1D analytical analysis for phase field with AT1, AT2, and Nucleation model

Basic

Starting from our energy functional

$$\Pi = \int_\Omega \psi(\varepsilon, d) + G_c \int_\Omega \gamma(d, \nabla d) + \int_{\Gamma_N} t \cdot u - \int_\Omega b \cdot u.$$

Crack surface density function reads

$$\gamma(d, \nabla d) := \frac{1}{c_0}\left(\frac{\alpha(d)}{l} + l \nabla d \cdot \nabla d\right),$$

where for $\psi = g(d)\psi_e^+ + \psi_e^-$,

In `AT-1`: $\alpha = d, c_0 = 8/3$, In `AT-2`: $\alpha = d^2, c_0 = 2$.

Jump to the strong form

$$\operatorname{div} \sigma = 0, g'(d)\psi_e^+ + \frac{G_c}{c_0}\left(\frac{\alpha'(d)}{l} - 2l \nabla \cdot \nabla d\right) = 0.$$

1D uniaxial homogenous

AT-1

For 1D, consider $\psi_e^+ = \frac{1}{2}E\varepsilon^2, \varepsilon = \frac{du}{dx}, \sigma = g(d)E\varepsilon, g(d) = (1-d)^2$, the strong form looks like

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} = 0, (d-1)E\varepsilon^2 + \frac{3G_c}{8}\left(\frac{1}{l} - 2l\frac{\mathrm{d}^2 d}{\mathrm{d}x^2}\right) = 0.$$

Consider uniaxial homogenous solution for d, we have

$$d = \max\left(0, 1 - \frac{3G_c}{8lE\varepsilon^2}\right),$$

and

$$\sigma = (1-d)^2 E\varepsilon = \begin{cases} E\varepsilon & \text{if } \varepsilon < \varepsilon_c \\ \dfrac{9G_c^2}{64l^2 E\varepsilon^3} & \text{if } \varepsilon \geq \varepsilon_c. \end{cases}$$

The critical strain is

$$\varepsilon_c = \sqrt{\frac{3G_c}{8El}},$$

and the corresponding critical stress is

$$\sigma_c = E\varepsilon_c = \sqrt{\frac{3EG_c}{8l}}.$$

AT-2

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} = 0, (d-1)E\varepsilon^2 + \frac{G_c}{2}\left(\frac{d}{l} - l\frac{\mathrm{d}^2 d}{\mathrm{d}x^2}\right) = 0.$$

Consider uniaxial homogenous solution for d, we have

$$d = \left(\frac{G_c}{2lE\varepsilon^2} + 1\right)^{-1},$$

and

$$\sigma = (1-d)^2 E\varepsilon = \left(\frac{2lE\varepsilon^2}{G_c} + 1\right)^{-2} E\varepsilon.$$

The critical strain is

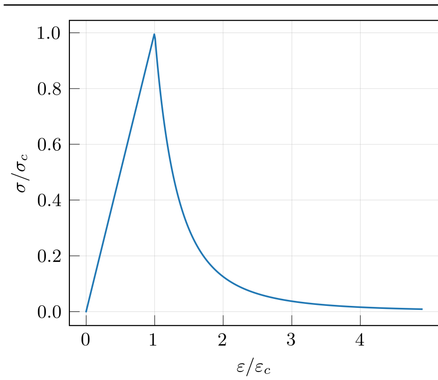
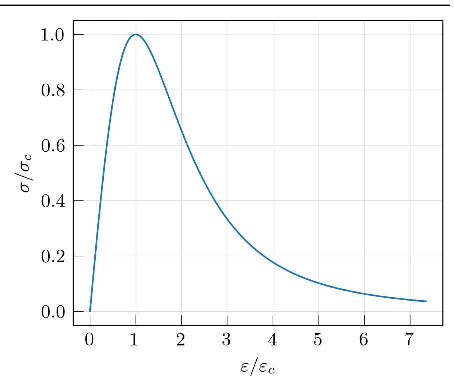
$$\varepsilon_c = \sqrt{\frac{G_c}{6El}},$$

The corresponding critical stress is

$$\sigma_c = \frac{9}{16}\sqrt{\frac{EG_c}{6l}}.$$

Plots for AT1 and AT2

Plot σ/σ_c over $\varepsilon/\varepsilon_c$:

AT1	AT2
	

The critical stress from Bobaru's paper, given $E = 72 \times 10^3$ MPa, $G_c = 3.8$ J/m^2,

$$\sigma_c(l = 1.2 \text{ mm}) = 3.47 \text{ (MPa)} \sigma_c(l = 0.6 \text{ mm}) = 4.90 \text{ (MPa)}$$

The tensile strength of soda-lime glass is 25~180 MPa

LDL nucleation model

$$g'(d)\psi_e^+ + \frac{\delta^l G_c}{c_0}\left(\frac{\alpha'(d)}{l} - 2l\nabla\cdot\nabla d\right) + c_e \geq 0,$$

where the external driving force reads

$$c_e = g(d)\widehat{c_e} = g(d)\left(\alpha_2\sqrt{J_2} + \alpha_1 I_1\right)$$

Consider uniaxial homogenous solution for d with `AT-1` settings, we have

$$d = \max\left(0, 1 - \frac{1}{E\varepsilon^2}\left(\frac{3\delta^l G_c}{8l} + c_e\right)\right).$$

The critical strain

$$\varepsilon_c = \sqrt{\frac{3\delta^l G_c}{8El} + \frac{c_e}{E}},$$

and the critical stress

$$\sigma_c = E\varepsilon_c = \sqrt{\frac{3\delta^l EG_c}{8l} + Ec_e}.$$

Verify with calculation Now given a set of material properties, e.g, graphite:

E	ν	G_c	σ_{ts}	σ_{cs}
9.8 GPa	0.13	91 N/m	27 MPa	77 MPa

1) Using standard `AT-1`, we have

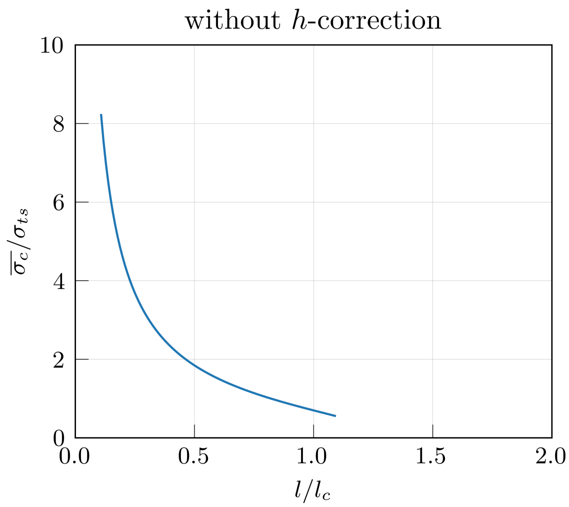
$$\sigma_c = E\varepsilon_c = \sqrt{\frac{3EG_c}{8l}} = \sigma_{ts},$$

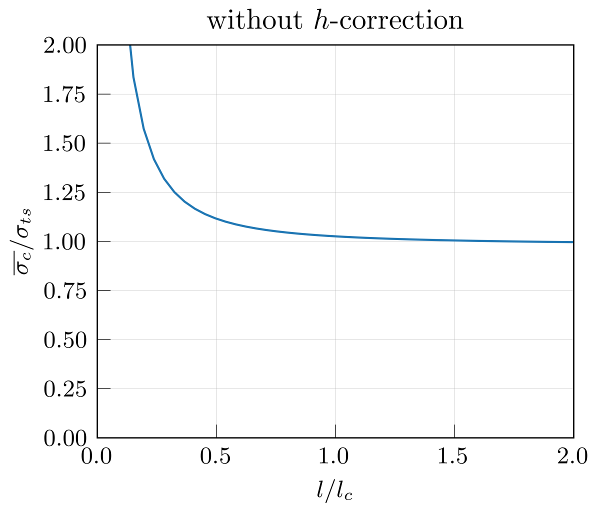
we derive $\varepsilon_c = \dfrac{\sigma_{ts}}{E} = 0.0027551$, $l_c = \dfrac{3EG_c}{8\sigma_{ts}^2} = 0.4587$ mm.

2) For the nucleation model, Now we can calculate c_e term at $\varepsilon = \varepsilon_c$ with various l

use δ without h correction term:

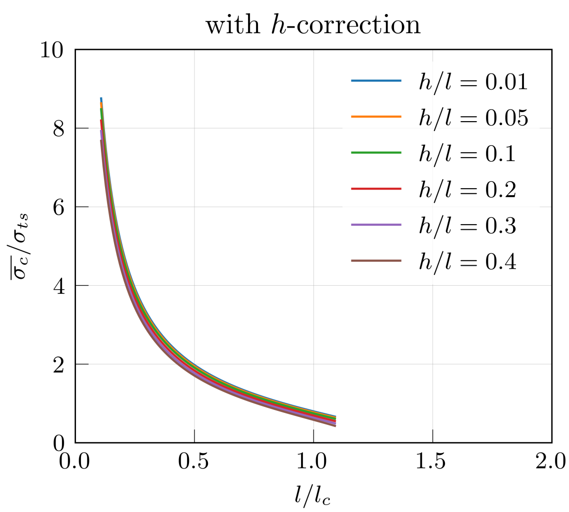
$+c_e$	$-c_e$

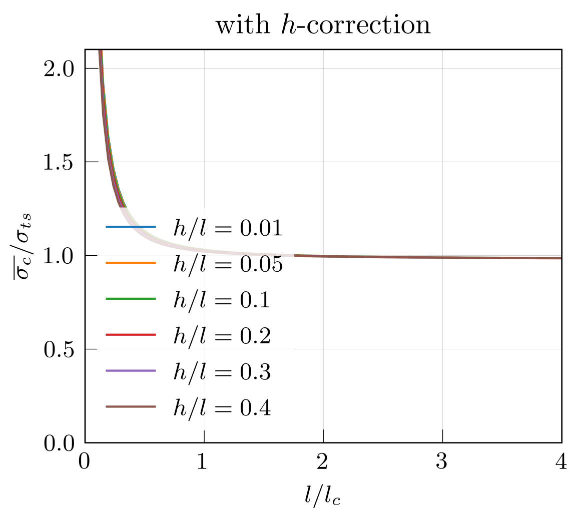




use δ with h correction term:

$+c_e$	$-c_e$

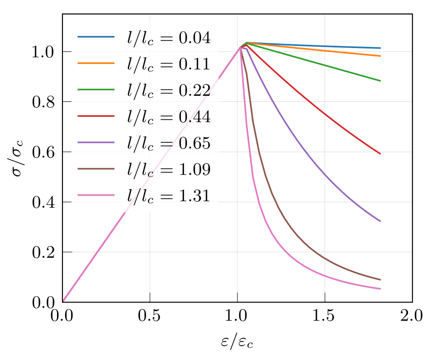
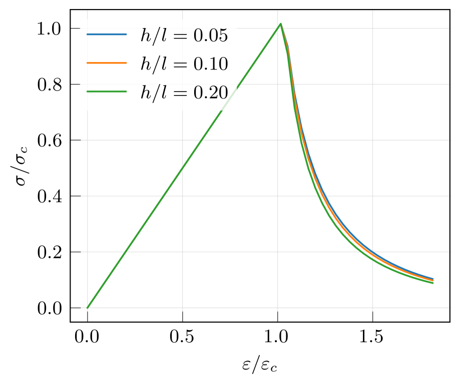




It turns out should actually be $-c_e$!

Single element simulation 1 element, 3D (set $\nu = 0$), uniaxial tension, $\Omega = \Omega_e = [0, 1] \times [0, 0.1] \times [0, 0.1]$, with $-c_e$

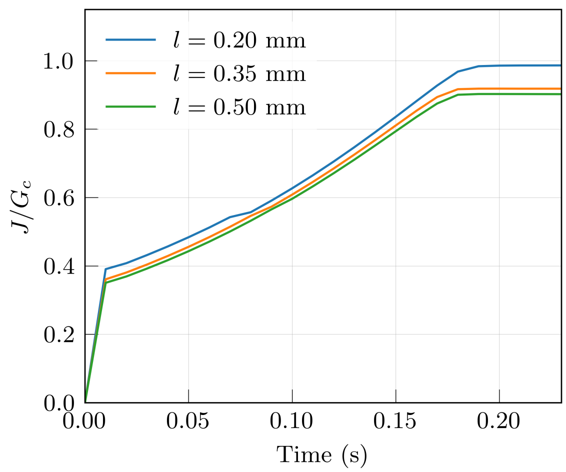
- without h correction, looking at $l = [0.02, 0.05, 0.1, 0.2, 0.3, 0.5, 0.6]$ mm.
- with h correction (actually change mesh, not single element), looking at $l = 0.5$ mm ($l/l_c = 1.09$), $h/l = [0.05, 0.1, 0.2]$

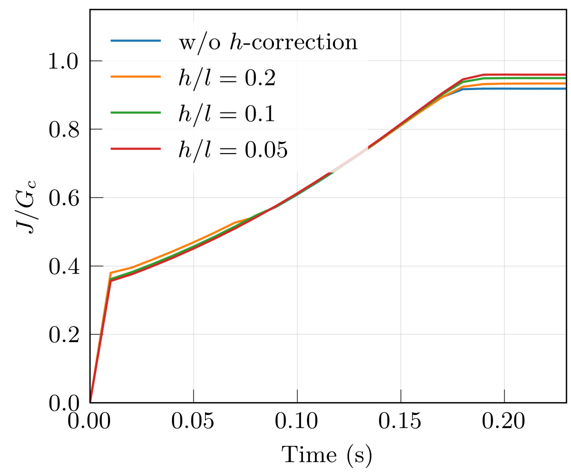
without h correction	with h correction
	

Surfing boundary problem Plot J/G_c without and with h correction

- without h correction, $l = [0.2, 0.35, 0.5]$ mm.
- with h correction, $l = 0.35$ mm, $h/l = [0.05, 0.1, 0.2]$.

without h correction	with h correction, e.g. $l = 0.35$ mm



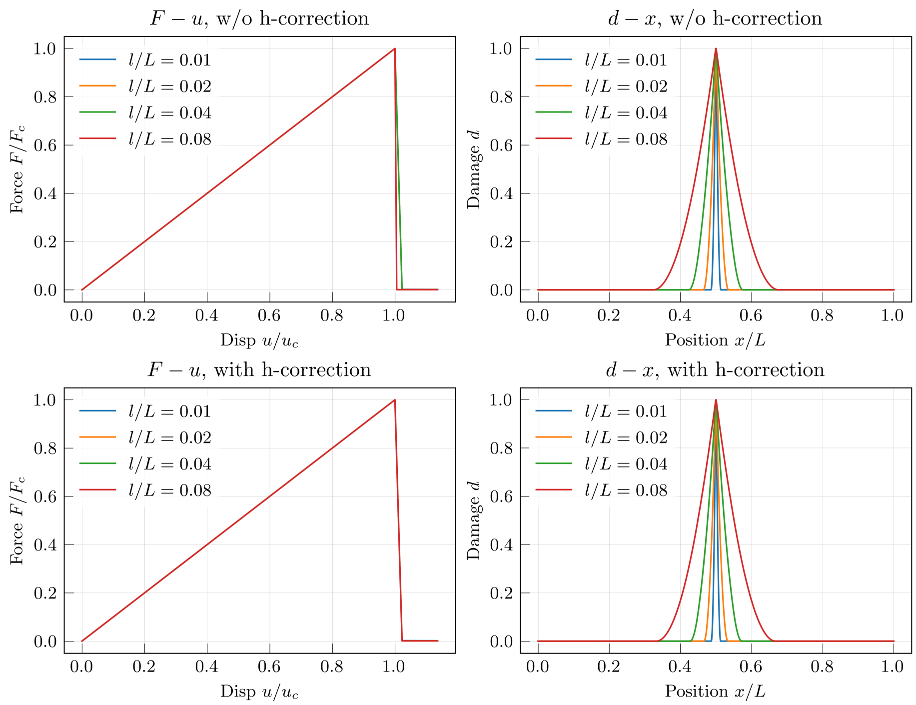


1D bar

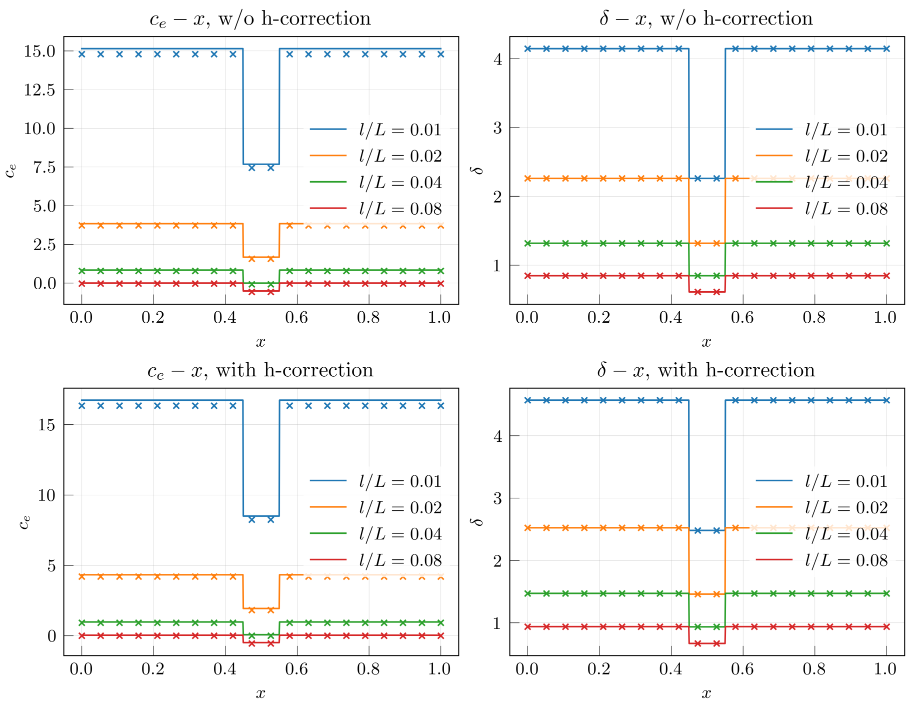
> see Rudy's paper on cohesive PFM, I run it with 1D element (with $\nu = 0$), changing E to $E/2$ in the middle.

Consider a bar of $\Omega = [0, L]$, where $L = 1$ mm, mesh size $h_e = L/2000 = 0.0005$ mm. $E = 10$ MPa at $x \in [0, 0.45] \cup [0.55, 1]$, $E = 5$ MPa at $x \in [0.45, 0.55]$. $G_c = 0.1$ N/mm, $\sigma_c = 2$ MPa

Load-displacement curves and damage profiles:



The plots of c_e and δ^l over x at $u = u_c$:


$c_e - x$, w/o h-correction
$\delta - x$, w/o h-correction
$c_e - x$, with h-correction
$\delta - x$, with h-correction
$l/L = 0.01$
$l/L = 0.02$
$l/L = 0.04$
$l/L = 0.08$
c_e
δ
x