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Triangular norm-based iterative compensatory operators

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Abstract

Aggregation operators based on a fixed t-norm and on suitable transformations of processed data are introduced. A special attention is paid to the class of iterative compensatory operators containing also the classical arithmetic, geometric and harmonic means. Several properties of introduced operators are studied, e.g., the symmetry, the associativity, the idempotency, the anihilation, etc. Several examples are given, including continuous idempotent bisymmetric iterative compensatory operators with zero anihilator. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The aggregation problem is crucial in most of the advanced information-based systems. Aggregation appears in several situations, e.g., when aggregating expert's opinions, objective functions, criteria evaluations, etc. [1,6,14]. Quantities to be aggregated are defined usually on a finite real interval. Without loss of generality, we will work on the interval $[0,1]$, since a linear transformation, mapping the original interval on $[0,1]$, is easy to be found.

There are several definitions of aggregation operators. The definition used in this paper agrees with the one in [5], see also [10,11].

Definition 1. An *aggregation operator* $\mathscr{A}$ on the unit interval is a mapping $\mathscr{A}: \bigcup_{n\in\mathcal{N}} [0,1]^n \to [0,1]$ with the following properties:

(A1) if $x\in[0,1]$ then $\mathscr{A}(x)=x$;

(A2) if $(x_1,\dots,x_n),(y_1,\dots,y_n)\in[0,1]^n$, $n\in\mathcal{N}$, such that $x_i\leqslant y_i$ for each $i\in\{1,2,\dots,n\}$ then $\mathscr{A}(x_1,\dots,x_n)\leqslant\mathscr{A}(y_1,\dots,y_n)$;

(A3) $\mathscr{A}\Big(\underbrace{1,\dots,1}_{n\text{-times}}\Big)=1$ for each $n\in\mathcal{N}$;

(A4) $\mathscr{A}\Big(\underbrace{0,\dots,0}_{n\text{-times}}\Big)=0$ for each $n\in\mathcal{N}$.

Usually, besides monotonicity defined by (A2) and boundary conditions (A1), (A3), (A4), sometimes some additional properties are required, for instance, continuity, symmetry, associativity, idempotency, bisymmetry, the existence of a neutral element, etc.

Recall that an aggregation operator $\mathscr{A}$ is idempotent if for each n-tuple $(x,\dots,x)\in[0,1]^n$, $n\in\mathcal{N}$, it holds

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that

$$\mathscr{A}(x,\dots,x)=x.$$

Equivalently, idempotent aggregation operators are averaging operators, i.e., for any n-tuple $(x_1,\dots,x_n)\in[0,1]^n$ it holds

$$\min(x_1,\dots,x_n)\leqslant\mathscr{A}(x_1,\dots,x_n)\leqslant\max(x_1,\dots,x_n).$$

An aggregation operator $\mathscr{A}$ is bisymmetric if for all $(x,y,u,v)\in[0,1]^4$ it holds that

$$\mathscr{A}(\mathscr{A}(x,y),\mathscr{A}(u,v))=\mathscr{A}(\mathscr{A}(x,u),\mathscr{A}(y,v)).$$

Well-known examples of aggregation operators are, e.g., triangular norms and conorms (with their natural extensions), OWA-operators introduced by Yager [15], γ-operators of Zimmermann and Zysno [16], etc.

A very often used aggregation operator on the unit interval is the arithmetic mean $\mathscr{M}_{\mathrm{a}}$, $\mathscr{M}_{\mathrm{a}}(x_1,\dots,x_n)=(1/n)\sum_{i=1}^n x_i$, $n\in\mathscr{N}$. It is evident that $\mathscr{M}_{\mathrm{a}}$ is a symmetric, continuous, bisymmetric and idempotent aggregation operator without a neutral element. Moreover, it is not associative.

An important property of the arithmetic mean is its ability to compensate the input data. Namely, for each n-tuple $(x_1,\dots,x_n)\in[0,1]^n$ such that $\mathscr{M}_{\mathrm{a}}(x_1,\dots,x_n)\in\,]0,1[$ there exist elements $y,z\in[0,1]$ such that

$$\mathscr{M}_{\mathrm{a}}(x_1,\dots,x_n,y)<\mathscr{M}_{\mathrm{a}}(x_1,\dots,x_n)<\mathscr{M}_{\mathrm{a}}(x_1,\dots,x_n,z).$$

Indeed, it is enough to choose any value $y\in[0,\mathscr{M}_{\mathrm{a}}(x_1,\dots,x_n)[$ and $z\in\,]\mathscr{M}_{\mathrm{a}}(x_1,\dots,x_n),1]$.

Note that also other types of means as the geometric mean $\mathscr{M}_{\mathrm{g}}$, $\mathscr{M}_{\mathrm{g}}(x_1,\dots,x_n)=(\prod_{i=1}^n x_i)^{1/n}$, the harmonic mean $\mathscr{M}_{\mathrm{h}}$, $\mathscr{M}_{\mathrm{h}}(x_1,\dots,x_n)=1/(1/n)\sum_{i=1}^n 1/x_i$ or the quadratic mean $\mathscr{M}_{\mathrm{q}}$, $\mathscr{M}_{\mathrm{q}}(x_1,\dots,x_n)=((1/n)\sum_{i=1}^n x_i^2)^{1/2}$, possess a similar compensation ability.

This has been the motivation for the following definition of a compensatory operator.

Definition 2. An aggregation operator $\mathscr{A}$ on the unit interval is called a *compensatory operator* if for each n-tuple $(x_1,\dots,x_n)\in[0,1]^n$, $n\in\mathscr{N}$, such that $\mathscr{A}(x_1,\dots,x_n)\in\,]0,1[$ there exist elements $y,z\in[0,1]$ such that

$$\mathscr{A}(x_1,\dots,x_n,y)<\mathscr{A}(x_1,\dots,x_n)<\mathscr{A}(x_1,\dots,x_n,z). \tag{1}$$

Since the purpose of this paper is to introduce iterative compensatory operators based on triangular norms, we recall the definition of triangular norms and conorms and the properties needed in the following sections. For more detailed information on triangular norms and conorms we refer, e.g., to [3,4], or [2].

Definition 3. (i) A triangular norm (t-norm) is a symmetric, associative, non-decreasing function $\boldsymbol{T}:[0,1]^2\to[0,1]$ that satisfies the boundary condition $\boldsymbol{T}(x,1)=x$ for each $x\in[0,1]$.

(ii) A triangular conorm (t-conorm) is a symmetric, associative, non-decreasing function $\boldsymbol{S}:[0,1]^2\to[0,1]$ that satisfies the boundary condition $\boldsymbol{S}(x,0)=x$ for each $x\in[0,1]$.

We will use continuous t-norms and continuous Archimedean t-norms. A continuous t-norm $\boldsymbol{T}$ is *Archimedean* if $\boldsymbol{T}(x,x)<x$ for each $x\in\,]0,1[$. A t-norm $\boldsymbol{T}$ is a continuous Archimedean t-norm if and only if there exists a continuous, strictly decreasing function $f:[0,1]\to[0,\infty]$ with $f(1)=0$, such that

$$\boldsymbol{T}(x,y)=f^{-1}(\min(f(0),f(x)+f(y))),\quad x,y\in[0,1].$$

We say that f is an *additive generator of* $\boldsymbol{T}$. A continuous Archimedean t-norm is *strict* if and only if $f(0)=\infty$. In that case, the last equality can be written in the form

$$\boldsymbol{T}(x,y)=f^{-1}(f(x)+f(y)),\quad x,y\in[0,1].$$

A continuous Archimedean t-norm is *nilpotent* if and only if $f(0)\in\,]0,\infty[$.

We can easily see that neither t-norms nor t-conorms are compensatory operators. Indeed, for any given $x\in\,]0,1[$ it holds that

$$\boldsymbol{T}(x,y)\leqslant x\quad\text{and}\quad\boldsymbol{S}(x,y)\geqslant x,$$

which means that applying any t-norm $\boldsymbol{T}$, we cannot increase the value x by adding a new value y and similarly, t-conorms are not able to decrease input values. Therefore, t-norms and t-conorms are not suitable in situations where some degree of compensation is needed. To avoid this property, several operators based simultaneously on t-norms and t-conorms were proposed. For instance, exponential compensatory operators defined for any parameter $\gamma \in]0,1[$ and all n-tuples $(x_1,\ldots,x_n) \in [0,1]^n$, $n \in \mathcal{N}$ by

$$\mathscr{E}_{T,S,\gamma}(x_1,\ldots,x_n) = [T(x_1,\ldots,x_n)]^{1-\gamma}[S(x_1,\ldots,x_n)]^{\gamma}.$$

Already mentioned γ-operators [16] are special types of exponential operators for the product t-norm and the probabilistic sum as t-conorm, i.e.,

$$\Gamma_\gamma(x_1,\ldots,x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1-\gamma} \left(1 - \prod_{i=1}^{n}(1-x_i)\right)^{\gamma}.$$

Operators constructed as linear convex combinations of t-norms and t-conorms were also introduced in [13]. They are defined for any parameter $\gamma \in]0,1[$ and all $(x_1,\ldots,x_n) \in [0,1]^n$, $n \in \mathcal{N}$ by

$$\mathscr{L}_{T,S,\gamma}(x_1,\ldots,x_n) = (1-\gamma)T(x_1,\ldots,x_n) + \gamma S(x_1,\ldots,x_n).$$

For instance, for $T = T_{\mathrm{M}}$, $T_{\mathrm{M}}(x,y) = \min(x,y)$ and $S = S_{\mathrm{M}}$, $S_{\mathrm{M}}(x,y) = \max(x,y)$, we have

$$\mathscr{L}_{T_{\mathrm{M}},S_{\mathrm{M}},\gamma}(x_1,\ldots,x_n) = (1-\gamma)\min(x_1,\ldots,x_n) + \gamma\max(x_1,\ldots,x_n),$$

and for $T = T_{\mathrm{M}}$ and $S = S_{\mathrm{L}}$, where $S_{\mathrm{L}}(x,y) = \min(1, x+y)$, we obtain

$$\mathscr{L}_{T_{\mathrm{M}},S_{\mathrm{L}},\gamma}(x_1,\ldots,x_n) = (1-\gamma)\min(x_1,\ldots,x_n) + \gamma\min\left(1, \sum_{i=1}^{n} x_i\right),$$

see [8].

For an overview of compensatory operators based on t-norms and t-conorms we recommend [9–11].

If we apply compensatory operators mentioned till now, we always aggregate in the first place the input values by means of a t-norm and t-conorm and then combine these partial results in some suitable way. Consequently, the compensation property of these operators fails whenever we reach 0 as the corresponding t-norm output or 1 as the corresponding t-conorm output.

Recently, Moser et al. [12] have proposed another type of operators (only as binary operators). They are defined for any parameter $\lambda \in [0,1]$ and all $(x,y) \in [0,1]^2$ by

$$O_{T,S,\lambda}(x,y) = T(S(x,1-\lambda), S(y,\lambda)),$$
$$O_{S,T,\lambda}(x,y) = S(T(x,1-\lambda), T(y,\lambda)). \tag{2}$$

Note that $O_{S,T,\lambda}$ is only a dual operator to $O_{T,S,1-\lambda}$.

A binary operator $O_{T,S,\lambda}$ fulfills properties (A2) and (A3). However, the boundary condition (A4) is fulfilled if and only if $T(1-\lambda,\lambda) = 0$. This is true for all $\lambda \in [0,1]$ if and only if T is a nilpotent t-norm with a symmetric additive generator f, $f(x) + f(1-x) = f(0)$ for all $x \in [0,1]$. As an example of such t-norm, the Lukasiewicz t-norm T_{L} defined by $T_{\mathrm{L}}(x,y) = \max(0, x+y-1)$ can be mentioned. Then the compensation property (1) holds whenever

$$S(x,1-\lambda) < S_{\mathrm{L}}(x,1-\lambda) = \min(1, x+1-\lambda), \quad x \in]0,1[.$$

This is fulfilled, e.g., for the probabilistic sum $S = S_{\mathrm{P}}$, $S_{\mathrm{P}}(x,y) = x+y-xy$.

Notice that operators $O_{T,S,\lambda}$ transform in the first place inputs by means of a t-conorm S and then aggregate the transformed inputs by means of a t-norm T.

In what follows, we introduce a new wide class of compensatory operators based on t-norms which are defined in a similar way as in (2). Since the dual operator $\mathscr{A}'$ to a compensatory operator $\mathscr{A}$:

$$\mathscr{A}'(x_1,\ldots,x_n) = 1 - \mathscr{A}(1-x_1,\ldots,1-x_n)$$

is again a compensatory operator, this class will cover by duality also t-conorm-based compensatory operators.

2. Triangular norm-based iterative compensatory operators

Definition 4. Let T be a continuous t-norm and $\mathscr{F} = \{f_{n,i} : [0,1] \to [0,1];\ n \in \mathcal{N},\ i \in \{1,2\}\}$ be a system of continuous, non-decreasing functions such that

$$f_{n,i}(1) = 1 \quad \text{and} \quad T(f_{n,1}(0), f_{n,2}(0)) = 0.$$

We define an operator $A_{T,\mathscr{F}}: \bigcup_{n\in\mathscr{N}} [0,1]^n \to [0,1]$ in the following way:

(i) if $x\in[0,1]$ then $A_{T,\mathscr{F}}(x)=x$;

(ii) if $n \geqslant 2$ and $(x_1,\dots,x_n) \in [0,1]^n$ then $A_{T,\mathscr{F}}(x_1,\dots,x_n)=\boldsymbol{T}(f_{n-1,1}(A_{T,\mathscr{F}}(x_1,\dots,x_{n-1})), f_{n-1,2}(x_n))$.

The operator $A_{T,\mathscr{F}}$ will be called an iterative aggregation operator.

Proposition 1. *The operator $A_{T,\mathscr{F}}$ introduced in Definition 4, is a continuous aggregation operator.*

Proof. It is enough to show that the operator $A_{T,\mathscr{F}}$ fulfills the properties (A1)–(A4) stated in Definition 1. The property (A1) is evident, the monotonicity of $A_{T,\mathscr{F}}$ follows from the monotonicity of functions $f_{n,i}$ and the t-norm $\boldsymbol{T}$.

Since $f_{n,i}(1)=1$, we obtain

$$A_{T,\mathscr{F}}(1,1)=\boldsymbol{T}(f_{1,1}(1),f_{1,2}(1))=1$$

and next, proceeding by induction, it can be proved that $A_{T,\mathscr{F}}\underbrace{(1,\dots,1)}_{n\text{-times}}=1$ for each $n\in\mathscr{N}$.

In a similar way, using the property $\boldsymbol{T}(f_{n,1}(0), f_{n,2}(0))=0$, we can see that the boundary condition $A_{T,\mathscr{F}}\underbrace{(0,\dots,0)}_{n\text{-times}}=0$ also holds for each $n\in\mathscr{N}$. □

It can be easily seen that any continuous t-norm $\boldsymbol{T}$ can be understood as an $A_{T,\mathscr{F}}$ operator, namely for $f_{n,i}(x)=x$ for each $n\in\mathscr{N}$, $i\in\{1,2\}$ and $x\in[0,1]$.

An iterative aggregation operator having the property (1) from Definition 2 will be called an *iterative compensatory operator.*

Proposition 2. *The operator $A_{T,\mathscr{F}}$ introduced in Definition 4 is an iterative compensatory operator if and only if for each $n\in\mathscr{N}$*

$$f_{n,1}(x)>x \quad \textit{for each } x\in]0,1[\tag{3}$$

and

$$\boldsymbol{T}(f_{n,1}(x),f_{n,2}(0))<x \quad \textit{for each } x\in]0,1[. \tag{4}$$

Proof. For simplicity, let us denote $A_{T,\mathscr{F}}=A$. The aggregation iterative operator $\mathscr{A}$ is an iterative compensatory operator if and only if Eq. (1) in Definition 2 is fulfilled. The sufficiency of conditions (3) and (4) for satisfying Eq. (1) is evident. It is enough to choose $z=1$ and $y=0$, and we obtain

$$\begin{aligned} A(x_1,\dots,x_n,1)&=\boldsymbol{T}(f_{n,1}(\mathscr{A}(x_1,\dots,x_n)), f_{n,2}(1))\\ &=f_{n,1}(\mathscr{A}(x_1,\dots,x_n))>\mathscr{A}(x_1,\dots,x_n).\end{aligned}$$

In a similar way, using the condition (4), we have $\mathscr{A}(x_1,\dots,x_n,0)<\mathscr{A}(x_1,\dots,x_n)$.

Let $(x_1,\dots,x_n)\in[0,1]^n$, $n\in\mathscr{N}$ be any n-tuple for which $\mathscr{A}(x_1,\dots,x_n)=u\in]0,1[$. If we want Eq. (1) to hold for some $z\in[0,1]$, from

$$u<\mathscr{A}(x_1,\dots,x_n,z)=\boldsymbol{T}(f_{n,1}(u),f_{n,2}(z))\leqslant f_{n,1}(u),$$

we obtain the necessary condition

$$f_{n,1}(u)>u \quad \text{for each } u\in]0,1[.$$

If we want Eq. (1) to hold for some $y\in[0,1]$, the relations

$$\begin{aligned} u&>\mathscr{A}(x_1,\dots,x_n,y)=\boldsymbol{T}(f_{n,1}(u),f_{n,2}(y))\\ &\geqslant \boldsymbol{T}(f_{n,1}(u),f_{n,2}(0))\end{aligned}$$

give the necessary condition $\boldsymbol{T}(f_{n,1}(u),f_{n,2}(0))<u$ for each $u\in]0,1[$.

Remark 1. (1) The condition $\boldsymbol{T}(f_{n,1}(x),f_{n,2}(0))<x$, $x\in]0,1[$, is immediately fulfilled if we use functions $f_{n,2}$ with $f_{n,2}(0)=0$ for each $n\in\mathscr{N}$.

(2) If $f_{n,1}=f_1$ and $f_{n,2}=f_2$ for each $n\in\mathscr{N}$, then the operator $A_{T,\mathscr{F}}$ is a natural extension of a binary operator $\mathscr{A}$, $\mathscr{A}(x,y)=\boldsymbol{T}(f_1(x),f_2(y))$ to an n-ary operator, i.e.,

$$A_{T,\mathscr{F}}(x_1,x_2)=\mathscr{A}(x_1,x_2)$$

and

$$A_{T,\mathscr{F}}(x_1,\dots,x_n)=\mathscr{A}(A_{T,\mathscr{F}}(x_1,\dots,x_n-1),x_n).$$

(3) For a given t-conorm $\boldsymbol{S}$ and $\lambda\in[0,1]$, put

$$f_{n,1}(x)=\boldsymbol{S}(x,1-\lambda), \qquad f_{n,2}(x)=\boldsymbol{S}(x,\lambda),$$
$$n\in\mathscr{N},\quad x\in[0,1].$$

Then

$$A_{T,\mathscr{F}}(x,y)=O_{T,S,\lambda}(x,y),$$

where $O_{T,S,\lambda}$ is an operator introduced by Moser et al. [12]; see expression (2).

Example 1. Let $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{M}}$ and $f_{n,1}(x) = \sqrt{x}$, $f_{n,2}(x) = x$ for each $n \in \mathcal{N}$ and $x \in [0,1]$. Then $A_{T_{\mathrm{M}},\mathcal{F}}$ is an iterative compensatory operator, which is non-symmetric. Moreover,

$$A_{T_{\mathrm{M}},\mathcal{F}}(x_1,\ldots,x_n) = \min(x_1^{1/2^{n-1}}, x_2^{1/2^{n-2}}, \ldots, x_{n-1}^{1/2}, x_n).$$

We can easily verify that all functions $f_{n,i}$ satisfy properties required in Definition 4, and sufficient conditions (3) and (4) for being an iterative compensatory operator are also fulfilled.

Example 2. Let $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{L}}$ be the Lukasiewicz t-norm. Put

$$f_{n,1}(x) = \frac{x+1}{2} \quad \text{and} \quad f_{n,2}(x) = \frac{3x+1}{4}$$
for each $n \in \mathcal{N}$ and $x \in [0,1]$.

Then $A_{T_{\mathrm{L}},\mathcal{F}}$ is an iterative compensatory operator.

All functions $f_{n,i}$ are continuous, increasing, $f_{n,1}(1) = 1$ and $\boldsymbol{T}_{\mathrm{L}}(f_{n,1}(0),\ f_{n,2}(0)) = \max(0, -\frac{1}{4}) = 0$. Moreover, $f_{n,1}(x) > x$ for each $x \in]0,1[$ and $\boldsymbol{T}_{\mathrm{L}}(f_{n,1}(x), f_{n,2}(0)) = \max(0, (2x-1)/4) < x$ for each $x \in]0,1]$. The assertion follows from Propositions 1 and 2.

For each $(x,y) \in [0,1]^2$ we obtain a binary operator $\mathcal{A}(x,y) = A_{T_{\mathrm{L}},\mathcal{F}}(x,y) = \max(0, (2x+3y-1)/4)$. Note that the natural extension of $\mathcal{A}$ to an n-ary operator gives just $A_{T_{\mathrm{L}},\mathcal{F}}$ (Remark 1 (2)):

$$A_{T_{\mathrm{L}},\mathcal{F}}(x,y,z) = \max\left(0, \frac{2x+3y+6z-3}{8}\right) = \mathcal{A}(\mathcal{A}(x,y),z)$$

for all $x,y,z \in [0,1]$, etc. hold.

Example 3. Let $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{Ho}}$ be the Hamacher product,

$$\boldsymbol{T}_{\mathrm{Ho}}(x,y) = \frac{xy}{x+y-xy} \quad \text{for } (x,y) \neq (0,0).$$

Let $f_{n,1}(x) = 3x/(x+2)$ and $f_{n,2}(x) = 3x/(2x+1)$ for each $n \in \mathcal{N}$ and $x \in [0,1]$. Then $A_{T_{\mathrm{Ho}},\mathcal{F}}$ is an iterative compensatory operator. Moreover, Remark 1(2) can be applied to an effective output computation.

Note that the operator $A_{T_{\mathrm{Ho}},\mathcal{F}}$ is a kind of weighted harmonic mean. We have

$$A_{T_{\mathrm{Ho}},\mathcal{F}}(x,y) = \boldsymbol{T}_{\mathrm{Ho}}(f_{1,1}(x), f_{1,2}(y)) = \frac{1}{(2/3x) + (1/3y)}, \quad x \neq 0,\ y \neq 0,$$

$$A_{T_{\mathrm{Ho}},\mathcal{F}}(x,y,z) = \frac{1}{(4/9x) + (2/9y) + (1/3z)},$$
$x \neq 0,\ y \neq 0,\ z \neq 0,$

etc.

Remark 2. There is another similar way to define compensatory operators, namely,

$$\mathcal{A}(x_1,\ldots,x_n) = \boldsymbol{T}(f_{n,1}(x_1), f_{n,2}(x_2), \ldots, f_{n,n}(x_n)),$$

where $f_{n,i}$, $n \in \mathcal{N}$, $i \in \{1,2,\ldots,n\}$, are suitable functions. We will not develop this point here.

Lemma 1. *Let $\boldsymbol{T}$ be a continuous Archimedean t-norm with additive generator f. Let $\mathcal{F} = \{f_{n,i} : [0,1] \to [0,1];\ n \in \mathcal{N},\ i \in \{1,2\}\}$ be a system of functions such that*

$$\begin{aligned} f_{n,1}(x) &= f^{-1}(\lambda_n f(x)), \\ f_{n,2}(x) &= f^{-1}((1-\lambda_n) f(x)), \end{aligned} \tag{5}$$

where $\lambda_n \in]0,1[$ are given parameters. Then $A_{T,\mathcal{F}}$ is a continuous iterative compensatory operator.

Proof. It can be easily shown that all functions $f_{n,i}$ are continuous and strictly increasing. Next, for each $n \in \mathcal{N}$ it holds that

$$f_{n,1}(1) = f^{-1}(\lambda_n f(1)) = f^{-1}(0) = 1$$

and similarly,

$$f_{n,2}(1) = 1.$$

For being an iterative aggregation operator, it is already enough to satisfy the condition $\boldsymbol{T}(f_{n,1}(0), f_{n,2}(0)) = 0$.

If $\boldsymbol{T}$ is a strict t-norm, i.e., $f(0) = \infty$, then $f_{n,1}(0) = f^{-1}(\infty) = 0$ and the same holds for $f_{n,2}(0)$. Hence, $\boldsymbol{T}(f_{n,1}(0), f_{n,2}(0)) = 0$.

For nilpotent t-norms we have $f(0) \in]0, \infty[$, therefore

$$\begin{aligned}
&\boldsymbol{T}(f_{n,1}(0), f_{n,2}(0)) \\
&\quad = f^{-1}(\min(f(0), f \circ f^{-1}(\lambda_n f(0)) \\
&\qquad\qquad + f \circ f^{-1}((1-\lambda_n) f(0)))) \\
&\quad = f^{-1}(f(0)) = 0.
\end{aligned}$$

This means that $A_{T,\mathscr{F}}$ is an iterative aggregation operator.

To prove that $A_{T,\mathscr{F}}$ is an iterative compensatory operator it is enough to show that conditions (3) and (4) are fulfilled.

Since $\lambda_n \in]0,1[$, for each $x \in]0,1[$ and $n \in \mathscr{N}$ we have $\lambda_n f(x) < f(x)$. The function f^{-1} is strictly decreasing, hence

$$f_{n,1}(x) = f^{-1}(\lambda_n f(x)) > x.$$

If $\boldsymbol{T}$ is a strict t-norm, then $f_{n,2}(0) = 0$ and therefore $\boldsymbol{T}(f_{n,1}(x), f_{n,2}(0)) < x$ for each $x \in]0,1]$ and $n \in \mathscr{N}$.

For nilpotent t-norms we have

$$\begin{aligned}
&\boldsymbol{T}(f_{n,1}(x), f_{n,2}(0)) \\
&\quad = f^{-1}(\min(f(0), f \circ f_{n,1}(x) + f \circ f_{n,2}(0))) \\
&\quad = f^{-1}(\min(f(0), \lambda_n f(x) + (1-\lambda_n) f(0))) \\
&\quad = f^{-1}(\lambda_n f(x) + (1-\lambda_n) f(0)) \\
&\quad < f^{-1}(f(x)) = x. \qquad \square
\end{aligned}$$

Example 4. The operator $A_{T,\mathscr{F}}$ defined as in Lemma 1 for $\lambda_n = n/(n+1)$, $n \in \mathscr{N}$, is

(1) the arithmetic mean if $\boldsymbol{T}$ is the Lukasiewicz t-norm $\boldsymbol{T}_{\mathrm{L}}$,

(2) the geometric mean if $\boldsymbol{T}$ is the product t-norm $\boldsymbol{T}_{\mathrm{P}}$,

(3) the harmonic mean if $\boldsymbol{T}$ is the Hamacher product $\boldsymbol{T}_{\mathrm{Ho}}$.

(1) The normed additive generator of the t-norm $\boldsymbol{T}_{\mathrm{L}}$ is the function f_{L}, $f_{\mathrm{L}}(x) = 1 - x$, $x \in [0,1]$. Therefore

$$f_{1,1}(x) = 1 - \frac{1}{2}(1-x) = \frac{1+x}{2}$$

and the same holds for $f_{1,2}$. Hence

$$\begin{aligned}
A_{T_{\mathrm{L}},\mathscr{F}}(x_1, x_2) &= \max\left(0, \frac{1+x_1}{2} + \frac{1+x_2}{2} - 1\right) \\
&= \frac{x_1 + x_2}{2}
\end{aligned}$$

for each $(x_1, x_2) \in [0,1]^2$.

By induction, we can deduce that the equality

$$A_{T_{\mathrm{L}},\mathscr{F}}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

holds for each $(x_1, \ldots, x_n) \in [0,1]^n$, $n \in \mathscr{N}$.

(2) An additive generator of the product t-norm $\boldsymbol{T}_{\mathrm{P}}$ is the function f_{P}, $f_{\mathrm{P}}(x) = -\log(x)$, $x \in [0,1]$ ($f(0) = \infty$), hence

$$f_{1,1}(x) = f_{1,2}(x) = \mathrm{e}^{1/2 \log x} = \sqrt{x}.$$

Therefore for each $(x_1, x_2) \in [0,1]^2$ we obtain

$$A_{T_{\mathrm{P}},\mathscr{F}}(x_1, x_2) = \sqrt{x_1 x_2},$$

and again, by induction, it can be proved that the formula

$$A_{T_{\mathrm{P}},\mathscr{F}}(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

holds for all $(x_1, \ldots, x_n) \in [0,1]^n$, $n \in \mathscr{N}$.

(3) The Hamacher t-norm $\boldsymbol{T}_{\mathrm{Ho}}$ has an additive generator f_{Ho}, $f_{\mathrm{Ho}}(x) = (1-x)/x$, $x \in [0,1]$, ($f(0) = \infty$). Therefore

$$f_{1,1}(x) = f_{\mathrm{Ho}}^{-1}\left(\frac{1}{2} f_{\mathrm{Ho}}(x)\right) = \frac{2x}{1+x}$$

and the same holds for $f_{1,2}(x)$. Hence

$$A_{T_{\mathrm{Ho}},\mathscr{F}}(x_1, x_2) = \frac{2x_1 x_2}{x_1 + x_2} = \frac{1}{1/2((1/x_1) + (1/x_2))},$$

for $x_1 \neq 0$, $x_2 \neq 0$. The desired result for $n > 2$ can be obtained by induction.

3. Properties of t-norm-based iterative compensatory operators $A_{T,\mathscr{F}}$

3.1. Associativity

The first property of iterative compensatory operators $A_{T,\mathscr{F}}$ introduced in the previous section, that will be discussed, is associativity. We prove the following assertion.

Proposition 3. *The iterative compensatory operators $A_{T,\mathscr{F}}$ are never associative.*

Proof. Suppose that $A_{T,\mathscr{F}}$ is an associative iterative compensatory operator. For any $x, y, z \in [0,1]$ we have

$$\begin{aligned} &A_{T,\mathscr{F}}(x, A_{T,\mathscr{F}}(y,z)) \\ &\quad = \boldsymbol{T}(f_{1,1}(x), f_{1,2}(A_{T,\mathscr{F}}(y,z))) \\ &\quad = \boldsymbol{T}(f_{1,1}(x), f_{1,2}(\boldsymbol{T}(f_{1,1}(y), f_{1,2}(z)))) \end{aligned}$$

and

$$\begin{aligned} &A_{T,\mathscr{F}}(A_{T,\mathscr{F}}(x,y), z) \\ &\quad = \boldsymbol{T}(f_{1,1}(A_{T,\mathscr{F}}(x,y)), f_{1,2}(z)) \\ &\quad = \boldsymbol{T}(f_{1,1}(\boldsymbol{T}(f_{1,1}(x), f_{1,2}(y))), f_{1,2}(z)). \end{aligned}$$

Put $y=1$. Using the property $f_{n,i}(1)=1$ and the boundary conditions for $\boldsymbol{T}$, we obtain

$$A_{T,\mathscr{F}}(x, A_{T,\mathscr{F}}(y,z)) = \boldsymbol{T}(f_{1,1}(x), f_{1,2}(f_{1,2}(z)))$$

and

$$A_{T,\mathscr{F}}(A_{T,\mathscr{F}}(x,y), z) = \boldsymbol{T}(f_{1,1}(f_{1,1}(x)), f_{1,2}(z)).$$

For $x=1$, the supposed associativity yields

$$f_{1,2}(f_{1,2}(z)) = f_{1,2}(z)$$

for each $z \in [0,1]$, i.e., $f_{1,2} = \mathrm{id}$, and in the same way, for $z=1$, we obtain $f_{1,1} = \mathrm{id}$, which contradicts property (3), $f_{1,1}(x) > x$ for each $x \in]0,1[$. □

3.2. Idempotency

The next property we will deal with, is the idempotency of the operator $A_{T,\mathscr{F}}$. We show that for any continuous t-norm $\boldsymbol{T}$ there exists a system $\mathscr{F}$ such that the corresponding iterative compensatory operator $A_{T,\mathscr{F}}$ is idempotent. We divide the proof of this claim into three steps.

We start with the case $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{M}}$. $\boldsymbol{T}_{\mathrm{M}}$ is a continuous, but not Archimedean t-norm. It has no additive generator.

Proposition 4. *Let $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{M}}$. The iterative compensatory operator $A_{T_{\mathrm{M}},\mathscr{F}}$ is idempotent if and only if $f_{n,2}(x) = x$ for each $n \in \mathscr{N}$, $x \in [0,1]$.*

Proof. Let $A_{T_{\mathrm{M}},\mathscr{F}}$ be an idempotent iterative compensatory operator. Then, for each n-tuple $(x,\dots,x) \in [0,1]^n$, $n \in \mathscr{N}$ it holds that $A_{T_{\mathrm{M}},\mathscr{F}}(x,\dots,x) = x$. Hence

$$x = A_{T_{\mathrm{M}},\mathscr{F}}(\underbrace{x,\dots,x}_{n+1}) = \min(f_{n,1}(x), f_{n,2}(x)).$$

Since $f_{n,1}(x) > x$ for each $n \in \mathscr{N}$, $x \in]0,1[$

$$f_{n,2}(x) = x$$

has to hold for each $n \in \mathscr{N}$, $x \in]0,1[$. The result for all $x \in [0,1]$ follows from the monotonicity of functions $f_{n,2}$.

The condition $f_{n,2}(x) = x$ for all $n \in \mathscr{N}$ and $x \in [0,1]$ is sufficient for an iterative compensatory operator to be idempotent. Namely,

$$A_{T_{\mathrm{M}},\mathscr{F}}(x,x) = \min(f_{1,1}(x), f_{1,2}(x)) = x,$$

because $f_{1,2}(x) = x$ and $f_{1,1}(x) > x$ for $x \in]0,1[$ implies $f_{1,1}(x) \geqslant x$ for $x \in [0,1]$.

The claim for any n-tuple $(x_1,\dots,x_n) \in [0,1]^n$, $n \in \mathscr{N}$ can be proved by induction. □

Proposition 5. *Let $\boldsymbol{T}$ be an Archimedean t-norm with additive generator f. The iterative compensatory operator $A_{T,\mathscr{F}}$ is idempotent if and only if for each $n \in \mathscr{N}$*

$$f = f \circ f_{n,1} + f \circ f_{n,2}. \tag{6}$$

Proof. Let $\boldsymbol{T}$ be an Archimedean t-norm with additive generator f and let Eq. (6) hold. Since f is strictly decreasing, $f(0) > f(x)$ for all $x \in]0,1[$. Hence

$$\begin{aligned} &A_{T,\mathscr{F}}(x,x) \\ &\quad = \boldsymbol{T}(f_{1,1}(x), f_{1,2}(x)) \\ &\quad = f^{-1}(\min(f(0), f \circ f_{1,1}(x) + f \circ f_{1,2}(x))) \\ &\quad = f^{-1}(\min(f(0), f(x))) = x. \end{aligned}$$

By induction, using the same arguments, it can be proved that Eq. (6) is the sufficient condition for the idempotency of $A_{T,\mathscr{F}}$.

If we suppose that $A_{T,\mathscr{F}}$, for a continuous Archimedean t-norm $\boldsymbol{T}$, is an idempotent iterative compensatory operator, we obtain

$$x = A_{T,\mathscr{F}}(\underbrace{x,\ldots,x}_{n+1}) = \boldsymbol{T}(f_{n,1}(x), f_{n,2}(x))$$

$$= f^{-1}(\min(f(0), f\circ f_{n,1}(x) + f\circ f_{n,2}(x))),$$

which means that

$$f(x) = \min(f(0), f\circ f_{n,1}(x) + f\circ f_{n,2}(x)).$$

The additive generator f is strictly decreasing on [0, 1], hence

$$f(x) = f\circ f_{n,1}(x) + f\circ f_{n,2}(x)$$

for each $x\in\,]0,1]$. The continuity of f ensures validity for all $x\in[0,1]$. □

Corollary 1. *Iterative compensatory operators introduced in Lemma* 1 *are idempotent iterative compensatory operators.*

Proof. By Eq. (5), it holds that $f\circ f_{n,1}(x) = \lambda_n f(x)$ and $f\circ f_{n,2}(x) = (1-\lambda_n)f(x)$. Hence

$$f(x) = f\circ f_{n,1}(x) + f\circ f_{n,2}(x)$$

for each $n\in\mathscr{N}$ and $x\in[0,1]$. Now, the assertion is a consequence of Proposition 5. □

Remark 3. By Corollary 1, the operators $A_{T,\mathscr{F}}$ defined as in Lemma 1 for t-norms $\boldsymbol{T}_{\mathrm{L}}, \boldsymbol{T}_{\mathrm{P}}, \boldsymbol{T}_{\mathrm{Ho}}$ and parameters $\lambda_n = n/(n+1)$ (Example 4), i.e., the arithmetic, geometric and harmonic means, are idempotent iterative compensatory operators.

Now, we will be interested in the idempotency of iterative compensatory operators $A_{T,\mathscr{F}}$ that are based on a continuous t-norm $\boldsymbol{T}$ which is neither $\boldsymbol{T}_{\mathrm{M}}$ nor an Archimedean t-norm.

In general, any continuous t-norm $\boldsymbol{T}$ can be expressed as an ordinal sum whose summands are continuous Archimedean t-norms $\boldsymbol{T}_k$, $\boldsymbol{T}\approx(\langle a_k, b_k, \boldsymbol{T}_k\rangle)_{k\in K}$, where K is a countable set and $\{]a_k,b_k[\}_{k\in K}$ is a system of open disjoint subintervals of [0, 1], whereby a_k, b_k are the idempotent elements of $\boldsymbol{T}$. In that case, the values of $\boldsymbol{T}$ are computed by

$$\boldsymbol{T}(x,y) = \begin{cases} f_k^{-1}(\min(f_k(a_k), f_k(x)+f_k(y))) & \text{if } (x,y)\in\,]a_k,b_k[^2 \text{ for some } k\in K,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{7}$$

where $f_k : [a_k, b_k] \to [0,\infty]$ are continuous strictly decreasing functions with $f_k(b_k) = 0$. For more information we refer to [4,7].

Proposition 6. *Let $\boldsymbol{T}$ be a continuous non Archimedean t-norm, $\boldsymbol{T}\neq\boldsymbol{T}_{\mathrm{M}}$ and let $\boldsymbol{T}\approx(\langle a_k,b_k,\boldsymbol{T}_k\rangle)_{k\in K}$. Let f_k, $k\in K$, be the generating functions from Eq.* (7). *The iterative compensatory operator $A_{T,\mathscr{F}}$ is idempotent if and only if for each $n\in\mathscr{N}$* and $x\in[0,1]$

$$f_k(x) = f_k\circ f_{n,1}(x) + f_k\circ f_{n,2}(x)$$

whenever $x\in\,]a_k,b_k[$ and $f_{n,1}(x)<b_k$ for some $k\in K$ and $f_{n,2}(x)=x$ in all other cases.

Proof. The operator $A_{T,\mathscr{F}}$ is idempotent if and only if for each $(n+1)$-tuple $(x,\ldots,x)\in[0,1]^{n+1}$, $n\in\mathscr{N}$,

$$x = A_{T,\mathscr{F}}(\underbrace{x,\ldots,x}_{n+1}) = \boldsymbol{T}(f_{n,1}(x), f_{n,2}(x))$$

holds. Due to Eq. (7) this holds if and only if either

$$x = f_k^{-1}(\min(f_k(a_k), f_k\circ f_{n,1}(x) + f_k\circ f_{n,2}(x))) \tag{8}$$

or

$$x = \min(f_{n,1}(x), f_{n,2}(x)). \tag{9}$$

Equality (8) together with conditions $f_{n,i}(x)\in\,]a_k,b_k[$, $x\in\,]a_k,b_k[$ for some $k\in K$ lead to the condition

$$f_k(x) = f_k\circ f_{n,1}(x) + f_k\circ f_{n,2}(x)$$

whenever $x\in\,]a_k,b_k[$ and $f_{n,1}(x)<b_k$. In other cases, since $f_{n,1}(x)>x$, we obtain from Eq. (9), $f_{n,2}(x)=x$. □

3.3. Symmetry

Now, we will look into the conditions under which the iterative compensatory operator $A_{T,\mathcal{F}}$ is symmetric. Recall that an aggregation operator $\mathcal{A}$ on the unit interval is symmetric if for each n-tuple $(x_1,\ldots,x_n)\in[0,1]^n$, $n\in\mathcal{N}$, it holds that

$$\mathcal{A}(x_1,\ldots,x_n)=\mathcal{A}(x_{i_1},\ldots,x_{i_n}),$$

where $(i_1,\ldots,i_n)$ is any permutation of the n-tuple $(1,\ldots,n)$.

Especially, for iterative compensatory operators $A_{T,\mathcal{F}}$ we can prove the following results.

Proposition 7. *The iterative compensatory operator $A_{T_M,\mathcal{F}}$ is symmetric if and only if for each $n\in\mathcal{N}$,*

$$f_{n,2}=f_{n,1}\circ f_{n-1,1}\circ f_{n-2,1}\circ\cdots\circ f_{1,1}$$

holds.

Proof. Consider the binary operator $A_{T_M,\mathcal{F}}$. It is evident that the equality

$$A_{T_M,\mathcal{F}}(x_1,x_2)=A_{T_M,\mathcal{F}}(x_2,x_1)$$

holds for each $(x_1,x_2)\in[0,1]^2$ if and only if $f_{1,2}=f_{1,1}$.

Now, let (x_1,x_2,x_3) be any triplet from $[0,1]^3$. Then we have

$$A_{T_M,\mathcal{F}}(x_1,x_2,x_3)$$
$$=\min(f_{2,1}\circ f_{1,1}(x_1), f_{2,1}\circ f_{1,2}(x_2), f_{2,2}(x_3)).$$

The requirement that the value $A_{T_M,\mathcal{F}}$ for inputs x_1,x_2,x_3 cannot depend on the order of inputs, leads to the condition

$$f_{2,1}\circ f_{1,1}=f_{2,1}\circ f_{1,2}=f_{2,2}$$

for each x, and next, to the conditions

$$f_{1,2}=f_{1,1}\quad\text{and}\quad f_{2,2}=f_{2,1}\circ f_{1,1}.$$

In general, the necessary and sufficient condition for symmetry of $A_{T_M,\mathcal{F}}$ is

$$f_{n,2}=f_{n,1}\circ f_{n-1,1}\circ f_{n-2,1}\circ\cdots\circ f_{1,1}$$

for each $n\in\mathcal{N}$. The complete proof can be done by induction. □

Corollary 2. *There is no symmetric idempotent iterative compensatory operator $A_{T_M,\mathcal{F}}$.*

Proof. The idempotency of $A_{T_M,\mathcal{F}}$ demands $f_{n,2}(x)=x$ for all $n\in\mathcal{N}$ and $x\in[0,1]$. Hence, for $n=1$, we have $f_{1,2}(x)=x$.

Simultaneously, from Proposition 7, we obtain $f_{1,2}(x)=f_{1,1}(x)$. This contradicts the property $f_{1,1}(x)>x$. □

Proposition 8. *The iterative compensatory operator $A_{T,\mathcal{F}}$ based on a continuous Archimedean t-norm $\boldsymbol{T}$ with additive generator f is symmetric if and only if*

$$f_{n,2}=f_{n,1}\circ f_{n-1,1}\circ f_{n-2,1}\circ\cdots\circ f_{1,1}$$
for each $n\in\mathcal{N}$

and

$$f\circ f_{n,1}=\lambda_n f\quad\textit{for each } n\geqslant 2$$

and any parameter $\lambda_n\in\,]0,1[$, whereby $f\circ f_{n,2}(0)\leqslant f(0)/(n+1)$.

The proof is a modification of the preceding one, we omit the details.

Let us note that the above requirements ensure the validity of the equality

$$A_{T,\mathcal{F}}(x_1,\ldots,x_n)=f^{-1}\left(\sum_{i=1}^{n} f\circ f_{n-1,2}(x_i)\right).$$

Corollary 3. *Let $\boldsymbol{T}$ be a continuous Archimedean t-norm with additive generator f. The iterative compensatory operator $A_{T,\mathcal{F}}$ is idempotent and symmetric if and only if for each $n\in\mathcal{N}$ and $x\in[0,1]$,*

$$f_{n,1}(x)=f^{-1}\left(\frac{n}{n+1}f(x)\right)$$

and

$$f_{n,2}(x)=f^{-1}\left(\frac{1}{n+1}f(x)\right)$$

holds.

Proof. Under the given assumption, $A_{T,\mathcal{F}}$ is idempotent if and only if

$$f(x) = f \circ f_{n,1}(x) + f \circ f_{n,2}(x)$$

for each $n \in \mathcal{N}$ and $x \in [0,1]$.

Combining this with the property $f \circ f_{n,1}(x) = \lambda_n f(x)$ from Proposition 8, we obtain

$$f \circ f_{n,2}(x) = (1 - \lambda_n) f(x). \tag{10}$$

Next, by Proposition 8

$$f \circ f_{n,2}(x) = f \circ f_{n,1} \circ f_{n-1,1} \circ f_{n-2,1} \circ \cdots \circ f_{1,1}(x),$$

which gives

$$f \circ f_{n,2}(x) = \lambda_n f(f_{n-1,1} \circ f_{n-2,1} \circ \cdots \circ f_{1,1}(x)).$$

Repeating the same procedure, we finally obtain

$$f \circ f_{n,2}(x) = \lambda_n \lambda_{n-1} \ldots \lambda_1 f(x). \tag{11}$$

Comparing Eqs. (10) and (11) yields

$$1 - \lambda_n = \lambda_n \lambda_{n-1} \ldots \lambda_1 \quad \text{for each } n \in \mathcal{N}.$$

Hence, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{2}{3}, \ldots, \lambda_n = n/(n+1)$.

The complete proof for each $n \in \mathcal{N}$ can be done by induction.

For $\lambda_n = n/(n+1)$, we have $1 - \lambda_n = 1/(n+1)$, and therefore

$$f_{n,1}(x) = f^{-1}\left(\frac{n}{n+1} f(x)\right)$$

and

$$f_{n,2}(x) = f^{-1}\left(\frac{1}{n+1} f(x)\right)$$

for each $n \in \mathcal{N}$ and $x \in [0,1]$. □

Note that conditions in Corollary 3 are equivalent to

$$A_{T,\mathcal{F}}(x_1, \ldots, x_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right).$$

Remark 3. By Corollary 3, the arithmetic mean is the only iterative compensatory operator $A_{T,\mathcal{F}}$ based on the t-norm $\boldsymbol{T}_{\mathrm{L}}$, that is idempotent and symmetric.

The same can be deduced for the geometric mean and the t-norm $\boldsymbol{T}_{\mathrm{P}}$ and also for the harmonic mean and the t-norm $\boldsymbol{T}_{\mathrm{Ho}}$ (see also Example 4 and Remark 2).

The quadratic mean $\mathcal{M}_{\mathrm{q}}$ corresponds to a nilpotent t-norm $\boldsymbol{T}$ with the normed additive generator $f : [0,1] \to [0,1]$ defined by $f(x) = 1 - x^2$.

3.4. Zero-anihilator property

In some engineering situations a breakdown of one component leads to the failure of the whole system. From a mathematical point of view this means that zero among input values has outcome zero output.

Definition 5. An aggregation operator $\mathcal{A}$ has the zero-anihilator property if for each n-tuple $(x_1, \ldots, x_n) \in [0,1]^n$, $n \in \mathcal{N}$,

$$\text{if } \min_{1 \leqslant i \leqslant n} x_i = 0 \quad \text{then } \mathcal{A}(x_1, \ldots, x_n) = 0.$$

Lemma 2. *An aggregation operator $\mathcal{A}$ has the zero-anihilator property if and only if for each n-tuple $(\delta_1, \ldots, \delta_n) \in \{0,1\}^n$, $n \in \mathcal{N}$, such that $\delta_i = 0$ for some $i \in \{1, \ldots, n\}$,*

$$\mathcal{A}(\delta_1, \ldots, \delta_n) = 0 \tag{12}$$

holds.

Proof. If $\mathcal{A}$ has the zero-anihilator property then Eq. (12) follows directly from Definition 5.

Let $(x_1, \ldots, x_n) \in [0,1]^n$, $n \in \mathcal{N}$, and let $x_i = 0$ for some $i \in \{1, \ldots, n\}$. Then from Eq. (12) and the monotonicity of $\mathcal{A}$ it follows that

$$0 \leqslant \mathcal{A}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \leqslant \mathcal{A}(\delta_1, \ldots, \delta_n) = 0,$$

where $\delta_i = 0$ and $\delta_j = 1$ whenever $j \neq i$. Hence $\mathcal{A}$ has the zero-anihilator property. □

For iterative aggregation operators $A_{T,\mathcal{F}}$ defined in Definition 4, we can prove the following assertion.

Proposition 9. *The iterative aggregation operator $A_{T,\mathcal{F}}$ has the zero-anihilator property if and only if*

$$f_{n,i}(0) = 0 \quad \text{for all } n \in \mathcal{N}, \quad i \in \{1,2\}.$$

Proof. The sufficiency of the condition is evident. Next, suppose that $A_{T,\mathcal{F}}$ has the zero-anihilator

Table 1

Outputs \ Inputs	x_1 0.6	x_2 0.4	x_3 0.6	x_4 0.4	x_5 0.2	x_6 0.1	x_7 0.8	x_8 0.8	x_9 0.8	x_{10} 0.0	x_{11} 0.6	x_{12} 0.4
Min	0.6	0.4	0.4	0.4	0.2	0.1	0.1	0.1	0.1	0.0	0.0	0.0
Max	0.6	0.6	0.6	0.6	0.6	0.6	0.8	0.8	0.8	0.8	0.8	0.8
$\mathcal{M}_a$	0.6	0.5	0.533	0.5	0.44	0.383	0.443	0.488	0.522	0.470	0.482	0.475
$\mathcal{M}_g$	0.6	0.49	0.524	0.49	0.41	0.324	0.368	0.406	0.438	0.0	0.0	0.0
$\mathcal{M}_h$	0.6	0.48	0.514	0.48	0.375	0.257	0.285	0.31	0.332	0.0	0.0	0.0
$\mathcal{A}_1$	0.6	0.4	0.6	0.4	0.2	0.1	0.316	0.562	0.75	0.0	0.0	0.0
$\mathcal{A}_2$	0.6	0.35	0.375	0.238	0.019	0.0	0.35	0.525	0.613	0.057	0.229	0.165
$\mathcal{A}_3$	0.6	0.514	0.54	0.484	0.329	0.187	0.251	0.325	0.405	0.0	0.0	0.0
$\mathcal{A}_4$	0.6	0.32	0.396	0.279	0.128	0.056	0.422	0.569	0.628	0.0	0.3	0.26

property. Then for $(n+1)$-tuple $(1,1,\ldots,1,0)$ we have

$$0 = A_{T,\mathcal{F}}(1,1,\ldots,1,0)$$
$$= T(f_{n,1}(A_{T,\mathcal{F}}(1,1,\ldots,1)), f_{n,2}(0))$$
$$= T(f_{n,1}(1), f_{n,2}(0)) = f_{n,2}(0).$$

This forces $f_{n,2}(0) = 0$ for each $n \in \mathcal{N}$. Analogously,

$$0 = A_{T,\mathcal{F}}(0,1,\ldots,1,1)$$
$$= T(f_{n,1}(A_{T,\mathcal{F}}(0,1,\ldots,1)), f_{n,2}(1))$$
$$= T(f_{n,1}(0), 1) = f_{n,1}(0),$$

which gives $f_{n,1}(0) = 0$ for each $n \in \mathcal{N}$. □

Example 5. From the introduced iterative compensatory operators that have the zero-anihilator property we mention, e.g.,

(1) the geometric mean $\mathcal{M}_g(x_1, \ldots, x_n) = (\prod_{i=1}^n x_i)^{1/n}$,

(2) the operator $A_{T,\mathcal{F}}$ for $T = T_M$ and $\mathcal{F} = \{f_{n,1}(x) = \sqrt{x}, f_{n,2}(x) = x;\ n \in \mathcal{N}\}$ defined in Example 1, i.e.,

$$A_{T_M,\mathcal{F}}(x_1,\ldots,x_n)$$
$$= \min(x_1^{1/2^{n-1}}, x_2^{1/2^{n-2}}, \ldots, x_{n-1}^{1/2}, x_n).$$

Note that this iterative compensatory operator is also bisymmetric though it is neither symmetric nor associative.

Finally, let us still show an interesting example of an iterative compensatory operator that does not have the zero-anihilator property, however; zero among its input values plays an important role, because it effects that all preceding input values can be omitted, and it is enough to take zero as the first input.

Example 6. Let T be the product t-norm T_P and put for each $n \in \mathcal{N}$ and $x \in [0,1]$

$$f_{n,1}(x) = \frac{x+1}{2}, \qquad f_{n,2}(x) = x.$$

It is evident that the operator $A_{T_P,\mathcal{F}}$ satisfies conditions given in Definition 4 and Proposition 2. Moreover, by Remark 1(2),

$$A_{T_P,\mathcal{F}}(x_1,\ldots,x_n) = A_{T_P,\mathcal{F}}(A_{T_P,\mathcal{F}}(x_1,\ldots,x_{n-1}), x_n)$$

holds which gives

$$A_{T_P,\mathcal{F}}(x_1,\ldots,x_n) = \frac{A_{T_P,\mathcal{F}}(x_1,\ldots,x_{n-1}) + 1}{2} x_n$$

for each $(x_1,\ldots,x_n) \in [0,1]^n$, $n \in \mathcal{N}$, hence the output values can be easily computed.

Let us note that if $x_i = 0$ for some $i \in \{1,2,\ldots,n\}$, then

$$A_{T_P,\mathcal{F}}(x_1,\ldots,x_n) = A_{T_P,\mathcal{F}}(x_i,\ldots,x_n),$$

i.e., the preceding values $x_1,\ldots,x_{i-1}$ can be omitted and the output value is the same. This property can be called a *temporary breakdown property*.

In general, any iterative compensatory operator $A_{T,\mathcal{F}}$ given by $\mathcal{F}$ such that $f_{n,1} = f_1$, $f_{n,2} = f_2$, for all $n \in \mathcal{N}$ and $f_2(0) = 0$, possess the above temporary breakdown property, i.e., if $x_i = 0$ for some $i \in \{1,\ldots,n\}$, then $A_{T,\mathcal{F}}(x_1,\ldots,x_n) = A_{T,\mathcal{F}}(x_i, x_{i+1},\ldots,x_n)$.

4. Conclusions

We have introduced a new type of compensatory operator which extends the classical mean operators (arithmetic, geometric, harmonic, quadratic, etc.). Several properties have been discussed and several examples have been introduced. We expect that the iterative compensatory operators of this type will be useful for practical applications, especially in engineering sciences.

Finally, we illustrate behaviour of some iterative compensatory operators mentioned above for a given data set. For simplicity let us denote by $\mathscr{A}_1$ the iterative compensatory operator introduced in Example 1, by $\mathscr{A}_2$ the iterative compensatory operator given in Example 2, by $\mathscr{A}_3$ iterative compensatory operator from Example 3 and by $\mathscr{A}_4$ the iterative compensatory operator defined in Example 6 (see Table 1).

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References

[1] J. Dombi, Basic concepts for a theory of evaluation: the aggregative operator, European J. Oper. Res. 10 (1982) 282–293.

[2] E.P. Klement, Some mathematical aspects of fuzzy sets: triangular norms, fuzzy logics and generalized measures, Fuzzy Sets and Systems 90 (1997) 133–140.

[3] E.P. Klement, R. Mesiar, Triangular norms, Tatra Mount. Math. Publ. 13 (1997) 169–194.

[4] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Monograph, in preparation.

[5] E.P. Klement, R. Mesiar, E. Pap, On the relationship of associative compensatory operators to triangular norms and conorms, J. Uncertainty, Fuzziness Knowledge-Based Systems 4 (1996) 129–144.

[6] G.J. Klir, B. Yuan, Fuzzy Sets and Applications, Prentice-Hall, Upper Saddle River, New York, 1995.

[7] C.M. Ling, Representation of associative functions, Publ. Math. Debrecen 12 (1965) 189–212.

[8] M.K. Luhandjula, Compensatory operators in fuzzy linear programming with multiple objectives, Fuzzy Sets and Systems 8 (1982) 245–252.

[9] R. Mesiar, Compensatory operators based on triangular norms and conorms, in: H.-J. Zimmermann (Ed.), Proc. EUFIT'95, Aachen, 1995, pp. 131–135.

[10] R. Mesiar, M. Komorníková, Triangular norm-based aggregation of evidence under fuzzinness, in: B. Bouchon-Meunier (Ed.), Aggregation and Fusion of Imperfect Information, Physica-Verlag, Wurzburg, 1997, pp. 11–35.

[11] R. Mesiar, M. Komorníková, Aggregation operators, in: D. Herceg, K. Surla (Eds.), Proc. XI. Conf. on Applied Mathematics, Institute of Mathematics, Novi Sad, pp. 193–211.

[12] B. Moser, E. Tsiporkova, E.P. Klement, Convex combinations in terms of triangular norms: a characterization of idempotent, bisymmetrical and self-dual compensatory operators, Fuzzy Sets and Systems 104 (1999).

[13] I.B. Turksen, Interval-valued fuzzy sets and 'compensatory AND', Fuzzy Sets and Systems 51 (1992) 295–307.

[14] R.R. Yager, On ordered averaging aggregation operators in multi-criteria decision making, IEEE Trans. Systems Man Cybernet. 18 (1988) 183–190.

[15] R.R. Yager, Aggregation operators and fuzzy systems modelling, Fuzzy Sets and Systems 67 (1994) 129–146.

[16] H.J. Zimmermann, P. Zysno, Latent connectives in human decision making, Fuzzy Sets and Systems 4 (1980) 37–51.