# Linear Programming: Gaussian Elimination

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Advanced Algorithms and Complexity
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## Learning Objectives

- Solve a system of linear equations.
- Implement a row reduction algorithm.
- Say something about what the set of solution to a system of linear equations looks like.

# Last Time

Linear programming: Dealing with systems of linear inequalities.

Slides not the same as shown in lectures

# Linear Algebra

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For example:

$$\begin{aligned} x + y &= 5 \\ 2x + 4y &= 12. \end{aligned}$$

# Method of Substitution

- Use first equation to solve for one variable in terms of the others.
- Substitute into other equations.
- Solve recursively.
- Substitute back in to first equation to get initial variable.

# Example

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2x + 4y &= 12.
\end{aligned}
$$

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Substituting into second:

$$12 = 2x + 4y = 2(5 - y) + 4y = 10 + 2y.$$

So $y = 1, x = 5 - 1 = 4.$

# Problem

What is the value of $x$ in the solution to the following linear system?

$$\begin{aligned} x + 2y &= 6 \\ 3x - y &= -3. \end{aligned}$$

# Solution

From the first equation, we get

$$x = 6 - 2y.$$

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If there are n variables and n+1 equations then first n equations will get the value of first n variables and the remaining equations will not be able to satisfy initial value

Substituting into the second,

$$-3 = 3(6 - 2y) - y = 18 - 7y.$$

If there are n+1 variables and n equations then we cannot get the values of all the variables and one variable will always be in terms of other Variables hence instead of having just one solution we will have a space of solutions or a line of solutions

Solving gives, $y = 3$, so $x = 6 - 2 \cdot 3 = 0$.

# Notation

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We just store coefficients of equations in an (augmented) matrix, like so:

$$\begin{array}{c} \begin{array}{cc} x & y = 1 \end{array} \\ \left[\begin{array}{cc|r} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \end{array}$$

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Subtract twice first row from second.

# Basic Row Operations

There are three basic ways to manipulate our matrix. These are called Basic row operations. Each of them gives us an equivalent system of equations.

# Adding

Add/subtract a multiple of one row to another.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right]$$

# Scaling

Multiply/divide a row by a non-zero constant.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

# Swapping

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

# Row Reduction

Row reduction uses row operations to put a matrix into a simple standard form. The idea is to simulate the substitution method.

# Example

Consider the system given by the matrix:

Leftmost non zero entry is called pivot and the rows are called pivot rows. If you don't understand this see algo

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Use first for to solve for first variable.

$$\left[\begin{array}{rrrr|r} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Divide first row by 2.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Substitute into other equations.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Add first row to second.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Subtract twice first row from third.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

# Example

Need to solve for next variable.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

# Example

Cannot use second row.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

# Example

Swap second and third rows.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Solve for second variable.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Divide second row by $-2$.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Subtract twice second row from first.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Can't solve for third variable.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Solve for fourth instead.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Divide last row by $-2$.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Substitute into other equations.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Subtract twice third row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Add third row to second.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Done.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Answer

Our matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

corresponds to equations:

$$\begin{aligned} x + z &= -1 \\ y - z &= 1 \\ w &= 0. \end{aligned}$$

# Solution

So for any value of $z$, we have solution:

$$\begin{aligned} x &= -1 - z \\ y &= 1 + z \\ w &= 0. \end{aligned}$$

## RowReduce($A$)

Leftmost non-zero

Swap row to top

Make entry pivot

Rescale to make pivot 1

Subtract row from others to make other entries in column 0

Repeat

## RowReduce($A$)

Leftmost non-zero in non-pivot row

Swap row to top of non-pivot rows

Make entry pivot

Rescale to make pivot 1

Subtract row from others to make other entries in column 0

Repeat until no more non-zero entries outside of pivot rows

https://onlinemschool.com/math/assistance/equation/gaus/

Visit this link to solve any equations using gaussian eliminations

# Reading off Answer

- Each row has one pivot and a few other non-pivot entries.
- Gives equation writing pivot variable in terms of non-pivot variables.
- If pivot in units column, have equation $0 = 1$, so no solutions. If 0 = 0 then it has infinite solutions
- Otherwise, set non-pivot variables to anything, gives answer.

Non pivot variables are the variables in rows with no pivot in them are actually free variables, we can set those to whatever we want and then once we've done that each of our rows tells us what pivot variables should be in terms of non-pivot variables

# Degrees of Freedom

- Your solution set will be a subspace.
- Dimension = number of non-pivot variables.
- Or $n$ minus the number of pivot variables.
- Generally, dimension equals

$$\text{num. variables} - \text{num. equations}.$$

# Runtime

- $m$ equations in $n$ variables.
- $\min(n, m)$ pivots.
- For each pivot, need to subtract multiple of row from each other row $O(nm)$ time.
- Total runtime: $O(nm \min(n, m))$.