Notes for Operations Research & More

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4.2.1 Absolute Value

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j |x_j|, \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & x_j \quad \text{unrestricted}, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j (x_j^+ + x_j^-), \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \gtreqless b_i, \quad \forall i \in I \\ & x_j^+, x_j^- \geq 0, \quad \forall j \in J \end{aligned}$$

4.2.2 A Minimax Objective

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \max_{k \in K} \sum_{j \in J} c_{kj} x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & \sum_{j \in J} c_{kj} x_j \leq z, \quad \forall k \in K \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

4.2.3 A fractional Objective

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \frac{\sum_{j\in J} c_j x_j + \alpha}{\sum_{j\in J} d_j x_j + \beta} \\ \text{s.t.} \quad & \sum_{j\in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j x_j t + \alpha t \\ \text{s.t.} \quad & \sum_{j\in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & \sum_{j\in J} d_j x_j t + \beta t = 1 \\ & t > 0 \\ & x_j \geq 0, \quad \forall j \in J \\ & (t = \frac{1}{\sum_{j\in J} d_j x_j + \beta}) \end{aligned}$$

4.2.4 A range Constraint

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j x_j \\ \text{s.t.} \quad & d_i \leq \sum_{j\in J} a_{ij} x_j \leq e_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j x_j, \quad c_j > 0 \\ \text{s.t.} \quad & u_i + \sum_{j\in J} a_{ij} x_j = e_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \\ & 0 \leq u_i \leq e_i - d_i, \quad \forall i \in I \end{aligned}$$

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Formulation

18.1 Typical Problems

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18.2.1 A Variable Taking Discontinuous Values

Description: In algebraic notation:

$$x = 0, \quad \text{or} \quad l \leq x \leq u$$

Modeling:

$$\begin{aligned} & x \leq uy \\ & x \geq ly \\ & y \in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } l \leq x \leq u \end{cases}$$

18.2.2 Fixed Costs

Description: In algebraic notation:

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

Modeling:

$$\begin{aligned} & C^*(x, y) = ky + cx \\ & x \leq My \\ & x \geq 0 \\ & y \in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

18.2.3 Either-or Constraints

Description: In algebraic notation:

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \text{ or } \sum_{j \in J} a_{2j} x_j \leq b_2$$

$\boxed{\textbf{Modeling:}}$

$$
\begin{aligned}
&\sum_{j\in J} a_{1j}x_j \le b_1 + M_1 y \\
&\sum_{j\in J} a_{2j}x_j \le b_2 + M_1(1-y) \\
&y \in \{0,1\}
\end{aligned}
$$

where

$$
y = \begin{cases} 0, & \text{if } \sum_{j\in J} a_{1j}x_j \le b_1 \\ 1, & \text{if } \sum_{j\in J} a_{2j}x_j \le b_2 \end{cases}
$$

Notice that the sign before M is determined by the inequality $\ge$ or $\le$, if it is "$\ge$", use "$-$", if it "$\le$", use "+".

18.2.4 Conditional Constraints

$\boxed{\textbf{Description:}}$ If constraint A is satisfied, then constraint B must also be satisfied

$$
\text{If } \sum_{j\in J} a_{1j}x_j \le b_1 \text{ then } \sum_{j\in J} a_{2j}x_j \le b_2
$$

The key part is to find the opposite of the first condition. We are using $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$
Therefore it is equivalent to

$$
\sum_{j\in J} a_{1j}x_j > b_1 \text{ or } \sum_{j\in J} a_{2j}x_j \le b_2
$$

Furthermore, it is equivalent to

$$
\sum_{j\in J} a_{1j}x_j \ge b_1 + \epsilon \text{ or } \sum_{j\in J} a_{2j}x_j \le b_2
$$

Where ϵ is a very small positive number.

$\boxed{\textbf{Modeling:}}$

$$
\begin{aligned}
&\sum_{j\in J} a_{1j}x_j \ge b_1 + \epsilon - M_2 y \\
&\sum_{j\in J} a_{2j}x_j \le b_2 + M_2(1-y) \\
&y \in \{0,1\}
\end{aligned}
$$

18.2.5 Special Ordered Sets

$\boxed{\textbf{SOS1 Description}}$ Out of a set of yes-no decisions, at most one decision variable can be yes.

$$
\begin{gathered}
x_1 = 1, x_2 = x_3 = \cdots = x_n = 0 \\
\text{or} \\
x_2 = 1, x_1 = x_3 = \cdots = x_n = 0 \\
\text{or ...}
\end{gathered}
$$

$\boxed{\textbf{Modeling:}}$

$$
\sum_i x_i = 1, \quad i \in N
$$

$\boxed{\textbf{SOS2 Description 1}}$ Out of a set of binary variables, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list.

$\boxed{\textbf{Modeling:}}$ If $x_1, x_2, ..., x_n$ is a SOS2, then

$$
\begin{aligned}
&\sum_{i=1}^{n} x_i \le 2 \\
&x_i + x_j \le 1, \forall i \in \{1, 2, ..., n\}, j \in \{i+2, i+3, ..., n\} \\
&x_i \in \{0,1\}
\end{aligned}
$$

SOS2 Description 2 There is another type of definition, that is out of a set of nonnegative variables **not binary here**, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list. All variables summing to 1.
This definition of SOS2 is used in the following section *Piecewise Linear Formulations*

18.2.6 Piecewise Linear Formulations

Description: The objective function is a sequence of line segments, e.g. $y = f(x)$, consists $k-1$ linear segments going through k given points $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$.
Denote

$$d_i = \begin{cases} 1, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Then the objective function is

$$\sum_{i \in \{1,2,...,k-1\}} y = d_i f_i(x)$$

Modeling: Given that objective function as a piecewise linear formulation, we can have these constraints

$$\begin{aligned}
&\sum_{i \in \{1,2,...,k-1\}} d_i = 1 \\
&d_i \in \{0,1\}, i \in \{1,2,...,k-1\} \\
&x = \sum_{i \in \{1,2,...,k\}} w_i x_i \\
&y = \sum_{i \in \{1,2,...,k\}} w_i y_i \\
&w_1 \leq d_1 \\
&w_i \leq d_{i-1} + di, i \in \{2,3,...,k-1\} \\
&w_k \leq d_{k-1}
\end{aligned}$$

In this case, $w_i \in SOS2$ (second definition)

18.2.7 Conditional Binary Variables

Description: Choose at most n binary variable to be 1 out of $x_1, x_2, ...x_m, m \geq n$. If $n = 1$ then it is SOS1.
Modeling:

$$\sum_{k \in \{1,2,...,m\}} x_k \leq n$$

Description: Choose exactly n binary variable to be 1 out of $x_1, x_2, ...x_m, m \geq n$
Modeling:

$$\sum_{k \in \{1,2,...,m\}} x_k = n$$

Description: Choose x_j only if $x_k = 1$
Modeling:

$$x_j = x_k$$

Description: "and" condition, iff $x_1, x_2, ..., x_m = 1$ then $y = 1$
Modeling:

$$\begin{aligned}
&y \leq x_i, i \in \{1,2,...,m\} \\
&y \geq \sum_{i \in \{1,2,...,m\}} x_i - (m-1)
\end{aligned}$$

18.2.8 Elimination of Products of Variables

Description: For variables x_1 and x_2,

$$y = x_1 x_2$$

Modeling: If x_1, x_2 are binary, it is the same as "and" condition of binary variables.
If x_1 is binary, while x_2 is continuous and $0 \leq x_2 \leq u$, then

$$\begin{aligned}
&y \leq u x_1 \\
&y \leq x_2 \\
&y \geq x_2 - u(1 - x_1) \\
&y \geq 0
\end{aligned}$$

If both x_1 and x_2 are continuous, it is non-linear, we can use Piecewise linear formulation to simulate.

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38.1 Relationship between Some Random Variables

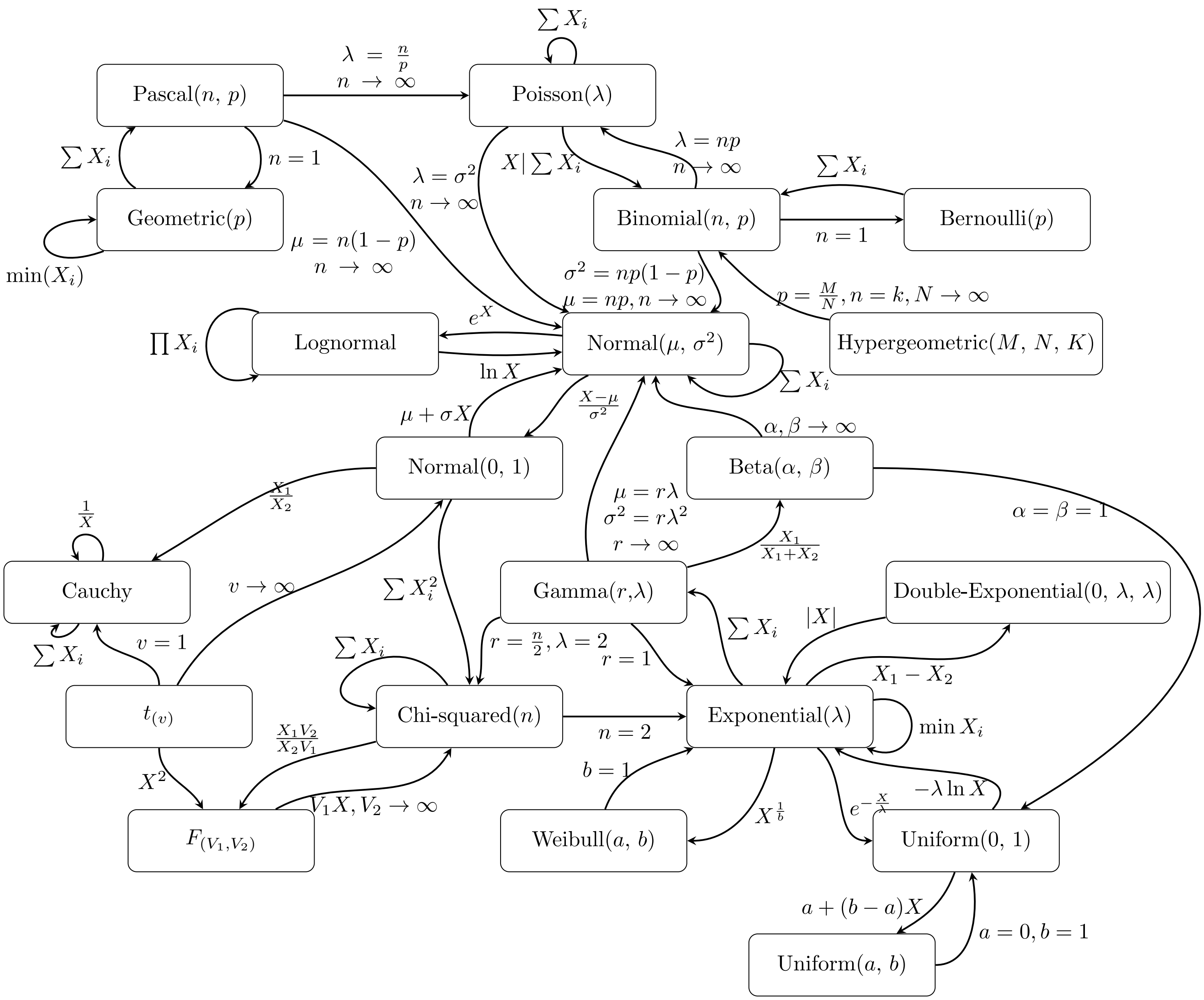

Figure 38.1: Relationship between Some Random Variables

38.2 Discrete Random Variables

Table 38.1: Discrete Random Variables

Distribution	PMF	CDF	Expectation	Variance	MGF
Discrete Uniform(a,b)	$f(x)=\frac{1}{b-a+1}$ $x=a,a+1,...,b$	$F(x)=\frac{x-a+1}{b-a+1}$ $x=a,a+1,...,b$	$E[X]=\frac{b-a}{2}$	$D[X]=\frac{(b-a+1)^2-1}{12}$	$M(t)=\frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$ $t\in\mathbb{R}$
Bernoulli(p)	$f(x)=p^x(1-p)^{1-x}$ $x\in\{0,1\}$	$F(x)=\begin{cases}0, & x<0\\ 1-p, & 0\le x\le 1\\ 1, & x>1\end{cases}$	$E[X]=p$	$D[X]=p(1-p)$	$M(t)=1-p+pe^t$ $t\in\mathbb{R}$
Binomial(n,p)	$f(x)=\binom{n}{x}p^x(1-p)^{n-x}$ $x=0,1,...,n$	$F(x)=\sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k}$ $x=0,1,...,n$	$E[X]=np$	$D[X]=np(1-p)$	$M(t)=(1-p+pe^t)^n$ $t\in\mathbb{R}$
Poisson(μ)	$f(x)=\frac{\mu^x e^{\mu}}{x!}$ $x=0,1,...,n,...$	$f(x)=\frac{\Gamma(x+1,\mu)}{\Gamma(x+1)}$ $x=0,1,...,n,...$	$E[X]=\mu$	$D[X]=\mu$	$M(t)=e^{\mu(e^t-1)}$ $t\in\mathbb{R}$
Geometric(p)	$f(x)=p(1-p)^x$ $x=0,1,...,n,...$	$F(x)=1-(1-p)^{x+1}$ $x=0,1,...,n,...$	$E[X]=\frac{1-p}{p}$	$D[X]=\frac{1-p}{p^2}$	$M(t)=\frac{p}{1-(1-p)e^t}$ $t<-\ln(1-p)$
Pascal(n,p)	$f(x)=\binom{n-1+x}{x}p^n(1-p)^x$ $x=0,1,2,...,n,...$	$F(x)=1-I_p(k+1,n)$ $x=0,1,2,...,n,...$	$E[X]=\frac{n(1-p)}{p}$	$D[X]=\frac{n(1-p)}{p^2}$	$M(t)=(\frac{p}{1-(1-p)e^t})^n$ $t<-\ln(1-p)$

38.3 Continuous Random Variables

Table 38.2: Continuous Random Variables

Distribution	PDF	CDF	Expectation	Variance	MGF
Uniform(a,b)	$f(x)=\frac{1}{b-a}$ $x=[a,b]$	$F(x)=\frac{x-a}{b-a}$ $x=[a,b]$	$E[X]=\frac{b-a}{2}$	$D[X]=\frac{(b-a)^2}{12}$	$M(t)=\begin{cases}1, & t=0\\ \frac{e^{bt}-e^{at}}{t(b-a)}, & t\neq 0\end{cases}$
Normal(μ,σ)	$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in\mathbb{R}$	$F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in\mathbb{R}$	$E[X]=\mu$	$D[X]=\sigma^2$	$e^{\frac{t(t\sigma^2+2\mu)}{2}}$ $t\in\mathbb{R}$
Exponential$\lambda)$	$f(x)=\lambda e^{-\lambda x}$ $x>0$	$F(x)=1-e^{\lambda x}$ $x>0$	$E[X]=\frac{1}{\lambda}$	$D[X]=\frac{1}{\lambda^2}$	$\frac{1}{1-\frac{t}{\lambda}}$ $t<\lambda$
Erlang(n,λ)	$f(x)=\frac{\lambda^n x^{n-1}e^{-\lambda x}}{(n-1)!}$ $x>0$	$F(x)=1-\sum_{i=0}^{n-1}\frac{\lambda^n x^n e^{-\lambda x}}{n!}$ $x>0$	$E[X]=\frac{n}{\lambda}$	$D[X]=\frac{n}{\lambda^2}$	$\frac{1}{(1-\frac{t}{\lambda})^n}$ $t<\lambda$

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