Regression Discontinuity Designs (RDD)

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1 Introduction

Oftentimes the policies and interventions that we are interested in evaluating are not randomly assigned, but rather delivered based on an observable threshold. In many cases, these thresholds are set by administrators or policymakers who roll out certain programs. In other words, units are sorted into treatment and control groups based on whether they meet or fail to meet some cut-off score (e.g., a jobs training program may be given to individual that exceed some cutoff measure of needs).

Although the "gold standard" of empirical research is the still randomized control trial, the inherent nature of some programs and policies require that we use some other method to estimate the effect of the intervention. To answer such questions, we can rely on the regression discontinuity design. In short, the regression discontinuity design (RDD) estimates the impact of a policy around the eligibility cutoff. As such, rather than estimating average treatment effects (ATE), the RDD answers treatment effects among those at or around the treatment cutoff point. RDD is particularly strong in internal validity because in essence, units on either side of the cutoff point are thought to be very similar to each other in their characteristics and only differ in their treatment status, so "jumps" in the outcome variable lends itself a very credible estimate of the intervention's treatment effect.

Consider a setting in which a sample of units each have some observed score of a continuous variable X, and that the treatment status for a unit i, D_i, is solely determined by whether X_i is above some threshold score of c. That is, $D_i = \mathbb{1}\{X_i > c\}$. We can now specify the following simplified model:

$$Y_i = \alpha + \tau D_i + \beta X_i + u_i \tag{1}$$

This equation effectively constructs two lines on either side of the cutoff point (this can take non-linear/functional forms): one representing the untreated units (intercept of α and slope of β for $X \leq c$) and the other representing the treated units (intercept of ($\alpha + \tau$) and a slope of β for $X > c$). Here, the cutoff point c represents the intercept of both regression lines, and the "jump" from one line to the other is τ, where τ is the effect of the intervention for those at the cutoff score (and hence a local average treatment effect). In short, we are interested in making the comparison between $\lim_{x \to c} \mathbb{E}[Y_i|X_i = x]$ and $\lim_{x \leftarrow c} \mathbb{E}[Y_i|X_i = x]$.

1.1 A Note on "Fuzzy" RDDs

In the sharp RD, it was the case that those above the treatment all received the treatment and those below it did not (X is deterministic of D). In a fuzzy regression discontinuity setting, as is in many RD settings, while there may be a discontinuity of treatment at the cutoff, there may be incomplete treatment compliance around the cutoff. In other words, rather than having a discrete jump in treatment status, the *probability* of treatment assignment changes as you cross into another side of the cutoff. In such a case, we have a "fuzzy" regression discontinuity design.

Unlike in sharp RD where the probability of treatment jumps from 0 to 1 at the cutoff score, in fuzzy RDs the probability jumps by less than one. In fuzzy RDDs, we only require that the conditional probability of treatment status, in the limit, is discontinuous as X approaches the cutoff:

$$\lim_{\epsilon \downarrow 0} \Pr(D = 1|X = c + \epsilon) \neq \lim_{\epsilon \uparrow 0} \Pr(D = 1|X = c + \epsilon)$$

The treatment effect can be recovered by dividing the jump in the relationship between Y and X at c by the portion of units induced to take-up the treatment at the threshold (i.e., "compliers"). Thus, the discontinuity is equivalent to an instrument for D (Lee and Lemieux, 2010). The treatment effect, then, can be expressed as:

$$\tau_{Fuzzy} = \frac{\lim_{\epsilon \downarrow 0} \mathbb{E}[Y|X = c + \epsilon] - \lim_{\epsilon \uparrow 0} \mathbb{E}[Y|X = c + \epsilon]}{\lim_{\epsilon \downarrow 0} \mathbb{E}[Y|D = c + \epsilon] - \lim_{\epsilon \uparrow 0} \mathbb{E}[D|X = c + \epsilon]} \tag{2}$$

Note that the above equation takes similar form to the Wald estimator. In the two-stage least squares framework, it is important to note that the estimating equations are as follows:

$$Y = \alpha + \tau D + f(X - c) + \varepsilon \tag{3}$$
$$D = \gamma + \delta T + g(X - c) + \upsilon, \tag{4}$$

In Equation 4, the dummy variable $T = \mathbb{1}\{X > c\}$ indicates whether the assignment variable exceeds the treatment eligibility threshold of c. When we substitute the Equation 4 into the outcome equation, we get the following reduced form equation:

$$Y = \alpha_r + \tau_r T + f_r(X - c) + \varepsilon_r, \tag{5}$$

where $\tau_r = \tau\delta$ and represents the intent-to-treat effect (ITT). In fuzzy RDs, estimation can be done with local linear regressions or polynomial regression approaches. 2SLS estimates are identical to the ratio of reduced form coefficients, τ_r/δ.[1]

2 Simulation & Reproduction Exercises

In this section I provide two uses of RDD, each of which representing either the sharp RD case or the fuzzy RD case. The purpose of this section is to help readers understand the intuition behind RDDs and how they are estimated. The accompanying STATA do-file and R code can be found in the same project folder with this write-up.

2.1 Sharp RD

In this simulation exercise, I randomly generate observations for sharp RD. I create a variable $X \sim N(50, 25)$ which will be used as the running variable. I then define an arbitrary cutoff score $c = 50$, and the treatment variable $D = \mathbb{1}\{X > c\}$. The outcome variable, Y, takes similar form to Equation (1), where I set the population parameters $\alpha = 25, \tau = 40$, and $\beta = 1.5$. Table 1 provides a summary table of the simulated data. In Figure 1, I create a scatterplot of outcome variable and running variable to demonstrate the sharp cutoff in outcome between both sides of the cutoff c:

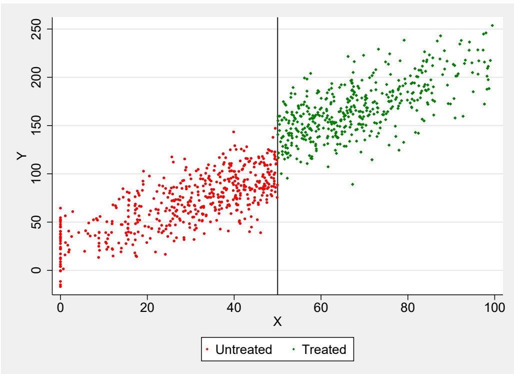
[1] Lee and Lemieux (2010) note that the same bandwidth be used for equations (4) and (5) in the local linear regression case, and that the same order of polynomials be used for $g(\cdot)$and $f(\cdot)$ in the polynomial regression case.

Table 1: Summary Table of Select Variables

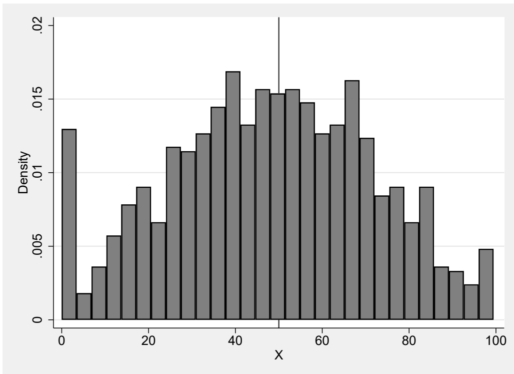
	mean	sd	min	max
Y	116.4353	56.15138	-16.7757	253.6825
D	.4736298	.4995625	0	1
X	48.03058	23.61583	0	99.45224
u	.4442312	20.51423	-76.7875	58.60476
N	967			

As can be seen in Figure 1a, we can see that there is a relatively sharp discontinuity in the outcome Y when X is either side of the cutoff $c = 50$. From simple visualization we can see that there is a noticeable jump between treated and untreated groups and that units in either group do not cross the other side of the cutoff. However, in some cases, it is plausible that individuals (or even program administrators) may *manipulate* treatment status.[2] In cases where there is "bunching around the cutoff", it is not appropriate to use the RD design since estimates would be biased. We can, however, perform another *density test* to see if there is successful treatment implementation by looking at the distribution of units. If the number of observations surprisingly different from either end of cutoff, it may perhaps be indicative of individuals (or some other administrative factor) manipulating treatment status. In Figure 1b we see that although is a slight jump in density from either end of the cutoff score, the general distribution of the running variable follows a bell-curve shape:

Figure 1: Sharp RD Plots



(a) Discontinuity in Outcome Y



(b) Distribution of Running Variable X

We now turn to obtaining our sharp-RD regression results. In this exercise, I use local polynomial regression using the STATA package *rdrobust*, however there exists a plethora of options available in STATA, R, and Python. Using the simulated data, I provide in Table 2 regression results from several alternative specifications.[3] In Figure 2, I also provide graphs that plot the regression results from Columns 1, 3, 5, and 7 with 95% confidence intervals.

As seen in Table 2, it is evident that the choice of polynomial order and bandwidth length (the range of values of X around the cutoff point which units are considered for analysis) is crucial to obtain credible treatment effect estimates. The researcher must exercise caution when choosing bandwidths and polynomial orders. Although higher-order polynomial terms may result in fitted values that better track the tendencies of our data, they can also lead to data over-fitting and can introduce bias to treatment effect estimates. In addition, narrow

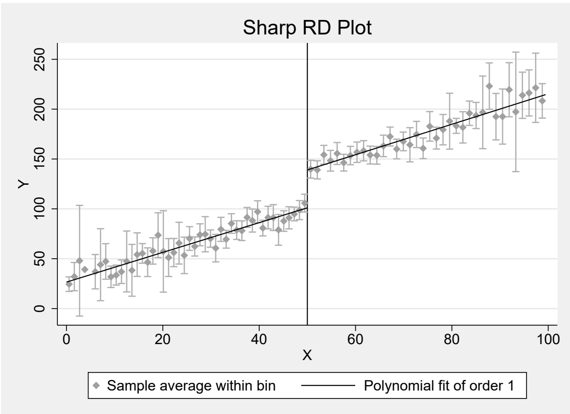
[2] A common example of this is sanitary scores of restaurants and food-serving institutions in New York City!

[3] Note that covariates may also be specified in the regression specification.

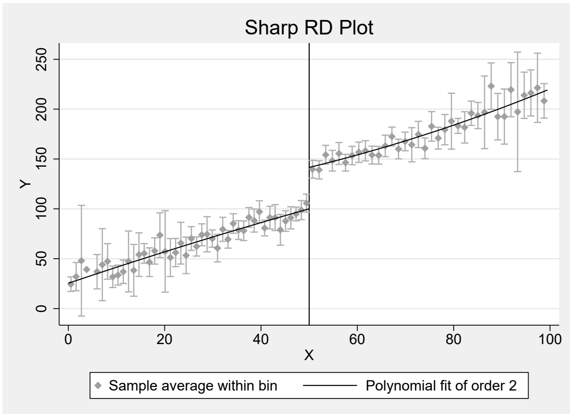
Table 2: Sharp RD - Local Polynomial Regression Estimates

VARIABLES	(1) Model 1	(2) Model 2	(3) Model 3	(4) Model 4	(5) Model 5	(6) Model 6	(7) Model 7	(8) Model 8
Conventional	41.17***	35.41***	29.68***	28.40***	27.46***	32.60***	25.01***	43.22***
	(3.578)	(4.559)	(6.026)	(6.694)	(6.874)	(8.581)	(7.387)	(10.67)
Bias-corrected	40.74***	28.40***	27.53***	32.60***	25.58***	43.22***	24.30***	44.12***
	(3.578)	(4.559)	(6.026)	(6.694)	(6.874)	(8.581)	(7.387)	(10.67)
Robust	40.74***	28.40***	27.53***	32.60***	25.58***	43.22***	24.30***	44.12***
	(4.221)	(6.694)	(6.513)	(8.581)	(7.414)	(10.67)	(7.955)	(12.87)
Observations	967	967	967	967	967	967	967	967
Covariates	NO	NO	NO	NO	NO	NO	NO	NO
Mean dep. var.	48.03	48.03	48.03	48.03	48.03	48.03	48.03	48.03
Bandwidth	17.03	10	12.71	10	16.81	10	22.14	10
Order polyn.	1	1	2	2	3	3	4	4

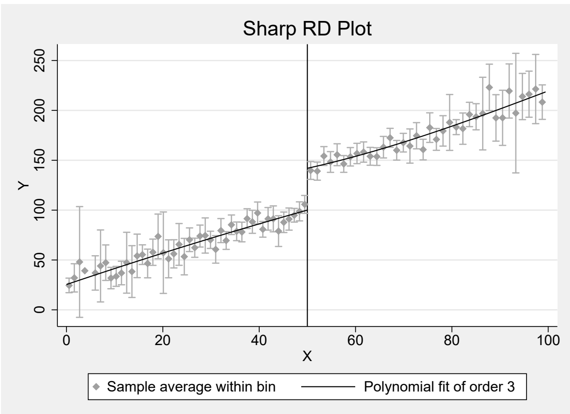
Figure 2: Sharp RD Plots with 95% Confidence Intervals



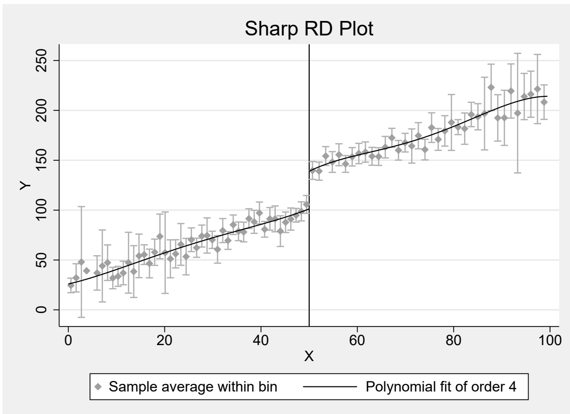
(a) Model 1



(b) Model 3



(c) Model 5



(d) Model 7

bandwidth might include very similar observations, leading to more precise estimates but fewer data points (thus an increase in variance). On the other hand, a wide bandwidth may include more observations at the expense of biasing results.

We can see that overall, the sharp RD coefficient estimate appears to drop as we move through Columns 1-8. Both the odd-numbered columns and even-numbered columns are ordered such that the polynomial order increases from 1 to 4. This is because we specified our outcome variable Y without the use of polynomials in our DGP. In the odd-numbered columns, bandwidths are selected through an MSE-optimal procedure, whereas in the even-numbered columns, bandwidths are selected (arbitrarily) 10 units above/below the cutoff score. We can see from comparing the even and odd columns that the coefficient estimates change and that the standard errors are larger in the latter.

2.2 Fuzzy RD

In this exercise, I reproduce the findings of Pons and Tricaud (2018), who studied the presence of third run-off candidates in French parliamentary and local elections and their effect on election outcomes. Although the original paper compiles data and analysis files in STATA, in this paper I produce the next several RD plots and tables in R. The accompanying R code can be found in this project's folder, and the replication package of the original article can be found here.

In this paper, the authors make use of French elections data, which includes a total of 7,257 observations: 3,458 (47.7%) from local elections and 3,799 (52.3%) from parliamentary elections. More specifically, they study the effect of a specific rule used for certain run-off elections, in which candidates who finish third-place get to advance to second run-off elections if they receive more than 12.5% of votes. Of course, candidates who came in third in the first round should be expected to have lower chances of winning or placing second in the run-off round than the other candidates who ranked first and second in the first round. However, the authors are interested in studying whether voters *adjust* expressively or strategically in response to the presence of a third candidate. Thus the research question of interest is: *how does electoral choice affect voter behavior?*

The authors obtain causal identification using a fuzzy regression discontinuity design based on the 12.5% threshold rule. They compare second-round results in French localities where the third candidate obtains a vote share just above/below the 12.5 percent threshold, which determines whether they participate in the run-off round. It is important to recognize why this rule is not a strict cut-off: even though candidates are not allowed to participate in second-round elections when they have less than 12.5% of votes, they do not necessarily have to participate in the second round even if they have at least 12.5% of votes. The following table presents Table 1 from the paper:

Table 3: Fuzzy RDD Summary Statistics

Variable	Mean	Median	Min.	Max.	No. Obs.
# registered citizens, round 1	45964	49046	883	189384	7257
% registered citizens who participated, round 1	0.58	0.59	0.13	0.91	7257
% registered citizens who voted for a candidate, round 1	0.56	0.57	0.13	0.89	7257
% registered citizens who voted blank or null, round 1	0.019	0.016	0.00091	0.094	7257
# candidates, round 1	7.8	6	3	29	7257
% registered citizens who participated, round 2	0.59	0.58	0.13	0.93	7257
% registered citizens who voted for a candidate, round 2	0.55	0.54	0.12	0.91	7257
% registered citizens who voted blank or null, round 2	0.035	0.029	0.0024	0.28	7257
# candidates, round 2	2	2	1	3	7257

They define the running variable X as the qualifying margin of the third candidate in the first round (i.e., the candidate's vote share expressed as a fraction of the number of registered citizens minus the 12.5 percent threshold). The assignment variable defines whether the candidate is eligible for the run-off, $D = \mathbb{1}\{X \geq 0\}$. Lastly, the treatment variable T represents if the candidate is actually present in the run-off election (i.e., $T = \mathbb{1}\{\text{present in run-off}\}$). Here, the "compliers" are those districts in which the third candidate qualifies ($D = 1$) and runs in the second round ($T = 1$).

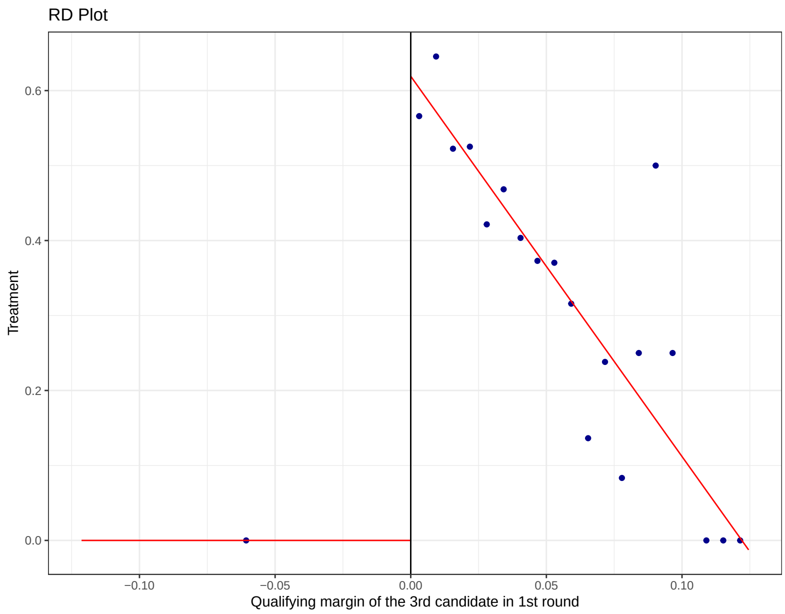
The authors aim to estimate the following equations, where T_i is instrumented with D_i:

$$Y_i = \alpha_1 + \tau T_i + \beta_1 X_i + \beta_2 X_i T_i + u_i \tag{6}$$

$$T_i = \alpha_0 + \gamma D_i + \delta_1 X_i + \delta_2 X_i D_i + \varepsilon_i \tag{7}$$

In Figure 3, I plot the treatment variable against the running variable to show the jump in the relationship between the 12.5% threshold rule and the treatment assignment. As can be sen from the figure, it is apparent that the probability of treatment assignment at the threshold jumps:

Figure 3: First Stage RD Plot



In Table 4, I present the first-stage results reproduced from the original article. In Column 1, the probability of treatment assignment at the threshold jumps from 0 to around 55.2%. Here, Columns 1 and 2 both use local linear regression to compute coefficient estimates. However, the results in Column 1 are under the MSERD optimal bandwidths, whereas the results in Column 2 are under IK-optimal bandwidths. Likewise, Columns 3 and 4 both use quadratic regression specifications with MSERD and IK-optimal bandwidths, respectively. Across all specifications, the first-stage results are statistically significant at the 1 percent level. Under satisfied assumptions, the second-stage results can be interpreted as the causal effect, at the threshold, of the treatment on complying districts.

Table 4: First Stage Results

	(1)	(2)	(3)	(4)
Conventional	0.552***	0.611***	0.509***	0.568***
	(0.042)	(0.030)	(0.051)	(0.042)
Bias-Corrected	0.535***	0.568***	0.493***	0.507***
	(0.042)	(0.030)	(0.051)	(0.042)
Robust	0.535***	0.568***	0.493***	0.507***
	(0.047)	(0.042)	(0.055)	(0.053)
Kernel	Triangular	Triangular	Triangular	Triangular
Bandwidth	mserd	Manual	mserd	Manual
nobs.effective.left	947	2664	1406	2664
nobs.effective.right	594	964	736	964

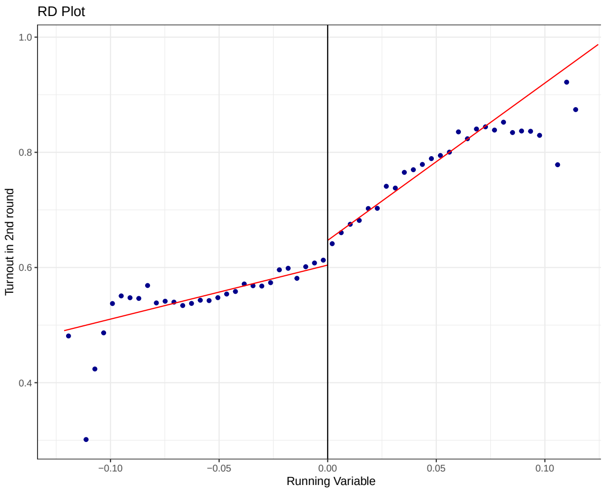
+ $p < 0.1$, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 5: Fuzzy RDD Results

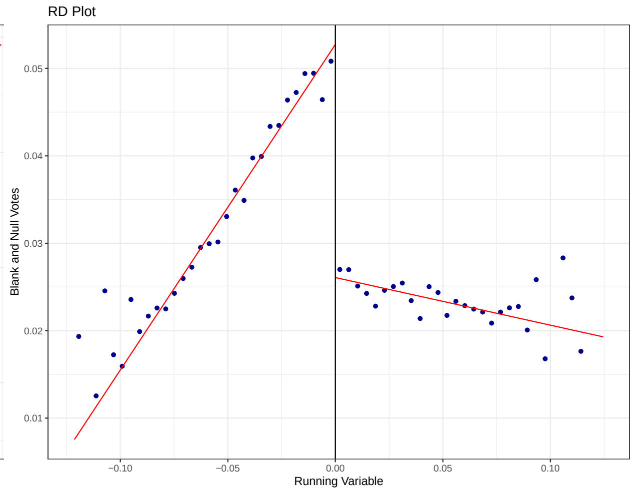
	Turnout	Blank and Null Votes	Candidate Votes
Conventional	0.040*	−0.037***	0.078***
	(0.017)	(0.004)	(0.019)
Bias-Corrected	0.042*	−0.036***	0.079***
	(0.017)	(0.004)	(0.019)
Robust	0.042*	−0.036***	0.079***
	(0.021)	(0.005)	(0.023)
Kernel	Triangular	Triangular	Triangular
Bandwidth	mserd	mserd	mserd
nobs.effective.left	1528	1797	1589
nobs.effective.right	770	833	785

+ $p < 0.1$, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

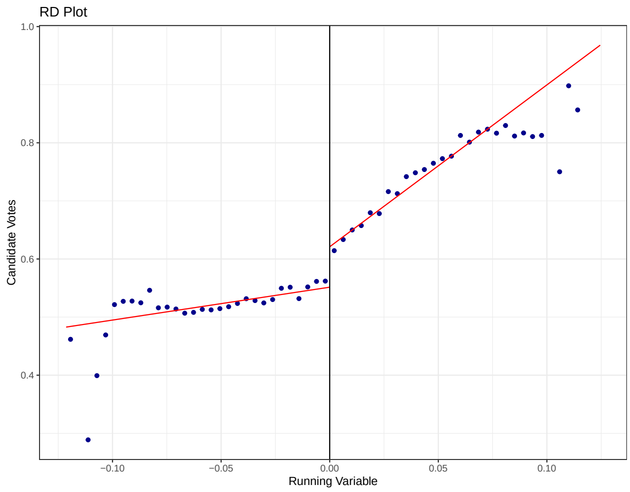
Recall that the running variable is the qualifying margin of the third candidate in the first round. In Figure 4 and Table 5, I present and plot the impact of the presence of a third candidate on voter participation and candidate votes.



(a)



(b)



(c)

Figure 4: Fuzzy RD Plots

3 Concluding Remarks

Overall, the regression discontinuity design is a powerful tool for empirical researchers interested in estimating treatment effects of interventions where treatments are assigned on a basis of eligibility of some score rather than random assignment. While sharp-RDD estimation is relatively straightforward, fuzzy RDDs on the other hand can be interpreted as a specific type of instrumental variables estimation, where the treatment assignment indicator is instrumented by treatment eligibility. Several threats to the validity of the regression discontinuity design include:

- **Bunching around the cutoff score.** If we examine the distribution of values in the running variable and notice bunching of units around the cutoff value, it may be indicative of manipulation of units' running variable values, either by program administrators or through the units' own awareness of the cutoff score. In this case it is not appropriate to use the RD design since estimates would be biased.

- **Continuous running variable.** Without a continuous treatment-determining variable, it is difficult to rank units observed in a smooth way. It also becomes difficult to argue for identification (recall that in RDD identification is at the limit).

- **Omitted variable bias.** For RD to work, we must assume that other potential determinants of the outcome variable do not themselves cause jumps at the threshold of the running variable.

References

Lee, D. S. and Lemieux, T. (2010). Regression Discontinuity Designs in Economics. *Journal of Economic Literature*, 48(2):281–355.

Pons, V. and Tricaud, C. (2018). Expressive Voting and Its Cost: Evidence From Runoffs With Two or Three Candidates. *Econometrica*, 86(5):1621–1649.