Linear Algebra Lecture 33

4 subspaces

Left-inverses

Right-inverses

Pseudo-inverse

2-sided inverse

$AA^{-1} = I = A^{-1}A$ $r = m = n$ full rank

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left inverse

full column rank $r = n$

null space $= \{0\}$ independent columns

0 or 1 solutions to $Ax = b$

A^TA is inverse as A is inverse

$\det(A) \neq 0$ $\det(A^T)\det(A) = \det(A^TA) \neq 0$

$$\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}} \quad n \times m \qquad A^{-1}_{left}A = I$$

$m \times n \qquad n \times n$

$$AA^{-1}_{left} = A(A^TA^{-1})A^T = P$$

Onto column space

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right-inverse

full row rank $r = m < n$

$N(A^T) = \{0\}$ independent rows

∞ solutions to $Ax = b$ $n-m$ free variables

$$A\underbrace{A^T(AA^T)^{-1}}_{A^{-1}_{right}} = I$$

$m \times n \qquad n \times m \qquad m \times m$

$$A^{-1}_{right}A = A^T(AA^T)^{-1}A = P$$

Onto row space

If $x \neq y$ in row space then $Ax \neq Ay$

$y = A^+(Ay) \quad x = A^+(Ax)$

proof: suppose $Ax = Ay$

$A(x-y) = 0$

$x-y$ in the null space also in row space

so $x-y$ must be $0 \Rightarrow x=$

Find the pseudoinverse A^+

① Start from SVD

$A = U\Sigma V^T$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 & \\ & \ddots & & \\ 0 & & \sigma_r & \\ & & & 0 \end{bmatrix} \begin{matrix} \text{rank } r \\ m \times n \end{matrix}$$

$A^+ = V\Sigma^+ U^T$

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & 0 & \\ & \ddots & & \\ & & 1/\sigma_r & \\ 0 & & & 0 \end{bmatrix} \; n \times m$$

$$\Sigma\Sigma^{+} = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}_{m\times m}$$

$$\Sigma^{+}\Sigma = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}_{n\times n}$$