ÉTALE COHOMOLOGY

These notes are an introduction to étale cohomology. The part about Grothendieck topologies is based on [Ols16], whereas for the part concerning étale cohomology we will mainly use as references [Mil80] and [CS21].

Morphism

Throughout this document, we will mention several classes of morphisms in the category of schemes, so we will give a quick reminder of the definitions of such morphisms.

Definition 0.1 (Flat morphism)**.** Let R be a ring and let M be a R-module. We say that M is flat if the functor

$$(-) \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$$

is exact. The module M is called faithfully flat is for every R-module A, B the induced map

$$\mathrm{Hom}_{\mathrm{Mod}_R}(A, B) \to \mathrm{Hom}_{\mathrm{Mod}_R}(A \otimes_R M, B \otimes_R M)$$

is injective.

A morphism of schemes $f : Y \to X$ is called flat if for $y \in Y$ the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat. The morphism f is called faithfully flat if f is flat and surjective.

Definition 0.2 (Unramified morphism)**.** Let A and B two Noetherian local rings. A homomorphism of local rings $f : A \to B$ is called unramified if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$.
(2) $\kappa(\mathfrak{m}_B)$ is a separable finite extension of $\kappa(\mathfrak{m}_A)$.
(3) B is essentially of finite type over A

Definition 0.3 (Smooth morphism)**.** A morphism $f : Y \to X$ is called smooth if it is flat, locally of finite presentation and for every geometric point $\bar{x} \to X$ tyhe fiber $Y_{\bar{x}}$ is regular.

Definition 0.4 (Étale morphism)**.** A morphism $f : Y \to X$ is called étale if it is an unramified and flat morphism or equivalently if it is unramified and smooth.

Definition 0.5 (Nisnevich morphism)**.** A morphism $f : Y \to X$ is called Nisnevich if it is an étale morphism such that for every point $x \in X$, there exists a point $y \in Y$ in the fiber $f^{-1}(x)$ such that the induced map of residue fields $k(x) \to k(y)$ is an isomorphism.

Recall that when we there are three abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, such that the first two have enough injectives, and left exact functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$.

Definition 0.6. Let $G : \mathcal{B} \to \mathcal{C}$ be a left exact functor. An object B of $\mathcal{B}$ is called G-acyclic if the derived functors of G vanish on B, i.e. $R^i F(B) = 0$ for $i \neq 0$.

Assume that F sends injective objects of $\mathcal{A}$ to G-acyclic objects of $\mathcal{B}$, then there exists a convergent first quadrant cohomological spectral sequence (Grothendieck spectral sequence [Wei94, Theorem 5.8.3]) for each $A \in \mathcal{A}$:

$$E_2^{p,q} = (R^p F)(R^q G)(A) \Longrightarrow R^{p+q}(FG)(A).$$

1. Grothendieck topologies

A Grothendieck topology, is the natural generalization of a topology in a topological space, but now if we consider a category $\mathcal{C}$ as a "space" and morphisms as "open subsets". In order to make this analogue, let us recall the following construction: consider a topological space X and let $\mathrm{Op}(X)$ be the collection of open subsets. This condition can be endowed with arrows between its objects: for two open subsets $U, V \in \mathrm{Op}(X)$ we set

$$\mathrm{Hom}_{\mathrm{Op}(X)}(U,V) = \begin{cases} \{*\} \text{ if } U \subset V \\ \emptyset \quad \text{if not.} \end{cases}$$

Here, a presheaf P with values in a category V, of a topological space X can be characterized as a contravariant functor

$$P : \mathrm{Op}(X)^{op} \to V.$$

In general V can be taken as the category of sets Set, abelian groups Ab, R-modules $R - \mathrm{Mod}$, etc. A presheaf P is a sheaf if and only if for every $U \in \mathrm{Op}(X)$ and covering $U = \bigcup_{i \in I} U_i$ the sequence

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer diagram.

Definition 1.1. Let $\mathcal{C}$ be a category. A Grothendieck topology[1] on the category $\mathcal{C}$ consists in the following data: for every object $X \in \mathrm{Ob}(\mathcal{C})$ and a set $\mathrm{Cov}(X)$ of collections of morphisms $\{X_i \to X\}_{i \in I}$ such that the following properties hold:

(1) If $V \to X$ is an isomorphism in $\mathcal{C}$, then $\{V \to X\} \in \mathrm{Cov}(X)$.
(2) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is an arrow in $\mathcal{C}$, then the fiber products $X_i \times_X Y$ exist in $\mathcal{C}$ and $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(X)$.
(3) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$, and if for every $i \in I$ we are given $\{V_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$, then the collection of composition $\{V_{ij} \to X_i \to X\}_{j \in J_i, i \in I}$ is in $\mathrm{Cov}(X)$.

If $\mathcal{C}$ has an associated Grothendieck we say that $\mathcal{C}$ is a site.

Example 1.2. (1) (Small classical topology) If X is a topological space, then we can associated a category and a Grothendieck topology to it. If X is a scheme, then the Zariski topology on it defines a Grothendieck topology, called the "*small Zariski site*". For a scheme, we denote the small Zariski site as X_{zar}.
(2) (Big Zariski site) Let X be a scheme and let $\mathcal{C} = \mathrm{Sch}/X$ be the category of schemes over X. For $U \to X$ we define $\mathrm{Cov}(U)$ to be the collections of X-morphisms $\{U_i \to U\}_{i \in I}$ with $U_i \to U$ open embeddings and $\bigcup_{i \in I} U_i = U$.
(3) (Small étale site) Let X be a scheme. Define $X_{\text{ét}}$ to be the full subcategory of the category of X-schemes whose objects are $f : U \to X$ with f étale. A collection of morphisms $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ if each $U_i \to U$ is étale and the map

$$\coprod_{i \in I} U_i \to U$$

is surjective.
(4) (Big étale site) Let X be a scheme and let $\mathcal{C} = \mathrm{Sch}/X$ be the category of schemes over X. For $U \to X$ we define $\mathrm{Cov}(U)$ to be the collections of X-morphisms $\{U_i \to U\}_{i \in I}$ with $U_i \to U$ étale morphism and $\coprod_{i \in I} U_i \to U$ is surjective.

[1] Or a pretopology in the most classical sense.

(5) (fppf site) Let X be a scheme and let $\mathcal{C} = \mathrm{Sch}/X$ be the category of schemes over X. For $U \to X$ we define $\mathrm{Cov}(U)$ to be the collections of X-morphisms $\{U_i \to U\}_{i \in I}$ with $U_i \to U$ flat and locally of finite type morphisms, and the morphism $\coprod_{i \in I} U_i \to U$ is surjective.

(6) (Smooth site) Let X be a scheme. Define $\mathcal{C}$ to be the full subcategory of the category of X-schemes whose objects are $f : U \to X$ with f smooth. A collection of morphisms $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ if each $U_i \to U$ is smooth and the map

$$\coprod_{i \in I} U_i \to U$$

is surjective.

(7) (Small Nisnevich site) Let X be a scheme. Define X_{Nis} to be the full subcategory of the category of X-schemes whose objects are $f : U \to X$ with f a Nisnevich morphism. A collection of morphisms $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ if each $U_i \to U$ is Nisnevich and the map $\coprod_{i \in I} U_i \to U$ is surjective.

(8) (h-topology) Let X be a scheme. The h-site of the category of X-scheme of finite presentation is generated by the fppf-coverings $\{U_i \to U\}_{i \in I}$ and diagramns of the form $\{U' \to U, Z \to U\}$ where
   - $U' \to U$ is a proper morphism of finite presentation,
   - $Z \to U$ is a closed immersion of finite presentation, and
   - $U' \to U$ is an isomorphism in $U \setminus Z$.

   If a diagram $\{U' \to U, Z \to U\}$ fulfils the previous conditions it is called an abstract blow-up.

(9) (fpqc topology) Let U be a scheme. A fpqc (*fidèlement plat quasi-compact*) covering of U is a family $\{U_i \to U\}_{i \in I}$ such that for each $U_i \to U$ is a flat morphism and for each affine open $V \subset U$ there exists a finite set $\{i_1, \ldots, i_m\} \subset I$, affine opens $V_{i_k} \subset U_{i_k}$ such that $\coprod_k V_{i_k} \to U$ is surjective. If we take X a scheme, by considering the Grothendieck topology given by the fpqc coverings in Sch/X we obtain the fpqc site of X, denoted by X_{fpqc}.

Remark 1.3. The difference between a small and big site is that in the small site we consider objects in Sch/X whose structural morphisms $U \to X$ are in the class of morphism considered (Zariski, étale or Nisnevich), while in the big site this is note required.

A morphism between sites $\mathcal{C}$ and $\mathcal{C}'$ is a continuous functor, c'est-à-dire, if for every $X \in \mathcal{C}$ and $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$, then $\{f(X_i) \to f(X)\}_{i \in I} \in \mathrm{Cov}(f(X))$, and if f commutes with fiber products when they exist in $\mathcal{C}'$.

Example 1.4. (1) For a scheme X, the identity morphism on X defines morphisms of sites

$$X_{\mathrm{fpqc}} \to X_{\mathrm{fppf}} \to X_{\text{Ét}} \to X_{\text{ét}} \to X_{\mathrm{Nis}} \to X_{\mathrm{zar}}$$

(2) Let k be a field and let K/k any field extension, and let X be a k-scheme, then the morphism $X_K \to X$ defines a morphism of sites.

Definition 1.5. Let $\mathcal{C}$ be a category. A presheaf on $\mathcal{C}$ with values in V is a contravariant functor

$$F : \mathcal{C}^{op} \to V.$$

In addition, if $\mathcal{C}$ is endowed with a Grothendieck topology, then

(1) a presheaf is called separated if for every $U \in \mathcal{C}$ and covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ the map $F(U) \to \prod_{i \in I} F(U_i)$ is injective.

(2) a presheaf is called a **sheaf** if for every $U \in \mathcal{C}$ and covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ the diagram

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

is an equalizer diagram. Here the two maps are induced by the projections $U_i \to U_i \times_U U_j$ and $U_j \to U_i \times_U U_j$.

Theorem 1.6. *Let $\mathcal{C}$ be a site, then the inclusion functor*

$$\{\textit{Sheaves on } \mathcal{C}\} \hookrightarrow \{\textit{Presheaves on } \mathcal{C}\}$$

has a left adjoint $F \mapsto F^s$, which is called the sheafification functor.

Definition 1.7. A category T equivalent to the category of sheaves on a site is called a topos.

Considering x the topos of sheaves of the space of one-point. A point in of a topos T is a moprhisms of topoi $f : x \to T$. We say that T has enough points if there exists a set of points $\{f_i : x_i \to T\}_{i\in I}$ of T such that the induced functor

$$\begin{aligned} T &\to \mathrm{Set}^I \\ F &\mapsto \{f_i^* F\}_{i\in I} \end{aligned}$$

if faithful.

Theorem 1.8. *[Ols16, Theorem 2.3.2] Let T be a topos and let R be a ring. Denote by R-Mod$_T$ the category of R-modules of T, then R-Mod$_T$ is an abelian category with enough injectives.*

Proof. Consider a topos T that has enough points. Since T has enough points, there exists a collection of morphisms $\{f_i : x_i \to T\}_{i\in I}$ of T (which we fix for the rest of the proof) such that the induced functor

$$\begin{aligned} T &\to \mathrm{Set}^I \\ F &\mapsto \{f_i^* F\}_{i\in I} \end{aligned}$$

if faithful. For $F \in R\text{-Mod}_T$ and $i \in I$ we fix $F_i := f_i^* F \in x_i$. The sheaf F_i is a R_i-module with R_i a ring. Choosing for each $i \in I$ and injective R_i-module I_i and an inclusion $F_i \hookrightarrow I_i$. The adjunction morphism induced by $(f_i)_*$ and f_i^* defines a morphism $p_i : F \to (f_i)_* f_i^* F \hookrightarrow (f_i)_* I_i$, taking the product over I we get a map

$$p : F \to \prod_{i\in I} (f_i)_* F \to \prod_{i\in I} (f_i)_* I_i.$$

The sheaf $\prod_{i\in I}(f_i)_* I_i$ is injective because $(f_i)_*$ has an exact left adjoint, preserves injectives and the product of injective is injective. The map p is an injection because $F_i \to I_i$ is an injection. □

We have a functor $\Gamma(T, -) : R\text{-Mod}_T \to \mathrm{Ab}$ where Ab is the category of abelian groups obtained by $\mathrm{Hom}_{R\text{-Mod}_T}(R, F)$. The cohomology groups of the site T with values in abelian groyps $H^i(T, -) : R\text{-Mod}_T \to \mathrm{Ab}$ are given by the i-th right derived functor of $\Gamma(T, -)$, which is left exact.

2. Étale sheaves and cohomology

We can define the local ring for the étale cohomology. We recall that for a point $x \to X$ the local ring of X at x is denoted by $\mathcal{O}_{X,x}$ and is obtained by a limit

$$\mathcal{O}_{X,x} = \varinjlim_{U \subset X} \mathcal{O}(U)$$

which is taken over all open subset $U \subset X$ containing x. The étale local ring of X in a point x is obtained as

$$\mathcal{O}^h_{X,x} = \varinjlim_{U \subset X} \mathcal{O}(U)$$

where the limit runs over all diagrams

$$\begin{tikzcd} & U \arrow[d, "\text{étale}"] \\ \bar{x} \arrow[ur] \arrow[r] & X \end{tikzcd}$$

This is called the hensenialization of the local ring $\mathcal{O}_{X,x}$. The residue field of this local ring is $k(x)$. The étale neighbourhood of a geometric point $\bar{x} \to X$ is an étale $X-$scheme U with a lifting point $u \to \bar{x}$.

Now let x be a point in X. One says that a geometric point $\bar{x}$ lies over x if the point x is the image of $\bar{x}$ i X (strictly saying that $k(x) \subset k(\bar{x})$). Define

$$\mathcal{O}^{\mathrm{sh}}_{X,x} = \varinjlim_{(U,\bar{x})} \mathcal{O}(U).$$

Where the limit runs over all étale neighbourhood of geometric points $\bar{x}$ which are over x. This is the strict henselianization of the local ring $\mathcal{O}_{X,x}$. The residue field of $\mathcal{O}^{\mathrm{sh}}_{X,x}$ is the separable closure of $k(x)$ in $k(\bar{x})$. The stalk of a presheaf $\mathcal{F}$ at a geometric point $\bar{x} \to X$ is defined as

$$\mathcal{F}_{\bar{x}} = \varinjlim \mathcal{F}(U)$$

where the limit is taken over all connected étale open $U \to X$ which lifts to $\bar{x}$.

With the notion of stalk, as in the classical case, we can obtain the following equivalent statements:

Proposition 2.1. *[Mil80] Let $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ be étale sheaves over X, then the following are equivalents*

(1) *the sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact in the category of étale sheaves over X.

(2) *the sequence of abelian groups*

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \to 0$$

is a short exact sequence for each geometric point $\bar{x} \to X$.

Remark 2.2. There is a direct link between étale cohomology and Galois cohomology. Let k be a field and $\mathrm{Spec}(k)_{\text{ét}}$ be the small étale site if k. The small category consists of finite dimensional étale k-algebras, i.e. finite products of finite separable field field extensions of k. A presheaf $\mathcal{P}$ on $\mathrm{Spec}(k)$ is a sheaf if for every disjoint union sends $\coprod_i \mathrm{Spec}(k_i)$ to a direct product of abalian groups and $\mathcal{P}(k') = \mathcal{P}(k'')^{\mathrm{Gal}(k''/k')}$ with $k \subset k' \subset k''$ finite Galois extensions. Choosing a separable closure k^s of k, and let $G_k = \mathrm{Gal}(k^s/k)$. For a sheaf $\mathcal{F}$ we associate a discrete G_k-module as follows

$$M_{\mathcal{F}} := \varinjlim_{k\subset k'\subset k^s} \mathcal{F}(k')$$

where k' runs over all finite separable extension of k. On the other hand, if M is a discrete G_k-module we can associate a sheaf over $\mathrm{Spec}(k)_{\text{ét}}$ in the following way

$$\mathcal{F}_M(A) := \mathrm{Hom}_{G_k\text{-Mod}}(F(A), M)$$

with $F(A) = \mathrm{Hom}_{k\text{-alg}}(A, k^s)$ and A is finite dimensional k-algebra. This correspondence defines an equivalence of categories between the étale sheaves over k and the discrete G_k-modules.

Since for an étale sheaf $\mathcal{F}$, we have $M_{\mathcal{F}}^{G_k} = \Gamma(k, \mathcal{F})$ then the étale cohomology groups $H^i_{\text{ét}}(k, \mathcal{F})$ are isomorphism to the group cohomology $H^i(G_k, M_{\mathcal{F}})$. Similarly the Ext-groups $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ in the category of étale sheaves over k are isomorphic to the Ext-groups $\mathrm{Ext}(M_{\mathcal{F}}, M'_{\mathcal{F}})$ in the category of discrete G_k-modules.

Example 2.3. Consider the following étale sheaves over X:

(1) $\mathbb{G}_{a,X}$ is the sheaf associated to the presheaf given by $\mathbb{G}_{a,X}(Y) = \Gamma(Y, \mathcal{O}_Y)$.
(2) $\mathbb{G}_{m,X}$ is the sheaf associated to the presheaf given by $\mathbb{G}_{m,X}(Y) = \Gamma(Y, \mathcal{O}_Y^*)$.
(3) $\mu_{n,X}$ for $n > 0$ is the sheaf associated to the presheaf given by $\mu_{n,X}(Y) = \{x \in \Gamma(Y, \mathcal{O}_Y^*) \mid x^n = 1\}$.

Suppose that we have continuous morphisms of sites $X'' \xrightarrow{\pi'} X' \xrightarrow{\pi} X$ and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are the categories of sheaves on X'', X', X respectively. The functor π^* is exact and has a right adjoint π_*, thus it sends injectives to injectives (in particular an injective object in $\mathcal{B}$ is G-acyclic) and hence, for every sheaf $\mathcal{F}$ on X'' we have a spectral sequence (given by the Grothendieck spectral sequence) called the Leray spectral sequence

$$E_2^{p,q} = (R^p\pi_*)(R^q\pi'_*)\mathcal{F} \Longrightarrow R^{p+q}(\pi\pi')_*\mathcal{F}$$

Some examples of étale cohomology groups:

Example 2.4. (1) The Picard group $\mathrm{Pic}(X)$ of a scheme X is the groups of invertible coherent sheaves of $\mathcal{O}_X$-modules, considered up to isomorphism. By this definition we have that

$$\mathrm{Pic}(X) = H^1_{\text{zar}}(X, \mathcal{O}_X^*) = H^1_{\text{zar}}(X, \mathbb{G}_{m,X})$$

By Hilbert's Theorem 90, see [Mil80, Prop. III.4.9], the canonical maps induced by the change of topology

$$H^1_{\text{zar}}(X, \mathbb{G}_{m,X}) \to H^1_{\text{ét}}(X, \mathbb{G}_{m,X}) \to H^1_{\text{fppf}}(X, \mathbb{G}_{m,X})$$

are isomorphisms.

(2) The Grothendieck-Brauer group or cohomological Brauer group of a scheme X is defined to be $H^2_{\text{ét}}(X, \mathbb{G}_{m,X})$

3. Descent theory

Descent theory has the following motivation: Consider a scheme X and a open covering $\mathcal{U} = \{U_i\}_{i\in I}$. Consider the following category $\{(F_i)_{i\in I}, (\sigma_{i,j})_{i,j\in I}\}$ where F_i is a coherent sheaf in U_i and for every $i,j \in I$ there is an isomorphism $\sigma_{i,j} : F_i\Big|_{U_i\cap U_j} \to F_j\Big|_{U_i\cap U_j}$ such that $\sigma_{i,i} = \mathrm{id}_{F_i}$ and for every $i,j,k \in I$ we have a commutative diagram

$$\begin{tikzcd} F_i\Big|_{U_{ijk}} \arrow[r, "\sigma_{i,j}|_{U_{ijk}}"] \arrow[rr, bend right, "\sigma_{i,k}|_{U_{ijk}}"'] & F_j\Big|_{U_{ijk}} \arrow[r, "\sigma_{j,k}|_{U_{ijk}}"] & F_k\Big|_{U_{ijk}} \end{tikzcd}$$

where $U_{ijk} = U_i \cap U_j \cap U_k$. To begin with descent theory we mention the following result: consider X and Y schemes and $Y \to X$ a morphism of schemes. Consider the functor

$$\begin{aligned} \underline{Y} : (\mathrm{Sch}/X)^{\mathrm{op}} &\to \mathrm{Sets} \\ (U \to X) &\mapsto \underline{Y}(U) := \mathrm{Hom}_{\mathrm{Sch}/X}(U,Y). \end{aligned}$$

This functor is clearly a presheaf over the category Sch/X, but also a fpqc sheaf for the category Sch/X.

Theorem 3.1. *For any morphism of schemes $Y \to X$, the functor $\underline{Y}$ defines a sheaf in the fpqc topology (and therefore is also an étale, fppf, Nisnevich, Zariski,... sheaf) on the category $(Sch/X)^{op}$.*

Bibliography

[CS21] Jean-Louis Colliot-Thélène and Alexei N Skorobogatov. *The Brauer-Grothendieck group*. Vol. 71. Springer, 2021.

[Mil80] James S Milne. *Etale cohomology (PMS-33)*. Princeton university press, 1980.

[Ols16] Martin Olsson. *Algebraic spaces and stacks*. Vol. 62. American Mathematical Soc., 2016.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450.