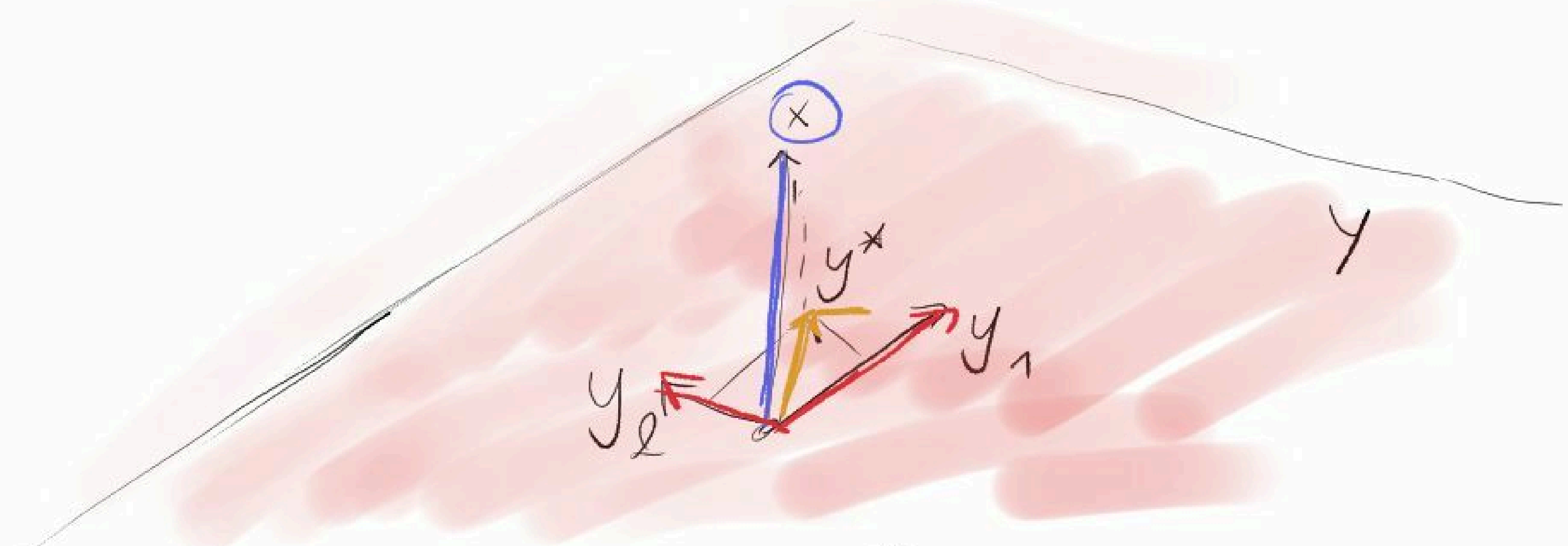
Notes on Section 4.3

"Best Approximation"



Let $Y = \text{span}\{y_1, \dots, y_n\} \subset X$

↳ orthonormal set

Given $x \in X$, find

$$y^* = \underset{y \in Y}{\arg\min} \|y - x\|$$

Relevant in:

- Numerical Analysis
- Computer Graphics
- Constrained optimization
- ...

Projection on a finite dimensional subspace

Let X be an inner product space, $x \in X$ and

$Y = \operatorname{span}\{u_1, \dots, u_n\} \subset X$

$\hookrightarrow$ orthonormal basis of Y

Then the solution y^* of

$$y^* = \arg\min_{y \in Y} \|x - y\|$$

is given by

$$y^* = \sum_{k=1}^{n} \langle x, u_k \rangle u_k$$

$$\left(\Leftrightarrow \forall y \in Y \quad \|x - y\| \geq \|x - y^*\| \right)$$

Example: least square approximation

Let $f \in C[a,b]$. We seek a polynomial φ_* of degree at most m s.t.

$$E(\varphi) = \int_a^b |f(t) - \varphi(t)|^2 \, dt$$

is minimized at φ_*

e.g. $[a,b] = [-1,1]$, $f(t) = e^t$ and $m = 2$

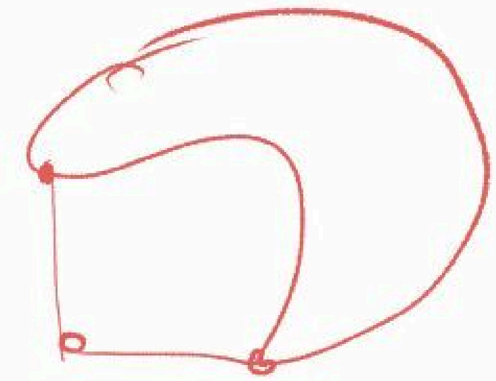
$$\rightsquigarrow \mathcal{P}_m = \operatorname{span}\{1, t, t^2\} = \operatorname{span}\Big\{\underbrace{\tfrac{1}{\sqrt{2}}}_{P_0}, \underbrace{\tfrac{\sqrt{3}}{\sqrt{2}} t}_{P_1}, \underbrace{\tfrac{3\sqrt{10}}{4}\left(t^2 - \tfrac{1}{3}\right)}_{P_2}\Big\}$$

$$\left.\begin{array}{l} \cdot \ \langle f, P_0 \rangle = \frac{1}{\sqrt{2}}\left(e - \frac{1}{e}\right) \\ \cdot \ \langle f, P_1 \rangle = \frac{\sqrt{6}}{e} \\ \cdot \ \langle f, P_2 \rangle = \frac{\sqrt{5}}{\sqrt{2}}\left(e - \frac{7}{e}\right) \end{array}\right\} \Rightarrow \varphi_*(t) = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e} t + \frac{15}{4}\left(e - \frac{7}{e}\right)\left(t^2 - \frac{1}{3}\right)$$

Projection on a convex set

Given a closed convex set $C \in H$, a Hilbert space, and some $x \in H$

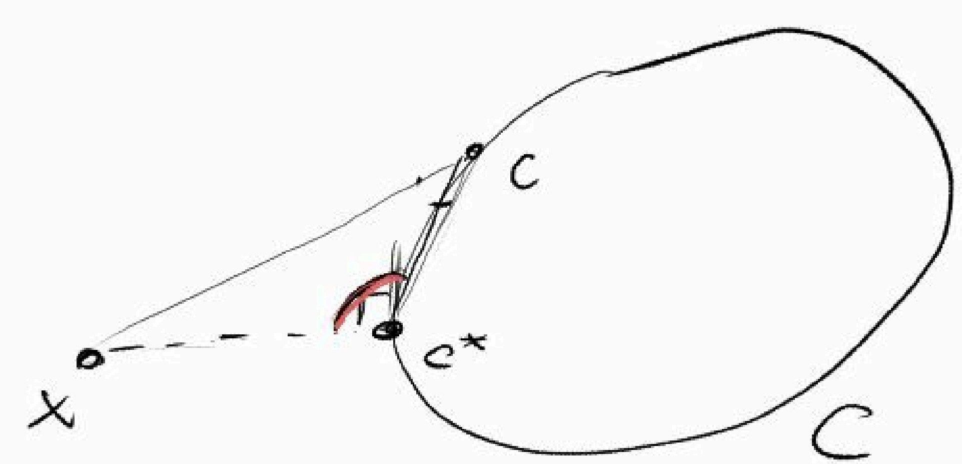
We seek to solve $\begin{cases} \min & \|x-c\| \\ \text{s.c.} & c \in C \end{cases}$



Theorem: 4.6

1) There is a unique $c_* \in C$ that minimizes $\|x-c_*\|$ over C

2) $\forall c \in C \quad \mathrm{Re}\langle x-c_*, c-c_*\rangle \leq 0$ ⟹ the angle between $x-c_*$ and $c-c_*$ is obtuse



Theorem 4.7 | Projection on a closed subspace |

Let Y be a closed subspace of a Hilbert space H and $x \in H$.

Then there exists a unique $y_* \in Y$ s.t. $\forall y \in Y$ $\|x - y_*\| \leq \|x - y\|$

The point y_* can be characterised by

$$\forall y \in Y, \ \langle x - y_*, y \rangle = 0 \qquad (x - y_* \perp Y)$$

· y_* is called the <u>orthogonal projection</u> of x on Y, $\underbrace{P_Y}_{\text{projection operator}} x$

<u>Theorem 4.8</u> Let Y be a closed subspace of H

1) $x \xmapsto{P_Y} P_Y x : H \to H \in \mathcal{CL}(H)$
2) $\operatorname{ran} P_Y = Y$
3) $\|P_Y\| = 1$, except if $Y = \{0\}$, in which case $P_Y = 0$
4) $\ker P_Y = Y^\perp$
5) $\forall x \in H, \exists y \in Y, z \in Y^\perp$ s.t. $x = y + z$
6) $P_{Y^\perp} = I - P_Y$
7) $P_Y^2 = P_Y$
8) P_Y is symmetric: $\langle P_Y x, x' \rangle = \langle x, P_Y x' \rangle$ $\forall x, x' \in H$

4.4 Generalized Fourier Series

Bases of vector spaces

Hamel basis

- only finite linear combinations
- uncountable for infinite dimensional spaces
- doesn't mesh with topology

CRINGE

Schauder Basis

- Infinite series* allowed
- always countable (for most reasonable spaces)
- * convergent series

BASED

+ Hilbert space

↓

Orthogonal Basis

$B \subset X$ is an <u>orthonormal basis</u> if

- span(B) dense in X
- B is orthonormal

Examples of orthonormal bases

ℓ^2: $B = \{ e_i = (\delta_{ij})_{j \in \mathbb{N}} \}$ (one-hot sequences)

$\mathrm{span}(B) = c_{00} \leftarrow$ sequences that are eventually zero (dense in ℓ^2)

$C[-1,1]$ ($L^2[-1,1]$) Legendre polynomials $\left\{ \sqrt{\frac{2n+1}{2}} P_n \right\}_{n \in \mathbb{N}}$ (Ex. 4.25)

$C[0,1]$ ($L^2[0,1]$) Fourier basis $\{ T_k = e^{i2\pi kt} \}$

Theorem: Let X be an inner product space with a countable orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$. Then

1) $\forall x \in X \qquad x = \sum_{n=0}^{+\infty} \langle x, u_n\rangle u_n$

2) $\forall x, y \in X \qquad \langle x, y\rangle = \sum_{n=0}^{\infty} \langle x, u_n\rangle \langle y, u_n\rangle^*$

3) $\forall x \in X \qquad \|x\|^2 = \sum_{n=0}^{+\infty} |\langle x, u_n\rangle|^2$

Convergent series

$\Rightarrow$ In practice, we can truncate to finitely many terms to compute

If $X = H$ is a Hilbert space w/ countable orth. basis $\{u_n\}_{n\in\mathbb{N}}$,

then $\forall (c_n)_{n\in\mathbb{N}} \in \ell^2$, $\sum_{n=0}^{+\infty} c_n u_n \in H$

[ℓ^2 can be embedded into any countably-based Hilbert space]

It's just ℓ^2 with extra steps

From finite dimensions we have

Any d-dimensional vector space (over $\mathbb{K}$) is isomorphic to $\mathbb{K}^d$

In infinite dimension:

Any Hilbert space with a countable orthonormal basis is isomorphic to ℓ^2

idea: $x \longmapsto (\langle x, u_0 \rangle, \langle x, u_1 \rangle, \dots)$ (Ex 4.26)

maps $x \in H$ to an element of ℓ^2

Fourier Series

$C[0,1]$ with basis $\left\{ e^{2\pi i n t} \right\}_{n \in \mathbb{N}}$ (T_n)

Any $f \in C[0,1]$ admits an expansion $f = \sum_{n=0}^{+\infty} \langle f, T_n \rangle T_n$

where $\langle f, T_n \rangle = \int_0^1 f(t) \left(e^{2\pi i n t}\right)^* dt = \int_0^1 f(t)\, e^{-2\pi i n t} dt = \hat{f}_n$

are the Fourier coefficients of f

NB. Convergence here means $\| \ \|_2$ convergence, which is different from pointwise convergence

[In particular, for non smooth f, we can have wierd artifacts around singularities]

$H = L^2(\mathbb{R})$ $\quad Y = \{ f \in L^2(\mathbb{R}) \mid f(t) = f(-t) \ \forall t \in \mathbb{R} \}$ 4.21

. Show that $P_Y f = \frac{f(t) + f(-t)}{2}$

. let $f \in L^2(\mathbb{R})$, $g \in Y$

$$\langle f - P_Y f , g \rangle = 0$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} \underbrace{\left(f(t) - \frac{f(t) + f(-t)}{2} \right)}_{\text{odd}} \underbrace{g(t)}_{\text{even}} \, dt = 0$$

. $P_Y^{\perp} f = (I - P_Y) f = f(t) - \frac{f(t) + f(-t)}{2} = \frac{f(t) - f(-t)}{2}$

4.22 H Hilbert space $S \in CL(H)$, $S^2 = I$, $\langle Sx, y\rangle = \langle x, Sy\rangle$

$Y = \{x \in H \mid Sx = x\}$ $Z = \{x \in H \mid Sx = -x\}$

Show that $Z = Y^\perp$, $Y = Z^\perp$, and compute P_Y and P_Z

- Let $y \in Y$, $z \in Z$

$$\langle y, z\rangle = \langle Sy, z\rangle = \langle y, Sz\rangle = -\langle y, z\rangle$$

$= 0$

- Let $x \in H \Rightarrow \boxed{x = y + z}$ $y \in Y$, $z \in Z$

$$Sx = Sy + Sz = y - z$$

$\rightarrow$ $y = \dfrac{x + Sx}{2}$ $\quad z = \dfrac{x - Sx}{2}$