Probability and Random Processes

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1 Information Theory

Defintion 1.1 (Entropy)**.**

$$H(X) = \mathbb{E}\left[-\log_2 p(x)\right] = \sum_{x \in X} -p(x) \log_2 p(x)$$

If $X \sim B(p)$, then

$$H(X) = p - \log_2 p + (1-p) \log_2 (1-p) \triangleq h(p)$$

*called the **binary entropy function**.*

Defintion 1.2 (Joint Entropy)**.**

$$H(X,Y) = \mathbb{E}\left[-\log_2 p(x,y)\right] = \sum_{x} \sum_{y} -p_{x,y}(x,y) \log_2 p(x,y)$$

If X, Y are independent, then $H(X,Y) = H(X) + H(Y)$.

Proof.

$$\begin{aligned} H(X,Y) &= \mathbb{E}\left[-\log_2 p(x,y)\right] \\ &= \mathbb{E}\left[-\log_2 p(x)\, p(y)\right] \\ &= \mathbb{E}\left[-\log_2 p(x) - \log_2 p(y)\right] \\ &= \mathbb{E}\left[-\log_2 p(x)\right] + \mathbb{E}\left[-\log_2 p(y)\right] \\ &= H(X) + H(Y) \end{aligned}$$

□

Defintion 1.3 (Conditional Entropy)**.**

$$H(Y \mid X) = \mathbb{E}_{XY}\left[-\log_2 p(y \mid x)\right] H(X,Y) - H(X)$$

Defintion 1.4 (Mutual Information)**.**

$$\begin{aligned} I(X;Y) &\triangleq H(X) - H(X \mid Y) \\ &= H(Y) - H(Y \mid X) \\ &= H(X) + H(Y) - H(X,Y) \end{aligned}$$

Theorem 1.5 (Asymptotic Equipartition)**.** *If* $X_1, X_2, \ldots, X_n$ *are iid* $\sim p(X)$*, then*

$$\frac{-\log_2 p(X_1, X_2, \ldots, X_n)}{n} \xrightarrow{p} H(X).$$