Special Topic - Formulation Tips

Lan Peng, Ph.D.

School of Management, Shanghai University, Shanghai, China

"Trick or Treat."

1 Linear Program Formulation Tips

Absolute Value Consider the following model statement:

$$\begin{aligned}
\min \quad & \sum_{j \in J} c_j |x_j|, \quad c_j > 0 \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& x_j \quad \text{unrestricted}, \quad \forall j \in J
\end{aligned}$$

Equivalent model:

$$\begin{aligned}
\min \quad & \sum_{j \in J} c_j (x_j^+ + x_j^-), \quad c_j > 0 \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \gtreqless b_i, \quad \forall i \in I \\
& x_j^+, x_j^- \geq 0, \quad \forall j \in J
\end{aligned}$$

Minimax Objective Consider the following model statement:

$$\begin{aligned}
\min \quad & \max_{k \in K} \sum_{j \in J} c_{kj} x_j \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}$$

Equivalent model:

$$\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& \sum_{j \in J} c_{kj} x_j \leq z, \quad \forall k \in K \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}$$

Fractional Objective Consider the following model statement:

$$\begin{aligned}
\min \quad & \frac{\sum_{j\in J} c_j x_j + \alpha}{\sum_{j\in J} d_j x_j + \beta} \\
\text{s.t.} \quad & \sum_{j\in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}$$

Equivalent model:

$$\begin{aligned}
\min \quad & \sum_{j\in J} c_j x_j t + \alpha t \\
\text{s.t.} \quad & \sum_{j\in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& \sum_{j\in J} d_j x_j t + \beta t = 1 \\
& t > 0 \\
& x_j \geq 0, \quad \forall j \in J \\
& (t = \frac{1}{\sum_{j\in J} d_j x_j + \beta})
\end{aligned}$$

Range Constraint Consider the following model statement:

$$\begin{aligned}
\min \quad & \sum_{j\in J} c_j x_j \\
\text{s.t.} \quad & d_i \leq \sum_{j\in J} a_{ij} x_j \leq e_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}$$

Equivalent model:

$$\begin{aligned}
\min \quad & \sum_{j\in J} c_j x_j, \quad c_j > 0 \\
\text{s.t.} \quad & u_i + \sum_{j\in J} a_{ij} x_j = e_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J \\
& 0 \leq u_i \leq e_i - d_i, \quad \forall i \in I
\end{aligned}$$

2 Integer Program Formulation Tips

A Variable Taking Discontinuous Values In algebraic notation:

$$x = 0, \quad \text{or} \quad l \leq x \leq u$$

Equivalent model:

$$
\begin{aligned}
&x \le uy \\
&x \ge ly \\
&y \in \{0,1\}
\end{aligned}
$$

where

$$
y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } l \le x \le u \end{cases}
$$

Fixed Costs In algebraic notation:

$$
C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}
$$

Equivalent model:

$$
\begin{aligned}
&C^*(x, y) = ky + cx \\
&x \le My \\
&x \ge 0 \\
&y \in \{0,1\}
\end{aligned}
$$

where

$$
y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \ge 0 \end{cases}
$$

Either-or Constraints In algebraic notation:

$$
\sum_{j \in J} a_{1j} x_j \le b_1 \text{ or } \sum_{j \in J} a_{2j} x_j \le b_2
$$

Equivalent model:

$$
\begin{aligned}
&\sum_{j \in J} a_{1j} x_j \le b_1 + M_1 y \\
&\sum_{j \in J} a_{2j} x_j \le b_2 + M_1 (1 - y) \\
&y \in \{0,1\}
\end{aligned}
$$

where

$$
y = \begin{cases} 0, & \text{if } \sum_{j \in J} a_{1j} x_j \le b_1 \\ 1, & \text{if } \sum_{j \in J} a_{2j} x_j \le b_2 \end{cases}
$$

Notice that the sign before M is determined by the inequality $\ge$ or $\le$, if it is "$\ge$", use "$-$", if it "$\le$", use "$+$".

Conditional Constraints If constraint A is satisfied, then constraint B must also be satisfied

$$\text{If} \quad \sum_{j \in J} a_{1j}x_j \leq b_1 \text{ then } \sum_{j \in J} a_{2j}x_j \leq b_2$$

The key part is to find the opposite of the first condition. We are using $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$
Therefore it is equivalent to

$$\sum_{j \in J} a_{1j}x_j > b_1 \text{ or } \sum_{j \in J} a_{2j}x_j \leq b_2$$

Furthermore, it is equivalent to

$$\sum_{j \in J} a_{1j}x_j \geq b_1 + \epsilon \text{ or } \sum_{j \in J} a_{2j}x_j \leq b_2$$

Where ϵ is a very small positive number.

Equivalent model:

$$\begin{aligned}
&\sum_{j \in J} a_{1j}x_j \geq b_1 + \epsilon - M_2 y \\
&\sum_{j \in J} a_{2j}x_j \leq b_2 + M_2(1-y) \\
&y \in \{0, 1\}
\end{aligned}$$

Special Ordered Sets SOS1 Description: Out of a set of yes-no decisions, at most one decision variable can be yes.

$$\begin{gathered}
x_1 = 1, x_2 = x_3 = \cdots = x_n = 0 \\
\text{or} \\
x_2 = 1, x_1 = x_3 = \cdots = x_n = 0 \\
\text{or ...}
\end{gathered}$$

Equivalent model:

$$\sum_i x_i = 1, \quad i \in N$$

SOS2 Description: Out of a set of binary variables, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list.

Equivalent model: If $x_1, x_2, ..., x_n$ is a SOS2, then

$$\begin{aligned}
&\sum_{i=1}^{n} x_i \leq 2 \\
&x_i + x_j \leq 1, \forall i \in \{1, 2, ..., n\}, j \in \{i+2, i+3, ..., n\} \\
&x_i \in \{0, 1\}
\end{aligned}$$

Piecewise Linear Formulations The objective function is a sequence of line segments, e.g. $y = f(x)$, consists $k-1$ linear segments going through k given points $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$.

Denote

$$d_i = \begin{cases} 1, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Then the objective function is

$$\sum_{i \in \{1,2,...,k-1\}} y = d_i f_i(x)$$

Equivalent model: Given that objective function as a piecewise linear formulation, we can have these constraints

$$\begin{aligned}
&\sum_{i \in \{1,2,...,k-1\}} d_i = 1 \\
&d_i \in \{0,1\}, i \in \{1,2,...,k-1\} \\
&x = \sum_{i \in \{1,2,...,k\}} w_i x_i \\
&y = \sum_{i \in \{1,2,...,k\}} w_i y_i \\
&w_1 \leq d_1 \\
&w_i \leq d_{i-1} + di, i \in \{2,3,...,k-1\} \\
&w_k \leq d_{k-1}
\end{aligned}$$

In this case, $w_i \in SOS2$ (second definition)

Conditional Binary Variables Choose at most n binary variable to be 1 out of $x_1, x_2, ...x_m, m \geq n$
If $n = 1$ then it is SOS1.

Equivalent model:

$$\sum_{k \in \{1,2,...,m\}} x_k \leq n$$

Choose exactly n binary variable to be 1 out of $x_1, x_2, ...x_m, m \geq n$

Equivalent model:

$$\sum_{k \in \{1,2,...,m\}} x_k = n$$

Choose x_j only if $x_k = 1$

Equivalent model:

$$x_j = x_k$$

"and" condition, iff $x_1, x_2, ..., x_m = 1$ then $y = 1$

Equivalent model:

$$y \leq x_i, i \in \{1,2,...,m\}$$

$$y \geq \sum_{i \in \{1,2,\ldots,m\}} x_i - (m-1)$$

Elimination of Products of Variables For variables x_1 and x_2,

$$y = x_1 x_2$$

Equivalent model: If x_1, x_2 are binary, it is the same as “and” condition of binary variables.
If x_1 is binary, while x_2 is continuous and $0 \leq x_2 \leq u$, then

$$\begin{aligned} &y \leq u x_1 \\ &y \leq x_2 \\ &y \geq x_2 - u(1 - x_1) \\ &y \geq 0 \end{aligned}$$

If both x_1 and x_2 are continuous, it is non-linear, we can use Piecewise linear formulation to simulate.