Strength surface of various phase field models

See Lorenzis IJF paper

Basic assumptions: use `AT-1`, $d_c = 0$, homogenous damage field ($\Delta d = 0$)

Standard phase field

Energy density

$$W(\varepsilon, d) = \varphi(\varepsilon, d) + G_c\gamma(d, \nabla d)$$

Stress and energy release rate

$$\sigma(\varepsilon, d) = \frac{\partial W}{\partial \varepsilon} = \frac{\partial \varphi}{\partial \varepsilon},$$

$$G(\varepsilon, d) = -\frac{\partial W}{\partial d} = -\frac{\partial \varphi}{\partial d} - G_c\gamma'(d).$$

Impose the damage criterion $G \geq 0$

$$-\frac{\partial \varphi(\varepsilon, d)}{\partial d} \geq G_c\gamma'(d, \nabla d)$$

Consider 1D case, right φ in terms of ε and σ:

$$\varphi(\varepsilon, d) = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}g(d)E\varepsilon^2 = g(d)\varphi_0(\varepsilon),$$

$$\varphi^*(\sigma, d) = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\frac{\sigma^2}{g(d)E} = \frac{\varphi_0^*(\sigma)}{g(d)}.$$

Actually, In Laura's paper, $\varphi^*(\sigma)$ is defined as the **complementary** elastic energy

$$\varphi^*(\sigma, d) := \sup_{\varepsilon \in Sym} \sigma \cdot \varepsilon - \varphi(\varepsilon, d)$$

In general, we have:

$$\varphi(\varepsilon, d) = g(d)\varphi_0(\varepsilon),$$

$$\varphi^*(\sigma, d) = \frac{\varphi_0^*(\sigma)}{g(d)},$$

where

$$\varphi_0(\varepsilon) = \kappa \mathrm{tr}(\varepsilon)^2/2 + \mu||\varepsilon_{dev}||^2,$$

$$\varphi_0^*(\sigma) = \mathrm{tr}(\sigma)^2/2n^2\kappa + ||\sigma_{dev}||^2/4\mu,$$

where n is number of dim.

$$\frac{\partial \varphi^*(\sigma, d)}{\partial d} = -\frac{\partial \varphi(\varepsilon, d)}{\partial d}$$

We have damage criterion for stress

$$\frac{\partial \varphi^*(\sigma, d)}{\partial d} \geq G_c \gamma'(d, \nabla d)$$

Consider `AT-1`, isotropic (no split), quadratic degradation function:

$$\gamma(d) = \frac{3}{8}\left(\frac{d}{l} + l\nabla d \cdot \nabla d\right), g(d) = (1-d)^2$$

We have

$$\frac{\partial \varphi^*(\sigma, d)}{\partial d} \geq \frac{3G_c}{8l}$$

$$\Rightarrow \frac{-\varphi_0^*(\sigma) g'(d)}{g^2(d)} \geq \frac{3G_c}{8l}$$

Consider $g(0) = 1, g'(0) = -2$,

$$\Rightarrow 2\varphi_0^*(\sigma) \geq \frac{3G_c}{8l}$$

$$\frac{\text{tr}(\sigma)^2}{n^2\kappa} + \frac{||\sigma_{dev}||^2}{2\mu} - \frac{3G_c}{8l} \geq 0$$

Substitute $\sigma_{ts} = \sqrt{\frac{3EG_c}{8l}}$, $I_1 = \text{tr}(\sigma), J_2 = ||\sigma_{dev}||^2/2, n = 3$,

$$\mathcal{F}(\sigma) = \frac{I_1}{9\kappa} + \frac{J_2}{\mu} - \frac{\sigma_{ts}}{E} = 0. \tag{1}$$

Nucleation model

Nuc2020 and Nuc 2022

We dont have c_e in the energy density

$$W(\varepsilon, d) = \varphi(\varepsilon, d) + G_c\gamma(d, \nabla d) + f(c_e)$$

But at the driving force for d

$$\frac{\partial W}{\partial d} = g'(d)\varphi_0(\varepsilon) + G_c\gamma'(d, \nabla d) + c_e$$

We can write $c_e = g(d)\hat{c_e}$, and

$$f(c_e) = \int_0^1 c_e \mathrm{d}d = \hat{c_e}\int_0^1 g(d)\mathrm{d}d$$

Now we can come back to the damage criteria

$$G(\varepsilon, d) = -\frac{\partial W}{\partial d} = -\frac{\partial \varphi}{\partial d} - G_c\gamma'(d) - g(d)\hat{c_e} \geq 0.$$

Change to stress $\varphi^*(\sigma, d)$, substitute $\gamma(d)$ and φ_0, consider homogenous d, and $d = 0$ for $g(d)$

$$\frac{\text{tr}(\sigma)^2}{n^2\kappa} + \frac{||\sigma_{dev}||^2}{2\mu} - \frac{3G_c}{8l} - g(d)\hat{c_e} \geq 0,$$

which correspond to

$$\mathcal{F}(\sigma) = \frac{I_1}{9\kappa} + \frac{J_2}{\mu} - \frac{3G_c}{8l} - \hat{c_e} = 0. \tag{2}$$

Note that in Aditya's 2020 and 2022 paper they just use $\hat{c_e}$ to represent the 'degraded' c_e, here I am being consistent with 2024 paper.

Nuc2024

Energy density

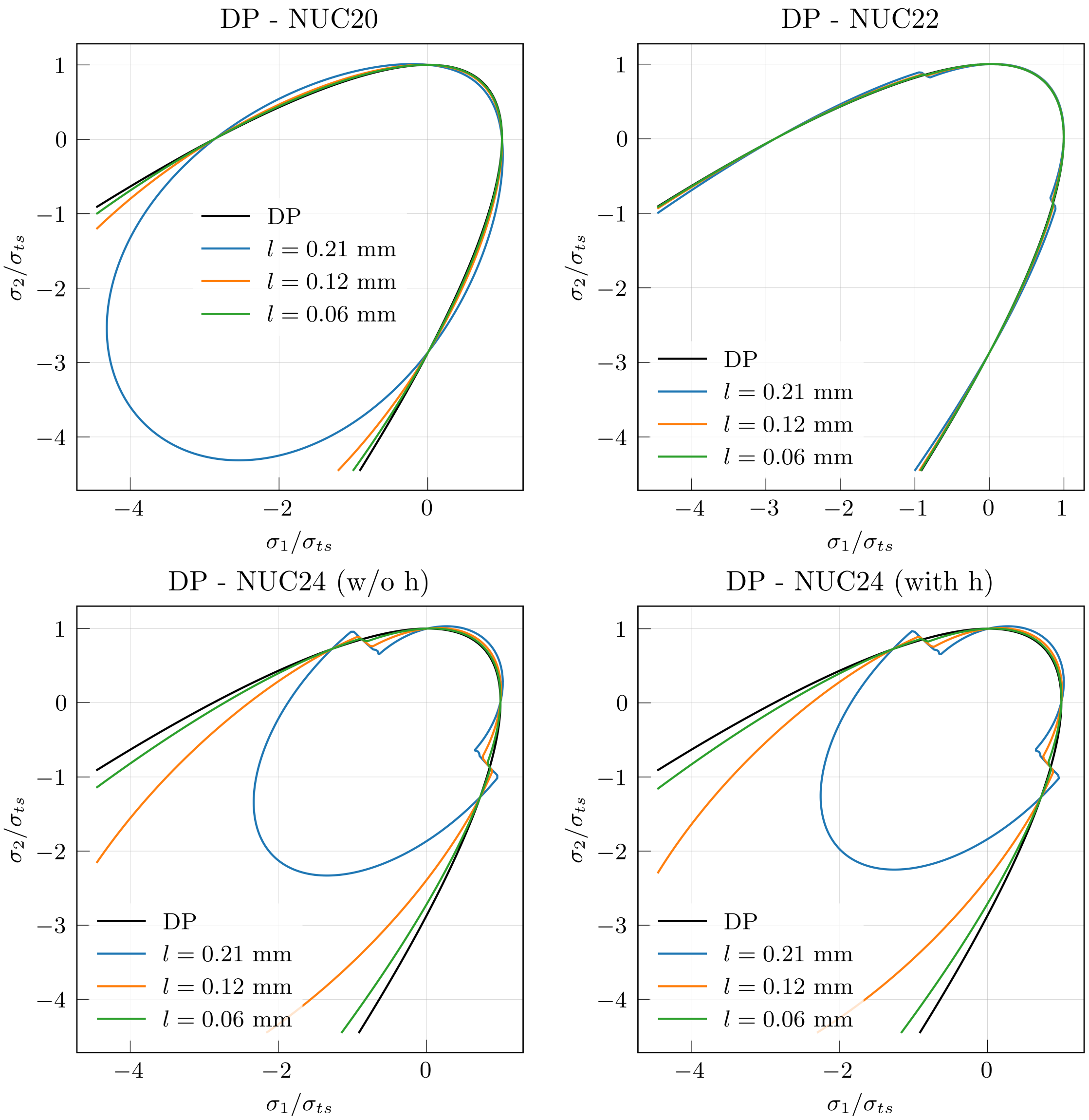
$$W(\varepsilon, d) = \varphi(\varepsilon, d) + \delta^l G_c \gamma(d, \nabla d) + \int_0^1 g(d)\hat{c_e}\,\mathrm{d}d$$

Similarly, consider `AT-1`, $g(d) = (1-d)^2$, $c_e = g(d)\hat{c_e} \Rightarrow c_e'(d) = g'(d)\hat{c_e}$

$$\mathcal{F}(\sigma) = \frac{I_1}{9\kappa} + \frac{J_2}{\mu} - \frac{3\delta G_c}{8l} - \hat{c_e} = 0. \tag{3}$$

Comparison

Use material properties of graphite (see Aditya's 2020 JMPS paper, Fig 2)



Looking at $l = 0.12$ mm, very $h/l = [0.5, 0.25, 0.1, 0.05, 0.01]$ with h-correction,