Flows in Networks: Maxflow-Mincut

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Advanced Algorithms and Complexity
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Learning Objectives

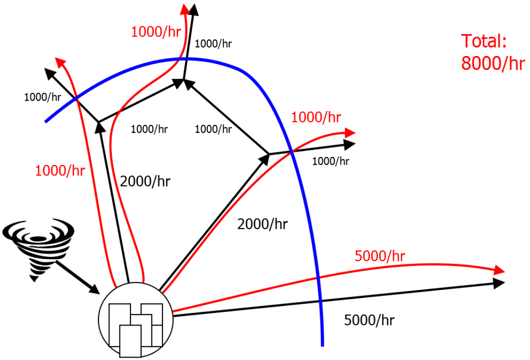
- Understand the relationship between flows and cuts.
- Produce a cut with size matching that of a maximum flow.
- Identify when a flow is maximum.

Problem

In order to find maxflows, we need a way of verifying that they are optimal.
In particular, we need techniques for bounding the size of the maxflow.

Idea

Recall our original example:



Idea

Find a bottleneck for the flow. All flow needs to cross the bottleneck.

Cuts

Definition

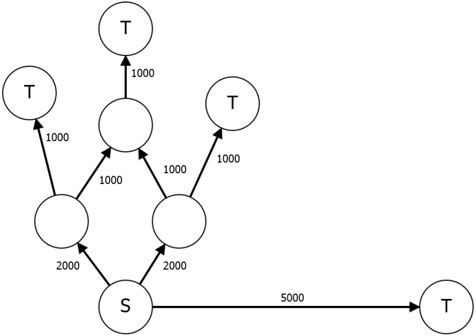
Given a network G, a cut $\mathcal{C}$, is a set of vertices of G so that $\mathcal{C}$ contains all sources of G and no sinks of G.

The size of a cut is given by

$$|\mathcal{C}| := \sum_{e \text{ out of } \mathcal{C}} C_e$$

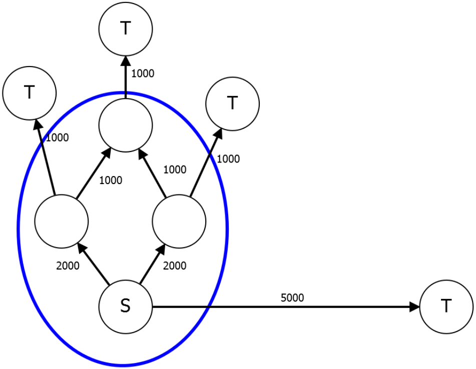
Example

Network G.



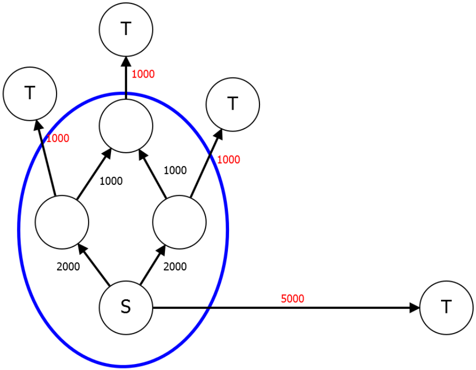
Example

Cut $\mathcal{C}$.



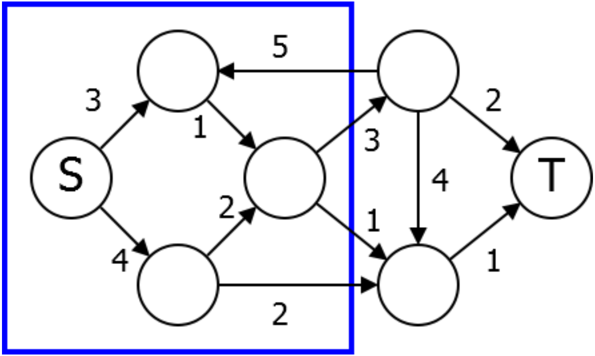
Example

Edges cut. Total size 8000.



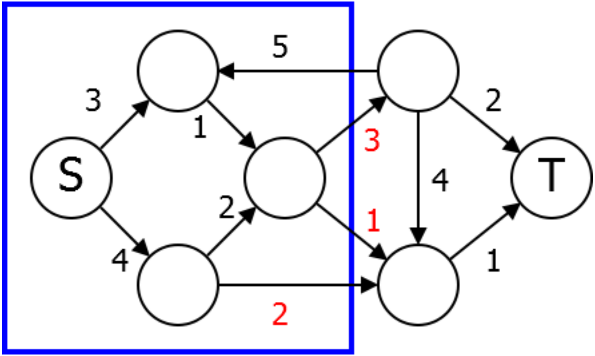
Problem

What is the size of the cut below?



Solution

$1 + 2 + 3 = 6.$



Bound

Lemma

Let G be a network. For any flow f and any cut $\mathcal{C}$,

$$|f| \leq |\mathcal{C}|.$$

Cuts provide upper bounds on the size of the flow. Any piece of flow needs to cross the cut, there's only so much capacity that lets you cross the cut, and so that's an upper bound on the flow.

Proof

$$
\begin{aligned}
|f| &= \sum_{v \text{ source}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\
&= \sum_{v \in \mathcal{C}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\
&= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\
&\leq \sum_{e \text{ out of } \mathcal{C}} C_e = |\mathcal{C}|.
\end{aligned}
$$

Bounds

In other words, for any cut $\mathcal{C}$, we get an upper bound on the maxflow. In particular,

$$\text{Maxflow} \leq |\mathcal{C}|.$$

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Bounds

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Question: Is this bound good enough?
Surprisingly, it is...

Maxflow-Mincut

Theorem

For any network G,

$$\max_{\text{flows } f} |f| = \min_{\text{cuts } \mathcal{C}} |\mathcal{C}|.$$

Maxflow-Mincut

Theorem

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$$\max_{\text{flows f}} |f| = \min_{\text{cuts } \mathcal{C}} |\mathcal{C}|.$$

In other words, there is always a cut small enough to give the correct upper bound. on maximum flows

A Special Case

What happens when Maxflow $= 0$?

- There is no path from source to sink.
- Let $\mathcal{C}$ be the set of vertices reachable from sources.
- There are no edges out of $\mathcal{C}$.
- So $|\mathcal{C}| = 0$.

The General Case

- Let f be a maxflow for G.
- Note that G_f has maxflow 0.
- There is a cut $\mathcal{C}$ with size 0 in G_f.
- Claim: $|\mathcal{C}| = |f|$.

Proof

$$
\begin{aligned}
|f| &= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\
&= \sum_{e \text{ out of } \mathcal{C}} C_e - \sum_{e \text{ into } \mathcal{C}} 0 \\
&= |\mathcal{C}|.
\end{aligned}
$$

Conclusion

- We have found an f and $\mathcal{C}$ with $|f| = |\mathcal{C}|$.
- By Lemma, cannot have larger $|f|$ or smaller $|\mathcal{C}|$.
- So $\max |f| = \min |\mathcal{C}|$.

Summary

- Can always check if flow is maximal by finding matching cut.
- f a maxflow only if there is no source-sink path in G_f.
- We will use this in an algorithm next time.