Disjoint Sets: Naive Implementations

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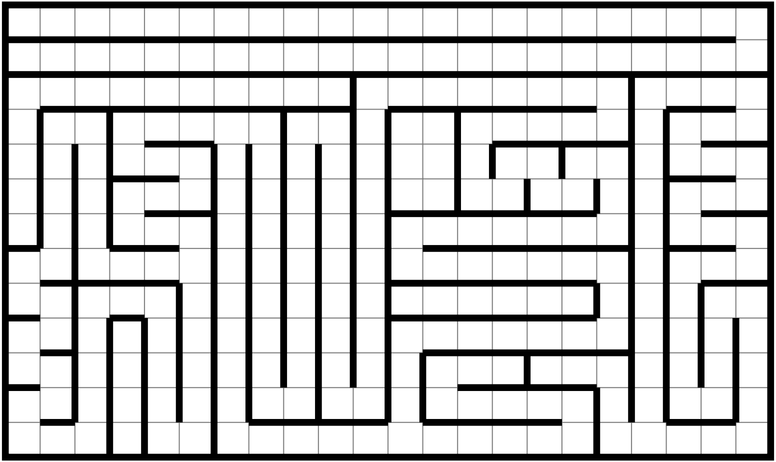
Data Structures
Data Structures and Algorithms

Outline

1. Overview
2. Naive Implementations

Maze: Is B Reachable from A?



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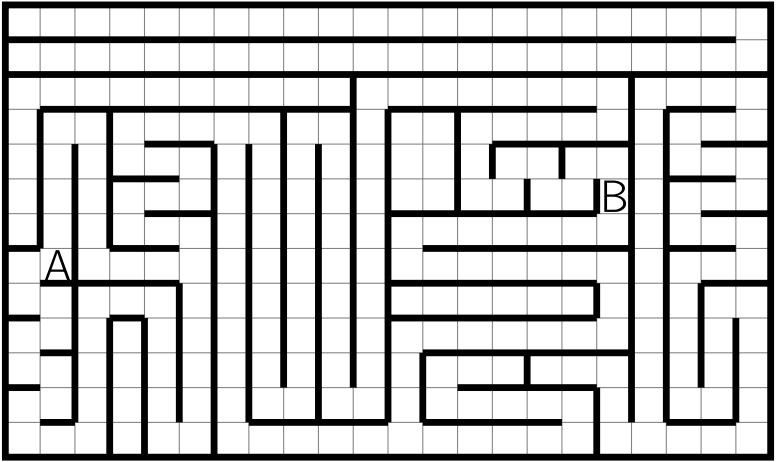

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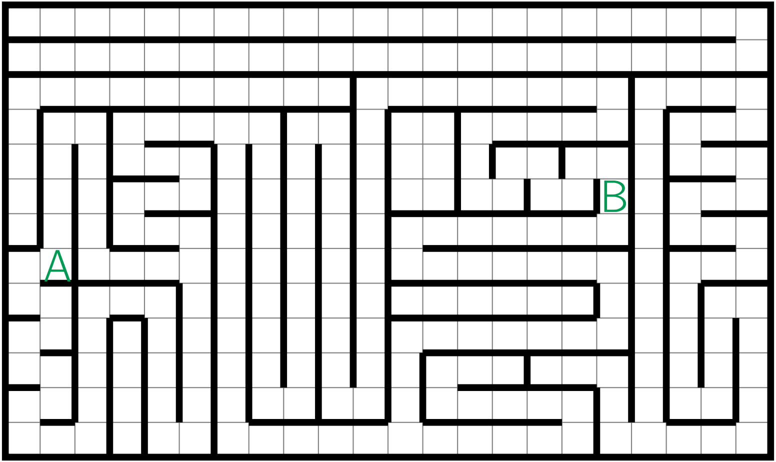

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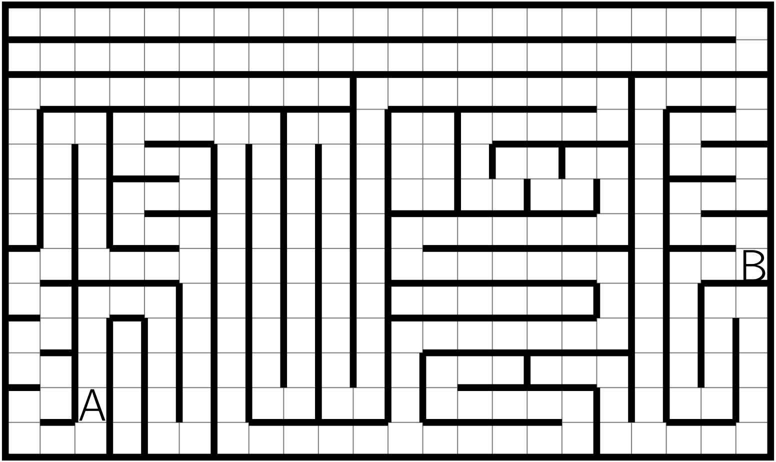

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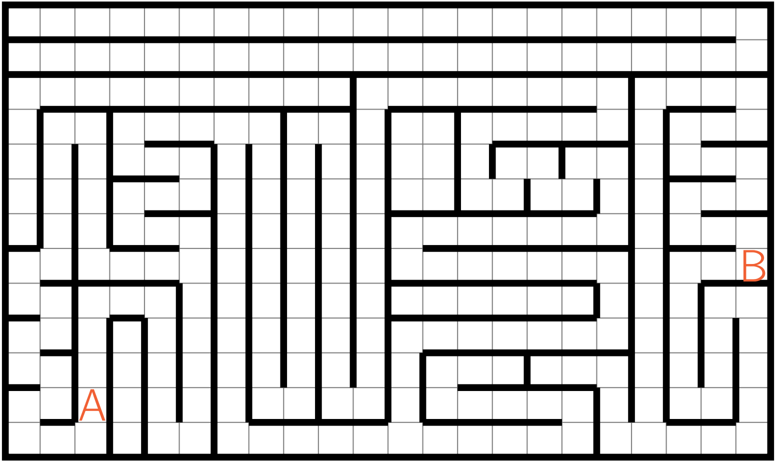

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  - otherwise, $\texttt{Find}(x) \neq \texttt{Find}(y)$
- $\texttt{Union}(x, y)$ merges two sets containing x and y

Preprocess($maze$)

```
for each cell c in maze:
  MakeSet(c)
for each cell c in maze:
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    Union(c, n)
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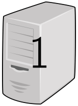
IsReachable(A, B)

```
return Find(A) = Find(B)
```

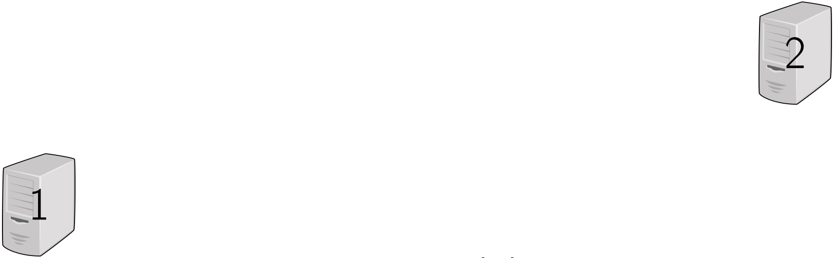
Building a Network

Building a Network



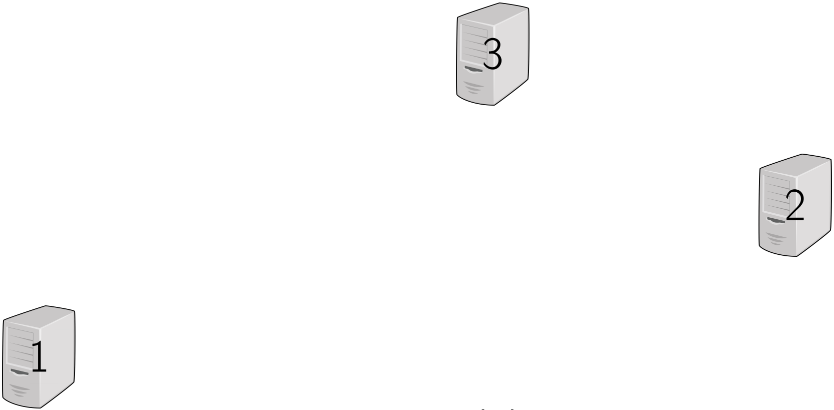
`MakeSet(1)`

Building a Network



`MakeSet(2)`

Building a Network



`MakeSet(3)`

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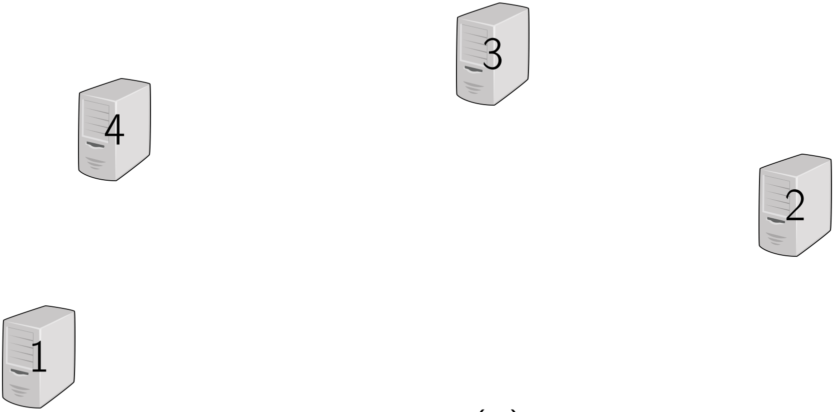

MakeSet(4)

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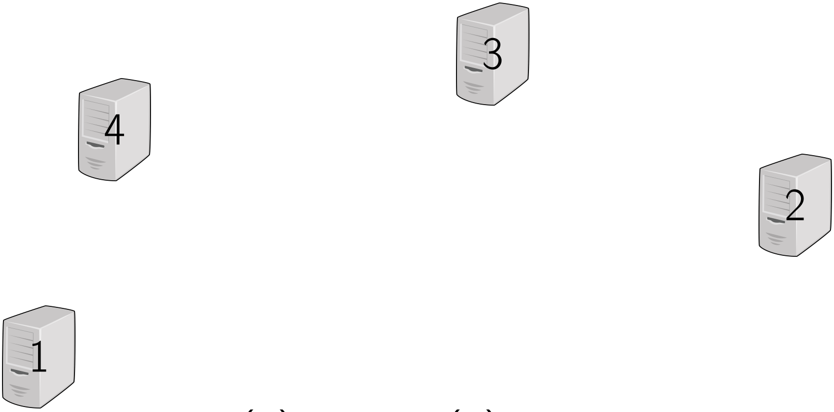

$$\texttt{Find}(1) = \texttt{Find}(2) \rightarrow \texttt{False}$$

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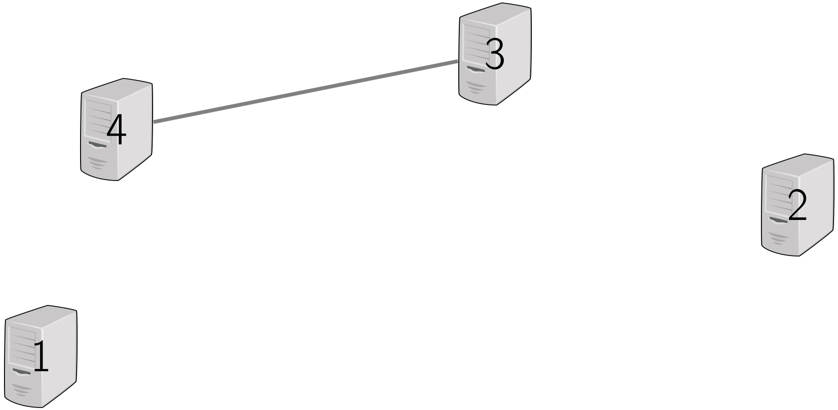

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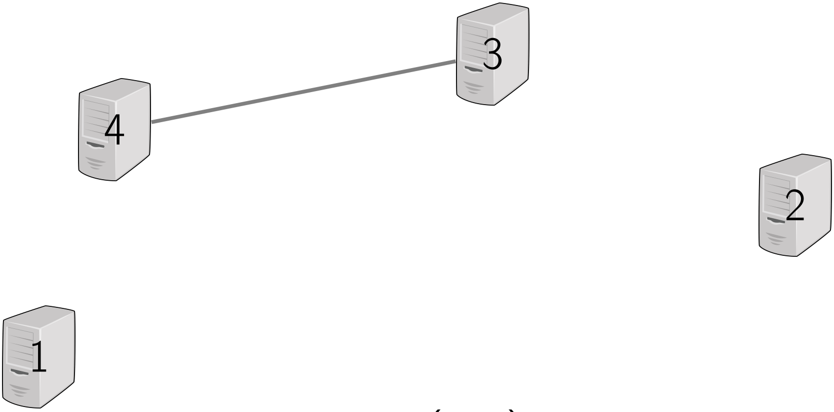

Union(3, 4)

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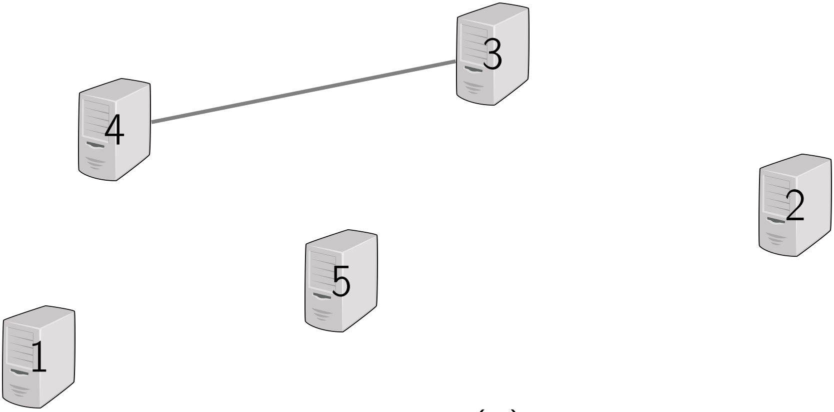
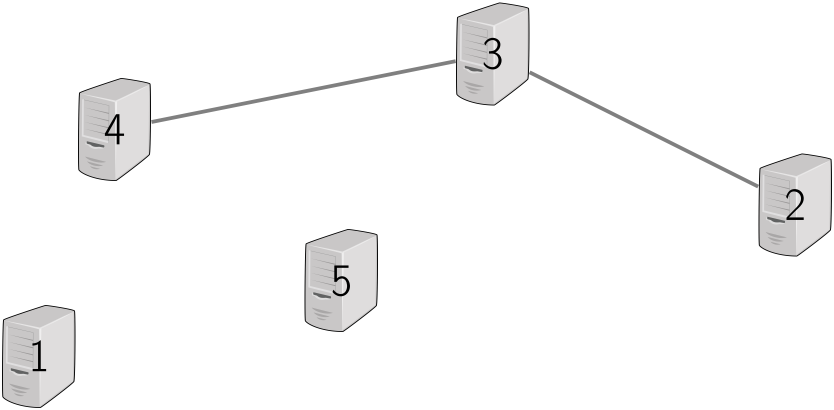


MakeSet(5)

Building a Network



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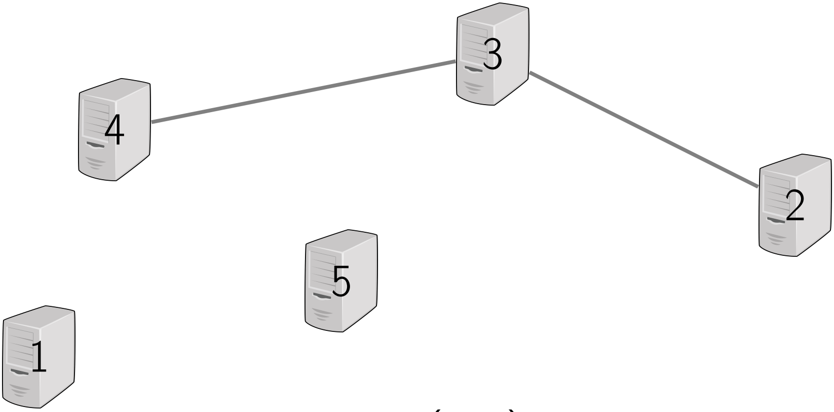
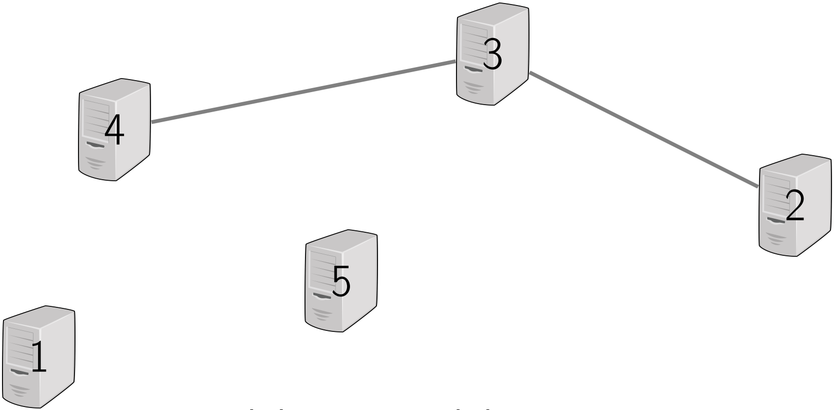


`Union(3, 2)`

Building a Network



$$\texttt{Find}(1) = \texttt{Find}(2) \rightarrow \texttt{False}$$

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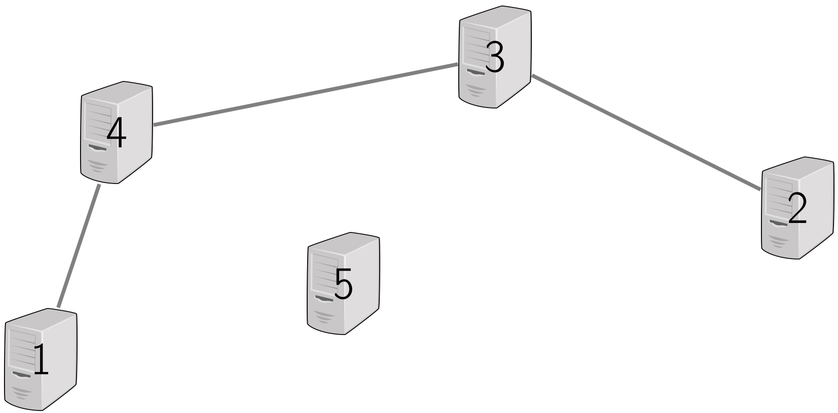

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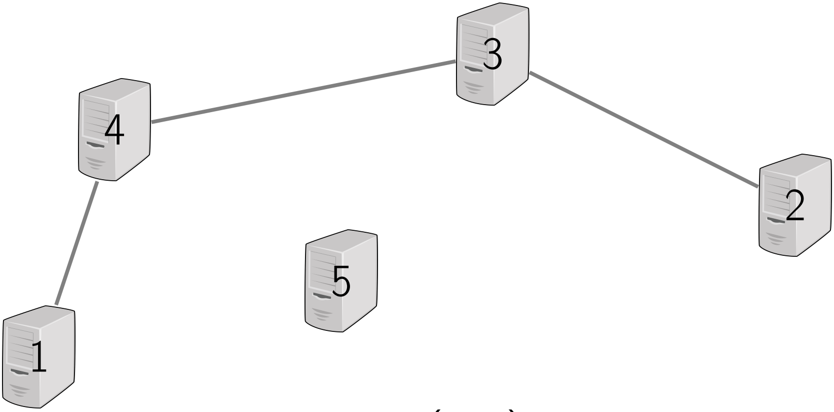
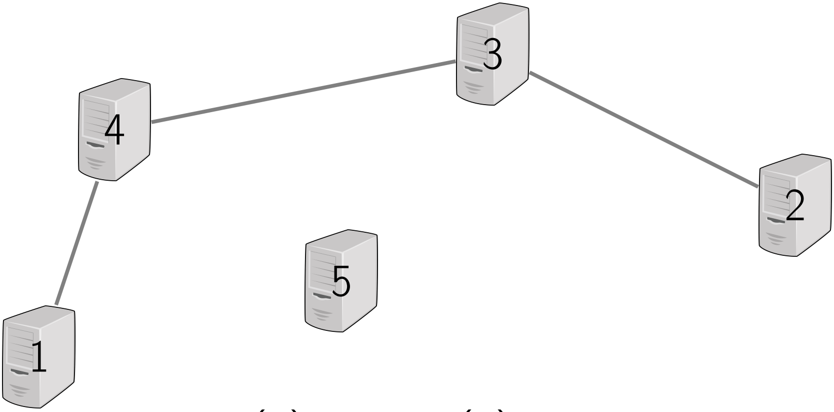


`Union(1,4)`

Building a Network



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For simplicity, we assume that our n objects are just integers $1, 2, \ldots, n$.

Using the Smallest Element as ID

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- Use array `smallest`$[1 \dots n]$:
  `smallest`$[i]$ stores the smallest element in the set i belongs to

Example

$$\{9, 3, 2, 4, 7\} \quad \{5\} \quad \{6, 1, 8\}$$

	1	2	3	4	5	6	7	8	9
`smallest`	1	2	2	2	5	1	2	1	2

MakeSet(i)

```
smallest[i] ← i
```

Find(i)

```
return smallest[i]
```

MakeSet(i)

smallest[i] $\leftarrow i$

Find(i)

return smallest[i]

Running time: $O(1)$

$\texttt{Union}(i, j)$

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i_id ← Find(i)
j_id ← Find(j)
if i_id = j_id:
  return
m ← min(i_id, j_id)
for k from 1 to n:
  if smallest[k] in {i_id, j_id}:
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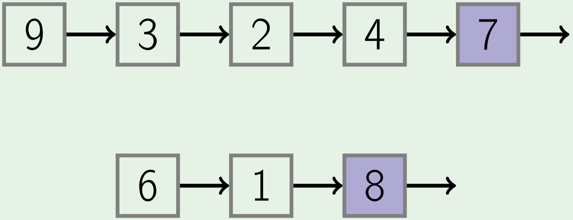
- What basic data structure allows for efficient merging?
- Linked list!
- Idea: represent a set as a linked list, use the list tail as ID of the set

Example: merging two lists



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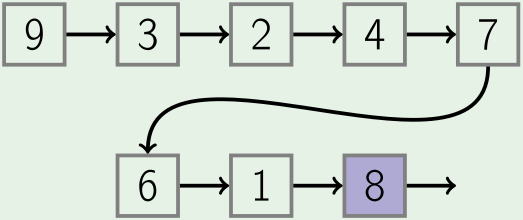

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  - Running time of `Find` is $O(n)$ as we need to traverse the list to find its tail
  - `Union`(x, y) works in time $O(1)$ only if we can get the tail of the list of x and the head of the list of y in constant time!

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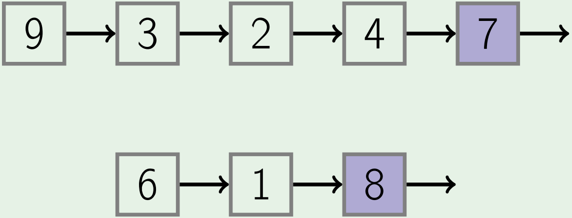

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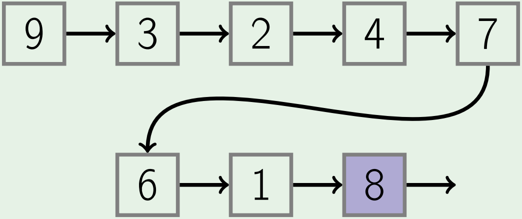

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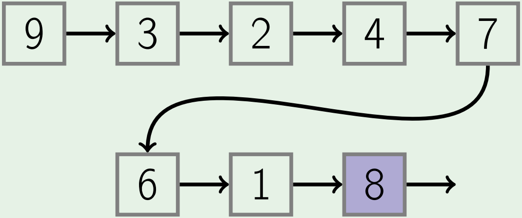

`Find`(9) goes through all elements

This means merging in this way makes find operation O(n), considering that Id is represented by the tail. Also, that making tail as ID gives the advantage of ID automatically updated to tail of the second list after merging.

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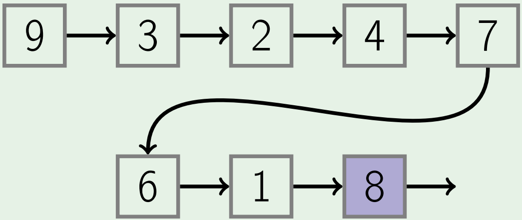

can we merge in a different way?

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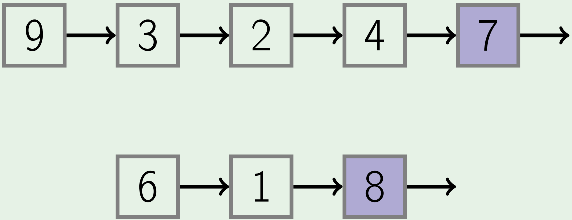

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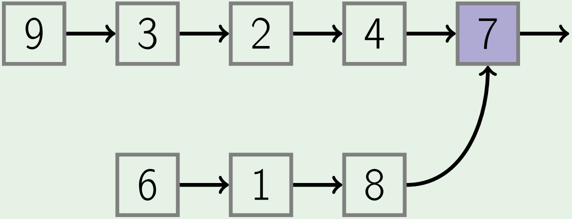

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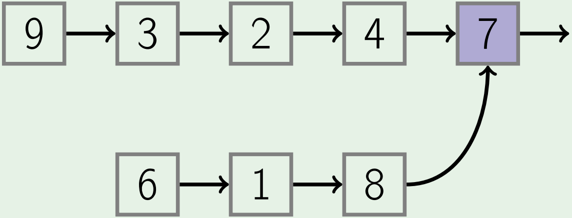
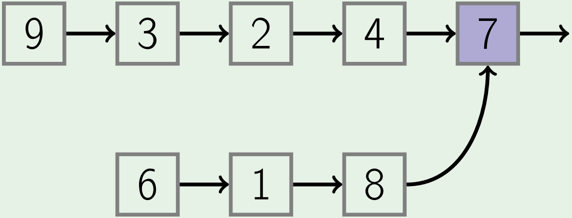


instead of a list we get a tree

Example: merging two lists



we'll see that representing sets as trees gives a very efficient implementation: nearly constant amortized time for all operations