Lecture Note - (Lecture)

Lan Peng, Ph.D.

School of Management, Shanghai University, Shanghai, China

"Life is a Crystal."

1 Preliminaries

1.1 Valid Inequalities and Faces

The inequality denoted by (π, π_o) is called a valid inequality for P if $\pi x \leq \pi_0, \forall x \in P$. Note that (π, π_0) is a valid inequality iff P lies in the half-space $\{x \in \mathbb{R}^n | Ax \leq b\}$

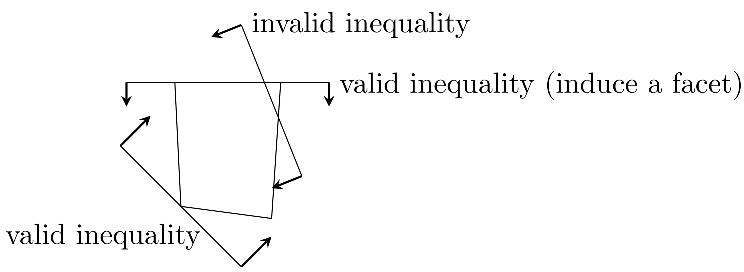


Figure 1: Example of valid/invalid inequality

- If (π, π_0) is a valid inequality for P and $F = \{x \in P | \pi x = x_0\}$, F is called a facet of P and we say that (π, π_0) represents or defines F
- A face is said to be proper if $F \neq \emptyset$ and $F \neq P$
- The face represented by (π, π_0) is nonempty iff $\max\{\pi x | x \in P\} = \pi_0$
- If the face F is nonempty, we say it supports P
- Let P be a polyhedron with equality set $M^=$. If

  $$F = \{x \in P | \pi^T x = \pi_0\}$$

  is not empty, then F is a polyhedron. Let

  $$M^= \subseteq M_F^=, M_F^\leq = M \setminus M_F^=$$

  then

  $$F = \{x | a_i^T x = b_i, \forall i \in M_F^=, a_i^T x \leq b_i, \forall i \in M_f^\leq\}$$

1.2 Facet

- A face F is said to be a facet of P if $dim(F) = dim(P) - 1$
- Facets are all we need to describe polyhedral

- If F is a facet of P, then in any description of P, there exists some inequality representing F
- Every inequality that represents a face that is not a facet is unnecessary in the description of P
- Every full-dimensional polyhedron P has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of P
- If $dim(P) = n - k$ with $k > 0$, then P is described by a maximal set of linearly independent rows of $(A^=, b^=)$, as well as one inequality representing each facet of P

1.3 Proving Facet

To prove an inequality $\sum_i a_i x_i \leq b_i$ is facet inducing for a D dimensional polyhedral, we need to prove there are D affinely independent vectors in $\sum_i a_i x_i = b_i$

2 Some Examples

2.1 Vertices Packing

Vertices Packing Given a graph $G = (V, E)$, with $|V| = n$. A vertices packing solution is that no two neighboring vertices can be chosen at the same time.

$$PACK(G) = \{x \in \mathbb{B}^n | x_i + x_j \leq 1, \forall (i,j) \in E\}$$

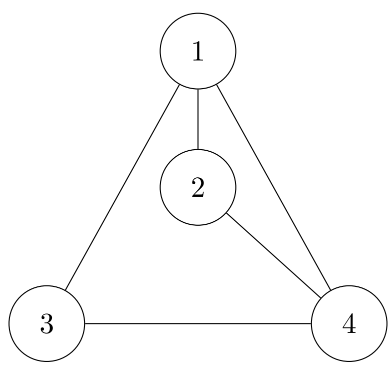


Figure 2: Example of vertices packing problem

The PACK of this graph is

$$PACK = conv \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right)$$

The dimension of PACK, i.e. $dim(PACK(G))$ is (full-dimensional)

$$dim(PACK(G)) = |V|$$

To prove that $dim(PACK(G)) = |V|$, we need to find $|V|+1$ affinely independent vectors.

Proof.

$$rank\left(\begin{bmatrix}0 & I_{|V|}\\ 1 & 1\end{bmatrix}\right) = |V|+1$$

□

Therefore, in PACK, $rank(A^{=}, b^{=}) = 0$

Inequalities and Facets of conv(VP) - Nonnegative Constraints $x_i \geq 0$ induce facets.

Proof.

$$rank\left(\begin{bmatrix}0 & 0\\ 0 & I_{|V|}\end{bmatrix}\right) = |V|+1$$

□

Inequalities and Facets of conv(VP) - Neighborhood Constraints $x_i + x_j \leq 1$ is a valid constraint, but it **DOES NOT** always induce facet.

Inequalities and Facets of conv(VP) - Odd Hole Consider a graph $G = (V, E)$, the covering problem is

$$\sum_{e\in\delta(i)} x_e \leq 1, i \in V, x_e \in \{0,1\}, e \in E$$

For $T \subset V$, denote $\delta(i)$ as all edges induce to $i \in V$, denote $E(T) \subset E$ as all the edges linked between $(i,j), i \in T, j \in T$, therefore we have

$$\sum_{i\in T}\sum_{e\in\delta(i)} x_e \leq |T|$$

For edges linking $i \in T, j \in T$, count them twice, for edges linking $i \in T, j \notin T$, count them once.We can have a new constraint

$$2\sum_{e\in E(T)} x_e + \sum_{e\in\delta(V\setminus T,T)} x_e \leq |T|$$

Perform the Gomory Cut, the following constraint is a valid:

$$\sum_{e\in E(T)} x_e \leq \lfloor\frac{|T|}{2}\rfloor$$

H is an odd hole if it contains circle of k nodes, such that k is odd and there is no cords. e.g. {1, 2, 5, 6, 3}. Then, the following inequality is valid,

$$\sum_{i\in H} x_i \leq \frac{|H|-1}{2}$$

Odd Hole inequality **DOES NOT** always induce facets.

Inequalities and Facets of conv(VP) - Maximum Clique A **clique** is a subset of a graph that in the clique every two vertices linked with each other (complete sub-graph). A **maximum clique** is a clique that any other vertice can not form a clique with all the points in this clique.

C is a maximum clique, then the following inequality is valid and induce a facet,

$$\sum_{i \in C} x_i \leq 1$$

Proof. First, if $C = V$

$$rank\left([I]\right) = |C| = |V|$$

Second, if C is a subset of V, for each vertice in $V \setminus C$, there should be at least one vertice in C that is not linked with it. Therefore for each vertice in C we can find a packing. □

2.2 Knapsack Problem

Knapsack Problem Formulation Consider the knapsack set KNAP

$$conv(KNAP) = conv(\{x \in \mathbb{B}^n | \sum_{j \in N} a_j x_j \leq b\})$$

in where

- $N = \{1, 2, ..., n\}$
- With out lost of generality, assume that $a_j > 0, \forall j \in N$ and $a_j < b, \forall j \in N$

Valid Inequalities for a Relaxation For $P = \{x \in \mathbb{B}^n | Ax \leq b\}$, each row can be regard as a Knapsack problem, i.e. for row i

$$P_i = \{x \in \mathbb{B}^n | a_i^T x \leq b_i\}$$

is a relaxation of P, therefore,

$$P \subseteq P_i, \forall i = 1, 2, ..., m$$

$$P \subseteq \cap_{i=1}^{m} P_i$$

So any inequality valid for a relaxation of an IP is also valid for IP itself.

Cover and Extended Cover A set $C \subseteq N$ is a cover if $\sum\limits_{j \in C} a_j > b$, a cover C is minimal cover if

$$C \subseteq N | \sum_{j \in C} a_j > b, \sum_{j \in C \setminus k} a_j < b, \forall k \in C$$

For a cover C, we can have the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

The inequality is trivial considering the pigeonhole principle.
$C \subseteq N$ is a minimal cover, then $E(C)$ is defined as following:

$$E(C) = C \cup \{j \in N | a_j \geq a_i, \forall i \in C\}$$

is called an extended cover. Then we have,

$$\sum_{i\in E(C)} x_i \le |C|-1 \text{ dominates } \sum_{i\in C} x_i \le |C|-1$$

and

$$\sum_{i\in E(C)} x_i \le |C|-1 \text{ dominates } \sum_{i\in E(C)} x_i \le |E(C)|-1$$

Hereby we need to prove that $\sum_{i\in E(C)} x_i \le |C|-1$ is valid, by contradiction.

Proof. Suppose $x^R \in KNAP$, R is a feasible solution, Where

$$x_j^R = \begin{cases} 1, & \text{if } j \in R \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\sum_{j\in E(C)} x_j^R \ge |C| \Rightarrow |R \cap E(C)| \ge |C|$$

therefore

$$\sum_{j\in R} a_j \ge \sum_{j\in R\cap E(C)} a_j \ge \sum_{j\in C} a_j > b$$

which means R is a cover, it is contradict to $\sum_{j\in E(C)} x_j^R \ge |C|$ so $x^R \notin KNAP$ □

Dimension of KNAP $conv(KNAP)$ is full dimension, i.e. $dim(conv(KNAP)) = n$.

Proof. $0, e_j, \forall j \in N$ are $n+1$ affinely independent points in $conv(KNAP)$ □

Inequalities and Facets of conv(KNAP) - Lower Bound and Upper Bound Constraints

- $x_k \ge 0$ is a facet of $conv(KNAP)$

  Proof. $0, e_j, \forall j \in N \setminus k$ are n affinely independent points that satisfied $x_k = 0$ □

- $x_k \le 1$ is a facet iff $a_j + a_k \le b, \forall j \in N \setminus k$

  Proof. $e_k, e_j + e_k, \forall j \in N \setminus k$ are n affinely independent points that satisfied $x_k = 1$ □

Inequalities and Facets of conv(KNAP) - Extended Cover Order the variables so that $a_1 \ge a_2 \ge \dots \ge a_n$, therefore $a_1 = a_{max}$
Let C be a cover with $\{j_1, j_2, \dots, j_r\}$ where $j_1 < j_2 < \dots < j_r$ so that $a_{j_1} \ge a_{j_2} \ge \dots \ge a_{j_r}$
Let $p = \min\{j | j \in N \setminus E(C)\}$
Then

$$\sum_{j\in E(C)} x_j \le |C|-1$$

is a facet of $conv(KNAP)$ if

- $C = N$

  Proof.
  $$R_k = C \setminus k, \forall k \in C = N \setminus k, \forall k \in N$$
  have $|N|$ affinely independent vectors $\square$

- $E(C) = N$ and $\displaystyle\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_{max} \leq b$

  Proof. (j_1, j_2 are two heaviest elements in C)
  $$S_k = C \setminus \{j_1, j_2\} \cup \{k\}, \forall k \in E(C) \setminus C$$
  $R_k \cup S_k$ have $|C| + |E(C) \setminus C| = |E(C)| = |N|$ affinely independent vectors $\square$

- $C = E(C)$ and $\displaystyle\sum_{j \in C \setminus j_1} a_j + a_p \leq b)$

  Proof. (j_1 is the heaviest element in C, k is the lightest element outside extended cover)
  $$T_k = C \setminus j_i \cup \{k\}, \forall k \in N \setminus E(C)$$
  $R_k \cup T_k$ have $|N \setminus E(C)| + |E(C)| = |N \setminus C| + |C| = |N|$ affinely independent vectors $\square$

- $C \subset E(C) \subset N$ and $\displaystyle\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_{max} \leq b$ and $\displaystyle\sum_{j \in C \setminus j_1} a_j + a_p \leq b$

  Proof. $S_k \cup T_k$ have $|E(C) \setminus C| + |N \setminus E(C)| = |N|$ affinely independent vectors $\square$

3 Generic Cutting Planes

3.1 General Approach

Cutting Planes

- Cutting planes are referred to inequalities valid for $conv(S)$, but which is violated by the solution obtained by solving the current LP relaxation.
- Cutting plane methods attempt to improve the bound produced by the LP relaxation by iteratively adding cutting planes to the initial LP relaxation.
- Adding such inequalities to the LP relaxation may improve the bound (this is not a guarantee).

Separation Problem Given a polygon $P \in \mathbb{R}^n$ and $\mathbf{x}^* \in \mathbb{R}^n$, determine whether $\mathbf{x}^* \in P$, and if not, determine (π, π_0), a valid inequality for P such that $\pi\mathbf{x}^* \geq \pi_0$

3.2 Generic Cutting Planes

Observation For integer program over a set of variables $x_1, x_2, \cdots, x_n \in \mathbb{Z}_+$

- If $\mathbf{ax} = \mathbf{b}$ is a constraint of P, then $\mathbf{ax} \leq \mathbf{b}$ is a valid constraint of P
- Suppose there are two valid constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

  and

$$a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

  then

$$\begin{aligned} & u_1(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) \\ +& u_2(a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n) \\ & \qquad \leq u_1b_1 + u_2b_2 \end{aligned}$$

  is a valid inequality for P

- If $a \leq b$ and a is an integer number, then $a \leq \lfloor b \rfloor$
- $x_i \in \mathbb{Z}_+ \Rightarrow -x_i \leq 0$
- WLOG, assume $\forall a_i$ in constraint $\sum_{i=1}^{n} a_i x_i \leq b$, a_i is a fractional number. Let $f_i = a_i - \lfloor a_i \rfloor$ be the fractional part, and $f_i \geq 0$. Then,

$$\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} f_i x_i \leq b - 0$$

  $\Rightarrow$

$$\sum_{i=1}^{n} \lfloor a_i \rfloor x_i \leq \lfloor b \rfloor$$

  is a valid inequality for P.

Chvátal-Gomory Rounding Method Let $X = P \cap \mathbb{Z}^n$ be the feasible set of an integer program, where

$$P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq 0\}$$

Also let $\mathbf{a}_i$ be the ith column of $\mathbf{A}$, then

$$\sum_{i=1}^{n} \mathbf{ua}_i x_i \leq \mathbf{ub}$$

is a valid inequality for P for $u \geq 0$, then

$$\sum_{i=1}^{n} \lfloor \mathbf{ua}_i \rfloor x_i \leq \mathbf{ub}$$

is a valid inequality for P, then

$$\sum_{i=1}^{n} \lfloor \mathbf{u}\mathbf{a}_i \rfloor x_i \leq \lfloor \mathbf{u}\mathbf{b} \rfloor$$

is a valid inequality for P, since x_i is an integer. Such procedures are called Chvátal-Gomory rounding procedure, and such cuts are called Chvátal-Gomory Cuts.

Example 1 Let P be

$$\begin{aligned} 2x_1 + 3x_2 &\leq 5 \\ -x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Then

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \Rightarrow \mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let

$$\mathbf{u} = \begin{bmatrix} 1/2 & 1/3 \end{bmatrix}$$

, then

$$\begin{bmatrix} 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_2 \leq \begin{bmatrix} 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

which is

$$\frac{2}{3}x_1 + \frac{11}{6}x_2 \leq \frac{19}{6}$$

then,

$$\left\lfloor \frac{2}{3} \right\rfloor x_1 + \left\lfloor \frac{11}{6} \right\rfloor x_2 \leq \left\lfloor \frac{19}{6} \right\rfloor \Rightarrow x_2 \leq 3$$

Example 2 Let P be

$$\begin{aligned} 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Let $u = \begin{bmatrix} \frac{2}{7} & \frac{37}{63} & 0 \end{bmatrix}$

Then $2x_1 + \frac{1}{63}x_2 \leq \frac{185}{21} \Rightarrow x_1 \leq 4$

Observation

- It is possible to generate $conv(X)$ using C-G procedures
- The crux lies on the choice of $\mathbf{u}$
- The number of iterations can be huge

- Nevertheless, the procedure has been proven to work well in practice in recent years to generate good inequalities
- One can try generating random multipliers and test the method

3.3 Gomory Cuts

Gomory cutting planes can also be derived directly from the tableau while solving an LP relaxation. Consider the set

$$P = \{(\mathbf{x}, \mathbf{s}) \in \mathbb{Z}_+^{n+m} | \mathbf{Ax} + \mathbf{Is} = \mathbf{b}\}$$

in which the LP relaxation of an ILP is put in standard form. Assume for A, all the coefficients are integers, so that the slack variables are integers. Clearly,

$$\mathbf{B^{-1}Ax} + \mathbf{B^{-1}Is} = \mathbf{B^{-1}b}$$

Let $\lambda = \mathbf{B^{-1}}_j$, then we obtain

$$\sum_{j=1}^{n} \lambda \mathbf{A_j} x_j + \sum_{i=1}^{m} \lambda_i s_i = \lambda \mathbf{b}$$

where $\mathbf{A}_j$ is the jth column of $\mathbf{A}$ and λ is a row if $\mathbf{B}^{-1}$. Then, the Gomory cut is define by

$$\sum_{j=1}^{n} (\lambda \mathbf{A}_j - \lfloor \lambda \mathbf{A}_j \rfloor) x_j + \sum_{i=1}^{m} (\lambda_i - \lfloor \lambda_i \rfloor) s_i \geq \lambda \mathbf{b} - \lfloor \lambda \mathbf{b} \rfloor$$

Gomory cut is a Chvátal-Gomory cut with weights $\mathbf{u}_i = \lambda_i - \lfloor \lambda_i \rfloor$.

Example For the following IP

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \leq 28 \\ & x_1 + 4x_2 \leq 27 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \in \mathbb{Z}_+ \end{aligned}$$

The optimal tableau of the LP relaxation is as follows

	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0	1	-2/30	8/30	0	16/3
s_3	0	0	-1/3	1/3	1	2/3
x_1	1	0	8/30	-2/30	0	17/3

For the first row, the Gomory cut is

$$\frac{28}{30} s_1 + \frac{8}{30} s_2 \geq \frac{1}{3}$$

Replace the slack variables, we have

$$4x_1 + 2x_2 \leq 33$$

For the second row, the Gomory cut is

$$\frac{2}{3}s_1 + \frac{1}{3}s_2 \leq \frac{2}{3}$$

Replace the slack variables, we have

$$3x_1 + 2x_2 \leq 27$$

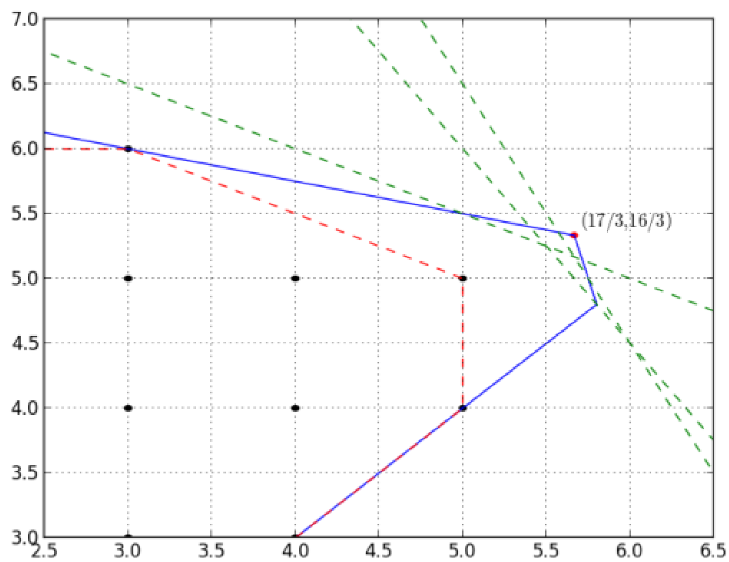
For the third row, the Gomory cut is

$$\frac{8}{30}s_1 + \frac{28}{30}s_2 \geq \frac{2}{3}$$

Replace the slack variables, we have

$$x_1 + 2x_2 \leq 16$$

This picture shows the effect of adding all Gomory cuts in the first round



4 Branch and Cut

Two types of valid inequalities The structural constraints are

- while this is not a standard term used in mathematical programming, we will use it in reference to the constraints that define a formulation
- these are constraints needs to define the problem. If removed, some infeasible integer solutions may become part of the solution space

The cutting planes are

- these are valid constraints that are not needed to define the problem but can be added to tighten the LP relaxation of a formulation
- in other words, they are used for trying to obtain a better LP bound

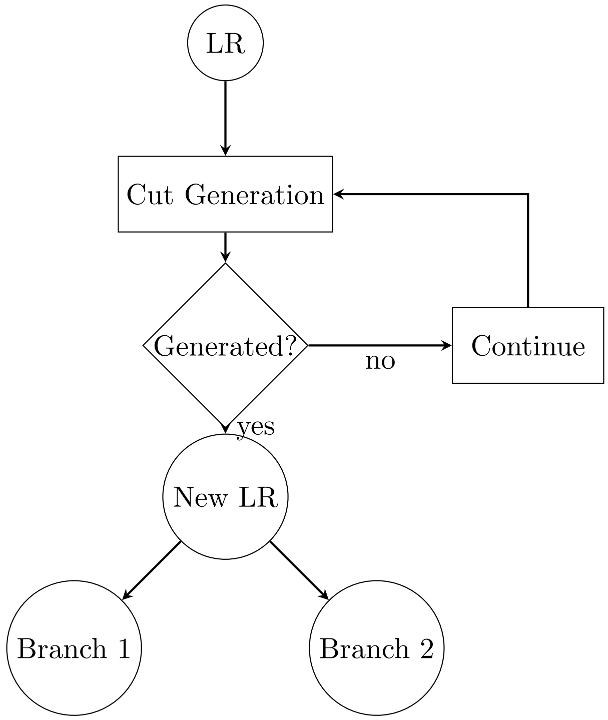


Figure 3: Branch and Cut for Optional Inequality

User cuts Consider the vertice packing problem, we have mentioned that the maximum cliques can be used for adding valid inequalities, this is how it works

- Given a fractional solution $\hat{x}$, we can find a clique for which $\sum_{i \in C} x_i \leq 1, C \in Clique(G)$ is violated
- Solve the following separation problem

$$\begin{aligned} \max \quad & \gamma = \sum_{i \in V} \hat{x}_i z_i \\ \text{s.t.} \quad & z_i + z_j \leq 1, \{i, j\} \notin E \\ & z_i \in \{0, 1\}, i \in V \end{aligned}$$

- if $\gamma > 1$ add the clique cut associate with C.

Lazy cuts A typical example for lazy cuts is the DFJ sub-tour elimination for TSP. We will discuss later in this semester.

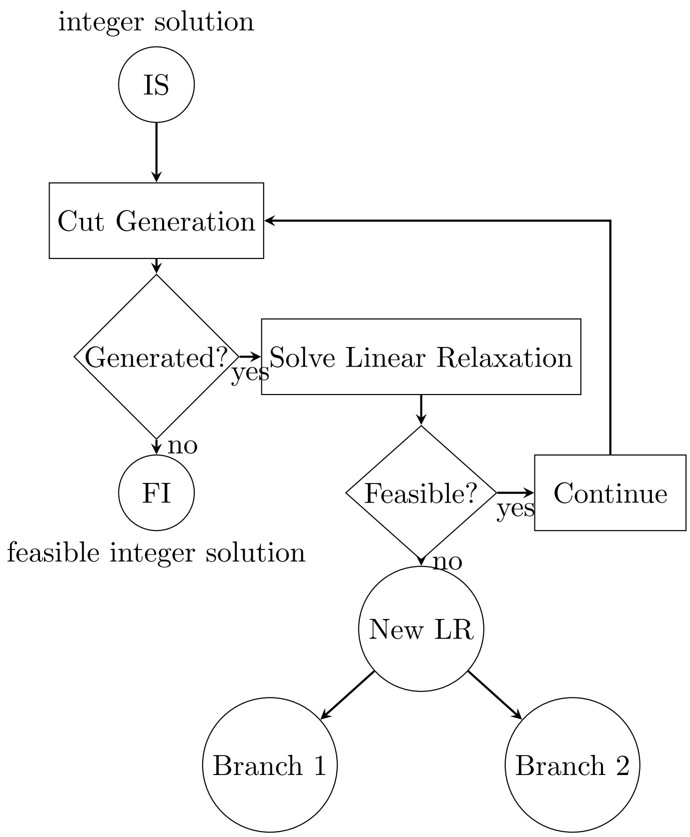


Figure 4: Branch and Cut for Essential Inequality