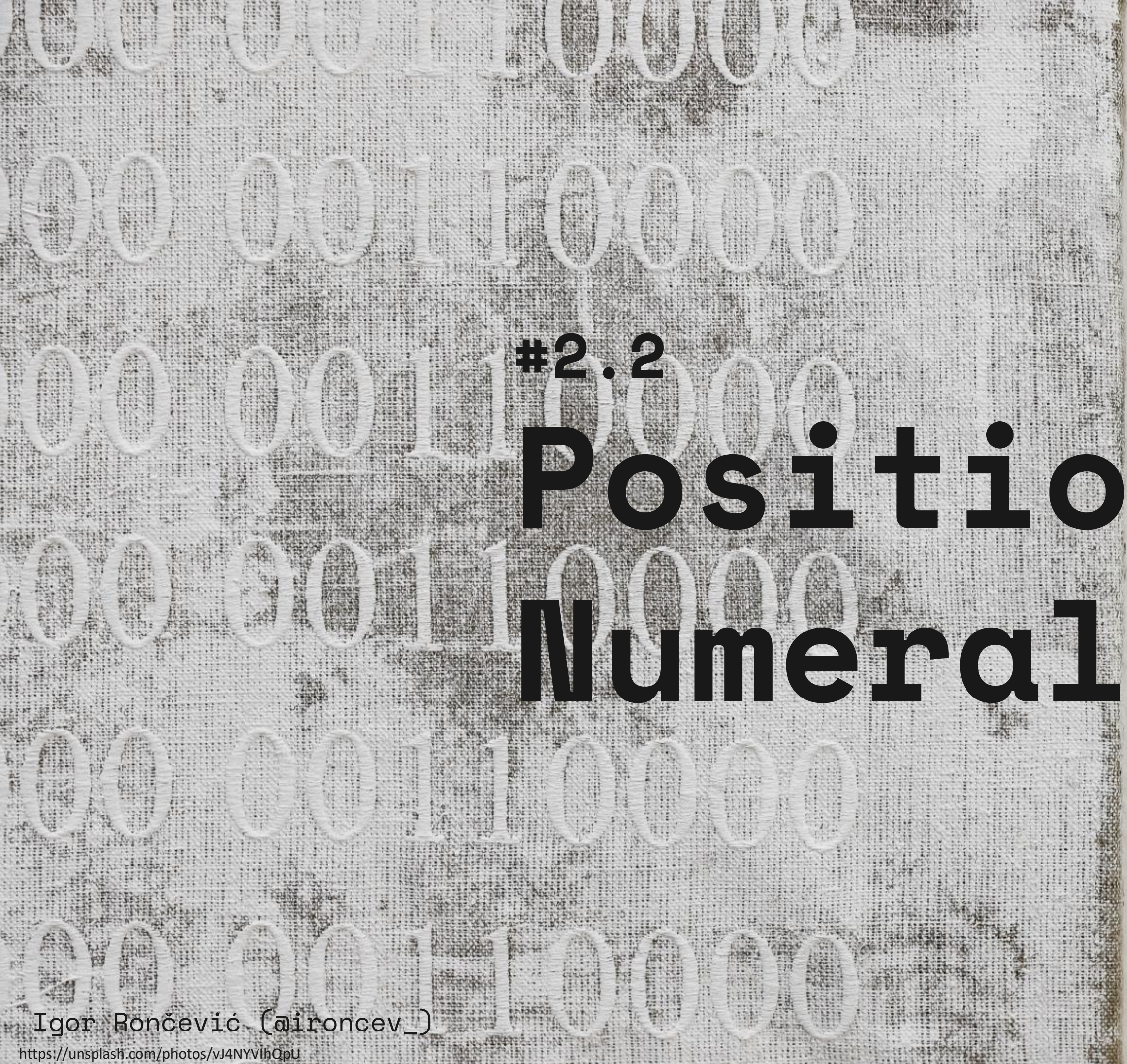

#2.2

Positional Numeral Systems

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https://unsplash.com/photos/vJ4NYVlhQpU

There are only 10 types of people in the world.

Those who understand binary and those who don’t.

“There are many different **numeral systems**, that is, **writing systems** for expressing **numbers**.”

List of numeral systems, Wikipedia

https://upload.wikimedia.org/wikipedia/commons/f/f8/La_Mojarra_Estela_1_%28Escritura_superior%29.jpg

https://commons.wikimedia.org/wiki/File:Handelskai_452_(Roman_numerals),_Vienna,_2019.jpg

$$
\begin{aligned}
1872 = {} & 2 \cdot 1 + \\
& 7 \cdot 10 + \\
& 8 \cdot 100 + \\
& 1 \cdot 1000
\end{aligned}
$$

$$
\begin{aligned}
(\overset{3}{1}\overset{2}{8}\overset{1}{7}\overset{0}{2})_{10} = {} & 2 \cdot 10^{0} + \\
& 7 \cdot 10^{1} + \\
& 8 \cdot 10^{2} + \\
& 1 \cdot 10^{3}
\end{aligned}
$$

$$\overset{3}{1}\overset{2}{8}\overset{1}{7}\overset{0}{2}$$

$$(1872)_{10} = 2 \cdot 10^0 + 7 \cdot 10^1 + 8 \cdot 10^2 + 1 \cdot 10^3$$

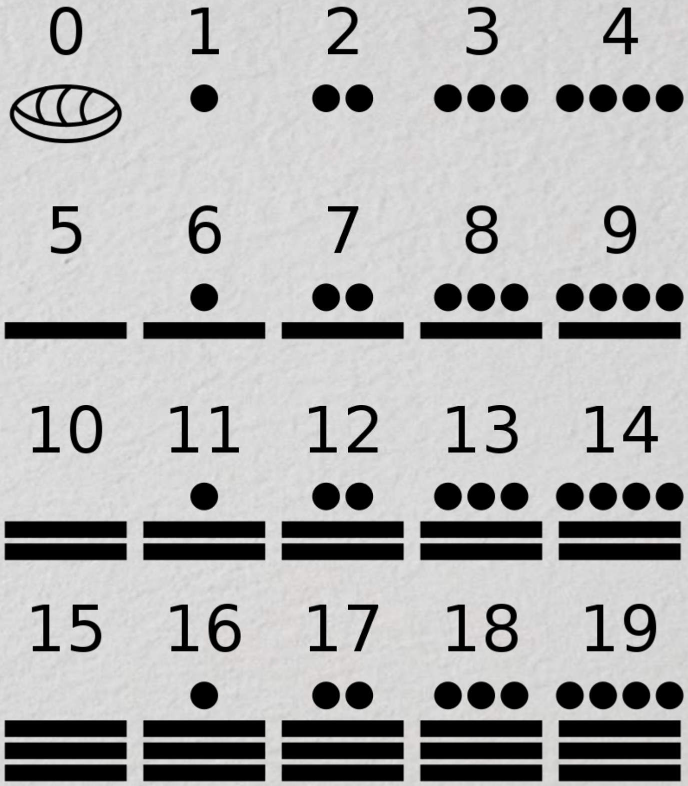
$$(\boxed{a_3}\boxed{a_2}\boxed{a_1}\boxed{a_0})_{10} = \sum_{i=0}^{3} a_i \cdot 10^i$$

$$(a_n \dots a_1 a_0)_{10} = \sum_{i=0}^{n} a_i \cdot 10^i$$

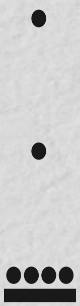
99

quatre-vingt-dix-neuf

$4 * 20 + 19$



https://en.wikipedia.org/wiki/Maya_numerals



2		$1 \cdot 20^{2}$	$12 \cdot 20^{2}$
1	$1 \cdot 20^{1}$	$1 \cdot 20^{1}$	$16 \cdot 20^{1}$
0	$13 \cdot 20^{0}$	$9 \cdot 20^{0}$	$5 \cdot 20^{0}$
	33	429	5125

https://en.wikipedia.org/wiki/Maya_numerals

https://upload.wikimedia.org/wikipedia/commons/f/f8/La_Mojarra_Estela_1_%28Escritura_superior%29.jpg

$$
\begin{aligned}
1872 &= 12 \cdot 1 + \\
&\quad 13 \cdot 20 + \\
&\quad 4 \cdot 400 \\
&= 12 \cdot 20^0 + \\
&\quad 13 \cdot 20^1 + \\
&\quad 4 \cdot 20^2
\end{aligned}
$$

$$1872 = 12 \cdot 20^0 + 13 \cdot 20^1 + 4 \cdot 20^2$$

??

digit symbols

1 2 3 4 5 6 7 8 9 A B C D E F G H I J

...

...

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

digit values

$$(1872)_{10} = (4DC)_{20}$$

$$(a_n \dots a_1 a_0)_{20} = \sum_{i=0}^{n} a_i \cdot 20^i$$

Positional Numeral System*

$r, r \in \mathbb{N}, r > 1$

Radix or **base** of the numeral system.

$D = \{\dots\}$

Set of symbols, called **digits**, that are used on a single position in a positional notation.

$|D| = r$

Numeral system with **radix** `r` has exactly `r` **digits** in `D`.

$v : D \to \{0,1,2,\dots,(r-1)\} \subset \mathbb{N}$

Bijective function that assigns a **numerical value** to each **digit** in `D`.

$(d_n \dots d_1 d_0)_r, d_i \in D, n \in \mathbb{N}$

A **numeral** consisting of one or more **digits** used for representing a **number** in a numeral system with **radix** `r`.

$$N = \sum_{i=0}^{n} v(d_i) \cdot r^i$$

The **number** represented by the above **numeral** in a numeral system with **radix** `r`.

* With positive integer bases and without fractions ;-) In a more general case, a base can be a negative integer, any real number like e.g., π, even a complex number, and of course we can have fractions as well.

E.g., Base-10 (Decimal) System

$r = 10$

$D = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$

$|D| = 10$

d	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$v(d)$	0	1	2	3	4	5	6	7	8	9

$(\mathbf{1872})_{10}$

$$N = 2 \cdot 10^0 + 7 \cdot 10^1 + 8 \cdot 10^2 + 1 \cdot 10^3 = 1872$$

🔄 E.g., Base-10 (Decimal) System [Devanagari]

$r = 10$

$D = \{०,१,२,३,४,५,६,७,८,९\}$

$|D| = 10$

d	०	१	२	३	४	५	६	७	८	९
$v(d)$	0	1	2	3	4	5	6	7	8	9

$(१८७२)_{10}$

$N = 2 \cdot 10^0 + 7 \cdot 10^1 + 8 \cdot 10^2 + 1 \cdot 10^3 = 1872$

E.g., Base-20 (Vigesimal) System

$r = 20$

$D = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}\}$

$|D| = 20$

d	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***	***J***
$v(d)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

$(\boldsymbol{4DC})_{20}$

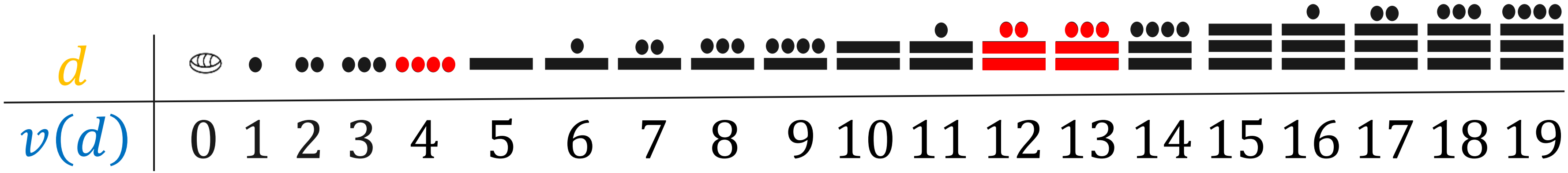
$N = 12 \cdot 20^0 + 13 \cdot 20^1 + 4 \cdot 20^2 = 1872$

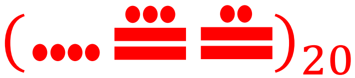
🗘 E.g., Base-20 (Vigesimal) System [Maya]

$r = 20$

$D = \{$  $\}$

$|D| = 20$





$N = 12 \cdot 20^0 + 13 \cdot 20^1 + 4 \cdot 20^2 = 1872$

🔄 E.g., Base-16 (Hexadecimal) System

$$r = 16$$

$$D = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}\}$$

$$|D| = 16$$

d	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	$\mathbf{7}$	$\mathbf{8}$	$\mathbf{9}$	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$	$\boldsymbol{E}$	$\boldsymbol{F}$
$v(d)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

$$(\mathbf{750})_{16}$$

$$N = 0 \cdot 16^0 + 5 \cdot 16^1 + 7 \cdot 16^2 = 1872$$

🔄 E.g., Base-8 (Octal) System

$r = 8$

$D = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}$

$|D| = 8$

d	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	$\mathbf{7}$
$v(d)$	0	1	2	3	4	5	6	7

$(\mathbf{3520})_8$

$N = 0 \cdot 8^0 + 2 \cdot 8^1 + 5 \cdot 8^2 + 3 \cdot 8^3 = 1872$

🔄 E.g., Base-2 (Binary) System

$r = 2$

$D = \{\mathbf{0}, \mathbf{1}\}$

$|D| = 2$

d	$\mathbf{0}$ $\mathbf{1}$
$v(d)$	0 1

$(\mathbf{11101010000})_2$

$N = 0 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + \cdots + 1 \cdot 2^{10} = 1872$

Some "Obvious" Identities

$$\min(v(d)) = 0$$

$$\max(v(d)) = r - 1$$

$$\max(\{\mathrm{N} = (d_n \dots d_1 d_0)_r : d_i \in D, n \in \mathbb{N}\}) = r^{n+1} - 1$$

E.g.:

$$\mathrm{r} = 10, \mathrm{n} = 2 \Rightarrow \max(N) = 10^{2+1} - 1 = (1000)_{10} - 1 = (999)_{10}$$

$$\mathrm{r} = 16, \mathrm{n} = 2 \Rightarrow \max(N) = 16^{2+1} - 1 = (1000)_{16} - 1 = (FFF)_{10}$$

$$\mathrm{r} = 2, \mathrm{n} = 2 \Rightarrow \max(N) = 2^{2+1} - 1 = (1000)_{2} - 1 = (111)_{2}$$

$(x_n \dots x_1 x_0)_r \rightarrow (y_m \dots y_1 y_0)_{10}$ Conversion

$$(4DC)_{20} = 12 \cdot 20^0 + 13 \cdot 20^1 + 4 \cdot 20^2 = (1872)_{10}$$

$$(750)_{16} = 0 \cdot 16^0 + 5 \cdot 16^1 + 7 \cdot 16^2 = (1872)_{10}$$

$$(3520)_8 = 0 \cdot 8^0 + 2 \cdot 8^1 + 5 \cdot 8^2 + 3 \cdot 8^3 = (1872)_{10}$$

$$(11101010000)_2 = 0 \cdot 2^0 + 0 \cdot 2^1 + \dots + 1 \cdot 2^{10} = (1872)_{10}$$

$(x_n \cdots x_1 x_0)_{10} \to (y_m \cdots y_1 y_0)_r$ Conversion

$$
\begin{array}{r l | l l}
 & & \text{reminder} & \\
1872 & / \ 20 & 12 & C \\
93 & / \ 20 & 13 & D \\
4 & / \ 20 & 4 & \\
0 & & &
\end{array}
$$

$(4DC)_{20}$