# FA Reading Group Section 3.1 and 3.2

## Functional Derivatives

## 07/08/2022

# Section contents

- Definition of the "derivative" of a map $f : X \to Y$ between generic normed spaces
- First order conditions for optimization problems (in possibly infinite dimensions)

Applications: Euler-Lagrange equations/Quantum mechanics/Optimal control, ...

# Functional derivative

Let $f : X \to Y$ be a map between two normed spaces $X$ and $Y$. $f$ is said to be *differentiable* at $x_0 \in X$ if there exists $L \in CL(X, Y)$, such that for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $\|x - x_0\|_X < \delta$, then

$$\frac{\|f(x) - f(x_0) - L(x - x_0)\|_Y}{\|x - x_0\|_X} < \varepsilon$$

Note: This is known as the *Fréchet derivative*. There are weaker notions of functional derivatives, such as the *Gateaux derivative*.

# Type signature

The derivative of a function at a point is a linear operator from $X$ to $Y$, i.e. the derivative has the "type signature"

$$f' : X \to CL(X, Y)$$

which contrasts with the good old derivative of a scalar function

$$f' : \mathbb{R} \to \mathbb{R}$$

# Type signature

| Primal | Derivative |
|---|---|
| $f : \mathbb{R} \to \mathbb{R}$ | $f' : \mathbb{R} \to \mathbb{R}$ |
| $f : \mathbb{R}^n \to \mathbb{R}$ | $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ |
| $f : \mathbb{R}^n \to \mathbb{R}^m$ | $\mathcal{J} f : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ |
| $f : X \to Y$ | $Df : X \to CL(X, Y)$ |
| $f : X \to \mathbb{R}$ | $Df : X \to X'$ |

Turns out we were just confused about the type signature of the classical derivative!

# Key results

- For scalar functions the (Fréchet) derivative is the same as the classical one
- Uniqueness of the derivative
- $f$ differentiable $\Rightarrow$ $f$ continuous (Ex. 3.2)
- Chain rule still works! (Ex. 3.4)

# Optimality conditions

From Undergrad Real analysis, remember that, for $f : \mathbb{R} \to \mathbb{R}$ (differentiable)

1. $x_* \in \mathbb{R}$ is a minimizer of $f \Rightarrow f'(x_*) = 0$
2. $f'(x_*) = 0$ and $f''(x) \geq 0, \forall x \Rightarrow x_*$ is a minimum of $f$

## When you don't have $f''$

For a real valued function $f : X \to \mathbb{R}$

1. $x_*$ is a minimizer of $f \Rightarrow f'(x_*) = 0$
2. $f$ convex and $f'(x_*) = 0 \Rightarrow x_*$ is a minimizer of $f$

# Extra: Gateaux derivative

$f : X \to Y$ is *Gateaux differentiable* at $x_0 \in X$ if there exists a map $g : X \to Y$ s.t. $\forall h \in X$

$$\lim_{\tau \to 0} \frac{f(x_0 + \tau h) - f(x_0)}{\tau} = g(h)$$

Generalization of the directional derivative!
Gateaux derivative: $df : X \times X \to Y$ (not necessarily linear, or continuous)

# Example: Optimal Control

Minimize

$$\mathcal{J}[x, u, t_0, t_f] = \mathcal{E}(x(t_0), t_0, x(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{F}(x(t), u(t), t) dt$$

with constraints

- $\dot{x}(t) = f(x(t), u(t), t)$
- $h(x(t), u(t), t) \leq 0$
- $e(x(t_0), t_0, x(t_f), t_f) = 0$

# Example: Shortest (differentiable) Path

Let $x, y \in \mathbb{R}^n$. Consider the curve length functional $\ell : C_0^1([0,1], \mathbb{R}^n) \to \mathbb{R}$

$$\ell(\gamma) = \int_0^1 \|\lambda'(t) + \gamma'(t)\| dt$$

where $\lambda(t) = (1-t)x + ty$

Problem: find the minimizer(s) of $\ell$ (i.e. the shortest curve from $x$ to $y$)