Functional Analysis Reading Group

Orthogonality (section 4.2)

02/10/2022

Today's menu

Now that we're living in inner-product spaces, we can define what it means for two vectors to be *orthogonal*, which entails

- Orthogonal/Orthonormal bases
- *Orthogonalizing* a basis

Orthonormal bases are very useful, both mathematically and numerically, so having them on function spaces is good©.

NB. Most of the results in this section hold for *inner product spaces*.

Orthogonal vectors / Orthonormal set

Let $(X, \langle\cdot,\cdot\rangle)$$ be an inner product space

1. Vectors $x, y \in X$ are *orthogonal* if $\langle x, y\rangle = 0$
2. A set $S \subset X$ is *orthonormal* if
   - $\forall x, y \in S$, s.t. $x \neq y$, $\langle x, y\rangle = 0$
   - $\forall x \in S$, $\|x\|^2 = \langle x, x\rangle = 1$

Examples of orthnormal sets (1)

- $\mathbb{R}^d$: $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$ (one-hot vectors)
- ℓ^2: $\{e_i \mid i \in \mathbb{N}\}$ where e_i is the i-th "one-hot sequence"

Examples of orthonormal sets (2)

In $C[0,1]$, the Fourier basis $\{F_n | n \in \mathbb{Z}\}$, defined as

$$F_n(t) = e^{2\pi int}$$

is an orthonormal set.

Indeed, it is an *orthonormal basis* of $C[0,1]$, and taking the Fourier series decomposition of a function f is equivalent to expressing f in that basis!

Orthonormal sets and basis expansions

Given an orthonormal set $\{u_k | k \in K\}$, and a vector x in its span, we can write

$$x = \sum_{k \in K} \alpha_k u_k$$

where

$$\alpha_k = \langle x, u_k \rangle$$

Moreover, any orthonormal set is always linearly independent.

Gram-Schmidt orthonormalisation

Given a linearly independent subset $\{x_1, x_2 \ldots\}$, we can construct an orthonormal set $\{u_1, u_2, \ldots\}$ by the following procedure:

1. $u_1 = \frac{x_1}{\|x_1\|}$
2. $u_n = \frac{x_n - \sum_{k=1}^{n-1} \langle x_n, u_k \rangle u_k}{\|x_n - \sum_{k=1}^{n-1} \langle x_n, u_k \rangle u_k\|}$ for $n \geq 2$

The constructed set $\{u_1, u_2, \ldots\}$ has the same span as the original set, i.e. for all $n \in \mathbb{N}$

$$\operatorname{span}\{u_1, \ldots, u_n\} = \operatorname{span}\{x_1, \ldots, x_n\}$$

Orthonormal sets of polynomials

The set of monomials $\{1, t, t^2, t^3, \ldots\}$ is a linearly independent set in $C[-1,1]$, but it is not orthonormal!

Applying Gram-Schmidt orthogonalization to it yields the *Legendre polynomials*, which satisfy

$$P_n(t) = \frac{1}{n!2^n}\left(\frac{d}{dt}\right)^n (t^2-1)^n$$

and are orthogonal, with norm $\|P_n\|^2 = \frac{2}{2n+1}$

Other orthonormal bases of polynomials

- Hermite polynomials (Ex 4.13) (orthogonal set of polynomials over $L^2(\mathbb{R})$)

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}$$

- Chebyshev polynomials ($x \in [-1, 1]$)

$$T_n(x) = \cos(n \arccos(x))$$

Note: These families of functions (Fourier, Legendre, Hermite, ...) tend to show up as fundamental solutions of classical ODEs from physics

Orthogonal complement

Given a subspace Y of an inner product space X, the *orthogonal complement* $Y^\perp$ of Y is defined as

$$Y^\perp = \{x \in X \mid \langle x, y \rangle = 0, \forall y \in Y\}$$

This generalizes the notion of "hyperplane"

Properties of the orthogonal complement

- $Y^\perp$ is a *closed subspace* of X
- $Y \cap Y^\perp = \{0\}$ (Ex. 4.16)
- $Y \subset Y^{\perp\perp} = (Y^\perp)^\perp$ (4.17)
- $Y \subset Z \ \Rightarrow\ Z^\perp \subset Y^\perp$
- $Y^\perp = (\bar{Y})^\perp$
- Y dense in $X \Rightarrow\ Y^\perp = \{0\}$

Concluding remarks

Stepping into inner product spaces is already yielding us benefits:

- Basis decomposition into an orthogonal basis is *tractable* (mathematically **and** numerically)
- We get "hyperplanes" of vectors in the *same space* as our vectors
- Connection to famous ODEs from physics

PSA

- From now on, the meetings will be scheduled by default on Sundays at the "usual time" (20:00 GMT)
- Next meeting on the 23/10/2022 (unless someone is willing to do the recap in my stead)
- Feedback wanted! How are you finding these meetings?