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CHOW-KÜNNETH DECOMPOSITION

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ECOLE DOCTORALE
CARNOT - PASTEUR

Décomposition de Chow-Künneth

Résumé: L'année dernière, Rosenschon et Srivinas ont prouvé une équivalence entre la conjecture de Hodge avec des coefficients rationnels et une version intégrale de la conjecture de Hodge utilisant la cohomologie motivique étale. Dans le même esprit, la question qui se pose naturellement est de savoir si l'on peut obtenir un résultat similaire concernant la décomposition de Chow-Künneth des motifs. Cette thèse est consacrée à l'étude de la décomposition de Chow-Künneth d'un point de vue motivique étale, présentant la décomposition intégrale du motif étale des variétés abéliennes.

Dans la première partie de la thèse, nous posons les bases de la théorie des motifs purs et mixtes, ainsi qu'une description complète de la cohomologie étale motivique, en donnant les principales similitudes et différences avec les groupes de Chow. Dans la deuxième partie, nous examinons certaines conséquences sur les aspects géométriques intégraux des motifs en utilisant la catégorie triangulée des motifs étales. Tout d'abord, nous obtenons une conjecture équivalente, utilisant des coefficients intégraux, de la conjecture de Hodge généralisée. Enfin, nous commençons à étudier la décomposition des motifs étales, dans un premier temps, en utilisant un analogue étale de l'application degré des 0-cycles. Puis, on continue avec l'étude de la décomposition des motifs en utilisant la propriété de conservativité sur le changement des coefficients intégraux vers coefficients rationnels et finis. Avec ce résultat, nous obtenons la décomposition du motif étale intégral d'un groupe commutatif lisse sur une base avec des propriétés suffisantes.

Mots clés : Cohomologie motivique, cycles algébriques, motifs étale, cohomologie étale, conjecture de Hodge généralisée, décomposition motivique.

Chow-Künneth decomposition

Abstract: In the past few years, Rosenschon and Srivinas proved an equivalence between the Hodge conjecture with rational coefficients and an integral version of the Hodge conjecture using étale motivic cohomology. Using the same spirit, the question that arises naturally is whether or not we can obtain a similar result concerning the Chow-Künneth decomposition of motives. This thesis is devoted to the study of the Chow-Künneth decomposition from an étale motivic point of view, presenting the integral decomposition of the étale motive of abelian varieties.

In the first part of the thesis, we set the basis for the theory of pure and mixed motives, together with a full description of the structure of étale motivic cohomology, giving the principal similarities and differences with the Chow groups. In the second part, we look at some consequences of the integral geometric aspects of motives using the triangulated category of étale motives. First, we obtain an equivalent conjecture, using integral coefficients, of the generalized Hodge conjecture. Finally, we start looking at the decomposition of an étale motive, in the first instance, using an étale analog of the degree map. After we continue the study of the decomposition of motives using the conservativity property about the change of coefficients from integral to rational and finite coefficients, with this result, we obtain the decomposition of the integral étale motive of a smooth commutative group over a base with good enough properties.

Keywords: Motivic cohomology, algebraic cycles, étale motives, étale cohomology, generalized Hodge conjecture, motivic decomposition.

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Contents

Introduction

Historical background

In different expressions of art, such as narrative, music, visual or textiles arts, the word *motif* appears as the definition of a recurring element which has an important role inside the piece of art in question, such as a reason or pattern and a fundamental element.

The theory of pure motives was introduced by Grothendieck in a letter to Serre in the middle of the 60's as an attempt to explain the structures underlying different Weil cohomology theories. such as Betti, de Rham (analytic and algebraic), ℓ-adic and crystalline cohomology. In some cases, there is a deep relation between them, for instance, if the base field is $k = \mathbb{C}$ and $X/\mathbb{C}$ is a smooth projective variety, then by the de Rham theorem one has

$$H^*_{dR}(X_{an};\mathbb{C}) \xrightarrow{\sim} H^*_B(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

On the other hand, under the same assumptions of X and the base field k, one has the theorem of Artin

$$H^*_{\text{ét}}(X,\mathbb{Q}_\ell) \xrightarrow{\sim} H^*_B(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

This evidence indicates that there should exist an underlying reason for the similar behaviour of these different cohomology theories in the complex case. Having this idea in mind, Grothendieck tried to define a "universal cohomology theory" which explains these connections between cohomology theories. In order to achieve that goal, he introduced the category of pure motives, whose construction, using smooth projective varieties over a field k, is fairly simple and unconditional. Grothendieck worked with numerical motives (motives modulo numerical equivalence), whereas in this thesis we mainly use *Chow motives* (motives modulo rational equivalence). The link between cohomology theories should be given by realizations of the objects in this category. Using the construction of the category and the existence of realization functors ρ, we find that any Weil cohomology theory with coefficients in a field F of characteristic zero factors uniquely through the category of Chow motives as follows:

$$\begin{tikzcd} \mathrm{SmProj}_k \arrow[rr, "H^*(-)"] \arrow[dr, "h(-)"'] & & \{\text{graded } F\text{-algebras}\} \\ & \mathrm{Chow}(k)_{\mathbb{Q}}. \arrow[ur, "\rho(-)"'] & \end{tikzcd} \tag{1}$$

where H is a Weil-cohomology theory, $h(-)$ is the functor which associates to a smooth projective variety X its Chow motive $h(X) = (X, \Delta_X)$.

Mimicking the cohomological behaviour, is conjectured that the Chow motive of a smooth projective variety carries a decomposition, known as the *Chow-Künneth decomposition*. This conjecture was described by Murre in [Mur93] and states that the diagonal cycle Δ_X in $X \times X$ is a sum of cycles $\Delta_X = \sum_{i=0}^{2\dim(X)} p_i(X)$ with $p_i(X) \in \mathrm{CH}^{\dim(X)}(X \times X)_{\mathbb{Q}}$ a cycle in the Chow groups of $X \times X$ of codimension $\dim(X)$, such that $p_i(X)$ are send to the Künneth projectors Δ_i^{topo}. Also the composition of correspondences $p_i(X) \circ p_j(X)$ should be zero if $i \neq j$ and $p_i(X)$ otherwise.

In some sense, due to the realization to de Rham cohomology in the complex case, we can see the theory of pure motives as an analogue of pure Hodge structures. Hence one can ask if this construction can be extended to the case of singular or noncompact varieties where we should obtain mixed motives by analogy with mixed Hodge structures. Several candidates for such a category of mixed motives have been proposed by Hanamura, Levine, Nori and Voevodsky. In the Voevodsky setting, is described in [Voe00], it is possible to obtain triangulated categories of mixed motives over a perfect field k with different coefficients and suitable topologies such as étale, Nisnevich and h-topology. One of the main successes of Voevodky's approach was his proof of the Milnor and Bloch-Kato conjectures. The triangulated category of mixed motives over a field k with coefficients in $\mathbb{Q}$, denoted by $\mathrm{DM}(k, \mathbb{Q})$, is a generalization of $\mathrm{Chow}(k)_{\mathbb{Q}}$ is the sense that there exists a fully-faithful embedding $\mathrm{Chow}(k)_{\mathbb{Q}}^{op} \hookrightarrow \mathrm{DM}(k, \mathbb{Q})$ and according to Bondarko [Bon14], $\mathrm{Chow}(k)_{\mathbb{Q}}$ appears as the element of weight zero in $\mathrm{DM}(k, \mathbb{Q})$.

Later, Morel and Voevodsky introduced motivic homotopy theory, This new approach is a link between algebraic topology and algebraic geometry, putting in a more general context the notion of $\mathbb{A}^1$-homotopy theory of schemes. Within this theory the concept of $\mathbb{P}^1$-stabilization process for motivic complexes was introduced.

In the second part of Ayoub's thesis [Ayo06], he gives a full description of Grothendieck's six functor formalism for systems of triangulated categories. This is equivalent to the properties of $\mathbb{A}^1$-localization, $\mathbb{P}^1$-stabilization and rigidity. Later, Cisinski and Déglise [CD19] studied fibered triangulated categories. They give a full description of the six functor formalism, construction problems and the relation between rational motivic complexes and the Beilinson program/Beilinson's motives.

By changing from Nisnevich to étale or h-topology, we can obtain different models for integral motives having a deep link with étale cohomology theory. The first description about the categories with étale or h-topology is given in [Voe00] and [MVW06]. Later, Ayoub in [Ayo14b] gave the functorial framework for the category $\mathbf{DA}^{\text{ét}}(k, \mathbb{Z})$ of motivic complexes without transfers. Another model for the étale category of motives, the category of $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$, is given by Cisinski and Déglise in [CD16], which consider étale sheaves with transfers and gives the equivalence with the category of h-motives $\mathrm{DM}_h(k, \mathbb{Z})$.

The triangulated category of étale or h-motives with integral coefficients is one of the main candidate for being the good framework for integral motives. As a example of the good properties of the integral étale motives, recently, Rosenschon and Srinivas [RS16] gave a new characterization of the Hodge and Tate conjectures using étale motivic cohomology, the analogue of Chow groups but using the category $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$. Recall that the Hodge and Tate conjectures are the following:

Conjecture (Hodge conjecture)**.** *For a complex smooth projective variety X and $n \in \mathbb{N}$ the image of the cycle class map $c^n : CH^n(X)_{\mathbb{Q}} \to H^{2n}_B(X, \mathbb{Q}(n))$ is the set of Hodge classes $Hdg^{2n}(X, \mathbb{Q}) = H^{n,n}(X) \cap H^{2n}(X, \mathbb{Q})$.*

Where $H^*_B(X, \mathbb{Q}(n))$ is the Betti cohomology ring of X with coefficients in $\mathbb{Q}$.

Conjecture (Tate conjecture)**.** *Let X be a smooth projective geometrically integral k-variety with k a finite field. Let $\bar{k}$ be a separable closure of k. If $\ell \neq char(k)$ is a prime then the cycle class map*

$$c^n_{\mathbb{Q}_\ell} : CH^n(X) \otimes \mathbb{Q}_\ell \to H^{2n}_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell(n))^{\Gamma_k}$$

is surjective

Here Γ_k represents the Galois group of k, $H^*_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$ is the ℓ-adic cohomology and $\bar{X} = X \otimes_k \bar{k}$.

These conjectures can be expressed in terms of motives and the realization functor as stated in [And04, Propositions 7.2.1.3 et 7.3.1.1].

Making a connection between the realization of Chow motives with rational coefficients and the category of integral étale motives, it was possible to conclude that the étale versions of the conjectures, i.e. changing Chow groups by an étale analogue, called Lichtenbaum cohomology groups, which is the étale hypercohomology of the complex of étale sheave given by the Bloch complex,

Conjecture (Lichtenbaum Hodge conjecture)**.** *For a complex smooth projective variety X and $n \in \mathbb{N}$ the image of the cycle class map $c^n_L : CH^n_L(X) \to H^{2n}_B(X, \mathbb{Z}(n))$ is $Hdg^{2n}(X, \mathbb{Z})$.*

Conjecture (Lichtenbaum Tate conjecture)**.** *Let X be a smooth projective geometrically integral k-variety with k a finite field. Consider $\bar{k}$ be a separable closure of k. If $\ell \neq char(k)$ is a prime then the cycle class map*

$$c^n_{L,\mathbb{Z}_\ell} : CH^n_L(X) \otimes \mathbb{Z}_\ell \to H^{2n}_{\acute{e}t}(\bar{X}, \mathbb{Z}_\ell(n))^{\Gamma_k}$$

is surjective.

With this idea in mind, is valid to ask whether or not is possible to obtain new results about algebraic or arithmetic properties of integral étale motives and relate them with its rational counterpart. This is due to the fact that there exist counter-examples for the integral Hodge conjecture when using Chow groups, therefore integrally, étale Chow groups should give us more information about X.

In this thesis, we aim to approach to the decomposition of integral étale motives using the same spirit given by Rosenschon and Srinivas: see if we can improve the decomposition of integral motives but working with the category of étale motives $\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})$. Given results about conservative family of functors associated to coefficients change:

$$\rho_{\mathbb{Q}} : \mathrm{DM}_{\text{ét}}(k,\mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(k,\mathbb{Q})$$
$$\rho_{\mathbb{Z}/\ell} : \mathrm{DM}_{\text{ét}}(k,\mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(k,\mathbb{Z}/\ell)$$

one can see whether or not it is possible to lift isomorphism of motives with rational coefficients to the integral case passing through an argument involving finite coefficients. As an underlying goal, we present a detailed description about the structure of étale motivic cohomology and develop the étale analogue of intersection theory using the motivic categorical formalism.

Trying to follow the historical development of motives, we structure the thesis as follows: with a total of four chapters and each chapter divided in several sections. The first two chapters work as an introduction to the theory of pure and mixed motives, setting the bases for a further generalization to the étale case. We continue by introducing the étale analogue to the classical theory, giving all the parallelism that we can give: in the level of Chow groups, induced morphism , equivalences on algebraic cycles and the category of étale Chow motives; and aspects about étale motives and étale motivic cohomology.

In the third chapter, using as a guideline the characterization of the *generalized Hodge conjecture* in terms of realization of effective motives, we revisit the main result of [RS16] giving a new characterization of the generalized Hodge conjecture in terms of the category of étale Chow motives, and the description of non-algebraic integral cohomology classes in term of étale motivic cohomology. In the last chapter we will present decomposition of étale motives using the conservative of the family of functors associated to change of coefficients in the following way

Definition. *Let k be a field and let $f : X \to k$ be a smooth projective variety, of dimension d. We say that $h_{ét}(X)$ admits an integral Chow-Künneth decomposition in $Chow_{ét}(k)$ if:*

- *$h(X)$ admits a rational Chow-Künneth decomposition*

$$h(X) \xrightarrow{\simeq} \bigoplus_{i=0}^{2d} h^i(X) \in Chow(k)_{\mathbb{Q}},$$

  and this map is induced by a morphism $g : h_{ét}(X) \to M = (Y,p)$ in $Chow_{ét}(k)$.

- *Consider the base change to the algebraic closure $\bar{g} : h_{ét}(X_{\bar{k}}) \to M_{\bar{k}}$. For every prime number $\ell \neq char(k)$, the induced map $\rho_\ell(\bar{g}) : R\bar{f}_*(\mathbb{Z}/\ell) \to M_{\bar{k}}/\ell \in D(\bar{k}_{ét},\mathbb{Z}/\ell)$ is an isomorphism and $\rho_\ell(\bar{p}) = p_1 + \ldots + p_{2d}$ with the following conditions*

$$p_i \circ p_j = \begin{cases} p_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \qquad \rho(\bar{g})^{-1} \circ p_i(M_{\bar{k}}/\ell) = R^i\bar{f}_*(\mathbb{Z}/\ell) \text{ for all } i.$$

using different approaches and points of view. The following is a more described outline of the thesis.

In this thesis, we present the result obtained by the author in the pre-prints [Ros22] and [Ros23b], together with the work in progress [Ros23a].

Outline of the thesis

Chapter 1

The first chapter is divided in two sections: the first one treats the construction of pure motives, giving main definition and results of such theory. The second part is devoted to the triangulated category of motives. We start giving a general overview of the theory of classical motives, focused in the algebraic properties of pure motives, for this, we revisit the main references of [MNP13], [And04] and [Sch94]. We discuss the main results and basics of the theory of pure motives, such as the Manin's identity principle [Man68] and the application of it to the computation of the motive of a projective bundle, blow-up with smooth center and varieties that admit cellular decomposition. Continuing with developing the theory of motives, going to the triangulated category of motivic complexes, visiting the references of [CD19] and presenting the basis of premotivic categories and the six functor formalism in the motivic context.

The goal of this chapter is refresh the theory of pure motives and establish the basis for the construction of étale Chow motives and the triangulated category of mixed motives part is there in order to give a proper introduction to the terminology and functoriality properties of the triangulated categories of motives and in that way be able to use important tools in the construction of our theory.

Chapter 2

Chapter 2 is the most extensive one, because is the one in which we treat in a deep way étale cohomology and étale motivic complexes. For that, we start by introducing two different models of the triangulated category of étale motives: the ones introduced in [CD16] and [Ayo14b], considering complexes of sheaves with and without transfers respectively. We also give a result about conservative functors, mimicking the proof given for [AHP16, Lemma A.6.]:

Lemma (Lemma 2.1.5)**.** *Let S be a scheme of finite Krull dimension and*

$$pcd_p(S) = \sup_{s\in S} \{cd_p(\kappa(s))\} \in \mathbb{N} \cup \{\infty\},$$

where $\kappa(s)$ is the residue field of a point $s \in S$, is bounded for all prime number p. Then the following holds:

1. *Let $M \in \boldsymbol{DA}^{ét}(S, \mathbb{Z})$ be a motive. Then M is zero if and only if the pullbacks $i_{\bar{s}}^{*}M$ to any geometric point $\bar{s} \to S$ is zero.*

2. *Let f be a morphism in $\boldsymbol{DA}^{ét}(S, \mathbb{Z})$. Then f is an isomorphism if and only if the pullback $i_{\bar{s}}^*(f)$ for any geometric point $\bar{s} \to S$ is an isomorphism.*

In section 2 of this chapter we explore the different notions of étale motivic cohomology. The first one is defined by means of the model for étale motivic complexes with transfers, while the second one is defined as the étale hypercohomology of a complex of étale sheaves (using the étale sheafification of Bloch's complex), known as *Lichtenbaum cohomology*. The advantages of giving the two definitions are that with the first one, we can describe the functorial behaviour of étale motivic cohomology and constructing the analogue of cycle's operations and maps like *specialization map*. Whereas using the second definition, we can use computational tools, such as *Hoschschild-Serre spectral sequence* and its relation with Galois cohomology.

Proposition (Lemma 2.2.16)**.** *Let $p : Y \to X$ be a finite Galois covering of X with Galois group G, then there exists a convergent Hochschild-Serre spectral sequence with abutment the Lichtenbaum cohomology group*

$$E_2^{p,q}(n) = H^p(G, H_L^q(Y, \mathbb{Z}(n))) \Longrightarrow H_L^{p+q}(X, \mathbb{Z}(n)).$$

In section 3 we discuss the birational properties of the étale analogue for 0-cycles, giving examples where the étale analogue is not an invariant for birational maps. We also linked the theory of étale Chow groups with the decomposition of the diagonal in the sense on Bloch-Srinivas [BS83].

The fourth section works as a parallel between classical theory of algebraic cycles and the one we defined in section 2. We define in the étale setting different equivalence relations such as *algebraic, homological and numerical*, establishing the similarities and differences with the properties obtained for the classical case.

During the fifth and last section, we construct the étale analogue of the category of pure motives with integral coefficients, which we call étale Chow motives, and which embeds full faithfully into the triangulated category of étale motives. For that, we given a description of étale correspondences and their actions as morphism of algebraic cycles. Using the theory of étale correspondences, we can construct the category of étale Chow motives, and since this category is an analogue of the one of pure motives, one can recover classical result such as Manin's identity principle.

We also introduce some result about conservative family of functors associated to change of base fields.

Chapter 3

After the results given in [RS16], we continue looking the consequences of such equivalence between Hodge conjectures in the integral étale and rational cases.

We prove a refined version of [RS16, Theorem 1.1] (which can be seen as a direct consequence of the previously cited theorem):

Proposition (see Corollary 3.1.8). *Let X be a complex smooth projective variety and consider a sub-Hodge structure $W \subset H_B^{2k}(X, \mathbb{Z}(k))$ of type (k,k). Then W is L-algebraic, i.e. $W \subset im(c_L^k)$, if and only if $W \otimes \mathbb{Q}$ is algebraic.*

Starting from [RS16, Remark 5.1.a] we use the étale analogue of the generalized Hodge conjecture given there in order to study the classical version. In Proposition 3.2.6 we give a complete proof of the equivalence between the different versions of the generalized Hodge conjecture (usual case and Lichtenbaum) in weight $2k-1$ and level 1, result that was stated in the same remark in [RS16]. For that, we split the proof in two parts: in the first, we prove that the L-generalized Hodge conjecture in weight $2k-1$ and level 1 is equivalent to the fact that a part of the Hodge conjecture for the product of $X \times C$ is true for all smooth and projective curve C, after that we invoke Corollary 3.1.8. To finalize, our main results are the following:

First, we obtain a characterization of the generalized conjecture (for all $X \in \mathrm{SmProj}_{\mathbb{C}}$) given in [RS16] which follows the idea of the classical case, that is, in term of realization of motives previously defined in section 2 and the Hodge conjecture:

Theorem (see Theorem 3.2.8). *The Lichtenbaum generalized Hodge conjecture for all $X \in SmProj_{\mathbb{C}}$ holds if and only if the following two conditions hold:*

- *the Lichtenbaum Hodge conjecture holds,*
- *a homological étale motive is effective if and only if its Hodge realization is effective.*

With this, we obtain as a corollary the following equivalence:

Corollary (see Corollary 3.2.9). *The generalized Hodge conjecture with $\mathbb{Q}$-coefficients holds if and only if the generalized integral L-Hodge conjecture holds.*

Concerning the counter-examples, in Claims 3.1.18 and 3.1.20 we give an explicit description of the torsion classes which arise as counter-examples to the integral Hodge conjecture given in [AH62] and [BO20] respectively. Since in both cases the class that is not algebraic is a torsion class, the main result that we used is the fact that for Lichtenbaum cohomology with finite coefficients we have the isomorphism $H_L^m(X, \mathbb{Z}/\ell^r(n)) \simeq H_{\text{ét}}^m(X, \mu_{\ell^r}^{\otimes n})$ which is a consequence of the Bloch-Kato conjecture proved by Voevodsky (see [CD16, Section 4] for an argument in terms of rigidity of étale motives). We need to remark that the way we use the rigidity theorem are different in both cases: in the first case we consider two things, that the counter-example comes from a Godeaux-Serre variety X, so there is a Serre spectral sequence associated fibration $BG \to Y \to X$, and the Steenrod operations for étale cohomology. For the second case, which comes from the product of a very general curve C of genus ≥ 1 and a smooth Enriques surface S, we used the fact that $\mathrm{Br}(S) = \mathbb{Z}/2$ and the Künneth formula for integral and finite coefficients. After that, in Proposition 3.1.23 we study general properties of the Lichtenbaum cohomology groups of smooth hypersurfaces in $\mathbb{P}_{\mathbb{C}}^{n+1}$ obtaining that their higher Brauer groups are zero and consequently $\mathrm{CH}_L^k(X) \otimes \mathbb{Z}/\ell^r \simeq H_{\text{ét}}^{2k}(X, \mu_{\ell^r}^{\otimes k})$. This allows us in Remark 3.1.25 to give a better description of the Lichtenbaum classes for the Kóllar counter-example and stating the differences with motivic cohomology and the failure of the Hodge conjecture with integral coefficients.

Chapter 4

In the last chapter we focus on the goal of the thesis; find an enrichment for the decomposition of integral motives. Roughly speaking, we can say that this chapter consists in two parts: definition of étale degree map and étale index of 0-cycles, giving examples where the later definition does not agree with the classical case, and decomposition of étale motives using different approaches.

The main results of the first part of the fourth chapter, concern the existence of smooth and projective varieties X over a field of cohomological dimension ≤ 1 whose index $I(X) > 1$ but with étale anaologue $I_{\text{ét}}(X) = 1$, as the following theorems show:

Theorem 1 (Theorem 4.2.4)**.** *There exists a smooth projective surface S over a field k, with $char(k) = 0$ of cohomological dimension ≤ 1, without zero cycles of degree one but $I_{ét}(X) = 1$.*

Theorem 2 (Theorem 4.2.5)**.** *For each prime $p \geq 5$ there exists a field k such that $char(k) = 0$ with $cd(k) = 1$ and a smooth projective hypersurface $X \subset \mathbb{P}^p_k$ with $I_{ét}(X) = 1$ but index $I(X) = p$.*

To find this kind of varieties, we use Proposition 4.2.3 which characterizes some smooth varieties X over a field k of cohomological dimension ≤ 1, the ones such that $\mathrm{Alb}(X_{\bar{k}})_{\mathrm{tors}} = 0$, whose étale degree map is surjective. The proof relies in the fact that the condition $\mathrm{Alb}(X_{\bar{k}})_{\mathrm{tors}} = 0$ impose that $\mathrm{CH}^L_0(X_{\bar{k}})_{\mathrm{hom}}$ is uniquely divisible, thus, with trivial Galois cohomology in positive degrees. After that we remark that the varieties presented in [CM04, Théorème 1.1], [CM04, Théorème 1.2] and [Col05, Theorem 5.1] fulfill the hypothesis of Proposition 4.2.3.

These results give us the first refinement for the existence of $h_{\text{ét}}(X) = h^0_{\text{ét}}(X) \oplus h^+_{\text{ét}}(X) \oplus h^{2d}_{\text{ét}}(X)$ in the category of integral étale motives but not in the category of integral Chow motives. Despite this new refinement of the index of a smooth projective variety, we give an example of how the property $I_{\text{ét}}(X) = 1$ is not always achieved. For Severi-Brauer varieties X, we show that $I_{\text{ét}}(X)$ is greater or equal to the order of the class $[X] \in \mathrm{Br}(k)$, as follows:

Theorem 3 (Theorem 4.2.10)**.** *Let X be a Severi-Brauer variety of dimension d over a field k. Then the image of $\deg_{ét} : CH^d_{ét}(X) \to \mathbb{Z}$ is isomorphic to a subgroup of $Pic(X)$, and in particular $I_{ét}(X) \geq ord([X])$ where $[X]$ is the Brauer class of X in $Br(k)$. Moreover, if $cd(k) \leq 4$ then this subgroup is isomorphic to $Pic(X)$ i.e. $I_{ét}(X) = ord([X])$.*

After that, we prove that this bound also holds for the product of Severi-Brauer varieties. In order to prove that, we give the following generalization of [GS06, Theorem 5.4.10]:

Lemma (Lemma 4.2.14)**.** *Let X be a Severi-Brauer variety of dimension d over a field k. For the product $X^{\times n} := \overbrace{X \times \ldots \times X}^{n-times}$ we then obtain an exact sequence*

$$0 \to Pic(X^{\times n}) \to Pic(\mathbb{P}^d_{\bar{k}} \times \ldots \times \mathbb{P}^d_{\bar{k}})^{G_k} \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \xrightarrow{s} Br(k) \to Br(X^{\times n})$$

where s sends $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i [X] \in Br(k)$.

With this lemma, we can state and prove the following result for a product of Severi-Brauer varieties:

Theorem 4 (Theorem 4.2.15)**.** *Let k be a field and let X be a Severi-Brauer variety over k of dimension d. Then $I_{ét}(X^{\times n}) \geq I_{ét}(X) \geq ord([X])$.*

After that, we move to the decomposition of étale motives in different settings, by using different tools and approaches. Between the ways that we can obtain decomposition of integral étale motives we will use three possible options: the first one is a consequence of [RS16] and the non-existence of transcendental cohomology classes for some complex algebraic varieties:

Proposition (Proposition 4.3.3)**.** *Fixing $k = \mathbb{C}$, let X be a smooth projective complex variety of dimension d such that the groups $H^i_B(X, \mathbb{Q})$ are algebraic for all $i \neq d$. Then $h_{ét}(X)$ admits an integral Chow-Künneth decomposition in $Chow_{ét}(\mathbb{C})$.*

We continue in a more general context with the decomposition of relative étale motives of a commutative group scheme in the category of $\mathrm{DM}_{\text{ét}}(S, \mathbb{Z})$. We define the **homotopy fixed points** and **homotopy orbits** of $\mathfrak{S}_n$ of a motive $M^S_{\text{ét}}(X)$ as follows: knowing that $\mathrm{DM}_{\text{ét}}(S, \mathbb{Z})^{\otimes}$ has a structure of an ∞-category which is monoidal and symmetric, thus we obtain adjunctions

$$\begin{aligned} (\ )^{\mathrm{triv}} : \mathrm{DM}_{\text{ét}}(S, \mathbb{Z}) \leftrightarrows \mathrm{DM}_{\text{ét}}(S, \mathbb{Z})^{B\mathfrak{S}_n} : (\ )^{h\mathfrak{S}_n} := \mathrm{holim}_{B\mathfrak{S}_n}, \\ \mathrm{hocolim}_{B\mathfrak{S}_n} =: (\ )_{h\mathfrak{S}_n} : \mathrm{DM}_{\text{ét}}(S, \mathbb{Z})^{B\mathfrak{S}_n} \leftrightarrows \mathrm{DM}_{\text{ét}}(S, \mathbb{Z}) : (\ )^{\mathrm{triv}}. \end{aligned}$$

With this definitions, one obtains the integral analogue of [AEH15] as follows:

Theorem 5 (Proposition 4.3.14)**.** *Let k be an algebraically closed field and G/k a connected commutative group scheme. Then the morphism*

$$\phi_G : M_{ét}(G) \to \bigoplus_{i=0}^{kd(G)} \left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}$$

is an isomorphism in $DM_{ét}(k, \mathbb{Z})$.

If we apply the following result which mimic the conclusion given in [AHP16, Lemma A.6.]

Lemma (Lemma 2.1.5)**.** *Let S be a scheme which has finite Krull dimension and the punctual p-cohomological dimension is bounded for all prime number p. Then the following holds:*

1. *Let $M \in \boldsymbol{DA}^{\acute{e}t}(S,\mathbb{Z})$ be a motive. Then M is zero if and only if the pullbacks $i_{\bar{s}}^{*}M$ to any geometric point $\bar{s} \to S$ is zero.*

2. *Let f be a morphism in $\boldsymbol{DA}^{\acute{e}t}(S,\mathbb{Z})$. Then f is an isomorphism if and only if the pullback $i_{\bar{s}}^{*}(f)$ for any geometric point $\bar{s} \to S$ is an isomorphism.*

we then obtain the relative version of Proposition 4.3.14:

Theorem 6 (Theorem 4.3.15)**.** *Let S be a scheme which has finite Krull dimension and the punctual p-cohomological dimension is bounded for all prime number p, and let G be a connected commutative scheme over S. Then the morphism*

$$\phi_G : M^S_{\acute{e}t}(G) \to \bigoplus_{i=0}^{kd(G/S)} \left(M_1(G/S)^{\otimes i}\right)^{h\mathfrak{S}_i}$$

is an isomorphism in $DM_{\acute{e}t}(S,\mathbb{Z})$.

We conclude the section about the decomposition of étale motives by giving a result involving the decompostion of the étale Chow motive of the product of Jacobian varieties:

Theorem 7 (Theorem 4.3.19)**.** *Let k be a field of finite cohomological dimension and consider C_i/k a projective smooth curve, for $i \in \{1,\ldots,n\}$. Then the variety $J(C_1) \times \ldots \times J(C_n)$ admits an integral Chow-Künneth decomposition.*

In the same spirit, for an algebrically closed field we can conclude that principally polarized varieties admit a Chow-Künneth decomposition in the étale setting:

Theorem 8 (Theorem 4.3.21)**.** *Let $k = \bar{k}$ be a field and let A be a principally polarized variety. Then there exists a Chow-Künneth decomposition of A.*

This result leads us to conditions that we can impose to a smooth projective variety X over an algebraically closed field in order to obtain the existence of the projectors $p_1^{\text{ét}}(X)$ and $p_{2d-1}^{\text{ét}}(X)$

Theorem 9 (Theorem 4.3.22)**.** *Let X be a smooth projective variety of dimension d over an algebraically closed field k. If $Pic^0(X)$ is a principally polarized variety, then there exists a decomposition of the motive $h_{\acute{e}t}(X)$ as*

$$h_{\acute{e}t}(X) = h^0_{\acute{e}t}(X) \oplus h^1_{\acute{e}t}(X) \oplus h^{+}_{\acute{e}t}(X) \oplus h^{2d-1}_{\acute{e}t}(X) \oplus h^{2d}_{\acute{e}t}(X)$$

Finally, we end the chapter four, with a characterization of isomorphism of étale Chow motives: we obtained an analogue to [Huy18, Lemma 1], which is the characterization of isomorphism in the category of Chow motives over algebraically closed fields.

Theorem 10 (Improved version of Manin's principle)**.** *[Theorem 4.3.43] Let $f : M \to N$ be a morphism in the category $Chow_{\acute{e}t}(k)$. Then f is a isomorphism of motives in $Chow_{\acute{e}t}(k)$ if and only if for Ω an universal domain over k, the induced map $(f_\Omega)_* : CH^*_{\acute{e}t}(M_\Omega) \to CH^*_{\acute{e}t}(N_\Omega)$ given by the base change $f_\Omega : M_\Omega \to N_\Omega$, is bijective.*

Conventions

For a field k we denote the n-dimensional k-projective space as $\mathbb{P}^n_k$ and SmProj_k is the category of smooth and projective reduced k-schemes. Let G be an abelian group, ℓ a prime number and $r \geq 1$, then we denote $G[\ell^r] := \{g \in G \mid \ell^r \cdot g = 0\}$, $G\{\ell\} := \bigcup_r G[\ell^r]$, G_{tors} denotes the torsion sub-group of G and $G_{\text{free}} := G/G_{\text{tors}}$ its torsion free quotient. The prefix "L-" indicates the respective version of some result, conjecture, group, etc. in the Lichtenbaum setting. $H^i_B(X, \mathbb{Z}(n))$ denotes the Betti cohomology groups of X. Continuing with the same hypothesis for G, for an integer p, we set $G[1/p] := G \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. If now G is a profinite group, i.e. can be written as $G = \varprojlim G_i$ with G_i finite groups, and A is a G-module we will consider its cohomology group $H^j(G, A)$ as the continuous cohomology group of G with coefficients in A defined as $H^j(G, A) := \varinjlim H^j(G_i, A^{H_i})$ with H_i running over the open normal subgroups of G such that $G/H_i \simeq G_i$.

Let k be a field, we denote as k^s and $\bar{k}$ the separable and algebraic closure of k respectively. For a prime number ℓ, we denote the $\ell-$cohomological dimension of k as $\mathrm{cd}_\ell(k)$, and we set the cohomological dimension of k to be $\mathrm{cd}(k) := \sup_\ell \{\mathrm{cd}_\ell(k)\}$. Sm_k will denote the category of smooth schemes over k and $X_{\text{ét}}$ denotes the small étale site of X.

Chapter 1

Motives: From pure to mixed motives

The aim of this chapter is to give an overview on the theory of motives, form the first definition of the category of Chow motives to the triangulated category of mixed motives.

We start by giving a quick overview about Chow motives, mentioning the principal definition and results about the theory of classical motives. We will follow as introduction for the theory of motives the references [And04], [MNP13] and [Sch94]. After that, we move to the notion of triangulated category of motives in different contexts. For this we follow the references [CD16] and [Ayo14b] for étale motives, and [MVW06] and [CD19] for a general context of motives in the Nisnevich setting.

In the first section of the chapter, we revisit well-known results about algebraic cycles of a smooth projective variety over a field k. We recall the classical operations on algebraic cycles and the notion of adequate equivalence relations, such as rational, algebraic, homological and numerical equivalence. We focus on rational equivalence, since it gives us the Chow groups. We continue by introducing the concept of correspondences, given the action of an specific algebraic cycle, that work as the morphisms in the category of pure motives.

The second section is devoted for the presentation of the theory of pure motives introduced by Grothendieck. We define the category of motives, depending on an adequate equivalence relation of algebraic cycles. Since the construction of such category is fairly easy but powerful, we present categorical consequences coming from this construction and the relation with realizations to Weil cohomology theories. We then give the description of the Manin principle, introduced by Manin in [Man68], which is a consequence of Yoneda's lemma and helps to characterize isomorphisms in the category of pure motives through an universal property. To illustrate this principle we compute the Chow motive of projective bundles, blow-ups of a smooth projective variety with smooth center and varieties that admit a cellular decomposition, such as Grassmannians. We finish section two by introducing the notion of Chow-Künneth decomposition, which gives the necessary terminology for the development of further techniques for the integral étale case.

Finally in the third section we present the theory of premotivic categories, which provides a more general framework than the theory presented in the second section. We start by giving the definition of Grothendieck's six functor formalism. After that we introduce the notion of premotivic category, give some examples and discuss the relation with the theory of pure motives. Next we present the theory of triangulated categories by giving examples of premotivic categories and their properties, and by giving an expression for the infinite suspension functor that allows us to invert Tate motives. In this section we also introduce several categories of motives which depend on a suitable Grothendieck topology, such as the h-topology, and the Nisnevich and qfh-topologies. We also recall a result of Bondarko [Bon14], which relates the category of pure motives $\mathrm{Chow}(k)_{\mathbb{Q}}$ to $\mathrm{DM}(k, \mathbb{Q})$.

1.1 Algebraic cycles and correspondences

For a field k, we denote as SmProj_k the category of smooth projective varieties over a field k. Throughout this thesis a variety will be a reduced scheme.

Let us give a quick introduction to algebraic cycles, using the references [Ful98] and [EH16]. After that we move on to the definition of correspondences. As we mainly want use them in the theory of pure motives, we follow the reference [MNP13].

Algebraic cycles

An algebraic cycle on a variety X is a formal finite linear combination $Z = \sum n_\alpha Z_\alpha$ of irreducible subvarieties Z_α, where $n_\alpha \in \mathbb{Z}$ for all α. Given an integer $i \geq 0$ we define the abelian group of codimension i-cycles of X, denoted by

$$Z^i(X) := \{\text{codim } i \text{ cycles on } X\}.$$

Also it is important to consider the group $Z^i(X)$ with coefficients in a field $\mathbb{K}$, which in almost all the cases will be $\mathbb{Q}$, denoted by $Z^i(X)_{\mathbb{K}} = Z^i(X) \otimes_{\mathbb{Z}} \mathbb{K}$.

Along with the sum of algebraic cycles, we can define other operations on cycles:

- *Cartesian product of cycles:* The usual cartesian product of subvarieties can be linearly extended to product of cycles, but now we shall consider this cycle on the product variety.

- *Pushforward:* Let $f : X \to Y$ be a proper morphism of $k-$varieties and $Z \subset X$ an irreducible subvariety. We define the degree as follows

$$\deg(Z/f(Z)) := \begin{cases} [k(Z) : k(f(Z))] & \text{if } \dim f(Z) = \dim Z \\ 0 & \text{otherwise.} \end{cases}$$

  Define the push-forward as the function $f_* : Z^i(X) \to Z^i(Y)$ which acts via $f_*(Z) = \deg(Z/f(Z))f(Z)$.

- *Intersection:* **(Not always defined)** Let V_1 and V_2 be two subvarieties of X of codimension i and j respectively. They intersect in a union of subvarieties Z_α of codimension greater or equal to $i+j$, see [Har77, Theorem 7.2, Section I]. If the codimension of Z_α is equal to $i+j$ for every α we say that the *intersection is proper*.

  In the case of a proper intersection, the *intersection number* is defined as follows:

  $$i(V_1 \cdot V_2; Z) := \sum_r (-1)^r \ell_A(\mathrm{Tor}_r^A(A/I(V_1), A/I(V_2)))$$

  where $AA = \mathcal{O}_{X,Z}$ is the local ring, $I(V_i)$ is the ideal of the variety V_i in the ring A. Then, the intersection product can be defined as

  $$V_1 \cdot V_2 = \sum_\alpha i(V_1 \cdot V_2; Z_\alpha) Z_\alpha$$

- *Pull-back:* **(Not always defined)** Let $f : X \to Y$ be a morphism in SmProj_k and $Z \subset Y$ any subvariety. Let $\Gamma_f \subset X \times Y$ be the graph of f. If Γ_f meets $X \times Z$ properly then we can define the pull-back function as

  $$f^*(Z) := [\mathrm{pr}_X]_*(\Gamma_f \cdot (X \times Z))$$

  where $\mathrm{pr}_X : X \times Y \to X$ is the projection. With the notion of pull-back we can define the intersection for cycles $V, W \in Z(X)$ as

  $$V \cdot W = \Delta_X^*(V \times W)$$

  where $\Delta_X : X \hookrightarrow X \times X$ is the diagonal embedding. Although for a general f the pull-back is not necessarily defined, if f is a flat morphism, then by [Ful98, Lemma 1.7.1], for a subscheme Z the pull-back f^*Z is the inverse image scheme $f^{-1}(Z)$ which is always defined, and this can be extended linearly to cycles inducing a homomorphism $f^* : Z^i(Y) \to Z^i(X)$.

- *Correspondences:* **(Not always defined)** A correspondence from X to Y is simply a cycle on the product $X \times Y$. A correspondence $Z \in Z^t(X \times Y)$ acts as follows:

  $$Z(T) := [\mathrm{pr}_X]_*(Z \cdot (T \times Y)) \in Z^{i+t-d}(Y)$$

  $T \in Z^i(X)$, $d = \dim(X)$ whenever this is defined.

The last three operations are not always defined because the intersection product is not always defined for any two projective varieties. As we will see in the next subsection, these operations become well-defined after taking the quotient of the group of algebraic cycles by an *adequate equivalence relation.*

Equivalence relations on algebraic cycles

Let us consider the graded group $Z(X) = \bigoplus_i Z^i(X)$. We can consider an equivalence relation $\sim$ on this group (which we are going to call "*good*" or "*adequate*") if it has the following properties:

1. Compatibility with grading and addition.

2. Compatibility with products: if $Z \sim 0$ then for all $Y \in \mathrm{SmProj}_k$ we have $Z \times Y \sim 0 \in Z(X \times Y)$.

3. Compatibility with intersection: if $X \sim 0$ and $X \cdot Y$ is defined, then $X \cdot Y \sim 0$.

4. Compatibility with projections: if $Z \sim 0$ in $Z(X \times Y)$, then $(\mathrm{pr}_X)_*(Z) \sim 0$ in $Z(X)$.

5. Moving lemma: given $Z, Y_1, \dots, Y_m \in Z(X)$ there exists $Z' \sim Z$ such that $Z' \cdot Y_i$ is defined for all $i \in \{1, \dots, m\}$.

For an adequate equivalence relation and a given integer i, we define the subgroup $Z^i_\sim(X) \subset Z^i(X)$ of $\sim$-trivial cycles as follows: $Z \in Z^i_\sim(X) \iff Z \in Z^i(X)$ and $Z \sim 0$. It follows from the first property of an adequate equivalence that $Z^i_\sim(X)$ is a subgroup of $Z^i(X)$, so we can define quotient $A^i_\sim = Z^i(X)/Z^i_\sim(X)$ of $\sim$-cycles of codimension i. This group has richer structure than $Z(X)$: because of the fifth property of an adequate relation $A^*_\sim(X)$ is a ring, with the product induced from intersection of cycles. Having a well defined intersection product means that every operation of algebraic cycles is now defined in the ring $A^*_\sim(X)$ as stated in the following lemma:

Lemma 1.1.1. *For any adequate equivalence relation $\sim$ we have:*

1. *$A^*_\sim(X)$ is a ring with a product operation induced from the intersection of cycles*

2. *For any morphism $f : X \to Y$ in SmProj$_k$ the maps f_* and f^* induce well defined ring homomorphisms $f_* : A^*_\sim(X) \to A^*_\sim(Y)$ and $f^* A^*_\sim(X) \to A^*_\sim(Y)$.*

3. *A correspondence Z from X to Y of degree r induces $Z_* : A^i_\sim(X) \to A^{i+r}_\sim(Y)$ and equivalent correspondences induce the same correspondence Z_*.*

In the following subsections we present five adequate equivalence relations, but in the sequel we mainly focus on rational equivalence.

Rational Equivalence

This is the adequate equivalence relation that give us the Chow groups for a variety X. Since we will use it throughout the following chapter, we recall the definitions and basic properties of this relation. Let $k(X)$ be the function field of X and consider $f \in k(X)$ The divisor of f is defined as follows:

- $\mathrm{div}(f) = \sum_{Y \subset X} \mathrm{ord}_Y(f) \cdot Y$ where Y is a subvariety of X of codimension 1, and the order is defined as $\mathrm{ord}_Y : k(X)^* \to \mathbb{Z}$, let $A = \mathcal{O}_{X,Y}$ local ring, $f \in A$ and $\mathrm{ord}_Y(f) = \ell_A(A/(f))$, where ℓ_A is the length of the $A-$module.

From this, it follows that the divisor $\operatorname{div}(f)$ of a function $f \in k(Y)^*$ on an irreducible subvariety $Y \subset X$ is a codimension 1 cycle on Y, and therefore if Y is of codimension $i-1$ in X, $\operatorname{div}(f) \in Z^i(X)$. Therefore, for a codimension i cycle $Z \subset X$, we have $Z \sim_{\text{rat}} 0$ if there exist (Y_α, f_α) codimension $i-1$ and irreducible cycles such that $Z = \sum \operatorname{div}(f_\alpha)$. Let $X^{(i)}$ be the collection of irreducible codimension i subvarieties of X. We have

$$Z^i_{\text{rat}}(X) = \operatorname{Im}\left\{ \bigoplus_{Y \in X^{(i-1)}} k(Y)^* \xrightarrow{\operatorname{div}} \bigoplus_{Z \in X^{(i)}} \mathbb{Z} \right\},$$

and the Chow group of codimension i cycles on X is defined as follows

$$\operatorname{CH}^i(X) = \operatorname{coker}\left\{ \bigoplus_{Y \in X^{(i-1)}} k(Y)^* \xrightarrow{\operatorname{div}} \bigoplus_{Z \in X^{(i)}} \mathbb{Z} \right\}$$

- We say that D is a principal divisor of X (cycle of codimension 1) if there exists $f \in k(X)$ such that $D = \operatorname{div}(f)$. We can define a equivalence between divisors called *linear equivalence*, denoted by $\sim_{\text{lin}}$, which is defined as follows: $D_1 \sim_{\text{lin}} D_2 \iff \exists\, f \in k(X)$ such that $D_1 - D_2 = \operatorname{div}(f)$.

- For divisors linear equivalence and rational equivalence coincide.

- If X is smooth, the quotient group $\operatorname{Div}(X)$ by the subgroup $\{\operatorname{div}(f) \mid f \in k(X)^\times\}$ (principal divisors) is the *Picard group* $\operatorname{Pic}(X)$.

Also, we can define the morphism

$$N : \bigoplus_{V \in X_{(n+1)}} k(V)^\times \to \bigoplus_{W \in Y_{(n+1)}} k(W)^\times$$

in the following way: If the field extension is of infinite degree we define $N|_{k(V)} = 0$, otherwise we can consider the usual norm of between fields $N : k(V)^\times \to k(W)^\times$. With those maps, the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_{V \in X} k(V)^\times \arrow[r, "\operatorname{div}"] \arrow[d, "N"'] & Z_n(X) \arrow[d, "f_*"] \\
\bigoplus_{W \in Y} k(W)^\times \arrow[r, "\operatorname{div}"'] & Z_n(Y)
\end{tikzcd}$$

This information an be summarized in the following theorem:

Theorem 1.1.2 ([Ful98, Proposition 1.4]). *Let $f : X \to Y$ be a proper surjective morphism of normal varieties, and let $r \in k(X)^*$. Then*

1. $f_*[div(r)] = 0$ *if* $dim(Y) < dim(X)$.

2. $f_*[div(r)] = [div(N(r))]$ *if* $dim(Y) = dim(X)$.

Definition 1.1.3 (Alternative definition)**.** *Suppose X is a smooth and projective variety, then $Z_1, Z_2 \in Z^i(X)$ are rationally equivalent if and only if there exist $W \in Z^i(X \times \mathbb{P}^1_k)$ and $a, b \in \mathbb{P}^1_k$ such that, defining $W(t) := (pr_X)_*(W \cdot (X \times t))$, we have $W(a) = Z_1$ and $W(b) = Z_2$.*

These two definition are equivalent:

Proposition 1.1.4 ([Ful98, Proposition 1.6])**.** *A cycle $\alpha \in Z^i(X)$ is rationally equivalent to zero if and only if there are subvarieties $V_1, \ldots, V_t$ of $X \times \mathbb{P}^1$ with codimension $i-1$ such that the projections from V_i to $\mathbb{P}^1$ are dominant, with*

$$\alpha = \sum_{i=1}^{t} [V_i(0)] - [V_i(\infty)]$$

in $Z^i(X)$.

Lemma 1.1.5 ([Blo10, Lemma 1A.1])**.** *Let X be a smooth variety over an algebraically closed field k, Y any $k-$variety. Let $i \geq 0$, then writing $K = k(Y)$ we have*

$$CH^i(X_K) \simeq \varinjlim_{U \subset Y \ open} CH^i(X \times U).$$

Theorem 1.1.6 ([MNP13, Theorem 1.2.6])**.**

1. *If $f : X \to Y$ is a morphism in $SmProj_k$, then $f^* : CH^*(Y) \to CH^*(X)$ is a graded ring homomorphism, and $f_* : \bigoplus_j CH_j(X) \to CH_j(Y)$ is an additive graded homomorphism of degree $dim(Y) - dim(X)$.*

2. *if $X, Y \in SmProj_k$, then $Z \in CH^{e+dim(X)}(X \times Y)$ induces an homomorphism $Z_* : CH^*(X) \to CH^*(Y)$ of degree e.*

3. *Local exact sequence: if $i : Y \hookrightarrow X$ is a closed embedding and $j : U := X - Y \hookrightarrow X$ the associated open embedding, then we have an exact sequence*

$$CH_q(Y) \xrightarrow{i_*} CH_q(X) \xrightarrow{j^*} CH_q(U) \to 0$$

4. *The homotopy property holds: the projection $pr_X : X \times \mathbb{A}^n_k \to X$ induces an isomorphism $pr_X^* : CH^i(X) \xrightarrow{\sim} CH^i(X \times \mathbb{A}^n_k)$*

We can say more about the structure of the Chow ring for some smooth projective varieties, such as projective bundles, Blow-ups with smooth center and varieties that admit a cellular decomposition.

Theorem 1.1.7 ([EH16, Theorem 9.6])**.** *Let E be a vector bundle of rank $r+1$ on a smooth projective scheme X, and let $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \in CH^1(\mathbb{P}(E))$. Let $\pi : \mathbb{P}(E) \to X$ be the projection. The map $\pi^* : CH^*(X) \to CH^*(\mathbb{P}(E))$ is an injection of rings, and via this map we have*

$$CH^*(\mathbb{P}(E)) \cong CH^*(X)[\xi]/(\xi^{r+1} + c_1(E)\xi^r + \ldots + c_{r+1}(E)).$$

In particular the group homomorphism

$$\bigoplus_{i=0}^{r} CH^*(X) \to CH^*(\mathbb{P}(E))$$
$$(a_0, \ldots, a_r) \to \sum_{i=0}^{r} \xi^i \pi^*(a_i)$$

is an isomorphism, which gives us $CH^*(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r} \xi^i CH^*(X)$.

Remark 1.1.8. Let $X, Y \in \mathrm{SmProj}_k$ and E be a locally free sheaf of rank $r+1>1$ on X. Let $\pi_1 : X \times Y \to X$ be the projection in the first component,which is dominant, then we have the following diagram:

$$\begin{tikzcd}
\mathbb{P}(\pi_1^* E) \arrow[r, "\pi_1'"] \arrow[d, "\pi'"'] & \mathbb{P}(E) \arrow[d, "\pi"] \\
X \times Y \arrow[r, "\pi_1"'] & X
\end{tikzcd}$$

and using compatibility of the Chern classes with pull-back, i.e., $c_i(f^*E) = f^*c_i(E)$, we obtain the isomorphism $\bigoplus_{i=0}^{r} \mathrm{CH}^*(X \times Y) \to \mathrm{CH}^*(\pi_1^*\mathbb{P}(E))$.

Theorem 1.1.9 ([Köc91, Appendix A]). *Suppose* $\pi : X \to S$ *is a flat morphism of relative dimension* n *and that* X *admits a filtration by closed subschemes* $X = X_0 \supset \ldots \supset X_k \supset$ *such that* $X_{i-1} - X_i \cong \mathbb{A}_S^{n-d_i}$ *for some* $d_i \in \mathbb{Z}$. *There is an isomorphisms of Chow groups*

$$\bigoplus_{i=0} CH^{i-d_i}(S) \to CH^i(X)$$

which is functorial with respect to cartesian squares

$$\begin{tikzcd}
X' \arrow[r, "\pi_1'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\
S' \arrow[r, "\pi_1"'] & S
\end{tikzcd}$$

Finally, let X be a smooth variety, let Z be a subvariety of X of codimension m. Let $i : Z \hookrightarrow X$ be the inclusion map. We associate $\pi : W := \mathrm{Bl}_Z X \to X$ the blow-up of X along Z. The exceptional divisor E of W is said to be $\pi^{-1}(Z)$ and $\mathcal{N}_{Z/X}$ the normal bundle of Z in X.

Theorem 1.1.10 ([EH16, Theorem 13.14]). *Let* $i : Z \to X$ *be the inclusion of a smooth subvariety of codimension* m *in a smooth variety* X, $\pi : W \to X$ *the blow-up of* X *along* Z *and* E *the exceptional divisor with inclusion* $j : E \to W$. *If* $\mathcal{Q}$ *is the universal quotient* *bundle on $E \cong \mathbb{P}\mathcal{N}_{Z/X}$, there is a split exact sequence of additives groups, preserving the grading by dimensions*

$$0 \to CH^*(Z) \xrightarrow{(i_*,h)} CH^*(X) \oplus CH^*(E) \xrightarrow{(\pi^* j_*)} CH^*(W) \to 0.$$

where $h : CH^*(Z) \to CH^*(E)$ is defined by $h(\alpha) = -c_{m-1}(\mathcal{Q})\pi_E^*(\alpha)$.

Algebraic equivalence

The definition is similar to the given in the rational equivalence, but instead of considering $\mathbb{P}^1$ we can consider any smooth curve C, such that $a, b \in C$.

Definition 1.1.11. *$Z_1 \sim_{alg} Z_2$ if and only if there is a smooth irreducible curve C, $W \in Z^i(C \times X)$ and two points $a, b \in C$ such that $Z_1 - Z_2 = W(a) - W(b)$.*

Remark 1.1.12. It is possible to define rational equivalence by using a smooth projective variety M instead of a curve, fulfilling the condition $a, b \in M$, but we can consider a curve C immersed in M such that $a, b \in C$.

Remark 1.1.13. The definition of algebraic equivalence gives us an important result: if $Z_1 \sim_{\text{rat}} Z_2 \Longrightarrow Z_1 \sim_{\text{alg}} Z_2$ i.e. $\sim_{\text{rat}}$ is finer than $\sim_{\text{alg}}$, but in general the two definitions do not coincide. Let X be an elliptic curve and let a, b be different points on $X(k)$, and define the cycle $a - b$. This cycle is not rationally equivalent to zero (see [Sil09, Corollary 3.5]), but it is algebraically equivalent to zero because its degree is zero.

Smash nilpotent equivalence

Let X be a smooth projective variety. We will denote the n-th cartesian product of itself as X^n (the same notation will be used for cycles of X). Then we will say that Z is smash nilpotent equivalent to zero, denoted $Z \sim_{\otimes} 0$, if and only if there exists $n \in \mathbb{N}$ such that $Z^n \sim_{\text{rat}} 0$ on X^n.

It is easy to see that from the definition we have $Z^i_{\text{rat}}(X) \subset Z^i_{\otimes}(X)$, but the assertion $Z^i_{\text{alg}}(X) \subset Z^i_{\otimes}(X)$ is not quite as straightforward.

Theorem 1.1.14 (Voevodsky-Voisin)**.** *[[MNP13, Theorem B-1.2]] We have that $Z^i_{alg}(X)_{\mathbb{Q}} \subset Z^i_{\otimes}(X)_{\mathbb{Q}}$.*

Homological equivalence

Let F be a field of characteristic 0. To define the *Homological equivalence* first we shall recall what is a *Weil-cohomology theory.* A Weil cohomology theory is a graded functor H between the category $\mathrm{SmProj}_k^{\mathrm{op}}$ and the category of finite dimensional graded vector spaces over the field F, which satisfies the following axioms:

1. There exists a cup product between $H(X) \times H(X) \to H(X)$, which is graded and super-commutative

2. We have the Poincaré duality: there is a trace isomorphism $\mathrm{Tr} : H^{2d}(X) \xrightarrow{\sim} F$ (X being irreducible and equidimensional) such that

$$H^i(X) \times H^{2d-i}(X) \xrightarrow{\cup} H^{2d}(X) \xrightarrow{\sim} F$$

is a perfect pairing

3. The Künneth formula holds:

$$H(X) \otimes H(Y) \xrightarrow{(\mathrm{pr}_X)^* \otimes (\mathrm{pr}_Y)^*} H(X \times Y)$$

is a graded isomorphism.

4. There is a *cycle class* map $\gamma_X : \mathrm{CH}^i(X) \to H^{2i}(X)$ which is:

   - functorial in the sense that for $f : X \to Y$ in SmProj_k, we have $f^* \circ \gamma_Y = \gamma_X \circ f^*$ and $f_* \circ \gamma_X = \gamma_Y \circ f_*$.
   - compatible with intersection product, i.e. $\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$.
   - compatible with points P, which means the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CH}^0(X) \arrow[r, "\gamma_P"] \arrow[d, "\deg"'] & H^0(P) \arrow[d, "\mathrm{Tr}"] \\
\mathbb{Z} \arrow[r, equal] & F
\end{tikzcd}$$

5. Weak Lefschetz property holds: if $i : Y_{d-1} \hookrightarrow X_d$ is a smooth hyperplane section, then

$$H^i(X) \xrightarrow{i^*} H^i(Y) \text{ is } \begin{cases} \text{an isomorphism for } i < d-1 \\ \text{injective for } i = d-1. \end{cases}$$

6. Hard Lefschetz property holds: the Lefschetz operator $L(\alpha) = \alpha \cup \gamma_X(Y)$ induces isomorphisms

$$L^{d-1} : H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X), \quad 0 \leq i \leq d.$$

Example 1.1.15. *1. Some examples of Weil cohomology theories (if the characteristic of k is equal to zero and $k \subset \mathbb{C}$):*

- *Singular cohomology group $H^i(X_{an})$ with $\mathbb{Q}$ or $\mathbb{C}$ coefficients. X_{an} denotes the complex manifold associated to X.*
- *The classical de Rham cohomology $H_{dR}(X_{an}, \mathbb{C})$.*
- *The algebraic de Rham cohomology $H^i_{dR}(X, \mathbb{C}) := \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k})$*

2. *If $X \in SmProj_k$, consider the base change to the algebraic closure $X_{\bar{k}} = X \otimes_k \bar{k}$. We define the étale cohomology of X as follows: let ℓ be a prime number different from the characteristic of the field k, let us recall the definitions*

$$H^i_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell) := \varprojlim_n H^i_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}/\ell^n)$$
$$H^i_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell) := H^i_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

For further details about étale cohomology see [Mil80].

3. *For a perfect field k and a smooth projective k-variety X, one has crystalline cohomology $H^i_{crys}(X/W(k)) \otimes K$, where K is the field of fractions of the Witt ring $W(k)$.*

Definition 1.1.16. *Let X be an equidimensional smooth projective variety over k and $Z \in Z^i(X)$. For a given a Weil cohomology theory H, we define $Z \in Z \sim_{hom} 0 \iff \gamma_X(Z) = 0$.*

Remark 1.1.17. It is important to say that the homological equivalence depends on the Weil cohomology theory we are working with.

An important fact, is that we obtain a second relation between different equivalences, if $Z_1 \sim_{\text{alg}} Z_2 \implies Z_1 \sim_{\text{hom}} Z_2$. This follows because two points on a curve are homologically equivalent and the properties of the cycle class map.

In the same way, we can obtain that $Z^i_{\otimes}(X) \subset Z^i_{\text{hom}}(X)$. Let us consider that $Z \sim_{\otimes} 0$, which means that exists $n \in \mathbb{N}$ positive, such that $Z^n \sim_{\text{alg}} 0$. Considering its cycle class

$$\gamma_{X^n}(Z^n) = \bigotimes_{i=1}^{n} \gamma_X(Z) = 0 \in H^{2in}(X^n)$$

then $\gamma_X(Z) = 0$.

Numerical equivalence

Let $X \in \text{SmProj}_k$ be a equidimensional and irreducible variety. For $Z \in Z^i(X)$ we say $Z \sim_{\text{num}} 0$ if and only if for every $W \in Z^{d-i}(X)$ (with d dimension of X) where the product $Z \cdot W$ is defined, we have $\deg(Z \cdot W) = 0$.

If $Z \in Z^i_{\text{hom}}(X)$ (for a given cohomology theory H) with $i < d$, and $W \in Z^{d-i}(X)$ such that $Z \cdot W$ is defined, then by functorial properties of the degree map we obtain

$$\begin{aligned} \deg(Z \cdot W) &= \text{Tr}(\gamma_X(Z \cdot W)) \\ &= \text{Tr}(\gamma_X(Z) \cup \gamma_X(W)) = 0, \end{aligned}$$

which gives us the assertion $Z^i_{\text{hom}}(X) \subset Z^i_{\text{num}}(X)$.

Theorem 1.1.18 (Matsusaka)**.** *Let k be an algebraically closed field, then for divisors we have the equality $Z^1_{alg}(X) = Z^1_{hom}(X) = Z^1_{num}(X)$.*

If k is an algebraically closed field, it is conjectured that $Z^i_{\text{hom}}(X) = Z^i_{\text{num}}(X)$ for all i. This is one of the standard conjectures known as the conjecture $D(X)$. If k is algebraically closed field of characteristic zero, the equality is known for $i = 2$, and for curves and abelian varieties.

Remark 1.1.19. The equivalence relations previously presented are adequate equivalence relations. For more details about algebraic, homological and numerical equivalence, see [Ful98, Chapter 19] and [MNP13, Chapter 1].

Finally, we can conclude the following chain of inclusions between the group of cycles that are equivalent to zero by different relations

$$Z^i_{\text{rat}}(X) \subset Z^i_{\text{alg}}(X) \subset Z^i_{\text{hom}}(X) \subset Z^i_{\text{num}}(X)$$
$$Z^i_{\text{rat}}(X)_{\mathbb{K}} \subset Z^i_{\text{alg}}(X)_{\mathbb{K}} \subset Z^i_{\otimes}(X)_{\mathbb{K}} \subset Z^i_{\text{hom}}(X)_{\mathbb{K}} \subset Z^i_{\text{num}}(X)_{\mathbb{K}} \text{ if } \mathbb{Q} \subset \mathbb{K}$$

Lemma 1.1.20 ([And04, Lemme 3.2.2.1])**.** *Rational equivalence $\sim_{rat}$ is the finest adequate equivalence relations, and numerical equivalence is the coarsest.*

Proof. Let $\sim$ be an adequate equivalence relation, if $Z \sim 0$, by properties of adequate equivalence relations $Z \cdot Y \sim 0$ for Y, whenever the intersection product is defined, which gave us $Z^i_{\sim}(X) \subset Z^i_{\text{num}}(X)$.

Now, we need to prove $Y \sim_{\text{rat}} 0 \implies Y \sim 0$. The general idea is prove that $[0] \sim [\infty]$ on $\mathbb{P}^1$ (in general any two points are related) by using the properties of an *adequate relation*. By the moving lemma, there is a cycle $\sum n_i[x_i] \sim [1]$ such that the intersection product $\sum n_i[x_i] \cdot [1]$ is well defined, i.e. $x_i \neq 1$ for all i. Let $\alpha = \sum n_i[x_i] - 1 \sim 0$ and let Γ_f be the graph cycle of $f(x) = 1 - \prod \left(\frac{x - x_i}{1 - x_i}\right)^{m_i}$ with $m_i > 0$. Then we have a sequence of implications of equivalences to zero of different algebraic cycles:

$$\alpha \sim 0 \implies \alpha \times \mathbb{P}^1 \sim 0 \implies \Gamma_f \cdot (\alpha \times \mathbb{P}^1) \sim 0 \implies (\text{pr}_X)_* \left({}^t\Gamma_f \cdot (\alpha \times \mathbb{P}^1)\right) \sim 0$$

we obtain that $mn[1] \sim m[0]$ with $n = \sum n_i$ and $m = \sum m_i$, since m is arbitrary we conclude that $n[1] \sim [0]$. Applying $x \to 1/x$ we obtain $n[1] \sim [\infty] \implies [0] \sim [\infty]$. □

Remark 1.1.21. Another important conjecture about algebraic cycles, known as Voevodsky's conjecture, which states that $Z^i_{\otimes}(X) = Z^i_{\text{num}}(X)$. Note that the last conjecture implies the standard conjecture $D(X)$ for every Weil cohomology theory, since Voevodsky's conjecture is independent of the choice of a Weil cohomology theory.

Correspondences

Let X and Y be in SmProj_k. For a given adequate equivalence relation $\sim$, we define $\text{Corr}^r_{\sim}(X, Y)$, the group of correspondences of degree r from X to Y, as follows: When X is an equi-dimensional variety of dimension d, $\text{Corr}^r_{\sim}(X, Y) = A^{d+r}_{\sim}(X \times Y)_{\mathbb{Q}}$. If $X = \amalg X_i$ where X_i is a connected variety, then

$$\text{Corr}^r_{\sim}(X, Y) = \bigoplus \text{Corr}^r_{\sim}(X_i, Y) \subset A^*_{\sim}(X \times Y)_{\mathbb{Q}}.$$

Let $\mathrm{pr}_{XZ} : X \times Y \times Z \to X \times Z$ be the projection (analogously for pr_{XY} and pr_{YZ}), $f \in \mathrm{Corr}_\sim(X, Y)$ and $g \in \mathrm{Corr}(Y, Z)$, then we define the composition of correspondences $g \circ f \in \mathrm{Corr}_\sim(X, Z)$ by the formula

$$g \circ f = [\mathrm{pr}_{XZ}]_* \{(f \times Z) \cdot (X \times g)\},$$

in the case when $X = Y = Z$ then $\mathrm{Corr}_\sim(X, X)$ has a ring structure, where the cycle of the diagonal $\Delta(X) \subset X \times X$ acts as the identity element. It is necessary to remark that the composition is well defined, because the intersection product in $A^*_\sim(X \times Y \times Z)$ is well defined, and in general is not commutative.

Definition 1.1.22. *A projector for X is an element (or also an idempotent element) $p \in Corr_\sim(X \times X)$ such that $p \circ p = p$. Note that a projector p has degree 0.*

Let $\phi : X \to Y$ be a morphism of varieties, with X and Y irreducible varieties of dimension d and e respectively. Let $\Gamma_\phi \subset X \times Y$ the associated graph of ϕ, this define $\phi_* := \Gamma_\phi \in \mathrm{Corr}^{e-d}_\sim(X, Y)$ and $\phi^* := {}^t\Gamma_\phi \in \mathrm{Corr}^0_\sim(Y, X)$.

Example 1.1.23. *Suppose ϕ is a generically finite morphism of degree r and $d = e$, then $\phi_* \circ \phi^*$ defines a correspondence from Y to Y of degree 0 which in fact can be described as*

$$\phi_* \circ \phi^* = [pr_{YY}]_* \left\{({}^t\Gamma_\phi \times Y) \cdot (Y \times \Gamma_\phi)\right\}$$

Let us notice that the cycle $[pr_{YY}]_* \left\{({}^t\Gamma_\phi \times Y) \cdot (Y \times \Gamma_\phi)\right\}$ can be see as the pushforward of the following map composition

$$\begin{array}{c} X \xrightarrow{\phi \times id_X \times \phi} Y \times X \times Y \xrightarrow{pr_{YY}} Y \times Y \\ x \xrightarrow{\phi \times id_X \times \phi} (\phi(x), x, \phi(x)) \xrightarrow{pr_{YY}} (\phi(x), \phi(x)) \end{array}$$

which is the same cycle resulted of the image of the morphism's composition $X \xrightarrow{\phi} Y \xrightarrow{\Delta_Y} Y \times Y$. Using the pushforward of the morphisms ϕ and Δ_Y we obtain

$$CH^*(X) \xrightarrow{\phi_*} CH^*(Y) \xrightarrow{(\Delta_Y)_*} CH^*(Y \times Y)$$

where $\phi_*([X]) = r[Y]$ and $(\Delta_Y)_*([Y]) = id_Y$, therefore $\Gamma_\phi \circ {}^t\Gamma_\phi = r id_Y$.

Any correspondence induces a homomorphism of groups between cycles of some codimension depending of the degree of the correspondence. Let $f \in \mathrm{Corr}^r_\sim(X, Y)$, then we define the induced homomorphism as follows:

$$\begin{aligned} f_* : A^i_\sim(X)_\mathbb{Q} &\to A^{i+r}_\sim(Y)_\mathbb{Q} \\ Z &\to f_*(Z) := (\mathrm{pr}_Y)_* \{f \cdot (\mathrm{pr}_X)^*(Z)\} \end{aligned}$$

If f has degree zero, then the homomorphism respects the degree. In the same way for $f \in \mathrm{Corr}^r_\sim(X, Y)$ it is possible to define an operation on a Weil cohomology (only in the cases when $\sim$ is finer or equal than $\sim_{\mathrm{hom}}$)

$$\begin{aligned} f_* : H^i(X) &\to H^{i+2r}(Y) \\ \alpha &\to f_*(\alpha) := \mathrm{pr}_Y\{\gamma_{X \times Y}(f) \cup (\mathrm{pr}_X)^*(\alpha)\} \end{aligned}$$

Lemma 1.1.24 (Lieberman's lemma, [MNP13, Lemma 2.1.3])**.** *Let $f \in Corr_\sim(X,Y)$, $\alpha \in Corr_\sim(X,X')$, $\beta \in Corr_\sim(Y,Y')$, then $(\alpha\times\beta)_*(f) = \beta\circ f\circ {}^t\alpha$.*

1.2 Theory of pure motives

Definition of motives

The construction of the category of classical motives is simple and does not depend on the *standard conjectures*. We will proceed by the definition of motives with respect to an adequate equivalence relation $\sim$.

Definition 1.2.1. *The category $\mathcal{M}_\sim(k)$ of k-motives with respect to the adequate equivalence relation $\sim$ is defined as follows: an object of this category is a triplet (X,p,m) where X is in $SmProj_k$, $p\in Corr^0_\sim(X,X)$ is a projector and $m\in\mathbb{Z}$. If (X,p,m) and (Y,q,n) are motives then*

$$Hom_{\mathcal{M}_\sim(k)}((X,p,m),(Y,q,n)) = q\circ Corr^{n-m}_\sim(X,Y)\circ p.$$

A morphism $f:(X,p,m)\to(Y,q,n)$ is of the form $q\circ g\circ p$ where g is a correspondence of degree $n-m$. There is another way to see the morphism of motives. Consider the subgroup

$$M((X,p,m),(Y,q,n)) := \{g\in \mathrm{Corr}^{n-m}_\sim(X,Y)\mid g\circ p = q\circ g\}$$

an we define an equivalence relation by declaring $g\approx 0$ if and only if $p\circ g = g\circ q = 0$, then $g\approx g\circ p\approx q\circ g\approx q\circ g\circ p$. In the subgroup $M((X,p,m),(X,p,m))$ we have $p\approx \mathrm{id}_X$, the same for (Y,q,n) and q.

Let $[g]$ be the equivalence class, then $[f]=[g]$, therefore

$$\mathrm{Hom}_{\mathcal{M}_\sim(k)}((X,p,m),(Y,q,n)) = M((X,p,m),(Y,q,n))/\approx .$$

Remark 1.2.2.

1. By properties of the group of correspondences and the idempotent elements p and q, the category $\mathcal{M}_\sim(k)$ is additive, $\mathbb{Q}-$linear and pseudoabelian, but in general is not abelian (see [Sch94, §3.5]). Another important fact of the category of motives comes from Jannsen's theorem, which states that $\mathcal{M}_\sim(k)$ is an abelian semi-simple category if and only if the equivalence relation used in the definition of the category is the *numerical* equivalence (see [Jan00] or [MNP13, Theorem 3.2.1]).

2. Notice that by changing the coefficients of the correspondences, to its integral version, we can define the integral version of the category $\mathcal{M}_\sim(k)$, that we denote by $\mathcal{M}_\sim(k)_\mathbb{Z}$.

3. When $\sim=\sim_{\mathrm{rat}}$ we denote the category $\mathcal{M}_\sim(k)$ (resp. $\mathcal{M}_\sim(k)_\mathbb{Z}$) by $\mathrm{Chow}(k)_\mathbb{Q}$ (resp. by $\mathrm{Chow}(k)_\mathbb{Z}$).

In the general case, we have a (contravariant) functor $h_\sim$ which acts as follows:

$$\begin{aligned} h_\sim : \mathrm{SmProj}_k &\to \mathcal{M}_\sim(k) \\ X &\to h_\sim(X) = (X, \Delta_X, 0) \\ (f : X \to Y) &\to h_\sim(f) = {}^t\Gamma_f \in \mathrm{Corr}^0(X,Y) = \mathrm{Hom}_{\mathcal{M}_\sim(k)}(h_\sim(X), h_\sim(Y)) \end{aligned}$$

Thanks to the definition of the category $\mathcal{M}_\sim(k)$, we obtain that two motives, $M = (X,p,0)$ and $N = (Y,q,0)$, are isomorphic if there exist two zero degree correspondences $f' : X \to Y$ and $g' : Y \to X$ such that the composition of morphism $f = q \circ f' \circ p$ and $g = p \circ g' \circ q$ one has $f \circ g = q = \mathrm{id}_N$ and $g \circ f = p = \mathrm{id}_M$. In general, for motives two $M = (X,p,m)$ and $N = (Y,q,n)$ in order to be isomorphic one should find f' and g' correspondences having degree $n-m$ and $m-n$, such that the relations holds. If $\sim = \sim_{\mathrm{rat}}$ we denote the functor $h_\sim$ simply by h.

Examples and properties of motives

Let k be a field, we can define the following motives: $\mathbf{1} := (\mathrm{Spec}(k), \mathrm{id}, 0)$ (motive of a point). This motive acts as the unity when we define a product structure in the category of motives. The Lefschetz motive, defined as $\mathbb{L} = (\mathrm{Spec}(k), \mathrm{id}, -1)$ and the Tate motive $\mathbb{T} = (\mathrm{Spec}(k), \mathrm{id}, 1)$. Assume $X(k) \neq \emptyset$ and consider $e \in X(k)$, this can be assured if we enlarge the field k. We can define two projectors of X

$$p_0(X) := e \times X, \quad p_{2d}(X) := X \times e.$$

Those projectors will be important in the developing of the decomposition of motives $h(X)$, because they define orthogonal projectors i.e. $p_0(X) \circ p_{2d}(X) = p_{2d}(X) \circ p_0(X) = 0$, because:

$$\begin{aligned} p_0(X) \circ p_{2d}(X) &= (\mathrm{pr}_{XX})_* \{(e \times X \times X) \cdot (X \times X \times e)\} \\ &= 0 \end{aligned}$$

by definition of the pushforward on algebraic cycles. In case that $X(k) = \emptyset$, then we choose a zero cycle $z \in \mathrm{CH}_0(X)$ of degree n and then define the projectors

$$p_0(X) := \frac{1}{n}[z \times X], \quad p_{2d}(X) := \frac{1}{n}[X \times z].$$

With these projectors, we have the definition of two new motives

$$h^0_\sim(X) = (X, p_0(X), 0), \quad h^{2d}_\sim(X) = (X, p_{2d}(X), 0).$$

Example 1.2.3. *The first example of an isomorphism of motives that we can construct by definition is $h^0(X) \cong \mathbf{1}$. Let $\alpha : X \to Spec(k)$ be the structural morphism and $e : Spec(k) \to X$ be the injection map with $e \in X(k)$. Those maps induces correspondences $\alpha^* = Spec(k) \times X$ and $e^* = e \times Spec(k)$. We have that $e^* \circ \alpha^* = id_{h(Spec(k))}$. On the other hand*

$$\begin{aligned} \alpha^* \circ e^* &= (pr_{XX})_* \{(e \times Spec(k) \times X) \cdot (X \times Spec(k) \times X)\} \\ &= e \times X \end{aligned}$$

The morphisms of motives $\alpha^* : \mathbf{1} \to h^0_\sim(X)$ *and* $e^* : h^0_\sim(X) \to \mathbf{1}$ *are mutually inverse because*

$$\begin{aligned} id \circ e^* \circ p_0(X) \circ \alpha^* \circ id = id \\ p_0(X) \circ \alpha^* \circ id \circ e^* \circ p_0(X) = p_0(X) \end{aligned}$$

therefore $\mathbf{1} \cong h^0_\sim(X)$.

There is a tensor product on the category $\mathcal{M}_\sim(k)$, which is defined on objects by

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n).$$

By the definition of tensor product of motives, we have immediately that $\mathbb{L} \otimes \mathbb{T} = \mathbf{1}$.

In the same way, we can define another operation such as the *direct sum of motives.* Let $M = (X, p, m)$ and $N = (Y, q, n)$ be motives, then one can define a motive $M \oplus N$. In the case that $m = n$ then

$$(X, p, m) \oplus (Y, q, m) := (X \amalg Y, p \amalg q, m).$$

For a general construction when $m \neq n$, let us assume $m < n$, then we can decompose

$$\begin{aligned} M &= (X, p, n) \otimes \mathbb{L}^{n-m} \\ &= (X, p, n) \otimes h^2_\sim(\mathbb{P}^1)^{n-m} \\ &= (X \times (\mathbb{P}^1)^{n-m}, \tilde{p}, n) \end{aligned}$$

where the projector $\tilde{p}$ is defined as $\tilde{p} = p \otimes (\mathbb{P}^1 \times \{x\})^{n-m}$. Therefore $M \oplus N = (X \times (\mathbb{P}^1)^{n-m} \amalg Y, \tilde{p} \amalg q, n)$.

If $\phi : X \to Y$ is a generically finite morphism of degree d we have that $\phi_* \circ \phi^* = d \cdot \mathrm{id}_Y$ and $p := (1/d) \cdot \phi^* \circ \phi_*$ is a projector on the variety X. In fact $(X, p, 0) \cong h_\sim(Y)$

Let $M = (X, p, 0)$ and $N = (Y, q, 0)$ be motives and assume that there exist morphisms of motives f, g such that $f : M \to N$, $g : N \to M$ and $f \circ g = \mathrm{id}_N = q$, then by the remarks in the section of category theory $p' = g \circ f$ defines an idempotent correspondence which is a projector on X and $N \cong (X, p', 0)$, and also $M \cong N \oplus (X, p - p', 0)$.

There exists a direct application to the direct sum of motives. Let us consider the projectors p_0 and p_{2d} defined previously. Let $p^+(X) := \Delta - p_0(X) - p_{2d}(X)$ be a correspondence, which by the properties of p_0 and p_{2d} is also a projector. If we put $h^+_\sim(X) := (X, p^+(X), 0)$ there is a decomposition of $h_\sim(X)$ as

$$h_\sim(X) = h^0_\sim(X) \oplus h^+_\sim(X) \oplus h^{2d}_\sim(X)$$

Example 1.2.4. *Let* $x \in \mathbb{P}^1(k)$. *One has a decomposition of the diagonal as the sum of* $\{x\} \times \mathbb{P}^1$ *and* $\mathbb{P}^1 \times \{x\}$, *actually, this decomposition is independent of* x, *then we can obtain the following decomposition of motives*

$$h_\sim(X) = \mathbf{1} \oplus \mathbb{L}.$$

This is the first example of the decomposition of the diagonal in the category of motives.

In general $h^{2d}_{\sim}(X) \cong (\mathrm{Spec}(k), \mathrm{id}, -d) \cong \mathbb{L}^d$. Let us prove the first isomorphism. Due to the definition of isomorphism of motives, we need to find two correspondences $f' : \mathrm{Spec}(k) \to X$ and $g' : X \to \mathrm{Spec}(k)$ such that

$$\begin{aligned} f' &\in \mathrm{Corr}^d_{\sim}(\mathrm{Spec}(k), X) = A^d_{\sim}(\mathrm{Spec}(k) \times X) \otimes \mathbb{Q} \\ g' &\in \mathrm{Corr}^{-d}_{\sim}(X, \mathrm{Spec}(k)) = A^0_{\sim}(X \times \mathrm{Spec}(k)) \otimes \mathbb{Q}. \end{aligned}$$

Using the correspondences $f' = e_*$ and $g' = \alpha_*$, on one hand we obtain $\alpha_* \circ e_* = 1$ as a correspondence on $\mathrm{Spec}(k)$, on the other hand $e_* \circ \alpha_* = X \times e$ because

$$\begin{aligned} e_* \circ \alpha_* &= \mathrm{pr}_{XX} \{(X \times \mathrm{Spec}(k) \times X) \cdot (X \times \mathrm{Spec}(k) \times e)\} \\ &= X \times e. \end{aligned}$$

The conclusion is similar to the conclusion of the isomorphism between $h^0_{\sim}(X) \cong \mathbf{1}$.

Proposition 1.2.5 ([Sch94, Proposition 1.12])**.** *Any motive M can be expressed as a direct factor of some $h(X') \otimes \mathbb{L}^n$ with X' equidimensional.*

Proof. Let $M = (X, p, m)$ then $M = ph_{\sim}(X) \otimes \mathbb{L}^{-m}$, so it is enough to prove the case when $M = h_{\sim}(X)$. Let $X = \amalg_{i=1}^r X_i$ be the decomposition of X into its equidimensional components. Let $d_i := \dim(X_i)$ and set $a_1, \ldots, a_r \in \mathbb{N}$ such that for some $k \in \mathbb{N}$, $d_i + a_i = k$ for all $i = 1, \ldots, r$. Then

$$\begin{aligned} h_{\sim}(X) &= \bigoplus_{i=1}^r h_{\sim}(X_i) = \bigoplus_{i=1}^r (h_{\sim}(X_i) \otimes \mathbf{1}) \\ &\cong \bigoplus_{i=1}^r \left(h_{\sim}(X_i) \otimes h^0_{\sim}(\mathbb{P}^{a_i})\right) \\ &\subseteq \bigoplus_{i=1}^r h_{\sim}\left(X_i \times \mathbb{P}^{a_i}\right) \end{aligned}$$

but $\bigoplus_{i=1}^r h_{\sim}\left(X_i \times \mathbb{P}^{a_i}\right) = h_{\sim}\left(\amalg_{i=1}^r X_i \times \mathbb{P}^{a_i}\right)$. ☐

It is possible to define Chow groups of motives (also for every adequate relation). For any projector $p : X \to X$, for all i one has induced maps $p_* : \mathrm{CH}^i(X)_{\mathbb{Q}} \to \mathrm{CH}^i(X)_{\mathbb{Q}}$ and for $M = (X, p, m)$ one defines

$$\mathrm{CH}^i(M) := p_*(\mathrm{CH}^{i+m}(X)_{\mathbb{Q}}) \subset \mathrm{CH}^{i+m}(X)_{\mathbb{Q}}$$

Proposition 1.2.6 ([MNP13, Proposition 2.5.1])**.** *If $M = (X, p, m)$, one has that*

$$CH^i(M) \cong Hom_{Chow(k)_{\mathbb{Q}}}(\mathbb{L}^i, M)$$

Remark 1.2.7. The proof of the last proposition uses Lieberman's lemma, which is valid for any adequate relation, so it is possible define $A^i_{\sim}(M)$ as $A^i_{\sim}(M) = \mathrm{Hom}_{\mathcal{M}_{\sim}(k)}(\mathbb{L}^i, M)$
.

Note that the Chow group of the motive $M \otimes \mathbb{L}^j$ is closely related to the motive of M by

$$\begin{aligned} A^i_\sim(M \otimes \mathbb{L}^j) &= A^i_\sim(X, p, m-j) \\ &= p_*(A^{i+m-j}_\sim(X)_\mathbb{Q}) \\ &= A^{i-j}_\sim(M) \end{aligned}$$

Now let $\sim$ be an equivalence relation finer or equal than homological equivalence. Then for a motive $M = (X, p, m)$ we can define the cohomology groups for it in a similar way as cycle groups. Considering the induced map $p_* : H^i(X) \to H^i(X)$ then

$$H^i(M) := p_*(H^{i+m}(X)) \subset H^{i+2m}(X).$$

If $M = (X, p, 0)$ is a motive, then this motive contains information about X, in the sense that is a "piece" of X which is responsible for a certain part of the geometrical and/or algebraical properties of X (for instance the Chow groups of the motive M as a subgroup of the Chow group of X), depending on the equivalence relation that we work with. When $M = (X, p, m)$ with $m \neq 0$, by Proposition 1.2.5 M is a direct summand of the motives $h_\sim(X \times (\mathbb{P}^1)^m)$, therefore the motive M can be realized as a part of different varieties.

Manin's identity principle

There exists the *duality operator* which acts as follows:

$$\begin{aligned} {}^\vee : \mathcal{M}_\sim(k)^{\mathrm{opp}} &\to \mathcal{M}_\sim(k) \\ M = (X, p, m) &\to (X, p, m)^\vee = (X, {}^t p, d - m) \end{aligned}$$

if X is purely $d-$dimensional. In particular, if we continue with the assumption of X purely $d-$dimensional, then $h(X)^\vee = h(X) \otimes \mathbb{L}^{-d}$. It is clear that the duality operator is an involution, i.e. $M^{\vee\vee} = M$, and we have the formula

$$\mathrm{Hom}_{\mathcal{M}_\sim(k)}(M \otimes N, P) = \mathrm{Hom}_{\mathcal{M}_\sim(k)}(M, N^\vee \otimes P).$$

For any motive $M \in \mathrm{ob}(\mathcal{M}_\sim(k))$ and $d \in \mathbb{Z}$ we define the cycle groups of M by $A^d_\sim(M) = \mathrm{Hom}_{\mathcal{M}_\sim(k)}(\mathbb{L}^d, M)$. Let $\mathrm{Vect}_\mathbb{Q}$ be the category of $\mathbb{Q}-$vector spaces, then $A^*_\sim(-) : \mathcal{M}_\sim(k) \to \mathrm{Vect}_\mathbb{Q}$ is a $\mathbb{Z}-$graded and additive functor. If M, $N \in \mathrm{ob}(\mathcal{M}_\sim(k))$ then

$$\mathrm{Hom}_{\mathcal{M}_\sim(k)}(N, M) = \mathrm{Hom}_{\mathcal{M}_\sim(k)}(\mathbf{1}, M \otimes N^\vee) = A^0_\sim(M \otimes N^\vee)$$

Using the properties of the category $\mathcal{M}_\sim(k)$, in particular that is a small category, by the Yoneda lemma and the remarks made in the subsection of preliminaries, the functor $F : \mathcal{M}_\sim(k) \to \mathbf{Sets}^{\mathcal{M}_\sim(k)^{opp}}$ which attaches $M \in \mathrm{ob}(\mathcal{M}_\sim(k))$ to the functor $\mathrm{Hom}_{\mathcal{M}_\sim(k)}(-, M)$ is fully-faithful.

As we know, any $N \in \mathrm{ob}(\mathcal{M}_\sim(k))$ can be seen as a direct factor of $h_\sim(Y) \otimes \mathbb{L}^n$ for some $Y \in \mathrm{SmProj}_k$ and $n \in \mathbb{Z}$, and along with the properties of duality, we have $A^0_\sim(M \otimes h_\sim(Y) \otimes \mathbb{L}^n) = A^{-n}_\sim(M \otimes h_\sim(Y))$. Denoting by ω_M the following functor

$$\begin{aligned} \omega_M : \mathrm{SmProj}_k^{\mathrm{opp}} &\to \mathrm{Vect}_{\mathbb{Q}} \\ Y &\to \omega_M(Y) := A^*_\sim(M \otimes h_\sim(Y)) \end{aligned}$$

then the functor which attaches $M \in \mathrm{ob}(\mathcal{M}_\sim(k))$ to ω_M is full faithful. This is because of the properties of the category of motives and Yoneda's lemma.

Theorem 1.2.8 ([Sch94, Section 2.3])**.** *[Manin's identity principle]*

- *Let f, $g : M \to N$ be morphism of motives. Then f is an isomorphism if and only if the induced map*
$$\omega_f(Y) : A^*_\sim(M \otimes h_\sim(Y)) \to A^*_\sim(N \otimes h_\sim(Y))$$
is an isomorphism for every $Y \in SmProj_k$, and $f = g$ if and only if $\omega_f(Y) = \omega_g(Y)$ for every $Y \in SmProj_k$.
- *A sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ in $\mathcal{M}_\sim(k)$ is exact if and only if, for every $Y \in SmProj_k$, the sequence*
$$0 \to A^*_\sim(M' \otimes h_\sim(Y)) \xrightarrow{\omega_f(Y)} A^*_\sim(M \otimes h_\sim(Y)) \xrightarrow{\omega_g(Y)} A^*_\sim(M'' \otimes h_\sim(Y)) \to 0$$
is exact.

Proof. The proof of the first assertion comes directly from the properties of the category of motives and Yoneda's lemma. The hypothesis are clearly fulfilled for this case because of the given arguments about the morphisms in the category of motives. The second property follows since the functors are fully-faithful. □

Motive of a projective bundle

The first example that we consider is the calculation of the motive of a projective bundle, because it gives us an important fact about the functor h and how it acts on the objects of $\mathrm{Chow}(k)_{\mathbb{Q}}$.

Remark 1.2.9. For the next example, we need to remember that the Chow group for $X \times \mathbb{P}^n$ where $X \in \mathrm{SmProj}_k$ is given by $\mathrm{CH}^*(X \times \mathbb{P}^n) \cong \mathrm{CH}^*(X)[t]/(t^{n+1})$.

Let us consider $X \in \mathrm{SmProj}_k$ and the free projective bundle over X of rank n, $X \times \mathbb{P}^n$. By definition we have $h(X \times \mathbb{P}^n) = (X \times \mathbb{P}^n, \Delta_{X \times \mathbb{P}^n}, 0)$ which is isomorphic to $h(X) \otimes h(\mathbb{P}^n) = (X, \Delta_X, 0) \otimes (\mathbb{P}^n, \Delta_{\mathbb{P}^n}, 0)$ (due to the decomposition of the diagonal). For the case of the motive $h(\mathbb{P}^n)$ one has the decomposition $h(\mathbb{P}^n) = \mathbf{1} \oplus \mathbb{L}^1 \oplus \ldots \oplus \mathbb{L}^n$ [1]. In other words, for the motive of the free projective bundle of rank we obtain

$$h(X \times \mathbb{P}^n) \cong \bigoplus_{i=0}^{n} h(X) \otimes \mathbb{L}^i.$$

[1] this comes from the fact that we construct projectors $p_i = \xi^{n-i} \times \xi^i$ for all $i \in \{0, \ldots, n\}$, where ξ is the class of a hyperplane in the Chow group of $\mathbb{P}^n$, such that the projectors p_i are pairwise orthogonal and $\Delta_{\mathbb{P}^n} = \sum_{i=0}^n p_i$. For more details see [Man68] p. 455.

Example 1.2.10. *Let E be a locally free sheaf of rank $r+1$ on $X \in SmProj_k$, and let $\pi : \mathbb{P}(E) \to X$ be the projective bundle and $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ the tautological line bundle. As we have seen before, there is an isomorphism λ for Chow groups*

$$CH^*(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r} \xi^i \, CH^*(X)$$

with inverse μ. Now, let us consider $T \in SmProj_k$, it is easy to see that $(id_T \times \mu) \circ (id_T \times \lambda) = id$ on the group $CH^*(T \times \mathbb{P}(E))$ i.e. the property remains true universally after an arbitrary base change $T \to Spec(k)$ (also considering $(id_T \times \lambda) \circ (id_T \times \mu) = id$ on the group $\bigoplus_{i=0}^{r} \xi^i \, CH^*(T \times X)$), having for all T an isomorphism

$$CH^*(T \times \mathbb{P}(E)) \cong \bigoplus_{i=0}^{r} \xi^i \, CH^*(T \times X)$$

therefore, we can conclude that $(\mathbb{P}(E), \Delta_{\mathbb{P}(E)}, 0) \cong \bigoplus_{i=0}^{r} (X, \Delta_X, -i)$ (with the usual notation $h(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r} h(X) \otimes \mathbb{L}^i$)

Remark 1.2.11.

1. This example shows two varieties, $X \times \mathbb{P}^r$ and $\mathbb{P}(E)$ with E a locally free sheaf of rank $r+1 > 1$ over X, that are not isomorphic and such that their respective motives are isomorphic. We obtain an important conclusion, the motive of a projective bundle over $X \in \mathrm{SmProj}_k$ only depends of the rank of E, so it can be constructed as if E were a free sheaf. In a language of category theory, the functor $h : \mathrm{SmProj}_k \to \mathrm{Chow}(k)_{\mathbb{Q}}$ is not conservative, i.e. is not injective on objects.

2. In case of the existence of a Chow-Künneth decomposition of X one has a decomposition of the motive of a projective bundle as

$$h(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r} \bigoplus_{j=0}^{2\dim(X)} \left(h^j(X) \otimes \mathbb{L}^i\right).$$

Motive of a Blow-up

Example 1.2.12 ([Sch94, 2.7, Theorem 2.8], [Man68, §9, p. 461])**.** *Another example of the Manin principle is the isomorphism that relates the motives of a variety X, a subvariety $Z \subset X$ of codimension $(m+1)$ and $W = Bl_Z X$ the blow-up of X along Z in the following way:*

$$h(X) \oplus h(E) \cong h(Z) \oplus h(W)$$

Due to Theorem 1.1.10, for $T \in SmProj_k$ the exact sequence can be extended to

$$0 \to CH^*(T \times Z) \xrightarrow{(i_* h)} CH^*(T \times X) \oplus CH^*(T \times E) \xrightarrow{(\pi^* j_*)} CH^*(T \times W) \to 0.$$

which gives an exact sequence

$$h(W) = h(X) \oplus \bigoplus_{i=1}^{m} h(Z) \otimes \mathbb{L}^i$$

in the category of motives $Chow(k)_{\mathbb{Q}}$.

Cellular decomposition

Example 1.2.13. *Let $X \to S$ be a smooth projective variety which admits a cellular decomposition. Since the isomorphism presented in Theorem 1.1.9 is preserved under base change $S' \to S$, by Manin's identity principle we conclude that*

$$h(X) \cong \bigoplus_{i=0}^{k} h(S) \otimes \mathbb{L}^{d_i}.$$

In all the examples, the principal argument was the functoriality of the isomorphism of Chow groups with respect cartesian products, which allows us to apply Manin's identity principle. So in order to know if a morphism is an isomorphism we have to check it in a universal way for all $Y \in \mathrm{SmProj}_k$. There exists an improved version of Manin principle, but for algebraically closed fields as is stated in [Huy18, Lemma 1.1]: Let $f : M \to N$ be a morphism in the category $\mathrm{Chow}(k)_{\mathbb{Q}}$. Then f is a isomorphism of motives in $\mathrm{Chow}(k)_{\mathbb{Q}}$ if and only if for Ω an universal domain over k, the induced map $(f_\Omega)_* : \mathrm{CH}^*(M_\Omega)_{\mathbb{Q}} \to \mathrm{CH}^*(N_\Omega)_{\mathbb{Q}}$ given by the base change $f_\Omega : M_\Omega \to N_\Omega$, is bijective. We will describe this principle with more details in chapter four.

Chow-Künneth decomposition

For certain motives $h(X)$ there exists a decomposition, called the *Chow-Künneth decomposition*. Consider the following example: let C be a connected, smooth and projective curve defined over a field $k = \bar{k}$. Fixing a k-rational point $e \in C(k)$, we define the correspondences $p_0(C) = e \times C$ and $p_2(C) = C \times e$ which are idempotent and mutually orthogonal. Defining $p_1(C) := p^+(C)$ and $h^1_\sim(C) = (C, p_1(C), 0)$, we obtain

$$h_\sim(C) = h^0_\sim(C) \oplus h^1_\sim(C) \oplus h^2_\sim(C).$$

Modulo isomorphism, $h^1_\sim(C)$ is well defined as a unique motive. As we have said before, the theory of the motive $h^1_\sim(C)$ is closely related to the theory of abelian varieties.

In general, we say that $X \in \mathrm{SmProj}_k$ admits a *Chow-Künneth decomposition* if there exists projectors $p_i(X) \in \mathrm{Corr}_{\mathrm{rat}}(X, X)$, for all $i = 0, \dots, 2d$ such that

1. $\displaystyle\sum_{i=0}^{2d} p_i(X) = \Delta(X)$,
2. $p_i(X) \circ p_j(X) = p_i(X)$ if $i = j$, otherwise $p_i(X) \circ p_j(X) = 0$,
3. $\gamma_{X \times X}(p_i(X)) = \Delta_i^{\mathrm{topo}}(X)$, where $\Delta_i^{\mathrm{topo}}(X) \in H^{2d-i} \otimes H^i(X)$ corresponds $i-$th Künneth component of the diagonal as topological cycle class in the decomposition of $\gamma_{X\times X}(\Delta(X)) = \sum_{i=0}^{2d} \Delta_i^{\mathrm{topo}}(X)$.

If such projectors exists, we obtain a decomposition of motives, called the *Chow-Künneth decomposition for motives*, by defining $h^i(X) = (X, p_i(X), 0)$, then

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X).$$

There is a conjecture, called the *Chow-Künneth conjecture*, which states that every $X \in \mathrm{SmProj}_k$ admits a *Chow-Künneth decomposition.* Working with rational coefficients, the projectors $p_0(X)$ and $p_{2d}(X)$ always exist. This is also the case for the projectors $p_1(X)$ and $p_{2d-1}(X)$ by a theorem of Murre, see [MNP13, Theorem 6.2.1]. Hence we obtain a decomposition of the motive $h(X)$ as follows

$$h(X) \simeq h^0(X) \oplus h^1(X) \oplus h^+(X) \oplus h^{2d-1}(X) \oplus h^{2d}(X).$$

The motives $h^1(X)$ and $h^{2d-1}(X)$ are called the *Picard* and *Albanese motives* and satisfy the properties

$$\mathrm{CH}^i(h^1(X))_{\mathbb{Q}} \simeq \begin{cases} 0 \text{ if } i \neq 1 \\ \mathrm{Pic}^0(X)(k)_{\mathbb{Q}} \text{ if } i = 1, \end{cases} \qquad \mathrm{CH}^i(h^{2d-1}(X))_{\mathbb{Q}} \simeq \begin{cases} 0 \text{ if } i \neq 2d-1 \\ \mathrm{Alb}(X)(k)_{\mathbb{Q}} \text{ if } i = 2d-1. \end{cases}$$

Some examples of varieties for which is known that the Chow-Kûnneth decomposition holds are the following:

1. For dimension reasons and the existence of the projectors $p_0(X)$, $p_1(X)$, $p_{2d-1}(X)$ and $p_{2d}(X)$, the Chow-Künneth decomposition holds for curves and surfaces.

2. Abelian varieties A over a base scheme S, given by [MD91] or [Kün93], and using the Fourier-Mukai transform.

3. Conic bundles over a surface, by [NS09].

4. By arguments involving the nonexistence of transcendental cohomology in all degrees but in the middle dimension (see [MNP13, Appendix C]) such as complete intersections in projective spaces and Calabi-Yau threefolds.

5. If X and Y are two projective varieties that admit a Chow-Künneth decomposition, the product $X \times Y$ admits a decomposition as well, by imposing the projectors

$$p_k(X \times Y) = \sum_{i+j=k} p_i(X) \times p_j(Y).$$

1.3 Triangulated motives

The main result that we use in this section is the so called *Grothendieck six functors formalism.* We recall some facts about premotivic theories and the six functor formalism presented in [Ayo14b], [CD16] and [CD19].

A *monoidal category* is a category $\mathcal{C}$ equipped with an associative product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbf{1}$. The associativity property is expressed in terms of imposing an isomorphism

$$\eta_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

and for the unit $\mathbf{1}$ we demand the existence of isomorphisms $\alpha_A : A \otimes \mathbf{1} \to A$ and $\beta_A : \mathbf{1} \otimes A \to A$. We require naturality about these maps as follows:

- the isomorphism $\eta_{A,B,C}$ depends functorially on the triple (A,B,C), i.e. it can be regarded as a natural isomorphism between functors

$$\begin{aligned}\mathcal{C}\times\mathcal{C}\times\mathcal{C} &\to \mathcal{C}\\ (A,B,C) &\mapsto (A\otimes B)\otimes C\\ (A,B,C) &\mapsto A\otimes(B\otimes C)\end{aligned}$$

  and similarly for the maps α_A and β_A.

- Given any four objects $A,B,C,D\in\mathcal{C}$ the following pentagram

$$\begin{tikzcd}
 & ((A\otimes B)\otimes C)\otimes D \arrow[dl, "\eta_{A,B,C}\otimes \mathrm{id}_D"'] \arrow[dr, "\eta_{A\otimes B,C,D}"] & \\
(A\otimes(B\otimes C))\otimes D \arrow[d, "\eta_{A,B\otimes C,D}"] & & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\eta_{A,B,C\otimes D}"] \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "\mathrm{id}_A\otimes\eta_{B,C,D}"] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}$$

  is commutative.

- For any pair (A,B) of objects in $\mathcal{C}$, the triangle

$$\begin{tikzcd}
(A\otimes \mathbf{1})\otimes B \arrow[rr, "\eta_{A,\mathbf{1},B}"] \arrow[dr, "\alpha_A\otimes\mathrm{id}_B"'] & & A\otimes(\mathbf{1}\otimes B) \arrow[dl, "\mathrm{id}_A\otimes\beta_B"] \\
 & A\otimes B &
\end{tikzcd}$$

  is commutative.

Grothendieck six functors formalism

We assume that all schemes are noetherian and of finite dimension. Consider a family of morphisms $\mathcal{P}$ which is one of the following families

- The class of étale morphisms which are separated of finite type, denoted by Ét.
- The class of smooth morphisms which are separated of finite type, denoted by Sm.
- The class of separated morphisms of finite type, denoted by $\mathcal{F}^{ft}$.

Let S be a given base scheme, then we denote $S_{\text{ét}}$, Sm_S and $\mathcal{F}^{ft}_S$ be the category of noetherian S-scheme having structural morphism $U\to S$ in the class of morphisms Ét, Sm and $\mathcal{F}^{ft}$ respectively. All of these families are*admissible families* in the sense of [CD19]. In general a family of morphisms $\mathcal{P}$ of a category $\mathcal{C}$ is called admissible if it has the following properties:

1. All isomorphisms are in $\mathcal{P}$.
2. The class $\mathcal{P}$ is stable by composition.

3. The class $\mathcal{P}$ is stable by fiber products.

Definition 1.3.1 (Triangulated premotivic category)**.** *We say that a fibred category $\mathcal{M}$ over Sch is a triangulated (resp. abelian) $\mathcal{P}$-premotivic category if satisfies the following properties:*

1. *For any scheme S, the fiber $\mathcal{M}_S$ is a well generated triangulated (resp. Grothendieck abelian) category with a closed monoidal structure.*

2. *For any morphism of schemes f, the functor f^* is triangulated (resp. additive), monoidal and admits a right adjoint denoted by f_*.*

3. *For any morphism $p \in \mathcal{P}$, the functor p^* admits a left adjoint denoted by $p_\#$.*

4. *For any cartesian square*

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] & S \end{tikzcd}$$

   there exists a canonical isomorphism $Ex(\Delta^*_\#) : q_\# g^* \to f^* p_\#$.

5. *For any morphism $p : T \to S$ in $\mathcal{P}$ and any object (M, N) of $\mathcal{M}_T \times \mathcal{M}_S$, there exists a canonical isomorphism $Ex(p^*_\#, \otimes) : p_\#(M \otimes_T p^*(N)) \to p_\#(M) \otimes_S N$.*

Let us consider a premotivic triangulated category $\mathcal{T}$. Given any smooth morphism $p : X \to S$, we define the homological Voevodsky premotive associated to X/S as the object $M_S(X) := p_\#(\mathbf{1}_X)$, which has a covariant nature. Let $p : \mathbb{P}^1_S \to S$ be the projection. We define the Tate premotive as the kernel of the map $p_* : M_S(\mathbb{P}^1_S) \to \mathbf{1}_S$ shifted by -2, denoted by $\mathbf{1}(1)$ and for a premotive $M \in \mathcal{T}$ we define the $n-$th Tate twist of M as the $n-$th tensor power of M by $\mathbf{1}(1)$. If $\mathbf{1}(1)$ is $\otimes$-invertible in $\mathcal{T}$ then it is possible to define the Tate twist for negative n.

We associate to the premotivic category $\mathcal{T}$ a bi-graded cohomology theory defined by

$$H^{m,n}_{\mathcal{T}}(S) := \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}_S, \mathbf{1}_S(n)[m]).$$

One also introduces the following properties of the premotivic triangulated category $\mathcal{T}$:

1. Homotopy property: For any S-scheme, the canonical projection of the affine line over S induces an isomorphism $M_S(\mathbb{A}^1_S) \to \mathbf{1}_S$.

2. Stability property: The Tate premotive is $\otimes$-invertible.

3. Orientation: an orientation of $\mathcal{T}$ is natural transformation of contravariant functors, such that for all schemes S the map

$$c_1 : \mathrm{Pic}(\mathbb{P}^1_S) \to H^{2,1}_{\mathcal{T}}(\mathbb{P}^1_S).$$

sends $\mathcal{O}_{\mathbb{P}^1_S}(-1)$ to 1 in $H^{0,0}_{\mathcal{T}}(S)$ via the decomposition $H^{2,1}_{\mathcal{T}}(\mathbb{P}^1_S) \simeq H^{2,1}_{\mathcal{T}}(S) \oplus H^{0,0}_{\mathcal{T}}(S)$, where the identification of the last factor uses the stability property.

When $\mathcal{T}$ is equipped with an orientation we say that $\mathcal{T}$ is oriented.

Remark 1.3.2. As one of the authors of [CD19] mentions, there is a typo in the definition given in the book forgetting the previous condition.

Definition 1.3.3. *Consider $\mathcal{T}$ a triangulated premotivic category which is oriented. We say that $\mathcal{T}$ satisfies the Grothendieck six functors formalism if it satisfies the stability property and for any $f : Y \to X \in \mathcal{F}^{ft}$ there exists a pair of adjoints functors*

$$f_! : \mathcal{T}(Y) \rightleftarrows \mathcal{T}(X) : f^!$$

with the following properties:

1. *There exists a structure of a covariant 2-functor on $f \mapsto f_!$ and of a contravariant 2-functor on $f \mapsto f^!$.*

2. *There exists a natural transformation $\alpha_f : f_! \to f_*$ which turns out to be an isomorphism when f is proper.*

3. *For any smooth morphism $f : X \to S$ of relative dimension d there are canonical natural isomorphisms*

$$\begin{aligned} \beta_f &: f_\# \to f_!(d)[2d] \\ \tilde{\beta}'_f &: f^* \to f^!(-d)[-2d] \end{aligned}$$

   which are dual to each other.

4. *For any cartesian square*

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\Delta"] & \widetilde{X} \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

   such that $f \in \mathcal{F}^{ft}$, there exist natural isomorphisms

$$\begin{aligned} g^* f_! &\xrightarrow{\sim} \tilde{f}_! \tilde{g}^* \\ \tilde{g}_* \tilde{f}^! &\xrightarrow{\sim} f^! g_* \end{aligned}$$

5. *For any separated morphism of finite type $f : Y \to X$ there exist natural isomorphisms*

$$\begin{aligned} Ex(f_!^*, \otimes) : (f_! K) \otimes_X L &\xrightarrow{\sim} f_!(K \otimes_Y f^* L), \\ \underline{Hom}_X(f_!(L), K) &\xrightarrow{\sim} f_* \underline{Hom}_Y(L, f^!(K)) \\ f^! \underline{Hom}_X(L, M) &\xrightarrow{\sim} \underline{Hom}_Y(f^*(L), f^!(M)) \end{aligned}$$

6. *For any closed immersion $i : Z \to S$ with complementary open immersion j, there exist distinguished triangles of natural transformations as follows:*

$$j_! j^! \xrightarrow{\alpha'_j} 1 \xrightarrow{\alpha_i} i_* i^* \xrightarrow{\partial_i} j_! j^! [1]$$
$$i_! i^! \xrightarrow{\alpha'_i} 1 \xrightarrow{\alpha_j} j_* j^* \xrightarrow{\widetilde{\partial}_i} i_! i^! [1]$$

where α'_j and α_i denote the co-unit and unit of the adjunctions.

We introduce the following definitions related to some of the axiomatic properties of premotivic categories:

- Given a closed immersion i, the fact that i_* is conservative and the existence of the first distinguished triangle in (6) is called the *localization property with respect to i.*
- The conjunction of properties (2) and (3) of Definition 1.3.3 gives, for a smooth proper morphism f, an isomorphism $\mathfrak{p}_f : f_\# \to f_*(d)[2d]$. Under the stability and weak localization properties, when such an isomorphism exists, we say that f is $\mathcal{T}$*-pure* (or simply *pure* when $\mathcal{T}$ is clear).

Definition 1.3.4. *Consider the notation and assumptions above. We say that $\mathcal{T}$ satisfies the localization property (resp weak localization property) if it satisfies the localization property with respect to any closed immersion i (resp. which admits a smooth retraction).*

We say that $\mathcal{T}$ satisfies the purity property (resp. weak purity property) if for any smooth proper morphism f (resp. for any scheme S and integer $n > 0$, the projection $p : \mathbb{P}^n_S \to S$) is $\mathcal{T}$-pure.

Premotivic categories

Let $\mathcal{P}$ be one of the classes defined before. The categories $\mathrm{Sh}_{\text{ét}}(\mathcal{P}_S, \Lambda)$ of étale sheaves of Λ-modules over $\mathcal{P}_S$ form the fibers of an abelian premotivic category. The derived categories $D(\mathrm{Sh}_{\text{ét}}(\mathcal{P}_S, \Lambda))$ for various schemes S form the fibers of a canonical triangulated premotivic category. Consider the *homotopy relation*, [CD19, Definition 5.2.16]: first consider $\mathcal{A}$ an abelian $\mathcal{P}$-premotivic category compatible with an admissible topology t. Let us consider $\mathcal{W}_{\mathbb{A}^1}$ to be the family of morphism $M_S(\mathbb{A}^1_X)\{i\} \to M_S(X)\{i\}$ for a $\mathcal{P}$-scheme X/S and a twist i in τ. We define $D^{\text{eff}}_{\mathbb{A}^1}(\mathcal{A}) := D(\mathcal{A})[\mathcal{W}^{-1}_{\mathbb{A}^1}]$. We called this category as the effective $\mathcal{P}$-premotivic $\mathbb{A}^1$-derived category with coefficients in $\mathcal{A}$.

With this notation, we can define the following categories:

1. Consider the class $\mathcal{P} = \mathrm{Sm}$, the admissible topology $t = \mathrm{Nis}$ and a commutative ring Λ. Consider the category defined as $D^{\text{eff}}_{\mathbb{A}^1,\Lambda} := D^{\text{eff}}_{\mathbb{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(Sm, \Lambda))$. We define the fibers $D^{\text{eff}}_{\mathbb{A}^1}(S, \Lambda) := D^{\text{eff}}_{\mathbb{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \Lambda))$ for a scheme S. If $t = \text{ét}$ then we denote the category $D^{\text{eff,ét}}_{\mathbb{A}^1,\Lambda} := D^{\text{eff,ét}}_{\mathbb{A}^1,\Lambda}(\mathrm{Sh}_{\text{ét}}(Sm, \Lambda))$ and the fibers as $D^{\text{eff,ét}}_{\mathbb{A}^1}(S, \Lambda) := D^{\text{eff}}_{\mathbb{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm/S, \Lambda))$

2. Assume that $\mathcal{P} = \mathcal{F}^{ft}$. Consider the admissible topology $t = h$ (resp. $t = qfh$) and define the premotivic category of effective h-motives (resp. effective qfh-motives) over our base S with coefficients in Λ as follows:

$$\underline{DM}_h^{\text{eff}}(S,\Lambda) = D_{\mathbb{A}^1}^{\text{eff}}(\mathrm{Sh}_h(\mathcal{F}^{ft}/S,\Lambda)),$$
$$\text{resp. } \underline{DM}_{qfh}^{\text{eff}}(S,\Lambda) = D_{\mathbb{A}^1}^{\text{eff}}(\mathrm{Sh}_{qfh}(\mathcal{F}^{ft}/S,\Lambda)).$$

We define the category of h-motives over a base S, denoted by $\mathrm{DM}_h^{\text{eff}}(S,\Lambda)$, as the smallest full subcategory of $\underline{DM}_h^{\text{eff}}(S,\Lambda)$ closed under arbitrary small sums and containing the objects of the form $\Lambda_S^h(X)$ for $X \to S$ smooth.

3. Consider $\mathcal{S}$ the category of noetherian finite-dimensional schemes and let $\mathcal{S}m_S$ be the category of smooth separated S-schemes. We define the Λ-linear category of motivic complexes as the category $\mathrm{DM}_\Lambda^{\text{eff}} := D_{\mathbb{A}^1}^{\text{eff}}(\mathrm{Sh}^{tr}(-,\Lambda))$ where $\mathrm{Sh}^{tr}(S,\Lambda)$ is the category of Nisnevich sheaves with transfers for a scheme S. For a given scheme S we put $\mathrm{DM}^{\text{eff}}(S,\Lambda) := D_{\mathbb{A}^1}^{\text{eff}}(\mathrm{Sh}^{tr}(S,\Lambda))$.

For an abelian $\mathcal{P}$-premotivic category $\mathcal{A}$. Consider any scheme S, we then have a split monomorphism of $\mathcal{A}-$premotives $\mathbf{1}_S \to M_S(\mathbb{G}_{m,S})$. Let us denote by $\mathbf{1}_S\{1\}$ the cokernel of this monomorphism called the *suspended Tate S-premotive* with coefficients in $\mathcal{A}$. For an integer $n \geq 0$, we denote by $\mathbf{1}_S\{n\}$ its n-th tensor product. We define the symmetric Tate spectrum over S as the symmetric sequence $\mathbf{1}_S\{*\} = \mathrm{Sym}(\mathbf{1}_S\{1\})$. We denote by $\mathrm{Sp}(\mathcal{A})$ the abelian $\mathcal{P}-$premotivic category of modules over $\mathbf{1}_S\{*\}$ in the category $\mathcal{A}^{\mathfrak{S}}$. Notice that we have adjunctions

$$\Sigma^\infty : \mathcal{A} \leftrightarrows \mathrm{Sp}(\mathcal{A}) : \Omega^\infty \tag{1.1}$$

of abelian $\mathcal{P}$-premotivic categories. We can introduce the $\mathbb{A}^1$-localization to the category $\mathrm{Sp}(\mathcal{A})$ having an adjunction of triangulated $\mathcal{P}$-premotivic categories

$$\Sigma^\infty : \mathcal{A} \leftrightarrows \mathrm{Sp}(\mathcal{A}) : \Omega^\infty \tag{1.2}$$

Now consider X a $\mathcal{P}$-scheme over S. From the definition of the functor Σ^∞, there is a canonical morphism of abelian Tate spectra $[\Sigma^\infty(\mathbf{1}_S\{1\})]\{-1\} \to \Sigma^\infty\mathbf{1}_S$. Tensoring this map by elements of the form $\Sigma^\infty M_S(X,\mathcal{A})\{-n\}$ for any $\mathcal{P}$-scheme X over S and any integer $n \in \mathbb{N}$ we obtain a family of morphisms

$$[\Sigma^\infty(M_S(X,\mathcal{A})\{1\})]\{-(n+1)\} \to \Sigma^\infty M_S(X,\mathcal{A})\{-n\}$$

We denote this family by $\mathcal{W}_\Omega$ and set $\mathcal{W}_{\mathcal{A}^1,\Omega} := \mathcal{W}_\Omega \cup \mathcal{W}_{\mathcal{A}^1}$.

Definition 1.3.5. *Let $\mathcal{A}$ be an abelian $\mathcal{P}$-premotivic category compatible with an admissible topology t. We define the stable $\mathcal{A}^1$-derived $\mathcal{P}$-premotivic category with coefficients in $\mathcal{A}$ as the derived $\mathcal{P}$-premotivic category*

$$D_{\mathbb{A}^1}(\mathcal{A}) := D(Sp(\mathcal{A}))\left[\mathcal{W}_{\mathcal{A}^1,\Omega}^{-1}\right].$$

Given a scheme S and a commutative ring Λ, we focus on $\mathrm{DM}^{\mathrm{eff}}(S,\Lambda)$. Using the previous construction of the infinite suspension functor Σ^∞, we obtain an adjunction of triangulated premotivic categories

$$\Sigma^\infty : \mathrm{DM}^{\mathrm{eff}}(S,\Lambda) \leftrightarrows \mathrm{DM}(S,\Lambda) : \Omega^\infty$$

where $\mathrm{DM}(S,\Lambda)$ is called the Λ-linear category of stable motivic complexes. In this context, for a scheme S and $(m,n) \in \mathbb{Z}^2$, we define the motivic cohomology of S in degree m and twist n with coefficients in Λ as the Λ-module

$$H^m_M(S,\Lambda(n)) := \mathrm{Hom}_{\mathrm{DM}(S,\Lambda)}\left(\mathbf{1}_S, \mathbf{1}_S(n)[m]\right)$$

Let $\Lambda = \mathbb{Z}$ and k be a perfect field. Given any smooth separated k-scheme S, the motivic cohomology groups coincide with higher Chow groups: $H^m_M(S,\Lambda(n)) = \mathrm{CH}^n(S, 2m-n)$ (see [MVW06, Theorem 19.1]).

According to [CD19, Proposition 11.1.5], if Λ' is a localization of Λ, then the change of coefficients induces adjunctions

$$\mathrm{DM}(S,\Lambda) \otimes_\Lambda \Lambda' \leftrightarrows \mathrm{DM}(S,\Lambda')$$

which are equivalences of triangulated premotivic categories. As a consequence we obtain that the morphism $H^m_M(S,\mathbb{Z}(n)) \otimes \mathbb{Q} \to H^m_M(S,\mathbb{Q}(n))$ is an isomorphism for every bidegree $(m,n) \in \mathbb{Z}^2$. Now we fix $\Lambda = \mathbb{Q}$ and consider the functor $i : \mathrm{Chow}(k)^{op}_{\mathbb{Q}} \to \mathrm{DM}(k,\mathbb{Q})$ given by $i(h(X)) = M(X)$. By [MVW06, Theorem 20.1] this functor is fully-faithful embedding, thus we can see pure motives as a subcategory of $\mathrm{DM}(k,\mathbb{Q})$. This has a important consequence, because the category $\mathrm{Chow}(k)^{op}_{\mathbb{Q}}$ appears in $\mathrm{DM}(k,\mathbb{Q})$ as the subcategory generated by elements of pure weight 0, in the sense of Bondarko given in [Bon14].

Chapter 2

Étale Chow motives

In this chapter we mainly focus on the definition of the étale analogue of the category of Chow motives. It should be noted that the category that we construct cannot be defined as the subcategory of $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$ generated by elements of pure weight 0, in the sense of Bondarko, contrary to the Nisnevich case. For the definition of weight structure see [Bon14, Section 1] and [Bon14, Theorem 2.1.1]. For a detailed explanation of the non-existence of a weight structure on $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$ see [CD16, Remark 7.2.26].

In the first section of the present chapter we review the triangulated category of étale motives, presenting two models for this category, working with sheaves with and without transfers. For the first model we mainly use the references [Ayo14a] and [Ayo14b], whereas for the second model we use [CD16]. We recall properties of conservativity of functors associated to change of coefficients, morphisms between schemes and duality functors. We prove also that under suitable conditions on the base S, we obtain an analogue of [AHP16, Lemma A.6].

The second section aims to introduce two different notions of *étale motivic cohomology*: the first one is defined by using $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$ as model for étale motives. We use this definition to establish the existence of pull-backs, pushforwards, intersection product and localization long exact sequences, giving an étale analogue of classical properties of Chow groups. The second definition is obtained by taking the hypercohomology of the étale sheafification of the Bloch's complex sheaf and leads to so-called *Lichtenbaum cohomology groups*. Together with the definition we mention the main facts about the structure of Lichtenbaum cohomology groups and comparison maps between these groups and motivic or étale motivic groups.

In the third section we look at the problem of birational invariance, and explain when this property fails for the étale analogue of zero cycles $\mathrm{CH}_0(X)$. Even though we cannot find an étale analogue of birational invariance for the whole category SmProj_k, we present some cases where this invariance is true and cases where this obstruction appears for $\mathrm{CH}_0^{\text{ét}}(X)$.

Continuing with the introduction to the chapter, the goal of the fourth section is to give a brief description of different equivalence relations on étale Chow groups, such as algebraic, homological, nilpotent and numerical equivalence by analogy with the classical case.

Finally, in the last section, using the properties that we give in section two and four, we construct the category of *étale Chow motives*, denoted by $\mathrm{Chow}_{\text{ét}}(k)$, which fits in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Chow}(k)^{op}_{\mathbb{Z}} & \xrightarrow{\Phi} & \mathrm{DM}(k,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Chow}_{\text{ét}}(k)^{op} & \xrightarrow{\Phi^{\text{ét}}} & \mathrm{DM}_{\text{ét}}(k,\mathbb{Z}), \end{array}$$

where $\mathrm{Chow}(k)$ is the category of integral Chow motives, $\mathrm{DM}(k,\mathbb{Z})$ and $\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})$ are the triangulated categories of motives over k and its étale counterpart respectively, and the horizontal arrows are full embeddings. As in the classical case, we obtain a version of the Manin principle for $\mathrm{Chow}_{\text{ét}}(k)$ and consequently the decomposition of étale motives for projective bundles, varieties with cellular decomposition and blow-ups with smooth center.

2.1 Etale motives

We recall the definition of two models for the category étale of étale motives: the first one $\mathbf{DA}_{\text{ét}}(S,\Lambda)$, uses étale sheaves without transfers, and the second one, $\mathbf{DM}_{\text{ét}}(S,\Lambda)$, uses sheaves with transfers. As we have mentioned, for the first model we mainly use the references [Ayo14a] and [Ayo14b]; for the second one we use [CD16].

Let Λ be a commutative ring which in this context is called the *ring of coefficients*. We are interested in the cases when $\Lambda = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/m$ (we will omit Λ in the notation when $\Lambda = \mathbb{Z}$). We fix a noetherian scheme S as our base scheme and we denote Sch/S and Sm/S the categories of schemes of finite type and smooth schemes over S respectively. We denote by $\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda)$ the category of étale sheaves with values in $\Lambda-$modules.

For a given object X in Sm/S we denote by $\Lambda^S_{\text{ét}}(X)$ the étale sheaf associated to the presheaf $U \mapsto \Lambda[\mathrm{Hom}_{\mathrm{Sm}/S}(U,X)]$ where $\Lambda[\mathrm{Hom}_{\mathrm{Sm}/S}(U,X)]$ is the free $\Lambda-$module generated by $\mathrm{Hom}_{\mathrm{Sm}/S}(U,X)$.

Consider the derived category of étale sheaves $\mathbf{D}(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S,\Lambda))$ and denote by $\mathcal{L}$ the subcategory of the derived category of étale sheaves that contains the two complexes

$$\ldots \to 0 \to \Lambda^S_{\text{ét}}(\mathbb{A}^1_U) \to \Lambda_{\text{ét}}(U) \to 0 \to \ldots$$

and is closed under arbitrary direct sums. Here $\mathbb{A}^1 := \mathrm{Spec}(\mathbb{Z}[t])$ and U is a smooth $S-$scheme, while the non-zero map is induced by the projection $\mathbb{A}^1_U \to U$.

Definition 2.1.1. *Define $\boldsymbol{DA}^{\mathit{eff}}_{\acute{e}t}(S,\Lambda)$ as the Verdier quotient of $\mathbf{D}(Sh_{\acute{e}t}(Sm/S,\Lambda))$ by $\mathcal{L}$. An object in $\boldsymbol{DA}^{\mathit{eff}}_{\acute{e}t}(S,\Lambda)$ is called an effective motivic sheaf over S with coefficients in Λ. The motivic sheaf $\Lambda^S_{\acute{e}t}(X)$ is called the effective homological motive of X and from now on we will denote it by $M^S_{\acute{e}t}(X)$.*

It is necessary to remark that the category $\mathbf{DA}^{\text{eff}}_{\text{ét}}(S,\Lambda)$ has the same objects of the category $\mathbf{D}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda))$, the difference lies in the morphisms of the category, since every morphism in $\mathbf{D}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda))$ whose cone is in $\mathcal{L}$ gets inverted in $\mathbf{DA}^{\text{eff}}_{\text{ét}}(S,\Lambda)$. In particular there is an isomorphism $M^S_{\text{ét}}(\mathbb{A}^1_X) \to M^S_{\text{ét}}(X)$ for all $X \in \mathrm{Sm}/S$ induced by $p : \mathbb{A}^1_X \to X$. Another important observation is that $\mathbf{DA}^{\text{eff}}_{\text{ét}}(S,\Lambda)$ inherits the monoidal structure of $\mathbf{D}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda))$, which at the same time comes from the monoidal structure of $\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda)$.

Let $\mathbf{L}$ be the Lefschetz motive defined as the cokernel of the inclusion $\Lambda^S_{\text{ét}}(\infty_S) \hookrightarrow \Lambda^S_{\text{ét}}(\mathbb{P}^1_S)$. The next step in the construction of the triangulated category of motivic étale sheaves is to invert the Lefschetz motive for the monoidal structure. The process used to formally invert the Lefschetz motive in [Ayo14b] is to consider $\mathbf{L}-$spectra for the tensor product.

Definition 2.1.2. *An $\mathbf{L}-$spectrum of étale sheaves on Sm/S is a collection of étale sheaves*

$$E = (E_n, \gamma_n)_{n\in\mathbb{N}}$$

where $\gamma_n : \mathbf{L} \otimes E_n \to E_{n+1}$ is a morphism of sheaves called the n-th assembly map. We call the sheaf E_n the n-th level of the $\mathbf{L}-$spectrum E.

A morphism of $\mathbf{L}-$spectra $f : E \to E'$ is a collection of morphism of sheaves $f = (f_n)_{n\in\mathbb{N}}$, where $f_n : E_n \to E'_n$ such that the diagram

$$\begin{array}{ccc} \mathbf{L}\otimes E_n & \xrightarrow{\mathrm{id}\otimes f_n} & \mathbf{L}\otimes E'_n \\ \big\downarrow{\scriptstyle \gamma_n} & & \big\downarrow{\scriptstyle \gamma'_n} \\ E_{n+1} & \xrightarrow{f_{n+1}} & E'_{n+1} \end{array}$$

commutes for all $n \in \mathbb{N}$. We denote by $\mathbf{Spt_L}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda))$ the category of $\mathbf{L}$-spectra. Consider an $\mathbf{L}-$spectrum E. The evaluation functor $\mathrm{Ev}_p : \mathbf{Spt_L}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda)) \to \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda)$ admits a left adjoint $\mathrm{Sus}^p_{\mathbf{L}}$ given by

$$\mathrm{Sus}^p_{\mathbf{L}}(K) = (\overbrace{0,\dots,0}^{p-1 \text{ times}}, K, \mathbf{L}\otimes K, \mathbf{L}^{\otimes 2}\otimes K,\dots,)$$

When $p = 0$ the suspension functor is called the infinite suspension functor and it is denoted by $\Sigma^\infty_{\mathbf{L}}$. Finally, we define $\mathbf{DA}^{\text{ét}}(S,\Lambda)$ as the Verdier quotient of the category $\mathbf{D}(\mathbf{Spt_L}(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S,\Lambda)))$ by the smallest triangulated subcategory $\mathcal{L}_{st}$ closed by arbitrary sums and containing the complexes

$$\dots \to 0 \to \mathrm{Sus}^p_{\mathbf{L}}\Lambda^S_{\text{ét}}(\mathbb{A}^1_U) \to \mathrm{Sus}^p_{\mathbf{L}}\Lambda^S_{\text{ét}}(U) \to 0 \to \dots$$
$$\dots \to 0 \to \mathrm{Sus}^{p+1}_{\mathbf{L}}(\mathbf{L}\otimes\Lambda^S_{\text{ét}}(U)) \to \mathrm{Sus}^p_{\mathbf{L}}\Lambda^S_{\text{ét}}(U) \to 0 \to \dots$$

for all $U \in \mathrm{Sm}/S$ and all $p \in \mathbb{N}$.

Definition 2.1.3. *The objects in the category $\boldsymbol{DA}^{ét}(S,\Lambda)$ are called motivic étale sheaves over S. Given a smooth $S-$scheme X, then $\Sigma^\infty_{\mathbf{L}}\Lambda^S_{ét}(X)$ is called the homological motive of X and will be denoted by $M^S_{ét}(X)$. We denote $\boldsymbol{DA}^{ét}_{ct}(S,\Lambda)$ the smallest triangulated* *subcategory of* $\boldsymbol{DA}^{ét}(S,\Lambda)$ *closed under direct summands and containing the motives* $M^S_{ét}(X)(-p)[-2p] := Sus^p_{\mathbf{L}}\Lambda^S_{ét}(X)$ *for* $p \in \mathbb{N}$ *and* X *an* $S-$*scheme of finite presentation. Those motivic sheaves are called constructible.*

The category $\mathrm{DM}_{\text{ét}}(S,\Lambda)$ is constructed in a similar way as $\mathbf{DA}^{\text{ét}}(S,\Lambda)$, but instead of considering the whole category $\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}/S,\Lambda)$ we consider the category of étale sheaves with transfers, i.e. sheaves that come from an étale presheaf that is an additive contravariant functor.

Denote as $\mathbf{SmCor}(S,\Lambda)$ the category of smooth correspondences over S with coefficients in Λ. The objects are the same ones of Sm/S, and for $U,V \in \mathrm{Sm}/S$ the morphisms are finite Λ-correspondences from $U \to V$. Let $\mathbf{Sh}_{\text{ét}}(\mathbf{SmCor}(S,\Lambda))$ be the category of additive presheaves of commutative groups on $\mathbf{SmCor}(S,\Lambda)$ whose restriction to Sm/S is an étale sheaf. We call this the category of étale shaves with transfers. According to [CD16, Corollary 2.1.12] there is an adjoint pair of functors

$$\gamma^* : \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda) \rightleftarrows \mathbf{Sh}_{\text{ét}}(\mathbf{SmCor}(S,\Lambda)) : \gamma_*.$$

Let X be a smooth S-scheme. We denote by $\Lambda^{tr}(X)$ the complex of sheaves given by $c(-,X)$ the finite correspondences and let $\underline{X}$ be the sheaf associated to X defined by the presheaf

$$U \mapsto \underline{X}(U) = \Lambda\mathrm{Hom}_{\mathrm{Sm}/S}(U,X).$$

of commutative groups. The functor γ_* forgets transfers and $\gamma^*(\underline{X}) = \mathbb{Z}^{tr}(X)$. We continue by considering the derived category $D(\mathbf{Sh}_{\text{ét}}(\mathbf{SmCor}(S,\Lambda)))$ as $\mathbf{Sh}_{\text{ét}}(\mathbf{SmCor}(S,\Lambda))$ is an abelian category. After that we take the $\mathbb{A}^1$-localization of the derived category $D(\mathbf{Sh}_{\text{ét}}(\mathbf{SmCor}(S,\Lambda)))$, giving us the triangulated category of effective étale motives $\mathrm{DM}^{\text{eff}}_{\text{ét}}(S,\Lambda)$. Finally, to this category we can associate a stable $\mathbb{A}^1$-derived category $\mathrm{DM}_{\text{ét}}(S,\Lambda)$, the category of triangulated étale motives, by $\otimes$-inverting the Tate object $\Lambda^{\text{tr}}_S(1) := \Lambda^{\text{tr}}_S(\mathbb{P}^1_S,\infty)[-2]$. This can be obtained by applying the functor Σ^∞.

The functor γ^* is a left Quillen functor, thus we have its derived version

$$L\gamma^* : D(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S,\Lambda)) \rightleftarrows D(\mathbf{Sh}_{\text{ét}}(\mathbf{SmCor}(S,\Lambda))) : \gamma_*.$$

which preserves $\mathbb{A}^1$-equivalences. With this we obtain an adjunction in the following way

$$L\gamma^* : \mathbf{DA}^{\text{ét}}(S,\Lambda) \rightleftarrows \mathrm{DM}_{\text{ét}}(S,\Lambda) : R\gamma_*$$

If S is a noetherian scheme of finite dimension, notice that by [Ayo14b, Théorème B.1] and [CD16, Remark 5.5.9], the categories above mentioned are equivalent. In the context of Voevodsky motives the constructible (compact) objects are called the geometrical motives; the corresponding category is denoted by $\mathrm{DM}^{gm}_{\text{ét}}(S,\Lambda)$. Also by [CD16, Corollary 5.5.5] for a quasi-excellent geometrically unibranch noetherian scheme of finite dimension S the adjunctions

$$\mathbf{L}\psi_! : \mathrm{DM}_{\text{ét}}(S,R) \rightleftarrows \mathrm{DM}_h(S,R) : \mathbf{R}\psi^*$$

give us an equivalence of monoidal triangulated categories.

Among the properties that we have to mention about the different models for the triangulated category of étale motives, is the one concerning conservative functors. Let $\mathfrak{TC}$ be the 2-category of triangulated categories. According to [Ayo14b, Théorème 3.9], for a commutative ring Λ the homotopic stable 2-functor $\mathbf{DA}^{\text{ét}}(-,\Lambda) : \mathrm{Sch}/S \to \mathfrak{TC}$ is separated, this means that for any S-morphism $f : X \to Y$, the induced functor f^* is conservative. Fixing a field k, let us consider a field extension K/k, and the induced map $p : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$. Since p is a surjective k-morphism, we have that

$$p^* : \mathbf{DA}^{\text{ét}}(k,\Lambda) \to \mathbf{DA}^{\text{ét}}(K,\Lambda)$$

is conservative. Using the equivalence of categories $\mathbf{DA}^{\text{ét}}(k,\Lambda) \simeq \mathrm{DM}_{\text{ét}}(k,\Lambda)$ (and the same for K), we obtain that $p^* : \mathrm{DM}_{\text{ét}}(k,\Lambda) \to \mathrm{DM}_{\text{ét}}(K,\Lambda)$ is also conservative.

The next examples of a conservative family: according to [Ayo14b, Proposition 3.24], for a scheme S such that the cohomological dimension of the residue fields is bounded, the family of functors $x^* : \mathbf{DA}^{\text{ét}}(S,\Lambda) \to \mathbf{DA}^{\text{ét}}(x,\Lambda)$,for $x \in S$, is conservative.

Now consider a noetherian scheme S, by [AHP16, Lemma A.6], a motive $M \in \mathbf{DA}^{\text{ét}}(S,\mathbb{Q})$ is zero if and only if the pullback to any geometric point $i_{\bar{s}} : \bar{s} \to S$ is zero. Even more, a morphism $f \in \mathbf{DA}^{\text{ét}}(S,\mathbb{Q})$ is an isomorphism if and only if $i_{\bar{s}}^*(f)$ is an isomorphism for any geometric point $\bar{s}$. This theorem can be extended to $\mathbb{Z}$-coefficients by imposing more restrictions on the base S. In order to do this, let us recall some definitions given in [Ayo14b, Définition 3.12]: for a prime number p, we define the *punctual p-cohomological dimension* of a scheme S as $\mathrm{pcd}_p(S) = \sup_{s\in S}\{\mathrm{cd}_p(\kappa(s))\} \in \mathbb{N}\cup\{\infty\}$, where $\kappa(s)$ is the residue field of a point $s \in S$.

Definition 2.1.4. *Let S be a scheme. We say that S is good enough for this purposes if it has finite Krull dimension and the punctual p-cohomological dimension is bounded for every prime p.*

We can now move on to the following lemma, by mimicking the proof given for [AHP16, Lemma A.6.]:

Lemma 2.1.5. *Let S be a good enough scheme. Then the following holds:*

1. *Let $M \in \boldsymbol{DA}^{\text{ét}}(S,\mathbb{Z})$ be a motive. Then M is zero if and only if the pullback $i_{\bar{s}}^*M$ to any geometric point $\bar{s} \to S$ is zero.*

2. *Let f be a morphism in $\boldsymbol{DA}^{\text{ét}}(S,\mathbb{Z})$. Then f is an isomorphism if and only if the pullback $i_{\bar{s}}^*(f)$ is an isomorphism for any geometric point $\bar{s} \to S$.*

Proof. This follows from arguments given in [AHP16]. By [Ayo14b, Proposition 3.24] we can assume that $S = \mathrm{Spec}(k)$ with k a field. Assuming that k is perfect, consider an algebraic closure $\bar{k}$. Let $N \in \mathbf{DA}^{\text{ét}}(k,\mathbb{Z})$ be a motive such that the pullback i^*N, with $i : \bar{k} \to k$, vanishes. Under the assumptions on S, $\mathbf{DA}^{\text{ét}}(S,\mathbb{Z})$ is compactly generated. Therefore we have to prove that all morphism $f : C \to N$ with C compact vanish. Using the assumptions, $i^*(f)$ vanishes, ans according to [Ayo14b, Lemme 3.4], there exists a finite extension K/k such that the pullback of f vanishes. By [Ayo14b, Théorème 3.9] the functor $i_K^* : \mathbf{DA}^{\text{ét}}(k, \mathbb{Z}) \to \mathbf{DA}^{\text{ét}}(K, \mathbb{Z})$ is conservative, therefore f vanishes. When k is not perfect, we consider a purely inseparable extension k^i and note that the pullback functor $\mathbf{DA}^{\text{ét}}(k, \mathbb{Z}) \to \mathbf{DA}^{\text{ét}}(k^i, \mathbb{Z})$ is an equivalence of categories, by [CD16, Proposition 6.3.16]. The second statement follows from the first one. □

The last result that we mention about conservativity is related to the family of functors induced by change of coefficients. Let S be a noetherian scheme of finite dimension. For such S, we recall that by [CD16, Proposition 5.4.12], the family of functors

$$\begin{aligned} \rho_{\mathbb{Q}} &: \mathrm{DM}_{\text{ét}}(S, \mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(S, \mathbb{Q}) \\ \rho_{\mathbb{Z}/\ell} &: \mathrm{DM}_{\text{ét}}(S, \mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(S, \mathbb{Z}/\ell), \;\; \ell \text{ prime number invertible in } S \end{aligned}$$

is conservative. Since $\mathrm{DM}_{\text{ét}}(S, \mathbb{Q}) \simeq \mathrm{DM}(S, \mathbb{Q})$ and by [CD16, Theorem 4.5.2], the so-called rigidity theorem, we have that $\mathrm{DM}_{\text{ét}}(S, \mathbb{Z}/\ell) \simeq D(S_{\text{ét}}, \mathbb{Z}/\ell)$, where the last category is the derived category of étale sheaves with coefficients over $\mathbb{Z}/\ell$.

By [CD16, Theorem 6.2.17] and [CD19, Corollary 4.4.24], in the category of étale motives we also have Verdier duality. We say that an object $U \in \mathrm{DM}_{\text{ét}}(X, R)$ is dualizing if has the following two properties:

1. U is constructible;
2. For any constructible element $M \in \mathrm{DM}_{\text{ét}}(X, R)$ the morphism
$$M \to \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(M, U), U)$$
is an isomorphism.

Following [CD16, Theorem 6.2.17], if S is a regular scheme an object U is dualizing if and only if is $\otimes$-invertible. Let $f : X \to S$ be a separated morphism of finite type, we define the duality operator $D_{X/S}$ as

$$D_{X/S}(-) := \underline{\mathrm{Hom}}(-, f^!U)$$

If R is a $\mathbb{Q}$-algebra or if $R = \mathbb{Z}/\ell^m$, with ℓ an invertible element in S and m a natural number, then

1. For any separated S-scheme of finite type X, and for all objects M, N in $\mathrm{DM}_{\text{ét}}(X, R)$, if N is constructible, then we have a canonical isomorphism
$$D_{X/S}(M \otimes D_{X/S}(N)) \simeq \underline{\mathrm{Hom}}_{\mathrm{DM}_{\text{ét}}(X,R)}(M, N).$$
2. For any morphism between separated S-schemes $f : Y \to X$ we have natural isomorphisms
$$\begin{aligned} D_{Y/S} \circ f^* &\simeq f^! \circ D_{X/S} \\ f^* \circ D_{X/S} &\simeq D_{Y/S} \circ f^! \\ D_{X/S} \circ f_! &\simeq f_* \circ D_{Y/S} \\ f_! \circ D_{Y/S} &\simeq D_{X/S} \circ f_* \end{aligned}$$
restricting to constructible elements.

The case when R is a $\mathbb{Q}$-algebra follows from [CD19, Corollary 4.4.24] and the ℓ-adic case follows from Gabber's work [ILO14, Exposé XVII].

Remark 2.1.6. The category $\mathrm{DM}_h(S,R)$ has all the good properties described before, without any hypothesis over S and the coefficient ring R. One has an equivalence of categories $\mathrm{DM}_h(S,R)$ and $\mathbf{DA}^{\text{ét}}(S,R)$, at least if the base S is good enough. With respect to the model of étale sheaves with transfers, $\mathrm{DM}_{\text{ét}}(S,R)$ coincides with $\mathrm{DM}_h(S,R)$ for an arbitrary base S if the R is a torsion ring whose characteristic is invertible in S by [CD16, Corollary 5.5.4] or for any ring R and S a quasi-excellent geometrically unibranch noetherian scheme of finite dimension, [CD16, Corollary 5.5.5].

2.2 Étale motivic cohomology

Étale motivic cohomology

In this subsection we use the category of étale motives, since we do not mention much more details about the construction and/or functorial behaviour of the category, for further details about these properties we refer the reader to [Ayo14b] and [CD16]. Let k be a field and let R be a commutative ring. We denote the category of effective motivic étale sheaves with coefficients in R over the field k as $\mathrm{DM}^{\text{eff}}_{\text{ét}}(k,R)$. If we invert the Lefschetz motive, we then obtain the category of motivic étale sheaves denoted by $\mathrm{DM}_{\text{ét}}(k,R)$. One defines the **étale motivic cohomology** group of X of bi-degree (m,n) with coefficients in a commutative ring R as

$$H^m_{M,\text{ét}}(X,R(n)) := \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k,R)}(M_{\text{ét}}(X),R(n)[m]).$$

where $M_{\text{ét}}(X)=\rho^*M(X)$ with ρ the canonical map associated to the change of topology $\rho : (\mathrm{Sm}_k)_{\text{ét}} \to (\mathrm{Sm}_k)_{\text{Nis}}$, which induces an adjunction $\rho^* := \mathbf{L}\rho^* : \mathrm{DM}(k,\mathbb{Z}) \rightleftarrows \mathrm{DM}_{\text{ét}}(k,\mathbb{Z}) : \mathbf{R}\rho_* =: \rho_*$. In particular we define the **étale Chow groups** of codimension n as the étale motivic cohomology in bi-degree $(2n,n)$ with coefficients in $\mathbb{Z}$, i.e.

$$\begin{aligned}\mathrm{CH}^n_{\text{ét}}(X) : &= H^{2n}_{M,\text{ét}}(X,\mathbb{Z}(n)) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M_{\text{ét}}(X),\mathbb{Z}(n)[2n]).\end{aligned}$$

Remark 2.2.1.

1. Let k be a field and let ℓ be a prime number different from the characteristic of k. By the rigidity theorem for torsion motives, see [CD16, Theorem 4.5.2], we have an isomorphism

$$H^m_{M,\text{ét}}(X,\mathbb{Z}/\ell^r(n)) \simeq H^m_{\text{ét}}(X,\mu_{\ell^r}^{\otimes n}).$$

2. Let $f : X \to \mathrm{Spec}(k)$ be a smooth scheme over a field k. Due to the six functor formalism we can define

$$H^m_{\text{ét}}(X,\mathbb{Z}(n)) := \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(X)}(\mathbf{1}_X,\mathbf{1}_X(n)[m])$$

because

$$\begin{aligned} H^m_{\text{ét}}(X, \mathbb{Z}(n)) &= \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(X)}(\mathbf{1}_X, \mathbf{1}_X(n)[m]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(X)}(\mathbf{1}_X, f^*(\mathbf{1}_k)(n)[m]) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(\mathbf{L}f_\#(\mathbf{1}_X), \mathbf{1}_k(n)[m]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M_{\text{ét}}(X), \mathbf{1}_k(n)[m]) \end{aligned}$$

3. It is possible to work with étale motivic homology for a singular scheme X over a base S with structural morphism $f : X \to S$. For that we need to introduce the Borel-Moore homology as follows

$$H^{BM,\text{ét}}_{m,n}(X/S) := \mathrm{Hom}_{\mathrm{DM}_h(X)}\left(\mathbf{1}_X(n)[m], f^!(\mathbf{1}_S)\right)$$

   with this notation $\mathrm{CH}^{BM,\text{ét}}_n(X/S) := H^{BM,\text{ét}}_{2n,n}(X/S)$. We also can recover a comparison map $\sigma_n : \mathrm{CH}_n(X/S) \to \mathrm{CH}^{BM,\text{ét}}_n(X/S)$.

Gysin morphism and functoriality properties

With respect to functoriality properties of the étale Chow groups we should mention that we can recover well-known properties analogous to that of classical Chow groups, such as pull-back and proper pushforwards of cycles. In particular, we get a degree map. All these properties will arise from the properties of the category $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$ (resp. $\mathrm{DM}(k, \mathbb{Z})$) and the covariant functor $M_{\text{ét}}(-)$ (resp. $M(-)$).

Let us recall that the canonical map $\rho : (\mathrm{Sm}_k)_{\text{ét}} \to (\mathrm{Sm}_k)_{\mathrm{Nis}}$ induces an adjunction of triangulated categories

$$\rho^* : \mathrm{DM}_{gm}(k, \mathbb{Z}) \rightleftarrows \mathrm{DM}_{gm,\text{ét}}(k, \mathbb{Z}) : \rho_*,$$

which leads us to express the étale Chow groups in terms of morphism in the category $\mathrm{DM}(k, \mathbb{Z})$ as follows

$$\begin{aligned} H^m_{M,\text{ét}}(X, \mathbb{Z}(n)) &:= \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M_{\text{ét}}(X), \mathbb{Z}_{\text{ét}}(n)[m]) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(k)}(M(X), \rho_*\mathbb{Z}_{\text{ét}}(n)[m]). \end{aligned}$$

Proposition 2.2.2. *The comparison map*

$$\sigma^{m,n} : H^m_M(X, \mathbb{Z}(n)) \to H^m_{M,\acute{e}t}(X, \mathbb{Z}(n))$$

coming from the adjunction of triangulated categories, is compatible with pullbacks, pushforward and intersection products.

Proof. Consider the adjunction of triangulated categories

$$\rho^* : \mathrm{DM}(k, \mathbb{Z}) \rightleftarrows \mathrm{DM}_{\text{ét}}(k, \mathbb{Z}) : \rho_*$$

where ρ^* is the functor induced by the étale sheafification and ρ_* is the right adjoint which is a forgetful functor which forgets that the complexes are étale. As we have said, the cycle class map is obtained by the following use of the adjunction

$$\mathrm{Hom}_{\mathrm{DM}(k,\mathbb{Z})}(M(X), \mathbb{Z}(n)[m]) \xrightarrow{\rho^*} \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})}(\rho^* M(X), \rho^*\mathbb{Z}(n)[m])$$

where $\mathbb{Z}(n)$ is the motivic complex of twist n and $M(X)$ is the triangulated motive associated with X. By adjunction we have

$$\mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})}(\rho^* M(X), \rho^*\mathbb{Z}(n)[m]) \simeq \mathrm{Hom}_{\mathrm{DM}(k,\mathbb{Z})}(M(X), \rho_*\rho^*\mathbb{Z}(n)[m])$$

so we obtain a canonical map $\mathbb{Z}(n) \to \rho_*\rho^*\mathbb{Z}(n) = \rho_*\mathbb{Z}_{\text{ét}}(n)$ given by the unit transformation associated to the adjunction. Now, the functorial properties of maps $f : X \to Y$ follow from the (covariant) functorial properties of the motive $M(X)$ and the existence of Gysin maps, for more details about the existence of Gysin morphisms we refer to [Dég12a] and [Dég08]. To be more precise: Let $f : Y \to X$ be a morphism of relative dimension d, then we have induced commutative squares

$$\begin{array}{ccc} M(X)(d)[2d] & \xrightarrow{f^!} & M(Y) \\ \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} \\ M_{\text{ét}}(X)(d)[2d] & \xrightarrow{f^!} & M_{\text{ét}}(Y) \end{array} \qquad\qquad \begin{array}{ccc} M(Y) & \xrightarrow{f_*} & M(X) \\ \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} \\ M_{\text{ét}}(Y) & \xrightarrow{f_*} & M_{\text{ét}}(X) \end{array}$$

which induce the pullback and pushforward for proper morphisms. In fact, any morphism of motivic complexes like the one given by the adjunction will yield a morphism of cohomology groups compatible both with pullbacks and pushforward.

Finally, we need to prove the compatibility with respect to products. This property comes from the fact that we have a quasi-isomorphism

$$\mathbb{Z}(i) \otimes \mathbb{Z}(j) \xrightarrow{\sim} \mathbb{Z}(i+j)$$

and that the functor ρ^* is monoidal, i.e. $\rho^*(M \otimes N) \simeq \rho^*(M) \otimes \rho^*(N)$. Therefore we also obtain that

$$\mathbb{Z}(i)_{\text{ét}} \otimes \mathbb{Z}(j)_{\text{ét}} \xrightarrow{\sim} \mathbb{Z}(i+j)_{\text{ét}}$$

For intersection products the remaining part is to consider that the product comes from the operation $\alpha \cdot \beta = \Delta^*(\alpha \otimes \beta)$. □

Lemma 2.2.3 (Projection formula). *Let $f : Y \to X$ be a projective morphism of codimension d between smooth schemes. Then for all $y \in CH^*_{\acute{e}t}(Y)$ and $x \in CH^*_{\acute{e}t}(X)$ we have that*

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y).$$

Proof. Consider an element $\alpha \in H^m_M(X, \mathbb{Z}(n))$ and any morphism $\phi : M(X) \to N$ in $\mathrm{DM}(k)$, viewing $\alpha : M(X) \to \mathbf{1}(n)[m]$ as a morphism of motives and $\Delta_* : M(X) \to M(X) \otimes M(X)$ the map induced by the diagonal. Define the product

$$\phi \boxtimes \alpha := (\phi \otimes \alpha) \circ \Delta_* : M(X) \to N(p)[m].$$

By [Dég08, Corollary 5.18] for such f there is an equality $((\mathbf{1}_Y)_* \boxtimes f_*) \circ f^* = f^* \boxtimes (\mathbf{1}_X)_*$ as morphisms of the motives $M(X) \to M(X \times Y)(d)[2d]$. This induce a map of étale higher Chow groups $H^{m-2d}_{M,\text{ét}}(X \times Y, \mathbb{Z}(n-d)) \to H^m_{M,\text{ét}}(X, \mathbb{Z}(n))$. On one hand, the map $((\mathbf{1}_Y)_* \boxtimes f_*) \circ f^*$ on the level of motives induces the map $f_*(f^*(-) \cdot \mathrm{id}_Y)$ on étale Chow groups and the map $f^* \boxtimes (\mathbf{1}_X)_*$ induces $\mathrm{id}_X \cdot f_*(-)$, and we obtain the desired equality. □

Localization sequence and specialization properties

Consider a base scheme S, we define the relative motivic cohomology of X as follows:

Definition 2.2.4. *Let X be a smooth S-scheme. We define the motivic cohomology of X relative to S in the following way*

$$H^m_M(X/S, \mathbb{Z}(n)) := Hom_{DM(S)}\left(M_S(X), \mathbf{1}_S(n)[m]\right).$$

For the special case $m = 2n$ we set $CH^n(X/S) := H^{2n}_M(X/S, \mathbb{Z}(n))$. In the same way, we define the relative étale motivic cohomology of X

$$H^m_{M,ét}(X/S, \mathbb{Z}(n)) := Hom_{DM_{ét}(S)}\left(M_S(X), \mathbf{1}_S(n)[m]\right).$$

In the special case $m = 2n$ we write $CH^n_{ét}(X/S) := H^{2n}_{M,ét}(X/S, \mathbb{Z}(n))$

Consider $i : Z \to X$ be a closed immersion of smooth schemes over a scheme S of pure codimension c and denote the open complement as $U := X - Z$. Consider the structural morphism $p : X \to S$, then we have that $M_S(X) := p_\#(\mathbf{1}_X)$. We have an associated Gysin triangle of the form

$$M_S(U) \xrightarrow{j_*} M_S(X) \xrightarrow{i^*} M_S(Z)(c)[2c] \xrightarrow{\partial_{X,Z}} M_S(U)[1].$$

This exact triangle give us the long exact sequence known as localization sequence, which is the following

$$\ldots \to \mathrm{CH}^n(U/S, 1) \to \mathrm{CH}^{n-c}(Z/S) \to \mathrm{CH}^n(X/S) \to \mathrm{CH}^n(U/S) \to 0.$$

Since the functor $\rho : \mathrm{DM}(S, \mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(S, \mathbb{Z})$ is exact, we obtain the following exact triangle

$$M^S_{\text{ét}}(U) \xrightarrow{j_*} M^S_{\text{ét}}(X) \xrightarrow{i^*} M^S_{\text{ét}}(Z)(c)[2c] \xrightarrow{\partial_{X,Z}} M^S_{\text{ét}}(U)[1]$$

in $\mathrm{DM}_{\text{ét}}(S, \mathbb{Z})$. Thus, it is easy to see that we have an associated long exact sequence, and we obtain the étale analogue of the localization long exact sequence for Chow groups

$$\ldots \to \mathrm{CH}^{n-c}_{\text{ét}}(Z/S) \to \mathrm{CH}^n_{\text{ét}}(X/S) \to \mathrm{CH}^n_{\text{ét}}(U/S) \to \mathrm{CH}^{n-c}_{\text{ét}}(Z/S, -1) \to \ldots$$

Remark 2.2.5. Notice that even when $S = \mathrm{Spec}(k)$, there are no arguments to assume that in general the map $j^* : \mathrm{CH}^n_{\text{ét}}(X) \to \mathrm{CH}^n_{\text{ét}}(U)$ is surjective. For instance, consider Z a smooth projective surface in X and $n = \dim(X) - 1$, then we have $\mathrm{CH}^{n-c}_{\text{ét}}(Z, -1) \simeq \mathrm{Br}(Z)$.

Let S be a regular scheme, X a smooth S-scheme with $f : X \to S$, and $i : \bar{S} \hookrightarrow S$ a closed embedding. We set $S^\circ := S - \bar{S}$ and $j : S^\circ \hookrightarrow S$ the open immersion. In this way, we obtain a commutative diagram with cartesian squares

$$\begin{tikzcd} \bar{X} \arrow[r, hook, "i_X"] \arrow[d, "\bar{f}"] & X \arrow[d, "f"] & X^\circ \arrow[l, hook', "j_X"'] \arrow[d, "f^\circ"] \\ \bar{S} \arrow[r, hook, "i"] & S & S^\circ \arrow[l, hook', "j"'] \end{tikzcd} \tag{2.1}$$

Let us recall some functorial properties about these maps through the six functor formalism given in Definition 1.3.3. Since the motivic category $\mathrm{DM}(S,\mathbb{Z})$ has the six functor formalism, the natural transformations $i_! \to i_*$, $i_\sharp \to i_!(c)[2c]$ and $i^* \to i^!(-c)[-2c]$ are isomorphisms (where the last two isomorphisms also hold for j by setting $c = 0$). There is a natural transformation $f^* i_! \xrightarrow{\sim} i_{X!}\bar{f}^*$ which is an isomorphism, while for the left-hand square the natural transformation $f^\circ_* j^!_X \xrightarrow{\sim} j^! f_*$ is also an isomorphism.

Since the structure morphism of X° as an S-scheme is $f \circ j_X$, we obtain that $M_S(X^\circ) = (f \circ j_X)_\sharp(\mathbf{1}_S) = (j \circ f^\circ)_\sharp(\mathbf{1}_{X^\circ})$ and $M_{S^\circ}(X^\circ) = f^\circ_\sharp(\mathbf{1}_{X^\circ})$ therefore we have the following isomorphism of relative Chow groups

$$\begin{aligned} \mathrm{CH}^n(X^\circ/S) &= \mathrm{Hom}_{\mathrm{DM}(S)}\left(j_\sharp \circ f^\circ_\sharp(\mathbf{1}_{X^\circ}), \mathbf{1}_S(n)[2n]\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(S)}\left(j_! \circ f^\circ_\sharp(\mathbf{1}_{X^\circ}), \mathbf{1}_S(n)[2n]\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(S^\circ)}\left(f^\circ_\sharp(\mathbf{1}_{X^\circ}), j^!(\mathbf{1}_S(n)[2n])\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(S^\circ)}\left(f^\circ_\sharp(\mathbf{1}_{X^\circ}), \mathbf{1}_{S^\circ}(n)[2n]\right) = \mathrm{CH}^n(X^\circ/S^\circ), \end{aligned}$$

while for the closed immersion $\bar{X}$, we have that $M_S(\bar{X}) = (i \circ \bar{f})_\sharp(\mathbf{1}_{\bar{X}})$ and $M_{\bar{S}}(\bar{X}) = \bar{f}_\sharp(\mathbf{1}_{\bar{X}})$. Therefore

$$\begin{aligned} \mathrm{CH}^n(\bar{X}/S) &= \mathrm{Hom}_{\mathrm{DM}(S)}\left(i_\sharp \circ \bar{f}_\sharp(\mathbf{1}_{\bar{X}}), \mathbf{1}_S(n)[2n]\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(S)}\left(i_!(\bar{f}_\sharp(\mathbf{1}_{\bar{X}}))(c)[2c], \mathbf{1}_S(n)[2n]\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(\bar{S})}\left(\bar{f}_\sharp(\mathbf{1}_{\bar{X}}), i^!(\mathbf{1}_S(n-c)[2n-2c])\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(\bar{S})}\left(\bar{f}_\sharp(\mathbf{1}_{\bar{X}}), i^*(\mathbf{1}_S(n)[2n])\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(\bar{S})}\left(\bar{f}_\sharp(\mathbf{1}_{\bar{X}}), \mathbf{1}_{\bar{S}}(n)[2n]\right) = \mathrm{CH}^n(\bar{X}/\bar{S}). \end{aligned}$$

Remark 2.2.6. We can obtain the same formalism for étale motivic cohomology, since the duality properties still hold for the categories $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})$ with S an integral scheme. Therefore, if we define the relative versions of étale Chow groups there are isomorphisms $\mathrm{CH}^n_{\text{ét}}(X^\circ/S) \simeq \mathrm{CH}^n_{\text{ét}}(X^\circ/S^\circ)$ and $\mathrm{CH}^n_{\text{ét}}(\bar{X}/S) \simeq \mathrm{CH}^n_{\text{ét}}(\bar{X}/\bar{S})$.

Consider the operation $i^! : \mathrm{CH}^n(X/S) \to \mathrm{CH}^{n-c}(\bar{X}/S) \simeq \mathrm{CH}^{n-c}(\bar{X}/\bar{S})$ and suppose that $i^! \circ i_* : \mathrm{CH}^{n-c}(\bar{X}/\bar{S}) \to \mathrm{CH}^{n-c}(\bar{X}/\bar{S})$ is the zero map. Then there is a unique map $\sigma : \mathrm{CH}^n(X^\circ/S^\circ) \to \mathrm{CH}^{n-c}(\bar{X}/\bar{S})$, called the specialization map, such that $\sigma(j^*\alpha) = i^!(\alpha)$ for all $\alpha \in \mathrm{CH}^n(X/S)$. For more details see [Ful98, Chapter 20].

For a more general setting where we would be able to work with singular schemes, we have to consider Borel-Moore homology and motivic homotopic theory. For a base scheme S, let us consider the adjunctions, in the following diagram

$$\begin{tikzcd} \mathrm{DM}(S,\mathbb{Z}) \arrow[rr, shift left, "\rho^*"] \arrow[dr, shift left, "\gamma^*"] & & \mathrm{DM}_{\text{ét}}(S,\mathbb{Z}) \arrow[ll, shift left, "\rho_*"] \arrow[dl, shift left, "\gamma_*^{\text{ét}}"] \\ & \mathrm{SH}(S) \arrow[ul, shift left, "\gamma_*"] \arrow[ur, shift left, "\gamma_{\text{ét}}^*"] & \end{tikzcd} \tag{2.2}$$

Define the étale motivic cohomology spectrum as $H_{M,\text{ét}}\mathbb{Z} := \gamma_*^{\text{ét}}(\mathbb{Z}_{\text{ét}}(0))$ and the Borel-Moore étale motivic cohomology and homology as follows

$$\begin{aligned} H^{m,n}_{M,\text{ét}}(X) &:= \mathrm{Hom}_{\mathrm{SH}(k)}\left(\Sigma^\infty X_+, H_{M,\text{ét}}\mathbb{Z}(n)[m]\right) \\ H^{BM,\text{ét}}_{m,n}(X) &:= \mathrm{Hom}_{\mathrm{SH}(k)}\left(f_!(\mathbf{1}_X)(n)[m], H_{M,\text{ét}}\mathbb{Z}\right) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}\left(f_!(\mathbf{1}_X)(n)[m], \mathbf{1}_k\right) \end{aligned}$$

where $f : X \to k$ is a separated scheme of finite type. Now consider $f : X \to S$ a separated S-scheme of finite type, then we define the relative étale Borel-Moore motivic homology groups of X as

$$H^{BM,\text{ét}}_{m,n}(X/S) := \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})}\left(f_!(\mathbf{1}_X)(n)[m], \mathbf{1}_S\right)$$

Consider again the cartesian diagram 2.1. For any object $A \in \mathcal{F}(X)$, from [DJK21, (4.5.6.a)] we obtain a natural transformation of the form

$$i_{X*}\left(i_X^* A \otimes \bar{f}^*\mathrm{Th}(-N_{\bar{S}}S)\right) \to j_{X!}j_X^! A,$$

for a pure i-spectrum $\mathbb{E} \in \mathcal{F}(S)$ and any point of the K-theory space $e \in K(X)$, we have a specialization map

$$\sigma : \mathbb{E}(X^\circ/S^\circ, e) \simeq \mathbb{E}(X^\circ/S, e) \to \mathbb{E}(\bar{X}/S, e - f_{\bar{S}}^*\langle N_{\bar{S}}S\rangle) \simeq \mathbb{E}(\bar{X}/\bar{S}, e)$$

which is the generalization of the isomorphism obtained before. From the commutative diagram of adjunctions 2.2, we have the following definition:

Definition 2.2.7. *With the above notation, if $S = Spec(R)$ with R a discrete valuation ring we take $S^\circ = K := Frac(R)$ and $\bar{S} = k := R/\mathfrak{m}$, so we define the localization map for étale Chow groups as*

$$\sigma : CH_n^{BM,\acute{e}t}(X_K) \to CH_n^{BM,\acute{e}t}(X_k)$$

which can be understood as a map from the generic fiber to a special one.

Under the assumptions of smoothness, we obtain the specialization map $\sigma : \mathrm{CH}^n_{\text{ét}}(X_K) \to \mathrm{CH}^n_{\text{ét}}(X_k)$. Notice that this map is compatible with proper push-forward, flat pullbacks and intersection products.

Lichtenbaum cohomology

We consider a second notion of the étale version of Chow groups, the well known Lichtenbaum cohomology groups defined as the hypercohomology of the étale sheafification of the Bloch complex. These groups are characterized by Rosenschon and Srinivas in [RS16] using étale hypercoverings. In this context, we consider Sm_k as the category of smooth separated $k-$schemes over a field k. For each integer $n \geq 0$, we define the n-simplex scheme as the affine k-scheme

$$\Delta^n = \mathrm{Spec}\left(k[t_0, \ldots, t_n]/\left(\sum_{i=0}^{n} t_i - 1\right)\right)$$

which is isomorphic (non-canonically) to $\mathbb{A}^n_k$. Given a non-decreasing map $\rho : \{1, \ldots, m\} \to \{1, \ldots, n\}$ we obtain an induced map $\tilde{\rho} : \Delta^m \to \Delta^n$ acting on the coordinates as $t_i \mapsto \sum_{\rho(j)=i} t_j$. If the map ρ is injective, we call $\tilde{\rho}$ a face map with $\tilde{\rho}(\Delta^m)$ a face of Δ^n. If ρ is surjective, then $\tilde{\rho}$ is called a degeneracy map. Given natural numbers n and i, we define the group $z^n(X, i) \subset z^n(X \times \Delta^i)$ as the n-codimensional cycles in $X \times \Delta^i$ which intersect properly all $X \times F$ with $F \subset \Delta^i$ a face. We denote $z^n(X, \bullet)$ the cycle complex of abelian groups defined by Bloch

$$z^n(X, \bullet) : \cdots \to z^n(X, i) \to \cdots \to z^n(X, 1) \to z^n(X, 0) \to 0$$

where the differentials are given by the alternating sum of the pull-backs of the face maps and whose homology groups define the higher Chow groups $\mathrm{CH}^n(X, 2n - m) = H_m(z^n(X, \bullet))$.

Let us recall that $z^n(X, i)$ and the complex $z^n(X, \bullet)$ are covariant functorial for proper maps and contravariant functorial for flat morphisms between smooth k-schemes, see [Blo86, Proposition 1.3]. Therefore for a topology $t \in \{\text{flat, ét, Nis, Zar}\}$ we have a complex of t-presheaves $z^n(-, \bullet) : U \mapsto z^n(U, \bullet)$. In particular the presheaf $z^n(-, i) : U \mapsto z^n(U, i)$ is a sheaf for $t \in \{\text{flat, ét, Nis, Zar}\}$, see [Gei04, Lemma 3.1], and then $z^n(-, \bullet)$ is a complex of sheaves for the small étale, Nisnevich and Zariski sites of X. We define the complex of t-sheaves

$$R_X(n)_t = (z^n(-, \bullet)_t \otimes R)\,[-2n]$$

where R is an abelian group. For our purposes, we just consider $t = \mathrm{Zar}$ or ét and then we compute the hypercohomology groups $\mathbb{H}^m_t(X, R_X(n)_t)$. For example, setting $t = \mathrm{Zar}$ and $R = \mathbb{Z}$ the hypercohomology of the complex allows us to recover the higher Chow groups $\mathrm{CH}^n(X, 2n - m) \simeq \mathbb{H}^m_{\mathrm{Zar}}(X, \mathbb{Z}(n))$ because the complex of presheaves $U \mapsto z^n(U, \bullet)$ on X has the Mayer-Vietoris property i.e. for every open $U \subset X$ and every open covering $U = U_1 \cup U_2$ the square

$$\begin{tikzcd} z^n(U, \bullet) \arrow[r] \arrow[d] & z^n(U_1, \bullet) \arrow[d] \\ z^n(U_2, \bullet) \arrow[r] & z^n(U_1 \cap U_2, \bullet) \end{tikzcd}$$

is homotopy cartesian (Brown-Gersten), then by [MVW06, Theorem 19.11] the Bloch complex satisfies Zariski descent, i.e. the maps $H^m(z^m(U,\bullet)[-2n]) \to \mathbb{H}^m_{\mathrm{Zar}}(U,\mathbb{Z}_U(n))$ are isomorphisms. We denote the motivic and Lichtenbaum cohomology groups with coefficients in R as

$$H^m_M(X,R(n)) = \mathbb{H}^m_{\mathrm{Zar}}(X,R(n)), \quad H^m_L(X,R(n)) = \mathbb{H}^m_{\text{ét}}(X,R(n))$$

and in particular we set $\mathrm{CH}^n_L(X) = H^{2n}_L(X,\mathbb{Z}(n))$. Let $\pi : X_{\text{ét}} \to X_{\mathrm{Zar}}$ be the canonical morphism of sites, then the associated adjunction formula $\mathbb{Z}_X(n) \to R\pi_*\pi^*\mathbb{Z}_X(n) = R\pi_*\mathbb{Z}_X(n)_{\text{ét}}$ induces *comparison morphisms*

$$\mathrm{H}^m_M(X,\mathbb{Z}(n)) \xrightarrow{\kappa^{m,n}} \mathrm{H}^m_L(X,\mathbb{Z}(n))$$

for all bi-degrees $(m,n) \in \mathbb{Z}^2$. We can say more about the comparison map: due to [Voe11, Theorem 6.18], the comparison map $\kappa^{m,n} : H^m_M(X,\mathbb{Z}(n)) \to H^m_L(X,\mathbb{Z}(n))$ is an isomorphism for $m \leq n+1$ and a monomorphism for $m \leq n+2$.

Proposition 2.2.8. *Let X be a smooth projective variety over a field k, then the comparison map between motivic and Lichtenbaum cohomology groups*

$$\kappa^{m,n} : H^m_M(X,\mathbb{Z}(n)) \to H^m_L(X,\mathbb{Z}(n))$$

is compatible with respect to pullbacks of morphism and is also compatible with the product of cycles for all bi-degrees (m,n).

Proof. Consider $\pi : X_{\text{ét}} \to X_{\mathrm{Zar}}$ the canonical map of sites. There are induced functors in the derived categories

$$\pi^* : D(\mathrm{AbShv}(X_{\mathrm{Zar}})) \rightleftarrows D(\mathrm{AbShv}(X_{\text{ét}})) : R\pi_*$$

where π^* is the étalification of the Zariski sheaf and $R\pi_*$ is a forgetful functor.

Consider the canonical map induced by the adjunction $\mathbb{Z}(n) \to R\pi_*\pi^*\mathbb{Z}(n)$, i.e.

$$\begin{aligned}\mathrm{Hom}_{D_{\mathrm{Zar}}}(\mathbb{Z}(X),\mathbb{Z}(n)[m]) &\xrightarrow{\pi^*} \mathrm{Hom}_{D_{\text{ét}}}(\pi^*\mathbb{Z}(X),\pi^*\mathbb{Z}(n)[m]) \\ &\simeq \mathrm{Hom}_{D_{\mathrm{Zar}}}(\mathbb{Z}(X),R\pi_*\pi^*\mathbb{Z}(n)[m])\end{aligned}$$

The induced map is contravariantly functorial with respect to any morphism of smooth projective varieties, and also compatible with products. □

In some cases it is possible to obtain more information about the Lichtenbaum cohomology groups and the comparison with higher Chow groups. For instance there is a quasi-isomorphism $A_X(0)_{\text{ét}} = A$, the latter as an étale sheaf, thus we obtain that the Lichtenbaum cohomology agrees with the usual étale cohomology, i.e. $\mathrm{H}^m_L(X,A(0)) \simeq H^n_{\text{ét}}(X,A)$ for all $m \in \mathbb{Z}_{\geq 0}$ and in particular $\mathrm{CH}^0_L(X) = \mathbb{Z}^{\pi_0(X)}$. In the next step, $n=1$, since there is a quasi-isomorphism of complexes $\mathbb{Z}_X(1)_{\text{ét}} \sim \mathbb{G}_m[-1]$ we obtain the following isomorphisms

$$\mathrm{CH}^1(X) \simeq \mathrm{CH}^1_L(X) = \mathrm{Pic}(X)$$
$$\mathrm{H}^3_L(X,\mathbb{Z}(1)) \simeq \mathrm{H}^3_{\text{ét}}(X,\mathbb{G}_m[-1]) = \mathrm{Br}(X)$$

where $\mathrm{Pic}(X)$ and $\mathrm{Br}(X)$ are the Picard and Grothendieck-Brauer groups of X respectively. In fact for bi-degree $(n,1)$ by [VSF00, Corollary 3.4.3] there exists an isomorphism $\mathrm{H}^n_M(X,\mathbb{Z}(1)) \simeq \mathrm{H}^{n-1}_{\mathrm{Zar}}(X,\mathbb{G}_m)$ because the quasi-isomorphism $\mathbb{Z}_X(1) \sim \mathbb{G}_m[-1]$ also holds in the Zariski topology. As a particular case we obtain $\mathrm{H}^3_M(X,\mathbb{Z}(1)) \simeq \mathrm{H}^2_{\mathrm{Zar}}(X,\mathbb{G}_m) = 0$ because $\mathrm{H}^m_M(X,\mathbb{Z}(n)) = 0$ if $m > 2n$, whereas the Grothendieck-Brauer group of X is not always zero (for instance consider X an Enriques surface).

In bi-degree $(4,2)$ the comparison map is known to be injective but in general not surjective; we have a short exact sequence

$$0 \to \mathrm{CH}^2(X) \xrightarrow{\kappa^2} \mathrm{CH}^2_L(X) \to \mathrm{H}^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}(2)) \to 0,$$

where $\mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))$ is the Zariski sheaf associated to $U \mapsto \mathrm{H}_{\text{ét}}(U,\mathbb{Q}/\mathbb{Z}(2))$. Its unramified part is $\mathrm{H}^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}(2)) = \Gamma(X,\mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2)))$, for a proof we refer to [Kah12, Proposition 2.9]. If $k=\mathbb{C}$ the latter group surjects onto the torsion of the obstruction, in codimension 4, to the integral Hodge conjecture, i.e.

$$\mathrm{H}^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}(2)) \twoheadrightarrow \left(\mathrm{Hdg}^4(X,\mathbb{Z})/\mathrm{im}\left\{c^2 : \mathrm{CH}^2(X) \to H^4_B(X,\mathbb{Z}(2))\right\}\right)_{\mathrm{tors}}$$

and this obstruction is not zero in general, hence the comparison map κ^2 is not surjective; for more details see [CV12, Théorème 3.7].

Remark 2.2.9. The adjunction formula for rational coefficients, the morphism $\mathbb{Q}_X(n) \to R\pi_*\mathbb{Q}_X(n)_{\text{ét}}$ turns out to be isomorphism (see [Kah12, Théorème 2.6]), thus $\mathrm{H}^m_M(X,\mathbb{Q}(n)) \simeq \mathrm{H}^m_L(X,\mathbb{Q}(n))$ for all $(m,n) \in \mathbb{Z}^2$.

Example 2.2.10. *Let k be a field and let $X = Spec(k)$, then we can calculate the Lichtenbaum cohomology for $Spec(k)$ and compare it with the motivic case. By the previous remarks we have*

$$\begin{aligned} H^n_L(Spec(k),R(0)) &\simeq H^n_{\acute{e}t}(Spec(k),R) \\ &\simeq H^n(G,M). \end{aligned}$$

Here $G = Gal(k^s/k)$ where k^s is the separable closure of k and $M = \varinjlim_{k\subset k'} R(Spec(k'))$ where k' is a separable finite extension of k and $H^n(G,M)$ is the Galois cohomology of G with values in M. On the other hand we have that

$$H^n_M(Spec(k),R(0)) = \begin{cases} R \text{ if } n=0, \\ 0 \text{ otherwise.} \end{cases}$$

The following result is a well known fact, known as the *Suslin rigidity theorem*, about the morphism $\mathbb{Z}_X(n) \to R\pi_*\mathbb{Z}_X(n)_{\text{ét}}$ for $n \geq \dim(X)$ over $k = \bar{k}$.

Theorem 2.2.11. *[VSF00, Section 6, Theo. 4.2],[Gei17, Section 2] Let X be a smooth projective variety of dimension d over an algebraically closed field k. Then for $n \geq d$ the canonical map $\pi : X_{\acute{e}t} \to X_{Zar}$ induces a quasi-isomorphism between complexes of Zariski sheaves $\mathbb{Z}_X(n) \to R\pi_*\mathbb{Z}_X(n)_{\acute{e}t}$.*

Proof. Since $\mathbb{Q}_X(n) \to R\pi_*\mathbb{Q}_X(n)_{\text{ét}}$ is a quasi-isomorphism for all $n \in \mathbb{N}$, we only have to focus on torsion coefficients. In characteristic zero this was already proved by Suslin in [VSF00, Prop. 4.1, Thm. 4.2], an in general away from the characteristic of the field k.

For the general case, assume that $n = d$ and $l \in \mathbb{N}$. If k has positive characteristic then by [Gei10, Lemma 2.4] for a constructible sheaf $\mathcal{F}$ we have that $R\text{Hom}(\mathcal{F}, \mathbb{Z}/l(d)[2d])[-1] \cong R\text{Hom}(\mathcal{F}, \mathbb{Z}(d)[2d])$ and also there exists a perfect pairing of finite groups

$$\text{Ext}^{1-m}(\mathcal{F}, \mathbb{Z}_X(d)[2d]) \times H^m_c(X_{\text{ét}}, \mathbb{Z}/l)) \to \mathbb{Q}/\mathbb{Z},$$

so this gives us an isomorphism $H^{2d-m}_M(X, \mathbb{Z}/l(d))^* \simeq H^m_c(X_{\text{ét}}, \mathbb{Z}/l)$. Since X is smooth, Poincaré duality holds for étale cohomology, see [Mil80, Chapter VI §11]. Hence

$$H^m_c(X_{\text{ét}}, \mathbb{Z}/l))^* \simeq H^{2d-m}_{\text{ét}}(X, \mathbb{Z}/l(d)),$$

and therefore we obtain the isomorphisms

$$H^{2d-m}_M(X, \mathbb{Z}/l(d)) \simeq H^{2d-m}_{\text{ét}}(X, \mathbb{Z}/l(d)) \simeq H^{2d-m}_L(X, \mathbb{Z}/l(d)).$$

As in [VSF00, Theorem. 4.2] for a general $n \geq d$ we use the homotopy invariance of the higher Chow groups

$$\begin{aligned} H^{2d-m}_M(X, \mathbb{Z}/l(d)) &\simeq H^{2d-m}_M(X \times \mathbb{A}^n_k, \mathbb{Z}/l(d)) \\ &\simeq H^m_c(X \times \mathbb{A}^n_k, \mathbb{Z}/l)^* \\ &\simeq H^{m-2(n-d)}_c(X, \mathbb{Z}/l(d-n))^*. \end{aligned}$$

To conclude, we have a quasi-isomorphism $(\mathbb{Z}/l)_X(n) \to R\pi_*(\mathbb{Z}/l)_X(n)_{\text{ét}}$ for all $l \in \mathbb{N}$, therefore $(\mathbb{Q}/\mathbb{Z})_X(n) \to R\pi_*(\mathbb{Q}/\mathbb{Z})_X(n)_{\text{ét}}$ as well. Thus from the commutative diagram

$$\begin{tikzcd} \arrow[r] & H^{m-1}_M(X, \mathbb{Q}/\mathbb{Z}(n)) \arrow[r] \arrow[d, "\simeq"] & H^m_M(X, \mathbb{Z}(n)) \arrow[r] \arrow[d] & H^m_M(X, \mathbb{Q}(n)) \arrow[r] \arrow[d, "\simeq"] & H^m_M(X, \mathbb{Q}/\mathbb{Z}(n)) \arrow[r] \arrow[d, "\simeq"] & {} \\ \arrow[r] & H^{m-1}_L(X, \mathbb{Q}/\mathbb{Z}(n)) \arrow[r] & H^m_L(X, \mathbb{Z}(n)) \arrow[r] & H^m_L(X, \mathbb{Q}(n)) \arrow[r] & H^m_L(X, \mathbb{Q}/\mathbb{Z}(n)) \arrow[r] & {} \end{tikzcd}$$

we conclude that $H^m_M(X, \mathbb{Z}(n)) \simeq H^m_L(X, \mathbb{Z}(n))$. □

If R is torsion then we can compute the Lichtenbaum cohomology as étale cohomology. To be more precise, for a prime number ℓ, $r \in \mathbb{N} \geq 1$ and $R = \mathbb{Z}/\ell^r$ then we have the following quasi-isomorphisms

$$(\mathbb{Z}/\ell^r)_X(n)_{\text{ét}} \xrightarrow{\sim} \begin{cases} \mu_{\ell^r}^{\otimes n} & \text{if char}(k) \neq \ell \\ \nu_r(n)[-n] & \text{if char}(k) = \ell \end{cases}$$

where $\nu_r(n)$ is the logarithmic de Rham-Witt sheaf. After passing to direct limit we have also quasi-isomorphisms

$$(\mathbb{Q}_\ell/\mathbb{Z}_\ell)_X(n)_{\text{ét}} \xrightarrow{\sim} \begin{cases} \varinjlim_r \mu_{\ell^r}^{\otimes n} & \text{if char}(k) \neq \ell \\ \varinjlim_r \nu_r(n)[-n] & \text{if char}(k) = \ell \end{cases}$$

and finally set $(\mathbb{Q}/\mathbb{Z})_X(n)_{\text{ét}} = \bigoplus (\mathbb{Q}_\ell/\mathbb{Z}_\ell)_X(n)_{\text{ét}} \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}(n)_{\text{ét}}$. In the case when $k = \bar{k}$, X a smooth projective variety and $n \geq \dim(X)$ the morphism $\mathbb{Z}_X(n) \to R\rho_*\mathbb{Z}_X(n)_{\text{ét}}$ is a quasi-isomorphism by the Suslin rigidity theorem. Another important reminder concerns the vanishing of higher Chow groups. Following [MVW06, Theorem 3.6] for every smooth scheme and any abelian group R, we have $H^m_M(X, R(n)) = 0$ when $m > n + \dim(X)$. Also we have a second vanishing theorem for motivic cohomology, presented in [MVW06, Theorem 19.2], for X and R under the same assumptions as before, we have that $H^m_M(X, R(n)) = 0$ when $m > 2n$.

Remark 2.2.12. Let $k = \bar{k}$. Since the map $\mathbb{Z}_k(n) \to R\rho_*\mathbb{Z}_k(n)_{\text{ét}}$ is a quasi-isomorphism for all $n \geq 0$ we obtain that $H^m_L(\operatorname{Spec}(k), \mathbb{Z}(n)) \simeq H^m_M(\operatorname{Spec}(k), \mathbb{Z}(n))$ for all $(m, n) \in \mathbb{Z} \times \mathbb{N}$. In particular $H^m_L(\operatorname{Spec}(\bar{k}), \mathbb{Z}(n)) = 0$ for $m > n \geq 0$.

For a bi-degree $(2n, n)$ with $n \geq 3$ is more difficult to give an expression like a short exact sequence, because the comparison map κ^3 could be neither injective nor surjective. This is a consequence of the existence of a quasi-isomorphism of sheaves $\mathbb{Z}_X(n) \xrightarrow{\sim} \tau_{\leq n+1} R\pi_*\mathbb{Z}_X(n)_{\text{ét}}$, which leads us to a distinguished triangle of Zariski sheaves

$$\mathbb{Z}_X(n) \to R\pi_*\mathbb{Z}_X(n)_{\text{ét}} \to \tau_{\geq n+2} R\pi_*\mathbb{Z}_X(n) \to \mathbb{Z}_X(n)[1].$$

So we have the following long exact sequence

$$\ldots \to \mathbb{H}^{2n-1}_{\text{Zar}}(X, \tau_{\geq n+2} R\pi_*\mathbb{Z}_X(n)_{\text{ét}}) \to \mathrm{CH}^n(X) \to \mathrm{CH}^n_L(X) \to \mathbb{H}^{2n}_{\text{Zar}}(X, \tau_{\geq n+2} R\pi_*\mathbb{Z}_X(n)_{\text{ét}}) \to \ldots$$

and a spectral sequence associated to the hypercohomology

$$E_2^{p,q} = H^p(X, R^q\tau_{\geq n+2} R\pi_*\mathbb{Z}_X(n)_{\text{ét}}) \Longrightarrow \mathbb{H}^{p+q}_{\text{Zar}}(X, \tau_{\geq n+2} R\pi_*\mathbb{Z}_X(n)_{\text{ét}}).$$

where the E_2-terms can be described in more detail and are related to the unramified cohomology groups of X. Because of [Kah12, Corollaire 2.8] for $i > n+1$ there exists an isomorphism of Zariski sheaves $\mathcal{H}^{i-1}(R\pi_*\mathbb{Q}/\mathbb{Z}(n)) \to \mathcal{H}^i(R\pi_*\mathbb{Z}_X(n)_{\text{ét}})$.

Thus we have the quasi-isomorphism

$$R^q\tau_{\geq n+2} R\pi_*\mathbb{Z}(n)_{\text{ét}} \xrightarrow{\sim} \begin{cases} 0 & \text{if } q \leq n+1 \\ \mathcal{H}^{q-1}_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(n)) & \text{if } q \geq n+2 \end{cases}$$

Example 2.2.13. *Since $CH^n(X, -1) = 0$ for all $X \in SmProj_k$ and for all n, the long exact sequence shows that*

$$coker(\kappa^n) \simeq \mathbb{H}^{2n}_{Zar}(X, \tau_{\geq n+2} R\pi_*\mathbb{Z}_X(n)_{\acute{e}t}).$$

These groups are torsion, but nonzero in general so the comparison map κ^n is not surjective in general.

By pursuing a similar vanishing theorem for Lichtenbaum cohomology, we obtain the following results about the vanishing of the cohomology groups:

Lemma 2.2.14. *Let k be a field and let X be in $SmProj_k$. Consider a bi-degree $(m, n) \in \mathbb{Z}^2$ we then have the following:*

1. *if $m > n$ and $m > cd(k) + 1$ we have that $H^m_L(Spec(k), \mathbb{Z}(n)) = 0$.*

2. *More generally if $m > n + cd(X)$ then $H^m_L(X, \mathbb{Z}(n)) = 0$.*

Proof. Let k be a field of characteristic exponent p and let $(m, n) \in \mathbb{Z}^2$. For (1) we use [Voe11, Theorem 6.18] to obtain that if $m \leq n + 1$ then $H^m_M(k, \mathbb{Z}[1/p](n)) \simeq H^m_{M,\text{ét}}(k, \mathbb{Z}(n))$ and in particular $H^{n+1}_{M,\text{ét}}(k, \mathbb{Z}(n)) = 0$. Now consider the exact triangle

$$\mathbb{Z}(n)_{\text{ét}} \to \mathbb{Q}(n)_{\text{ét}} \to \mathbb{Q}/\mathbb{Z}(n)_{\text{ét}} \xrightarrow{+1}$$

which induces a long exact sequence

$$\ldots \to H^m_L(k, \mathbb{Z}(n)) \to H^m_L(k, \mathbb{Q}(n)) \to H^m_L(k, \mathbb{Q}/\mathbb{Z}(n)) \to H^{m+1}_L(k, \mathbb{Z}(n)) \to \ldots$$

Considering the previous remark concerning the vanishing of higher Chow group and Lichtenbaum cohomology, we obtain an isomorphism $H^m_L(k, \mathbb{Z}(n)) \simeq H^{m-1}_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(n))$ where $n \in \mathbb{N}$ and $m > n$, with the later isomorphism we conclude that $H^m_L(k, \mathbb{Z}(n)) = 0$ if $m > n$ and $m > \text{cd}(k) + 1$.

For the more general case presented in (2) consider X be SmProj_k and the motivic complex $\mathbb{Z}(n)$. This complex vanishes for degrees greater than n. Let us consider the canonical map $\rho : X_{\text{ét}} \to X_{\text{Zar}}$, the functor that is induced by the change

$$\rho^* : D(\text{AbShv}_{\text{Zar}}(\text{Sm}_k)) \leftrightarrows D(\text{AbShv}_{\text{ét}}(\text{Sm}_k)) : R\rho_*.$$

Recall that $H^m_L(X, \mathbb{Z}(n))$ is the hypercohomology of the complex of étale sheaves $\mathbb{Z}_X(n)_{\text{ét}}$. Since the functor ρ^* is exact, the étale cohomology sheaves of $\mathbb{Z}_X(n)_{\text{ét}}$ vanish in cohomological degree $> n$. Thus, we conclude that $H^m_L(X, \mathbb{Z}(n)) = 0$ for $m > n + \text{cd}(X)$. $\square$

If $\text{char}(k) = 0$ there is an explicit relation between motivic and Lichtenbaum cohomology groups, which is analogue to the case of étale and Zariski cohomology of sheaves: X a smooth quasi-projective $k-$variety, then by [RS16, Theorem 4.2] the canonical map of sites induces an isomorphism

$$\mathrm{H}^m_L(X, \mathbb{Z}(n)) \simeq \varinjlim_{X_\bullet} \mathrm{H}^m_M(X_\bullet, \mathbb{Z}(n)),\ m \in \mathbb{Z},\ n \geq 0,$$

where the direct limit is taken over all étale hypercoverings $X_\bullet \to X$.

Let us denote the Suslin-Voevodsky motivic complex of Nisnevich sheaves in Sm_k as $\mathbb{Z}_{SV}(n)$. Since $\mathbb{Z}_X(n)_{\text{ét}} \xrightarrow{\sim} \mathbb{Z}_{SV}(n)\Big|_{X_{\text{ét}}}$ is a quasi-isomorphism we have a comparison map

$$\rho^{m,n} : H^m_L(X, \mathbb{Z}(n)) \to H^m_{M,\text{ét}}(X, \mathbb{Z}(n))$$

which is induced by the quasi-isomorphisms $\mathbb{Z}_X(n)_{\text{ét}} \xrightarrow{\sim} \mathbb{Z}_{SV}(n)\Big|_{X_{\text{ét}}}$ and $\mathbb{Z}_{SV}(n)_{\text{ét}} \to L_{\mathbb{A}^1}(\mathbb{Z}_{SV}(n)_{\text{ét}})$ where $L_{\mathbb{A}^1}$ is the $\mathbb{A}^1-$localization functor of étale motivic complexes. According to [CD16, Theorem 7.1.2] the morphism $\rho^{m,n}$ becomes an isomorphism after inverting the characteristic exponent of k. If p equals the field characteristic, therefore by using $\mathbb{Z}[1/p]_X(n)_{\text{ét}}$ we can recover the functorial properties of étale motivic cohomology for Lichtenbaum cohomology.

The latter isomorphism gives us an important tool for the study of the étale motivic cohomology, after inverting the characteristic exponent of the field, which is the relationship between Galois cohomology and Lichtenbaum cohomology via the Hochschild-Serre spectral sequence for Lichtenbaum cohomology. In order to present this important result that we will use throughout the following chapters, let us recall some definitions and results about profinite cohomology groups: If G is a profinite group, i.e. $G = \varprojlim G_i$ with G_i finite groups, and A is a G-module, we will consider its cohomology group $H^j(G, A)$ as the continuous cohomology group of G with coefficients in A defined as $H^j(G, A) := \varinjlim H^j(G_i, A^{H_i})$ with H_i running over the open normal subgroups of G such that $G/H_i \simeq G_i$. We start by presenting a useful fact about continuous cohomology of profinite groups with coefficients in a uniquely divisible module, which will be used several times in this thesis.

Lemma 2.2.15. *Let G be a profinite commutative group and let A be a G-module which is uniquely divisible. Then $H^n(G, A) = 0$ for all $n \geq 1$.*

Proof. Let G be a profinite group and let H be an open normal subgroup of G. By definition we have that

$$H^j(G, A) = \varinjlim_{H} H^j(G/H, A^H),$$

as G/H is a finite group, using [Wei94, Proposition 6.1.10] we have that the result holds for H-modules where multiplication is an isomorphism , in particular uniquely divisible modules, therefore $H^j(G/H, A^H) = 0$ for all H and all $j > 0$. The result then follows from the definition of continuous cohomology. □

Let us recall the definition of Galois cohomology. Let k be a field, fix a separable closure denoted by k^s and denote by G_k its Galois group. Our main interest is the study of the cohomology of the group G_k. For a finite Galois extension K/k we denote the Galois group of K by $\mathrm{Gal}(K/k)$ and recall that $G_k \simeq \varprojlim \mathrm{Gal}(K/k)$, where K runs through the finite Galois extensions of k, is a profinite group. The importance of this fact throughout the paper is reflected in the relationship between Galois cohomology and Lichtenbaum cohomology groups via a Hoschschild-Serre spectral sequence, stated in [CK13] without a proof which was done in [RS18, Pages 6-7].

Lemma 2.2.16. *[CK13, P. 31] Let $p : Y \to X$ be a finite Galois covering of X with Galois group G. There exists a convergent Hochschild-Serre spectral sequence with abutment the Lichtenbaum cohomology group*

$$E_2^{p,q}(n) = H^p(G, H^q_L(Y, \mathbb{Z}(n))) \Longrightarrow H^{p+q}_L(X, \mathbb{Z}(n)).$$

Proof. Let $p : Y \to X$ be a Galois covering with X be a smooth projective $k-$variety and G the Galois group associated to the covering. Let $\mathcal{C}_X := \mathbf{Ch}(\mathrm{Shv}_{\text{ét}}(X))$ be the category of cochain complexes of abelian étale sheaves. Consider the composite functor

$$\begin{aligned} \mathrm{Shv}_{\text{ét}}(X) &\to \mathbb{Z}[G]\text{-mod} \to \mathbf{Ab} \\ F &\mapsto F(Y) \mapsto F(Y)^G \end{aligned}$$

which is $\Gamma(X,-)$ by [Mil80, Proposition II.1.4], therefore for $C^\bullet \in \mathcal{C}_X$ we have a spectral sequence, [Wei94, Section 5.7] associated with such functor

$$E_2^{p,q} = H^p(G, \mathbb{H}^q_{\text{ét}}(Y, C^\bullet)) \Longrightarrow \mathbb{H}^{p+q}_{\text{ét}}(X, C^\bullet).$$

Our main interest is the case when we consider $C^\bullet$ as the étale sheafification of the Bloch complex $\mathbb{Z}_X(n)_{\text{ét}}$ for some n, so from now on we consider $C^\bullet = \mathbb{Z}_X(n)_{\text{ét}}$. To show the convergence of the spectral sequence we use the arguments given in [Kah12, Section 2]. We have $\mathbb{H}^m_{\text{ét}}(X, \mathbb{Z}_X(n)_{\text{ét}}) \simeq \mathbb{H}^m_{\text{Zar}}(X, R\rho_*\mathbb{Z}_X(n)_{\text{ét}})$ with $\rho : X_{\text{ét}} \to X_{\text{Zar}}$ and consider the exact triangle $R\rho_*\mathbb{Z}_X(n) \to R\rho_*\mathbb{Q}_X(n) \to R\rho_*\mathbb{Q}/\mathbb{Z}_X(n) \xrightarrow{+1}$. Since $\mathbb{Q}_X(n) \simeq R\rho_*\mathbb{Q}_X(n)_{\text{ét}}$, the hypercohomology of the second and third terms are convergent and so are their respective hypercohomology spectral sequences. $\square$

Remark 2.2.17. Let k be a field and k^s be a separable closure. Since cohomology commutes with inverse limits, and the absolute Galois group of k is the inverse limit of $\mathrm{Gal}(K/k)$ over the finite separable field extensions $k \subset K \subset k^s$, the convergent spectral sequences $H^p(\mathrm{Gal}(K/k), H^q_L(X_K, \mathbb{Z}(n))) \Longrightarrow H^{p+q}_L(X, \mathbb{Z}(n))$ for $[K:k] < \infty$ induce a spectral sequence for the absolute Galois group $H^p(G_k, H^q_L(X_{k^s}, \mathbb{Z}(n))) \Longrightarrow H^{p+q}_L(X, \mathbb{Z}(n))$.

Lemma 2.2.18. *Let $X, Y \in SmProj_k$ with k be a field of finite cohomological dimension or characteristic zero, then for $i \in \mathbb{N}$ we have an isomorphism*

$$CH^i_{\acute{e}t}(X_{k(Y)}) \simeq \varinjlim_{\substack{U \subset Y \\ U \text{ open}}} CH^i_{\acute{e}t}(X \times_k U)$$

Proof. Considering the projective system $X_{k(Y)} = \varprojlim_{\substack{U \subset Y \\ U \text{ open}}} (X \times_k U)$ if k has finite cohomological dimension then the result follows from [CD16, Proposition 6.3.7] and [CD16, Remark 6.3.8].

If $\mathrm{char}(k) = 0$ then by [RS16, Theorem 4.2] we have that $\mathrm{CH}^i_{\text{ét}}(X) = \varinjlim_{X_\bullet} H^{2i}_M(X_\bullet, \mathbb{Z}(i))$ where the limit is taken over all étale hypercoverings $X_\bullet \to X$. By the same proposition we have that $\mathrm{CH}^i_{\text{ét}}(X_{k(Y)}) = \varinjlim_{Y_\bullet} H^{2i}_M(Y_\bullet, \mathbb{Z}(i))$ with $Y_\bullet \to X_{k(Y)}$. Then we have to prove that

$$\mathrm{CH}^i_{\text{ét}}(X_{k(Y)}) = \varinjlim_{Y_\bullet} H^{2i}_M(Y_\bullet, \mathbb{Z}(i)) \simeq \varinjlim_{\substack{U \subset Y \\ U \text{ open}}} \varinjlim_{\substack{X'_\bullet \to X \times_k U \\ X'_\bullet \text{ étale hyp.}}} H^{2i}_M(X'_\bullet, \mathbb{Z}(i)) = \varinjlim_{\substack{U \subset Y \\ U \text{ open}}} \mathrm{CH}^i_{\text{ét}}(X \times_k U)$$

Let us denote

- $\mathrm{Op}(Y)$ the category of open sub-schemes of Y.
- J_U the category of étale hypercoverings of $X \times U$ for a fixed $U \in \mathrm{Op}(Y)$.
- $J_{\lim}$ the category of étale hypercoverings of $X_{k(Y)}$.
- $\mathcal{C}$ be the category whose objects are defined as pairs $(U, U_\bullet)$ with $U \in \mathrm{Op}(Y)$ and $U_\bullet \subset J_U$ with morphisms

$$\mathrm{Hom}_{\mathcal{C}}((U, V_\bullet), (W, W_\bullet)) = \begin{cases} \mathrm{Hom}_{J_U}(V_\bullet, W_\bullet) \text{ if } U = W \\ \quad \emptyset \quad \text{ otherwise.} \end{cases}$$

and consider the functor $F : \mathcal{C} \to J_{\lim}$ which acts $(U, X_\bullet) \mapsto \widetilde{X}_\bullet$ with $\widetilde{X}_\bullet$ being the fiber product

$$\begin{tikzcd} \widetilde{X}_\bullet \arrow[r] \arrow[d] & X_{k(Y)} \arrow[d] \\ X_\bullet \arrow[r] & X \times U \end{tikzcd} \tag{2.3}$$

We can see that this functor is co-final and apply [AM06, Appendix, Proposition 1.8]: consider a hypercovering $\widetilde{X}_\bullet \to X_{k(Y)}$, using the canonical map for an open U, $X_{k(Y)} \to X \times U$ we can find an element $(U, Y_\bullet)$ with $Y_\bullet := \widetilde{X}_\bullet \to X_{k(Y)} \to X \times U$ in the category $\mathcal{C}$.

Now consider two hypercoverings of $X_{k(Y)}$, denoted by $X_\bullet$ and $Y_\bullet$ such that there exist two morphism $X_\bullet \underset{g}{\overset{f}{\rightrightarrows}} Y_\bullet$, where $F(U, Y'_\bullet) = Y_\bullet$ for some $(U, Y'_\bullet) \in \mathcal{C}$. Consider the constant hypecovering $(X \times U)_\bullet \to X \times U$, then we have $F(U, (X \times U)_\bullet) = \left(X_{k(Y)}\right)_\bullet$. It is clear that we have a canonical map $Y'_\bullet \xrightarrow{i_U} (X \times U)_\bullet$, which is sent to the canonical map

$$\left(Y'_\bullet \xrightarrow{i_U} (X \times U)_\bullet\right) \mapsto \left(Y_\bullet \xrightarrow{i} \left(X_{k(Y)}\right)_\bullet\right).$$

By the construction of F, we obtain that i_U equalizes $X_\bullet \underset{g}{\overset{f}{\rightrightarrows}} Y_\bullet$ as

$$X_\bullet \underset{g}{\overset{f}{\rightrightarrows}} F(U, Y'_\bullet) = Y_\bullet \xrightarrow{F(i_U)=i} \left(X_{k(Y)}\right)_\bullet = F(U, (X \times U)_\bullet).$$

□

Lemma 2.2.19. *Let X and Y be smooth projective varieties over an algebraically closed field k of characteristic $p \geq 0$. Then for $\ell \neq p$ and for every bi-degree $(m, n) \in \mathbb{Z}^2$ such that $2m + 1 \neq n$ we have an isomorphism*

$$H^m_L(X_{k(Y)}, \mathbb{Z}(n))\{\ell^r\} \simeq H^{m-1}_{\acute{e}t}(X_{k(Y)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)).$$

Proof. Consider a smooth open $U \subset Y$, we have a short exact sequence

$$0 \to H^{m-1}_L(X \times U, \mathbb{Z}(n)) \otimes \mathbb{Z}/\ell^r \to H^{m-1}_{\text{ét}}(X \times U, \mu_{\ell^r}^{\otimes n}) \to H^m_L(X \times U, \mathbb{Z}(n))[\ell^r] \to 0.$$

Taking the direct limit over r, by [RS16, Proposition 3.1] we have that $H^m_L(X \times U, \mathbb{Z}(n))\{\ell^r\} \simeq H^{m-1}_{\text{ét}}(X \times U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. On the other hand, if we take the limit over the open subsets $U \subset Y$ instead of r (and using again direct limit is an exact functor), we obtain the short exact sequence

$$0 \to H^{m-1}_L(X_{k(Y)}, \mathbb{Z}(n)) \otimes \mathbb{Z}/\ell^r \to H^{m-1}_{\text{ét}}(X_{k(Y)}, \mu_{\ell^r}^{\otimes n}) \to H^m_L(X_{k(Y)}, \mathbb{Z}(n))[\ell^r] \to 0.$$

This result is obtained by the continuity properties described in [CD16, Proposition 6.3.7]. Using the isomorphism of functors $\varinjlim_r \varinjlim_{U \subset Y} \simeq \varinjlim_{U \subset Y} \varinjlim_r$, we can conclude that

$$H^m_L(X_{k(Y)}, \mathbb{Z}(n))\{\ell^r\} \simeq H^{m-1}_{\text{ét}}(X_{k(Y)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)).$$

□

An important result about Lichtenbaum cohomology concerns the change of base fields, when the fields are algebraically closed or purely inseparable extensions. The following result is similar to [Via17, Lemma 1.2]:

Proposition 2.2.20. *Let k be a field, X a smooth projective k-scheme and K a field extension of k. Let $i \geq 0$ be an integer.*

1. *If k is an algebraically closed field and also $K = \bar{K}$, then the map $CH^i_L(X) \to CH^i_L(X_K)$ induced by the base change is injective.*
2. *If K is a finite purely inseparable extension then the maps $CH^i_L(X) \to CH^i_L(X_K)$ and $CH^i_L(X_K) \to CH^i_L(X)$ are isomorphisms.*

Proof. First, let k be a perfect field with $k = \bar{k}$ and consider a field extension K which is again algebraically closed. By the smooth base change, we have that $H^m_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n}) \to H^m_{\text{ét}}(X_K, \mu_{\ell^r}^{\otimes n})$ is an isomorphism when ℓ is prime to the characteristic of k and then so it is the morphism $H^m_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \to H^m_{\text{ét}}(X_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ and from the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^m_L(X,\mathbb{Z}(n))\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r] \arrow[d] & H^m_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \arrow[r] \arrow[d, "\simeq"] & H^{m+1}_L(X,\mathbb{Z}(n))\{\ell\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^m_L(X_K,\mathbb{Z}(n))\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r] & H^m_{\text{ét}}(X_K,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \arrow[r] & H^{m+1}_L(X_K,\mathbb{Z}(n))\{\ell\} \arrow[r] & 0
\end{tikzcd}$$

we conclude that $H^m_L(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^m_L(X_K, \mathbb{Z}(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is an injective morphism. Recall that for a separably closed field as in our case we have that for $m \neq 2n+1$ an isomorphism $H^m_L(X, \mathbb{Z}(n))\{\ell\} \simeq H^{m-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$.

Let $A^{m,n} = H^m_L(X, \mathbb{Z}(n))$ and $A^{m,n}_K = H^m_L(X_K, \mathbb{Z}(n))$, then we have that in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^{m,n}_{\text{tor}} \arrow[r] \arrow[d, "\simeq"] & A^{m,n} \arrow[r] \arrow[d] & A^{m,n}\otimes\mathbb{Q} \arrow[r] \arrow[d, hook] & A^{m,n}\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & A^{m,n}_{K,\text{tor}} \arrow[r] & A^{m,n}_K \arrow[r] & A^{m,n}_K\otimes\mathbb{Q} \arrow[r] & A^{m,n}_K\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

the arrow $A^{m,n} \otimes \mathbb{Q} \to A^{m,n}_K \otimes \mathbb{Q}$ is an injection by classical arguments, therefore $A^{m,n} \to A^{m,n}_K$ is an injective map as well.

For the second part we proceed in a similar way. The isomorphism for the torsion part is a consequence of the map $X_K \to X$ which is finite surjective radiciel (see [Fu15, Proposition 5.7.1]), therefore $H^{m-1}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n}) \to H^{m-1}_{\text{ét}}(X_K, \mu_{\ell^r}^{\otimes n})$ is an isomorphism. The isomorphism of the torsion free part is a consequence of [Via17, Lemma 1.2]. We then conclude as in the previous case. $\square$

In a more general setting for field extension we have the following result

Proposition 2.2.21. *Let X be a $k-$smooth projective variety with k an algebraically closed field, and let $K \supset k$ be a field extension, $n \in \mathbb{N}$ and $\varepsilon \in \{L, ét\}$, then the map $f^* : CH^n_\varepsilon(X) \to CH^n_\varepsilon(X_K)$ induced by $f : X_K \to X$ has torsion kernel.*

Proof. Due to the contravariant functoriality of Lichtenbaum and étale Chow groups we have a commutative square

$$\begin{tikzcd} \mathrm{CH}^n(X) \arrow[r, "\sigma^n"] \arrow[d, "f^*"] & \mathrm{CH}^n_\varepsilon(X) \arrow[d, "f^*"] \\ \mathrm{CH}^n(X_K) \arrow[r, "\sigma^n_K"] & \mathrm{CH}^n_\varepsilon(X_K) \end{tikzcd}$$

Notice that by [Blo86, Lemma (1A.3)] the map $\mathrm{CH}^n(X) \to \mathrm{CH}^n(X_K)$ has torsion kernel, hence with rational coefficients it becomes an injective morphism. On the other hand, with rational coefficients the horizontal arrows are isomorphisms, therefore $\mathrm{CH}^n_\varepsilon(X) \to \mathrm{CH}^n_\varepsilon(X_K)$ has torsion kernel. □

We conclude this section by mentioning some well-known results about the structure of étale motivic and Lichtenbaum cohomology groups of projective bundles, smooth blow-ups and varieties with cellular decomposition:

Lemma 2.2.22. *Let k be a field of characteristic $p \geq 0$ and let X be a smooth projective scheme over k. Let $\varepsilon \in \{L, \{M, ét\}\}$ and consider a bi-degree $(m,n) \in \mathbb{Z}^2$, then there exists the following characterizations:*

1. *If $r \geq 0$ and let $\mathbb{P}^r_X$ be the projective space of dimension r over X, then the canonical map $\mathbb{P}^r_X \to X$ induces an isomorphism:*
$$H^m_\varepsilon(\mathbb{P}^r_X, \mathbb{Z}(n)) \simeq \bigoplus_{i=0}^{r} H^{m-2i}_\varepsilon(X, \mathbb{Z}(n-i)).$$
2. *Let Z be a smooth projective sub-scheme of X of codimension $c \geq 2$. Denote the blow-up of X along Z as $Bl_Z(X)$, then*
$$H^m_\varepsilon(Bl_Z(X), \mathbb{Z}(n)) \simeq H^m_\varepsilon(X, \mathbb{Z}(n)) \oplus \bigoplus_{i=1}^{c-1} H^{m-2i}_\varepsilon(Z, \mathbb{Z}(n-i)).$$
3. *Assume that $k = k^s$ and that there exists a map $f : X \to S$ which is a flat of relative dimension r over a smooth base S. Assume as well that X has a filtration $X = X_p \supset X_{p-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$ where X_i is smooth and projective for all i and $U_i := X_i - X_{i-1} \simeq \mathbb{A}^{r-d_i}_S$ then we obtain the following formula:*
$$H^m_L(X, \mathbb{Z}[1/p](n)) \simeq \bigoplus_{i=0}^{p} H^{m-2d_i}_L(S, \mathbb{Z}[1/p](n-d_i)).$$

Proof. The statements (1) and (2) are obtained in a similar way: first notice that by properties of $\mathrm{DM}(k,R)$ with R a commutative ring, see [MVW06, Section 14 & 15], we have canonical isomorphisms of motives

$$\bigoplus_{i=0}^{r} M(X)(i)[2i] \xrightarrow{\simeq} M(\mathbb{P}^r_X) \quad \text{and} \quad M(\mathrm{Bl}_Z(X)) \simeq M(X) \oplus \left(\bigoplus_{i=1}^{c-1} M(Z)(i)[2i] \right).$$

When $\varepsilon = L$ both formulas (1) holds because for $R = \mathbb{Q}$ we recover the formulas for rational coefficients whereas for finite coefficients we invoke [Mil80, VI, Lemma 10.2] for coefficients away from the characteristic and [Gro85, I, Théorème 2.1.11] for the logarithmic Hodge-Witt complex. The formula (2) holds again because it holds for $R = \mathbb{Q}$ and for finite coefficients by the proper base change [Mil80, VI, Corollary 2.3] and [Gro85, IV, Corollaire 1.3.6] for the logarithmic Hodge-Witt complex.

Meanwhile for $\varepsilon = \{M, \text{ét}\}$ the statement holds because of the previous isomorphisms when $R = \mathbb{Z}$ and the fact that the functor $\rho^* : \mathrm{DM}(k, \mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$ is exact.

For (3) we have to invert the characteristic of k. We will proceed as in [Köc91, Appendix], by induction and use the localization long exact sequence. We have $\dim(X_j) = \dim(S) + n - d_j$ and for $k < j$ we put $c_{k,j} = \operatorname{codim}(X_k, X_j) = d_k - d_j$. Notice that d_j is the codimension of X_j in X. By the homotopy invariance of higher Chow groups and [Mil80, VI, Corollary 4.20], the map $\pi_0^* : H_L^m(S, \mathbb{Z}[1/p](n)) \to H_L^m(X_0, \mathbb{Z}[1/p](n))$ is an isomorphism for all bi-degree. Denote $\pi_j : U_j \to S$. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{j-1} H_L^{m-2c_{j,i}, n-c_{j,i}}(S) \arrow[r] \arrow[d, "\simeq"] & \bigoplus_{i=0}^{j} H_L^{m-2c_{j,i}, n-c_{j,i}}(S) \arrow[r] \arrow[d] & H_L^{m,n}(S) \arrow[r] \arrow[d, "(\pi_j)^*"] & 0 \\
\dots \arrow[r] & H_L^{m-2c_{j-1,j}, n-c_{j-1,j}}(X_{j-1}) \arrow[r, "i_*"] & H_L^{m,n}(X_j) \arrow[r, "j^*"] & H_L^{m,n}(U_j) \arrow[r] & \dots
\end{tikzcd}$$

where $H_L^{m,n}(Y) := H_L^m(Y, \mathbb{Z}[1/p](n))$. By the inductive hypothesis the right vertical arrow is an isomorphism, and the left one is an isomorphism because of the homotopy invariance of étale motivic cohomology, therefore the map j^* is surjective and i_* is injective. Thus we obtain the desired formula. □

Remark 2.2.23.
1. Consider a cycle module M is the sense of Rost, for further details see [Ros96]. By using the same kind of arguments as the ones in [Köc91] and applying the homotopy invariance given in [Ros96, Theorem 8.6], which says that if $\pi : Y \to X$ is an affine bundle of dimension n and M is a cycle module, then

$$\pi^* : A_p(X; M) \to A_{p+n}(Y; M)$$

is bijective for all n, we can recover the formula

$$\bigoplus_{i=0}^{n} A_{p-d_i}(S; M) \to A_p(Y; M)$$

for Y having a cellular decomposition.

2. For any $\epsilon \in \{L, \{M, \text{ét}\}\}$ the isomorphisms described in Lemma 2.2.22 are functorial with respect to base change $T \to X$. For this, one has the functoriality for Chow groups as is described in subsection 1.1.1, while the torsion different to the characteristic is given by [Mil80, §, Theorem 4.1]. For the logarithmic Hodge-Witt complex, the functoriality for the projective bundle and flag varieties is given by [Gro85, Chap. III, Théorème 1.1.1].

In the sequel we recall a few facts about the structure of Lichtenbaum cohomology group of smooth projective varieties over an algebraically closed field. For further details about the structure and properties about Lichtenbaum cohomology we refer the reader to [Kah12, Proposition 4.17], [Gei17, Theorem 1.1] and [RS16, Theorem 3.1]. Consider $X \in \mathrm{SmProj}_k$ with $k = \bar{k}$ of characteristic exponent p and consider a bi-degree $(m, n) \in \mathbb{Z}^2$. If $m \neq 2n$ then according to [RS16, Theorem 3.1] $H^m_L(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ for all $\ell \neq p$. Denoting $(\mathbb{Q}/\mathbb{Z})' = \bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ we have that $H^m_L(X, \mathbb{Z}[1/p](n)) \otimes (\mathbb{Q}/\mathbb{Z})' = 0$ and then

$$0 \to H^m_L(X, \mathbb{Z}(n))_{\mathrm{tors}} \to H^m_L(X, \mathbb{Z}(n)) \to H^m_L(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \to 0.$$

In fact this short exact sequence splits, so for $m \neq 2n$, $H^m_L(X, \mathbb{Z}(n))$ is the direct sum of a uniquely divisible group and a torsion group. For the case when $m \neq 2n+1$ we have an isomorphism $H^m_L(X, \mathbb{Z}(n))\{\ell\} \simeq H^{m-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$, again considering $\ell \neq p$.

Since for any n we have an exact triangle

$$\mathbb{Z}_X(n)_{\text{ét}} \to \mathbb{Q}_X(n)_{\text{ét}} \to (\mathbb{Q}/\mathbb{Z})_X(n)_{\text{ét}} \xrightarrow{+1}$$

and for $m < 0$ the group $H^m_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(n))$ vanishes, we conclude that for such m we have isomorphisms $H^m_L(X, \mathbb{Z}(n)) \simeq H^m_L(X, \mathbb{Q}(n))$ i.e. the Lichtenbaum cohomology groups with integral coefficients are $\mathbb{Q}$-vector spaces, thus uniquely divisible groups.

Now let us come-back to the Hochschild-Serre spectral sequence for Lichtenbaum cohomology. Assume that X is a smooth projective geometrically integral $k-$variety of dimension d with k a perfect field of characteristic exponent $\tilde{p}$, and let $\bar{k}$ be an algebraic closure of k with Galois group G_k and define $X_{\bar{k}} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$. For such X consider the Hochschild-Serre spectral sequence

$$E_2^{p,q}(n) := H^p(G_k, H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))) \Longrightarrow H^{p+q}_L(X, \mathbb{Z}[1/\tilde{p}](n)).$$

Using the previous results, we can give information about the vanishing of some $E_2^{p,q}(n)-$terms:

- $E_2^{p,q}(n) = 0$ for $p < 0$ because we work with the cohomology of a profinite group.
- $E_2^{p,q}(n) = 0$ for $p > 0$ and $q < 0$ since $H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))$ is uniquely divisible.
- $E_2^{p,q}(n) = 0$ for $p > \mathrm{cd}(k)$ and $q \neq 2n$. Indeed, as $q \neq 2n$ then

  $$H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n)) \simeq H^q_L(X_{\bar{k}}, \mathbb{Q}(n)) \oplus H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))_{\mathrm{tors}},$$

  since $H^q_L(X_{\bar{k}}, \mathbb{Q}(n))$ is uniquely divisible, so for a pair (p, q) satisfying the above restrictions, we have that

  $$H^p(G_k, H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))) \simeq H^p(G_k, H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))_{\mathrm{tors}}).$$

  Now, if $p > \mathrm{cd}(k)$, the group $H^p(G_k, H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))_{\mathrm{tors}})$ vanishes.

Example 2.2.24. *If we assume that $cd(k) \leq 2$ and $q < 2n$, then we have the following isomorphisms*

$$\begin{aligned}
E_\infty^{0,q}(n) &= ker\left\{d_2 : E_2^{0,q}(n) \to E_2^{2,q-1}(n)\right\} \\
&= ker\left\{d_2 : H^q(X_{\bar{k}}, \mathbb{Z}(n))^{G_k} \to H^2(G_k, H_L^{q-1}(X_{\bar{k}}, \mathbb{Z}(n)))\right\} \\
E_\infty^{1,q}(n) &\simeq E_2^{1,q}(n) \\
E_\infty^{2,q}(n) &\simeq E_2^{2,q}(n)/im\left\{E_2^{0,q+1}(n) \to E_2^{2,q}(n)\right\} \\
&= H^2(G_k, H_L^q(X_{\bar{k}}, \mathbb{Z}(n)))/im\left\{H_L^{q+1}(X_{\bar{k}}, \mathbb{Z}(n))^{G_k} \to H^2(G_k, H_L^q(X_{\bar{k}}, \mathbb{Z}(n)))\right\}.
\end{aligned}$$

2.3 Birational invariance

Let us recall some definitions from birational geometry. Let X, Y be smooth k-varieties. We say that a rational map $f : X \to Y$ is birational if there exist open subsets $U \subset X$ and $V \subset Y$ such that $f : U \to V$ is an isomorphism. We say that X is stably birational to Y if there exist $r, s \in \mathbb{N}$ such that $X \times \mathbb{P}_k^r \to Y \times \mathbb{P}_k^s$ is a birational morphism. The importance of $\mathrm{CH}_0(X)$ lies in its birational invariance, for which we refer to [Ful98, Example 16.1.11]. If $X \to Y$ is stably birational then there exist r, s such that

$$\mathrm{CH}_0(X \times \mathbb{P}_k^r) \xrightarrow{\simeq} \mathrm{CH}_0(Y \times \mathbb{P}_k^s),$$

but by the projective bundle formula for Chow groups and the vanishing properties we obtain that $\mathrm{CH}_0(X \times \mathbb{P}_k^r) \simeq \mathrm{CH}_0(X)$ and $\mathrm{CH}_0(Y \times \mathbb{P}_k^s) \simeq \mathrm{CH}_0(Y)$ so $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y)$. So CH_0 is also a stable birational invariant.

Remark 2.3.1. The proof of birational invariance of $\mathrm{CH}_0(X)$ in [Ful98, Example 16.1.11] is given for algebraically closed fields, but the same argument works for any field.

The first question that arises is whether or not $\mathrm{CH}_0^L(X)$ (or $\mathrm{CH}_0^{\text{ét}}(X)$) is a birational invariant or a stably birational invariant. Let X be a smooth projective variety over a field k, because of the comparison map $\mathrm{CH}_0(X) \to \mathrm{CH}_0^L(X)$ we can say a few words about the invariance depending on the field and the dimension of X: if $k = \bar{k}$ then $\mathrm{CH}^d(X) \simeq \mathrm{CH}_L^d(X)$, thus we can use the stable birational invariance of zero cycles in the classical setting cited above, for the category SmProj_k. If the field is not algebraically closed, then we lose many of the properties. For example, consider k a number field which can be embedded into $\mathbb{R}$ and $d \geq 2$, by invoking Lemma 2.2.22 and the vanishing properties of Lemma 2.2.14, we see immediately that

$$\mathrm{CH}_L^0(\mathrm{Spec}(k)) \neq \mathrm{CH}_L^d(\mathbb{P}_k^d) \simeq \bigoplus_{i=0}^{d} \mathrm{CH}_L^i(\mathrm{Spec}(k)).$$

Thus CH_0^L is not a stable birational invariant. If now we focus on the birational invariance of $\mathrm{CH}_0^L(X)$, we have the following result:

Proposition 2.3.2. *Let k be an arbitrary field and let X be a smooth projective scheme of dimension d over k. Then CH_0^L is a birational invariant if $d \in \{0, 1, 2\}$.*

Proof. The case $d = 0$ is trivial. If $d = 1$, we use the isomorphism $\mathrm{CH}^1(X) \simeq \mathrm{CH}^1_L(X)$ and the birational invariance of zero cycles in the classical case. For $d = 2$ we have a short exact sequence

$$0 \to \mathrm{CH}^2(X) \to \mathrm{CH}^2_L(X) \to H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \to 0.$$

The group $\mathrm{CH}^2(X)$ is a birational invariant for surfaces and the unramified cohomology groups $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ is birational invariant for any dimension. (This is a consequence of the Gersten's conjecture, see [CV12, Théorème 2.8]). Therefore $\mathrm{CH}^2_L(X)$ is a birational invariant. □

In higher dimensions the argument using the comparison map fails. To illustrate this consider the following: Let X be a smooth projective variety of dimension three over a field k. There is a long exact sequence

$$\to \mathbb{H}^5_{\mathrm{Zar}}(X, \tau_{\geq 5} R\pi_* \mathbb{Z}(3)_{\mathrm{\acute{e}t}}) \to \mathrm{CH}^3(X) \to \mathrm{CH}^3_L(X) \to \mathbb{H}^6_{\mathrm{Zar}}(X, \tau_{\geq 5} R\pi_* \mathbb{Z}(3)_{\mathrm{\acute{e}t}}) \to 0.$$

We have that $\mathbb{H}^5_{\mathrm{Zar}}(X, \tau_{\geq 5} R\pi_* \mathbb{Z}(3)_{\mathrm{\acute{e}t}}) \simeq H^4_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(3))$ is a birational invariant. Therefore $\mathrm{CH}^3_L(X)$ is a birational invariant if and only if $\mathbb{H}^6_{\mathrm{Zar}}(X, \tau_{\geq 5} R\pi_* \mathbb{Z}(3)_{\mathrm{\acute{e}t}})$ is a birational invariant. We obtain a short exact sequence,

$$0 \to H^1_{\mathrm{Zar}}(X, \mathcal{H}^4_{\mathrm{\acute{e}t}}(\mathbb{Q}/\mathbb{Z}(3))) \to \mathbb{H}^6_{\mathrm{Zar}}(X, \tau_{\geq 5} R\pi_* \mathbb{Z}(3)_{\mathrm{\acute{e}t}}) \to E_\infty^{0,6} \to 0$$

where $E_\infty^{0,6} = \ker \left\{ H^5_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(3)) \to H^2_{\mathrm{Zar}}(X, \mathcal{H}^4_{\mathrm{\acute{e}t}}(\mathbb{Q}/\mathbb{Z}(3))) \right\}$. In fact, one can find the first counter-example in dimension 3. Recall that by the formulas given in Lemma 2.2.22 we have the following: let X be a smooth projective variety and let $Z \subset X$ a smooth sub-variety of codimension c. Then for the blow-up $\tilde{X}_Z$ of X along Z, Lichtenbaum cohomology decomposes as follows

$$\mathrm{CH}^d_L(\tilde{X}_Z) \simeq \mathrm{CH}^d_L(X) \oplus \bigoplus_{j=1}^{c-1} \mathrm{CH}^{d-j}_L(Z).$$

Notice that $d - j > d - c = \dim(Z)$, therefore the groups $\mathrm{CH}^{d-j}_L(Z)$ are just torsion isomorphic to $\mathbb{H}^{2(d-j)}_{\mathrm{Zar}}(X, \tau_{\geq d-j+2} R\pi_* \mathbb{Z}(d-j)_{\mathrm{\acute{e}t}})$. The next example shows how to exploit this fact to get a counter-example.

Example 2.3.3. *Consider X a smooth threefold with a rational point over K, with K an algebraic number field which is not totally imaginary, and let $Z = Spec(K)$. Let $\tilde{X}_Z$ be the blow-up with center Z, then we have that*

$$CH^3_L(\tilde{X}_Z) = CH^3_L(X) \oplus CH^2_L(Spec(K)).$$

Since $CH^2_L(Spec(K)) \simeq H^3_{\acute{e}t}(Spec(K), \mathbb{Q}/\mathbb{Z}(2))$ we can conclude that $CH^3_L(\tilde{X}_Z) \neq CH^3_L(X)$.

In general we have the proposition:

Proposition 2.3.4. *Let k be a field and assume that there exists $n \geq 2$ such that $H^{2n-1}_{\acute{e}t}(Spec(k), \mu_{\ell^r}^{\otimes n}) \neq 0$ for some prime number ℓ and $r \in \mathbb{N}$, then CH_0^L is not a birational invariant for $SmProj_k$.*

Proof. Let us consider the field k such that $H^{2n-1}_{\text{ét}}(\text{Spec}(k), \mu_{\ell^r}^{\otimes n}) \neq 0$ for some prime number ℓ, $r \in \mathbb{N}$ and $n \geq 2$. Consider X a smooth projective variety over k of dimension $d \geq n+1$ such that X has a k-rational point. Let $\tilde{X}$ be the blow-up of X along a point $Z = \text{Spec}(k) \to X$. Invoking Lemma 2.2.22 we obtain

$$\text{CH}^d_L(\tilde{X}) \simeq \text{CH}^d_L(X) \oplus \bigoplus_{j=1}^{d-1} \text{CH}^{d-j}_L(Z)$$

As $\text{CH}^i_L(Z) \simeq H^{2i-1}_{\text{ét}}(Z, \mathbb{Q}/\mathbb{Z}(i))$ for $i \geq 2$, the hypothesis implies that $\text{CH}^n_L(Z) \neq 0$ and thus $\text{CH}^d_L(\tilde{X}) \neq \text{CH}^d_L(X)$. □

Remark 2.3.5. Note that the hypothesis of the last proposition implies that the cohomological dimension of k should be ≥ 3. Thus the previous argument does not give a counter-example for fields with cohomological dimension ≤ 2.

We have the following étale analogue of the results of Bloch-Srinivas.

Proposition 2.3.6. *Let X be such smooth projective variety over k of dimension d_X such that $CH_0(X_\Omega) \simeq CH^{d_X}_{\acute{e}t}(X_\Omega) = \mathbb{Z}$ for a universal domain Ω (an extension of k of infinite transcendence degree). Consider the diagonal $\Delta_{\acute{e}t} \in CH^{d_X}_{\acute{e}t}(X \times X)$. There exist an integer N, a closed sub-scheme $T \subset X$ and cycles Γ_1, $\Gamma_2 \in CH^{d_X}_{\acute{e}t}(X \times X)$ with*

$$\Gamma_1 \in im\Big\{CH^{d_X - c_V}_{\acute{e}t}(V \times X) \xrightarrow{i_*} CH^{d_X}_{\acute{e}t}(X \times X)\Big\}$$

and

$$\Gamma_2 \in im\Big\{CH^{d_X - c_T}_{\acute{e}t}(X \times T) \xrightarrow{i_*} CH^{d_X}_{\acute{e}t}(X \times X)\Big\}$$

such that $N\Delta_{\acute{e}t} = \Gamma_1 + \Gamma_2$.

Proof. Denote $L := k(X)$ with an immersion in Ω which extends $k \hookrightarrow \Omega$. Recall that we have an isomorphism $\text{CH}^{d_X}_{\text{ét}}(X_L) \simeq \varinjlim_U \text{CH}^{d_X}_{\text{ét}}(X \times U)$ where U runs over all nonempty Zariski open subsets $U \subset X$. As mentioned before, we have an isomorphism $\text{CH}^{d_X}(X_\Omega) \simeq \text{CH}^{d_X}_{\text{ét}}(X_\Omega)$. Therefore $\text{CH}^{d_X}_{\text{ét}}(U_L)$ is torsion as in the classical case.

From the localization sequence

$$\ldots \to \text{CH}^{d_X - c_V}_{\text{ét}}(V_L) \xrightarrow{i_*} \text{CH}^{d_X}_{\text{ét}}(X_L) \xrightarrow{j^*} \text{CH}^{d_X}_{\text{ét}}(U_L) \to \text{CH}^{d_X - c_V}_{\text{ét}}(V_L, -1) \to \ldots$$

we conclude that for every cycle $z \in \text{im}(j^*) \subset \text{CH}^{d_X}_{\text{ét}}(U_L)$ there exists $N \in \mathbb{N}$ and $y \in \text{CH}^{d_X - c_V}_{\text{ét}}(V_L)$ such that $Nz = i_*(y) \in \text{CH}^{d_X}_{\text{ét}}(X_L)$.

Consider $\eta_{\text{ét}}$, the image of the generic point η of X_L, and set $z = \eta_{\text{ét}}$. Then we have $N\eta_{\text{ét}} = \delta \in \text{CH}^{d_X}_{\text{ét}}(X_L)$, by the same kind of argument as in [BS83, Proposition 1]. Since the closure of η in $X \times X$ is the diagonal, we have that $\Delta_{\text{ét}}$ maps to $\eta_{\text{ét}}$ through the map

$\gamma : \mathrm{CH}^{d_X}_{\text{ét}}(X \times X) \to \mathrm{CH}^{d_X}_{\text{ét}}(X_L)$, which is the pullback of the map $p : X_L \to X \times X$ given by the cartesian square

$$\begin{array}{ccc} X_L & \xrightarrow{\quad p \quad} & X \times X \\ \big\downarrow & & \big\downarrow \mathrm{pr} \\ \mathrm{Spec}(k(X)) & \xrightarrow{\qquad} & X. \end{array}$$

With the same arguments we find an element

$$\Gamma_1 \in \mathrm{im}\left\{\mathrm{CH}^{d_X - c_V}_{\text{ét}}(V \times X) \xrightarrow{i_*} \mathrm{CH}^{d_X}_{\text{ét}}(X \times X)\right\}$$

whose image through γ is δ. Then it is easy to see that $N\Delta_{\text{ét}} - \Gamma_1 \in \ker(\gamma)$. Since $\mathrm{CH}^{d_X}_{\text{ét}}(X_L) \simeq \varinjlim_U \mathrm{CH}^{d_X}_{\text{ét}}(X \times U)$, there exists $U \subset X$ open sub-scheme such that $j^*_{X \times U}(N\Delta_{\text{ét}} - \Gamma_1) = 0 \in \mathrm{CH}^{d_X}_{\text{ét}}(X \times U)$ so setting $T := X - U$, from the localization sequence we conclude that there exists

$$\Gamma_2 \in \mathrm{im}\left\{\mathrm{CH}^{d_X - c_T}_{\text{ét}}(X \times T) \xrightarrow{i_*} \mathrm{CH}^{d_X}_{\text{ét}}(X \times X)\right\}$$

such that $N\Delta_{\text{ét}} - \Gamma_1 = \Gamma_2 \in \mathrm{CH}^{d_X}_{\text{ét}}(X \times X)$. □

2.4 Equivalence relations on étale cycles

As in the classical theory of algebraic cycles it is possible to define étale cycles which are algebraically, homologically or numerically equivalent to zero.

Algebraic equivalence

We say that a cycle $z \in \mathrm{CH}^i_{\text{ét}}(X)$ is algebraically equivalent to zero if there exists a smooth connected projective curve C and distinct points $t_1, t_2 \in C$ such that z is in the image of the map

$$\mathrm{CH}^i_{\text{ét}}(C \times X) \xrightarrow{t_2^* - t_1^*} \mathrm{CH}^i_{\text{ét}}(X).$$

We denote the subgroup of those elements as $\mathrm{CH}^i_{\text{ét}}(X)_{\mathrm{alg}}$.

Proposition 2.4.1. *The comparison map induces a map $CH^i(X)_{alg} \to CH^i_{ét}(X)_{alg}$.*

Proof. Let $z \in \mathrm{CH}^i(X)_{\mathrm{alg}}$ then there exists a curve C, a cycle $W \in \mathrm{CH}^i(C \times X)$ and two points $a, b \in C$ such that the action

$$W(t) := (\mathrm{pr}_X)_*(W \cdot (X \times t))$$

evaluated in these points gives us $z = W(a) - W(b)$. Since the comparison map is compatible with push-forwards and intersection product we conclude. □

Smash Nilpotent equivalence

Assume that X is a smooth projective variety. We define the étale analogue of the smash nilpotent equivalence.

Definition 2.4.2. *Let $z \in CH^k_{ét}(X)$, we say that z is étale smash nilpotent equivalent to zero if and only if exists $n \in \mathbb{N}$ such that $z^{\otimes n} = 0 \in CH^k_{ét}(X^n)$. The subgroup of these elements is denoted by $CH^k_{ét}(X)_\otimes$.*

Remark 2.4.3.

1. $\mathrm{CH}^k_{\text{ét}}(X)_\otimes$ is a subgroup because if we take two elements z_1, $z_2 \in \mathrm{CH}^k_{\text{ét}}(X)_\otimes$ then there exists n_1 and n_2 such that $z_i^{\otimes n_i} = 0$. If we consider that

$$(z_1 + z_2)^n = \sum_{i=0}^{n} \binom{n}{i} z_1^i \otimes z_2^{n-i}$$

just consider $n = n_1 + n_2$.

2. Due to the compatibility of the comparison map with products we have a morphism $\kappa : \mathrm{CH}^k(X)_\otimes \to \mathrm{CH}^k_{\text{ét}}(X)_\otimes$

Homological equivalence

At this point we should state some conventions, because we can define several cycle class maps. These depend on the characteristic of the ground field, and on the use of étale Chow groups or Lichtenbaum cohomology groups. If we work over the complex numbers, then we use the isomorphism $\mathrm{CH}^n_{\text{ét}}(X) \simeq \mathrm{CH}^n_L(X)$ and the cycle class map constructed in [RS16].

In this case we have a cycle class map $c_L^{m,n} : H^m_L(X, \mathbb{Z}(n)) \to H^m_B(X, \mathbb{Z}(n))$ to Betti cohomology and we define the Lichtenbaum (or étale) cycles equivalent to zero as the kernel of the L-cycle class map

$$\mathrm{CH}^n_{\text{ét}}(X)_{\text{hom}} := \ker \left\{ c^n_{\text{ét}} : \mathrm{CH}^n_{\text{ét}}(X) \to H^{2n}_B(X, \mathbb{Z}(n)) \right\}.$$

Lemma 2.4.4. *Let X be a complex smooth projective variety, then:*

1. *$c_L^{1,1}$ is the zero map,*
2. *$c_L^{3,1}$ induces a surjection $Br(X) \to H^3(X, \mathbb{Z}(1))_{tors}$.*

Proof. Consider the exponential sheaf sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{f \mapsto e^f} \mathcal{O}_X^* \to 0$$

which arises a long exact sequence

$$\begin{aligned} 0 \to \Gamma(X, \mathbb{Z}(1)) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X^*) \xrightarrow{h} H^1(X, \mathbb{Z}(1)) \\ \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}(1)) \to \dots \end{aligned}$$

we know that $\Gamma(X, \mathbb{Z}(1)) \simeq 2\pi i\mathbb{Z}$, $\Gamma(X, \mathcal{O}_X) \simeq \mathbb{C}$ and $\Gamma(X, \mathcal{O}_X^*) \simeq \mathbb{C}^*$ and therefore h is zero. On the other hand, by [Voe11, Theorem 6.18] $H^1_L(X, \mathbb{Z}_X(1)) \simeq H^1_M(X, \mathbb{Z}_X(1)) \simeq$

$\Gamma(X, \mathcal{O}_X^*)$ and then the induced map h coincide with the cycle class map $c_L^{1,1}$. The second statement follows from the isomorphism $H_L^3(X, \mathbb{Z}(1)) \simeq H^2_{\text{ét}}(X, \mathbb{G}_m) \simeq \mathrm{Br}(X)$ (which is torsion) and [RS16, Theorem 1.1]. $\square$

If we consider as a base an algebraically closed field $k \neq \mathbb{C}$ of characteristic $p \geq 0$, the quasi-isomorphism $(\mathbb{Z}/\ell^r\mathbb{Z})_X(n)_{\text{ét}} \xrightarrow{\sim} \mu_{\ell^r}^{\otimes n}$, where $\ell \neq \mathrm{char}(k)$, and the short exact sequence of complexes of étale sheaves

$$0 \to \mathbb{Z}_X(n)_{\text{ét}} \xrightarrow{\ell^r} \mathbb{Z}_X(n)_{\text{ét}} \to (\mathbb{Z}/\ell^r\mathbb{Z})_X(n)_{\text{ét}} \to 0$$

we obtain a map $c_{L,\ell^r}^{m,n} : H_L^m(X, \mathbb{Z}(n)) \to H_L^m(X, (\mathbb{Z}/\ell^r\mathbb{Z})_X(n)) \simeq H^m_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n})$. After taking the inverse limit $c_{L,\ell}^{m,n} := \varprojlim_r c_{L,\ell^r}^{m,n}$ we obtain a Lichtenbaum $\ell-$adic cycle class map

$$c_{L,\ell}^{m,n} : H_L^m(X, \mathbb{Z}(n)) \to H^m_{\text{ét}}(X, \mathbb{Z}_\ell(n)).$$

We then define the étale algebraic cycles homologically equivalent to zero as

$$\mathrm{CH}_L^n(X)_{\text{hom}} := \ker \left\{ \prod_{\ell \neq \mathrm{char}(k)} c_{L,\ell}^{m,n} : \mathrm{CH}_L^n(X) \to \prod_{\ell \neq \mathrm{char}(k)} H^{2n}_{\text{ét}}(X, \mathbb{Z}_\ell(n)) \right\}$$

Notice that by compatibility with the comparison maps the classical Betti and $\ell-$adic cycle class maps factor through Lichtenbaum cohomology: ùhaving the following compositions

$$\mathrm{CH}^n(X) \to \mathrm{CH}^n_{\text{ét}}(X) \to H^{2n}(X, \mathbb{Z}(n)) \text{ if } k = \mathbb{C},$$
$$\mathrm{CH}^n(X) \to \mathrm{CH}^n_L(X) \to \prod_{\ell \neq \mathrm{char}(k)} H^{2n}_{\text{ét}}(X, \mathbb{Z}_\ell(n)).$$

Hence there exists a homomorphism $\mathrm{CH}^n(X)_{\text{hom}} \to \mathrm{CH}^n_L(X)_{\text{hom}}$. There is a big difference concerning the algebraic properties of the homologically trivial L-cycles and the usual case. For example there exist algebraic varieties where $\mathrm{Griff}(X) \otimes \mathbb{Q}/\mathbb{Z} \neq 0$ whereas in the Lichtenbaum case if we define its analogue we have that $\mathrm{Griff}_L(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$.

Lemma 2.4.5. *[Gei17, Lemma 3.2] Let X be a smooth and projective $k-$variety with k and algebraically closed field, then the subgroup $CH^n_L(X)_{hom}$ is the maximal divisible subgroup inside $CH^n_L(X)$.*

Proof. Let k be a field and X be a smooth projective k-variety. Consider the groups of homologically trivial Lichtenbaum cycles

$$\mathrm{CH}_L^n(X)_{\text{hom}} := \ker \left\{ \prod_{\ell \neq \mathrm{char}(k)} c_{L,\ell}^{n} : \mathrm{CH}_L^n(X) \to \prod_{\ell \neq \mathrm{char}(k)} H^{2n}_{\text{ét}}(X, \mathbb{Z}_\ell(n)) \right\}$$

Since $H^{2n}_{\text{ét}}(X, \mathbb{Z}_\ell(n)) = \varprojlim_i H^{2n}_{\text{ét}}(X, \mathbb{Z}/\ell^i(n))$ and the cycle class map $c^n_{L,\ell}$ factors trough $\varprojlim_i H^{2n}_{\text{ét}}(X, \mathbb{Z}(n))/\ell^i$, the kernel is ℓ-divisible for each $\ell \neq \mathrm{char}(k)$, so it is divisible. The maximality comes from the factorization $c^n_{L,\ell}$ through $H^{2n}_{\text{ét}}(X, \mathbb{Z}(n))/\ell^i$. $\square$

Numerical equivalence

If we fix an algebraically closed field k as a base, recall that according to Theorem 2.2.11 we have an isomorphism $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0^L(X)$. So in this case we can use the same definition of degree map as in the classical case. If the field is not algebraically closed of characteristic p, then one has to consider an étale version of the degree map $\deg_{\text{ét}} := p_* : \mathrm{CH}_{\text{ét}}^{\dim(X)}(X) = \mathrm{CH}_0^{\text{ét}}(X) \to \mathbb{Z}[1/p]$.[1]

Let X be a smooth projective variety of dimension d One has a pairing

$$\begin{aligned} \mathrm{CH}^i_{\text{ét}}(X) \times \mathrm{CH}^{d-i}_{\text{ét}}(X) &\to \mathbb{Z}[1/p] \\ (\alpha, \beta) \qquad &\mapsto \deg_{\text{ét}}(\alpha \cdot \beta). \end{aligned}$$

For a fixed $\alpha \in \mathrm{CH}^i_{\text{ét}}(X)$ define $\deg_{\text{ét},\alpha}(\beta) := \deg_{\text{ét}}(\alpha \cdot \beta)$.

Definition 2.4.6. *Let X be a smooth projective variety of dimension d over a field k, and let $\alpha \in CH^i_{ét}(X)$ be a fixed but arbitrary étale cycle. We say that α is numerically equivalent to zero if and only if ker $\left(\deg_{ét,\alpha}(\cdot)\right) = CH^{d-i}_{ét}(X)$. We will denote the group of the elements of codimension i numerically equivalent to zero as $CH^i_{ét}(X)_{num} \subset CH^i_{ét}(X)$.*

Proposition 2.4.7. *Let $\kappa : CH^i(X) \to CH^i_{ét}(X)$ be comparison map, tthe image of $CH^i(X)_{num}$ under κ is contained in $CH^i_{ét}(X)_{num}$.*

Proof. We have to prove that the diagram

$$\begin{tikzcd} \mathrm{CH}^{d-i}(X) \arrow[rr, "\kappa"] \arrow[dr, "\deg_\alpha"'] & & \mathrm{CH}^{d-i}_{\text{ét}}(X) \arrow[dl, "\deg_{\text{ét},\kappa(\alpha)}"] \\ & \mathbb{Z} & \end{tikzcd}$$

is commutative, but this comes immediately from the fact that κ is compatible with pushforward and products of algebraic cycles.

□

Consider the group $\mathrm{CH}^i_{\text{ét}}(X)_{\text{num}}$ previously defined, then we define

$$\mathrm{NM}^i_{\text{ét}}(X) := \mathrm{CH}^i_{\text{ét}}(X)/\mathrm{CH}^i_{\text{ét}}(X)_{\text{num}}$$

With this definition we wanted to follow the spirit of the numerical cycles in the classical sense of Chow groups in order to obtain a similar result for the nilpotent properties and try to extend the conjecture to étale Chow groups.

Proposition 2.4.8. *Let X be a smooth projective variety over and algebraically closed field k. Then for $n \geq 0$ the induced map*

$$NM^n(X) \to NM^n_{ét}(X)$$

is an isomorphism of finitely generated free abelian groups.

[1] In chapter 4 we will return to the degree map in more detail, giving functoriality properties and the definition of the étale analogue for the index of a scheme.

Proof. In order to prove that $\mathrm{NM}^n(X)$ is a finitely generated free abelian group, first notice that $\mathrm{NM}^n(X)$ is torsion free since by definition we have a non-degenerated pairing, then we use the well-known fact that $\mathrm{NM}^n(X)_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}-$vector space of dimension $\leq b_{2n} = \dim_{\mathbb{Q}_\ell} H^{2n}_{\text{ét}}(X, \mathbb{Q}_\ell(n))$. Choose generators $\{\alpha_1, \ldots, \alpha_i\} \subset \mathrm{NM}^n(X)$ of $\mathrm{NM}^n(X)_{\mathbb{Q}}$ and let $\widetilde{\mathrm{NM}}^n(X)$ be the $\mathbb{Z}$-submodule of $\mathrm{NM}^n(X)$ generated by the $\alpha_i' s$. Define the dual $\mathrm{NM}^n(X)^\vee := \mathrm{Hom}(\mathrm{NM}^n(X), \mathbb{Z})$ and notice that $\mathrm{NM}^n(X)/\widetilde{\mathrm{NM}}^n(X)$ is torsion, therefore its dual as $\mathbb{Z}$ is zero. Hence the map

$$\mathrm{NM}^{d-n}(X) \subset \mathrm{NM}^n(X)^\vee \to \widetilde{\mathrm{NM}}^n(X)^\vee$$

is injective, and since $\widetilde{\mathrm{NM}}^n(X)^\vee$ is free, $\mathrm{NM}^{d-n}(X)$ is a finitely generated free abelian group. For the isomorphism, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{NM}^n(X) \arrow[r] \arrow[d, equal] & \mathrm{NM}^n_{\text{ét}}(X) \arrow[d, equal] \\
\mathrm{NM}^{d-n}(X)^\vee & \mathrm{NM}^{d-n}_{\text{ét}}(X)^\vee . \arrow[l]
\end{tikzcd}$$

Since the above map is surjective after tensor product with rational numbers, all the groups have the same rank. $\square$

This is the same proof given in [Gei17, Proposition 3.1] which states the same result but using Lichtenbaum cohomology instead of étale Chow groups. Now let $0 \leq i \leq d = \dim(X)$, the previous result leads us to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^i(X)_{\text{num}} \arrow[r] \arrow[d, "\kappa|_{\text{num}}"] & \mathrm{CH}^i(X) \arrow[r] \arrow[d, "\kappa"] & \mathrm{NM}^i(X) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathrm{CH}^i_{\text{ét}}(X)_{\text{num}} \arrow[r] & \mathrm{CH}^i_{\text{ét}}(X) \arrow[r] & \mathrm{NM}^i_{\text{ét}}(X) \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma we have that $\ker\left(\kappa\Big|_{\text{num}}\right) \to \ker(\kappa)$ is an isomorphism, as well as the map $\mathrm{coker}\left(\kappa\Big|_{\text{num}}\right) \to \mathrm{coker}(\kappa)$. Then, and after noticing that by [RS16, Proposition 5.1] the groups $\ker(\kappa)$ and $\mathrm{coker}(\kappa)$ are torsion since κ is an isomorphism with rational coefficients, the cycles numerically equivalent to zero which map to zero through the comparison map are just torsion elements.

Lemma 2.4.9. *Let X be a smooth projective variety. The map*

$$CH^n(X)_{num}/CH^n(X)_{hom} \to CH^n_{\acute{e}t}(X)_{num}/CH^n_{\acute{e}t}(X)_{hom}$$

is injective with torsion kernel and cokernel.

Proof. Let $z \in \mathrm{CH}^n(X)$ such that $\kappa(z) \in \mathrm{CH}^n_{\text{ét}}(X)_{\text{hom}}$ since the cycle class map factors through κ then is contained in $\mathrm{CH}^n(X)_{\text{hom}}$ which implies the injectivity. Again the kernel and cokernel are torsion because the groups agree rationally. $\square$

2.5 The category of étale Chow motives

Correspondences

We first introduce the concept of correspondences, which play an important role in the definition of the morphisms in the category of étale Chow motives. For this construction we will use étale Chow groups, but always keep in mind the following: consider a field k of characteristic exponent p and a smooth k-scheme X which is of finite type. For every bi-degree $(m,n) \in \mathbb{Z}^2$ there exists a map $\rho_X^{m,n} : H_L^m(X, \mathbb{Z}(n)) \to H^m_{M,\text{ét}}(X, \mathbb{Z}(n))$ which is induced by the $\mathbb{A}^1$-localization functor of effective étale motivic sheaves. If we tensor by $\mathbb{Z}[1/p]$, then $\rho_X^{m,n}$ becomes an isomorphism, for a proof we refer to [CD16, Theorem 7.1.2]. In particular the two definitions coincide in characteristic zero.

Definition 2.5.1. *Let X and Y be smooth projective varieties. An étale correspondence from X to Y of degree r is defined as follows: if X is purely of dimension d*

$$Corr^r_{\acute{e}t}(X,Y) = CH^{r+d}_{\acute{e}t}(X \times Y).$$

For the general case

$$Corr^r_{\acute{e}t}(X,Y) = \bigoplus_{i=1}^{n} CH^{r+d_i}_{\acute{e}t}(X_i \times Y)$$

where $X = \coprod_{i=1}^n X_i$ and d_i is the dimension of X_i.

For $\alpha \in \mathrm{Corr}^r_{\text{ét}}(X,Y)$ and $\beta \in \mathrm{Corr}^s_{\text{ét}}(Y,Z)$ we define the composition $\beta \circ \alpha \in \mathrm{Corr}^{r+s}_{\text{ét}}(X,Z)$ of correspondences as

$$\beta \circ \alpha = (\mathrm{pr}_{13})_* \left(\mathrm{pr}_{12}^* \alpha \cdot \mathrm{pr}_{23}^* \beta\right)$$

where $\mathrm{pr}_{12} : X \times_k Y \times_k Z \to X \times_k Y$ (similar definition for pr_{23} and pr_{13} with the respective change in the projection's components).

Proposition 2.5.2. *The composition of correspondences is an associative operation.*

Proof. To see that this operation is associative, we recall the Gysin morphism for étale motives. Consider X, Y and S smooth schemes over k such that there exists a cartesian square of smooth schemes

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "g"] & Y \arrow[d, "f"] \\ X \arrow[r, "p"] & S \end{tikzcd} \tag{2.4}$$

with p and q are projective morphism and $\dim(X/S) = \dim(X \times_S Y/Y)$, thus by [Dég08, Proposition 5.17] we have the following commutative diagrams

$$\begin{tikzcd} M(X \times_S Y)(-n)[-2n] \arrow[d, "g_*"] & M(Y) \arrow[l, "q^*"'] \arrow[d, "f_*"] \\ M(X)(-n)[-2n] & M(S) \arrow[l, "p^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}^{i+n}_{\text{ét}}(X \times_S Y) \arrow[r, "q_*"] & \mathrm{CH}^i_{\text{ét}}(Y) \\ \mathrm{CH}^{i+n}_{\text{ét}}(X) \arrow[u, "g^*"] \arrow[r, "p_*"] & \mathrm{CH}^i_{\text{ét}}(S) \arrow[u, "f^*"] \end{tikzcd} \tag{2.5}$$

where $n = \dim(X/S)$.

Consider the following commutative diagram

$$\begin{tikzcd}
X \times Y \times Z \times W \arrow[r, "\mathrm{pr}_{XYZ}^{XYZW}"] \arrow[d, "\mathrm{pr}_{XZW}^{XYZW}"] & X \times Y \times Z \arrow[d, "\mathrm{pr}_{XZ}^{XYZ}"] \\
X \times Z \times W \arrow[r, "\mathrm{pr}_{XZ}^{XZW}"] & X \times Z
\end{tikzcd}$$

by (2) we have that $\left(\mathrm{pr}_{XYZ}^{XYZW}\right)_* \left(\mathrm{pr}_{XZW}^{XYZW}\right)^* = \left(\mathrm{pr}_{XZ}^{XYZ}\right)^* \left(\mathrm{pr}_{XZ}^{XZW}\right)_*$, so the rest of the proof is similar to the proof of [Ful98, 16.1.1.(a)]: the formula $\left(\mathrm{pr}_{XYZ}^{XYZW}\right)_* \left(\mathrm{pr}_{XZW}^{XYZW}\right)^* = \left(\mathrm{pr}_{XZ}^{XYZ}\right)^* \left(\mathrm{pr}_{XZ}^{XZW}\right)_*$ gives us the following

$$\begin{aligned}
\gamma \circ (\beta \circ \alpha) &= \mathrm{pr}_{XW*}^{XZW} \left(\mathrm{pr}_{XZ}^{XZW*} \left(\mathrm{pr}_{XZ*}^{XYZ} \left(\mathrm{pr}_{XY}^{XYZ*}\alpha \cdot \mathrm{pr}_{YZ}^{XYZ*}\beta\right)\right) \cdot \mathrm{pr}_{ZW}^{XZW*}\gamma\right) && (2.6) \\
&= \mathrm{pr}_{XW*}^{XZW} \left(\mathrm{pr}_{XZW*}^{XYZW} \left(\mathrm{pr}_{XYZ}^{XYZW*} \left(\mathrm{pr}_{XY}^{XYZ*}\alpha \cdot \mathrm{pr}_{YZ}^{XYZ*}\beta\right)\right) \cdot \mathrm{pr}_{ZW}^{XZW*}\gamma\right) && (2.7) \\
&= \mathrm{pr}_{XW*}^{XZW} \left(\mathrm{pr}_{XZW*}^{XYZW} \left(\left(\mathrm{pr}_{XY}^{XYZW*}\alpha \cdot \mathrm{pr}_{YZ}^{XYZW*}\beta\right) \cdot \mathrm{pr}_{XZW}^{XYZW*}\mathrm{pr}_{ZW}^{XZW*}\gamma\right)\right) && (2.8) \\
&= \mathrm{pr}_{XW*}^{XYZW} \left(\left(\mathrm{pr}_{XY}^{XYZW*}\alpha \cdot \mathrm{pr}_{YZ}^{XYZW*}\beta\right) \cdot \mathrm{pr}_{ZW}^{XYZW*}\gamma\right) && (2.9) \\
&= \mathrm{pr}_{XW*}^{XYZW} \left(\mathrm{pr}_{XY}^{XYZW*}\alpha \cdot \left(\mathrm{pr}_{YZ}^{XYZW*}\beta \cdot \mathrm{pr}_{ZW}^{XYZW*}\gamma\right)\right).
\end{aligned}$$

Here (2.6) is the definition of composition of correspondences, (2.7) is obtained by the argument given in the second point of the remark 2.5, (2.8) by the projection formula and (2.9) by the functoriality of pullbacks.

□

Remark 2.5.3.

- The composition of correspondences gives $\mathrm{Corr}_{\text{ét}}(X, X)$ a ring structure. In general it is not a commutative ring.
- Let X be a smooth projective scheme of dimension n, then the étale cycle $\Delta_X^{\text{ét}}$, induced by the diagonal, is the identity for the composition operation, i.e. for $\alpha \in \mathrm{Corr}_{\text{ét}}^r(X, Y)$ and $\beta \in \mathrm{Corr}_{\text{ét}}^r(Y, X)$ we obtain that $\alpha \circ \Delta_X^{\text{ét}} = \alpha$ and $\Delta_X^{\text{ét}} \circ \beta = \beta$.

Operations on correspondences

We define the addition and product of correspondences in the following way: suppose that $\alpha \in \mathrm{Corr}_{\text{ét}}(X, X)$ and $\beta \in \mathrm{Corr}_{\text{ét}}(Y, Y)$, then we define the element $\alpha + \beta$ as the element resulting from the following operation on cycles:

$$\begin{aligned}
\mathrm{CH}_{\text{ét}}(X \times X) \oplus \mathrm{CH}_{\text{ét}}(Y \times Y) &\hookrightarrow \mathrm{CH}_{\text{ét}}\left(\left(X \coprod Y\right) \times \left(X \coprod Y\right)\right) \\
(\alpha, \beta) &\mapsto (i_1)_*\alpha + (i_2)_*\beta
\end{aligned}$$

where $i_1 : X \times X \hookrightarrow (X \coprod Y) \times (X \coprod Y)$ is the usual closed immersion map (similar definition for i_2 and Y). In a similar way we define the tensor product of correspondences as

$$\begin{aligned}
\mathrm{CH}_{\text{ét}}(X \times X) \otimes \mathrm{CH}_{\text{ét}}(Y \times Y) &\to \mathrm{CH}_{\text{ét}}(X \times Y \times X \times Y) \\
(\alpha, \beta) &\mapsto \mathrm{pr}_{XX}^*\alpha \cdot \mathrm{pr}_{YY}^*\beta
\end{aligned}$$

where $\mathrm{pr}_{XX} : X \times Y \times X \times Y \to X \times X$, similar definition for pr_{YY}. Both structures will play a big role in the definition of operations in the category of Chow étale motives. Another important operation is the transposition of cycles.

Definition 2.5.4. *Let X and Y be smooth projective varieties and let $\tau : X \times Y \to Y \times X$ which permutes the components $(x, y) \mapsto (y, x)$. Let $\Gamma \in CH^n_{ét}(X \times Y)$, we define the transpose cycle as $\Gamma^t := \tau_*(\Gamma)$.*

Due to some functoriality properties in $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$ we recover an étale version of Lieberman's lemma:

Lemma 2.5.5 (Lieberman's lemma)**.** *Let X, Y, Z and $W \in SmProj_k$. Consider $f \in Corr_{ét}(X, Y)$, $\alpha \in Corr_{ét}(X, Z)$ and $\beta \in Corr_{ét}(Y, W)$. Then $(\alpha \times \beta)_*(f) = \beta \circ f \circ \alpha^t$.*

Proof. We follow the proof of [MNP13, Lemma 2.1.3]. By definition of the action of cycles

$$(\alpha \times \beta)_*(f) = (\mathrm{pr}^{XZYW}_{ZW})_* \left(\alpha \times \beta \cdot (\mathrm{pr}^{XZYW}_{XY})^*(f)\right)$$

where $\mathrm{pr}^{ABC\dots}_{XY\dots} : A \times B \times C \dots \to X \times Y \dots$ denotes the projection. Note that we have the isomorphism $\tau : X \times Z \times Y \times W \to Z \times X \times Y \times W$, then we have

$$(\mathrm{pr}^{XZYW}_{XY})^* = \tau^* \circ (\mathrm{pr}^{ZXYW}_{XY})^* \text{ and } (\mathrm{pr}^{XZYW}_{ZW})_* = (\mathrm{pr}^{ZXYW}_{ZW})_* \circ \tau_*.$$

Using this and the projection formula thus we have the following

$$\begin{aligned}(\alpha \times \beta)_*(f) &= (\mathrm{pr}^{XZYW}_{ZW})_* \left(\alpha \times \beta \cdot (\mathrm{pr}^{XZYW}_{XY})^*(f)\right) \\ &= (\mathrm{pr}^{ZXYW}_{ZW})_* \left(\tau_* \left(\alpha \times \beta \cdot \tau^* \left((\mathrm{pr}^{ZXYW}_{XY})^*(f)\right)\right)\right) \\ &\simeq (\mathrm{pr}^{ZXYW}_{ZW})_* \left(\tau_*(\alpha \times \beta) \cdot (\mathrm{pr}^{ZXYW}_{XY})^*(f)\right)\end{aligned}$$

since τ just permutes the first two coordinates we obtain that $\tau_*(\alpha \times \beta) = \alpha^t \times \beta$. As pr^{ZXYW}_{ZW} factors through the canonical projections $Z \times X \times Y \times W \xrightarrow{p} Z \times Y \times W \xrightarrow{q} Z \times W$, where $p = \mathrm{pr}^{ZXYW}_{ZYW}$, $q = \mathrm{pr}^{ZYW}_{ZW}$ and also that $\alpha^t \times \beta = (\mathrm{pr}^{ZXYW}_{ZX})^*(\alpha^t) \cdot (\mathrm{pr}^{ZXYW*}_{YW})^*(\beta)$, then we replace in the previous expression

$$(\alpha \times \beta)_*(f) \simeq q_* \circ p_* \left((\mathrm{pr}^{ZXYW}_{ZX})^*(\alpha^t) \cdot (\mathrm{pr}^{ZXYW}_{YW})^*(\beta) \cdot (\mathrm{pr}^{ZXYW}_{XY})^*(f)\right).$$

By similar arguments we find that $\mathrm{pr}^{ZXYW}_{YW} = q' \circ p$ and then $(\mathrm{pr}^{ZXYW}_{YW})^* = p^* \circ q'^*$ with $q' = \mathrm{pr}^{ZYW}_{YW}$ so

$$\begin{aligned}(\alpha \times \beta)_*(f) &\simeq q_* \left\{p_* \left((\mathrm{pr}^{ZXYW}_{ZX})^*(\alpha^t) \cdot (\mathrm{pr}^{ZXYW}_{XY})^*(f) \cdot p^* \left(q'^*(\beta)\right)\right)\right\} \\ &\simeq q_* \left\{p_* \left((\mathrm{pr}^{ZXYW}_{ZX})^*(\alpha^t) \cdot (\mathrm{pr}^{ZXYW}_{XY})^*(f)\right) \cdot q'^*(\beta)\right\}.\end{aligned}$$

Let us focus in the part $p_* \left((\mathrm{pr}^{ZXYW}_{ZX})^*(\alpha^t) \cdot (\mathrm{pr}^{ZXYW}_{XY})^*(f)\right)$. Consider the projections $r : Z \times X \times Y \times W \to Z \times X \times Y$, $s_1 : Z \times X \times Y \to Z \times X$ and $s_2 : Z \times X \times Y \to X \times Y$. Then we obtain

$$\begin{aligned}p_* \left((\mathrm{pr}^{ZXYW}_{ZX})^*(\alpha^t) \cdot (\mathrm{pr}^{ZXYW}_{XY})^*(f)\right) &\simeq p_* \left(r^*(s_1^*(\alpha^t)) \cdot r^*(s_2^*(f))\right) \\ &\simeq p_* \circ r^* \left(s_1^*(\alpha^t) \cdot s_2^*(f).\right)\end{aligned}$$

Considering the commutative diagram

$$\begin{tikzcd} Z \times X \times Y \times W \arrow[r, "p"] \arrow[d, "r"] & Z \times Y \times W \arrow[d, "\mathrm{pr}_{ZY}^{ZYW}"] \\ Z \times X \times Y \arrow[r, "\mathrm{pr}_{ZY}^{ZXY}"] & Z \times Y. \end{tikzcd}$$

By (2) of remark 2.5 we have $p_* \circ r^* = \left(\mathrm{pr}_{ZY}^{ZYW}\right)^* \left(\mathrm{pr}_{ZY}^{ZXY}\right)_*$, which gives us the equality $p_* \circ r^* \left(s_1^*(\alpha^t) \cdot s_2^*(f)\right) = \left(\mathrm{pr}_{ZY}^{ZYW}\right)^* \left(\mathrm{pr}_{ZY}^{ZXY}\right)_* \left(s_1^*(\alpha^t) \cdot s_2^*(f)\right)$ and by definition of composition we obtain $\left(\mathrm{pr}_{ZY}^{ZYW}\right)^* \left(f \circ \alpha^t\right)$. The last part is just a direct consequence of the definition

$$\begin{aligned}(\alpha \times \beta)_*(f) &= q_* \left\{p_* \left((\mathrm{pr}_{ZX}^{ZXYW})^*(\alpha^t) \cdot (\mathrm{pr}_{XY}^{ZXYW})^*(f)\right) \cdot q'^*(\beta)\right\} \\ &= \left(\mathrm{pr}_{ZW}^{ZYW}\right)_* \left\{\left(\mathrm{pr}_{ZY}^{ZYW}\right)^* \left(f \circ \alpha^t\right) \cdot (\mathrm{pr}_{YW}^{ZYW})^*(\beta)\right\} \\ &= \beta \circ f \circ \alpha^t.\end{aligned}$$

□

Action on cycles and cohomology groups

Let X and Y be smooth projective varieties. For a correspondence $\Gamma \in \mathrm{Corr}_{\text{ét}}^r(X, Y)$ we define the action $\Gamma_* : \mathrm{CH}_{\text{ét}}^i(X) \to \mathrm{CH}_{\text{ét}}^{i+r}(Y)$ as

$$\Gamma_* Z = \mathrm{pr}_{Y*}\left(\Gamma \cdot \mathrm{pr}_X^*(Z)\right) \in \mathrm{CH}_{\text{ét}}^{i+r}(Y)$$

for $Z \in \mathrm{CH}_{\text{ét}}^i(X)$. Here we need to work with étale Chow groups because of their functoriality properties for proper maps, instead of Lichtenbaum cohomology. In order to use an action considering Lichtenbaum cohomology, it would be necessary to invert the characteristic exponent of the base field.

Classical correspondences have a natural action over on their étale analogue using the comparison map:

$$\begin{aligned}\mathrm{Corr}^0(X, X) \times \mathrm{Corr}_{\text{ét}}^r(X, Y) &\to \mathrm{Corr}_{\text{ét}}^r(X, Y) \\ (\alpha, Z) &\mapsto \mathrm{pr}_{XY*}\left(\mathrm{pr}_{XX}^*(\kappa(\alpha)) \cdot \mathrm{pr}_{XY}^*(Z)\right)\end{aligned}$$

Let $\Gamma \in \mathrm{Corr}_{\text{ét}}^r(X, Y)$ be an étale correspondence of degree r. Let us assume that there exist a cohomology theory (not necessarily a Weil cohomology theory) H with a cycle class map $c_{\text{ét},H}^i : \mathrm{CH}_{\text{ét}}^i(X) \to H^{2i}(X)$. We recall that this choice depends on the base field. For example if $k = \mathbb{C}$ we can consider $H^i(X) = H_B^i(X, \mathbb{Z})$ or if $k = \bar{k}$ one can consider $H^i(X) = H_{\text{ét}}^i(X, \mathbb{Z}/\ell)$ or $H^i(X) = H_{\text{ét}}^i(X, \mathbb{Z}_\ell)$, with $\ell \neq \mathrm{char}(k)$. As in the classical case, the correspondence gives us an action $\Gamma_* : H^i(X) \to H^{i+2r}(Y)$ defined by

$$\Gamma_* z := \mathrm{pr}_{Y*}\left(c_{\text{ét},H}^{d_X+r}(\Gamma) \cup \mathrm{pr}_X^*(z)\right) \in H^{i+2r}(Y)$$

with $z \in H^i(X)$. As we will see in the following chapter, this action will be the cornerstone for a well-defined version Hodge conjecture and generalized Hodge conjecture in the Lichtenbaum setting.

Étale Chow motives

Let SmProj_k be the category of smooth projective varieties over k. We construct the category of effective étale motives over k, denoted by $\mathrm{Chow}^{\mathrm{eff}}_{\text{ét}}(k)$, as follows:

- The elements are pairs (X, p) where X is a smooth projective variety and $p \in \mathrm{Corr}^0_{\text{ét}}(X, X)$ is an idempotent element, i.e. $p \circ p = p$.
- Morphism $(X, p) \to (Y, q)$ are the elements of the form $f = q \circ g \circ p$ where $g \in \mathrm{Corr}^0_{\text{ét}}(X, Y)$, therefore

$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{eff}}_{\text{ét}}(k)}((X, p), (Y, q)) = q \circ \mathrm{Corr}^0_{\text{ét}}(X, Y) \circ p$$

Finally, the category $\mathrm{Chow}_{\text{ét}}(k)$ of Chow étale motives is defined in the following way: the objects are triplets (X, p, m) where X is a smooth projective variety, p is a correspondence of degree 0 and idempotent and $m \in \mathbb{Z}$. The morphisms $(X, p, m) \to (Y, q, n)$ are defined as

$$\mathrm{Hom}_{\mathrm{Chow}_{\text{ét}}(k)}((X, p, m), (Y, q, n)) = q \circ \mathrm{Corr}^{n-m}_{\text{ét}}(X, Y) \circ p$$

As in the theory of Chow motives, for étale motives there is an obvious fully-faithful functor $\mathrm{Chow}^{\mathrm{eff}}_{\text{ét}}(k) \hookrightarrow \mathrm{Chow}_{\text{ét}}(k)$.

We define a functor $h_{\text{ét}} : \mathrm{SmProj}^{\mathrm{op}}_k \to \mathrm{Chow}_{\text{ét}}(k)$ as

$$\begin{aligned} h_{\text{ét}} : \mathrm{SmProj}_k &\to \mathrm{Chow}_{\text{ét}}(k) \\ X &\mapsto h_{\text{ét}}(X) := (X, \mathrm{id}_X, 0) \\ \left(X \xrightarrow{f} Y\right) &\mapsto \left(h_{\text{ét}}(Y) \xrightarrow{h_{\text{ét}}(f)} h_{\text{ét}}(X)\right) \end{aligned}$$

where id_X is the element that acts as the identity on the correspondences from X to itself and $h_{\text{ét}}(f) = \kappa([\Gamma^t_f])$.

Here there are some examples of étale Chow motives:

1. *Lefschetz motive* is

$$\mathbb{L} := (\mathrm{Spec}(k), \mathrm{id}, -1)$$

   and $\mathbb{L}^d := \mathbb{L}^{\otimes d} \simeq (\mathrm{Spec}(k), \mathrm{id}, -d)$.

2. The unit motive $\mathbf{1}$ defined as $\mathbf{1} := (\mathrm{Spec}(k), \mathrm{id}, 0)$.

3. The Tate motive is defined as

$$\mathbb{T} := (\mathrm{Spec}(k), \mathrm{id}, 1)$$

Remark 2.5.6. 1. Let us remark that there was another construction of a category of Chow étale motives, which here we denote $\mathrm{Chow}^K_{\text{ét}}(k)$, given in [Kah02, §5] by Kahn. This category is pseudo-abelian and rigid symmetric monoidal. The definition of such (effective) category is similar as the one we gave, but just considering elements (X, p) where $p = p^2 \in \mathrm{Corr}_{\text{ét}}(X, X) \otimes \mathbb{Q}$ and morphisms between (X, p) and (Y, q) are correspondences $f \in \mathrm{Corr}_{\text{ét}}(X, Y)$ such that $f \otimes \mathbb{Q} = q \circ \tilde{f} \circ p = \tilde{f} \in \mathrm{Corr}_{\text{ét}}(X, Y) \otimes \mathbb{Q}$.

2. If k is an algebraically closed field of characteristic zero, then by Theorem 2.2.11 for every $X \in \mathrm{SmProj}_k$ of dimension d we have that $\mathrm{CH}^{d+n}(X) \simeq \mathrm{CH}^{d+n}_{\text{ét}}(X)$ for all $n \geq 0$. If k is not algebraically closed then $\mathrm{CH}^n_{\text{ét}}(\mathrm{Spec}(k))$ is torsion for all $n \geq 1$.

Lemma 2.5.7. *There exists a monoidal functor $c_{\acute{e}t} : Chow(k) \to Chow_{\acute{e}t}(k)$ coming from the comparison map from classical to étale Chow groups.*

Proof. We define $c_{\text{ét}} : \mathrm{Chow}(k) \to \mathrm{Chow}_{\text{ét}}(k)$ as follows: for an element $M = (X, p, m) \in \mathrm{Chow}(k)$we have $c_{\text{ét}}(M) = (X, \sigma(p), m)$, its action on morphisms is given by

$$\left(M \xrightarrow{f} N\right) \mapsto \left(c_{\text{ét}}(M) \xrightarrow{c_{\text{ét}}(f)} c_{\text{ét}}(N)\right)$$
$$= (X, \sigma(p), m) \xrightarrow{\sigma(f)} (Y, \sigma(q), n).$$

The monoidal property comes from the compatibility of the cycle class map with the product of cycles. $\square$

Remark 2.5.8. Along with the category $\mathrm{Chow}_{\text{ét}}(k)$ we can define the étale analogue of the categories $\mathcal{M}_\sim(k)$ for an adequate equivalence relation. We denote as $\mathcal{M}^{\text{ét}}_\sim(k)$ if we replace the étale Chow groups for $\sim$-étale groups. If the base field k is algebraically closed, we then obtain the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Chow}(k)_{\mathbb{Z}} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{alg}}(k)_{\mathbb{Z}} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{hom}}(k)_{\mathbb{Z}} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{num}}(k)_{\mathbb{Z}} \arrow[d, equal] \\
\mathrm{SmProj}_k^{op} \arrow[ur, "h_{\mathbb{Z}}(-)"] \arrow[dr, "h_{\text{ét}}(-)"'] & & & & \\
 & \mathrm{Chow}_{\text{ét}}(k) \arrow[r] & \mathcal{M}^{\text{ét}}_{\mathrm{alg}}(k) \arrow[r] & \mathcal{M}^{\text{ét}}_{\mathrm{hom}}(k) \arrow[r] & \mathcal{M}^{\text{ét}}_{\mathrm{num}}(k).
\end{tikzcd}$$

Proposition 2.5.9. *Similar to the theory of pure Chow motives, there exists a fully-faithful embedding functor $F : Chow_{\acute{e}t}(k)^{op} \hookrightarrow DM_{\acute{e}t}(k)$*

Proof. Let $X, Y, Z \in \mathrm{SmProj}_k$. The map $\epsilon_{X,Y} : \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M(X), M(Y)) \xrightarrow{\simeq} \mathrm{Corr}^0_{\text{ét}}(Y, X)$ is an isomorphism, which can be obtained with the same arguments as in [MVW06, Proposition 20.1]. We proceed as in case of the Chow motives. Let X and Y be two equidimensional smooth projective varieties, with dimension d_X and d_Y respectively, then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M(X), M(Y)) &\simeq \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M(X) \otimes DM(Y), \mathbb{Z}) \\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M(X) \otimes M(Y)(-d_Y)[-2d_Y], \mathbb{Z}) \\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M(X) \otimes M(Y), \mathbb{Z}(d_Y)[2d_Y]) \\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M(X \times Y), \mathbb{Z}(d_Y)[2d_Y]) \\
&= \mathrm{CH}^{d_Y}_{\text{ét}}(X \times Y) \\
&= \mathrm{Corr}^0_{\text{ét}}(Y, X) = \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(h(Y), h(X))
\end{aligned}$$

Denote as $\epsilon_{X,Y} : \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k)}(M(X), M(Y)) \xrightarrow{\sim} \mathrm{CH}^{d_Y}_{\text{ét}}(Y \times X)$, it remains to prove that the composition is compatible with $\epsilon_{X,Y}$., but the compatibility obtained as in [Fan16, Theorem 3.17] using [Fan16, Proposition 2.39]. $\square$

Definition 2.5.10. *Let $M = (X, p, m)$ be an étale motive. We define the $i-$th étale Chow group of M as the image of the action of the correspondence p, i.e.*

$$CH^i_{\acute{e}t}(M) = im\left\{p_* : CH^{i+m}_{\acute{e}t}(X) \to CH^{i+m}_{\acute{e}t}(X)\right\}.$$

Similarly the $i-$th cohomology group of M is define as the action of the projector p

$$H^i(M) = im\left\{p_* : H^{i+2m}(X) \to H^{i+2m}(X)\right\}$$

where H^i can be $H^i(X, \mathbb{Z})$ if X is a complex variety and $H^i_{\acute{e}t}(X, \mathbb{Z}_\ell)$ for an algebraically closed field (not necessarily of characteristic zero), but always $\ell \neq char(k)$. These are the Betti and $\ell-$adic realizations of M.

Let us consider the functor F^i defined as follows $F^i : \mathrm{Chow}_{\text{ét}}(k) \to \mathbb{Z}\text{-mod}$, $M \mapsto F^i(M) := \mathrm{Hom}_{\mathrm{Chow}_{\text{ét}}(k)}(\mathbb{L}^i, M)$, with M of the form $M = (X, p, m)$, and consider the $\mathbb{Z}$-graded functor $F := \oplus_{i\in\mathbb{Z}} F^i : \mathrm{Chow}_{\text{ét}}(k) \to \mathbb{Z}\text{-modGr}$. By definition of F^i we have that

$$\begin{aligned}
F^i(M) &= \mathrm{Hom}_{\mathrm{Chow}_{\text{ét}}(k)}(\mathbb{L}^i, M) \\
&= p \circ \mathrm{CH}^{i+m}_{\text{ét}}(\mathrm{Spec}(k) \times X) \\
&\simeq p_* \mathrm{CH}^{i+m}_{\text{ét}}(X) = \mathrm{CH}^i_{\text{ét}}(M).
\end{aligned}$$

By definition F is an additive functor, and by duality

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Chow}_{\text{ét}}(k)}((X, p, m), (Y, q, n)) &= q \circ \mathrm{Corr}^{n-m}_{\text{ét}}(X, Y) \circ p \\
&= q \circ \mathrm{CH}^{n-m+d_X}_{\text{ét}}(X \times Y) \circ p \\
&\simeq F^0(M \otimes N^\vee).
\end{aligned}$$

Notice that N is a sub-motive of $h(Y) \otimes \mathbb{L}^n$ for some $Y \in \mathrm{SmProj}_k$ and $n \in \mathbb{Z}$ and by duality $F^0(M \otimes h(Y) \otimes \mathbb{L}^n) \simeq F^{-n}(M \otimes h_{\text{ét}}(Y))$. For a fixed $M \in \mathrm{Chow}_{\text{ét}}(k)$ define the following functor

$$\begin{aligned}
\omega_M : \mathrm{SmProj}_k^{op} &\to \mathbb{Z}\text{-modGr} \\
Y &\mapsto \omega_M(Y) := F(M \otimes h_{\text{ét}}(Y))
\end{aligned}$$

then Yoneda embedding implies that the functor

$$\omega : \mathrm{Chow}_{\text{ét}}(k) \to \mathbb{Z}\text{-modGr}^{\mathrm{SmProj}_k^{op}}$$
$$M \mapsto \omega_M$$

is fully-faithful. Hence we recover the classical Manin principle but in the étale setting!

Proposition 2.5.11. *[Manin's identity principle] Let $f,\, g : M \to N$ be morphism of étale motives then:*

1. *f is an isomorphism if and only if the induced map*
$$\omega_f(Y) : \omega_M(h(Y)) \to \omega_N(h(Y))$$
is an isomorphism for all $Y \in SmProj_k$ and $f = g$ is and only if $\omega_f(Y) = \omega_g(Y)$ for all $Y \in SmProj_k$.

2. *A sequence*
$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$
is exact if and only if, for every $Y \in SmProj_k$ the sequence
$$0 \to \omega_{M_1}(h(Y)) \xrightarrow{\omega_f(h(Y))} \omega_{M_2}(h(Y)) \xrightarrow{\omega_g(h(Y))} \omega_{M_3}(h(Y)) \to 0$$

Proof. This properties is a consequence of the faithfulness of the functor ω and the fact that fully-faithful functor reflects monic, epi and isomorphisms. □

The following isomorphisms in $\mathrm{Chow}_{\text{ét}}(k)$ are obtained as a consequence of Lemma 2.2.22 about the structure of étale motivic cohomology groups: we can obtain decomposition for motives of a projective bundle, blow-ups with smooth center and flag varieties.

Example 2.5.12. 1. *Consider E a locally free sheaf of rank $(n+1)$ over X, and $\pi : \mathbb{P}_X(E) \to X$ its associated projective bundle. Then*
$$CH^i_{ét}(\mathbb{P}_X(E)) \simeq \bigoplus_{j=0}^{n} CH^{i-j}_{ét}(X).$$
Since this isomorphism is functorial with respect to base change, for all $Y \in SmProj_k$ we have an isomorphism $CH^i_{ét}(Y \times \mathbb{P}_X(E)) \simeq \bigoplus_{j=0}^{n} CH^{i-j}_{ét}(Y \times X)$, therefore we have a decomposition of the motive of $\mathbb{P}_X(E)$ as
$$h_{ét}(\mathbb{P}_X(E)) \simeq \bigoplus_{i=0}^{n} h_{ét}(X)(-i).$$

2. *Consider $Y = Bl_Z X$ the Blow-up of $X \in SmProj_k$ along a smooth sub-scheme Z of codimension $(d+1)$. Since the isomorphism described in Lemma 2.2.22 is functorial with respect fo base change, then we have a decomposition of the motive of Y as follows*
$$h_{ét}(Y) \simeq h_{ét}(X) \oplus \bigoplus_{i=1}^{m} h_{ét}(Z)(-i).$$

3. *Let S be a smooth k-scheme and let $X \to S$ be a flat morphism of relative dimension n such that X has a decomposition in smooth projective varieties $X = X_p \supset X_{p-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$ with $X_i - X_{i-1} \simeq \mathbb{A}_S^{n-d_i}$ for some $d_i \in \mathbb{Z}$. Since the characterization of the étale Chow groups of X given in Lemma 2.2.22 is functorial with respect to base change $S \to S \times Y$, then*

$$h_{\acute{e}t}(X) \simeq \bigoplus_{i=0}^{p} h_{\acute{e}t}(S)(d_i).$$

In chapter 4 we will continue with another results concerning the decomposition of étale Chow motives and some generalized version of Manin principle. For the moment, let us mention an analogue of [Kim05, Theorem 6.8].

Definition 2.5.13. *Let $f : M \to N$ be a morphism of étale Chow motives. We say that f is a surjective morphism if for all $Z \in SmProj_k$ the induced map*

$$(f \otimes id_Z)_* : CH^n_{\acute{e}t}(M \otimes h(Z)) \to CH^n_{\acute{e}t}(N \otimes h(Z))$$

is surjective for all n.

Lemma 2.5.14. *Let $f : M = (X, p, m) \to N = (Y, q, n)$ be a morphism of étale Chow motives. The following conditions are equivalent:*

1. *f is surjective.*
2. *There exists a right inverse $g : N \to M$ i.e. $f \circ g = id_N$.*
3. *$q = f \circ s$ for some $s \in Corr^0_{\acute{e}t}(Y, X)$.*

Proof. (1. $\implies$ 2.) For this implication, we use Lieberman's lemma (see Lemma 2.5.5) for étale correspondences. Assuming point 1. consider the particular case $Z = Y$ and $q^t \in \mathrm{Corr}^0_{\text{ét}}(Y, Y)$. By Lieberman's lemma $q^t = (q \times \mathrm{id}_Y)_* \mathrm{id}_Y$ then $q^t \in \mathrm{CH}^*_{\text{ét}}(N \otimes h(Y))$. By assumption there exists an element $r \in \mathrm{CH}^*_{\text{ét}}(M \otimes h(Y)) \subset \mathrm{CH}^*_{\text{ét}}(X \times Y)$ such that $(f \times \mathrm{id}_Y)_* r = q^t$, and again by Lieberman $r \circ f^t = q^t$. Take $g = p \circ r^t \circ q$.

(2. $\implies$ 1.) As $f \circ g = \mathrm{id}_N$ after base change using $Z \in \mathrm{SmProj}_k$ we obtain that $(f \times \mathrm{id}_Z)_* \circ (g \times \mathrm{id}_Z)_* = \mathrm{id}_{N \otimes Z}$. Therefore $(f \times \mathrm{id}_Z)_* : \mathrm{CH}^n_{\text{ét}}(M \otimes h(Z)) \to \mathrm{CH}^n_{\text{ét}}(N \otimes h(Z))$ is surjective.

(2. $\implies$ 3.) For that just take the element s as the correspondence associated to $f \in \mathrm{Corr}^0_{\text{ét}}(Y, X)$.

(3. $\implies$ 2.) Consider the morphism defined by the correspondence $g = p \circ s \circ q$. □

Now again, we get an étale analogue of [Via17, Lemma 3.2]:

Proposition 2.5.15. *Let $f : M \to N$ be a morphism of étale motives defined over an algebraically closed field k:*

1. *Assume that for some field extension K (with $K = \bar{K}$) the map $(f_K)_* : CH^i_{\acute{e}t}(M_K) \to CH^i_{\acute{e}t}(N_K)$ is injective. Then $f_* : CH^i_{\acute{e}t}(M) \to CH^i_{\acute{e}t}(N)$ is injective.*

2. *Assume that for some field extension K (with $K = \bar{K}$) the map $(f_K)_* : CH^i_{\acute{e}t}(M_K) \to CH^i_{\acute{e}t}(N_K)$ is surjective. Then $f_* : CH^i_{\acute{e}t}(M) \to CH^i_{\acute{e}t}(N)$ is surjective.*

Proof. The first statement follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^i_{\text{ét}}(M) & \longrightarrow & \mathrm{CH}^i_{\text{ét}}(M_K) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (f_K)_*} \\ \mathrm{CH}^i_{\text{ét}}(N) & \longrightarrow & \mathrm{CH}^i_{\text{ét}}(N_K) \end{array}$$

and the fact that $\mathrm{CH}^i_{\text{ét}}(X) \to \mathrm{CH}^i_{\text{ét}}(X_K)$ is an injection by Proposition 2.2.20. For the surjectivity, notice that under assumptions about the base field, the map $H^{m-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \to H^{m-1}_{\text{ét}}(X_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ is an isomorphism for every bi-degree, therefore if the map after tensor with the rational is surjective (which is the result of [Via17, Lemma 3.2]), we then obtain that the map is surjective from a similar argument of Proposition 2.2.20. ☐

Lemma 2.5.16. *Let k be a field and let $(u_i : k_i \to k)_i$ be finite Galois extensions of the field k. Then the associated family of functors $u_i^* : Chow_{\acute{e}t}(k) \to Chow_{\acute{e}t}(k_i)$ is conservative.*

Proof. By Proposition 2.5.9, the functor $\Phi^{\text{ét}}_k : \mathrm{Chow}_{\text{ét}}(k)^{op} \to \mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$ is fully-faithful, hence conservative. According to [Ayo14b, Théorème 3.9] the family of functors $u_i^* : \mathrm{DM}_{\text{ét}}(k, \mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(k_i, \mathbb{Z})$ is also conservative. The commutative diagram

$$\begin{array}{ccc} \mathrm{Chow}_{\text{ét}}(k)^{op} & \xrightarrow{\Phi^{\text{ét}}_k} & \mathrm{DM}_{\text{ét}}(k, \mathbb{Z}) \\ \big\downarrow{\scriptstyle u_i^*} & & \big\downarrow{\scriptstyle u_i^*} \\ \mathrm{Chow}_{\text{ét}}(k_i)^{op} & \xrightarrow{\Phi^{\text{ét}}_{k_i}} & \mathrm{DM}_{\text{ét}}(k_i, \mathbb{Z}), \end{array}$$

then shows that the family of functors $u_i^* : \mathrm{Chow}_{\text{ét}}(k) \to \mathrm{Chow}_{\text{ét}}(k_i)$ is conservative. ☐

Lemma 2.5.17. *Let $M = (X, p, m)$ be an étale Chow motive over a field k. Then $M = 0$ if and only if $M_K = 0$ for some field extension K.*

Proof. This is a direct consequence of Lemma 2.5.16. ☐

Remark 2.5.18. Notice that [Via17, Proposition 1.3] is also a direct consequence of the condition of separateness of $\mathrm{DM}_{\text{ét}}(k, \mathbb{Q})$ and the fully-faithful embedding $\mathrm{Chow}(k)^{\mathrm{op}} \to \mathrm{DM}(k, \mathbb{Q}) \simeq \mathrm{DM}_{\text{ét}}(k, \mathbb{Q})$.

Proposition 2.5.19. *Let k be a field and let K be an inseparable extension of k. Then the associated functor $p^* : Chow_{\acute{e}t}(k) \to Chow_{\acute{e}t}(K)$ is fully-faithful.*

Proof. This is a consequence of Lemma 2.2.20. ☐

Chapter 3

Hodge and generalized Hodge conjecture

In this chapter we study the consequences of [RS16] on the level of Hodge structures. Let X be a smooth projective variety over $\mathbb{C}$, $k \in \mathbb{N}$ and consider the cycle class map $c^k : \mathrm{CH}^k(X) \to H^{2k}_B(X, \mathbb{Z}(k))$ where $\mathbb{Z}(k) = (2\pi i)^k\mathbb{Z}$. Its image is a subgroup of the Hodge classes $\mathrm{Hdg}^{2k}(X, \mathbb{Z})$. The integral Hodge conjecture asks whether or not this map is surjective. Putting $n = \dim_{\mathbb{C}}(X)$, then for $k = 0$ and $k = n$ the conjecture is immediately true and also for $k = 1$ by the Lefschetz (1,1) theorem, however for $k = 2$ the statement is not true as is shown by the counterexamples given by Atiyah and Hirzebruch in [AH62] (a torsion class which is not algebraic) and by Kóllar in [BCC92] (a non-algebraic non-torsion class) respectively. Even with rational coefficients the validity of the statement regarding the surjectivity of the cycle class map is still an open question, and is known as the Hodge conjecture. In a more general and ambitious framework, there exists another conjecture, called the *generalized Hodge conjecture*, which deals with sub-Hodge structures of smooth projective varieties of different weights and levels. To be more precise the conjecture for weight k and level $k - 2c$ (or equivalently for weight k and coniveau c) says that for any rational sub-Hodge structure $H \subset H^k(X, \mathbb{Q})$ of level at most $k - 2c$ there exists a closed subvariety $Z \hookrightarrow X$ of codimension $\geq c$ such that

$$H \subset \mathrm{im}\left\{H^{k-2c}(\widetilde{Z}, \mathbb{Q}(-c)) \xrightarrow{\gamma_*} H^k(X, \mathbb{Q})\right\}$$

where $\gamma_* = i_* \circ d_*$, i_* is the Gysin map associated to the inclusion $i : Y \hookrightarrow X$ and $d : \widetilde{Z} \to Z$ is a resolution of singularities.

The aim of this chapter is to find an analogue of [RS16, Theorem 1.1] for the generalized Hodge conjecture. In the first section, we attack two problems: throughout the first subsection, we present a refined version of [RS16, Theorem 1.1]. We show that if we restrict to a sub-Hodge structure $W \subset H^{2k}_B(X, \mathbb{Z}(k))$ and ask whether $W \otimes \mathbb{Q}$ is algebraic in the usual sense if and only if W is L-algebraic. In subsection 3.2 we give an explicit description of the torsion classes that are not algebraic in the classical sense, for the counterexamples presented in [AH62] and [BO20]. We then study the Lichtenbaum cohomology groups for hypersurfaces in subsection 3.2.3 and explain the torsion-free counter-example of Kóllar given in [BCC92].

With respect to the generalized Hodge conjecture, in section 2 of this chapter, we show several equivalences between the classical case and the L-version (involving Lichtenbaum cohomology and integral Hodge structures) in different weights and levels using characterizations through the Hodge conjecture (étale and classical setting) and the effectiveness of étale Chow motives, the category that we introduced in chapter 2. In the last subsection, we consider the equivalence between the classical and étale version of the generalized Hodge conjecture in Bardelli's example in [Bar91].

3.1 Hodge conjecture

Hodge conjecture and Lichtenbaum cohomology

Fix an integer $k \in \mathbb{Z}$. An integral pure Hodge structure H of weight k is a finitely generated $\mathbb{Z}$-module such that $H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$ where $H^{p,q}$, is a complex vector space with $H^{q,p} = \overline{H^{p,q}}$. For $m \in \mathbb{Z}$ we denote by $\mathbb{Z}(m)$ the Tate Hodge structure of weight $-2m$ whose Hodge decomposition is concentrated in bi-degree $(-m,-m)$. For a pure Hodge structure H of weight k its Tate twist $H(m)$ is defined to be the tensor product $H \otimes_{\mathbb{Z}} \mathbb{Z}(m)$ which is a Hodge structure of weight $k-2m$ and its decomposition is

$$H(m) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k-2m} H(m)^{p,q} = \bigoplus_{p+q=k-2m} H^{p-m,q-m}$$

If X is a complex smooth projective variety of dimension d, we denote by $\mathrm{Hdg}^{2n}(X,\mathbb{Z})$ the Hodge classes of X of weight $2n$, defined as

$$\mathrm{Hdg}^{2n}(X,\mathbb{Z}) := \left\{\alpha \in H^{2n}_B(X,\mathbb{Z}(n)) \;\middle|\; \rho(\alpha) \in F^n H^{2n}(X,\mathbb{C})\right\}$$

where $\rho : H^{2n}_B(X,\mathbb{Z}) \to H^{2n}(X,\mathbb{C})$ and $F^p H^{2n}(X,\mathbb{C}) = \bigoplus_{i\geq p} H^{i,2n-i}(X)$. Notice that by definition $H^{2n}_B(X,\mathbb{Z})_{\mathrm{tors}} \subset \mathrm{Hdg}^{2n}(X,\mathbb{Z})$. The image of the cycle class map to Betti cohomology $c^n : \mathrm{CH}^n(X) \to H^{2n}_B(X,\mathbb{Z}(n))$ is contained in $\mathrm{Hdg}^{2n}(X,\mathbb{Z})$. We denote as $\mathrm{HC}^n(X)$ the following statement:

Conjecture 3.1.1 (Hodge conjecture with integral coefficients)**.** *For a smooth complex projective variety X and $n \in \mathbb{N}$, the image of the cycle class map $c^n : CH^n(X) \to H^{2n}_B(X,\mathbb{Z}(n))$ is $Hdg^{2n}(X,\mathbb{Z})$.*

Under the above hypothesis for X, by trivial arguments we have that $\mathrm{HC}^0(X)$ and $\mathrm{HC}^d(X)$ holds. The validity of $\mathrm{HC}^1(X)$ is a consequence of the Lefschetz (1,1) theorem. For $n \notin \{0,1,d\}$ it is known that the Hodge conjecture (with integral coefficients) does not hold, even if we work with torsion free classes. We define the obstruction to the integral Hodge conjecture as $Z^{2i}(X) := \mathrm{Hdg}^{2i}(X,\mathbb{Z}(i))/\mathrm{im}(c^i)$. In [AH62] it is proved that for every prime number p, there exists a smooth variety X such that $Z^4(X)[p] \neq 0$. If we replace in the conjecture $\mathbb{Z}$ by $\mathbb{Q}$ coefficients, we will denote this new statement as $\mathrm{HC}^n(X)_{\mathbb{Q}}$ the following statement:

Conjecture 3.1.2 (Hodge conjecture with rational coefficients)**.** *For a smooth complex projective variety X and $n \in \mathbb{N}$, the image of the cycle class map $c^n_{\mathbb{Q}} : CH^n(X)_{\mathbb{Q}} \to H^{2n}_B(X, \mathbb{Q}(n))$ is isomorphic to $Hdg^{2n}(X, \mathbb{Q})$.*

Thanks to the Hard Lefschetz theorem, the statement $\mathrm{HC}^{d-1}(X)_{\mathbb{Q}}$ is true, but oly if we work with rational coefficients. More generally, if $\mathrm{HC}^n(X)_{\mathbb{Q}}$ holds for some $n < d/2$, then $\mathrm{HC}^{d-n}(X)_{\mathbb{Q}}$ holds. Of course $\mathrm{HC}^n(X)_{\mathbb{Q}}$ is still an open problem; there is no known counter-example for the Hodge conjecture with rational coefficients up to this day.

The Hodge conjecture can be stated in terms of motives as well. By using the Hodge realization, we can characterize the validity of the conjecture for the category $\mathrm{SmProj}_{\mathbb{C}}$:

Proposition 3.1.3. *Consider $k = \mathbb{C}$ and let ρ_H the Hodge realization for $Chow(\mathbb{C})$ (with rational coefficients), then $HC(X)_{\mathbb{Q}}$ holds for all $X \in SmProj(\mathbb{C})$ if and only if ρ_H is a full functor.*

Proof. Suppose that the Hodge conjecture holds. Then the Hodge classes of X are algebraic. By the Künneth formula and Poincaré duality, the Hodge classes in $H^{2k}_B(X \times Y, \mathbb{Q})$ are in bijection with $\bigoplus_{i=0}^{2k} \mathrm{Hom}_{\mathrm{HS}\mathbb{Q}}(H^{2(d_X-k)+i}(X, \mathbb{Q}), H^i(Y, \mathbb{Q}))$. Therefore we have the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Chow}(\mathbb{C})}(h(X), h(Y)) \arrow[r, "\rho_H"] \arrow[d, equal] & \mathrm{Hom}_{\mathrm{HS}\mathbb{Q}}(H^p(X, \mathbb{Q}), H^p(Y, \mathbb{Q})) \arrow[d, equal] \\
\mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q}} \arrow[r, "\rho_H"] \arrow[d, equal] & \left(H^{2d_X-p}(X, \mathbb{Q}) \otimes H^p(Y, \mathbb{Q})\right) \cap H^{d_X,d_X}(X \times X) \arrow[d, hook] \\
\mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q}} \arrow[r, two heads] & \mathrm{Hdg}^{2d_X}(X \times Y, \mathbb{Q})
\end{tikzcd}$$

which implies that ρ_H is full.

On the other hand, suppose that the Hodge realization is full, so in particular for all smooth projective variety X and $n \in \mathbb{N}$ the following map

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Chow}(\mathbb{C})}(\mathbf{1}(-n), (X, \Delta_X)) \arrow[r, "\rho_H"] \arrow[d, equal] & \mathrm{Hom}_{\mathrm{HS}\mathbb{Q}}(\mathbb{Q}(-n), H^{2n}(X, \mathbb{Q})) \arrow[d, equal] \\
\mathrm{CH}^n(X)_{\mathbb{Q}} \arrow[r] & \mathrm{Hdg}^{2n}(X, \mathbb{Q})
\end{tikzcd}$$

is surjective, which is the exact statement of the Hodge conjecture. □

Before going into the proof of the equivalences of the weaker version of the equivalence between the Hodge conjecture with rational coefficients and the Lichtenbaum Hodge conjecture let us recall the definitions of Deligne cohomology and intermediate Jacobians. Fixing an integer $k \geq 0$ one defines the $k-$th intermediate Jacobian $J^k(X)$ as the complex torus

$$J^k(X) := H^{2k-1}(X, \mathbb{C})/(F^k H^{2k-1}(X, \mathbb{C}) \oplus H^{2k-1}(X, \mathbb{Z})).$$

Consider the Deligne complex $\mathbb{Z}(p)_D$ of a complex manifold X defined as

$$0 \to \mathbb{Z}(p) \to \mathcal{O}_X \to \Omega^1_X \to \ldots \to \Omega^{p-1}_X \to 0.$$

We then define the **Deligne cohomology groups** as the hypercohomology groups of the Deligne complex i.e.

$$H^k_D(X, \mathbb{Z}(p)) := \mathbb{H}^k_{\text{an}}(X, \mathbb{Z}(p)_D).$$

We have an exact sequence relating Hodge classes and intermediate Jacobians

$$0 \to J^k(X) \to H^{2k}_D(X, \mathbb{Z}(k)) \to \mathrm{Hdg}^{2k}(X, \mathbb{Z}) \to 0.$$

Remark 3.1.4. The definition of intermediate Jacobians can be extended to pure Hodge structures of odd weight. Assume that H is a Hodge structure of weight $2k-1$ then we define the complex torus $J^k(H) := H_{\mathbb{C}}/(F^kH \oplus H)$. This construction is functorial with respect to morphisms of Hodge structures. For more details about these facts see [Voi02, Remarque 12.3] and [PS08, Section 3.5].

There exist maps $c^k_D : \mathrm{CH}^k(X) \to H^{2k}_D(X, \mathbb{Z}(k))$ and $\Phi^k_X : \mathrm{CH}^k(X)_{\text{hom}} \to J^k(X)$ called the **Deligne cycle class** and the **Abel-Jacobi map** respectively. There is a useful relation between the Deligne cycle class map, the Abel-Jacobi map and the cycle class map given by the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^k(X)_{\text{hom}} \arrow[r] \arrow[d, "\Phi^k_X"] & \mathrm{CH}^k(X) \arrow[r, "c^k"] \arrow[d, "c^k_D"] & I^k(X) \arrow[r] \arrow[d, "\text{into}"] & 0 \\
0 \arrow[r] & J^k(X) \arrow[r] & H^{2k}_D(X, \mathbb{Z}(k)) \arrow[r] & \mathrm{Hdg}^{2k}(X, \mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

For Lichtenbaum cohomology groups we have analogous maps, $c^k_{L,D} : \mathrm{CH}^k_L(X) \to H^{2k}_D(X, \mathbb{Z}(k))$ and $\Phi^k_{X,L} : \mathrm{CH}^k_L(X)_{\text{hom}} \to J^k(X)$ (the construction of the first one is done in [RS16, Theorem 4.4]) which fit in a similar commutative diagram as the one given before.

Remark 3.1.5. Let ℓ be a prime number and $r \in \mathbb{N}$. Notice that the exact triangle $0 \to \Omega^{\leq n-1}[-1] \to \mathbb{Z}_D(n) \to \mathbb{Z}(n) \to 0$ induces maps $c^{m,n}_{D,B} : H^m_D(X, \mathbb{Z}(n)) \to H^m_B(X, \mathbb{Z}(n))$ which fit in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{m-1}_D(X, \mathbb{Z}(n)) \otimes \mathbb{Z}/\ell^r \arrow[r] \arrow[d] & H^{m-1}_D(X, \mathbb{Z}/\ell^r(n)) \arrow[r, "\beta_D"] \arrow[d, "\simeq"] & H^m_D(X, \mathbb{Z}(n))[\ell^r] \arrow[r] \arrow[d, "c^{m,n}_{D,B}"] & 0 \\
0 \arrow[r] & H^{m-1}_B(X, \mathbb{Z}(n)) \otimes \mathbb{Z}/\ell^r \arrow[r] & H^{m-1}_B(X, \mathbb{Z}/\ell^r(n)) \arrow[r, "\beta"] & H^m_B(X, \mathbb{Z}(n))[\ell^r] \arrow[r] & 0
\end{tikzcd}$$

where β_D is the morphism induced by the exact triangle $0 \to \mathbb{Z}_D(n) \xrightarrow{\cdot \ell^r} \mathbb{Z}_D(n) \to$

$(\mathbb{Z}/\ell^r)_D(n) \to 0$. Also we obtain another commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_L^{m-1}(X,\mathbb{Z}(n))\otimes\mathbb{Z}/\ell^r \arrow[r]\arrow[d] & H_{\text{ét}}^{m-1}(X,\mu_{\ell^r}^{\otimes n}) \arrow[r]\arrow[d, "\simeq"] & H_L^m(X,\mathbb{Z}(n))[\ell^r] \arrow[r]\arrow[d, "c_{D,L}^{m,n}"] & 0 \\
0 \arrow[r] & H_D^{m-1}(X,\mathbb{Z}(n))\otimes\mathbb{Z}/\ell^r \arrow[r]\arrow[d] & H_D^{m-1}(X,\mathbb{Z}/\ell^r(n)) \arrow[r, "\beta_D"]\arrow[d, "\simeq"] & H_D^m(X,\mathbb{Z}(n))[\ell^r] \arrow[r]\arrow[d, "c_{D,B}^{m,n}"] & 0 \\
0 \arrow[r] & H_B^{m-1}(X,\mathbb{Z}(n))\otimes\mathbb{Z}/\ell^r \arrow[r] & H_B^{m-1}(X,\mathbb{Z}/\ell^r(n)) \arrow[r, "\beta"] & H_B^m(X,\mathbb{Z}(n))[\ell^r] \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma the arrows

$$H_L^{m-1}(X,\mathbb{Z}(n))/\ell^r \to H_D^{m-1}(X,\mathbb{Z}(n))/\ell^r \text{ and } H_D^{m-1}(X,\mathbb{Z}(n))/\ell^r \to H_B^{m-1}(X,\mathbb{Z}(n))/\ell^r$$

are injective while the arrows

$$H_L^m(X,\mathbb{Z}(n))[\ell^r] \to H_D^m(X,\mathbb{Z}(n))[\ell^r] \text{ and } H_D^m(X,\mathbb{Z}(n))[\ell^r] \to H_B^m(X,\mathbb{Z}(n))[\ell^r]$$

are surjective. Also the image of the composite of the right vertical arrows is equal to the image of $c_L^{m,n}$ restricted to ℓ^r-torsion elements.

The following results are immediate corollaries obtained after [RS16, Prop 5.1 (b) and Theo. 1.1]:

Corollary 3.1.6. *Let X be a smooth projective variety over $\mathbb{C}$, and fix an integer k such that $1 \le k \le \dim_{\mathbb{C}}(X)$. Then the restriction of the Abel-Jacobi map to torsion groups $\Phi_X^k\Big|_{tors} : \left(CH_L^k(X)_{hom}\right)_{tors} \to J^k(X)_{tors}$ is an isomorphism.*

Proof. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}_L^k(X)_{\mathrm{hom}} \arrow[r]\arrow[d, "c_{D,L}^k|_{\mathrm{hom}}"] & \mathrm{CH}_L^k(X) \arrow[r, "c_L^k"]\arrow[d, "c_{D,L}^k"] & I_L^k(X) \arrow[r]\arrow[d, "\text{into}"] & 0 \\
0 \arrow[r] & J^k(X) \arrow[r] & H_D^{2k}(X,\mathbb{Z}(k)) \arrow[r] & \mathrm{Hdg}^{2k}(X,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{CH}_L^k(X)_{\mathrm{hom}} \otimes \mathbb{Q}/\mathbb{Z} = 0$ by [RS16, Proposition 5.1 (b)] and $J^k(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ because $J^k(X)$ is divisible, we have then a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Big(\mathrm{CH}_L^k(X)_{\mathrm{hom}}\Big)_{\mathrm{tors}} \arrow[r]\arrow[d, "c_{D,L}^k|_{\mathrm{hom}}"] & \mathrm{CH}_L^k(X)_{\mathrm{tors}} \arrow[r]\arrow[d, "c_{D,L}^k"] & I_L^k(X)_{\mathrm{tors}} \arrow[r]\arrow[d, "\text{into}"] & 0 \\
0 \arrow[r] & J^k(X)_{\mathrm{tors}} \arrow[r] & H_D^{2k}(X,\mathbb{Z}(k))_{\mathrm{tors}} \arrow[r] & \mathrm{Hdg}^{2k}(X,\mathbb{Z})_{\mathrm{tors}} \arrow[r] & 0
\end{tikzcd} \tag{3.1}$$

Since $\mathrm{CH}_L^k(X)_{\mathrm{tors}} \simeq H_D^{2k}(X,\mathbb{Z}(k))_{\mathrm{tors}}$ by [RS16, Proposition 5.1 (a)] and the map $\mathrm{CH}_L^k(X)_{\mathrm{tors}} \to H^{2k}(X,\mathbb{Z})_{\mathrm{tors}}$ is surjective (see [RS16, Remark 3.2]), the middle arrow is an isomorphism as well as the right one. Therefore the left arrow is an isomorphism. □

Remark 3.1.7. Notice that if we set $k = \dim(X)$, by Proposition 2.2.11 Chow groups and Lichntebaum cohomology coincide. Then we recover the classical Roitman theorem

$$\mathrm{CH}_0^{\mathrm{hom}}(X)_{\mathrm{tors}} \simeq \mathrm{Alb}_X(\mathbb{C})_{\mathrm{tors}}.$$

We say that $W \subset H_B^{2k}(X, \mathbb{Z}(k))$ is a sub-Hodge structure if W is s sub-lattice of $H_B^{2k}(X, \mathbb{Z}(k))$ such that it has an induced Hodge decomposition $W_{\mathbb{C}} = \bigoplus_{p+q=2k} W^{p,q}$ with $W^{p,q} = W_{\mathbb{C}} \cap H^{p,q}$. Let $W \subset H_B^{2k}(X, \mathbb{Z}(k))$ be a sub-Hodge structure, we define the partial Hodge conjecture with rational coefficients related to W as the following statement: for every element $\alpha \in W$ there exists $N \in \mathbb{N}$ and an algebraic cycle $\widetilde{\alpha} \in \mathrm{CH}^k(X)$ such that $c(\widetilde{\alpha}) = N\alpha$. It is clear that for $W = \mathrm{Hdg}^{2k}(X, \mathbb{Z})$ we recover the usual Hodge conjecture. For a fixed W we denote the previous statement by $\mathrm{HC}^k(X, W)_{\mathbb{Q}}$.

Similarly we denote by $\mathrm{HC}_L^k(X, W)_{\mathbb{Z}}$ the statement that for every element of $\alpha \in W$ there exists a Lichtenbaum cycle $\widetilde{\alpha} \in \mathrm{CH}_L^k(X)$ such that $c_L(\widetilde{\alpha}) = \alpha$. Then, inspired by the proof of [RS16, Theorem 1.1], we obtain the following result:

Corollary 3.1.8. *Let X be a complex smooth projective variety and let $W \subset H_B^{2k}(X, \mathbb{Z}(k))$ be a sub-Hodge structure. Then $HC_L^k(X, W)_{\mathbb{Z}}$ holds if and only if $HC^k(X, W)_{\mathbb{Q}}$ holds.*

Proof. Let $W \subset \mathrm{H}_B^{2k}(X, \mathbb{Z}(k))$ be a sub-Hodge structure and let $c_L^k : \mathrm{CH}_L^k(X) \to \mathrm{H}_B^{2k}(X, \mathbb{Z}(k))$ be the Lichtenbaum cycle class map constructed in [RS16] (similarly we can consider the classical cycle class map $c^k : \mathrm{CH}^k(X) \to \mathrm{H}_B^{2k}(X, \mathbb{Z})$). Define $\mathrm{CH}_{W,L}^k(X) := (c_L^k)^{-1}(W)$ as the preimage of W in $\mathrm{CH}_L^k(X)$. It is easy to see that $\mathrm{CH}_L^k(X)_{\mathrm{hom}} \hookrightarrow \mathrm{CH}_{W,L}^k(X)$. Following with this notation, we will denote $I_{W,L}^k(X) := \mathrm{im}(c_L^k) \cap W$, therefore W is Lichtenbaum algebraic if and only if $Z_{W,L}^k(X) := W/I_{W,L}^k(X) = 0$. In the classical case, this is equivalent to say that $W_{\mathbb{Q}}$ is algebraic if and only if $Z_W^k(X)$ is a finite group. Since $I_W^k(X) \subset I_{W,L}^k(X)$ we have an exact sequence

$$0 \to I_{W,L}^k(X)/I_W^k(X) \to Z_W^k(X) \to Z_{W,L}^k(X) \to 0$$

Denote $\pi : H_D^{2k}(X, \mathbb{Z}(k)) \to \mathrm{Hdg}^{2k}(X, \mathbb{Z})$ the surjective map coming from the short exact sequence of Deligne-Beilinson cohomology, intermediate Jacobian and Hodge classes and denote $H_{W,D}^{2k}(X, \mathbb{Z}(k)) := \pi^{-1}(W)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}_L^k(X)_{\mathrm{hom}} \arrow[r] \arrow[d, "c_{D,L}^k|_{\mathrm{hom}}"] & \mathrm{CH}_{W,L}^k(X) \arrow[r, "c_L^k"] \arrow[d, "c_{D,L}^k|_{W_L^{-1}}"] & I_{W,L}^k(X) \arrow[r] \arrow[d, "\text{into}"] & 0 \\
0 \arrow[r] & J^k(X) \arrow[r] & H_{W,D}^{2k}(X, \mathbb{Z}(k)) \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{CH}_L^k(X)_{\mathrm{hom}} \otimes \mathbb{Q}/\mathbb{Z} = 0$ by [RS16, Proposition 5.1 (b)] and $J^k(X)$ is divisible, we obtain the commutative diagram but with the torsion part of each group

$$\begin{tikzcd}
0 \arrow[r] & \Big(\mathrm{CH}_L^k(X)_{\mathrm{hom}}\Big)_{\mathrm{tors}} \arrow[r] \arrow[d, "c_{D,L}^k|_{\mathrm{hom}}"] & \Big(\mathrm{CH}_{W,L}^k(X)\Big)_{\mathrm{tors}} \arrow[r, "c_L^k"] \arrow[d, "c_{D,L}^k|_{W_L^{-1}}"] & I_{W,L}^k(X)_{\mathrm{tors}} \arrow[r] \arrow[d, "\text{into}"] & 0 \\
0 \arrow[r] & J^k(X)_{\mathrm{tors}} \arrow[r] & H_{W,D}^{2k}(X, \mathbb{Z}(k))_{\mathrm{tors}} \arrow[r] & W_{\mathrm{tors}} \arrow[r] & 0.
\end{tikzcd}$$

Due to the surjectivity of $\mathrm{CH}_L^k(X)_{\mathrm{tors}} \to H^{2k}(X, \mathbb{Z})_{\mathrm{tors}}$, the right vertical arrow is an isomorphism. If we can prove that the arrow in the middle is surjective, then we can conclude with similar arguments as in [RS16], but this comes from the fact that $\left(\mathrm{CH}^k_L(X)_{\mathrm{hom}}\right)_{\mathrm{tors}} \simeq J^k(X)_{\mathrm{tors}}$ by Corollary 3.1.6, and therefore $c^k_{D,L}|_{\mathrm{hom}}$ induces an isomorphism in the torsion part.

Since we have an isomorphism $\left(\mathrm{CH}^k_{W,L}(X)\right)_{\mathrm{tors}} \simeq H^{2k}_{W,D}(X,\mathbb{Z})_{\mathrm{tors}}$, we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A_{\mathrm{tors}} \arrow[r] \arrow[d, "\simeq"] & A \arrow[r] \arrow[d, "c^k_{D,L}|_A"] & A\otimes\mathbb{Q} \arrow[r] \arrow[d] & A\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\text{into}"] & 0 \\ 0 \arrow[r] & B_{\mathrm{tors}} \arrow[r] & B \arrow[r] & B\otimes\mathbb{Q} \arrow[r] & B\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{3.2}$$

where $A = \mathrm{CH}^k_{W,L}(X)$ and $B = H^{2k}_{W,D}(X,\mathbb{Z}(k))$ and $A\otimes\mathbb{Q}/\mathbb{Z} \hookrightarrow B\otimes\mathbb{Q}/\mathbb{Z}$ is an injection, this can be seen in the computations done in Corollary 3.1.6. We can split diagram (3.2) into two diagrams with short exact sequences as rows:

$$\begin{tikzcd} 0 \arrow[r] & A_{\mathrm{tors}} \arrow[r] \arrow[d, "\simeq"] & A \arrow[r] \arrow[d, "c^k_{D,L}"] & A_{\mathrm{free}} \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & B_{\mathrm{tors}} \arrow[r] & B \arrow[r] & B_{\mathrm{free}} \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & A_{\mathrm{free}} \arrow[r] \arrow[d, "f"] & A\otimes\mathbb{Q} \arrow[r] \arrow[d, "c^k_{D,L}"] & A\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\text{into}"] & 0 \\ 0 \arrow[r] & B_{\mathrm{free}} \arrow[r] & B\otimes\mathbb{Q} \arrow[r] & B\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{3.3}$$

The cokernel from the induced map $A\otimes\mathbb{Q} \to B\otimes\mathbb{Q}$ is torsion free as a quotient of $\mathbb{Q}$-vector spaces. Thus from diagram (3.3), we obtain that $\mathrm{coker}(f)$ is torsion free because it injects into a torsion free group, which, implies that $\mathrm{coker}(c^k_{D,L}|_A)$ is torsion free and, along with the divisibility of $J^k(X)$, so $Z^k_{W,L}(X)$ as well.

The remaining part of the proof consists in proving that $I^k_{W,L}(X)/I^k_W(X)$ is a torsion group, but this comes from the fact that $I^k_{W,L}(X)$ and $I^k_W(X)$ have the same $\mathbb{Z}-$rank and therefore the quotient should be a finite group, so

$$Z^k_W(X)\otimes\mathbb{Q} = 0 \iff Z^k_{W,L}(X)\otimes\mathbb{Q} = 0 \iff Z^k_{W,L}(X) = 0.$$

□

Künneth conjecture

Let us remark a consequence of Corollary 3.1.8. Fixing $k = \mathbb{C}$, it is possible to find an equivalence between the Künneth conjecture in the classical and Lichtenbaum setting, but before state this equivalence we need to define the Künneth conjecture in the general setting.

Let X be a smooth projective variety over a field k of dimension d and consider $\Delta_X \in \mathrm{CH}^d(X\times X)_{\mathbb{Q}}$ the diagonal. Fix a Weil cohomology theory H^*, thanks to the Künneth decomposition of H^* we have

$$c^d(\Delta_X) \in H^{2d}(X\times X) \simeq \bigoplus_{i=0}^{2d} H^{2d-i}(X)\otimes H^i(X).$$

We write $\Delta_i^{\text{topo}} \in H^{2d-i}(X) \otimes H^i(X)$ for the i-th Künneth component.

Conjecture 3.1.9 (Künneth conjecture)**.** *Fixing a Weil cohomology theory H^*, then the Künneth components Δ_i^{topo} are algebraic, i.e. there are algebraic cycle classes $\Delta_i \in CH^d(X \times X)_{\mathbb{Q}}$ such that $c^d(\Delta_i) = \Delta_i^{topo}$.*

Now, consider the Betti realization for the étale setting, this is well-defined with integral coefficients due to the existence of a Lichtenbaum cycle class map, but, in general is not a $\otimes-$functor, because the Künneth decomposition would carry torsion elements, it becomes a $\otimes-$functor after modding out the torsion of the cohomology groups. With integral coefficients, or in a more general setting with coefficients in a principal ideal domain R, the Künneth formula is given by the following short exact sequence

$$0 \to \bigoplus_{p+q=n} H_B^p(X,R) \otimes H_B^q(Y,R) \to H_B^n(X \times Y, R) \to \bigoplus_{p+q=n+1} \operatorname{Tor}_1^R(H_B^p(X,R), H_B^q(Y,R)) \to 0$$

which is natural on X and Y, and also splits, but not canonically (see [Hat02, Theorem 3B.6]). Notice that if R is a field, then the Tor_1^R functor vanishes and we obtain the classical Künneth formula. The same happens if one of the cohomology groups of X or Y is torsion free. The case in which we are interested is when $R = \mathbb{Z}$.

Remark 3.1.10. For $\ell-$adic cohomology there exists a similar Künneth formula, see [Mil80, Theorem 8.21]. Let X, Y two varieties over a field k and let $\ell \neq \operatorname{char}(k)$ then there exists a short exact sequence

$$0 \to \bigoplus_{p+q=n} H_{\text{ét}}^p(X,\mathbb{Z}_\ell) \otimes H_{\text{ét}}^q(Y,\mathbb{Z}_\ell) \to H_{\text{ét}}^n(X \times Y, \mathbb{Z}_\ell) \to \bigoplus_{n+1} \operatorname{Tor}_1^{\mathbb{Z}_\ell}(H_{\text{ét}}^p(X,\mathbb{Z}_\ell), H_{\text{ét}}^q(Y,\mathbb{Z}_\ell)) \to 0$$

which also holds for cohomology with compact support.

For the case when $k = \mathbb{C}$, the restriction to torsion subgroups $c_L^n : \mathrm{CH}_L^n(X)_{\text{tors}} \to H_B^{2n}(X,\mathbb{Z})_{\text{tors}}$ is surjective, therefore it is possible (for this case) to work with a version of the Künneth conjecture modulo torsion.

Conjecture 3.1.11 (Lichtenbaum-Künneth conjecture)**.** *Let X be a smooth projective coplex variety. Then the integral Künneth components $\Delta_i^{topo} \in H_B^{2d-i}(X,\mathbb{Z}) \otimes H_B^i(X,\mathbb{Z})$ are étale algebraic, i.e. there exists a Lichtenbaum cycle $\Delta_i \in CH_{\acute{e}t}^d(X \times X)$ such that $c_{\acute{e}t}^d(\Delta_i) = \Delta_i^{topo}$.*

Proposition 3.1.12. *Over the complex field, the Künneth conjecture holds if and only if the Lichtenbaum version of the Künneth conjecture holds.*

Proof. Consider $X \in \mathrm{SmProj}_{\mathbb{C}}$ of dimension d and let us consider $H_B^{2d}(X \times X, \mathbb{Z})$ modulo torsion. As we consider it modulo torsion, we apply the Künneth decomposition $H_B^{2d}(X \times X, \mathbb{Z}) \simeq \bigoplus H_B^{2d-i}(X,\mathbb{Z}) \otimes H_B^i(X,\mathbb{Z})$ and let $\Delta_i \in H_B^{2d-i}(X,\mathbb{Z}) \otimes H_B^i(X,\mathbb{Z})$ be the i-th component of the diagonal. Let W_i be the sub-Hodge structure generated by Δ_i. By Corollary 3.1.8 W_i is L-algebraic if and only if $W_i \otimes \mathbb{Q}$ is algebraic, therefore the rational Künneth conjecture for X holds if and only if the Künneth components are L-algebraic. ◻

Sign conjecture

This is a weaker version of Künneth conjecture, for more details we refer to [And04, Chapitre 5]. We set $\Delta_+ = \sum_{i=0}^{d} \Delta_{2i}^{\mathrm{topo}}$ called the *even Künneth projector* and its odd counterpart $\Delta_- = \Delta - \Delta_+$. For the following, we fix the base field to $k = \mathbb{C}$, but the following conjecture can be defined for any field and a Weil cohomology theory.

Conjecture 3.1.13. *The even Künneth projector Δ_+ is algebraic, i.e. exists an algebraic cycle $Y \in CH^d(X \times X)_{\mathbb{Q}}$ such that $c^d(Y) = \Delta_+$.*

As in Künneth conjecture, in the complex case the sign conjecture has a Lichtenbaum equivalent:

Conjecture 3.1.14. *The even Künneth projector Δ_+ is Lichtenbaum algebraic, i.e. exists an algebraic cycle $Y \in CH_L^d(X \times X)$ such that $c_L^d(Y) = \Delta_+$.*

The previous conjecture gives the next proposition making the link between both version of the conjecture of signs:

Proposition 3.1.15. *Over the complex field, the sign conjecture holds if and only if the Lichtenbaum version of the conjectur of sings holds.*

Proof. The proof goes in the same way as in the Künneth conjecture using Corollary 3.1.8. □

Standard conjectures of Lefschetz type

Consider a smooth projective variety X over $\mathbb{C}$ and let Y be a smooth hyperplane section. Consider the cohomology class $c^1(Y) \in H_B^2(X, \mathbb{Q})$ and the *Lefschetz operator*

$$\begin{aligned} L : H_B^i(X,\mathbb{Q}) &\to H_B^{i+2}(X,\mathbb{Q}) \\ \alpha &\mapsto \alpha \cup c^1(Y). \end{aligned}$$

The iteration of the operation if denoted by L^r. By the hard Lefschetz theorem

$$L^j : H_B^{d-j}(X,\mathbb{Q}) \xrightarrow{\sim} H_B^{d+j}(X,\mathbb{Q})$$

with $0 \le j \le d$. The hard Lefschetz property defines an unique linear map $\Lambda : H_B^i(X,\mathbb{Q}) \to H_B^{i-2}(X,\mathbb{Q})$, where $2 \le i \le 2d$, in cohomology which makes the following diagram commutes. For $i = d-j$ and $0 \le j \le d-2$ we have

$$\begin{tikzcd} H_B^{d-j}(X,\mathbb{Q}) \arrow[r, "L^j", "\sim"'] \arrow[d, dashed, "\Lambda"] & H_B^{d+j}(X,\mathbb{Q}) \arrow[d, "L"] \\ H_B^{d-j-2}(X,\mathbb{Q}) \arrow[r, "L^{j+2}", "\sim"'] & H_B^{d+j+2}(X,\mathbb{Q}). \end{tikzcd}$$

For $i = d+1$ we obtain the isomorphism $H_B^{d-1}(X,\mathbb{Q}) \underset{\Lambda}{\overset{L}{\rightleftarrows}} H_B^{d+1}(X,\mathbb{Q})$. Lastly, for $i = d+j$ and $2 \leq j \leq d$:

$$\begin{tikzcd} H_B^{d-j+2}(X,\mathbb{Q}) \arrow[r, "L^{j-2}", "\sim"'] & H_B^{d+j-2}(X,\mathbb{Q}) \\ H_B^{d-j}(X,\mathbb{Q}) \arrow[u, "L"] \arrow[r, "L^j", "\sim"'] & H_B^{d+j}(X,\mathbb{Q}). \arrow[u, dashed, "\Lambda"'] \end{tikzcd}$$

Conjecture 3.1.16. *The linear map $\Lambda : H_B^i(X,\mathbb{Q}) \to H_B^{i-2}(X,\mathbb{Q})$ is induced by an algebraic cycle i.e. there exists some $Z \in CH^{d-1}(X \times X)_{\mathbb{Q}}$ such that $\Lambda = c^{d-1}(Z) \in H_B^{2d-2}(X \times X, \mathbb{Q})$.*

Due to the equivalence given in Corollary 3.1.8, we have the following consequence:

Proposition 3.1.17. *There exists an algebraic cycle $Z \in CH^{d-1}(X \times X)_{\mathbb{Q}}$ such that $\Lambda = c^{d-1}(Z)$ if and only exists $Z' \in CH_L^{d-1}(X \times X)$ with $\Lambda = c_L^{d-1}(Z')$.*

Examples

We study two counterexamples of the Hodge conjecture with integral coefficients: the ones presented in [AH62] and [BO20]. Both cases deal with torsion Hodge classes that do not come from algebraic cycles, but the constructions of are different: the first example uses arguments of K-theory, the second one is based on a degeneration argument.

Atiyah-Hirzebruch's countexample

Let us start by giving a quick overview of the construction of Atiyah-Hirzebruch's counterexample presented in [AH62]. They consider a smooth projective quotient variety with a non-algebraic torsion class, that is constructed using the Steenrod algebra and classifying spaces. By [AH62, Theorem 6.1], if a class $\alpha \in H_B^{2p}(X,\mathbb{Z})$ is algebraic then $Sq^i(\bar{\alpha}) = 0$ for all i odd prime, where $\bar{\alpha}$ is the reduction mod 2 and $Sq^i : H_B^{2p}(X,\mathbb{Z}/2) \to H_B^{2p+i}(X,\mathbb{Z}/2)$ is the i-th the Steenrod operation. Also considering [AH62, Proposition 6.6], for every finite group G and $r \in \mathbb{N}_{\geq 1}$ there exists a complete intersection variety Y with $\dim_{\mathbb{C}}(Y) = r$ and G acting freely on Y, such that for the Godeaux-Serre type variety $X = Y/G$, the group cohomology $H^i(G,\mathbb{Z})$ is a direct factor of $H_B^i(X,\mathbb{Z})$ for all $i \leq r$.

As a particular case, consider $G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ and $r = 7$, thus there exists $X \in \mathrm{SmProj}_{\mathbb{C}}$ such that $H^i(G,\mathbb{Z}) \hookrightarrow H_B^i(X,\mathbb{Z})$ as a direct factor for $i \leq 7$. As $H^*(\mathbb{Z}/2,\mathbb{Z}/2) \simeq \mathbb{Z}/2[u]$ with $\deg(u) = 1$, the Künneth formula shows that $H^*(G,\mathbb{Z}/2) \simeq \mathbb{Z}/2[u_1,u_2,u_3]$. Consider the element $u_1u_2u_3 \in H^3(G,\mathbb{Z}/2)$ and $\beta(u_1u_2u_3) =: \alpha \in H^4(G,\mathbb{Z}) \hookrightarrow H_B^4(X,\mathbb{Z})$, where β is the Bockstein's morphism $\beta : H^3(G,\mathbb{Z}/2) \to H^4(G,\mathbb{Z})$, and the following commutative diagram

$$\begin{tikzcd} H^3(G,\mathbb{Z}/2) \arrow[r, "\beta"] \arrow[d, "Sq^1"] & H^4(G,\mathbb{Z}) \arrow[r, hook] & H^4_B(X,\mathbb{Z}(2)) \arrow[d, "\mathrm{red}_2"] \\ H^4(G,\mathbb{Z}/2) \arrow[rr] \arrow[d, "Sq^3"] & & H^4_B(X,\mathbb{Z}/2(2)) \arrow[d, "Sq^3"] \\ H^7(G,\mathbb{Z}/2) \arrow[rr] & & H^7_B(X,\mathbb{Z}/2(4)). \end{tikzcd}$$

A direct computation gives that $Sq^3(Sq^1(u_1u_2u_3)) = Sq^3(\bar{\alpha}) \neq 0 \in H^7_B(X,\mathbb{Z}/2(4))$, and consequently α is a 2-torsion class which is not algebraic. However, we have a short exact sequence

$$0 \to J^2(X) \to H^4_D(X,\mathbb{Z}(2)) \xrightarrow{g} \mathrm{Hdg}^4(X,\mathbb{Z}) \to 0$$

which after tensoring by $\mathbb{Z}/2$, and considering that $J^2(X)$ is divisible, induces a short exact sequence or torsion groups

$$0 \to J^2(X)[2] \to H^4_D(X,\mathbb{Z}(2))[2] \xrightarrow{g} \mathrm{Hdg}^4(X,\mathbb{Z})[2] \to 0$$

and therefore the composite map $\mathrm{CH}^2_L(X)[2] \to H^4_D(X,\mathbb{Z}(2))[2] \to \mathrm{Hdg}^4(X,\mathbb{Z})[2]$ is surjective. Specifically, we have the following result, which gives an explicit representative of the Lichtenbaum class that maps to α.

Claim 3.1.18. *Let X be a Godeaux-Serre variety as the one described previously. Then there exists a class $x \in CH^2_L(X)[2]$ such that $c^2_L(x) = \alpha$ and*

$$red_2(x) = \bar{\alpha} \in ker\left\{H^4_{ét}(X,\mu_2^{\otimes 2}) \to H^5_L(X,\mathbb{Z}(2))\right\}.$$

Also there exists $x \in CH^2_L(X)$ which maps to α; it is the image of $u_1u_2u_3 \in H^3(G,\mathbb{Z}/2)$ in $CH^2_L(X)$.

Proof. Let X be a smooth projective quotient variety coming from the action of $G = (\mathbb{Z}/2)^3$ over Y, with Y satisfying the above hypothesis (a complete intersection variety of dimension 7). We consider the fibration $Y \to X \to BG$ with its associated the Serre spectral sequence $E_2^{p,q} = H^p(BG, H^q(Y,\mathbb{Z}/2)) \Longrightarrow H^{p+q}(X,\mathbb{Z}/2)$, where the differentials are graded derivations. Since Y is a smooth complete intersection variety

$$H^q_{\text{ét}}(Y,\mathbb{Z}/2) \simeq \begin{cases} \mathbb{Z}/2, & \text{if } q \text{ even and } q \neq 7 \\ 0, & \text{if } q \text{ odd and } q \neq 7, \end{cases}$$

therefore if $q \neq 7$, the terms of the Serre spectral sequence are either $H^p(BG,\mathbb{Z}/2) \cong H^p(G,\mathbb{Z}/2)$ or 0. Notice that due to the structure of the $E_2^{p,q}$-terms we have isomorphisms $E_2^{p,q} \simeq E_3^{p,q}$ for $q < 7$. Since d_3 is a graded derivation, then $d_3 : E_3^{0,2} \simeq \mathbb{Z}/2 \to E_3^{3,0}$ is the trivial map, so $E_\infty^{3,0} \simeq E_3^{3,0}$ and therefore $0 \to E_3^{3,0} \to H^3_{\text{ét}}(X,\mathbb{Z}/2)$, which gives us the existence of an injection $H^3(G,\mathbb{Z}/2) \hookrightarrow H^3_B(X,\mathbb{Z}/2(2)) \simeq H^3_{\text{ét}}(X,\mathbb{Z}/2)$. Consider $u_1u_2u_3 \in H^3_B(X,\mathbb{Z}/2(2))$ and the short exact sequence

$$0 \to J^2(X) \to H^4_D(X,\mathbb{Z}(2)) \xrightarrow{g} \mathrm{Hdg}^4(X,\mathbb{Z}) \to 0.$$

Let $\beta(u_1u_2u_3) = \alpha \in H^4_B(X, \mathbb{Z}(2))[2]$ be the non-algebraic torsion class. It can be lifted to an element $\alpha_D \in g^{-1}(\alpha) \subset H^4_D(X, \mathbb{Z}(2))[2]$ because $J^2(X)$ is divisible. It follows from [RS16, Proposition 5.1 (a)] that it has a unique preimage $\widetilde{x} \in \mathrm{CH}^2_L(X)[2]$. Consider the exact triangle $\mathbb{Z}_X(2)_{\text{ét}} \xrightarrow{\cdot 2} \mathbb{Z}_X(2)_{\text{ét}} \to (\mathbb{Z}/2)_X(2)_{\text{ét}} \xrightarrow{+1}$ and the resulting commutative diagram with exact rows

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{CH}^2_L(X) \arrow[r, "\mathrm{red}_2"] \arrow[d] & H^4_{\text{ét}}(X, \mu_2^{\otimes 2}) \arrow[r] \arrow[d, "\simeq"] & H^5_L(X, \mathbb{Z}(2)) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^4_D(X, \mathbb{Z}(2)) \arrow[r] & H^4_B(X, \mathbb{Z}/2(2)) \arrow[r] & H^5_D(X, \mathbb{Z}(2)) \arrow[r] & \dots
\end{tikzcd}$$

It shows that $\mathrm{red}_2(\mathrm{CH}^2_L(X)) = \ker\left\{H^4_{\text{ét}}(X, \mu_2^{\otimes 2}) \to H^5_L(X, \mathbb{Z}(2))\right\}$ and $\mathrm{red}_2(\widetilde{x}) = \bar{\alpha}$. Take again the element $u_1u_2u_3 \in H^3_B(X, \mathbb{Z}/2(2))$ and consider its image in $\mathrm{CH}^2_L(X)$ via the map $p : H^3_B(X, \mathbb{Z}/2(2)) \xrightarrow{\sim} H^3_{\text{ét}}(X, \mu_2^{\otimes 2}) \to \mathrm{CH}^2_L(X)$ (map which is surjective over the 2-torsion of $\mathrm{CH}^2_L(X)$), denoted by $x = p(u_1u_2u_3) \in \mathrm{CH}^2_L(X)[2]$. The last assertion to be proved is that $c^2_L(x) = \alpha$ and $c^2_{L,D}(x) \in g^{-1}(\alpha)$. For this, considering the morphisms and commutative diagrams of remark 3.1.5, we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^2_L(X,1) \otimes \mathbb{Z}/2 \arrow[r] \arrow[d] & H^3_{\text{ét}}(X, \mu_2^{\otimes 2}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{CH}^2_L(X)[2] \arrow[r] \arrow[d, "c^2_{D,L}"] & 0 \\
0 \arrow[r] & H^3_D(X, \mathbb{Z}(2)) \otimes \mathbb{Z}/2 \arrow[r] & H^3_D(X, \mathbb{Z}/2(2)) \arrow[r, "\beta_D"] \arrow[d, "\simeq"] & H^4_D(X, \mathbb{Z}(2))[2] \arrow[r] \arrow[d, "c^2_{D,B}"] & 0 \\
 & & H^3_B(X, \mathbb{Z}/2(2)) \arrow[r, "\beta"] & H^4_B(X, \mathbb{Z}(2))[2] \arrow[r] & 0.
\end{tikzcd}$$

Notice that the image of the map $c^2_{D,B}$ restricted to 2-torsion classes is isomorphic to the image of g restricted to such classes. Hence $c^2_L(x) = \beta(u_1u_2u_3) = \alpha$. ☐

Remark 3.1.19.
1. In [Tot97] Totaro revisited Atiyah-Hirzebruch's example and gave an explanation in terms of complex cobordism: the cycle class map $\mathrm{CH}^i(X) \to H^{2i}_B(X, \mathbb{Z})$ admits a factorization $\mathrm{CH}^i(X) \to \mathrm{MU}^{2i}(X) \otimes_{\mathrm{MU}^{2i}} \mathbb{Z} \to H^{2i}_B(X, \mathbb{Z})$ where $\mathrm{MU}^{2i}(X)$ is the cobordism group of X. Therefore, if a torsion class is not in the image of the map $\mathrm{MU}^{2i}(X) \otimes_{\mathrm{MU}^{2i}} \mathbb{Z} \to H^{2i}_B(X, \mathbb{Z})$ cannot be algebraic. Notice that through cobordism, one can only explain the obstruction to the integral Hodge conjecture when it comes from torsion classes, since in the torsion free part of the cobordism group is isomorphic to the free part of the cohomology group.

2. Totaro used Godeaux-Serre type varieties as an example of a smooth projective variety of dimension 7 such that $\mathrm{CH}^2(X)/2 \to H^4_B(X, \mathbb{Z}/2)$ is not injective and a variety dimension of dimension 15 such that there exists an element $\alpha \in \mathrm{CH}^3(X)$ of order 2 that is mapped to 0 in $H^6_B(X, \mathbb{Z})$ and $J^3(X)$. Here we can find differences with Lichtenbaum cohomology groups.

   a) In Totaro's example $\mathrm{CH}^2(X)/2 \to H^4(X, \mathbb{Z}/2)$ is not injective, while in the Lichtenbaum case, and for all $X \in \mathrm{SmProj}_{\mathbb{C}}$, the sequence

$$0 \to \mathrm{CH}^2_L(X)/2 \to H^4_L(X, \mu_2^{\otimes 2}) \to H^5_L(X, \mathbb{Z}(2))[2] \to 0$$

is exact and also $H^4_L(X,\mathbb{Z}/2(2)) \simeq H^4_{\text{ét}}(X,\mu_2^{\otimes 2}) \simeq H^4_B(X,\mathbb{Z}/2(2))$ by the comparison theorem, therefore $\mathrm{CH}^2_L(X)/2 \to H^4_B(X,\mathbb{Z}/2(2))$ is always injective.

b) In the second of Totaro's examples the condition can't hold for a Lichtenbaum cycle because if $\alpha \in \mathrm{CH}^3_L(X)[2]$ and $\alpha \in \mathrm{CH}^3_L(X)_{\mathrm{hom}}$ then by Corollary 3.1.6 its image in $J^3(X)_{\mathrm{tors}}$ is not zero unless the intermediate Jacobian is zero itself.

Benoist-Ottem counterexample

Let us recall [BO20, Theorem 0.1]. Let S be an arbitrary but fixed complex Enriques surface and let $g \geq 1$ be an integer. Then if B is a very general smooth projective complex curve of genus g, the integral Hodge conjecture for codimension 2 cycles does not hold on the product $B \times S$, and the non-algebraic class is a 2-torsion class. Since the non-algebraic cycle is torsion, it comes from a Lichtenbaum class. In the sequel we give an explicit construction of such a Lichtenbaum cycle.

Let $C \in \mathrm{SmProj}_{\mathbb{C}}$ be a connected curve of genus $g \geq 1$ and S be the previous Enriques surface, we have then a cycle class map $c^2_L : \mathrm{CH}^2_L(C \times S) \to H^4_B(C \times S, \mathbb{Z}(2))$. As is mentioned in [BO20, Proposition 1.1], $H^{2,0}(C \times S) = 0$ because $H^{1,0}(S) = H^{2,0}(S) = 0$, thus the validity of the L-Hodge conjecture for $C \times S$ relies on the surjectivity of the map $c^2_L : \mathrm{CH}^2(C \times S) \to H^4_B(C \times S, \mathbb{Z}(2))$. Since $H^*_B(C,\mathbb{Z})$ is torsion free, we have the following isomorphism obtained from the Künneth formula:

$$H^4_B(C \times S, \mathbb{Z}(2)) \simeq H^0(C,\mathbb{Z}) \otimes H^4(S,\mathbb{Z}) \oplus H^1(C,\mathbb{Z}) \otimes H^3(S,\mathbb{Z}) \oplus H^2(C,\mathbb{Z}) \otimes H^2(S,\mathbb{Z}),$$

where $H^0(C,\mathbb{Z}) \otimes H^4(S,\mathbb{Z})$ is algebraic and by the Lefschetz (1,1) theorem $H^2(C,\mathbb{Z}) \otimes H^2(S,\mathbb{Z})$ is algebraic as well, so L-algebraic. In particular if there exists a non-algebraic class, it should come from $H^1(C,\mathbb{Z}) \otimes H^3(S,\mathbb{Z})$.

Consider the exact sequence of abelian groups

$$0 \to \mathbb{Z}(1) \xrightarrow{\cdot 2} \mathbb{Z}(1) \to \mathbb{Z}/2(1) \to 0$$

which induces a short exact sequence

$$0 \to H^1_B(C,\mathbb{Z}(1)) \otimes \mathbb{Z}/2 \to H^1_B(C,\mathbb{Z}/2(1)) \to H^2_B(C,\mathbb{Z}(1))[2] \to 0.$$

In the case of Lichtenbaum cohomology the sequence of complexes of étale sheaves

$$0 \to \mathbb{Z}_C(1)_{\text{ét}} \xrightarrow{\cdot 2} \mathbb{Z}_C(1)_{\text{ét}} \to (\mathbb{Z}/2)_C(1)_{\text{ét}} \to 0$$

induces a short exact sequence

$$0 \to H^1_L(C,\mathbb{Z}(1)) \otimes \mathbb{Z}/2 \to H^1_L(C,\mathbb{Z}/2(1)) \to \mathrm{CH}^1_L(C)[2] \to 0.$$

Moreover $H^1_L(C,\mathbb{Z}(1)) \otimes \mathbb{Z}/2 = 0$ because $H^1_L(C,\mathbb{Z}(1)) \simeq \mathbb{C}^*$ is divisible, and $H^1_L(C,\mathbb{Z}/2(1)) \simeq H^1_{\text{ét}}(C,\mu_2) \simeq H^1_B(C,\mathbb{Z}/2)$ because of the comparison theorem of cohomologies of complex varieties, and because the cohomology groups of a smooth and projective curve are torsion free $H^2_B(C,\mathbb{Z}(1))[2]=0$, therefore $\mathrm{CH}^1_L(C)[2]\simeq H^1_B(C,\mathbb{Z}(1))\otimes\mathbb{Z}/2$. For the Enriques surface S consider the short exact sequence

$$0\to H^2_L(S,\mathbb{Z}(1))\otimes\mathbb{Z}/2\to H^2_L(S,\mathbb{Z}/2(1))\to H^3_L(S,\mathbb{Z}(1))[2]\to 0$$

where $H^3_L(S,\mathbb{Z}(1))[2]=\mathrm{Br}(S)[2]\simeq\mathrm{Br}(S)\simeq\mathbb{Z}/2$ (see [Bea09, p. 2]) and $H^2_L(S,\mathbb{Z}/2(1))\simeq H^2_{\text{ét}}(S,\mu_2)$. We have a composite map

$$p: H^1_{\text{ét}}(C,\mu_2)\otimes H^2_{\text{ét}}(S,\mu_2)\hookrightarrow H^3_{\text{ét}}(C\times S,\mu_2^{\otimes 2})\to H^4_L(C\times S,\mathbb{Z}(2))[2]$$

where the first inclusion is the one given by the Künneth formula with finite coefficients and the second map is obtained from the short exact sequence

$$0\to\mathrm{CH}^2(C\times S,1)\otimes\mathbb{Z}/2\to H^3_{\text{ét}}(C\times S,\mu_2^{\otimes 2})\to H^4_L(C\times S,\mathbb{Z}(2))[2]\to 0 \tag{3.4}$$

induced by the exact triangle $\mathbb{Z}_{C\times S}(2)_{\text{ét}}\xrightarrow{\cdot 2}\mathbb{Z}_{C\times S}(2)_{\text{ét}}\to(\mathbb{Z}/2)_{C\times S}(2)_{\text{ét}}\xrightarrow{+1}$.

Finally, we need to find an element which is not contained in the image of the induced injection $H^3_B(C\times S,\mathbb{Z}(2))\otimes\mathbb{Z}/2\hookrightarrow H^3_{\text{ét}}(C\times S,\mu_2^{\otimes 2})$. So we can take $c\in H^1_{\text{ét}}(C,\mu_2)$ and denote by $b_S\in\mathrm{Br}(S)$ the non-zero element of the Brauer group of S. We then fix an element $s\in H^2_{\text{ét}}(S,\mu_2)$ such that s maps to b_S through the map $H^2_{\text{ét}}(S,\mu_2)\to\mathrm{Br}(S)$ and define $\gamma_c:=p(c\otimes s)\in\mathrm{CH}^2_L(C\times S)[2]$. In the following result we give an explicit description of the Lichtenbaum classes that maps to a given element of $H^1_B(C,\mathbb{Z})\otimes H^3_B(S,\mathbb{Z})$ in terms of γ_c.

Claim 3.1.20. *Let S be an Enriques surface and C be a smooth projective and connected curve curve of genus $g\geq 1$, let $\widetilde{c}\otimes\widetilde{b}_S\in H^1(C,\mathbb{Z})\otimes H^3(S,\mathbb{Z})$ be an arbitrary class and let $c\in H^1_B(C,\mathbb{Z}/2)$ be the reduction mod 2 of $\widetilde{c}$. Then $c^2_L(\gamma_c)=\widetilde{c}\otimes\widetilde{b}_S$.*

Proof. Let $\mathrm{pr}_1: C\times S\to C$ and $\mathrm{pr}_2: C\times S\to S$ be the canonical projections and consider the induced pull-backs and Bockstein homomorphisms, then we have the following commutative squares

$$\begin{array}{ccc} H^i_B(C,\mathbb{Z}/2) & \xrightarrow{\beta} & H^{i+1}_B(C,\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{pr}_1^*} & & \downarrow{\scriptstyle \mathrm{pr}_1^*} \\ H^i_B(C\times S,\mathbb{Z}/2) & \xrightarrow{\beta} & H^{i+1}_B(C\times S,\mathbb{Z}) \end{array} \qquad \begin{array}{ccc} H^j_B(S,\mathbb{Z}/2) & \xrightarrow{\beta} & H^{j+1}_B(S,\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{pr}_2^*} & & \downarrow{\scriptstyle \mathrm{pr}_2^*} \\ H^j_B(C\times S,\mathbb{Z}/2) & \xrightarrow{\beta} & H^{j+1}_B(C\times S,\mathbb{Z}). \end{array}$$

From now on fix $i=1$, $j=2$ together with $a=\mathrm{pr}_1^*(c)$ and $b=\mathrm{pr}_2^*(s)$ in order to have $a\cup b=c\otimes s$. Since Bockstein homomorphisms satisfy derivation properties (see [Hat02, Section 3.E]), it follows that

$$\begin{aligned}\beta(a\cup b)&=\beta(a)\cup b-(-1)^{\deg(a)}a\cup\beta(b)\\&=\beta(\mathrm{pr}_1^*(c))\cup b-(-1)^{\deg(a)}a\cup\beta(\mathrm{pr}_2^*(s))\\&=\mathrm{pr}_1^*(\beta(c))\cup b-(-1)^{\deg(a)}a\cup\mathrm{pr}_2^*(\beta(s))\\&=\Big(c\times[S]\cup[C]\times\widetilde{b}_S\Big)=c\otimes\widetilde{b}_S\in H^1_B(C,\mathbb{Z})\otimes H^3_B(S,\mathbb{Z})\subset H^4_B(C\times S,\mathbb{Z})[2].\end{aligned}$$

As $c \otimes s \notin \operatorname{im}\left\{H_B^3(C \times S, \mathbb{Z}) \otimes \mathbb{Z}/2 \to H_B^3(C \times S, \mathbb{Z}/2)\right\}$, then neither can be lifted to $H_D^3(C \times S, \mathbb{Z}(2)) \otimes \mathbb{Z}/2$ nor $H_L^3(C \times S, \mathbb{Z}(2)) \otimes \mathbb{Z}/2$ thus $\gamma_c \neq 0$ for all $c \neq 0 \in H_B^1(C, \mathbb{Z}/2)$. Therefore from the commutative diagrams of Remark 3.1.5, we obtain $c_L^2(\gamma_c) = \beta(c \otimes s)$ giving the characterization of the preimages of the elements in $H_B^1(C, \mathbb{Z}) \otimes H_B^3(S, \mathbb{Z})$. □

Remark 3.1.21. By the same kind of arguments as in [BO20, Proposition 1.1], mutatis mutandis, we can obtain an equivalence between the action of Lichtenbaum correspondences and the L-Hodge conjecture, i.e. the Lichtenbaum Hodge conjecture holds for codimension 2-cycles in the product if and only if for every $c \in H^1(C, \mathbb{Z}/2)$ there exists a correspondence $Z \in \mathrm{CH}_L^2(C \times S)$ such that $Z^*\alpha = c$, where $\alpha \in H_B^1(S, \mathbb{Z}/2)$ is the class corresponding to the degree 2 étale cover of S by a K3-surface. Since $c_L^2(\gamma_c) = c \otimes \widetilde{b}_S$ we have that

$$\begin{aligned}\gamma_c^*\alpha &= \mathrm{pr}_{1*}(c_L^2(\gamma_c) \cup \mathrm{pr}_2^*(\alpha))\\ &= \mathrm{pr}_{1*}(\mathrm{pr}_1^*(c) \cup \mathrm{pr}_2^*(\widetilde{b}_S) \cup \mathrm{pr}_2^*(\alpha))\\ &= \mathrm{pr}_{1*}(\mathrm{pr}_1^*(c) \cup \mathrm{pr}_2^*(\widetilde{b}_S \cup \alpha)).\end{aligned}$$

By Poincaré duality we have that $\widetilde{b}_S \cup \alpha$ is a non-zero element in $H^4(S, \mathbb{Z}/2)$, then $\gamma_c^*\alpha = c$.

Kollár's counterexample

As we have said, the situation with the Hodge conjecture does not improve if we consider just the free part of the cohomology, due to Kollár's example [BCC92]. We will start by giving some general facts about smooth hypersurfaces. Consider a smooth hypersurface $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ of degree d. By the Lefschetz hyperplane theorem ([Voi02, Théorème 13.23]) if $k < n$ then we have the isomorphism

$$H_B^k(\mathbb{P}^{n+1}, \mathbb{Z}) \xrightarrow{i^*} H_B^k(X, \mathbb{Z})$$

and if $k = n$, the map i^* is an injection. Here $H_B^{2k}(\mathbb{P}^{n+1}, \mathbb{Z}) \simeq \mathbb{Z}H^k$ with $H = c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$ and $H_B^{2k+1}(\mathbb{P}^{n+1}, \mathbb{Z}) = 0$. Since Betti cohomology groups of hypersurfaces (with integral coefficients) are torsion free, by Poincaré duality we obtain the isomorphisms

$$H_B^{2k}(X, \mathbb{Z})^* \simeq H_B^{2(n-k)}(X, \mathbb{Z}).$$

In particular if $2k > n$ then $H_B^{2k}(X, \mathbb{Z}) \simeq \mathbb{Z}\alpha$ where $\langle \alpha, h^{n-k} \rangle = 1$ with $\langle \cdot, \cdot \rangle$ being the intersection product and $h = c_1(\mathcal{O}_X(1)) = H\big|_X$.

Remark 3.1.22. By the Lefschetz hyperplane section in étale cohomology (see [Mil80, Chapter VI, §7]) the map $H^i(X, \mu_{\ell^r}^{\otimes k}) \xrightarrow{i_*} H^{i+2}(\mathbb{P}_{\mathbb{C}}^{n+1}, \mu_{\ell^r}^{\otimes k+1})$ is an isomorphism if $i > n$ and a surjection if $i = n$.

In the following proposition, we give characterizations for some of the Lichtenbaum cohomology groups of a smooth hypersurfaces X in $\mathbb{P}_{\mathbb{C}}^{n+1}$ and study the close relation with étale cohomology groups with finite coefficients.

Proposition 3.1.23. *Let $i : X \hookrightarrow \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth and projective hypersurface of degree d and let $j : (U := \mathbb{P}^{n+1}_{\mathbb{C}} \backslash X) \to \mathbb{P}^{n+1}_{\mathbb{C}}$ be the open complement. Let k be an integer with $0 \leq k \leq n$ such that $2k \notin \{n-1, n\}$, then:*

1. *The higher Brauer group $Br^k(X) := H^{2k+1}_L(X, \mathbb{Z}(k))$ and the group $H^{2k+1}_D(X, \mathbb{Z}(k))_{tors}$ are trivial.*
2. *If $2k > n$ then $CH^{k+1}(U) \simeq Z^{2k}(X)$ and $CH^{k+1}_{ét}(U) \simeq Z^{2k}_L(X)$ and in particular the group $CH^{k+1}_L(U)$ is trivial.*

Proof. Let us start with the first statement. Since our base field is of characteristic zero, fix an arbitrary prime number ℓ and an (also arbitrary) natural number r. As in [RS16, Proposition 3.1] consider the long exact sequence

$$\ldots \to \mathrm{CH}^k_{\text{ét}}(X) \xrightarrow{\cdot \ell^r} \mathrm{CH}^k_{\text{ét}}(X) \to H^{2k}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k}) \to \mathrm{Br}^k(X) \xrightarrow{\cdot \ell^r} \mathrm{Br}^k(X) \to H^{2k+1}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k}) \to \ldots$$

By assumption $2k+1 \neq n$, hence $H^{2k+1}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k}) = 0$. Therefore the map $\mathrm{Br}^k(X) \xrightarrow{\cdot \ell^r} \mathrm{Br}^k(X)$ is surjective for every prime number ℓ and for all r, thus $\mathrm{Br}^k(X)$ is divisible and a torsion group. For the remaining part, we consider the commutative diagram given in Remark 3.1.5 and the short exact sequence

$$0 \to J^k(X) \to H^{2k}_D(X, \mathbb{Z}(k)) \to \mathrm{Hdg}^{2k}(X, \mathbb{Z}) \to 0.$$

Since $J^k(X)$ is divisible, then $H^{2k}_D(X, \mathbb{Z}(k))/\ell^r \simeq \mathrm{Hdg}^{2k}(X, \mathbb{Z})/\ell^r$. Under the conditions for k we have the isomorphisms $H^{2k}_B(X, \mathbb{Z}(k)) \simeq \mathrm{Hdg}^{2k}(X, \mathbb{Z}(k))$ and since $H^{2k+1}_B(X, \mathbb{Z}) = 0$ then by the diagram of Remark 3.1.5 we conclude that $H^{2k+1}_D(X, \mathbb{Z}(k))[\ell^r] = 0$.

The short exact sequence

$$0 \to \mathrm{CH}^k_L(X)/\ell^r \to H^{2k}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k}) \to \mathrm{Br}^k(X)[\ell^r] \to 0$$

gives a surjective map $\mathbb{Z}/\ell^r \to \mathrm{Br}^k(X)[\ell^r]$. Taking the direct limit, we obtain a surjection $\mathbb{Q}/\mathbb{Z} \twoheadrightarrow \mathrm{Br}^k(X)$. As $\mathrm{Br}^k(X) \simeq (\mathbb{Q}/\mathbb{Z})^r$ (for the structure of Lichtenbaum cohomology see [Gei17, Theorem 1.1]) for some $r \in \mathbb{N}$ we have that $r = 0$ or 1. Since $I^{2k}(X) \neq 0$ we have $I^{2k}_L(X) \neq 0$ and hence there are isomorphisms $\mathrm{CH}^k_L(X) \otimes \mathbb{Q}/\mathbb{Z} \simeq I^{2k}_L(X) \otimes \mathbb{Q}/\mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z}$ so $\mathrm{Br}^k(X) = 0$.

For part 2. consider the localization sequence for Chow groups and its étale analogue. By functoriality of the comparison map with Gysin morphisms and pull-backs we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^{k+1}(U,1) \arrow[r, "\partial"] \arrow[d] & \mathrm{CH}^k(X) \arrow[r, "i_*"] \arrow[d] & \mathrm{CH}^{k+1}(\mathbb{P}^{n+1}_{\mathbb{C}}) \arrow[r, "j^*"] \arrow[d, "\simeq"] & \mathrm{CH}^{k+1}(U) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}^{k+1}_L(U,1) \arrow[r, "\partial_{\text{ét}}"] & \mathrm{CH}^k_L(X) \arrow[r, "i_*"] & \mathrm{CH}^{k+1}_L(\mathbb{P}^{n+1}_{\mathbb{C}}) \arrow[r, "j^*"] & \mathrm{CH}^{k+1}_L(U) \arrow[r] & 0.
\end{tikzcd}$$

Notice that the map $\mathrm{CH}^{k+1}_L(\mathbb{P}^{n+1}_{\mathbb{C}}) \xrightarrow{j^*} \mathrm{CH}^{k+1}_L(U)$ is not surjective in general, but in this case it is surjective as a consequence of part 1. By the functorial properties of the usual cycle class map we have the following commutative square

$$\begin{array}{ccc} \mathrm{CH}_{n-k}(X) & \xrightarrow{i_*} & \mathrm{CH}_{n-k}(\mathbb{P}^{n+1}_{\mathbb{C}}) \\ \downarrow{\scriptstyle c_{n-k}} & & \downarrow{\scriptstyle \simeq} \\ H_{2(n-k)}(X,\mathbb{Z}) & \xrightarrow{i_*} & H_{2(n-k)}(\mathbb{P}^{n+1}_{\mathbb{C}},\mathbb{Z}) \end{array}$$

where if $2(n-k) < n$ then the map $i_* : H_{2(n-k)}(X,\mathbb{Z}) \to H_{2(n-k)}(\mathbb{P}^{n+1}_{\mathbb{C}},\mathbb{Z})$ is an isomorphism. Therefore $i_*\left(\mathrm{CH}^k(X)\right) \simeq \mathrm{im}(c^k)$, hence using the previous commutative diagrams, which relates localization exact sequences, it is easy to see that

$$\begin{aligned} \mathrm{CH}^{k+1}(U) &\simeq \mathrm{coker}(i_*) \\ &\simeq H_{2(n-k)}(X,\mathbb{Z})/\mathrm{im}(c_{n-k}) = Z^{2k}(X). \end{aligned}$$

For the étale case we proceed in the same way. The last part of the second statement is due to the fact that $i_*\left(\mathrm{CH}^k_L(X)\right) \simeq \mathrm{CH}^{k+1}_L(\mathbb{P}^{n+1}_{\mathbb{C}}) \simeq \mathbb{Z}$ therefore $j^* : \mathrm{CH}^{k+1}_L(\mathbb{P}^{n+1}_{\mathbb{C}}) \to \mathrm{CH}^{k+1}_L(U)$ has trivial image, thus we conclude that $\mathrm{CH}^{k+1}_L(U)$ injects into $\mathrm{Br}^k(X)$, which is trivial by the first point. □

Corollary 3.1.24. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth projective hypersurface of degree d and let k be an integer such that $2k \notin \{n-1, n\}$. Then we have the following*

1. *For all prime numbers ℓ and all r the cycle class map $c^k_{L,\ell^r} : CH^k_L(X) \to H^{2k}_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes k})$ is surjective.*

2. *For all prime numbers ℓ, all r and k such that $2k \notin \{n-1, n, n+1\}$ the pairing*

$$CH^{n-k}_L(X)/\ell^r \otimes CH^k_L(X)/\ell^r \to CH^n_L(X)/\ell^r \simeq \mathbb{Z}/\ell^r$$

is non-degenerate. The result also holds for the particular case $n = 3$ and $k = 2$.

3. *If $2k > n$ there exists a class $z \in CH^k_L(X)$ such that $i_*(z) = H^{k+1} \in CH^{k+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ where H^{k+1} is the generator of $CH^{k+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$. Furthermore $c^k_{\ell^r}(z)$ and $c^k_L(z)$ are the generators of the groups $H^{2k}_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes k})$ and $H^{2k}_B(X, \mathbb{Z}(k))$ respectively.*

Proof. This is a direct consequence of Proposition 3.1.23. Fix arbitrary prime and natural numbers denoted ℓ and r respectively. For part 1. use the long exact sequence

$$\ldots \to \mathrm{CH}^k_L(X) \xrightarrow{\cdot \ell^r} \mathrm{CH}^k_L(X) \to H^{2k}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k}) \to \mathrm{Br}^k(X) \xrightarrow{\cdot \ell^r} \mathrm{Br}^k(X) \to \ldots$$

and obtain the surjectivity since $\mathrm{Br}^k(X) = 0$. The second part follows from the vanishing of $\mathrm{Br}^k(X)$. Hence we obtain an isomorphism $\mathrm{CH}^k_L(X) \otimes \mathbb{Z}/\ell^r \xrightarrow{\simeq} H^{2k}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k})$ and the same for codimension $n-k$. Thus the non-degeneracy comes from Poincaré duality in étale cohomology. For the case $n = 3$ and $k = 2$, use that $\mathrm{CH}^1_L(X) \simeq \mathrm{CH}^1(X) \simeq \mathbb{Z} \cdot c_1(\mathcal{O}_X(1))$, then $\mathrm{CH}^1_L(X) \otimes \mathbb{Z}/\ell^r \simeq H^2_{\text{ét}}(X, \mu_{\ell^r})$. While $\mathrm{CH}^2_L(X) \otimes \mathbb{Z}/\ell^r \simeq H^4_{\text{ét}}(X, \mu_{\ell^r}^{\otimes 2})$ by Corollary 3.1.24.1. The last assertion follows from the localization sequence, the vanishing of $\mathrm{Br}^k(X)$ and the compatibility of the cycle class maps with push-forwards. □

Remark 3.1.25. 1. If $n = 3$ and $k = 2$ then $\mathrm{Hdg}^4(X, \mathbb{Z}) = \mathbb{Z}$ while the image of the cycle class map $I^4(X) = I \subset \mathbb{Z}$ is determined by the degrees of the curves $C \subset X$, i.e. $I = \gcd(\{\deg(C) \mid C \subset X\})\,\mathbb{Z}$. The strategy for the counter-example to the integral Hodge conjecture is as follows: consider a very general hypersurface X of degree $d = sp^3$ with p a prime number ≥ 5, Kollár proved that under these assumptions for every curve $C \subset X$ its degree $\deg(C)$ is divisible by p and therefore $Z^4(X) = \mathbb{Z}/m \neq 0$ with m divisible by p. Notice that if $d > 6$ the Griffiths-Harris conjecture would imply that $m = d$. Here the class α is not algebraic, whereas $d\alpha = h^2$.

2. Motivic and Lichtenbaum cohomology behave differently when we work with finite coefficients. In general, for $j \in \mathbb{N}$, $H^{2j+1}_M(X, \mathbb{Z}(j)) = 0$ so $\mathrm{CH}^j(X) \otimes \mathbb{Z}/\ell^r \xrightarrow{\simeq} H^{2j}_M(X, \mathbb{Z}/\ell^r(j))$. By Bloch-Ogus we know that $\mathrm{CH}^j(X) \otimes \mathbb{Z}/\ell^r \simeq A^j(X) \otimes \mathbb{Z}/\ell^r$, where $A^j(X)$ is the group of codimension j cycles of X modulo algebraic equivalence. Again consider $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ and k as in part 2 of Corollary 3.1.24. We obtain a commutative diagram

$$\begin{tikzcd}
H^{2k}_M(X, \mathbb{Z}/\ell^r(k)) \otimes H^{2(n-k)}_M(X, \mathbb{Z}/\ell^r(n-k)) \arrow[r, "\cup"] \arrow[d] & H^{2n}_M(X, \mathbb{Z}/\ell^r(n)) \arrow[r, "\deg_{\ell^r}"] \arrow[d, "\simeq"] & \mathbb{Z}/\ell^r \arrow[d, equal] \\
H^{2k}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k}) \otimes H^{2(n-k)}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n-k}) \arrow[r, "\cup"] & H^{2n}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n}) \arrow[r, "\mathrm{tr}_{\ell^r}"] & \mathbb{Z}/\ell^r
\end{tikzcd}$$

where the pairing in the lower row is non-degenerate because of Poincaré duality, whereas the one in the upper row could be degenerate as Kóllar's example shows or as Griffiths-Harris' conjecture states. By Proposition 3.1.23 there is an isomorphism $\mathrm{CH}^k_L(X) \otimes \mathbb{Z}/\ell^r \xrightarrow{\simeq} H^{2k}_{\text{ét}}(X, \mu_{\ell^r}^{\otimes k})$, thus (if $2k > n$) we always have an element of degree 1 in the Lichtenbaum groups.

3.2 Generalized Hodge conjecture

Let H be a pure Hodge structure of weight n and let $0 \neq H_{\mathbb{C}} = H \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$. We say that H is effective if and only if $H^{p,q} = 0$ for $p < 0$ or $q < 0$. The level of l of H is defined as $l = \max\{|p - q| \mid H^{p,q} \neq 0\}$. Let X be a smooth projective complex variety, we write $\mathrm{GHC}(n, c, X)_{\mathbb{Q}}$ for the generalized Hodge conjecture in weight n and level $n - 2c$:

Conjecture 3.2.1 ([Gro69, Generalized Hodge conjecture])**.** *For every $\mathbb{Q}-$sub-Hodge structure $H \subset H^n(X, \mathbb{Q})$ of level $n - 2c$ there exists a subvariety $Y \subset X$ of pure codimension c such that H is supported on Y, i.e. H is contained in the image of*

$$H \subset im\left\{H^l(\widetilde{Y}, \mathbb{Q}(-c)) \xrightarrow{\gamma_*} H^n(X, \mathbb{Q})\right\}$$

where $\gamma_* = i_* \circ d_*$, i_* is the Gysin map associated to the inclusion $i : Y \hookrightarrow X$ and $d : \widetilde{Y} \to Y$ is a resolution of singularities.

There is an equivalent assertion of the generalized Hodge conjectures, in terms of algebraic cycles; for a proof, we refer to [Sch89, Lemma 0.1].

Conjecture 3.2.2. *If $H \subset H^n(X,\mathbb{Q})$ is a $\mathbb{Q}$-sub-Hodge structure of level $l = n-2c$, then GHC(n,c,X) holds for H if and only if there exist a smooth projective complex variety Y and a correspondence $z \in Corr^c(Y,X)$ such that H is contained in $z_*H^l(Y,\mathbb{Q})$.*

Note that this conjecture, similarly to the Hodge conjecture, can be stated in terms of classical motives over $\mathbb{C}$:

Proposition 3.2.3. *[Gro69, Page 301] The generalized Hodge conjecture for all $X \in SmProj_{\mathbb{C}}$ is equivalent to the following statement: the Hodge conjecture holds and a homological motive is effective if and only if its Hodge realization is effective.*

Proof. Suppose that the generalized Hodge conjecture holds, this implies immediately the Hodge conjecture. Consider a pure motive M such that its realization $H := \rho_H(M)$ is effective of weight k and coniveau c, or equivalently its level is $l = k-2c$. Then there exists a closed subvariety $Y \hookrightarrow X$ of codimension c such that H is contained in the image of the composition map

$$H \subset \operatorname{im}\left\{H^l(\widetilde{Y},\mathbb{Q}(-c)) \xrightarrow{\gamma_*} H^k(X,\mathbb{Q})\right\},$$

where $\gamma_* = i_* \circ d_*$, i_* is the Gysin map associated to the inclusion $i : Y \hookrightarrow X$ and $d : \widetilde{Y} \to Y$ is a resolution of singularities. There exists an integer n such that $M(n)$ is effective. Hence we can recover $M(n)$ as a sub-object of $h(\widetilde{Y})$ which then implies that M is effective because is a sub-object of $h(X)$.

On the other hand, suppose that $H \subset H^n(X,\mathbb{Q})$ is a sub-Hodge structure of weight n and level $l = n-2c$, then $H(c)$ is still an effective Hodge structure, by effectiveness hypotheses there exists Y smooth and projective variety such that $H(c)$ is a quotient Hodge structure of $H^{n-2c}(Y,\mathbb{Q})$. Since the category of $\mathbb{Q}$-polarized Hodge structures is semi-simple we obtain a decomposition

$$H^{n-2c}(Y,\mathbb{Q}) \simeq H(c) \oplus R$$

which gives us a morphism of Hodge structures $f : H^{n-2c}(Y,\mathbb{Q}(-c)) \to H^n(X,\mathbb{Q})$ defined by

$$H^{n-2c}(Y,\mathbb{Q}) \xrightarrow{\mathrm{pr}_1} H(c) \xrightarrow{\mathrm{id}\otimes\mathbb{Q}(-c)} H \hookrightarrow H^n(X,\mathbb{Q}).$$

Such a morphism f contains H in its image. Furthermore, note that there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{HS}\mathbb{Q}}(H^{n-2c}(Y,\mathbb{Q}), H^n(X,\mathbb{Q})) \simeq \mathrm{Hdg}^{2(d_Y+c)}(Y \times X,\mathbb{Q}).$$

Therefore by the assumption of the Hodge conjecture, the map f is induced by a correspondence $\gamma \in \mathrm{CH}^{d_Y+c}(Y \times X)_{\mathbb{Q}}$. Thus the generalized Hodge conjecture holds. $\square$

Generalized Hodge conjecture and Lichtenbaum cohomology

Based on the previous reformulation of the generalized Hodge conjecture, Rosenschon and Srinivas proposed the following variant of the generalized Hodge conjecture for integral coefficients, using Lichtenbaum cohomology groups:

Conjecture 3.2.4 (L-Generalized Hodge Conjecture)**.** *Let X be a smooth projective complex variety. If $H \subset H^n(X,\mathbb{Z})$ is a $\mathbb{Z}$-sub-Hodge structure of level $l = n-2c$, then $GHC_L(n,c,X)$ holds for H if and only if there exist a smooth projective complex variety Y and an element $z \in Corr_L^c(Y,X)$ such that H is contained in $z_*H^l(Y,\mathbb{Z})$.*

For a smooth projective complex variety X, conjecture 3.2.4 is denoted by $\mathrm{GHC}_L(n,c,X)_{\mathbb{Q}}$. In some particular cases it is known to be equivalent to $\mathrm{GHC}(n,c,X)_{\mathbb{Q}}$. For instance if we consider $\mathrm{GHC}(2k-1,k-1,X)_{\mathbb{Q}}$ in [Gro69, §2] it was mentioned that with this level and weights it is related to the usual Hodge conjecture:

Proposition 3.2.5. *[Lew99, Remark 12.30] Let X be a smooth projective complex variety. Then $GHC(2k-1,k-1,X)_{\mathbb{Q}}$ holds if and only if $\big(H^{2k-1}(X,\mathbb{Q}) \otimes H^1(\Gamma,\mathbb{Q})\big) \cap H^{k,k}(\Gamma \times X)$ is algebraic for every smooth projective complex curve Γ.*

The Lichtenbaum version of the previous result still holds as is stated in [RS16, Remarks 5.2]; the proof uses arguments similar to the ones presented in [Lew99, Remark 12.30]. Before we go into the proof of the proposition, it is necessary to introduce some notation and conventions. First Betti cohomology is considered modulo torsion. Define

$$H^{2k-1}_{\text{L-alg}}(X,\mathbb{Z}) := \Big\{\sigma_* : H^1(Y,\mathbb{Z}) \to H^{2k-1}(X,\mathbb{Z}) |\ \sigma \in \mathrm{Corr}_L^{k-1}(Y,X),\ \dim Y = 1\Big\}/\mathrm{tors}$$

where Y is smooth and projective, and recall

$$H^{2k-1}_{\max}(X,\mathbb{Z}) = \Big\{\text{the largest } \mathbb{Z}\text{-sub HS in } \Big\{H^{k,k-1}(X) \oplus H^{k-1,k}(X)\Big\} \cap H^{2k-1}(X,\mathbb{Z})\Big\}$$

The generalized Hodge conjecture $\mathrm{GHC}_L(2k-1,k-1,X)$ states that these are equal. Note that $H^{2k-1}_{\text{L-alg}}(X,\mathbb{Z}) \otimes \mathbb{C} = H^{k,k-1}_{\text{L-alg}}(X) \oplus H^{k-1,k}_{\text{L-alg}}(X)$ because of the Hodge decomposition. Also there exists a partial version of the previous result, which asks whether or not a sub-Hodge structure $W \subset H^{2k-1}(X,\mathbb{Z})$ is contained in the image of the action of some Lichtenbaum correspondence over cohomology groups.

In the following proposition, we characterize this partial étale version of the generalized Hodge conjecture of a Hodge structure of weight $2k-1$ and level 1, and give a general description of the $\mathrm{GHC}_L(2k-1,k-1,X)$ and its equivalence to $\mathrm{GHC}(2k-1,k-1,X)_{\mathbb{Q}}$, as is stated in [RS16, Remark 5.2]:

Proposition 3.2.6. *Let $X \in SmProj_{\mathbb{C}}$, $k \in \mathbb{N}_{\geq 1}$ and let $W \subset H^{2k-1}(X,\mathbb{Z})$ be a sub-Hodge structure of level 1. Then:*

(i.) there exist $Y \in SmProj_{\mathbb{C}}$ and a Lichtenbaum correspondence $z \in CH_L^{d_Y+1}(Y \times X)$ such that $W \subset z_*H^1(Y,\mathbb{Z})$ if and only if for all curves $C \in SmProj_{\mathbb{C}}$ the Hodge classes $H^{k,k}(C \times X) \cap \big\{H^1(C,\mathbb{Z}) \otimes W\big\}$ are algebraic.

(ii.) In particular $GHC_L(2k-1,k-1,X)$ holds if and only if for all curves $C \in SmProj_{\mathbb{C}}$ the Hodge classes $H^{k,k}(C\times X)\cap\{H^1(C,\mathbb{Z})\otimes H^{2k-1}(X,\mathbb{Z})\}$ are L-algebraic, i.e.

$$H^{k,k}(C\times X)\cap\Big\{H^1(C,\mathbb{Z})\otimes H^{2k-1}(X,\mathbb{Z})\Big\}\subset im\Big\{c_L^k : CH_L^k(C\times X)\to H_B^{2k}(X,\mathbb{Z}(k))\Big\}.$$

(iii.) $GHC(2k-1,k-1,X)_{\mathbb{Q}}$ holds if and only if $GHC_L(2k-1,k-1,X)$ holds.

Proof. For (i.), let $W\subset H^{2k-1}(X,\mathbb{Z})$ be a sub-Hodge structure of weight $2k-1$ and level 1 and assume that there exists $Y\in \mathrm{SmProj}_{\mathbb{C}}$ and a Lichtenbaum correspondence $z\in \mathrm{CH}_L^{d_Y+k-1}(Y\times X)$ (correspondence of degree $k-1$) such that $W\subset z_*H^1(Y,\mathbb{Z})$. Consider C a smooth complex projective curve and consider an element $h\in\{H^1(Y,\mathbb{Z})\otimes W\}\cap H^{k,k}(C\times X)\simeq \mathrm{Hom}_{\mathrm{HS}}(H^1(C,\mathbb{Z}),W)$. Let $h_*: H^1(C,\mathbb{Z})\to W$ be the map of Hodge structures induced by h. Define $V=\ker(z_*)$, which by the theory of Hodge structures, is a Hodge structure itself. We know that the image of $\mathrm{im}(z_*)$ is a Hodge structures of the same weight (see [Voi02, Lemme 7.23 et 7.25]), then $H^1(Y,\mathbb{Z})=V\oplus R$ where $R=V^{\perp}$. Then we have a morphism $\lambda:=(z_*|_R)^{-1}\Big|_W : \mathrm{im}(z_*)\cap W\subset H^{2k-1}(X,\mathbb{Z})\to R$ which fits into the following commutative diagram

$$\begin{tikzcd}
H^1(C,\mathbb{Z}) \arrow[r, "h_*"] \arrow[d, "h_*"] & W \arrow[r, equal] & \mathrm{im}(z_*)\cap W \arrow[r, "\lambda"] & H^1(Y,\mathbb{Z}) \arrow[d, "z_*"] \\
W \arrow[rrr, equal] & & & W.
\end{tikzcd}$$

The map obtained obtained from the upper arrows is induced by a Hodge class by Lefschetz (1,1) and therefore h is algebraic.

Conversely, suppose that for all smooth and projective curve C the Hodge classes $H^{k,k}(C\times X)\cap\{H^1(C,\mathbb{Z})\otimes W\}$ are algebraic. Let $W\subset H^{2k-1}(X,\mathbb{Z})$ be a sub-Hodge structure of level 1 and notice that W has a decomposition as $W\otimes\mathbb{C}=W^{k,k-1}\oplus W^{k-1,k}$. Then its associated k-th intermediate Jacobian is of the form $J^k(W)=W^{k-1,k}/W$ which is an abelian variety. Since $J^k(W)$ is a complex torus, then its holomorphic tangent bundle is $W^{k-1,1}$ and the fundamental group is isomorphic to the lattice W, thus $\pi_1(J^k(W))\simeq H_1(J^k(W),\mathbb{Z})=W$. Set $m=\dim(J^k(W))$ then $H^{2m-1}(J^k(W),\mathbb{C})=H^{m-1,m}(J^k(W))\oplus H^{m,m-1}(J^k(W))$ and

$$\begin{aligned}
H^{m-1,m}(J^k(W)) &\simeq H^{1,0}(J^k(W))^*\\
&= H^0(J^k(W),\Omega^1_{J^k(W)})^*\\
&\simeq H^0(J^k(W),\Omega^1_{J^k(W)})^*\\
&\simeq H^0(J^k(W),T^*_{J^k(W)})^*\simeq W^{k-1,k}.
\end{aligned}$$

then $H^{2m-1}(J^k(W),\mathbb{C})=W^{k-1,k}\oplus W^{k,k-1}$.

Taking hyperplane sections of $J^k(W)$ and applying Bertini's theorem, we find a smooth projective curve $\Gamma\subset J^k(W)$ and a surjective map $H_1(\Gamma,\mathbb{Z})\to H_1(J^k(W),\mathbb{Z})\simeq W$. Also by Poincaré duality $H_1(\Gamma,\mathbb{Z})\simeq H^1(\Gamma,\mathbb{Z})$ so we have a surjective map $f: H^1(\Gamma,\mathbb{Z})\to W$. Since the map f is a morphism of Hodge structures, then it is an element in $H^{k,k}(\Gamma\times X)\cap\{H^1(\Gamma,\mathbb{Z})\otimes W\}$ which by hypothesis is L-algebraic. Therefore there exists a class $z\in \mathrm{CH}_L^{2k}(\Gamma\times X)$ such that $W\subset z_*H^1(\Gamma,\mathbb{Z})\subset H^{2k-1}(X,\mathbb{Z})$.

The statement (ii.) is a direct consequence of (i.) taking $W = H^{2k-1}(X, \mathbb{Z})$ and the maximal sub-Hodge structure of it $H^{2k-1}_{\max}(X, \mathbb{Z})$. For (iii.) notice that for a complex smooth projective curve C the Betti cohomology groups are torsion free. Thus Künneth formula holds for the product $C \times X$ and then

$$H^{k,k}(C \times X) \cap \Big\{H^1(C, \mathbb{Z}) \otimes H^{2k-1}(X, \mathbb{Z})\Big\} \subset H^{k,k}(C \times X) \cap H^{2k}(C \times X, \mathbb{Z}) = \mathrm{Hdg}^{2k}(C \times X, \mathbb{Z}).$$

Invoking Corollary 3.1.8, the Hodge classes $H^{k,k}(C \times X) \cap \{H^1(C, \mathbb{Z}) \otimes H^{2k-1}(X, \mathbb{Z})\}$ are L-algebraic if and only if $H^{k,k}(C \times X) \cap \{H^1(C, \mathbb{Q}) \otimes H^{2k-1}(X, \mathbb{Q})\}$ are algebraic in the usual sense, which gives us the equivalences

$$\begin{aligned}
&\mathrm{GHC}(2k-1, k-1, X)_{\mathbb{Q}} \text{ holds} \\
&\qquad \iff H^{k,k}(C \times X) \cap \Big\{H^1(C, \mathbb{Q}) \otimes H^{2k-1}(X, \mathbb{Q})\Big\} \text{ is alg. } \forall \text{ curve } C \\
&\qquad \iff H^{k,k}(C \times X) \cap \Big\{H^1(C, \mathbb{Z}) \otimes H^{2k-1}(X, \mathbb{Z})\Big\} \text{ is L-alg. } \forall \text{ curve } C \\
&\qquad \iff \mathrm{GHC}_L(2k-1, k-1, X) \text{ holds.}
\end{aligned}$$

□

In the sequel, we give more subtle relations between the Hodge conjecture and the generalized one, following the proof of the classical case given in [Fu12, Lemma 2.3]:

Lemma 3.2.7. *Let X be a smooth projective variety of dimension n and $H \subset H^k(X, \mathbb{Z})$ be a sub-Hodge structure of coniveau at least c and assume that there exists a smooth projective variety Y of dimension d_Y, such that $H(c)$ is a sub-Hodge structure of $H^{k-2c}(Y, \mathbb{Z})$. If $H^{d_Y+c, d_Y+c}(Y \times X) \cap \{H^{2(d_Y+c)-k}(Y, \mathbb{Z}) \otimes H^k(X, \mathbb{Z})\}$ is L-algebraic, the generalized L-Hodge conjecture for H holds.*

Proof. Since torsion classes come from Lichtenbaum cycles, for simplicity we will neglect torsion Hodge classes. Suppose that H is a sub-Hodge structure of $H^k(X, \mathbb{Z})$ of weight k and coniveau c. We know that $H(c)$ is still an effective Hodge structure, then there is a smooth projective variety Y such that $H(c)$ is a sub-Hodge structure of $H^{k-2c}(Y, \mathbb{Z})$, which by polarization can be decomposed as $H^{k-2c}(Y, \mathbb{Z}) \simeq H(c) \oplus R$. Consider $f : H^{k-2c}(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$ the morphism resulting from the composition of the following maps

$$H^{k-2c}(Y, \mathbb{Z}) \xrightarrow{\mathrm{pr}_1} H(c) \xrightarrow{\mathrm{id} \otimes \mathbb{Z}(-c)} H \hookrightarrow H^k(X, \mathbb{Z})$$

Since $\mathrm{Hom}_{\mathrm{HS}\mathbb{Z}}(H^{k-2c}(Y, \mathbb{Z}), H^k(X, \mathbb{Z})) \simeq \mathrm{Hdg}^{2d_Y+2c}(X \times Y)$ the hypothesis implies that f comes from a Lichtenbaum algebraic cycle $\gamma \in \mathrm{CH}^{d_Y+c}_L(Y \times X)$ and $H \subset \gamma_* H^{k-2c}(Y, \mathbb{Z})$. Thus the generalized L-Hodge conjecture holds for H. □

Using the same kind of arguments, and adding an hypothesis of effectiveness it is possible to characterize the generalized Hodge conjecture in terms of the integral Hodge conjecture in the étale setting.

Theorem 3.2.8. *The Lichtenbaum generalized Hodge conjecture for all $X \in SmProj_{\mathbb{C}}$ holds if and only if the following two conditions hold:*

- *the Lichtenbaum Hodge conjecture holds,*
- *a homological étale motive is effective if and only if its Hodge realization is effective.*

Proof. The generalized L-Hodge conjecture immediately implies the L-Hodge conjecture. Suppose that M has an effective realization and let $H := \rho_H(M)$ be its associated Hodge structure of weight n and coniveau c. By the generalized L-Hodge conjecture there exists $Y \in \mathrm{SmProj}_{\mathbb{C}}$ and $\gamma \in \mathrm{CH}_L^{d_Y+c}(Y \times X)$ such that $H \subset \gamma_* H^{n-2c}(Y, \mathbb{Z}) \subset H^n(X, \mathbb{Z})$. The motive $M(c)$ is effective in $h(Y)$, thus M is effective because it is a sub-object of the effective motive $h(X)$.

Assume that the L-Hodge conjecture holds for every $X \in \mathrm{SmProj}_{\mathbb{C}}$ and that a homological motive is effective if and only if its realization is effective. We can neglect torsion Hodge classes because they always come from torsion algebraic cycles. Suppose that H is a sub-Hodge structure of $H^n(X, \mathbb{Z})$ of weight n and coniveau c. We know that $H(c)$ is still an effective Hodge structure. Then there is a smooth projective variety Y such that $H(c)$ is a sub-Hodge structure of $H^{n-2c}(Y, \mathbb{Z})$ which by polarization can be decomposed as $H^{n-2c}(Y, \mathbb{Z}) \simeq H(c) \oplus R$. Consider $f : H^{n-2c}(Y, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ the morphism resulting from the composition of the following maps

$$H^{n-2c}(Y, \mathbb{Z}) \xrightarrow{\mathrm{pr}_1} H(c) \xrightarrow{\mathrm{id} \otimes \mathbb{Z}(-c)} H \hookrightarrow H^n(X, \mathbb{Z})$$

Since $\mathrm{Hom}_{\mathrm{HS}\mathbb{Z}}(H^{n-2c}(Y, \mathbb{Z}), H^n(X, \mathbb{Z})) \simeq \mathrm{H}dg^{2d_Y+2c}(X \times Y)$, f comes from a Lichtenbaum algebraic cycle $\gamma \in \mathrm{CH}_L^{d_Y+c}(Y \times X)$ and $H \subset \gamma_* H^{n-2c}(Y, \mathbb{Z})$. Thus the generalized L-Hodge conjecture holds. $\square$

Then we have the following corollary coming from the previous characterizations of the Generalized Hodge conjecture (classical and Lichtenbaum setting)

Corollary 3.2.9. *The generalized Hodge conjecture holds if and only if the generalized L-Hodge conjecture holds.*

Bardelli's example

Let us recall the example presented in [Bar91] of a threefold X where $\mathrm{GHC}(3, 1, X)_{\mathbb{Q}}$ holds. Let $\sigma : \mathbb{P}^7 \to \mathbb{P}^7$ be the involution defined as $\sigma(x_0 : \ldots : x_3 : y_0, \ldots, y_3) = (x_0 : \ldots : x_3 : -y_0, \ldots, -y_3)$ and let $X = V(Q_0, Q_1, Q_2, Q_3)$ be a smooth complete intersection of four σ-invariant quadrics. There exists a smooth irreducible curve C, of genus 33, obtained as the intersection of two nodal surfaces, and an étale double covering $\widetilde{C} \to C$ such that $H^1(\widetilde{C}, \mathbb{Q})^- \to H^3(X, \mathbb{Q})^-$ is surjective, where the first group is the anti-invariant part of the involution $\tau : \widetilde{C} \to \widetilde{C}$ associated to the double covering and the later group is the anti-invariant part associated to the involution σ. Notice that by [Bar91, Fact 2.4.1] if we assume that X is a very general threefold, then $H^3(X, \mathbb{Q})^+$ and $H^3(X, \mathbb{Q})^-$ are perpendicular with respect to the cup product on $H^3(X, \mathbb{Q})$ and

$H^{3,0}(X) \subset H^3(X,\mathbb{C})^+$ therefore $H^3(X,\mathbb{Q})^-$ is a polarized Hodge structure perpendicular to $H^{3,0}(X)$ i.e. a polarized sub-Hodge structure of $H^3(X,\mathbb{Q})$ of level 1. The isogeny $\alpha : \mathrm{Prym}(\widetilde{C} \to C) \to J(X)^-$, where $J(X)^-$ is the projection of $H^{1,2}(X)^-$ into $J^2(X)$, is the correspondence that induces the isomorphism $H^1(\widetilde{C},\mathbb{Q})^- \to H^3(X,\mathbb{Q})^-$, but in the case of integral coefficients the image of the correspondence is a subgroup of index 2. From the previous results we have the following equivalences:

$$
\begin{aligned}
&\mathrm{GHC}(3,1,X)_{\mathbb{Q}} \text{ holds for } H^3(X,\mathbb{Q})^- \\
&\qquad \iff H^{2,2}(\Gamma \times X) \cap \left\{H^1(\Gamma,\mathbb{Q}) \otimes H^3(X,\mathbb{Q})^-\right\} \text{ is alg. } \forall \text{ curve } \Gamma \\
&\qquad \iff H^{2,2}(\Gamma \times X) \cap \left\{H^1(\Gamma,\mathbb{Z}) \otimes H^3(X,\mathbb{Z})^-\right\} \text{ is L-alg. } \forall \text{ curve } \Gamma \\
&\qquad \iff \mathrm{GHC}_L(3,1,X) \text{ holds for } H^3(X,\mathbb{Z})^-
\end{aligned}
$$

so there exists a smooth projective curve Γ' and a correspondence $z \in \mathrm{CH}^2_L(\Gamma' \times X)$ such that $H^3(X,\mathbb{Z})^- \subset z_* H^1(\Gamma',\mathbb{Z})$.

Chapter 4

Decomposition of integral étale motives

The following chapter is devoted to the decomposition of étale motives and the existence of projectors in étale motivic cohomology groups. Let us recall some facts about the decomposition of motives in the category of integral motives in the classical sense. If there exists a zero cycle of degree 1 in a smooth projective variety X, then one can define the projectors $p_0(X)$ and $p_{2d}(X)$ integrally, otherwise it is necessary to invert some integer in the coefficient ring. In the étale setting, one may be able to improve this result and define new integral projectors.

In the following two sections we focus on the étale degree map, in order to see when it is possible to obtain integral projectors $p_0^{\text{ét}}(X)$ and $p_{2d}^{\text{ét}}(X)$ for a smooth projective variety X over a field k. In the first section, we define the étale analogue of the degree map on $\mathrm{CH}^d_{\text{ét}}(X)$, $d = \dim(X)$. We then study varieties over a field k of characteristic zero for which the étale degree map is surjective. Also, we show that surjectivity does not always hold for Severi-Brauer varieties that do not split over the field k.

The last section, which is divided into three subsections, is devoted to the study of the decomposition of integral étale motives. In the first part, we use the result of [RS16] to construct a projector in étale motivic cohomology and then use this to find an integral decomposition of complex varieties that do not have transcendental cohomology in degrees different from the dimension, extending the result given in [MNP13, Appendix C] to the case of integral coefficients. In the second part, we give an étale analogue of a result of Huybrechts in [Huy18, Lemma 1.1]. The last part is concerned to the integral étale decomposition of smooth commutative groups schemes G over a base S, as a consequence of the good properties of the family of functors associated to the change of coefficients and the results given in [AEH15], [AHP16] and [BS13].

4.1 Étale degree map

Let X be a smooth projective variety over a field k. One defines the zero cycles of X, denoted by $Z_0(X)$, as the free the abelian group generated by sums $\sum_x n_x x$ with x a closed point of X and $n_x = 0$ for all but finitely many x. The degree map is defined by

$$\begin{aligned} \deg : Z_0(X) &\to \mathbb{Z} \\ \sum_x n_x x &\mapsto \sum_x n_x [k(x) : k], \end{aligned}$$

see [Ful98, Defintion 1.4] for more details.

This map descends to the quotient $\mathrm{CH}_0(X)$. By definition, it coincides with the push-forward along the structural map $p : X \to \mathrm{Spec}(k)$ as $p_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Spec}(k)) = \mathbb{Z}$. We define the index of a variety X over k as follows

$$I(X) := \gcd \{[k(x) : k] \mid x \in X\}.$$

If the field is algebraically closed then there exists a k-rational point and the map is surjective. However if the base field is not algebraically closed the existence of a k-rational point, or even of a zero cycle of degree 1, is not guaranteed. Let us remark that the existence of a k-rational point implies the existence of a zero cycle of degree 1, but the converse does not always hold. As it is shown in [CM04] for $d = 2, 3, 4$ there exist del Pezzo surfaces of degree d over a field of cohomological dimension 1 which do not have a zero cycle of degree 1. Or as is presented in [Col05, Theorem 5.1] a hypersurface whose index $I(X) = p$, for a prime $p \geq 5$.

We can reformulate this definition due to the existence of Gysin morphisms in $\mathrm{DM}(k, \mathbb{Z})$ as is described in [Dég12a] and [Dég08]. With this formalism we obtain the pull-back of the morphism p defined as $p^* : M(\mathrm{Spec}(k))(d)[2d] = \mathbb{Z}(d)[2d] \to M(X)$ in the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k, \mathbb{Z})$. Applying the contravariant functor $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k,\mathbb{Z})}(-, \mathbb{Z}(d)[2d])$ we re-obtain the previous definition of push-forward in the case of Chow groups, [Dég12a, Proposition 4.9] . From this, we can extend the existence of Gysin morphisms for $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$, giving us an étale analogue of the degree map for étale Chow groups:

Definition 4.1.1. *Let X be a smooth and projective scheme of dimension d over k, where k is a field of characteristic exponent equal to p. Then we define the étale degree map $\deg_{\acute{e}t} : CH^d_{\acute{e}t}(X) \to CH^0_{\acute{e}t}(Spec(k)) \simeq \mathbb{Z}[1/p]$ as $\deg_{\acute{e}t} := p_*$, where p is the structure morphism $p : X \to Spec(k)$. We define the étale index of X as the greatest common divisor of the subgroup $\deg_{\acute{e}t}(CH^d_{\acute{e}t}(X)) \cap \mathbb{Z}$, denoted by $I_{\acute{e}t}(X)$.*

Remark 4.1.2. 1. Let k be a field of characteristic exponent p. Due to functoriality properties we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{DM}(k,\mathbb{Z})}(M(Y), \mathbb{Z}(d)[2d]) & \xrightarrow{p_*} & \mathrm{Hom}_{\mathrm{DM}(k,\mathbb{Z})}(\mathbb{Z}(d)[2d], \mathbb{Z}(d)[2d]) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})}(M_{\text{ét}}(Y), \mathbb{Z}(d)[2d]) & \xrightarrow{p_*} & \mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})}(\mathbb{Z}(d)[2d], \mathbb{Z}(d)[2d]) \end{array}$$

where for $\mathrm{CH}^0_\tau(\mathrm{Spec}(k))$ with $\tau \in \{\mathrm{Nis}, \text{ét}\}$, there are isomorphisms

$$\mathrm{Hom}_{\mathrm{DM}(k,\mathbb{Z})}(\mathbb{Z}(d)[2d], \mathbb{Z}(d)[2d]) = H^{0,0}_M(\mathrm{Spec}(k)) \simeq \mathbb{Z}$$

and

$$\mathrm{Hom}_{\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})}(\mathbb{Z}(d)[2d], \mathbb{Z}(d)[2d]) = H^{0,0}_{M,\text{ét}}(\mathrm{Spec}(k)) \simeq \mathbb{Z}[1/p]$$

2. By the previous point, if $\mathrm{char}(k) = 0$, K/k is a finite Galois extension and $X \to k$ a smooth projective k-scheme then the morphism $f : X_K \to X$ is a finite étale morphism. As f is proper, there exists an induced map $f_* : \mathrm{CH}^d_{\text{ét}}(X_K) \to \mathrm{CH}^d_{\text{ét}}(X)$ which fits into the following commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^d(X_K) \arrow[r, "f_*"] \arrow[dd] \arrow[dr, "\deg"] & \mathrm{CH}^d(X) \arrow[dd, dashed, bend left] \arrow[dr, "\deg"] & \\
& \mathbb{Z} \arrow[r, "{[K:k]\cdot}"] & \mathbb{Z} \\
\mathrm{CH}^d_{\text{ét}}(X_K) \arrow[r, "f_*"] \arrow[ur, "\deg_{\text{ét}}"] & \mathrm{CH}^d_{\text{ét}}(X) \arrow[ur, "\deg_{\text{ét}}"'] &
\end{tikzcd}$$

with $[K : k]$ the degree of the extension.

3. It is possible to define the étale degree map for Lichtenbaum cohomology over a field $k = \bar{k}$. If X is a smooth and proper projective variety of dimension d, there is a quasi-isomorphism $\mathbb{Z}_X(n)_{\mathrm{Zar}} \to R\pi_*\mathbb{Z}_X(n)_{\text{ét}}$ for $n \geq d$, see Theorem 2.2.11. If this is not the case, we then can invert the characteristic exponent of k and use the isomorphism between Lichtenbaum and étale Chow groups.

Since in the following part we will use spectral sequences, for the sake of legibility, from the following we will denote the characteristic exponent of a field k by $\tilde{p}$. The letter p is reserved for the bi-degrees of the spectral sequence.

Let $f : X \to Y$ be a projective morphism of smooth varieties of relative dimension c. Again by the existence of Gysin morphisms in $\mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$, we obtain a push-forward map for étale motivic cohomology

$$f_* : H^{m+2c}_{M,\text{ét}}(X, \mathbb{Z}(n + c)) \to H^m_{M,\text{ét}}(Y, \mathbb{Z}(n)).$$

Combining the existence of push-forward maps for étale motivic cohomology and the functoriality of the Hochschild-Serre spectral sequence we obtain the following diagram

$$\begin{tikzcd}
H^p(G_k, H^{q+2c}_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n+c))) \arrow[r, Rightarrow] \arrow[d, "\tilde{f}_*"] & H^{p+q+2c}_L(X, \mathbb{Z}[1/\tilde{p}](n+c)) \arrow[d, "f_*"] \\
H^p(G_k, H^q_L(Y_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))) \arrow[r, Rightarrow] & H^{p+q}_L(Y, \mathbb{Z}[1/\tilde{p}](n))
\end{tikzcd}$$

where $\tilde{p}$ is the characteristic exponent of k and $\tilde{f} : X_{\bar{k}} \to Y_{\bar{k}}$. For the particular case of the étale degree map we have the following:

Proposition 4.1.3. *Let X be a smooth and projective of dimension d over a field k with characteristic exponent $\tilde{p}$. Then the map $\deg_{\text{ét}} : CH^d_{\text{ét}}(X) \to \mathbb{Z}[1/\tilde{p}]$ factors through a subgroup of $CH^d(X_{\bar{k}})[1/\tilde{p}]^{G_k}$.*

Proof. We will prove that the subgroup in question is given by the $E^{0,2d}_\infty$-term of the Hochschild-Serre spectral sequence associated to X. To see this, consider the structural morphism $f : X \to k$, then we have an induced morphism of E_2-terms

$$E^{p,q}_2 := H^p(G_k, H^q_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](d))) \to H^p(G_k, H^{q-2d}_L(\mathrm{Spec}(\bar{k}), \mathbb{Z}[1/\tilde{p}](0)))$$

if we look at the cases when $q - 2d \leq 0$, we get that

$$H_L^{q-2d_X}(\mathrm{Spec}(\bar{k}), \mathbb{Z}[1/\tilde{p}](0)) \simeq \begin{cases} 0 \text{ for } q \neq 2d \\ \mathbb{Z}[1/\tilde{p}] \text{ for } q = 2d. \end{cases}$$

This gives us $H_L^p(k, \mathbb{Z}[1/\tilde{p}](0)) \simeq H^p(G_k, H^0(\bar{k}, \mathbb{Z}[1/\tilde{p}](0)))$ and hence we conclude that $\deg_{\text{ét}} : \mathrm{CH}^d_{\text{ét}}(X) \to \mathbb{Z}[1/\tilde{p}]$ factors as

$$\begin{tikzcd} \mathrm{CH}^d_{\text{ét}}(X) \arrow[r] \arrow[dr, "\deg_{\text{ét}}"'] & E_\infty^{0,2d} \arrow[d, "\widetilde{\deg}"] \\ & \mathbb{Z}[1/\tilde{p}] \end{tikzcd}$$

where $\widetilde{\deg}$ is the composite map

$$E_\infty^{0,2d} \hookrightarrow E_2^{0,2d} = \mathrm{CH}^d(X_{\bar{k}})[1/\tilde{p}]^{G_k} \hookrightarrow \mathrm{CH}^d(X_{\bar{k}})[1/\tilde{p}] \xrightarrow{\deg} \mathbb{Z}[1/\tilde{p}].$$

□

4.2 Lichtenbaum zero cycles

Varieties where $I_{\text{ét}}(X) = 1$

The aim of this subsection is to construct examples where the étale degree map is surjective but its classical counterpart is not. In order to achieve this, we start by giving a lemma about the divisibility of the zero cycles of degree zero of a variety over an algebraically closed field:

Lemma 4.2.1. *Let X be a complete scheme over an algebraically closed field k of characteristic $p \geq 0$. Define $A_0(X) = ker\{\deg : CH_0(X) \to \mathbb{Z}\}$, then $A_0(X)$ is a divisible group. If X is a smooth quasi-projective scheme and $H^{2d-1}_{ét}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = 0$ for $\ell \neq p$ then $A_0(X) \xrightarrow{\cdot \ell^r} A_0(X)$ is an isomorphism for all $r \in \mathbb{N}$.*

Proof. The first statement is known, see [Ful98, Example 1.6.6]. The argument goes as follows: since $A_0(X)$ is generated by the image of the maps of the form:

$$\begin{aligned} f_* : A_0(C) &\to A_0(X) \\ [P] - [Q] &\mapsto f_*([P] - [Q]) \end{aligned}$$

where $f : C \to X$ a smooth projective curve with P, Q points in C. Since $A_0(C) \simeq J(C)$ and the Jacobian of a smooth projective curve is divisible over an algebraically closed field k, we obtain the desired result. We prove the second assertion. Notice that by the assumption that k is an algebraically closed field, one gets that $\mathrm{CH}^d(X) \simeq \mathrm{CH}^d_L(X)$ and that $\mathrm{CH}^d_L(X)\{\ell\} \simeq H^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$. Therefore

$$\mathrm{CH}_0(X)\{\ell\} = \mathrm{CH}^{2d}(X)\{\ell\} \simeq H^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = 0$$

and $\mathrm{CH}_0(X)\{\ell\} \simeq A_0(X)\{\ell\}$, so one deduces that under the assumption, $A_0(X)$ is ℓ^r-divisible for any $r > 0$. □

Remark 4.2.2. Notice that with the previous statement, if $H^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = 0$ for all ℓ different from the characteristic of k, we conclude that $A_0(X)$ is uniquely ℓ^r-divisible.

For X a smooth and projective variety over a field k of characteristic exponent equal to p, we set

$$A_0^{\text{ét}}(X) := \ker \left\{ \deg_{\text{ét}} : \mathrm{CH}^d_{\text{ét}}(X) \to \mathbb{Z}[1/p] \right\}.$$

Notice that if k is algebraically closed then we have an isomorphism $A_0^{\text{ét}}(X) \simeq A_0(X)[1/p]$.

Proposition 4.2.3. *Let X be a geometrically integral smooth projective variety of dimension $d \geq 2$ over a perfect field k with $cd(k) \leq 1$ and $\tilde{p}$ the characteristic exponent of k. Let $\bar{k}$ be the algebraic closure of k and assume that $H^{2d-1}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = 0$ for every prime $\ell \neq char(k)$, then $\deg_{\acute{e}t} : CH^d_{\acute{e}t}(X) \to \mathbb{Z}[1/\tilde{p}]$ is surjective.*

Proof. First assume that $\mathrm{char}(k) = 0$, then $\mathrm{CH}^n_L(X) \simeq \mathrm{CH}^n_{\text{ét}}(X)$ for all $n \in \mathbb{N}$. Using the notation given in Lemma 2.2.16, if $\mathrm{cd}(k) \leq 1$ then $E_2^{2,q}(n) = 0$ for $1 < q < 2n$, so by the characterizations of the infinity terms given in Example 2.2.24 we obtain a short exact sequence $0 \to H^1(G, H^{2n-1}_L(X_{\bar{k}}, \mathbb{Z}(n))) \to \mathrm{CH}^n_L(X) \to \mathrm{CH}^n_L(X_{\bar{k}})^{G_k} \to 0$. For $n = d$ we have that $\mathrm{CH}^d_L(X) \to \mathrm{CH}^d_L(X_{\bar{k}})^{G_k}$ is always surjective. Now consider the short exact sequence

$$0 \to A_0(X_{\bar{k}}) \to \mathrm{CH}^d_L(X_{\bar{k}}) \xrightarrow{\deg_{\text{ét}}} \mathbb{Z} \to 0$$

where $A_0(X_{\bar{k}}) := \ker \left\{ \deg_{\text{ét}} : \mathrm{CH}^d_{\text{ét}}(X_{\bar{k}}) \to \mathbb{Z} \right\}$, i.e. the numerically trivial zero cycles of $X_{\bar{k}}$, which induces a long exact sequence

$$0 \to A_0(X_{\bar{k}})^{G_k} \to \mathrm{CH}^d_L(X_{\bar{k}})^{G_k} \xrightarrow{\widetilde{\deg}} \mathbb{Z} \to H^1(G_k, A_0(X_{\bar{k}})) \to \dots$$

where the factor $\mathbb{Z}$ is obtained by using the fact that $\mathrm{CH}^0(\mathrm{Spec}(\bar{k}))^{G_k} \simeq \mathrm{CH}^0(\mathrm{Spec}(k))$. By [RS16, Proposition 3.1(a)] we have that $\mathrm{CH}^d_L(X_{\bar{k}})\{\ell\} \simeq H^{2d-1}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$ so $A_0(X_{\bar{k}})_{\text{tors}} \simeq \mathrm{CH}^d_L(X_{\bar{k}})_{\text{tors}} = 0$ and then the group $A_0(X_{\bar{k}})$ is uniquely divisible, so we conclude that $H^1(G, A_0(X_{\bar{k}})) = 0$. Consequently the map $\deg_{\text{ét}} : \mathrm{CH}^d_L(X) \to \mathrm{CH}^d_L(X_{\bar{k}})^{G_k} \to \mathbb{Z}$ is surjective.

Now assume that $\mathrm{char}(k) = p > 0$, in this case it is necessary to invert the characteristic exponent $\tilde{p}$ of the field. For an abelian group A we put $A[1/\tilde{p}] := A \otimes_{\mathbb{Z}} \mathbb{Z}[1/\tilde{p}]$. Setting $q \neq 2d$, we have that $H^q_L(X_{\bar{k}}, \mathbb{Z}(d))$ is a extension of a divisible groups D by a torsion groups T. Using the convention for tensor product, we notice that

$$0 \to D \to H^q_L(X_{\bar{k}}, \mathbb{Z}(d))[1/p] \to T[1/p] \to 0$$

where the last map kills the p-primary part of the torsion group T. Also the spectral sequence holds for the complex of étale sheaves $\mathbb{Z}[1/p](n)_{\text{ét}}$, for the convergence we use the same arguments with the exact triangle $\mathbb{Z}[1/p]_X(d)_{\text{ét}} \to \mathbb{Q}_X(d)_{\text{ét}} \to \bigoplus_{\ell \neq \mathrm{char}(k)} \mathbb{Q}_\ell/\mathbb{Z}_\ell(d) \xrightarrow{+1}$ therefore we have a similar short exact sequence $0 \to H^1(G_k, H^{2n-1}_L(X_{\bar{k}}, \mathbb{Z}[1/\tilde{p}](n))) \to \mathrm{CH}^n_L(X)[1/\tilde{p}] \to \mathrm{CH}^n_L(X_{\bar{k}})[1/\tilde{p}]^{G_k} \to 0$ and also $0 \to A_0(X_{\bar{k}})[1/\tilde{p}] \to \mathrm{CH}^d_L(X_{\bar{k}})[1/\tilde{p}] \xrightarrow{\deg_L} \mathbb{Z}[1/\tilde{p}] \to 0$, therefore we can conclude. $\square$

Theorem 4.2.4. *There exists a smooth projective surface S over a field k of characteristic zero and cohomological dimension ≤ 1, such that X does not admit a zero cycle of degree one but $I_{ét}(X) = 1$.*

Proof. By [CM04, Théorème 1.1] and [CM04, Théorème 1.2] there exist del Pezzo surfaces of degree 2, 3 and 4 over a field k of characteristic zero and $\mathrm{cd}(k) = 1$ without zero cycles of degree 1. Let S be one of such surfaces of degree $d \in \{2, 3, 4\}$. Since S is a del Pezzo surface, thus for all field extension K/k the variety S_K is a del Pezzo surface of degree d as well, so in particular for $K = \bar{k}$. As $S_{\bar{k}}$ is del Pezzo, we have that $H^1(S_{\bar{k}}, \mathcal{O}_{S_{\bar{k}}}) = H^2(S_{\bar{k}}, \mathcal{O}_{S_{\bar{k}}}) = 0$ therefore $\mathrm{Alb}(S_{\bar{k}}) = 0$. Since we are working over an algebraically closed field, $\mathrm{CH}^2(S_{\bar{k}}) \simeq \mathrm{CH}^2_L(S_{\bar{k}})$ and then by Roitman's theorem $\mathrm{CH}^2_L(S_{\bar{k}})_{\mathrm{tors}} = A_0(S_{\bar{k}})_{\mathrm{tors}} = 0$ so the group $A_0(S_{\bar{k}})$ is uniquely divisible and consequently by Proposition 4.2.3 the map $\mathrm{CH}^2_L(S) \to \mathrm{CH}^2_L(S_{\bar{k}})^{G_k} \to \mathbb{Z}$ is surjective, while $\mathrm{CH}^2(S) \to \mathbb{Z}$ is not a surjective map. □

Theorem 4.2.5. *For each prime $p \geq 5$ there exist a field F such that $char(F) = 0$ with $cd(F) = 1$ and a smooth projective hypersurface $X \subset \mathbb{P}^p_F$ with $I_{ét}(X) = 1$ but $I(X) = p$.*

Proof. Let us consider $n \geq 2$, a field k such that $\mathrm{cd}(k) \leq 1$ and a hypersurface $X \subset \mathbb{P}^{n+1}_k$ that is geometrically integral. Consider the hypersurface $X_{\bar{k}} \subset \mathbb{P}^{n+1}_{\bar{k}}$. By the Lefschetz hyperplane theorem [Mil80, Theorem 7.1], we have

$$H^{2n-1}_{\text{ét}}(X_{\bar{k}}, \mu_{\ell^r}^{\otimes n}) \simeq H^{2n+1}_{\text{ét}}(\mathbb{P}^{n+1}_{\bar{k}}, \mu_{\ell^r}^{\otimes n+1}) = 0$$

for all $\ell \neq \mathrm{char}(k)$, thus $H^{2n-1}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = 0$ so by Proposition 4.2.3 the morphism $\deg_{\text{ét}} : \mathrm{CH}^n_{\text{ét}}(X) \to \mathbb{Z}$ is surjective. Now if we fix a prime number $p \geq 5$ then by [Col05, Theorem 1.1] there exist a field F with $\mathrm{cd}(F) = 1$ and a smooth projective hypersurface $X \subset \mathbb{P}^p_F$ with index equal to p. □

Remark 4.2.6. Assume that k is a field with $\mathrm{cd}(k) \leq 1$. Consider S a smooth geometrically integral k-surface with $H^1(S, \mathcal{O}_S) = 0$, therefore $\mathrm{Alb}(S) = 0$ so again by Roitman's theorem $\mathrm{CH}^2_L(S_{\bar{k}})$ is torsion free and uniquely divisible, so $H^1(G, A_0(S_{\bar{k}})) = 0$ and then $\mathrm{CH}^2_L(S) \to \mathbb{Z}$ is surjective. In general if $A_0(X_{\bar{k}})$ is a divisible group then $\mathrm{CH}^d_L(X) \to \mathbb{Z}$ is surjective.

Étale degree of Severi-Brauer varieties

In the following, we will see non-trivial examples where the étale degree map is not surjective. For this we will study the Lichtenbaum cohomology groups of Severi-Brauer varieties by giving an explicit characterization of the zero cycles of Lichtenbaum groups of Severi-Brauer varieties.

Definition 4.2.7. *A variety X over a field k is called a Severi-Brauer variety of dimension n if and only if $X_{\bar{k}} \simeq \mathbb{P}^n_{\bar{k}}$. If X is a Severi-Brauer variety of dimension n and there exists an algebraic extension $k \subset k' \subset \bar{k}$ such that $X_{k'} \simeq \mathbb{P}^n_{k'}$ we say that X splits over k'.*

If $\mathrm{Br}(k) = 0$, there exists a unique Severi-Brauer variety modulo isomorphisms to $\mathbb{P}^n_k$. Some cases of such fields are the following:

- a field k with $\mathrm{cd}(k) \leq 1$. In this category we can find fields such as algebraically and separable closed fields, finite fields, extensions of transcendence degree 1 of an algebraically closed field.
- If k is a field extension of $\mathbb{Q}$ containing all the roots of unity, see [Ser68, §7] and [Ser02, II.§3, Proposition 9].

Lemma 4.2.8. *Let X be a Severi-Brauer variety of dimension d over k which splits over a field k'. Then for all $0 \leq n \leq d$ the group $CH^n(X_{k'}) \simeq CH^n(\mathbb{P}^d_{k'})$ is a trivial $Gal(k'/k)$-module.*

Proof. First consider $d = 1$, then $\mathrm{CH}^1(\mathbb{P}^d_{k'}) \simeq \mathrm{Pic}(\mathbb{P}^d_{k'}) \simeq \mathbb{Z}$. Following the argument given in [GS06, Proposition 5.4.4] the action of $\mathrm{Gal}(k'/k)$ over $\mathrm{Pic}(\mathbb{P}^d_{k'})$ is trivial as the only non-trivial action would permute 1 with -1. The vector bundles of $\mathbb{P}^d_{k'}$ with Chern class 1 cannot permute with the ones in the class -1 due to the existence of global sections for the first case. For the general cases when $n \neq 1$ we consider the isomorphisms $\mathrm{CH}^1(\mathbb{P}^d_{k'}) \simeq \mathrm{CH}^n(\mathbb{P}^d_{k'})$ given by the intersection with hyperplanes. □

Remark 4.2.9.

1. We can similarly deduce that for all $m, n \in \mathbb{N}$ we have $\mathrm{Pic}(\mathbb{P}^m_{\bar{k}} \times \mathbb{P}^n_{\bar{k}}) \simeq \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta]$, where α and β are the generators of $\mathrm{Pic}(\mathbb{P}^m_{\bar{k}})$ and $\mathrm{Pic}(\mathbb{P}^n_{\bar{k}})$ respectively, is a trivial G_k-module.

2. Let k be a perfect field of characteristic exponent $\tilde{p}$ and let X be a Severi-Brauer variety of dimension d over k. The fact $X_{\bar{k}} \simeq \mathbb{P}^d_{\bar{k}}$ simplifies several computations for the Hochschild-Serre spectral sequence given in Lemma 2.2.16. For instance if $m \neq 2n+1$, then for $\ell \neq \tilde{p}$ we can characterize the ℓ-primary torsion groups as follows
$$H^m_L(X_{\bar{k}}, \mathbb{Z}(n))\{\ell\} \simeq H^{m-1}_{\text{ét}}(\mathbb{P}^d_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \simeq \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell \text{ if } m \text{ is odd} \\ 0 \text{ otherwise.} \end{cases}$$
Therefore for m even and $m < 2n$ the group $H^m_L(X_{\bar{k}}, \mathbb{Z}(n))$ is uniquely divisible, thus some of the $E_2(n)$-terms associated to the Hochschild-Serre spectral sequence of $H^{p+q}_L(X, \mathbb{Z}[1/\tilde{p}](n))$ can be characterized in the following way
$$E_2^{p,q}(n) = \begin{cases} H^q(\mathbb{P}_{\bar{k}}, \mathbb{Z}(n))^{G_k} \quad \text{if } p = 0, \\ H^p(G_k, H^{q-1}_{\text{ét}}(\mathbb{P}_{\bar{k}}, (\mathbb{Q}/\mathbb{Z})'(n))) \;\; \text{if } q \text{ is odd and } p > 0, \\ 0 \quad \text{if } q \text{ is even and } p > 0. \end{cases}$$

Now let us set $n = 1$ and let X be a Severi-Brauer variety over k of dimension d. If we use the Hoschschild-Serre spectral sequence given in Lemma 2.2.16 and Lemma 4.2.8, we recover a classical result of Lichtenbaum [GS06, Theorem 5.4.10] concerning the Picard group of X and Brauer groups: there is an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{P}^d_{\bar{k}})^{G_k} \xrightarrow{\delta} \mathrm{Br}(k) \to \mathrm{Br}(k(X)), \tag{4.1}$$

where the map δ sends 1 to the class of X in $\mathrm{Br}(k)$. For an arbitrary integer n, if we apply the projective bundle formula to obtain

$$H^m_L(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(n)) \simeq \bigoplus_{i=0}^{d} H^{m-2i}_L(\mathrm{Spec}(\bar{k}), \mathbb{Z}(n-i)).$$

After base change to the algebraic closure we have $H^m_L(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)) \simeq H^m_M(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d))$ for all $m \in \mathbb{Z}$ and in particular $H^{m-2i}_L(\mathrm{Spec}(\bar{k}), \mathbb{Z}(d-i)) = 0$ if $m-d > i$. For instance if $m = 2d-1$ then $H^{2d-1}_L(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)) \simeq K^M_1(\bar{k})$ or for $m = 2d-2$ we have $H^{2d-2}_L(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)) \simeq K^M_2(\bar{k})$ and hence for a Severi-Brauer variety and applying Lemma 2.2.16, we obtain that $E^{1,2d-1}_2(d) = H^1(G_k, \bar{k}^*) = 0$, by Hilbert 90, and $E^{2,2d-1}_2(d) = H^2(G_k, \bar{k}^*) = \mathrm{Br}(k)$.

Theorem 4.2.10. *Let X be a Severi-Brauer variety of dimension d over a field k. Then the image of $\deg_{\acute{e}t} : CH^d_{\acute{e}t}(X) \to \mathbb{Z}$ is isomorphic to a subgroup of $Pic(X)$ and in particular $I_{\acute{e}t}(X) \geq ord([X])$ where $[X]$ is the Brauer class of X in $Br(k)$. Moreover, if $cd(k) \leq 4$ then this subgroup is isomorphic to $Pic(X)$ i.e. $I_{\acute{e}t}(X) = ord([X])$.*

Proof. Let X be a Severi-Brauer variety of dimension d, and consider the Hochschild-Serre spectral sequence for Lichtenbaum cohomology in two cases: when $n = 1$ and $n = d$. For $n = 1$ we recover (4.1), where some of the terms of the exact sequence come from $E^{0,2}_2(1) = \mathrm{Pic}(\mathbb{P}^d_{\bar{k}})^{G_k}$ and $E^{2,1}_2(1) \simeq \mathrm{Br}(k)$. For the case when $n = d$, and using the computations from the previous discussion, we obtain the following terms: $E^{0,2d}_2(d) = \mathrm{CH}^d(\mathbb{P}^d_{\bar{k}})^{G_k}$ and $E^{2,2d-1}_2(d) \simeq \mathrm{Br}(k)$. Notice that the isomorphisms

$$H^{2n}_L(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(n)) \simeq \bigoplus_{i=0}^{d} H^{2(n-i)}_L(\bar{k}, \mathbb{Z}(n-i))$$

for $n = 1$ and d are induced by the map $\mathbb{P}^d_{\bar{k}} \to \mathrm{Spec}(\bar{k})$. This gives us a commutative diagram where the vertical arrows are given by the intersection with the hyperplane section of $\mathbb{P}^d_{\bar{k}}$

$$\begin{tikzcd}
\mathrm{Pic}(\mathbb{P}^d_{\bar{k}})^{G_k} \arrow[r, "\delta"] \arrow[d, "\simeq"] & \mathrm{Br}(k) \arrow[d, "\simeq"] \\
\mathrm{CH}^d(\mathbb{P}^d_{\bar{k}})^{G_k} \arrow[r, "d_2^{0,2d}(d)"] & \mathrm{Br}(k)
\end{tikzcd}$$

Since the vertical arrows are isomorphisms, $E^{0,2d}_3(d) = \ker(d^{0,2d}_2(d)) \simeq \ker(\delta) \simeq \mathrm{Pic}(X)$. Now by Proposition 4.1.3, the map $\deg_{\acute{e}t}$ factors through $E^{0,2d}_\infty(d)$ which is a subgroup of $E^{0,2d}_3(d)$. The assumption about the cohomological dimension of k gives us that $E^{0,2d}_\infty(d) \simeq E^{0,2d}_3(d)$. For further details about this computation see the next example and proposition. $\square$

Remark 4.2.11. Notice the following: Consider $1 \leq n \leq d$ and consider the Hochschild-Serre spectral sequence associated to $\mathrm{CH}^n_L(X)$. From the projective bundle formula we have that $H^{2n-1}_L(\mathbb{P}^d_{\bar{k}}) \simeq K^M_1(\bar{k})$, thus $E^{0,2n}_2(n) \simeq \mathrm{Br}(k)$ and consequently by the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\mathbb{P}^d_{\bar{k}})^{G_k} \arrow[r, "\delta"] \arrow[d, "\simeq"] & \mathrm{Br}(k) \arrow[d, "\simeq"] \\
\mathrm{CH}^n(\mathbb{P}^d_{\bar{k}})^{G_k} \arrow[r, "d_2^{0,2n}(n)"] & \mathrm{Br}(k)
\end{tikzcd}$$

the term $E^{0,2n}_\infty(n)$ is isomorphic to a subgroup of $\mathrm{Pic}(X)$.

Example 4.2.12. *Let X be a Severi-Brauer variety of dimension $d = 2$ over a perfect field k with Galois group G_k. Using the previous characterizations through the projective bundle formula, we then describe the E_2-terms associated to X in the following way:*

$$\begin{aligned}
&E^{p,0}_2 = H^p(G_k, H^0_M(\mathit{Spec}(\bar{k}), \mathbb{Z}(2))),\ E^{p,1}_2 = H^p(G_k, H^1_M(\mathit{Spec}(\bar{k}), \mathbb{Z}(2))),\\
&E^{p,2}_2 = H^p(G_k, K^M_2(\bar{k})),\ E^{p,3}_2 = H^p(G_k, K^M_1(\bar{k})),\\
&E^{p,4}_2 = H^p(G_k, CH^2_L(\mathbb{P}^2_{\bar{k}}))\ \text{and}\ E^{p,q}_2 = 0\ \text{for}\ q \geq 5.
\end{aligned}$$

By Remark 4.2.9 (2), we have that $E^{p,0}_2 = E^{p,2}_2 = 0$ for $p > 0$, also $E^{1,3}_2 = 0$ by Hilbert 90 theorem and $E^{2,3}_2 = Br(k)$, obtaining with this the following terms: for trivial reasons $E^{1,3}_\infty = E^{2,2}_\infty = E^{4,0}_\infty = 0$ and:

$$E^{3,1}_\infty = H^3(G_k, H^1_M(\bar{k}, \mathbb{Z}(2)))/im\left\{K^M_1(\bar{k})^{G_k} \to H^3(G_k, H^1_M(\bar{k}, \mathbb{Z}(2)))\right\}$$

The only remaining piece of the filtration of $CH^2_L(X)$ that we need to study is $E^{0,4}_\infty$. By definition we have that $E^{0,4}_3 = ker\left\{CH^2(\mathbb{P}^2_{\bar{k}})^{G_k} \to Br(k)\right\}$ and as $E^{3,2}_2 = 0$ then $E^{0,4}_4 = E^{0,4}_3$. Finally, we observe that $E^{4,1}_4 = E^{4,1}_3 = E^{4,1}_2$ and thus again by definition

$$\begin{aligned}
E^{0,4}_\infty &= ker\left\{E^{0,4}_4 \to E^{4,1}_4\right\}\\
&= ker\left\{E^{0,4}_4 \to H^4(G_k, H^1_M(\mathit{Spec}(\bar{k}), \mathbb{Z}(2)))\right\}.
\end{aligned}$$

Therefore $CH^2_L(X)$ fits into a short exact sequence given by the filtration induced by the Hochschild-Serre spectral sequence

$$0 \to E^{3,1}_\infty \to CH^2_L(X) \to E^{0,4}_\infty \to 0.$$

If we want to generalize this result for higher dimension, we need to impose a condition on the cohomological dimension of k:

Proposition 4.2.13. *Let X be a Severi-Brauer variety of dimension d over a perfect field k of cohomological dimension $cd(k) \leq 4$. Then the group $CH^L_0(X)$ fits in an exact sequence*

$$0 \to E^{3,2d-1}_\infty \to CH^d_L(X) \to E^{0,2d}_\infty \to 0$$

with $E^{0,2d}_\infty = ker\left\{CH^d_{\acute{e}t}(\mathbb{P}^d_{\bar{k}})^{G_k} \to Br(k)\right\}$. In particular $I_{\acute{e}t}(X) = ord([X])$.

Proof. We follow the arguments given in example 4.2.12. Consider k and X as above, then by hypothesis $X_{\bar{k}} \simeq \mathbb{P}^d_{\bar{k}}$. By the projective bundle formula for Lichtenbaum cohomology we have

$$H^m(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)) \simeq \bigoplus_{i=0}^{d} H_L^{m-2i}(\mathrm{Spec}(\bar{k}), \mathbb{Z}(d-i)).$$

Notice that by divisibility arguments we have that $E_2^{p,2k} = 0$ for $0 \leq k \leq d$ and $p > 0$. Under the assumption about the cohomological dimension of k we have that $E_2^{p,q} = 0$ for $p > 4$ and $q < 2n$, this results that $E_\infty^{0,2d} \simeq E_3^{0,2d} = \ker\left\{\mathrm{CH}^d(\mathbb{P}^d_{\bar{k}})^{G_k} \to H^2(G_k, \bar{k}^*)\right\}$ and the other $E_2^{p,q}$-terms with $p+q = 2d$ that could not vanish are $E_2^{1,2d-1}$ and $E_2^{3,2d-3}$, but $H_L^{2d-1}(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)) \simeq K_1^M(\bar{k})$ therefore $E_2^{1,2d-1} = 0$. On the other hand, the remaining piece of the filtration, which is $E_\infty^{3,2d-3} = E_4^{3,2d-3}$, is defined as

$$\begin{aligned} E_4^{3,2d-3} &= E_3^{3,2d-3}/\mathrm{im}\left\{E_3^{0,2d-1} \to E_3^{3,2d-3}\right\} \\ &= H^3(G_k, H_M^{2d-3}(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)))/\mathrm{im}\left\{K_1^M(\bar{k})^{G_k} \to H^3(G_k, H_M^{2d-3}(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)))\right\}. \end{aligned}$$

Using the recursive formula

$$H_L^m(\mathbb{P}^n_{\bar{k}}, \mathbb{Z}(n)) \simeq H_L^m(\bar{k}, \mathbb{Z}(n)) \oplus H_L^{m-2}(\mathbb{P}^{n-1}_{\bar{k}}, \mathbb{Z}(n-1)).$$

we obtain

$$H_M^{2d-3}(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(d)) \simeq \begin{cases} 0 \text{ if } d = 1 \\ H_M^1(\bar{k}, \mathbb{Z}(2)) \text{ if } d = 2 \\ H_M^1(\bar{k}, \mathbb{Z}(2)) \oplus K_3^M(\bar{k}) \text{ if } d \geq 3. \end{cases}$$

Again as in Example 4.2.12, the group $\mathrm{CH}^d_{\text{ét}}(X)$ fits into the following short exact sequence

$$0 \to E_\infty^{3,2d-3} \to \mathrm{CH}^d_{\text{ét}}(X) \to E_\infty^{0,2d} \to 0.$$

As mentioned in Proposition 4.1.3, the étale degree map factors through $E_\infty^{0,2d}$. This gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E_\infty^{3,2d-3} \arrow[r] & \mathrm{CH}^d_{\text{ét}}(X) \arrow[r] \arrow[dr, "\deg_{\text{ét}}"'] & E_\infty^{0,2d} \arrow[r] \arrow[d, "\widetilde{\deg}"] & 0 \\ & & & \mathbb{Z} & \end{tikzcd}$$

where $\widetilde{\deg} : E_\infty^{0,2d} \to \mathbb{Z}$ is the composition of the maps

$$E_\infty^{0,2d} \hookrightarrow \mathrm{CH}^d(\mathbb{P}^d_{\bar{k}})^{G_k} \xrightarrow{\simeq} \mathrm{CH}^d(\mathbb{P}^d_{\bar{k}}) \xrightarrow{\deg} \mathbb{Z}.$$

□

As we may expect, the étale index of a product of Severi-Brauer is again bounded by the order of the Brauer class in $\mathrm{Br}(k)$. For the sequel we denote $X^{\times n} := \overbrace{X \times \ldots \times X}^{\text{n-times}}$

Lemma 4.2.14. *Let X be a Severi-Brauer variety of dimension d over a field k. Then there exists an exact sequence*

$$0 \to Pic(X \times X) \to Pic(\mathbb{P}^d_{\bar{k}} \times \mathbb{P}^d_{\bar{k}})^{G_k} \simeq \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{s} Br(k) \to Br(X \times X).$$

The map s sends (a, b) to $(a + b)\,[X] \in Br(k)$, where $[X]$ is the Brauer class associated to X. In general for a product $X^{\times n}$ we then obtain an exact sequence

$$0 \to Pic(X^{\times n}) \to Pic(\mathbb{P}^d_{\bar{k}} \times \ldots \times \mathbb{P}^d_{\bar{k}})^{G_k} \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \xrightarrow{s} Br(k) \to Br(X^{\times n})$$

with $s(a_1, \ldots, a_n) = \sum_{i=1}^n a_i\,[X] \in Br(k)$.

Proof. Let Y a smooth projective variety over k. Considering the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_k, H^q_L(Y_{\bar{k}}, \mathbb{Z}(1))) \Longrightarrow H^{p+q}_L(Y, \mathbb{Z}(1))$$

we obtain the following exact sequence $0 \to E^2_\infty \to E_2^{0,2} \to E_2^{2,1} \to E^3_\infty$. If $Y = X^{\times n}$ then $Y_{\bar{k}} \simeq \mathbb{P}^d_{\bar{k}} \times \ldots \times \mathbb{P}^d_{\bar{k}}$ and consequently $\mathrm{Pic}(\mathbb{P}^d_{\bar{k}} \times \ldots \times \mathbb{P}^d_{\bar{k}}) \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$. By remark 4.2.9 we obtain an isomorphism $\mathrm{Pic}(\mathbb{P}^d_{\bar{k}} \times \ldots \times \mathbb{P}^d_{\bar{k}})^{G_k} \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ that gives us the exact sequences of the statement.

Now let us see the easiest case for $Y = X \times X$. Consider the maps

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times X \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \end{tikzcd}$$

where $\Delta : X \to X \times X$ is the diagonal embedding and $\mathrm{pr}_i : X \times X \to X$ is the projection to the i-th component. Notice that the composition gives the identity on X. Notice that the morphism $\mathrm{pr}_i : X \times X \to X$ induces a morphism

$$\mathrm{pr}_i^* : H^m_L(X, \mathbb{Z}(n)) \to H^m_L(X \times X, \mathbb{Z}(n)) \ \text{ and } \ \mathrm{pr}_i^* : H^m_L(\mathbb{P}^d_{\bar{k}}, \mathbb{Z}(n)) \to H^m_L(\mathbb{P}^d_{\bar{k}} \times \mathbb{P}^d_{\bar{k}}, \mathbb{Z}(n))$$

for every bi-degree (m, n). By functoriality properties of the Hochschild-Serre spectral sequence we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "f"] & \mathrm{Br}(k) \arrow[r] \arrow[d, "\tilde{f}"] & \mathrm{Br}(X) \arrow[d] \\ 0 \arrow[r] & \mathrm{Pic}(X \times X) \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "s"] & \mathrm{Br}(k) \arrow[r] & \mathrm{Br}(X \times X) \end{tikzcd}$$

where the vertical arrows are induced by pr_i^*. The composition $\mathrm{pr}_i \circ \Delta$ is the identity on X, thus $\mathrm{id}^* = \Delta^* \circ \mathrm{pr}_i^*$ therefore we obtain that the maps $f : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ and $\tilde{f} : \mathrm{Br}(k) \to \mathrm{Br}(k)$ are injective and then, the elements of the form $(a, 0)$ and $(0, b)$ are mapped to $a\,[X]$ and $b\,[X] \in \mathrm{Br}(k)$ respectively. For the general case we consider the maps

$$\begin{tikzcd} X \arrow[r, "\tilde{\Delta}"] & \overbrace{X \times \ldots \times X}^{\text{n-times}} \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\mathrm{pr}_n"'] & X \end{tikzcd}$$

where $\tilde{\Delta}$ is the n-diagonal morphism and pr_i is the projection to the i-th component, and conclude as in the case of $X \times X$. $\square$

Theorem 4.2.15. *Let k be a field and let X be a Severi-Brauer variety over k of dimension d. Then $I_{ét}(X^{\times n}) \geq I_{ét}(X) \geq ord([X])$.*

Proof. Fix an integer $n \geq 1$. By hypothesis we have that $X_{\bar{k}} \simeq \mathbb{P}^d_{\bar{k}}$, hence $(X^{\times n})_{\bar{k}} \simeq \mathbb{P}^d_{\bar{k}} \times \ldots \times \mathbb{P}^d_{\bar{k}}$. Consider the Hochschild-Serre spectral sequence for the Lichtenbaum cohomology of $X^{\times n}$

$$E_2^{p,q} = H^p(G_k, H^q_L((\mathbb{P}^d_{\bar{k}})^{\times n}, \mathbb{Z}(nd))) \Longrightarrow H^{p+q}_L(X^{\times n}, \mathbb{Z}(nd)).$$

By the projective bundle formula for Lichtenbaum cohomology we have

$$H^{2nd-1}_L\left((\mathbb{P}^d_{\bar{k}})^{\times n}, \mathbb{Z}(nd)\right) \simeq \bigoplus_{0\leq a_1,\ldots a_n\leq d} H_L^{2nd-1-2\sum_{j=1}^n a_j}\left(\mathrm{Spec}(\bar{k}), \mathbb{Z}\left(nd - \sum_{j=1}^n a_j\right)\right).$$

If $2nd-1-2\sum_{j=1}^d a_j > nd-\sum_{j=1}^n a_j$ then $H_L^{2nd-1-2\sum_{j=1}^n a_j}(\mathrm{Spec}(\bar{k}), \mathbb{Z}(nd-\sum_{j=1}^n a_j)) = 0$, this give us a vanishing condition for $nd-1 > \sum_{j=1}^n a_j$. As $0 \leq a_j \leq d$ for all j, then the only n-tuples $(a_1, \ldots, a_n)$ which do not satisfy such condition are

$$\epsilon_i = (d, \ldots, d, \overbrace{d-1}^{i\text{-th pos.}}, d, \ldots, d) \text{ for all } i, \text{ and } (d, \ldots, d).$$

For such cases, if $a_j = d$ for all j then

$$H_L^{2nd-1-2nd}(\mathrm{Spec}(\bar{k}), \mathbb{Z}(nd-nd)) = H_L^{-1}(\mathrm{Spec}(\bar{k}), \mathbb{Z}(0)) = 0,$$

and if $(a_1, \ldots, a_n) = \epsilon_i$, then

$$H_L^{2nd-1-2\sum_{j=1}^n a_j}(\mathrm{Spec}(\bar{k}), \mathbb{Z}(nd-\sum_{j=1}^n a_j)) = H^1(\mathrm{Spec}(\bar{k}), \mathbb{Z}(1)) \simeq K_1^M(\bar{k}) = \bar{k}^*.$$

Hence $H_L^{2nd-1}((\mathbb{P}^d_{\bar{k}})^{\times n}, \mathbb{Z}(nd)) \simeq \bigoplus_{i=1}^n \bar{k}^*$ and consequently $E_2^{2,2nd-1} \simeq \bigoplus_{i=1}^n \mathrm{Br}(k)$. The term $E_3^{0,2nd}$ is isomorphic to $\ker\left\{\mathrm{CH}^{nd}((\mathbb{P}^d_{\bar{k}})^{\times n})^{G_k} \xrightarrow{g} \bigoplus_{i=1}^n \mathrm{Br}(k)\right\}$,. Consider the element

$$\delta = c_1\left(\mathcal{O}_{\mathbb{P}^d_{\bar{k}}\times\ldots\times\mathbb{P}^d_{\bar{k}}}(1)\right)^{nd-1} = \sum_{\substack{a_1,\ldots,a_n\in\{d-1,d\}\\ a_1+\ldots+a_n=nd-1}} x_1^{a_1}\cdots x_n^{a_n}$$

and let x_i be the pull-back of the generator of $\mathrm{Pic}(\mathbb{P}^d_{\bar{k}})$ through the map $\mathrm{pr}_i : X^{\times n} \to X$. The intersection product with δ defines morphisms

$$\mathrm{Pic}((\mathbb{P}^d_{\bar{k}})^{\times n}) \xrightarrow{\cup\delta} \mathrm{CH}^{nd}((\mathbb{P}^d_{\bar{k}})^{\times n}) \quad \text{and} \quad H^1_L((\mathbb{P}^d_{\bar{k}})^{\times n}), \mathbb{Z}(1)) \xrightarrow{\cup\delta} H_L^{2nd-1}((\mathbb{P}^d_{\bar{k}})^{\times n}), \mathbb{Z}(nd)).$$

By the functoriality of the Hochschild-Serre spectral sequence we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}((\mathbb{P}^d_{\bar{k}})^{\times n})^{G_k} \arrow[r, "s"] \arrow[d] & \mathrm{Br}(k) \arrow[d] \\
\mathrm{CH}^{nd}((\mathbb{P}^d_{\bar{k}})^{\times n})^{G_k} \arrow[r, "g"] & \bigoplus_{i=1}^n \mathrm{Br}(k),
\end{tikzcd}$$

where the vertical arrows are induced by δ. According to Lemma 4.2.14 the map s sends $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ to $\sum_{i=1}^{d} \alpha_i [X] \in \mathrm{Br}(k)$. Note that the map $x_i \mapsto x_1^d \cdots x_i^{d-1} \cdots x_n^d$ induces an isomorphism $\mathrm{CH}^1((\mathbb{P}^d_{\bar{k}})^{\times n}) \simeq \mathrm{CH}^{nd-1}((\mathbb{P}^d_{\bar{k}})^{\times n})$ and that we have an isomorphism

$$\mathrm{CH}^{nd-1}((\mathbb{P}^d_{\bar{k}})^{\times n}) \otimes H^1((\mathbb{P}^d_{\bar{k}})^{\times n}, \mathbb{Z}(1)) \simeq H_L^{2nd-1}((\mathbb{P}^d_{\bar{k}})^{\times n}, \mathbb{Z}(nd))$$

given by the map $(\alpha_1, \dots, \alpha_n) \otimes \beta \mapsto \beta(\alpha_1, \dots, \alpha_n)$ which is the cup product.

Therefore g maps $a \in \mathrm{CH}^{nd}((\mathbb{P}^d_{\bar{k}})^{\times n})^{G_k}$ to $(a[X], \dots, a[X]) \in \mathrm{Br}(k)$ giving us that $\ker(g) = \mathrm{ord}([X])\mathbb{Z}$. Since $E_\infty^{0,2nd} \hookrightarrow E_3^{0,2nd} = \ker(g)$ and $\deg_{\text{ét}}$ factors through $E_\infty^{0,2nd}$ we conclude the proof. $\square$

The natural question that arises is when this bound is reached. This is the case for the product $C \times C$ when C is a smooth, geometrically connected curve of genus 0 over a field k such that $C_{\bar{k}} \simeq \mathbb{P}^1_{\bar{k}}$ as the following proposition shows:

Proposition 4.2.16. *Let k be a perfect field of characteristic $p \geq 0$ with Galois group G_k, and let C be a smooth, geometrically connected curve of genus 0 over the field k such that $C_{\bar{k}} \simeq \mathbb{P}^1_{\bar{k}}$, then $I_{\acute{e}t}(C \times C) = ord([C])$.*

Proof. By our assumptions we have that $C_{\bar{k}} \simeq \mathbb{P}^1_{\bar{k}}$ then $(C \times C)_{\bar{k}} \simeq \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$. Considering the Hochschild-Serre spectral sequence for Lichtenbaum cohomology

$$E_2^{p,q} = H^p(G_k, H_L^q(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2))) \Longrightarrow H_L^{p+q}(C \times C, \mathbb{Z}(2)).$$

Since $H_L^m(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) \simeq H_M^m(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2))$ for $m \leq 3$, using again the projective bundle formula for motivic cohomology we obtain that

$$\begin{aligned}
H_L^3(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) &\simeq H_M^3(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) \oplus H_M^1(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}(1)) \simeq K_1(\bar{k}) \oplus K_1(\bar{k}) \\
H_L^2(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) &\simeq H_M^2(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) \oplus H_M^0(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}(1)) \simeq K_2(\bar{k}) \\
H_L^1(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) &\simeq H_M^1(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) \simeq H_M^1(\mathrm{Spec}(\bar{k}), \mathbb{Z}(2)) \\
H_L^0(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) &\simeq H_M^0(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) \simeq H_M^0(\mathrm{Spec}(\bar{k}), \mathbb{Z}(2)).
\end{aligned}$$

As we have mentioned before, $H_M^0(\mathrm{Spec}(\bar{k}), \mathbb{Z}(2))$ and $K_2(\bar{k})$ are uniquely divisible, hence for $p > 0$ we have $E_2^{p,0} = E_2^{p,2} = 0$. Due to the compatibility of étale cohomology with colimits, and in particular with direct sums, we obtain $E_2^{p,3} \simeq H^p(G_k, \bar{k}^*) \oplus H^p(G_k, \bar{k}^*)$. In particular, notice that again Hilbert's theorem 90 gives us that $E_2^{1,3} = 0$ and that by definition $E_2^{2,3} \simeq \mathrm{Br}(k) \oplus \mathrm{Br}(k)$.

With this information about the E_2-terms, we obtain $E_\infty^{1,3} = E_\infty^{2,2} = E_\infty^{4,0} = 0$, $E_\infty^{0,4} = \ker\left\{\mathrm{CH}^2(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}})^{G_k} \to \mathrm{Br}(k) \oplus \mathrm{Br}(k)\right\}$ and $E_\infty^{3,1} = E_2^{3,1}/\mathrm{im}\left\{E_2^{0,3} \to E_2^{3,1}\right\}$. Hence we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_\infty^{3,1} \arrow[r] & \mathrm{CH}^2_{\text{ét}}(C \times C) \arrow[r] \arrow[dr, "\deg_{\text{ét}}"'] & E_\infty^{0,4} \arrow[r] \arrow[d, "\widetilde{\deg}"] & 0 \\
& & & \mathbb{Z} &
\end{tikzcd}$$

where $\widetilde{\deg} : E_\infty^{0,4} \to \mathbb{Z}$ is the composition of the following maps:

$$E_\infty^{0,4} \hookrightarrow \mathrm{CH}^2(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}})^{G_k} \xrightarrow{\simeq} \mathrm{CH}^2(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}) \xrightarrow{\deg} \mathbb{Z}.$$

Let us give more information about the term $E_\infty^{0,4}$. Mimicking the proof of Theorem 4.2.15, we have an isomorphism $\mathrm{Pic}(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}) \simeq \mathbb{Z}[x] \oplus \mathbb{Z}[y]$. The Chern class $\delta = c_1(\mathcal{O}_{\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}}(1)) = x + y$ induces morphisms

$$H^1_L(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(1)) \xrightarrow{\cup\delta} H^3_L(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) \ \text{ and } \ \mathrm{CH}^1(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}) \xrightarrow{\cup\delta} \mathrm{CH}^2(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}).$$

Consider the isomorphism

$$\begin{aligned} \mathrm{CH}^1(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}) \otimes H^1_L(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(1)) &\xrightarrow{\simeq} H^3_L(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}, \mathbb{Z}(2)) \\ (a,b) \otimes \alpha &\mapsto \alpha(a,b) \end{aligned}$$

induced by the cup product. Hence the cup product with the diagonal induces a map $\mathrm{Br}(k) \to \mathrm{Br}(k) \oplus \mathrm{Br}(k)$ defined by $a \mapsto (a,a)$ and then we can deduce that $\mathrm{CH}^2(\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}})^{G_k} \to \mathrm{Br}(k) \oplus \mathrm{Br}(k)$ sends the $1 \mapsto ([C],[C])$. Since $E_\infty^{0,4} \simeq \mathrm{ord}([C])\mathbb{Z}$ we conclude that $I_{\text{ét}}(C \times C) = \mathrm{ord}([C])$. □

Remark 4.2.17. If k is a field with $\mathrm{Br}(k) = 0$, then the Severi-Brauer varieties X over k split and $I(X) = I_{\text{ét}}(X) = 1$. Hence Theorem 4.2.15 shows that $\mathrm{Br}(k)$ is an obstruction for the existence of an étale zero cycle of degree 1.

4.3 Decomposition of étale motives

We apply Theorems 4.2.4 and 4.2.5 to the decomposition of integral étale motives. Even though by Theorem 4.2.15 there exists X such that $I_{\text{ét}}(X) \neq 1$, at least we have that $I(X) \geq I_{\text{ét}}(X)$ and in the particular case of Theorems 4.2.4 and 4.2.5 we obtain the existence of integral projectors in the following sense: if there exists an element $e \in \mathrm{CH}^d_{\text{ét}}(X)$ of étale degree 1 then we define

$$p_0^{\text{ét}}(X) = \mathrm{pr}_1^*(e) \cdot \mathrm{pr}_2^*(X) \ \text{ and } \ p_{2d}^{\text{ét}}(X) = \mathrm{pr}_1^*(X) \cdot \mathrm{pr}_2^*(e)$$

where $\mathrm{pr}_i : X \times X \to X$ is the projection to the i-th factor, this lead us to a decomposition of the integral motive $h_{\text{ét}}(X)$ as follows

$$h_{\text{ét}}(X) = h^0_{\text{ét}}(X) \oplus h^+_{\text{ét}}(X) \oplus h^{2d}_{\text{ét}}(X)$$

where $h^0_{\text{ét}}(X) = (X, p_0^{\text{ét}}(X), 0)$ and $h^{2d}_{\text{ét}}(X) = (X, p_{2d}^{\text{ét}}(X), 0)$. These projectors do not exist in the integral classical Chow groups and then we have an improvement in the existence of integral projectors by changing from Chow to étale motivic cohomology as expected.

Notice that in general this improvement is not always possible, for example, if C is a projective curve over k without a zero cycle of degree 1, then the projectors $p_0(X)$ and

$p_2(C)$ do not exist and consequently there is no chance of obtaining an integral decomposition in the classical nor the étale setting. This is a consequence of the isomorphism $\mathrm{CH}^1(X) \simeq \mathrm{CH}^1_{\text{ét}}(X)$ when X is smooth and projective over a field k of characteristic zero.

Our goal is to study the existence of an integral decomposition of the motive $h_{\text{ét}}(X)$. Let us start by giving the definition of an integral decomposition in $\mathrm{Chow}_{\text{ét}}(k)$:

Definition 4.3.1. *Let k be a field and let $f : X \to k$ be a smooth projective variety, of dimension d. We say that $h_{ét}(X)$ admits an integral Chow-Künneth decomposition in $Chow_{ét}(k)$ if:*

- *$h(X)$ admits a rational Chow-Künneth decomposition, see [MNP13, Definition 6.1.1],*

$$h(X) \xrightarrow{\simeq} \bigoplus_{i=0}^{2d} h^i(X) \in Chow(k)_{\mathbb{Q}},$$

  and this map is induced by a morphism $g : h_{ét}(X) \to M = (Y, p)$ in $Chow_{ét}(k)$.

- *Consider the base change to the algebraic closure $\bar{g} : h_{ét}(X_{\bar{k}}) \to M_{\bar{k}}$. For every prime number $\ell \neq char(k)$, the induced map $\rho_\ell(\bar{g}) : R\bar{f}_*(\mathbb{Z}/\ell) \to M_{\bar{k}}/\ell \in D(\bar{k}_{ét}, \mathbb{Z}/\ell)$ is an isomorphism and $\rho_\ell(\bar{p}) = p_1 + \ldots + p_{2d}$ with the following conditions*

$$p_i \circ p_j = \begin{cases} p_i \text{ if } i = j \\ 0 \text{ if } i \neq j, \end{cases} \qquad \rho(\bar{g})^{-1} \circ p_i(M_{\bar{k}}/\ell) = R^i \bar{f}_*(\mathbb{Z}/\ell) \text{ for all } i.$$

This is nothing but a direct translation of the conservativity properties of the family of functors associated to the change of coefficients in [CD16, Proposition 5.4.12] and combining the results about conservativity [Ayo14b, Théorème 3.9] and continuity [CD16, Proposition 6.3.7] applied to $\bar{k} = \varprojlim k_i$ where k_i runs over the finite fields extensions of k.

Proposition 4.3.2. *Consider a field k of finite cohomological dimension and let $h_{ét}(X) \in Chow_{ét}(k)$. Then $h_{ét}(X_{\bar{k}})$ has an integral Chow-Künneth decomposition if and only if there exists a field extension K/k such that $h_{ét}(X_K)$ has an integral Chow-Künneth decomposition.*

Proof. For simplicity, up to tensoring with Lefschetz motive, which is a direct summand of a geometric motive, we may assume that $M = (X, p)$. If $M_{\bar{k}}$ has an integral Chow-Künneth decomposition then the result is trivial.

Conversely, assume that there exists a Chow-Künneth decomposition for some field extension K/k. For the rational part we invoke [Via17, Proposition 1.5]. For the torsion part, let $\ell \neq \mathrm{char}(k)$ be a prime number and consider the field extension K/k. Consider the morphism of $s : \mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(\bar{k})$. Let us assume that $h_{\text{ét}}(X_K)$ has a Chow-Künneth decomposition, thus we have that $\rho_\ell(\bar{g_K}) : R\bar{f}_K(\mathbb{Z}/\ell) \to M_{\bar{K}}/\ell$ with the above properties for $\rho_\ell(\bar{p}_K)$. The induced functor $s^* : D(\bar{k}_{\text{ét}}, \mathbb{Z}/\ell) \to D(\bar{K}_{\text{ét}}, \mathbb{Z}/\ell)$ is an equivalence of categories, hence $\rho_\ell(\bar{g}) : R\bar{f}_*(\mathbb{Z}/\ell) \to M_{\bar{k}}/\ell \in D(\bar{k}_{\text{ét}}, \mathbb{Z}/\ell)$ is an isomorphism, thus we have the same results for $\rho_\ell(\bar{p})$ and conclude the proof.

□

Étale decomposition of complex varieties

We present some applications of [RS16, Theorem 1.1] to the integral decomposition of étale Chow motives. The simplest case is the one described in [MNP13, Appendix C] for varieties without transcendental cohomology classes in degrees different from the dimension.

Proposition 4.3.3. *Fixing $k = \mathbb{C}$, let X be a smooth projective complex variety of dimension d such that the groups $H^i_B(X, \mathbb{Q})$ are algebraic for all $i \neq d$. Then $h_{\acute{e}t}(X)$ admits an integral Chow-Künneth decomposition in Chow$_{\acute{e}t}(\mathbb{C})$.*

Proof. We will use the equivalence given in [RS16, Theorem 1.1] and [MNP13, Appendix C]. Let us start by saying that according to [RS16, Theorem 1.1.a] the map Lichtenbaum cycle class map

$$c_L^{m,n} : H^m_L(X, \mathbb{Z}(n)) \to H^m_B(X, \mathbb{Z}(n))$$

restricted to the torsion subgroup $H^m_L(X, \mathbb{Z}(n))_{\text{tors}} \to H^m_B(X, \mathbb{Z}(n))_{\text{tors}}$ is surjective. With this in mind we consider that the groups $H^i_B(X, \mathbb{Z})$ are torsion free and then Poincaré duality holds, i.e. the pairing

$$\begin{aligned} H^i_B(X, \mathbb{Z}) \otimes H^{2d-i}_B(X, \mathbb{Z}) &\to \mathbb{Z} \\ (\alpha, \beta) &\xrightarrow{\cup} \alpha \cup \beta \end{aligned}$$

is perfect. By [RS16, Theorem 1.1] we have $H^{2i}_B(X, \mathbb{Q})$ is algebraic if and only if $H^{2i}_B(X, \mathbb{Z})$ is L-algebraic, thus there exists a set of cycles which are send to the generators $\left\{e^{2i}_j\right\}_{1 \leq j \leq b_{2i}(X)}$ of $H^i_B(X, \mathbb{Z})$ and notice that by Poincaré duality we have a dual basis $\left\{\hat{e}^{2(d-i)}_j\right\}_{1 \leq j \leq b_{2(d-i)}(X)}$ for the dual of $H^{2i}_B(X, \mathbb{Z})$. Let us remark that we have the following

$$e^{2i}_j \cup \hat{e}^{2(d-i)}_l = \begin{cases} 0 \text{ if } j \neq l \\ 1 \text{ if } j = l \end{cases}$$

By hypothesis, there exists L-algebraic cycles $\left\{\alpha^i_j\right\}_{1 \leq j \leq b_{2i}(X)} \subset \mathrm{CH}^i_L(X)$ and

$$\left\{\hat{\alpha}^{d-i}_l\right\}_{1 \leq l \leq b_{2(d-i)}(X)} \subset \mathrm{CH}^{d-i}_L(X)$$

such that

$$c^i_L(\alpha^i_j) = e^{2i}_j, \quad c^{d-i}_L(\hat{\alpha}^{d-i}_l) = \hat{e}^{2(d-i)}_l$$

for all $1 \leq j \leq b_{2i}(X)$ and $1 \leq l \leq b_{2(d-i)}(X)$. Due to the compatibility of the cycle class map with intersection products we have that

$$\alpha_j^i \cdot \hat{\alpha}_l^{d-i} = \begin{cases} 0 \text{ if } j \neq l \\ 1 \text{ if } j = l. \end{cases}$$

Let us define the elements

$$p_{2i,j} = \alpha_j^i \times \hat{\alpha}_j^{d-i} \quad q_{2i,j} = \hat{\alpha}_j^{d-i} \times \alpha_j^i$$

and note that $p_{2i,j} = q_{2i,j}^t$. Even more, these are orthogonal projectors. For $i < d$, define the projectors

$$p_{2i}(X) := \sum_{1 \leq j \leq b_{2i}(X)} p_{2i,j} \quad p_{2(d-i)}(X) := \sum_{1 \leq j \leq b_{2i}(X)} q_{2i,j}$$

and for $2i-1 \neq d$ we put $p_{2i-1}(X) = 0$. The remaining part should involve torsion classes. As the groups $H_B^{2j+1}(X, \mathbb{Z})$ are torsion for all $j \in \mathbb{N}$, the groups $H_B^{2k+1}(X \times X, \mathbb{Z})$ are torsion for all $k \in \mathbb{N}$ by the Künneth formula, this implies that all intermediates Jacobians $J^{k+1}(X \times X)$ vanish for all $k \in \mathbb{N}$. Combining [RS16, Theorem 1.1.b] and [Ros23b, Proposition 3.1.5] we obtain an isomorphism $\mathrm{CH}_L^k(X \times X)_{\mathrm{tors}} \xrightarrow{\simeq} H_B^{2k}(X \times X, \mathbb{Z}(k))_{\mathrm{tors}}$ for all $k \in \mathbb{N}$, so in particular for the degree $k = d$. Consider that we have the diagonal element Δ and let us denote the torsion free part as $\Delta_{\mathrm{tf}} = \sum_{j=0}^{2d} p_i(X)$ and consider $\Delta_{\mathrm{tors}} = \Delta - \Delta_{\mathrm{tf}}$. As this element $\Delta_{\mathrm{tors}} \in H_B^{2d}(X \times X, \mathbb{Z})$, then it has a unique preimage in $\mathrm{CH}_L^d(X \times X)_{\mathrm{tors}}$, which is denoted as Δ_{tors} again, thus we have the following decomposition of the diagonal

$$\Delta = \sum_{j=0}^{2d} p_i(X) + \Delta_{\mathrm{tors}}.$$

Since the isomorphism $\mathrm{CH}_L^k(X \times X)_{\mathrm{tors}} \xrightarrow{\simeq} H_B^{2k}(X \times X, \mathbb{Z})_{\mathrm{tors}}$ is an isomorphism for all k, the projectors in $H_B^{2k}(X \times X, \mathbb{Z})_{\mathrm{tors}}$ can be lifted to $\mathrm{CH}_L^k(X \times X)$.

□

Example 4.3.4. *1. Let X be a smooth complex complete intersection in projective space. As all the cohomology groups are algebraic and torsion free, we have a decomposition of étale integral motives as follows:*

$$h_{ét}(X) \simeq \mathbf{1} \oplus \mathbb{L} \oplus \ldots \oplus h_{ét}^d(X) \oplus \ldots \oplus \mathbb{L}^d.$$

where $\mathbb{L}$ is the Lefschetz motive and $h_{ét}^d(X) = (X, p_d^{ét}(X), 0)$ with

$$p_d^{ét}(X) = \Delta - \sum_{i=0, 2i \neq d}^{2d} p_i^{ét}(X).$$

2. Let X be a smooth K3 surface. For such X we have the following isomorphisms

$$H^0(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z}) \simeq \mathbb{Z}, \;\; H^1(X, \mathbb{Z}) \simeq H^3(X, \mathbb{Z}) = 0, \quad H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$$

and Pic(X) $= \mathbb{Z}^{\rho(X)}$, with $\rho(X)$ the Picard rank of X and $0 \leq \rho(X) \leq 20$. Since the cohomology is torsion free, we apply Proposition 4.3.3 to obtain a decomposition of the étale motive

$$h_{\acute{e}t}(X) \simeq h^0_{\acute{e}t}(X) \oplus h^2_{\acute{e}t}(X) \oplus h^4_{\acute{e}t}(X).$$

3. *Let S be an Enriques surface. As $H^i(\mathcal{O}_S) = 0$ for $i = 1, 2$ we have an isomorphism $Pic(S) \to H^2_B(S, \mathbb{Z}) \simeq \mathbb{Z}^{10} \oplus \mathbb{Z}/2$ while the other cohomology groups are characterized by*

$$H^0(S, \mathbb{Z}) = H^4(S, \mathbb{Z}) = \mathbb{Z},\ H^1(S, \mathbb{Z}) = 0 \quad and \quad H^3(S, \mathbb{Z}) = \mathbb{Z}/2.$$

as we can lift the torsion free part, we have to care about the torsion part of the cohomology. By Künneth formula, we have that $H^5_B(S \times S, \mathbb{Z}) \simeq (\mathbb{Z}/2)^{\oplus 23}$ and $H^3_B(S \times S, \mathbb{Z}) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ thus we conclude that the intermediate Jacobians $J^2(S \times S) = 0$ and $J^3(S \times S) = 0$ vanish. Combining [RS16, Proposition 5.1] and Corollary 3.1.6, we have an isomorphism $CH^2_L(S \times S)_{tors} \xrightarrow{\simeq} H^4_B(S \times S, \mathbb{Z}(2))_{tors}$ which acts as the identity on the torsion part.

4. *For a Calabi-Yau threefold X (for example a quintic threefold) X the Betti numbers are $h^1(X) = h^5(X) = 0$ and $h^0(X) = h^2(X) = h^4(X) = h^6(X) = 1$, thus we obtain a decomposition of the motive $h_{\acute{e}t}(X)$ as*

$$h_{\acute{e}t}(X) \simeq \mathbf{1} \oplus \mathbb{L} \oplus h^3_{\acute{e}t}(X) \oplus \mathbb{L}^2 \oplus \mathbb{L}^3.$$

Commutative group schemes

Let S be a noetherian finite dimensional scheme and let G/S a smooth commutative group scheme of finite type over S. We start with the definition of the 1-motive associated to G/S, for that we define the étale sheaf induced by G/S:

Definition 4.3.5. *Let $\underline{G/S}$ be the étale sheaf of abelian groups on Sm_S defined by G:*

$$\underline{G/S}(U) = Hom_{Sm_S}(U, G)$$

for $U \in Sm_S$. We say that $M_1(G/S)$ is the 1-motive associated to $\underline{G/S}$ and is defined as

$$M_1(G/S) := \Sigma^\infty M_1^{eff}(G/S) \in \boldsymbol{DA}^{\acute{e}t}(S, \mathbb{Z}),$$

where $M_1^{eff}(G/S)$ is the effective étale motive in $\boldsymbol{DA}^{\acute{e}t}_{eff}(S, \mathbb{Z})$ induced by $\underline{G/S}$.

According to [AHP16, Theorem 3.7], we have a decomposition of the relative motive $M_S(G)$ in the motivic category $\mathrm{DM}_{\acute{e}t}(S, \mathbb{Q})$ in the following way

$$M_S(G) \xrightarrow{\simeq} \left(\bigoplus_{n \geq 0}^{\mathrm{kd}(G/S)} \mathrm{Sym}^n M_1(G/S) \right) \otimes M(\pi_0(G/S)),$$

where $M_1(G/S)$ is the 1-motive induced by the étale sheaf represented by $\underline{G/S} \otimes \mathbb{Q}$ and $\mathrm{kd}(G/S) := \max\{2g_s + r_s \mid s \in S\}$ is the Kimura dimension (g_s is the abelian rank of G_s and r_s is the torus rank).

Definition 4.3.6. *The order of $\pi_0(G/S)$, denoted by $o(\pi_0(G/S))$ is defined as the least common multiple of the order of all the elements of the groups $\pi_0(G_{\bar{s}}/\bar{s})$, with $\bar{s}$ geometric point of S.*

The aim of this subsection is to see if we can lift this isomorphism to integral coefficients in $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})$. If we want to construct such a morphism in $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})$, we have to define the integral analogue of the *symmetric algebra*. For this we consider the homotopy fixed points of a group action, where the group is finite.

$\mathfrak{S}_n$-actions on $\mathbf{DM}_{\text{ét}}(S,\mathbb{Z})$

In this subsection we will present some aspects about the action of a finite group G in the category $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})$ of integral étale motives. In this context, an ∞-category will be an $(\infty,1)$-category in the sense of Lurie [Lur09]. An ∞-functor between two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is simply a map $F:\mathcal{C}\to\mathcal{D}$ of simplicial sets.

Consider the group of permutations of n elements $\mathfrak{S}_n$ and let $B\mathfrak{S}_n$ be the category of a single object and morphism the elements of the groups $\mathfrak{S}_n$. Define the **homotopy fixed points** and **homotopy orbits** of $\mathfrak{S}_n$ of a motive $M^S_{\text{ét}}(X)$ as follows: we know that $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})$ carries a structure of an ∞-category. Let $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})^{B\mathfrak{S}_n}$ be the category of étale motives with a $\mathfrak{S}_n-$action, i.e. ∞-functors $B\mathfrak{S}_n\to\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})$. We obtain adjunctions

$$(\ )^{\mathrm{triv}}:\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})\leftrightarrows\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})^{B\mathfrak{S}_n}:(\ )^{h\mathfrak{S}_n}:=\mathrm{holim}_{B\mathfrak{S}_n},$$
$$\mathrm{hocolim}_{B\mathfrak{S}_n}=:(\ )_{h\mathfrak{S}_n}:\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})^{B\mathfrak{S}_n}\leftrightarrows\mathrm{DM}_{\text{ét}}(S,\mathbb{Z}):(\ )^{\mathrm{triv}}$$

where $(\ )^{\mathrm{triv}}$ represents the trivial action. Let $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})^{\otimes}$ be the underlying symmetric monoidal category of $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})$. We can give an explicit description of $(-)^{h\mathfrak{S}_n}$ for some motives by using the monoidal structure of $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})^{\otimes}$. Notice that for $X\in\mathrm{Sm}_k$ we have an action of $\mathfrak{S}_n$ given by

$$\sigma_*:X^n\to X^n$$
$$(x_1,\dots,x_n)\mapsto(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

where $\sigma\in\mathfrak{S}_n$ and $X^n:=\overbrace{X\times\ldots\times X}^{\text{n-times}}$. For such X and n, consider the functor

$$F^n_X:B\mathfrak{S}_n\to\mathrm{Sm}_S$$
$$*\mapsto X^n$$
$$(*\xrightarrow{\sigma}*)\mapsto(X^n\xrightarrow{\sigma_*}X^n)$$

We can consider the motive $M^S_{\text{ét}}(X)^{\otimes n}$ as an ∞-functor from the category Sm_S to $\mathrm{DM}_{\text{ét}}(S,\mathbb{Z})^{\otimes}$. Therefore we obtain the homotopy fixed points of $M^S_{\text{ét}}(X)^{\otimes n}$ as

$$\left(M^S_{\text{ét}}(X)^{\otimes n}\right)^{h\mathfrak{S}_n}\simeq\mathrm{holim}_{B\mathfrak{S}_n}M^S_{\text{ét}}\circ F^n_X(-). \tag{4.2}$$

If we $\mathbb{Q}$-linearize the homotopy fixed points, then we have the following result relating them with the usual fixed points of a group action:

Lemma 4.3.7. *Let $M_S(X^n) \simeq M_S(X)^{\otimes n} \in DM_{ét}(S, \mathbb{Q})$, then*

$$\left(M_S(X)^{\otimes n}\right)^{h\mathfrak{S}_n} \simeq \left(M_S(X)^{\otimes n}\right)^{\mathfrak{S}_n}$$

and equals the image of the projector $\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \sigma_* M_S(X)^{\otimes n}$.

Proof. This holds in greater generality, see [CD19, p. 3.3.21]. If $\mathcal{V}$ is a $\mathbb{Q}$-linear stable model category and G is a finite group that acts on an object $E \in \mathcal{V}$ we define

$$E^{hG} := \operatorname{holim}_{BG} E,$$

and define $E^G \in Ho(\mathcal{V})$ as the image of

$$p(x) = \frac{1}{|G|}\sum_{g\in G} g.x.$$

Then the morphism $E^G \xrightarrow{\sim} E^{hG}$ induced by the inclusion $E^G \to E$ is an isomorphism in $Ho(\mathcal{V})$. □

Remark 4.3.8.

1. The same argument works in a category $\mathrm{DM}_{\text{ét}}(S, \Lambda)$ if n is invertible in the ring Λ. A very important remark is that the proof of the previous lemma relies in the commutative structure of the $\mathbb{Q}$-linear vector space. If we consider the homotopy fixed points using an anti-commutative structure (and n is invertible), then we obtain that $E^{h\mathfrak{S}_n}$ equals the image of the projector $\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma_* E$.

2. In the same way, we define the homotopy orbits of $\mathfrak{S}_n$ as the co-invariants of $M^S_{\text{ét}}(X)^{\otimes n}$, i.e. in the following way $(M^S_{\text{ét}}(X)^{\otimes n})_{h\mathfrak{S}_n} = \operatorname{hocolim}_{B\mathfrak{S}_n} M^S_{\text{ét}}(X)^{\otimes n}$. By the definition of homotopy colimit, we have a map $M^S_{\text{ét}}(X)^{\otimes n} \to (M^S_{\text{ét}}(X)^{\otimes n})_{h\mathfrak{S}_n}$.

3. Let $\mathrm{DM}_h(S, \hat{\mathbb{Z}}_\ell)$ be localizing subcategory of $\mathrm{DM}_h(S, \mathbb{Z})$ generated by the objects of the form $M/\ell = \mathbb{Z}/\ell \otimes^R M$. According to [CD16, p. 7.2.10] we have an adjunction $\hat{\rho}^*_\ell : \mathrm{DM}_h(S, \mathbb{Z}) \leftrightarrows \mathrm{DM}_h(S, \hat{\mathbb{Z}}_\ell) : \hat{\rho}_{\ell *}$, where $\hat{\rho}^*_\ell$ is called the ℓ-adic realization functor, which by [CD16, Theorem 7.2.11] it is compatible with the six functors formalism of Grothendieck, and preserves colimits. Let $D(S_{\text{ét}}, \mathbb{Z}_\ell)$ be the derived category of ℓ-adic sheaves as in [Eke90]. Consider the equivalence of categories given in [CD16, Proposition 7.2.21], then $\mathrm{DM}_h(S, \hat{\mathbb{Z}}_\ell) \simeq D(S_{\text{ét}}, \mathbb{Z}_\ell)$, so we define the realization functor $\rho_\ell : \mathrm{DM}_h(S, \mathbb{Z}) \to D(S_{\text{ét}}, \mathbb{Z}_\ell)$. This functor again is compatible with the six functors formalism of Grothendieck, and preserves colimits (as it is the composition of a left adjoint with an), thus we have that

$$\rho_\ell\left((M^S_{\text{ét}}(X)^{\otimes n})_{h\mathfrak{S}_n}\right) \simeq \left((\rho_\ell M^S_{\text{ét}}(X))^{\otimes n}\right)_{h\mathfrak{S}_n}.$$

For the sake of completeness, we will present a reminder about the theory of 1-motives, with such goal in mind, we present some of the main results of [Org04]. Consider a commutative group scheme G over a perfect field k. According to [Org04, Lemme 3.1.1], the sheaf $\underline{G}$ is an étale presheaf which admits transfers. Notice that as a presheaf $\underline{G}$ is homotopy invariant i.e. $\underline{G}(U) \xrightarrow{\sim} \underline{G}(U \times \mathbb{A}^1_k)$ is an isomorphism for any k-smooth variety U, for a proof see [Org04, Lemme 3.3.1]. As we have a morphism of a complex of sheaves $C_*^{\mathrm{Sus}}\mathbb{Z}^{tr}(G) \to C_*^{\mathrm{Sus}}(\underline{G})$ and a quasi-isomorphism $\underline{G} \to C_*^{\mathrm{Sus}}(\underline{G})$, finally we obtain a morphism in $\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})$ between the motives $M_{\text{ét}}(G) \xrightarrow{\alpha_A} M_1(G)$.

If we want to work over a noetherian base S, and simplify the condition about étale sheaves with transfers, we work with the category $\mathbf{DA}^{\text{ét}}(S,\mathbb{Z})$. Consider a commutative group scheme G over a noetherian base S and let $\underline{G/S}$ the associated abelian sheaf and $M_1(G/S)$ the 1-motive described in Definition 4.3.5. As we have a morphism of a complex of pre-sheaves $a_{G/S} : \mathbb{Z}\mathrm{Hom}_S(\cdot, G) \to \underline{G/S}$, then after sheafification we obtain a morphism $\alpha_{G/S}^{\mathrm{eff}} : M_{\text{ét}}^{S,\mathrm{eff}}(G) \to M_1^{\mathrm{eff}}(G/S) \in \mathbf{DA}_{\mathrm{eff}}^{\text{ét}}(S,\mathbb{Z})$. Finally we obtain a morphism in $\mathbf{DA}^{\text{ét}}(S,\mathbb{Z})$ between the motives $\alpha_{G/S} = \Sigma^\infty \alpha_{G/S}^{\mathrm{eff}} : M_{\text{ét}}^S(G) \xrightarrow{\alpha_A} M_1(G/S)$ in $\mathbf{DA}^{\text{ét}}(S,\mathbb{Z})$.

As the functor $M_{\text{ét}}^S$ is monoidal and commutative, we obtain an isomorphism $M_{\text{ét}}^S(G\times G) \simeq M_{\text{ét}}^S(G) \otimes M_{\text{ét}}^S(G)$. For the general case we denote $M_{\text{ét}}^S(G)^{\otimes n} := M_{\text{ét}}^S(\overbrace{G\times \ldots \times G}^{\text{n-times}})$. For a fixed n and using the n-diagonal morphism $\delta_{G/S}^n : G \to G \times \ldots \times G$, we obtain an induced morphism of motives

$$M_{\text{ét}}^S(G) \xrightarrow{(\delta_{G/S}^n)_*} M_{\text{ét}}^S(G)^{\otimes n}.$$

Together with the map α_G we construct a map

$$\phi_n : M_{\text{ét}}^S(G) \xrightarrow{(\delta_{G/S}^n)_*} M_{\text{ét}}^S(G)^{\otimes n} \xrightarrow{\alpha_{G/S}^{\otimes n}} M_1(G/S)^{\otimes n}.$$

Notice that $M_{\text{ét}}^S(G)^{\otimes n}$ admits an action of the permutation group $\mathfrak{S}_n$, and this action leaves invariant the diagonal map $\delta_{G/S}^n$. Therefore we can apply the functor of homotopy fixed points $h\mathfrak{S}_n$. With this, we have a commutative diagram

$$\begin{tikzcd}
M_{\text{ét}}^S(G) \arrow[r, "(\delta_{G/S}^n)_*"] \arrow[dr, "(\delta_{G/S}^n)_*"'] & M_{\text{ét}}^S(G)^{\otimes n} \arrow[r, "\alpha_{G/S}^{\otimes n}"] & M_1(G/S)^{\otimes n} \\
& \left(M_{\text{ét}}^S(G)^{\otimes n}\right)^{h\mathfrak{S}_n} \arrow[u] \arrow[r, "\alpha_{G/S}^{\otimes n}"] & \left(M_1(G/S)^{\otimes n}\right)^{h\mathfrak{S}_n} \arrow[u]
\end{tikzcd}.$$

We denote the composite map $\alpha_{G/S}^{\otimes n} \circ (\delta_{G/S}^n)_* : M_{\text{ét}}^S(G) \to \left(M_1(G/S)^{\otimes n}\right)^{h\mathfrak{S}_n}$ as $\phi_{G/S}^n$.

Definition 4.3.9. *Let G be a smooth commutative group scheme over a noetherian scheme S. Then we define the following:*

$$\phi_{G/S} := \bigoplus_{i\geq 0} \phi_{G/S}^i : M_{\acute{e}t}^S(G) \to \bigoplus_{i\geq 0} \left(M_1(G/S)^{\otimes i}\right)^{h\mathfrak{S}_i}$$

in the category $DM_{\acute{e}t}(S,\mathbb{Z})$. We define the weak symmetric algebra of $M_1(G)$ as

$$wSym(M_1(G)) := \bigoplus_{i\geq 0} \left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}.$$

Let us give some information about the realization of the morphism $\phi_{G/S}$ presented in Definition 4.3.9, in the category $D(S_{\text{ét}}, \mathbb{Z}/\ell^n)$ for a prime number ℓ invertible in S and $n \in \mathbb{N}$. We have a realization $\rho_{\mathbb{Z}/\ell^n} : \mathrm{DM}_h(S, \mathbb{Z}) \to D(S_{\text{ét}}, \mathbb{Z}/\ell^n)$ and denote $M_1(G/S, \ell^n) := \rho_{\mathbb{Z}/\ell^n}(M_1(G/S))$, which is a complex in degree -1. We use $\mathcal{H}_1(G/S, \mathbb{Z}/\ell^n)$ for the homology of $M_1(G/S, \ell^n)$ is degree 1 and $\mathcal{H}^1(G/S, Z/\ell^n)$ for the cohomology of the complex $M_1(G/S, \ell^n)$ in degree -1. If the base is $S = \mathrm{Spec}(k)$ with $\bar{k} = k$ then $D(k_{\text{ét}}, \mathbb{Z}/\ell) \simeq (\mathbb{F}_\ell - v.s.)^{\mathbb{Z}}$, where $(\mathbb{F}_\ell - v.s.)^{\mathbb{Z}}$ is the category of $\mathbb{Z}$-graded $\mathbb{F}_\ell$-vector spaces, $\mathcal{H}_1(G/S, \mathbb{Z}/\ell^n)$ is a finite dimensional $\mathbb{F}_\ell$-vector space.

Lemma 4.3.10. *Let ℓ be a prime number invertible in S, $n \in \mathbb{N}$ and consider the realization functor $\rho_{\mathbb{Z}/\ell^n} : DM_{\text{ét}}(S, \mathbb{Z}) \to D(S_{\text{ét}}, \mathbb{Z}/\ell^n)$. Then:*

1. *if $\phi : DM_{\text{ét}}(S, \mathbb{Z}) \to DM_{\text{ét}}(S, \mathbb{Z})$ is an additive functor and $\bar{\phi} : D(S_{\text{ét}}, \mathbb{Z}/\ell^n) \to D(S_{\text{ét}}, \mathbb{Z}/\ell^n)$ its associated counterpart with finite coefficients, then the functor $\rho_{\mathbb{Z}/\ell^n}$ commutes with ϕ, in the sense that $\bar{\phi} = \rho_{\mathbb{Z}/\ell^n} \circ \phi$.*

2. *If $S = Spec(k)$ for some field k, then $\rho_{\mathbb{Z}/\ell^n}(M_1(G)) = \mathcal{H}_1(G, \mathbb{Z}/\ell^n) \simeq G[\ell^n][1]$[1].*

3. *There exists $N >> 0$ such that $(M_1(G/S)^{\otimes m})^{h\mathfrak{S}_m} = 0$ for all $m > N$.*

Proof. 1. Recall that the functor is defined as $\rho_{\mathbb{Z}/\ell^n}(M) = \mathbb{Z}/\ell^n \otimes^L M = \mathrm{coker}(M \xrightarrow{\cdot \ell^n} M)$, therefore we have a canonical isomorphism $\rho_{\mathbb{Z}/\ell^n}(M) \simeq \mathrm{Cone}(\ell^n \cdot \mathrm{id}_M)$. Let ϕ be an additive functor, then in the commutative diagram

$$\begin{tikzcd}
\phi(M) \arrow[r, "\phi(\ell^n \cdot \mathrm{id}_M)"] \arrow[d, equal] & \phi(M) \arrow[r] \arrow[d, equal] & \bar{\phi}\left(\rho_{\mathbb{Z}/\ell^n}(M)\right) \arrow[r, "+1"] \arrow[d] & {} \\
\phi(M) \arrow[r, "\ell^n \cdot \mathrm{id}_{\phi(M)}"] & \phi(M) \arrow[r] & \rho_{\mathbb{Z}/\ell^n}(\phi(M)) \arrow[r, "+1"] & {}
\end{tikzcd}$$

the right vertical arrow is an isomorphism as well.

2. Let us consider the 1-motive $M_1(G) = \underline{G}$ which is concentrated in degree 1. Recall that the ℓ-adic realization of $M_1(G)$, integral or rational, is given by the Tate module $T_\ell(G) = \varprojlim_n G[\ell^n]$, thus $\rho_{\mathbb{Z}/\ell^n}(M_1(G)) \simeq G[\ell^n][1]$ by the transition maps.

3. Using Lemma 4.3.7, we see that the weak symmetric algebra of $M_1(G/S)$ with rational coefficients coincides with the symmetric algebra of $M_1(G/S)$. In particular $\mathrm{Sym}^n(M_1(G/S)) = 0$ in $\mathrm{DM}_{\text{ét}}(k, \mathbb{Q})$ if $n > \mathrm{kd}(G/S)$ by [AHP16, Proposition 4.1]. The only argument that remains to be given is for the torsion part. For this, consider a prime number ℓ invertible in S. Notice that $\mathrm{wSym}(\mathcal{H}_1(G/S, \mathbb{Z}/\ell))$ is anticommutative by the cup-product, see [Fu15, Proposition 7.4.10], therefore according to the first point in Remark 4.3.8, if n and ℓ are coprime, we have that $(\mathcal{H}_1(G/S, \mathbb{Z}/\ell)^{\otimes n})^{h\mathfrak{S}_n} \simeq \bigwedge^n \mathcal{H}_1(G/S, \mathbb{Z}/\ell)$, and in particular vanishes if $n > \mathrm{kd}(G/S)$. If n and ℓ are not coprime, then we proceed as follows: we can reduce to the case where $S = \mathrm{Spec}(k)$ for an algebraically closed field k. Then by the point 2, we have that $M_1(G/S, \ell^n)$ is a complex in

[1] Here the first square bracket is associated to the ℓ^n-torsion of G and the second is associated to the translation functor.

degree 1 whose first homology group $\mathcal{H}_1(G/S,\mathbb{Z}/\ell)$ is a finite dimensional vector space over $\mathbb{F}_\ell$. Let r_ℓ be the dimension of $\mathcal{H}_1(G/S,\mathbb{Z}/\ell)$ and let $\{e_1,\dots,e_{r_\ell}\}$ be a base, then if we consider $m > r_\ell$, then there will always be at least one e_i repeated in $e_{i_1}\otimes\ldots\otimes e_{i_m}$ as a base of $\mathcal{H}_1(G/S,\mathbb{Z}/\ell)^{\otimes m}$. By the alternating action of $\mathfrak{S}_m$ in $\mathcal{H}_1(G/S,\mathbb{Z}/\ell)^{\otimes m}$ and the description in this particular case of the homotopy fixed points given in (4.2), we conclude that $(\mathcal{H}_1(G/S,\mathbb{Z}/\ell)^{\otimes m})^{h\mathfrak{S}_m} = 0$ for all $m > r_\ell$.

Since the family of functors associated to the change of coefficient is conservative, then one concludes that $N = \max_\ell\{\mathrm{kd}(G), r_\ell\}$. □

Remark 4.3.11.

1. Consider $S = \mathrm{Spec}(k)$ with k an algebraically closed field, then for a commutative algebraic group G/k. By an argument given in [BS13, Proposition 4.1] involving reduction to the assumption that G is semi-abelian, we may assume that G is the extension of an abelian variety A by a torus T, we have a short exact sequence $1 \to T[\ell^n] \to G[\ell^n] \to A[\ell^n] \to 1$ obtaining that $G[\ell^n] \simeq (\mathbb{Z}/\ell^n)^{2g+r}$, where g is the dimension of A and r is the rank of T.
2. Under the same assumptions for S, thank to the second point of Lemma 4.3.10 we get that $\rho_{\mathbb{Z}/\ell}(M_1(G)) \in D(k_{\text{ét}},\mathbb{Z}/\ell) \simeq (\mathbb{F}_\ell - v.s.)^{\mathbb{Z}}$, where $(\mathbb{F}_\ell - v.s.)^{\mathbb{Z}}$ is the category of $\mathbb{Z}$-graded $\mathbb{F}_\ell$-vector spaces, is a finite dimensional $\mathbb{F}_\ell$-vector space. Since the dimension of the vector space depends only on G and not on ℓ, we can say that the N described in point 3 of Lemma 4.3.10 corresponds to the Kimura dimension $\mathrm{kd}(G)$.

Lemma 4.3.12. *Let G be a smooth group scheme over a field $k = \bar{k}$ and let ℓ be a prime number different from char(k). Then we have an isomorphism in $D(k_{\acute{e}t},\mathbb{Z}/\ell) \simeq (\mathbb{F}_\ell - e.v.)^{\mathbb{Z}}$ given by*

$$\rho_{\mathbb{Z}/\ell}(M_{\acute{e}t}(G)) = M_{\acute{e}t}(G)/\ell \xrightarrow{\sim} \bigoplus_{i=0}^{kd(G)} \left(\bigwedge^{i} \mathcal{H}_1(G,\mathbb{Z}/\ell)\right)[i]$$

Proof. First, we have that $M_1(G)$ is a geometric motive and is $\mathbb{Z}$-additive, therefore we have $M_1(G\times H) \simeq M_1(G)\oplus M_1(H)$. Let us recall that the weak symmetric algebra of $M_1(G/S)$ is defined as

$$\mathrm{wSym}(M_1(G)) := \bigoplus_{i=0}^{\mathrm{kd}(G)} \left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}.$$

By point 1 of Lemma 4.3.10, $\rho_{\mathbb{Z}/\ell}$ commutes with any additive functor, so we get

$$\rho_{\mathbb{Z}/\ell}\left(\bigoplus_{i=0}^{\mathrm{kd}(G)} \left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}\right) \simeq \bigoplus_{i=0}^{\mathrm{kd}(G)} \rho_{\mathbb{Z}/\ell}\left(\left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}\right).$$

Notice that by definition $\rho_{\mathbb{Z}/\ell}(M) = M\otimes^L \mathbb{Z}/\ell$. Since $-\otimes^L\mathbb{Z}/\ell$ commutes with colimits and is monoidal, see [Ayo14b, Definition 5.6], we obtain an isomorphism

$$\begin{aligned}\rho_{\mathbb{Z}/\ell}\left(\left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}\right) &\simeq \left(\rho_{\mathbb{Z}/\ell}\left(M_1(G)^{\otimes i}\right)\right)^{h\mathfrak{S}_i}\\ &\simeq \left(\mathcal{H}_1(G,\mathbb{Z}/\ell)^{\otimes i}\right)^{h\mathfrak{S}_i}.\end{aligned}$$

So in terms of realization we obtain $\rho_{\mathbb{Z}/\ell}(\mathrm{wSym}(M_1(G))) \simeq \mathrm{wSym}(\mathcal{H}_1(G,\mathbb{Z}/\ell))$. Notice that if $\ell > \mathrm{kd}(G)$, then we get immediately (using the proof in point 3 of Lemma 4.3.10) that $\mathrm{wSym}(\mathcal{H}_1(G,\mathbb{Z}/\ell)) \simeq \bigoplus_{i=0}^{\mathrm{kd}(G)} \left(\bigwedge^i \mathcal{H}_1(G,\mathbb{Z}/\ell)\right)[i]$.

In the following part we will prove that for a prime $\ell \neq \mathrm{char}(k)$ and G, H two smooth commutative algebraic groups we have an isomorphism $\mathrm{wSym}(\mathcal{H}_1(G\times H,\mathbb{Z}/\ell)) \simeq \mathrm{wSym}(\mathcal{H}_1(G,\mathbb{Z}/\ell))\otimes\mathrm{wSym}(\mathcal{H}_1(H,\mathbb{Z}/\ell))$, which will allow us to conclude when $\ell \leq \mathrm{kd}(G)$. By definition

$$\begin{aligned}\mathrm{wSym}(M_1(G\times H)) &\simeq \mathrm{wSym}(M_1(G)\oplus M_1(H))\\ &= \bigoplus_{i=0}^{\mathrm{kd}(G\times H)} \left((M_1(G)\oplus M_1(H))^{\otimes i}\right)^{h\mathfrak{S}_i}\end{aligned}$$

Since the homotopy fixed points of a motive M are defined as a homotopy limit, they commute with finite sums. For simplicity we write $M := M_1(G)$ and $N := M_1(H)$, thus we have $\left(M^{\otimes n}\oplus N^{\otimes n}\right)^{h\mathfrak{S}_n} \simeq \left(M^{\otimes n}\right)^{h\mathfrak{S}_n} \oplus \left(N^{\otimes n}\right)^{h\mathfrak{S}_n}$. Moreover we have a canonical morphism

$$\mathrm{holim}_{B\mathfrak{S}_n}(M\oplus N)^{\otimes n} \to (M\oplus N)^{\otimes n} \simeq \bigoplus_{i=0}^{n}\bigoplus^{\binom{n}{i}} M^{\otimes i}\otimes N^{\otimes n-i},$$

where the last isomorphism is obtained by the distributive and commutative properties. Since $\rho_{\mathbb{Z}/\ell}$ commutes with additive functors and limits and is monoidal, we obtain $\rho_{\mathbb{Z}/\ell}\left(\left((M\oplus N)^{\otimes n}\right)^{h\mathfrak{S}_n}\right) \simeq \left((M/\ell\oplus N/\ell)^{\otimes n}\right)^{h\mathfrak{S}_n}$, passing to the realization $\rho_{\mathbb{Z}/\ell}$ and due to the anticommutative structure given by the cup-product, if n and ℓ are coprimes, then

$$\left((M/\ell\oplus N/\ell)^{\otimes n}\right)^{h\mathfrak{S}_n} \simeq \bigwedge^{n}(M/\ell\oplus N/\ell).$$

Notice that the functor $B\mathfrak{S}_i\times B\mathfrak{S}_j \to B(\mathfrak{S}_i\times\mathfrak{S}_j)$ is an equivalence of categories, then

$$\begin{aligned}((M/\ell)^{\otimes i}\otimes(N/\ell)^{\otimes n-i})^{h(\mathfrak{S}_i\times\mathfrak{S}_{n-i})} &\simeq \mathrm{holim}_{B\mathfrak{S}_i\times B\mathfrak{S}_{n-i}}(M/\ell)^{\otimes i}\otimes(N/\ell)^{\otimes n-i}\\ &\simeq ((M/\ell)^{\otimes i})^{h\mathfrak{S}_i}\otimes((N/\ell)^{\otimes n-i})^{h\mathfrak{S}_{n-i}}.\end{aligned}$$

Due to the anticommutativity of the weak algebra, we have an isomorphism of $\mathbb{F}_\ell$-vector spaces

$$\begin{tikzcd}
\mathrm{holim}_{B\mathfrak{S}_n}\overbrace{M^{\otimes i}\otimes N^{n-i}\oplus\ldots\oplus M^{\otimes i}\otimes N^{n-i}}^{\binom{n}{i}\text{-times}} \arrow[r] \arrow[d, "\simeq"] & \bigoplus^{\binom{n}{i}} M^{\otimes i}\otimes N^{\otimes n-i} \\
\mathrm{holim}_{B(\mathfrak{S}_i\times\mathfrak{S}_{n-i})}M^{\otimes i}\otimes N^{n-i} \arrow[ur] &
\end{tikzcd}$$

Therefore, we have an isomorphism of graded (anticommutative) algebras $\mathrm{wSym}(\mathcal{H}_1(G\times H,\mathbb{Z}/\ell)) \simeq \mathrm{wSym}(\mathcal{H}_1(G,\mathbb{Z}/\ell))\otimes\mathrm{wSym}(\mathcal{H}_1(H,\mathbb{Z}/\ell))$.

So the final argument is that the addition map $m : G \times G \to G$ induces a morphism of graded algebras over the field $\mathbb{F}_\ell$

$$\begin{aligned} m^* : \mathrm{wSym}(M_1(G)/\ell) &\to \mathrm{wSym}(M_1(G)/\ell) \otimes \mathrm{wSym}(M_1(G)/\ell) \\ x &\mapsto x \otimes 1 + 1 \otimes x + \sum x_i \otimes y_i. \end{aligned}$$

By a lemma about the fundamental structure of such algebras, [Mil86, Lemma 15.2], we have that $\mathrm{wSym}(M_1(G)/\ell) \simeq \bigoplus_{i=0}^{\mathrm{kd}(G)} \left(\bigwedge^i \mathcal{H}_1(G, \mathbb{Z}/\ell)\right)[i]$. □

By using the properties of conservative functors associated to change of coefficients described in [CD16, Proposition 5.4.12], which for the sake of completeness, we recall such proposition:

Proposition 4.3.13 ([CD16, Proposition 5.4.12])**.** *Let $\mathcal{P}$ be the set of prime integers and S be a noetherian scheme of finite dimension. If R is a flat ring over $\mathbb{Z}$, then the family of change of coefficients functors:*

$$\begin{aligned} &\rho_{\mathbb{Q}} : DM_h(S, R) \to DM_h(S, R \otimes \mathbb{Q}) \\ &\rho_{\mathbb{Z}/p} : DM_h(S, R) \to DM_h(S, R/p), \quad p \in \mathcal{P} \end{aligned}$$

is conservative.

With this proposition, we get an improvement of the results obtained in [AEH15], getting the following theorem:

Theorem 4.3.14. *Let k be an algebraically closed field and G/k a connected commutative group scheme. Then the morphism*

$$\phi_G : M_{\acute{e}t}(G) \to \bigoplus_{i=0}^{kd(G)} \left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}$$

is an isomorphism in $DM_{\acute{e}t}(k, \mathbb{Z})$.

Proof. We split the proof into two steps: first we start by looking at the functor $\rho_{\mathbb{Q}}$. Applying Lemma 4.3.7, we obtain that the induced morphism by $\rho_{\mathbb{Q}}(\phi_G)$ is the morphism φ_G given in [AHP16, Definition 3.1] and [AHP16, Theorem 3.3], with $S = k$, which is shown to be an isomorphism in $\mathrm{DM}_{\acute{e}t}(k, \mathbb{Q})$. The reason behind this is the following: $\rho_{\mathbb{Q}}(M_{\acute{e}t}(G)) = M(G)$ and by Lemma 4.3.7 we have $\rho_{\mathbb{Q}}\left(\bigoplus_{i=0}^{\mathrm{kd}(G)} \left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}\right) \simeq \bigoplus_{i=0}^{\mathrm{kd}(G)} \mathrm{Sym}^n(M_1(G))$ and finally, by construction of the morphisms ϕ_G and φ_G of Definition 4.3.9 and [AHP16, Definition 3.1], and the uniqueness part of [AHP16, Theorem 2.8] we get that $\rho_{\mathbb{Q}}(\phi_G) = \varphi_G$.

For the second step, we fix a prime number $\ell \neq \mathrm{char}(k)$. Let us consider the functor $\rho_{\mathbb{Z}/\ell}$ and let us compute the elements of

$$\rho_{\mathbb{Z}/\ell}(\phi_G) : \rho_{\mathbb{Z}/\ell}\left(M_{\acute{e}t}(G)\right) \to \rho_{\mathbb{Z}/\ell}\left(\bigoplus_{i=0}^{\mathrm{kd}(G)} \left(M_1(G)^{\otimes i}\right)^{h\mathfrak{S}_i}\right).$$

Here we are assuming that the realization functor is covariant (sending the elements to homological instead of cohomological objects). By [BS13, Theorem 4.1] we have that

$$H^*_{\text{ét}}(G, \mathbb{Z}/\ell) \simeq \bigoplus_{i=0}^{\text{kd}(G)} \left(\bigwedge^i \mathcal{H}^1(G, \mathbb{Z}/\ell) \right) [i],$$

where ℓ is a prime number not equal to $\text{char}(k)$. Consider the duality operator D_k on $D(k_{\text{ét}}, \mathbb{Z}/\ell)$, let $f : G \to k$ be the structure morphism and d the dimension of G. By definition we have

$$\begin{aligned} D_k\left(\rho_{\mathbb{Z}/\ell}\left(M_{\text{ét}}(G)\right)\right) &\simeq D_k f_! f^! (\mathbb{Z}/\ell)_k \\ &\simeq f_* D_G (\mathbb{Z}/\ell)_G \\ &\simeq (M_{\text{ét}}(G)_\ell)(d)[2d]. \end{aligned}$$

Here the first isomorphism $\rho_{\mathbb{Z}/\ell}\left(M_{\text{ét}}(G)\right) \simeq f_! f^! (\mathbb{Z}/\ell)_k$ is because the realization functor commutes with the six functors formalism, see [CD16, A.1.16], while the second are third are given by [ILO14, Exposé XVII]

As we stated in Lemma 4.3.12, one has that $\rho_{\mathbb{Z}/\ell}(M_{\text{ét}}(G)) \simeq \bigoplus_{i=0}^{\text{kd}(G)} \left(\bigwedge^i \mathcal{H}_1(G, \mathbb{Z}/\ell) \right) [i]$ for all $\ell \neq \text{char}(k)$, whose dual is isomorphic to $\bigoplus_{i=0}^{\text{kd}(G)} \left(\bigwedge^i \mathcal{H}^1(G, \mathbb{Z}/\ell) \right) [i]$, thus by conservative properties given in [CD16, Proposition 5.4.12], we conclude that

$$M_{\text{ét}}(G) \xrightarrow{\simeq} \bigoplus_{i=0}^{\text{kd}(G)} \left(M_1(G)^{\otimes i} \right)^{h\mathfrak{S}_i} \in \text{DM}_{\text{ét}}(k, \mathbb{Z}).$$

□

Theorem 4.3.15. *Let S be a good enough scheme in the sense of Definition 2.1.4, and let G be a connected commutative scheme over S. Then the morphism ϕ_G given in Definition 4.3.9 is an isomorphism.*

Proof. Consider a morphism of good enough schemes $f : T \to S$, we have that $f^* M_1(G/S) \simeq M_1(G_T/T) \in \text{DM}_{\text{ét}}(T, \mathbb{Z})$. As we have done before, we will split the proof in two: first for rational coefficients $\text{DM}_{\text{ét}}(T, \mathbb{Q})$ and then for $\text{DM}_{\text{ét}}(T, \mathbb{Z}/\ell)$ for all prime integer ℓ invertible in T. According to [AHP16, Proposition 2.7], one has that $f^* M_1(G/S)_{\mathbb{Q}} \simeq M_1(G_T/T)_{\mathbb{Q}} \in \text{DM}_{\text{ét}}(T, \mathbb{Q})$. On the other hand, having shown that $\rho_{\mathbb{Z}/\ell^n}(M_1(G/S)) \simeq \underline{G/S}[\ell^n][-1]$, and that by the universal property of fibre product we have $f^* \underline{G/S} \simeq \underline{G_T/T}$. Invoking [Ayo14b, Théorème 6.6(A)] one gets for a quasi-projective morphism $f^* \circ \rho_{\mathbb{Z}/\ell^r} \simeq \rho_{\mathbb{Z}/\ell^r} \circ f^*$, thus we get the following isomorphism

$$\begin{aligned} \rho_{\mathbb{Z}/\ell^n}(f^* M_1(G/S)) &\simeq f^* \left(\rho_{\mathbb{Z}/\ell^n}(M_1(G/S)) \right) \\ &\simeq f^* \left(\underline{G/S}[\ell^n][1] \right) \\ &\simeq f^* \left(\underline{G/S} \right) [\ell^n][1] \\ &\simeq \underline{G_T/T}[\ell^n][1] = \rho_{\mathbb{Z}/\ell^n}(M_1(G_T/T)). \end{aligned}$$

As ℓ is any prime number, then we conclude that $f^*M_1(G/S) \simeq M_1(G_T/T) \in \mathrm{DM}_{\text{ét}}(T, \mathbb{Z})$. In particular we obtain that ϕ_G is natural over the base. By the previous fact, we get that the morphism $\phi_{G_T} := f^*(\phi_G)$ acts as follows:

$$M^S_{\text{ét}}(G) \xrightarrow{\phi_G} \bigoplus_{i=0}^{\mathrm{kd}(G/S)} \left(M_1(G/S)^{\otimes i}\right)^{h\mathfrak{S}_i} \rightsquigarrow M^T_{\text{ét}}(G_T) \xrightarrow{\phi_{G_T}} \bigoplus_{i=0}^{\mathrm{kd}(G_T/T)} \left(M_1(G_T/T)^{\otimes i}\right)^{h\mathfrak{S}_i}.$$

In this way, for any geometric point $i_{\bar{s}} : \bar{s} \to S$ we have an isomorphism of motives $i^*_{\bar{s}}M_1(G/S) \simeq M_1(G_{\bar{s}}/\bar{s})$, and then by the previous remark, $i^*_{\bar{s}}(\phi_G) = \phi_{G_{\bar{s}}}$ for any geometric point $\bar{s}$ of S, the map

$$M^{\bar{s}}_{\text{ét}}(G_{\bar{s}}) \xrightarrow{\phi_{G_{\bar{s}}}} \bigoplus_{i=0}^{\mathrm{kd}(G_{\bar{s}}/\bar{s})} \left(M_1(G_{\bar{s}}/\bar{s})^{\otimes i}\right)^{h\mathfrak{S}_i}$$

turns out to be an isomorphism by Theorem 4.3.14. By Lemma 2.1.5, the family of functors $i^*_{\bar{s}}$ is conservative, therefore $\phi_{G/S}$ is an isomorphism. □

Remark 4.3.16. The direct factor $h_n(G/S) = \phi^{-1}_{G/S}\left((M_1(G/S)^{\otimes n})^{h\mathfrak{S}_n}\right)$ of $M^S_{\text{ét}}(G)$ is characterized as follows: for $m \in \mathbb{Z}$ that is equal to 1 modulo $o(\pi_0(G/S))$ (see Definition 4.3.6), the map $M_{\text{ét}}([m])$ operates on $h_n(G/S)$ as $m^n \cdot \mathrm{id}$. This is a consequence of [AHP16, Lemma 2.6(1)]. If we tensorize by $\mathbb{Q}$, we recover the following fact about decomposition of the motivic cohomology groups of G: Suppose that S is a good enough scheme. Then for every bi-degree $(m, n) \in \mathbb{Z}^2$ the relative étale cohomology groups of G in degrees (m, n) with integral coefficients decomposes as

$$H^m_{M,\text{ét}}(G/S, \mathbb{Q}(n)) \simeq \bigoplus_{j=0}^{\mathrm{kd}(G/S)} H^{m,j}_{M,\text{ét}}(G/S, \mathbb{Q}(n)),$$

where

$$H^{m,j}_{M,\text{ét}}(G/S, \mathbb{Q}(n)) = \left\{Z \in H^m_{M,\text{ét}}(G/S, \mathbb{Q}(n)) \mid [n]^*Z = n^j Z,\ \forall n \equiv 1(\mathrm{mod}\ o(\pi_0(G/S)))\right\}.$$

As is stated in [AHP16, Theorem 3.9].

Let A be an abelian variety over an algebraically closed field k, the question which arises naturally is if the isomorphism $M_{\text{ét}}(A) \to \bigoplus_{i=0}^{\mathrm{kd}(A)} \left(M_1(A)^{\otimes i}\right)^{h\mathfrak{S}_i}$ comes from an morphism in $\mathrm{Chow}_{\text{ét}}(k)$.

Proposition 4.3.17. *Let A be an abelian variety of dimension g over a field $k = \bar{k}$, then the following are equivalent:*

1. *the isomorphism $M_{\text{ét}}(A) \to \bigoplus_{i=0}^{2g} \left(M_1(A)^{\otimes i}\right)^{h\mathfrak{S}_i}$ is a morphism in the category $Chow_{\text{ét}}(k)$.*

2. *The exist an element $h \in Chow_{\text{ét}}(k)$ such that $h \simeq M_1(A) \in Chow_{\text{ét}}(k)$.*

Proof. Let A be an abelian variety over k of dimension g, and consider the map

$$M_{\text{ét}}(A) \to \bigoplus_{i=0}^{2g} \left(M_1(A)^{\otimes i}\right)^{h\mathfrak{S}_i} \in \mathrm{DM}_{\text{ét}}(k, \mathbb{Z}).$$

Let us remark that $h_{\text{ét}}(A) \simeq M_{\text{ét}}(A)$ by the full embedding of $\mathrm{Chow}_{\text{ét}}(k)^{op} \hookrightarrow \mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$. If we assume (1), then (2) follows immediately since $M_1(A)$ is a direct factor of the finite sum $\bigoplus_{i=0}^{2g} \left(M_1(A)^{\otimes i}\right)^{h\mathfrak{S}_i} \in \mathrm{Chow}_{\text{ét}}(k)$.

If we assume (2), there exists an element $h \in \mathrm{Chow}_{\text{ét}}(k)$ such that $h \simeq M_1(A)$, then $h^{\otimes i} \in \mathrm{Chow}_{\text{ét}}(k)$ for all $i \geq 0$. We will assume that $h = (X, p)$ for some $X \in \mathrm{SmProj}_k$ and $p \in \mathrm{Corr}^0_{\text{ét}}(X, X)$. In other words, $p \in \mathrm{End}_{\mathrm{Chow}_{\text{ét}}(k)}(h(X)) \simeq \mathrm{End}_{\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})}(M(X))$, therefore $p^{\otimes n} \in \mathrm{End}_{\mathrm{Chow}_{\text{ét}}(k)}(h(X)^{\otimes n})$, thus we define $(p^{\otimes n})^{h\mathfrak{S}_n}$ as the image of $p^{\otimes n}$ in $\mathrm{End}_{\mathrm{DM}_{\text{ét}}(k,\mathbb{Z})}((M(X)^{\otimes n})^{h\mathfrak{S}_n})$, we then define the motive

$$\left(h^{\otimes n}\right)^{h\mathfrak{S}_n} := (\overbrace{X \times \ldots \times X}^{n\text{-times}}, (p^{\otimes n})^{h\mathfrak{S}_n}) \in \mathrm{Chow}_{\text{ét}}(k)$$

Then we see that morphism $M_{\text{ét}}(A) \to \bigoplus_{i=0}^{2g} \left(M_1(A)^{\otimes i}\right)^{h\mathfrak{S}_i}$ is in $\mathrm{Chow}_{\text{ét}}(k)$. $\square$

Changing the coefficients in the proof of Theorem 4.3.14, we obtain that $h_{\text{ét}}(A)$ admits an integral Chow-Künneth decomposition in $\mathrm{Chow}_{\text{ét}}(k)$ if the 1-motive $M_1(A)$ belongs to the category $\mathrm{Chow}_{\text{ét}}(k)$.

In order to give an example of an integral étale motive with Chow-Künneth decomposition, we should recall some results coming from the classical theory of abelian varieties. We have the following results:

Lemma 4.3.18. *Let C be a smooth projective curve over an algebraically closed field k, and let $J(C)$ be the Jacobian of C, then:*

1. *the motive $M_{\acute{e}t}(C)$ can be decomposed as $M_{\acute{e}t}(C) \simeq \mathbf{1} \oplus h^1_{\acute{e}t}(C) \oplus \mathbf{1}(1)[2]$.*
2. *If C' is another smooth projective curve over k, then*

$$Hom_{Chow_{\acute{e}t}(k)}\left(h^1_{\acute{e}t}(C), h^1_{\acute{e}t}(C')\right) \simeq Hom_{AV}\left(J(C), J(C')\right)[1/p].$$

3. *The motives $h^1_{\acute{e}t}(C)$ and $M_1(J(C))$ are isomorphic.*

Proof. 1. It is a classic result, for instance see [MNP13, Theorem 2.7.2] and the fully-faithful functor of 1-motives to the étale cohomology with integral coefficients.

2. This is a consequence of the isomorphism

$$\mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Z}}}\left(h^1(C), h^1(C')\right)[1/p] \simeq \mathrm{Hom}_{\mathrm{Chow}_{\text{ét}}(k)}\left(h^1_{\text{ét}}(C), h^1_{\text{ét}}(C')\right)$$

and [MNP13, Theorem 2.7.2.(b)].

3. The argument is the same as in [AEH15, Lemma 4.3.2]. Consider the 1-motive $M_{\text{ét}}(C)$ which is cohomologically concentrated in degrees 0 and -1 as is given in [Voe00, Theorem 3.4.2]. The cohomology in degree 0 is $\underline{\mathrm{Pic}_{C/k}}[1/p]$ while in degree 0 is equal to $\underline{\mathbb{G}_m}[1/p]$. Since $\mathbb{Z}_C(1)_{\text{ét}} \sim \underline{\mathbb{G}_m}[1/p]_{\text{ét}}[-1]$ we obtain that $\mathbf{1} \oplus \mathbf{1}(1)[2] \simeq \mathbb{Z}[1/p] \oplus \underline{\mathbb{G}_m}[1/p][1]$. The remaining object is given by the kernel of the map $\underline{\mathrm{Pic}_{C/k}}[1/p] \to \mathbb{Z}[1/p]$, which is isomorphic to $M_1(J(C))$. $\square$

Theorem 4.3.19. *Let $k = \bar{k}$ be a field and consider C_i/k a projective smooth curve, for $i \in \{1, \ldots, n\}$. Then the variety $J(C_1) \times \ldots \times J(C_n)$ admits an integral Chow-Künneth decomposition.*

Proof. By point 3. of Lemma 4.3.18, one has an isomorphism $h^1_{\text{ét}}(C_i) \simeq M_1(J(C_i))$. Since M_1 is an additive functor, we obtain $M_1(J(C_1) \times \ldots \times J(C_n)) \simeq \bigoplus_{i=1}^n M_1(J(C_i))$, thus $M_1(J(C_1) \times \ldots \times J(C_n)) \simeq \bigoplus_{i=1}^n h^1_{\text{ét}}(C_i)$ in $\mathrm{Chow}_{\text{ét}}(k)$. Therefore $M_1(J(C_1) \times \ldots \times J(C_n))$ is isomorphic to the motive $h = \left(\coprod_i C_i, \sum_{i=1}^n p^1_{\text{ét}}(C_i)\right) \in \mathrm{Chow}_{\text{ét}}(k)$ $\square$

Recall that thanks to [Kün93] we have Chow-Künneth decomposition with rational coefficients for abelian varieties and that by [AEH15, Proposition 4.3.3] the $h_1(A)$ part is isomorphic to $M_1(A)_{\mathbb{Q}}$ in $\mathrm{Chow}(k)_{\mathbb{Q}}$. Given the part $h_1(A)$ of the motive $h(A)$ and its associated projector $p_1(A)$, then we can characterize the existence of an integral étale Chow-Künneth decomposition in the following way:

Theorem 4.3.20. *Let a field $k = \bar{k}$ and consider A an abelian variety of dimension g over k. Then the following statements are equivalents:*

1. *$h_{\acute{e}t}(A)$ admits an integral étale Chow-Künneth decomposition.*
2. *The projector associated to $h_1(A) \in Chow(k)_{\mathbb{Q}}$ can be lifted to a projector in $CH^g_{\acute{e}t}(A \times A)$.*

Proof. 1. $(\Longrightarrow)$ 2. is immediate. If we assume 2. then there exists an element $p \in \mathrm{CH}^g_{\text{ét}}(A \times A)$ such that $h_1(A) = (A, p_{\mathbb{Q}}) \in \mathrm{Chow}(k)_{\mathbb{Q}}$ where $p_{\mathbb{Q}}$ is the image of p in $\mathrm{CH}^g_{\text{ét}}(A \times A)_{\mathbb{Q}}$. Then the realization of the motive $h = (A, p)$ coincides with $h_1(A)$ if we change to rational coefficients. If $\ell \neq \mathrm{char}(k)$, then $H_1(A_{\text{ét}}, \mathbb{Z}_\ell) = T_\ell(A)$ is $\mathbb{Z}_\ell$-torsion free since $A[\ell^n] \simeq (\mathbb{Z}/\ell^n)^{2g}$, so we have an injection $H_1(A_{\text{ét}}, \mathbb{Z}_\ell) \hookrightarrow T_\ell(A) \otimes \mathbb{Q}_\ell = H_1(A_{\text{ét}}, \mathbb{Q}_\ell)$, therefore $p_{\mathbb{Q}}$ acts as the identity on $H_1(A_{\text{ét}}, \mathbb{Z}_\ell)$.

Consider the realization $p_{\ell^n} \in \mathrm{CH}^g_{\text{ét}}(A \times A, \mathbb{Z}/\ell^n)$, where the last group is isomorphic to $H^{2g}_{\text{ét}}(A \times A, \mathbb{Z}/\ell^n)$. As $\mathbb{Q}_\ell$ is a flat $\mathbb{Z}_\ell$-module and $\varprojlim_{n\in\mathbb{N}}$ is a right exact functor we have that p_{ℓ^n} acts as the identity over $H_1(A_{\text{ét}}, \mathbb{Z}/\ell^n)$. $\square$

Theorem 4.3.21. *Let $k = \bar{k}$ be a field and let A be a principally polarized variety. Then there exists a Chow-Künneth decomposition of A.*

Proof. Consider an abelian variety A/k, then we have that

$$\mathrm{End}_{\mathrm{Chow}_{\text{ét}}(k)}(h_{\text{ét}}(A)) \simeq \mathrm{End}_{\mathrm{DM}_{\text{ét}}(k)}(M_{\text{ét}}(A)).$$

Since $M_1(A)$ is a direct factor of $M_{\text{ét}}(A)$, then it defines an endomorphism p of $M_{\text{ét}}(A)$, as the endomorphism of the motive $h_{\text{ét}}(A)$ is defined as $\mathrm{CH}^g_{\text{ét}}(A \times A)$ where $g = \dim(A)$. Since $p \in \mathrm{CH}^g_{\text{ét}}(A \times A)$ such that $p^2 = p$, thus we define the motive $h_1(A) := (A, p, 0)$. The functor $\mathrm{Chow}_{\text{ét}}(k)^{op} \hookrightarrow \mathrm{DM}_{\text{ét}}(k)$, sends $h_1(A) \mapsto M_1(A)$, therefore, as $M_1(A) \in \mathrm{Chow}_{\text{ét}}(k)$ we conclude that $h_{\text{ét}}(A)$ admits a Chow-Künneth decomposition. $\square$

Theorem 4.3.22. *Let X be a smooth projective variety of dimension d over an algebraically closed field k. If $Pic^0(X)$ is a principally polarized variety, then there exists a decomposition of the motive $h_{ét}(X)$ as*

$$h_{ét}(X) = h^0_{ét}(X) \oplus h^1_{ét}(X) \oplus h^+_{ét}(X) \oplus h^{2d-1}_{ét}(X) \oplus h^{2d}_{ét}(X)$$

Proof. An abelian variety A admits a principal polarization if and only $\widehat{A}$ admits one. So if $A = \mathrm{Pic}^0(X)$ then the Picard variety is principally polarized if and only if $\mathrm{Alb}_X(k)$ admits one. Let k be an algebraically closed field and let A be a principally polarized abelian variety $\lambda : A \xrightarrow{\sim} \widehat{A}$ induced by a symmetric ample line bundle. We have an injection

$$\mathrm{Hom}_{AV}(A, \widehat{A}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(A), T_\ell(\widehat{A}))$$

Therefore λ induce an isomorphism of Tate modules. Since we have the following isomorphisms

$$H^{2g-1}_{\text{ét}}(A, \mathbb{Z}_\ell) \simeq T_\ell(A) \xrightarrow{\sim} T_\ell(\widehat{A}) \simeq H^{2g-1}_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell)$$

considering the isomorphism $H^1_{\text{ét}}(A, \mathbb{Z}_\ell) \simeq H^{2g-1}_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell)$. Since λ is induced by a cycle (thanks to the integral étale Fourier transform). Now take an hyperplane H in X and intersect it with itself $g-1$ times, then it induces a Lefschetz operator

$$L_A^{g-1} : H^1_{\text{ét}}(X, \mathbb{Z}_\ell) \to H^{2g-1}_{\text{ét}}(X, \mathbb{Z}_\ell)$$

which turns out to be an injection. We will see that there exists and étale cycle in $\mathrm{CH}^g_{\text{ét}}(A \times A)$ whose multiple by an integer equals the Lefschetz operator. We recall that there exists isomorphisms

$$\begin{aligned}\mathrm{Hom}^0_{\mathrm{SmProj}_k}(X, \mathrm{Pic}^0(X)) &\simeq \mathrm{Hom}_{AV}(\mathrm{Alb}_X(k), \mathrm{Pic}^0(X)) \\ &\simeq \mathrm{CH}^1_{\text{ét}}(X \times X)/\mathrm{CH}^1_{\equiv}(X \times X)\end{aligned}$$

where $\mathrm{CH}^1_{\equiv}(X \times X) = \mathrm{pr}_1^*(\mathrm{CH}^1_{\text{ét}}(X)) \oplus \mathrm{pr}_2^*(\mathrm{CH}^1_{\text{ét}}(X))$ and $\mathrm{Hom}^0_{\mathrm{SmProj}_k}$ stands for the pointed morphisms of smooth projective varieties over k. Thus the polarization $\lambda : \mathrm{Alb}_X(k) \to \mathrm{Pic}^0(X)$ is induced by a divisor in $X \times X$. Now consider the abelian variety $A = \mathrm{Pic}^0(X)$, then there is a morphism $\lambda^{-1} : \mathrm{Pic}^0(X) \to \mathrm{Alb}_X(k)$, thanks to the existence of a Fourier transform with integral coefficients which is motivic. The cycle $c_1(\mathcal{P}_{\mathrm{Pic}^0(X)})^{2g-1}/(2g-1)! \in \mathrm{CH}^{2g-1}_{\text{ét}}(\mathrm{Pic}^0(X) \times \mathrm{Alb}_X(k))$ where $g = \dim(\mathrm{Pic}^0(X))$ induce an isomorphism $H^1(X, \mathbb{Z}/\ell^n) \to H^{2d-1}(X, \mathbb{Z}/\ell^n)$, since we have isomorphisms $f : \mathrm{Pic}^0(X)[\ell^n] \simeq H^1(X, \mathbb{Z}/\ell^n)$ and $g : \mathrm{Alb}_X(k)[\ell^r] \simeq H^{2d-1}(X, \mathbb{Z}/\ell^n)$ which are induced by divisors in $\mathrm{CH}^1_{\text{ét}}(X \times X)$, see [Mil80, Chap. III, Cor. 4.18], thus we associate the cycle

$$\lambda^{-1} := g \circ \frac{c_1(\mathcal{P}_{\mathrm{Pic}^0(X)})^{2g-1}}{(2g-1)!} \circ f^{-1} \in \mathrm{CH}^d_{\text{ét}}(X \times X).$$

By arguments given in [MNP13, Lemma 6.2.3], one has that L_X^{g-1} defines an isogeny $\alpha : \mathrm{Pic}^0(X) \to \mathrm{Alb}_X(k)$ and another one $\beta : \mathrm{Alb}_X(k) \to \mathrm{Pic}^0(X)$ such that $\alpha \circ \beta =$

$m\cdot \mathrm{id}_{\mathrm{Alb}_X(k)}$ and $\beta\circ\alpha = m\cdot\mathrm{id}_{\mathrm{Pic}^0(X)}$ for some $m \in \mathbb{N}$. If we take the isogeny $\lambda : \mathrm{Alb}_X(k) \to \mathrm{Pic}^0(X)$ and $\lambda^{-1} : \mathrm{Pic}^0(X) \to \mathrm{Alb}_X(k)$, with this, we have that $\lambda^{-1}\circ\lambda = \mathrm{id}_{\mathrm{Alb}_X(k)}$ and $\lambda\circ\lambda^{-1} = \mathrm{id}_{\mathrm{Pic}^0(X)}$, since $\alpha\circ\beta$ is induced by an algebraic cycle and also $\lambda^{-1}\circ\lambda$ (but in this specific case it is induced by an étale cycle). Therefore m is invertible in $\mathrm{CH}^g_{\text{ét}}(X\times X)$, thus we obtain the existence of the projector $p_1^{\text{ét}}(X)$ and $p_{2d-1}^{\text{ét}}(X)$. □

PD-structure

Let X be a quasi-projective scheme over a field k. For an integer $d \geq 1$, we define the d-th symmetric power $\mathrm{Sym}^d(X)$ of X (over k) as the quotient of X^d by the natural actions of the symmetric group $\mathfrak{S}_d$ (this quotient always exists for a finite group, see [DG70, II, §, n°6]). This quotient is functorial in the sense that for a morphism $f : X \to Y$ between quasi-projective k-schemes we have that $\mathrm{Sym}^d(f) : \mathrm{Sym}^d(X) \to \mathrm{Sym}^d(Y)$.

Lemma 4.3.23 ([MP10, Lemma 1.1]). *Let X be a quasi-projective scheme over k.*

1. *The quotient morphism $q_{d,X} : X^d \to Sym^d(X)$ is again quasi-projective.*
2. *Assume that X is equidimensional of dimension $n > 0$. Then X^d and $Sym^d(X)$ are equidimensional of dimension dn, and there exists a dense open subset in $Sym^d(X)$ over which $q_{d,X}$ is étale of degree $d!$.*
3. *Assume that X is equi-dimensional and that there exist non-negatives integers $d_1, \dots, d_r$ such that $d_1 + \ldots + d_r = d$. Then the natural map*
$$\alpha_{d_1,\ldots,d_r} : Sym^{d_1}(X) \times \ldots \times Sym^{d_r}(X) \to Sym^d(X)$$
is finite, and there is a dense open subset in $Sym^d(X)$ over which it is étale of degree $\dfrac{d!}{d_1! \cdot \ldots \cdot d_r!}$. For $d, e \geq 1$, the natural map $Sym^d(Sym^e(X)) \to Sym^{de}(X)$ is finite, and there is a dense open subset in $Sym^{de}(X)$ over which it is étale of degree $\dfrac{(de)!}{d!(e!)^d}$.

Let us consider a quasi-projective scheme X over a field k. Consider the d-diagonal embedding $X \xrightarrow{\delta^d} X^d$ and the composite map $p_d : X \xrightarrow{\delta^d} X^d \xrightarrow{q_{d,X}} \mathrm{Sym}^d(X)$. Since δ^d and $q_{d,X}$ are proper morphism, then we have a push-forward map
$$(p_d)_* : \mathrm{CH}^{\text{ét}}_m(X) \to \mathrm{CH}^{\text{ét}}_{dm}(\mathrm{Sym}^d(X))$$
In the same way we define the Pontryagin product as
$$\mathrm{CH}^{\text{ét}}_*(\mathrm{Sym}^{d_1}(X)) \times \mathrm{CH}^{\text{ét}}_*(\mathrm{Sym}^{d_2}(X)) \to \mathrm{CH}^{\text{ét}}_*(\mathrm{Sym}^{d_1+d_2}(X))$$
using the formula $x*y := (\alpha_{d_1,d_2})_*(x\times y)$. For a cycle $\xi = \sum_{j=1}^r n_j Z_j$ with $Z_j \in \mathrm{CH}^{\text{ét}}_*(X)$, we define
$$\gamma_d(\xi) := \sum_{d_1+\ldots+d_r=d} n_1^{d_1}\cdots n_r^{d_r}\cdot\gamma_{d_1}(Z_1) * \ldots * \gamma_{d_r}(Z_r)$$
For $d = 0$ let us set for an element $a \in \mathrm{CH}^{\text{ét}}_*(X)$ the $\gamma_0(a) = [\mathrm{Spec}(k)] \in \mathrm{CH}^{\text{ét}}_0(X)$, which is the unit element in $\mathrm{CH}^{\text{ét}}_0(X)$.

Lemma 4.3.24. *Let X be a quasi-projective scheme over k, then:*

1. *If $Z \in CH^{ét}_{>0}(X)$ and $d_1, \ldots, d_t$ are non-negative integers with $d_1 + \ldots + d_t = d$ then*
$$\gamma_{d_1}(Z) * \cdots * \gamma_{d_t}(Z) = \frac{d!}{d_1! d_2! \cdots d_t!} \cdot \gamma_d(Z).$$
2. *Let $\xi_1, \ldots, \xi_t$ be cycles in $CH^{ét}_{>0}(X)$, and let $\xi = \sum_{i=1}^t \xi_i$. Then*
$$\gamma_d(\xi) = \sum_{d_1+\ldots+d_t=d} \gamma_{d_1}(\xi_1) * \cdots * \gamma_{d_t}(\xi_t).$$
3. *If $i : V \hookrightarrow X$ is a closed immersion and let ξ be a cycle in $CH^{ét}_m(V)$, then*
$$\gamma_d(i_*\xi) = Sym^d(i)_*(\gamma_d(\xi)) \in CH^{ét}_{dm}(Sym^d(X))$$
4. *Let $V \subset X$ be a closed subscheme, equidimensional of positive dimension. Then*
$$\gamma_d([V]_{ét}) = [Sym^d(V)]_{ét}$$
where we view $Sym^d(V)$ as a closed subscheme of $Sym^d(X)$ and $[Sym^d(V)]_{ét}$ represents the image of the algebraic cycle $[Sym^d(V)] \in CH_{dm}(Sym^d(X))$ in $CH^{ét}_{dm}(Sym^d(X))$

Proof. (1) is a direct consequence of [MP10, Lemma 1.1.(iii)], [CD19, Proposition 11.2.5] and the existence of a functor $\mathrm{DM}(X, \mathbb{Z}) \to \mathrm{DM}_{\text{ét}}(X, \mathbb{Z})$. The same argument works for (2) and the compatibility of the comparison map.

(3) This again is obtained by the definition of Pontryagin product using push-forward. The push-forward via the map $\mathrm{Sym}^d(i) : \mathrm{Sym}^d(V) \to \mathrm{Sym}^d(X)$ respects Pontryagin products.

(4) Is a direct consequence of [MP10, Lemma 1.3.4] and the compatibility of the comparison map with proper push-forwards. □

Lemma 4.3.25. *Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then for all $x \in CH^{ét}_{>0}(X)$ and all $d \geq 0$ one has*
$$Sym^d(f)_*(\gamma_d(x)) = \gamma_d(f_*x)$$

Proof. This follows from [MP10, Proposition 1.5] and [CD19, Proposition 11.2.5]. □

Let k be a field and let $(M_n)_{n\in\mathbb{N}}$ be a commutative graded monoid in the category of quasi-projective schemes. Recalling such definition: M_n is a quasi-projective k-scheme for all $n \geq 0$, and that we have product maps $\mu_{m,n} : M_m \times M_n \to M_{m+n}$ which satisfy commutativity and associativity. Assuming that there exists a k-rational point $e \in M_0(k)$ which is a unit for these products and that the maps $\mu_{n,m}$ are proper morphism, then we can define the Pontryagin product on the ring
$$\mathrm{CH}^{\text{ét}}_*(M_\bullet) := \bigoplus_{n\in\mathbb{N}} \mathrm{CH}^{\text{ét}}_*(M_n)$$
by the formula $x * y := (\mu_{m,n})_*(x \times y)$ for $x \in \mathrm{CH}^{\text{ét}}_*(M_m)$ and $y \in \mathrm{CH}^{\text{ét}}_*(M_n)$. Something that we ought to notice is that the iteration of the multiplication map $\mu_{n,\ldots,n} : M_n^d \to M_{dn}$ factors through the proper map $p_d : \mathrm{Sym}^d(M_n) \to M_{dn}$. We set p_0 to be the map $e \to M_0$.

Theorem 4.3.26. *For a commutative graded monoid $(M_n)_{n\in\mathbb{N}}$ with identity and with proper product morphisms, the maps*

$$\gamma_d^M : CH_{>0}^{\acute{e}t}(M_n) \to CH_*^{\acute{e}t}(M_{dn})$$

given by $x \mapsto (p_d)_*\gamma_d(x)$ extends uniquely to a PD-structure $\{\gamma_d^M\}_{d\geq 0}$ on the ideal $CH_{>0}^{\acute{e}t}(M_\bullet) \subset CH_*^{\acute{e}t}(M_\bullet)$. The PD-structure is functorial with respect to $(f_n : M_n \to N_n)$ which are proper for all $n \in \mathbb{N}$.

Proof. Let $x = \sum_{n\in\mathbb{N}} x_n$, $x_n \in \mathrm{CH}_{>0}^{\acute{e}t}(M_\bullet)$ and x_n is non-zero for finitely many n. Thus we define

$$\gamma_d^M(x) := \sum_{d_1+d_2+\ldots=d} \gamma_{d_1}^M(x_1) * \gamma_{d_2}^M(x_2) * \ldots$$

Clearly by definition we get $\gamma_d^M(\lambda x) = \lambda^d \gamma_d^M(x)$, and by Lemma 4.3.24 we have

$$\gamma_d^M(x+y) = \sum_{d_1+d_2=d} \gamma_{d_1}^M(x) * \gamma_{d_2}^M(y)$$

for all x, $y \in \mathrm{CH}_{>0}^{\acute{e}t}(M_\bullet)$ and for all $d \geq 0$. As this formula holds, for $x \in \mathrm{CH}_{>0}^{\acute{e}t}(X)$ and d, $e \geq 0$ we obtain the following relation

$$\gamma_d^M(x) * \gamma_e^M(x) = \binom{d+e}{d} \cdot \gamma_{d+e}^M(x).$$

For the other property

$$\gamma_d^M\left(\gamma_e^M(x)\right) = \frac{(de)!}{d!(e!)^d}\gamma_{de}^M(x)$$

we apply [CD19, Proposition 11.2.5], the functorial properties of the topology change $\rho : X_{\acute{e}t} \to X_{\mathrm{Zar}}$ and [MP10, Lemma 1.1.(iii)]. □

If X is a smooth quasi-projective k-scheme then the previous construction gives a PD-structure on the graded ideal $\bigoplus_{i\geq 0} \mathrm{CH}_{>0}^{\acute{e}t}(\mathrm{Sym}^i(X))$. If $M_n = \emptyset$ for all $n > 0$ (like for example an abelian variety and the multiplication) we obtain the following version (which is ungraded):

Corollary 4.3.27. *Let M be a commutative monoid with identity in the category of quasi-projective k-schemes, such that the product morphism $\mu : M \times M \to M$ is proper. Let $p_d : Sym^d(M) \to M$ be the morphism induced by the iterated multiplication map $M^d \to M$. Then the maps $\gamma_d^M : CH_{>0}^{\acute{e}t}(M) \to CH_*^{\acute{e}t}(M)$ defined by $x \mapsto (p_d)_*\gamma_d(x)$ define a PD-structure on the ideal $CH_{>0}^{\acute{e}t}(M) \subset CH_*^{\acute{e}t}(M)$.*

Corollary 4.3.28. *Let $k = \bar{k}$ be a field. Let A be an abelian variety over k, then there is a canonical PD-structure, with respect to the Pontryagin product, on the augmentation ideal in $CH_*^{\acute{e}t}(A)$, generated by $CH_{>0}^{\acute{e}t}(A)$ together with the 0-cycles of degree 0.*

Proof. Let $I \subset \mathrm{CH}_0^{\acute{e}t}(A)$ be the ideal of 0-cycles of degree 0 on A. After noticing that over algebraically closed fields we have an isomorphism $\mathrm{CH}_0(A) \simeq \mathrm{CH}_0^L(A)$, then the existence of the PD structure is due to [MP10, Corollary 1.8]. □

Etale Fourier transform

Let A be an abelian variety over a field k. The Fourier transform on the level of Chow groups is the groups homomorphism

$$\mathcal{F}_A : \mathrm{CH}(A)_{\mathbb{Q}} \to \mathrm{CH}(\widehat{A})_{\mathbb{Q}}$$

induced by the correspondence $\mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$, where $\mathrm{ch}(\mathcal{P}_A)$ is the Chern character of $\mathcal{P}_A$. One has the Fourier transform on the level of étale cohomology:

$$\mathfrak{F}_A : H^\bullet_{\text{ét}}(A_{k^s}, \mathbb{Q}_\ell(\bullet)) \to H^\bullet_{\text{ét}}(\widehat{A}_{k^s}, \mathbb{Q}_\ell(\bullet))$$

which preserves integral cohomology classes and induces, for each i with $0 \leq i \leq 2g$, an isomorphism

$$\mathfrak{F}_A : H^i_{\text{ét}}(A_{k^s}, \mathbb{Z}_\ell(n)) \to H^{2g-i}_{\text{ét}}(\widehat{A}_{k^s}, \mathbb{Z}_\ell(n+g-i)),$$

and if $k = \mathbb{C}$, then $\mathrm{ch}(\mathcal{P}_A)$ induces, for each $0 \leq i \leq 2g$, an isomorphism of Hodge structures

$$\mathfrak{F}_A : H^i(A, \mathbb{Z}) \to H^{2g-i}(\widehat{A}, \mathbb{Z}(g-i)).$$

Definition 4.3.29. *Let A be an abelian variety over k and let $\mathcal{F}_{\acute{e}t} : CH_{\acute{e}t}(A) \to CH_{\acute{e}t}(\widehat{A})$ be a group homomorphism. We call $\mathcal{F}_{\acute{e}t}$ a weak integral étale Fourier transform if the following diagram commutes*

$$\begin{tikzcd} CH_{\acute{e}t}(A) \arrow[r, "\mathcal{F}_{\acute{e}t}"] \arrow[d] & CH_{\acute{e}t}(\widehat{A}) \arrow[d] \\ CH_{\acute{e}t}(A)_{\mathbb{Q}} \arrow[r, "\mathcal{F}_A"] & CH_{\acute{e}t}(\widehat{A})_{\mathbb{Q}}. \end{tikzcd}$$

We call a weak integral Fourier transform $\mathcal{F}_{\acute{e}t}$ algebraic if it is induced by a cycle $\Gamma \in CH_{\acute{e}t}(A \times \widehat{A})$ that satisfies $\Gamma_{\mathbb{Q}} = ch(\mathcal{P}_A)$. A group homomorphism $\mathcal{F}_{\acute{e}t} : CH_{\acute{e}t}(A) \to CH_{\acute{e}t}(\widehat{A})$ is an integral étale Fourier transform up to homology if the following diagram commutes:

$$\begin{tikzcd} CH_{\acute{e}t}(A) \arrow[r, "\mathcal{F}_{\acute{e}t}"] \arrow[d] & CH_{\acute{e}t}(\widehat{A}) \arrow[d] \\ \bigoplus_{i=0}^{2g} H^{2i}_{\acute{e}t}(\widehat{A}_{k^s}, \mathbb{Z}_\ell(i)) \arrow[r, "\mathfrak{F}_A"] & \bigoplus_{i=0}^{2g} H^{2i}_{\acute{e}t}(\widehat{A}_{k^s}, \mathbb{Z}_\ell(i)). \end{tikzcd}$$

Finally an integral étale Fourier transform up to homology $\mathcal{F}_{\acute{e}t}$ is called algebraic if it is induced by a cycle $\Gamma \in CH_{\acute{e}t}(A \times \widehat{A})$ such that $cl(\Gamma) = ch(\mathcal{P}_A) \in \bigoplus_{i=0}^{4g} H^{2i}_{\acute{e}t}((A \times \widehat{A})_{k^s}, \mathbb{Z}_\ell(i))$. Similarly, a $\mathbb{Z}_\ell$-module homomorphism $\mathcal{F}_{\acute{e}t,\ell} : CH_{\acute{e}t}(A)_{\mathbb{Z}_\ell} \to CH_{\acute{e}t}(\widehat{A})_{\mathbb{Z}_\ell}$ is called an ℓ-adic integral Fourier transform up to homology if $\mathcal{F}_{\acute{e}t,\ell}$ is compatible with $\mathfrak{F}_A$ and the ℓ-adic cycle class map. If such homomorphism exists and is induced by a cycle $\Gamma_\ell \in CH_{\acute{e}t}(A \times \widehat{A})_{\mathbb{Z}_\ell}$ and $cl(\Gamma_\ell) = ch(\mathcal{P}_A)$ is called an algebraic ℓ-adic integral étale Fourier transform.

If $\mathcal{F}_{\text{ét}} : \mathrm{CH}_{\text{ét}}(A) \to \mathrm{CH}_{\text{ét}}(\widehat{A})$ is a weak integral étale Fourier transform, then $\mathcal{F}_{\text{ét}}$ is an integral étale Fourier transform up to homology. If $k = \mathbb{C}$, then $\mathcal{F}_{\text{ét}} : \mathrm{CH}_{\text{ét}}(A) \to \mathrm{CH}_{\text{ét}}(\widehat{A})$ is an integral étale Fourier transform up to homology if and only if $\mathcal{F}_{\text{ét}}$ is compatible with the Fourier transform $\mathfrak{F}_A : H_B^\bullet(A, \mathbb{Z}) \to H_B^\bullet(\widehat{A}, \mathbb{Z})$ on Betti cohomology.

Lemma 4.3.30. *Let A be a complex abelian variety and let $\mathcal{F}_{\acute{e}t} : CH_{\acute{e}t}(A) \to CH_{\acute{e}t}(\widehat{A})$ be an integral étale Fourier transform up to homology.*

1. *For each $i \in \mathbb{N}$ the integral étale Hodge conjecture for degree $2i$ classes on A implies the integral étale Hodge conjecture for degree $2(g-i)$ classes on $\widehat{A}$.*
2. *If $\mathcal{F}_{\acute{e}t}$ is algebraic, then $\mathfrak{F}_A$ induces a group isomorphism $Z^{2i}_{\acute{e}t}(A) \to Z^{2(g-i)}_{\acute{e}t}(\widehat{A})$, where $Z^{2i}_{\acute{e}t}(A)$ is the image of the Lichtenbaum cycle class map.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathrm{CH}^i_{\text{ét}}(A) \arrow[r] \arrow[d, "c^i_{\text{ét}}"] & \mathrm{CH}_{\text{ét}}(A) \arrow[r, "\mathcal{F}_{\text{ét}}"] \arrow[d] & \mathrm{CH}_{\text{ét}}(\widehat{A}) \arrow[r] \arrow[d] & \mathrm{CH}^{g-i}_{\text{ét}}(\widehat{A}) \arrow[d, "c^{g-i}_{\text{ét}}"] \\
H^{2i}_B(A, \mathbb{Z}) \arrow[r] & H^\bullet_B(A, \mathbb{Z}) \arrow[r] & H^\bullet_B(\widehat{A}, \mathbb{Z}) \arrow[r] & H^{2(g-i)}_B(\widehat{A}, \mathbb{Z})
\end{tikzcd}$$

The composition of the bottom line $H^{2i}_B(A, \mathbb{Z}) \to H^{2(g-i)_B}(\widehat{A}, \mathbb{Z})$ is an isomorphism of Hodge structures, then we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^i_{\text{ét}}(A) \arrow[r] \arrow[d, "c^i_{\text{ét}}"] & \mathrm{CH}^{g-i}_{\text{ét}}(\widehat{A}) \arrow[d, "c^{g-i}_{\text{ét}}"] \\
\mathrm{Hdg}^{2i}(A, \mathbb{Z}) \arrow[r, "\simeq"] & \mathrm{Hdg}^{2(g-i)}(\widehat{A}, \mathbb{Z})
\end{tikzcd}$$

Thus the surjectivity of $c^i_{\text{ét}}$ implies the surjectivity of $c^{g-i}_{\text{ét}}$. Arguing in the same way for $\widehat{A}$ and $\widehat{\widehat{A}}$ we obtain the desired equivalence. □

For an abelian variety A over k we define the following cycles:

$$\ell = c_1(\mathcal{P}_A) \in \mathrm{CH}^1_{\text{ét}}(A \times \widehat{A})_{\mathbb{Q}}, \quad \widehat{\ell} = c_1(\mathcal{P}_{\widehat{A}}) \in \mathrm{CH}^1_{\text{ét}}(\widehat{A} \times A)_{\mathbb{Q}}$$
$$\mathcal{R}_A = \frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} \in \mathrm{CH}^{2g-1}_{\text{ét}}(A \times \widehat{A})_{\mathbb{Q}}, \quad \mathcal{R}_{\widehat{A}} = \frac{c_1(\mathcal{P}_{\widehat{A}})^{2g-1}}{(2g-1)!} \in \mathrm{CH}^{2g-1}_{\text{ét}}(\widehat{A} \times A)_{\mathbb{Q}}.$$

For $a \in \mathrm{CH}_{\text{ét}}(A)_{\mathbb{Q}}$ we define $E(a) \in \mathrm{CH}_{\text{ét}}(A)_{\mathbb{Q}}$ as the exponential element using $*$-operation:

$$E(a) := \sum_{n \geq 0} \frac{a^{*n}}{n!} \in \mathrm{CH}_{\text{ét}}(A)_{\mathbb{Q}}.$$

The following theorem is the same one as [BG23, Theorem 3.8] but changing Chow groups to its étale analogue,

Theorem 4.3.31. *Let A be an abelian variety over k of dimension g. The following statements are equivalent:*

1. *The one cycle* $\dfrac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} \in CH_{ét}(A \times \widehat{A})_{\mathbb{Q}}$ *lifts to* $CH^{2g-1}_{ét}(A \times \widehat{A})$.

2. *The abelian variety A admits an étale motivic weak integral Fourier transform.*

3. *The abelian variety $A \times \widehat{A}$ admits an étale motivic weak integral Fourier transform.*

If we assume that A carries a symmetric ample line bundle which induces a principal polarization $\lambda : A \xrightarrow{\sim} \widehat{A}$, therefore the previous statements are equivalent to the following

(4) *The two cycle* $\dfrac{c_1(\mathcal{P}_A)^{2g-2}}{(2g-2)!} \in CH_{ét}(A \times \widehat{A})_{\mathbb{Q}}$ *lifts to* $CH^{2g-2}_{ét}(A \times \widehat{A})$.

(5) *Denoting as $\Theta \in CH^1_{ét}(A)_{\mathbb{Q}}$ to the symmetric ample class attached to λ, then the one cycle* $\Gamma_\Theta = \dfrac{\Theta^{g-1}}{(g-1)!} \in CH_{ét}(A)_{\mathbb{Q}}$ *lifts to a one cycle in* $CH^{g-1}_{ét}(A)$.

(6) *The abelian variety A admits a weak integral étale Fourier transform.*

(7) *The Fourier transform $\mathcal{F}_A$ satisfies $\mathcal{F}_A(CH_{ét}(A)_{tf}) \subset CH_{ét}(\widehat{A})_{tf}$.*

(8) *There exists a PD-structure on the ideal $CH^{>0}_{ét}(A)_{tf} \subset CH_{ét}(A)_{tf}$.*

Proof. Assuming (1), then there exists a cycle $Z \in \mathrm{CH}^{2g-1}_{\text{ét}}(A \times \widehat{A})$ such that $Z_{\mathbb{Q}} \in \mathrm{CH}^{2g-1}_{\text{ét}}(A \times \widehat{A})_{\mathbb{Q}}$ equals $\dfrac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!}$. Consider the cycle $(-1)^g \cdot E((-1)^g \cdot Z) \in \mathrm{CH}_{\text{ét}}(A \times \widehat{A})$, by [BG23, Lemma 3.4] we have that

$$(-1)^g \cdot E((-1)^g \cdot Z)_{\mathbb{Q}} = (-1)^g \cdot E\left((-1)^g \cdot \frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!}\right) = \mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}_{\text{ét}}(A \times \widehat{A})_{\mathbb{Q}}$$

then follows (2). By the same principle, the line bundle $\mathcal{P}_{A \times \widehat{A}}$ on the abelian variety $X = A \times \widehat{A} \times \widehat{A} \times A$, we have that $\mathcal{P}_{A \times \widehat{A}} \simeq \pi_{13}^* \mathcal{P}_A \otimes \pi_{24}^* \mathcal{P}_{\widehat{A}}$, then

$$\begin{aligned}
\mathcal{R}_{A \times \widehat{A}} &= \frac{(\pi_{13}^* c_1(\mathcal{P}_A) + \pi_{24}^* c_1(\mathcal{P}_{\widehat{A}}))^{4g-1}}{(4g-1)!} \\
&= \pi_{13}^* \left(\frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!}\right) \cdot \pi_{24}^*([0]_{A \times \widehat{A}}) + \pi_{13}^*([0]_{\widehat{A} \times A}) \cdot \pi_{24}^* \left(\frac{c_1(\mathcal{P}_{\widehat{A}})^{2g-1}}{(2g-1)!}\right)
\end{aligned}$$

therefore we conclude that $\mathcal{R}_{A \times \widehat{A}}$ lifts to $\mathrm{CH}^{4g-1}_{\text{ét}}(X)$, this implies that $A \times \widehat{A}$ admits a motivic weak integral Fourier transform. (3)$\Longrightarrow$(1) follows from the fact that $(-1)^g \mathcal{F}_{\widehat{A} \times A}(-\widehat{\ell}) = \mathcal{R}_A$.

From now on, we assume that A is a principally polarized variety $\lambda : A \to \widehat{A}$, with $\mathcal{L}$ be the symmetric ample line bundle. Assuming that (4) holds and denoting $s_A \in \mathrm{CH}_2^{\text{ét}}(A \times A) = \mathrm{CH}^{2g-2}_{\text{ét}}(A \times A)$ such that $(s_A)_{\mathbb{Q}} = \frac{c_1(\mathcal{P}_A)^{2g-2}}{(2g-2)!}$. Consider the symmetric line bundles $\mathrm{CH}^1_{\mathrm{Sym}}(A) \subset \mathrm{CH}^1(A)$ and the homomorphism $\mathcal{F} : \mathrm{CH}^1_{\mathrm{Sym}}(A) \to \mathrm{CH}_1^{\text{ét}}(A)$ defined as the composition

$$\mathrm{CH}^1_{\mathrm{Sym}}(A) \hookrightarrow \mathrm{CH}^1(A) \xrightarrow{\mathrm{pr}_1^*} \mathrm{CH}^1_{\text{ét}}(A \times A) \xrightarrow{\cdot s_A} \mathrm{CH}^{2g-2}_{\text{ét}}(A \times A) \xrightarrow{\mathrm{pr}_{2*}} \mathrm{CH}_1^{\text{ét}}(A)$$

As the line bundle $\mathcal{L}$ is symmetric, we have the following

$$\begin{aligned}\Theta &= \frac{1}{2} \cdot (\mathrm{id}, \lambda)^* c_1(\mathcal{P}_A) \\ &= \frac{1}{2} \cdot c_1((\mathrm{id}, \lambda)^* \mathcal{P}_A) \\ &= \frac{1}{2} \cdot c_1(\mathcal{L} \otimes \mathcal{L}) = c_1(\mathcal{L}) \in \mathrm{CH}^1(A)_{\mathbb{Q}}\end{aligned}$$

The Chern class of $\mathcal{L}$ is sent to Θ, therefore $\mathcal{F}(c_1(\mathcal{L}))_{\mathbb{Q}} = \Gamma_\Theta$, therefore (5) holds. If we assume that (5) holds, then by [BG23, Lemma 3.5] we obtain (1).

If (2) holds then immediately holds (4), so we obtain that (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4). Under the assumptions about polarization, we see that (2) $\Longrightarrow$ (6) $\Longrightarrow$ (7). If we assume that (7), since $\Theta = c_1(\mathcal{L})$ is lifted to $\mathrm{CH}^1(A)$, then $\mathcal{F}_A(\Theta) = (-1)^{g-1}\Gamma_\Theta$ is lifted to $\mathrm{CH}_1^{\text{ét}}(A)$, thus (5) holds. Again if (7) holds, then $\mathcal{F}_A$ defines an isomorphism

$$\mathcal{F}_A : \mathrm{CH}_{\text{ét}}(A)_{\mathrm{tf}} \xrightarrow{\sim} \mathrm{CH}_{\text{ét}}(A)_{\mathrm{tf}}.$$

The ideal $\mathrm{CH}^{\text{ét}}_{>0}(A)_{\mathrm{tf}} \subset \mathrm{CH}_{\text{ét}}(A)_{\mathrm{tf}}$ admits a PD-structure for the Pontryagin product. As $\mathcal{F}_A$ exchanges Pontryagin product by intersection product, we obtain (8).

□

Using the same arguments, we can obtain the following equivalences for the different notions of étale Fourier transform:

Proposition 4.3.32. *Let A be an abelian variety of dimension g over a field k. The following assertions are equivalent:*

1. *The class $\dfrac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} \in H^{4g-2}_{\acute{e}t}((A \times \widehat{A})_{k^s}, \mathbb{Z}_\ell(2g-1))$ is the class of a cycle in $CH^{2g-1}_{\acute{e}t}(A \times \widehat{A})$.*

2. *The abelian variety A admits an étale integral Fourier transform up to homology which is algebraic.*

3. *The abelian variety $A \times \widehat{A}$ admits an étale integral Fourier transform up to homology which is algebraic.*

If we assume that A carries a symmetric ample line bundle which induces a principal polarization $\lambda : A \xrightarrow{\sim} \widehat{A}$, therefore the previous statements are equivalent to the following:

(4) The class $\dfrac{c_1(\mathcal{P}_A)^{2g-2}}{(2g-2)!} \in H^{4g-4}_{\acute{e}t}((A \times \widehat{A})_{k^s}, \mathbb{Z}_\ell(2g-2))$ is the class of a cycle in $CH^{2g-2}_{\acute{e}t}(A \times \widehat{A})$.

(5) The class $\theta^{g-1}/(g-1)! \in H^{2g-2}_{\acute{e}t}(A_{k^s}, \mathbb{Z}_\ell(g-1))$ lifts to a cycle in $CH^{g-1}_{\acute{e}t}(A)$.

(6) The abelian variety A admits an integral étale Fourier transform up to homology.

If $k = \mathbb{C}$ then the previous (1)-(6) is equivalent to the same statement replacing étale cohomology by Betti cohomology.

Proof. The proof of the equivalences is analogue to Theorem 4.3.31. If the base field is $k = \mathbb{C}$, we have an isomorphism $H^i_{\text{ét}}(A, \mathbb{Z}_\ell) \simeq H^i_B(A, \mathbb{Z}_\ell)$ and the fact that $\beta \in H^{2i}_B(A, \mathbb{Z})$ is in the image of the cycle class map if and only if $\beta_\ell \in H^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell)$ is in the image of the cycle class map. □

Remark 4.3.33. Since the PD-sctructure on $(\mathrm{CH}^{\text{ét}}_{>0}(A), *)$ induces a PD-structure on $(\mathrm{CH}^{\text{ét}}_{>0}(A)_{\mathbb{Z}_\ell}, *)$, Proposition 4.3.32 remains true if we change $\mathrm{CH}_{\text{ét}}(A)$ by $\mathrm{CH}_{\text{ét}}(A)_{\mathbb{Z}_\ell}$ and "étale integral Fourier transform up to homology" by "étale ℓ-adic integral Fourier transform up to homology".

Corollary 4.3.34. *Let k be an algebraically closed field and let $\ell \neq char(k)$ be a prime integer. Considering an abelian variety A/k, then for ℓ there exists an ℓ-adic integral étale Fourier transform up to homology which is algebraic.*

Proof. Consider a smooth projective variety X over k, then we have a cycle class map $c^i_{\text{ét},\ell} : \mathrm{CH}^i_{\text{ét}}(X)_{\mathbb{Z}_\ell} \to H^{2i}_{\text{ét}}(X, \mathbb{Z}_\ell)$. Let us consider a finitely generated sub $\mathbb{Z}_\ell$-module $G \subseteq H^{2i}_{\text{ét}}(X, \mathbb{Z}_\ell)$. Let $\mathrm{CH}^i_{\text{ét}}(X)_{\mathbb{Z}_\ell} \supseteq W := c^{i,-1}_{\text{ét},\ell}(G)$ and take the map f as $c^i_{\text{ét},\ell}$ restricted to W, thus we have $f : W \to G$. Denoting by $I^{2i}_{W,\text{ét},\ell}(X) = \mathrm{im}(f)$ and $I^{2i}_{\text{ét},\ell}(X) : \mathrm{im}(c^i_{\text{ét},\ell})$, then we have that $\left(G/I^{2i}_{W,\text{ét},\ell}(X)\right)\{\ell\} \hookrightarrow \left(H^{2i}_{\text{ét}}(X, \mathbb{Z}_\ell)/I^{2i}_{\text{ét},\ell}(X)\right)\{\ell\} = 0$. Thus we can conclude that $\left(G/I^{2i}_{W,\text{ét},\ell}(X)\right)$ is a torsion free $\mathbb{Z}_\ell$-module, thus

$$\left(G/I^{2i}_{W,\text{ét},\ell}(X)\right) \otimes \mathbb{Q} = 0 \iff \left(G/I^{2i}_{W,\text{ét},\ell}(X)\right) = 0$$

So this implies that G is in the preimage of $\mathrm{CH}^i_{\text{ét}}(X)_{\mathbb{Z}_\ell}$ if and only if $G \otimes \mathbb{Q}_\ell$ is in the preimage of $\mathrm{CH}^i_{\text{ét}}(X)_{\mathbb{Q}_\ell}$. In particular, consider $X = A \times \widehat{A}$ and the integral ℓ-adic class $\dfrac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} \in H^{4g-2}_{\text{ét}}(A \times \widehat{A}, \mathbb{Z}_\ell(2g-1))$ and let G be the sub $\mathbb{Z}_\ell-$module of $H^{4g-2}_{\text{ét}}(A \times \widehat{A}, \mathbb{Z}_\ell(2g-1))$ generated by $\dfrac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!}$ (which is algebraic with rational coefficients), thus by the previous remark we obtain that lifts to $\mathrm{CH}^{\text{ét}}_1(A \times \widehat{A})_{\mathbb{Z}_\ell}$. □

Remark 4.3.35. Even though we can lift the Chern class of the Poincaré bundle $\mathcal{P}_A$ it is not clear whether or not the Tate conjecture holds for abelian varieties due to the obstruction $T_\ell(\mathrm{Br}^n(A \times \widehat{A}))$.

Corollary 4.3.36. *Let k be any of the following fields: one finitely generated over $\mathbb{Q}$ or $\mathbb{F}_{p^r}$ for p a prime number and $r \in \mathbb{N}$. Then any abelian variety A/k admits anétale ℓ-adic integral Fourier transform up to homology for the other cases.*

Proof. The other cases are a consequence of [RS16, Theorem 1.3 and 1.4] respectively using the argument described beforehand in the proof of last corollary. □

Theorem 4.3.37. *Let k be an algebraically closed field, then for any abelian variety A over k there exists an integral algebraic étale Fourier transform if and only if for all $\ell \neq char(k)$, A/k admits an étale ℓ-adic integral Fourier transform up to homology.*

Proof. One way is clear. For the other one we shall split the proof in several parts: First, let us prove that the Fourier transform preserves torsion classes. Notice that as $H^i_{\text{ét}}(A, \mathbb{Z}_\ell(j))$ is a torsion free $\mathbb{Z}_\ell$-module, then we have a short exact sequence

$$0 \to H^i_{\text{ét}}(A, \mathbb{Z}_\ell(j)) \to H^i_{\text{ét}}(A, \mathbb{Q}_\ell(j)) \to H^i_{\text{ét}}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) \to 0,$$

assuming that $i \neq 2j+1$ by [RS16, Proposition 5.1] then we have an isomorphism $H^i_{\text{ét}}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) \simeq H^{i+1}_{M,\text{ét}}(A, \mathbb{Z}(j))\{\ell\}$, the same holds for $\widehat{A}$. With this we have obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^i_{\text{ét}}(A, \mathbb{Z}_\ell(j)) \arrow[r] \arrow[d, "\mathfrak{F}_A"] & H^i_{\text{ét}}(A, \mathbb{Q}_\ell(j)) \arrow[r] \arrow[d, "\mathfrak{F}_A"] & H^i_{\text{ét}}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) \arrow[r] \arrow[d, "\mathfrak{F}_A^{q,\ell}"] & 0 \\
0 \arrow[r] & H^{2g-i}_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell(a)) \arrow[r] & H^{2g-i}_{\text{ét}}(\widehat{A}, \mathbb{Q}_\ell(a)) \arrow[r] & H^{2g-i}_{\text{ét}}(\widehat{A}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(a)) \arrow[r] & 0
\end{tikzcd}$$

where $a = j + g - i$ and $\mathfrak{F}_A^{q,\ell}$ is the induced map by the quotient, therefore we have an morphism of torsion groups, therefore we have that $\mathfrak{F}_A^{q,\ell} : \mathrm{CH}^i_{\text{ét}}(A)\{\ell\} \xrightarrow{\simeq} \mathrm{CH}^{g-i+1}_{\text{ét}}(\widehat{A})\{\ell\}$. Assuming that for each prime number $\ell \neq \mathrm{char}(k)$, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^i_{\text{ét}}(A)_{\mathbb{Z}_\ell} \arrow[r] \arrow[d, "c^i_{\text{ét},\ell}"] & \mathrm{CH}_{\text{ét}}(A)_{\mathbb{Z}_\ell} \arrow[r, "\mathcal{F}_{\text{ét}}"] \arrow[d] & \mathrm{CH}_{\text{ét}}(\widehat{A})_{\mathbb{Z}_\ell} \arrow[r] \arrow[d] & \mathrm{CH}^{g-i}_{\text{ét}}(\widehat{A})_{\mathbb{Z}_\ell} \arrow[d, "c^{g-i}_{\text{ét},\ell}"] \\
H^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell) \arrow[r] & H^\bullet_{\text{ét}}(A, \mathbb{Z}_\ell) \arrow[r] & H^\bullet_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell) \arrow[r] & H^{2(g-i)}_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell)
\end{tikzcd}$$

exchanging the Fourier transform $\widehat{A}$ with the double dual of A, we can conclude that $c^i_{\text{ét},\ell}(\mathrm{CH}^i_{\text{ét}}(A)_{\mathbb{Z}_\ell}) \simeq c^{g-i}_{\text{ét},\ell}(\mathrm{CH}^{g-i}_{\text{ét}}(\widehat{A})_{\mathbb{Z}_\ell})$. Since the kernel of the cycle class map $c^i_{\text{ét},\ell}$ is ℓ-divisible, then $\mathrm{CH}^i_{\text{ét}}(A) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \simeq I^{2i}_{\text{ét},\ell}(A) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$.

Consider the short exact sequence $0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$, then by the previous remark, we obtain a quotient map $F^\ell_A : \mathrm{CH}_{\text{ét}}(A)_{\mathbb{Q}_\ell/\mathbb{Z}_\ell} \to \mathrm{CH}_{\text{ét}}(\widehat{A})_{\mathbb{Q}_\ell/\mathbb{Z}_\ell}$ which is an isomorphism. First let us assume that $\mathrm{char}(k) = 0$, with that we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}_{\text{ét}}(A)_{\text{tors}} \arrow[r] \arrow[d, "\mathfrak{F}_A^q"] & \mathrm{CH}_{\text{ét}}(A) \arrow[r] \arrow[d, "\mathfrak{F}_A"] & \mathrm{CH}_{\text{ét}}(A)_{\mathbb{Q}} \arrow[r] \arrow[d, "\mathcal{F}_A"] & \mathrm{CH}_{\text{ét}}(A)_{\mathbb{Q}/\mathbb{Z}} \arrow[r] \arrow[d, "F_A"] & 0 \\
0 \arrow[r] & \mathrm{CH}_{\text{ét}}(\widehat{A})_{\text{tors}} \arrow[r] & \mathrm{CH}_{\text{ét}}(\widehat{A}) \arrow[r] & \mathrm{CH}_{\text{ét}}(\widehat{A})_{\mathbb{Q}} \arrow[r] & \mathrm{CH}_{\text{ét}}(\widehat{A})_{\mathbb{Q}/\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

where $\mathfrak{F}_A^q = \oplus_{\ell \neq \mathrm{char}(k)} \mathfrak{F}_A^{q,\ell}$ and $F_A = \oplus_{\ell \neq \mathrm{char}(k)} F^\ell_A$. In particular we found $\mathfrak{F}_A \otimes \mathbb{Q} = \mathcal{F}_A$, so it maintains integral étale cycles. If we work over positive characteristic p, then we take $\ell \neq p$ and use the fact that $\mathrm{CH}_{\text{ét}}(A) \simeq \mathrm{CH}_L(A)[1/p]$. $\square$

Decomposition of motives over an algebraically closed field

In this subsection we aim to obtain an analogue of [Huy18, Lemma 1.1] for the category $\mathrm{Chow}_{\text{ét}}(k)$. Roughly speaking, this result is an improved version of Manin's principle, but only when one works over an algebraically closed field.

Manin principle says that a morphism $f : M \to N$ between Chow motives is an isomorphism if and only if the associated map $(f \times \mathrm{id}_Z)_* : \mathrm{CH}^*(M \otimes h(Z))_{\mathbb{Q}} \to \mathrm{CH}^*(N \otimes h(Z))_{\mathbb{Q}}$ is an isomorphism for every smooth projective variety Z. There are few cases where the structure of the Chow groups are maintained in an easy way, such as projective bundles or blow-ups, but in general it is not an easy task to obtain this property. Recalling that a universal domain Ω over k is an algebraically closed field extension of infinite transcendence degree (for example $k = \bar{\mathbb{Q}}$ and $\Omega = \mathbb{C}$), the improved Manin priciple states the following:

Theorem 4.3.38 ([Huy18, Lemma 1.1])**.** *Consider an algebraically closed field k. Let $f : M \to N$ be a morphism in the category Chow$(k)_{\mathbb{Q}}$. Then f is an isomorphism of motives in Chow$(k)_{\mathbb{Q}}$ if and only if for Ω a universal domain over k, the induced map $(f_\Omega)_* : CH^*(M_\Omega)_{\mathbb{Q}} \to CH^*(N_\Omega)_{\mathbb{Q}}$ given by the base change $f_\Omega : M_\Omega \to N_\Omega$, is bijective.*

Therefore for an algebraically closed field, is not necessary to test a morphism indexed by the objects in SmProj_k, only for a huge field extension of k. The improved version of Manin principle is a direct consequence of the results [GG12, Lemma 1], [Via17, Theorem 3.18] and [BP20, Lemma 2.4].

Example 4.3.39. *Consider a conic bundle $X \to \mathbb{P}^2_{\bar{Q}}$, we have that the Chow groups of X are characterized by*

$$CH^0(X)_{\mathbb{Q}} \simeq CH^3(X)_{\mathbb{Q}} \simeq \mathbb{Q},\ \ CH^1(X) \simeq \mathbb{Q} \oplus \mathbb{Q} \text{ and } CH^2(X) \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus Prym(\bar{C}/C)_{\mathbb{Q}},$$

so for this case we can recover the motivic decomposition of X obtained in [NS09]. In this context, C is called the discriminant curve of X, $\sigma_C : \bar{C} \to C$ is a double covering and Prym$(\bar{C}/C)$ is the Prym variety.

In the following, we will present the analogue of [Huy18, Lemma 1.1] for the category $\mathrm{Chow}_{\text{ét}}(k)$. To obtain that, we will prove the analogue of [GG12, Lemma 1]:

Lemma 4.3.40. *Let $M = (X, p, m)$ be an étale Chow motive defined over an algebraically closed field k. Let Ω be a universal domain of k and assume that $CH^i_{\acute{e}t}(M_\Omega) = 0$ for all $i \geq 0$. Then $M \simeq 0$ in Chow$_{\acute{e}t}(k)$.*

Proof. We proceed with similar arguments as in [GG12, Lemma 1]. Consider $Y \in \mathrm{SmProj}_k$ and let $i : Z \hookrightarrow Y$ be a smooth closed immersion of codimension c_Z and let $U := Y - Z$ be the open complement

$$\ldots \to \mathrm{CH}^{i-c_Z}_{\text{ét}}(X \times Z) \to \mathrm{CH}^i_{\text{ét}}(X \times Y) \to \mathrm{CH}^i_{\text{ét}}(X \times U) \to \mathrm{Br}^{i-c_Z}(X \times Z) \to \ldots$$

now take the direct limit over opens $U \subset Y$ we obtain that

$$\ldots \to \bigoplus_{Z \subset Y} \mathrm{CH}^{i-c_Z}_{\text{ét}}(X \times Z) \to \mathrm{CH}^i_{\text{ét}}(X \times Y) \to \varinjlim_{U \subset Y} \mathrm{CH}^i_{\text{ét}}(X \times U) \to \bigoplus_{Z \subset Y} \mathrm{Br}^{i-c_Z}(X \times Z) \to \ldots$$

since we have the isomorphism $\varinjlim_{U\subset Y} \mathrm{CH}^i_{\text{ét}}(X \times U) \simeq \mathrm{CH}^i_{\text{ét}}(X_{k(Y)})$ and consider the morphism (defined through the action of correspondences) $p \otimes Y : \mathrm{CH}^i_{\text{ét}}(X \times Y) \to \mathrm{CH}^i_{\text{ét}}(X \times Y)$ defined as follows

$$(p \otimes Y)(\alpha) := (\mathrm{pr}_{23})_* \left(\mathrm{pr}_{13}^*(\alpha) \times p \cdot \Gamma_{\mathrm{pr}_{12}}\right)$$

where $\mathrm{pr}_{ij} : X \times X \times Y \to X_i \times X_j$ and $\Gamma_{\mathrm{pr}_{12}}$ is the graph of the projection morphism. We apply the morphisms $\bigoplus_Z p \otimes Z$, $p \otimes Y$ and $p \otimes k(Y)$ we then obtain the following exact sequence

$$\bigoplus_{Z\subset Y} \mathrm{im}(p \otimes Z) \to \mathrm{im}(p \otimes Y) \to \mathrm{CH}^i_{\text{ét}}(M_{k(Y)}) \to \bigoplus_{Z\subset Y} \mathrm{im}(p \otimes Z)_{-1}.$$

Notice the following facts about the étale Chow groups of the motive M:

- If Y is irreducible with $\dim(Y) = 0$, then $\mathrm{im}(p \otimes Y) = \mathrm{CH}^i_{\text{ét}}(M)$, and consider Ω a field extension of k which is algebraically closed. Then we have that $\mathrm{CH}^i_{\text{ét}}(M) \to \mathrm{CH}^i_{\text{ét}}(M_\Omega)$ is injective, so by the hypothesis $\mathrm{CH}^i_{\text{ét}}(M) = 0$ for all $i \geq 0$.

- By induction, assume that for Z of dimension $0, \dots, n-1$ we have that $\mathrm{im}(p \otimes Z)$ vanish, then $\mathrm{im}(p \otimes Y)$ injects in $\mathrm{CH}^i_{\text{ét}}(M_{k(Y)})$ by the localization sequence. By [GG12, Lemma 1], the action of $p \otimes k(Y)$ over the torsion free part of $\mathrm{CH}^i_{\text{ét}}(X_{k(Y)})$, then $\mathrm{CH}^i_{\text{ét}}(M_{k(Y)}) \simeq (p \otimes k(Y))_* H^{2i-1}_{\text{ét}}(X_{k(Y)}, (\mathbb{Q}/\mathbb{Z})'(i))$.

  To conclude we will use a specialization argument. Consider a open subset $U \subset Y$ and consider the motive (X_U, p_U). Now let u be a closed point of U therefore we can define the regular embedding $j_u : u \hookrightarrow U$. Notice that the closed fibers of U are isomorphic to (X, p) over k. Since the specialization map commutes with products, pull-backs and pushforwards, we obtain that the projector $p \otimes U$ acts as zero over $\mathrm{CH}^i_{\text{ét}}(X \times U)$, therefore we conclude that $p \otimes k(Y)$ acts as zero over $\mathrm{CH}^i_{\text{ét}}(X_{k(Y)})$. Finally, we conclude that $\mathrm{CH}^i_{\text{ét}}(M_{k(Y)}) = 0$ for all integer $i \geq 0$.

Since we have that $\mathrm{CH}^i_{\text{ét}}(M_{k(Y)}) = 0$ and $\mathrm{im}(p \otimes Y)$ injects into $\mathrm{CH}^i_{\text{ét}}(M_{k(Y)})$ for all $Y \in \mathrm{SmProj}_k$, since $\mathrm{im}(p \otimes Y) \simeq \mathrm{CH}^i_{\text{ét}}(M \otimes h(Y))$ by the Manin principle for étale motives we can conclude that $M = 0$. ☐

Along with Definition 2.5.13 and Lemma 2.5.14 we obtain the analogue of [Via17, Theorem 3.18] for étale motivic cohomology:

Lemma 4.3.41. *Let $f : M \to N$ be a morphism of motives over k with $k = \bar{k}$, $k \hookrightarrow \Omega$, with Ω an universal domain, such that the induced morphism $(f_\Omega)_* : CH^*_{\text{ét}}(M_\Omega) \to CH^*_{\text{ét}}(N_\Omega)$ is surjective. Then f is surjective.*

Proof. Let f be a morphism of motives over k and let Ω be a universal domain such that $k \hookrightarrow \Omega$. Consider $Z \in \mathrm{SmProj}_k$, we will prove that the morphism $f \otimes Z : \mathrm{CH}^*_{\text{ét}}(X \times Z) \to \mathrm{CH}^*_{\text{ét}}(Y \times Z)$ has the same image as $q \otimes Z : \mathrm{CH}^*_{\text{ét}}(Y \times Z) \to \mathrm{CH}^*_{\text{ét}}(Y \times Z)$, for all Z. In order to prove this, we proceed by induction over the dimension of Z.

If $\dim(Z) = 0$ then the result is clear. So let us assume that works for $\dim(Z) \leq n$. Consider the following commutative diagram induced by the localization sequence

$$\begin{tikzcd}
\dots \arrow[r] & \bigoplus_{D\subset Z} \mathrm{CH}^*_{\text{ét}}(X \times D) \arrow[r] \arrow[d, "\bigoplus f\otimes D"] & \mathrm{CH}^*_{\text{ét}}(X \times Z) \arrow[r] \arrow[d, "f\otimes Z"] & \mathrm{CH}^*_{\text{ét}}(X_K) \arrow[r] \arrow[d, "(f_K)_*"] & \dots \\
\dots \arrow[r] & \bigoplus_{D\subset Z} \mathrm{CH}^*_{\text{ét}}(Y \times D) \arrow[r] & \mathrm{CH}^*_{\text{ét}}(Y \times Z) \arrow[r] & \mathrm{CH}^*_{\text{ét}}(Y_K) \arrow[r] & \dots
\end{tikzcd}$$

where $K = k(Z)$. By assumption, we have that the map $(f_\Omega)_* : \mathrm{CH}^*_{\text{ét}}(X_\Omega) \to \mathrm{CH}^*_{\text{ét}}(Y_\Omega)$ is surjective, then we have that in the level of torsion $\mathrm{CH}^*_{\text{ét}}(X)_{\text{tors}} \to \mathrm{CH}^*_{\text{ét}}(X_\Omega)_{\text{tors}}$ is an isomorphism, so the map $\mathrm{CH}^*_{\text{ét}}(X_K)_{\text{tors}} \to \mathrm{CH}^*_{\text{ét}}(X_{\bar{K}})_{\text{tors}} \xrightarrow{\simeq} \mathrm{CH}^*_{\text{ét}}(X_\Omega)_{\text{tors}}$ is surjective because it factors through the previous map. This gives us that the map $(f_K)_*$: $\mathrm{CH}^*_{\text{ét}}(X_K)_{\text{tors}} \to \mathrm{CH}^*_{\text{ét}}(Y_K)_{\text{tors}}$ is surjective, and then also the induced map $(f_K)_*$: $\mathrm{CH}^*_{\text{ét}}(X_K) \to \mathrm{CH}^*_{\text{ét}}(Y_K)$.

By induction hypothesis $f \otimes D$ it has the same image as $q \otimes D$, for all $D \subset Z$. In the same way, we have a similar commutative diagram involving $\bigoplus_{D\subset Y} q \otimes D$, $q \otimes Z$ and $q \otimes K$.

$$\begin{tikzcd}
\dots \arrow[r] & \bigoplus_{D\subset Z} \mathrm{CH}^*_{\text{ét}}(Y \times D) \arrow[r] \arrow[d, "\bigoplus q\otimes D"] & \mathrm{CH}^*_{\text{ét}}(Y \times Z) \arrow[r] \arrow[d, "q\otimes Z"] & \mathrm{CH}^*_{\text{ét}}(Y_K) \arrow[r] \arrow[d, "(q_K)_*"] & \dots \\
\dots \arrow[r] & \bigoplus_{D\subset Z} \mathrm{CH}^*_{\text{ét}}(Y \times D) \arrow[r] & \mathrm{CH}^*_{\text{ét}}(Y \times Z) \arrow[r] & \mathrm{CH}^*_{\text{ét}}(Y_K) \arrow[r] & \dots
\end{tikzcd}$$

Finally, as we have $\mathrm{im}(q \otimes Z) = \mathrm{im}(f \otimes Z)$, then $(f \times \mathrm{id}_Z)_* : \mathrm{CH}^*_{\text{ét}}(M \otimes h_{\text{ét}}(Z)) \to \mathrm{CH}^*_{\text{ét}}(N \otimes h_{\text{ét}}(Z))$ is a surjective map for all Z smooth projective variety, therefore by Lemma 2.5.14 we have that f is surjective.

□

Finally, we can get the extension to the integral étale case of [Huy18, Lemma 1.1] using the following lemma:

Lemma 4.3.42. *Let $f : M \to N$ be a morphism of motives over k such that for a universal domain Ω, the induced morphism $(f_\Omega)_* : CH^i_{ét}(M_\Omega) \to CH^i_{ét}(N_\Omega)$ is an isomorphism for all $i \geq 0$. Then f_Ω is an isomorphism in the category $Chow_{ét}(\Omega)$.*

Proof. Let Ω be an universal domain of k. By assumption, we have an isomorphism $(f_\Omega)_* : \mathrm{CH}^i_{\text{ét}}(M_\Omega) \xrightarrow{\simeq} \mathrm{CH}^i_{\text{ét}}(N_\Omega)$, so by Lemma 4.3.41, there exists a morphism $g : N_\Omega \to M_\Omega$ such that $f_\Omega \circ g = \mathrm{id}_{N_\Omega}$. Therefore, we have a sub-oject of M_Ω, denoted by T, and an isomorphism $f_\Omega : M_\Omega \to N_\Omega \oplus T$. Since $\mathrm{CH}^i_{\text{ét}}(M_\Omega) \xrightarrow{\simeq} \mathrm{CH}^i_{\text{ét}}(N_\Omega)$, then we obtain that $\mathrm{CH}^i_{\text{ét}}(T) \simeq 0$ for all $i \geq 0$, so invoking Lemma 4.3.40, we obtain that $T = 0$, so we obtain that $f_\Omega : M_\Omega \to N_\Omega$ is an isomorphism. □

Theorem 4.3.43 (Improved version of Manin's principle). *Let $f : M \to N$ be a morphism in the category $Chow_{ét}(k)$. Then f is a isomorphism of motives in $Chow_{ét}(k)$ if and only if for Ω an universal domain over k, the induced map $(f_\Omega)_* : CH^*_{ét}(M_\Omega) \to CH^*_{ét}(N_\Omega)$ given by the base change $f_\Omega : M_\Omega \to N_\Omega$, is bijective.*

Proof. Assume that $f : M \to N$ is a isomorphism and K/k a field extension of k, then it is clear that $f_K : M_K \to N_K$ is an isomorphism in $\mathrm{Chow}_{\text{ét}}(K)$. Now let us assume that $(f_\Omega)_* : \mathrm{CH}^*_{\text{ét}}(M_\Omega) \to \mathrm{CH}^*_{\text{ét}}(N_\Omega)$ is an isomorphism. By Lemma 4.3.42, then the map $f_\Omega : M_\Omega \to N_\Omega$ is bijective, then if we invoke [Ayo14b, Théorème 3.9] and the full-faithfulness of the functor $\mathrm{Chow}_{\text{ét}}(k)^{op} \hookrightarrow \mathrm{DM}_{\text{ét}}(k, \mathbb{Z})$, we obtain that the associated functor $i^* : \mathrm{Chow}_{\text{ét}}(k) \to \mathrm{Chow}_{\text{ét}}(\Omega)$ is conservative, since $i^*(f) = f_\Omega$, we conclude that f is an isomorphism in $\mathrm{Chow}_{\text{ét}}(k)$. ☐

Remark 4.3.44. We can obtain a fully characterization of the étale Chow groups a conic bundle $X \to S$ with S a smooth surface. Since $J^2(X) \simeq \mathrm{Alb}_S(S) \oplus \mathrm{Pic}^0(S) \oplus P_{\widetilde{C}}$ by [Bel85, Theorem 3.5] and using the results of [Bea77] and [Bel85], we obtain the following characterization

$$\begin{aligned}
&\mathrm{CH}^0_{\text{ét}}(X) \simeq \mathbb{Z},\\
&\mathrm{CH}^1_{\text{ét}}(X) \simeq \mathrm{CH}^1(X),\\
&\mathrm{CH}^2_{\text{ét}}(X) \simeq \mathrm{CH}^2_{\text{ét}}(S) \oplus \mathrm{CH}^1_{\text{ét}}(S) \oplus P_{\widetilde{C}},\\
&\mathrm{CH}^3_{\text{ét}}(X) \simeq \mathrm{CH}^3(X).
\end{aligned}$$

Since we have an isomorphism $\mathrm{CH}^*(X) \simeq \mathrm{CH}^*_{\text{ét}}(X)$, by [RS16, Theorem 1.1] we can conclude that the classical integral Hodge conjecture holds for smooth conic bundles $X \to S$.

Open question about decomposition

We address to a problem about the decomposition of motives which is induced by a filtration of dimension. We start with general facts about triangulated categories and the representation of functors in this contexts. After these facts, we recall the definition of n-motives and give a reason why starting from 0 and 1-motives we can think that this filtration can be related to the Chow-Künneth decomposition.

Before that, we ought to say a few words about the existence of a reasonable t-structure in the category $\mathrm{DM}^{\text{eff}}_{gm}(k, \mathbb{Z})$, which again is linked with the decomposition of a motive, pointing out differences with the étale case and why the argument of Voevodsky for the nonexistence of a reasonable t-structure cannot be used in the étale analogue of the category.

t-structure with integral coefficients and decomposition for integral motives

In [VSF00], Voevodsky provides a counterexample for the existence of a reasonable t-structure for the triangulated category of geometrical motives $\mathrm{DM}^{\text{eff}}_{gm}(k, \mathbb{Z})$ with integral coefficients. A t-structure $\tau = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathrm{DM}^{\mathrm{eff}}_{gm}(k, \mathbb{Z})$ is called reasonable if the following conditions hold:

1. τ is compatible with Tate twist, i.e. $M \in \mathcal{D}^{\leq 0}$ (similarly if $M \in \mathcal{D}^{\geq 0}$) if and only if $M(1) \in \mathcal{D}^{\leq 0}$ (respectively $M(1) \in \mathcal{D}^{\geq 0}$).

2. For a smooth affine scheme X of dimension n one has

$$H_i^\tau(M_{gm}(X)) = 0 \quad \text{for i} < 0 \text{ or i} > n$$
$$H_i^\tau(M^c_{gm}(X)) = 0 \quad \text{for i} < n \text{ or i} > 2n$$

Corollary 4.3.45 ([VSF00, Corollary 3.4.3]). *Let X be a smooth scheme over k. Then one has*

$$Hom_{DM^{eff}_{gm}(k,\mathbb{Z})}(M_{gm}(X), \mathbb{Z}(1)[j]) = H^{j-1}_{Zar}(X, \mathbb{G}_m)$$

Now we can state the counterexample provided by Voevodsky.

Proposition 4.3.46. *Let k be a field such that there exists a conic X over k with no $k-$rational points. Then $DM^{eff}_{gm}(k, \mathbb{Z})$ has no reasonable t-structure.*

Proof. Suppose that exists a reasonable t-structure $\tau = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, then for any smooth plane curve $X \subset \mathbb{P}^2_k$ we have

$$H_i^\tau(M_{gm}(X)) = \begin{cases} 0 & \text{for } i \neq 0,\ 1,\ 2 \\ \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}(1) & \text{for } i = 2 \end{cases} \tag{4.3}$$

and for a smooth hypersurface Y in $\mathbb{P}^3_k$ we have the exact triangles in $\mathrm{DM}^{eff}_{gm}(k)$ of the form

$$M^c_{gm}(Y) \to M^c_{gm}(\mathbb{P}^3_k) \to M^c_{gm}(\mathbb{P}^3_k - Y) \to M^c_{gm}(Y)[1]$$
$$M_{gm}(\mathbb{P}^3_k - Y) \to M_{gm}(\mathbb{P}^3_k) \to M_{gm}(Y)(1)[2] \to M_{gm}(\mathbb{P}^3_k - Y)[1]$$

then $H_1^\tau(M_{gm}(Y)) = 0$ because of the definition of the cohomology functor and the translation of $M_{gm}(Y)(1)$ in the second exact triangle. Let X be a conic over k with no rational points and consider $X \hookrightarrow X \times X$ the diagonal embedding in $\mathbb{P}^3_k$, because every curve C can be embedded in $\mathbb{P}^3_k$ (for example see [Har77, Corollary 3.6, sect. IV]). Then we can conclude that

$$H_i^\tau(M_{gm}(X)) = \begin{cases} 0 & \text{for } i \neq 0,\ 2 \\ \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}(1) & \text{for } i = 2 \end{cases} \tag{4.4}$$

since $M_{gm}(X)$ is clearly a direct summand of $M_{gm}(X \times X)$. We have a distinguished triangle

$$\mathbb{Z}(1)[2] \to M_{gm}(X) \to \mathbb{Z} \to \mathbb{Z}(1)[3].$$

Due to the previous corollary we have that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{gm}(k,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(1)[3]) &= \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{gm}(k,\mathbb{Z})}(M_{gm}(\mathrm{Spec}(k)),\mathbb{Z}(1)[3]) \\ &= H^2_{\mathrm{Zar}}(\mathrm{Spec}(k),\mathbb{G}_m) = 0\end{aligned}$$

therefore the last arrow is the zero map, giving us that the triangle splits $M_{gm}(X) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$. Then the map

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{gm}(k,\mathbb{Z})}(\mathbb{Z}, M_{gm}(X)) \to \mathbb{Z} \iff \mathrm{CH}_0(X) \xrightarrow{\deg} \mathbb{Z}$$

is surjective which contradicts the hypothesis of X about the nonexistence of a rational point. □

Remark 4.3.47.

1. This counterexample does not work for the contradiction of the existence of a reasonable t-structure for the étale case because of the following argument: by definition we have

   $$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{gm,\text{ét}}(k,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(1)[3]) &= H^3_{\text{ét}}(\mathrm{Spec}(k),\mathbb{Z}(1)) \\ &= Br(k)\end{aligned}$$

   which is the Brauer group of k, and therefore related with the existence of a k-rational point. This later group may be non zero and thus we obtain a non splitting exact triangle. In this direction we can think that in the étale setting we can obtain a decomposition of the motive which does not split. To continue developing this idea we need to introduce the notion of $n-$motives and focus in the cases when $n = 0$ or $n = 1$.

2. The dependence of the existence of a $k-$rational point is crucial in this example and in general to define the projectors $\pi_0(X)$ and $\pi_{2d}(X)$ where X is smooth projective variety of dimension d. With rational coefficients this can be bypassed if we define the projectors $p_0(X)$ and $p_{2d}(X)$ with a 0-cycle of degree n (and inverting the degree). Another way of solving this problem would be enlarging our base field k to K where it is possible to find a $K-$rational point.

Triangulated categories

We have to say a few words about triangulated categories that are compactly generated, for that we mainly focus in [Nee01] and [Ayo06]. Let us recall the definition of a compact object:

Definition 4.3.48. *Let $\mathcal{T}$ be a triangulated category with small sums. An object $A \in \mathcal{T}$ is compact if and only if the functor $Hom_{\mathcal{T}}(A,-)$ commutes with small sums, i.e. for every small family of objects $(B_i)_{i\in I}$ in $\mathcal{T}$ the canonical homomorphism*

$$\bigoplus_{i\in I} Hom_{\mathcal{T}}(A,B_i) \to Hom_{\mathcal{T}}(A,\oplus_{i\in I} B_i)$$

is invertible.

Definition 4.3.49. *Let $\mathcal{T}$ be a triangulated category with small sums. We say that $\mathcal{T}$ is compactly generated if there is a set of compact objects $\Lambda \subset \mathcal{T}$ such that $\mathcal{T}$ is generated by Λ.*

Proposition 4.3.50. *[Nee01, Theorem 8.3.3], [Ayo06, Proposition 2.1.21] Let $\mathcal{T}$ be a compactly generated triangulated category with small sums. Let $h : \mathcal{T} \to \mathbf{Ab}$ an exact contravariant functor that transforms small sums in small products. Then h is representable.*

Remark 4.3.51. A consideration on the notation and used names should be done. In the context of triangulated categories, the term exact functor is another name for triangulated functor, i.e. an additive functor that commutes with translations and preserves distinguished triangles.

The previous result is a criterion of representability of Brown which implies that for a triangulated functor of a compactly generated category there exists a right adjoint, under some technical properties:

Proposition 4.3.52. *[Ayo06, Corollaire 2.1.22] Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories with small sums. Suppose that $\mathcal{T}$ is compactly generated. Let $F : \mathcal{T} \to \mathcal{T}'$ a covariant triangulated functor which commutes with small sums. Then F admits a right adjoint.*

Proof. Let B be an object in the category $\mathcal{T}'$. Define h_B as follows

$$\begin{aligned} h_B : \mathcal{T} &\to \mathbf{Ab} \\ A &\mapsto h_B(A) := \mathrm{Hom}_{\mathcal{T}'}(f(A), B). \end{aligned}$$

By the hypothesis of f the functor h_B transforms small sums in small products. This functor is represented by an object $g(B)$ in $\mathcal{T}$, then we have an isomorphism

$$h_B(A) = \mathrm{Hom}_{\mathcal{T}'}(f(A), B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(A, g(B)).$$

Consider the association of elements $B \to g(B)$. There is a way of understanding this association such that the isomorphism is natural in A and B. □

The fact that the previously defined right adjoint functor is a triangulated functor comes immediately as a consequence of [Ayo06, Lemme 2.1.23]. An immediate conclusion from the previous theorems of existence of adjoints is the existence of universal object with respect to a subcategory.

Lemma 4.3.53. *Let $\mathcal{A}$ be a compactly generated triangulated category stable under small sums, and $\mathcal{B}$ be a full subcategory and let $i : \mathcal{B} \to \mathcal{A}$ be the full embedding. Then for every compact object $A \in \mathcal{A}$ there exist an object $M_A \in B$ and a map $i_A : M_A \to A$ such that for every object $B \in \mathcal{B}$ that admits a morphism $f : B \to A$, the f factors through M_A, such that $f : B \to M_A \xrightarrow{i_A} A$.*

Proof. By the previous lemmas the full embedding admits a right adjoint, denoted by ν. If we consider the co-unit transformation $\delta : i \circ \nu \Longrightarrow \mathrm{id}$ then we define $M_A := i(\nu(A))$ giving us the existence of the arrow $M_A \to A$. The universality of M_A comes from the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(B, A) &\simeq \mathrm{Hom}_{\mathcal{B}}(B, \nu(A))\\ &\xrightarrow{i} \mathrm{Hom}_{\mathcal{A}}(B, B_A)\end{aligned}$$

given by the adjunction and the full embedding. □

Remark 4.3.54. Notice that this properties give us that for a compact object in $B \in \mathcal{B}$ there exists an isomorphism $M_B \xrightarrow{\sim} B$. We say that M_B is the stabilizer of B.

Filtration induced by n-motives

We recall the definition of n-motives, where we use mainly the references of [VSF00], [Org04] and [BK16].

For a perfect field k, in [Voe00, Section 3.4], Voevodsky defined and constructed a filtration of the category of (effective) geometrical motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{Nis,gm}}(k, \mathbb{Z})$ induced by the dimension of the generating classes of geometric motives (i.e. the compact elements in the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{Nis},-}(k, \mathbb{Z})$), where he studied the cases of the filtration of dimension at most 1.

Concerning the category of $0-$motives, the so called category Artin motives and denoted in [Voe00] as $d_{\leq 0}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Z})$, it was proved in [Org04, Proposition 2.2 et 2.7] that it is equivalent to pseudo-abelian envelope of $\mathcal{H}^b(\mathrm{Perm}(k))$, which is the full subcategory of $\mathbb{Z}[G_k]-$modules that are permutational representations with $G_k = \mathrm{Gal}(\bar{k}/k)$. The case for $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{Nis},-}(k, \mathbb{Z})$ has also an equivalence category to $D^-(\mathrm{Shv}(\mathrm{Perm}(k)))$ where $\mathrm{Shv}(\mathrm{Perm}(k))$ is the category of additive contravariant functors from $\mathrm{Perm}(k)$ to the **Ab**, all of this fitting in the following commutative diagram

$$\begin{array}{ccc}\mathcal{H}^b(\mathrm{Perm}(k)) & \longrightarrow & D^-(\mathrm{Shv}(\mathrm{Perm}(k)))\\ \downarrow & & \downarrow\\ d_{\leq 0}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Z}) & \longrightarrow & d_{\leq 0}\mathrm{DM}^{\mathrm{eff}}_{-}(k, \mathbb{Z}).\end{array}$$

In a more general context it is possible to give a description of the category of 1-motives. In [BK16, Theorem 2.1.2] it is shown that there exists a functor

$$T : D^b(\mathcal{M}_1[1/p]) \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k, \mathbb{Z})$$

whose essential image is $d_{\leq 1}\mathrm{DM}^{\mathrm{eff}}_{\text{ét,gm}}(k, \mathbb{Z})$. Here, $D^b(\mathcal{M}_1[1/p])$ represents the derived category of bounded Deligne's 1-motives and p is the characteristic exponent of the field. We can generalize the construction of 0 and 1-motives by taking the category of cohomological motives, which is generated by motives of cohomological type defined as follows:

Definition 4.3.55. *Let S be a noetherian scheme and X a proper $S-$scheme. We define the cohomological motive of X as follows*

$$h_S(X) := (\pi_X)_*(\pi_X)^*\mathbf{1}_S \in \boldsymbol{DA}_{\acute{e}t}(S,\mathbb{Z}),$$

where $\pi_X : X \to S$.

Note that by the properties of the structural morphism, this is equivalent to [AZ12, Definition 1.3], where the cohomological motive of X over S is defined to be $h_S(X) = (\pi_X)_*\mathbf{1}_X$. This is a consequence of fact that if $f : X \to Y$ then f^* takes the homological motive of Y and sends it to the homological motive of the $X-$scheme $X \times_S Y$.

Definition 4.3.56. *Let S be a noetherian scheme. We denote by $\boldsymbol{DA}^{coh}_{\acute{e}t}(S,\Lambda)$ the category of constructible cohomological motives, which is the smallest triangulated subcategory of $\boldsymbol{DA}_{\acute{e}t}(S,\Lambda)$ stable under finite sums and containing $h_S(X)$ for all quasi-projective $S-$scheme X. Respectively, the category of Artin motives over S with coefficients in Λ, which is denoted $\boldsymbol{DA}^{0}_{\acute{e}t}(S,\Lambda)$, as the smallest subcategory containing $h_S(X)$ where X is a zero dimensional $S-$scheme.*

Remark 4.3.57. We use the same definition for cohomological motives in the model of étale motives with transfers: we denote by $\mathrm{DM}^{\mathrm{coh}}_{\mathrm{\acute{e}t}}(S,\Lambda)$ the category of cohomological motives, which is the smallest triangulated subcategory of $\mathrm{DM}_{\mathrm{\acute{e}t}}(S,\Lambda)$ stable under finite sums and containing $h_S(X)$ for all quasi-projective $S-$schemes X.

Definition 4.3.58. *If we take a noetherian scheme S, we define the category of n-motives relative to S with coefficients in Λ, denoted by $DM^{n}_{\acute{e}t}(S,\Lambda)$ as the smallest subcategory of $DM^{coh}_{\acute{e}t}(S,\Lambda)$ containing the elements $h_S(X)$ where X is a m-dimensional S-scheme with $m \leq n$. We define in the same way the analogue for the category of étale motives without transfers $\boldsymbol{DA}^{n}_{\acute{e}t}(S,\Lambda)$.*

The full subcategory $\mathrm{DM}^{n}_{\mathrm{\acute{e}t}}(S,\Lambda)$ is called the category of $n-$motivic étale sheaves over S with coefficients in Λ. Along with these definitions, we use the following notations:

- i^m_n will stand for the full embedding $\mathrm{DM}^{n}_{\mathrm{\acute{e}t}}(S,\Lambda) \hookrightarrow \mathrm{DM}^{m}_{\mathrm{\acute{e}t}}(S,\Lambda)$ where $m > n$,
- $i_n =$ "i^∞_n" is the full embedding $\mathrm{DM}^{n}_{\mathrm{\acute{e}t}}(S,\Lambda) \hookrightarrow \mathrm{DM}^{\mathrm{coh}}_{\mathrm{\acute{e}t}}(S,\Lambda)$.
- The functors ν^m_n and ν_n are the right adjoints of i^m_n and i_n respectively

$$i^m_n : \mathrm{DM}^{n}_{\mathrm{\acute{e}t}}(S,\Lambda) \leftrightarrows \mathrm{DM}^{m}_{\mathrm{\acute{e}t}}(S,\Lambda) : \nu^m_n, \quad i_n : \mathrm{DM}^{n}_{\mathrm{\acute{e}t}}(S,\Lambda) \leftrightarrows \mathrm{DM}^{\mathrm{coh}}_{\mathrm{\acute{e}t}}(S,\Lambda) : \nu_n.$$

- Let $\omega^m_n := i^m_n \circ \nu^m_n$ and $\omega^n := i_n \circ \nu_n$ be the functors associated with the co-unit transformations.

With this construction, we have a sequence of full embeddings of categories indexed by the bound of the dimension for the generators n:

$$\mathrm{DM}^{0}_{\mathrm{\acute{e}t}}(S,\Lambda) \hookrightarrow \mathrm{DM}^{1}_{\mathrm{\acute{e}t}}(S,\Lambda) \hookrightarrow \ldots \hookrightarrow \mathrm{DM}^{n}_{\mathrm{\acute{e}t}}(S,\Lambda) \hookrightarrow \ldots \hookrightarrow \mathrm{DM}^{\mathrm{coh}}_{\mathrm{\acute{e}t}}(S,\Lambda).$$

Consider the graded pieces of the category $\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)$ as

$$\mathrm{gr}_n\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda) := \mathrm{DM}^{n}_{\text{ét}}(S,\Lambda)/\mathrm{DM}^{n-1}_{\text{ét}}(S,\Lambda)$$

where for $M \in \mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)$ the n-th graded pieces are defined as

$$\mathrm{gr}_p(M) = \mathrm{coker}(\omega^{p-1}(M) \to \omega^p(M)).$$

Notice that the construction of the graded pieces by definition is functorial. Now let us prove some basic results about $n-$motives and the underlying filtration of $\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)$ that we defined previously.

Lemma 4.3.59. *Let S be a noetherian scheme and Λ a commutative ring. Assume that M is a cohomological motive in $DM^{coh}_{\acute{e}t}(S,\Lambda)$. For all $\delta^n : \omega^n \to id$ with $n \geq 0$ and $\delta^l_m : \omega^l_m \to id$, with $l > m$, the co-augmentation associated with the co-unit:*

(i) If $M \in DM^n_{\acute{e}t}(S,\Lambda)$, we have an isomorphism $\delta^n : \omega^n(M) \xrightarrow{\sim} M$. In particular $\delta^n(\omega^n) : \omega^n \circ \omega^n \xrightarrow{\sim} \omega^n$. Moreover $\delta^n \circ \omega^n = \omega^n \circ \delta^n$.

(ii) For $k \in \mathbb{N}$ there exists a natural transformation δ^{n+k}_n between the functors $\delta^{n+k}_n : \omega^n \to \omega^{n+k}$.

(iii) For $k \in \mathbb{N}$ the natural transformations δ^n have an induced compatibility expressed in term of $\delta^n = \delta^{n+k} \circ \delta^{n+k}_n$, also there exists an isomorphism $\omega^{n-1}(\omega^n(M)) \simeq \omega^n(\omega^{n-1}(M)) \simeq \omega^{n-1}(M)$.

Proof. In (i) the first statement follows a general proposition of triangulated categories given in Lemma 4.3.53, meanwhile the second is a consequence of the universality of the elements $\omega^n(M)$. The last statement of (i) is a consequence of the commutative diagram

$$\begin{tikzcd}
\omega^n(\omega^n(M)) \arrow[r, "\delta^n(\omega^n(M))"] \arrow[d, "\omega^n(\delta^n(M))"] & \omega^n(M) \arrow[d, "\delta^n(M)"] \\
\omega^n(M) \arrow[r, "\delta^n(M)"] & M
\end{tikzcd}$$

because $\omega^n(\delta^n(M))$ and $\delta^n(\omega^n(M))$ are isomorphisms due to the stabilization property. Therefore we obtain that $\omega^n(\delta^n(M)) = \delta^n(\omega^n(M))$.

In (ii) and for $k = 1$ the existence of the map $\omega^n(M) \xrightarrow{\delta^{n+1}_n} \omega^{n+1}(M)$ comes from the universality of $\omega^{n+1}(M)$ and the fact that $\omega^n(M)$ is a $(n+1)-$motive, the remaining cases arises from an induction on k.

The compatibility of the natural transformations is a consequence of the universality of the elements $\omega^n(M)$ in the category of $n-$motives. Concerning the isomorphisms of (iii), the second one is obtained as a consequence of (i), meanwhile for the first one consider the following commutative diagrams

$$\begin{tikzcd}
\omega^n(M) \arrow[r, "\delta^n"] \arrow[d, "\delta^{n+1}_n"] & M \arrow[d, equal] \\
\omega^{n+1}(M) \arrow[r, "\delta^{n+1}"] & M
\end{tikzcd}$$

and apply to it the functor ω^n along with result (i) we conclude.

$$\begin{tikzcd} \omega^n(M) \arrow[r, "\omega^n\circ\delta^n"] \arrow[d, "\omega^n\circ\delta_n^{n+1}"] & \omega^n(M) \arrow[d, equal] \\ \omega^n(\omega^{n+1}(M)) \arrow[r, "\omega^n\circ\delta^{n+1}"] & \omega^n(M) \end{tikzcd}$$

□

Proposition 4.3.60. *For any constructible cohomological motive $M \in DM^{coh}_{ét}(S,\Lambda)$ there exists $n \in \mathbb{N}$ such that $\omega^n(M) = M$.*

Proof. Immediate after the previous observations and the construction of the category $\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)$. □

Definition 4.3.61. *The sequence of transformations*

$$\omega^0 \to \omega^1 \to \ldots \to \omega^n \to \ldots \to id$$

is called the filtration by dimension.

Let M be a cohomological motive in $\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)$, we define the object $\omega^{>n}(M) = \omega^{\geq n+1}(M)$ which fits into the distinguished triangle

$$\omega^n(M) \to M \to \omega^{>n}(M) \to \omega^n(M)[1].$$

The first thing we must do is to show that this element is defined uniquely up to isomorphism, for that we use the axioms of a triangulated category and the universality of the elements due to the filtration.

Lemma 4.3.62. *Let $M \in DM^{coh}_{ét}(S,\Lambda)$ then the object $\omega^{>n}(M)$ is defined uniquely up to isomorphism and $\omega^{>n}$ defines a functor.*

Proof. For all $M_1,\ M_2 \in \mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)$ and $k \in \mathbb{N}$ then by the universality of ω^{n+k} we have

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)}(\omega^n(M_1),\omega^{n+k}(M_2)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)}(\omega^n(M_1),M_2)$$

which is equivalent to

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)}(\omega^n(M_1),\omega^{>n+k}(M_2)) = 0$$

proving (i). Notice that there are natural transformations $\mathrm{id} \to \omega^{>n}$ and $\omega^{>n} \to \omega^{>n+1}$. □

Let M be cohomological motive in $\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)$ and $(n,p,q) \in \mathbb{N}\times\mathbb{Z}^2$ where n is fixed but arbitrary. Define the following terms

$$D^{p,q} := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)}(\omega^p(M),\mathbf{1}_S(n)[p+q])$$
$$E^{p,q} := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)}(\mathrm{gr}_p(M),\mathbf{1}_S(n)[p+q])$$

which induce an exact couple, where the transition maps arises naturally from the exact triangle

$$\omega^{n-1}(M) \to \omega^n(M) \to \mathrm{gr}_n(M) \to \omega^{n-1}(M)[1]$$

and the contravariant functor $\mathrm{Hom}(-, \mathbf{1}_S(n)[p+q])$ the first map $D^{p,q} \to D^{p-1,q+1}$ of grading $(-1,1)$ comes from the map $\omega^p(M) \to \omega^{p+1}(M)$, the map $D^{p-1,q+1} \to E^{p,q+1}$ of degree $(1,0)$ comes from map $\mathrm{gr}_p(M) \to \omega^{p-1}(M)[1]$.

Recalling [Dég12b, Definition 1.2], in a triangulated category $\mathcal{T}$, a tower $X_\bullet$ over X is the data of a sequence $(X_p \to X)_{p\in\mathbb{Z}}$ of objects over X and a sequence of morphism over X

$$\ldots \to X_{p-1} \to X_p \to \ldots$$

Now let M be an object in $\mathrm{DM}^n_{\text{ét}}(S,\Lambda))$ we know that:

1. for n big enough ω^n stabilizes for M and by convention we can put $\omega^n(M) = 0$ for $n < 0$.

2. There is a family of objects and morphism $(\omega^n(M) \to M)_{n\in\mathbb{Z}}$ such that

$$0 \to \omega^0(M) \to \omega^1(M) \to \ldots \to M.$$

So in other words the family of functors $(\omega^n)_{n\in\mathbb{Z}}$ defines a bounded and exhaustive tower for each M, in the sense of [Dég12b, Definition 1.2], thus for a fixed $n \in N$, we get a convergent spectral sequence

$$E_1^{p,q} = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)}(\mathrm{gr}_p(M), \mathbf{1}_S(n)[p+q]) \Longrightarrow \mathrm{Hom}_{\mathrm{DM}^{\mathrm{coh}}_{\text{ét}}(S,\Lambda)}(M, \mathbf{1}_S(n)[p+q]) \quad (4.5)$$

Remark 4.3.63. Notice that if $\Lambda = \mathbb{Q}$, $S = \mathrm{Spec}(k)$ and $M = M(X)$ where X is a smooth k-scheme of relative dimension d then $D^{0,q} = \mathrm{CH}^q_{\text{ét}}(\pi_0(X))_{\mathbb{Q}}$, where $X \to \pi_0(X/k) \to k$ is the Stein factorization, and $D^{d,q} = \mathrm{CH}^{2d+q}_{\text{ét}}(X)_{\mathbb{Q}}$.

Proposition 4.3.64. *Let S be a noetherian schemes and let $f : S \to T$ be a morphism of schemes, and Λ a commutative ring, then*

1. $- \otimes - : DM^n_{\acute{e}t}(S,\Lambda) \otimes DM^n_{\acute{e}t}(S,\Lambda) \to DM^{n+m}_{\acute{e}t}(S,\Lambda)$

2. $f^* : DM^n_{\acute{e}t}(T,\Lambda) \to DM^n_{\acute{e}t}(S,\Lambda)$,

3. *if f is separated of finite type of relative dimension m, then we have $f_! : DM^n_{\acute{e}t}(S,\Lambda) \to DM^{n+m}_{\acute{e}t}(T,\Lambda)$,*

4. *Assume that $X \to S$ is proeper, then $D_S(\pi_X)_*\mathbf{1}_X$ is a cohomological object.*

Proof. For the first assertion consider $\pi_X : X \to S$ and $\pi_Y : Y \to S$ two proper morphism and the product of cohomological motives $(\pi_X)_*\mathbf{1}_X \otimes (\pi_Y)_*\mathbf{1}_Y$. Using the six functors formalism on $\mathrm{DM}_{\text{ét}}(S,\Lambda)$, we have that

$$\begin{aligned}(\pi_X)_*\mathbf{1}_X \otimes (\pi_Y)_*\mathbf{1}_Y &\simeq (\pi_X)_!\mathbf{1}_X \otimes (\pi_Y)_!\mathbf{1}_Y\\ &\simeq (\pi_X)_!\left(\mathbf{1}_X \otimes (\pi_X)^*(\pi_Y)_!\mathbf{1}_Y\right)\\ &= (\pi_X)_!(\pi_X)^*(\pi_Y)_!\mathbf{1}_Y\\ &\simeq (\pi_X)_!h_!g^*\mathbf{1}_Y\\ &\simeq (\pi_X)_*h_*\mathbf{1}_{X\times_S Y} \simeq (\pi_X\circ h)_*\mathbf{1}_{X\times_S Y}\end{aligned}$$

where h and g are morphisms associated to the cartesian square

$$\begin{tikzcd} X\times_S Y \arrow[r, "g"] \arrow[d, "h"'] & Y \arrow[d, "\pi_Y"] \\ X \arrow[r, "\pi_X"'] & S. \end{tikzcd}$$

Therefore $(\pi_X)_*\mathbf{1}_X \otimes (\pi_Y)_*\mathbf{1}_Y$ is a cohomological motive and $\pi_X\circ h : X\times_S Y \to S$ has relative dimension at most $m+n$, concluding the proof of 1.

2. It follows from the base change formula because then its action is characterized by $f^*(\pi_X)_*\mathbf{1}_X \simeq h(Z\times X)$.

3. For the last one we consider the morphism $f : S \to T$ and a proper morphism $\pi_X : X \to S$, consider the cohomological motive $(\pi_X)_*\mathbf{1}_X$, then we consider $f_!(\pi_X)_*\mathbf{1}_X$, since $f_! \simeq f_*$ then we have that $f_!(\pi_X)_*\mathbf{1}_X \simeq f_*(\pi_X)_*\mathbf{1}_X \simeq (f\circ\pi_X)_*\mathbf{1}_X$ which is a T-morphism of relative dimension $n+m$. □

4. Let $M=(\pi_X)_*\mathbf{1}_X$ be a cohomological motive, then $D_S(M)=f_*(D_X\mathbf{1}_X)\simeq f_*\mathbf{1}_X$. Moreover, if M is a n-motive, then the dual is also a n-motive.

Example 4.3.65. *Consider a smooth curve C over a field k and let $h(C)$ be its cohomological motive i.e. $h(C)=M(C)^\vee = M(C)(-1)[-2]$. Using the fact that $gr_p(M) = coker(\omega^{p-1}(M)\to\omega^p(M))$ and $\omega^1(h(C))\cong h(C)$ we obtain that the graded pieces are characterized as*

$$\begin{aligned}&gr_0(h(C)) = \omega^0(h(C)) \cong h(\pi_0(C))\\ &gr_1(h(C)) = coker(\omega^0(h(C))\to\omega^1(h(C))) \cong coker(h(\pi_0(C))\to h(C))\end{aligned}$$

which fit in the exact triangle

$$h(\pi_0(C)) \to h(C) \to coker(h(\pi_0(C))\to h(C)) \to h(\pi_0(C))[1].$$

If C is connected and has a $k-$rational point then $h(\pi_0(C))\cong\mathbb{Z}$ and the motive of the curve is $h(C)\simeq\mathbb{Z}\oplus Jac(C)\oplus\mathbb{Z}(1)[2]$. Hence $gr_1(h(C)) = coker\left(\omega^0(h(C))\to\omega^1(h(C))\right)$ is isomorphic to $Jac(C)\oplus\mathbb{Z}(1)[2]$.

We conclude with a list of open questions:

1. If we set $\Lambda=\mathbb{Q}$, can we relate n-th graded piece of the motive $h(X)$ to the $h^n(X)$ part of the Chow-Künneth decomposition? Is $\mathrm{gr}_2(h(X))$ linked to $h^2(X)$, or does it give a good candidate for this motive and by duality, for $h^{2d-2}(X)$?

2. Is it possible that this filtration leads to a motivic equivalent of the conjectural descending filtration of Chow groups called the Bloch-Beilinson filtration? See [MNP13, 7.1 and 7.2]

3. In the spirit of [RS16], are there other conjectures about motives and algebraic cycles that can be reformulated in terms of étale motives?

4. Does the spectral sequence given in 4.5 induce a filtration which corresponds to a niveau filtration? For example, the niveau filtration of Bloch and Ogus presented in [BO74] ?

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