

Divide-and-Conquer: Quick Sort

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

- 1 Overview
- 2 Algorithm
- 3 Random Pivot
- 4 Running Time Analysis
- 5 Equal Elements
- 6 Final Remarks

Quick Sort

- comparison based algorithm
- running time: $O(n \log n)$ (on average)
In the worst case it is $O(n^2)$
- efficient in practice

Example: quick sort

6	4	8	2	9	3	9	4	7	6	1
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partition with respect to $x = A[1]$
in particular, x is in its final position

1	4	2	3	4	6	6	9	7	8	9
---	---	---	---	---	---	---	---	---	---	---

≤ 6

> 6

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sort the two parts recursively

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QuickSort(A, ℓ, r)

if $\ell \geq r$:

 return

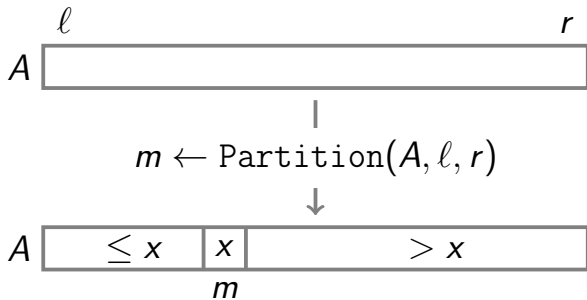
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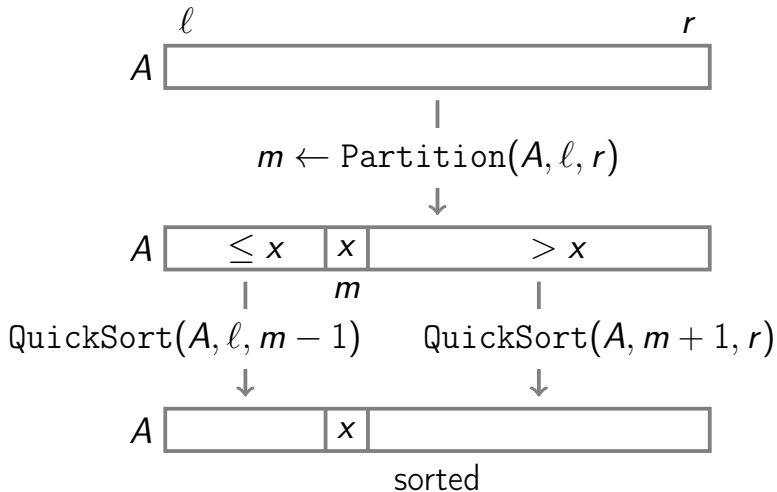
$\{A[m] \text{ is in the final position}\}$

QuickSort($A, \ell, m - 1$)

QuickSort($A, m + 1, r$)







Partitioning: example

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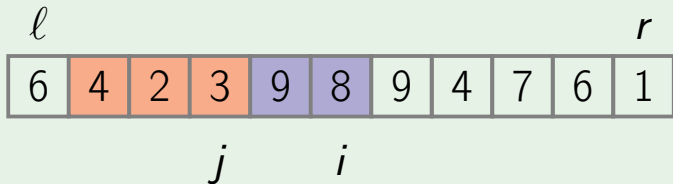
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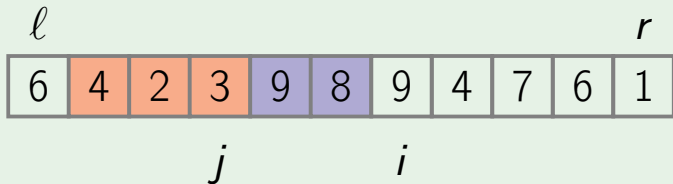
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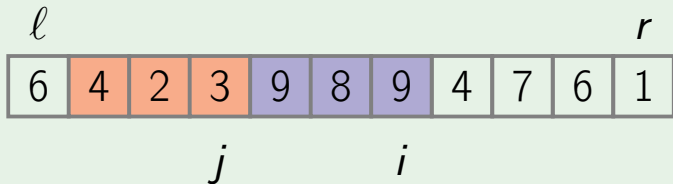
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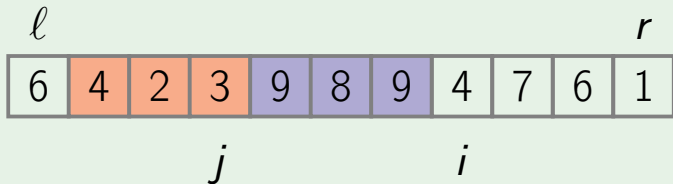
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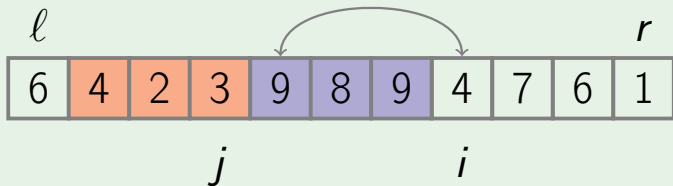
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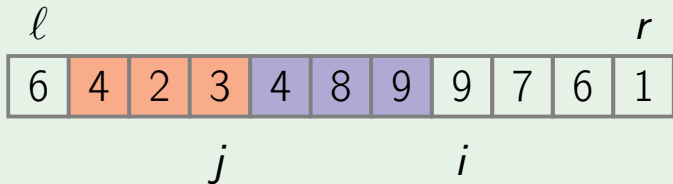
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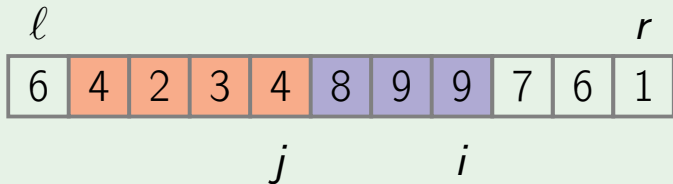
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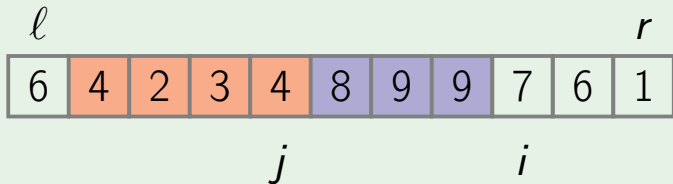
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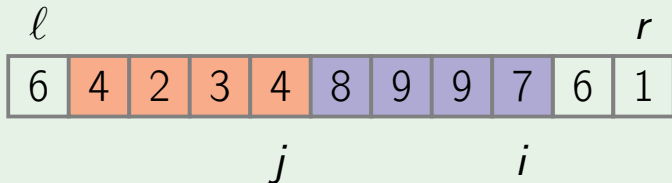
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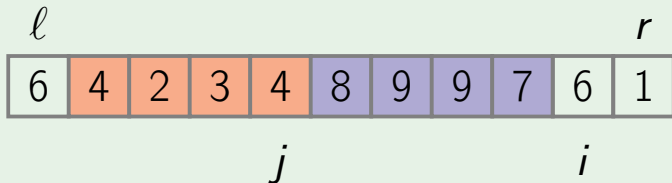
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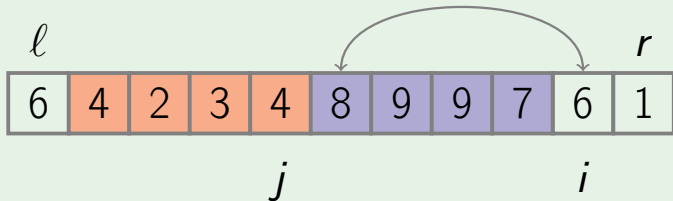
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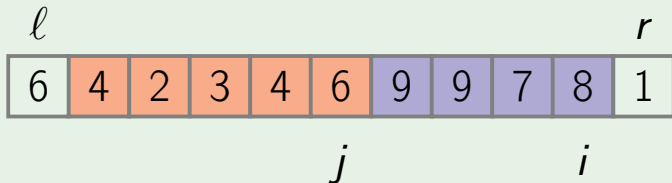
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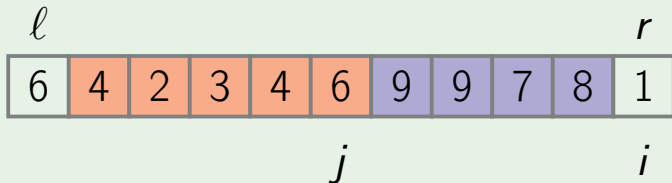
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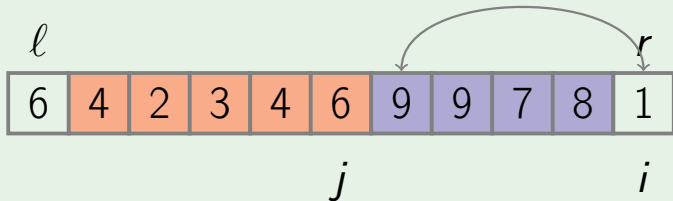
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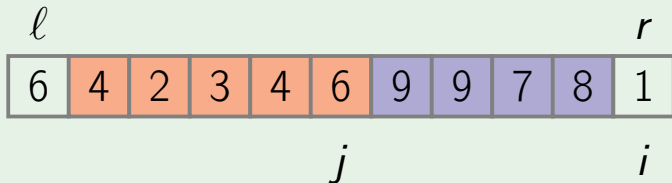
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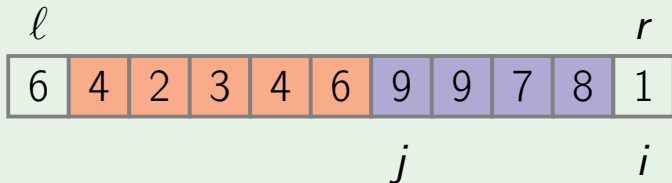
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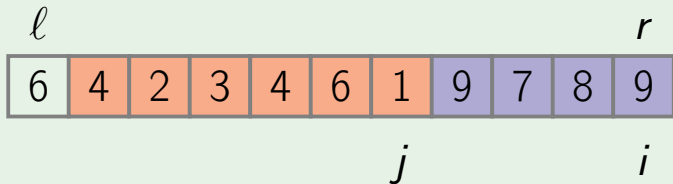
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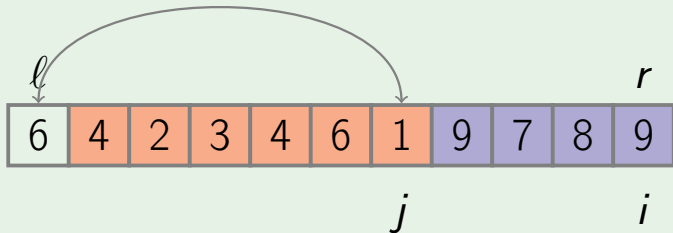
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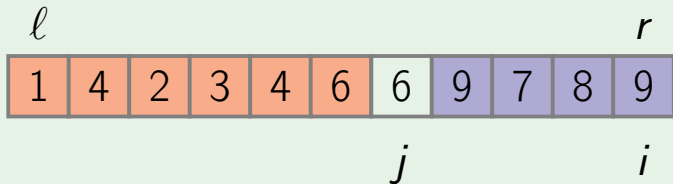
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All the elements before x are less than x and all the elements after x are greater than x



Partition(A, ℓ, r)

$x \leftarrow A[\ell]$ {pivot}

$j \leftarrow \ell$

for i from $\ell + 1$ to r :

 if $A[i] \leq x$:

$j \leftarrow j + 1$

 swap $A[j]$ and $A[i]$

 { $A[\ell + 1 \dots j] \leq x$, $A[j + 1 \dots i] > x$ }

swap $A[\ell]$ and $A[j]$

return j

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Unbalanced Partitions

- $T(n) = n + T(n - 1)$:

$$T(n) = n + (n-1) + (n-2) + \dots = \Theta(n^2)$$

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- $T(n) = n + T(n - 5) + T(4):$

$$T(n) \geq n + (n-5) + (n-10) + \dots = \Theta(n^2)$$

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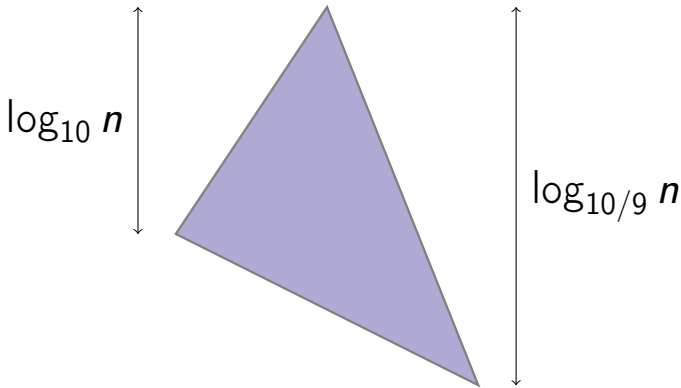
$$T(n) = \Theta(n \log n)$$

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$$T(n) = T(n/10) + T(9n/10) + O(n)$$

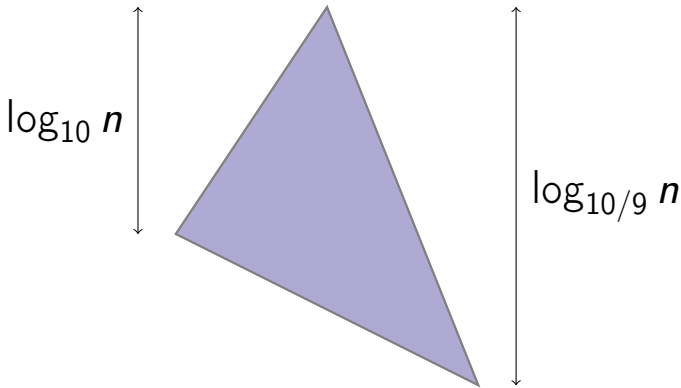
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$$T(n) = O(n \log n)$$

Random Pivot

RandomizedQuickSort(A, ℓ, r)

if $\ell \geq r$:

 return

$k \leftarrow$ random number between ℓ and r

 swap $A[\ell]$ and $A[k]$

$m \leftarrow \text{Partition}(A, \ell, r)$

$\{A[m] \text{ is in the final position}\}$

 RandomizedQuickSort($A, \ell, m - 1$)

 RandomizedQuickSort($A, m + 1, r$)

Why Random?

half of the elements of A guarantees a balanced partition:



Theorem

Assume that all the elements of $A[1 \dots n]$ are pairwise different. Then the average running time of $\text{RandomizedQuickSort}(A)$ is $O(n \log n)$ while the worst case running time is $O(n^2)$.

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Remark

Averaging is over random numbers used by the algorithm, but not over the inputs.

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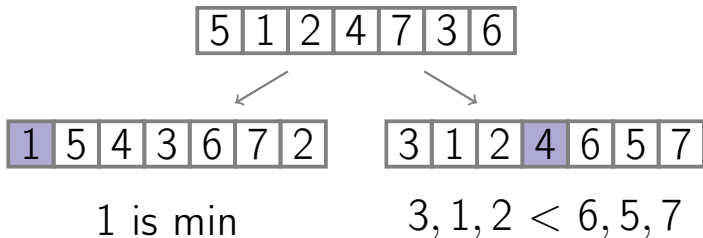
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Proof Ideas: Comparisons

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- balanced partition are better since they reduce the number of comparisons needed:



Proof Ideas: Probability

A	5	1	8	9	2	4	7	3	6
-----	---	---	---	---	---	---	---	---	---

A'	1	2	3	4	5	6	7	8	9
------	---	---	---	---	---	---	---	---	---

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$$\text{Prob}(3 \text{ and } 4 \text{ are compared}) =$$

Proof Ideas: Probability

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A'	1	2	3	4	5	6	7	8	9

$$\text{Prob}(1 \text{ and } 9 \text{ are compared}) = \frac{2}{9}$$

$$\text{Prob}(3 \text{ and } 4 \text{ are compared}) = 1$$

Proof

- let, for $i < j$,

$$\chi_{ij} = \begin{cases} 1 & A'[i] \text{ and } A'[j] \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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- for all $i < j$, $A'[i]$ and $A'[j]$ are either compared exactly once or not compared at all (as we compare with a pivot)
- this, in particular, implies that the worst case running time is $O(n^2)$

Proof (continued)

- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \dots j]$ is $A'[i]$ or $A'[j]$

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- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \dots j]$ is $A'[i]$ or $A'[j]$
- then $\text{Prob}(\chi_{ij}) = \frac{2}{j-i+1}$ and
$$\mathbb{E}(\chi_{ij}) = \frac{2}{j-i+1}$$

Proof (continued)

Then (the expected value of) the running time is

$$\begin{aligned} \mathbb{E} \sum_{i=1}^n \sum_{j=i+1}^n \chi_{ij} &= \sum_{i=1}^n \sum_{j=i+1}^n \mathbb{E}(\chi_{ij}) \\ &= \sum_{i < j} \frac{2}{j-i+1} \\ &\leq 2n \cdot \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) \\ &= \Theta(n \log n) \end{aligned}$$

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- the array is always split into two parts of size 0 and $n - 1$
- $T(n) = n + T(n - 1) + T(0)$ and hence $T(n) = \Theta(n^2)$!

To handle equal elements, we replace the line

$$m \leftarrow \text{Partition}(A, \ell, r)$$

with the line

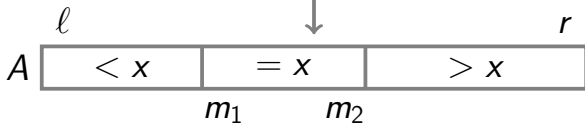
$$(m_1, m_2) \leftarrow \text{Partition3}(A, \ell, r)$$

such that

- for all $\ell \leq k \leq m_1 - 1$, $A[k] < x$
- for all $m_1 \leq k \leq m_2$, $A[k] = x$
- for all $m_2 + 1 \leq k \leq r$, $A[k] > x$



$(m_1, m_2) \leftarrow \text{Partition3}(A, \ell, r)$



RandomizedQuickSort(A, ℓ, r)

if $\ell \geq r$:

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$\{A[m_1 \dots m_2] \text{ is in final position}\}$

 RandomizedQuickSort($A, \ell, m_1 - 1$)

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Tail Recursion Elimination

QuickSort(A, ℓ, r)

while $\ell < r$:

$m \leftarrow \text{Partition}(A, \ell, r)$

 QuickSort($A, \ell, m - 1$)

$\ell \leftarrow m + 1$

QuickSort(A, ℓ, r)

while $\ell < r$:

$m \leftarrow \text{Partition}(A, \ell, r)$

 if $(m - \ell) < (r - m)$:

 QuickSort($A, \ell, m - 1$)

$\ell \leftarrow m + 1$

 else:

 QuickSort($A, m + 1, r$)

$r \leftarrow m - 1$

QuickSort(A, ℓ, r)

```
while  $\ell < r$ :  
     $m \leftarrow \text{Partition}(A, \ell, r)$   
    if  $(m - \ell) < (r - m)$ :  
        QuickSort( $A, \ell, m - 1$ )  
         $\ell \leftarrow m + 1$   
    else:  
        QuickSort( $A, m + 1, r$ )  
         $r \leftarrow m - 1$ 
```

Worst-case space requirement: $O(\log n)$

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- if the recursion depth exceeds a certain threshold $c \log n$ the algorithm switches to heap sort
- the running time is $O(n \log n)$ in the worst case

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- Efficient in practice