{14}$$

Also $m_2 = \frac{K_1}{K_2} m_1$. Hence,

$$\implies \boxed{m_2 \sin^3(i) = \frac{K_1}{K_2} m_1 \sin^3(i) = \frac{P}{2\pi G} K_1 (K_1 + K_2)^2 (1 - e^2)^{\frac{3}{2}}} \tag{15}$$

Mass function

In case of SB1 systems, the RV curve of only one star is available. Then, the mass function is defined as (when the RV curve of the primary star is only available),

$$f(m) = \frac{(m_2)^3 \sin^3(i)}{(m_1 + m_2)^2} \tag{16}$$

Now replacing $m_1 = \frac{K_2}{K_1} m_2$,

$$f(m) = \frac{(m_2)^3 \sin^3(i)}{(m_1 + m_2)^2} = \frac{(m_2)^3 \sin^3(i)}{(\frac{K_2}{K_1} m_2 + m_2)^2} = \frac{(K_1)^2}{(K_1 + K_2)^2} \cdot m_2 \sin^3(i)$$

Using (15), we can write,

$$f(m) = \frac{(K_1)^2}{(K_1 + K_2)^2} \cdot \frac{P}{2\pi G} K_1 (K_1 + K_2)^2 (1 - e^2)^{\frac{3}{2}}$$

$$\implies \boxed{f(m) = \frac{P}{2\pi G} (K_1)^3 (1 - e^2)^{\frac{3}{2}}} \tag{17}$$