# Flows in Networks: Network Flows

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Advanced Algorithms and Complexity
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## Learning Objectives

- Provide the definitions of a network and a flow.
- Give some examples of real world situations in which network flow problems might arise.

# Last Time

- Last time: Discussed disaster management problem.
- Today: Talk about formalization of this and similar problems.

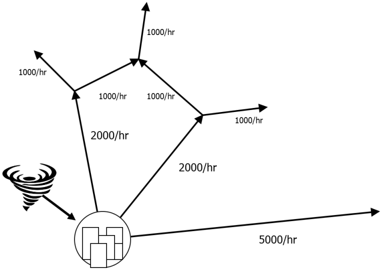
# Network

## Definition

A network is a directed graph $G$ with:

- Each edge, $e$, is assigned a positive real capacity, $C_e$.
- One (or more) vertex is labelled a source.
- One (or more) vertex is labelled a sink.

# Example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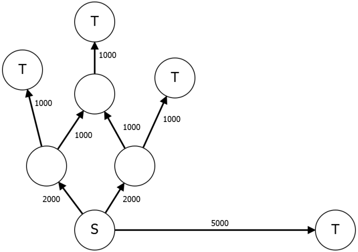
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# Flows

- Next we want to be able to talk about flows (traffic) through a network.
- Rather than talking about where each car goes, we will instead concern ourselves with the total flow, $f_e$, through each edge $e$.

# Flows

- Next we want to be able to talk about flows (traffic) through a network.
- Rather than talking about where each car goes, we will instead concern ourselves with the total flow, $f_e$, through each edge $e$.
- This must satisfy two conditions:

# Rate Limitation

For each edge $e$,

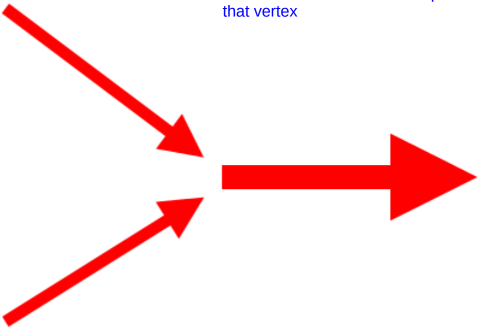
$$0 \leq f_e \leq C_e.$$

# Conservation of Flow

For all $v$ not a source or sink:

$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$

Total flow into the vertex is equal to total flow coming out of that vertex



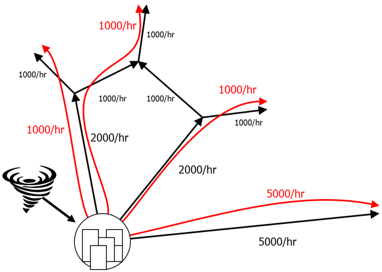
# Formal Definition

## Definition

A flow in a network is an assignment of a real number flow, $f_e$ to each edge $e$ so that

- For all $e$
$$0 \leq f_e \leq C_e.$$
- For all $v$ not a source or sink
$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$

# Example



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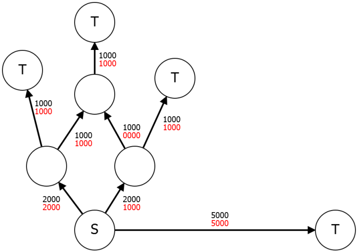

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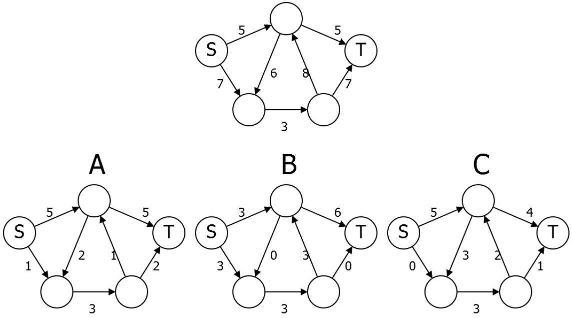
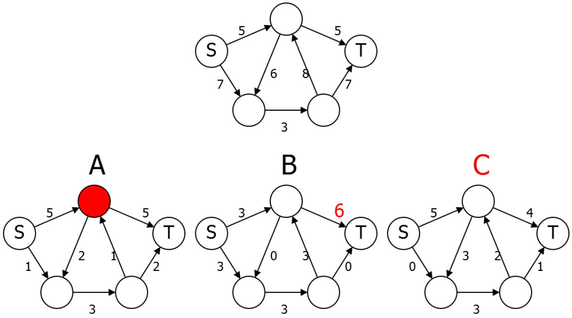
# Problem

Which of the following is a valid flow for the given network?



# Solution

Only flow $C$ is valid. $A$ fails to conserve flow at one vertex, and $B$ exceeds.



# Examples of Network Flows

Network flows are useful to study since they can model a number of real-life phenomena.

# Flows of Goods on a Transportation Network

# Flows of Electricity Through the Power Grid

# Flows of Water Through Pipes

# Flows of Information Through a Communications Network

# Flow Size

One thing to know about a flow is how much stuff is actually flowing. through the entire network

# Flow Size

One thing to know about a flow is how much stuff is actually flowing. This can be computed by looking at the sources.

## Definition

For a flow, $f$, the size of the flow is given by

$$|f| := \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ into a source}} f_e.$$

But how can flow go into the source ??

# Sinks

You can also compute this by looking at sinks.

## Lemma

$$|f| = \sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e.$$

## Proof

$$\begin{aligned} 0 &= \sum_{e} f_e - \sum_{e} f_e \\ &= \sum_{v} \left( \sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\ &= \sum_{v \text{ source or sink}} \left( \sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \end{aligned}$$

## Proof

$$
\begin{aligned}
&= \sum_{v \text{ source or sink}} \left( \sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\
&= \sum_{e \text{ into a source}} f_e + \sum_{e \text{ into a sink}} f_e \\
&\quad - \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ out of a sink}} f_e \\
&= -|f| + \left( \sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e \right).
\end{aligned}
$$

# Problem

How large a flow can we fit through a network?

## Maxflow

Input: A network $G$

Output: A flow $f$ for $G$ with $|f|$ as large as possible.