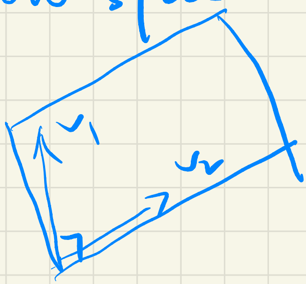
Linear Algebra Lecture 29

Singular Value Decomposition = SVD

$A = U\Sigma V^T$ // Σ diagonal

U, V orthogonal

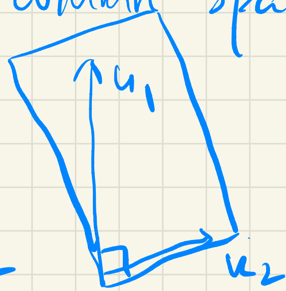
row space



$\sigma_1 u_1 = A v_1$

$\sigma_2 u_2 = A u_2$

column space



$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \end{bmatrix}$$

$$AV = U\Sigma = U\Sigma V^{-1} = U\Sigma V^T$$

$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ v_1, v_2 in row space R^2

u_1, u_2 in col space R^n

$\sigma_1 > 0$ $\sigma_2 > 0$ $Av_1 = \sigma u_1$

$$\downarrow \; A^TA = V\Sigma^T U^T U \Sigma V^T = V\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix} V^T$$

$$\uparrow$$

$$A^TA$$

V is the eigenvector of A^TA

$\sigma_1^2, \sigma_2^2, \dots$ is the eigenvalue of A^TA

U is the eigenvector of AA^T

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \lambda_1 = 32 \quad v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad \lambda_2 = 18$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(there is problem)

Find u :

$$AA^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma\Sigma^T U^T$$

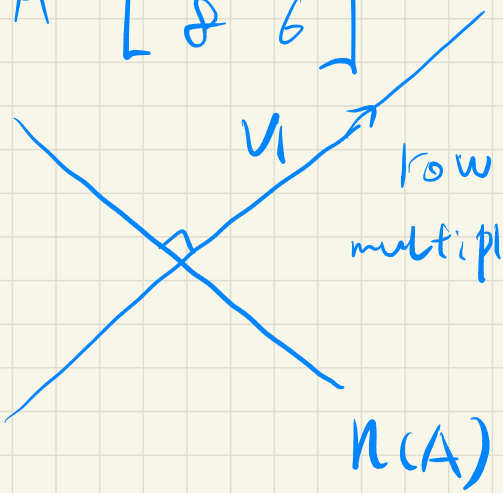
$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

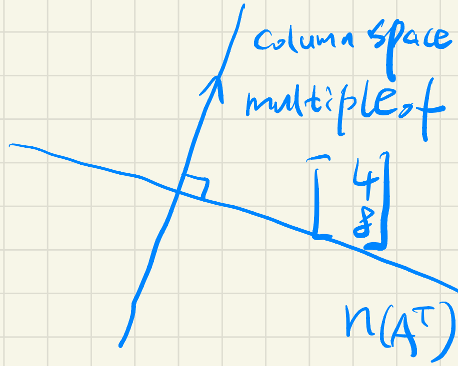
$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \lambda_1 = 32 \qquad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda_2 = 18$$

$$u = \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix}$$

Example 2:

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \qquad v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$





$$\underbrace{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}}_{A} = \frac{1}{\sqrt{5}} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}}_{V^T}$$

$$A^T A = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

$$\lambda_1 = 0 \quad \lambda_2 = 125$$

$v_1, \ldots\ldots, v_r$ Orthonormal basis for row space

$u_1, \ldots\ldots, u_r$ orthonormal basis for column space

$v_{r+1}, \ldots\ldots v_n$ - - - - - - - - - for null space

$u_{r+1} \ldots\ldots u_m$ - - - - - - - for $N(A^T)$