P223 - A2

ISAAC PREYSER

1. find eigenvalues + associated eigenvectors for Pauli Matrices:

a) $\sigma_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_I |\lambda\rangle = \lambda |\lambda\rangle$: for $|\lambda\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \overset{?}{=} \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\Rightarrow$ $\left.\begin{matrix} c_1 = \lambda c_1 \\ c_2 = \lambda c_2 \end{matrix}\right\}$ $\lambda = 1$;

all vectors $\vec{v} \in \mathbb{C}^2$ (work shown on next page.)
are eigenvectors of σ_I for $\lambda = 1$.

b) $\sigma_x \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\Rightarrow$ $\left.\begin{matrix} c_2 = \lambda c_1 \\ c_1 = \lambda c_2 \end{matrix}\right\}$ $\begin{matrix} \lambda c_1 - c_2 = 0 \\ c_1 - \lambda c_2 = 0 \end{matrix}$ $\Rightarrow \lambda^2 - 1 = 0$; $\therefore$ $\lambda = \pm 1$.

for $\lambda = 1$: we search for vector $\vec{v}$ s.t.

$$(\sigma_x - \lambda I)\vec{v} = \vec{0} .$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} v_2 - v_1 = 0 \\ v_1 - v_2 = 0 \end{matrix}\right\} \underline{v_1 = v_2},$$

to be normalized, $\langle v | v \rangle = 1$, hence $v_1 = v_2 = e^{i\phi} \cdot \frac{1}{\sqrt{2}}$, for arbitrary ϕ.
by convention, $\phi \equiv 0$.

$$\boxed{\therefore \quad \vec{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \text{ for } \lambda = 1.$$

for $\lambda = -1$:

$$(\sigma_x - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $v_1 + v_2 = 0$ $\Rightarrow$ $\vec{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, //

$$i \cdot -i = -(i^2) = 1$$

c) $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\det(\sigma_y - \lambda I) = 0$$

$$\det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm 1.$$

$\lambda = 1$:

$$(\sigma_y - \lambda I)\vec{v} = \vec{0} = \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} -v_1 - i v_2 = 0 \\ i v_1 - v_2 = 0 \end{matrix}\right\} -v_1 = i v_2 \Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix} = \vec{v}$$

for $\vec{v}$ to be normalized, take $\vec{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ for $\lambda = 1$. //

$\lambda = -1$:

$$\vec{v}(\sigma_y - \lambda I) = \vec{0} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} v_1 - i v_2 = 0 \\ i v_1 + v_2 = 0 \end{matrix}\right\} \begin{matrix} v_1 = i v_2 \\ i v_1 = -v_2 \end{matrix} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \text{ for } \lambda = -1. //$$

d) $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\det(\sigma_z - \lambda I) = 0 = \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\lambda = 1$: $(\sigma_z - \lambda I)\vec{v} = \vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow v_2 \cdot -2 = 0$
v_1 free. $\therefore \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, for $\lambda = 1$ //

$\lambda = -1$: $(\sigma_z - \lambda I)\vec{v} = \vec{0} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow 2v_1 = 0$
v_2 free. $\therefore \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, for $\lambda = -1$. //

a) find eigenvectors for σ_I ; $\lambda = 1$.

$\lambda = 1$: $(\sigma_I - \lambda I)\vec{v} = \vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

v_1, v_2 free variables.

$\Rightarrow$ $\therefore$ all vectors $\in \mathbb{C}^2$ are eigenvectors.

P223
A2.

2. A is normal iff. $A^{\dagger}A = AA^{\dagger}$.

a) Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. then, $A^{\dagger} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

$$AA^{\dagger} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^{\dagger}A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$AA^{\dagger} \neq A^{\dagger}A \Rightarrow$ Not Normal.

b) first we find eigenvalues of A:

$$\det(A - \lambda I) = 0 = \det\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2$$
$$= (-(\lambda-1))^2 = (\lambda-1)^2$$
$$\Rightarrow \lambda = 1, \text{ w/ multiplicity } 2.$$

* Recall: Similar matrices share a characteristic polynomial, ∴ have same eigenvalues & algebraic multiplicities.

∴ the only diagonal matrix D able to satisfy $A \sim D$ is:

$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So, if A is diagonizable, then we must have

$A = P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}P^{-1}$ where P is an invertible $n \times n$ matrix, by defn of similarity & diaganalization.

Calculating the R.S. of the eqn, we get:

$$P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}P^{-1} = PIP^{-1} = (1)\cdot PP^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

but $I \neq A$. ∴ A is not diagonalizable.

3. $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cdot e^{i\phi} \end{bmatrix}$

$\Rightarrow$ Show that $|\psi\rangle$ is represented by

$$\hat{n} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$$ when plotted in the Bloch sphere.

Recall:

$$\hat{n} = \begin{pmatrix} \langle\psi|X|\psi\rangle \\ \langle\psi|Y|\psi\rangle \\ \langle\psi|Z|\psi\rangle \end{pmatrix}$$ for pauli matrices X, Y, Z.

$\therefore$ we must show $\langle\psi|X|\psi\rangle = \cos\phi\sin\theta$,
$\langle\psi|Y|\psi\rangle = \sin\phi\sin\theta$,
$\langle\psi|Z|\psi\rangle = \cos\theta$.

i) Hence:

$$\begin{bmatrix} \cos\frac{\theta}{2} & \sin\left(\frac{\theta}{2}\right)\cdot e^{-i\phi} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\left(\frac{\theta}{2}\right)\cdot e^{i\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} & \sin\left(\frac{\theta}{2}\right)e^{-i\phi} \end{bmatrix} \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \cos\theta/2 \end{bmatrix}$$

$$= \cos\frac{\theta}{2}\sin\left(\frac{\theta}{2}\right)e^{i\phi} + \cos\frac{\theta}{2}\sin\left(\frac{\theta}{2}\right)e^{-i\phi}$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{i\phi} + e^{-i\phi}\right)$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(\cos\phi + \cancel{i\sin\phi} + \cos\phi - \cancel{i\sin\phi}\right)$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cdot 2\cos\phi$$

$$= \left(\frac{1}{\cancel{2}}\left(\sin\theta + \cancel{\sin 0}^{\,0}\right)\right)\cdot\cancel{2}\cos\phi$$ by product-to-sum

$$= \sin\theta\cos\phi //$$

ii) $\langle\psi|Y|\psi\rangle = \left[\cos\frac{\theta}{2} \quad e^{-i\phi}\sin\frac{\theta}{2}\right]\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}$

$= \left[\cos\frac{\theta}{2} \quad \sin\left(\frac{\theta}{2}\right)e^{-i\phi}\right]\begin{bmatrix}-i\sin\left(\frac{\theta}{2}\right)e^{i\phi}\\ i\cos\frac{\theta}{2}\end{bmatrix}$

$= -i\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} + i\cos\frac{\theta}{2}\sin\left(\frac{\theta}{2}\right)e^{-i\phi}$

$= i\cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(-e^{i\phi}+e^{-i\phi}\right)$

$= \frac{i\sin\theta}{2}\left(e^{-i\phi}-e^{i\phi}\right)$

$= \frac{i\sin\theta}{2}\left(\cos\phi - i\sin\phi - (\cos\phi + i\sin\phi)\right)$

$= \frac{i\sin\theta}{2}\left(\cancel{\cos\phi} - i\sin\phi - \cancel{\cos\phi} - i\sin\phi\right)$

$= \frac{\cancel{i}\sin\theta}{\cancel{2}}\left(\cancel{-2}\cancel{i}\sin\phi\right)$

$= \sin\theta\sin\phi$ //

iii) $\langle\psi|Z|\psi\rangle = \left[\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}e^{-i\phi}\right]\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix}$

$= \left[\cos\frac{\theta}{2} \quad \sin\left(\frac{\theta}{2}\right)e^{-i\phi}\right]\begin{bmatrix}\cos\theta/2\\ -\sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}$

$= \left(\cos\frac{\theta}{2}\right)^2 + \sin\left(\frac{\theta}{2}\right)e^{-i\phi} \cdot -\sin\left(\frac{\theta}{2}\right)e^{i\phi}$

$= \cos^2\frac{\theta}{2} + -\sin^2\left(\frac{\theta}{2}\right) \cdot \cancelto{1}{\left(e^{-i\phi} \cdot e^{i\phi}\right)}$

$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= \cos\theta$ //

by i), ii), iii) we demonstrate that

$$\hat{n} = \begin{pmatrix}\langle\psi|X|\psi\rangle\\ \langle\psi|Y|\psi\rangle\\ \langle\psi|Z|\psi\rangle\end{pmatrix} = \begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix}$$

When plotted in the Bloch Sphere.

$$R_z(\gamma) = e^{-i\frac{\gamma}{2}Z}.$$

4.

a) As Z is diagonal, we can do the following:

$$e^{-i\frac{\gamma}{2}Z} = \begin{bmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{bmatrix} //$$

b) let $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

$$\therefore \quad |\psi'\rangle = R_z(\gamma)\,|\psi\rangle = \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\gamma}{2}} \cdot \psi_1 \\ e^{i\gamma/2} \cdot \psi_2 \end{pmatrix} //$$

$\Rightarrow$ Now to plot in Bloch sphere, following Q3 defn of $|\psi\rangle$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$$

b)

$$\therefore \quad |\psi'\rangle = R_z(\gamma)\,|\psi\rangle = \begin{pmatrix} e^{-i\gamma/2} \cdot \cos\frac{\theta}{2} \\ e^{i\gamma/2} \cdot \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$$

$$\begin{pmatrix} \langle\psi'|X|\psi'\rangle \\ \langle\psi'|Y|\psi'\rangle \\ \langle\psi'|Z|\psi'\rangle \end{pmatrix} = \hat{n}'..$$

i) $\langle \psi' | X | \psi' \rangle = \left[e^{i\gamma/2} \cos\left(\frac{\theta}{2}\right) \quad e^{-i\gamma/2} \cdot e^{-i\phi} \sin\left(\frac{\theta}{2}\right) \right] \ldots$

$$\cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} e^{-i\gamma/2} \cos\left(\frac{\theta}{2}\right) \\ e^{i\gamma/2} \cdot e^{i\phi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \left[e^{i\gamma/2} \cos\left(\frac{\theta}{2}\right) \quad e^{-i\left(\frac{\gamma}{2}+\phi\right)} \sin\left(\frac{\theta}{2}\right) \right] \begin{bmatrix} e^{i\left(\frac{\gamma}{2}+\phi\right)} \sin\left(\frac{\theta}{2}\right) \\ e^{-i\frac{\gamma}{2}} \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= e^{i(\gamma+\phi)} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) + e^{-i(\gamma+\phi)} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)$$

$$= \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \left(e^{i(\gamma+\phi)} + e^{-i(\gamma+\phi)} \right)$$

$$= \frac{\sin\theta}{2} \left(\cos(\gamma+\phi) + \cancel{i \sin(\gamma+\phi)} + \cos(\gamma+\phi) - \cancel{i \sin(\gamma+\phi)} \right)$$

$$= \frac{\sin\theta}{\cancel{2}} \cdot \cancel{2} \cos(\gamma+\phi)$$

$$= \sin\theta \cos(\gamma+\phi)$$

ii) $\langle \psi' | Y | \psi' \rangle = \left[e^{i\gamma/2}\cos\left(\frac{\theta}{2}\right) \quad e^{-i\gamma/2} \cdot e^{-i\phi}\sin\left(\frac{\theta}{2}\right) \right] \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{bmatrix} e^{-i\gamma/2}\cos\left(\frac{\theta}{2}\right) \\ e^{i\gamma/2} \cdot e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$

$$= \left[e^{i\gamma/2}\cos\left(\frac{\theta}{2}\right) \quad e^{-i\gamma/2} \cdot e^{-i\phi}\sin\left(\frac{\theta}{2}\right) \right] \begin{bmatrix} -i e^{i(\gamma/2 + \phi)} \sin\left(\frac{\theta}{2}\right) \\ i e^{-i(\gamma/2)} \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= -i e^{i(\gamma+\phi)} \sin\theta/2 \cos\theta/2 + i e^{-i(\gamma+\phi)} \cos\theta/2 \sin\theta/2$$

$$= \frac{i\sin\theta}{2}\left(-e^{i(\gamma+\phi)} + e^{-i(\gamma+\phi)} \right)$$

$$= \frac{i\sin\theta}{2}\left(e^{-i(\gamma+\phi)} - e^{i(\gamma+\phi)} \right)$$

$$= \frac{i\sin\theta}{2}\left(\cancel{\cos(\gamma+\phi)} - i\sin(\gamma+\phi) - \cancel{\cos(\gamma+\phi)} - i\sin(\gamma+\phi) \right)$$

$$= \sin\theta \sin(\gamma+\phi)$$

iii) $\langle \psi' | Z | \psi' \rangle = \left[e^{i\gamma/2} \cos\left(\frac{\theta}{2}\right) \;\; e^{i\gamma/2} e^{-i\phi} \sin\left(\frac{\theta}{2}\right) \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} \cos\left(\frac{\theta}{2}\right) \\ e^{i(\frac{\gamma}{2}+\phi)} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$

$$= \left[e^{i\gamma/2} \cos\left(\frac{\theta}{2}\right) \;\; e^{-i(\frac{\gamma}{2}+\phi)} \sin\left(\frac{\theta}{2}\right) \right] \begin{bmatrix} e^{-i\gamma/2} \cos\left(\frac{\theta}{2}\right) \\ -e^{i(\gamma/2+\phi)} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= e^{i(\gamma/2)} \cos\left(\frac{\theta}{2}\right) e^{-i(\gamma/2)} \cos\left(\theta/2\right) - e^{-i(\gamma/2+\phi)} \sin\left(\frac{\theta}{2}\right) e^{i(\frac{\gamma}{2}+\phi)} \sin\left(\frac{\theta}{2}\right)$$

$$= \cos^2\frac{\theta}{2} \left(\cancelto{1}{e^{i\gamma/2} \cdot e^{-i\gamma/2}} \right) - \sin^2\frac{\theta}{2} \left(\cancelto{1}{e^{-i(\frac{\gamma}{2}+\phi)} \cdot e^{i(\frac{\gamma}{2}+\phi)}} \right)$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= \cos\theta$$

by i), ii), iii), we derive $\hat{n}'$:

$$\hat{n}' = \begin{pmatrix} \langle \psi' | X | \psi' \rangle \\ \langle \psi' | Y | \psi' \rangle \\ \langle \psi' | Z | \psi \rangle \end{pmatrix} = \begin{pmatrix} \sin\theta \cos(\phi+\gamma) \\ \sin\theta \sin(\phi+\gamma) \\ \cos\theta \end{pmatrix} .$$

$\Rightarrow$ this suggests R_z is a function that shifts the azimuthal angle by γ.

☆ find attached plots displaying this transformation.