Scaling and criticality in a stochastic multi-agent model of a financial market

Thomas Lux* **& Michele Marchesi†**

* *Department of Economics, University of Bonn, Adenauerallee 24–42, 53113 Bonn, Germany*
† *Department of Electrical and Electronic Engineering, University of Cagliari, Piazza d'Armi, 09123 Cagliari, Italy*

Financial prices have been found to exhibit some universal characteristics[1–6] that resemble the scaling laws characterizing physical systems in which large numbers of units interact. This raises the question of whether scaling in finance emerges in a similar way—from the interactions of a large ensemble of market participants. However, such an explanation is in contradiction to the prevalent 'efficient market hypothesis'[7] in economics, which assumes that the movements of financial prices are an immediate and unbiased reflection of incoming news about future earning prospects. Within this hypothesis, scaling in price changes would simply reflect similar scaling in the 'input' signals that influence them. Here we describe a multi-agent model of financial markets which supports the idea that scaling arises from mutual interactions of participants. Although the 'news arrival process' in our model lacks both power-law scaling and any temporal dependence in volatility, we find that it generates such behaviour as a result of interactions between agents.

In our model, the pool of traders is divided into two groups: the first group ('fundamentalists') follows the premise of the efficient market hypothesis in that they expect the price (p) to follow the so-called fundamental value of the asset (p_f), which is the discounted sum of expected future earnings (for example, dividend payments). A fundamentalist trading strategy consists of buying (selling) when the actual market price is believed to be below (above) the fundamental value. The second group (called 'noise traders' following established terminology in economics[8]), however, does not believe in an immediate tendency of the price to revert to its underlying fundamental value. Instead of focusing on fundamentals, these agents attempt to identify price trends and patterns (charts), and also consider the behaviour of other traders as a source of information, which results in a tendency towards herding behaviour. Furthermore (because it is important for the resulting market operations whether a noise trader believes in a rising or declining market), we further distinguish between optimistic and pessimistic individuals in this group: optimists will buy additional units of the asset, whereas the pessimists will sell part of their actual holdings of the asset.

The main building blocks of the model are movements of individuals from one group to another together with the (exogenous) changes of the fundamental value and the (endogenous) price changes resulting from the agents' market operations. A distinguishing feature of our approach as compared with other recent simulation models[9–14] is that we adopt a mass-statistical formalization inspired by statistical physics[15,16]: individuals react to certain economic forces by changing their behaviour with a certain (endogenous) probability. As a simple formalization of movements into and out of the three groups we use exponential functions, so that a switch from one group to another occurs with a certain endogenous and time-varying probability $\nu e^{U(t)}\Delta t$ within some small increment Δt. Here the coefficient ν is a parameter for the frequency of revaluation of opinion or strategy by the agents (possibly assuming different values for different types of switches), and the function $U(t)$ is a forcing term covering those factors that are decisive for the pertinent changes of behaviour. The dynamics of our artificial market incorporates the following elements:

Changes of opinion of the noise traders. These changes, from a pessimistic to an optimistic mood and vice versa, are governed by the development of the 'noise' factors: the actual price trend is used as a proxy for the influence of 'chartist' practices while herding effects are formalized by computing the majority opinion among noise traders as the (scaled) difference between optimistic and pessimistic individuals. A weighted combination of both factors appears in the U function that governs changes of opinion. If both components are in harmony, a dominant trend of switches will ensue. For example, observation of price increases together with a prevalence of optimistic individuals would be seen as a strong indication of a continuation of a rising market, and would result in a tendency of formerly pessimistic individuals to convert to the optimistic group. Conflicting signals, of course, would reduce the disposition to follow either the majority opinion or the actual price trend.

Switches between the noise traders and the fundamentalists. Such switches are driven by the difference between the (momentary) profits earned by individuals in both groups: profits of noise traders from the optimistic group consist in short-term capital gains due to the price change (or losses in the case of a fall of the market price). Fundamentalists, on the other hand, consider the deviation between price and fundamental value as the source of arbitrage opportunities. The difference between both strategies is that the gains of chartists are immediately realized whereas those claimed by fundamentalists occur only in the future and depend on the uncertain time for reversal to the fundamental value. To take account of this, fundamentalists' arbitrage profits are discounted by a factor <1. The difference between profits of both groups enters the U-function in the transition probability, with switches from the fundamentalist group to the (optimistic) noise traders dominating if the profit

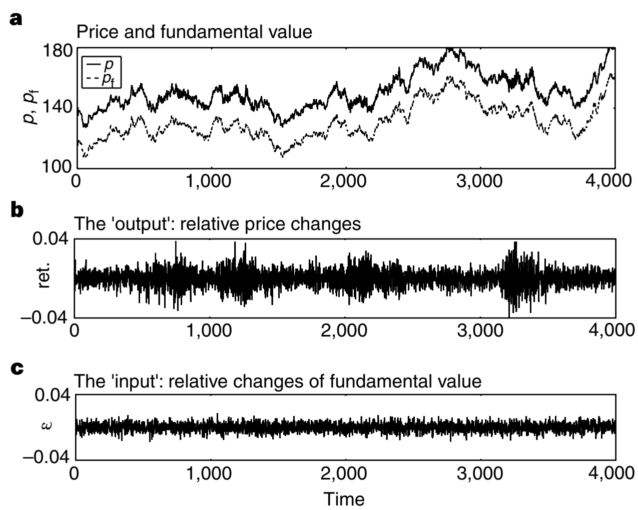

Figure 1 Typical 'snapshot' from a longer simulation run. Panel **a** shows the development with time of both the market price p (solid line) and the fundamental value p_f (dotted line). (We have shifted the latter series vertically for better visibility.) Panel **b** shows returns (log changes of the market price: $\mathrm{ret.}_t = \ln(p_t) - \ln(p_{t-1})$); panel **c** shows log changes of the fundamental value ($\epsilon_t = \ln(p_{f,t}) - \ln(p_{f,t-1})$), respectively; both series have been computed from the time series shown in **a**. The increments of $\ln(p_f)$ follow a normal distribution.

differential is in favour of the latter group and vice versa. For comparison of profits between pessimistic noise traders and fundamentalists, the point of view has to be changed appropriately: because the former rush out of the market in order to avoid losses, their gain is given by the difference between the average profit rate from alternative investments (assumed to be constant) minus the price change (which, when negative, amounts to a capital loss) of the asset they sell. Again, a profit differential in favour of one of the two groups tends to induce changes of behaviour among members of the other group.

Price changes. These are endogenous responses of the market to imbalances between demand and supply—excess demand (supply) leading to an increase (decrease) of the prevailing price. Demand and supply, however, originate from the decisions of our agents: assuming a constant average trading volume of noise traders, their demand and supply is readily determined by the actual numbers of optimistic and pessimistic individuals. Fundamentalists' sensitivity to the relative deviation of the price from the fundamental value, on the other hand, amounts to an excess demand of this group depending on the difference $p - p_f$. Overall excess demand is the sum of both components.

Changes of fundamental value. These constitute the external driving force which affects the market through the operations of fundamentalist traders. In order to ensure that none of the typical characteristics of financial prices can be traced back to exogenous factors, we assume that the relative changes of p_f are Gaussian random variables, i.e. $\ln(p_{f,t}) - \ln(p_{f,t-1}) = \epsilon_t$ with ϵ_t following a normal distribution with mean zero and time-invariant variance σ_ϵ^2.

A more detailed description of our model and some theoretical results are available as Supplementary Information. Theoretical analysis reveals that stationary states of our dynamics are characterized by a price which on average equals the fundamental value. Hence, at least in the long term, we have an 'efficient' market which incorporates all new information into market prices: no 'pathological' situations, with persistent deviations from the economy's fundamentals, occur. A typical simulation (Fig. 1a) shows how closely the price tracks the development of the fundamental value. But comparing the time paths of returns extracted from the price path: $\text{ret.}(\tau) = \ln(p_t) - \ln(p_{t-\tau})$ with relative changes of p_f (Fig. 1b and c) it is evident that the distributional characteristics of ϵ_t are not reflected in similarly normally distributed returns—despite the close association of the integrated time paths in Fig. 1a, the statistical properties of the increments differ fundamentally. In particular, the time series of returns exhibits a higher frequency of extreme events and clustering of volatility.

Figure 2 shows the differences of the unconditional distributions between the input (logarithm of changes of p_f) and the output (relative price changes). As compared with the exponential fall-off of the density of the input, one observes a clear widening of the distribution of large price fluctuations which roughly follows a power law. Determining the exponent α of the Pareto distribution for the tails (that is, $F(\text{ret.} > x) \approx cx^{-\alpha}$) from a regression in logarithmic coordinates yields an estimate of 2.64 for data with a unit time step; this is in good agreement with results obtained for empirical data at daily frequencies[17,18]. However, for returns at lower frequencies (that is, under time aggregation), we also observe a cross-over to the normal distribution with increasing time lag τ. The same happens for empirical financial data.

Turning to the issue of temporal dependence: we estimated the self-similarity parameter H for both raw returns and absolute returns using the approach of Peng *et al.*[19] (Fig. 3). For raw returns, differences in scaling behaviour between the input and output time series are small, yielding $H = 0.49$ and $H = 0.48$, respectively. This is in accordance with absence of long memory ($H = 0.5$) in empirical financial returns. As a consequence, the degree of predictability of price changes is small. However, the picture changes

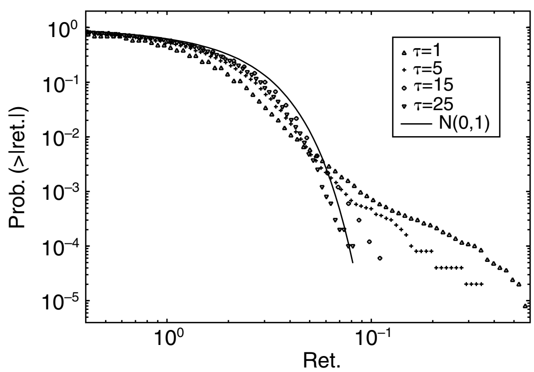

Figure 2 Log-log plot of the complement of the cumulative distribution of returns (ret.) at different levels of time aggregation: $\text{ret.}(\tau) = \ln(p_\tau) - \ln(p_{t-\tau})$. All time series are scaled by their sample standard deviation and the positive and negative tails have been merged by using absolute returns. For comparison, the solid line gives the complement of the cumulative distribution of the standard normal distribution on which the scaled changes of the fundamental value, $\epsilon_t = \ln(p_{f,t}) - \ln(p_{f,t-\tau})$, would collapse at all levels of aggregation. For the highest frequencies, one observes clear deviations from the exponential decay with approximate power-law scaling. Performing a log-log regression on the largest 30% of the observations, the estimated slope is -2.64 ± 0.077 at unit time steps ($\tau = 1$). This is close to results obtained for various financial prices at daily frequencies. Increasing the time step τ, a cross-over to the normal distribution is observed. This is also known to occur with financial data when proceeding from daily frequency to lower (weekly, monthly) frequencies.

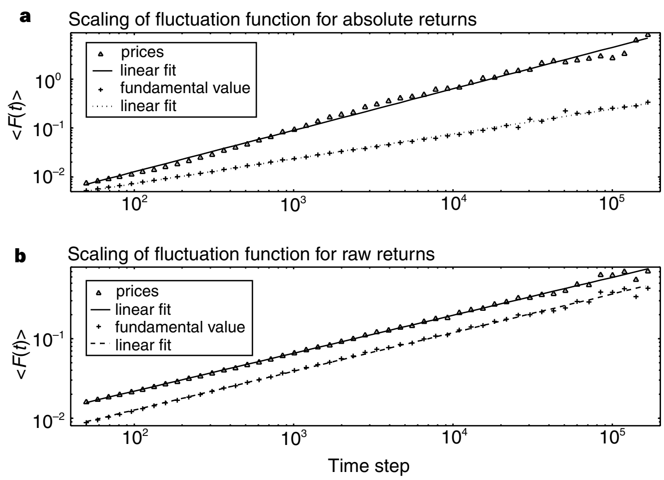

Figure 3 Estimation of self-similarity parameter H. **a**, Absolute returns; **b**, raw returns. Using the approach of Peng *et al.*[19] the exponent H was estimated from the behaviour of the average fluctuation $\langle F(t) \rangle$ of a random variable about its local trend in intervals of size t. The expected behaviour is a power law, $\langle F(t) \rangle \propto t^H$, from which H can be extracted performing a regression in log coordinates. Panel **b** compares the scaling of raw returns and changes of $\ln(p_f)$. For the latter, the self-similarity parameter H is estimated to be $H = 0.49 \pm 0.03$ (slope of dashed line), which is close to the theoretically expected value of 0.5 for a white-noise process. The scaling of returns yields $H = 0.48 \pm 0.003$, which differs only slightly from the results for the input. (The curve for p_f has been shifted vertically for better visibility.) Panel **a** depicts the development of the fluctuation function of absolute returns. Here we see clear differences between the behaviour of the exogenous force ($H = 0.51 \pm 0.004$, the slope of the dotted line) and the series of absolute returns, the latter being characterized by strong persistence with estimated $H = 0.85 \pm 0.010$ (solid line).

dramatically when considering absolute returns as a measure of volatility. Here we see that the transformed price data behave differently from their counterpart derived from the input series, and exhibit $H = 0.85$ which is a sign of strong persistence in volatility. The exponent is again very close to the scaling found for empirical data[6,20].

As these scaling properties are absent in the external driving force, they are generated by the interaction of economic agents with heterogeneous beliefs and strategies in our simulated market. Can we explain the emergence of power laws in these simulations? A closer investigation reveals that the alternation between tranquil and turbulent periods comes about through the changes of agents between groups. In particular, in periods of high volatility we also find a large fraction of agents in the noise trader group. Theoretical analysis shows that a critical value for the number of noise traders exists where the system loses its stability. Volatility is above average when the fraction of noise traders comes close to this critical point, or even increases beyond it, for some time. However, the ensuing destabilization is only temporary, and turbulent phases are overcome quickly by endogenous mechanisms: large deviations from the fundamental value are seen as profit opportunities by fundamentalists whose operations then tend to stabilize the market again. This temporal instability is similar to mechanisms found recently in various models in physics where this phenomenon has been denoted on-off intermittency[21–23]. As the possibility of temporal destabilization (with ensuing bursts of volatility) exists for an open set of parameter values, the qualitative outcome of our model seems to be extremely robust. This has been confirmed by simulations with many different parameter sets (data not shown), which all led to endogenous emergence of power-law tails and temporal dependence of volatility, albeit with varying coefficients α and H. □

Received 29 October; accepted 27 November 1998.

1. Mandelbrot, B. The variation of certain speculative prices. *J. Bus.* **35,** 394–419 (1963).
2. Fama, E. Mandelbrot and the stable Paretian hypothesis. *J. Bus.* **35,** 420–429 (1963).
3. Mantegna, R. N. & Stanley, E. Scaling behaviour in the dynamics of an economic index. *Nature* **376,** 46–49 (1995).
4. Ghashgaie, S., Breymann, W., Peinke, J., Talkner, P. & Dodge, Y. Turbulent cascades in foreign exchange markets. *Nature* **381,** 767–770 (1996).
5. Galluccio, S., Caldarelli, G., Marsili, M. & Zhang, Y.-C. Scaling in currency exchange. *Physica A* **245,** 423–436 (1997).
6. Liu, Y., Cizeau, P., Meyer, M., Peng, C.-K. & Stanley, H. E. Correlations in economic time series. *Physica A* **245,** 437–440 (1997).
7. Fama, E. Efficient capital markets: a review of theory and empirical work. *J. Fin.* **25,** 383–417 (1970).
8. Shleifer, A. & Summers, L. H. The noise trader approach to finance. *J. Econ. Perspect.* **4,** 19–33 (1990).
9. Palmer, R. G., Arthur, W. B., Holland, J. H., LeBaron, B. & Tayler, P. Artificial economic life: a simple model for a stockmarket. *Physica D* **75,** 264–274 (1994).
10. Levy, M., Levy, H. & Solomon, S. Microscopic simulation of the stock market: the effect of microscopic diversity. *J. Phys. I (France)* **5,** 1087–1107 (1995).
11. Bak, P., Paczuski, M. & Shubik, M. Price variations in a stock market with many agents. *Physica A* **246,** 430–453 (1997).
12. Caldarelli, G., Marsili, M. & Zhang, Y.-C. A prototype model of stock exchange. *Europhys. Lett.* **40,** 479–484 (1997).
13. Brock, W. & LeBaron, B. A dynamical structural model for stock return volatility and trading volume. *Rev. Econ. Stat.* **78,** 94–110 (1996).
14. Brock, W. & Hommes, C. Rational routes to randomness. *Econometrica* **65,** 1059–1095 (1997).
15. Lux, T. Time variation of second moments from a noise trader/infection model. *J. Econ. Dyn. Control* **22,** 1–38 (1997).
16. Lux, T. The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions. *J. Econ. Behav. Organizat.* **33,** 143–165 (1998).
17. Guillaume, D. M. *et al.* From the bird's eye to the microscope: a survey of new stylized facts of the intra-daily foreign exchange markets. *Fin. Stoch.* **1,** 95–129 (1997).
18. Gopikrishnan, P., Meyer, M., Amaral, L. A. N. & Stanley, H. E. Inverse cubic law for the distribution of stock price variations. *Eur. Phys. J. B* **3,** 139–140 (1998).
19. Peng, C.-K. *et al.* Mosaic organization of DNA nucleotides. *Phys. Rev. E* **49,** 1685–1689 (1994).
20. Lux, T. Long-term stochastic dependence in financial prices: evidence from the German stock market. *Appl. Econ. Lett.* **3,** 701–706 (1996).
21. Platt, N., Spiegel, E. A. & Tresser, C. On-off intermittency: a mechanism for bursting. *Phys. Rev. Lett.* **70,** 279–282 (1993).
22. Haegy, J. F., Platt, N. & Hammel, S. M. Characterization of on-off intermittency. *Phys. Rev. E* **49,** 1140–1150 (1994).
23. Youssefmir, M. & Huberman, B. Clustered volatility in multiagent dynamics. *J. Econ. Behav. Organizat.* **32,** 101–118 (1997).

Supplementary information is available on *Nature*'s World-Wide Web site (http://www.nature.com) or as paper copy from the London editorial office of *Nature*.

Acknowledgements. Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn is acknowledged.

Correspondence and requests for materals should be addressed to T.L. (e-mail: lux@iiw.uni-bonn.de).

A single-photon turnstile device

J. Kim*, O. Benson*, H. Kan† & Y. Yamamoto*‡

* *ERATO Quantum Fluctuation Project, Edward L. Ginzton Laboratory, Stanford University, Stanford, California 94305, USA*
† *ERATO Quantum Fluctuation Project, Hamamatsu Photonics Inc., Hamamatsu, Shizuoka, 434-0041, Japan*
‡ *NTT Basic Research Laboratories, 3-1 Morinosato-Wakamiya Atsugi, Kanagawa, 243-01, Japan*

Quantum-mechanical interference between indistinguishable quantum particles profoundly affects their arrival time and counting statistics. Photons from a thermal source tend to arrive together (bunching) and their counting distribution is broader than the classical Poisson limit[1]. Electrons from a thermal source, on the other hand, tend to arrive separately (anti-bunching) and their counting distribution is narrower than the classical Poisson limit[2–4]. Manipulation of quantum-statistical properties of photons with various non-classical sources is at the heart of quantum optics: features normally characteristic of fermions—such as anti-bunching, sub-poissonian and squeezing (sub-shot-noise) behaviours—have now been demonstrated[5]. A single-photon turnstile device was proposed[6–8] to realize an effect similar to conductance quantization. Only one electron can occupy a single state owing to the Pauli exclusion principle and, for an electron waveguide that supports only one propagating transverse mode, this leads to the quantization of electrical conductance: the conductance of each propagating mode is then given by $G_Q = e^2/h$ (where e is the charge of the electron and h is Planck's constant; ref. 9). Here we report experimental progress towards generation of a similar flow of single photons with a well regulated time interval.

When a light-emitting p–n junction is driven with a high-impedance constant-current source, injection of electron–hole pairs can be regulated to below the classical shot-noise limit and light with sub-shot-noise intensity fluctuations can be generated[10]. This is possible because the inelastic scattering of electrons in a highly dissipative resistor can suppress the current noise by means of the Pauli exclusion principle[11,12], and the Coulomb repulsive interaction between electrons in a p–n junction can suppress the electron injection noise by way of the collective Coulomb blockade effect[13–15]. In these squeezing experiments with a macroscopic p–n junction, however, only large numbers of photons (of the order of $\sim 10^8$) can be regulated owing to a small single charging energy.

It has been demonstrated in mesoscopic physics that an ultra-small tunnel junction regulates the electron transport one by one owing to a single charging energy that is large compared to the thermal background energy[16–18]. If such a single-electron control technique could be extended to simultaneous control of electron and hole in a p–n junction, a single photon would be regularly emitted, one by one[6].

Our single-photon turnstile device utilizes simultaneous Coulomb blockade for electrons and holes in a mesoscopic double barrier p–n junction (Fig. 1a). The structure consists of an intrinsic central quantum well (QW) in the middle of a p–n junction and the n-type and p-type side QWs isolated by tunnel barriers from the central QW. The lateral size of the device is reduced to increase the single charging energy $e^2/2C_i$, where C_i (i is n or p) is the capacitance between the central QW and the i-side QW. At a certain bias voltage V_0, the conditions for electron resonant tunnelling are fulfilled, and the mth electron can tunnel into an electron sub-band in the central QW. When the mth electron tunnels, the Coulomb repulsive interaction between electrons shifts