Flows in Networks: Residual Networks

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Construct the residual network associated to a flow.
- Understand why edges to reverse existing flow are necessary.

Last Time

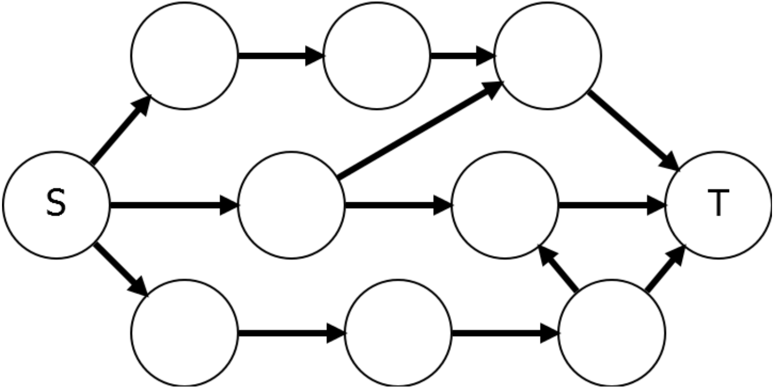
- Defined network.
- Defined flows.
- Defined maxflow problem.

Technique for Solving Maxflow

Build up flows a little bit at a time.

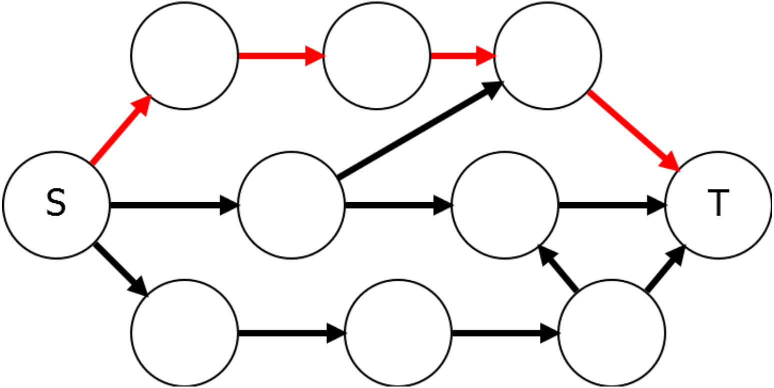
Example

[All capacities are 1]



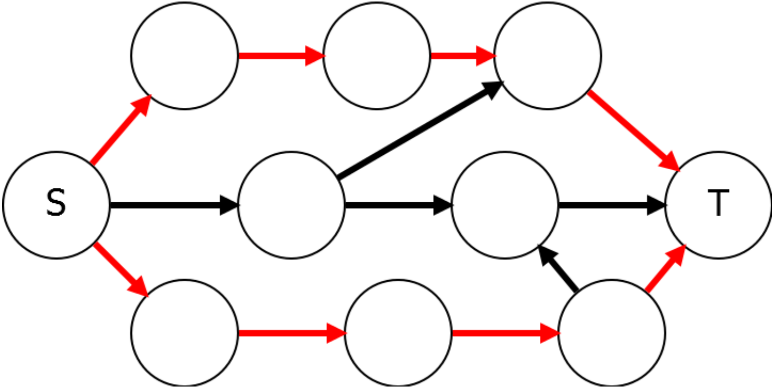
Example

[All capacities are 1]



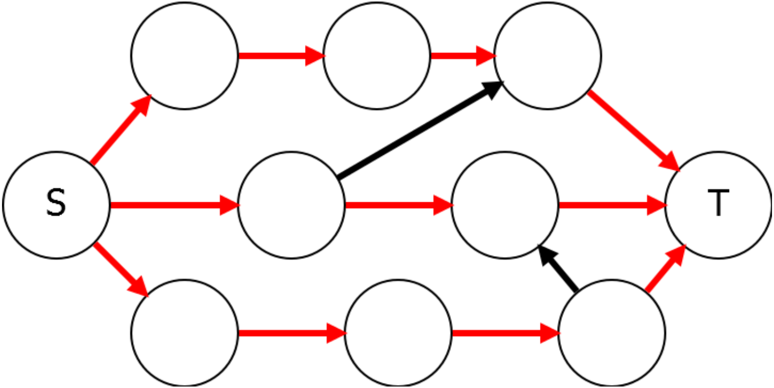
Example

[All capacities are 1]



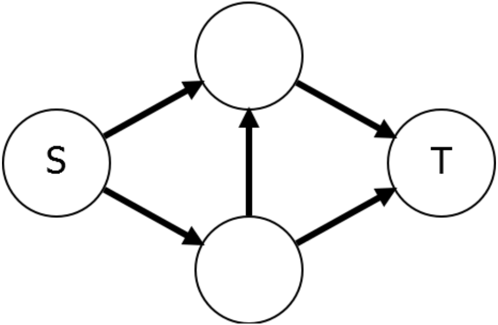
Example

[All capacities are 1]



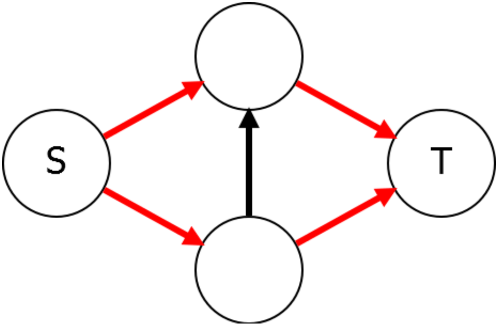
Example II

Consider another example.



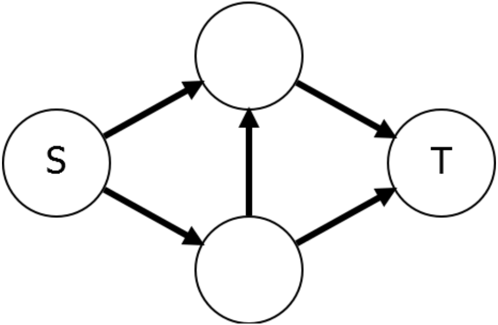
Example II

Maximum flow of 2.



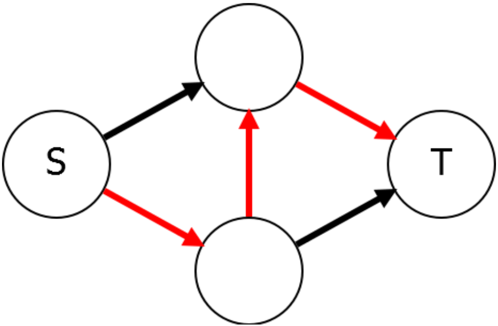
Example II

Consider adding flow incrementally.



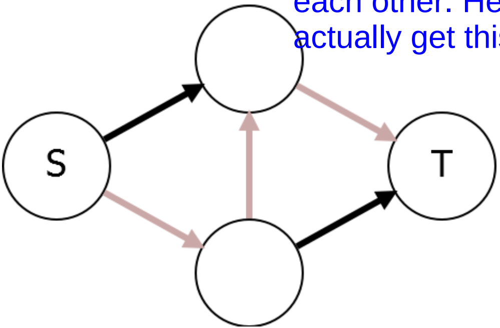
Example II

Add flow here.



Example II

Cannot add second unit. because we've already used up these edges, the remaining edges just don't connect to each other. Hence we can't actually get this flow to work

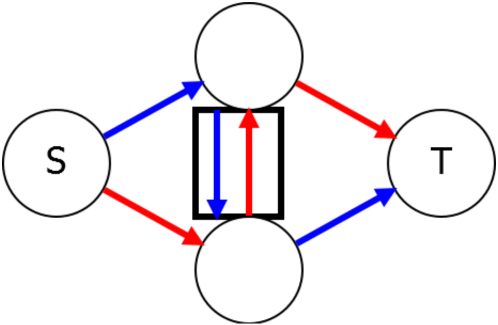


Example II

Need to add flow here, cancelling flow in the middle.

Since the Max flow is 2, we can add a blue middle path and route flow along this blue path

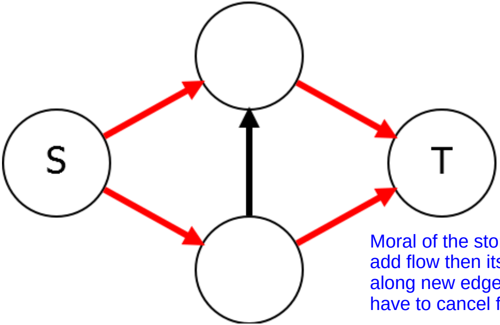
We can think of the middle blue path as cancelling the flow we are currently sending in the up direction and hence the total flow between two middle nodes is thus zero.



Example II

End up with this.

black edge here indicates no flow running over the middle edges



Moral of the story is that if you want to add flow then its not enough to add flow along new edges but sometimes you also have to cancel flow along existing edges

Residual Network

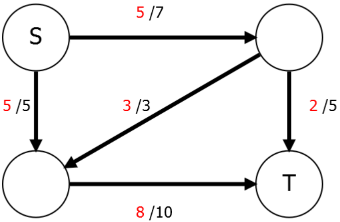
Given a network G and flow f, we construct a residual network G_f, representing places where flow can still be added to f, including places where existing flow can be cancelled.

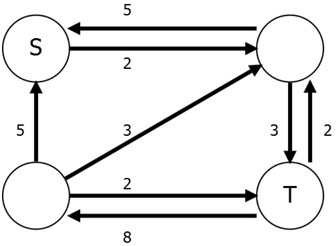
Residual Network

For each edge e of G, G_f has edges:

- e with capacity $C_e - f_e$ (unless $f_e = C_e$).
- opposite e with capacity f_e (unless $f_e = 0$).

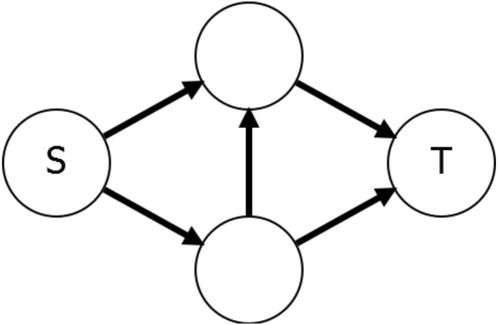
Example





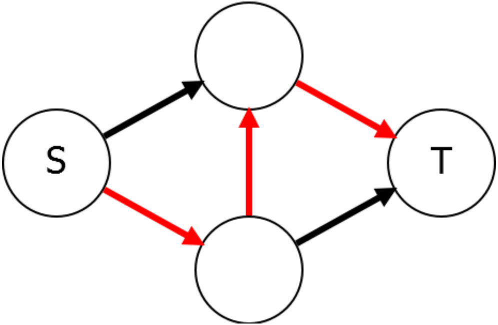
Review Example

Recall our previous example.



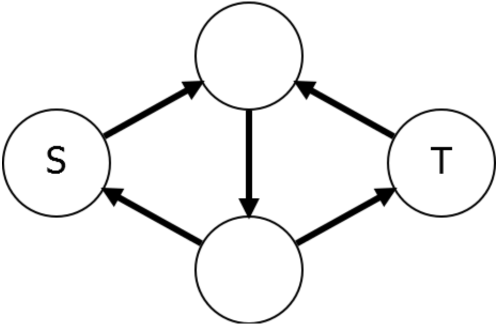
Review Example

This flow could not be added to directly.



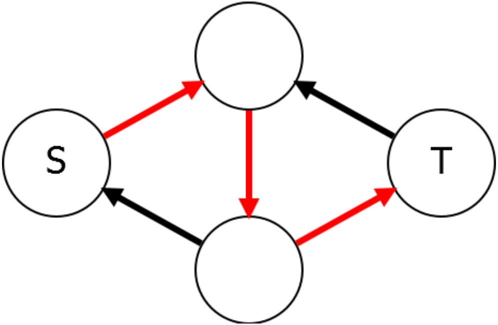
Review Example

But the residual graph is as shown.



Review Example

Which can support flow.



Residual Flow

Given network G, flow f. Any flow, g on G_f can be added to f to get a new flow on G.

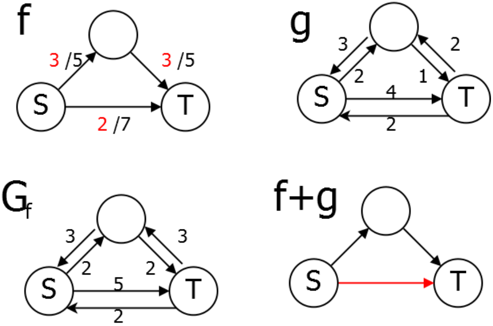
- g_e adds to f_e.
- $g_{e'}$ subtracts from f_e. see next slide to understand this

Problem

What is the flow of $f + g$ along the highlighted edge?

g is a network from residual network Gf

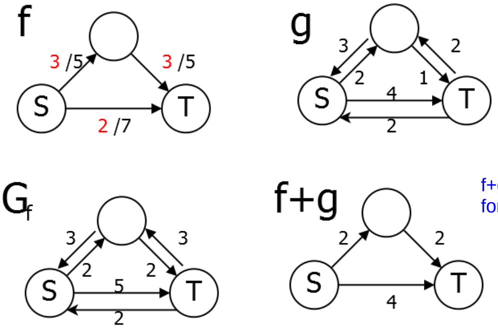
Gf here is residual network corresponding to f



Solution

Flow is given by:

$$f_e + g_e - g_{e'} = 2 + 4 - 2 = 4.$$



f+g does give you a valid flow
for the original network

Theorem

Given G, a flow f, and flow g on G_f:

- $f + g$ is a flow on G.
- $|f + g| = |f| + |g|$. Size of f+g
- All flows on G are of this form for some g.

Proof I

- Conservation of flow of f and Conservation of flow of g imply Conservation of flow of $f+g$.
- $f_e + g_e \leq f_e + (C_e - f_e) = C_e$.
- $f_e - g_{e'} \geq f_e - f_e = 0$.
- So $f+g$ is a flow.

Proof II

- Flow of $f + g$ out of s is flow of f out of s plus flow of g out of s. So $|f + g| = |f| + |g|$.
- For any flow h for G, it is not hard to show that $g := h - f$ is a flow on G_f.
- So $h = f + g$.

Summary

Flows on G_f are exactly ways to add flow to f.

Thus residual graph gives us small flows which we can add to our original network G and increase the flow