Probability & Random Processes

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1 Discrete Probability

At it's core probability is a principled way of reasoning about uncertainty,

> Probability is common sense reduced to calculations — Laplace

Definition (Sample Space, Ω)**.** The sample space of an experiment is the set of all possible outcomes of said experiment.

Definition (Event)**.** An event A is any allowable subset of Ω.

Definition (Probability Space)**.** A probability space $(\Omega, \mathcal{F}, P)$ is a mathematical construct used to model experiments.

- Ω: set of all possible outcomes
- $\mathcal{F}$: set of all events
- P: probability measure

1.1 Probability Axioms

There exists three fundamental rules that make probability work,

1. The probability of nothing occurring is zero,
$$P(\emptyset) = 0.$$
2. The probability of everything occurring is one,
$$P(\Omega) = 1.$$
This comes as a direct consequence from the previous statement.
3. Given a sequence of disjoint events $A_1, A_2, \ldots$, then
$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Regarding the third probability axiom, if we are given a finite sequence of events $A_1, A_2, \ldots, A_n$ (whether or not they are disjoint) we have

$$P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i).$$

In other words, the probability of $\bigcup_{i=1}^{n} A_i$ is at most the sum of the individual probabilities $\sum_{i=1}^{n} P(A_i)$. This is known as the **union bound**.

1.2 Discrete Probability Law

For any event A in some Ω,

$$P(A) = \sum_{\omega \in A} P(\omega)$$

and if Ω is uniform then

$$P(A) = \frac{|A|}{|\Omega|}.$$

This fact is essential when computing probabilities over discrete sets of outcomes.

Example (Birthday Paradox)**.** Suppose that there are $n \geq 2$ people in a room. We want to find the probability that *at least two people have the same birthday.* Ignoring leap years, we accomplish this by first assuming that every person's birthday is assigned uniformly at random and that there are k number of distinct birthdays (less abstractly k can be at most 365). Let

$A = \{\textit{at least two people share the same birthday}\}$ and $A^C = \{\textit{no two people share the same birthday}\}$

then

$$P(A^C) = \frac{k \times (k-1) \times (k-2) \times \ldots (k-n+1)}{k \times k \times k \times \ldots k} = \prod_{i=0}^{n-1} \frac{k-i}{k} = \prod_{i=0}^{n-1} \left(1 - \frac{i}{k}\right)$$

where for $n > k$ the product becomes zero (by the pigeonhole principle) and $P(A) = 1$. Otherwise for $n < k$ we can take the log of the probability and use the fact that $\log(1-x) \approx -x$ (given small enough x),

$$\log P(A^C) = \sum_{i=0}^{n-1} \log(1 - \frac{i}{k}) \approxeq -\frac{1}{k} \sum_{i=0}^{n-1} i = -\frac{n(n-1)}{2k}$$

exponentiating both sides yields

$$P(A^C) = \exp\left(-\frac{n(n-1)}{2k}\right).$$

So $P(A) = 1 - e^{-n(n-1)/2k}$ and notice that as the number of people increases $e^{-n(n-1)/2k} \to 0$ for a fixed value of k.