Functional Analysis Reading Group

The Euler-Lagrange Equations

28/08/2022

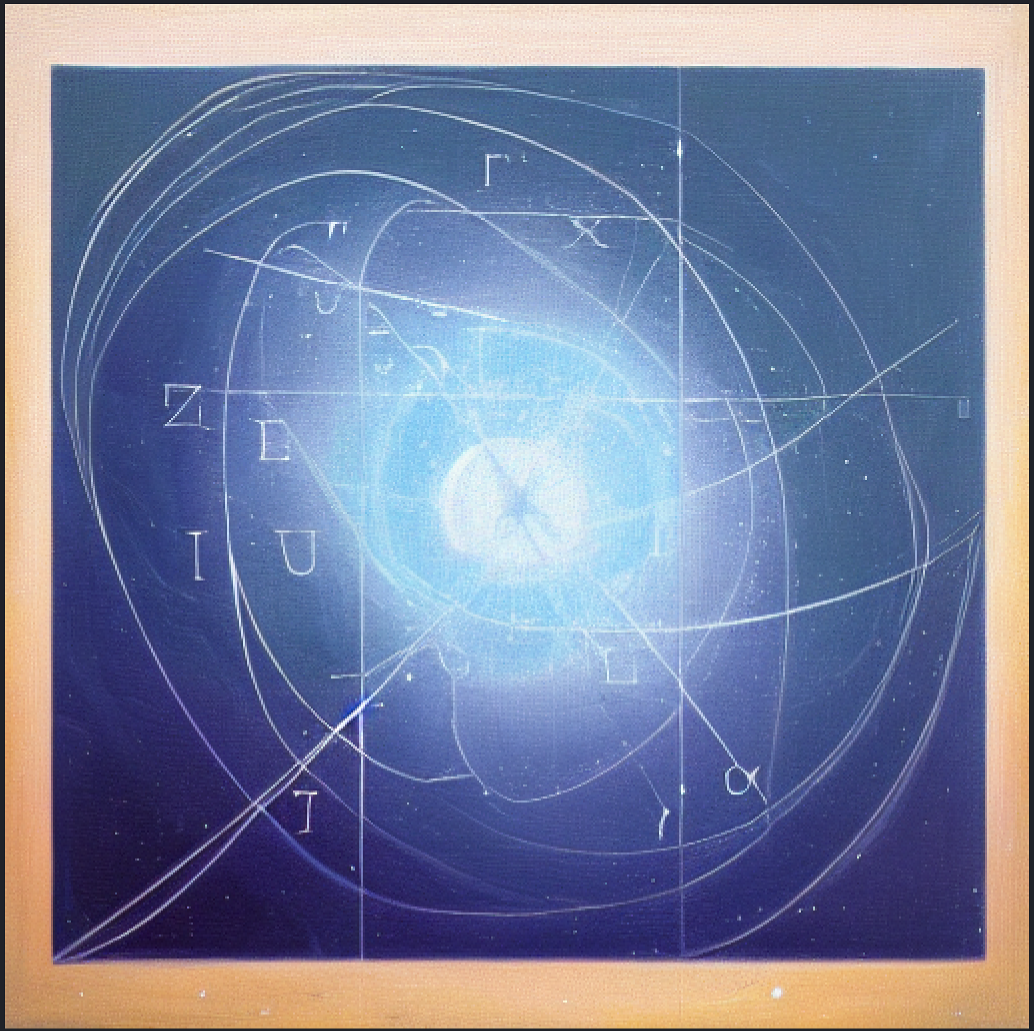
Last meeting

- Generalized derivative of maps $f : X \to Y$
- Optimality conditions (stationary point + convexity)

Finding stationary points is generally not easy, though

Today

- A way to turn a broad class of functional minimization problems into differential equations
- Secret sauce behind modern physics



"The Euler-Lagrange Equations, oil painting, trending on artstation HQ" generated with Stable Difusion

Euler-Lagrange Equations

Given a functional of the form

$$f(x) = \int_a^b F(x(t), x'(t), t)dt$$

Stationary points x_* of f satisfy the *Euler-Lagrange equation*

$$\frac{\partial F}{\partial x}(x_*(t), x_*'(t), t) - \frac{d}{dt}\frac{\partial F}{\partial x'}(x_*(t), x_*'(t), t) = 0$$

Link with the functional derivative

Given a candidate stationary point x_*, consider the Banach space of perturbations

$$X = \{h \in C^1[a,b] : h(a) = h(b) = 0\}$$

and the modified functional $\tilde{f} : X \to \mathbb{R}$

$$\tilde{f}(h) = f(x_* + h)$$

Then x_* is a stationary point of f iff $D\tilde{f}(0) = 0$

Example: Shortest path (euclidian norm)

Let $S = \{\gamma \in C^1([0,1], \mathbb{R}^n) \mid \gamma(0) = x_a, \gamma(1) = x_b\}$

$$L_1(\gamma) = \int_0^1 \|\gamma'(t)\|_2 dt$$

The E-L equations yield

$$\frac{d}{dt} \frac{\gamma_i'(t)}{\|\gamma'(t)\|_2} = 0$$

which admits the solution

$$\gamma(t) = (1-t)x_a + tx_b$$

Newtonian mechanics

Given a point mass particle of mass m, subject to some force F, Newton's law describes the motion of the particle:

$$mq''(t) = F(q(t))$$

This can be reformulated using the Euler-Lagrange equations

Lagrangian mechanics

Given some (1D) trajectory q, define its potential energy V by

$$V(x) = -\int_{x_0}^{x} F(d\xi)d\xi$$

equivalently

$$\frac{dV}{dx} = F(x)$$

and its kinetic energy as

$$K = \frac{m}{2}(q'(t))^2$$

Principle of stationary action

Define the *Action* of the system as

$$A(q) = \int_{t_i}^{t_f} L(q(t), q'(t))dt$$

where $L(q(t), q'(t)) = \frac{m}{2}(q'(t))^2 - V(q(t))$ is the *Lagrangian* of the system

Principle of Stationary action: The motion $q(t)$ of the particle is a *stationary point* of the Action

From Euler-Lagrange to Newton

The Euler-Lagrange equations are

$$-\frac{dV}{dx}(q(t)) - \frac{d}{dt}(\frac{m}{2}.dq'(t)) = 0$$

Injecting $\frac{dV}{dx}(x) = -F(x)$, we recover Newton's law

$$mq''(t) = F(q(t))$$

Example: 2-body problem

Given a body of mass m, orbiting around a body of mass M, with polar coordinates r, φ

$$f(r,\varphi) = \int_a^b \frac{m}{2}(r'(t))^2 + \frac{m}{2}(r(t)\varphi'(t)) + \frac{GMm}{r(t)} dt$$

yields the equations

$$r''(t) = r(t)(\varphi'(t)) - \frac{GMm}{(r(t))^2}$$

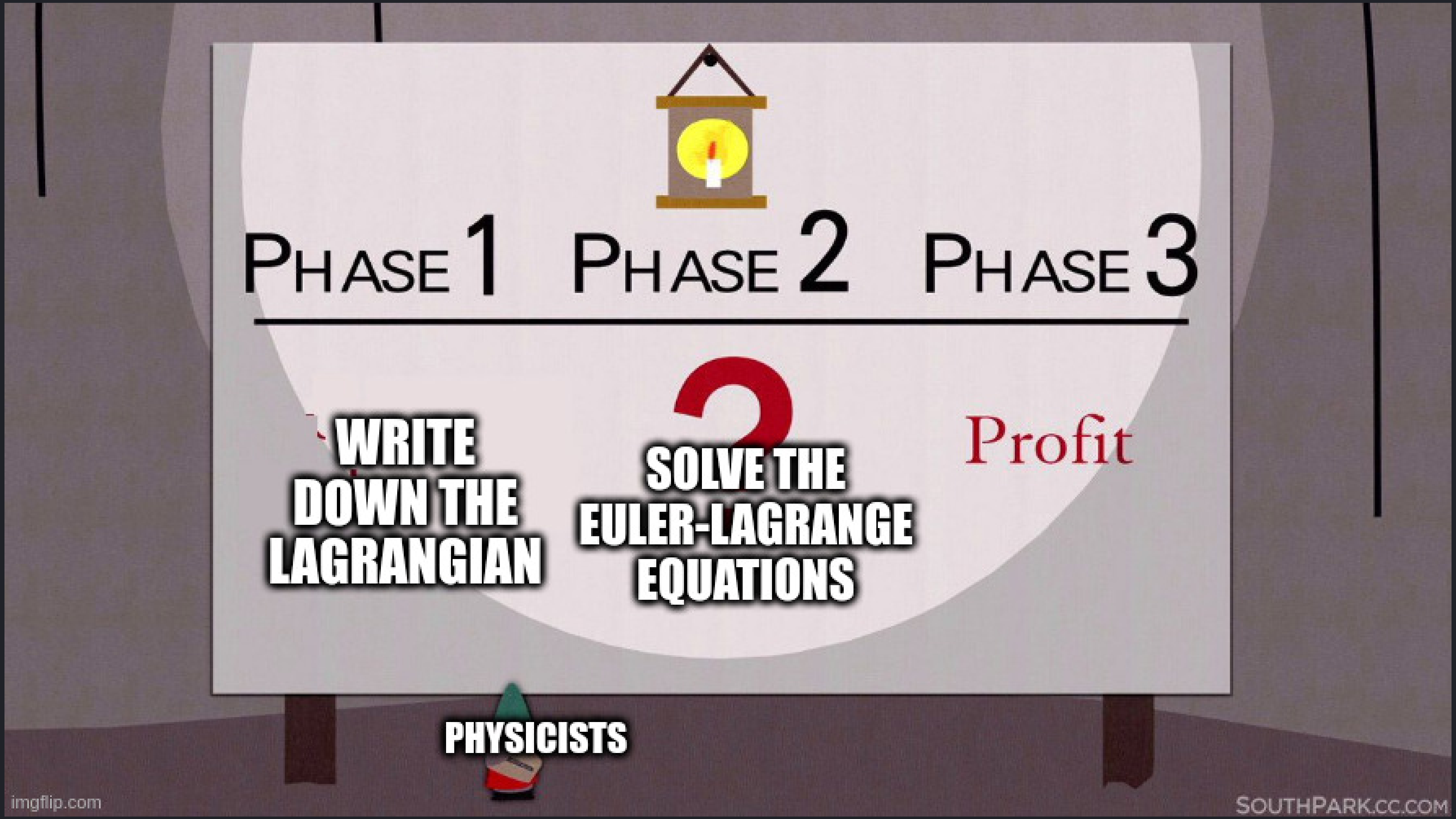
$$\frac{d}{dt}\left(m(r(t))^2\varphi'(t)\right) = 0$$

Lagrangian mechanics

Beyond classical mechanics, most of modern physics can be framed in terms of the stationary action principle:

- Electromagnetism: $\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J^{\mu} A_{\mu}$
- General Relativity $\mathcal{L} = \frac{1}{2\kappa} R + \mathcal{L}_M$
- Quantum mechanics*

Lagrangian mechanics provide a unifying language for physics!


PHASE 1 PHASE 2 PHASE 3
WRITE DOWN THE LAGRANGIAN
SOLVE THE EULER-LAGRANGE EQUATIONS
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1

$$-\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \frac{1}{2}ig_s^2(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu$$

2

$$- \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h\left[\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right] + \frac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \frac{1}{8}g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] - gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \frac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \frac{1}{4}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-] - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+\phi^-$$

3

$$- \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_wA_\mu\left[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right] + \frac{ig}{4c_w}Z^0_\mu\left[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2 - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \frac{8}{3}s_w^2 - \gamma^5)d^\lambda_j)\right] + \frac{ig}{2\sqrt{2}}W^+_\mu\left[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right] + \frac{ig}{2\sqrt{2}}W^-_\mu\left[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right]$$

4

$$+ \frac{ig}{2\sqrt{2}}\frac{m^\lambda_e}{M}\left[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)\right] - \frac{g}{2}\frac{m^\lambda_e}{M}\left[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)\right] + \frac{ig}{2M\sqrt{2}}\phi^+\left[-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right] + \frac{ig}{2M\sqrt{2}}\phi^-\left[m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right] - \frac{g}{2}\frac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)$$

5

$$+ \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \frac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_wW^+_\mu(\partial_\mu\bar{X}^0X^- - \partial_\mu\bar{X}^+X^0) + igs_wW^+_\mu(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+Y) + igc_wW^-_\mu(\partial_\mu\bar{X}^-X^0 - \partial_\mu\bar{X}^0X^+) + igs_wW^-_\mu(\partial_\mu\bar{X}^-Y - \partial_\mu\bar{Y}X^+) + igc_wZ^0_\mu(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) + igs_wA_\mu(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) - \frac{1}{2}gM\left[\bar{X}^+X^+H + \bar{X}^-X^-H + \frac{1}{c_w^2}\bar{X}^0X^0H\right] + \frac{1-2c_w^2}{2c_w}igM\left[\bar{X}^+X^0\phi^+ - \bar{X}^-X^0\phi^-\right] + \frac{1}{2c_w}igM\left[\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right] + igMs_w\left[\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right] + \frac{1}{2}igM\left[\bar{X}^+X^+\phi^0 - \bar{X}^-X^-\phi^0\right]$$

Conservation laws

Symmetries of the Lagrangian give rise to conservation laws

- The Energy $E = K + V$ is conserved

$$E'(t) = (mq''(t) - F(q(t))).\, q'(t) = 0$$

- If $L = L(q')$, the momentum $p = mq'(t) is conserved

$$p'(t) = mq''(t) = -\frac{\partial L}{\partial x}(q(t), q'(t)) = 0$$

See also: Noether's theorem

Beyond Euler-Lagrange

Lagrangian mechanics can be further refined into *Hamiltonian mechanics*
(We won't get into it today, but they are fundamental in e.g. Quantum Mechanics)