Linear Algebra Lecture 28

A^TA is positive definite!

SIMILAR MATRICES A, B

$B = M^{-1}AM$ / JORDAN FORM

If A and B are positive def

$$x^T(A+B)x = x^TAx + x^TBx > 0$$

$A+B$ is positive def.

Now A is m by n A is not square

But A^TA is square, symmetric

$$x^TA^TAx = (Ax)^T(Ax) = |Ax|^2 \geq 0$$

Ax is column vector

If A's rank is n, then $|Ax|^2 > 0$

A and B are similiar means: for some M

$$B = M^{-1}AM.$$

Example: A is similiar to Λ

$$S^{-1}AS=\Lambda$$

$$A=\begin{bmatrix}2&1\\1&2\end{bmatrix}\quad \Lambda=\begin{bmatrix}3&0\\0&1\end{bmatrix}$$

$$\overset{M^{-1}}{\begin{bmatrix}1&-4\\0&1\end{bmatrix}}\overset{A}{\begin{bmatrix}2&1\\1&2\end{bmatrix}}\overset{M}{\begin{bmatrix}1&4\\0&1\end{bmatrix}}=\begin{bmatrix}1&-4\\0&1\end{bmatrix}\begin{bmatrix}2&9\\1&6\end{bmatrix}$$

$$=\begin{bmatrix}-2&-15\\1&6\end{bmatrix}=B$$

$\text{trace}(B)=4 \quad \det B=-12+15=3$

Similar metrices have same eigenvalues.

$Ax=\lambda x \quad (B=M^{-1}AM)$ (not eigenvectors)

$$M^{-1}AMM^{-1}x=\lambda M^{-1}x$$

$$BM^{-1}x=\lambda M^{-1}x \longrightarrow \lambda \text{ is also } B\text{'s evalue}$$

Eigenvector of B is M^{-1}(eigenvectors of A)

BAD CASE $\lambda_1 = \lambda_2 = 4$ (matrix can not diagonal)

one ~~family~~ has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

small (only one)

big family incudes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ ← Jordan form

more memebers of family

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \text{ no similiar to } \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Jordan block: $J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$

Every square A is similiar to a Jordan matrix J

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_J} \end{bmatrix}$$

blocks = # eigenvectors