LONDON METROPOLITAN UNIVERSITY

London Metropolitan University - Coursework Coversheet

Part 1 - To Remain with the Assignment After Marking

Student ID:

16027432

Module Code:

MA6P52

Module Name:

Academic Independent Study

Component:

001

Description:

Written Report (3000 words max.)

Module Leader:

Zhanyuan Hou

Due Date:

23/May/2019 3pm

Group Number:

ZZZ

Please enter tutor name:

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LONDON METROPOLITAN UNIVERSITY

London Metropolitan University - Coursework Coversheet

Part 2 - To be returned to the student by the Module Leader

Student ID:

16027432

Module Code:

MA6P52

Module Name:

Academic Independent Study

Component:

001

Description:

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Please enter tutor name:

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Further Comments

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Module tutor: Date:

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Report of the Module
MA6P52 Academic Independent Study

LONDON metropolitan university

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Student ID: 16027432
Supervisor: Edward Kissin

Project Title: Laplace Transforms – Theory and Applications

2018-2019

School of Computing

Contents

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Dr Zhanyuan Hou is also thanked for igniting my interest in applied mathematics.

Isa Ahmad

May 2019

Abstract

This study focuses on the Laplace Transform of functions of a real variable; and the application to obtaining solutions of ordinary differential equations.

The project starts with important results from calculus and analysis, needed for the project. It then commences with a review of the properties of Laplace Transforms and transforms of standard functions, as studied in the second year taught course.

Once defined this is then applied as a powerful differential equation solving technique. Initially as a mode of solution for constant coefficient equations, then progressing to systems of differential equations.

1 Introduction

Laplace Transforms are mainly due to Pierre-Simon Laplace (23 March 1749 – 5 March 1827). He was born in the Normandy region of France, to affluent parents. His father owned and managed many types of estates.

He was initially sent to the University of Caen to read theology, upon the wishes of his father. As part of these studies, he was taught mathematics. It was here that his mathematical prowess was discovered and nurtured by two of his mentors, Christophe Gadbled and Pierre Le Canu were quick to notice his promise. This created much excitement for the young Laplace who started developing a real thirst for the subject. Discontented by his theological studies, Laplace did not graduate in this discipline.

Having been given an introduction by Le Canu to Jean le Rond d'Alembert, he left for Paris to pursue mathematics. There he was met with resistance from d'Alembert who tried his best to extinguish his interest in mathematics by setting what appeared to be a series of unrealistic tasks. d'Alembert however, had not counted on Laplace's persistence, tenacity and brilliance. Upon this realisation, he decided to take the role of being Laplace's mentor. d'Alembert eventually gave Laplace a recommendation for a teaching position at École Militaire, a large military training complex in Paris. Having taken the position, Laplace now enjoyed a steady income, as well as a light teaching schedule. This allowed Laplace to dedicate himself to academic research, which spanned the next seventeen years of his life, from 1771 to 1787.

Unlike current times, where academics specialise in narrow fields of mathematics, mathematicians from the past were masters of many subjects. In addition to expertise in mathematics, a significant part of Laplace's self studies were astronomy related. One of his more famous publications was that in which he further developed the Laplace Transform.

The attraction of this subject is that the theory relies on mathematical analysis - both real and complex, while its application base spans applied and engineering mathematics.

In this project we will take a closer look at them and how they can be used to solve differential equations with both constant and variable coefficients.

The discrete analogue of the Laplace Transform, namely $z-$Transform is also studied. This allows consideration of the sum of infinite series and associated convergence results.

The shorthand L.T refers to Laplace Transform and cf. is *comparing with.*

The project has been typeset using Scientific Word; the diagrams were created using MS Excel.

2 Mathematical Preliminaries

In this first mathematical section, we commence by presenting some fundamental results required for later use. These are topics that have not been covered in earlier modules.

A very important result used extensively in many branches of mathematics is

$$\int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi}. \tag{1}$$

Here the notation $\displaystyle\int_{\mathbb{R}}$ is used to write $\int_{-\infty}^{\infty}$.

There are two ways to obtain this. The first can be thought of as the 'poor man's' derivation.

We begin by recalling a result from probability. The cumulative distribution function or **CDF** for the Normal Distribution $N(x)$ is

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$$

For a continuous random variable X this represents the probability of it being less than an observed value x, written $p(X < x)$. If $x \longrightarrow \infty$ then we know (by the fact that the area under a probability distribution function has to sum to unity) that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/2} ds = 1.$$

Make the substitution $x = s/\sqrt{2}$ to give $dx = ds/\sqrt{2}$, hence the integral becomes

$$\sqrt{2} \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{2\pi}$$

and hence we obtain

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

The second method requires double integration.

Put

$$I = \int_{\mathbb{R}} e^{-x^2} dx$$

so that

$$I^2 = \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\left(x^2+y^2\right)} dxdy$$

The region of integration is a square centred at the origin of infinite dimension

$$\begin{aligned} x &\in (-\infty, \infty) \\ y &\in (-\infty, \infty) \end{aligned}$$

i.e. the complete 2D plane. Introduce plane polars

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} dxdy \rightarrow rdrd\theta$$

The region of integration is now a circle centred at the origin of infinite radius

$$\begin{aligned} 0 &\leq r < \infty \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

so the problem becomes

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

Hence

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Because e^{-x^2} is an even function, we also note that

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{2}$$

2.1 The Gamma Function

When considering improper integrals, the integrability condition for convergence is very important. Even if not formally stated, we assume its implicit presence in all results involving integration.

Recall (if $a > 0$), then

$$\begin{aligned} \int_0^a x^p dx \quad \text{exists} \quad \text{for } p &> -1 \\ \int_a^\infty x^p dx \quad \text{exists} \quad \text{for } p &< -1 \end{aligned}$$

The **Gamma Function** $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \ \ (x > 0)$$

So we can write

$$\Gamma(x) = \int_0^1 t^{x-1} dt - \int_0^1 \left(\frac{1-e^{-t}}{t}\right) t^x dt + \int_1^\infty e^{-t} t^{x-1} dt$$
$$\int_0^\infty e^{-t} dt = 1$$

Integration by parts gives us

$$\int_0^\infty e^{-t} t^x dt = x \int_0^\infty e^{-t} t^{x-1} dt = x(x-1) \int_0^\infty e^{-t} t^{x-2} dt = \ .... \ = x! \tag{3}$$

Important results:

$$\begin{aligned} \Gamma(n+1) &= n! \ \ (n \geq 0) \rightarrow \Gamma(n) = (n-1)! \ \ (n \geq 1) \\ \Gamma(1) &= 1 \end{aligned}$$

and also from eqn. 3

$$\Gamma(x+1) = x\Gamma(x).$$

If we make the substitution $t = u^2$ in $\Gamma(x) = \displaystyle\int_0^\infty e^{-t} t^{x-1} dt$ we obtain

$$\Gamma(x) = 2\int_0^\infty e^{-u^2} u^{2x-1} du$$

and put $x = 1/2$ so that

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du$$

and we know from eqn. 2 the value of this integral, hence

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

This acts as a base case from which other values of the gamma function can be obtained.

Examples:

1. $\Gamma(4) = 3! = 6; \quad \dfrac{\Gamma(4)}{\Gamma(5)} = \dfrac{3!}{4!} = \dfrac{1}{4};$

2. $\Gamma\left(\dfrac{5}{2}\right)$ – use $\Gamma(x+1) = x\Gamma(x)$ with $x = 3/2$

$$\begin{aligned}\Gamma\left(\frac{5}{2}\right) &= \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} \\ &= \frac{3}{4}\sqrt{\pi}\end{aligned}$$

3. $\Gamma\left(-\dfrac{3}{2}\right)$ – now use $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$

$$\begin{aligned}\Gamma\left(-\frac{3}{2}\right) &= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-3/2} = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) \\ &= -\frac{2}{3}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{-1/2}\right] = -\frac{2}{3}\times -2\times\sqrt{\pi} \\ &= \frac{4}{3}\sqrt{\pi}\end{aligned}$$

2.2 The Dirac delta function

The *delta* function denoted $\delta(x)$; due to *Dirac* is a very useful 'object' in applied maths and more recently in quant finance. It is the mathematical representation of a point source/impulse e.g. force, payment. Although labelled a function, it is more of a distribution or *generalised function.* Consider the following definition for a piecewise function

$$f_\eta(x) = \begin{cases} \frac{1}{\eta}, & x \in \left[-\frac{\eta}{2}, \frac{\eta}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

Now put the delta function equal to the above for the following limiting value

$$\delta_\eta(x) = \lim_{\eta \longrightarrow 0} f_\eta(x).$$

What is happening here? As η decreases we note the 'hat' narrows whilst becoming taller eventually becoming a spike. Due to the definition, the area under the curve (i.e. rectangle) is fixed at 1, i.e. $\eta \times \frac{1}{\eta}$; which is independent of the value of η. So mathematically we can write this in integral terms as

$$\int_{-\infty}^{\infty} f_\eta(x)\, dx = \eta \times \frac{1}{\eta} = 1 \text{ for all } \eta.$$

Looking at what happens in the limit $\eta \longrightarrow 0$, the spike like (singular) behaviour at the origin gives the following definition

$$\delta_\eta(x) \to \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

with the property

$$\int_{-\infty}^{\infty} \delta_\eta(x)\, dx = 1.$$

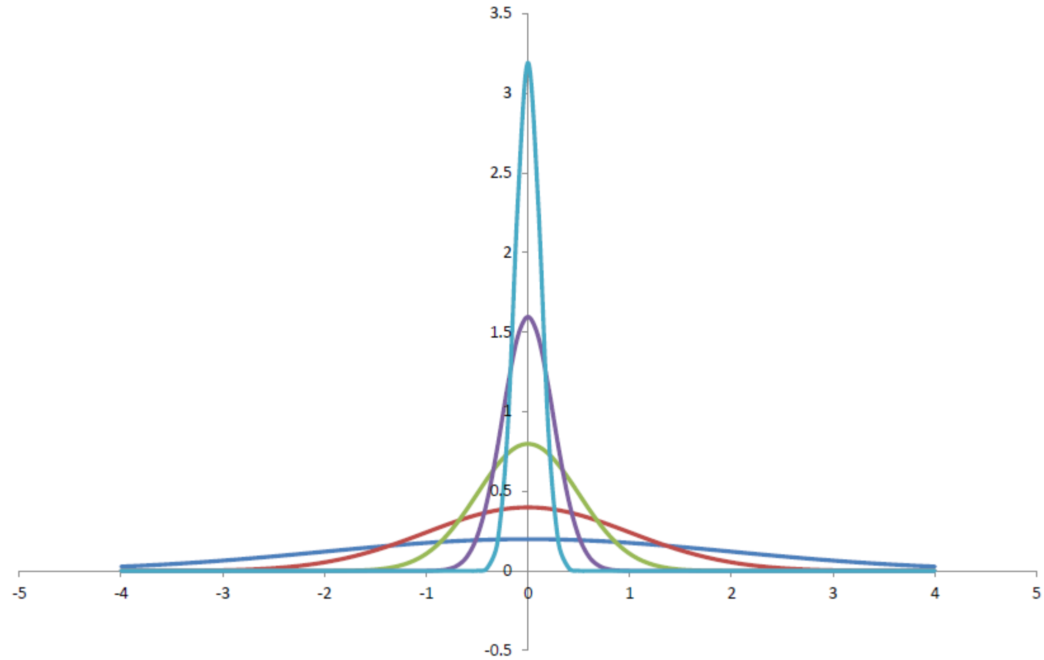
There are many ways to define $\delta(x)$. Consider the Gaussian/Normal distribution with pdf

$$G_\eta(x) = \frac{1}{\eta\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\eta^2}\right).$$

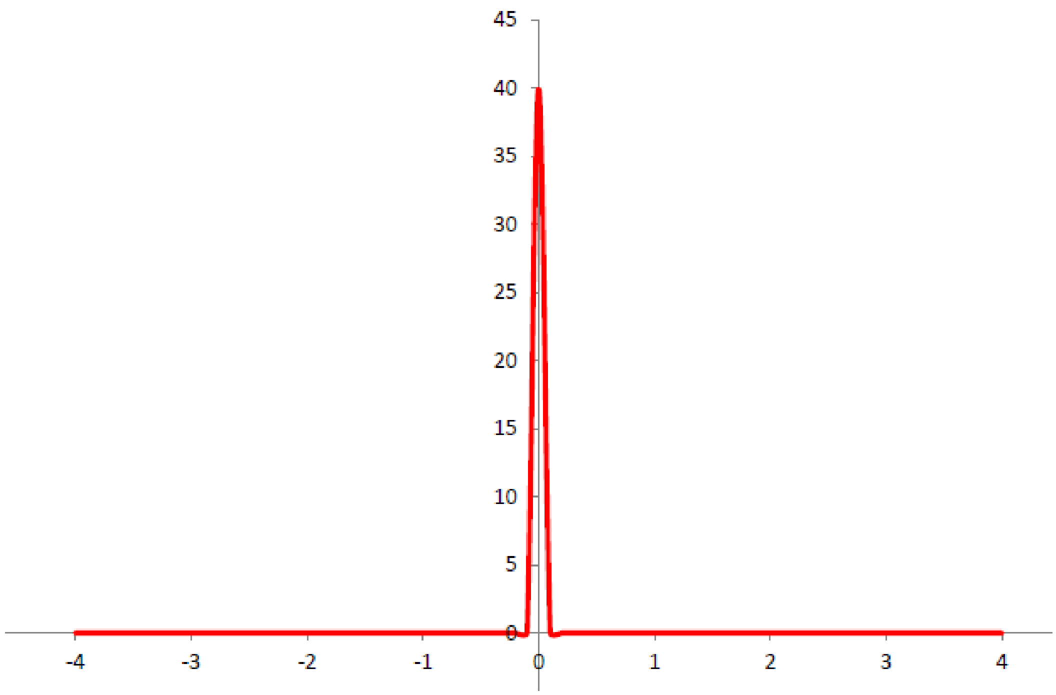
The function takes its highest value at $x = 0$;as $|x| \to \infty$ there is exponential decay away from the origin. If we stay at the origin, then as η decreases, $G_\eta(x)$ exhibits the earlier spike (as it shoots up to infinity), so

$$\lim_{\eta \longrightarrow 0} G_\eta(x) = \delta(x).$$

The normalising constant $\frac{1}{\eta\sqrt{2\pi}}$ ensures that the area under the curve will always be unity. The graph below shows $G_\eta(x)$ for values $\eta = 2.0$ (royal blue), 1.0 (red), 0.5 (green), 0.25 (purple), 0.125 (turquoise); the Gaussian curve becomes slimmer and more peaked as η decreases.



$G_\eta(x)$ is plotted for $\eta = 0.001$



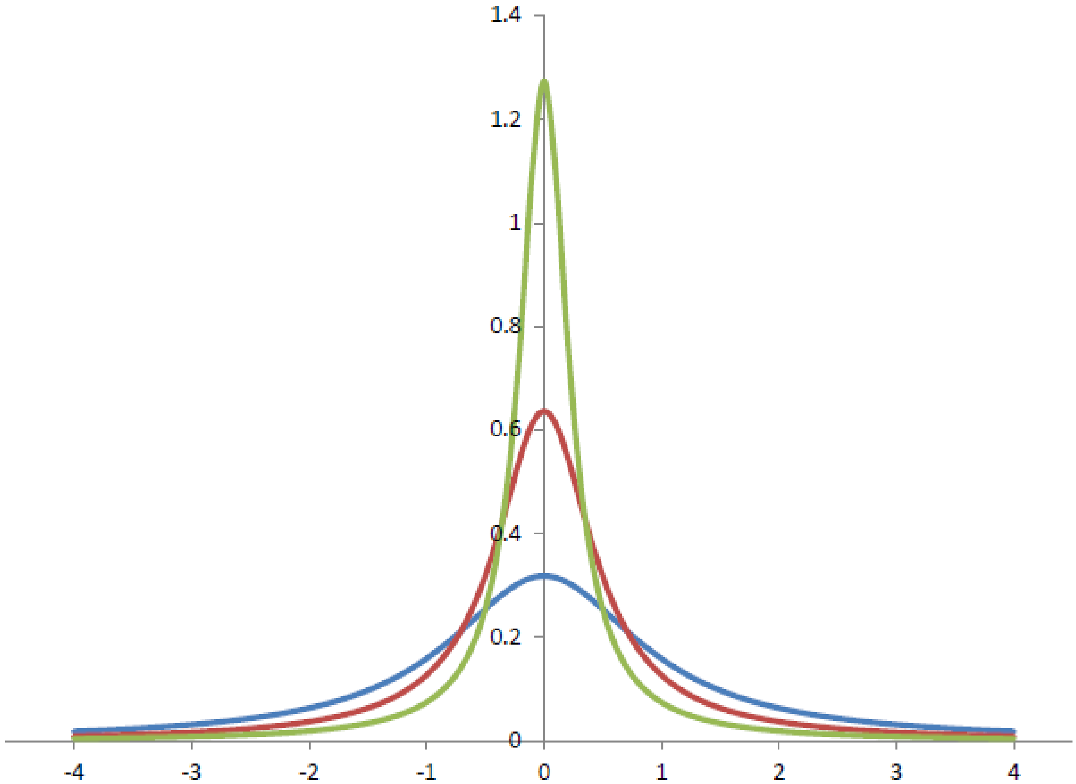
Now generalise this definition by centring the function $f_\eta(x)$ at any point x'. So

$$\delta(x - x') = \lim_{\eta \longrightarrow 0} f_\eta(x - x')$$

$$\int_{-\infty}^{\infty} \delta(x - x')\, dx = 1.$$

The figure will be as before, except that now centred at x' and not at the origin as before.

So we see two definitions of $\delta(x)$. Another is the Cauchy distribution (also called the Lorentzian function)



$$L_\eta(x) = \frac{1}{\pi}\frac{\eta}{x^2+\eta^2}$$

Now suppose we have a smooth function $g(x)$ and consider the following integral problem

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x')\,dx = \int_{-\infty}^{\infty} g(x)\,f_\eta(x-x')\,dx = \int_{x'-\frac{\eta}{2}}^{x'+\frac{\eta}{2}} g(x)\,\frac{1}{\eta}dx = \frac{1}{\eta}\int_{x'-\frac{\eta}{2}}^{x'+\frac{\eta}{2}} g(x)\,dx = \frac{1}{\eta}$$

2.2.1 Translation of the delta function

When the delta function represents an impulse, it is typically written in terms of time t. Consider the case of the impulse now being located at an arbitrary time (say) a, rather than $t = 0$. We can rewrite the earlier definitions as

$$\delta(t-a) = 0, \text{ if } t \neq c$$

and

$$\int_{-\infty}^{\infty} \delta(t-c)\,dt = 1.$$

2.3 Heaviside Function

We frequently encounter functions that change abruptly at specified values of the underlying variable. One common example of this switching process can be described mathematically by the unit step function denoted $\mathcal{H}(x)$ and described by

$$\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

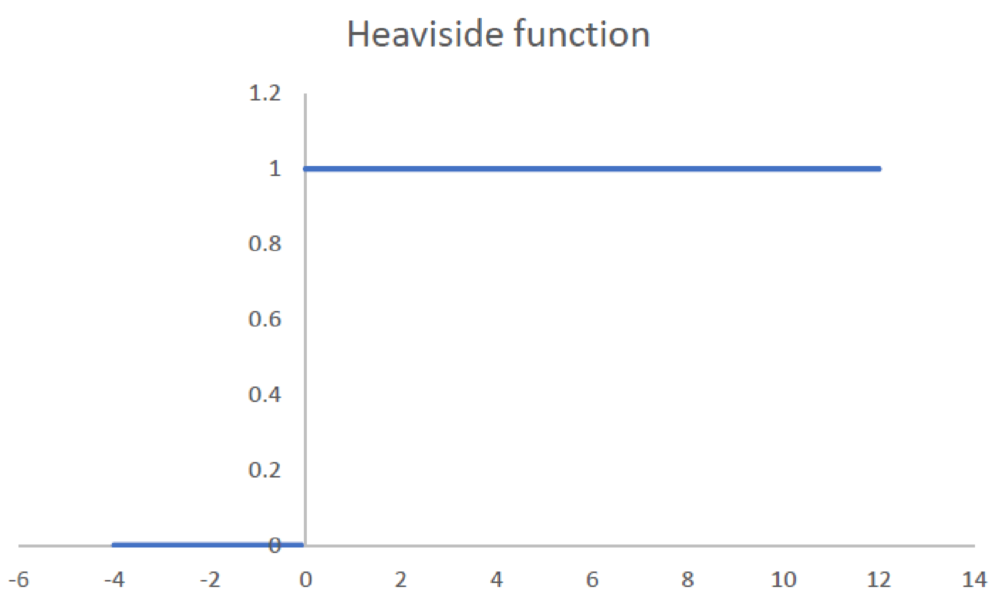
Mathematicians refer to this as the Heaviside function, named after Oliver Heaviside. It is an example of a function that is discontinuous at the origin. Some definitions have

$$\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$

and

$$\mathcal{H}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} .$$

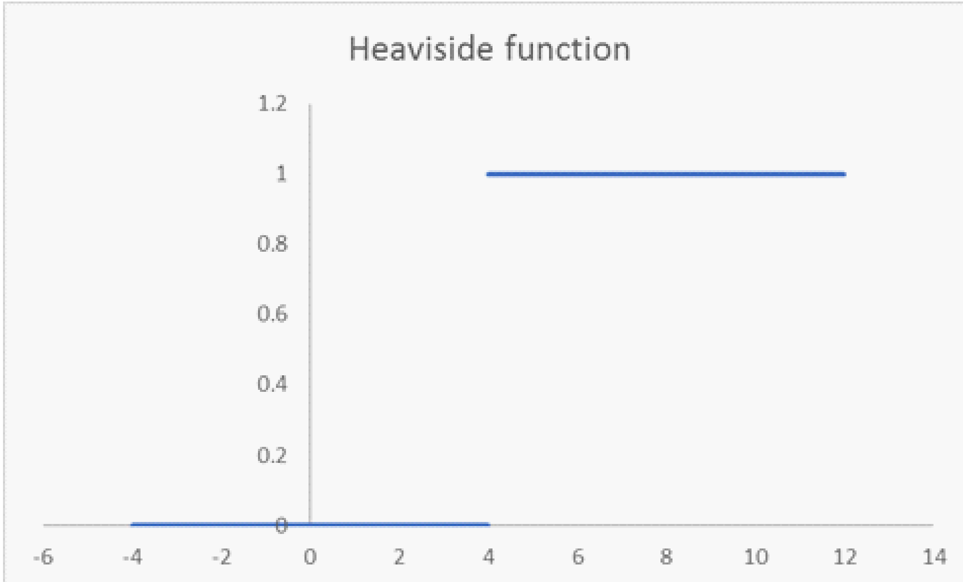
For this project, the first definition will be assumed.



Of greater use is the general shifted/delayed unit step function

$$\mathcal{H}(x-a) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases}$$

So a function which has value 0 up to $x = a$ and thereafter has value 1.

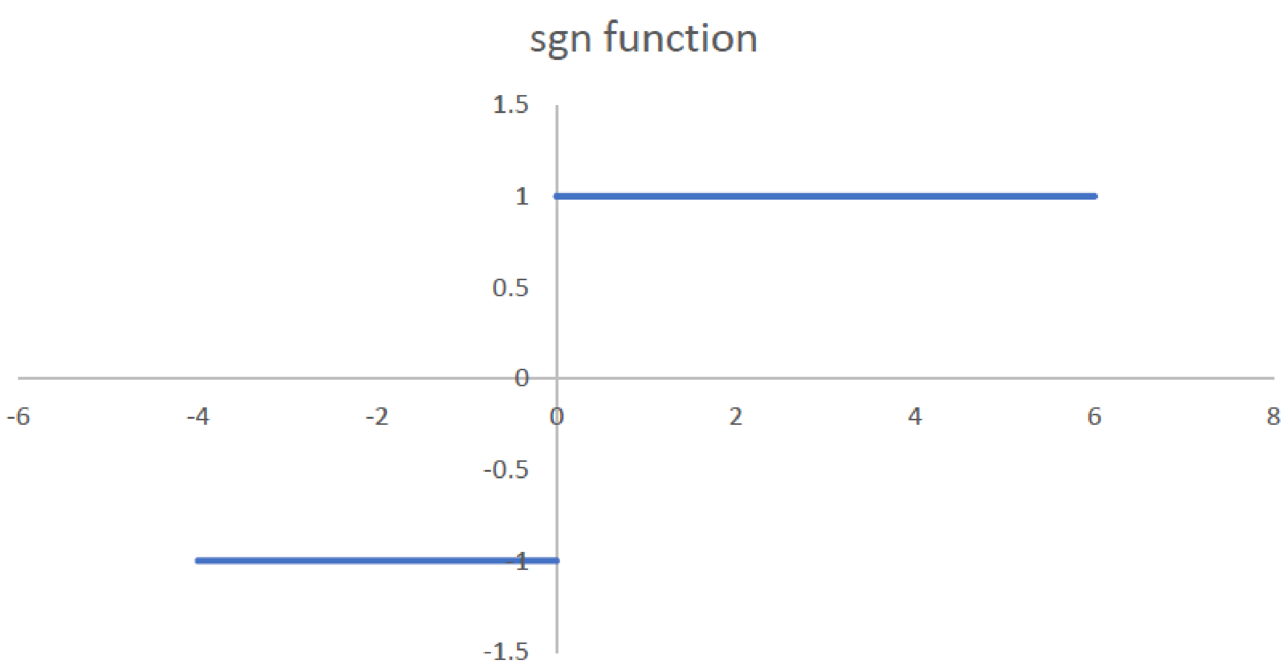


We now define another function that returns the sign of a real number. This is the **sgn** or **signum** function defined

$$\mathbf{sgn}\,(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} = \frac{|x|}{x}.$$

It is related to the Heaviside function by the relationship

$$\mathbf{sgn}\,(x) = \mathcal{H}\,(x) - \mathcal{H}\,(-x)$$



3 The Laplace Transform

Given a function defined $\forall t > 0$ then the *Laplace Transform* (LT) of $f(t)$, written $\mathcal{L}\{F(t)\}$ is defined by the improper integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

provided this integral exists.

In engineering applications the parameter s represents a complex number frequency parameter.

If $\displaystyle\int_0^\alpha f(t)\, dt$ exists for all $\alpha > 0$ and there exists β such that $f(t)\, e^{-\beta t} \to 0$ as $t \to \infty$, then $F(s)$ exists, for $s > \beta$.

$\mathcal{L}\{F(t)\}$ is a function of s which we denote by $F(s)$. In general the convention is given by

Lower case letters	Upper case letters
$f(t)$	$F(s)$
$g(t)$	$G(s)$
$h(t)$	$H(s)$

We begin by providing the working for a number of transforms which we consider as standard/fundamental results. From these we can build a table of important transforms.

1. If $f(t) = 1$ then $\mathcal{L}\{1\} = \displaystyle\int_0^\infty 1 \times e^{-st} dt$. Look at the right hand side

$$\begin{aligned} \lim_{T\to\infty} \int_0^T e^{-st} dt &= \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^T \quad (s \neq 0) \\ &= \lim_{T\to\infty} \left(\frac{1}{s} \left[1 - e^{-sT} \right] \right) \\ &\to \frac{1}{s} \text{ if } \; s > 0 \end{aligned}$$

The limit does not exist for $s < 0$.
What about $s = 0$?

$$\int_0^\infty e^{0t} dt = \int_0^\infty 1 dt$$

which does not exist. So for $f(t) = 1$, $F(s) = \dfrac{1}{s}$ $(s > 0)$ and the L.T. does not exist for $s \leq 0$.

2. $\mathcal{L}\{e^{at}\} = \displaystyle\int_0^\infty e^{at} e^{-st} dt =$

$$\begin{aligned} \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\, dt &= \lim_{T\to\infty} \left[\frac{1}{s-a} \left(1 - e^{-(s-a)T} \right) \right] \\ &= \frac{1}{s-a} \qquad (s > a) \end{aligned}$$

3. $\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt$ (integration by parts)

$$= -\frac{t}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt$$

Now $te^{-st} \to 0$ as $t \to \infty \because s > 0$ therefore

$$\begin{aligned}\mathcal{L}\{t\} &= \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_0^\infty \\ &= \frac{1}{s^2}\end{aligned}$$

Note:

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt$$

put $u = st \to dt = \frac{1}{s}du$

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty \frac{u}{s}e^{-u}.\frac{1}{s}du = \frac{1}{s^2}\int_0^\infty ue^{-u}.du \\ &= \frac{1}{s^2}\Gamma(2) = \frac{1!}{s^2} \\ &= \frac{1}{s^2} \quad \text{from the gamma function}\end{aligned}$$

4. $\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt \qquad (s > 0)$, so again put $u = st$ to give

$$\begin{aligned}\int_0^\infty \left(\frac{u}{s}\right)^n e^{-u}\frac{du}{s} &= \frac{1}{s^{n+1}}\int_0^\infty u^n e^{-u}du \\ &= \frac{\Gamma(n+1)}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}}\end{aligned}$$

5. Similarly $\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \qquad (s > 0)$

6. $\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st}\sin kt dt$

$$\begin{aligned}&= \left[-\frac{e^{-st}}{s}\sin kt\right]_0^\infty - \left(-\frac{1}{s}\right)\int_0^\infty e^{-st}k\cos kt dt \\ &= 0 + \frac{k}{s}\int_0^\infty e^{-st}\cos kt dt\end{aligned}$$

Set $I = \mathcal{L}\{\sin t\} = \frac{k}{s}\int_0^\infty e^{-st}\cos kt.dt$

$$= \frac{k}{s}\left\{\left[-\frac{e^{-st}}{s}\cos kt\right]_0^\infty - \left(-\frac{1}{s}\right)\int_0^\infty e^{-st}(-k\sin kt)\,dt\right\}$$

$$I=\frac{k}{s}\left(\frac{1}{s}-\frac{k}{s}I\right)\therefore I=\frac{k}{s^2}-\frac{k^2}{s^2}I$$

Hence

$$I\left(1+\frac{k^2}{s^2}\right)=\frac{k}{s^2}$$

and

$$\mathcal{L}\left\{\sin t\right\}=\frac{k}{k^2+s^2}$$

Using Euler's Identity is much quicker and robust. So a more elegant way of obtaining the result for trigonometric functions is to consider

$$\begin{aligned}\mathcal{L}\left\{e^{ikt}\right\} &= \frac{1}{s-ik}=\frac{s+ik}{s^2+k^2}\equiv\mathcal{L}\left\{\cos kt+i\sin kt\right\}\\ &= \frac{s}{s^2+k^2}+ik\frac{1}{s^2+k^2}\end{aligned}$$

Hence

$$\mathcal{L}\left\{\cos kt\right\}=\frac{s}{s^2+k^2}\quad,\qquad\mathcal{L}\left\{\sin kt\right\}=\frac{k}{s^2+k^2}$$

7. The delta function

a. $\mathcal{L}\left\{\delta\left(t\right)\right\}=1$
This can be achieved by integrating e^{-st} with $\delta\left(t\right)$, amounting to evaluating e^{-st} at $t=0$, and $e^0=1$.

$$\mathcal{L}\left\{\delta\left(t\right)\right\}=\int_0^\infty\delta\left(t\right)e^{-st}dt=1.$$

b. $\mathcal{L}\left\{\delta\left(t-a\right)\right\}=e^{-sa}$ for $a>0$.
We are now integrating e^{-st} with $\delta\left(t-a\right)$, amounting to evaluating e^{-st} at $t=a$.

$$\mathcal{L}\left\{\delta\left(t-a\right)\right\}=\int_0^\infty\delta\left(t-a\right)e^{-st}dt=e^{-sa}.$$

We note that setting $a=0$ recovers the formula in part a.

c. a. and b. then give the result

$$\mathcal{L}\left\{\delta\left(t-a\right)f\left(t\right)\right\}=\int_0^\infty\delta\left(t-a\right)f\left(t\right)e^{-st}dt=f\left(a\right)e^{-sa}.$$

8. Heaviside function $\mathcal{H}\left(t\right)$

$$\begin{aligned}\mathcal{L}\left\{\mathcal{H}\left(t\right)\right\} &= \int_0^\infty 1\times e^{-st}dt\\ &= \frac{1}{s}\end{aligned}$$

9. signum function $\mathbf{sgn}\left(t\right)$

$$\begin{aligned}\mathcal{L}\left\{\mathbf{sgn}\left(t\right)\right\} &= \mathcal{L}\left(\mathcal{H}\left(x\right)-\mathcal{H}\left(-x\right)\right)\\ &= \mathcal{L}\left(\mathcal{H}\left(x\right)\right)-\mathcal{L}\left(\mathcal{H}\left(-x\right)\right)\\ &= \frac{1}{s}-\left(-\frac{1}{s}\right)=\frac{2}{s}\end{aligned}$$

3.1 Simple Properties

1. $\mathcal{L}\{f(t)+g(t)\}=\mathcal{L}\{f(t)\}+\mathcal{L}\{g(t)\}$
2. $\mathcal{L}\{\lambda f(t)\}=\lambda\mathcal{L}\{f(t)\};\ \lambda$ constant
3. $\mathcal{L}\{f(t)\cdot g(t)\}\neq\mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$

The first two properties state that the Laplace Transform is a linear operation. This should be obvious, given that it is a definite (and improper) integral. This can be verified

$$\begin{aligned}\mathcal{L}\{\lambda f(t)+g(t)\} &= \int_0^\infty e^{-st}(\lambda f(t)+g(t))\,dt \\ &= \lambda\int_0^\infty e^{-st}f(t)\,dt+\int_0^\infty e^{-st}g(t)\,dt \\ &= \lambda\mathcal{L}\{f(t)\}+\mathcal{L}\{g(t)\}\end{aligned}$$

3.2 Shift Theorem

If $\mathcal{L}\{f(t)\}=F(s)$, then

$$\mathcal{L}\left\{e^{at}f(t)\right\}=F(s-a).$$

To show this, we return to the definition

$$\begin{aligned}&\int_0^\infty e^{-st}\left(e^{at}f(t)\right)dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,dt=\int_0^\infty e^{-ut}f(t)\,dt \\ &= F(u)\end{aligned}$$

where $u=s-a$, and result follows.

Hence if $f(t)=t^n$ with $F(s)=\dfrac{n!}{s^{n+1}}$, then

$$\mathcal{L}\left\{e^{at}t^n\right\}=F(s-a)=\frac{n!}{(s-a)^{n+1}}$$

We now look at other useful results, consider

$$\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\}$$

$$=\int_0^\infty e^{-st}f\left(\frac{t}{a}\right)dt$$

writing $u=\dfrac{t}{a}\rightarrow t=au$ and $adu=dt$ so the line above becomes

$$
\begin{aligned}
&= a\int_0^\infty e^{-sau} f(u)\,du \\
&= a\int_0^\infty e^{-(as)u} f(u)\,du \\
&= aF(as).
\end{aligned}
$$

The *delay* property is

$$\mathcal{L}\{f(t-a)\} = e^{-as}F(s).$$

Applying this to the following gives

$$\mathcal{L}\{\cos(t-a)\} = e^{-as}\frac{s}{s^2+1}$$

This can be derived as follows. Firstly we know the standard result involving circular functions

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

If $k = 1/a$, then

$$
\begin{aligned}
\mathcal{L}\left\{\cos\left(\frac{1}{a}t\right)\right\} &= \frac{s}{s^2+\left(\frac{1}{a}\right)^2} \\
&= \frac{a^2s^2}{a^2s^2+1}
\end{aligned}
$$

Now use the standard result

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F\left(\frac{s}{k}\right)$$

where $k = 1/a$, to give

$$\frac{1}{1/a}F\left(\frac{s}{1/a}\right) = aF(as)$$

$$\mathcal{L}\left\{\cos\left(\frac{t}{a}\right)\right\} = a\frac{as}{s^2+1} = \frac{a^2s}{a^2s^2+1}$$

We can also derive the result

3.3 Table of Standard Forms

We are now able to construct a table of standard results

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$e^{\pm at}$	$\frac{1}{s\mp a}$
t^n	$\frac{n!}{s^{n+1}}$
t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
$\sin kt$	$\frac{k}{k^2+s^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$e^{at}f(t)$	$F(s-a)$
$f\left(\frac{t}{a}\right)$	$aF(as)$
$f(t-a)$	$e^{-as}F(s)$

3.4 The Inverse Laplace Transform

As with all transforms and substitutions, a clear technique for returning to the original variables is required. Given $F(s)$:

$$\mathcal{L}^{-1}(F(s)) = \text{the function } f(t) \text{ for which } \mathcal{L}\{f(t)\} = F(s).$$

Therefore we can read the table of Laplace Transforms in two ways

$$\begin{array}{cc} f(t) & F(s) \\ \overset{\mathcal{L}}{\longrightarrow} & \\ & \overset{\mathcal{L}^{-1}}{\longleftarrow} \end{array}$$

For example reading the earlier table from right to left we have

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s}\right) &= 1 \\ \mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) &= \frac{t^n}{n!} \end{aligned}$$

Example 1 :

Evaluate $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+2s+2}\right)$.

The structure should suggest some variation of

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) \text{ or } \mathcal{L}^{-1}\left(\frac{1}{s^2+k^2}\right).$$

Now $s^2+2s+2=(s+1)^2+1^2$. Hence

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+2}\right) \equiv \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1^2}\right) \text{ cf. } \mathcal{L}^{-1}\left(\frac{1}{s^2+k^2}\right)$$

with s replaced by $s+1$.

From the table:

$$\mathcal{L}\left\{e^{at}\sin kt\right\} = \frac{k}{(s+a)^2+k^2}$$

$$\therefore \mathcal{L}\left\{e^{-t}\sin t\right\} = \frac{1}{(s+1)^2+1^2} \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+2s+2}\right) = e^{-t}\sin t$$

Example 2 : $\mathcal{L}^{-1}\left(\dfrac{1}{(s+1)(s+2)}\right)$

Using partial fractions this becomes $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)} - \dfrac{1}{(s+2)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\}$$

We know

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{(s-a)} \therefore \mathcal{L}\left\{e^{-t}\right\} = \frac{1}{s+1} \quad \& \quad \mathcal{L}\left\{e^{-2t}\right\} = \frac{1}{s+2}$$

Hence we have

$$e^{-t} - e^{-2t}$$

3.5 The Convolution Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then there is a particular kind of product called the *convolution* given by

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

such that

$$\mathcal{L}\{f(t) * g(t)\} = F(s)\, G(s).$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\, G(s)\} &= f(t) * g(t) \\ &= \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\} \end{aligned}$$

Example: Given

$$F(s) = \frac{1}{s+1} \longrightarrow \mathcal{L}^{-1}(F(s)) = e^{-t}$$

$$G(s) = \frac{1}{s+2} \longrightarrow \mathcal{L}^{-1}(G(s)) = e^{-2t}$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s+1} \cdot \frac{1}{s+2}\right) &= e^{-t} * e^{-2t} \\ &= \int_0^t e^{-\tau} e^{-2(t-\tau)} d\tau = \int_0^t e^{-2t} e^{\tau}\, d\tau \\ &= e^{-2t} \left[e^{\tau}\right]_0^t = e^{-2t}\left(e^t - 1\right) \\ &= \left(e^{-t} - e^{-2t}\right). \end{aligned}$$

The correctness of this is easily verifiable by taking the Laplace Transform of $(e^{-t} - e^{-2t})$.

4 Solving Ordinary Differential Equations

4.1 Introduction

Laplace Transforms provide us with a very useful technique for solving IVP's.

This is based on two results (which are also theorems):

1. $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
2. $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$\mathcal{L}\{f'(t)\} = \displaystyle\int_0^\infty e^{-st} f'(t)\,dt$ and we integrate by parts to give

$$f(t)\,e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt \quad = \quad -f(0) + sF(s)$$

$\mathcal{L}\{f''(t)\} = \displaystyle\int_0^\infty e^{-st} f''(t)\,dt$ which requires integration by parts twice (and includes use of the earlier result)

$$\begin{aligned} f'(t)\,e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt &= -f'(0) + s(-f(0) + sF(s)) \\ &= s^2F(s) - sf(0) - f'(0) \end{aligned}$$

Repeated use allows us to write e.g.

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

Theoretically, we can also extend this to an n^{th} order IVP (with constant coefficients) by finding

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \ldots - sF^{(n-2)}(0) - F^{(n-1)}(0).$$

We can use these to transform an IVP such as

$$\left.\begin{array}{l} ay'' + by' + cy = f(t) \\ y(0) = \alpha; \quad y'(0) = \beta \end{array}\right\}$$

into an algebraists problem by taking Laplace Transforms of both sides of the ode. Let $Y(s) = \mathcal{L}\{y(t)\}$ then

$$a\left(s^2Y(s) - sy\ \ (0) - y'(0)\right) + b\left(sY(s) - y\ \ (0)\right) + cY(s) = F(s)$$

and simplifying

$$Y(s) = \frac{F(s) + ay'(0) + (as+b)\,y(0)}{as^2 + bs + c}.$$

The final step in the working entails taking inverse Laplace Transforms to have $Y(s) \to y(t)$.

Example: Solve the IVP

$$\left.\begin{array}{c} y'' + 4y' + 6y = 1 + e^{-t} \\ y(0) = y'(0) = 0. \end{array}\right\}$$

Start by taking LT of the whole equation, with $Y(s) = \mathcal{L}\{y(t)\}$

$$\begin{aligned} \mathcal{L}\{y'' + 4y' + 6y\} &= \mathcal{L}\left\{1 + e^{-t}\right\} \\ \mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} &= \mathcal{L}\{1\} + \mathcal{L}\left\{e^{-t}\right\} \end{aligned}$$

The initial condition simplifies the problem considerably to give

$$\begin{aligned} s^2 Y(s) + 4sY(s) + 6Y(s) &= \left(\frac{1}{s} + \frac{1}{s+1}\right) \\ Y(s) &= \frac{1}{s^2 + 4s + 6}\left(\frac{1}{s} + \frac{1}{s+1}\right) \end{aligned}$$

Now taking inverse LT's

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + 4s + 6)} + \frac{1}{(s^2 + 4s + 6)(s+1)}\right\}$$

after firstly using partial fractions, i.e. decompose as

$$\frac{1}{6s} + \frac{1}{3(s+1)} - \frac{s+2}{2(s^2 + 4s + 6)} - \frac{2}{3(s^2 + 4s + 6)}$$

therefore

$$y(t) = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2 + 4s + 6)}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + 4s + 6)}\right\}$$

We know $s^2 + 4s + 6 \equiv (s+2)^2 + 2$. So use the Shift Theorem

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s - a)$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}; \quad \mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}; \quad \mathcal{L}\{1\} = \frac{1}{s}$$

So the solution is

$$y(t) = \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{2}{3}e^{-2t}\cos\sqrt{2}t.$$

4.2 Solving Simultaneous Differential Equations

In this subsection we look at extending the work of the previous section to consider simultaneous differential equations by reducing them to linear systems.

We study these by way of solving the following example.

Example:

$$\begin{aligned} x'' - x + y' - y &= 0 \\ 2x' + 2x - y'' + y + e^{-t} &= 0 \end{aligned}$$

subject to the conditions

$$x(0) = 0,\ x'(0) = -1,\ y(0) = 1,\ y'(0) = 1.$$

Start by taking Laplace transforms of the system given, by

$$\begin{aligned} s^2 X(s) - sx(0) - x'(0) - X(s) + sY(s) - y(0) - Y(s) &= 0 \\ s^2 X(s) - X(s) + sY(s) - Y(s) &= 0 \\ Y(s) &= -X(s)(s+1) \end{aligned}$$

and the second one is

$$\begin{aligned} 2(sX(s) - x(0)) + 2X(s) - \left(s^2 Y(s) - sy(0) - y'(0)\right) + Y(s) + \frac{1}{s+1} &= 0 \\ 2sX(s) + 2X(s) - \left(s^2 Y(s) - s - 1\right) + Y(s) + \frac{1}{s+1} &= 0 \\ 2sX(s) + 2X(s) - s^2 Y(s) + Y(s) + s + 1 + \frac{1}{s+1} &= 0 \\ 2\underbrace{X(s)(s+1)}_{-Y(s)} - s^2 Y(s) + Y(s) + s + 1 + \frac{1}{s+1} &= 0 \\ -Y(s) - s^2 Y(s) + (s+1) + \frac{1}{s+1} &= 0 \\ \left(1 + s^2\right) Y(s) &= (s+1) + \frac{1}{s+1} \end{aligned}$$

After rearranging and using partial fractions

$$Y(s) = \frac{s+1}{s^2+1} + \frac{1}{2(s+1)} - \frac{s-1}{2(s^2+1)}$$

Therefore upon inverting

$$y(t) = \frac{1}{2}\cos t + \frac{3}{2}\sin t + \frac{1}{2}e^{-t}$$

After some more messy algebra we have

$$X(s) = -\frac{1}{s^2+1} - \frac{1}{2(s+1)^2} - \frac{1}{2(s+1)} + \frac{s}{2(s^2+1)}$$

which is inverted to give

$$x(t) = -\sin t - \frac{1}{2}te^{-t} - \frac{1}{2}e^{-t} + \frac{1}{2}\cos t$$

The earlier techniques and extensive practice of manipulating inverse Laplace Transforms is used effectively to solve coupled ordinary differential equations. Given the careful algebraic handling, the full steps in simplification are presented above.

4.3 Solving Variable Coefficient Differential Equations

We now introduce a 2^{nd} order equation in which the coefficients are variable in x. We begin by considering a simple but classic differential equation. An equation of the form

$$Ly = ax^2\frac{d^2y}{dx^2} + \beta x\frac{dy}{dx} + cy = g(x)$$

is called a Cauchy-Euler equation. Note the relationship between the coefficient and corresponding derivative term, i.e. $a_n(x) = ax^n$ and $\frac{d^ny}{dx^n}$, i.e. both power and order of derivative are n.

The equation is still linear. To solve the homogeneous part, we look for a solution of the form

$$y = x^{\lambda}$$

So

$$y' = \lambda x^{\lambda-1} \rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2},$$

which upon substitution (with $g(x) = 0$) gives

$$\begin{aligned} ax^2\frac{d^2y}{dx^2} + \beta x\frac{dy}{dx} + cy &= 0 \\ ax^2\left(\lambda(\lambda-1)x^{\lambda-2}\right) + \beta x\left(\lambda x^{\lambda-1}\right) + cx^{\lambda} &= 0 \\ \left(a\lambda^2 + (\beta-a)\lambda + c\right)x^{\lambda} &= 0, \end{aligned}$$

yielding the quadratic, A.E.

$$a\lambda^2 + b\lambda + c = 0$$

[where $b = (\beta - a)$] which can be solved in the usual way - there are 3 cases to consider, depending upon the nature of $b^2 - 4ac$.

$\boxed{\text{Case 1:}}$ $b^2 - 4ac > 0 \rightarrow \lambda_1, \lambda_2 \in \mathbb{R}$ - 2 real distinct roots

$$\text{GS } y = Ax^{\lambda_1} + Bx^{\lambda_2}$$

$\boxed{\text{Case 2:}}$ $b^2 - 4ac = 0 \rightarrow \lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$ - 1 real (double fold) root

$$\text{GS } y = x^{\lambda}(A + B\ln x)$$

$\boxed{\text{Case 3:}}$ $b^2 - 4ac < 0 \rightarrow \lambda = \alpha \pm i\beta \in \mathbb{C}$ - pair of complex conjugate roots

$$\text{GS } y = x^{\alpha}(A\cos(\beta\ln x) + B\sin(\beta\ln x))$$

$\boxed{\textbf{Example 1}}$ Solve $x^2y'' - 2xy' - 4y = 0$

Put $y = x^{\lambda} \Rightarrow y' = \lambda x^{\lambda-1} \Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2}$ and substitute in differential equation to obtain (upon simplification) the A.E. $\lambda^2 - 3\lambda - 4 = 0 \rightarrow (\lambda-4)(\lambda+1) = 0$

$\Rightarrow \lambda = 4\ \&\ -1$: 2 distinct $\mathbb{R}$ roots. So GS is

$$y(x) = Ax^4 + Bx^{-1}$$

$\boxed{\textbf{Example 2}}$ Solve $x^2y'' - 7xy' + 16y = 0$

So assume $y = x^\lambda$

A.E $\qquad \lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda = 4\ ,\ 4$ (2 fold root)

'go up one', i.e. instead of $y = x^\lambda$, take $y = x^\lambda \ln x$ to give

$$y(x) = x^4 (A + B \ln x)$$

$\boxed{\textbf{Example 3}}$ Solve $\qquad x^2y'' - 3xy' + 13y = 0$

Assume existence of solution of the form $y = x^\lambda$

A.E becomes $\qquad \lambda^2 - 4\lambda + 13 = 0 \to \lambda = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm 6i}{2}$

$\lambda_1 = 2 + 3i,\ \ \lambda_2 = 2 - 3i\ \equiv \alpha \pm i\beta \qquad (\alpha = 2,\ \ \beta = 3)$

$$y = x^2 \left(A \cos(3 \ln x) + B \sin(3 \ln x)\right)$$

Laplace Transforms provide a robust method for solving variable coefficient problems where the nth derivative term is of the form

$$t^m \frac{dy^n}{dt^n}$$

where m and n are positive integer values. This is a more generalised equation than the Euler problem described above.

We now look at Laplace transforms of these types of results.

Show that

$$\mathcal{L}\left\{t\frac{d^2x}{dt^2}\right\} = -\frac{d}{ds}\left(s^2X(s)\right) + x(0)\,.$$

and establish that the variable coefficient differential equation

$$\mathcal{L}\left\{t\frac{d^2x}{dt^2}\right\} = \int_0^\infty e^{-st}tx''dt$$

$$\begin{array}{rr} v = e^{-st}t & u' = x'' \\ v' = (1-st)\,e^{-st} & u = x' \end{array}$$

$$\begin{aligned} & e^{-st}tx'\Big|_0^\infty - \int_0^\infty (1-st)\,e^{-st}x'dt \\ = & -\int_0^\infty (1-st)\,e^{-st}x'dt \end{aligned}$$

Integration by parts again

$$\begin{array}{rr} v = -(1-st)\,e^{-st} & u' = x \\ v' = 2se^{-st} - s^2t & u = x \end{array}$$

$$-(1-st)\,e^{-st}x\Big|_0^\infty + \int_0^\infty \left(-2s + s^2t\right)e^{-st}xdt$$

The first part yields $x(0)$. Looking at the integral, we show the working due to a subtle step involved

$$\begin{aligned}
\int_0^\infty \left(-2s + s^2 t\right) e^{-st} x dt &= -2s \int_0^\infty e^{-st} x dt + s^2 \int_0^\infty e^{-st} t x dt \\
&= -2sX(s) + s^2 \int_0^\infty \frac{d}{ds}\left(-e^{-st}\right) x dt \\
&= -2sX(s) - s^2 \frac{d}{ds}\left\{\int_0^\infty e^{-st} x dt\right\} \\
&= -2sX(s) - s^2 \frac{d}{ds}\mathcal{L}\{x(t)\} \\
&= -\frac{d}{ds}\left(s^2 X(s)\right)
\end{aligned}$$

which gives

$$\mathcal{L}\left\{t\frac{d^2x}{dt^2}\right\} = x(0) - \frac{d}{ds}\left(s^2 X(s)\right).$$

In a similar fashion, but with less work, we obtain

$$\mathcal{L}\left\{t\frac{dx}{dt}\right\} = -X(s) - s\frac{d}{ds}X(s).$$

After these messy calculations we consider solving the variable coefficient problem.

Example Establish that the equation

$$tx'' + (1-t)x' + \alpha x = 0,$$

where $\alpha \in \mathbb{R}$, has a solution

$$L_\alpha(t) = \mathcal{L}^{-1}\left[s^{-\alpha-1}(s-1)\alpha\right].$$

We use the results obtained above. So taking Laplace Transforms yields

$$\left(x(0) - \frac{d}{ds}\left(s^2 X(s)\right)\right) + (sX(s) - x(0)) + \left(X(s) + s\frac{d}{ds}X(s)\right) + \alpha X(s) = 0$$

Simplifying and rearranging this allows us to write

$$\frac{dX(s)}{X(s)} = \left(\frac{\alpha}{s-1} - \frac{(\alpha+1)}{s}\right) ds$$

which is a variable separable equation, so integrating

$$\ln X(s) = \alpha \ln(s-1) - (\alpha+1)\ln s + k$$

where k is an arbitrary constant

$$\ln X(s) = \ln\frac{(s-1)^\alpha}{s^{\alpha+1}} + k$$

hence taking exponentials and writing $C = e^k$

$$X(s) = C(s-1)^\alpha s^{-(\alpha+1)}$$

5 The $z-$Transform

5.1 Introduction

This is the discrete analogue of the Laplace Transform. Let $x(n)$ be a discrete function of $n = 0, 1, 2, ...$

We define the $z-$transform of $x(n)$, $Z(x(n))$ also written

$$X(z) = \sum_{n=0}^{\infty} \frac{x(n)}{z^n}.$$

A Power Series centred at $x = x_0 \in \mathbb{R}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + ... + a_n (x - x_0)^n + ...$$

where each a_i is a real number and a_0 is called the **constant** term.

Here we will consider series with $x_0 = 0$, without loss of generality.

Let the power series $\sum_{n=0}^{\infty} a_n x^n$ converge for all x in the interval $(-R,\ R)$. Define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

Then R is called the *Radius of Convergence* of the power series and, for $-R < x < R$, then the following results are true:

i)

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + ... + n a_n x^{n-1} + ...$$

(meaning, that the power series may be differentiated 'term by term', for $-R < x < R$)

ii)

$$\begin{aligned} \int_0^x f(t)\,dt &= \sum_{n=0}^{\infty} \int_0^x (a_n t^n)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \\ &= a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + ... + \frac{1}{n+1} a_n x^{n+1} + ... \end{aligned}$$

(meaning, that the power series maybe integrated 'term by term', for $-R < x < R$).

There is clearly interest in ascertaining convergence conditions for infinite sums. One well know test is the **Ratio Test**.

We work with

$$\sum_{n=0}^{\infty} a_n x^n.$$

Define

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

Then

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$, more work is required. The series may be

   i. divergent
   ii. conditionally convergent
   iii. absolutely convergent

We can use the ratio test to establish convergence criteria:

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \left| \frac{x(n+1)/z^{n+1}}{x(n)/z^n} \right| \\ &= \left| \frac{x(n+1)}{x(n)} \right| \times \left| \frac{1}{z} \right| \end{aligned}$$

Under a mild restriction

$$\left| \frac{x(n+1)}{x(n)} \right| \to \alpha,$$

a constant, then the series will be absolutely convergent for $|z| > \alpha$. In general for our $x(n)$ the infinite sum will be convergent for $|z| >$ some constant α, which we need not specify.

Examples

1. $x(n) = 1;\ n \in \mathbb{N}$. the $z-$transform is

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \frac{1}{z^n} \\ &= \frac{1}{1-1/z} = \frac{z}{z-1}. \end{aligned}$$

2. $x(n) = n;\ n \in \mathbb{N}$. the $z-$transform is

$$X(z) = \sum_{n=0}^{\infty} \frac{n}{z^n}.$$

   We know from above

$$\sum_{n=0}^{\infty} \frac{1}{z^n} = \frac{z}{z-1}$$

   and differentiate with respect to z

$$\begin{aligned} \frac{d}{dz} \sum_{n=0}^{\infty} \frac{1}{z^n} &= \frac{d}{dz} \left(\frac{z}{z-1} \right) \\ &= \frac{d}{dz} \left(1 + \frac{1}{z-1} \right), \text{ inside radius of convergence.} \end{aligned}$$

$$\begin{aligned}
-\sum_{n=0}^{\infty}\frac{n}{z^{n+1}} &= \frac{d}{dz}\left(1+\frac{1}{z-1}\right)\\
\frac{1}{z}\sum_{n=0}^{\infty}\frac{n}{z^{n}} &= \frac{1}{\left(z-1\right)^{2}}
\end{aligned}$$

So,

$$\sum_{n=0}^{\infty}\frac{n}{z^{n}}=\frac{z}{\left(z-1\right)^{2}}.$$

3. $x\left(n\right)=a^{n}$; for constant a. $n\in\mathbb{N}$. the $z-$transform is

$$X\left(z\right)=\sum_{n=0}^{\infty}\frac{a^{n}}{z^{n}}.$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\frac{1}{1-a/z}\\
&= \sum_{n=0}^{\infty}\frac{z}{z-a}
\end{aligned}$$

which is inside the radius of convergence.

5.2 Properties:

1. $Z\left(x\left(n\right)+y\left(n\right)\right)=Z\left(x\left(n\right)\right)+Z\left(y\left(n\right)\right)$
2. $Z\left(\lambda x\left(n\right)\right)=\lambda Z\left(x\left(n\right)\right)$

If two function $x(n)$ and $y(n)$ have the same $z-$transform then $x(n)=y(n)$, because

$$X\left(z\right)=\sum_{n=0}^{\infty}\frac{x\left(n\right)}{z^{n}},\quad Y\left(z\right)=\sum_{n=0}^{\infty}\frac{y\left(n\right)}{z^{n}}$$

therefore

$$\begin{aligned}
\left(X\left(z\right)-Y\left(z\right)\right) &= 0\Rightarrow\left(x\left(n\right)-y\left(n\right)\right)\\
&\Rightarrow x\left(n\right)=y\left(n\right)
\end{aligned}$$

5.3 Special Transforms

If $x(n)$ has $z-$transform $X\left(z\right)$, then what is the $z-$transform of $x(n+1)$ and $x(n+2)$?

For $x(n+1) : Z\left(x(n+1)\right) =$

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\frac{x\left(n+1\right)}{z^{n}} \\
= \quad & z\sum_{n=0}^{\infty}\frac{x\left(n+1\right)}{z^{n+1}} \\
= \quad & z\left(\frac{x\left(1\right)}{z}+\frac{x\left(2\right)}{z^{2}}+...\right) \\
= \quad & z\left(X\left(z\right)-\frac{x\left(0\right)}{z^{0}}\right) \\
= \quad & zX\left(z\right)-zx\left(0\right)
\end{aligned}
$$

For $x(n+2) : Z\left(x(n+2)\right) =$

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\frac{x\left(n+2\right)}{z^{n}} \\
= \quad & z^{2}\sum_{n=0}^{\infty}\frac{x\left(n+2\right)}{z^{n+2}} \\
= \quad & z^{2}\left(\frac{x\left(2\right)}{z^{2}}+\frac{x\left(3\right)}{z^{3}}+...\right) \\
= \quad & z^{2}\left(X\left(z\right)-\frac{x\left(0\right)}{z^{0}}-\frac{x\left(1\right)}{z^{1}}\right) \\
= \quad & z^{2}X\left(z\right)-z^{2}x\left(0\right)-zx\left(1\right)
\end{aligned}
$$

Compare these to the Laplace transforms $\mathcal{L}\left\{f'\left(t\right)\right\}$ and $\mathcal{L}\left\{f''\left(t\right)\right\}$

5.4 The Inverse $z-$transform

The problem here is given $X(z)$, find the corresponding $x(n)$.

Solution:

$$X(z) = \sum_{n=0}^{\infty} \frac{x(n)}{z^n}$$

Therefore we know $x(n)$ is the coefficient of $\dfrac{1}{z^n}$ in the expansion of $X(z)$ in powers of $1/z$.

Example

Find $x(n)$ given $X(z) = \dfrac{1}{(z-1)^3}$.

$$\begin{aligned}
\frac{1}{(z-1)^3} &= \frac{1}{z^3(1-1/z)^3} \\
&= \frac{1}{z^3}\left(1-\frac{1}{z}\right)^{-3} \\
&= \frac{1}{z^3}\left(1+\frac{3}{z}+\frac{3\cdot 4}{1\cdot 2}\frac{1}{z^2}+\frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}\frac{1}{z^3}+...\right) \\
&= \left(0+\frac{0}{z}+\frac{0}{z^2}+\frac{1}{z^3}+\frac{3}{z^4}+\frac{3\cdot 4}{1\cdot 2}\frac{1}{z^5}+\frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}\frac{1}{z^6}+...\right)
\end{aligned}$$

in powers of $1/z$.

$$\begin{aligned}
x(0) &= x(1) = x(2) = 0,\ x(3) = 1 \\
x(n) &= \frac{3\cdot 4\cdot 5...(n-1)}{1\cdot 2\cdot 3...(n-3)},\ n \geq 4.
\end{aligned}$$

5.5 Solving Linear Difference Equations with $z-$transforms

Example: Solve

$$\begin{cases} x(n+2)+5x(n+1)+6x(n)=0 \\ x(0)=1,\ x(1)=-2 \end{cases}$$

to find $x(n)$.

Solution: Start by taking the $z-$transform of the difference equation

$$Z\left(x(n+2)\right)+5Z\left(x(n+1)\right)+6Z\left(x(n)\right)=Z\left(0\right)$$

$$\begin{aligned} & \left(z^2X\left(z\right)-z^2x\left(0\right)-zx\left(1\right)\right) \\ & +5\left(zX\left(z\right)-zx\left(0\right)\right) \\ & +6X\left(z\right) \\ = & \ 0 \end{aligned}$$

$$\begin{aligned} X\left(z\right)\left(z^2+5z+6\right) &= z^2+3z \\ X\left(z\right) &= \frac{z^2+3z}{z^2+5z+6} \\ &= \frac{z}{z+2}. \end{aligned}$$

So required $x(n)$ is the inverse $z-$transform of $\dfrac{z}{z+2}$. So expand this in powers of $1/z$.

$$\begin{aligned} & \left(1+\frac{2}{z}\right)^{-1} \\ = & \ 1+\left(-1\right)\frac{2}{z}+\frac{\left(-1\right)\left(-1-1\right)}{2!}\frac{2^2}{z^2}+\frac{\left(-1\right)\left(-1-1\right)\left(-1-2\right)}{3!}\frac{2^3}{z^3}+... \end{aligned}$$

for which the general n^{th} term is given by

$$\frac{\left(-1\right)\left(-1-1\right)...\left(-1-\left(n-1\right)\right)}{n!}\frac{2^n}{z^n}=\left(-1\right)^n\frac{2^n}{z^n}$$

Therefore

$$x\left(n\right)=\left(-1\right)^n 2^n : n=0,1,2,...$$

6 Conclusions

This one term undertaking centres on the Laplace Transform, some of its underlying theory and powerful applications. The most rewarding aspect of the project is the additional mathematical topics studied in order to consider the main theme.

Chapter 2 introduces a variety of topics in advanced calculus which are required for an initial discussion of the Laplace Transform. These topics while important results in their own right, appear extensively in applied maths and mathematical physics.

Chapter 3 introduces the Laplace transform in detail and derives a number of important standard transforms. The inverse Laplace transform is demonstrated and examples of its use are discussed.

Chapter 4 forms a major part of this work by applying this transform to its possible "reason for being" or *raison d'être.* This allows us to solve a number of ordinary differential equations. Both constant and variable coefficient equations are solved. For the latter, some background information is used by introducing the Cauchy-Euler differential equation. A system of differential equations is also solved using the Laplace transform.

In Chapter 5 we consider the discrete analogy of the Laplace transform, namely the $z-$transform. The $z-$transform plays an important role in electronic engineering and recently has a central position in digital signal processing. This is again a very gratifying section. The prerequisite is an appreciation of the conditions for convergence of infinite sums. Further practice of manipulating infinite sums is achieved by calculating transforms of sequences. The section ends with the solution of difference equations using the $z-$transform.

It is perhaps a good point at which to mention that completion of this work concurrently with examination preparation for the degree finals was challenging. If there had been more time I would have studied complex analysis topics to include more contour integration and further residue calculus to solve integrals of the form

$$\int_{-\infty}^{\infty} f(x)\, dx,$$

where $f(x)$ is a polynomial fraction. This would allow a more theoretical approach to discussing the inversion integral.

On the application side, any future study would also include solving partial differential equations using the Laplace transform, in particular the Heat equation which was studied in Mathematical Modelling.

7 References

Kissin, Edward. (2018/19), MA6010 Algebra and Analysis course notes

Hou, Zhanyuan (2017/18), MA5052 Differential Equations course notes

Spiegel, Murray. (2012), "Schaum's Outline of Laplace Transforms; 1st Edition (Schaum's Outlines)" (McGraw-Hill Education). ISBN: 007060231X

Bronson, Richard. (2014), "Schaum's Outline of Differential Equations; 4th Edition (Schaum's Outlines)" (McGraw-Hill Education). ISBN: 0071824855

Kreyszig, Erwin. (2006), "Advanced Engineering Mathematics" (John Wiley and Sons Ltd).

Zill, D. and *Cullen, M.* (2001), "Differential Equations with Boundary-Value Problems"; 5th Edition (Brooks Cole)

Spivak, Michael. (2008), "Calculus"; 5th Edition (Publish or Perish, Inc.). ISBN: 0914098918

https://ocw.mit.edu/courses/mathematics MIT OpenCourseWare: Laplace Transforms