Linear Algebra lecture 22

1. Diagonalizing a matrix $S^{-1}AS = \Lambda$
2. Powers of A / equation $u_{k+1} = Au_k$

Suppose n indep eginvector of A,
Put them in column of S

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}$$

punchline

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \\ \vdots & 0 \\ 0 & \lambda_n \end{bmatrix} \rightarrow \text{diagonal eigenvalue matrix } \Lambda$$

$$= S\Lambda$$

$$AS = S\Lambda$$

$$S^{-1}\Lambda S = \Lambda \qquad A = S\Lambda S^{-1}$$

If $Ax = \lambda x$, $\underline{A^2x = \lambda Ax = \lambda^2 x}$

$$A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

$$A^k = S\Lambda^k S^{-1}$$

Theorem: $A^k \to 0$ $k \to \infty$, if all $|\lambda_i| < 1$

A is sure to have n indep evectors (and be diagonalizable)

if all λ's are different (no repeated λ's)

Repeated eigenvalues // may or may not have n indep evectors.

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$

$\lambda = 2, 2$. $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Equation $u_{k+1} = Au_k$ start with given vector u_0

$u_1 = Au_0$, $u_2 = A^2 u_0$, $\boxed{u_k = A^k u_0}$

To really solve: write

$$u_0 = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n = Sc$$

$$Au_0 = c_1 \lambda_1 X_1 + c_2 \lambda_2 X_2 + \cdots + c_n \lambda_n X_n$$

$$A^{100}u_0 = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2 + \cdots + c_n\lambda_n^{100}x_n$$

$$u_{100} = S\Lambda^{100}C$$

Fibonacci example: 0, 1, 1, 2, 3, 5, 8, 13, ...

$F_{100} = ?$

TRICK

$$F_{k+2} = F_{k+1} + F_k \qquad u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$F_{k+1} = F_{k+1} \qquad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \quad \lambda_2 = \frac{1}{2}(1-\sqrt{5})$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad c_1x_1 + c_2x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1\lambda_1 + c_2\lambda_2 = 1$$

$$c_1 + c_2 = 0$$

$$c_1\lambda_1 - c_1\lambda_2 = 1$$

$$c_1(\lambda_1 - \lambda_2) = 1$$

$$c_1 = \frac{1}{\sqrt{5}}$$

$$c_2 = -\frac{1}{\sqrt{5}}$$

$$u_0 = \frac{1}{\sqrt{5}}x_1 - \frac{1}{\sqrt{5}}x_2$$

$$u_1 = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = Au_0$$

$$u_2 = \begin{bmatrix} F_3 \\ F_2 \end{bmatrix} = A^2 u_0$$

$$\vdots$$

$$u_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix} = A^{100} u_0$$

$$A^{100}u_0 = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1^{100} & 0 \\ 0 & \lambda_2^{100} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^{100}x_1 & \lambda_2^{100}x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2$$

$$\begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{100} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{100} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$$