

Notes for Operations Research & More

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Part I

Preliminary Topics

Chapter 1

Review of Linear Algebra

Chapter 2

Convex Sets

Chapter 3

Convex Functions and Generalizations

Part II

Linear Programming

Chapter 4

Formulation

4.1 Typical Problems

4.2 Formulation Skills

4.2.1 Absolute Value

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j |x_j|, \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtrless b_i, \quad \forall i \in I \\ & x_j \text{ unrestricted}, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j (x_j^+ + x_j^-), \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \gtrless b_i, \quad \forall i \in I \\ & x_j^+, x_j^- \geq 0, \quad \forall j \in J \end{aligned}$$

4.2.2 A Minimax Objective

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \max_{k \in K} \sum_{j \in J} c_{kj} x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtrless b_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtrless b_i, \quad \forall i \in I \\ & \sum_{j \in J} c_{kj} x_j \leq z, \quad \forall k \in K \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

4.2.3 A fractional Objective

Description: Consider the following model statement:

$$\begin{aligned}
 \min \quad & \frac{\sum_{j \in J} c_j x_j + \alpha}{\sum_{j \in J} d_j x_j + \beta} \\
 \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq b_i, \quad \forall i \in I \\
 & x_j \geq 0, \quad \forall j \in J
 \end{aligned}$$

Modeling:

$$\begin{aligned}
 \min \quad & \sum_{j \in J} c_j x_j t + \alpha t \\
 \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq b_i, \quad \forall i \in I \\
 & \sum_{j \in J} d_j x_j t + \beta t = 1 \\
 & t > 0 \\
 & x_j \geq 0, \quad \forall j \in J \\
 & (t = \frac{1}{\sum_{j \in J} d_j x_j + \beta})
 \end{aligned}$$

4.2.4 A range Constraint

Description: Consider the following model statement:

$$\begin{aligned}
 \min \quad & \sum_{j \in J} c_j x_j \\
 \text{s.t.} \quad & d_i \leq \sum_{j \in J} a_{ij} x_j \leq e_i, \quad \forall i \in I \\
 & x_j \geq 0, \quad \forall j \in J
 \end{aligned}$$

Modeling:

$$\begin{aligned}
 \min \quad & \sum_{j \in J} c_j x_j, \quad c_j > 0 \\
 \text{s.t.} \quad & u_i + \sum_{j \in J} a_{ij} x_j = e_i, \quad \forall i \in I \\
 & x_j \geq 0, \quad \forall j \in J \\
 & 0 \leq u_i \leq e_i - d_i, \quad \forall i \in I
 \end{aligned}$$

Chapter 5

Simplex Method

5.1 Basic Feasible Solutions and Extreme Points

5.2 Simplex Method

5.2.1 Simplex Method Algorithm

5.2.2 Simplex Method Tableau

5.2.3 Simplex Method as a Search Algorithm

5.3 Revised Simplex Method

5.4 Simplex Method with Bounded Variables

5.5 Artificial Variable

5.5.1 Two-Phase Method

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5.5.3 Single Artificial Variable Technique

5.6 Degeneracy and Cycling

5.6.1 Degeneracy

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5.6.3 Cycling Prevention Rules

Lexicographic Rule

Bland's Rule

Successive Ratio Rule

Chapter 6

Duality Theory and Sensitivity Analysis

Chapter 7

Decomposition Principle

Chapter 8

Ellipsoid Algorithm

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Projective Algorithm

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Interior-Point Algorithm

Part III

Graph and Network Theory

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Paths, Trees, and Cycles

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Shortest-Path Problem

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Graph Algorithms

Part IV

Integer and Combinatorial Programming

Chapter 18

Formulation

18.1 Typical Problems

18.2 Integer Programming Formulation Skills

18.2.1 A Variable Taking Discontinuous Values

Description: In algebraic notation:

$$x = 0, \quad \text{or} \quad l \leq x \leq u$$

Modeling:

$$\begin{aligned} x &\leq uy \\ x &\geq ly \\ y &\in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } l \leq x \leq u \end{cases}$$

18.2.2 Fixed Costs

Description: In algebraic notation:

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

Modeling:

$$\begin{aligned} C^*(x, y) &= ky + cx \\ x &\leq My \\ x &\geq 0 \\ y &\in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

18.2.3 Either-or Constraints

Description: In algebraic notation:

$$\sum_{j \in J} a_{1j}x_j \leq b_1 \quad \text{or} \quad \sum_{j \in J} a_{2j}x_j \leq b_2$$

Modeling:

$$\begin{aligned}
\sum_{j \in J} a_{1j}x_j &\leq b_1 + M_1y \\
\sum_{j \in J} a_{2j}x_j &\leq b_2 + M_1(1 - y) \\
y &\in \{0, 1\}
\end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } \sum_{j \in J} a_{1j}x_j \leq b_1 \\ 1, & \text{if } \sum_{j \in J} a_{2j}x_j \leq b_2 \end{cases}$$

Notice that the sign before M is determined by the inequality \geq or \leq , if it is “ \geq ”, use “ $-$ ”, if it “ \leq ”, use “ $+$ ”.

18.2.4 Conditional Constraints

Description: If constraint A is satisfied, then constraint B must also be satisfied

$$\text{If } \sum_{j \in J} a_{1j}x_j \leq b_1 \text{ then } \sum_{j \in J} a_{2j}x_j \leq b_2$$

The key part is to find the opposite of the first condition. We are using $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$. Therefore it is equivalent to

$$\sum_{j \in J} a_{1j}x_j > b_1 \text{ or } \sum_{j \in J} a_{2j}x_j \leq b_2$$

Furthermore, it is equivalent to

$$\sum_{j \in J} a_{1j}x_j \geq b_1 + \epsilon \text{ or } \sum_{j \in J} a_{2j}x_j \leq b_2$$

Where ϵ is a very small positive number.

Modeling:

$$\begin{aligned}
\sum_{j \in J} a_{1j}x_j &\geq b_1 + \epsilon - M_2y \\
\sum_{j \in J} a_{2j}x_j &\leq b_2 + M_2(1 - y) \\
y &\in \{0, 1\}
\end{aligned}$$

18.2.5 Special Ordered Sets

SOS1 Description Out of a set of yes-no decisions, at most one decision variable can be yes.

$$\begin{aligned}
x_1 = 1, x_2 = x_3 = \dots = x_n = 0 \\
\text{or} \\
x_2 = 1, x_1 = x_3 = \dots = x_n = 0 \\
\text{or } \dots
\end{aligned}$$

Modeling:

$$\sum_i x_i = 1, \quad i \in N$$

SOS2 Description 1 Out of a set of binary variables, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list.

Modeling: If x_1, x_2, \dots, x_n is a SOS2, then

$$\begin{aligned}
\sum_{i=1}^n x_i &\leq 2 \\
x_i + x_j &\leq 1, \forall i \in \{1, 2, \dots, n\}, j \in \{i+2, i+3, \dots, n\} \\
x_i &\in \{0, 1\}
\end{aligned}$$

SOS2 Description 2 There is another type of definition, that is out of a set of nonnegative variables **not binary here**, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list. All variables summing to 1.

This definition of SOS2 is used in the following section *Piecewise Linear Formulations*

18.2.6 Piecewise Linear Formulations

Description: The objective function is a sequence of line segments, e.g. $y = f(x)$, consists $k - 1$ linear segments going through k given points $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$.

Denote

$$d_i = \begin{cases} 1, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Then the objective function is

$$\sum_{i \in \{1, 2, \dots, k-1\}} y = d_i f_i(x)$$

Modeling: Given that objective function as a piecewise linear formulation, we can have these constraints

$$\begin{aligned} \sum_{i \in \{1, 2, \dots, k-1\}} d_i &= 1 \\ d_i &\in \{0, 1\}, i \in \{1, 2, \dots, k-1\} \\ x &= \sum_{i \in \{1, 2, \dots, k\}} w_i x_i \\ y &= \sum_{i \in \{1, 2, \dots, k\}} w_i y_i \\ w_1 &\leq d_1 \\ w_i &\leq d_{i-1} + d_i, i \in \{2, 3, \dots, k-1\} \\ w_k &\leq d_{k-1} \end{aligned}$$

In this case, $w_i \in SOS2$ (second definition)

18.2.7 Conditional Binary Variables

Description: Choose at most n binary variable to be 1 out of $x_1, x_2, \dots, x_m, m \geq n$. If $n = 1$ then it is SOS1.

Modeling:

$$\sum_{k \in \{1, 2, \dots, m\}} x_k \leq n$$

Description: Choose exactly n binary variable to be 1 out of $x_1, x_2, \dots, x_m, m \geq n$

Modeling:

$$\sum_{k \in \{1, 2, \dots, m\}} x_k = n$$

Description: Choose x_j only if $x_k = 1$

Modeling:

$$x_j = x_k$$

Description: “and” condition, iff $x_1, x_2, \dots, x_m = 1$ then $y = 1$

Modeling:

$$\begin{aligned} y &\leq x_i, i \in \{1, 2, \dots, m\} \\ y &\geq \sum_{i \in \{1, 2, \dots, m\}} x_i - (m - 1) \end{aligned}$$

18.2.8 Elimination of Products of Variables

Description: For variables x_1 and x_2 ,

$$y = x_1 x_2$$

Modeling: If x_1, x_2 are binary, it is the same as “and” condition of binary variables. If x_1 is binary, while x_2 is continuous and $0 \leq x_2 \leq u$, then

$$y \leq ux_1$$

$$y \leq x_2$$

$$y \geq x_2 - u(1 - x_1)$$

$$y \geq 0$$

If both x_1 and x_2 are continuous, it is non-linear, we can use Piecewise linear formulation to simulate.

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Branch and Bound

Chapter 20

Branch and Cut

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Packing and Matching

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Traveling Salesman Problem

Chapter 23

Knapsack Problem

Part V

Nonlinear Programming

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KKT Optimality Conditions

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Lagrangian Duality

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Unconstrained Optimization

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Penalty and Barrier Functions

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Games with Cardinal Payoffs

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Part X

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Part XI

Markov Chains

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Discrete-Time Markov Chains

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Continuous-Time Markov Chains

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Queuing Model

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Birth-and-Death Queuing Models

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Multidimensional Birth-and-Death Queuing Models

Chapter 42

Phase-Type Queue

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Bulk Queue

Chapter 44

Imbedded-Markov-Chain Queuing Models

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Queuing Network

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Inventory Theory

Part XIV

Reliability Theory

Part XV

Maintenance Policy

Part XVI

Bayesian Statistic

Part XVII

Classical Statistic

Part XVIII

Simulation

