Linear Programming: Convex Polytopes

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Advanced Algorithms and Complexity
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Learning Objectives

- Understand what a convex polytope is and why it is relevant to linear programming.
- Get a feel for what a convex polytope looks like.
- Prove some basic facts about convex polytopes.

Linear Programs

Optimize linear function given linear inequality constraints.

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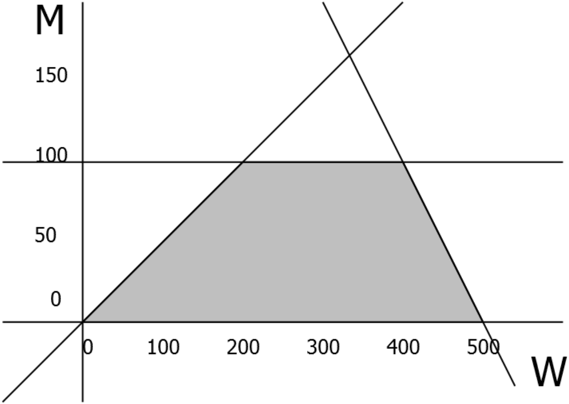
Want to understand region of points defined by inequalities.

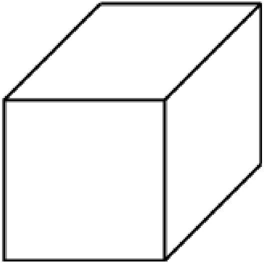
Example

From factory example



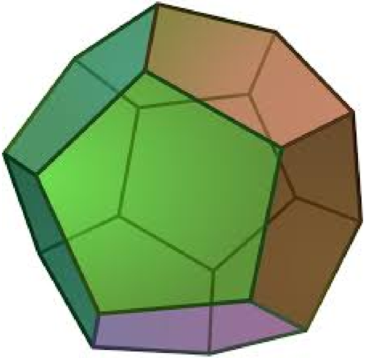
Example II

Equations: $0 \leq x, y, z \leq 1$ give cube.



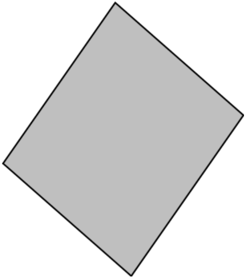
In General

Get what's called a convex polytope



Hyperplanes

A single linear equation defines a hyperplane.



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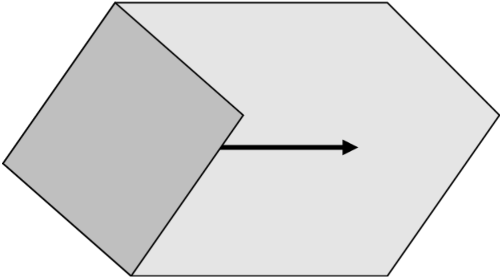

An inequality, defines a halfspace.

Polytopes

So a system of linear inequalities, defines a region bounded by a bunch of hyperplanes.

Definition

A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces.

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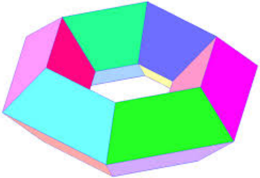
Definition

A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces. These surfaces may intersect in lower dimensional facets (like edges), with zero-dimensional facets called vertices. Imagine cube here

More Conditions

But not *every* polytope is possible. as a set of solutions to such a system of linear inequalities

Ex DONUT is not a system of solutions to one of these systems because if you look at some of these inward pointing faces , these faces lie in a hyper plane but you've got portions of your region on both sides of that hyper plane whereas, if you have a polytope defined by one of these systems of linear inequalities each bounding hyperplane is actually coming from one of those linear inequalities and you can only have points on one side that hyperplane or the other. [Imagine a Cube]. Hence condition is everything must be on one side of each face.

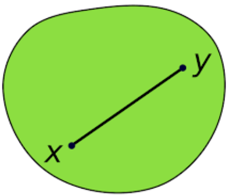


Everything must be on one side of each face.

Convexity

Definition

A region $\mathcal{C} \subset \mathbb{R}^n$ is convex, if for any $x, y \in \mathcal{C}$, the line segment connecting x and y is contained in $\mathcal{C}$.



Convexity

Lemma

An intersection of halfspaces is convex.

Proof

- Defined by $Ax \geq b$.
- Need for $x, y \in \mathcal{C}$ and $t \in [0, 1]$, $tx + (1-t)y \in \mathcal{C}$.

$$
\begin{aligned}
A(tx + (1-t)y) &= tAx + (1-t)Ay \\
&\geq tb + (1-t)b \\
&= b.
\end{aligned}
$$

SKIP

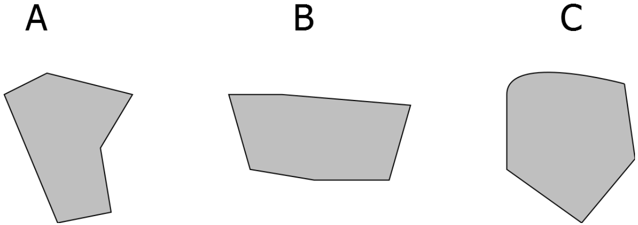
Convex Polytope

Theorem

The region defined by a system of linear inequalities is always a convex polytope.

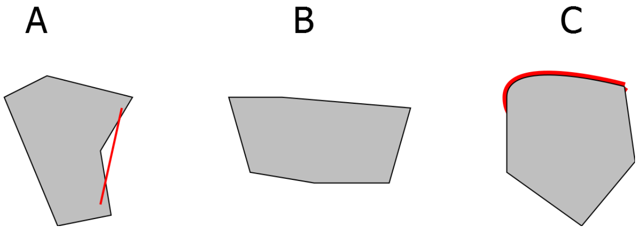
Problem

Which of these figures is a convex polytope?



Solution

Only B.



A is not because end points of the segment are in A but some points in the middle are not

C is not because there's this region of the boundary here which is a curved region whereas in polytope all that bound your regions would have to be straight lines

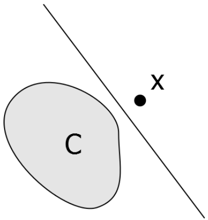
Lemmas

We will conclude with a couple important lemmas about convex polytopes.

Separation

Lemma

Let $\mathcal{C}$ be a convex region and $x \notin \mathcal{C}$ a point. Then there is a hyperplane H separating x from $\mathcal{C}$.



Separation

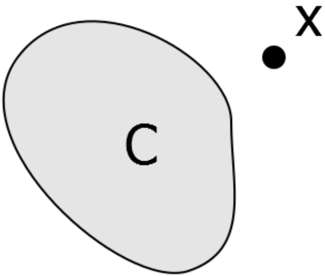
Lemma

Let $\mathcal{C}$ be a convex region and $x \notin \mathcal{C}$ a point. Then there is a hyperplane H separating x from $\mathcal{C}$.

Note that if $\mathcal{C}$ is given by a system of linear inequalities, we can just find one of the defining inequalities that x violates.

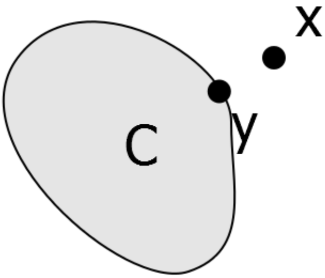
Proof (Optional)

Start with x.



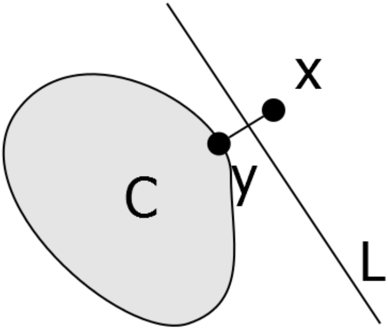
Proof (Optional)

Let y be closest point in $\mathcal{C}$



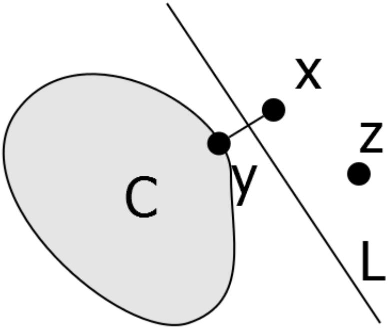
Proof (Optional)

Let L be the perpendicular bisector of xy.



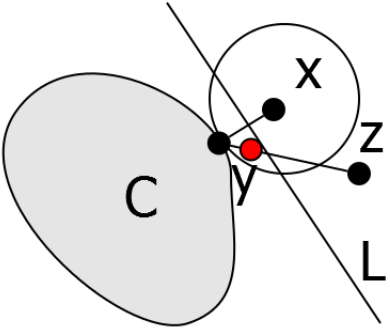
Proof (Optional)

If $z \in \mathcal{C}$ on wrong side of L,



Proof (Optional)

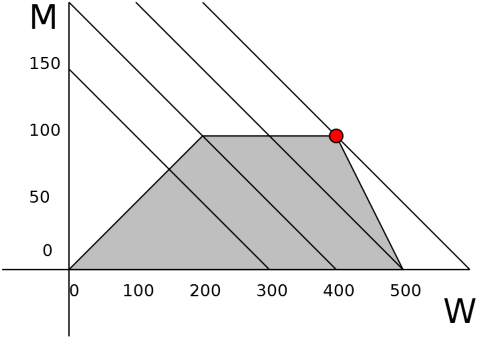
yz contains point closer to x. Contradiction.



Extreme Points

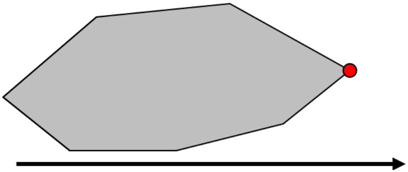
Lemma

A linear function on a polytope takes its minimum/maximum values on vertices.



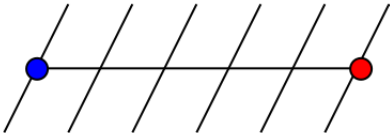
Intuition

The corners are the only extreme points.
Optima must be there.



Idea

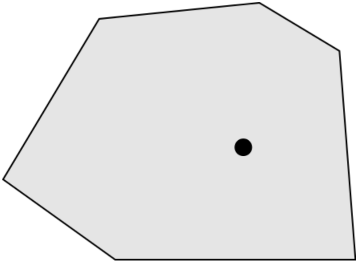
Linear function on segment takes extreme values on ends.



Use to push towards corners.

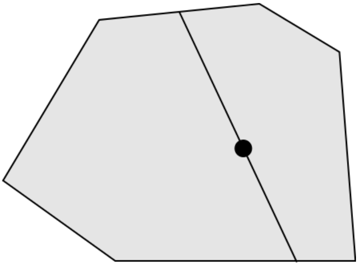
Proof

Start at any point.



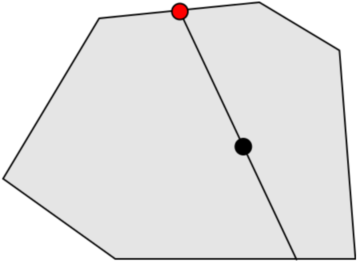
Proof

Pick line through point.



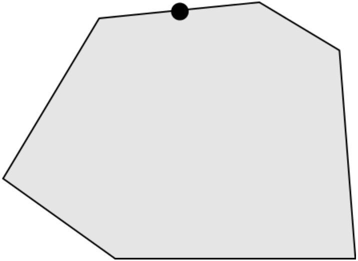
Proof

Extreme values at endpoint.



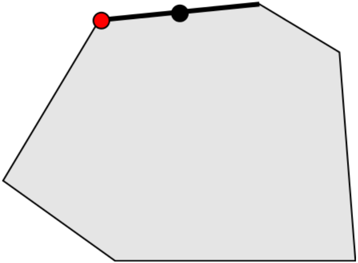
Proof

Point is on a facet.



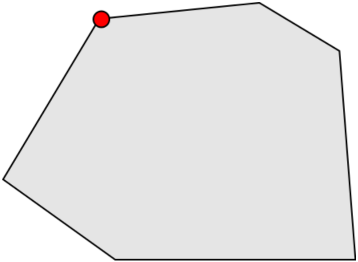
Proof

Repeat to push to lower dimensional facet.



Proof

Eventually on a vertex.



Summary

- Region determined by LP always convex polytope.
- Optimum always at vertex.
- Can separate from outside points by hyperplanes.