

# Notes for Operations Research & More

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Part I

**Preliminary Topics**





## Chapter 1

# Review of Linear Algebra



## Chapter 2

# Convex Sets



## Chapter 3

# Convex Functions and Generalizations



Part II

Linear Programming





# Chapter 4

## Formulation

### 4.1 Typical Problems

### 4.2 Formulation Skills

#### 4.2.1 Absolute Value

**Description:** Consider the following model statement:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j |x_j|, \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtrless b_i, \quad \forall i \in I \\ & x_j \text{ unrestricted}, \quad \forall j \in J \end{aligned}$$

**Modeling:**

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j (x_j^+ + x_j^-), \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \gtrless b_i, \quad \forall i \in I \\ & x_j^+, x_j^- \geq 0, \quad \forall j \in J \end{aligned}$$

#### 4.2.2 A Minimax Objective

**Description:** Consider the following model statement:

$$\begin{aligned} \min \quad & \max_{k \in K} \sum_{j \in J} c_{kj} x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtrless b_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

**Modeling:**

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtrless b_i, \quad \forall i \in I \\ & \sum_{j \in J} c_{kj} x_j \leq z, \quad \forall k \in K \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

### 4.2.3 A fractional Objective

**Description:** Consider the following model statement:

$$\begin{aligned}
 \min \quad & \frac{\sum_{j \in J} c_j x_j + \alpha}{\sum_{j \in J} d_j x_j + \beta} \\
 \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq b_i, \quad \forall i \in I \\
 & x_j \geq 0, \quad \forall j \in J
 \end{aligned}$$

**Modeling:**

$$\begin{aligned}
 \min \quad & \sum_{j \in J} c_j x_j t + \alpha t \\
 \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq b_i, \quad \forall i \in I \\
 & \sum_{j \in J} d_j x_j t + \beta t = 1 \\
 & t > 0 \\
 & x_j \geq 0, \quad \forall j \in J \\
 & (t = \frac{1}{\sum_{j \in J} d_j x_j + \beta})
 \end{aligned}$$

### 4.2.4 A range Constraint

**Description:** Consider the following model statement:

$$\begin{aligned}
 \min \quad & \sum_{j \in J} c_j x_j \\
 \text{s.t.} \quad & d_i \leq \sum_{j \in J} a_{ij} x_j \leq e_i, \quad \forall i \in I \\
 & x_j \geq 0, \quad \forall j \in J
 \end{aligned}$$

**Modeling:**

$$\begin{aligned}
 \min \quad & \sum_{j \in J} c_j x_j, \quad c_j > 0 \\
 \text{s.t.} \quad & u_i + \sum_{j \in J} a_{ij} x_j = e_i, \quad \forall i \in I \\
 & x_j \geq 0, \quad \forall j \in J \\
 & 0 \leq u_i \leq e_i - d_i, \quad \forall i \in I
 \end{aligned}$$

## Chapter 5

# Simplex Method

### 5.1 Basic Feasible Solutions and Extreme Points

### 5.2 Simplex Method

#### 5.2.1 Simplex Method Algorithm

#### 5.2.2 Simplex Method Tableau

#### 5.2.3 Simplex Method as a Search Algorithm

### 5.3 Revised Simplex Method

### 5.4 Simplex Method with Bounded Variables

### 5.5 Artificial Variable

#### 5.5.1 Two-Phase Method

#### 5.5.2 Big-M Method

#### 5.5.3 Single Artificial Variable Technique

### 5.6 Degeneracy and Cycling

#### 5.6.1 Degeneracy

#### 5.6.2 Cycling

#### 5.6.3 Cycling Prevention Rules

Lexicographic Rule

Bland's Rule

Successive Ratio Rule



## Chapter 6

# Duality Theory and Sensitivity Analysis



## Chapter 7

# Decomposition Principle





## Chapter 8

# Ellipsoid Algorithm



## Chapter 9

# Projective Algorithm



## Chapter 10

# Interior-Point Algorithm



## Part III

# Graph and Network Theory





## Chapter 11

# Paths, Trees, and Cycles



## Chapter 12

# Shortest-Path Problem



## Chapter 13

# Minimum Spanning Tree Problem



## Chapter 14

# Maximum Flow Problem





## Chapter 15

# Minimum Cost Flow Problem



## Chapter 16

# Assignment and Matching Problem



## Chapter 17

# Graph Algorithms



## Part IV

# Integer and Combinatorial Programming





# Chapter 18

## Formulation

### 18.1 Typical Problems

### 18.2 Integer Programming Formulation Skills

#### 18.2.1 A Variable Taking Discontinuous Values

**Description:** In algebraic notation:

$$x = 0, \quad \text{or} \quad l \leq x \leq u$$

**Modeling:**

$$\begin{aligned} x &\leq uy \\ x &\geq ly \\ y &\in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } l \leq x \leq u \end{cases}$$

#### 18.2.2 Fixed Costs

**Description:** In algebraic notation:

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

**Modeling:**

$$\begin{aligned} C^*(x, y) &= ky + cx \\ x &\leq My \\ x &\geq 0 \\ y &\in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

#### 18.2.3 Either-or Constraints

**Description:** In algebraic notation:

$$\sum_{j \in J} a_{1j}x_j \leq b_1 \quad \text{or} \quad \sum_{j \in J} a_{2j}x_j \leq b_2$$

**Modeling:**

$$\begin{aligned}
\sum_{j \in J} a_{1j}x_j &\leq b_1 + M_1y \\
\sum_{j \in J} a_{2j}x_j &\leq b_2 + M_1(1 - y) \\
y &\in \{0, 1\}
\end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } \sum_{j \in J} a_{1j}x_j \leq b_1 \\ 1, & \text{if } \sum_{j \in J} a_{2j}x_j \leq b_2 \end{cases}$$

Notice that the sign before  $M$  is determined by the inequality  $\geq$  or  $\leq$ , if it is “ $\geq$ ”, use “ $-$ ”, if it “ $\leq$ ”, use “ $+$ ”.

### 18.2.4 Conditional Constraints

**Description:** If constraint A is satisfied, then constraint B must also be satisfied

$$\text{If } \sum_{j \in J} a_{1j}x_j \leq b_1 \text{ then } \sum_{j \in J} a_{2j}x_j \leq b_2$$

The key part is to find the opposite of the first condition. We are using  $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$ . Therefore it is equivalent to

$$\sum_{j \in J} a_{1j}x_j > b_1 \text{ or } \sum_{j \in J} a_{2j}x_j \leq b_2$$

Furthermore, it is equivalent to

$$\sum_{j \in J} a_{1j}x_j \geq b_1 + \epsilon \text{ or } \sum_{j \in J} a_{2j}x_j \leq b_2$$

Where  $\epsilon$  is a very small positive number.

**Modeling:**

$$\begin{aligned}
\sum_{j \in J} a_{1j}x_j &\geq b_1 + \epsilon - M_2y \\
\sum_{j \in J} a_{2j}x_j &\leq b_2 + M_2(1 - y) \\
y &\in \{0, 1\}
\end{aligned}$$

### 18.2.5 Special Ordered Sets

**SOS1 Description** Out of a set of yes-no decisions, at most one decision variable can be yes.

$$\begin{aligned}
x_1 = 1, x_2 = x_3 = \dots = x_n = 0 \\
\text{or} \\
x_2 = 1, x_1 = x_3 = \dots = x_n = 0 \\
\text{or } \dots
\end{aligned}$$

**Modeling:**

$$\sum_i x_i = 1, \quad i \in N$$

**SOS2 Description 1** Out of a set of binary variables, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list.

**Modeling:** If  $x_1, x_2, \dots, x_n$  is a SOS2, then

$$\begin{aligned}
\sum_{i=1}^n x_i &\leq 2 \\
x_i + x_j &\leq 1, \forall i \in \{1, 2, \dots, n\}, j \in \{i+2, i+3, \dots, n\} \\
x_i &\in \{0, 1\}
\end{aligned}$$

**SOS2 Description 2** There is another type of definition, that is out of a set of nonnegative variables **not binary here**, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list. All variables summing to 1.

This definition of SOS2 is used in the following section *Piecewise Linear Formulations*

### 18.2.6 Piecewise Linear Formulations

**Description:** The objective function is a sequence of line segments, e.g.  $y = f(x)$ , consists  $k - 1$  linear segments going through  $k$  given points  $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$ .

Denote

$$d_i = \begin{cases} 1, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Then the objective function is

$$\sum_{i \in \{1, 2, \dots, k-1\}} y = d_i f_i(x)$$

**Modeling:** Given that objective function as a piecewise linear formulation, we can have these constraints

$$\begin{aligned} \sum_{i \in \{1, 2, \dots, k-1\}} d_i &= 1 \\ d_i &\in \{0, 1\}, i \in \{1, 2, \dots, k-1\} \\ x &= \sum_{i \in \{1, 2, \dots, k\}} w_i x_i \\ y &= \sum_{i \in \{1, 2, \dots, k\}} w_i y_i \\ w_1 &\leq d_1 \\ w_i &\leq d_{i-1} + d_i, i \in \{2, 3, \dots, k-1\} \\ w_k &\leq d_{k-1} \end{aligned}$$

In this case,  $w_i \in SOS2$  (second definition)

### 18.2.7 Conditional Binary Variables

**Description:** Choose at most  $n$  binary variable to be 1 out of  $x_1, x_2, \dots, x_m, m \geq n$ . If  $n = 1$  then it is SOS1.

**Modeling:**

$$\sum_{k \in \{1, 2, \dots, m\}} x_k \leq n$$

**Description:** Choose exactly  $n$  binary variable to be 1 out of  $x_1, x_2, \dots, x_m, m \geq n$

**Modeling:**

$$\sum_{k \in \{1, 2, \dots, m\}} x_k = n$$

**Description:** Choose  $x_j$  only if  $x_k = 1$

**Modeling:**

$$x_j = x_k$$

**Description:** “and” condition, iff  $x_1, x_2, \dots, x_m = 1$  then  $y = 1$

**Modeling:**

$$\begin{aligned} y &\leq x_i, i \in \{1, 2, \dots, m\} \\ y &\geq \sum_{i \in \{1, 2, \dots, m\}} x_i - (m - 1) \end{aligned}$$

### 18.2.8 Elimination of Products of Variables

**Description:** For variables  $x_1$  and  $x_2$ ,

$$y = x_1 x_2$$

**Modeling:** If  $x_1, x_2$  are binary, it is the same as “and” condition of binary variables. If  $x_1$  is binary, while  $x_2$  is continuous and  $0 \leq x_2 \leq u$ , then

$$y \leq ux_1$$

$$y \leq x_2$$

$$y \geq x_2 - u(1 - x_1)$$

$$y \geq 0$$

If both  $x_1$  and  $x_2$  are continuous, it is non-linear, we can use Piecewise linear formulation to simulate.

## Chapter 19

# Branch and Bound



## Chapter 20

# Branch and Cut





## Chapter 21

# Packing and Matching



## Chapter 22

# Traveling Salesman Problem



## Chapter 23

# Knapsack Problem



Part V

**Nonlinear Programming**





## Chapter 24

# KKT Optimality Conditions



## Chapter 25

# Lagrangian Duality



## Chapter 26

# Unconstrained Optimization



## Chapter 27

# Penalty and Barrier Functions





## Part VI

# Algorithms and Computational Complexity



## Chapter 28

# Computational Complexity



## Chapter 29

# Sorting

29.1 Elementary Sorting Algorithms

29.2 Heap-sort

29.3 Quick-sort

29.4 Sorting in Linear Time

29.5 Medians and Order Statistics



## Chapter 30

# Data Structures

30.1 Elementary Data Structures

30.2 Hash Tables

30.3 Binary Search Trees

30.4 Red-Black Trees

30.5 B-Trees

30.6 Fibonacci Heaps

30.7 van Emde Boas Trees





## Chapter 31

# Design and Analysis Techniques

31.1 Dynamic Programming

31.2 Greedy Algorithms

31.3 Amortized Analysis

31.4 Multi-threaded Algorithms

31.5 Matrix Operations

31.6 Polynomials and the FFT

31.7 Number-Theoretic Algorithms

31.8 String Matching

31.9 Computational Geometry

31.10 NP-Completeness

31.11 Approximation Algorithms



## Part VII

# Heuristics and Meta-heuristics



Part VIII

Game Theory



## Chapter 32

# Games with Ordinal Payoffs

**32.1 Ordinal Games in Strategic Form**

**32.2 Perfect-information Games**

**32.3 General Dynamic Games**





## Chapter 33

# Games with Cardinal Payoffs

33.1 Expected Utility Theory

33.2 Strategic-form Games

33.3 Extensive-form Games



## Chapter 34

# Knowledge, Common Knowledge, Beliefs

34.1 Common Knowledge

34.2 Adding Beliefs to Knowledge

34.3 Rationality



## Chapter 35

# Refinements of Subgame-perfect Equilibrium

35.1 Weak Sequential Equilibrium

35.2 Sequential Equilibrium

35.3 Perfect Bayesian Equilibrium



## Chapter 36

# Incomplete Information

36.1 Static Games

36.2 Dynamic Games

36.3 The Type-Space Approach





Part IX

Decision Analysis



## Part X

# Probability, Stochastic Processes and Markov Chains



## Chapter 37

# Discrete-Time Markov Chains



## Chapter 38

# Continuous-Time Markov Chains





Part XI

Queuing Theory



## Chapter 39

# Queuing Model



## Chapter 40

# Birth-and-Death Queuing Models



## Chapter 41

# Multidimensional Birth-and-Death Queuing Models





## Chapter 42

# Phase-Type Queue



## Chapter 43

# Bulk Queue



## Chapter 44

# Imbedded-Markov-Chain Queuing Models



## Chapter 45

# Queuing Network





Part XII

Inventory Theory



Part XIII

Reliability Theory



## 45.1 Maintenance Optimization



Part XIV

Bayesian Statistic





Part XV

Classical Statistic



**Part XVI**

**Simulation**

