1. $u = x^3 + 3xy^2 + z^2 - 39x - 36y + 2z + 26$

$u'_x = 3x^2 + 3y^2 - 39$

$u''_{xy} = 6y$

$u''_{xz} = 0$

$u''_{xx} = 6x$

$u'_y = 6xy - 36$

$u''_{yx} = 6y$

$u''_{yz} = 0$

$u''_{yy} = 6x$

$u'_z = 2z + 2$

$u'_{zx} = 0$

$u'_{zy} = 0$

$u''_{zz} = 2$

смешанные производные равны

2. $u = \frac{256}{x} + \frac{x^2}{y} + \frac{y^2}{z} + z^2$

$u'_x = -\frac{256}{x^2} + \frac{2x}{y}$

$u''_{xy} = -\frac{2x}{y^2}$ $u''_{xz} = 0$ $u''_{xx} = \frac{256 \cdot 2}{x^3} + \frac{2}{y}$

$u'_y = -\frac{x^2}{y^2} + 2\frac{y}{z}$

$u''_{yx} = -\frac{2x}{y^2}$ $u''_{yz} = -\frac{2y}{z^2}$ $u''_{yy} = \frac{2x^2}{y^3} + \frac{2}{z}$

$u'_z = -\frac{y^2}{z^2} + 2z$

$u''_{zx} = 0$ $u''_{zy} = -\frac{2y}{z^2}$ $u''_{zz} = \frac{2y^2}{z^3} + 2$

смешанные производные равны

3. $u = x^2 + y^2 + z^2$; $\vec{c}(-9; 8; -12)$; $M(8; -12; 9)$

$u'_x = 2x \quad u'_y = 2y \quad u'_z = 2z$

$\operatorname{grad} u = (2x; 2y; 2z)$

$\operatorname{grad} u\Big|_{(8;-12;+9)} = (16; -24; 18)$

$|\vec{c}| = \sqrt{x_0^2 + y_0^2 + z_0^2} = \sqrt{81 + 64 + 144} = 17$

$\vec{c_0} = \frac{\vec{c}}{|\vec{c}|} = \left(\frac{-9}{17}; \frac{8}{17}; -\frac{12}{17}\right)$

$u'_{\operatorname{grad} u}\Big|_{(8;-12;9)} = -\frac{9}{17} \cdot 16 - \frac{8}{17} \cdot 24 - \frac{12}{17} \cdot 18 =$

$= \frac{-144 - 192 - 216}{17} = -\frac{552}{17}$

4. $u = e^{x^2+y^2+z^2}$, $\vec{d} = (4, -13, -16)$, $L(-16; 4; -13)$

$u'_x = 2x e^{x^2+y^2+z^2}$

$u'_y = 2y e^{x^2+y^2+z^2}$

$u'_z = 2z e^{x^2+y^2+z^2}$

$$\operatorname{grad} u\Big|_{(-16;4;-13)} = \left(-32e^{441}; 8e^{441}; -26e^{441}\right)$$

$$|\vec{d}| = \sqrt{16 + 16\cdot 9 + 256} = 21$$

$$\vec{d_0} = \frac{\vec{d}}{|\vec{d}|} = \left(\frac{4}{21}; -\frac{13}{21}; -\frac{16}{21}\right)$$

$$u'_{\operatorname{grad} u}\Big|_{(-16;4;-13)} = -32e^{441}\cdot\frac{4}{21} - 8e^{441}\cdot\frac{13}{21} +$$

$$+ 26e^{441}\cdot\frac{16}{21} = \frac{2e^{441}}{21}\left(-16\cdot 4 - 4\cdot 13 + 13\cdot 16\right) =$$

$$= \frac{8e^{441}}{21}\left(-16 - 13 + 52\right) = \frac{184e^{441}}{21}$$

5. $u = \log_{21}(x^2+y^2+z^2)$, $F(-19; 8; -4)$.

$$u'_x = \frac{2x}{(x^2+y^2+z^2)\ln 21}$$

$$u'_y = \frac{2y}{(x^2+y^2+z^2)\ln 21}$$

$$u'_z = \frac{2z}{(x^2+y^2+z^2)\ln 21}$$

$$\operatorname{grad} u\big|_{(-19;8;-4)} = \left(\frac{2\cdot(-19)}{((-19)^2+64+16)\ln 21};\right.$$

$$\left.; \frac{2\cdot 8}{441\cdot\ln 21}; \frac{-8}{441\ln 21}\right) = \left(\frac{-38}{441\ln 21}; \frac{16}{441\ln 21}; \frac{-8}{441\ln 21}\right)$$

$\vec{a}_0 = \left(\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$ — возьмем единичный вектор, т.к. самое быстрое направление — это направление градиентного спуска

$$u'_{\vec{a}}\big|_{(-19;8;-4)} = -\frac{38}{441\ln 21}\cdot\frac{\sqrt{2}}{2} + \frac{16}{441\ln 21}\cdot\frac{\sqrt{2}}{2} - \frac{8}{441\ln 21}\cdot\frac{\sqrt{2}}{2} =$$

$$= \sqrt{2}\left(\frac{-19+8-4}{441\ln 21}\right) = -\frac{15\sqrt{2}}{441\ln 21} = -\frac{5\sqrt{2}}{147\ln 21}.$$

6. $u = x^2y + \frac{1}{3}y^3 + 2x^2 + 3y^2 - 1$

$u'_x = 2xy + 4x \qquad u'_y = x^2 + y^2 + 6y$

$\cancel{u'_x}\ \begin{cases} 2xy + 4x = 0 \\ x^2 + y^2 + 6y = 0 \end{cases} \Rightarrow \begin{cases} x(2y+4) = 0 \\ x^2 + y^2 + 6y = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x_1 = 0 \\ y_1 = -2 \end{cases} \Rightarrow x^2 + (-2)^2 + 6(-2) = 0 \Rightarrow$

$\cancel{x^2 + y^2 + 6y = 0 + 4 - 12}$

$\Rightarrow x_{2,3} = \pm\sqrt{8}$

при $x = 0 \Rightarrow 0 + y^2 + 6y = 0 \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = -6 \end{cases}$

получаем точки:

$(0, 0),\ (0, -6),\ (-\sqrt{8}, -2),\ (-\sqrt{8}, -2)$

$u''_{xx} = 2y + 4$

$\cancel{u''_{xy} = 2y}\quad u''_{xy} = u''_{yx} = 2x$

$u''_{yy} = 2y + 6$

$$\begin{pmatrix} u''_{xx} & u''_{xy} \\ u''_{yx} & u''_{yy} \end{pmatrix} = \begin{pmatrix} 2y+4 & 2x \\ 2x & 2y+6 \end{pmatrix} \qquad \Delta_1 = 2y+4$$

$$\Delta_2 = \begin{vmatrix} 2y+4 & 2x \\ 2x & 2y+6 \end{vmatrix} = (2y+4)(2y+6) - 4x^2$$

при $(0;0)$:

$\Delta_1 = 4$, $\Delta_2 = 24$

$(0,0)$ – точка минимума

при $(0;-6)$:

$\Delta_1 = -8$ $\quad \Delta_2 = (-12+4)(-12+6) = 48$

$(0,-6)$ – точка максимума

при $(\sqrt{8};-2)$:

$\Delta_1 = 0$ $\quad$ эту точку можно дальше не считать по теории Гессе

при $(-\sqrt{8};-2)$ – аналогично, т.к. $\Delta_1 = 0$

7. $u = e^{-\frac{x}{2}}(x^2+y^2)$

$u'_x = -\frac{e^{-\frac{x}{2}}}{2}(x^2+y^2) + 2xe^{-\frac{x}{2}}$

$u'_y = 2ye^{-\frac{x}{2}}$

$u''_{xx} = \frac{e^{-x/2}}{4}(x^2+y^2) + (-e^{\frac{x}{2}} \cdot x) + 2e^{-\frac{x}{2}} + (-\frac{e^{-\frac{x}{2}}}{2} \cdot 2x) =$

$= \frac{e^{-x/2}}{4}(x^2+y^2) - 2x \cdot e^{-\frac{x}{2}} + 2e^{-\frac{x}{2}}$

$u''_{xy} = u''_{yx} = -ye^{-\frac{x}{2}}$

$u''_{yy} = 2e^{-\frac{x}{2}}$

$\begin{cases} e^{-\frac{x}{2}}(-\frac{x^2+y^2}{2} + 2x) = 0 \\ 2ye^{-\frac{x}{2}} = 0 \end{cases} \Rightarrow e^{-\frac{x}{2}} = 0$ не имеет корней

$\begin{cases} y = 0 \\ -\frac{x^2+y^2}{2} + 2x = 0 \end{cases} \Rightarrow -\frac{x^2}{2} + 2x = 0 \Rightarrow$

$\Rightarrow \begin{cases} x_1 = 0 \\ -\frac{x_2}{2} + 2 = 0 \end{cases} \quad x_2 = 4$

получаем точки:

$(0;0)$, $(4;0)$