NP-complete Problems: Reductions

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Outline

1. Reductions
2. Showing **NP**-completeness
3. Independent Set $\rightarrow$ Vertex Cover
4. 3-SAT $\rightarrow$ Independent Set
5. SAT $\rightarrow$ 3-SAT
6. All of **NP** $\rightarrow$ SAT
7. Using SAT-solvers

Informally

We say that a search problem A is reduced to a search problem B and write $A \to B$, if a polynomial time algorithm for B can be used (as a black box) to solve A in polynomial time.

Reduction: $A \to B$

instance I of A

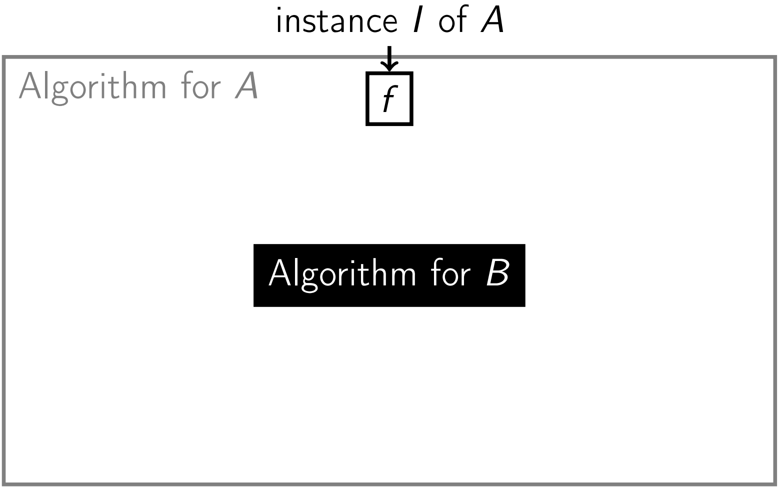
Reduction: $A \to B$

instance I of A

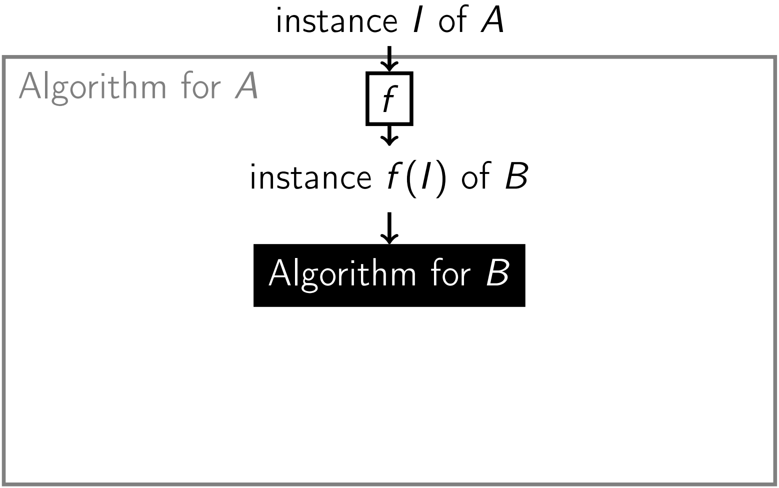
Algorithm for A

Algorithm for B

Reduction: $A \to B$



Reduction: $A \to B$



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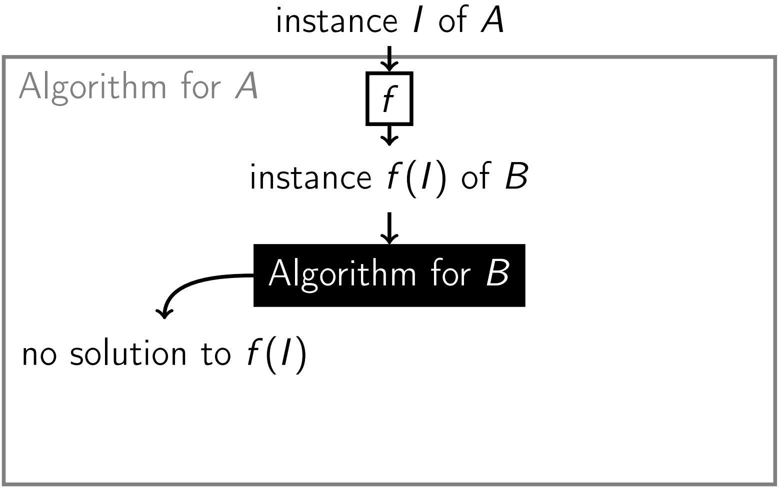

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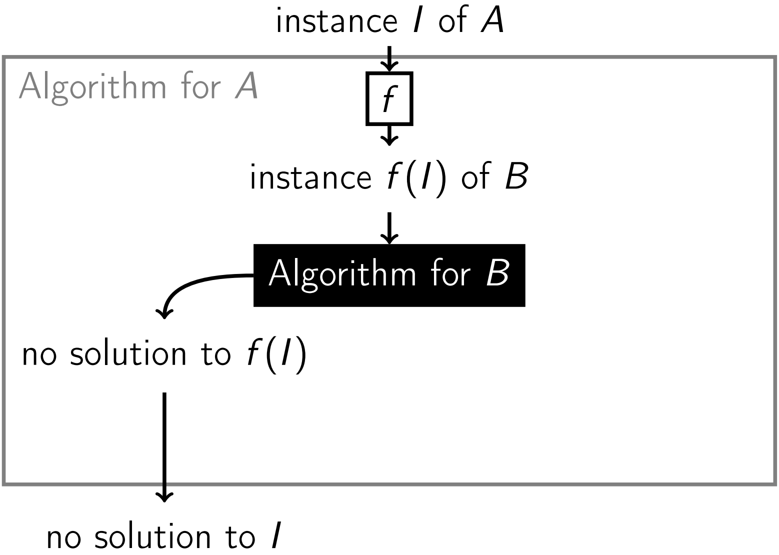
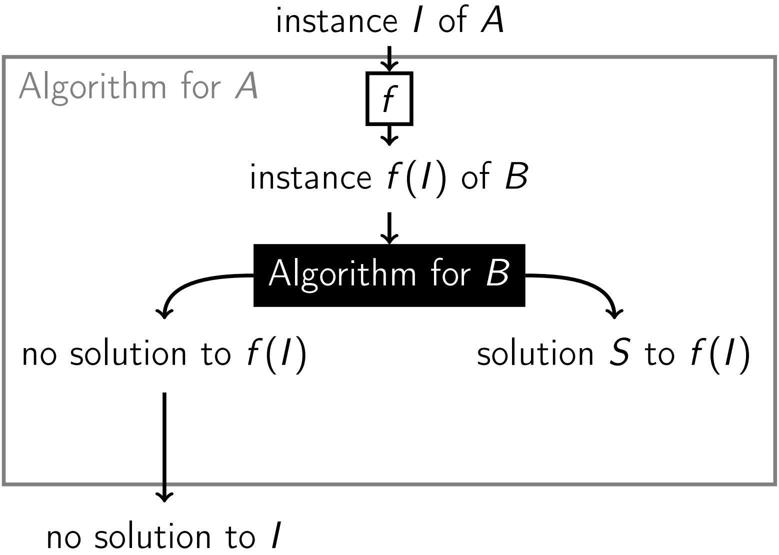


Reduction: $A \rightarrow B$



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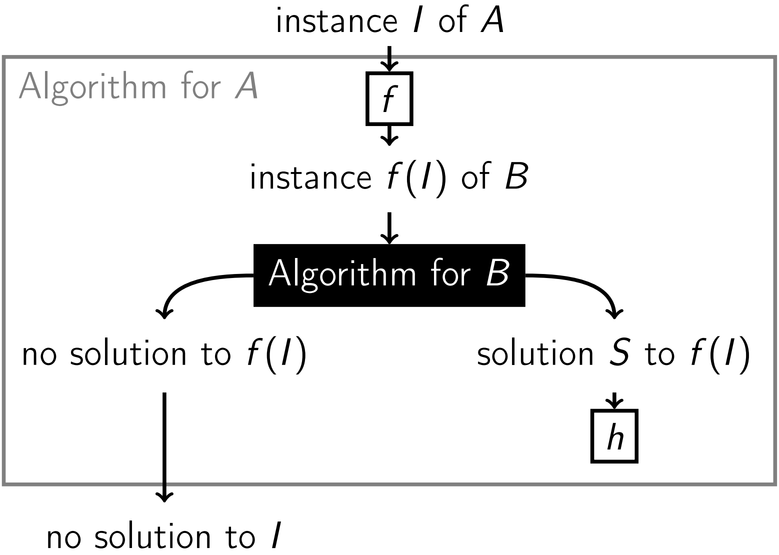
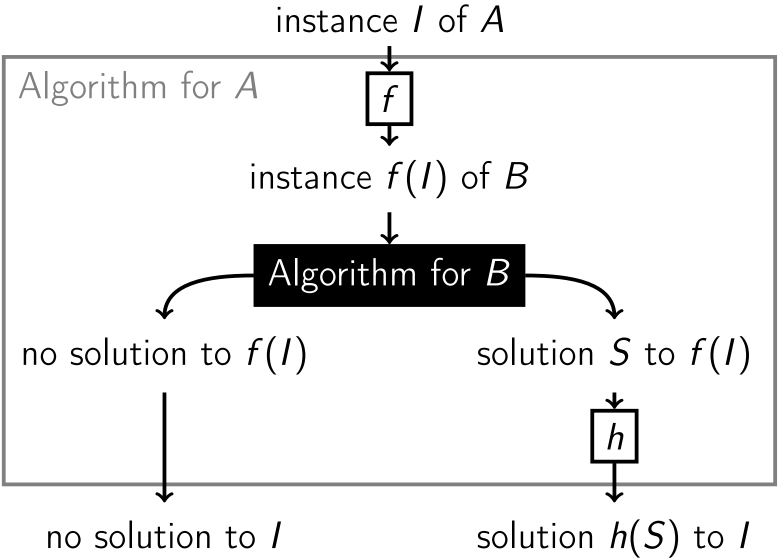


Reduction: $A \to B$

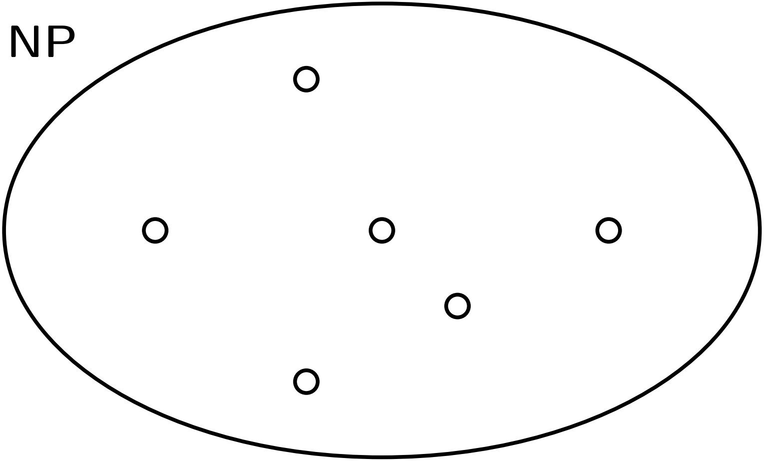


Formally

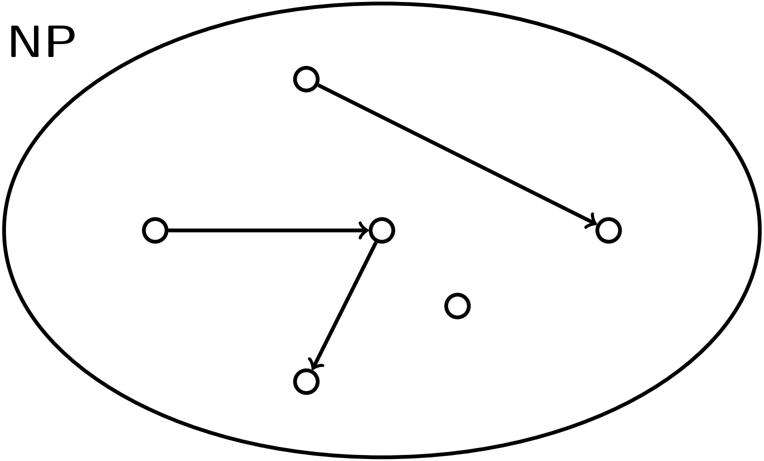
Definition

We say that a search problem A is reduced to a search problem B and write $A \to B$, if there exists a polynomial time algorithm f that converts any instance I of A into an instance $f(I)$ of B, together with a polynomial time algorithm h that converts any solution S to $f(I)$ back to a solution $h(S)$ to A. If there is no solution to $f(I)$, then there is no solution to I.

Graph of Search Problems



Graph of Search Problems



NP-complete Problems

Definition

A search problem is called **NP**-complete if all other search problems reduce to it.

Meaning if you can solve one such P problem you can solve all the search problems of a big NP problem

NP-complete Problems

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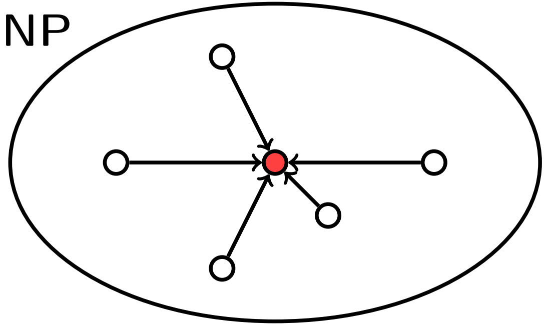

Do they exist?

It's not at all immediate that **NP**-complete problems even exist. We'll see later that all hard problems that we've seen in the previous part are in fact **NP**-complete!

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Two ways of using $A \to B$:

1. if B is easy (can be solved in polynomial time), then so is A
2. if A is hard (cannot be solved in polynomial time), then so is B

Reductions Compose

Lemma

If $A \to B$ and $B \to C$, then $A \to C$.

Proof

- The reductions $A \to B$ and $B \to C$ are given by pairs of polytime algorithms (f_{AB}, h_{AB}) and (f_{BC}, h_{BC}).

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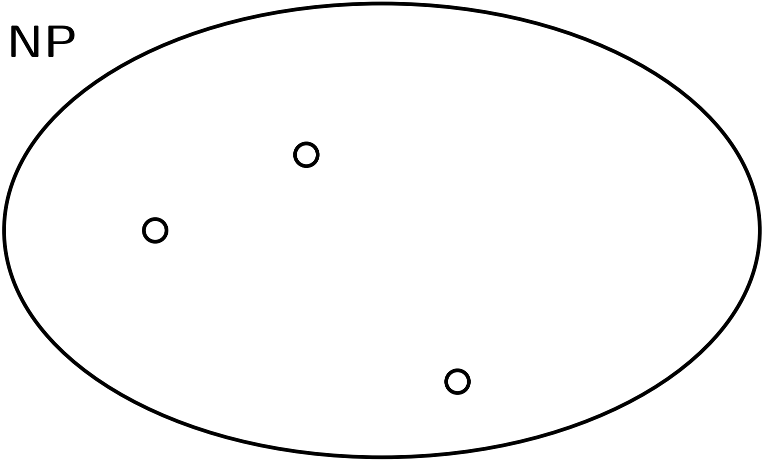
- To transform an instance I_A of A to an instance I_C of C we apply a polytime algorithm $f_{BC} \circ f_{AB}$: $I_C = f_{BC}(f_{AB}(I_A))$.

Proof

- The reductions $A \to B$ and $B \to C$ are given by pairs of polytime algorithms (f_{AB}, h_{AB}) and (f_{BC}, h_{BC}).
- To transform an instance I_A of A to an instance I_C of C we apply a polytime algorithm $f_{BC} \circ f_{AB}$: $I_C = f_{BC}(f_{AB}(I_A))$.
- To transform a solution S_C to I_C to a solution S_A to I_A we apply a polytime algorithm $h_{AB} \circ h_{BC}$: $S_A = h_{AB}(h_{BC}(S_C))$. $\square$

Pictorially



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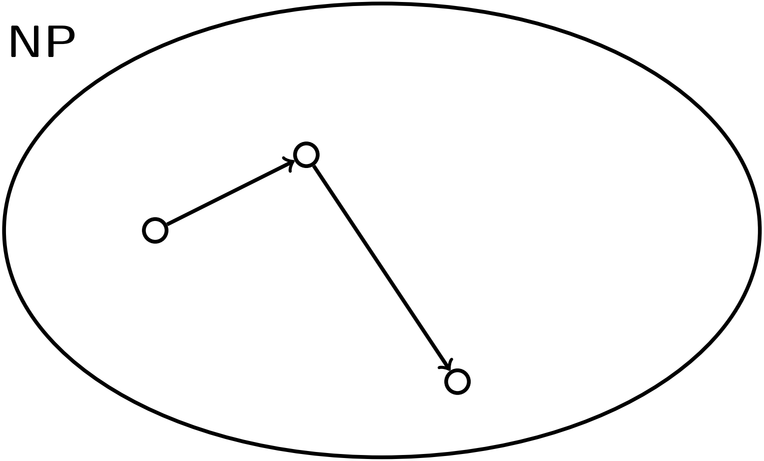
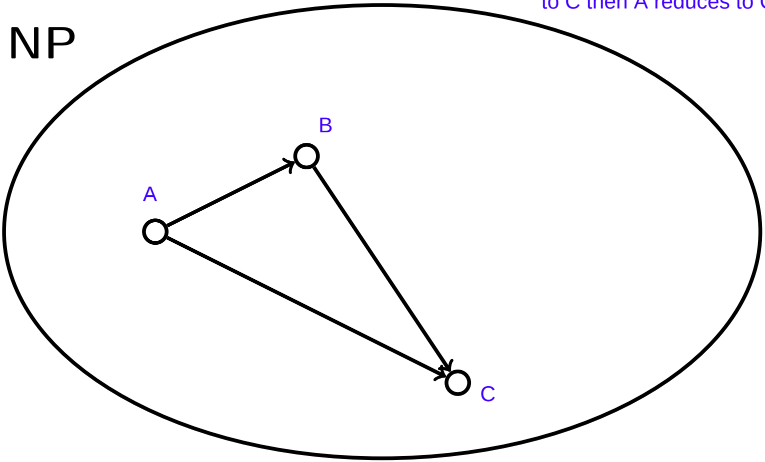


Pictorially

If A reduces to B and B reduces to C then A reduces to C



Showing **NP**-completeness

Corollary

If $A \to B$ and A is **NP**-complete, then so is B.

Showing $\mathbf{NP}$-completeness

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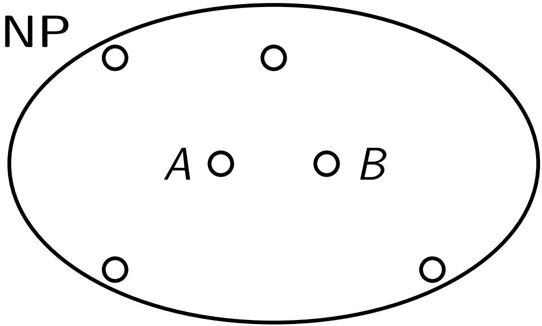

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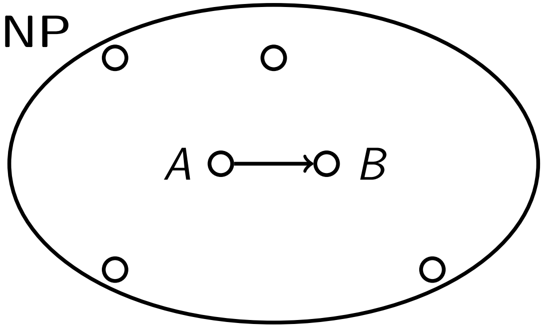
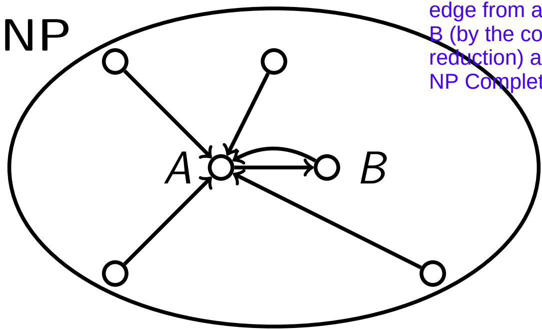


Showing **NP**-completeness

Corollary

If $A \to B$ and A is **NP**-complete, then so is B.

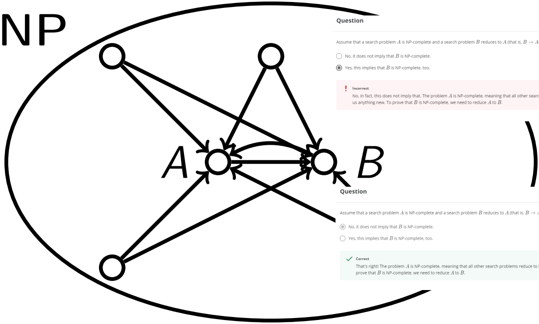
There is an edge from any other problem to A and at the same time there is an edge from A to B then it means that there is an edge from any other problem to B (by the composition of reduction) and hence B is also NP Complete



Showing **NP**-completeness

Corollary

If $A \to B$ and A is **NP**-complete, then so is B.



Question

Assume that a search problem A is NP-complete and a search problem B reduces to A (that is, $B \to A$). Does this mean that B is also NP-complete?

- ○ No, it does not imply that B is NP-complete.
- ◉ Yes, this implies that B is NP-complete, too.

! Incorrect
No, in fact, this does not imply that. The problem A is NP-complete, meaning that all other search problems reduce to it — in particular, the problem B. Hence the fact that B reduces to A does not tell us anything new. To prove that B is NP-complete, we need to reduce A to B.

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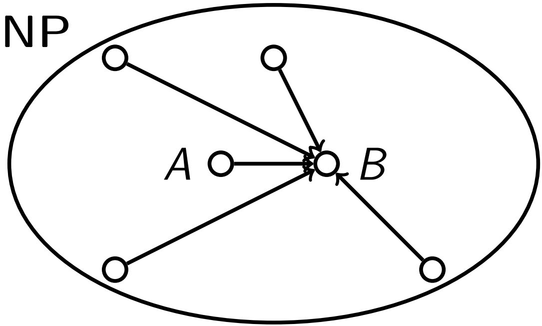
- ◉ No, it does not imply that B is NP-complete.
- ○ Yes, this implies that B is NP-complete, too.

✓ Correct
That's right! The problem A is NP-complete, meaning that all other search problems reduce to it — in particular, the problem B. Hence the fact that B reduces to A does not tell us anything new. To prove that B is NP-complete, we need to reduce A to B.

Showing **NP**-completeness

Corollary

If $A \to B$ and A is **NP**-complete, then so is B.



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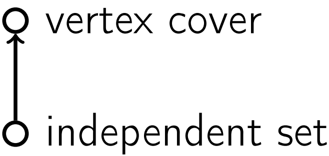

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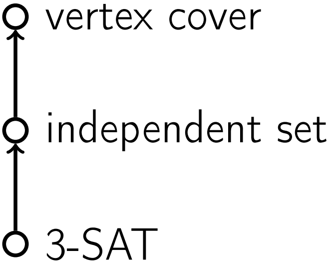

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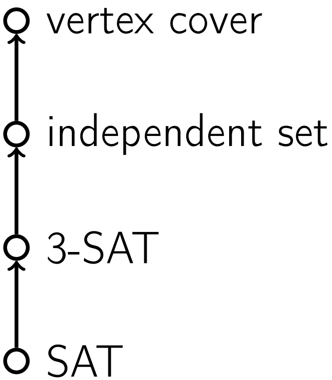

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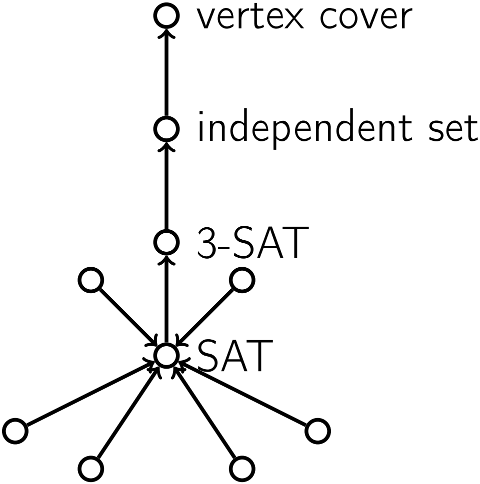

Outline

Independent set

Input: A graph and a budget b.

Output: A subset of at least b vertices such that no two of them are adjacent.

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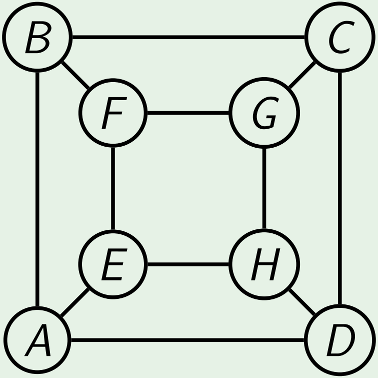
Vertex cover

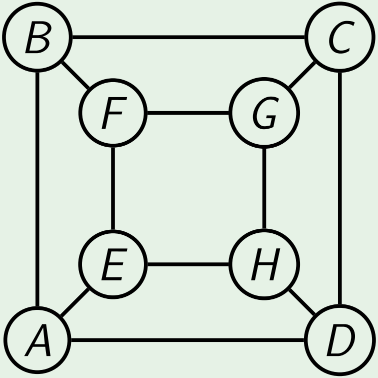
Input: A graph and a budget b.

Output: A subset of at most b vertices that touches every edge.

Example

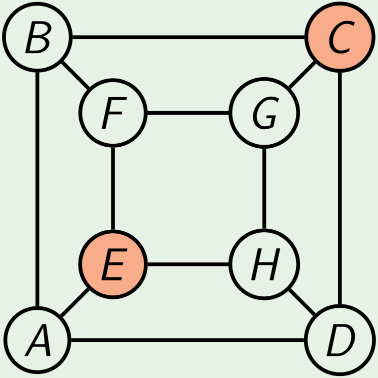


Example



Independent sets:

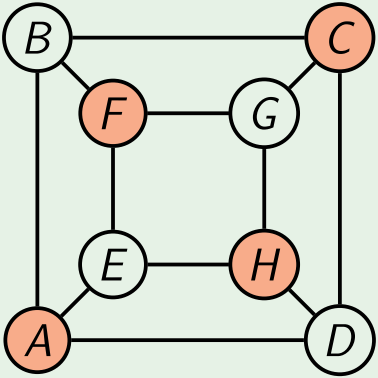
Example



Independent sets:

$\{E, C\}$

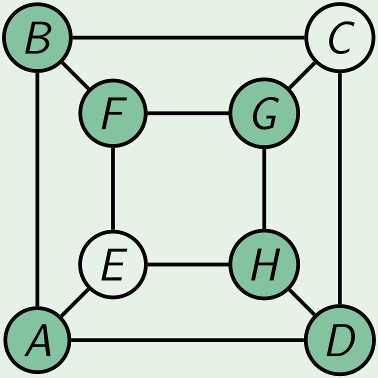
Example



Independent sets:

$\{E, C\}$ $\{A, C, F, H\}$

Example



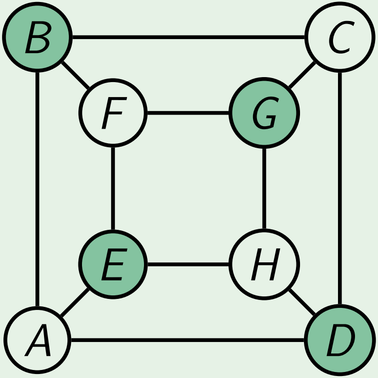
Independent sets:

$\{E, C\}$ $\{A, C, F, H\}$

Vertex covers:

$\{A, B, D, F, G, H\}$

Example



Independent sets:

$\{E, C\}$ $\{A, C, F, H\}$

Vertex covers:

$\{A, B, D, F, G, H\}$

$\{B, D, E, G\}$

Lemma

I is an independent set of $G(V, E)$, if and only if $V - I$ is a vertex cover of G.

Proof

- $\Rightarrow$ If I is an independent set, then there is no edge with both endpoints in I. Hence $V - I$ touches every edge.
- $\Leftarrow$ If $V - I$ touches every edge, then each edge has at least one endpoint in $V - I$. Hence I is an independent set. $\square$

Reduction

Independent set $\rightarrow$ vertex cover: to check whether $G(V, E)$ has an independent set of size at least b, check whether G has a vertex cover of size at most $|V| - b$:

f needs to transfer instance of independent set problem to instance of vertex cover, so we need to transfer budget b to |V|-b now we need to call .

- $f(G(V, E), b) = (G(V, E), |V| - b)$

Then we call algorithm to solve vertex cover problem on the following instance

- $h(S) = V - S$

If it finds that there is no vertex cover of size atmost |V|-b then we immediately report that there is no independent set of size atleast b in our graph otherwise it returns some solution S which is the vertex cover in graph G of size atmost |V|-b by taking complementary of it we get the independent set of size atleast b

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3-SAT

Input: Formula F in 3-CNF (a collection of clauses each having at most three literals).

Output: An assignment of Boolean values to the variables of F satisfying all clauses, if exists.

Goal

Design a polynomial time algorithm that, given a 3-CNF formula F, outputs a graph G and an integer b, such that:

> *F is satisfiable, if and only if G has an independent set of size at least b.*

We need to find an assignment of Boolean values to variables, such that each clause contains at least one satisfied literal.

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Example

- Setting $x = 1, y = 1, z = 1$ satisfies a formula $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.

One entire equation is called a clause
x,y,z are variables
satisfied literal is a "TRUE" variable

We need to find an assignment of Boolean values to variables, such that each clause contains at least one satisfied literal.

Example

- Setting $x = 1, y = 1, z = 1$ satisfies a formula $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.
- Setting $x = 1, y = 0, z = 0$ also satisfies it: $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.

Alternatively, we need to select at least one literal from each clause, such that the set of selected literals is consistent: it does not contain a literal ℓ together with its negation $\overline{\ell}$.

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Like over here we cannot select y, y' and z'. because these selected literals has to be satisfied meaning "TRUE" and y can have only one value (specific) either TRUE or FALSE. So it is not possible to have y TRUE in one clause and also y' TRUE in other clause at the same time

Example: $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$

- Consistent: $\{x, x, \overline{z}\}$, $\{x, x, y\}$, $\{x, \overline{y}, \overline{z}\}$.

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- Consistent: $\{x, x, \overline{z}\}$, $\{x, x, y\}$, $\{x, \overline{y}, \overline{z}\}$.
- Inconsistent: $\{y, \overline{y}, \overline{z}\}$, $\{z, x, \overline{z}\}$.

Using Alternative Statement

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$$

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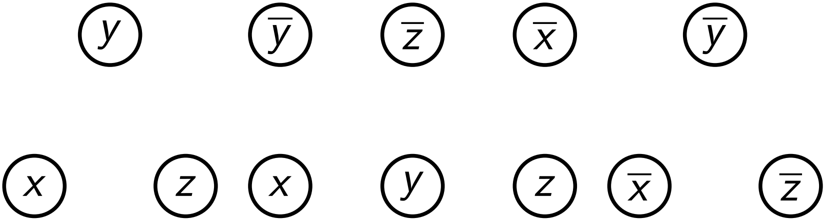

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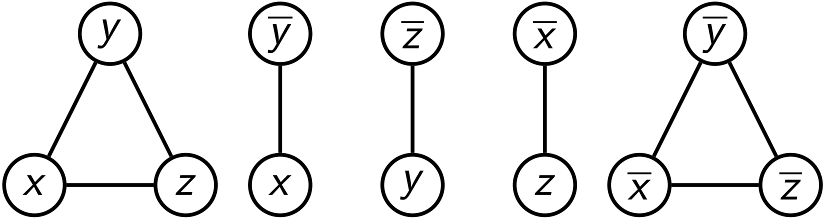

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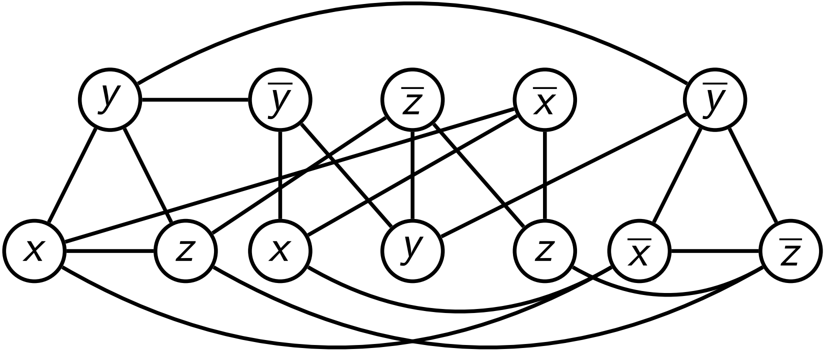

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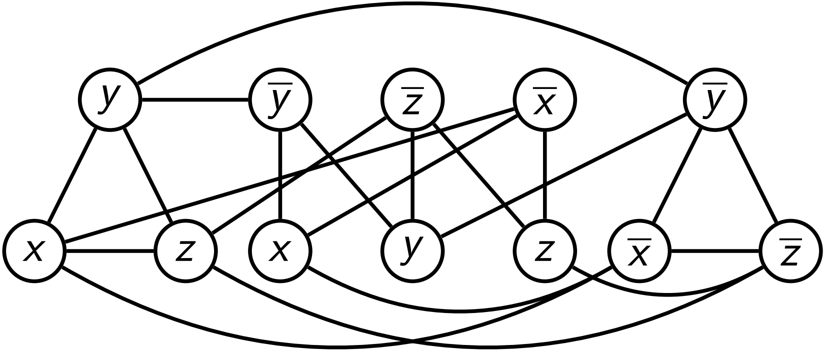

the formula is satisfiable iff the resulting graph has independent set of size 5

Transforming an Instance

- For each clause of the input formula F, introduce three (or two, or one) vertices in G labeled with the literals of this clause. Join every two of them.

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- F is satisfiable if and only if G has independent set of size equal to the number of clauses in F.

Transforming an Instance

Question is interesting. If some assignment satisfies CNF F it does not
mean that the corresponding set having vertices equal to m clauses in

Question

Given a 3-CNF formula F with m clauses, we constructed a graph G that has an independent set of size m if and only if the initial formula is satisfiable. Does this reduction preserve the number of solutions? In other words, is it true that the number of satisfying assignments of F is equal to the number of independent sets of G of size m?

- (•) Yes, this is true.
- ( ) No, it is not true.

Incorrect

No, this reduction does not preserve the number of solutions. For example, for a formula $(x \vee y)$ the reduction produces a graph consisting of a single edge between x and y. This graph has two

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- ( ) Yes, this is true.
- (•) No, it is not true.

Correct

That's right! For example, for a formula $(x \vee y)$ the reduction produces a graph consisting of a single edge between x and y. This graph has two independent sets of size 1, but the formula has three satisfying assignments.

- For each claus introduce three in G labeled w clause. Join ev
- Join every pair of vertices labeled with complementary literals.
- F is satisfiable if and only if G has independent set of size equal to the number of clauses in F.
- Transformation takes polynomial time.

Transforming a Solution

- Given a solution S for G, just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).

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- Given a solution S for G, just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).
- If there is no solution for G, then F is unsatisfiable: indeed, a satisfying assignment for F would give a required independent set in G.

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Goal

Transform a CNF formula into an equisatisfiable 3-CNF formula. That is, reduce a problem to its special case.

Equisatisfiable means F is satisfiable for CNF formula if and only if F' is satisfiable for 3 CNF

Transforming an Instance

- We need to get rid of clauses of length more than 3 in an input formula

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- Introduce a fresh variable y and replace C with the following two clauses: $(\ell_1 \vee \ell_2 \vee y)$, $(\overline{y} \vee A)$
- The second clause is shorter than C
- Repeat while there is a long clause

Running time

The running time of the transformation is polynomial: at each iteration we replace a clause with a shorter clause and a 3-clause. Hence the total number of iterations is at most the total number of literals of the initial formula.

Correctness

To prove that the constructed reduction is correct, we're going to show that the initial formula F with a long clause is satisfiable if and only if the resulting formula (where we replaced a long clause with a 3-clause) is also satisfiable

Lemma

The formulas $F = (\ell_1 \lor \ell_2 \lor A)\ldots$ and $F' = (\ell_1 \lor \ell_2 \lor y)(\bar{y} \lor A)\ldots$ are equisatisfiable.

Question

We've reduced SAT to 3-SAT. This means that if we have an algorithm solving 3-SAT in polynomial time, then we can use it as a black box to solve SAT in polynomial time.

The reverse reduction also holds. That is, 3-SAT can be reduced to SAT. The corresponding reduction is straightforward: given a 3-CNF formula, we just leave it untouched and call an algorithm solving SAT on it. Is it a correct reduction?

- ○ No, this is not correct, of course.
- ◉ Yes, sure, this is correct.

✓ **Correct**

That's right! The reduction is so simple, because 3-SAT is a special case of SAT.

Proof

$F = (\ell_1 \vee \ell_2 \vee A) \ldots$

$F' = (\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A) \ldots$

$\Rightarrow$ If either ℓ_1 or ℓ_2 is satisfied, set $y = 0$. Otherwise A must be satisfied. Then set $y = 1$.

$\Leftarrow$ If $(\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A)$ are satisfied, then so is $(\ell_1 \vee \ell_2 \vee A)$. □

Transforming a Solution

Given a satisfying assignment for F', just throw away the values of all new variables (y's) to get a satisfying assignment of the initial formula.

Outline

1. Reductions
2. Showing **NP**-completeness
3. Independent Set $\rightarrow$ Vertex Cover
4. 3-SAT $\rightarrow$ Independent Set
5. SAT $\rightarrow$ 3-SAT
6. All of **NP** $\rightarrow$ SAT
7. Using SAT-solvers

Goal

Show that every search problem reduces to SAT.

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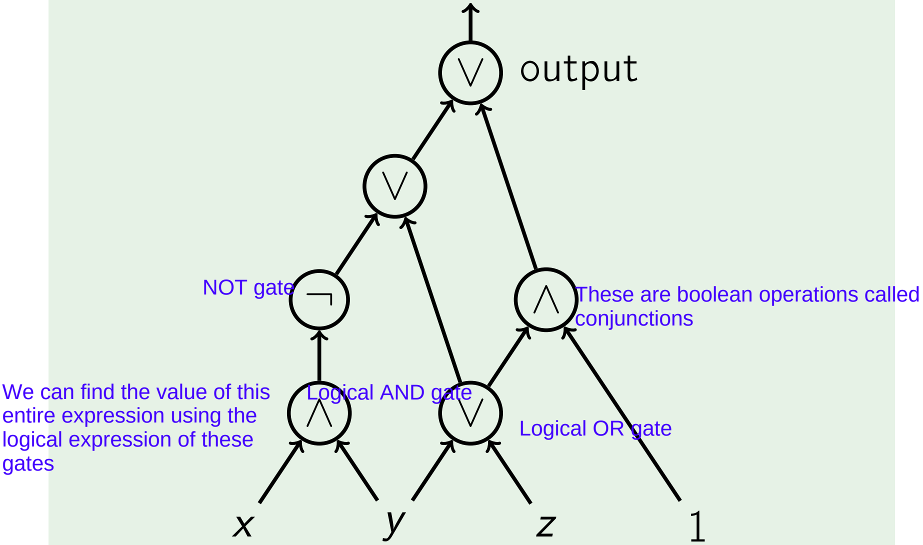
Show that every search problem reduces to SAT.

Instead, we show that any problem reduces to Circuit SAT problem, which, in turn, reduces to SAT.

Circuit


output
NOT gate
These are boolean operations called conjunctions
We can find the value of this entire expression using the logical expression of these gates
Logical AND gate
Logical OR gate
x
y
z
1

Definition

A circuit is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called inputs and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called gates: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as output.

Circuit-SAT

Input: A circuit.

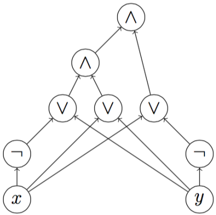
Output: An assignment of Boolean values to the input variables of the circuit that makes the output true.

SAT is a special case of Circuit-SAT as a CNF formula can be represented as a circuit:

Example: $(x \vee y \vee z)(y \vee \overline{x})$

Question

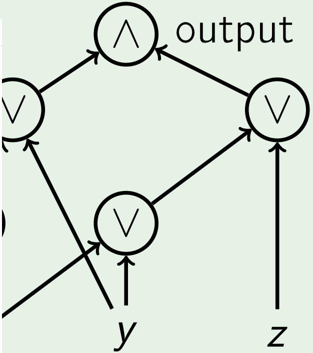
Mark all assignments that satisfy the given circuit.



- [ ] $x = 1, y = 0$
- [ ] $x = 0, y = 1$
- [ ] $x = 0, y = 0$
- [x] $x = 1, y = 1$

Correct

This assignment satisfies the circuit. The circuit encodes a 2-CNF formula $(x \vee y)(\overline{x} \vee y)(x \vee \overline{y})$.



Circuit-SAT $\to$ SAT

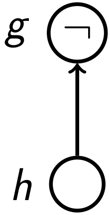
To reduce Circuit-SAT to SAT, we need to design a polynomial time algorithm that for a given circuit outputs a CNF formula which is satisfiable, if and only if the circuit is satisfiable

Idea

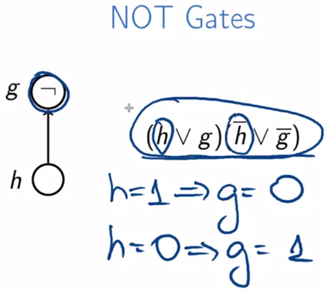
- Introduce a Boolean variable for each gate
- For each gate, write down a few clauses that describe the relationship between this gate and its direct predecessors

NOT Gates

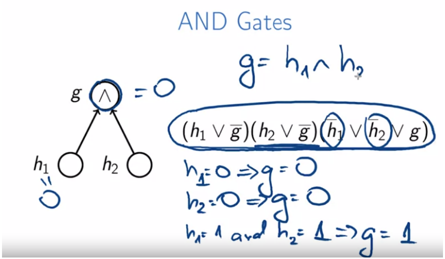
Here g is always negation of h. To satisfy each of these clause h and g has to be negation of each other

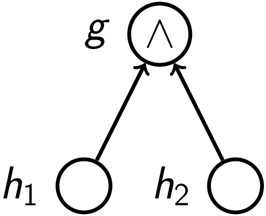


$$(h \vee g)(\overline{h} \vee \overline{g})$$



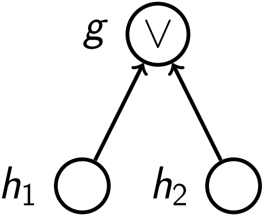
AND Gates





$$(h_1 \vee \overline{g})(h_2 \vee \overline{g})(\overline{h}_1 \vee \overline{h}_2 \vee g)$$

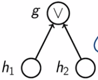
OR Gates



$$(\overline{h}_1 \vee g)(\overline{h}_2 \vee g)(h_1 \vee h_2 \vee \overline{g})$$

OR Gates

$g = h_1 \vee h_2$



$$(\overline{h}_1 \vee g)(\overline{h}_2 \vee g)(h_1 \vee h_2 \vee \overline{g})$$

$h_1 = 1 \Rightarrow g = 1$

$h_2 = 1 \Rightarrow g = 1$

$h_1 = 0, h_2 = 0 \Rightarrow g = 0$

Output Gate

$$g \bigcirc \text{output} \qquad (g)$$

(g) means that we we would like this gate to be 1 that is we want our output 1

- The resulting CNF formula is consistent with the initial circuit: in any satisfying assignment of the formula, the value of g is equal to the value of the gate labeled with g in the circuit

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- Therefore, the CNF formula is equisatisfiable to the circuit
- The reduction takes polynomial time

Goal

Reduce every search problem to Circuit-SAT.

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Reduce every search problem to Circuit-SAT.

- Let A be a search problem

Note that we don't even know whether the input, whether the instance of this problem are graphs or boolean formulae, or boolean circuits or just numbers or systems of linear inequalities. The only thing we know is that A is a search problem.

Goal

Reduce every search problem to Circuit-SAT.

- Let A be a search problem
- We know that there exists an algorithm $\mathcal{C}$ that takes an instance I of A and a candidate solution S and checks whether S is a solution for I in time polynomial in $|I|$

Goal

Reduce every search problem to Circuit-SAT.

- Let A be a search problem
- We know that there exists an algorithm $\mathcal{C}$ that takes an instance I of A and a candidate solution S and checks whether S is a solution for I in time polynomial in $|I|$
- In particular, $|S|$ is polynomial in $|I|$

Turn an Algorithm into a Circuit

- Note that a computer is in fact a circuit (of constant size!) implemented on a chip

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- Each step of the algorithm $\mathcal{C}(I, S)$ is performed by this computer's circuit

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- Note that a computer is in fact a circuit (of constant size!) implemented on a chip
- Each step of the algorithm $\mathcal{C}(I, S)$ is performed by this computer's circuit
- This gives a circuit of size polynomial in $|I|$ that has input bits for I and S and outputs whether S is a solution for I (a separate circuit for each input length)

Reduction

To solve an instance I of the problem A:

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$

Reduction

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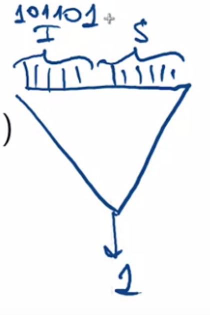
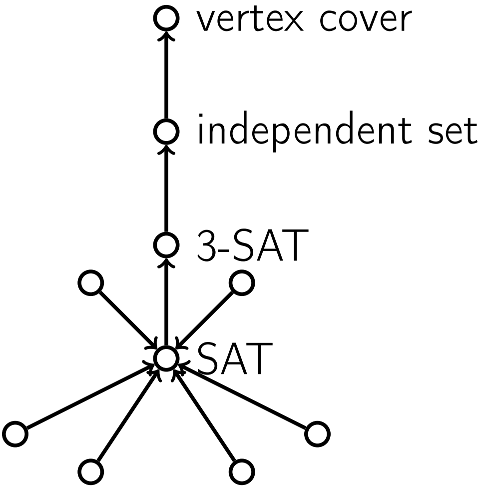
Reduction



To solve an instance *I* of the probl

- take a circuit corresponding tc
- the inputs to this circuit encode candidate solutions
- use a Circuit-SAT algorithm for this circuit to find a solution (if exists)

Summary



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Sudoku Puzzle

This part

A simple and efficient Sudoku solver

SAT: Theory and Practice

Theory: we have no algorithm checking the satisfiability of a CNF formula F with n variables in time $\text{poly}(|F|) \cdot 1.99^n$

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Theory: we have no algorithm checking the satisfiability of a CNF formula F with n variables in time $\mathrm{poly}(|F|) \cdot 1.99^n$

Practice: SAT-solvers routinely solve instances with thousands of variables

Solving Hard Problems in Practice

An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)

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An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)
- Use a SAT solver

Sudoku Puzzle

Goal: fill in with digits the partially completed 9×9 grid so that each row, each column, and each of the nine 3×3 subgrids contains all the digits from 1 to 9.

Example

Variables

There will be $9 \times 9 \times 9 = 729$ Boolean variables: for $1 \leq i, j, k \leq 9$, $x_{ijk} = 1$, if and only if the cell $[i, j]$ contains the digit k

This means 9 rows and 9 columns and it can contain k = 9 values and xijk = 1 if [i,j] contains the intended k

Now we will reduce the soduko puzzle in to CBF formulae which can be fed to SAT solvers

Exactly One Is True

Clauses expressing the fact that exactly one of the literals ℓ_1, ℓ_2, ℓ_3 is equal to 1:

$$(\ell_1 \lor \ell_2 \lor \ell_3)(\overline{\ell}_1 \lor \overline{\ell}_2)(\overline{\ell}_1 \lor \overline{\ell}_3)(\overline{\ell}_2 \lor \overline{\ell}_3)$$

shows that only one is true

shows l1 and l2 cannot be true simultaneously

Constraints

- Cell $[i, j]$ contains exactly one digit: $\texttt{ExactlyOneOf}(x_{ij1}, x_{ij2}, \ldots, x_{ij9})$

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  $\texttt{ExactlyOneOf}(x_{ij1}, x_{ij2}, \ldots, x_{ij9})$

This is to check if the box contains only one digit

- k appears exactly once in row i:
  $\texttt{ExactlyOneOf}(x_{i1k}, x_{i2k}, \ldots, x_{i9k})$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Constraints

- Cell $[i, j]$ contains exactly one digit:
  $\texttt{ExactlyOneOf}(x_{ij1}, x_{ij2}, \ldots, x_{ij9})$ i and j remains fixed only the variable shown changes
- k appears exactly once in row i:
  $\texttt{ExactlyOneOf}(x_{i1k}, x_{i2k}, \ldots, x_{i9k})$
- k appears exactly once in column j:
  $\texttt{ExactlyOneOf}(x_{1jk}, x_{2jk}, \ldots, x_{9jk})$

Constraints

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- k appears exactly once in column j: $\texttt{ExactlyOneOf}(x_{1jk}, x_{2jk}, \ldots, x_{9jk})$
- k appears exactly once in a 3×3 block: $\texttt{ExactlyOneOf}(x_{11k}, x_{12k}, \ldots, x_{33k})$

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- k appears exactly once in a 3×3 block: $\texttt{ExactlyOneOf}(x_{11k}, x_{12k}, \ldots, x_{33k})$
- $[i, j]$ already contains k: (x_{ijk})

When some cell is already filled that is the cell [i,j] already contains digit k then we need just one clause stating that corresponding variable must be equal to true

Resulting Formula

State-of-the-art SAT-solvers find a satisfying assignment for the resulting formula in blink of an eye, though the corresponding search space has size about $2^{729} \approx 10^{220}$

This means that for i=9 x j=9 x k=9 possibilities if we find for a given number then it can be true or false this means there can be about 2^729 possibilities