INTRODUCTION TO THE METHODS
MULTI-CRITERIA OF OVERCLASSING

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Among the multi-criteria Decision Aid methods (also known as Multi-Criteria Analysis methods), the class of Outclassing methods has been developed to address problems of choice (best action among several alternatives), of sorting (attribution of the actions considered to several classes of whose characteristics are known) and ordering (construction of an order of preference on the set of possible actions to be undertaken). The goal is to provide decision makers with tools that help them face decision-making problems characterized by a multiplicity of significant points of view and often by a limited level of structuring, in decision-making processes that develop in an organizational or multi-organizational context.

There are two main families: the methods *ELECTRE (* acronym meaning *ELimination Et Choix TRaduisant la REalité* - Elimination and choice translating reality), oriented to choice or to sorting, ed the methods of *selection / segmentation,* which address the problem of sorting. Referring for more details to Ostanello (1992), Vincke (1992), Roy and Bouyssou (1993) and Roy (1996), some definitions will be provided here, to then introduce the concepts and some procedures that characterize the ELECTRE methods and the methods of selection / segmentation.

1. Actions and criteria

The development of a multi-criteria model is aimed at defining a set A of *actions* possible and a coherent family F of *criteria,* which allows to evaluate the actions and then face the decision-making problem recognized as significant and formulated through a problem of choice, sorting or classification.

A' *action* it is “a representation of a possible contribution to the global decision” (Roy, 1996). The set A of actions *potentials (* strategic choices of innovation or possible sites for a location, alternative projects or candidates for a job, ...) can be *finished and stable,* that is, defined a priori, for example by means of a procedure, tests or the clarification of rigid boundaries. It can be *evolutionary,* as new actions are identified and developed over time, and in some cases it may initially be an empty set.

A possible definition of criterion (cf Roy and Bouyssou, 1993) is the following: "a criterion is a tool that allows to compare any two potential actions according to a certain point of view or dimension of the problem". Criteria they must be: *significant,* with respect to the context of the problem and the objectives expressed by the decision maker; *common to all alternative actions* and sufficient to characterize them with respect to the situation considered; *suitable* to represent preferences. In order for the family of criteria to be coherent, it is necessary to verify some conditions, including mainly those of completeness, cohesion and non-redundancy.

A criterion 'g' is a function from the set 'A' of potential actions to a totally ordered set 'E' called a rating scale such that:

$\forall$ to $\in$ TO $\Rightarrow$ g (a) $\in$ IS,

so it is assumed that it is well founded to compare any two actions **to** ed **to',**
according to a certain dimension, based on the evaluations g (a) and g (a ').

If the evaluation model consists of a single criterion, the resulting comparisons are interpreted in terms of overall preference over A. If the criteria are multiple, the comparisons deriving from each criterion are interpreted in terms of
partial preference, in the sense that they are limited only to the aspects taken into consideration from the point of view underlying the definition of the criterion. It can therefore be said that a criterion g is a *template* which allows you to establish preference relationships between actions, consistent with the preferences of decision makers.

Each criterion is associated with an evaluation scale, which can be ordinal (both quantitative and qualitative or relational) if only the position counts for the purposes of the relationship between the alternatives, or cardinal, i.e. quantitative with attribution of relevance to the numerical difference. The quality of the data used to evaluate the actions on the criterion (and therefore their different level of uncertainty, imprecision or bad determination) must be analyzed to choose the type of criterion to be used. We talk about *true criterion* if any difference between two evaluations on it implies one *preference in the strict sense (* or *narrow).* Conversely, thresholds can be introduced by defining other types of criteria, to blur the indication when it is reasonable to admit that small differences between the evaluations do not express preference but still a substantial situation. *indifference* between assessments (in relation to the level of uncertainty of the data used). The threshold **q,** called indifference, it is introduced to indicate the maximum gap between evaluations still compatible with a situation of indifference. A deviation greater than **q** it can indicate a situation of clear preference between one evaluation and another or, in conditions of preferential uncertainty, a zone of *weak preference,* which translates a hesitation between indifference and strict preference. In this case, a new threshold can be introduced **s,** said of presumption of preference, and the value of **s** is greater than or equal to that of **q.**

If it only exists **s** which discriminates a weak preference zone from that of narrow preference, we are in the presence of a *pre-criterion.* If you have both thresholds, but they coincide, a weak preference is not perceived but only a narrow one and an interval of indifference, in this case it is in the presence of a *quasi-criterio*
Finally, in the presence of two distinct thresholds, and therefore of three intervals

(indifference and weak preference, which together constitute the "presumption of preference", in addition to strict preference) we speak of *pseudo-criterion.*

A *coefficient of relative importance (* said *weight)*
which constitutes one of the most "delicate" parts of the model, because it is the most direct and explicit expression of decision-making preferences and can significantly influence the results of the application of a Multicriteria Analysis method. Care in the choice of criteria, the explicit expression of coefficients that indicate the different importance of the criteria (with the possibility of representing different scenarios of importance of the criteria) and the operational definition of different decision rules (both to classify alternative actions and to assign them to qualification classes) allow to reproduce with the necessary attention the different needs of a specific decision-making situation and therefore to provide support in a more adequate way.

1.1 Multi-criteria methods of outclassing

In the multi-criteria methods proposed here, the actions are compared in pairs on each single criterion to establish whether one of the two is preferable to the other or whether they are indifferent. The problem of aggregating the results of the comparisons is faced by constructing a mixed binary relation S, called di *Outclassing,*

expressed as the union of three elementary relations, indifference (I), weak preference (Q) and strict preference (P): S = IUQU P. In this context, the possibility of *incomparability* between actions (N), different from indifference as it is caused by the existence of conflicting preferences on the different criteria, which make it impossible to establish which of the two actions is better, even knowing that they are not equal.

It can be said that action ***to*** *outclasses* the action ***to' (*** aSa ') if "there are sufficient reasons to believe that **to** is at least as good as **to'** and there are no good reasons to reject this claim ". Operationally, this definition translates into the verification of conditions of *concordance (* of sufficient reasons to believe that **to** is at least as good as **to')** and of *no discord (* that is, good reasons to reject the claim). As this relation is defined, union of three elementary relations characterized by different properties, it is not possible to infer anything about the outclassing between **to'** ed **to,** known the one between **to** ed **to'.** If there are no out-of-class relations between two actions, either in one sense or the other, they are said to be *incomparable.*

The outclass can be *defined,* and in this case affirm that **to**
outclasses **to'** corresponds to indicating a certain outclassing while the statement ' **to** does not outclass **to'** it implies a certain non-outclassing. In this case it is possible to indicate with certainty the preference of one action over the other (one of the two outclasses the other and not vice versa) or their indifference (they outclass each other) or their incomparability (neither of them outclasses the other). other).

We speak instead of outclassing *faded (* or *fuzzy* in English or *soft focus* in French), when a *degree of credibility,* indicated with δ (a, a ') is

between 0 and 1, with which to express a different credibility in affirming that there is an outclassing relationship between two specific actions.

2. The ELECTRE methods

I'm methods multi-criteria of aggregation of the preferences by outclassing, developed by Bernard Roy the first in 1968, the second in 1972 (together with P. Bertier), the third in 1979 and the fourth in 1982. The methods differ in the problems addressed (chosen for the first, ordering for the others), the nature of the data processed and therefore the type of criteria (true for the first and second, with cardinal scales the first and cardinal or ordinal the second; pseudo-criteria instead for the last two, which use cardinal scales with thresholds) and for the modeling procedure of outclassing, defined or fuzzy.

All the ELECTRE methods are structured in two phases: in the first phase (of modeling the outclassing relationship) the actions on each criterion are compared in pairs and the results obtained are aggregated, by means of the construction of indices or the application of tests that verify the presence of conditions of concordance and non-disagreement, at the basis of the concept of outclassing; in the second, the procedure for classifying the actions relating to the problem under examination and the modeled decision 'rule' is activated. The choice between the different methods is motivated by indications connected both to the nature of the data available, therefore to the criteria that can be used, and to the precise decision rule that you want to make operational.

A brief description and some application examples of the ELECTRE II method (Roy and Bertier, 1972) are provided below, while only some of the characterizing elements of the ELECTRE III method (Roy, 1979) are presented. For a more complete description of these and other ELECTRE methods, see (Vincke, 1992; Roy and Bouyssou,
1993).

2.1 ELECTRE II

ELECTRE II uses *true-criteria,* those, that is, in which any difference in evaluation indicates a preference in the strict sense. The scales can be cardinal or ordinal.

The outclassing employed in this method is *definite (certain);* this implies that the associated feature function δ (a, a '), defined on AxA, take values in the set {0, 1}:

a S a ' ⇔ δ (a, a ') = 1 (Certain outclassing)
to S / ⇔ δ (a, a ') = 0 (Non-outclassing
to' of course).

The result of the modeling of the outclassing is represented by a graph in which the nodes are the compared actions and the oriented arcs express the relationships between the pairs of nodes. The edges from appear in the outclassing graph **to**
to **to'** for which δ (a, a ') = 1.

Phase I

The outclassing is modeled in the first step of the method using the tests of *concordance* and of *no discord (* Figure 1).

To carry out the *concordance test* it is necessary to compare all the pairs in pairs actions, criterion by criterion, and subdivide, for each pair of actions, the criteria g_j in three subsets: criteria in accordance with the statement "the evaluation of **to** is better than that of **to'** "(J +), criteria for which" the evaluation of **to** is the same as that of **to'" (** J =) and criteria in accordance with the statement "the evaluation of **to** is worse than that of **to'** "
(J-). That is, they are defined:

J + (a, a ') = {j ∈ J: $g_{j(}$ a) $g_{j(}$ a ')} J = (a, a') =

{j ∈ J: $g_{j(}$ a) = $g_{j(}$ a ')} $J_{-(}$ a, a ') = {j ∈ J: $g_{j(}$ to) ≺

$g_{j(}$ to') }.

Notice the weights ($p_{J)}$ associated with each criterion, it is possible to calculate the relative importance of each subset by adding the weights of the individual criteria belonging to the
subset.
From which:

$$P + (a, a ') = \sum_{j \in J+} p_j$$

$$P = (a, a ') = \sum_{j \in J=} p_j$$

$$P._{-(} a, a ') = \sum_{j \in J-} p_{j.}$$

To pass the concordance test it is necessary that:

the) $c (a, a ') = \frac{P + (a, a ') + P = \cancel{(a, a')}}{P.} \geq c$

ii) $\frac{P + (a, a ')}{P._{-(} a, a ')} \geq 1$

where c (a, a ') is the *concordance index* is **c** is a parameter that represents the *threshold of concordance* or *majority.* You can choose the *thresholds,* said *natural (* $\mathbf{c}_{f}$ = 3/4 said *strong threshold* is $\mathbf{c}_{d}$ = 2/3 said *weak threshold)* or in any case threshold values close to these.

With the *non-discordance test* it occurs that, on the criteria belonging to $J_{-(}$ a, a '), a *veto* to the outclassing of **to'** from **to,** veto that is triggered in correspondence with expressed discrepant values (for each criterion j * ∈ J *) [1] from *sets of discordance (* $D._{j*)}$, that is, sets of pairs of values (e, e ') of the scale of

[1] Where J * ⊆ J is the subset of criteria, identified by the decision maker, on which there are vetoes.

criterion j * with e 'much worse' than e '(e ≺ ≺ e '), which indicate the limits beyond which the veto becomes operative.

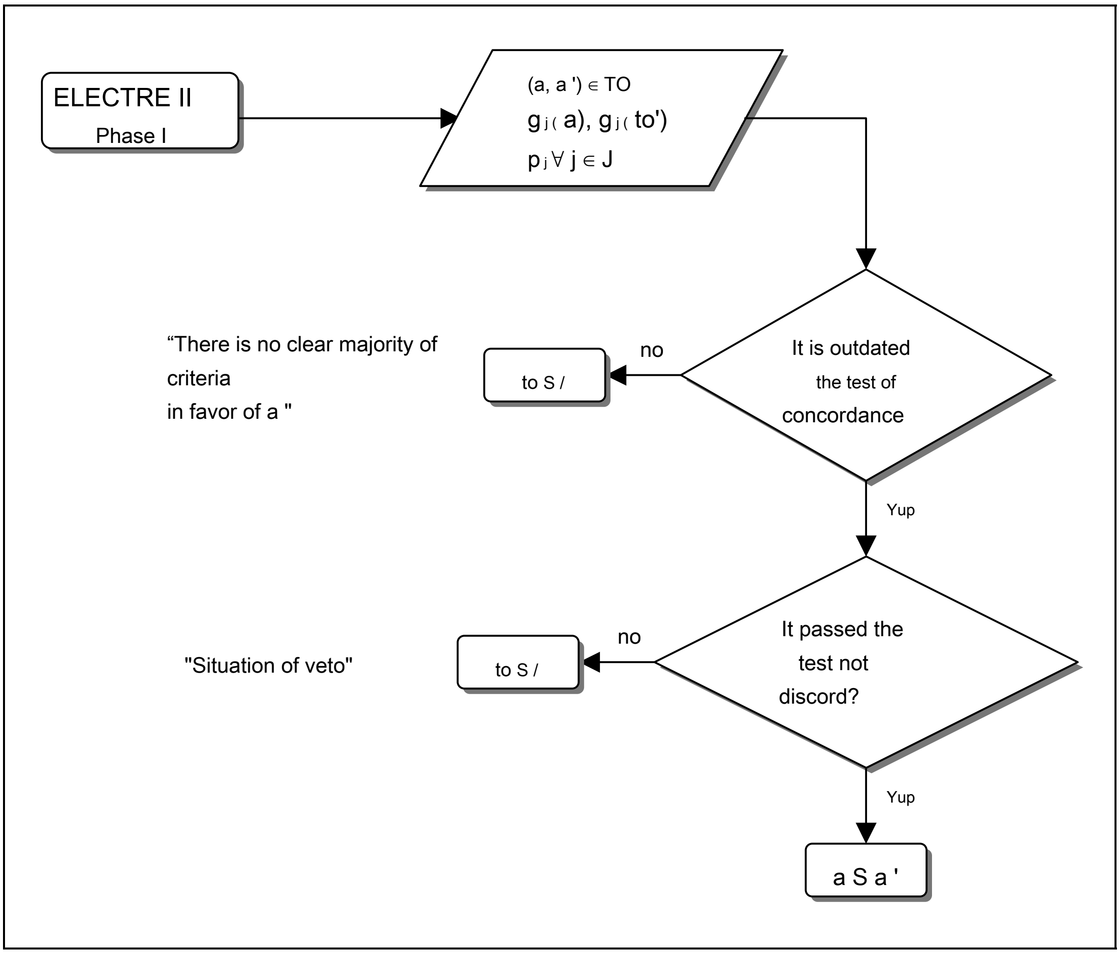


Figure 1: Modeling of outclassing in ELECTRE II

The non-discordance test fails, and therefore a S / a ', if for at least one criterion j * ∈ J * it occurs that:

$g_{j^*}(a)$ = **is** is $g_{j^*}(a')$ = **is'** with (**e, e ')** ∈ D_{j^*}

The result of Phase I of the method is represented on the *outclassing graph* G (A, S_f), to which one can be associated *incidence matrix* whose elements m_{ij} are given by
function δ (to the, to j), on which the Phase II classification procedure will apply. An example of application of Phase I of ELECTRE II is presented in the report
to a problem of industrial settlement. The number of settlement actions examined and the criteria used to evaluate them has been reduced to speed up the calculations as much as possible.

Industrial settlement problem: application of Phase I of ELECTRE II

Five zones in different territorial areas (a 1, to 2, to 3, to 4, to 5) have been evaluated in order to be able to recognize their different suitability for a possible
industrial settlement. The six criteria make it possible to assess the provision of services and manpower in the various areas (table 1). Applying Phase I of ELECTRE II, we get to obtain an outclassing graph.

Table 1: Evaluations and parameters

Criteria	Streets Infrastructure	Energy Endowment electric railway		is water	Offer labor banking	Services
Weights	0.26	0.14	0.20	0.12	0.21	0.07
Zones						
to 1	0	10.3	3	5	2.1	0.35
to 2	0.4	52.5	3	5	1.5	0.30
to 3	0.3	1.4	0	3	0	1
to 4	0.7	28.1	3	7	3.2	0.39
to 5	0.7	12.2	4	10	2.8	1.6
Ins. Disc. (0, 0.7)						

Remarks

- The example is taken from a real case from 1978 ('Multi-criteria analysis of an industrial location problem: experimentation on the Turin area' by Ostanello, Simoni and Vernoni) in which both the number of areas and the criteria is much higher than of the model shown in table 1. Each criterion required different measurement methods that should be described, as well as the information sources used. The case can be analyzed in detail by using the model provided among the examples of real applications.
- By analyzing the criteria it is possible to recognize the preferred direction and develop the calculations relating to the first phase of the method.
- The coefficients of relative importance of the criteria, called weights, are already normalized to one; if they weren't, they would need to be normalized.
- The set of discordance is defined on a single criterion.
- The natural thresholds are $c_f = 3/4$ ec $c_d = 2/3$. It is possible to choose these thresholds or others that do not differ much from the natural thresholds.

Phase I

By verifying that for each criterion proposed in the table the direction of preference increases as the numerical values increase, it is possible to move on to the comparison between pairs of areas of possible settlement and to the tests of concordance and non-discordance.

Table 2: Phase I of ELECTRE II

	J +	J =	J -	P + ≥ P. - P	+ + P = V ETO		S.
to$_1$to$_2$	5, 6	3, 4	1, 2	no			
to$_1$to$_3$	2, 3, 4, 5	/	1, 6	Yup	0.67		
to$_1$to$_4$	/	3	1, 2, 4, 5, 6	no		Yup	
to$_1$to$_5$	/	/	1 2 3 4 5, 6	no		Yup	
to$_2$to$_1$	1, 2	3, 4	5, 6	Yup	0.72		
to$_2$to$_3$	1 2 3 4 5	/	6	Yup	0.93		to$_2$ S a$_3$
to$_2$to$_4$	2	3	1, 4, 5, 6	no			
to$_2$to$_5$	2	/	1, 3, 4, 5, 6	no			
to$_3$to$_1$	1, 6	/	2, 3, 4, 5	no			
to$_3$to$_2$	6	/	1 2 3 4 5	no			
to$_3$to$_4$	6	/	1 2 3 4 5	no			
to$_3$to$_5$	/	/	1 2 3 4 5, 6	no			
to$_4$to$_1$	1, 2, 4, 5, 6	3	/	Yup	1		to$_4$ S a$_1$
to$_4$to$_2$	1, 4, 5, 6	3	2	Yup	0.86		to$_4$ S a$_2$
to$_4$to$_3$	1 2 3 4 5	/	6	Yup	0.93		to$_4$ S a$_3$
to$_4$to$_5$	2, 5	1	3, 4, 6	no			
to$_5$to$_1$	1 2 3 4 5, 6	/		Yup	1		to$_5$ S a$_1$
to$_5$to$_2$	1, 3, 4, 5, 6	/	2	Yup	0.86		to$_5$ S a$_2$
to$_5$to$_3$	1 2 3 4 5, 6	/	/	Yup	1		to$_5$ S a$_3$
to$_5$to$_4$	3, 4, 6	1	2, 5	Yup	0.65		

The choice of the natural threshold as a strong threshold is practically mandatory. Many outclasses are obtained (seven) and nothing authorizes us to propose a different choice. For the threshold weak it is possible to either choose the natural threshold (equal to 2/3) or to adopt c_{d} = 0.65, a very close value that allows us to consider also the weak outclassing between a$_5$ ed to$_4$. In this case, the choice of the latter value was preferred. From Phase I we obtain the outclassing graph of figure 2.

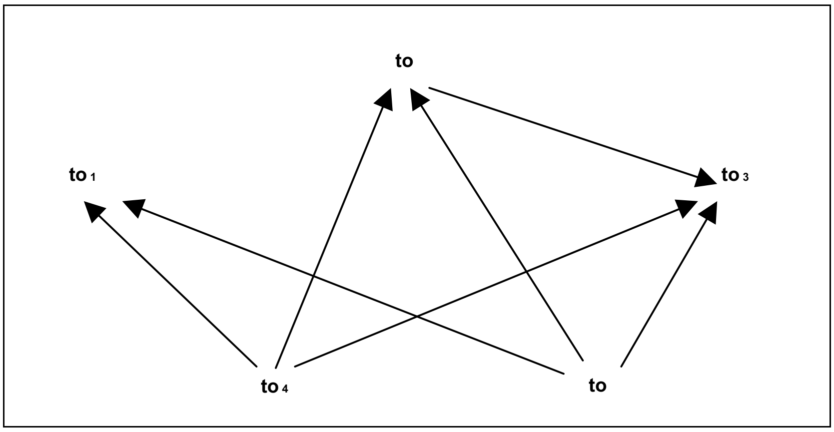


Figure 2: Outclass graph of the example

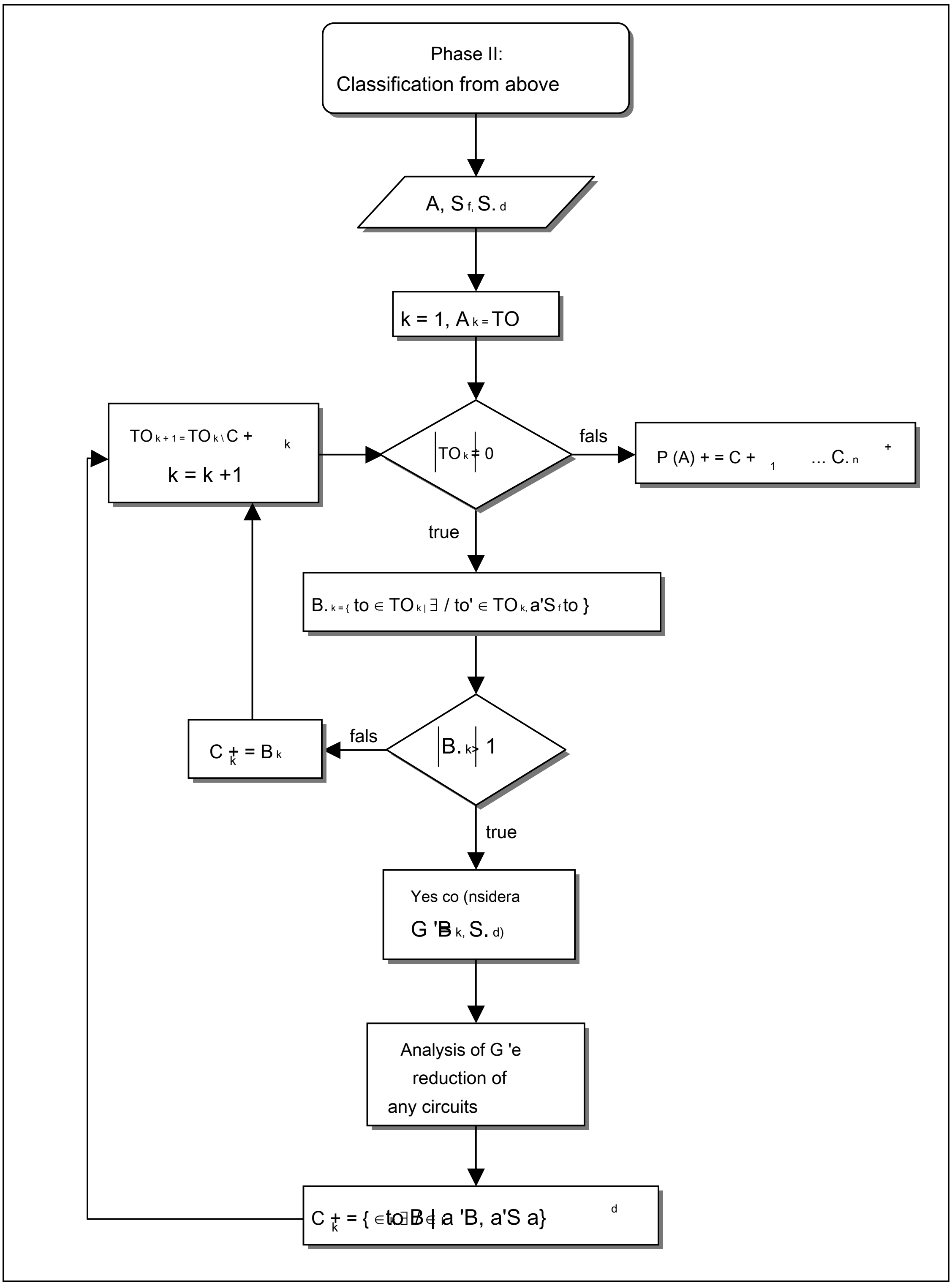


Figure 3: Construction of the pre-order from above P (A) + in ELECTRE II

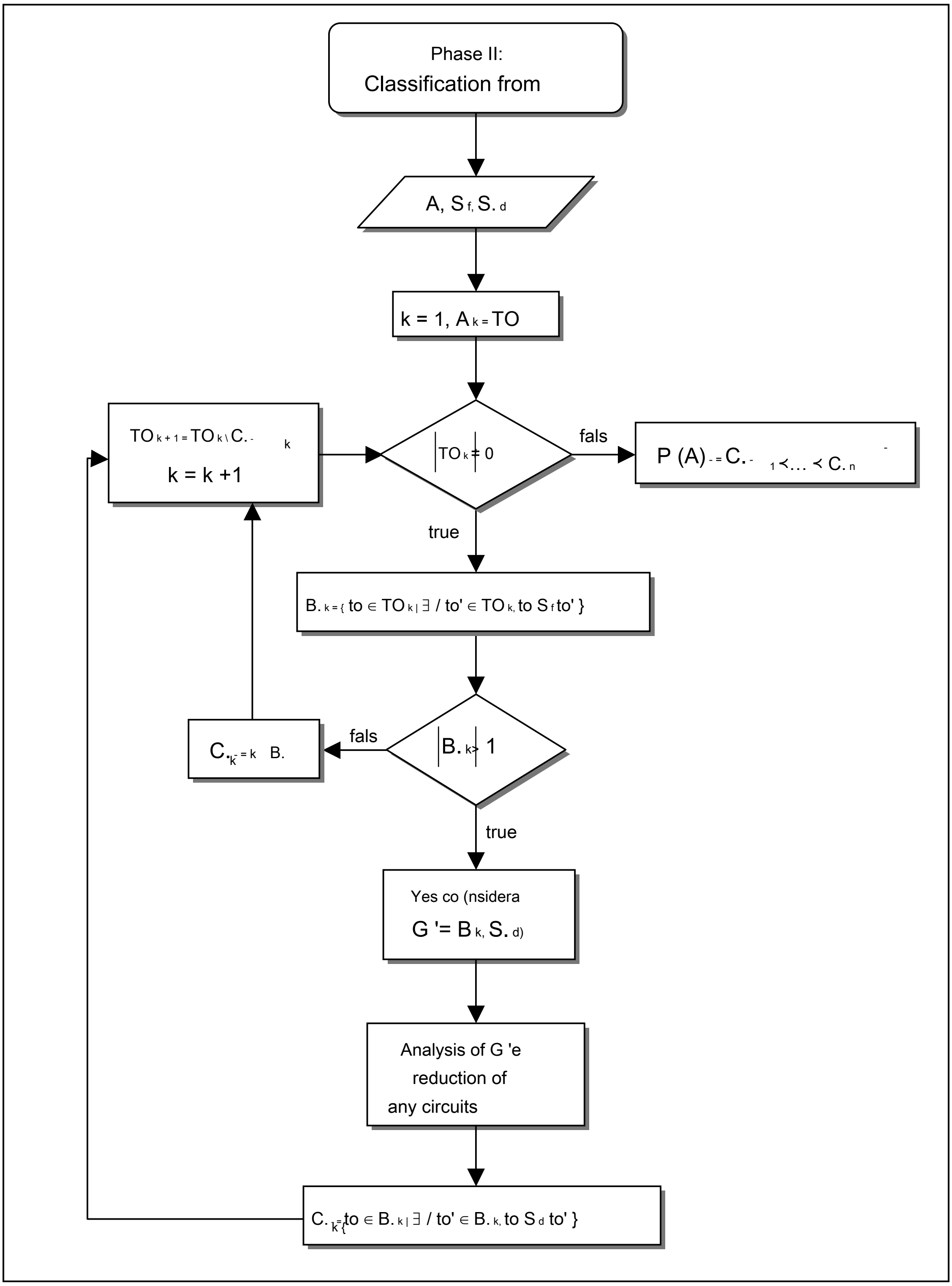


Figure 4: Construction of the pre-order from below P (A) - in ELECTRE II

Phase II of ELECTRE II

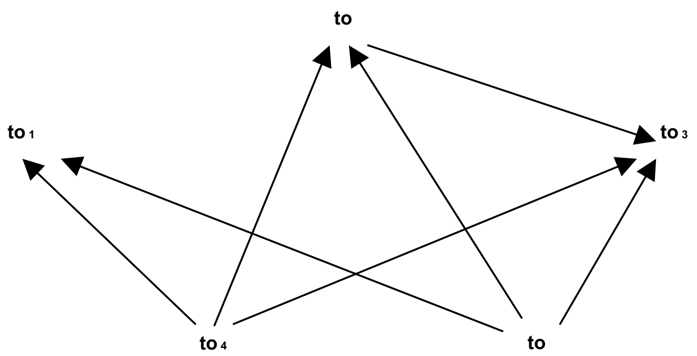
In order to apply the 'rules' for identifying the best or worst actions, used iteratively by ELECTRE II in the second phase of the method, it is necessary to verify that there are no circuits [2] on the graph. If they exist it is necessary to eliminate them by contracting the graph itself, if it is not possible to revise the model [3] and return to apply Phase I.

To sort shares in descending preference classes, 'strong' outclass is considered **S.**$_f$, obtained with the 'strong' concordance threshold (c_f), and then, possibly and locally, the 'weak' one **S.**$_d$, obtained with a 'weak' threshold (c_d).

The method involves the construction of a classification from above, P + (A), and one from below, P - (A), through the procedures outlined in Figures 3 and 4. The two results, expressed in the form of the complete preorder (structure which corresponds to the intuitive notion of classification with the possibility of ex æquo) or the complete order (which corresponds to the intuitive notion of classification without the possibility of ex æquo), can coincide and thus directly provide the final ranking of the actions. If the pre-orders or the complete orders of the actions do not coincide but are “sufficiently close”, the final partial pre-order is constructed as the intersection of the two complete pre-orders obtained with the two classification procedures from above and from below.

Application example of Phase II

Taking up the example introduced for Phase I, we start from the outclassing graph previously obtained, a graph that does not have circuits. The weak threshold adopted is equal to
0.65.



[2] A circuit is a path, that is, a sequence of arcs oriented in the same direction whose initial node coincides with the final node.

[3] The presence of a circuit may be an indication of the model's poor ability to clearly distinguish actions.

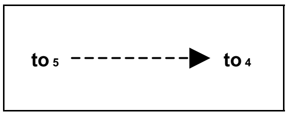
Top classification P (A) +

Iteration 1

$TO_1 = TO$

There are two elements of A not outclassed: $D^+_1 = \{to_4, to_5\}$.

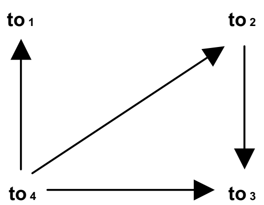
To try to distinguish them and assign only one to the first class from above, we pass to weak outclassing. to_5, which weakly outclasses a_4, it is the only element not outclassed, therefore it is attributable to the first class from above.



$C^+_1 = \{to_5\}$

Iteration 2

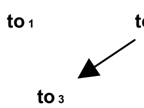
$TO_2 = TO_1 \setminus C^+_1 = \{to_1, to_2, to_3, to_4\}$



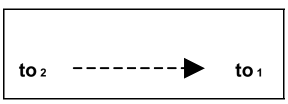
$C^+_2 = \{to_4\}$

Iteration 3

$TO_3 = TO_2 \setminus C^+_2 = \{to_1, to_2, to_3\}$



$D^+_3 = \{to_1, to_2\}$



$C^+_3 = \{to_2\}$

Iteration 4

$TO_4 = TO_3 \setminus C^+_3 = \{ to_1, to_3 \}$

to_1 **to_3** $D^+_4 = \{ to_1, to_3 \}$

$$\boxed{\mathbf{to_1} \dashrightarrow \mathbf{to_3}}$$

$C^+_4 = \{ to_1 \}$

$TO_5 = TO_4 \setminus C^+_4 = \{ to_3 \} \Rightarrow TO_5 = 1| \Rightarrow \$.$ TOP

At the end of the procedure, the result is the following:

$P(A)^+ = \{a_5\}\{ to_4\}\{ to_2 \}\{ to_1\}\{ to_3\}$.

Bottom classification P (A) -

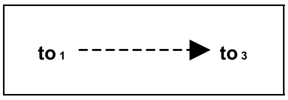
Iteration 1

We start from the initial outclassing graph.

$TO_1 = TO$

$D_1 = \{ to_1, to_3 \}$

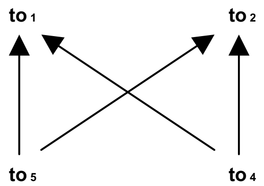
There is a weak outclass from to $_1$ towards a $_3$, therefore only a $_3$ it is unable to outperform other elements; it is therefore attributed to the first class from below.



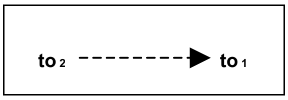
$C_1 = \{ to_3 \}$

Iteration 2

$TO_2 = TO_1 \setminus C_1 = \{ to_1, to_2, to_4, to_5 \}$



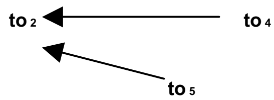
$D_2 = \{ to_1, to_2 \}$



$C_2 = \{ to_1 \}$

Iteration 3

$TO_3 = TO_2 \setminus C_2 = \{ to_2, to_4, to_5 \}$

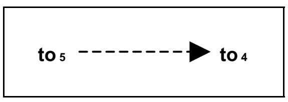


$C_3 = \{ to_2 \}$

Iteration 4

$TO_4 = TO_3 \setminus C_3 = \{ to_4, to_5 \}$

$D_4 = \{ to_4, to_5 \}$



$C_4 = \{ to_4 \}$

$TO_5 = TO_4 \setminus C_4 = \{ to_5 \} \Rightarrow A = 1 \Rightarrow S_{TOP}^{5}$

At the end of the procedure, the result obtained is represented here in the order from best to worst action:

P (A) $_{-}$ = { **to** $_5$} { **to** $_4$} { **to** $_2$ } { **to** $_1$} { **to** $_3$}.

A final ranking P (A) is easily obtained, since the two results P (A) + and P (A) $_{-}$ coincide.

2.2 Construction of the final partial pre-order

If P (A) + and P (A) $_{-}$ coincide, defining the final order is immediate. The problem arises when it is necessary to define a final partial pre-order starting from two pre-orders P (A) + and P (A) $_{-}$ which, in the most favorable case, can present two actions, placed in consecutive classes to each other, which swap positions, or such that different actions are "dancers", that is, they appear in distant classes in the two pre-orders . In this situation, a procedure is necessary that allows to obtain a final result without losing the information deriving from these different positions, an expression of incomparability between the actions themselves.

The procedure proposed in (Schärlig, 1996) to define a *final partial pre-order* it's a *intersection,* according to the meaning of set theory, which is based on the following rules:

- a final pre-order action cannot be placed before another unless it is before the latter in one of the two complete pre-orders-orders (P (A) + or P (A) $_{-}$) and before this one or ex æquo in the other;
- two shares cannot be ex æquo in the final pre-order unless they belong to the same class in both the top and bottom classifications;
- two actions are incomparable in the final preorder if one is before the other in one classification (either from above or from below) and follows it in the other.

The result can only be represented in the form of a graph. By examining the following two pre-orders it is possible to exemplify the proposed procedure:

P (A) + = {a $_3$, **to** $_7$} { **to** $_2$ } { **to** $_6$, **to** $_8$} { **to** $_5$} { **to** $_1$} { **to** $_4$}

P (A) $_{-}$ = { **to** $_7$} { **to** $_3$, **to** $_6$} { **to** $_5$} { **to** $_1$, **to** $_2$ } { **to** $_4$, **to** $_8$}.

With the first step, the application of the rules leads to the graph represented in figure 5.

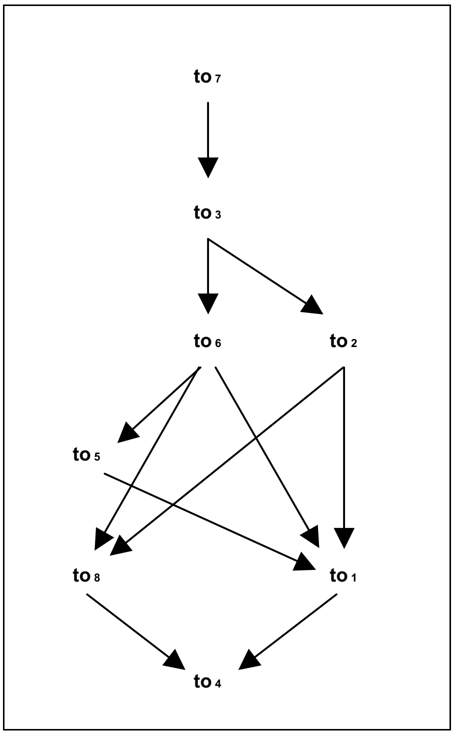


Figure 5: A first representation of the final partial pre-order

This graph is constructed by observing for each action how it is placed in relation to the others. For example, a $_7$ he is first class in both pre-orders and in P (A) + he is ex æquo with a $_{3,}$ consequently it can be placed at the head of the partial pre-order. After $_7$ is placed at $_3$ since once it is ex æquo with a $_7$ and the other is in second class. Then in P (A) + appears a $_2$ and then to $_{6,}$ together with a $_{8,}$ while in P (A) $_-$

appears at $_{6,}$ which is with a $_3$ in second class, a $_2$ it's at $_8$ appear in subsequent classes (a $_2$ in the fourth ea $_8$ even in the last). The two pre-orders are therefore quite different. However, wanting to build the final partial pre-order, from a $_3$ Yes
two distinct paths branch off, one towards a $_2$ and one verse a $_{6,}$ thereby indicating an incomparability between the two elements.
Instead, after noting that a $_6$ comes after a $_7$ in both pre-orders, the direct arrow from to is no longer drawn $_7$ to a $_{6,}$ because there is already a path, from one to the other of these two actions, which passes for a $_{3.}$ The procedure goes on similarly for the other actions.

The first result is sometimes not very legible, especially due to the crossing of the arrows. Then we can organize ourselves according to a simple principle: we look for the longest path, that is, the one that passes through the greatest number of actions, between the first and the last action. This path is then traced vertically, as shown in figure 6, and then the other paths are added.

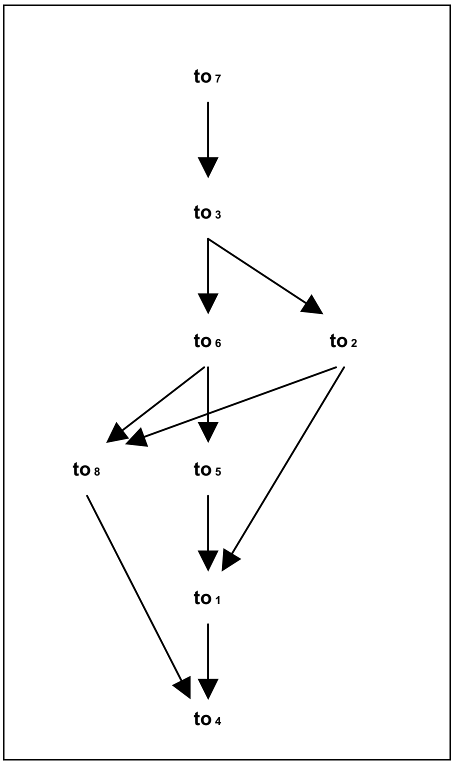


Figure 6: The definitive representation of the final order

This operation is not always immediate, as incomparable couples sometimes create considerable difficulties and it may even be necessary to resort to some conventions. For example we see, in the graph of figure 5, that a $_2$ is incomparable with a $_6$ and with a $_5$. The question is whether to draw a $_2$ at the height of a $_5$ or at the height of a $_6$. When only one action is “skipped” the problem does not arise, here instead the convention intervenes which establishes: 'you choose the highest level'. Arises then to $_2$ at the height of a $_6$ and, by the same agreement, a $_8$ at the level of a $_5$. The final partial pre-order highlights that two of the eight actions feature numerous incomparability with the others; it is a useful indication because it signals the presence of possible anomalies linked either to the inadequacy of the model or to the lack of information or to the fact that the actions are by their nature incomparable and this can be very important informationally, in relation to the type of decision to be made. to take.

2.3 Variant of Phase II

In more recent applications of ELECTRE II, the use of a variant of the second phase has become widespread. This variant starts from the observation that, if in the outclassing graph there are no circuits (or in any case if the existing circuits have been eliminated), it is possible to extract from A all the elements of a subset B $_1$ of elements characterized by the fact of " *not to be outclassed by any other element of A* ", Using strong outclassing. We can then proceed to define the subset B_2 of the remaining elements, such as " *not be outclassed by any other element* ". By repeating this procedure, it is easy to construct the partition of A: {$B_{1,}$ $B._{2,}$ …}.

On this basis we define what can be called one *coarse version* of the complete preorder (Roy and Bouyssou, 1993), placing all the elements of B at the head and in the position of ex æquo $_{1,}$ below (i.e. in second position) those of B_2 and so on. We then move on to examine whether it is possible to refine this pre-order, using the weak outclassing. This refinement consists in using the information that this less credible outclassing brings to distinguish the different elements of a subset B_j whenever this contains several.

In the first place it is a question of highlighting any circuits of $B._{j,}$ according to $S_{D.}$
(weak outclass), and then to determine the full preorder on $B._{j.}$ For the
pre-order from below, the argument is analogous, replacing " *not be outclassed by any other element* " the condition " *do not outclass any other element* ".

Other variants are used, one in particular in the numerous applications developed in Switzerland in predominantly environmental management problems (Maystre et al., 1994), and described in detail in (Schärlig, 1996). Particular mention should be made of the Regulations implementing the framework law on public works (Merloni ter Law), the full text of which is on

http://www.llpp.it/NuovoSito/legislativo/merloni_ter/ , where the use of the ELECTRE method is suggested for the calculation of the most economically advantageous offer and an application scheme is proposed (Annex B).

Exercise 2.4.1 - Advertising Campaign

For the launch of a product, a company has decided to carry out a campaign advertising and, for this purpose, has identified five sector magazines (a_1, to $_2$, to $_3$, to $_4$, to $_5$), which were evaluated against four criteria (the evaluations are reported in table E2.4.1). Indicate which magazine or magazines should be chosen for the advertising campaign. Each criterion is associated with a coefficient of relative importance (weights); we defined a pair of values in discordance (5.10) on " *Editorial context* "And one (3, 9) on the criterion" *Prestige* ".

Table E2.4.1: Evaluations and parameters

Editorial criteria	Context is	Cost	Regularity	Prestige
Weights	7	5	4	2
Magazines				
to $_1$	5	84	6	9
to $_2$	8	51	4	9
to $_3$	5	74	8	3
to $_4$	10	60	4	9
to $_5$	5	90	7	3
Ins. Disc.	(5.10)			(3, 9)

Remarks and suggestions for resolution

Reflect on the preferred direction of the criteria and the nature of the weights. Develop the calculations of the first phase, choosing a pair of thresholds. Go to the second step of the method, analyze the results and provide a final answer.

Phase I

Table IS 2.4.2: Phase I of E LECTR AND II

	J +	J =	J -	P + ≥ P -	P + + P =	V. ETO	S.
to 1 to 2							
to 1 to 3							
to 1 to 4							
to 1 to 5							
to 2 to 1							
to 2 to 3							
to 2 to 4							
to 2 to 5							
to 3 to 1							
to 3 to 2							
to 3 to 4							
to 3 to 5							
to 4 to 1							
to 4 to 2							
to 4 to 3							
to 4 to 5							
to 5 to 1							
to 5 to 2							
to 5 to 3							
to 5 to 4							

Result

At the end of the procedure, the result is the following:

P (A) + = {a $_{4}$} { to $_{2}$ } { to $_{1}$, to $_{3}$} { to $_{5}$}

P (A) $_{-}$ = { to $_{4}$} { to $_{2}$ } { to $_{1}$, to $_{3}$} { to $_{5}$}.

Since P (A) + ≡ P (A) $_{-}$, the final result coincides with the previous two. This means that the newspaper, to whom to preferably entrust the advertising campaign, is a $_{4}$. If unforeseen difficulties arise, it may be useful to contact a $_{2}$. The newspaper a $_{5}$ it is certainly the least reliable, while it is difficult to distinguish a $_{1}$ from to $_{3}$.

Example 2.4.2 - Waste Treatment Strategies

Having to adopt an overall urban waste treatment strategy capable of solving the problems of the entire area in the medium-long term, five criteria have been identified and the five different strategies have been evaluated on them.
to $_1$, to $_2$, to $_3$, to $_4$, to $_5$ (table E2.4.3). Considering a set of discrepancies on the "Environmental impact" criterion consisting of the pairs of levels with a difference equal to or
greater than 0.3, provide guidance to help make decisions about the problem.

Table E2.4.3: Evaluations and parameters

Criteria	Costs for year	Recovery energetic	Impact environmental is	Flexibility of the system	Reliability of the system
Unit of measure	million FS	GWh	level	degree	degree
Weights	0.25	0.20	0.15	0.20	0.20
Strategies					
to $_1$	45	122	0.5	10	9
to $_2$	18	- 75	0.7	5	6
to $_3$	35	95	0.3	6	5
to $_4$	32	88	0.5	5	7
to $_5$	35	95	0.6	7	5

Observation

The wording of the veto is worded differently than the example proposed in the text. In correspondence with the difference, it is necessary to define the pairs of values that make up the set of discordance.

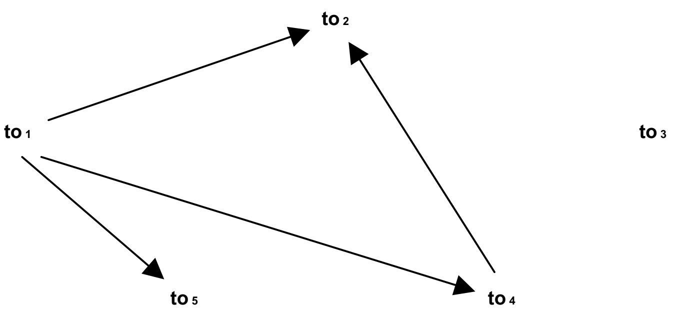
This example refers to a real case developed in Switzerland (Simos, 1990) with multiple criteria and strategies. Having reduced the number of criteria can lead to more uncertain results, because there are no elements of comparison between the actions.

Phase I

Table E2.4.4: Phase I of ELECTRE II

	J +	J =	J -	P + ≥ P. -	P + + P =	V ETO	S.
to 1, to 2	2, 3, 4, 5	/	1	Yup	0.75		to 1 S a 2
to 1, to 3	2, 4, 5	/	1, 3	Yup	0.60		
to 1, to 4	2, 4, 5	3	1	Yup	0.75		to 1 S a 4
to 1, to 5	2, 3, 4, 5	/	1	Yup	0.75		to 1 S a 5
to 2, to 1	1	/	2, 3, 4, 5	no			
to 2, to 3	1, 5	/	2, 3, 4	no		Yup	
to 2, to 4	1	4	2, 3, 5	no			
to 2, to 5	1, 5	/	2, 3, 4	no			
to 3, to 1	1, 3	/	2, 4, 5	no			
to 3, to 2	2, 3, 4	/	1, 5	Yup	0.55		
to 3, to 4	2, 3, 4	/	1, 5	Yup	0.55		
to 3, to 5	3	1, 2, 5	4	no			
to 4, to 1	1	3	2, 4, 5	no			
to 4, to 2	2, 3, 5	4	1	Yup	0.75		to 4 S a 2
to 4, to 3	1, 5	/	2, 3, 4	no			
to 4, to 5	1, 3, 5	/	2, 4	Yup	0.60		
to 5, to 1	1	/	2, 3, 4, 5	no			
to 5, to 2	2, 3, 4	/	1, 5	Yup	0.55		
to 5, to 3	4	1, 2, 5	3	Yup	0.85	Yup	
to 5, to 4	2, 4	/	1, 3, 5	no			

A strong threshold is adopted $c_{f} =$ 0.75 and the weak threshold to be adopted is 0.60, in the absence of higher values close to the natural threshold of 2/3.



Phase II

The outclassing graph has no circuits. The presence of an isolated node makes the classification procedure slower.

Top classification P (A) +

Iteration 1

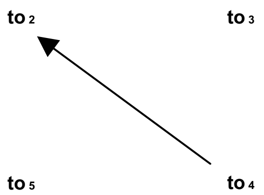
$TO_1 = TO$

$D^+_1 = \{to_1, to_3\}$

With the weak outclassing we obtain: $C^+_1 = \{to_1\}$.

Iteration 2

$TO_2 = TO_1 \setminus C^+_1 = \{to_2, to_3, to_4, to_5\}$



$D^+_2 = \{to_3, to_4, to_5\}$

With the weak outclassing we obtain: $C^+_2 = \{to_3, to_4\}$.

Iteration 3

$TO_3 = TO_2 \setminus C^+_2 = \{to_2, to_5\}$

$D^+_3 = \{to_2, to_5\}$

to 2 **to** 5 Even activating the weak override, the two actions cannot be distinguished: $C^+_3 = \{to_2, to_5\}$.

$TO_4 = TO_3 \setminus C^+_3 = \varnothing \Rightarrow$ S. TOP

At the end of the procedure, the result is:

P (A) + = $\{a_1\}\{to_3, to_4\}\{to_2, to_5\}$.

Bottom classification P (A) -

Iteration 1

$TO_1 = TO$

$D._1 = \{to_2, to_3, to_5\}$

Even activating the weak override, the two actions cannot be distinguished: $C._1 = \{to_2, to_3, to_5\}$.

Iteration 2

$TO_2 = TO_1 \setminus C._1 = \{to_1, to_4\}$

$\mathbf{to_1} \longrightarrow \mathbf{to_4}$ $C._2 = \{to_4\}$

$TO_3 = TO_2 \setminus C._2 = \{to_1\} \Rightarrow A = 1 \Rightarrow |S.^3| TOP$

The pre-order from below P (A) - therefore appears to be:

P (A) $_{-} = \{ \mathbf{to_1} \} \{ \mathbf{to_4} \} \{ \mathbf{to_2}, \mathbf{to_3}, \mathbf{to_5} \}$.

Result

to
↓
to
↓
to
↓
to $_{2,}$

P (A) + = $\{ \mathbf{a_1} \} \{ \mathbf{to_3}, \mathbf{to_4} \} \{ \mathbf{to_2}, \mathbf{to_5} \}$

P (A) $_{-} = \{ \mathbf{to_1} \} \{ \mathbf{to_4} \} \{ \mathbf{to_2}, \mathbf{to_3}, \mathbf{to_5} \}$

As the top and bottom pre-orders are similar (the only difference is that one presents to_3 and from $_4$ in the same position, while the other places them in two successive classes), we can define the final preorder (shown here on the side) without particular difficulties.

In the application it was adopted as a weak threshold $c_d = 0.60$, minimum acceptable value considering the data in the table. Appearing few values corresponding to 0.60 and many equal to 0.55, you can try to use this last value as a weak threshold, to carry out a robustness analysis of the result just obtained. To this end it becomes necessary to verify also what result would be obtained for 0.67; it is proposed as an exercise.

Exercise 2.4.3 - Location of a waste disposal plant

Possible locations of waste disposal plants were evaluated in relation to five criteria, as shown in table E2.4.5. Discrepancies on the
criterion g_1, corresponding to the pairs of extreme values on the scale, and on the criterion g_3,
for deviations equal to or greater than 100. It is requested to apply the ELECTRE II method and
propose a classification of possible sites from best to worst, starting from the results obtained.

Table E2.4.5: Evaluations and parameters

Criteria	g_1 (-)	g_2 (+)	g_3 (+)	g_4 (+)	g_5 (-)
Weights / Locations	0.21	0.18	0.20	0.16	0.25
to 1	Medium	2	100	Good	3
to 2	Medium-high	5	55	Sufficient is	2.6
to 3	Limited	1	55	Poor	4.2
to 4	High	5	180	Poor	5
to 5	Limited	4	50	Sufficient is	4.7
Ins. Disc.	(TO THE)		≥ 100		

Phase I

Table E2.4.6: Phase I of ELECTRE II

	J +	J =	J -	P + ≥ P. - P	+ + P = V ETO		S.
to $_1$ to $_2$	1, 3, 4	/	2, 5				
to $_1$ to $_3$	2, 3, 4, 5	/	1				
to $_1$ to $_4$	1, 4, 5	/	2, 3				
to $_1$ to $_5$							
to $_2$ to $_1$	2, 5	/	1, 3, 4				
to $_2$ to $_3$	2, 4, 5	3	1				
to $_2$ to $_4$	1, 4, 5	2	3				
to $_2$ to $_5$							
to $_3$ to $_1$	1	/	2, 3, 4, 5				
to $_3$ to $_2$	1	3	2, 4, 5				
to $_3$ to $_4$	1, 5	4	2, 3				
to $_3$ to $_5$							
to $_4$ to $_1$	2, 3	/	1, 4, 5				
to $_4$ to $_2$	3	2	1, 4, 5				
to $_4$ to $_3$	2, 3	4	1, 5				
to $_4$ to $_5$							
to $_5$ to $_1$							
to $_5$ to $_2$							
to $_5$ to $_3$							
to $_5$ to $_4$							

Result

to $_{1,}$

↓

to

↓

to

↓

to

P (A) + = {a $_{1,}$ to $_2$ } { to $_{3,}$ to $_4$} { to $_5$}

P (A) $_{-=\{}$ to $_{1,}$ to $_2$ } { to $_3$} { to $_{4,}$ to $_5$}

Exercise 2.4.4 - More information on the example given in the text

To make the indication provided by the obtained result more robust, one could move on to the sensitivity analysis of the result, in which it is advisable to check whether by varying the weak threshold the results do not change and then, by simulating an analysis carried out in interaction with the decision maker, define possible ranges of variability of the weights and then make them vary in a combined manner and compare the results.

2.5 ELECTRE III method

This method differs from ELECTRE II mainly because it uses pseudo-criteria, i.e. criteria to which elements of informational and preferential uncertainty can be associated, and because it therefore models, in the first phase of the method, an outclassing *nuanced (* or fuzzy, or flou), which associates a characteristic function to each relationship between ordered pairs of shares δ (a, a '), which expresses the *degree of credibility* of the outclassing relation and which can vary in the interval [0,1]. The outclassing graph resulting from Phase I of the method will therefore have all the nodes connected by evaluated arcs, which is associated with a degree of credibility of

outclassing; the incidence matrix associated with the graph will have as element m_{ij} the value assumed by δ (to $_{i,}$ to $_{j)}$. The procedure for defining δ (a, a ') is shown in Figure 7 and provides that several indices are calculated and obtained from these δ (a, a ').

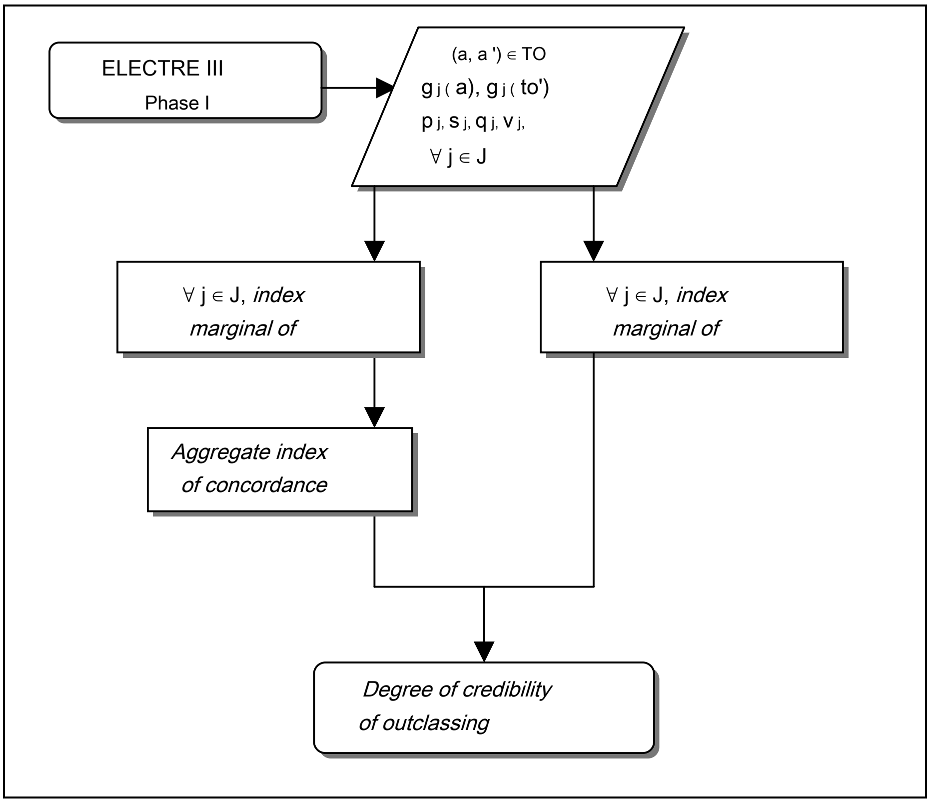


Figure 7: Procedure for defining the degree of credibility of the outclassing

In relation to a pseudo-criterion, a threshold of *preference (* $s_{j)}$, which serves to distinguish between a situation of preference in the strict sense and one of
presumption of preference, mixed relationship that includes the elementary relations of Indifference and weak preference (Q) without distinguishing them, and a threshold of
indifference ($q_{j)}$, which characterizes situations of information uncertainty in which it is impossible to distinguish (Indifference) between pairs of possible values.

To calculate the marginal concordance index $c_j(a, a')$, given the pair of shares (a, a'), the following situations may arise:

- $\forall j \in J$, if $g_j(to) \geq g_j(to')$, **to** marginally outclasses **to',** to S_j to' $\Rightarrow c_j(a, a') = 1$;

- self $g_j(a) < g_j(a')$, a marginal outclassing statement of **to** up **to',** and precisely, given $u = g_j(a') - g_j(a) > 0$, if $u \leq q_j[g_j(to')]$

  $\Rightarrow \quad c_j(a, a') = 1$,

  if $u \geq s_j[g_j(to')]$ $\Rightarrow \quad c_j(a, a') = 0$,

  if $q_j[g_j(a')] < u < s_j[g_j(to')] \Rightarrow c_j(a, a')$ decreases from 1 to 0

(for example linearly) as g increases $_j(a')$ from $[g_j(a') + q_j]$ a $[g_j(a') + s_j]$, as in figure 8.

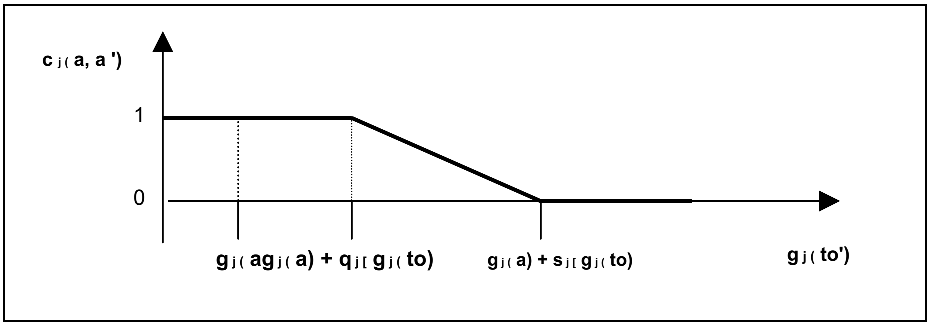


Figure 8: Marginal index of concordance

L' **aggregate concordance index c (a, a ')** is given by the following weighted sum:

$$c(a, a') = \sum_{j \in J} p_j \cdot c_j(a, a').$$

L' **marginal index of discrepancy $D_j(a, a')$** is a function of $g_j(a')$, monotonous not decreasing, which can assume values included in the interval [0,1] and, in particular, different from zero only when $c_j(a, a')$ is null.

A veto threshold was defined on the criterion (see $_j$),

$v_j[g_j(to)] \geq s_j[g_j(to)]$,

self $g_j(to') \geq g_j(a) + v_j[g_j(a)]$ $v_j[g_j(a)] > s_j[g_jD_j(a)]$ a_j $s_j[g_j(a')] < g_j(a') - g_j(a) < v_j[g_j$
self $(to')]$

$\Rightarrow$ $D_j(a, a')$ grows from 0 to 1 (for example linearly) as g increases $_j(a')$ from $[g_j(a') + s_j]$ a $[g_j(a') + v_j]$, as shown in figure 9.

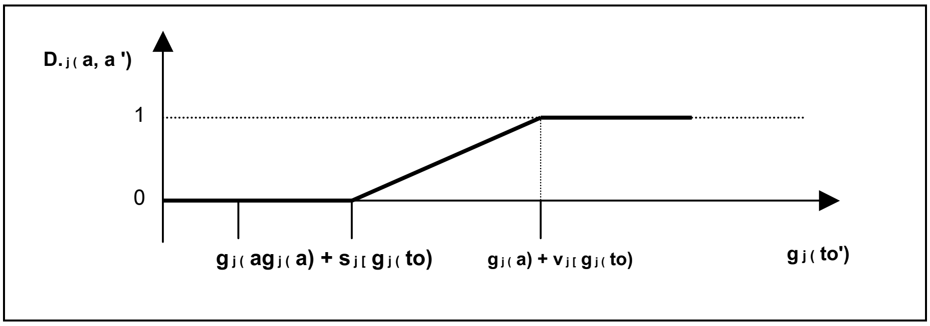


Figure 9: Marginal index of discordance

The **degree of credibility** δ (a, a '), known these indices, is calculated as shown below.

Self $\forall$ j $\in$ JD $_{j(}$ a, a ') = 0 assumes δ (a, a ') = c (a, a');

self $\exists$ j $\in$ J: D $_{j(}$ a, a ')> 0 then:

- if D $_{j(}$ a, a ') <c (a, a') assumes δ (a, a ') = c (a, a'),
- self $\exists$ j * $\in$ J * $\subseteq$ J: D $_{j^*(}$ a, a ') ≥ c (a, a ') is assumed

$$\delta(a, a') = c(a, a') \cdot \prod_{j^* \in J^*} \frac{1 - \text{From to'})}{1 - c(a, a')} .$$

Example

Five alternatives of a set A were evaluated, as reported in the table 3, on a pseudo-criterion g $_j$, with ladder IS $_{j=[}$ 1, 4]. To develop a calculation example, the concordance and discordance indices of the outclassing of a are examined $_1$ and of a $_3$ compared to a $_{the}$ and of a $_{the}$ compared to a $_1$ and to a $_3$, on criterion g $_j$, on which they are defined as thresholds q (a) = 15% g $_{j(}$ to) is s (a) = v (a) = 30% g $_{4(}$ to) .

Table 3: Evaluation of alternatives on criterion g $_j$

Actions	to $_1$	to $_2$	to $_3$	to $_4$	to $_5$
g $_{j(}$ to)	1.35	1.54	1.86	2.05	2.78

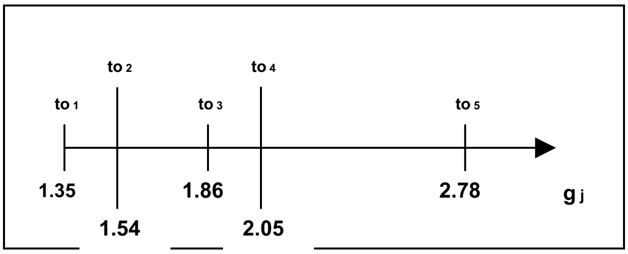


Figure 10: Position of the actions according to criterion g_j

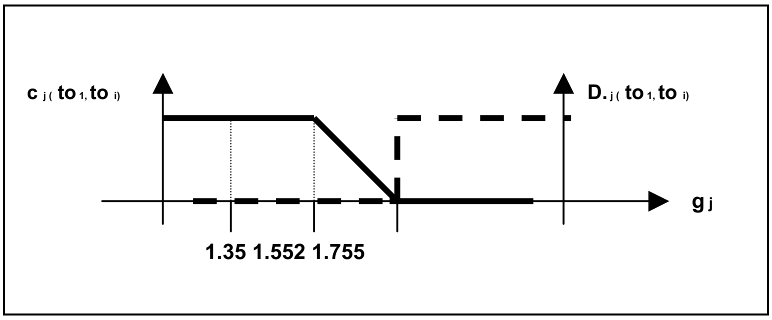


Figure 11: Position of the thresholds with respect to $g_j(to_1)$

In figure 11 the values shown correspond to the following thresholds:

$1.35 = g_j(to_1)$

$1.5525 = g_j(to_1) + q(a_1) = g_j(to_1) + 0.15 \cdot g_j(to_1) = 1.35 + 0.15 \cdot 1.35$

$1.755 = g_j(to_1) + s(a_1) = g_j(to_1) + \text{it goes}_1) = g_j(to_1) + 0.3 \cdot g_j(to_1) = 1.35 + 0.3 \cdot 1.35$.

Table 4: Thresholds and indices for pairs (to_1, to_i)

	$c_j(to_1, to_i)$	$D_j(to_1, to_i)$
to_2	1	0
to_3	0	1
to_4	0	1
to_5	0	1

Table 5: Thresholds and indices for pairs (to_{i}, to_{1})

	$g_{j}(to_{i}) + 0.15 \cdot g_{j}(to_{i})$	$g_{j}(to_{i}) + 0.3 \cdot g_{j}(to_{i})\, c_{j}(to_{i}$	$to_{1})\, D._{j}(to_{i}, to_{1})$	
to_{2}	1,771	2.002	1	0
to_{3}	2.139	2.418	1	0
to_{4}	2.3575	2.665	1	0
to_{5}	3.197	3.614	1	0

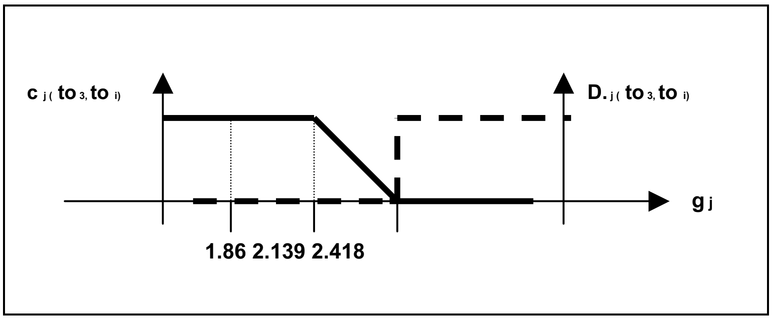


Figure 12: Position of the thresholds with respect to $g_{j}(to_{3})$

In figure 12 the values shown correspond to the following thresholds:

$1.86 = g_{j}(to_{3})$

$2.139 = g_{j}(to_{3}) + q(a_{3}) = g_{j}(to_{3}) + 0.15 \cdot g_{j}(to_{3}) = 1.86 + 0.15 \cdot 1.86$

$2.418 = g_{j}(to_{3}) + s(a_{3}) = g_{j}(to_{3})$ + it goes $_{3)} = g_{j}(to_{3}) + 0.3 \cdot g_{j}(to_{3}) = 1.86 + 0.3 \cdot 1.86$.

Table 6: Thresholds and indices for pairs (to_{3}, to_{i})

	$c_{j}(to_{3}, to_{i})$	$D._{j}(to_{3}, to_{i})$
to_{1}	1	0
to_{2}	1	0
to_{4}	1	0
to_{5}	0	1

Table 7: Thresholds and indices for pairs (to_{i}, to_{3})

	$g_{j}(to_{i}) + 0.15 \cdot g_{j}(to_{i})\, g_{j}(to$	$_{i)} + 0.3 \cdot g_{j}(to_{i})\, c_{j}(to_{i}, to_{3})\, D._{j}($	$to_{i}, to_{3})$	
to_{1}	1.5525	1,755	0	1
to_{2}	1,771	2.002	0.615	0
to_{4}	2.3575	2.665	1	0
to_{5}	3.197	3.614	1	0

The index $c_{j}(to_{2}, to_{3})$ was obtained from the proportion x: 1 = 0.142: 0.231, based on two similar right-angled triangles, as seen in Figure 12.

Phase II of ELECTRE III

ELECTRE III also foresees, in order to obtain the global ordering of the actions, the construction of classifications both from above and from below. A *distillation algorithm* which builds classes of preference by activating in succession lower and lower levels of credibility, said *of separation,* which allow to consider, at each level, only the significant relationships for the examined level. These levels depend on the *threshold of*

discrimination s (δ), that is, the maximum distance between two credibility that allows them to still be considered of the same order of magnitude. The distillation algorithm distinguishes actions based on theirs *qualification,* which is greater 'if the action has the capacity to outclass many others and not to be outclassed or to be outclassed by a few'. *δ- qualification,* defined as follows for each iteration of the procedure in both top and bottom sorting, which ends when there is no longer any element in the set A of the actions or when

achieved a level of separation equal to zero ($\delta_1 = 0$).

In distillation, both from above and from below, the first value to be identified is the *maximum degree of credibility* δ_0, identifiable in the matrix

$$\delta_0 = \max_{(a, a') \in TO_k} \delta(a, a').$$

Subtract from this the discrimination threshold s (δ) and so it is calculated $\boldsymbol{\delta'_0}$:

$$\delta'_0 = \delta_0 - s(\delta).$$

δ'_0 allows you to identify the first *level of separation* δ_1, ie the highest degree of credibility, present in the matrix, lower than δ'_0; if there is no longer any value in the matrix, the separation level is set equal to zero.

$$\delta_1 = \begin{cases} \max\limits_{(a, a') \in \Omega} \delta(a, a'), \text{ with } \Omega = \{(a, a') \mid \delta(a, a') < \delta'_0\} \neq \varnothing \\ 0, \text{ if } \Omega = \varnothing \end{cases}$$

where is it Ω is the subset of all pairs of shares with a lower degree of credibility to δ'_0.

Compared to δ_1, you can calculate the δ_1- qualification of each action.

For *δ- qualification* $\mathbf{q^{\delta}_{B}(to)}$ of a share “a”, belonging to a set B generic, we mean the value

$$q^{\delta}_{B}(a) = p^{\delta}_{B}(to) - d^{\delta}_{B}(to),$$

where is it $\mathbf{p^{\delta}_{B}(to)}$ and the *δ- power* of the share “a”, ie the number of shares in B outclassed from "to" with a degree of credibility higher than δ (the current level of separation) and with a difference, between the relative degrees of credibility, greater than the discrimination threshold:

$$p^{\delta}_{B}(a) = |\{a' \in B: \delta(a, a') > \delta \text{ is } \delta(a, a') - \delta(a', a) > s(\delta)\}|,$$

and where $d_{\delta B}$ (**to)** and the δ- weakness of the share "a", ie the number of shares in B that outclass "a", with a degree of credibility higher than δ (the current level of separation) and with a difference, between the relative degrees of credibility, greater than the discrimination threshold:

$$d_{\delta B}(a) = \{a' \in B: \delta(a', a) > \delta \text{ is } \delta(a', a) - \delta(a, a') > s(\delta)\}.$$

The top distillation algorithm classifies the actions according to the maximum qualification, according to the rule:

$$q^{+} = \max_{to \in TO_k} q_{\delta 1}(to), \qquad D^{+}_{1} = \{to \in A: q_{\delta 1}(a) = q^{+}\}$$

where is it D^{+}_{1} it will be there *first distillate from above* and each class C^{+}_{k} it will be built starting from above on this distillate; so far as D^{+}_{1} contains only one action, then yes poses $C^{+}_{k} = D^{+}_{1}$, otherwise the procedure described up to now is repeated, but applied to actions included in D^{+}_{1}, until it contains only an action or until it reaches null separation level. The set of actions for the next iteration is

$$TO_{k+1} = TO_k \setminus C^{+}_{k}.$$

For bottom distillation, the procedure is similar to the previous one; the selection, however, is made on the basis of the minimum qualification according to the rule:

$$q^{-} = \min_{to \in TO_k} q_{\delta 1}(to), \qquad D^{-}_{1} = \{to \in A: q_{\delta 1}(a) = q^{-}\}.$$

In this case D^{-}_{1} it will be there *first distillate from below,* and each class C^{-}_{k} Sara built starting from the bottom. The set of actions for the next iteration is

$$TO_{k+1} = TO_k \setminus C^{-}_{k}.$$

Obtained the two pre-orders P (A) + and P (A) - from the distillation algorithms, we will proceed as in ELECTRE II to identify the final pre-order P (A).

Example

Table 8 shows the incidence matrix associated with the outclassing graph obtained with the first phase of the ELECTRE III method. The degrees of credibility of the outclassing δ (a, a ') of the matrix are to be used as input for Phase II of ELECTRE III, in which the distillation algorithm allows to obtain a ranking (i.e. an ordering of the shares in distinct positions or classes; we will speak of pre-order if some shares appear on equal merit in the same position or class) both from above (from best to worst) and from below (from worst to best). Distillation is therefore activated twice, the first - from above - in search of the best actions, the second - from below - in search of the worst.

Table 8: Credibility degrees of outclassing

$\delta(a, a')$	to 1	to 2	to 3	to 4	to 5	to 6
to 1	-	1	0.60 0.75 0.30			0
to 2	0.50	-	0.90 0.65 0.90 0.65			
to 3	0	0.30	-	0.65 0.75 0.15		
to 4	0.50 0.85 0.75			-	1	0.75
to 5	0	0	0.75 0.10		-	0.30
to 6	0.15 0.80 0.10 0.30 0.80					-

Development with the threshold $s(\delta) = 0.15$

Distillation from above

We start from the whole A = {a_1, to $_2$, to $_3$, to $_4$, to $_5$, to $_6$}.

Iteration 1

$TO_1 = TO$

The highest degree of credibility δ_0 identifiable in table 4:

$$\delta_0 = \max_{(a, a') \in TO_1} \delta(a, a') = 1.$$

We subtract the discrimination threshold $s(\delta)$ and so it is calculated δ'_0:

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.15 = 0.85.$

δ'_0 allows to calculate the first separation level δ_1 for this iteration:

$$\delta_1 = \begin{cases} \max_{(a, a') \in \Omega} \delta(a, a'), \text{ with } \Omega = \{(a, a') \mid \delta(a, a') < \delta'_0\} \neq \varnothing. \\ 0, \text{ if } \Omega = \varnothing \end{cases}$$

Ω is the subset of all pairs of values such that their degree of credibility is less than δ'_0.

From the matrix, proposed in table 8, it appears that

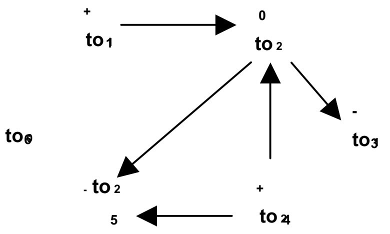
$\delta_1 = 0.80.$

With respect to this first level of separation, the outclassing graph is created for the actions belonging to A. $_1$, inserting an arc oriented from "a" towards "a '" between the pairs (a, a') for which

$\delta(a, a') > \delta_1$ is $\delta(a, a') - \delta(a', a) > s(\delta)$.

This oriented arc represents the outclassing between the two actions at the agreed separation level.

The qualification "q" of an action is expressed as
q = (number of outgoing arcs) - (number of incoming arcs) and indicated by the degree associated with each action node of the graph.



The action from the highest qualification is a $_4$, which is placed in the first class from above.

$C^+_1 = \{ to_4 \}$

Iteration 2

The set of actions for this iteration is obtained from the starting one, having excluded the action, or actions, just attributed to the first class from above.

$TO_2 = TO_1 \setminus C^+_1 = \{ to_1, to_2, to_3, to_5, to_6 \}$

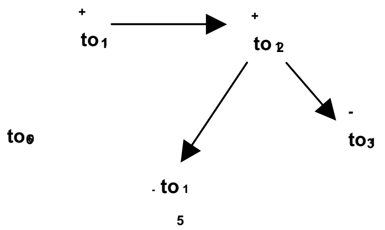
δ (a, a ')	to 1	to 2	to 3	to 5	to 6
to 1	-	1	0.60	0.30	0
to 2	0.50	-	0.90	0.90	0.65
to 3	0	0.30	-	0.75	0.15
to 5	0	0	0.75	-	0.30
to 6	0.15	0.80	0.10	0.80	-

$\delta_0 = \max_{(a, a') \in TO_2} \delta (a, a ') = 1$

$\delta'_0 = \delta_0 - s (\delta) = 1 - 0.15 = 0.85$

$\delta_1 = 0.80$

Based on this level of separation, the following outclassing graph.



There are two shares of the highest qualification, a $_1$ and from $_2$.

$D^+_2 = \{ to_1, to_2 \}$

$|D^+_2| \neq 1$

It is necessary to carry out a sub-distillation between these two actions. It starts from δ_1 and is taken from him the discrimination threshold.

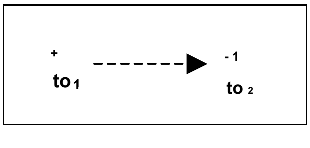
$\delta_1 = 0.80$

$\delta_1' = \Delta_1 - s (\delta) = 0.80 - 0.15 = 0.65$

Based on this value, the separation level for the sub-distillation between a $_1$ it's at $_2$ is

$\delta_2 = 0.50$.

With respect to this level, the outclassing graph is drawn between a $_1$ it's at $_2$.



The action with the highest qualification is a $_1$.

$C^+_2 = \{ to_1 \}$

Iteration 3

$TO_3 = TO_2 \setminus C^+_2 = \{to_2, to_3, to_5, to_6\}$

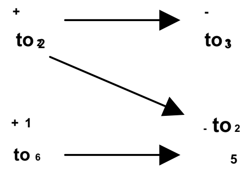
δ (a, a ')	to 2	to 3	to 5	to 6
to 2	-	0.90	0.90	0.65
to 3	0.30	-	0.75	0.15
to 5	0	0.75	-	0.30
to 6	0.80	0.10	0.80	-

$\delta_0 = \max_{(a, a') \in TO_3} \delta(a, a') = 0.90$

$\delta'_0 = \delta_0 - s(\delta) = 0.90 - 0.15 = 0.75$

$\delta_1 = 0.65$

It should be noted that the difference between the degrees relative to couple (a_3, to_5) and that of the couple (a_5, to_3) is null, while the one between the degrees relative to the couple (a_6, to_2) is equal to the discrimination threshold.



$C^+_3 = \{to_2\}$

Iteration 4

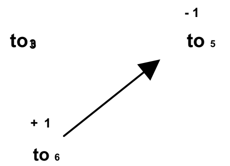
$TO_4 = TO_3 \setminus C^+_3 = \{to_3, to_5, to_6\}$

δ (a, a ')	to 3	to 5	to 6
to 3	-	0.75	0.15
to 5	0.75	-	0.30
to 6	0.10	0.80	-

$\delta_0 = \max_{(a, a') \in TO_4} \delta(a, a') = 0.80$

$\delta'_0 = \delta_0 - s(\delta) = 0.80 - 0.15 = 0.65$

$\delta_1 = 0.30$



$C^+_4 = \{to_6\}$

Iteration 5

$TO_5 = TO_4 \setminus C^+_4 = \{to_3, to_5\}$

δ (a, a ')	to 3	to 5
to 3	-	0.75
to 5	0.75	-

$\delta_0 = \max_{(a, a') \in TO_5} \delta(a, a') = 0.75$

$\delta'_0 = \delta_0 - s(\delta) = 0.75 - 0.15 = 0.60$

$\delta_1 = 0$

Since the two degrees of credibility present a zero gap, the two actions end up in the same class.

to 3 to 5

$C^+_5 = \{to_3, to_5\}$

Iteration 6

$TO_6 = TO_5 \setminus C^+_5 = \varnothing \Rightarrow$ S. TOP

At the end of the procedure, it is possible to write the result P (A) + of the distillation from above, which sorts the actions from best to worst:

P (A) + = {a $_{4}$} { to $_{1}$} { to $_{2}$ } { to $_{6}$} { to $_{3,}$ to $_{5}$}.

Bottom distillation

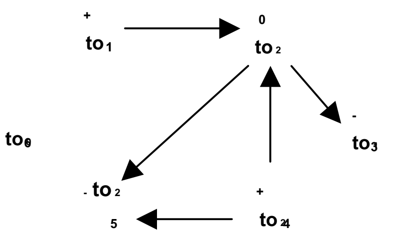
Now, following the bottom-up distillation procedure, at each iteration the action with the minimum qualification is sought. It starts again from
A = {a $_{1,}$ to $_{2,}$ to $_{3,}$ to $_{4,}$ to $_{5,}$ to $_{6}$} and the first iteration coincides with that of distillation from above.

Iteration 1

$\delta_0 = \max_{(a, a') \in TO_1} \delta(a, a') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.15 = 0.85$

$\delta_1 = 0.80$



$C^-_1 = \{$ to $_5\}$

Iteration 2

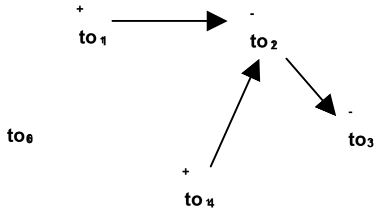
$TO_2 = TO_1 \setminus C^-_1 = \{$ to $_{1,}$ to $_{2,}$ to $_{3,}$ to $_{4,}$ to $_{6}\}$

δ (a, a ')	to 1	to 2	to 3	to 4	to 6
to 1	-	1	0.60	0.75	0
to 2	0.50	-	0.90	0.65	0.65
to 3	0	0.30	-	0.65	0.15
to 4	0.50	0.85	0.75	-	0.75
to 6	0.15	0.80	0.10	0.30	-

$\delta_0 = \max_{(a, a') \in TO_1} \delta(a, a') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.15 = 0.85$

$\delta_1 = 0.80$



$D_2 = \{to_2, to_3\}$

$|D_2| \neq 1$

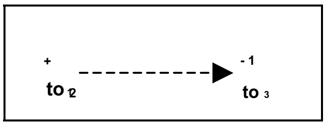
There is a need for under-distilling between these actions.

$\delta_1 = 0.80$

$\delta_1' = \Delta_1 - s(\delta) = 0.80 - 0.15 = 0.65$

$\delta_2 = 0.30$

With respect to this second level of distillation, the following outclassing graph is drawn.



$C_2 = to \{\}$

Iteration 3

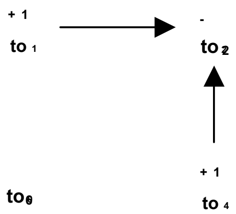
$TO_3 = TO_2 \setminus C_2 = \{to_1, to_2, to_4, to_6\}$

δ (a, a ')	to 1	to 2	to 4	to 6
to 1	-	1	0.75	0
to 2	0.50	-	0.65	0.65
to 4	0.50	0.85	-	0.75
to 6	0.15	0.80	0.30	-

$\delta_0 = \max_{(a, a') \in TO_3} \delta(a, a') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.15 = 0.85$

$\delta_1 = 0.80$



$C_3 = \{to_2\}$

Iteration 4

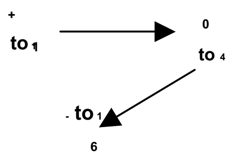
$TO_4 = TO_3 \setminus C_3 = \{to_1, to_4, to_6\}$

δ (a, a ')	to 1	to 4	to 6
to 1	-	0.75	0
to 4	0.50	-	0.75
to 6	0.15	0.30	-

$\delta_0 = \max_{(a, a') \in TO_4} \delta(a, a') = 0.75$

$\delta'_0 = \delta_0 - s(\delta) = 0.75 - 0.15 = 0.60$

$\delta_1 = 0.50$



$C_4 = \{to_6\}$

Iteration 5

$TO_5 = TO_4 \setminus \bar{C}_4 = \{ to_1, to_4 \}$

$\delta(a, a')$	to_1	to_4
to_1	-	0.75
to_4	0.50	-

$\delta_0 = \max_{(a, a') \in TO_4} \delta(a, a') = 0.75$

$\delta' = \delta_0 - s(\delta) = 0.75 - 0.15 = 0.60$

$\delta_1 = 0.50$

$\overset{+}{to_1} \longrightarrow \overset{-}{to_4}$ $\quad \bar{C}_5 = \{ to_4 \}$

Iteration 6

$TO_6 = TO_5 \setminus \bar{C}_5 = \{ to_1 \} \Rightarrow |TO_6| = 1 \Rightarrow \bar{C}_6 = \{ to_1 \} \Rightarrow$ S. TOP

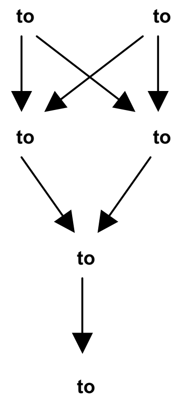
At the end of the procedure, the result is

P (A) $\bar{ } = \{ \mathbf{to}_1 \} \{ \mathbf{to}_4 \} \{ \mathbf{to}_6 \} \{ \mathbf{to}_2 \} \{ \mathbf{to}_3 \} \{ \mathbf{to}_5 \}$.

Result

$P(A)^{+} = \{a_4\}\{to_1\}\{to_2\}\{to_6\}\{to_3, to_5\}$

$P(A)^{-} = \{to_1\}\{to_4\}\{to_6\}\{to_2\}\{to_3\}\{to_5\}$



The orders are different at the top, while at the bottom one has two ex æquo shares and the other classifies them. This implies that the final pre-order, the result of which is reported here at side, presents an incomparability in the lead between a_1 and from $_4$; an other incomparability is achieved in the central part between the actions a_2 and from $_6$, that swap places in the two systems.

Procedure with threshold $s(\delta) = 0.10$

Table 9: Credibility degrees of outclassing

$\delta(a, a')$	to_1	to_2	to_3	to_4	to_5	to_6
to_1	-	1	0.60	0.75	0.30	0
to_2	0.50	-	0.90	0.65	0.90	0.65
to_3	0	0.30	-	0.65	0.75	0.15
to_4	0.50	0.85	0.75	-	1	0.75
to_5	0	0	0.75	0.10	-	0.30
to_6	0.15	0.80	0.10	0.30	0.80	-

Distillation from above

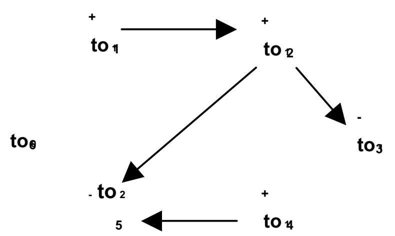
Iteration 1

$TO_1 = TO$

$\delta_0 = \max_{(a, a') \in TO_1} \delta(a, a') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.10 = 0.90$

$\delta_1 = 0.85$



Three actions result from the highest qualification: a_1, to_2 and from $_4$.

$D^{+}_{1} = \{to_1, to_2, to_4\}$

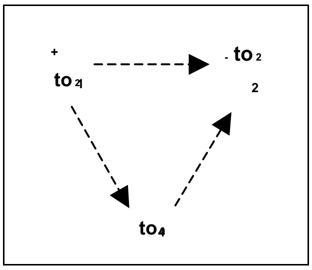
$|D^{+}_{1}| \neq 1 \Rightarrow$ Sub-distillation is required between these actions.

$\delta_1 = 0.85$

$\delta_1' = \Delta_1 - s(\delta) = 0.85 - 0.10 = 0.75$

$\delta_2 = 0.65$

δ_2 is the second separation level, which is now used for the outclassing graph in the subdistillation.



$C^+_1 = \{ to_1 \}$

Iteration 2

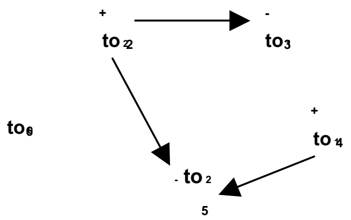
$TO_2 = TO_1 \setminus C^+_1 = \{ to_2, to_3, to_4, to_5, to_6 \}$

δ (a, a ')	to 2	to 3	to 4	to 5	to 6
to 2	-	0.90	0.65	0.90	0.65
to 3	0.30	-	0.65	0.75	0.15
to 4	0.85	0.75	-	1	0.75
to 5	0	0.75	0.10	-	0.30
to 6	0.80	0.10	0.30	0.80	-

$\delta_0 = \max_{(a, a') \in TO_2} \delta (a, a ') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.10 = 0.90$

$\delta_1 = 0.85$



$C^+_2 = \{ to_2 \}$

Iteration 3

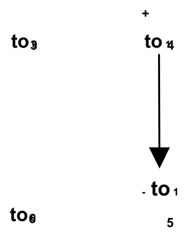
$TO_3 = TO_2 \setminus C^+_2 = \{ to_3, to_4, to_5, to_6 \}$

δ (a, a ')	to 3	to 4	to 5	to 6
to 3	-	0.65	0.75	0.15
to 4	0.75	-	1	0.75
to 5	0.75	0.10	-	0.30
to 6	0.10	0.30	0.80	-

$\delta_0 = \max_{(a, a') \in TO_3} \delta (a, a ') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.10 = 0.90$

$\delta_1 = 0.80$



$C^+_3 = \{ to_4 \}$

Iteration 4

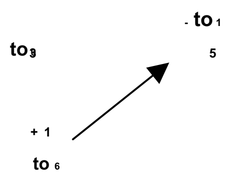
$TO_4 = TO_3 \setminus C^+_3 = \{to_3, to_5, to_6\}$

$\delta(a, a')$	to_3	to_5	to_6
to_3	-	0.75	0.15
to_5	0.75	-	0.30
to_6	0.10	0.80	-

$\delta_0 = \max_{(a, a') \in TO_4} \delta(a, a') = 0.80$

$\delta'_0 = \delta_0 - s(\delta) = 0.80 - 0.10 = 0.70$

$\delta_1 = 0.30$



Among the actions a_3 and from $_5$ it is not possible to distinguish an outclass, as the degrees of credibility relative to theirs reciprocal outclassings have the same value.

$C^+_4 = \{to_6\}$

Iteration 5

$TO_5 = TO_4 \setminus C^+_4 = \{to_3, to_5\}$

$\delta(a, a')$	to_3	to_5
to_3	-	0.75
to_5	0.75	-

$\delta_0 = \max_{(a, a') \in TO_5} \delta(a, a') = 0.75$

$\delta'_0 = \delta_0 - s(\delta) = 0.75 - 0.15 = 0.60$

$\delta_1 = 0$

Since the two degrees of credibility present a zero gap, the two actions end up in the same class.

$C^+_5 = \{to_3, to_5\}$

Iteration 6

$TO_6 = TO_5 \setminus C^+_5 = \varnothing \Rightarrow$ S. TOP

At the end of the procedure, the distillation result is written, ordering the actions from best to worst:

$$\mathbf{P(A)^+ = \{a_1\}\{to_2\}\{to_4\}\{to_6\}\{to_3, to_5\}.}$$

Bottom distillation

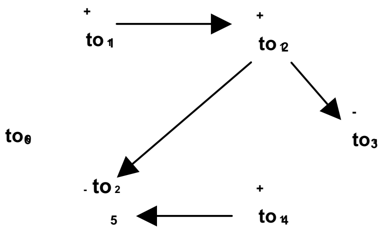
Iteration 1

$TO_1 = TO$

$\delta_0 = \max_{(a, a') \in TO_1} \delta(a, a') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.10 = 0.90$

$\delta_1 = 0.85$



$C_1 = \{to_5\}$

Iteration 2

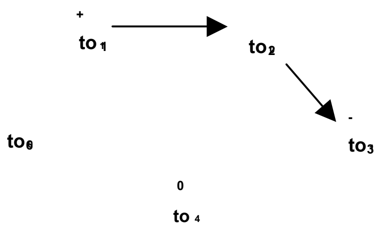
$TO_2 = TO_1 \setminus C_1 = \{to_1, to_2, to_3, to_4, to_6\}$

δ (a, a ')	to 1	to 2	to 3	to 4	to 6
to 1	-	1	0.60	0.75	0
to 2	0.50	-	0.90	0.65	0.65
to 3	0	0.30	-	0.65	0.15
to 4	0.50	0.85	0.75	-	0.75
to 6	0.15	0.80	0.10	0.30	-

$\delta_0 = \max_{(a, a') \in TO_2} \delta(a, a') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.10 = 0.90$

$\delta_1 = 0.85$



$C_2 = \{to_3\}$

Iteration 3

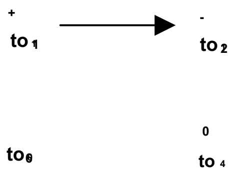
$TO_3 = TO_2 \setminus C_2 = \{to_1, to_2, to_4, to_6\}$

δ (a, a ')	to 1	to 2	to 4	to 6
to 1	-	1	0.75	0
to 2	0.50	-	0.65	0.65
to 4	0.50	0.85	-	0.75
to 6	0.15	0.80	0.30	-

$\delta_0 = \max_{(a, a') \in TO_3} \delta(a, a') = 1$

$\delta'_0 = \delta_0 - s(\delta) = 1 - 0.10 = 0.90$

$\delta_1 = 0.85$



$C_3 = \{to_2\}$

Iteration 4

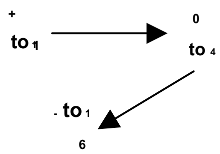
$TO_4 = TO_3 \setminus \bar{C}_3 = \{to_1, to_4, to_6\}$

δ (a, a ')	**to 1**	**to 4**	**to 6**
to 1	-	0.75	0
to 4	0.50	-	0.75
to 6	0.15	0.30	-

$\delta_0 = \max_{(a, a') \in TO_4} \delta(a, a') = 0.75$

$\delta'_0 = \delta_0 - s(\delta) = 0.75 - 0.10 = 0.65$

$\delta_1 = 0.50$



$\bar{C}_4 = \{to_6\}$

Iteration 5

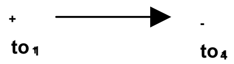
$TO_5 = TO_4 \setminus \bar{C}_4 = \{to_1, to_4\}$

δ (a, a ')	**to 1**	**to 4**
to 1	-	0.75
to 4	0.50	-

$\delta_0 = \max_{(a, a') \in TO_5} \delta(a, a') = 0.75$

$\delta'_0 = \delta_0 - s(\delta) = 0.75 - 0.10 = 0.65$

$\delta_1 = 0.50$



$\bar{C}_5 = \{to_4\}$

Iteration 6

$TO_6 = TO_5 \setminus \bar{C}_5 = \{to_1\} \Rightarrow$ STOP

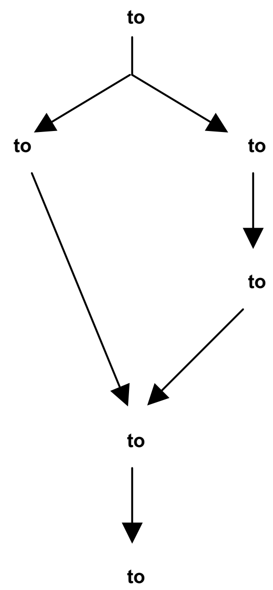
At the end of the procedure, the distillation result is determined, ordering the actions from worst to best:

P (A) $_- = \{to_1\}\{to_4\}\{to_6\}\{to_2\}\{to_3\}\{to_5\}$.

Result

P (A) + = {a $_{1\}\{}$ **to** $_{2\}\{}$ **to** $_{4\}\{}$ **to** $_{6\}\{}$ **to** $_{3,}$ **to** $_{5\}}$

P (A) $_{-=\{}$ **to** $_{1\}\{}$ **to** $_{4\}\{}$ **to** $_{6\}\{}$ **to** $_{2\}\{}$ **to** $_{3\}\{}$ **to** $_{5\}}$



The orderings obtained differ in the position of a $_{2,}$ which in distillation from above outclasses a $_4$ and from $_{6,}$ while in the one from below it is outclassed. The end result, that it springs from it, it is traced here alongside and shows the incomparability of action a $_2$ with a $_4$ and from $_{6.}$

Comment on the final rankings obtained with the two thresholds

As can be seen, the reduction of the discrimination threshold implies a refinement of distillation, which is shown here with a better perception of the incomparability between the candidate shares: in fact, with s (δ) = 0.15, a $_2$ it is incomparable only with a $_{6,}$ while with s (δ) = 0.10 is added to the incomparability with a $_{4.}$ Furthermore, the reduction of the threshold eliminated the incompatibility between a $_1$ and from $_{4.}$

In other cases (not in this one), refinement leads to a classification of the former æquo, which appear with a threshold s (δ) = 0.15.

2.6 Exercises proposed by ELECTRE III - Phase II Exercise

2.6.1

In table E2.6.1, the results of a modeling of outclass on a set of five projects, A = {a $_1$, to $_2$, to $_3$, to $_4$, to $_{5}$}. It is asked to construct a ranking of the elements of A on A, using a threshold constant s (δ) = 0.10.

Table E2.6.1: G. sparse of cred the bility del sur class ament

δ (a, a ')	to $_1$	to $_2$	to $_3$	to $_4$	to $_5$
to $_1$	-	0.76	0.85	0.41	0.27
to $_2$	0.61	-	0.35	0.50	0.61
to $_3$	0.50	0.37	-	0.81	0.90
to $_4$	0.27	0.42	0.31	-	0.52
to $_5$	0.69	0.60	0.32	0.45	-

Pre-order from above P (A) +

P (A) + = {a $_{1\}}$ { to $_{3\}}$ { to $_2$, to $_4$, to $_{5\}}$

Pre-order from below P (A) -

P (A) $_{-=\{}$ to $_4$, to $_{5\}}$ ≺ { to $_2$, to $_{3\}}$ ≺ { to $_{1\}}$

Result

to $_1$

↓

to $_3$

↓

to $_2$

↓

to $_4$, to $_5$

Since the two distillates are similar, we can define the pre-order depicted without particular problems of incomparability; to $_4$ and from $_5$ they are in the same class both in the pre-order from above and in the one from below.

Exercise 2.6.2

Table E2.6.2 shows the modeling results of the fuzzy outclass on a set of seven designs, A = {a_1, to_2, to_3, to_4, to_5, to_6, to_7}.
We are asked to construct the final partial preorder on A, using a constant threshold of discrimination s (δ) = 0.10.

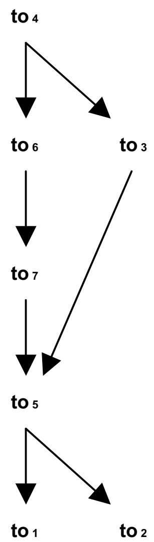
Table E2.6.2: Credibility degrees of outclassing

δ (a, a ')	to_1	to_2	to_3	to_4	to_5	to_6	to_7
to_1	-	0.30	0.27	0.52	0.41	0.30	0.20
to_2	0.41	-	0.15	0.20	0.25	0.30	0.50
to_3	0.15	0.20	-	0.78	0.60	0.10	0.10
to_4	0.37	0.80	0.70	-	0.90	0.88	0.50
to_5	0.56	0.45	0.25	0.15	-	0.60	0.50
to_6	0.35	0.40	0.37	0.20	0.40	-	0.80
to_7	0.83	0.90	0.60	0.10	0.76	0.20	-

Result

P (A) + = {a_4} { to_6} { to_7} { to_3} { to_5} { to_2 } { to_1}

P (A) - = { to_3, to_4} { to_6} { to_7} { to_5} { to_1} { to_2 }



P (A) + and P (A) - they are different and give rise to a final partial pre-order that presents some incomparability.