1 Out-of-Kilter algorithm

This algorithm is a Primal-dual method and is applied to the minimum weight circulation problem.

For LP optimality conditions we need primal feasibility, dual feasibility and complementary slackness, i.e., KKT conditions. Primal and dual feasibility are obvious so we need to show complementary slackness through following theorem.

Theorem 1.1. *Let x be a feasible circulation flow for (D, l, u, w). And suppose there exists a real value vector $\{y_i : i \in V\}$ which we called* ***vertex-numbers****. For all edges $e \in A$*

$$y_{h(e)} - y_{t(e)} > w_e \text{ implies } x_e = u_e \tag{1}$$

$$y_{h(e)} - y_{t(e)} < w_e \text{ implies } x_e = l_e \tag{2}$$

Then x is optimal to the circulation problem.

Proof. For each $e \in A$ define

$$\gamma_e = \max\{y_{h(e)} - y_{t(e)} - w_e, 0\} \tag{3}$$

$$\mu_e = \max\{w_e - y_{h(e)} + y_{t(e)}, 0\} \tag{4}$$

Then

$$\gamma_e - \mu_e = y_{h(e)} - y_{t(e)} - w_e \tag{5}$$

Furthermore

$$\sum_{e \in A} (\mu_e l_e - \gamma_e u_e) \tag{6}$$

$$= \sum_{e \in A} (\mu_e l_e - \gamma_e u_e) + \sum_{i \in V} y_i \Big(\sum_{h(e)=i} x_e - \sum_{t(e)=i} x_e \Big) \tag{7}$$

$$= \sum_{e \in A} (\mu_e l_e - \gamma_e u_e + x_e(y_{h(e)} - y_{t(e)})) \tag{8}$$

$$= \sum_{e \in A} (\mu_e l_e - \gamma_e u_e + x_e(\gamma_e - \mu_e + w_e)) \tag{9}$$

$$= \sum_{e \in A} (\gamma_e (x_e - u_e) + \mu_e (l_e - x_e) + x_e w_e) \tag{10}$$

$$\leq \sum_{e \in A} x_e w_e \tag{11}$$

The last inequality will be satisfied as equality iff the first two hold. □

The following is the formulation of circulation problem

$$\text{(P)} \quad \min \quad wx \tag{12}$$

$$\text{s.t.} \quad Nx = 0 \quad y \tag{13}$$

$$x \geq l \quad z^l \tag{14}$$

$$-x \leq -u \quad z^u \tag{15}$$

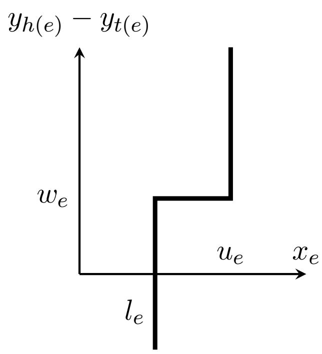
$$\text{(D)} \quad \max \quad lz^l - uz^u \tag{16}$$

$$\text{(s.t.)} \quad yN^{-1} + z^l - z^u \leq w \tag{17}$$

$$
\begin{aligned}
& y \quad free && (18)\\
& z^l, z^u \geq 0 && (19)\\
\text{(CS)} \quad & y_{h(e)} - y_{t(e)} > w_e \Rightarrow x_e = u_e && (20)\\
& y_{h(e)} - y_{t(e)} < w_e \Rightarrow x_e = l_e && (21)
\end{aligned}
$$

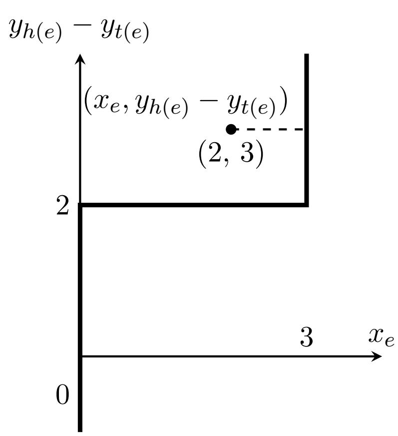
There is an alternative way of circulation optimality for a circulation problem. We define a **kilter-diagram** as follows.

For every edge construct the following:



For each point $(x_e, y_{h(e)} - y_{t(e)})$ we define a **kilter-number** k_e, be the minimum positive distance change in x_e required to put in on the kilter line.

Example. For edge $e : w_e = 2, l_e = 0, u_e = 3$, assume $x_e = 2, y_{h(e)} - y_{t(e)} = 3$, then $k_e = 1$



Lemma 1.2. *If for every circulation x and vertex number y we have $\sum_{e \in A} k_e = 0$, then x is optimal.*

Proof. Since k_e is a nonnegative number, then the only way that $\sum_{e \in A} k_e = 0$ is $k_e = 0, \forall e \in A$, which means $\forall e \in A, l_e \leq x_e \leq u_e$. Furthermore, the complementary slackness are satisfied. $\square$

General idea of algorithm is as follows. Suppose we are given a circulation x and vertex-numbers y (we do not require feasibility). Usually we pick $x = 0, y = 0$. If every edge is in kilter-line then we are optimal.

Otherwise there is at least one edge e^* that is out-of kilter. The algorithm consist of two phases, one called **flow-change** phase (horizontally), then other **number-change** phase (vertically).

In the flow-change phase, we want to find a new circulation for an out-of-kilter edge e^* say $\hat{e}$ such that we reduce the kilter number k_{e^*}, without increasing any other kilter number for other edges.

To do this, denote the edges of e^* to be s and t, where such that k_{e^*} will be decreased by increasing the flow from s to t on e^*.

If $e^* = (s,t)$ this will accomplished by increasing x_{e^*} and if $e = (t,s)$ it is accomplished by decreasing x_{e^*}.

To do this we look for an (s,t)-path p of the following edges.

- If e is forward in p, then increasing x_e does not increase k_e and
- If e is reversed in p, then decreasing x_e dose not increase k_e

In terms of kilter diagram, an arc satisfies "forward" if it is forward and in left side of kilter line, and it satisfies "reversed" if it is reverse and in right side of kilter line.

Suppose we can not find such a path. From s to t, let x be the vertices that can decrease by an augmenting path. Then either we can change the vertex numbers y so that $\sum_{e \in A} k_e$ does not increase but x does, or we can show that problem is infeasible.

The sketch of Out-of-kilter algorithm is as follows:

- **INPUT** a minimum circulation problem (D, l, u, w) a circulation x and vertex-numbers y
- **OUTPUT** conclusion that (D, l, u, w) is infeasible or an minimum weighted flow.
- Step 1: If every arc is in kilter ($k_e = 0, \forall e \in A$). Stop with x is optimal. Otherwise let e^* be an out-of-kilter arc. If increasing x_{e^*} decreases k_{e^*} set $s = h(e^*)$ and $t(e^*)$ otherwise set $s = t(e^*)$ and $t = h(e^*)$
- Step 2: If there exists an (s,t) augmenting path p then goto Step 3, otherwise goto Step 4.
- Step 3: Set $y_e = y_{h(e)} - y_{t(e)}, e \in A \begin{cases} \Delta_1 & = \min\{u_e - x_e : e \text{ is forward and } y_e \geq w_e\} \\ \Delta_2 & = \min\{l_e - x_e : e \text{ is forward and } y_e < w_e\} \\ \Delta_3 & = \min\{x_e - l_e : e \text{ is reverse and } y_e \leq w_e\} \\ \Delta_4 & = \min\{x_e - u_e : e \text{ is reverse and } y_e > w_e\} \end{cases}$
  and $\Delta = \min\{\Delta_i, i = 1,2,3,4\}$. Increase x_e by Δ on each forward arc in p, decrease x_e by Δ on each reverse arc in p.
  If $e^* = (s,t)$ decrease x_{e^*} by Δ, otherwise increase x_{e^*} by Δ
  If $k_{e^*} > 0$ goto Step 2. otherwise goto Step 1.
- Step 4: Let X be the set of vertices reachable from s by augmenting paths, then $t \notin X$, if every arc e with $h(e) \in X$ has $x_e \leq l_e$ and every arc e with $t(e) \in X$ has $x_e \geq u_e$, and at least one of the above inequality is strict, then Stop with problem infeasible. Otherwise set $\delta_1 = \min\{w_e - y_e : t(e) \in X, y_e < w_e, x_e \leq u_e \neq l_e\}$ $\delta_2 = \min\{y_e - w_e : h(e) \in X, y_e > w_e, x_e \geq u_e \neq l_e\}$ $\delta = \min\{\delta_1, \delta_2\}$ Set $y_i = y_i + \delta$ for $i \notin X$. If $k_{e^*} > 0$, goto Step 2, otherwise goto Step 1.

Out-of-kilter takes $O(|E||V|K)$ where $K = \sum_{e \in A} k_e$. However, there is an algorithm called **scaling algorithm** that uses out-of-kilter as subroutine that runs in $O(R|E|^2|V|)$ where $R = \lceil \max\{\log_2 u_e : e \in A\} \rceil$