Decomposition of Graphs: Connectivity

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Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Understand the importance of connected components of a graph.
- Compute the connected components of a graph.

Outline

1. Connected Components
2. Algorithm

Reachability

Want to understand which vertices in G are reachable from which others.

Connected Components

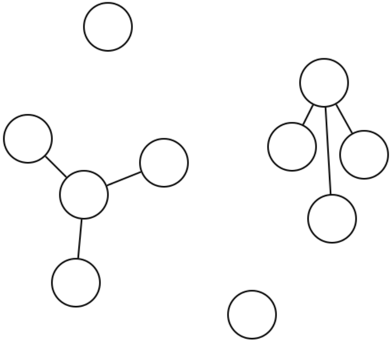
Theorem

The vertices of a graph G can be partitioned into Connected Components so that v is reachable from w if and only if they are in the same connected component.

Problem

How many connected components does the graph below have?

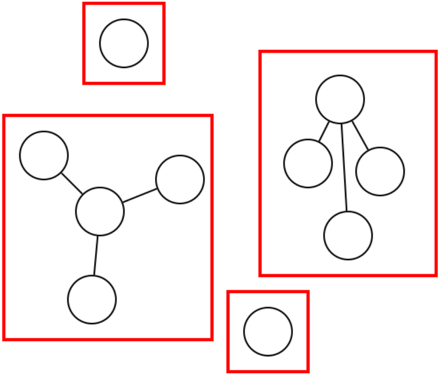
.



Solution

How many connected components does the graph below have?

4.



Proof

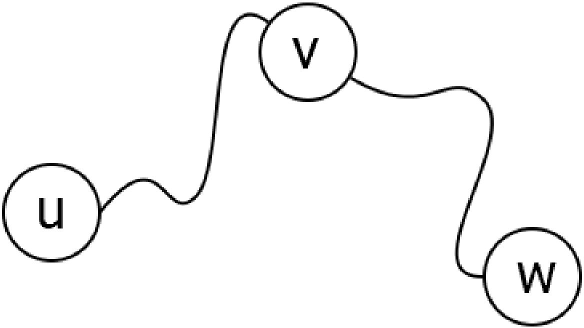
Proof.

Need to show reachability is an equivalence relation. Namely:

- v is reachable from v.
- If v reachable from w, w reachable from v.
- If v reachable from u, and w reachable from v, w reachable from u.

□

Proof (continued)



Problem

Connected Components

Input: Graph G

Output: The connected components of G

Outline

1. Connected Components

2. Algorithm

Idea

`Explore`(v) finds the connected component of v. Just need to repeat to find other components.

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`Explore`(v) finds the connected component of v. Just need to repeat to find other components.
Modify DFS to do this.
Modify goal to label connected components.

Modification of Explore

$\texttt{Explore}(v)$

$\texttt{visited}(v) \leftarrow \texttt{true}$
$\texttt{CCnum}(v) \leftarrow cc$
$\texttt{for } (v, w) \in E:$
$\quad \texttt{if not visited}(w):$
$\qquad \texttt{Explore}(w)$

Each island of connected component have same cc number

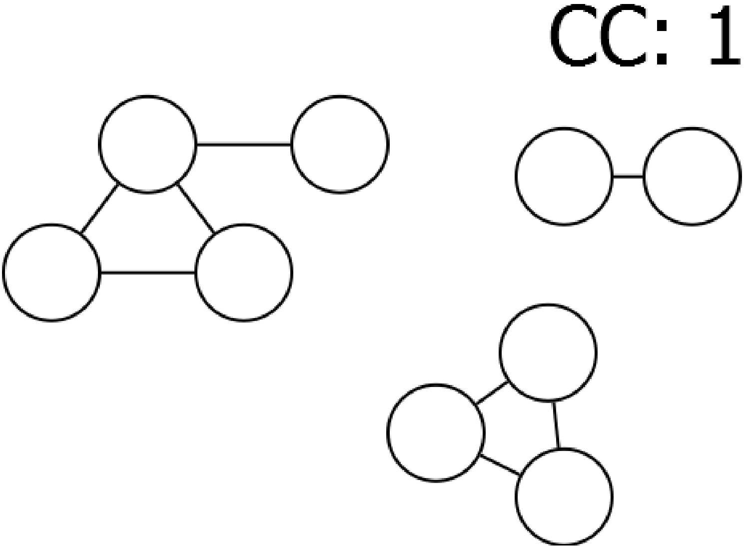
Modifications of DFS

DFS(G)

```
for all v ∈ V mark v unvisited
cc ← 1
for v ∈ V:
   if not visited(v):
      Explore(v)
      cc ← cc + 1
```

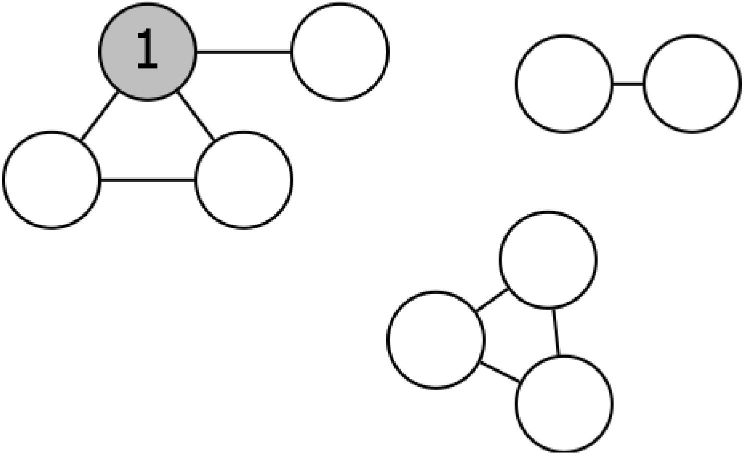
Hence, each different island of connected components have a different cc number incremented by 1

Example



Example

CC: 1



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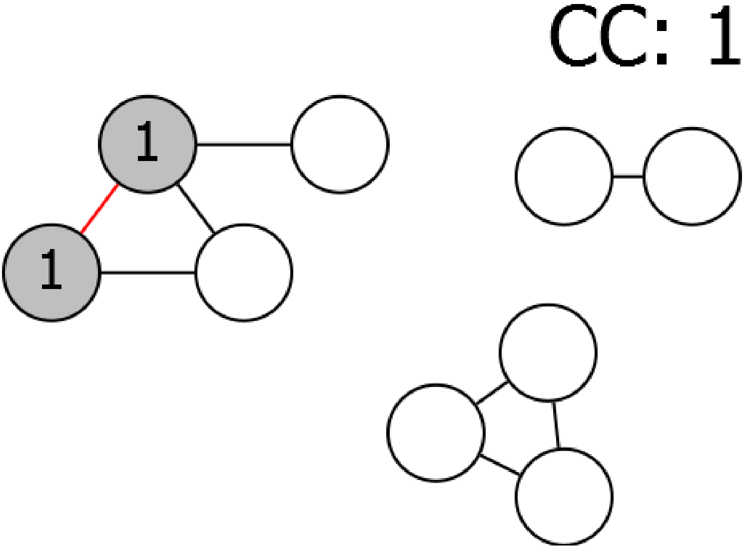

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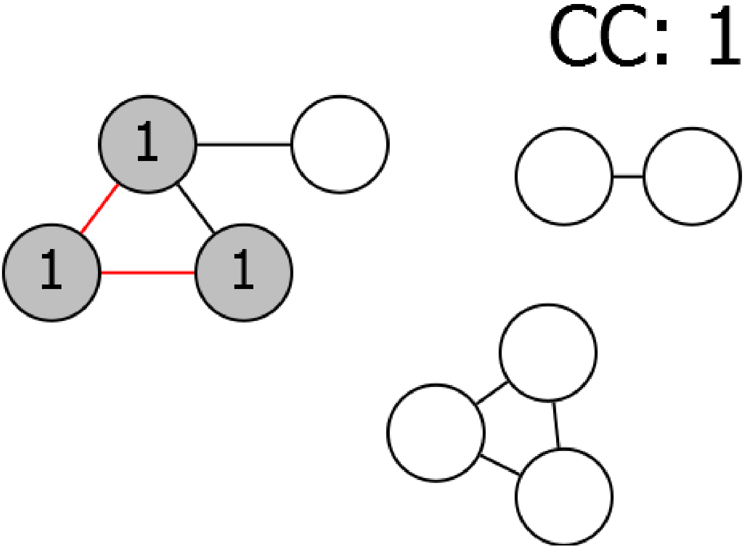
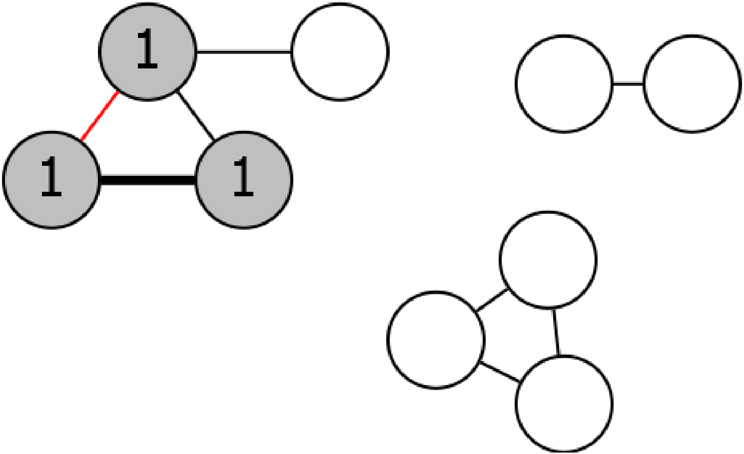


Example

CC: 1



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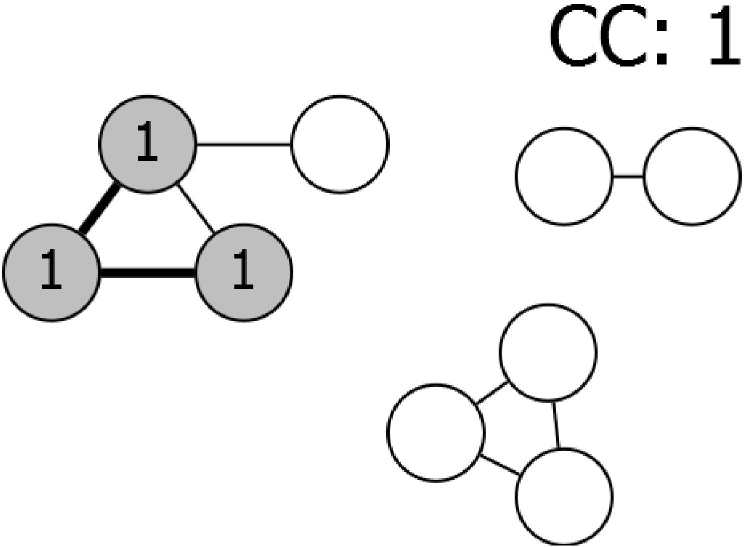

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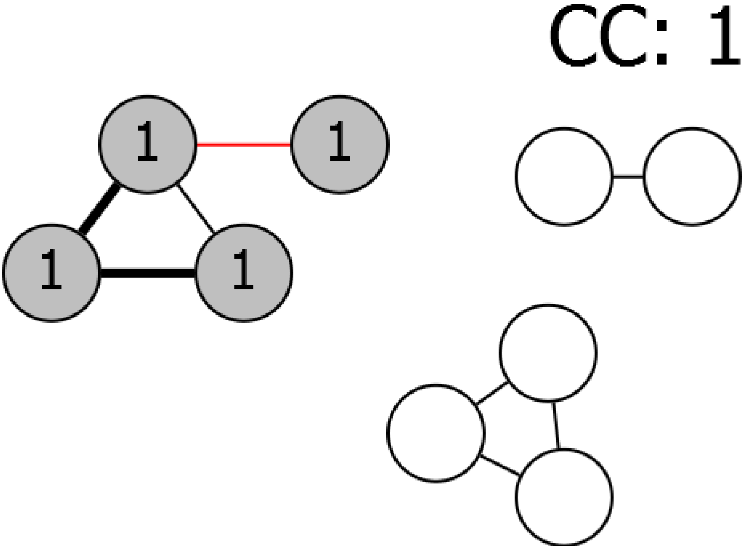

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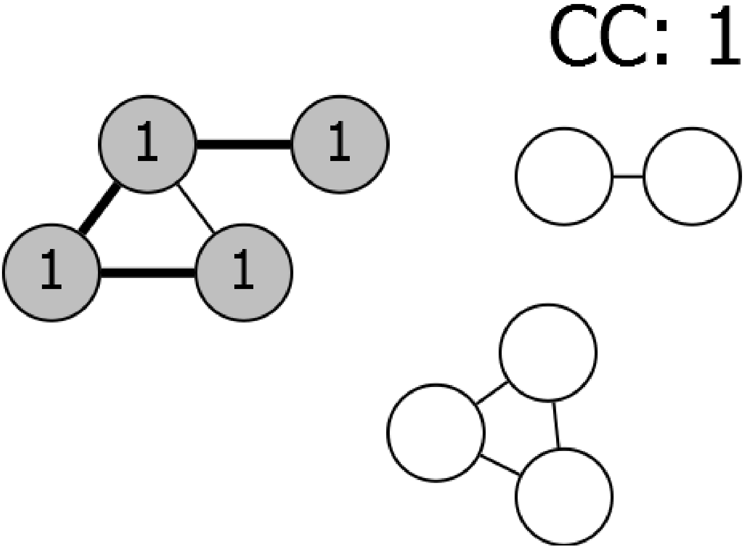

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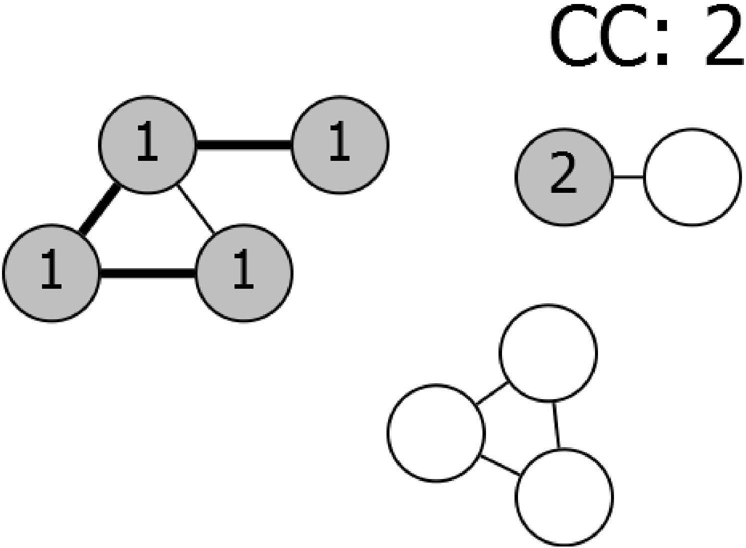

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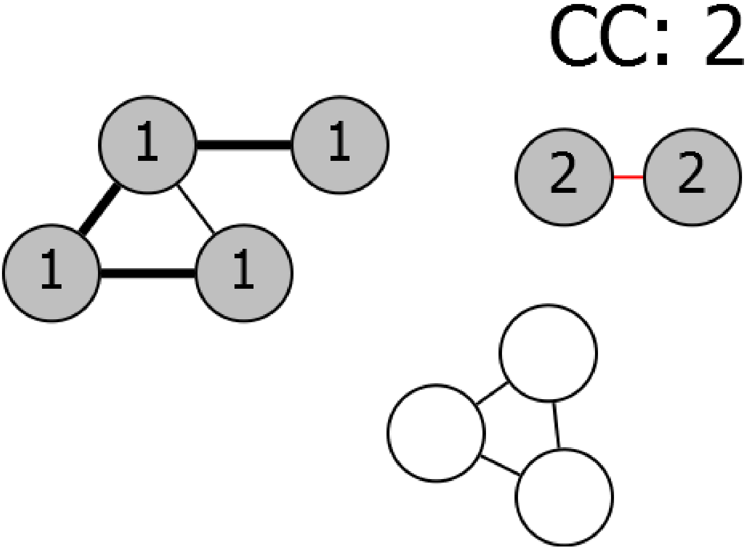

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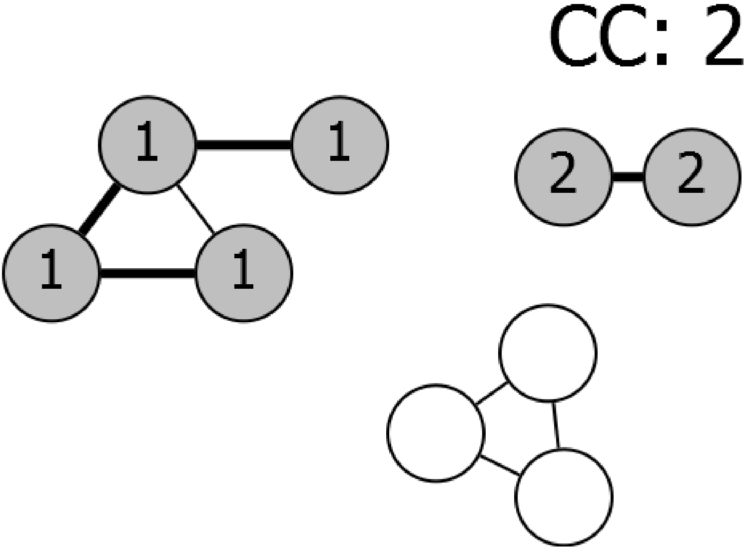
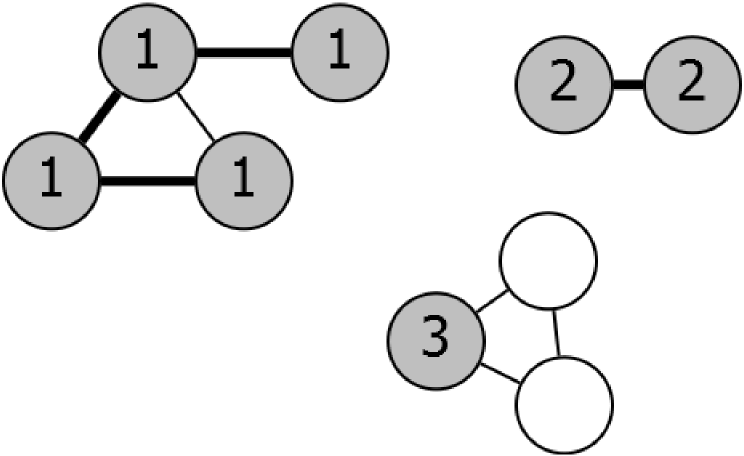


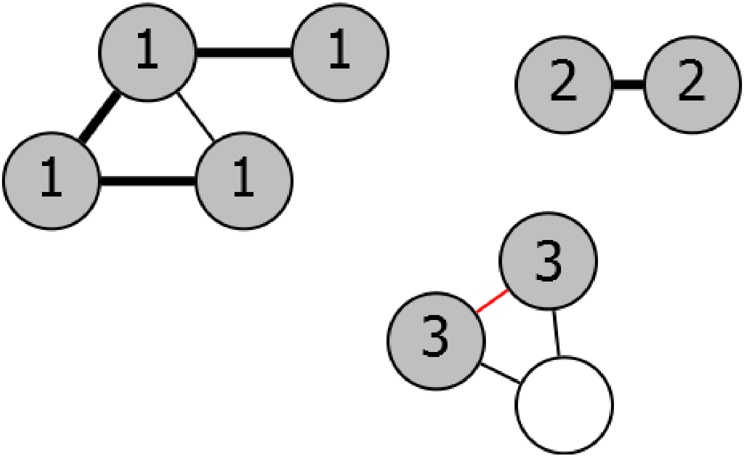
Example

CC: 3



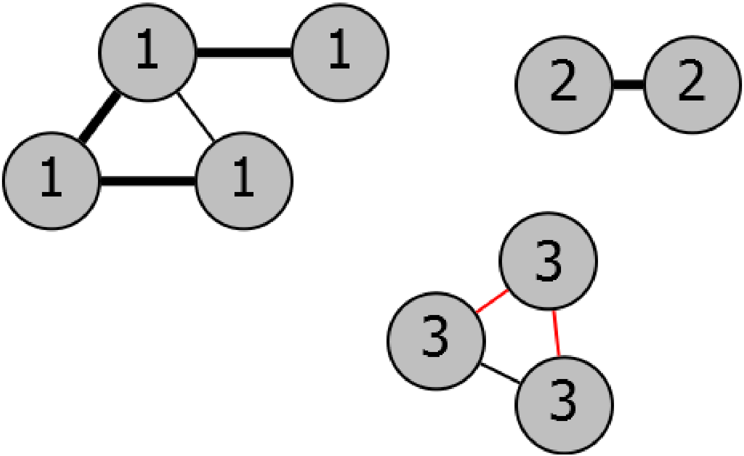
Example

CC: 3



Example

CC: 3



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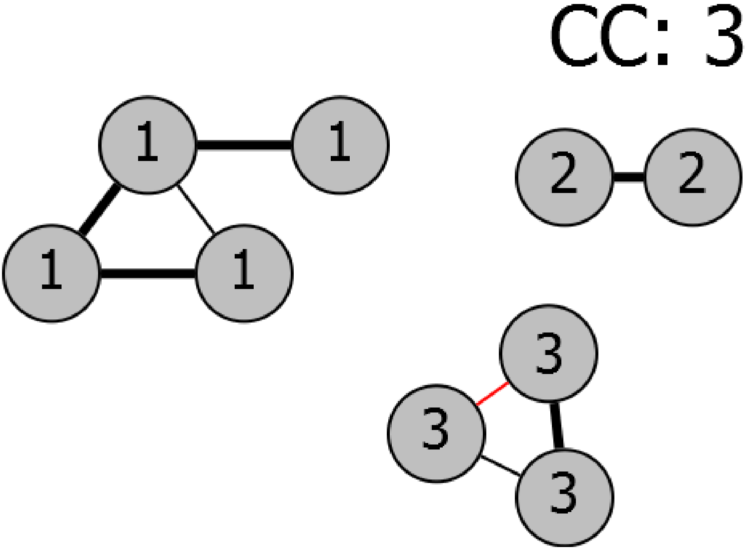

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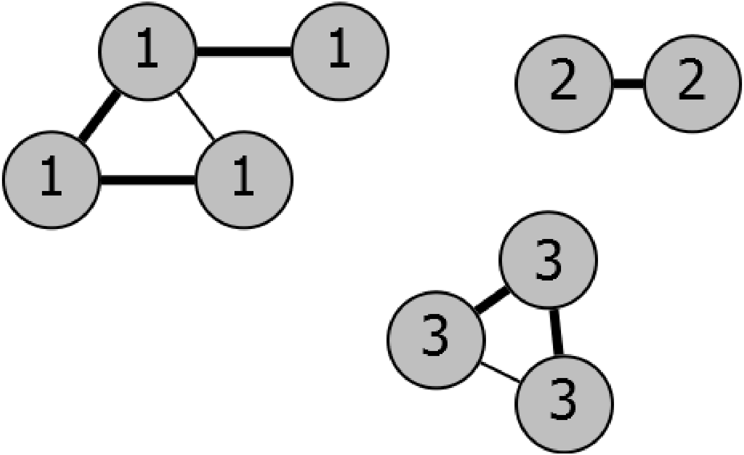
CC: 3



Correctness

- Each new `explore` finds new connected component.
- Eventually find every vertex.
- Runtime still $O(|V| + |E|)$.

Next Time

Other applications of `DFS`.