Probability and Random Processes

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1 Information Theory

Defintion 1.1 (Entropy)**.**

$$H(X) = \mathbb{E}\left[-\log_2 p(x)\right] = \sum_{x \in X} -p(x) \log_2 p(x)$$

If $X \sim B(p)$, then

$$H(X) = p - \log_2 p + (1-p)\log_2(1-p) \triangleq h(p)$$

*called the **binary entropy function**.*

Defintion 1.2 (Joint Entropy)**.**

$$H(X,Y) = \mathbb{E}\left[-\log_2 p(x,y)\right] = \sum_x \sum_y -p_{x,y}(x,y) \log_2 p(x,y)$$

If X, Y are independent, then $H(X,Y) = H(X) + H(Y)$.

Proof.

$$\begin{aligned} H(X,Y) &= \mathbb{E}\left[-\log_2 p(x,y)\right] \\ &= \mathbb{E}\left[-\log_2 p(x)\,p(y)\right] \\ &= \mathbb{E}\left[-\log_2 p(x) - \log_2 p(y)\right] \\ &= \mathbb{E}\left[-\log_2 p(x)\right] + \mathbb{E}\left[-\log_2 p(y)\right] \\ &= H(X) + H(Y) \end{aligned}$$

□

Defintion 1.3 (Conditional Entropy)**.**

$$H(Y \mid X) = \mathbb{E}_{XY}\left[-\log_2 p(y \mid x)\right] H(X,Y) - H(X)$$

Defintion 1.4 (Mutual Information)**.**

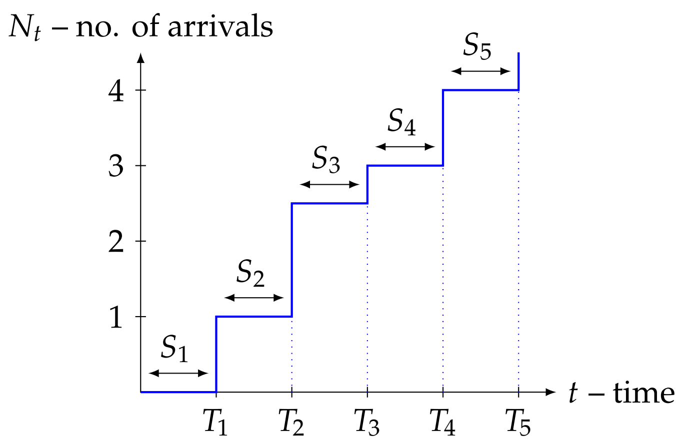
$$\begin{aligned} I(X;Y) &\triangleq H(X) - H(X \mid Y) \\ &= H(Y) - H(Y \mid X) \\ &= H(X) + H(Y) - H(X,Y) \end{aligned}$$

Theorem 1.5 (Asymptotic Equipartition)**.** *If $X_1, X_2, \ldots, X_n$ are iid $\sim p(X)$, then*

$$\frac{-\log_2 p(X_1, X_2, \ldots, X_n)}{n} \xrightarrow{p} H(X).$$

2 Poisson Process

A poisson process is the continuous time analog of "coin flipping" or Bernoulli processes. This makes it a good model for arrival processes: photons hitting a detector, packets in a network, number of emails per hour, etc.



Each T_i for $i = 1, 2, 3, 4, 5$ represents an arrival and generally, each arrival time is defined as

$$T_n = \sum_{i=1}^{n} S_i$$

where the interarrival times $S_1, S_2, \ldots, S_n \overset{\text{iid}}{\sim} Exponential(\lambda)$. Thus, every S_i has the probability density function

$$f_{S_i}(t) = \lambda e^{\lambda t};\ t > 0;\ i = 1, 2, 3, \ldots$$

and cumulative distribution function

$$F_{S_i}(t) = 1 - e^{-\lambda t}.$$

Defintion 2.1 (Number of Arrivals)**.**

$$N_t = \begin{cases} \max_{n \geq 1}\{n \mid T_n \leq t\} & t \geq 0 \\ 0 & t < T_1 \end{cases}$$

- Recall: $Exponential(\lambda)$ is a memoryless RV

1. $F_\tau(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} \Rightarrow f_\tau(t) = \lambda e^{-\lambda t}$
2. $\mathbb{E}[\tau] = \frac{1}{\lambda}$ and $\mathrm{Var}(\tau) = \frac{1}{\lambda^2}$
3. $P(\tau > t + s \mid \tau > s) = P(\tau > t)$
4. $P(\tau \leq t + \epsilon \mid \tau > t) = \lambda\epsilon + o(\epsilon)$; $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$

Proof.

$$\begin{aligned} P(\tau > t + \epsilon \mid \tau > t) &= P(\tau > \epsilon) \\ &= e^{-\lambda\epsilon} \\ &= 1 - \lambda\epsilon + o(\epsilon) \end{aligned}$$

$\square$