Decomposition of Graphs: Computing Strongly Connected Components

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Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Efficiently compute the strongly connected components of a directed graph.

Last Time

- Connectivity in directed graphs.
- Strongly connected components.
- Metagraph.

Problem

Strongly Connected Components

Input: A directed graph G

Output: The strongly connected components of G.

Easy Algorithm

EasySCC(G)

```
for each vertex v:
  run explore(v) to determine
    vertices reachable from v
for each vertex v:
  find the u reachable from v that
    can also reach v
these are the SCCs
```

Lets name this list as X

From the list X

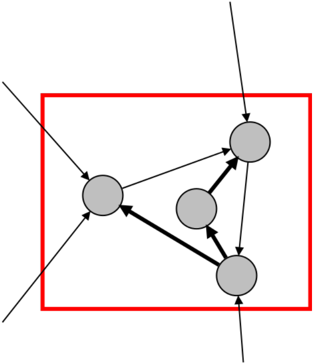
Runtime $O(|V|^2 + |V||E|)$. Want faster.

Outline

1. Sink Components
2. Algorithm

Sink Components

Idea: If v is in a sink SCC, `explore`(v) finds vertices reachable from v. This is exactly the SCC of v.



Finding Sink Components

Need a way to find a sink SCC.

Theorem

Theorem

If $\mathcal{C}$ and $\mathcal{C}'$ are two strongly connected components with an edge from some vertex of $\mathcal{C}$ to some vertex of $\mathcal{C}'$, then largest post in $\mathcal{C}$ bigger than largest post in $\mathcal{C}'$.

Proof

Cases:

- Visit $\mathcal{C}$ before visit $\mathcal{C}'$
- Visit $\mathcal{C}'$ before visit $\mathcal{C}$

Case I

Visit $\mathcal{C}$ first.

- Can reach everything in $\mathcal{C}'$ from $\mathcal{C}$.
- Explore all of $\mathcal{C}'$ while exploring $\mathcal{C}$.
- $\mathcal{C}$ has largest post.

Case II

Visit $\mathcal{C}'$ first.

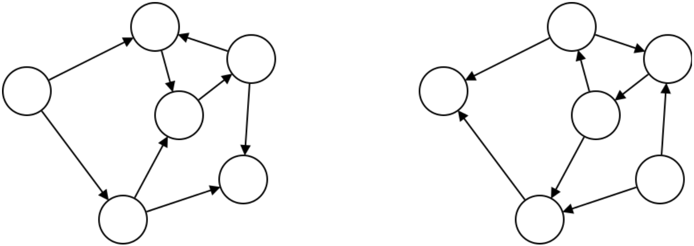
- Cannot reach $\mathcal{C}$ from $\mathcal{C}'$
- Must finish exploring $\mathcal{C}'$ before exploring $\mathcal{C}$
- $\mathcal{C}$ has largest post.

Conclusion

The vertex with the largest postorder number is in a source component!
Problem: We wanted a sink component.

Reverse Graph

Let G^R be the graph obtained from G by reversing all of the edges.



Reverse Graph Components

- G^R and G have same SCCs.
- Source components of G^R are sink components of G.

Find sink components of G by running `DFS` on G^R.

Problem

Which of the following is true?

- The vertex with largest postorder in G^R is in a sink SCC of G.
- The vertex with the largest preorder in G is in a sink SCC of G.
- The vertex with the smallest postorder in G is in a sink SCC of G.

Solution

Which of the following is true?

- The vertex with largest postorder in G^R is in a sink SCC of G.
- The vertex with the largest preorder in G is in a sink SCC of G.
- The vertex with the smallest postorder in G is in a sink SCC of G.

Outline

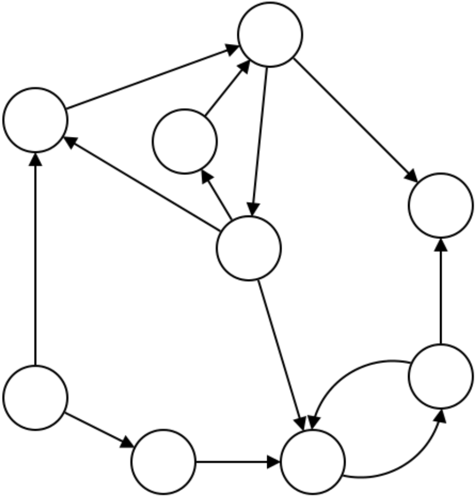
1. Sink Components
2. Algorithm

Basic Algorithm

SCCs(G)

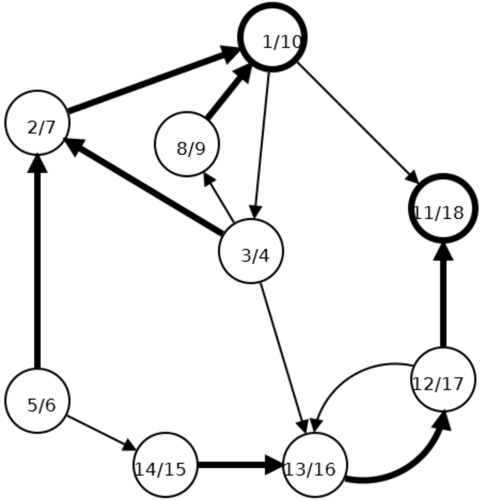
run DFS(G^R)
let v have largest post number
run Explore(v)
vertices found are first SCC
Remove from G and repeat

Example

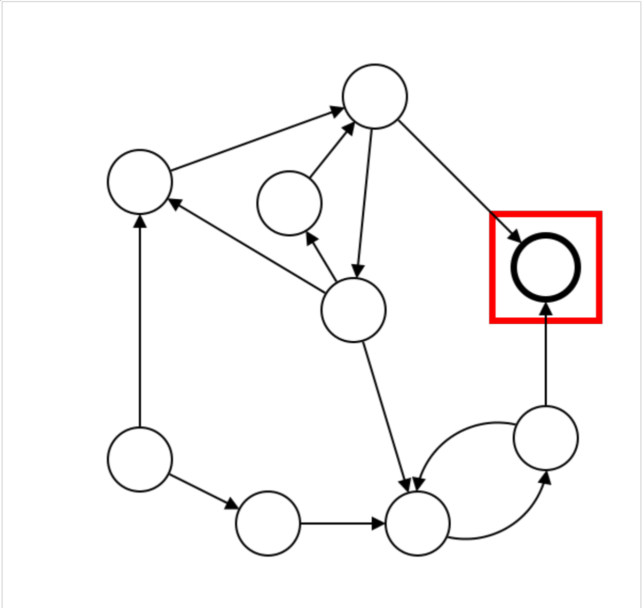


Example

This is G while the numberings are GR



Example



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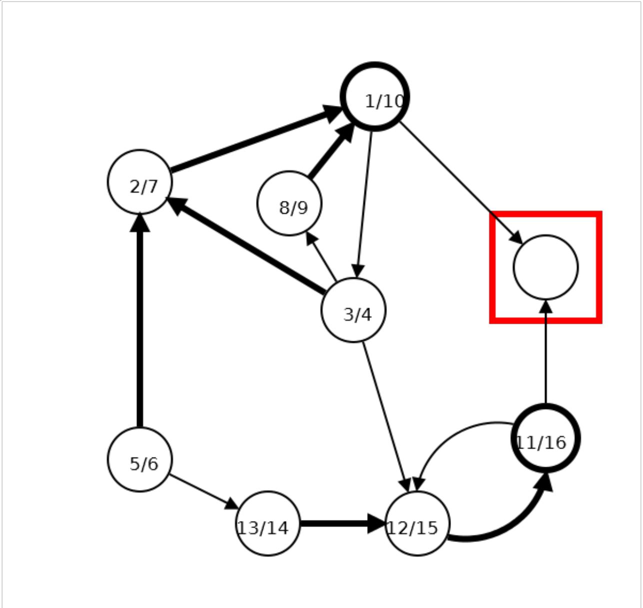

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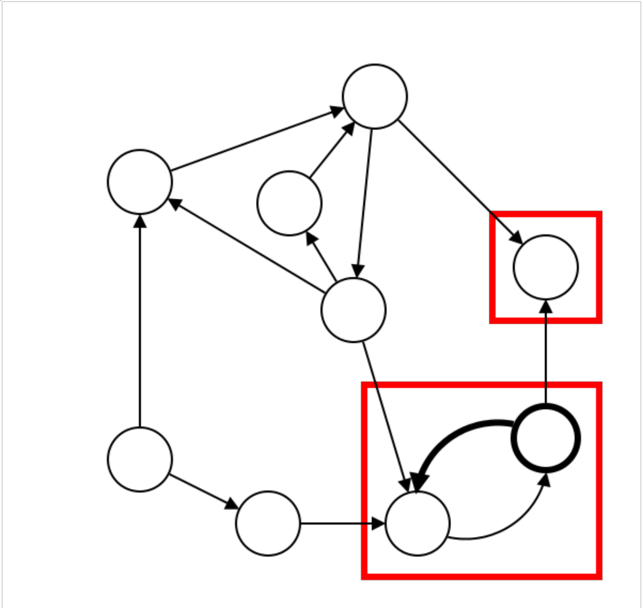

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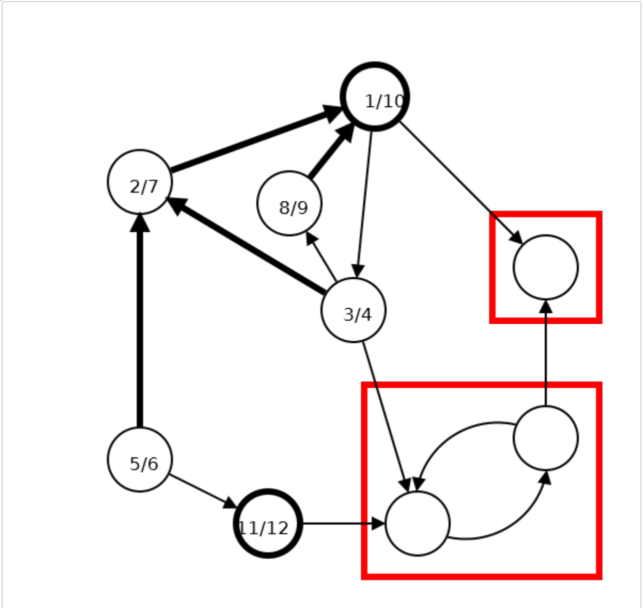

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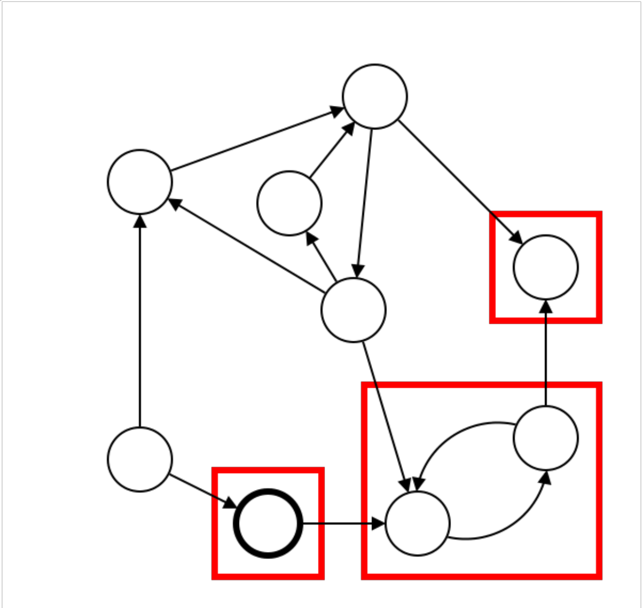

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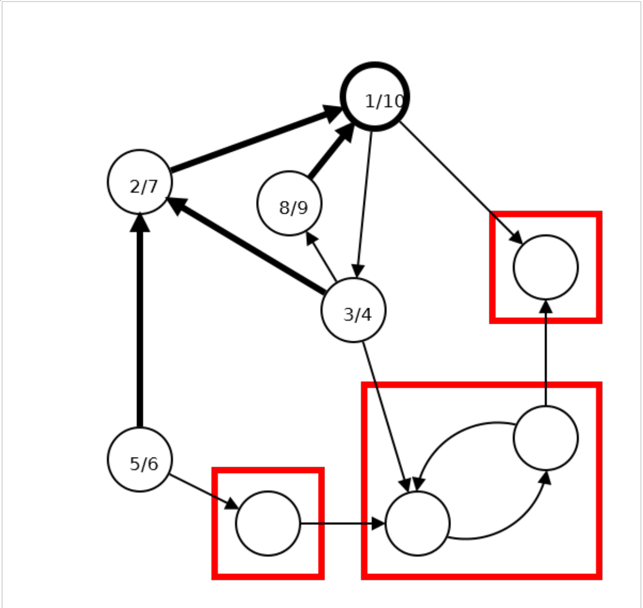

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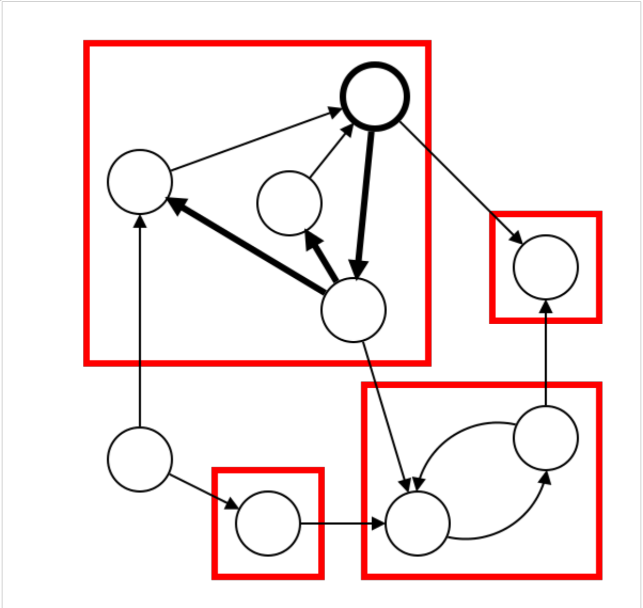

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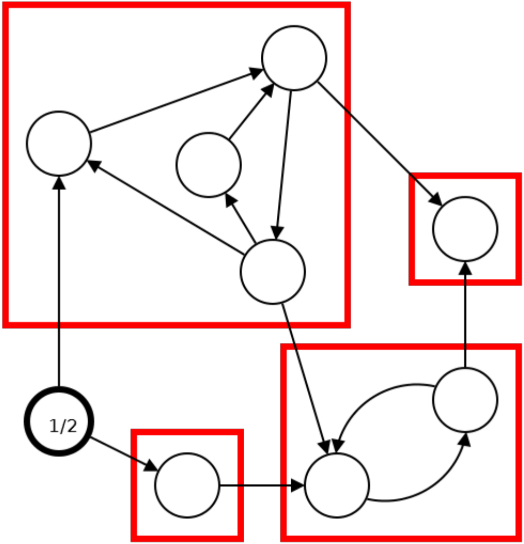

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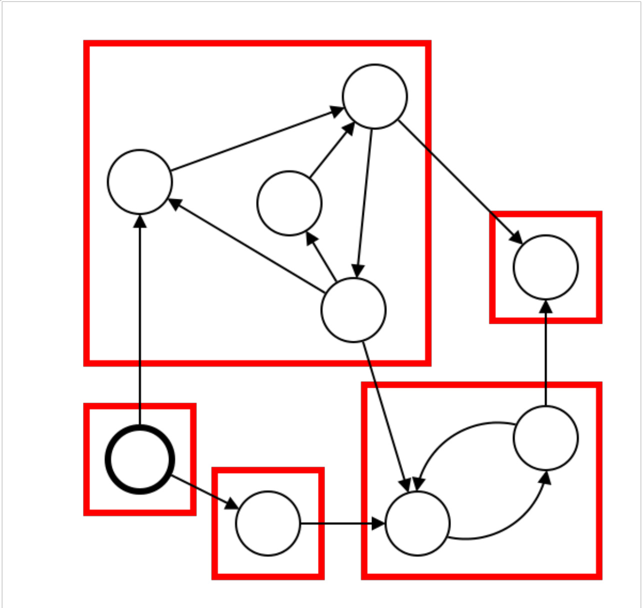


Improvement

- Don't need to rerun `DFS` on G^R.
- Largest remaining post number comes from sink component.

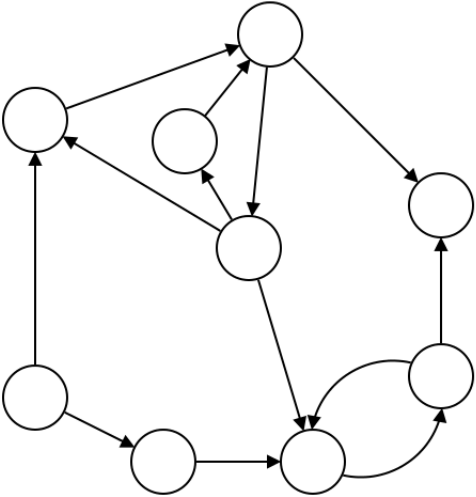
New Algorithm

SCCs(G)

```
Run DFS(G^R)   And Mark all the pre and post order numbers
for v ∈ V in reverse postorder:
   if not visited(v):
      Explore(v)   Now Run DFS on G
      mark visited vertices
         as new SCC
```

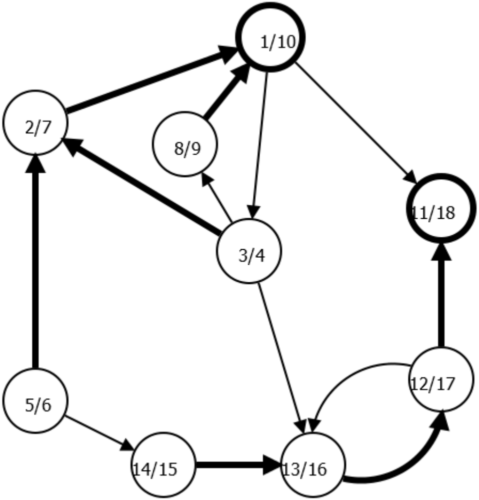
Kosaraju's algo is exactly similar to this algo but post order is found on G and connected components is found on GR in reverse post order of G

Example



Example

Notice that these numberings are postorder numbers obtained by DFS on GR



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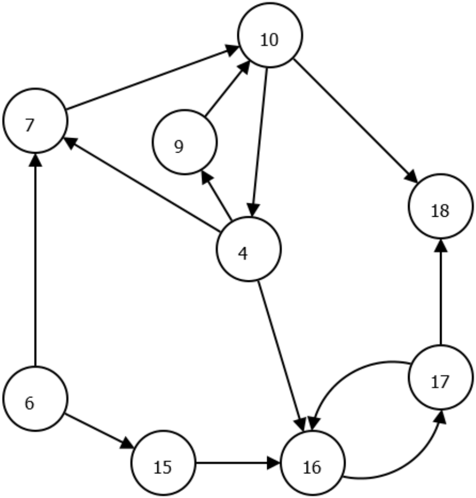

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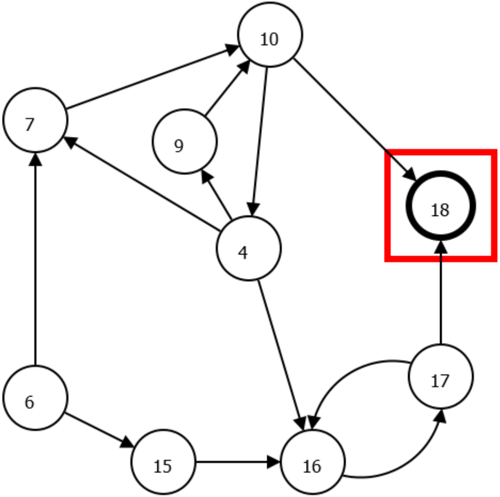

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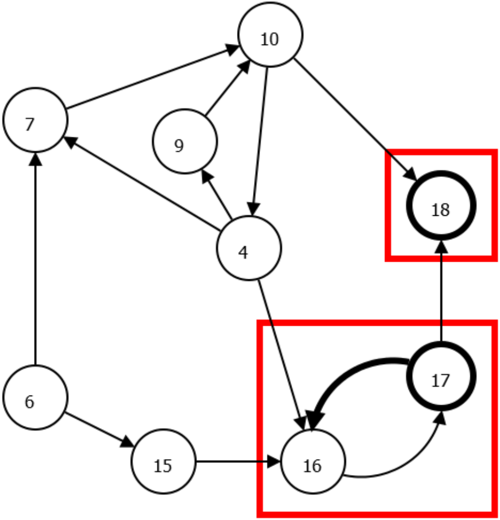

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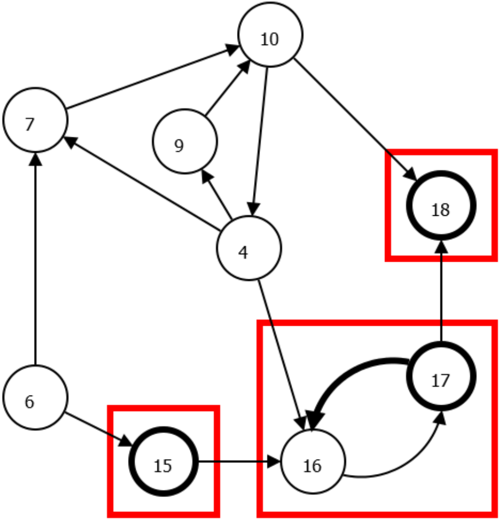

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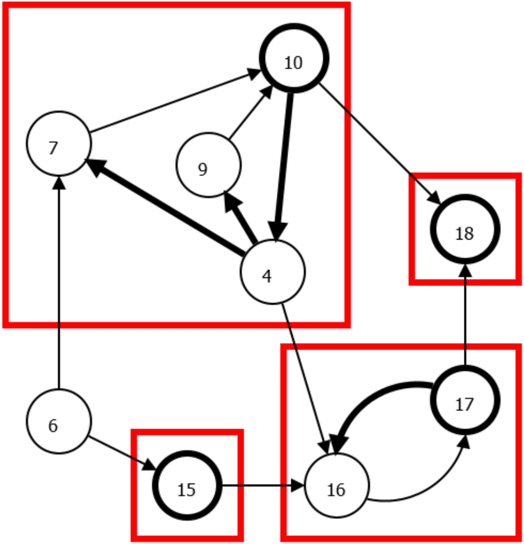

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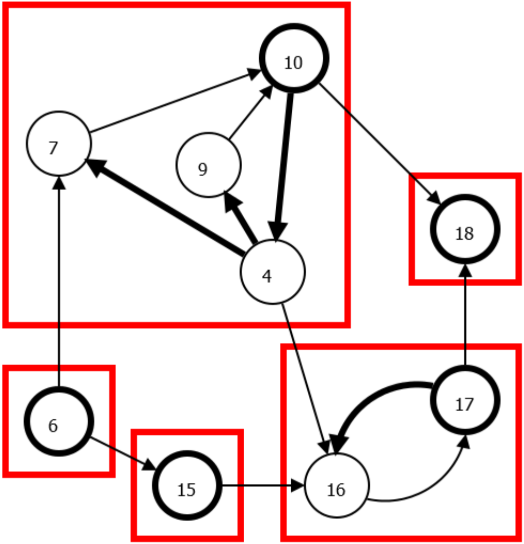

Runtime

- Essentially DFS on G^R and then on G.
- Runtime $O(|V| + |E|)$.

Because this is just two DFS first on Gr and then on G the run time is O(|V| + |E|) + O(|V| + |E|) =~ O(|V| + |E|)