1 Geometry

1.1 Triangles

- $c^2 = a^2 + b^2 - 2ab\cos C$
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$
- $\Delta = \frac{1}{2}ah_a = \frac{1}{2}ab\sin C = \dfrac{abc}{4R} = sr$
- $\Delta = \sqrt{(s(s-a)(s-b)(s-c))}$
- Median, $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$
- Angle Bisector, $b_a = \sqrt{bc\left(1 - \left(\frac{a}{b+c}\right)^2\right)}$

1.2 Baricentric Coordinates

- Centroid, $G = [1, 1, 1]$
- Incenter, $I = [a, b, c]$
- Excenter, $I_a = [-a, b, c]$
- Circumcenter, $O = [a^2(b^2 + c^2 - a^2)]$
- Orthocenter, $= [(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)]$

1.3 Polygons

- Area, $A = \frac{1}{2}\sum_{i=0}^{n-1}(x_i\ y_{i+1} - x_{i+1}\ y_i)$.
- $Cen_{\text{x}} = \frac{1}{6A}\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i\ y_{i+1} - x_{i+1}\ y_i)$
- $Cen_{\text{y}} = \frac{1}{6A}\sum_{i=0}^{n-1}(y_i + y_{i+1})(x_i\ y_{i+1} - x_{i+1}\ y_i)$,

1.4 Miscallaneous

- **Pick's Theorem:** $A = i + \frac{b}{2} - 1$
- **Euler:** $V - E + F = C + 1$
- A connected planar graph with n vertices has at most $3n - 6$ faces and $2n - 4$ edges.

2 Combinatorics

2.1 Sequences

1. **Binomial Coefficients** $\binom{n}{k} = k$ element subsets of a n element set.
   - $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{n!}{k!(n-k)!}$
   - $\sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m+1}$
   - $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$
2. **Derangements** Perms with no fixed points
   - $D(n) = (n-1)(D(n-1) + D(n-2))$
   - $D(n) = nD(n-1) + (-1)^n = \left[\frac{n!}{e}\right]$
   - $D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) = \left[\frac{n!}{e}\right]$
3. **Stirling numbers of the first kind** $s(n,k) =$ Permutations on n items with k cycles.
   - $s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k)$
   - $\sum_{k=0}^{n} s(n,k)x^k = x(x+1)\cdots(x+n-1)$
4. **Stirling numbers of the second kind** $S(n,k)$ = Partitions of n distinct elements into k groups.
   - $S(n,k) = S(n-1,k-1) + kS(n-1,k)$
   - $S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$
5. **Eulerian numbers** $E(n,k) =$ Permutations with exactly k indices i, st, $a_i > a_{i-1}$
   - $E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$
   - $E(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n$
6. **Catalan numbers** $C_n = 1, 1, 2, 5, 14, 42, 132, ..$
   - Balanced bracket sequences with length $2n$.
   - n-permutations with LIS ≤ 2
   - $C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$
   - $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$
7. **Partition function** Ways of writing n as a sum of positive integers, disregarding order, $p(n) = 1, 1, 2, 3, 5, 7, 11, 15, 22, 30\cdots$
   - $p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}}(-1)^{k+1}p(n - k(3k-1)/2)$

2.2 Principle of Inclusion Exclusion

Let A be a set, $c_1, c_2, \cdots, c_n$ be n conditions, and A_k be the set of elements satisfying c_k. Let E_m be the set satisfying exactly m conditions, and L_m be the set satisfying at least m conditions. Let $S_k = \sum_{|I|=k}\left|\bigcap_{i\in I} A_i\right|$. Then,

- $|E_m| = \sum_{k=m}^{n}(-1)^{k-m}\binom{k}{m}S_k$
- $|L_m| = \sum_{k=m}^{n}(-1)^{k-m}\binom{k-1}{m-1}S_k$

2.3 Permutation Cycles

- Exp. number of cycles in an n permutation is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$
- Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S.
  $$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

2.4 Burnside's lemma

- Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals $\dfrac{1}{|G|}\sum_{g\in G}|X^g|$, where X^g are elements fixed by g.
- If $f(n)$ counts "configurations" of length n, we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get $g(n) = \dfrac{1}{n}\sum_{k=0}^{n-1} f(\gcd(n,k)) = \dfrac{1}{n}\sum_{k|n} f(k)\phi(n/k)$.

2.5 Lucas' Theorem

Let n, m be non-negative integers and p a prime. Write $n = (n_k\cdots n_1 n_0)_p$ and $m = (m_k\cdots m_1 m_0)_p$ in base p. Then $\binom{n}{m} \equiv \prod_{i=0}^{k}\binom{n_i}{m_i} \pmod{p}$.

3 Graphs

3.1 Cayley's Theorem

- Labeled trees on n vertices: n^{n-2}
- Connect k trees of size n_i: $n_1 n_2\cdots n_k n^{k-2}$
- Trees with degrees d_i: $\frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!}$

3.2 Chromatic Polynomials

Let G be a graph with n vertices. Let $P(G,x)$ be the number of vertex colorings of G with x colors. Then P is a monic integer polynomial of degree n.

- $P(G,x) = P(G-uv,x) - P(G/uv,x)$ if $uv \in G$
- $P(G,x) = P(G+uv,x) + P(G/uv,x)$ if $uv \notin G$
- $P(G,x) = x(x-1)^{n-1}$ if G is a tree
- $P(G,x) = (x-1)^n + (-1)^n(x-1)$ if G is a cycle

3.3 Number of spanning trees

Let $D =$ degree matr, $A =$ adj. matr. Delete one row, col of $D - A$ to get L. Then no of spanning trees, $t(G) = t(G-uv) + t(G/uv) = det(L)$

3.4 Erdos Gallai Theorem

A simple graph with degree seq., $d_1 \geq \cdots \geq d_n$ exists iff $d_1 + \cdots + d_n$ is even and for all k, $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$

4 Flow and Matching

4.1 Marriage Theorem and Generlizations

• **Hall's Marriage Theorem** In a bipartite graph $G = A \cup B$, a matching saturating A exists iff $|N(S)| \geq |S|$ for all $S \subset A$

• **Generalization (Unproven)** In a bipartite graph $G = A \cup B$, a matching with size $|A| - x$ exists iff $|N(S)| \geq |S| - x$ for all $S \subset A$

• **Generalization (Sabbir)** In a bipartite (U, V) flow graph with $C()$, then maxflow is equal to $\sum\limits_{u \in U} C(u)$ iff for every set $S \subseteq U$, $\sum\limits_{u \in S} C(u) \leq F(S)$ where $F(S) = \sum\limits_{v \in V} min(C(v), \sum\limits_{u \in S} C(u, v))$

4.2 Bipartite Matching

Let M be a max matching of a graph with bipartitions L and R. Let U be unmatched vertices in L, Z be vertices reachable from U via alternating paths.

• **Konig's Theorem** Min Vertex Cover = Max Matching. $(L \setminus Z) \cup (R \cap Z)$ is such a cover. In fact, $L \setminus Z$ are the only nodes which is in at least one vertex cover. A vertex is part of all cover if its partner in M is part of no cover.

• **M**ax indpendent set is the compliment of Min Vertex Cover. Max BiClique is Max independent set in complement graph.

4.3 Feasible Flow

• **Feasible Flow**: flow of all old edge = upper cap - lower cap, $acc[u]$ = sum of flow into u - sum of flow out from u. Add **supersource**, **supersink**, edge from supersource to u (if $acc[u] > 0$), edge from u to supersink (if $acc[u] < 0$), capacity $|acc(u)|$ and an edge sink to source (capacity: ∞). if maxflow from supersouce to supersink = sum of outgoing cap from supersource. Then feasible flow exists.

• **Maxflow** remove all new edges (but not flows/caps in old edges) and apply maxflow from source to sink. this is the maxflow (with lower caps hidden), so max flow = $\sum flow(e) + low(e)$ where e is adjacent to source, (incomings are negative)

4.4 Miscallaneous

• **Dilworth's Theorem** In a poset, the size of a maximal antichain equals the size of a minimal chain cover. Thus minimum path cover in a DAG equals maximum flow in the transitive closure.

• **Tutte's Theorem** Let $o(S)$ be the number of odd components in S. A graph G has a perfect matching iff $\forall S \subset V, o(G \setminus S) \leq |S|$

5 Math

• $det(M + uv^T) = det(M) + u^T adj(M)v$, u, v are column matrices

• Given $x_i - x_j \leq C_{i,j}$ inequalities, construct graph with node for each variable and a ground node. put edge $j \to i$ with cost $c_{i,j}$ for the above ineq. also put $ground \to i$ for each i (with cost generally 0). then bellmand ford from ground will give value for all x_i. If we use w_i as cost for $ground \to i$, then that serves as $x_i \leq w_i$.

• **Lagrange Interpolation**
$P(x) = \sum y_i P_i(x)$ where $P_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$

• **Newton Interpolation** $P(x) = \sum a_i P_i(x)$ where $a_i = [y_0, \cdots, y_i]$ and $P_i(x) = \prod_{j<i}(x - x_j)$
$[y_a, \cdot, y_b] = ([y_{a+1}, \cdot, y_b] - [y_a, \cdot, y_{b-1}])/(x_b - x_a)$

• **simpson** $\int_a^b f(x)dx \approx$
$\frac{b-a}{3n}(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + f(x_n))$
$x_{i+1} - x_i = (b-a)/n$

• **Polynomial inverse** $B_0 = 1/A[0]$,
$B_{k+1} = B_k(2 - AB_k)$

6 SegTree beats

for min-update, keep max and second max (distinct) in each node. during update with x, in the case when (l, r) is inside update range (L, R), if $x \geq max$ do nothing, if $max > x \geq max22$ set lazy and return, if $max2 > x$ recurse deeper. (count of max needed for sum).