Paths in Graphs: Fastest Route

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Higher School of Economics

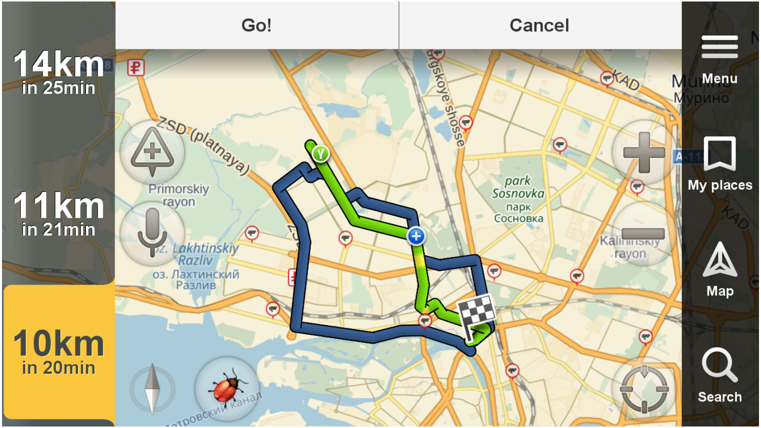
Graph Algorithms
Data Structures and Algorithms

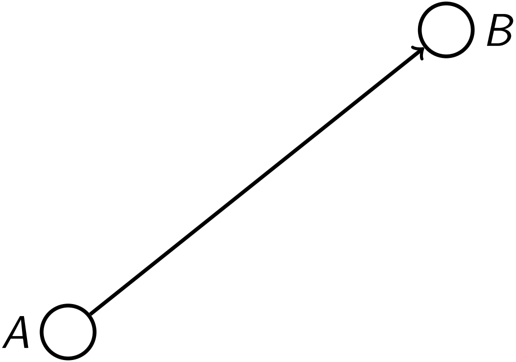
Outline

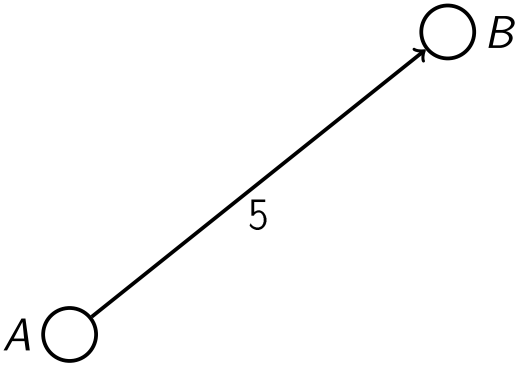
1. Fastest Route
2. Naive Algorithm
3. Dijkstra's Algorithm

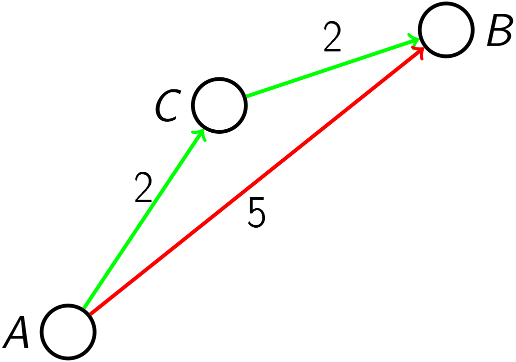
Fastest Route

What is the fastest route to get home from work?

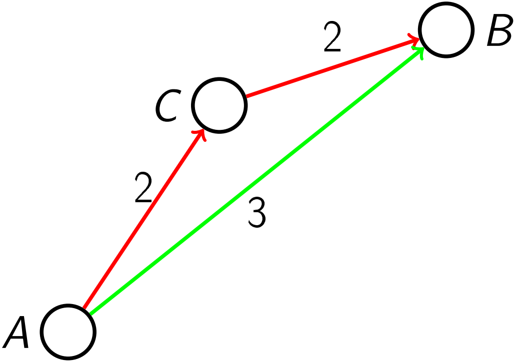



A
B


A
B
5



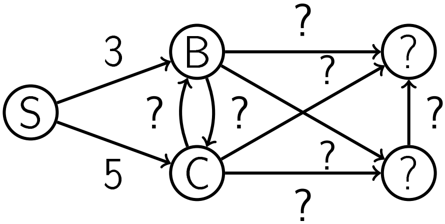
A
B
C
2
2
5



B
2
C
2
3
A

Intuition

- Assume that we stay at S and observe two outgoing edges:



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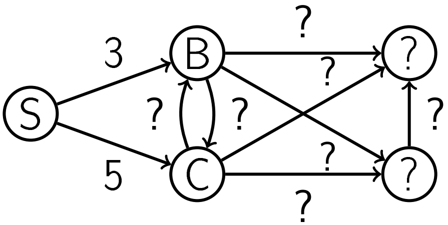

- Can we be sure that the distance from S to C is 5?

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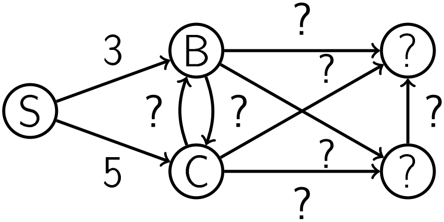

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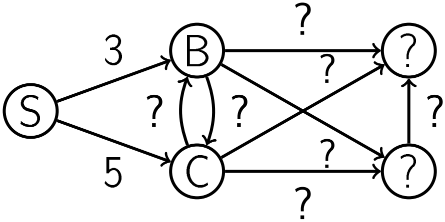

- No, because the weight of the edge (B, C) might be equal to, say, 1.

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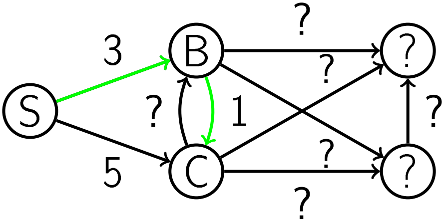

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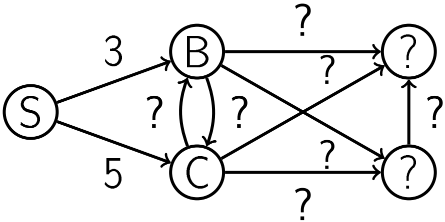
Intuition

- Can we be sure that the distance from S to B is 3?



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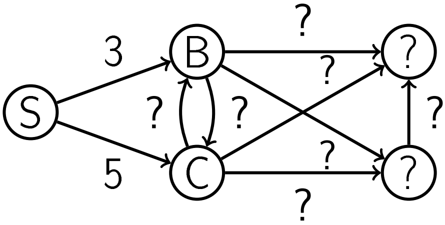

- Yes, because there are no negative weight edges.

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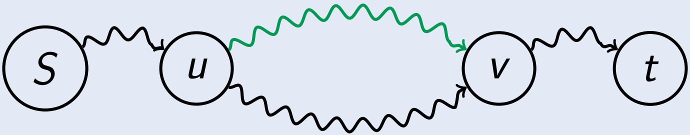
Optimal substructure

Observation

Any subpath of an optimal path is also optimal.

Proof

Consider an optimal path from S to t and two vertices u and v on this path. If there were a shorter path from u to v we would get a shorter path from S to t.



Corollary

If $S \to \ldots \to u \to t$ is a shortest path from S to t, then

$$d(S,t) = d(S,u) + w(u,t)$$

Here u is the previous destination hence d(S, u) is th previous distance

Edge relaxation

- `dist`[v] will be an upper bound on the actual distance from S to v.

Edge relaxation

- `dist`[v] will be an upper bound on the actual distance from S to v.
- The edge relaxation procedure for an edge (u, v) just checks whether going from S to v through u improves the current value of `dist`[v].

$\texttt{Relax}((u, v) \in E)$

$\texttt{if}\;\; dist[v] > dist[u] + w(u, v):$

$\quad dist[v] \leftarrow dist[u] + w(u, v)$

$\quad prev[v] \leftarrow u$ Storing the node from which we got to u that is the previous node u

Naive approach

Naive(G, S)

for all $u \in V$:

  $dist[u] \leftarrow \infty$

  $prev[u] \leftarrow nil$

$dist[S] \leftarrow 0$

do: Run the while loop until we can no longer relax any edge

  relax all the edges

$while$ at least one $dist$ changes

Correct distances

Lemma

After the call to `Naive` algorithm all the distances are set correctly.

Proof

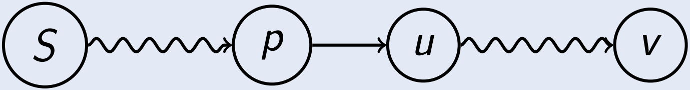
- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex v such that $\text{dist}[v] > d(S, v)$.

Assume for contradiction that even after applying the naive algorithm there exists a node such that dist[v] is not optimal

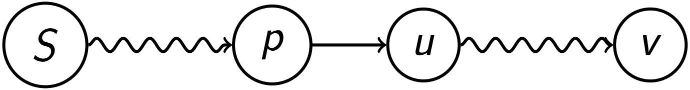
Proof

No need to break the head

- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex v such that $\text{dist}[v] > d(S, v)$.
- Consider a shortest path from S to v and let u be the first vertex on this path with the same property. Let p be the vertex right before u.



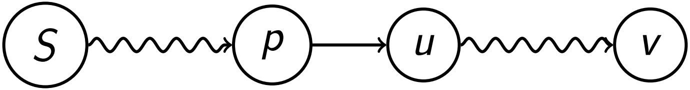
Proof (continued)



No need to break the head

- Then $d(S, p) = \text{dist}[p]$ and hence $d(S, u) = d(S, p) + w(p, u) = \text{dist}[p] + w(p, u)$

Proof (continued)



No need to break the head

- Then $d(S,p) = \mathrm{dist}[p]$ and hence $d(S,u) = d(S,p) + w(p,u) = \mathrm{dist}[p] + w(p,u)$
- $\mathrm{dist}[u] > d(S,u) = \mathrm{dist}[p] + w(p,u) \Rightarrow$ edge (p,u) can be relaxed — a contradiction. $\square$

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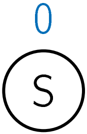

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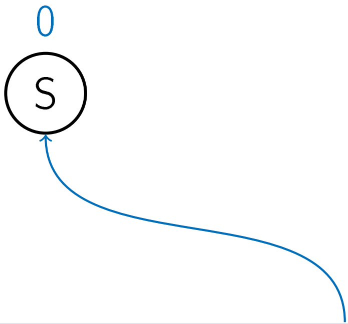
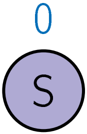


initially, we only know the distance to S

Intuition



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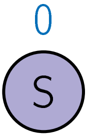
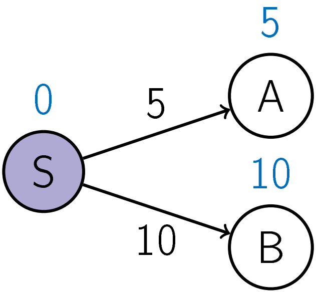


let's relax all the edges from S

Intuition



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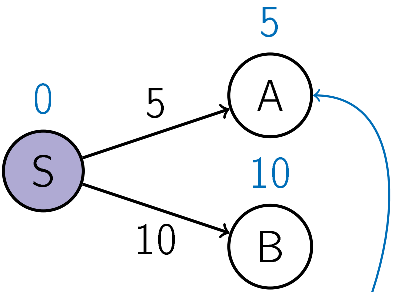
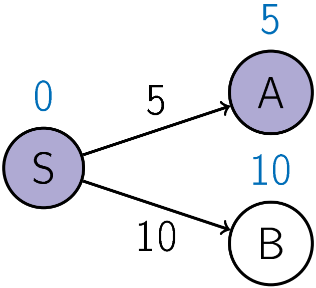


we now know the distance for A

Intuition



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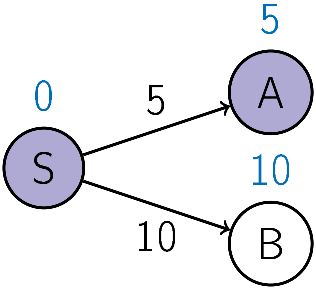

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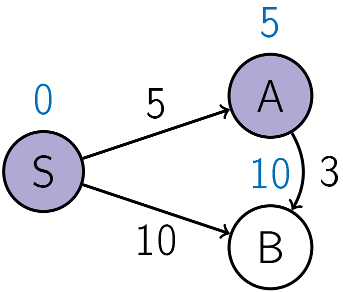

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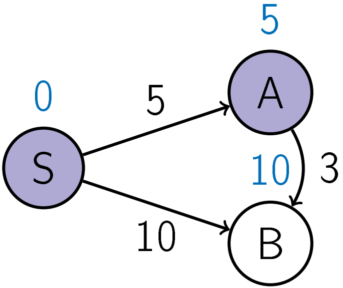
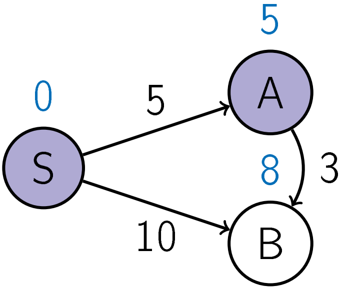


we discover an edge (A, B) of weight 3 that updates dist$[B]$

Intuition



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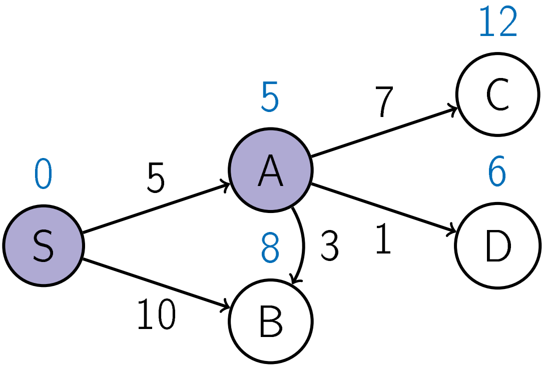
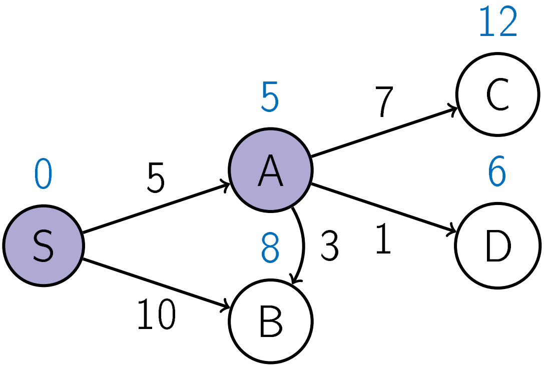


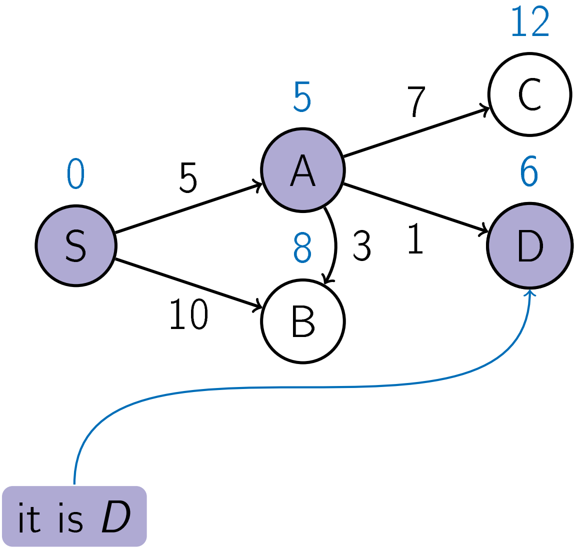
we also discover a few more outgoing edges

Intuition



what is the next vertex for which we already know the correct distance?

Intuition



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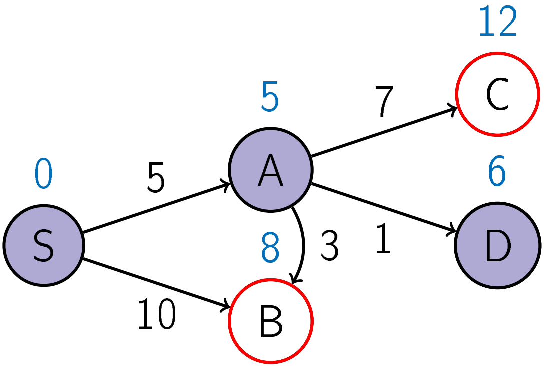

Because When we go to D from B or C then the distance cannot be less than 6. It can be atleast 8 when we go from B

If there is an edge from D to B or D to C with value 1 then their dist value will get improved

while for B and C it is possible that their dist values are larger than actual distances

Hence we know that at any moment of time if we relax the all the edges outgoing from some known set of nodes for which the distances are known correclty then the node with smallest dist value estimate is also node for which we know the distance correctly

Main ideas of Dijkstra's Algorithm

- We maintain a set R of vertices for which `dist` is already set correctly ("known region").

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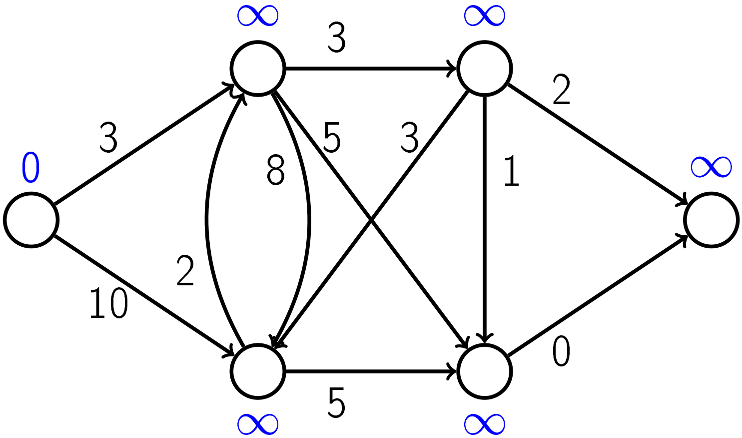
- We maintain a set R of vertices for which `dist` is already set correctly ("known region").
- The first vertex added to R is S.
- On each iteration we take a vertex outside of R with the minimal `dist`-value, add it to R, and relax all its outgoing edges.

After no of iterations equal to no of nodes in the graph we know that correct distance for all the nodes

Example



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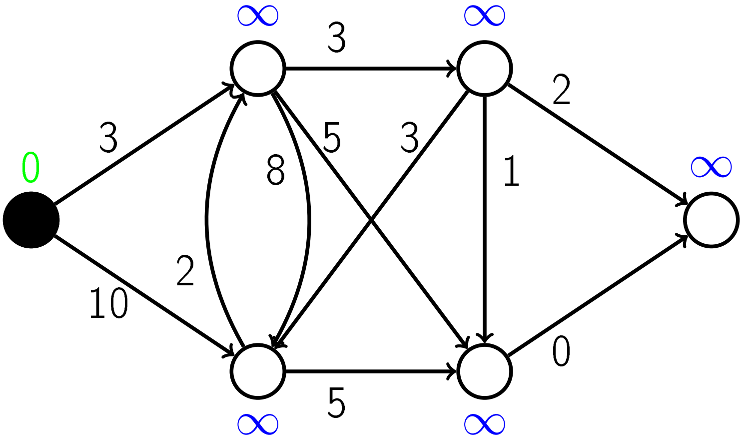

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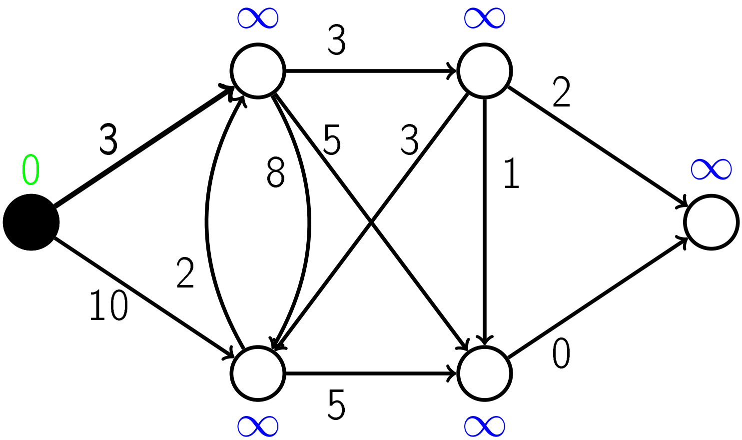

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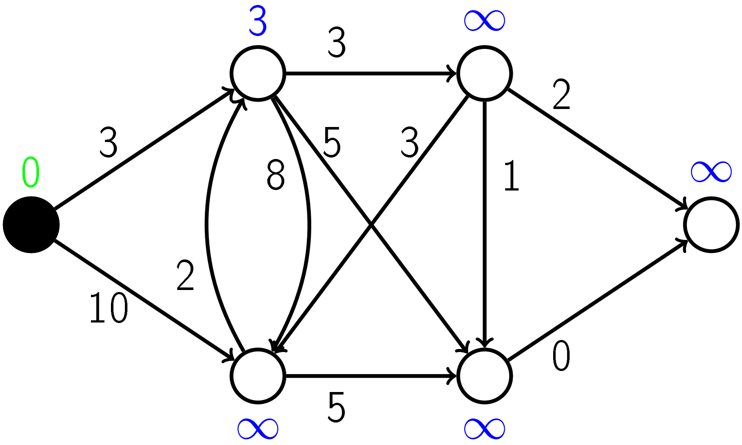

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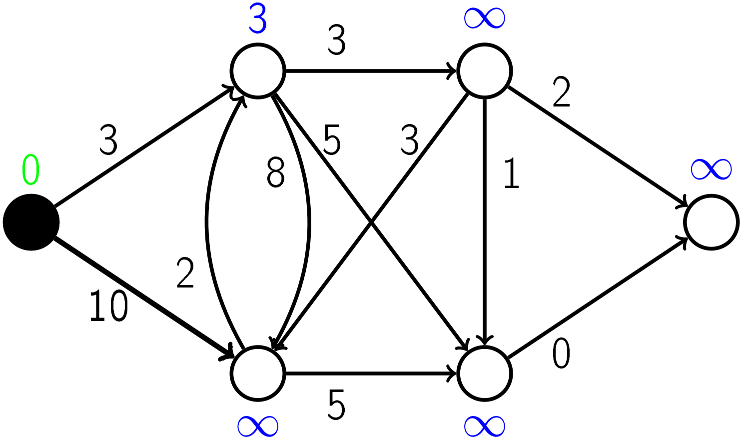

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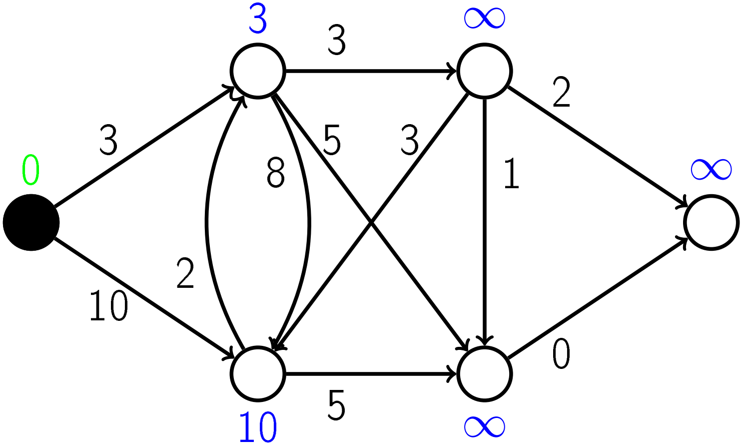

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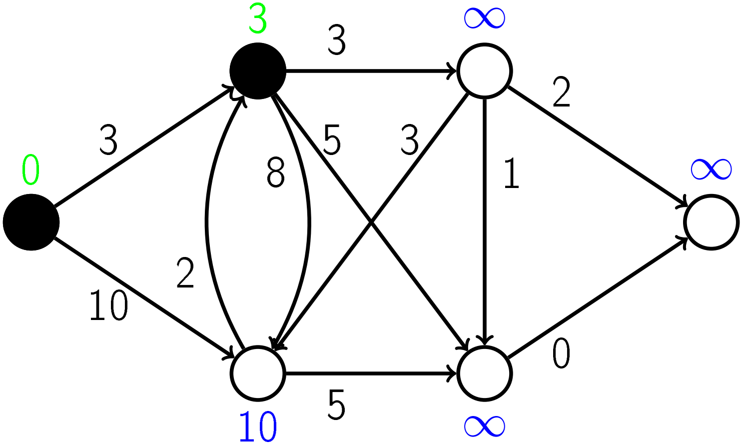

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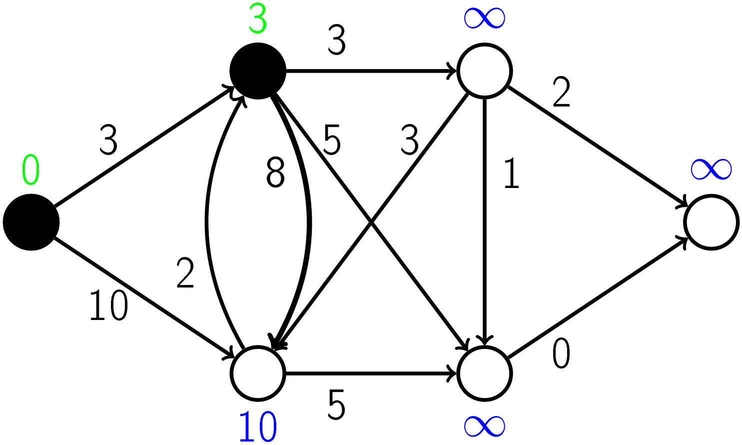

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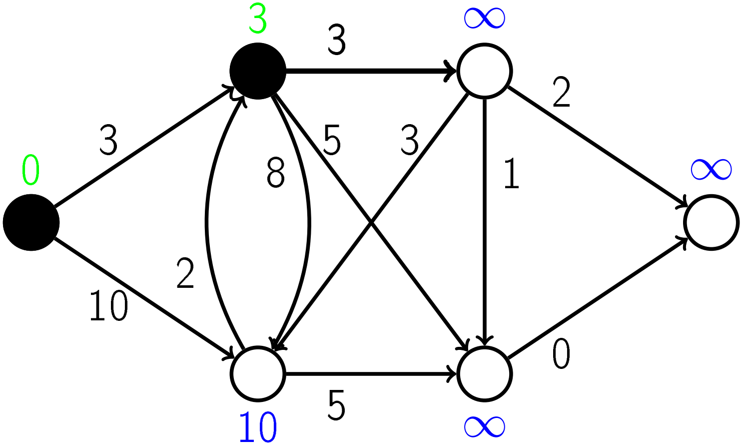

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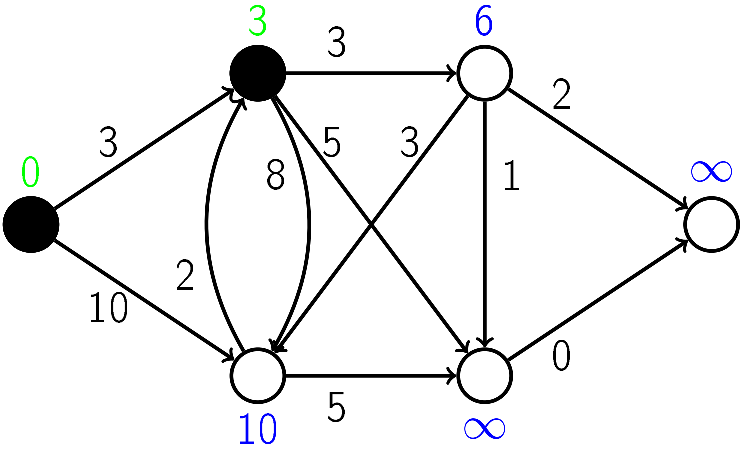

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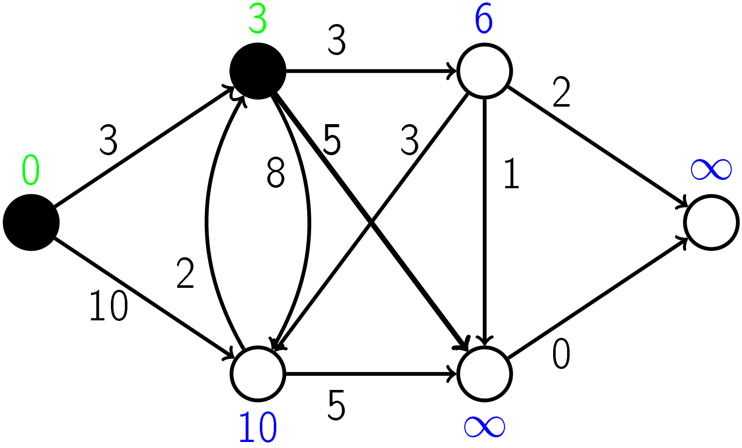

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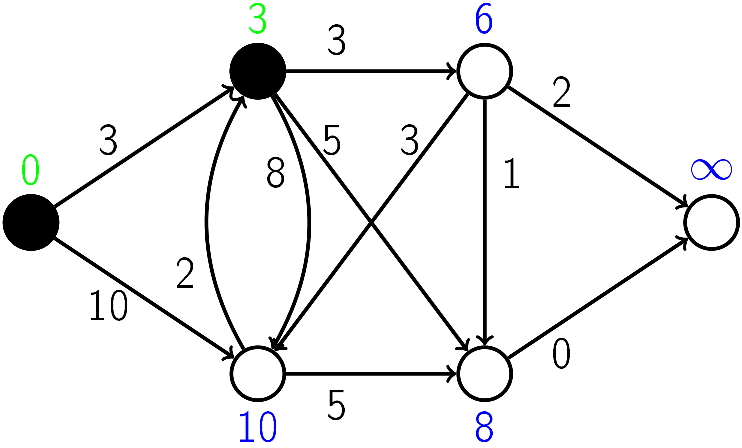

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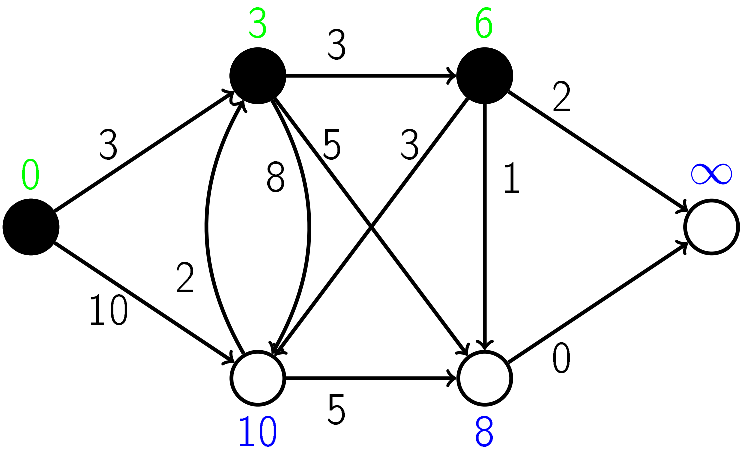

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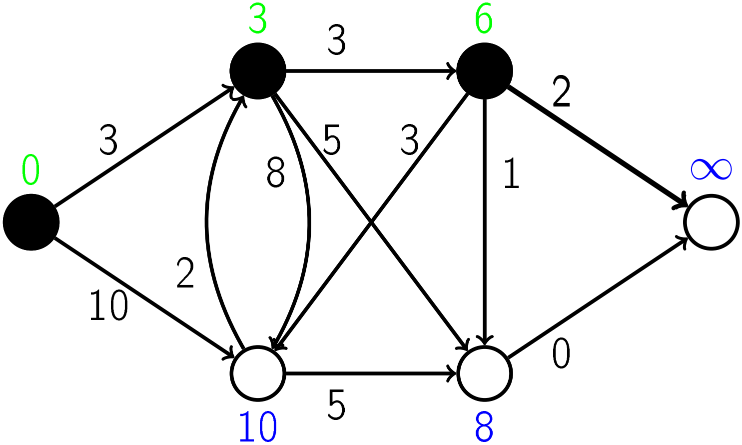

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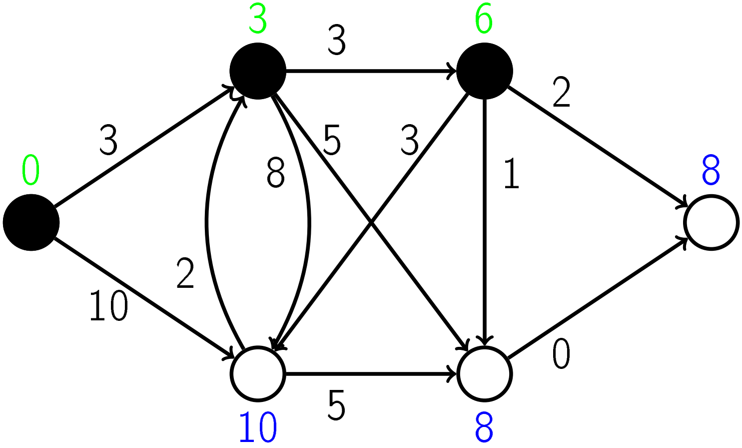

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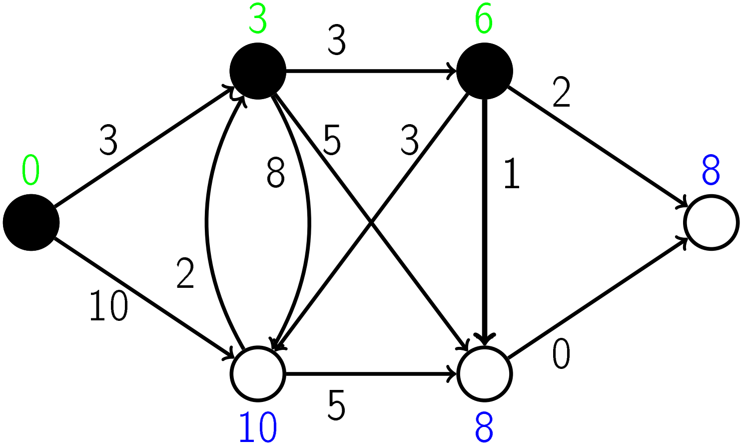

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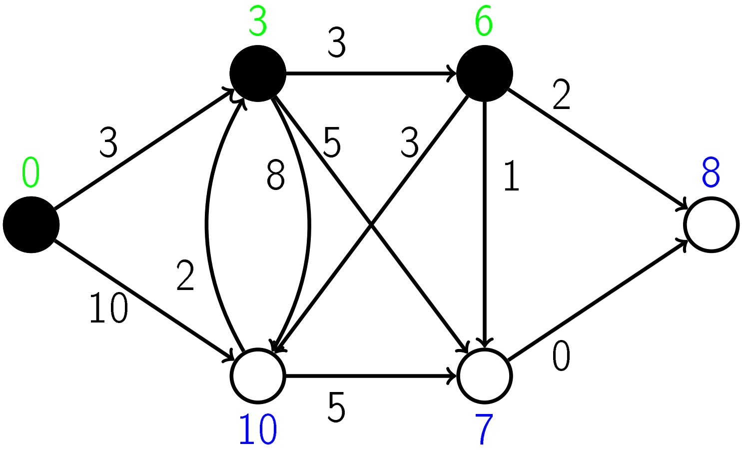

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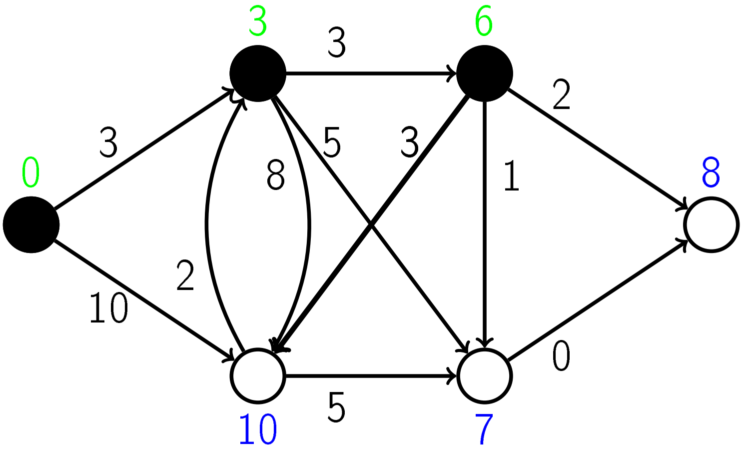

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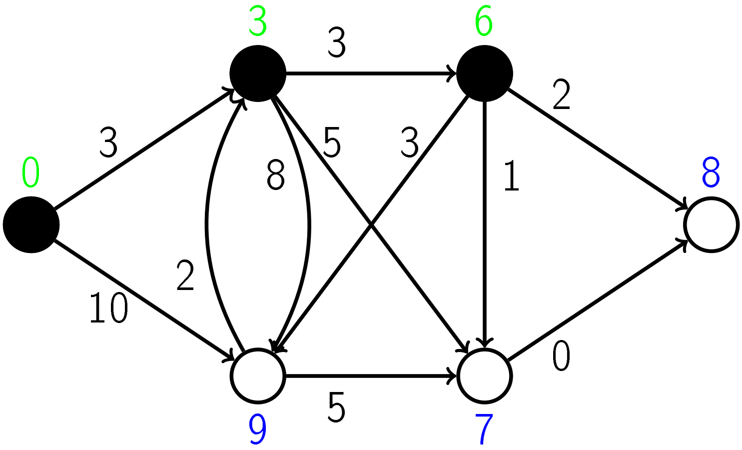

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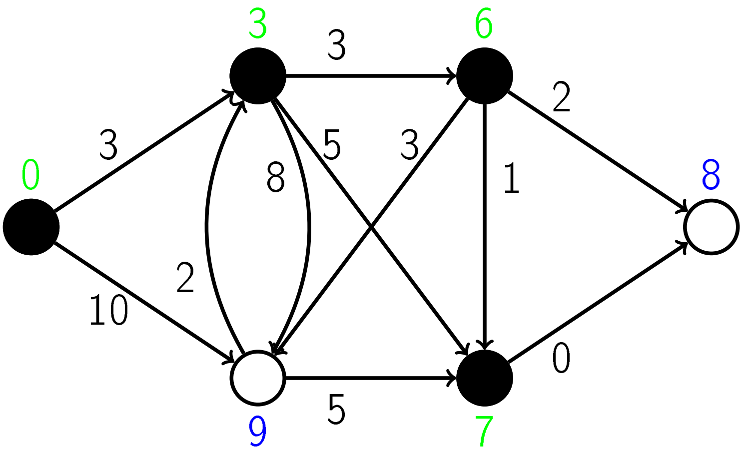

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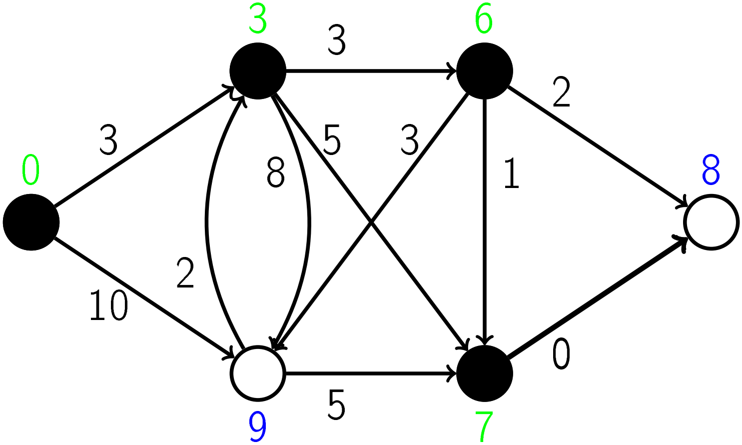

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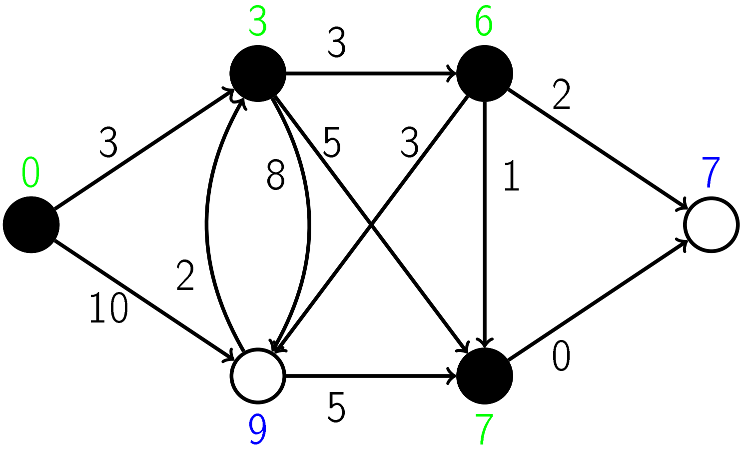

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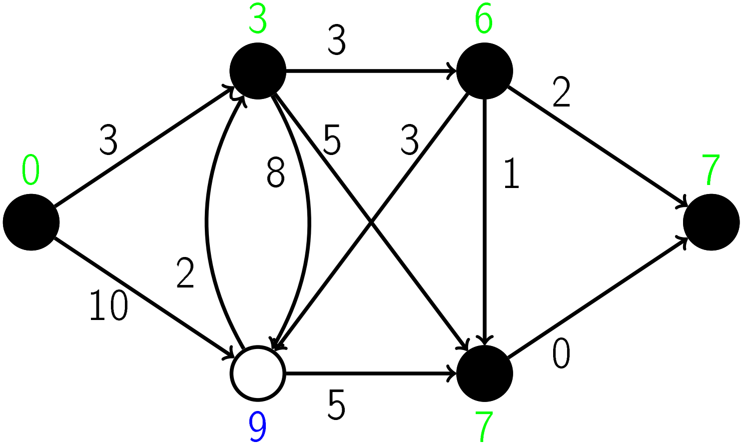

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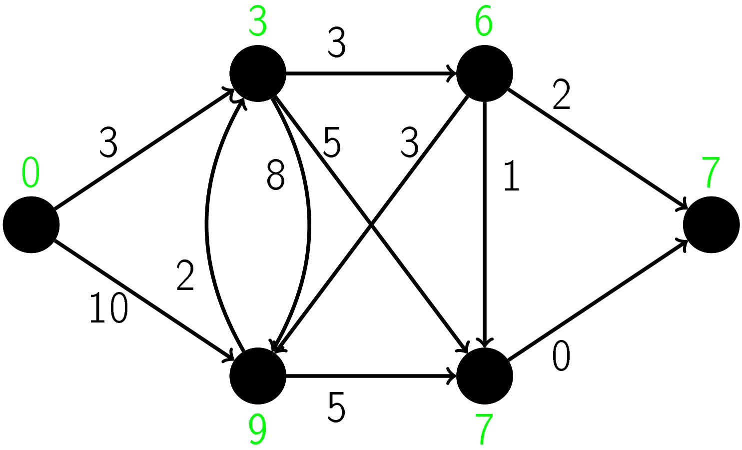

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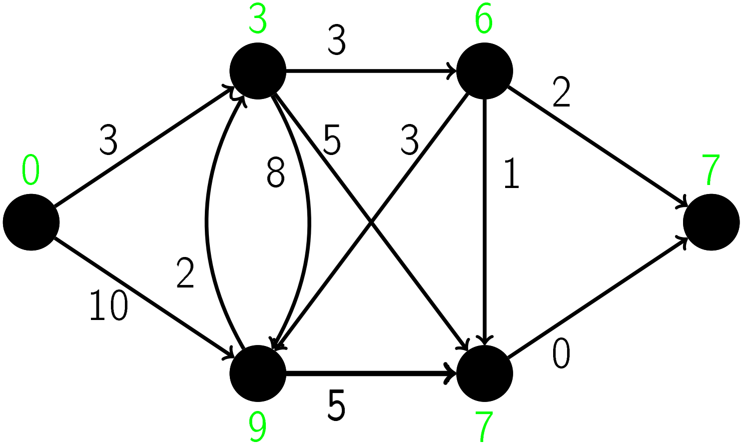

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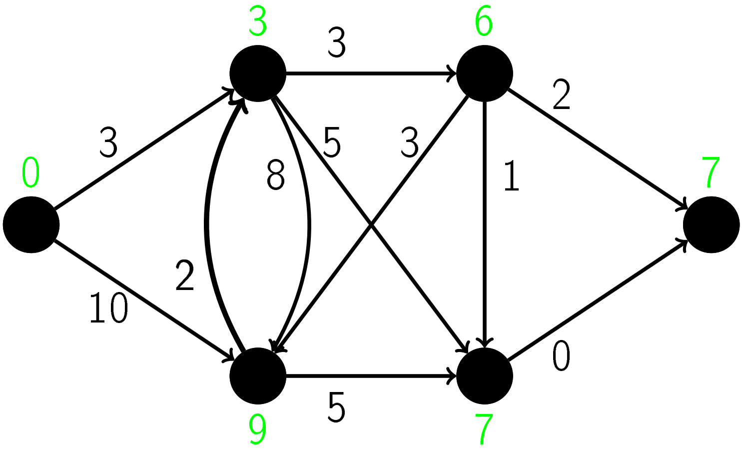

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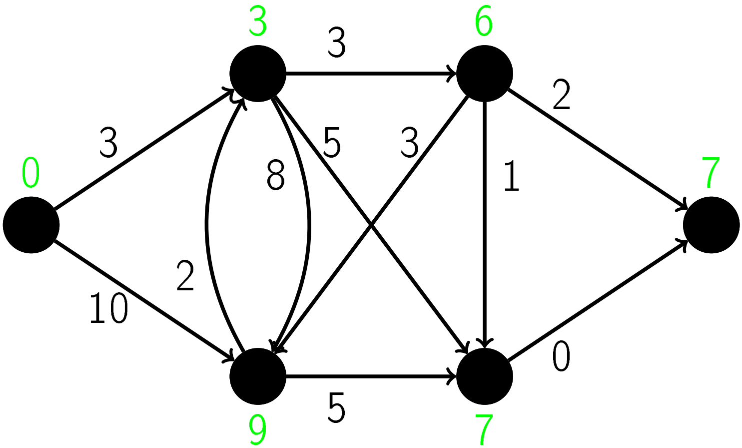

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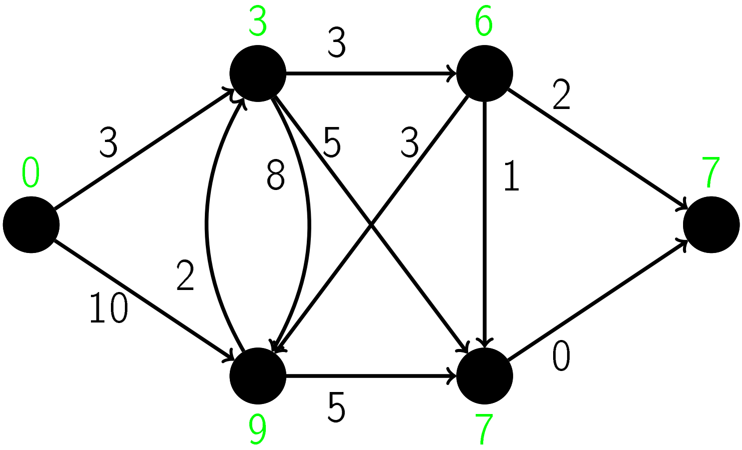


Pseudocode

Dijkstra(G, S)

for all $u \in V$:
  dist[u] $\leftarrow \infty$, prev[u] $\leftarrow$ nil
dist[S] $\leftarrow 0$
$H \leftarrow$ MakeQueue(V) {dist-values as keys} Runs |V| times
while H is not empty:
  $u \leftarrow$ ExtractMin(H)

ExtractMin should take the min value and remove it from Data structure. H is |V| long and we scan it |V| times hence total time is |V|^2

  for all $(u, v) \in E$: Runs |E| times
    if $dist[v] > dist[u] + w(u, v)$:
      $dist[v] \leftarrow dist[u] + w(u, v)$
      $prev[v] \leftarrow u$
      ChangePriority($H, v, dist[v]$)

Meaning update the min distance of node v in array H from which we are going to ExtractMin and process next

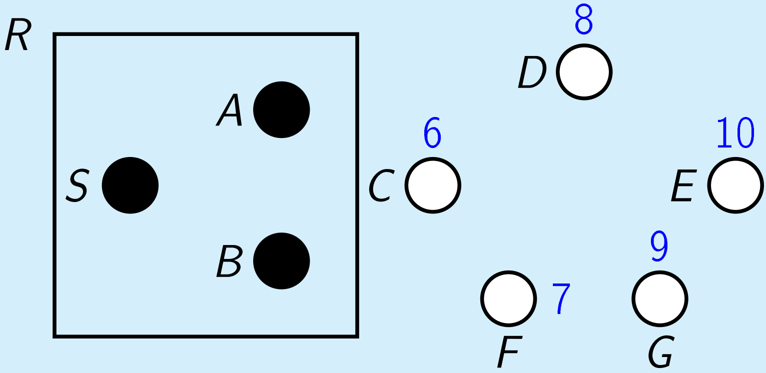
Correct distances

Lemma

When a node u is selected via `ExtractMin`, $\text{dist}[u] = d(S, u)$.

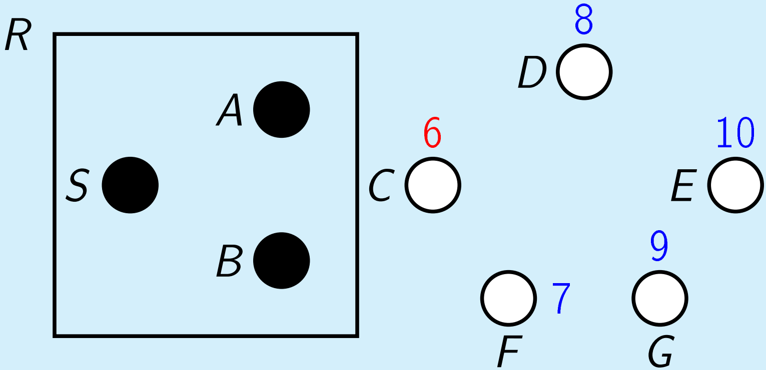
Proof

No need to break the head



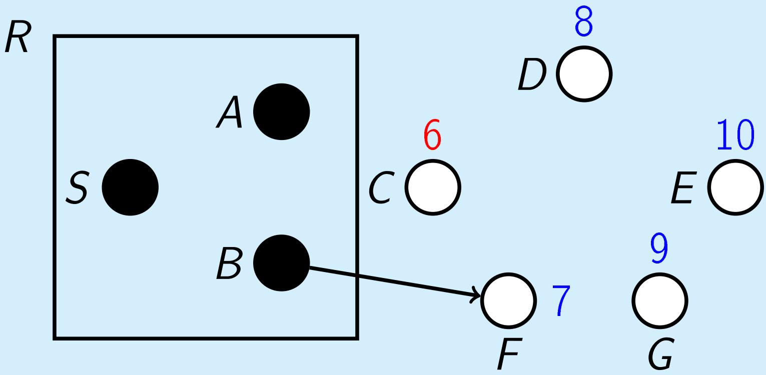
Proof

No need to break the head



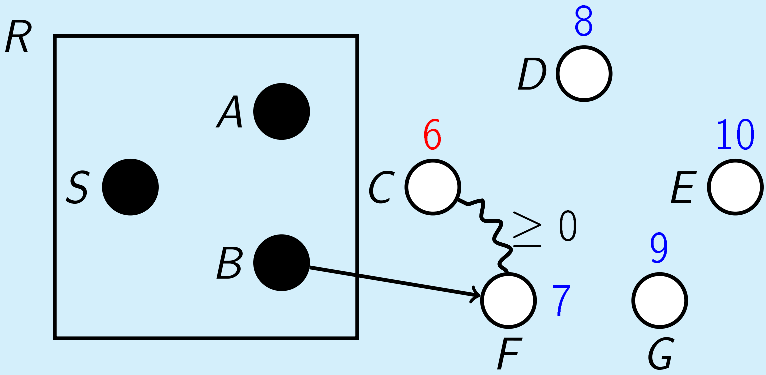
Proof

No need to break the head



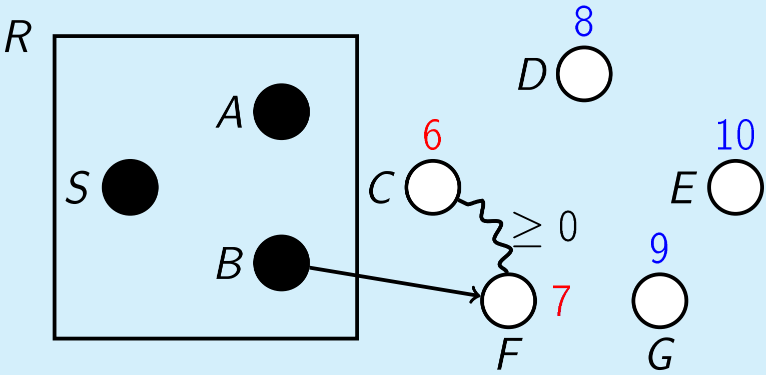
Proof

No need to break the head



Proof

No need to break the head



Running time

Total running time:

$$
\begin{aligned}
T(\texttt{MakeQueue}) &+ |V| \cdot T(\texttt{ExtractMin}) \\
&+ |E| \cdot T(\texttt{ChangePriority})
\end{aligned}
$$

Running time

Total running time:

$$T(\texttt{MakeQueue}) + |V| \cdot T(\texttt{ExtractMin}) + |E| \cdot T(\texttt{ChangePriority})$$

Priority queue implementations:

- array:

$$O(|V| + |V|^2 + |E|) = O(|V|^2)$$

Running time

Total running time:

$$T(\texttt{MakeQueue}) + |V| \cdot T(\texttt{ExtractMin}) + |E| \cdot T(\texttt{ChangePriority})$$

Priority queue implementations:

- array:

$$O(|V| + |V|^2 + |E|) = O(|V|^2)$$

- binary heap:
$O(|V| + |V| \log |V| + |E| \log |V|) = O((|V| + |E|) \log |V|)$

Conclusion

- Can find the minimum time to get from work to home
- Can find the fastest route from work to home
- Works for any graph with non-negative edge weights
- Works in $O(|V|^2)$ or $O((|V| + |E|)\log(|V|))$ depending on the implementation