Coping with NP-completeness: Exact Algorithms

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Exact algorithms or intelligent exhaustive search: finding an optimal solution without going through all candidate solutions

Brute force search algorithm finds an optimal solution enumerating through all possible candidate solutions and selecting the best one

Outline

3-Satisfiability (3-SAT)

Input: A set of clauses, each containing at most three literals (that is, a 3-CNF formula).

Output: Find a satisfying assignment (if exists).

Examples

- The formula

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$$

  is satisfiable: set $x = y = z = 1$ or $x = 1, y = z = 0$.

- The formula

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$$

  is unsatisfiable.

A brute force search algorithm checking satisfiability of a 3-CNF formula F with n variables, goes through all assignments and has running time $O(|F| \cdot 2^n)$.

It goes through all the variables of a formula and checks whether any of them satisfies the input formula . Running time is proportional to length of F times 2^n because for 2^n assignments we have to scan the formula to check whether all of the clauses are satisfied or not

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Goal

Avoid going through all 2^n assignments

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Main Idea of Backtracking

- Construct a solution piece by piece

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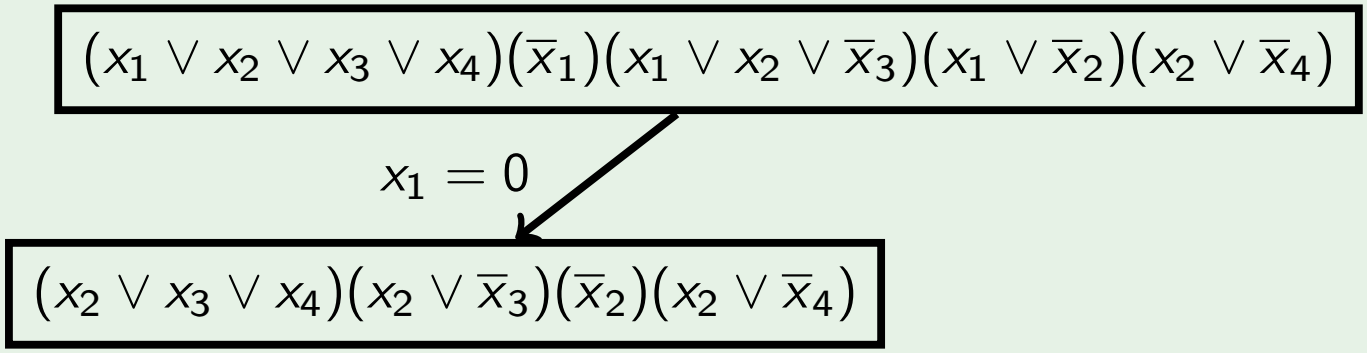
- Construct a solution piece by piece
- Backtrack if the current partial solution cannot be extended to a valid solution

Similar idea to that of playing game to solve a maze

Example

$$(x_1 \vee x_2 \vee x_3 \vee x_4)(\overline{x}_1)(x_1 \vee x_2 \vee \overline{x}_3)(x_1 \vee \overline{x}_2)(x_2 \vee \overline{x}_4)$$

Example


$(x_1 \vee x_2 \vee x_3 \vee x_4)(\overline{x}_1)(x_1 \vee x_2 \vee \overline{x}_3)(x_1 \vee \overline{x}_2)(x_2 \vee \overline{x}_4)$
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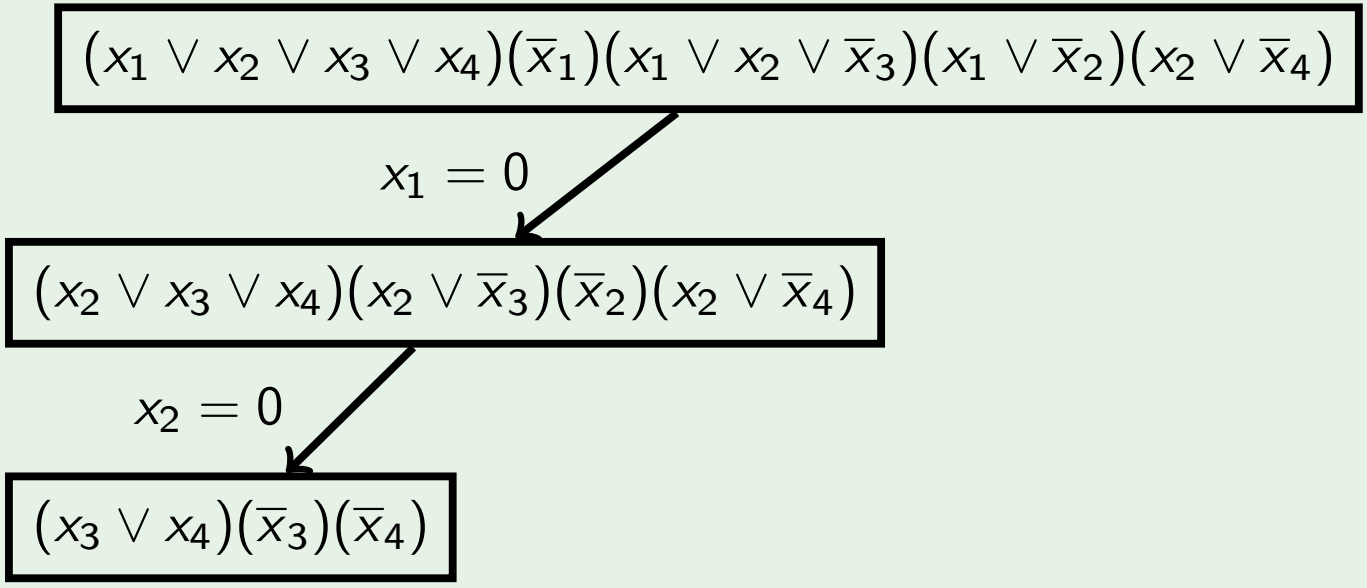
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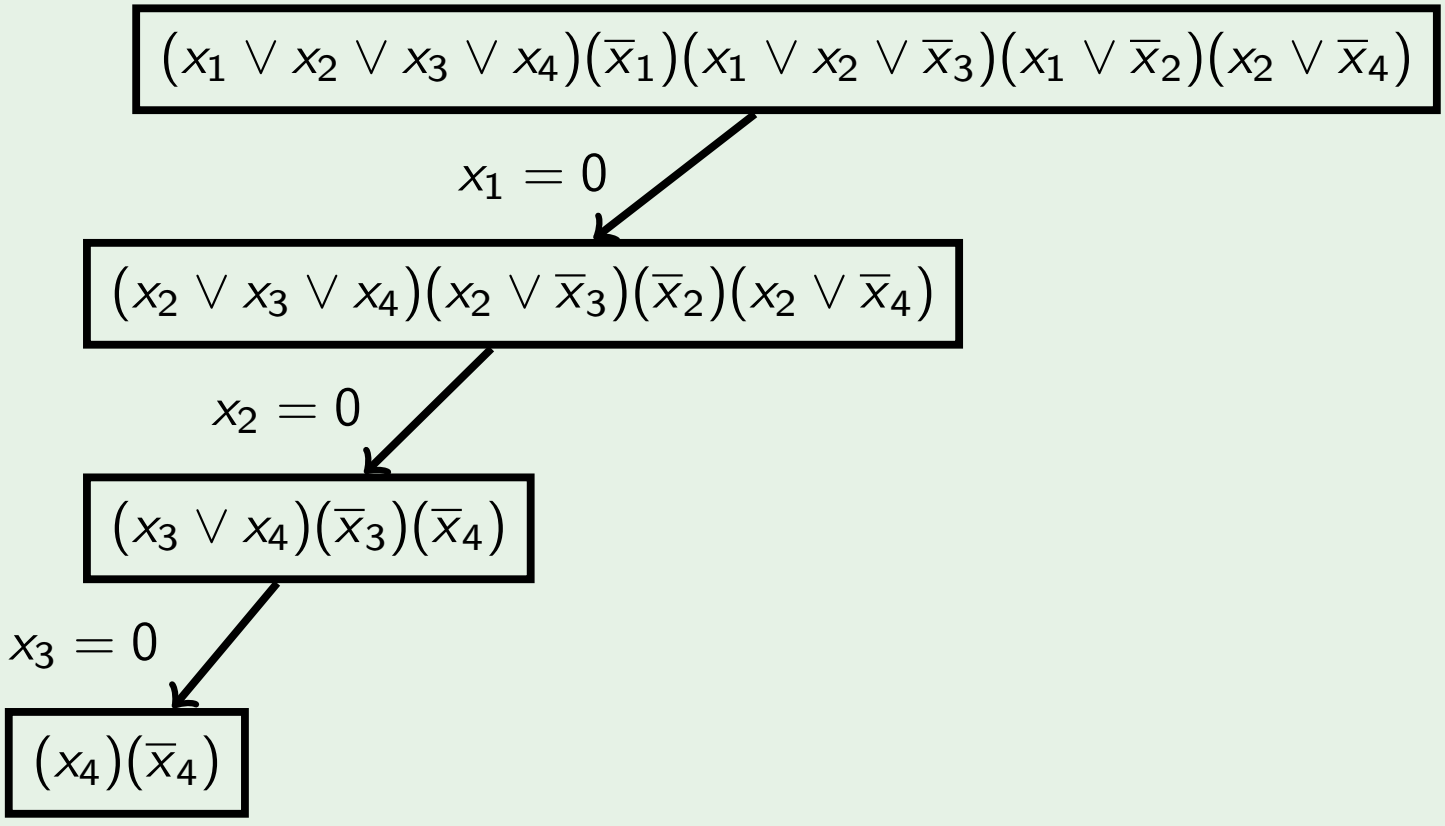
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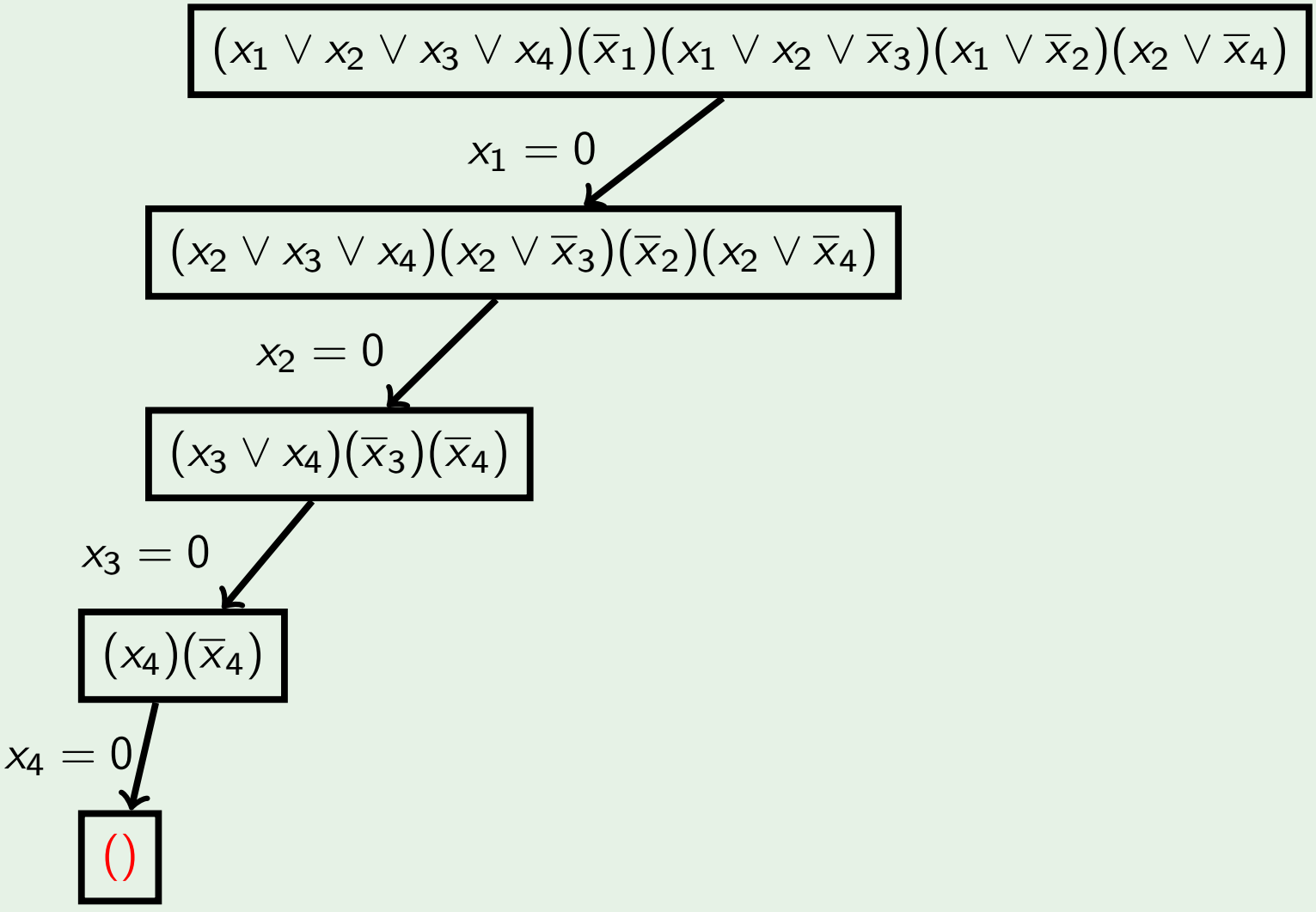
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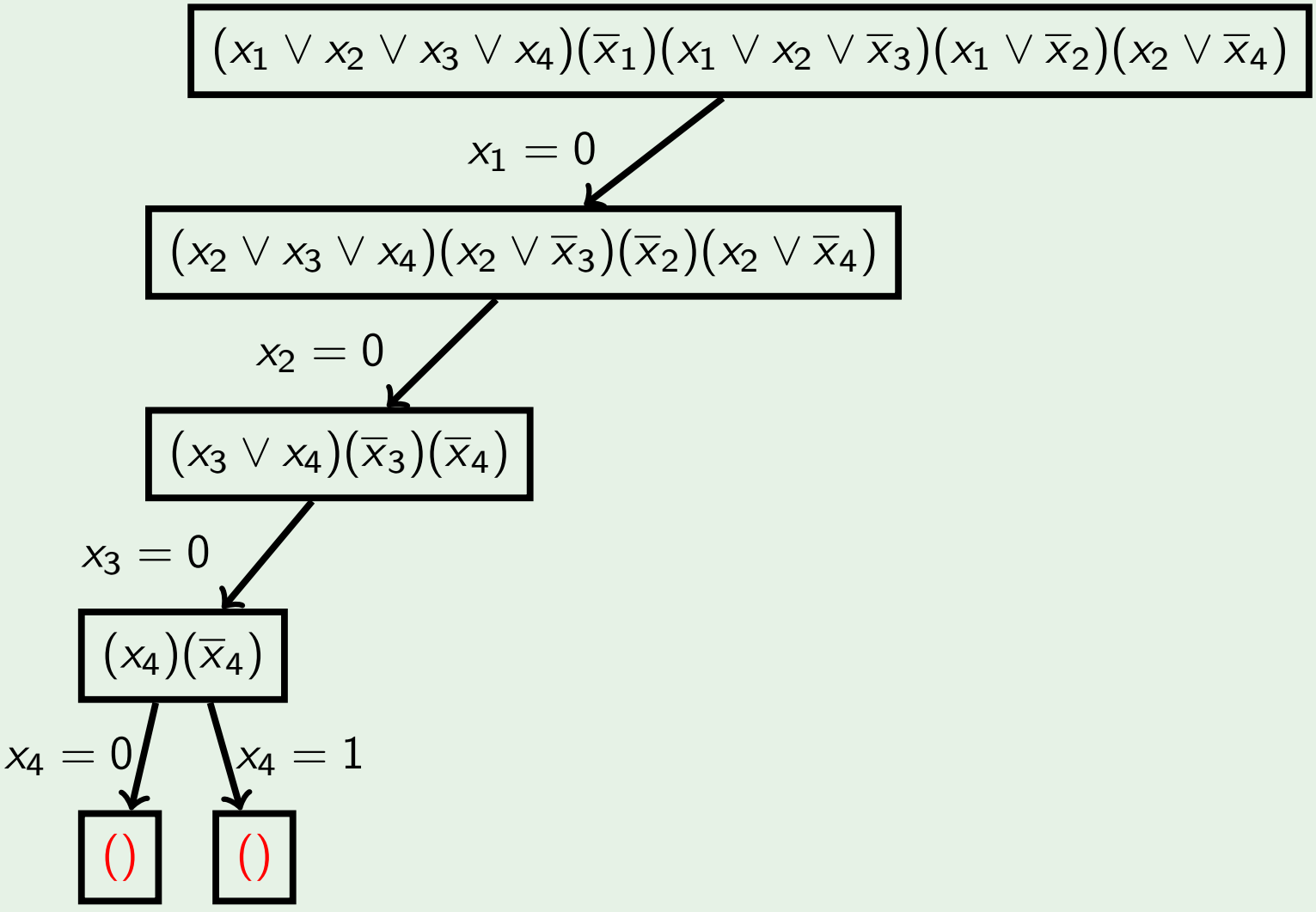
$x_4 = 0$
()

Example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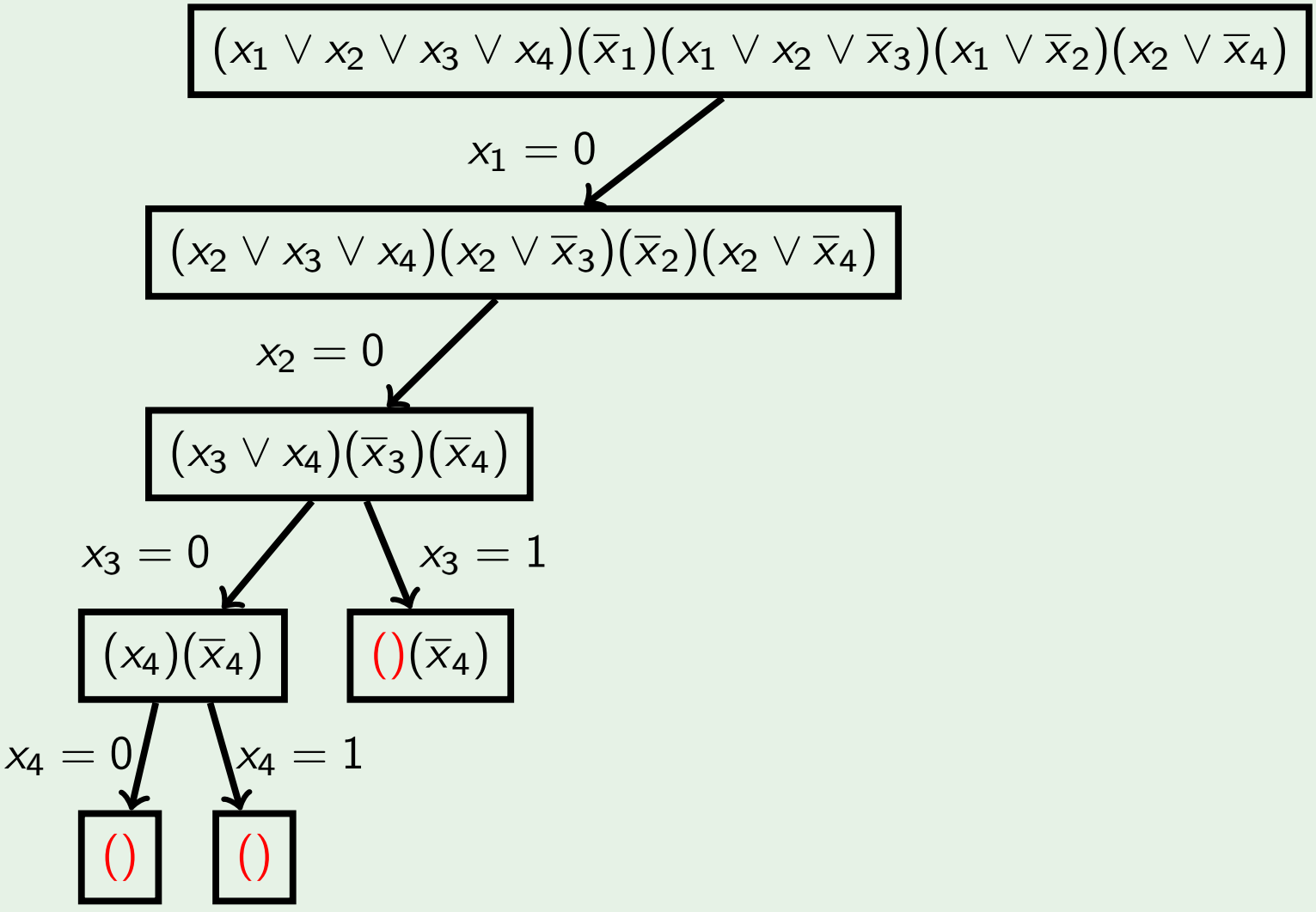
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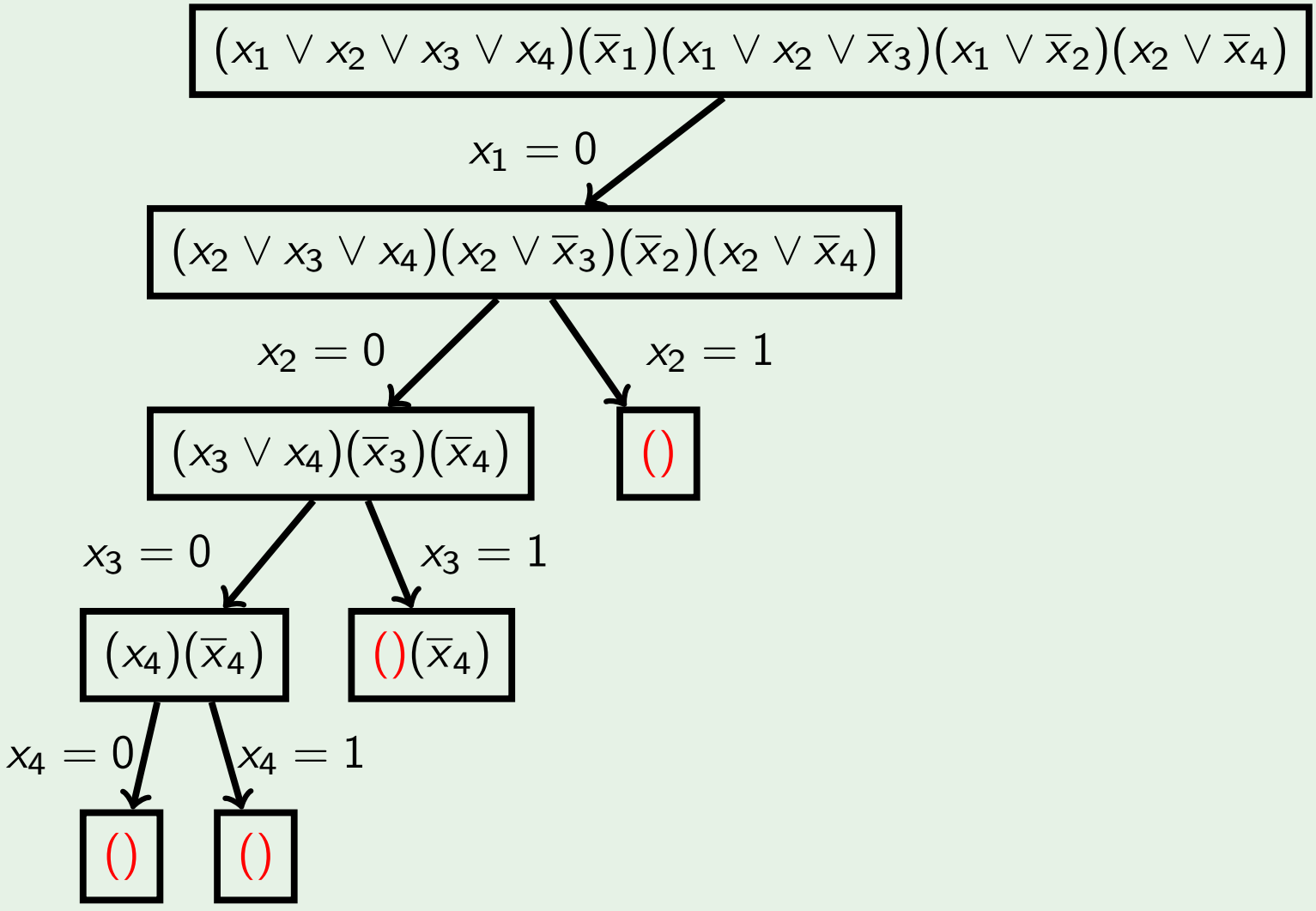
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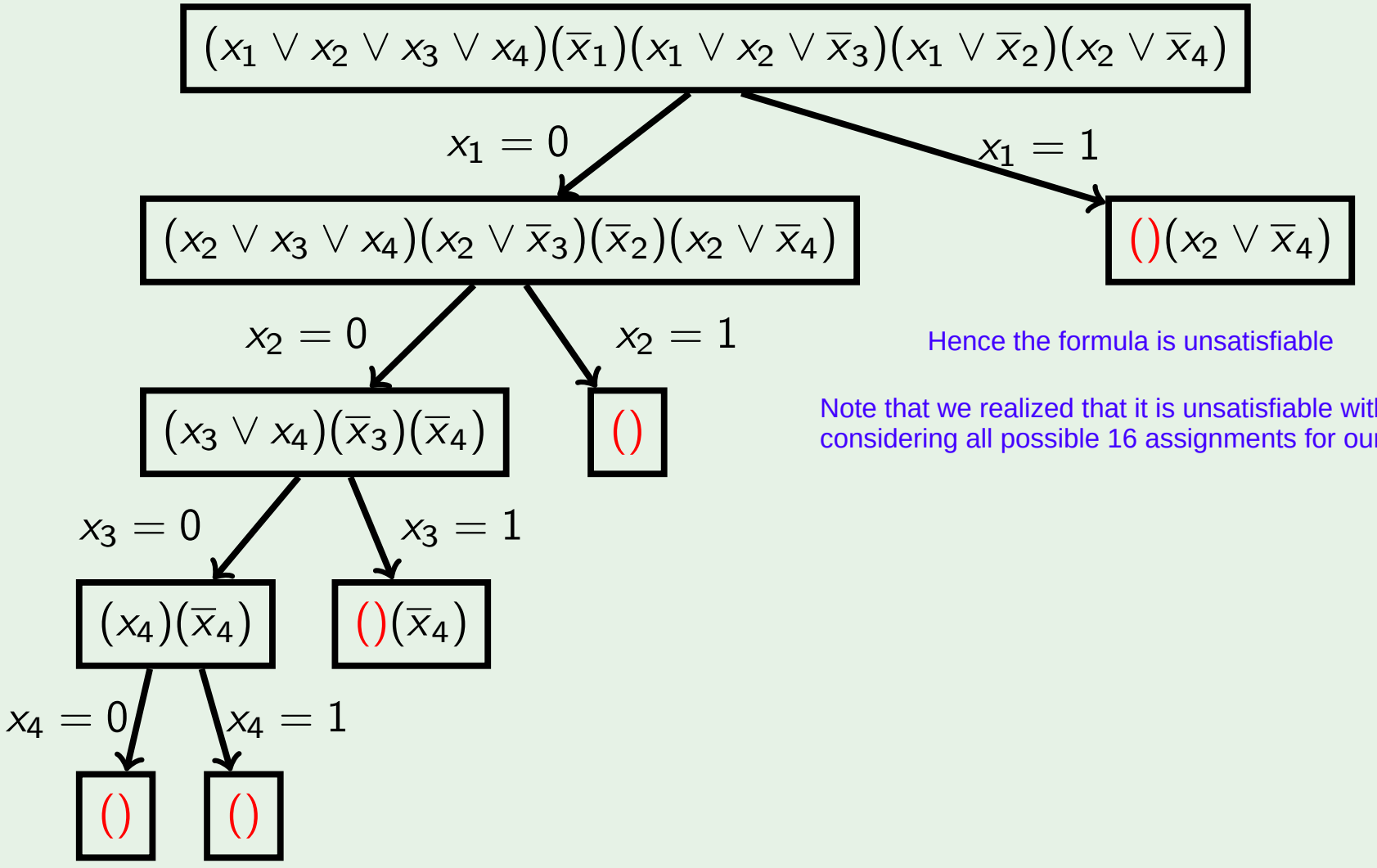

Hence the formula is unsatisfiable

Note that we realized that it is unsatisfiable without considering all possible 16 assignments for our 4 variables

SolveSAT(F)

```
if F has no clauses:
  return “sat”
```

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if F has no clauses:
  return "sat"
if F contains an empty clause:
  return "unsat"
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Empty clause is a clause which cannot be satisfied meaning all the literals has already been falsified AND THE ANSWER TURNS to 0

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```

Assigning 0 to x. This means all x! are satisfied. After this step we remove all occurrences of x! from the formula. Also we remove all occurrences of x from all other clauses because x is falsified and now we are left with SAT formula without x

SolveSAT(F)

if F has no clauses:
  return ‘‘sat’’
if F contains an empty clause:
  return ‘‘unsat’’
$x \leftarrow$ unassigned variable of F
if SolveSAT($F[x \leftarrow 0]$) = ‘‘sat’’:
  return ‘‘sat’’
if SolveSAT($F[x \leftarrow 1]$) = ‘‘sat’’:
  return ‘‘sat’’

Otherwise we backtrack and assign x = 1

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If the CNF is satisfied with x=1 or x=0 then the next formula will have x removed from it as shown in the previous slide

- Thus, instead of considering all 2^n branches of the recursion tree, we track carefully each branch

Question

Assume that a CNF formula with n variables has k satisfying assignments. What is the maximum possible number of leaves in the recursion tree of a backtracking algorithm solving satisfiability of this formula?

- ○ $2^n - k$
- ◉ $2^n - k + 1$
- ○ 2^k
- ○ k

✓ **Correct**

That's right! Each leaf of the recursion tree corresponds to a partial assignment that either satisfies the formula or falsifies it. The algorithm stops if it finds a satisfying assignment. Hence, in the worst case it will first go through all $2^n - k$ falsifying assignments, then will find a satisfying assignment and halt.

- Thus, instead of considering all 2^n branches of the recursion tree, we track carefully each branch
- When we realize that a branch is dead (cannot be extended to a solution), we immediately cut it

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We didn't prove the upper bound on the running time of this algorithm. As expected it must be exponential because if it is polynomial then it is no longer a NP problem

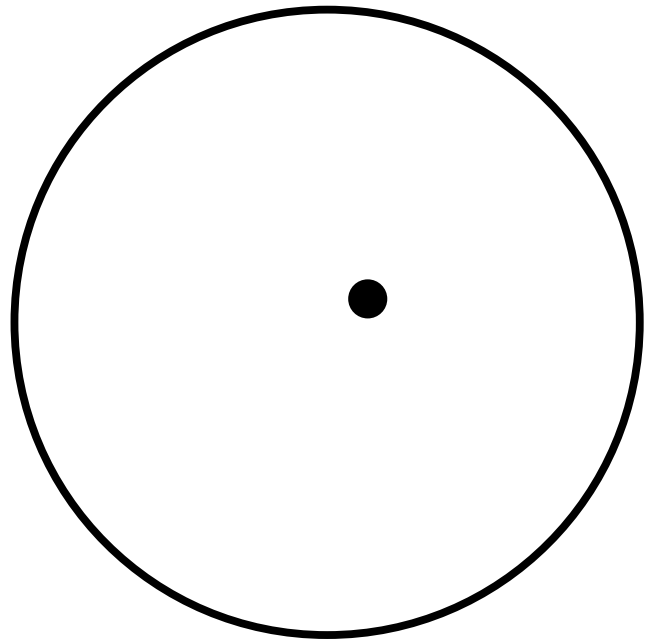
- Backtracking is used in many state-of-the-art SAT-solvers
- SAT-solvers use tricky heuristics to choose a variable to branch on and to simplify a formula before branching
- Another commonly used technique is local search — will consider it in the next part

Note that the upper bound on the running time of the algorithm might be exponential

Outline

Main Idea of Local Search

- Start with a candidate solution



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- Start with a candidate solution
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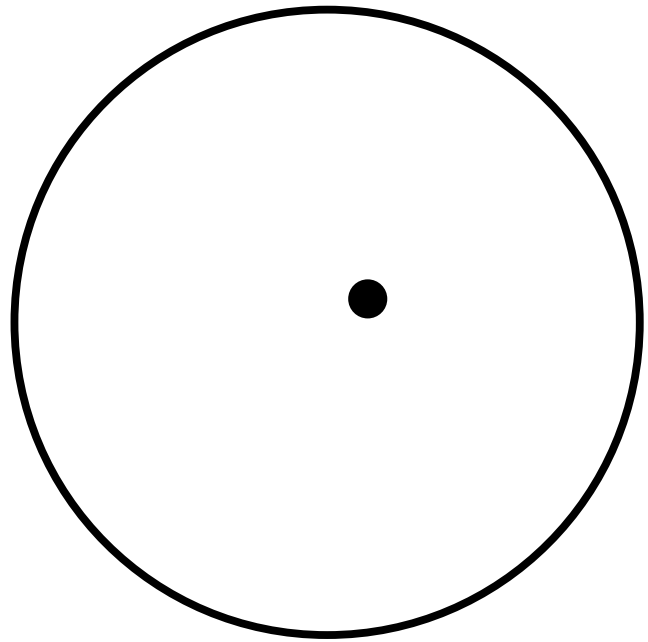

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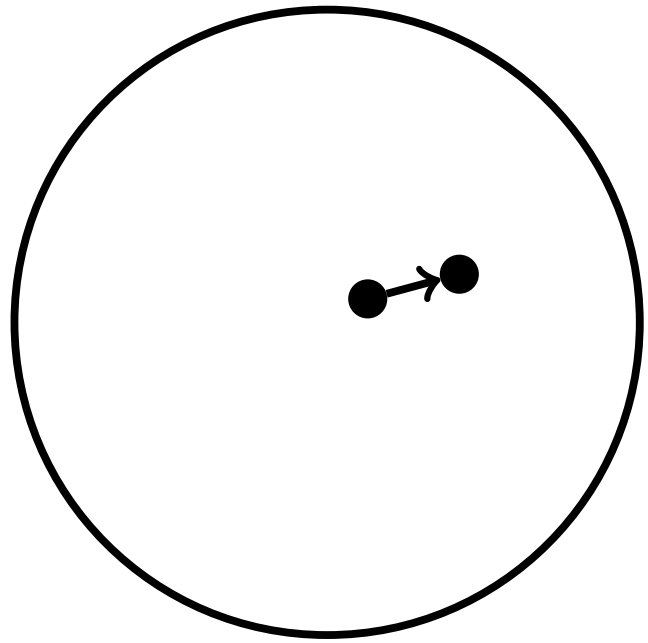

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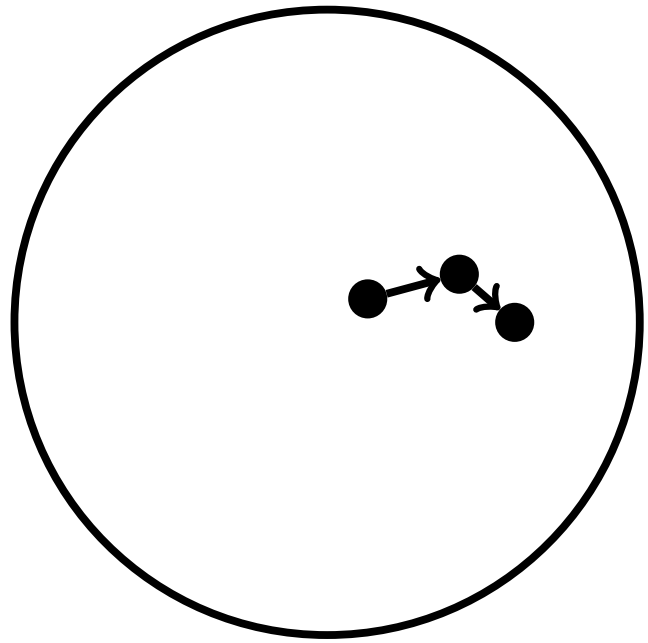

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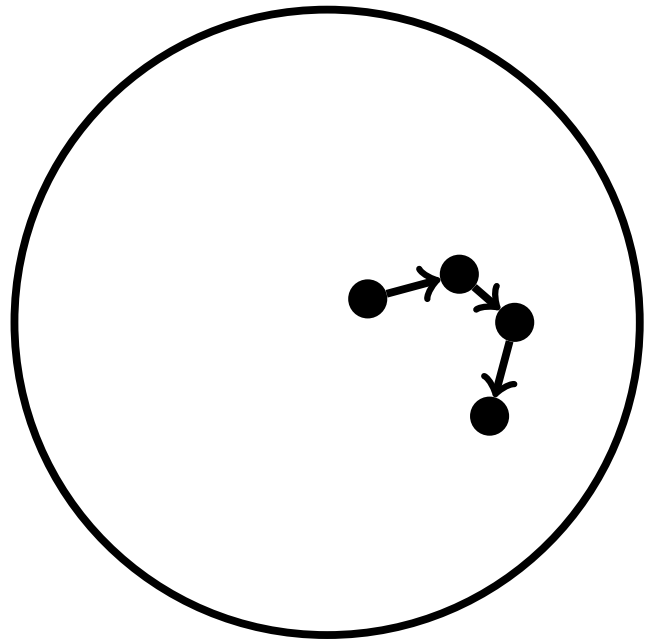

- Let F be a 3-CNF formula over variables $x_1, x_2, \ldots, x_n$

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- A candidate solution is a truth assignment to these variables, that is, a vector from $\{0,1\}^n$

Definition

Hamming distance (or just distance) between two assignments $\alpha, \beta \in \{0,1\}^n$ is the number of bits where they differ:
$\text{dist}(\alpha, \beta) = |\{i : \alpha_i \neq \beta_i\}|$.

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Hamming ball with center $\alpha \in \{0,1\}^n$ and radius r, denoted by $\mathcal{H}(\alpha, r)$, is the set of all truth assignments from $\{0,1\}^n$ at distance at most r from α.

Example

- $\mathcal{H}(1011, 0) = \{1011\}$

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- $\mathcal{H}(1011, 2) =$
  $\{1011, 0011, 1111, 1001, 1010,$
  $0111, 0001, 0010, 1101, 1110, 1000\}$

Searching a Ball for a Solution

Lemma

We already know that it contains a satisfying assignment

Assume that $\mathcal{H}(\alpha, r)$ contains a satisfying assignment β for F. We can then find a (possibly different) satisfying assignment in time $O(|F| \cdot 3^r)$.

The satisfying assignment that we are going to find might coincide with B or might be a completely different one but if we know that it contains satisfying assignment then we can find it in this running time

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take unsatisfied clause from the CNF Formula

- α assigns $x_i = 0, x_j = 1, x_k = 0$
- Let $\alpha^i, \alpha^j, \alpha^k$ be assignments resulting from α by flipping the i-th, j-th, k-th bit, respectively

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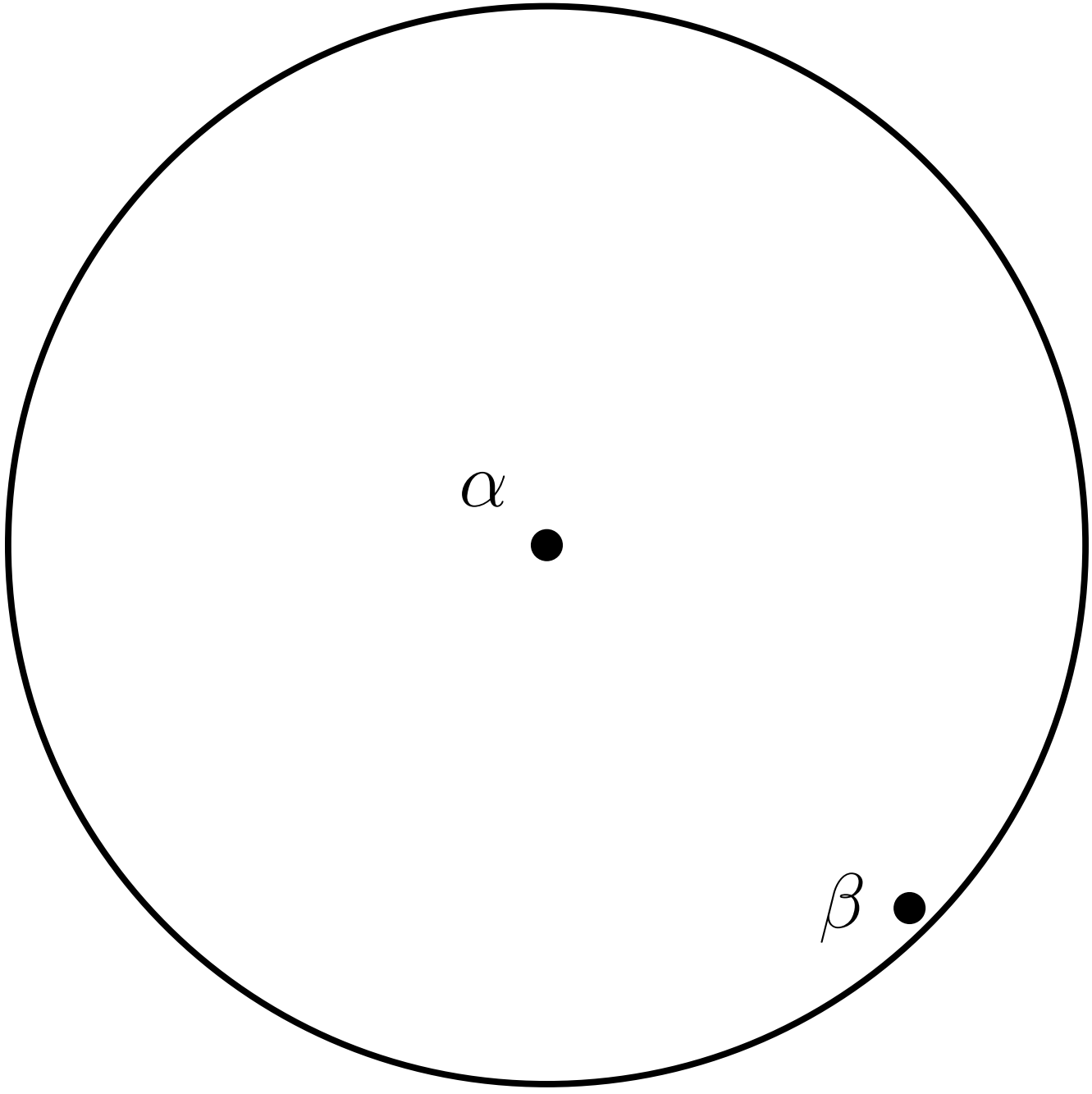
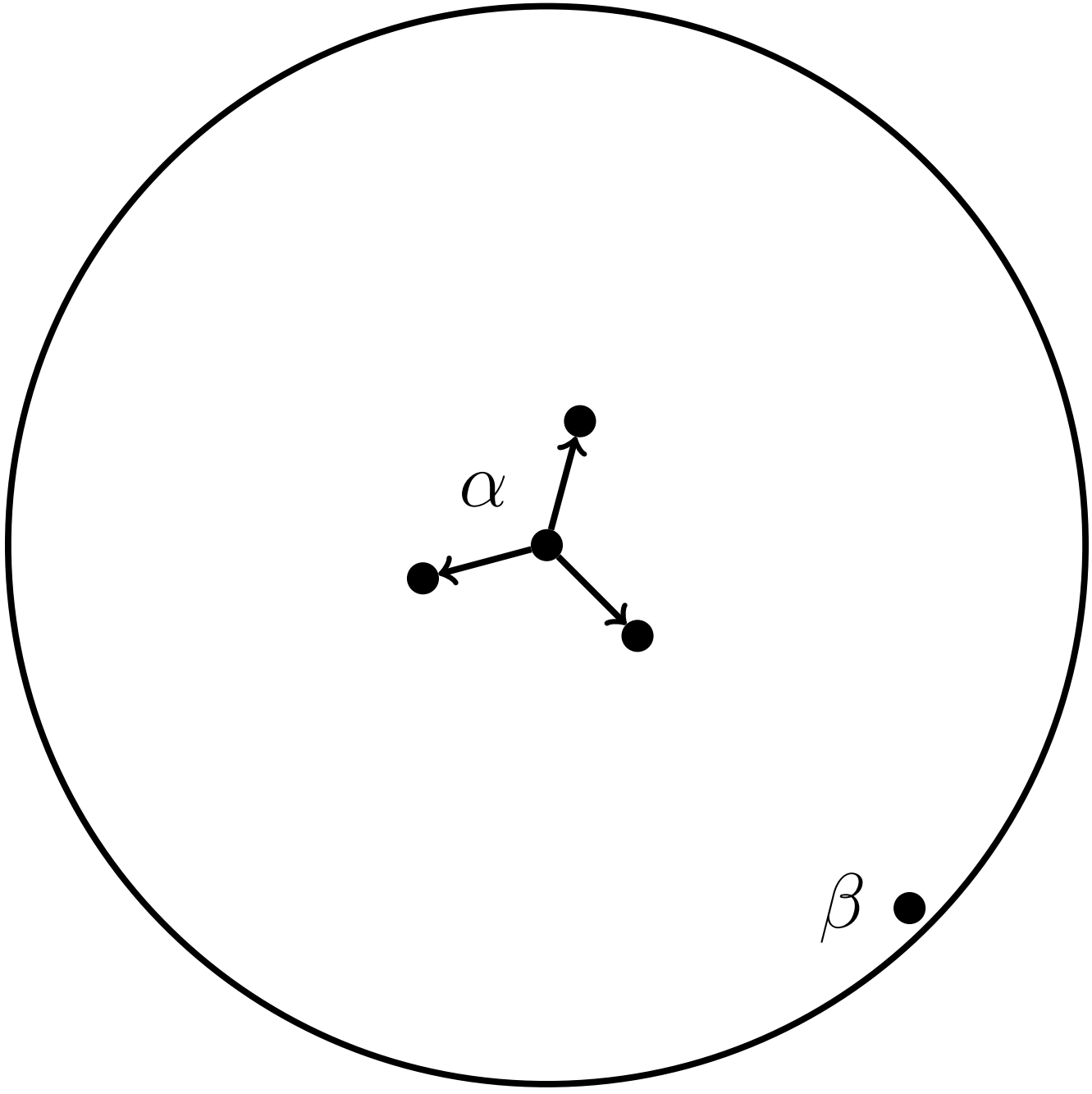
- Crucial observation: at least one of them is closer to β than α
- Hence there are at most 3^r recursive calls □

because there are 3 variables and the difference between alpha and beta is atmost r


α
β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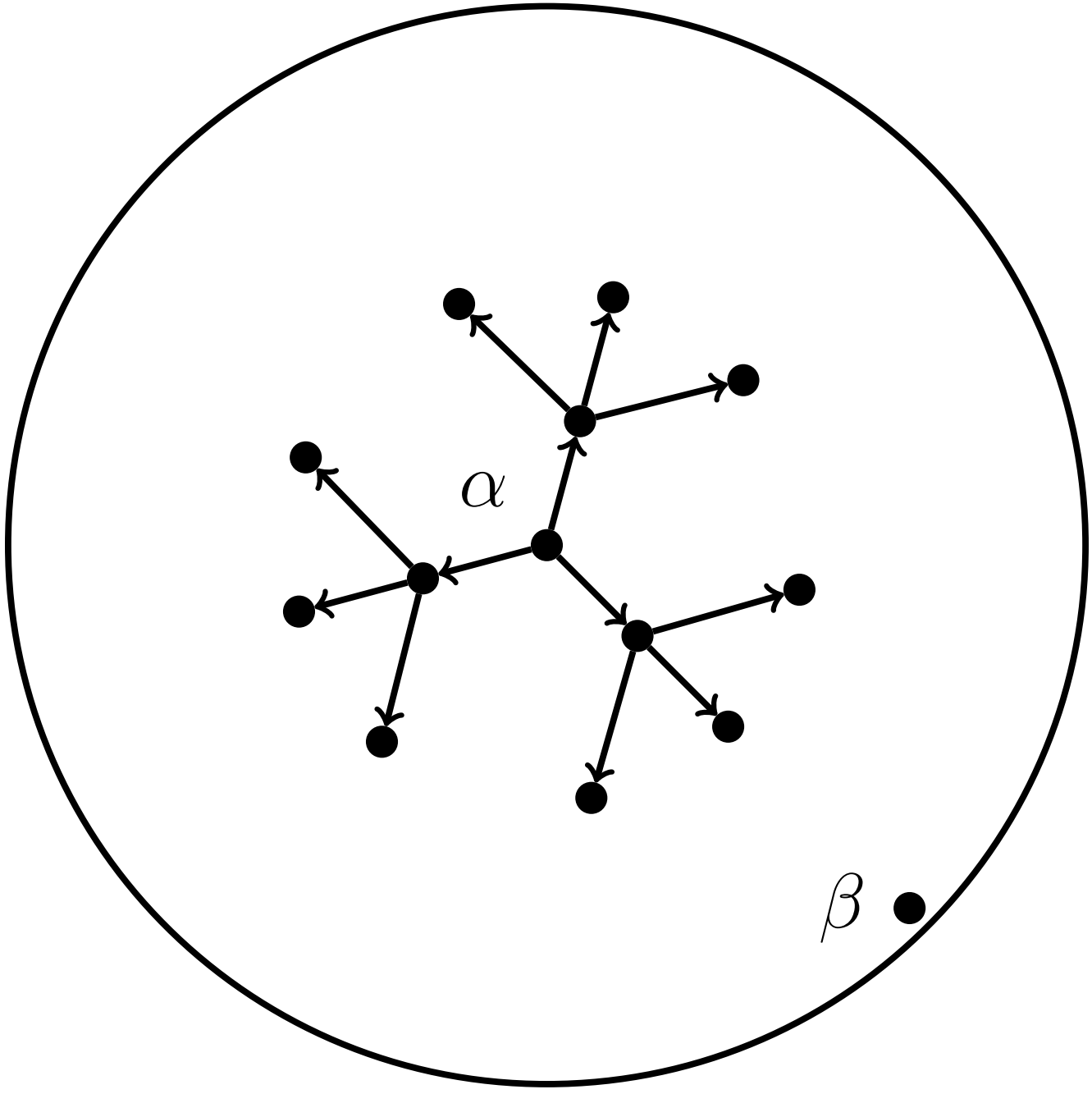
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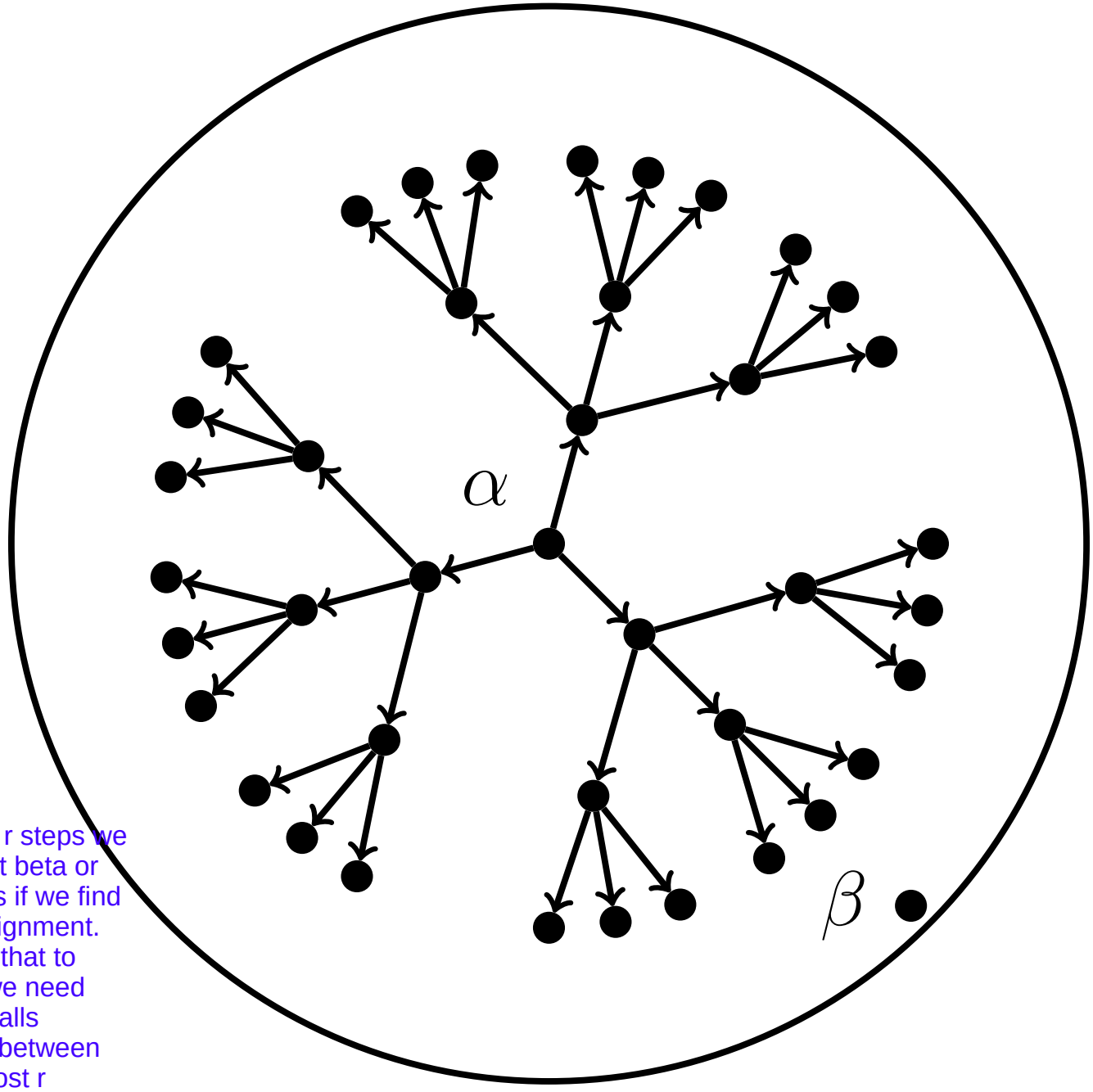

Eventually if we make r steps we will find an assignment beta or we will stop before this if we find another satisfying assignment. Crucial observation is that to eventually find beta, we need atmost 3^r recursive calls because the distance between alpha and beta is atmost r

CheckBall(F, α, r)

```
if α satisfies F:
  return α
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take any unsatisfied clause from the CNF Formula

$x_i, x_j, x_k \leftarrow$ variables of unsatisfied clause

$\alpha^i, \alpha^j, \alpha^k \leftarrow \alpha$ with bits i, j, k flipped

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CheckBall$(F, \alpha^i, r-1)$ If Satisfying assignment is found with alpha i return immediately and so on

CheckBall$(F, \alpha^j, r-1)$

CheckBall$(F, \alpha^k, r-1)$

CheckBall(F, α, r)

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if α satisfies F:
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x_i, x_j, x_k ← variables of unsatisfied clause
α^i, α^j, α^k ← α with bits i, j, k flipped
CheckBall(F, α^i, r − 1)
CheckBall(F, α^j, r − 1)
CheckBall(F, α^k, r − 1)
if a satisfying assignment is found:
  return it
else:
  return "not found"
```

- Assume that F has a satisfying assignment β

Now we need to understand HOW TO USE this procedure which checks whether a ball contains a satisfying assignment

- Assume that F has a satisfying assignment β
- If it has more 1's than 0's then it has distance at most $n/2$ from all-1's assignment

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Make only these two calls to the original CheckBall Algorithm

- Thus, it suffices to make two calls: `CheckBall`$(F, 11\ldots1, n/2)$ and `CheckBall`$(F, 00\ldots0, n/2)$

Thus total running time will be 3^r + 3^r = 2(3^r) =2(3^(n/2)) ~= 3^(n/2)

Running Time

- The running time of the resulting algorithm is $O(|F| \cdot 3^{n/2}) \approx O(|F| \cdot 1.733^n)$

Question

The branch-and-bound algorithm from the previous video can be used for general CNF formulas, not only for 3-CNF formulas. Is it true that the local search algorithm considered in this video also finds a satisfying assignment for any CNF formula with n variables in time roughly $3^{n/2}$?

- ○ Yes, this is true.
- ◉ No, this is not true.

✓ **Correct**
That's right! It is essential for the algorithm that each clause has at most three literals. This ensures that the recursion tree has branching factor 3. This, in turn, implies a $3^{n/2}$ upper bound on its running time.

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- The running time of the resulting algorithm is
  $O(|F| \cdot 3^{n/2}) \approx O(|F| \cdot 1.733^n)$
- On one hand, this is still exponential
- On the other hand, it is exponentially faster than a brute force search algorithm that goes through all 2^n truth assignments!

Outline

Traveling salesman problem (TSP)

Input: A complete graph with weights on edges and a budget b.

Output: A cycle that visits each vertex exactly once and has total weight at most b.

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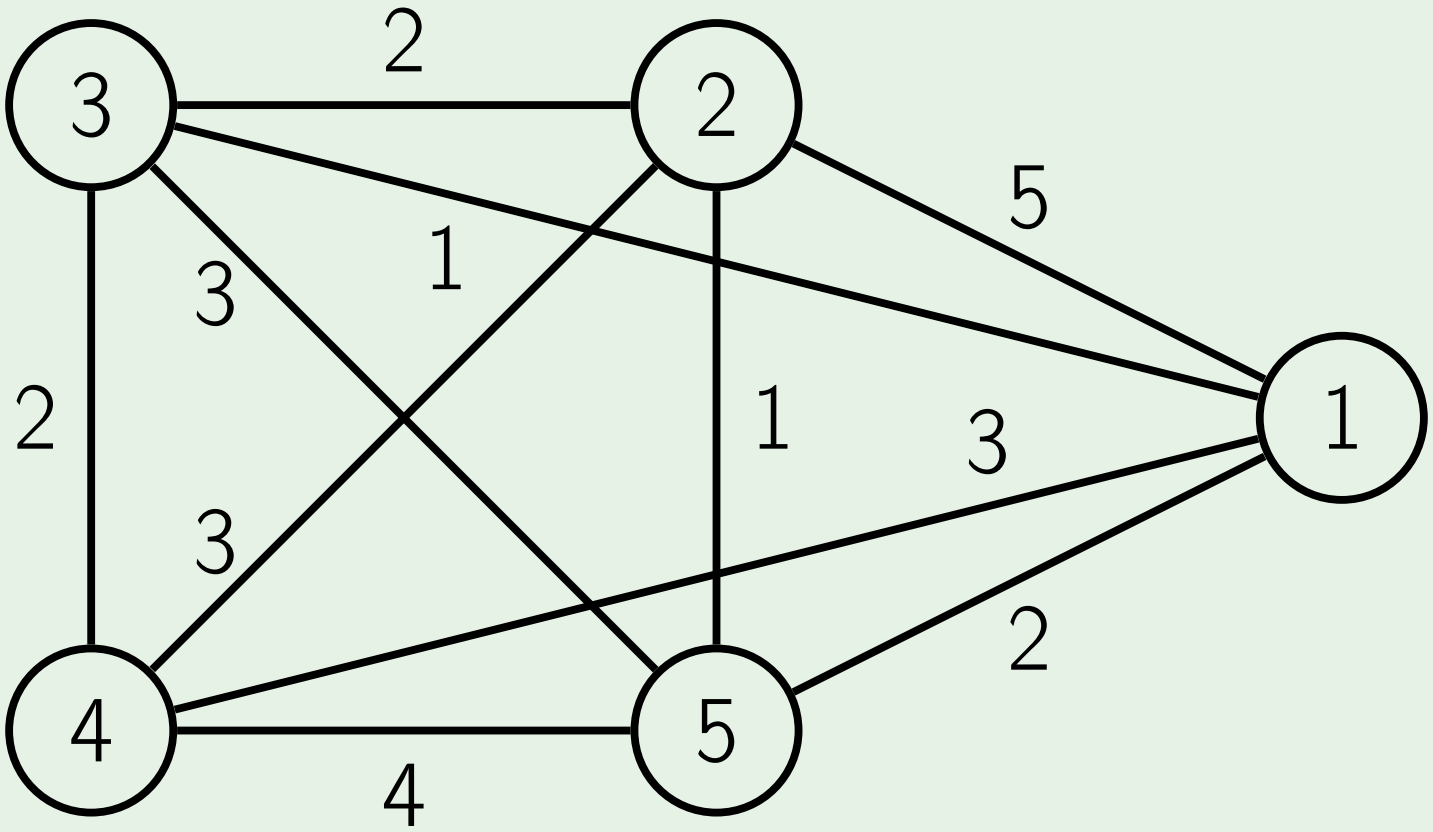
It will be convenient to assume that vertices are integers from 1 to n and that the salesman starts his trip in (and also returns back to) vertex 1.

Example



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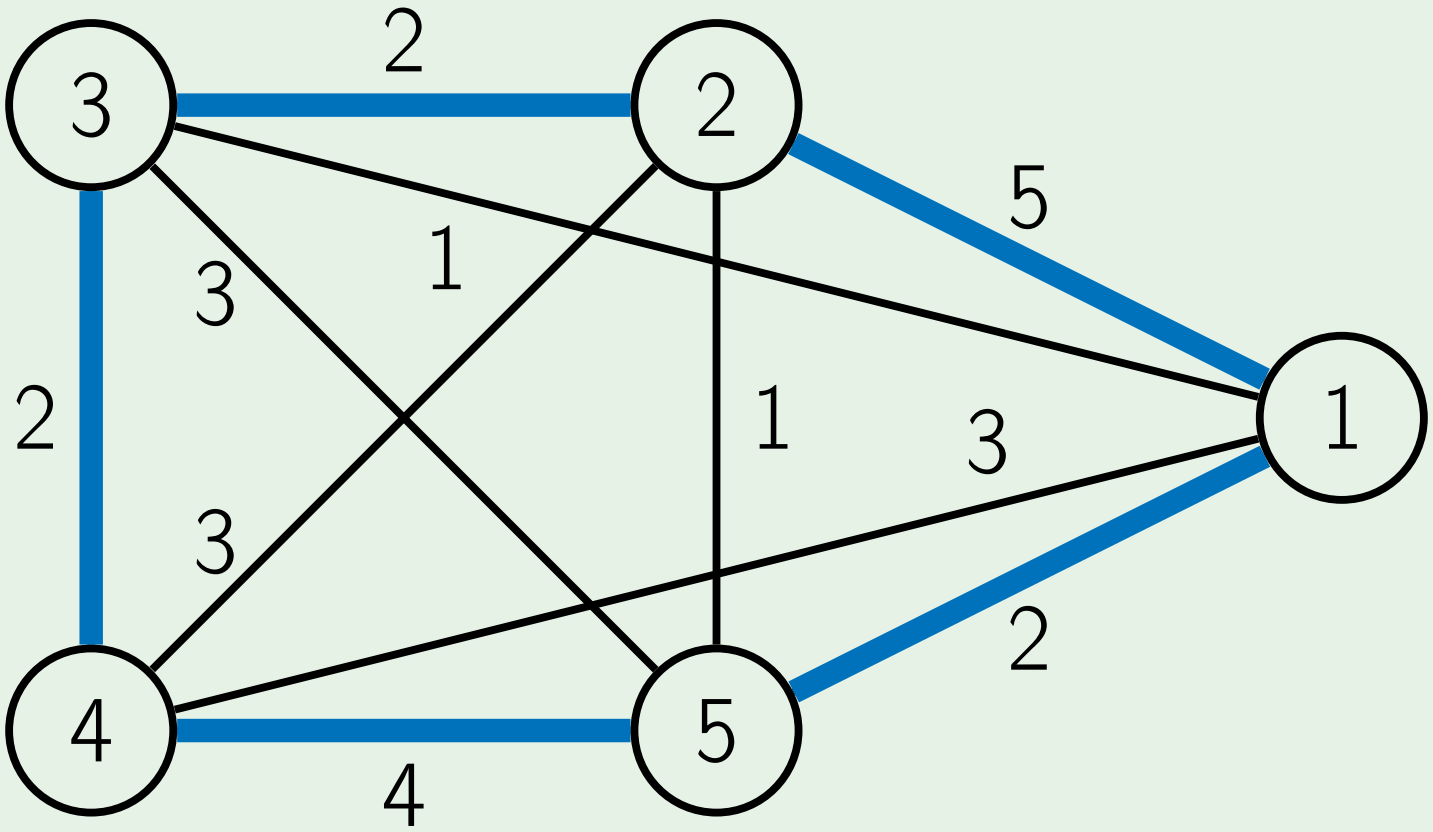
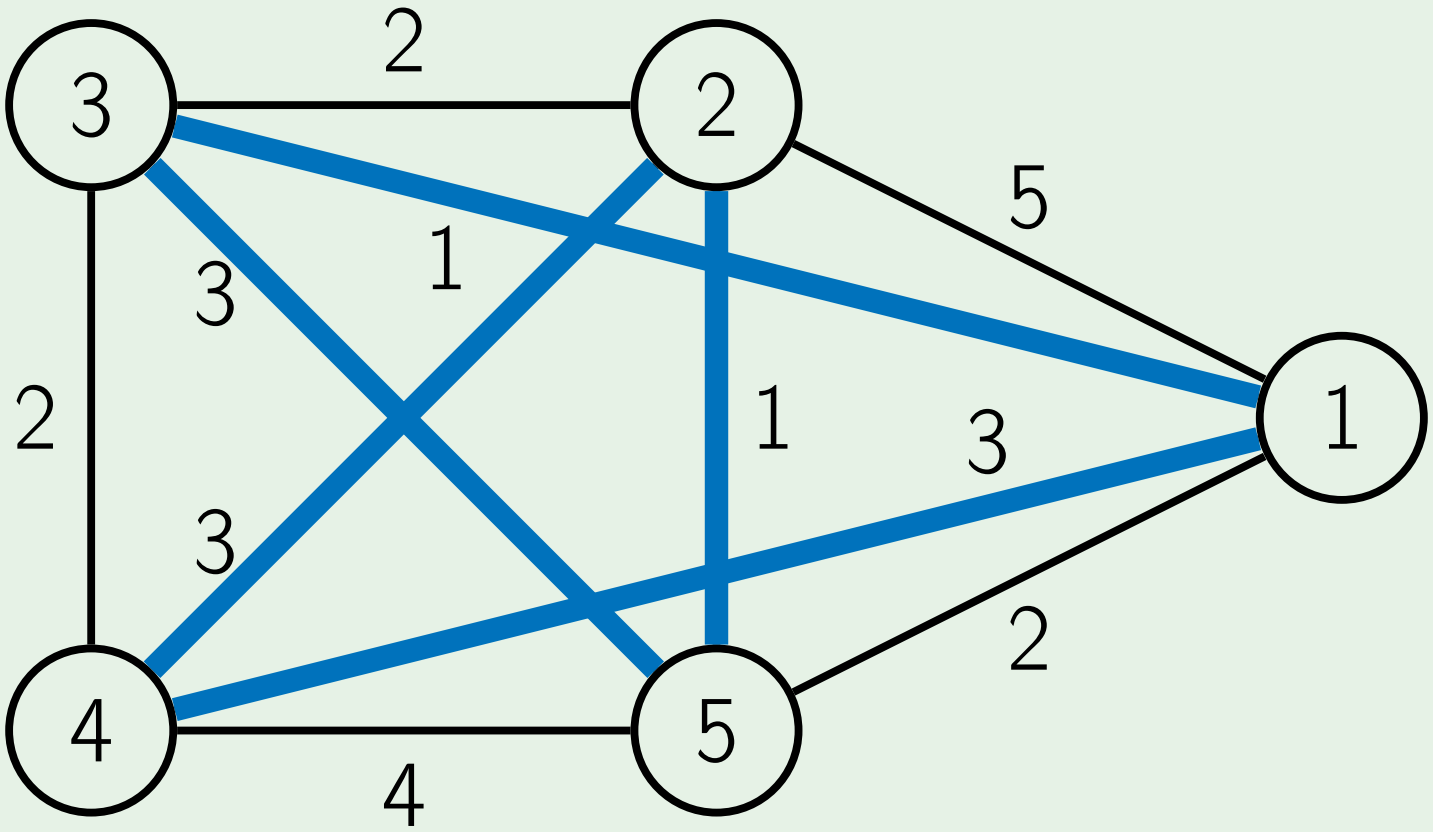


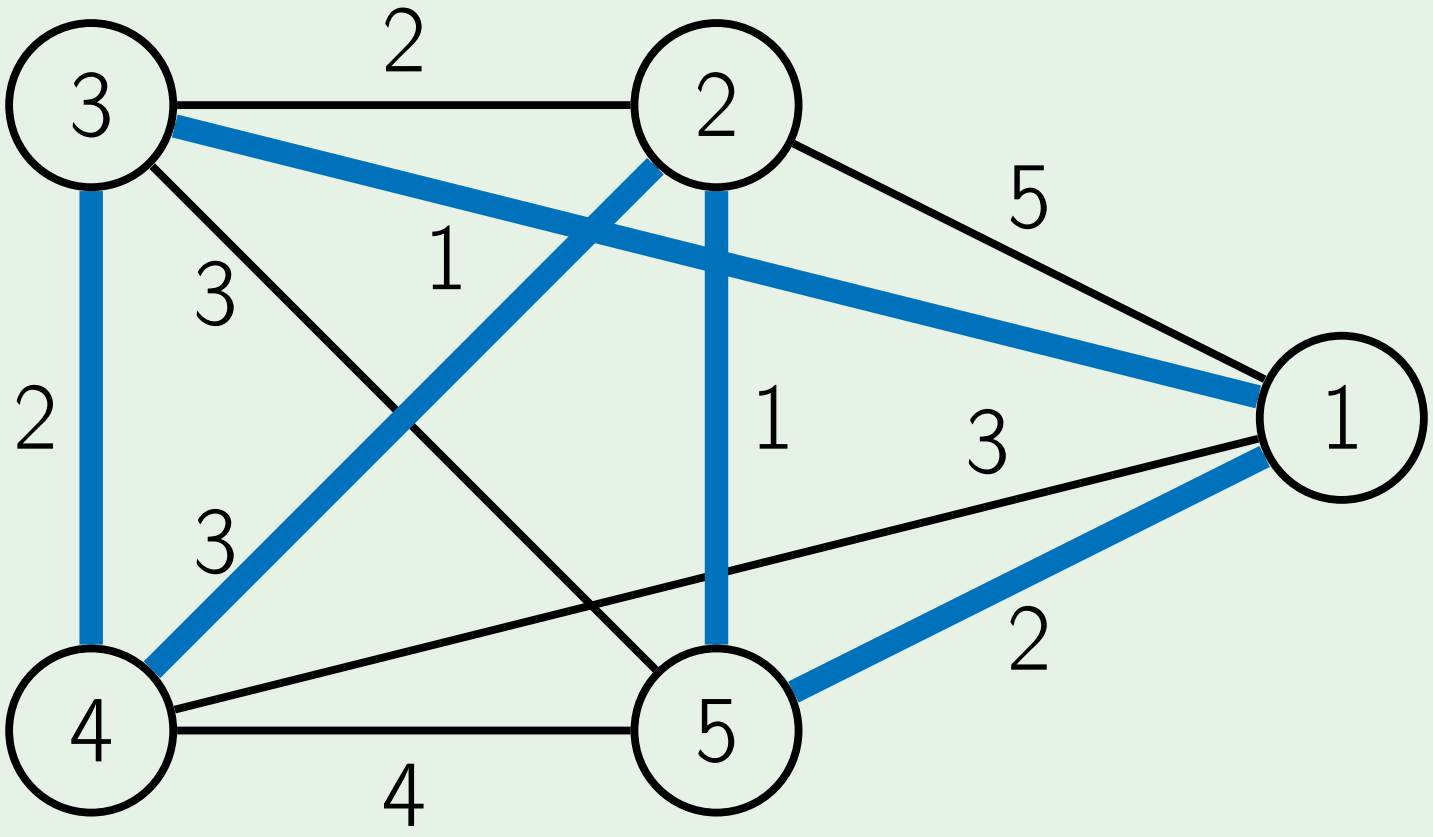
length: 15

Example



length: 11

Example



length: 9

Brute Force Solution

A naive algorithm just checks all possible $(n-1)!$ cycles.

because initially we start with vertex 1 and then we have (n-1)! choices. Of these (n-1)! vertices we have (n-2)! choices and so on.

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This part

- Use dynamic programming to solve TSP in $O(n^2 \cdot 2^n)$

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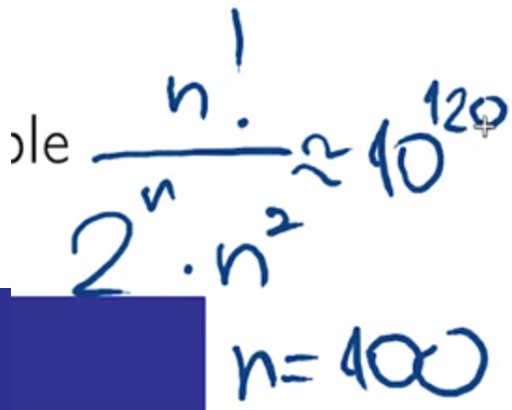

This part

- Use dynamic programming to solve TSP in $O(n^2 \cdot 2^n)$
- The running time is exponential, but is much better than $(n-1)!$.

Outline

Dynamic Programming

- We are going to use dynamic programming: instead of solving one problem we will solve a collection of (overlapping) subproblems

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- A subproblem refers to a partial solution
- A reasonable partial solution in case of TSP is the initial part of a cycle
- To continue building a cycle, we need to know the last vertex as well as the set of already visited vertices

Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ containing the vertex 1 and a vertex $i \in S$, let $C(S, i)$ be the length of the shortest path that starts at 1, ends at i and visits all vertices from S exactly once

Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ containing the vertex 1 and a vertex $i \in S$, let $C(S, i)$ be the length of the shortest path that starts at 1, ends at i and visits all vertices from S exactly once
- $C(\{1\}, 1) = 0$ and $C(S, 1) = +\infty$ when $|S| > 1$

For any other vertices in S the path cannot end in vertex 1 hence the distance is infinite

Recurrence Relation

- Consider the second-to-last vertex j on the required shortest path from 1 to i visiting all vertices from S

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- The subpath from 1 to j is the shortest one visiting all vertices from $S - \{i\}$ exactly once
- Hence
  $C(S, i) = \min\{C(S - \{i\}, j) + d_{ji}\}$,
  where the minimum is over all $j \in S$ such that $j \neq i$

Order of Subproblems

- Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in an order that guarantees that when computing the value of $C(S, i)$, the values of $C(S - \{i\}, j)$ have already been computed

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  Sp if we just go through all sub sets in order of increase in size then we are guaranteed to have the values of C(S-{i},j) when computing the value of C(S,i)

- For example, we can process subsets in order of increasing size

$\mathrm{TSP}(G)$

$C(\{1\}, 1) \leftarrow 0$

TSP(G)

$C(\{1\}, 1) \leftarrow 0$

```
for s from 2 to n:
  for all 1 ∈ S ⊆ {1, ..., n} of size s:
    C(S, 1) ← +∞
```

TSP(G)

$C(\{1\}, 1) \leftarrow 0$

```
for s from 2 to n:
  for all 1 ∈ S ⊆ {1,...,n} of size s:
    C(S,1) ← +∞
    for all i ∈ S, i ≠ 1:
      for all j ∈ S, j ≠ i:
        C(S,i) ← min{C(S,i), C(S−{i},j)+d_ji}
```

TSP(G)

$C(\{1\}, 1) \leftarrow 0$
for s from 2 to n:
  for all $1 \in S \subseteq \{1, \ldots, n\}$ of size s:
    $C(S, 1) \leftarrow +\infty$
    for all $i \in S,\ i \neq 1$:
      for all $j \in S,\ j \neq i$:
        $C(S, i) \leftarrow \min\{C(S, i), C(S - \{i\}, j) + d_{ji}\}$
return $\min_i \{C(\{1, \ldots, n\}, i) + d_{i,1}\}$

Question

What is the space requirement of this algorithm?

- ○ $\Theta(n)$
- ○ $\Theta(n^2)$
- ○ $\Theta(2^n)$
- ◉ $\Theta(n \cdot 2^n)$
- ○ $\Theta(n^2 \cdot 2^n)$

✓ **Correct**
That's right! The dynamic programming table has $n \cdot 2^n$ cells.

Implementation Remark

- How to iterate through all subsets of $\{1, \ldots, n\}$?

Implementation Remark

- How to iterate through all subsets of $\{1, \ldots, n\}$?
- There is a natural one-to-one correspondence between integers in the range from 0 and $2^n - 1$ and subsets of $\{0, \ldots, n-1\}$:

$$k \leftrightarrow \{i : i\text{-th bit of } k \text{ is } 1\}$$

Example

k	bin(k)	$\{i : i$-th bit of k is 1$\}$
0	000	$\emptyset$
1	001	$\{0\}$
2	010	$\{1\}$
3	011	$\{0,1\}$
4	100	$\{2\}$
5	101	$\{0,2\}$
6	110	$\{1,2\}$
7	111	$\{0,1,2\}$

- If k corresponds to S, how to find out the integer corresponding to $S - \{j\}$ (for $j \in S$)?

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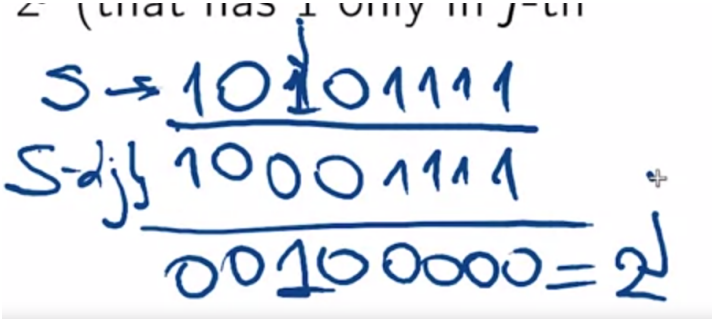
- For this, in turn, we compute a bitwise XOR of k and 2^j (that has 1 only in j-th position)
- In C/C++, Java, Python: `k ^ (1 << j)`

This computes 2^j



k is 1010111 in the example shown

Outline

- The branch-and-bound technique can be viewed as a generalization of backtracking for optimization problems

Backtracking is usually used for solving decision problems while branch and bound is used to solve optimization problems

- The branch-and-bound technique can be viewed as a generalization of backtracking for optimization problems
- We grow a tree of partial solutions

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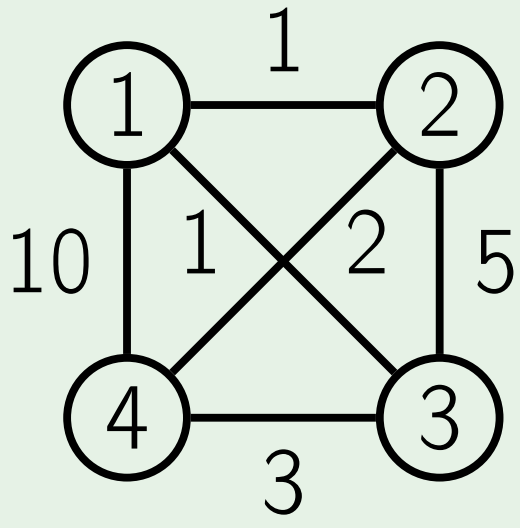
- The branch-and-bound technique can be viewed as a generalization of backtracking for optimization problems
- We grow a tree of partial solutions
- At each node of the recursion tree we check whether the current partial solution can be extended to a solution which is better than the best solution found so far
- If not, we don't continue this branch

Example: brute force search



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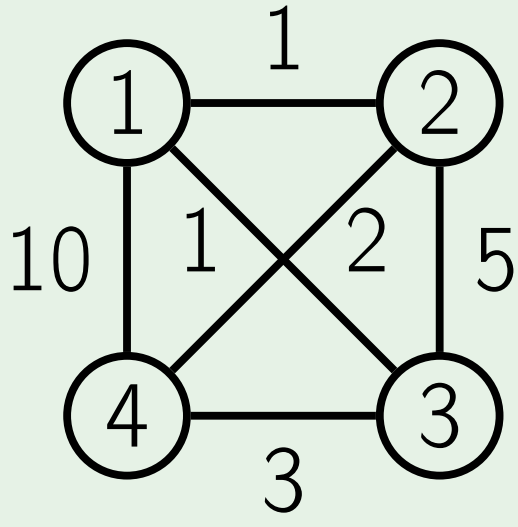
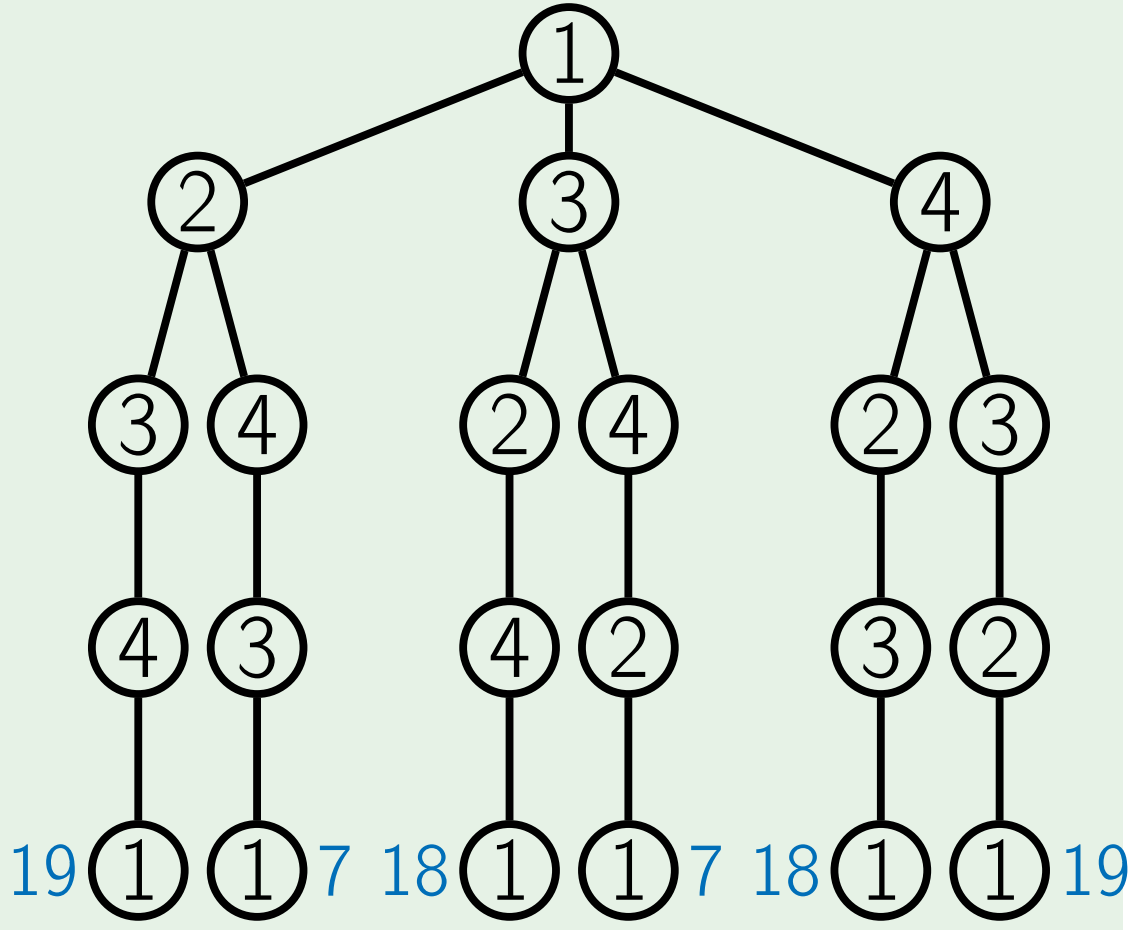
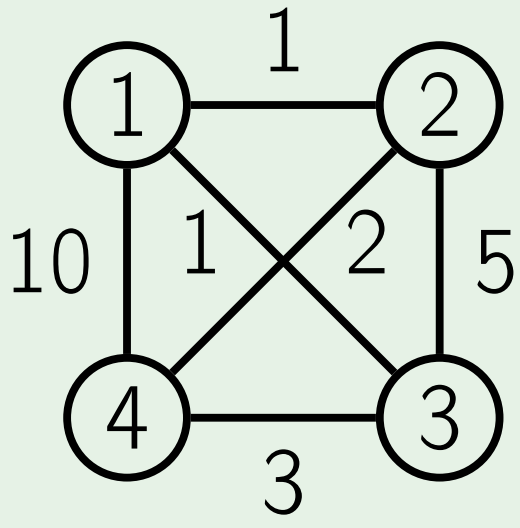




Example: pruned search



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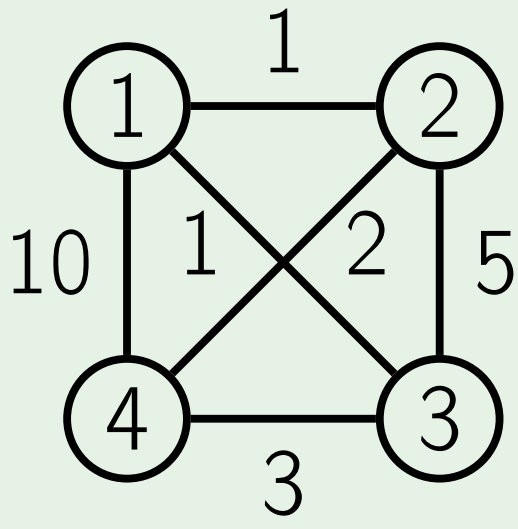

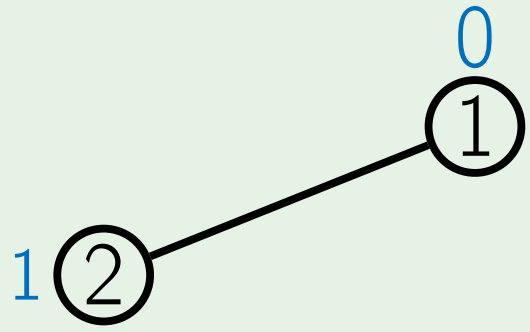

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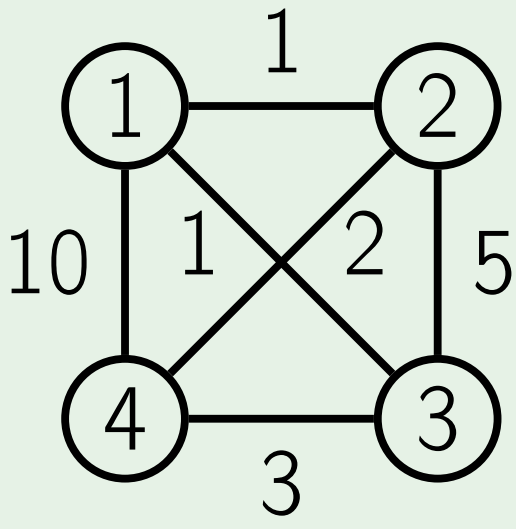

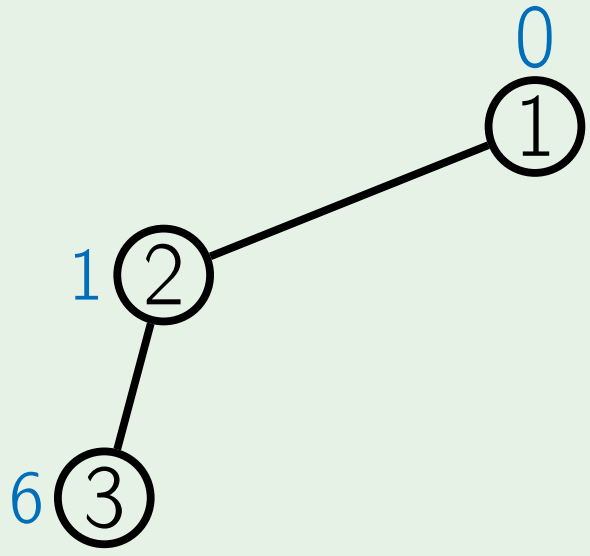

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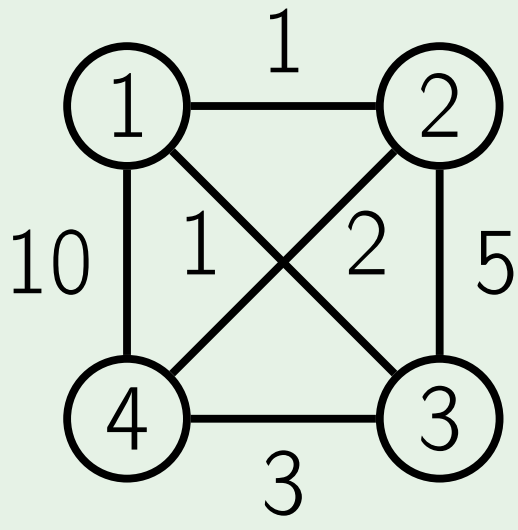

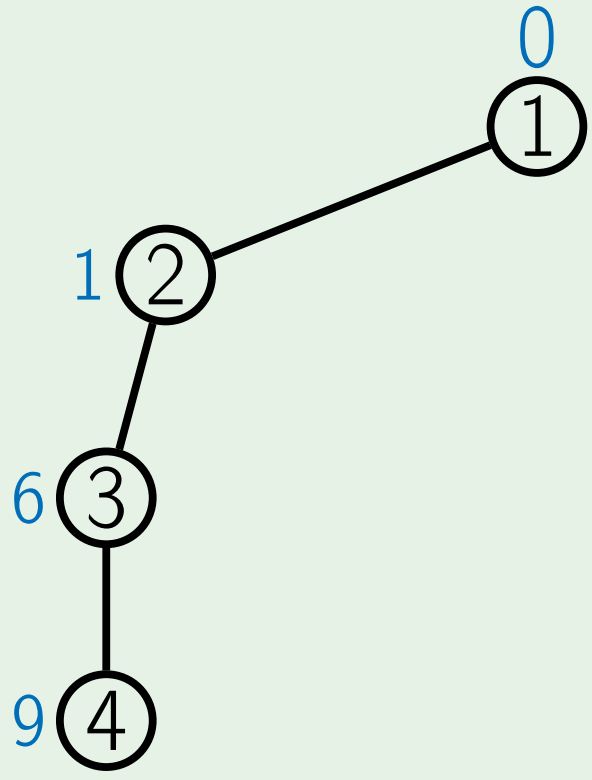

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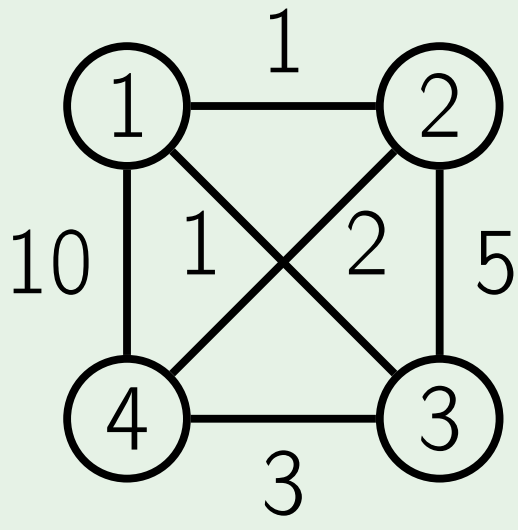
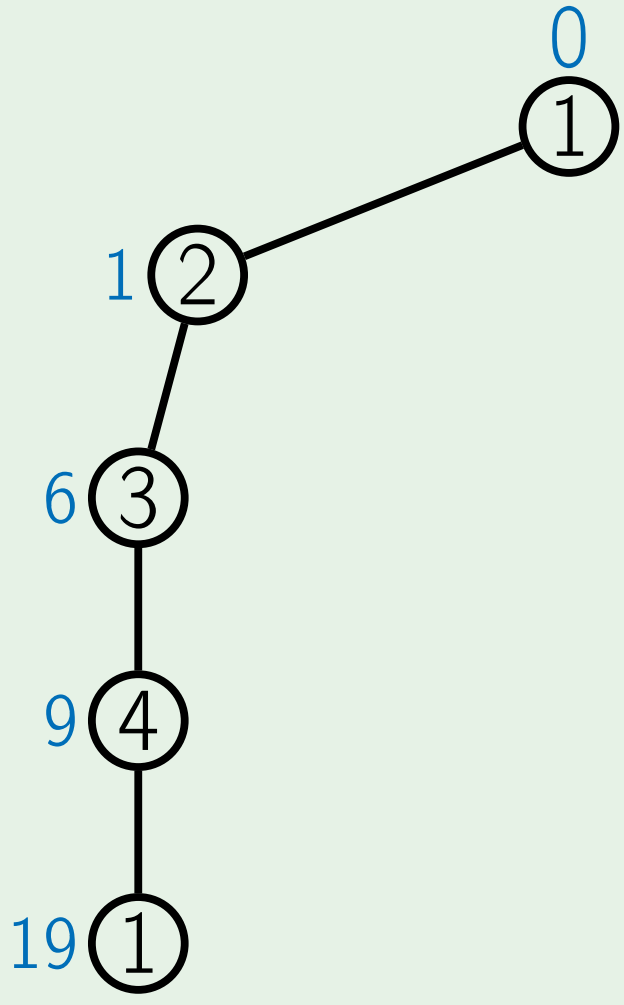
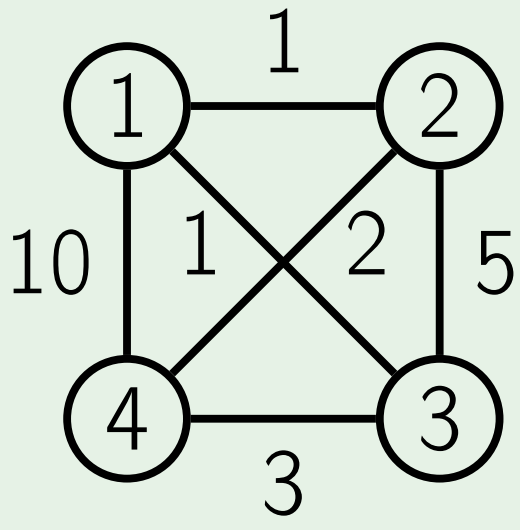
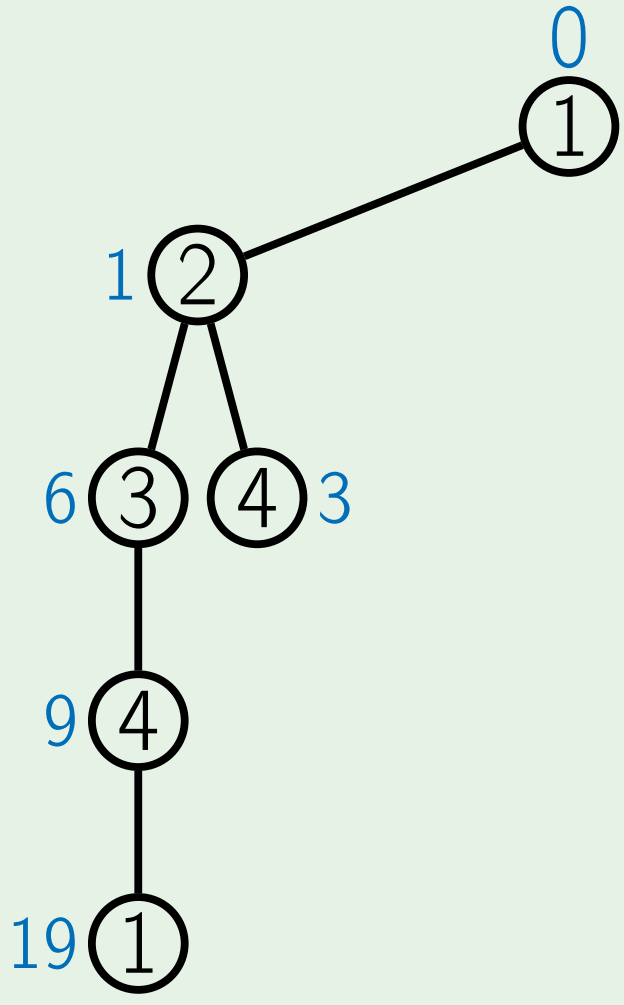




best so far: 19

Example: pruned search





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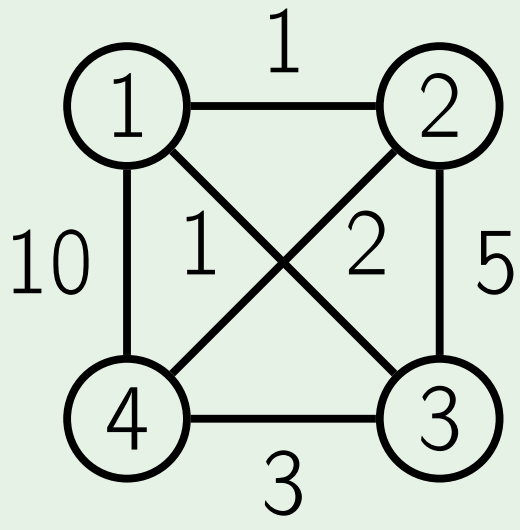

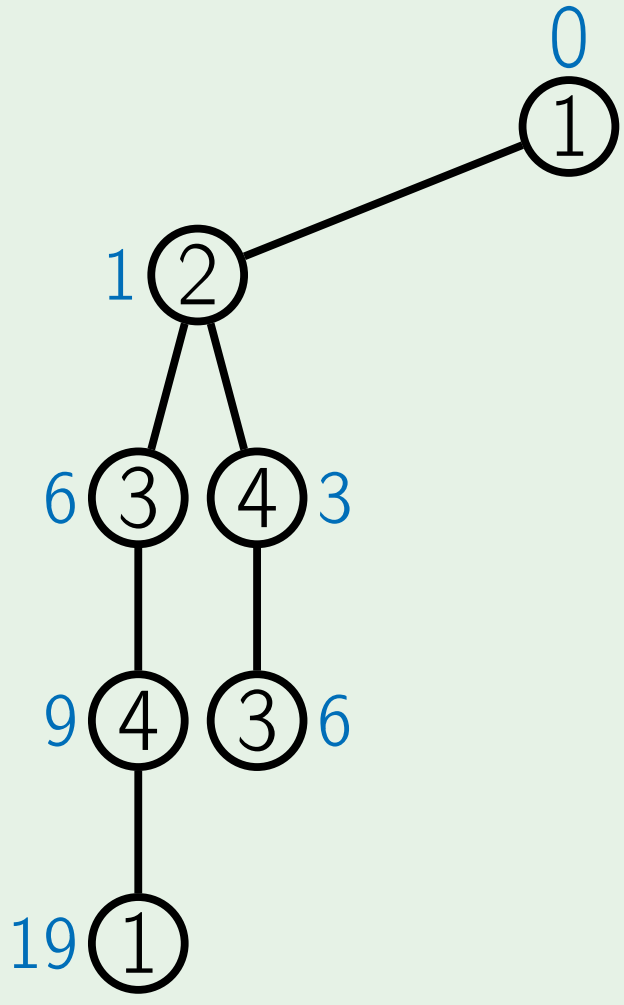

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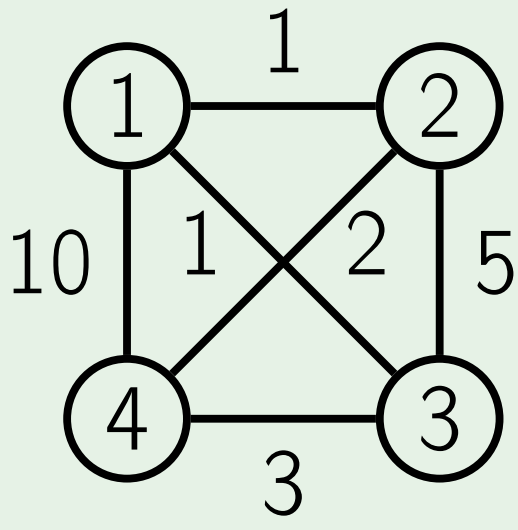

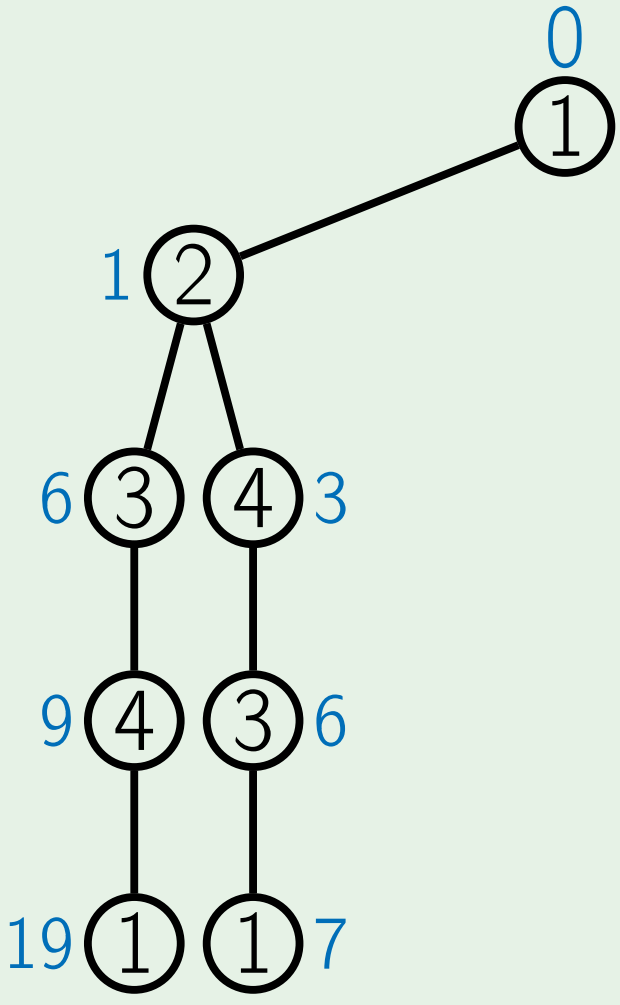

best so far: 7

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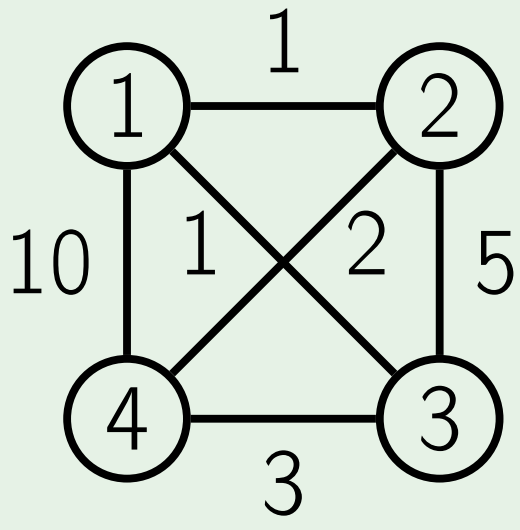

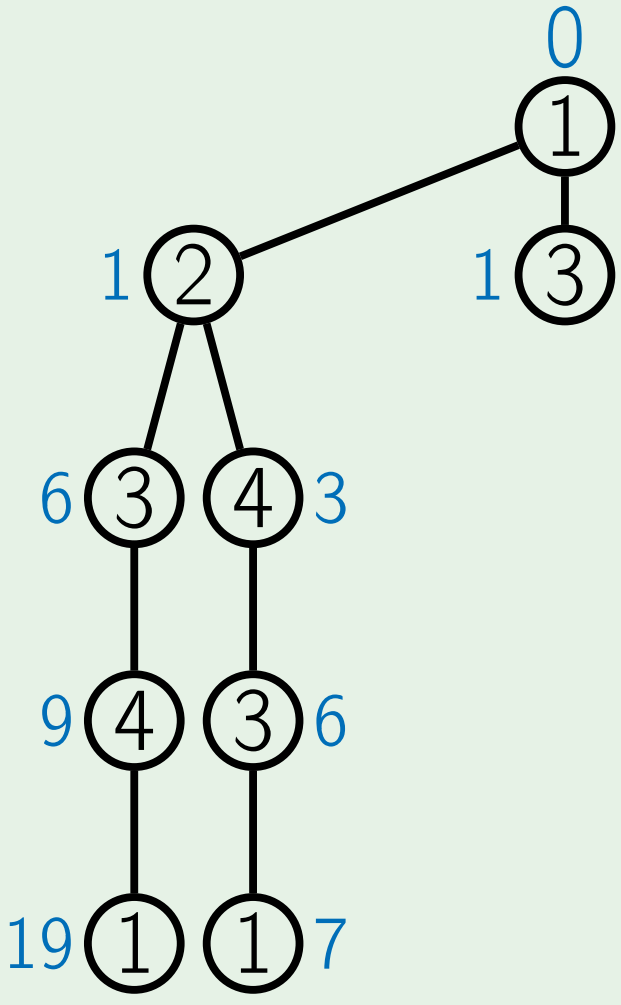

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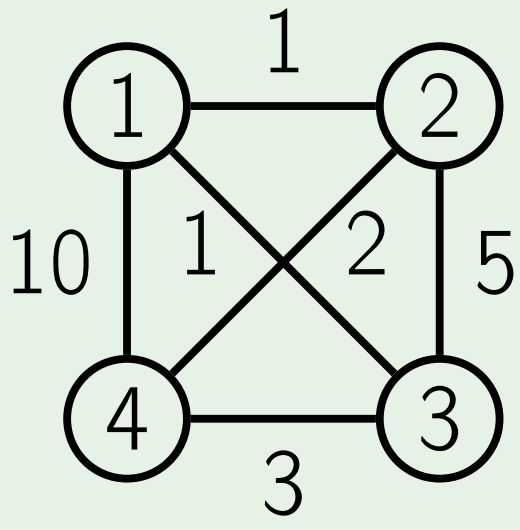

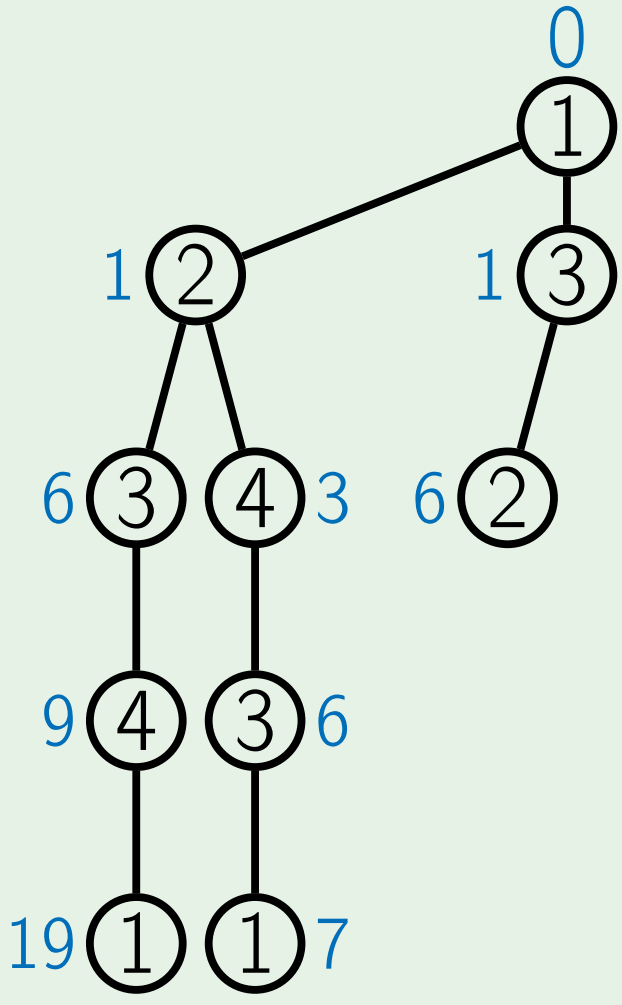

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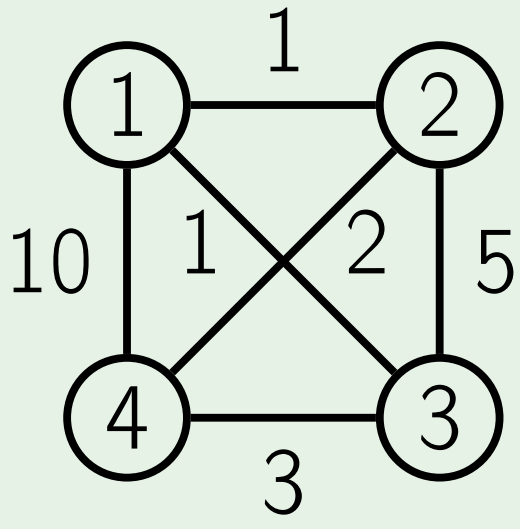

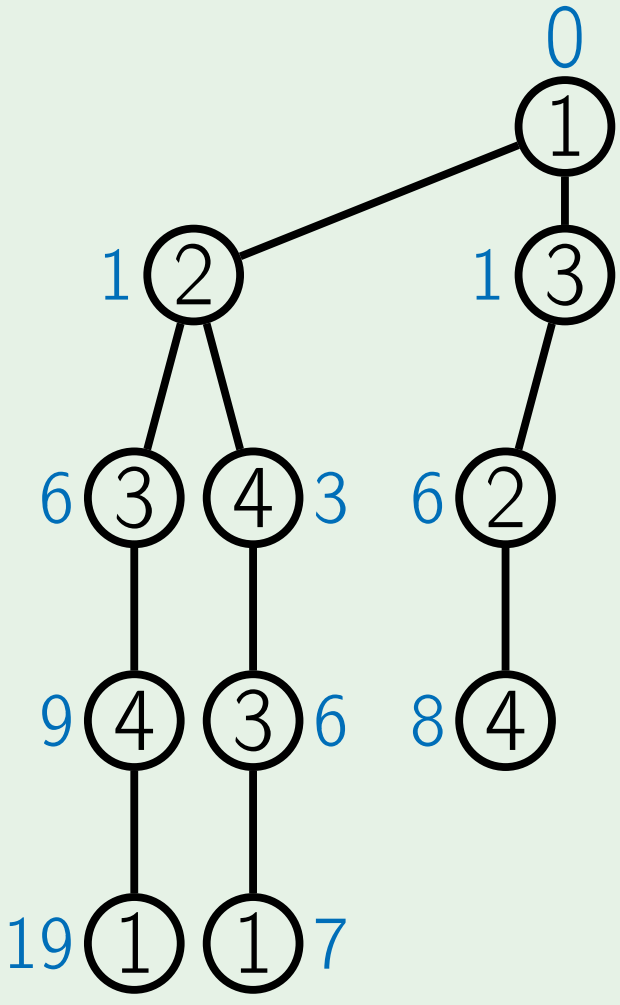

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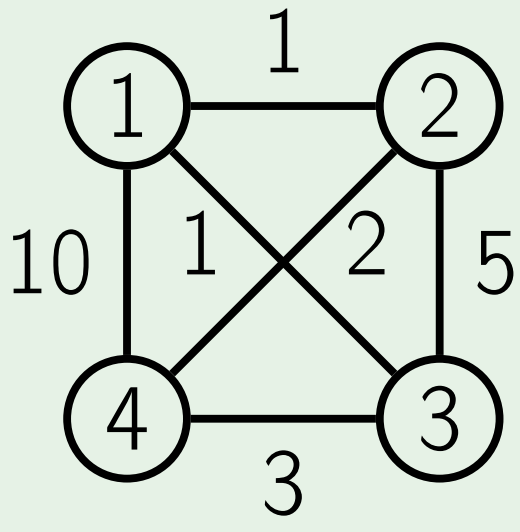

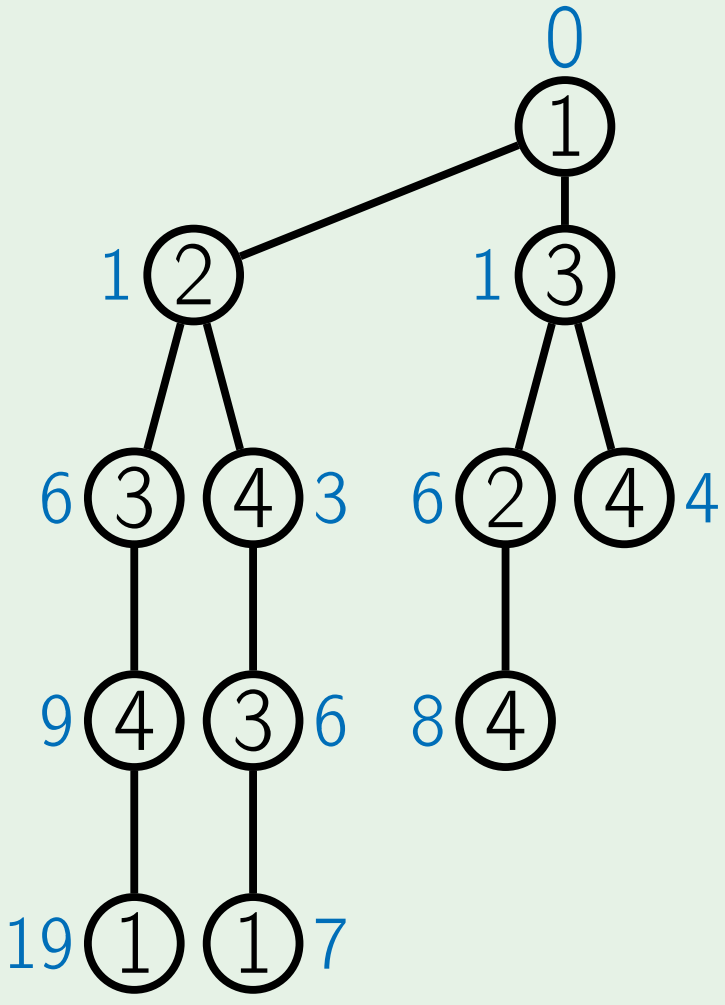

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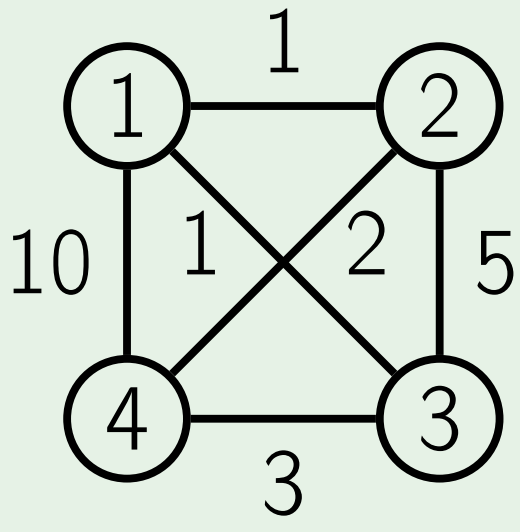

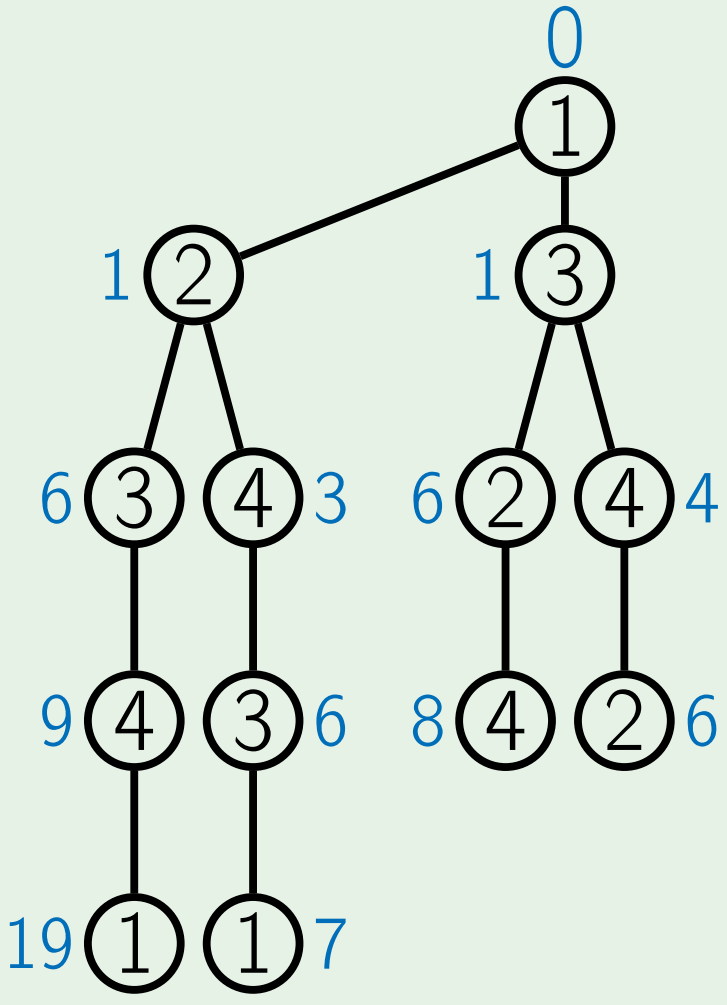

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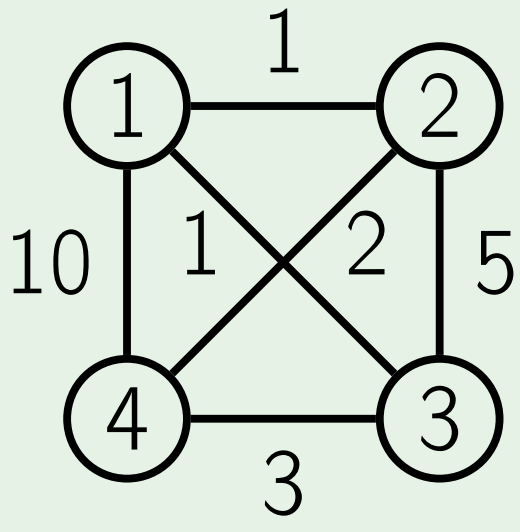

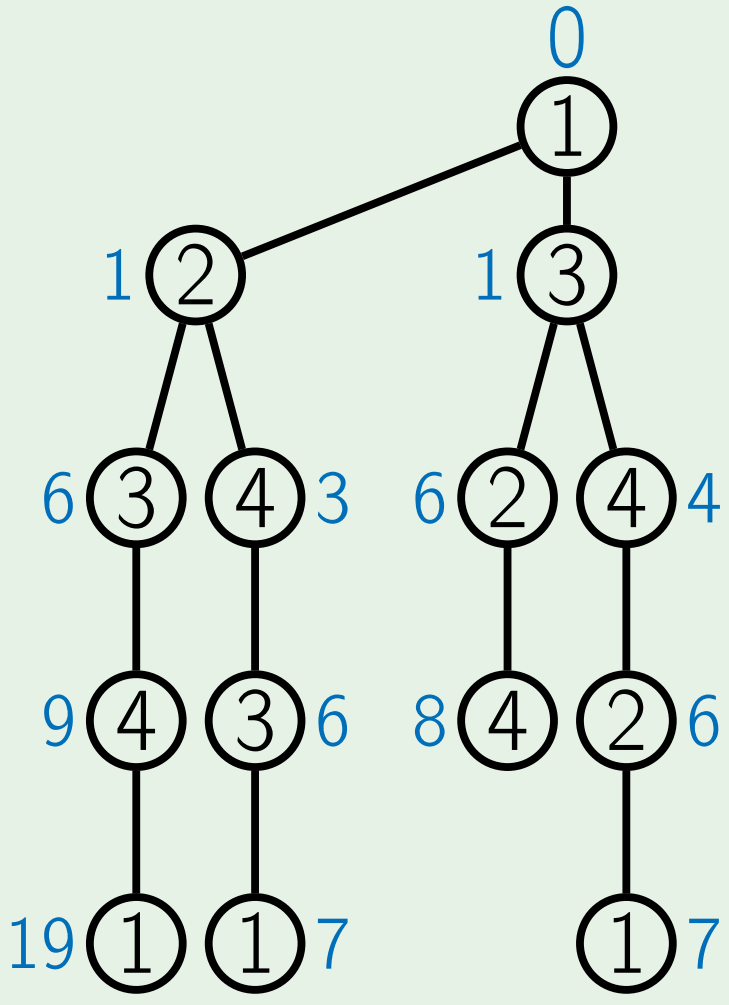

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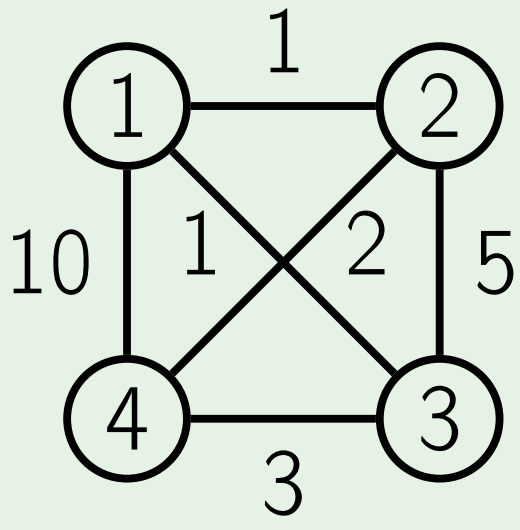

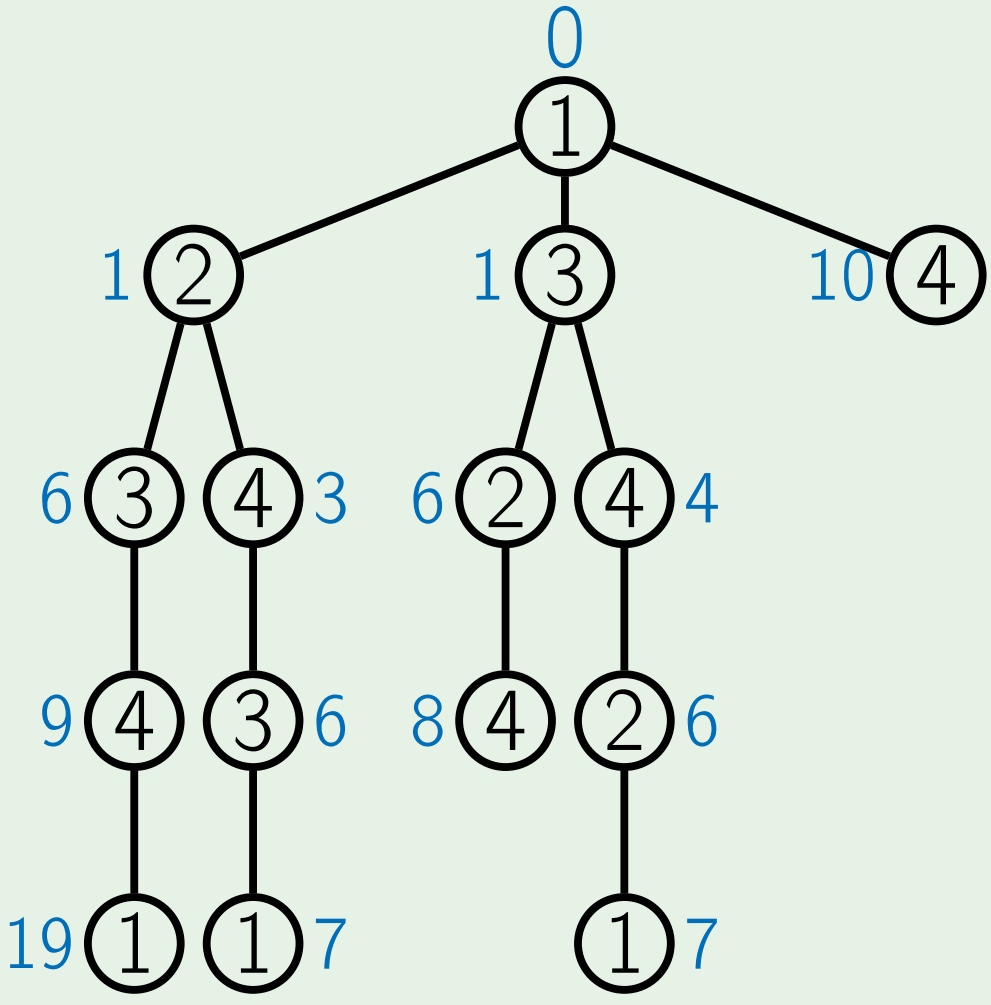

best so far: 7

- We used the simplest possible lower bound: any extension of a path has length at least the length of the path

- We used the simplest possible lower bound: any extension of a path has length at least the length of the path
- Modern TSP-solvers use smarter lower bounds to solve instances with thousands of vertices

Example: lower bounds (still simple)

The length of an optimal TSP cycle is at least

- $\frac{1}{2} \sum_{v \in V}$(two min length edges adjacent to v)

Example: lower bounds (still simple)

The length of an optimal TSP cycle is at least

- $\frac{1}{2}\sum_{v \in V}$(two min length edges adjacent to v)
- the length of a minimum spanning tree

Next time

Approximation algorithms: polynomial algorithms that find a solution that is not much worse than an optimal solution