17/10/25

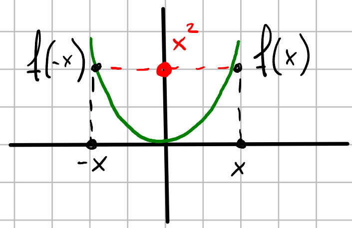
DEFINITION: $f : \mathbb{R} \to \mathbb{R}$. WE SAY THAT f IS:

- **EVEN**: IF $f(x) = f(-x) \ \forall x \in \mathbb{R}$ — Y-AXIS SYMMETRY
- **ODD**: IF $f(x) = -f(-x)$ OR $f(-x) = -f(x) \ \forall x \in \mathbb{R}$ — ORIGIN SYMMETRY

EX:

- $f(x) = x^2$

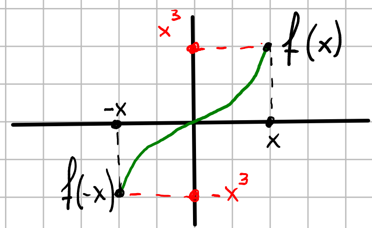
  $f(-x) = (-x)^2 = x^2 = f(x) \leftarrow$ EVEN



- $g(x) = x^3$

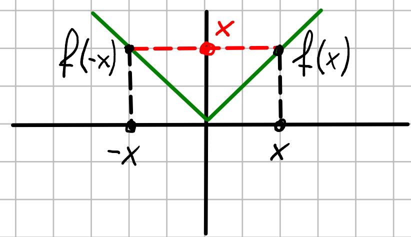
  $g(-x) = (-x)^3 = -x^3 = -g(x)$

  $\uparrow$ ODD



- $f(x) = |x|$

  $f(-x) = |-x| = |x| = f(x) \leftarrow$ EVEN



WE'VE DEFINED THE ODD AND EVEN SYMMETRY FOR $f : \mathbb{R} \to \mathbb{R}$ FUNCTIONS. YOU CAN EASILY EXTEND THE DEFINITION TO $f : A \to \mathbb{R}$, WHERE A IS A SET SYMMETRIC ABOUT THE ORIGIN.

EX: $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x} \to$ ODD BECAUSE: $\boxed{f(-x) = -\frac{1}{x} = -f(x)}$

THE DOMAIN IS NOT $\mathbb{R}$, BUT IF I HAVE x I EVEN HAVE ITS SYMMETRY $-x$.

DEF: $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ IS LIMITED IF $f(A)$, THE IMAGE, IS A LIMITED SUBSET OF $\mathbb{R}$. THAT IS : $\exists M > 0$ t.c.

$$\boxed{|f(x)| \leq M \quad \forall x \in A}$$

$$\boxed{-M \leq f(x) \leq M}$$

EX: • $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \longrightarrow$ NOT LIMITED

$$\boxed{\forall M > 0 \quad \exists x \in \mathbb{R} \quad t.c. \quad f(x) > M \quad x^2 > M} \Rightarrow \underline{x = \sqrt{M+1}}$$

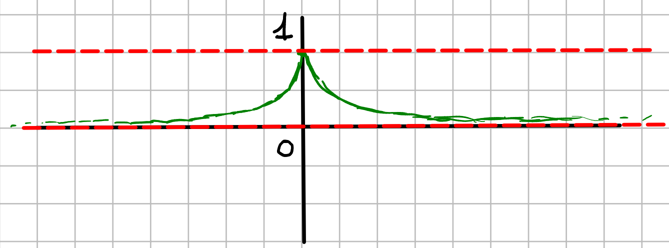
WE CAN ALWAYS TAKE A LARGER X BY ADDING $\underline{1 \text{ TO } M}$.

• $f: \mathbb{R} \to \mathbb{R}$, $g(x) = \dfrac{1}{1+x^2} \longrightarrow$ LIMITED

$D: 1+x^2 \neq 0 \quad \forall x \in \mathbb{R}$

$f(0) = 1$

$0 < f(x) \leq 1$

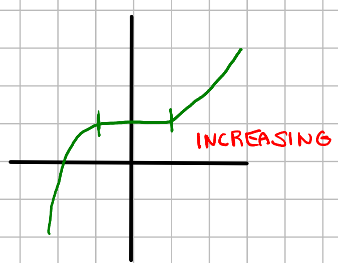


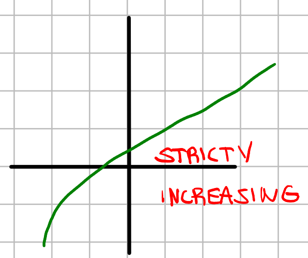
DEF: $f: \mathbb{R} \to \mathbb{R}$, IS CALLED:

• INCREASING (STRICTLY INCREASING): IF $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.
$(<) \to$ STRICTLY

• DECREASING (STRICTLY DECREASING): IF $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.
$(>)$

• MONOTONIC (STRICTLY MONOTONIC): IF $f(x)$ IS INCREASING OR DECREASING.
(STRICTLY)





ETC... $\Big\}$ ALL OF THEM ARE $\underline{\text{MONOTONIC}}$.

DEF: A STRICTLY MONOTONIC FUNCTION IS INJECTIVE, INDEED IF WE TAKE AS EXAMPLE A STRICTLY INCREASING FUNCTION AND WE TAKE $x_1 \neq x_2$, WE CAN SUPPOSE $x_1 < x_2$, THEN:

$$\boxed{f(x_1) \neq f(x_2) \longrightarrow \text{INJECTIVE}}$$

IN PARTICULAR, A STRICTLY MONOTONIC FUNCTION IS ALWAYS INVERTIBLE (BY RESTRICT THE CODOMAIN OF f TO ITS IMAGE)

OBS: IN CASE OF "$\mathbb{R}$ FUNCTIONS" IS POSSIBLE THAT THE DOMAIN IS NOT SPECIFIED. IN THIS CASE THE DOMAIN IS THE GREATEST SUBSET OF $\mathbb{R}$ t.c. $f(x)$ HAS A MEANING.

EX: $f(x) = \log(1+x^2)$ THE DOMAIN IS $\mathbb{R}$

$1+x^2 > 0 \quad \forall x \in \mathbb{R}$

$g(x) = \log(x^3) \Rightarrow x^3 > 0 \quad x > 0$

THE DOMAIN IS $(0, +\infty)$

$h(x) = \dfrac{1}{1+x} \Rightarrow 1+x \neq 0 \quad x \neq -1 \quad \mathbb{R} \setminus \{-1\}$

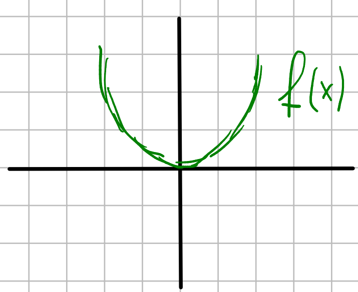
$K(x) = \sqrt{x+3} \Rightarrow x+3 \geq 0 \quad x \geq -3$

$[-3, +\infty)$

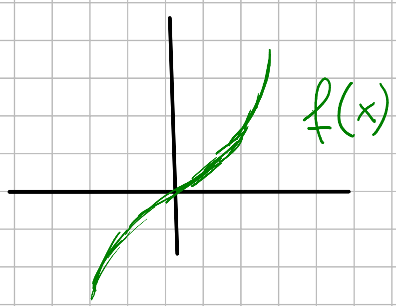
ELEMENTARY FUNCTIONS

1) POWER: IF $\alpha \in \mathbb{N}$, $f(x) = x^\alpha \quad \forall x \in \mathbb{R}$

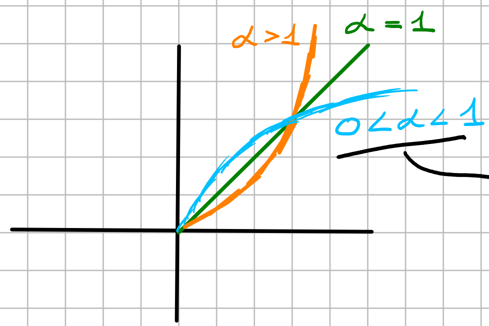
α IS EVEN



α IS ODD

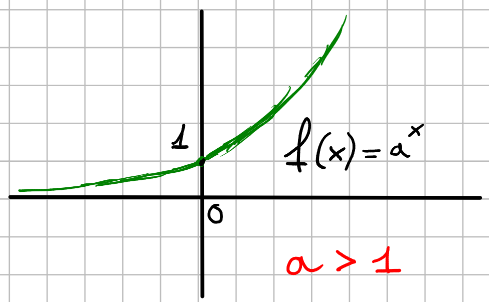


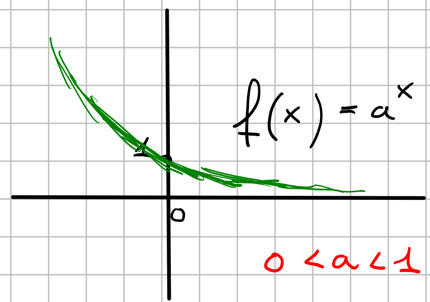
IF $\alpha \in \mathbb{R}$, $\alpha > 0$, $f(x) = x^{\alpha} \Rightarrow x \geq 0$



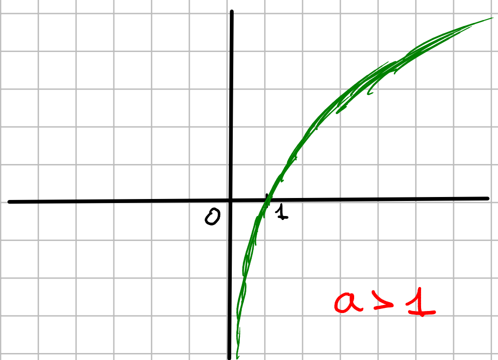
$\rightarrow$ ROOT $\left(x^{\frac{1}{2}} = \sqrt[2]{x} \right)$

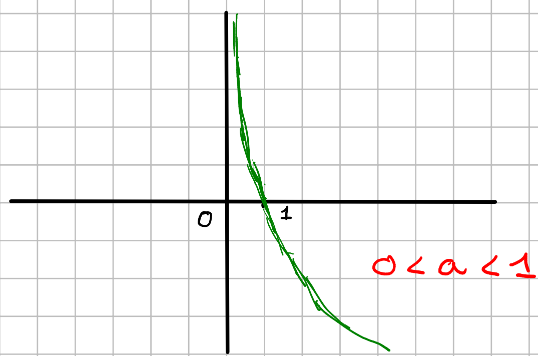
2) EXPONENTIAL: For $a > 0$, $a \neq 1$, $f(x) = a^x \quad \forall x \in \mathbb{R}$





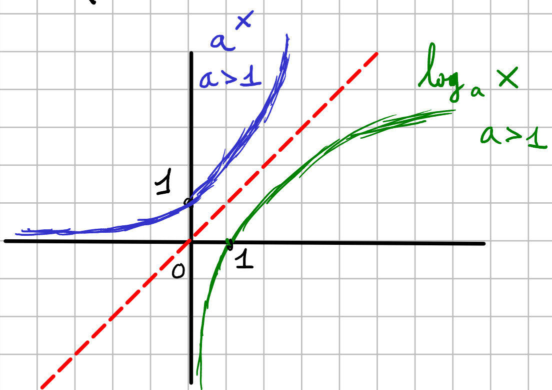
3) LOGARITHM: IF $a > 0$, $a \neq 1$, $f(x) = \log_a x \quad \forall x > 0$

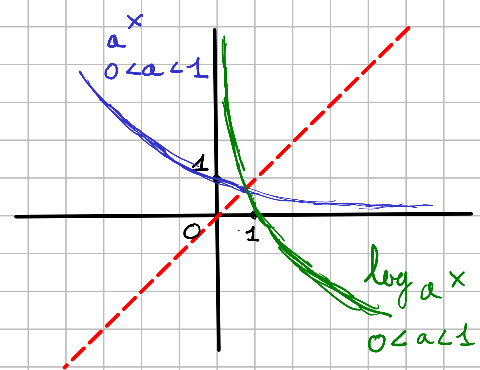




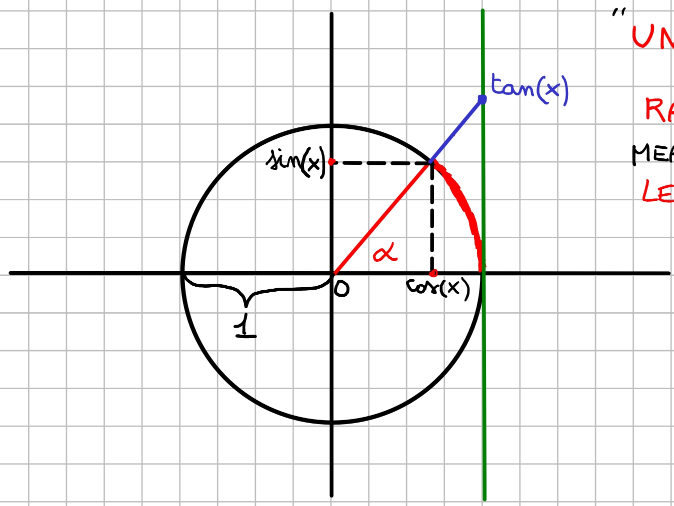
N.B: THE LOGARITHM IS THE INVERSE OF THE EXPONENTIAL

$$\left(a^{\log_a x} = x \ , \ \log_a a^x = x \right)$$





4) GONIOMETRIC FUNCTIONS:



"UNIT CIRCLE"

RADIAN = A UNIT OF ANGULAR MEASURE DEFINED AS THE ARC LENGTH CREATED BY THE ANGLE α.

EX:

$\alpha = 0°$ IN RAD $= 0$

$\alpha = 30°$ IN RAD $= \frac{\pi}{6}$

$\alpha = 45°$ IN RAD $= \frac{\pi}{4}$

$\alpha = 60°$ IN RAD $= \frac{\pi}{3}$

$\alpha = 90°$ IN RAD $= \frac{\pi}{2}$

$\alpha = 180°$ IN RAD $= \pi$

$\alpha = 270°$ IN RAD $= \frac{3}{4}\pi$

$\alpha = 360°$ IN RAD $= 2\pi$

SIN AND COS PROPERTY:

$\boxed{\sin(x) \text{ AND } \cos(x) \quad \forall x \in \mathbb{R}}$

1) SIN AND COS OF KNOWN ANGLES:

ANGLE	SIN	COS
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
π	0	-1

2) SIN AND COS ARE FUNCTION WITH A PERIOD OF 2π.

$$\sin(x+2\pi) = \sin(x)$$

$$\cos(x+2\pi) = \cos(x)$$

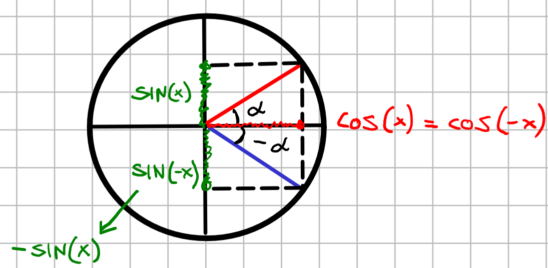
$$\forall x \in \mathbb{R}$$

3) $\boxed{\sin^2 x + \cos^2 x = 1 \quad \forall x \in \mathbb{R}}$

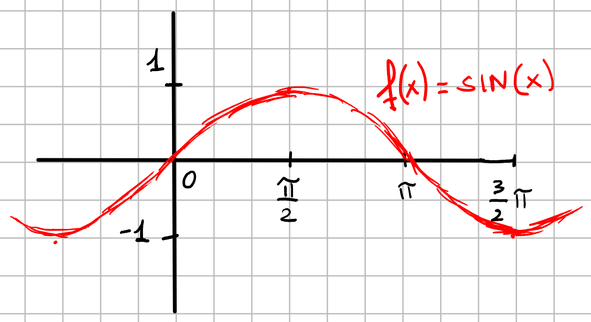
$\boxed{-1 \leq \cos(x), \sin(x) \leq 1}$

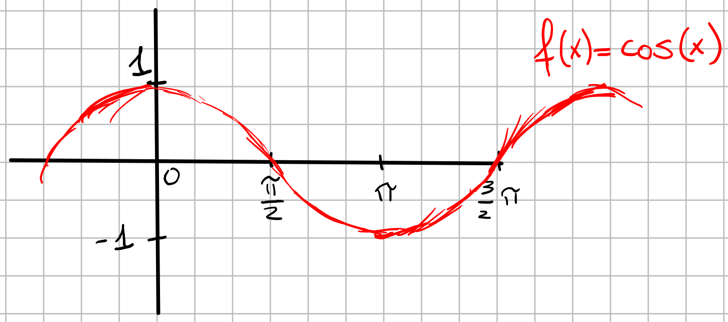
4) $\sin(-x) = -\sin(x)$ ODD

$\cos(-x) = \cos(x)$ EVEN



5) GRAPHIC :





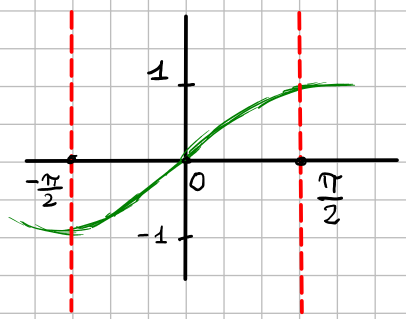
THESE FUNCTIONS ARE NOT INJECTIVE, AND ARE NOT INVERTIBLE IN $\mathbb{R}$, BUT WE CAN RESTRICT THE DOMAIN TO DO IT.

NOTATION: $f : A \to B,\ A' \subseteq A$:

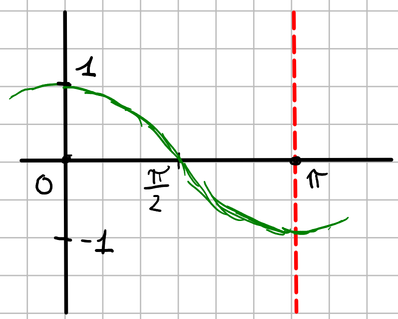
$$\boxed{f_{|A'} : A' \longrightarrow B \quad \text{t.c.} \quad f_{|A'}(x) = f(x) \quad \forall x \in A'}$$

RESTRICTION OF f IN A'.

LET'S CONSIDER :



$\sin_{|[-\frac{\pi}{2}, \frac{\pi}{2}]}(x)$



$\cos_{|[0, \pi]}(x)$

THE INVERSE FUNCTION OF $\sin(x)$ IS $\arcsin(x)$:

$$\arcsin(x) : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

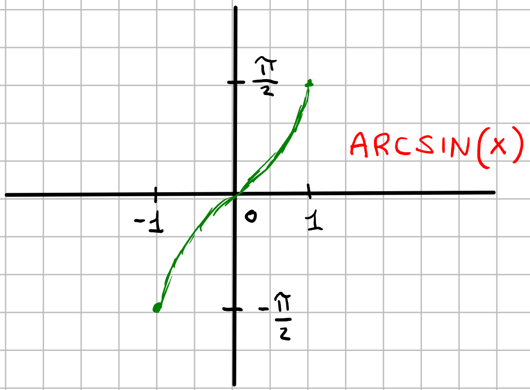
DEFINED BY $\arcsin(y) = x \iff \sin(x) = y$

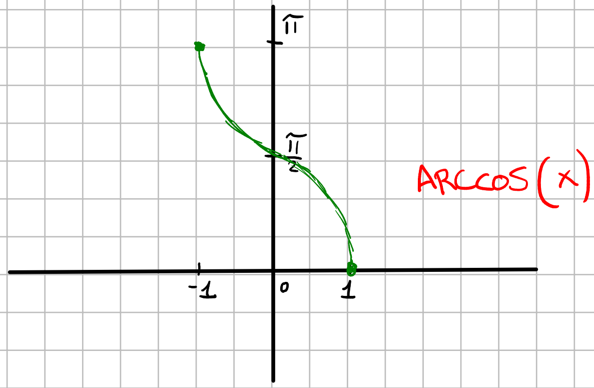
THE INVERSE FUNCTION OF $\cos(x)$ IS $\arccos(x)$:

$$\arccos(x) : [-1, 1] \longrightarrow [0, \pi]$$

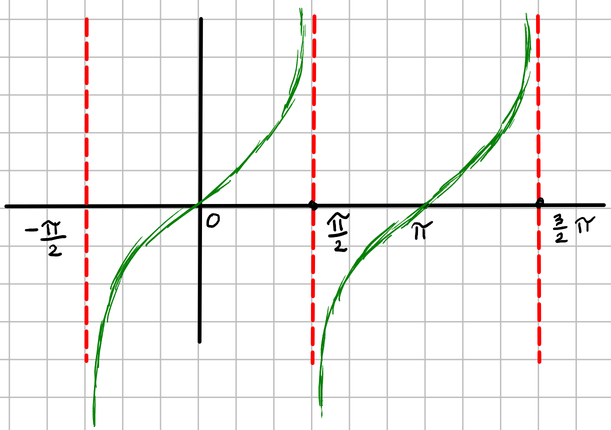
DEFINED BY $\arccos(y) = x \iff \cos(x) = y$

THE GRAPHICS OF ARCSIN AND ARCCOS :





• TANGENT : $TAN(x) = \frac{SIN(x)}{COS(x)}$ → $\cos(x) \neq 0 \Rightarrow$ $\boxed{\begin{array}{l} \forall x \in \mathbb{R} \text{ t.c} \\ x \neq \frac{\pi}{2} + K\pi \quad K \in \mathbb{Z} \end{array}}$

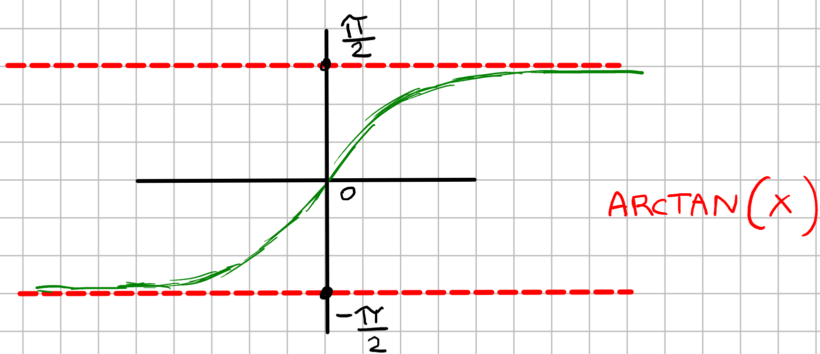


THE TANGENT FUNCTION CAN BE RESTRICT TO : $\tan|_{(-\frac{\pi}{2}, \frac{\pi}{2})}$

THE INVERSE FUNCTION OF TAN (x) IS ARCTAN (x) :

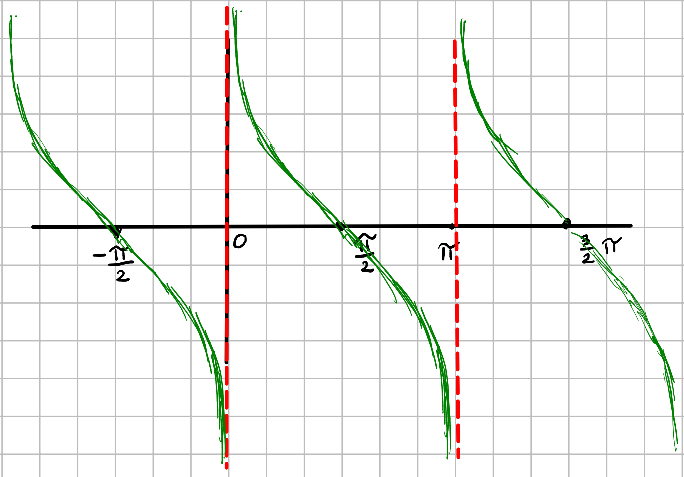
$ARCTAN(x) : \mathbb{R} \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

DEFINED BY $ARCTAN(y) = x$ me $TAN(x) = y$



- COTANGENT : $\text{COTAN}(x) = \dfrac{\cos(x)}{\sin(x)}$

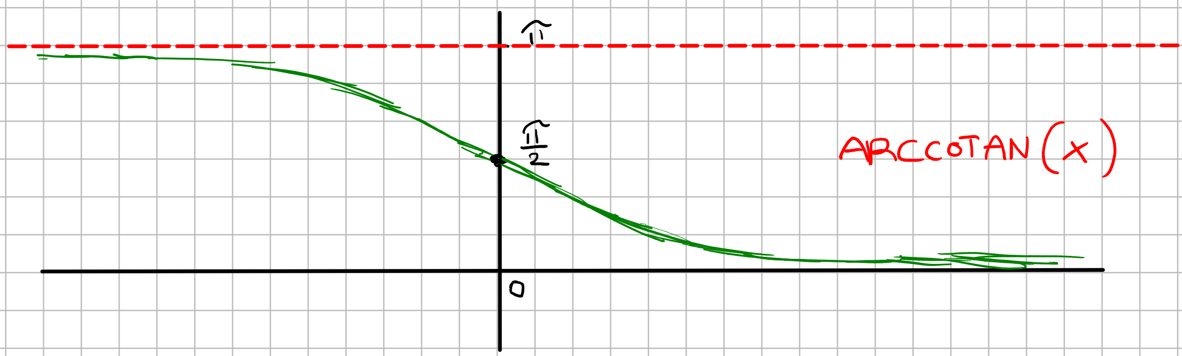
$\sin(x) \neq 0 \Rightarrow$ $\boxed{\forall x \in \mathbb{R} \text{ t.c } x \neq k\pi \quad k \in \mathbb{Z}}$



THE INVERSE FUNCTION OF COTAN IS ARCCOTAN(x) :

$\text{ARCCOTAN}(x) : \mathbb{R} \longrightarrow [0, \pi]$

DEFINED BY $\text{ARCCOTAN}(y) = x$ iff $\text{COTAN}(x) = y$



6) HYPERBOLIC FUNCTIONS :

- HYPERBOLIC SINE : $\sinh = \dfrac{e^{x} - e^{-x}}{2} \quad \forall x \in \mathbb{R}$

- HYPERBOLIC COSINE : $\cosh = \dfrac{e^{x} + e^{-x}}{2} \quad \forall x \in \mathbb{R}$

PROPERTY :

1) $\cosh(0) = 1 , \quad \sinh(0) = 0$

2) $(\cosh(x))^2 - (\sinh(x))^2 = 1 \implies \boxed{\cosh(x) \geq 1 \quad \forall x \in \mathbb{R}}$

3) $\cosh(x)$ IS EVEN

$$\cosh(-x) = \frac{e^{-x}+e^{x}}{2} = \frac{e^{x}+e^{-x}}{2} = \cosh(x)$$

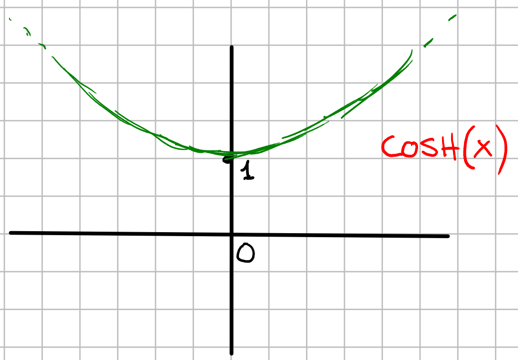
$\sinh(x)$ IS ODD

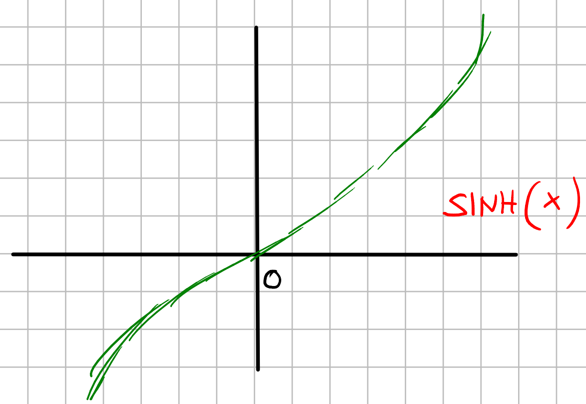
$$\sinh(-x) = \frac{e^{-x}-e^{x}}{2} = \frac{-e^{x}+e^{-x}}{2} = -\left(\frac{e^{x}-e^{-x}}{2}\right) = -\sinh(x)$$

4) $\sinh(x+y) = \sinh(x)\cdot\cosh(y) + \sinh(y)\cdot\cosh(x)$

$\cosh(x+y) = \cosh(x)\cdot\cosh(y) + \sinh(x)\cdot\sinh(y)$

5) GRAPHICS:





INVERSE HYPERBOLIC FUNCTIONS:

THE INVERSE FUNCTION OF SINH IS SETTSINH:

$$\boxed{\text{settsinh} = \log\left(y+\sqrt{y^2+1}\right)}$$

DIM:

INDEED $\sinh(x) = y$ we $\frac{e^{x}-e^{-x}}{2} = y \Rightarrow e^{x}-e^{-x} = 2y$

MULTIPLY BY e^{x}:

$$e^{2x} - e^{0} - 2ye^{x} = 0 \Rightarrow e^{2x} - 2ye^{x} - 1 = 0$$

$$t = e^{x} \Rightarrow t^2 - 2yt - 1 = 0 \qquad t_{1,2} = \frac{+2y \pm \sqrt{4y^2+4}}{2} =$$

$$= \frac{\cancel{2}\left(y \pm \sqrt{y^2+1}\right)}{\cancel{2}} = t = y \pm \sqrt{y^2+1} \Rightarrow e^{x} = y + \sqrt{y^2+1}$$

t HAS SENSE IN POSITIVE RANGE

$$\boxed{x = \log\left(y+\sqrt{y^2+1}\right)}$$

THE INVERSE FUNCTION OF $\cosh\big|_{[0,+\infty]}$ IS SETTCOSH :

$$\boxed{\text{settcosh} = \log\left(y+\sqrt{y^2-1}\right)} \quad \forall y \geq 1$$

DIM :

$$\cosh(x) = y \Rightarrow \frac{e^x+e^{-x}}{2} = y \Rightarrow e^x+e^{-x} = 2y$$

MULTIBY BY e^x :

$$e^{2x}+1-2ye^x = 0 \Rightarrow t = e^x$$

$$t^2-2yt+1 = 0 \Rightarrow t_{1,2} = y \pm\sqrt{y^2-1} \qquad y^2-1\geq 0$$

$$y \leq -1 \lor y\geq 1$$

$$e^x = y+\sqrt{y^2-1}$$

$$-1-\sqrt{0} = -1$$

$$-2-\sqrt{5} < 0$$

$$\boxed{x = \log\left(y+\sqrt{y^2-1}\right)}$$

- HYPERBOLIC TANGENT : $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

HOW TO FIND THE RIGHT DOMAIN OF A FUNCTION $f(x)$

FIND THE DOMAIN OF THIS FUNCTION : $f(x) = \sqrt{\dfrac{3^x-2}{2^x-3}}$

C.E.

$$\begin{cases} 2^x-3 \neq 0 \\ \dfrac{3^x-2}{2^x-3} \geq 0 \end{cases}$$

$$2^x-3\neq 0 \Rightarrow 2^x \neq 3 \Rightarrow \boxed{x \neq \log_2 3}$$

N: $3^x-2 \geq 0 \Rightarrow x \geq \log_3 2$

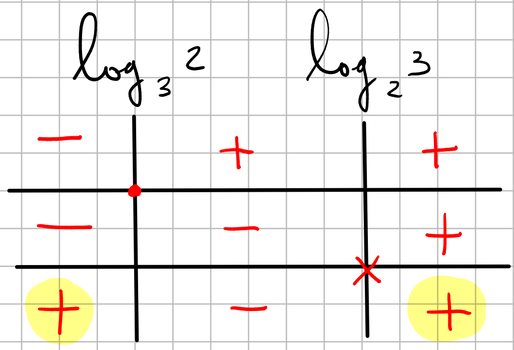
D: $2^x-3 > 0 \Rightarrow x > \log_2 3$

WHO IS GREATER BETWEEN $\log_3 2$ AND $\log_2 3$?

$$\log_3 2 = \frac{\log 2}{\log 3} < \log_2 3 = \frac{\log 3}{\log 2}$$

$N \geq 0$

$D > 0$



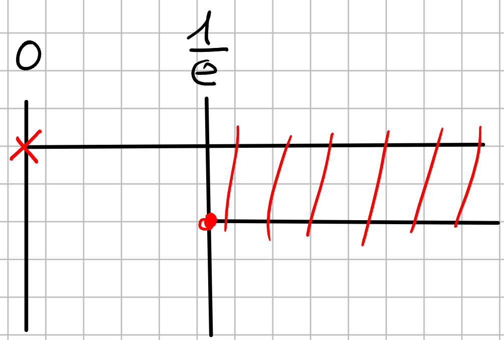
$$\boxed{x \leq \log_3 2 \ \vee \ x > \log_2 3}$$

DOMAIN : $(-\infty, \log_3 2] \cup (\log_2 3, +\infty)$

EX 2: $f(x) = \sqrt{1 + \log(x)}$

C.E.

$$\begin{cases} 1 + \log(x) \geq 0 \\ x > 0 \end{cases} \qquad \begin{cases} \log(x) \geq -1 \\ \log(x) \geq \log(e^{-1}) \end{cases} \qquad \begin{cases} \log(x) \geq \log(e^{-1}) \\ x \geq e \end{cases}$$

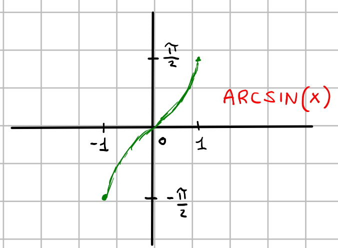


$x \geq \frac{1}{e}$

DOMAIN: $\left[\frac{1}{e}, +\infty\right)$

THE DOMAIN IS BETWEEN $[-1, 1]$

EX 3 : $f(x) = \arcsin\left(\frac{x}{x^2 - 1}\right) \longrightarrow$



C.E.

$$\begin{cases} x^2 - 1 \neq 0 \\ -1 \leq \frac{x}{x^2 - 1} \leq 1 \end{cases}$$

1 $\quad \frac{x}{x^2 - 1} \geq -1 \implies \frac{x^2 + x - 1}{x^2 - 1} \geq 0$

$N: \quad x^2 + x - 1 \geq 0 \qquad x_{1,2} = \frac{-1 \pm \sqrt{5}}{2} \qquad \boxed{x \leq \frac{-1-\sqrt{5}}{2} \ \vee \ x \geq \frac{-1+\sqrt{5}}{2}}$

$D: \quad x^2 - 1 > 0 \qquad \boxed{x < -1 \ \vee \ x > 1}$

	$\frac{-1-\sqrt{5}}{2}$		-1		$\frac{-1+\sqrt{5}}{2}$		1	
$N \geq 0$	+		−	−		+		+
$D > 0$	+		+	−		−		+
	+		−	+		−		+

$$\boxed{\begin{array}{c} x \leq \frac{-1-\sqrt{5}}{2} \ \vee \\ -1 < x \leq \frac{-1+\sqrt{5}}{2} \ \vee \\ x > 1 \end{array}}$$

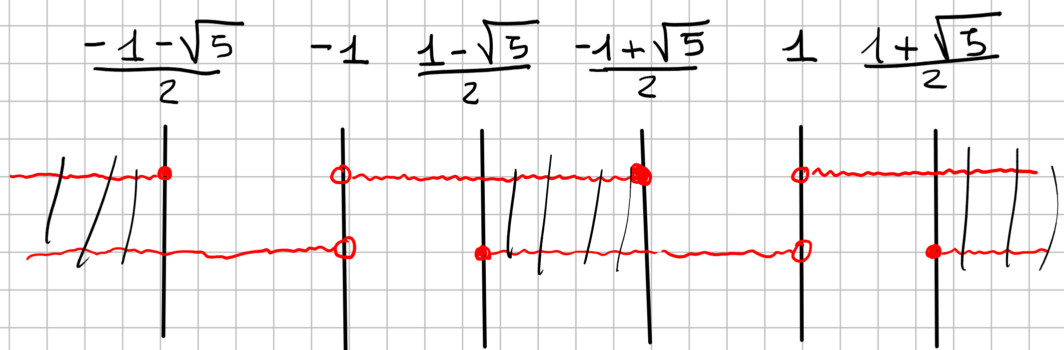
② $\frac{x}{x^2-1} \leq 1 \implies \frac{-x^2+x+1}{x^2-1} \leq 0 \implies \frac{x^2-x-1}{x^2-1} \geq 0$

$N: \quad x^2 - x - 1 \geq 0 \qquad \boxed{x \leq \frac{1-\sqrt{5}}{2} \ \vee \ x \geq \frac{1+\sqrt{5}}{2}}$

$D: \quad x^2 - 1 > 0 \qquad \boxed{x < -1 \ \vee \ x > 1}$

	-1	$\frac{1-\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	
$N \geq 0$	+	+	−	−	+
$D > 0$	+	−	−	+	+
	+	−	+	−	+

$$\boxed{\begin{array}{c} x < -1 \ \vee \\ \frac{1-\sqrt{5}}{2} \leq x < 1 \ \vee \\ x \geq \frac{1+\sqrt{5}}{2} \end{array}}$$



DOMAIN: $\left(-\infty, \frac{-1-\sqrt{5}}{2}\right] \cup \left[\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right] \cup \left[\frac{1+\sqrt{5}}{2}, +\infty\right)$