Derivation of gonverning equations of quasi-static phase field evolution

Consider the total potential energy (ignoring the body force and traction):

$$\Pi(\boldsymbol{u}, d) = \int_\Omega g(d)\psi_e^+ + \psi_e^- \,\mathrm{dV} + \int_\Omega \mathcal{G}_c \gamma(d, \nabla d)\,\mathrm{dV}.$$

In the `AT-1` model, we have $\gamma(d, \nabla d) = \frac{3}{8\ell}(d + \ell^2 \nabla d \cdot \nabla d)$,

$$\Pi(\boldsymbol{u}, d) = \int_\Omega g(d)\psi_e^+ + \psi_e^- \,\mathrm{dV} + \int_\Omega \frac{3\mathcal{G}_c}{8\ell}(d + \ell^2 \nabla d \cdot \nabla d)\,\mathrm{dV}.$$

When d increases, $g(d)$ decreases, and so does Π, we have

$$\Psi_f = -\frac{\partial \Pi}{\partial d} \le 0,\ \dot{d} \ge 0,\ \Psi_f \dot{d} = 0$$

where

$$\Psi_f = \int_\Omega -g'(d)\psi_e^+ - \frac{3\mathcal{G}_c}{8\ell}(1 - 2\ell^2 \nabla \cdot \nabla d) \le 0$$

Rewrite it

$$\nabla \cdot (\nabla d \mathcal{G}_c \ell) \le \frac{3}{4} g'(d)\psi_e^+ + \frac{\mathcal{G}_c}{2\ell}, \quad \text{if } \dot{d} = 0$$

The KKT condition:

1. if $\dot{d} > 0$, the damage is increasing $\to \frac{\partial \Pi}{\partial d} = 0$ (crack propagating)
2. if $\dot{d} = 0$, the damage is not increasing $\to \frac{\partial \Pi}{\partial d} > 0$ (have load, crack have'nt propagates)

Starting from Oscar's form

Eq(15) in Brazilian paper:

$$\nabla \cdot (\ell \mathcal{G}_c \nabla v) \ge \frac{8}{3} v\psi_e - \frac{4}{3} c_e - \frac{\mathcal{G}_c}{2\ell}$$

In Coh-PFM we don't have teh nucleation term $c_e = 0$, and in our convention, $v = 1 - d$,

$$\nabla \cdot (\ell \mathcal{G}_c \nabla (1 - d)) \ge \frac{8}{3}(1 - d)\psi_e - \frac{\mathcal{G}_c}{2\ell},$$

where $g(d) = (1-d)^2, g'(d) = -2(1-d)$.

$$-\nabla\cdot(\ell\mathcal{G}_c\nabla(d)) \geq -\frac{4}{3}g'(d)\psi_e - \frac{\mathcal{G}_c}{2\ell},$$

$$\Rightarrow \nabla\cdot(\ell\mathcal{G}_c\nabla(d)) \leq \frac{4}{3}g'(d)\psi_e + \frac{\mathcal{G}_c}{2\ell}.$$