Linear Algebra Lecture 23

1. Differential Equs $\frac{du}{dt} = Au$
2. Exponential e^{At} of a matrix

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \quad \lambda = 0, -3$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 3\lambda = 0$$

$$\lambda_1 = 0 \qquad A - \lambda_1 I = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} x = 0$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3 \qquad A - \lambda_2 I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} x = 0$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

As $u(0)$ is given as $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 = \frac{1}{3} \qquad c_2 = \frac{1}{3}$$

$$\frac{du}{dt} = Au \qquad \text{set } u = Sv$$

$$S\frac{dv}{dt} = ASv \Rightarrow \frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

$$v(t) = e^{\Lambda t} v(0) \qquad v_0 = S^{-1}u(0)$$

$$u(t) = Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0)$$

Matrix exponential e^{At}

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots + (At)^n + \cdots$$

$$e^{At} = SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \cdots$$

$$= Se^{\Lambda t}S^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix}$$