Advanced Shortest Paths: Bidirectional Dijkstra

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Graph Algorithms
Data Structures and Algorithms

Outline

1. Bidirectional Search

2. Bidirectional Dijkstra

Shortest Path

Input: A graph G with *non-negative* edge weights, a source vertex s and a target vertex t.

Graph can be directional or undirectional

Output: The shortest path between s and t in G.

Can solve the shortest path problem using the undirectional graphs in "linear time". Such linear time Algo for directional graph is not known we can do better compared to conventional Djikstra's running time

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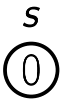
- Millions of users of Google Maps want the result in a blink of an eye, all at the same time
- Need something significantly faster

Dijkstra Progression



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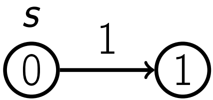

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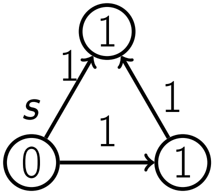

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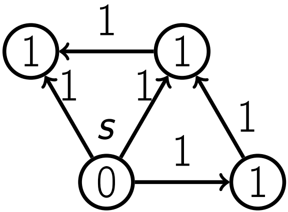

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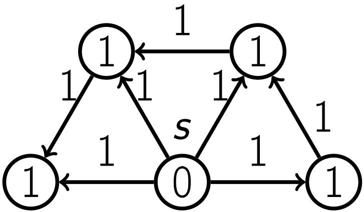

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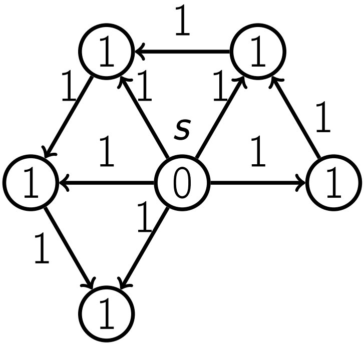

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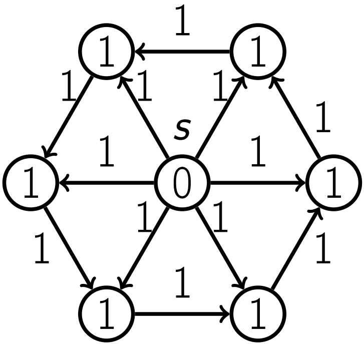

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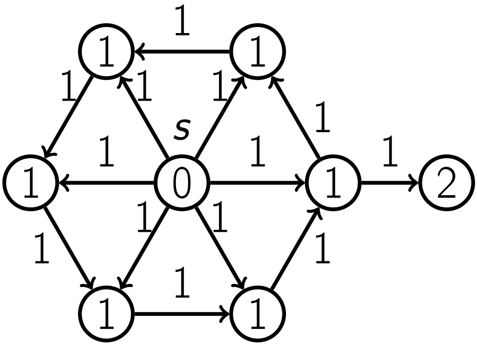

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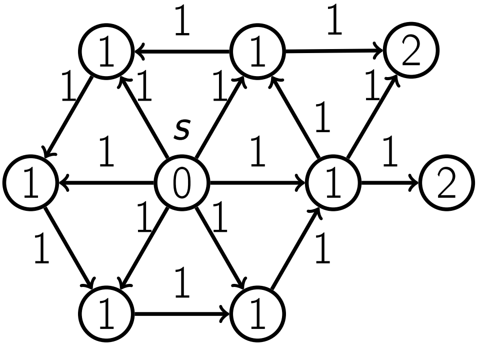

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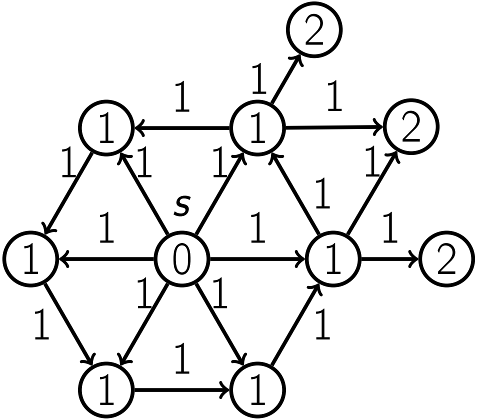

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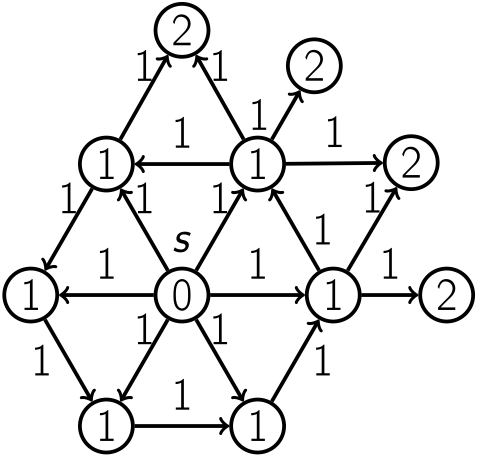

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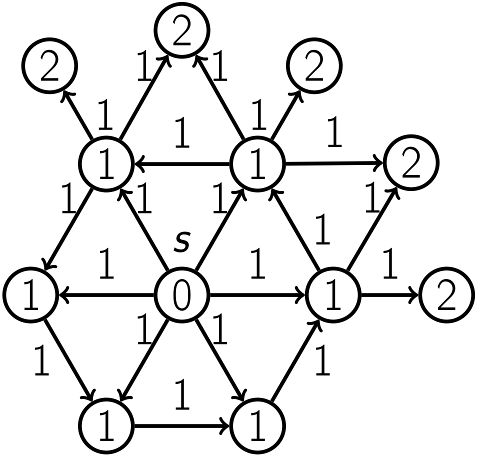

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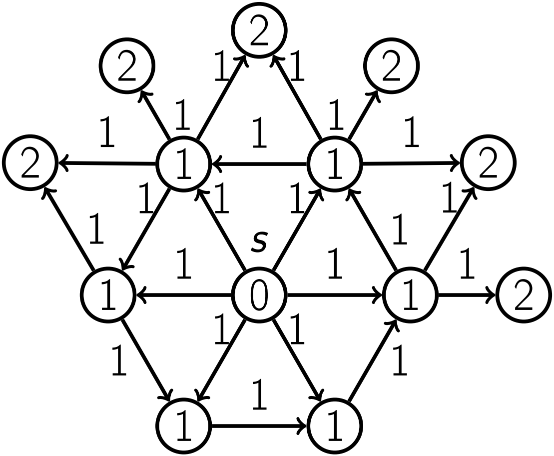

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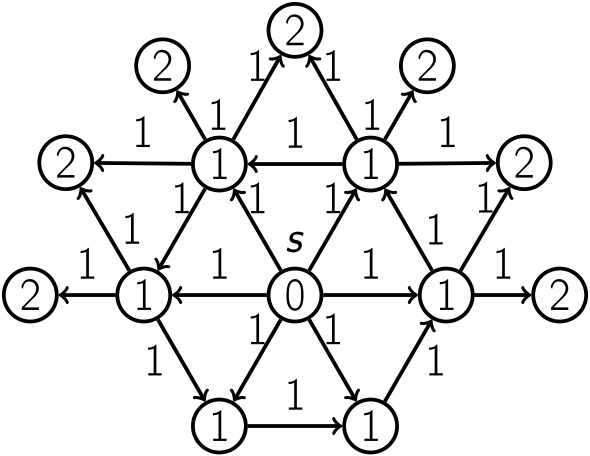

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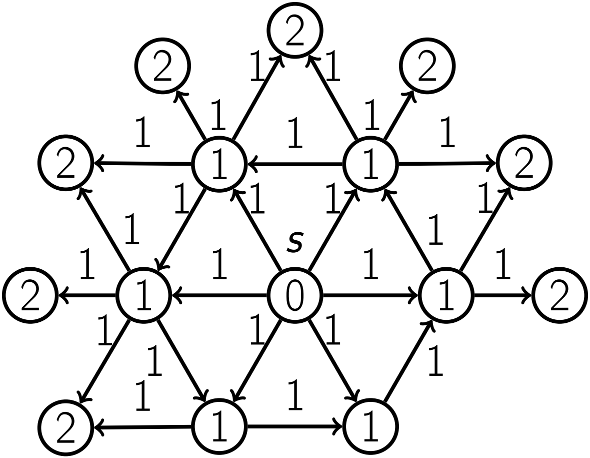

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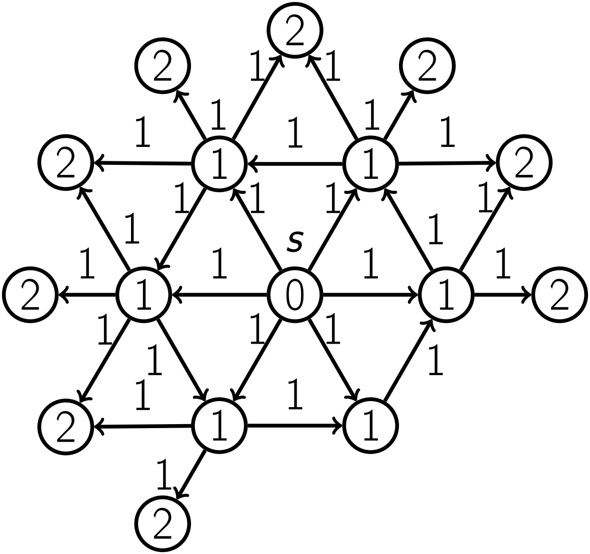

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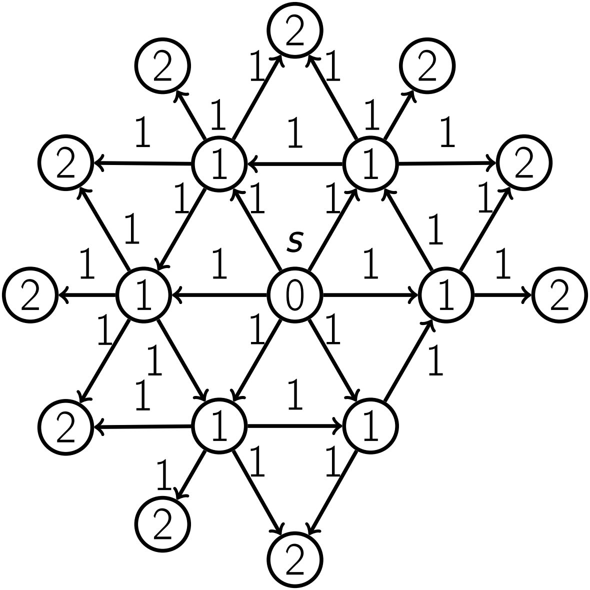

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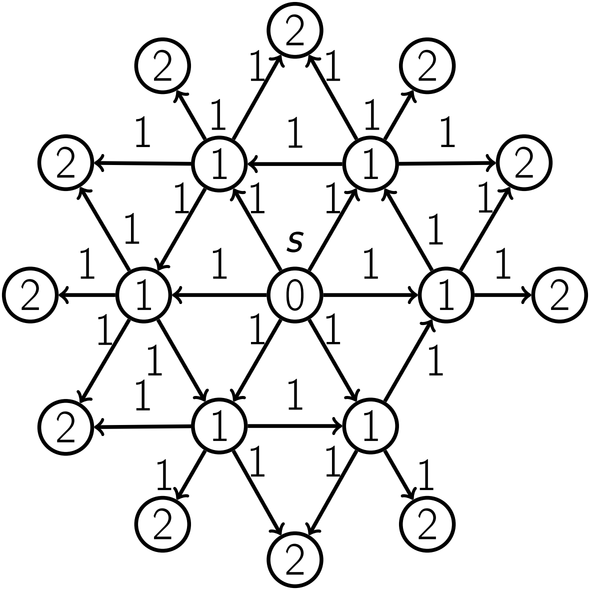

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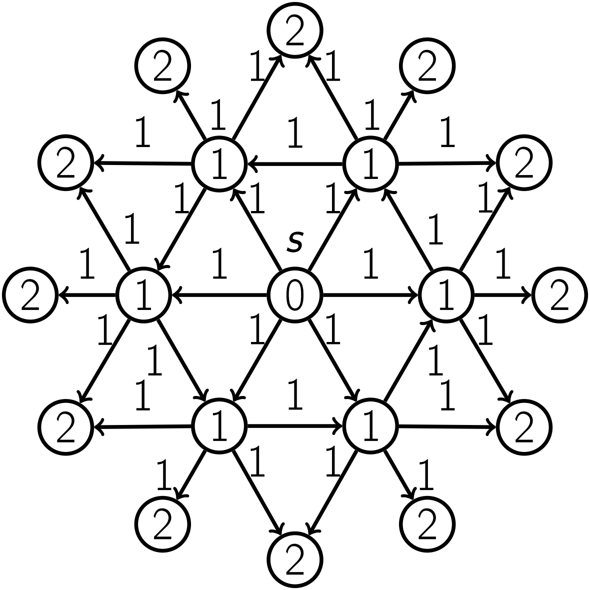

Idea: Growing Circle

Lemma

When a vertex u is selected via `ExtractMin`, $\text{dist}[u] = d(s, u)$.

- When a vertex is extracted from the priority queue for processing, all the vertices at smaller distances have already been processed

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- When a vertex is extracted from the priority queue for processing, all the vertices at smaller distances have already been processed
- A "circle" of processed vertices grows

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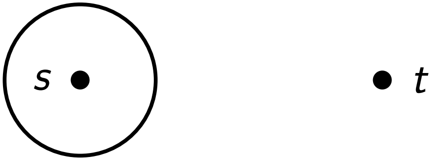

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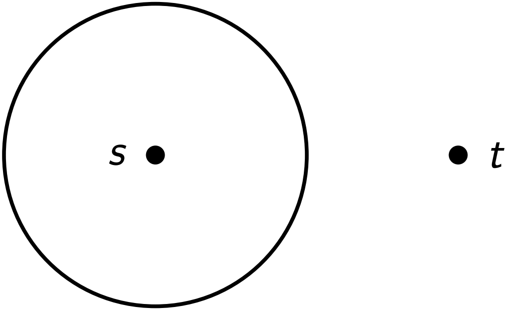

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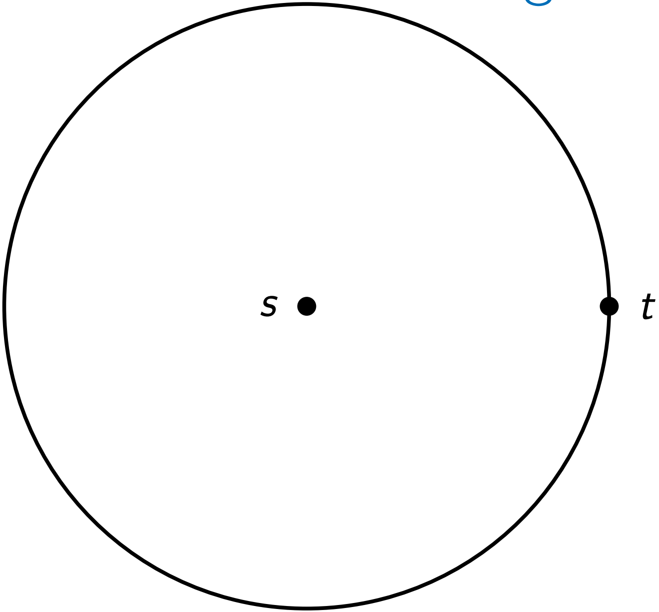

This Indicates calculating distance only till t. Returning as soon as t is found

Idea: Bidirectional Search

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Instead of growing from s to t, Grow from both the directions. From s grow forwards and from t grow backwards. And as soon as common node is found stop the algo. In Algo we cannot grow paths from s and t simultaneously hence we will alternate growing paths from s and t

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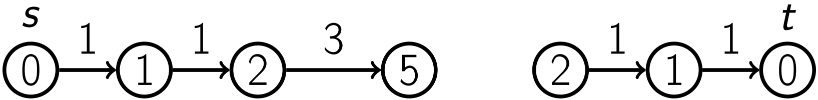

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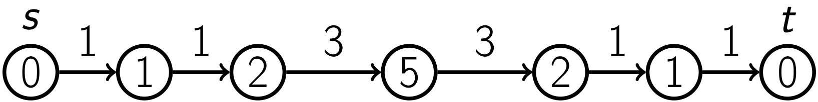

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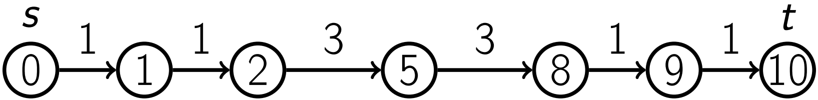

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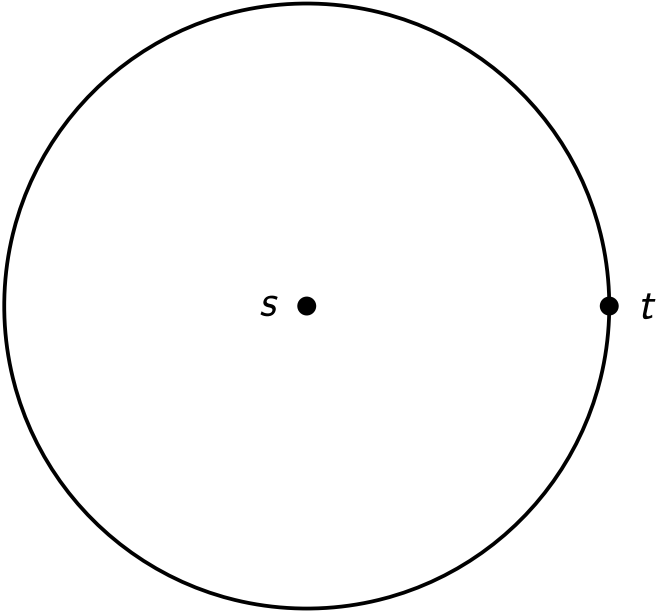

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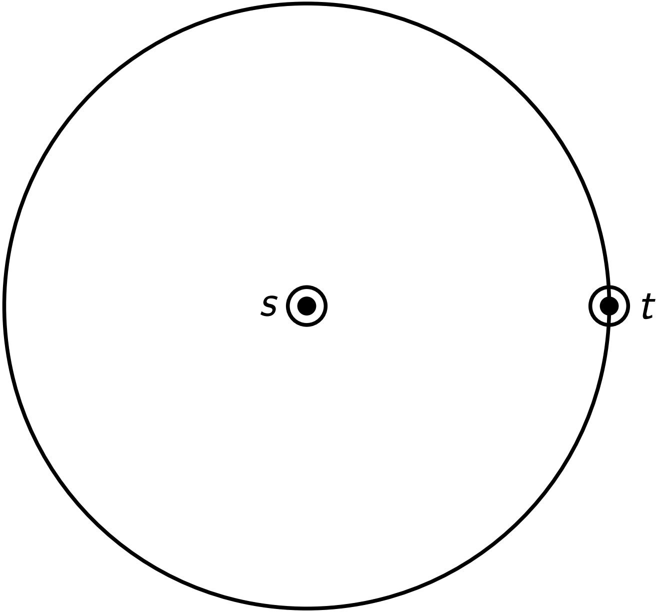

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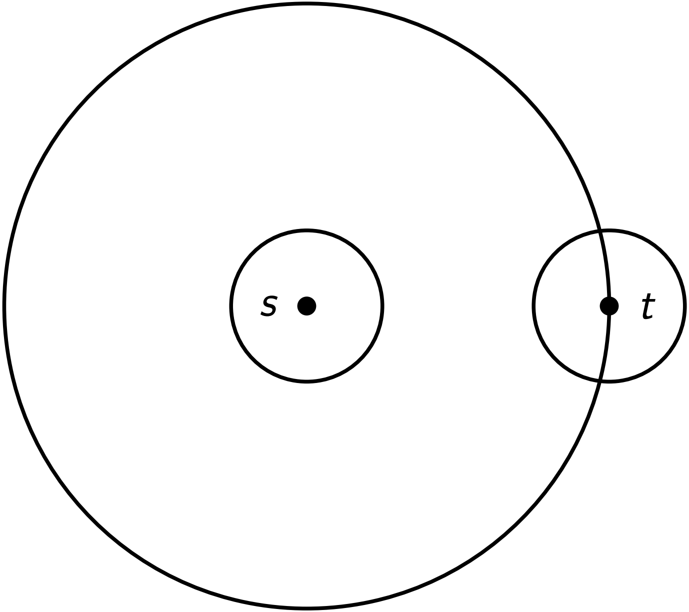

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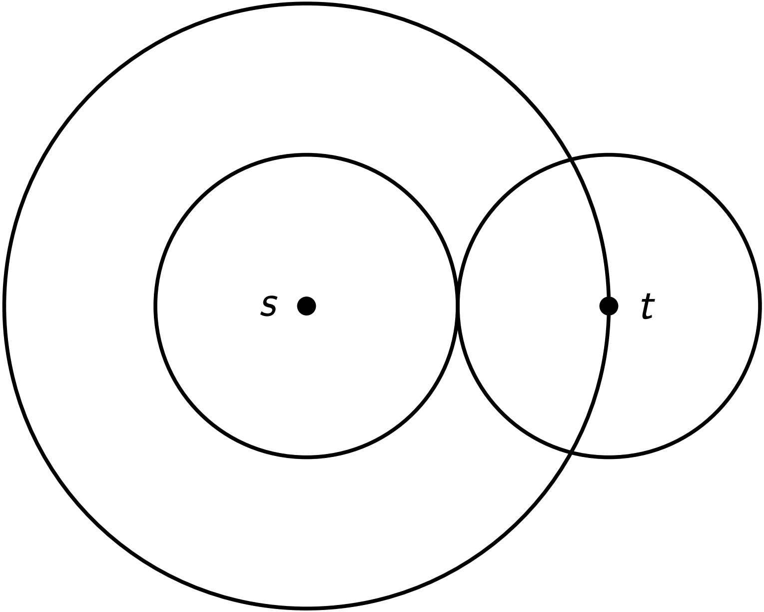

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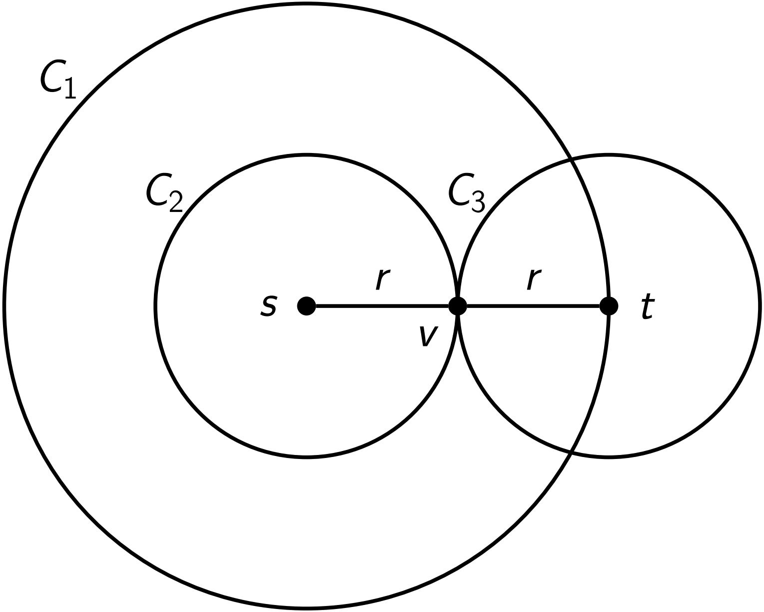
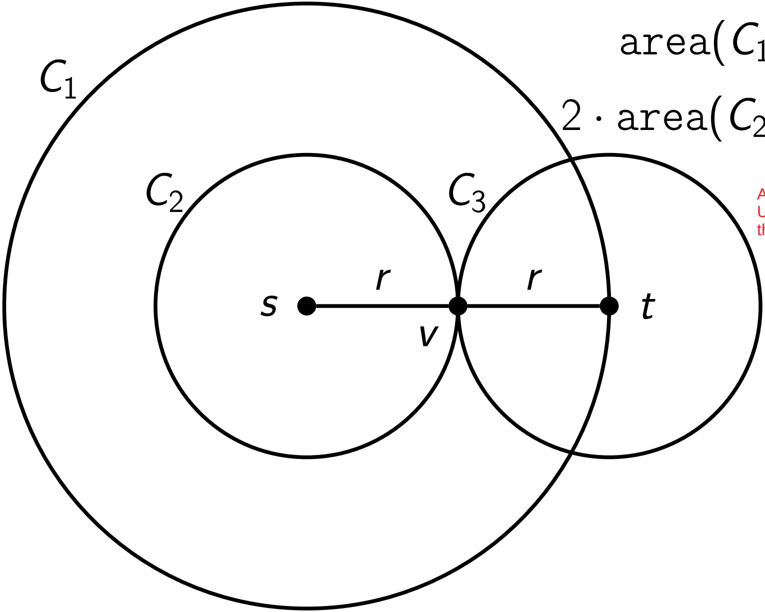


Idea: Bidirectional Search

$$\texttt{area}(C_1) = 4\pi r^2$$

$$2 \cdot \texttt{area}(C_2) = 2\pi r^2$$



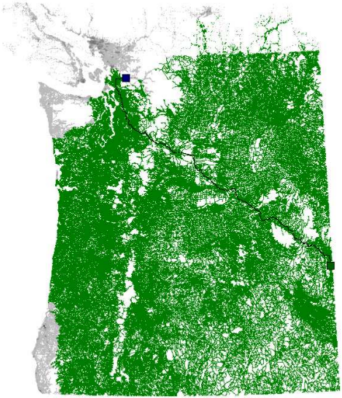
Area covered (Nodes) by Unidirectional twice to that of Bidirectional

Road networks

1.6M vertices, 3.8M arcs, travel time metric.

Picture by Andrew Goldberg.

Road networks

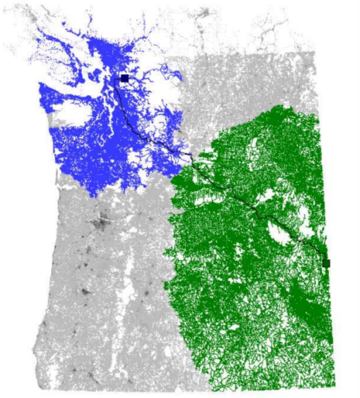


Unidirectional Djikstra's

Searched area

Picture by Andrew Goldberg.

Road networks



forward search/ reverse search

Bidirectional Djikstra's

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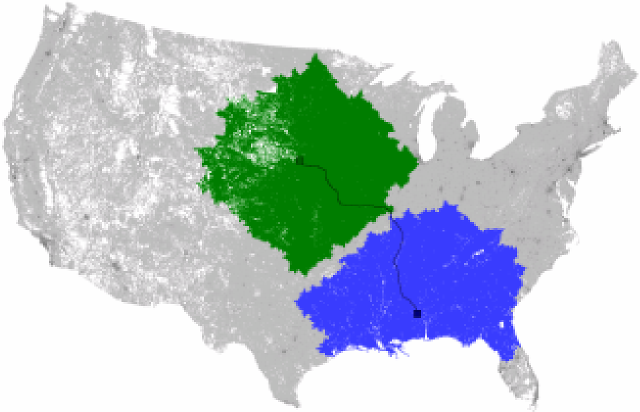

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- Let's look at social networks

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- This is close to truth according to experiments and is called a "six handshakes" or "six degrees of separation" idea

Facebook

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Facebook

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- Then 10000 friends of friends

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- 1 trillion people at six handshakes

Facebook

- Suppose an average person has around 100 Facebook friends
- Then 10000 friends of friends
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- ...
- 1 trillion people at six handshakes
- Not possible, as there are only about 7 billion people on earth

Facebook

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- Roughly 1M friends of friends of friends
- $1M + 1M = 2M$ people — 1000 times less

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- Instead of searching for all possible objects, search for first halves and for second halves separately
- Then find "compatible" halves
- Typically roughly $O(\sqrt{N})$ instead of $O(N)$, including the previous Facebook example

Conclusion

- Dijkstra goes in "circles"
- Bidirectional search idea can reduce the search space
- Roughly 2x speedup for road networks

Meet in Middle Applicable on social network or many other problems and not roads

- Meet-in-the-middle — $\sqrt{N}$ instead of N
- 1000 times faster for social networks
- Next video — Bidirectional Dijkstra algorithm

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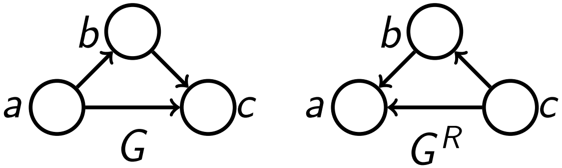
- `ExtractMin` — choose *unprocessed* u with the smallest $\texttt{dist}[u]$
- Process u — `Relax` the edges outgoing from u
- Repeat until t is processed

Reversed Graph

Definition

Reversed graph G^R for a graph G is the graph with the same set of vertices V and the set of reversed edges E^R, such that for any edge $(u, v) \in E$ there is an edge $(v, u) \in E^R$ and vice versa.



Bidirectional Dijkstra

- Build G^R

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Bidirectional Dijkstra

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- Start Dijkstra from s in G and from t in G^R
- Alternate between Dijkstra steps in G and in G^R
- Stop when some vertex v is processed both in G and in G^R
- Compute the shortest path between s and t

Computing Distance

Let v be the first vertex which is processed both in G and in G^R. Does it follow that there is a shortest path from s to t going through v?

Question

Let v be the first vertex which is processed both in the forward search from s in G and in the backward search from t in G^R in the process of Bidirectional Dijkstra algorithm launched to find the shortest path between s and t in graph G. Can we guarantee then that there exists a shortest path from s to t that goes through v?

- ○ No
- ◉ Yes

✗ **Incorrect**
See the counterexample in the lecture.

Skip Retry

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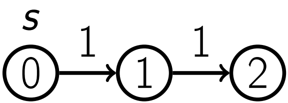

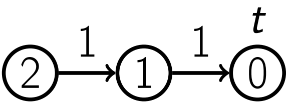

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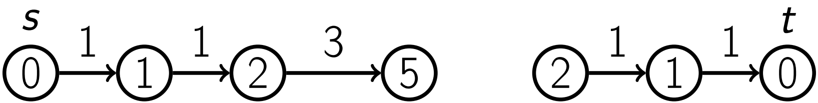

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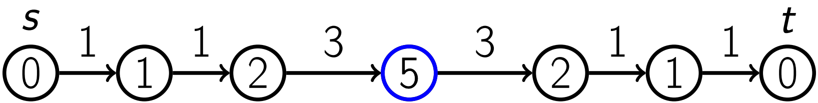

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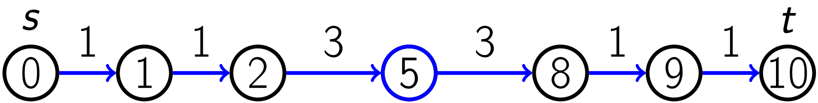

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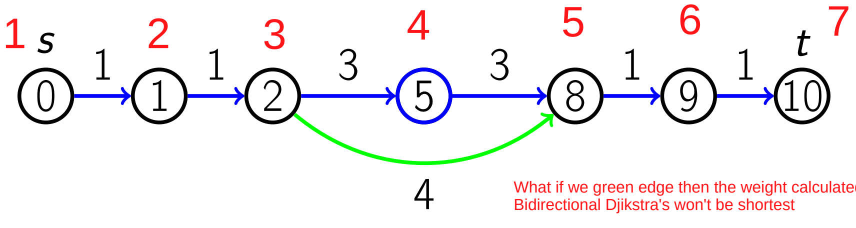

What if we green edge then the weight calculated with Bidirectional Djikstra's won't be shortest

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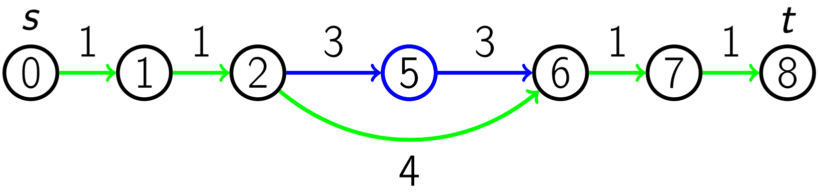
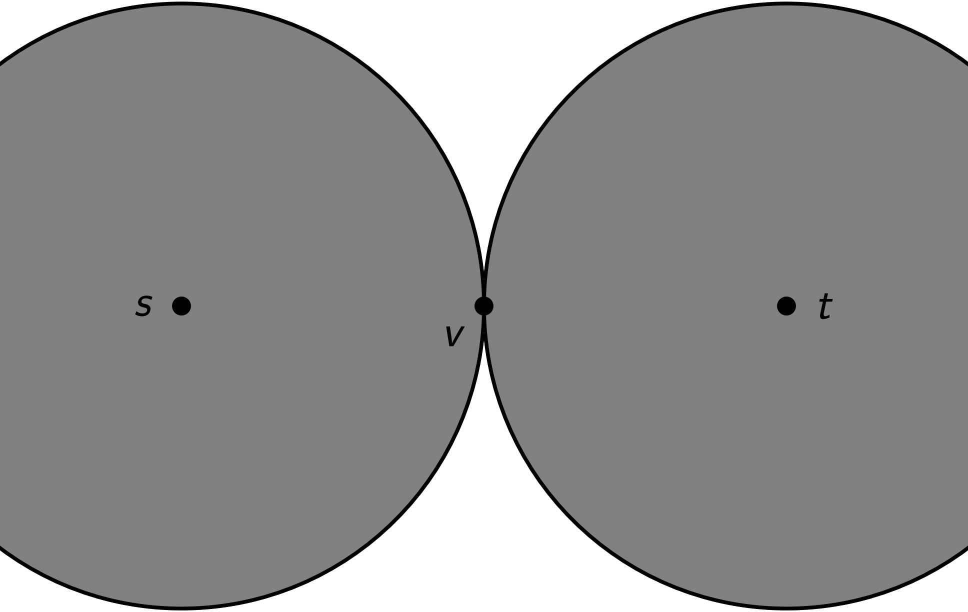


Computing Distance

Lemma

Let $\texttt{dist}[u]$ be the distance estimate in the forward Dijkstra from s in G and $\texttt{dist}^R[u]$ — the same in the backward Dijkstra from t in G^R. After some node v is processed both in G and G^R, some shortest path from s to t passes through some node u which is processed either in G, in G^R, or both, and $d(s,t) = \texttt{dist}[u] + \texttt{dist}^R[u]$.

Proof



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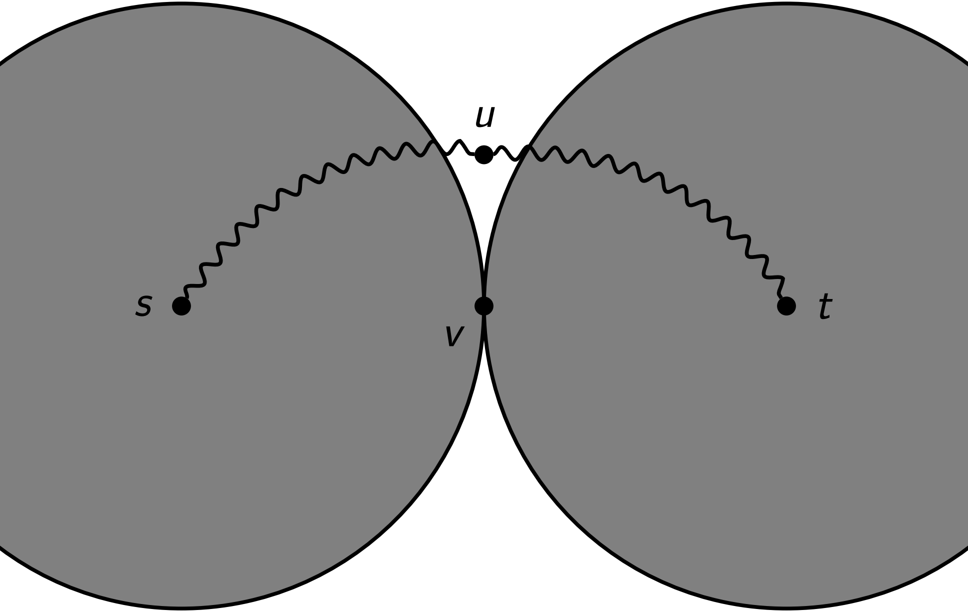
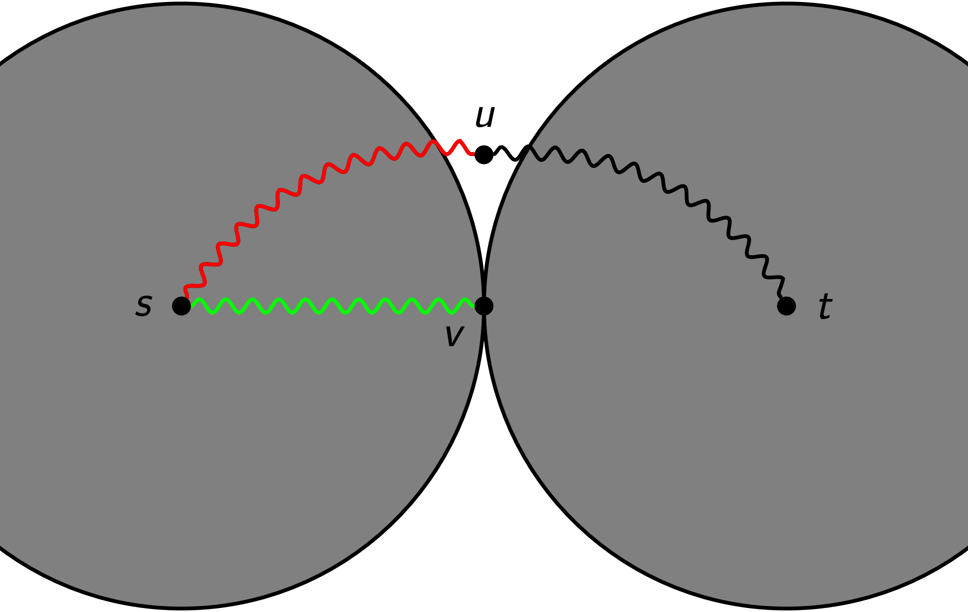
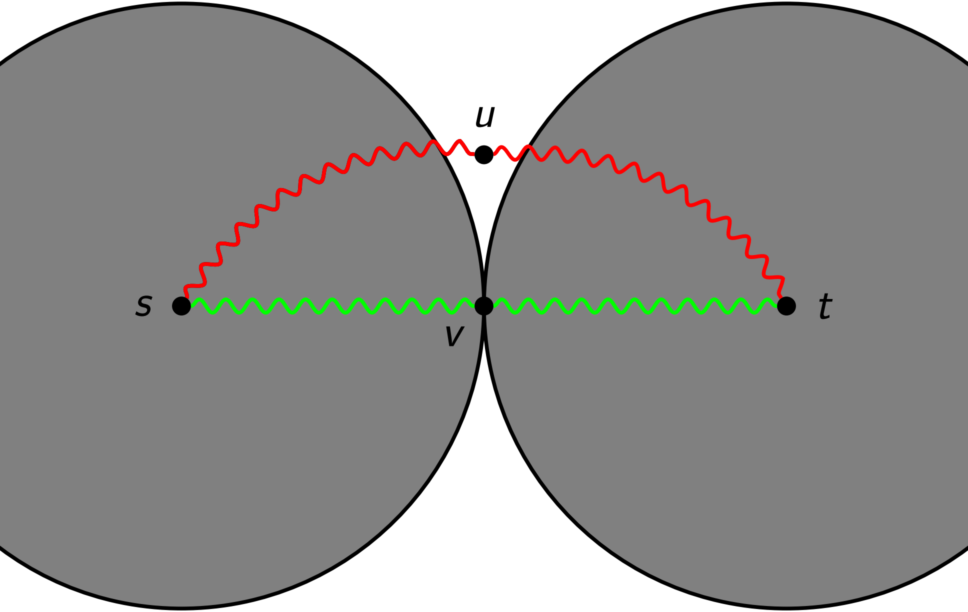


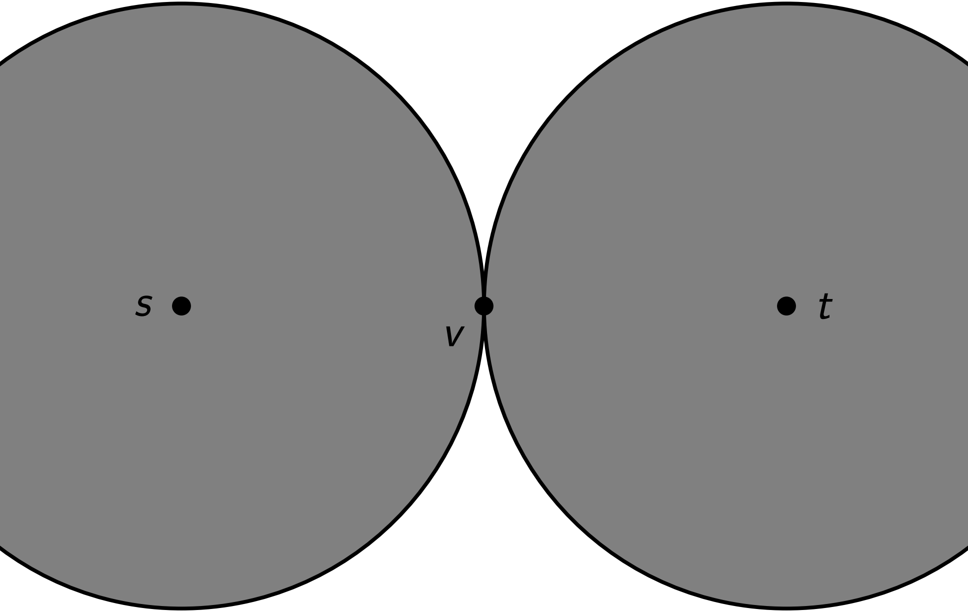
Proof



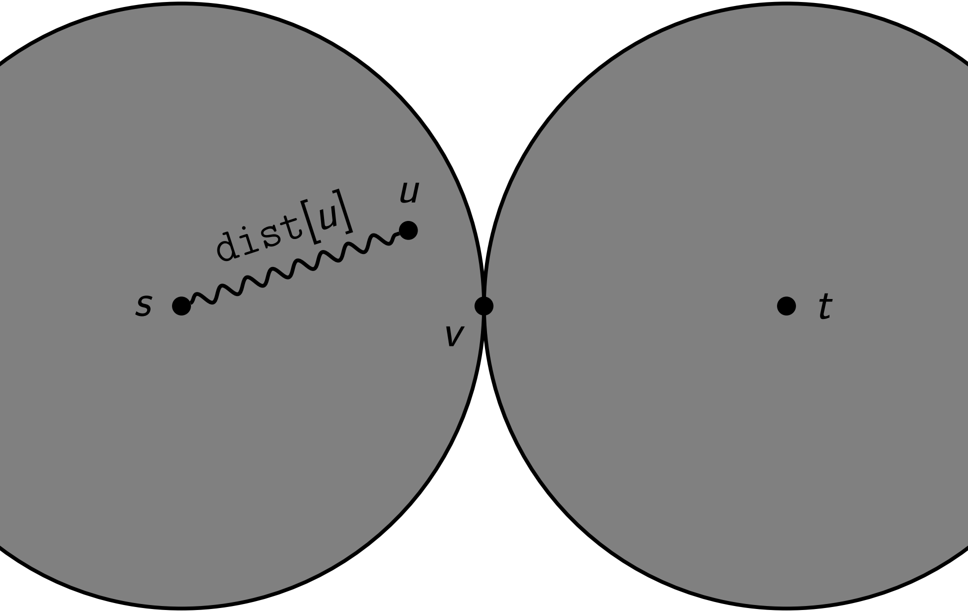
Proof



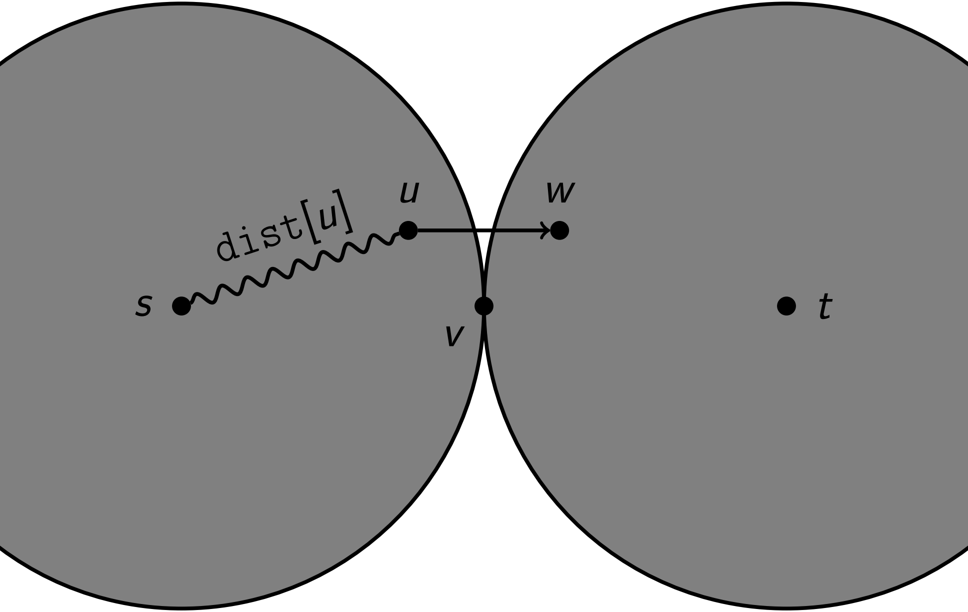
Proof



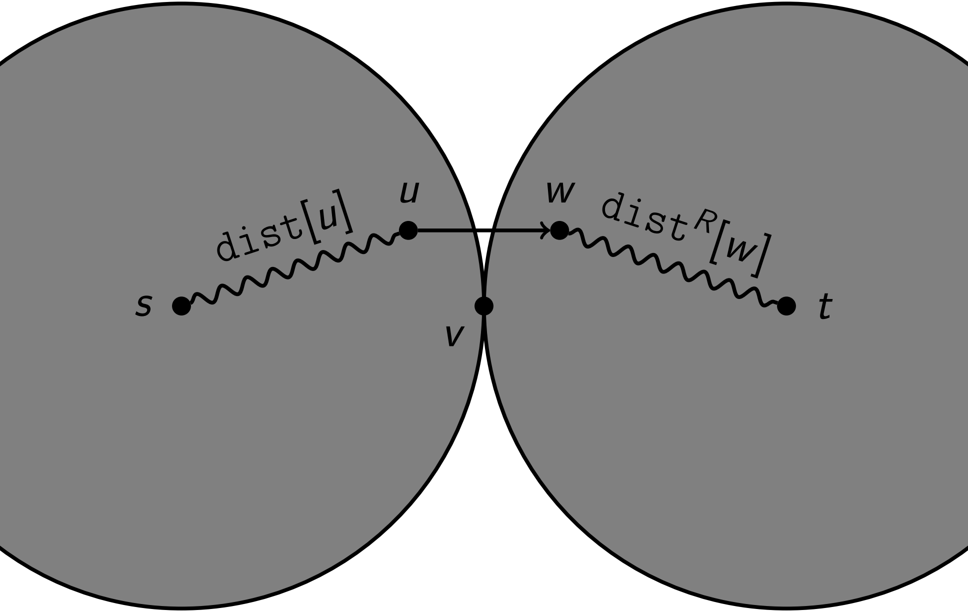
Proof



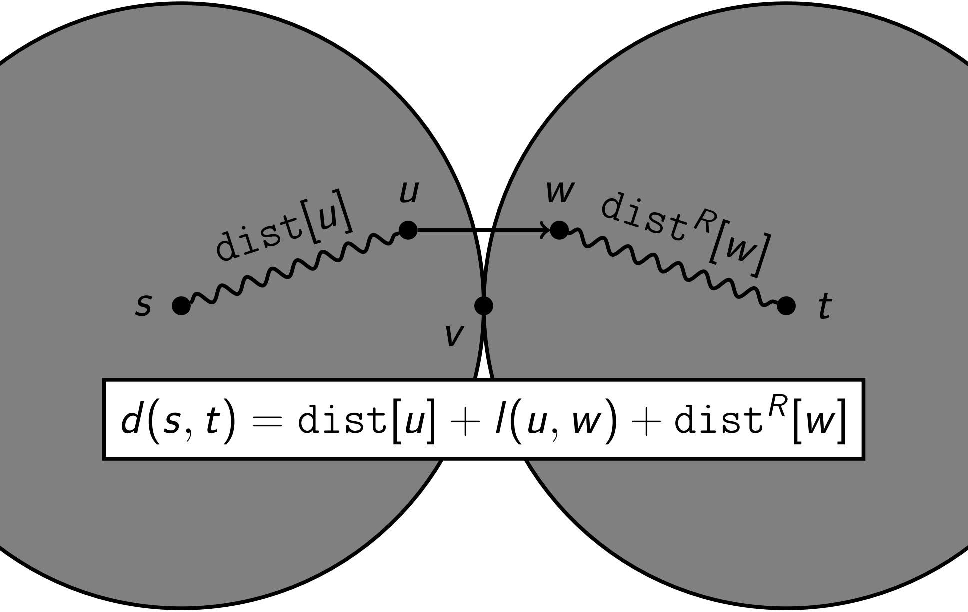
Proof



Proof

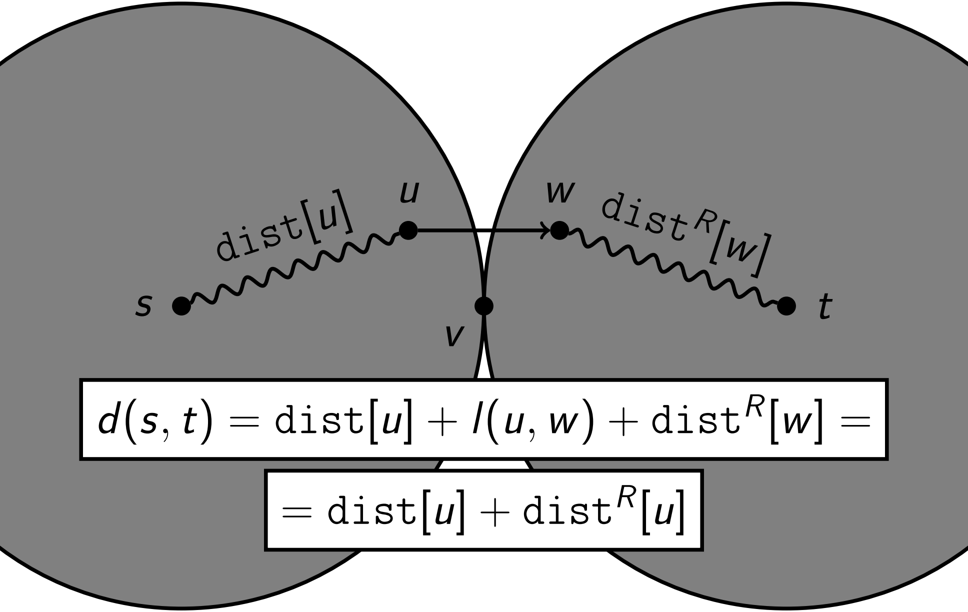


Proof



$$\boxed{d(s,t) = \mathtt{dist}[u] + l(u,w) + \mathtt{dist}^R[w]}$$

Proof



$$\boxed{d(s,t) = \texttt{dist}[u] + l(u,w) + \texttt{dist}^R[w] =}$$

$$\boxed{= \texttt{dist}[u] + \texttt{dist}^R[u]}$$

BidirectionalDijkstra(G, s, t)

```
G^R ← ReverseGraph(G)
Fill dist, dist^R with +∞ for each node
dist[s] ← 0, dist^R[t] ← 0
Fill prev, prev^R with None for each node
proc ← empty, proc^R ← empty
do:
  v ← ExtractMin(dist)
  Process(v, G, dist, prev, proc)
  if v in proc^R:
    return ShortestPath(s, dist, prev, proc, t, ...)
  v^R ← ExtractMin(dist^R)
  repeat symmetrically for v^R as for v
while True
```

$\texttt{Relax}(u, v, \texttt{dist}, \texttt{prev})$

$\texttt{if dist}[v] > \texttt{dist}[u] + w(u, v)$:

$\quad \texttt{dist}[v] \leftarrow \texttt{dist}[u] + w(u, v)$

$\quad \texttt{prev}[v] \leftarrow u$

```
Process(u, G, dist, prev, proc)

for (u, v) ∈ E(G):
  Relax(u, v, dist, prev)
proc.Append(u)
```

$\texttt{ShortestPath}(s, \texttt{dist}, \texttt{prev}, \texttt{proc}, t, \texttt{dist}^R, \texttt{prev}^R, \texttt{proc}^R)$

```
distance ← +∞,  u_best ← None
for u in proc + proc^R:
    if dist[u] + dist^R[u] < distance:
        u_best ← u
        distance ← dist[u] + dist^R[u]
path ← empty
last ← u_best
while last ≠ s:
    path.Append(last)
    last ← prev[last]
path ← Reverse(path)
last ← u_best
while last ≠ t:
    last ← prev^R[last]
    path.Append(last)
return (distance, path)
```

Conclusion

- Worst-case running time of Bidirectional Dijkstra is the same as for Dijkstra
- Speedup in practice depends on the graph For Road Networks 2X but for social networks it can be 1000 times
- Memory consumption is 2x to store G and G^R
- You'll see the speedup on social network graph in the Programming Assignment