Part I

Preliminary Topics

Chapter 1

Introduction to Optimization

1.1 Optimization Model

The following is the basic forms of terminology:

$$
\begin{aligned}
\text{(P)} \quad \min \quad & f(x) && (1.1)\\
\text{s.t.} \quad & g_i(x) \le 0, \quad i = 1, 2, ..., m && (1.2)\\
& h_j(x) = 0, \quad j = 1, 2, ..., l && (1.3)\\
& x \in X && (1.4)
\end{aligned}
$$

We have

- - $x \in R^n \to X \subseteq R^m$
- $g_i(x)$ are called inequality constraints
- $h_j(x)$ are called equality constraints
- X is the domain of the variables (e.g. cone, polygon, $\{0, 1\}^n$, etc.)
- Let F be the feasible region of (P):
  - x^0 is a feasible solution iff $x^0 \in F$
  - x^* is an optimized solution iff $x^* \in F$ and $f(x^*) \le f(x^0), \forall x^0 \in F$ (for minimized problem)

Notice: Not every (P) has a feasible region, we can have $F = \emptyset$. Even if $F \ne \emptyset$, there might not be an solution to P, e.g. unbounded. If (P) has optimized solution(s), it could be 1) Unique 2) Infinite number of solution 3) Finite number of solution

Types of Optimization Problem

- $m = l = 0$, $x \in R^n$, unconstrained problem
- $m + l > 0$, constrained problem
- $f(x), g_i(x), h_j(x)$ are linear, Linear Optimization
  - If $X = R^n$, Linear Programming
  - If X is discrete, Discrete Optimization
  - If $X \subseteq Z^n$, Integer Programming
  - If $X \in \{0, 1\}^n$, Binary Programming
  - If $X \in Z^n \times R^m$, Mixed Integer Programming

1.2 Problem Manipulation

1.2.1 Inequalities and Equalities

An inequality can be transformed into an equation by adding or subtracting the nonnegative slack or surplus variable

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j - x_{n+1} = b_i \tag{1.5}$$

or

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j + x_{n+1} = b_i \tag{1.6}$$

Although it is not the practice, equality can be transformed into inequality too

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Rightarrow \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \leq b_i \\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i \end{cases} \tag{1.7}$$

Also, in linear programming, we only care about close set, so we will not have $<$, $>$ in the formulation, we can use the following

$$\sum_{j=1}^{n} a_{ij}x_j > b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j \geq b_i + \epsilon \tag{1.8}$$

where ϵ is a small number.

1.2.2 Minimization and Maximization

To convert a minimization problem into a maximization problem, we can use the following to define a new objective function

$$\min \sum_{j=1}^{n} c_j x_j = -\max \sum_{j=1}^{n} c_j x_j \tag{1.9}$$

1.2.3 Standard Form and Canonical Form

Standard Form

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j && (1.10) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} && (1.11) \\ & \mathbf{x} \geq \mathbf{0} && (1.12) \end{aligned}$$

Canonical Form

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j && (1.13) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} && (1.14) \\ & \mathbf{x} \geq \mathbf{0} && (1.15) \end{aligned}$$

1.3 Typical Linear Programming Problems

1.4 Linear Programming Formulation Skills

1.4.1 Absolute Value

Consider the following model statement:

$$\min \quad \sum_{j\in J} c_j|x_j|, \quad c_j > 0 \tag{1.16}$$

$$\text{s.t.} \quad \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I \tag{1.17}$$

$$x_j \quad \text{unrestricted}, \quad \forall j \in J \tag{1.18}$$

Modeling:

$$\min \quad \sum_{j\in J} c_j(x_j^+ + x_j^-), \quad c_j > 0 \tag{1.19}$$

$$\text{s.t.} \quad \sum_{j\in J} a_{ij}(x_j^+ - x_j^-) \gtreqless b_i, \quad \forall i \in I \tag{1.20}$$

$$x_j^+, x_j^- \geq 0, \quad \forall j \in J \tag{1.21}$$

1.4.2 A Minimax Objective

Consider the following model statement:

$$\min \quad \max_{k\in K} \sum_{j\in J} c_{kj}x_j \tag{1.22}$$

$$\text{s.t.} \quad \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I \tag{1.23}$$

$$x_j \geq 0, \quad \forall j \in J \tag{1.24}$$

Modeling:

$$\min \quad z \tag{1.25}$$

$$\text{s.t.} \quad \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I \tag{1.26}$$

$$\sum_{j\in J} c_{kj}x_j \leq z, \quad \forall k \in K \tag{1.27}$$

$$x_j \geq 0, \quad \forall j \in J \tag{1.28}$$

1.4.3 A Fractional Objective

Consider the following model statement:

$$\min \quad \frac{\sum_{j\in J} c_jx_j + \alpha}{\sum_{j\in J} d_jx_j + \beta} \tag{1.29}$$

$$\text{s.t.} \quad \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I \tag{1.30}$$

$$x_j \geq 0, \quad \forall j \in J \tag{1.31}$$

Modeling:

$$\min \quad \sum_{j\in J} c_j x_j t + \alpha t \tag{1.32}$$
$$\text{s.t.} \quad \sum_{j\in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in J \tag{1.33}$$
$$\sum_{j\in J} d_j x_j t + \beta t = 1 \tag{1.34}$$
$$t > 0 \tag{1.35}$$
$$x_j \geq 0, \quad \forall j \in J \tag{1.36}$$
$$(t = \frac{1}{\sum_{j\in J} d_j x_j + \beta}) \tag{1.37}$$

For the following statement:

$$\min \quad z^P = \frac{\mathbf{c}^\top \mathbf{x} + d}{\mathbf{e}^\top \mathbf{x} + f} \tag{1.38}$$
$$\text{s.t.} \quad \mathbf{G}\mathbf{x} \leq \mathbf{h} \tag{1.39}$$
$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1.40}$$

Modeling

$$\min \quad z^R = \mathbf{c}^\top \mathbf{y} + dz \tag{1.41}$$
$$\text{s.t.} \quad \mathbf{G}\mathbf{y} - \mathbf{h}z \leq 0 \tag{1.42}$$
$$\mathbf{A}\mathbf{y} - \mathbf{b}z = 0 \tag{1.43}$$
$$\mathbf{e}^\top \mathbf{y} + fz = 1 \tag{1.44}$$
$$z \geq 0 \tag{1.45}$$

1.4.4 A Range Constraint

Consider the following model statement:

$$\min \quad \sum_{j\in J} c_j x_j \tag{1.46}$$
$$\text{s.t.} \quad d_i \leq \sum_{j\in J} a_{ij} x_j \leq e_i, \quad \forall i \in I \tag{1.47}$$
$$x_j \geq 0, \quad \forall j \in J \tag{1.48}$$

Modeling:

$$\min \quad \sum_{j\in J} c_j x_j, \quad c_j > 0 \tag{1.49}$$
$$\text{s.t.} \quad u_i + \sum_{j\in J} a_{ij} x_j = e_i, \quad \forall i \in I \tag{1.50}$$
$$x_j \geq 0, \quad \forall j \in J \tag{1.51}$$
$$0 \leq u_i \leq e_i - d_i, \quad \forall i \in I \tag{1.52}$$

1.5 Typical Integer Programming Problems

1.6 Integer Programming Formulation Skills

1.6.1 A Variable Taking Discontinuous Values

In algebraic notation:

$$x = 0, \quad \text{or} \quad l \leq x \leq u \tag{1.53}$$

Modeling:

$$x \leq uy \tag{1.54}$$
$$x \geq ly \tag{1.55}$$
$$y \in \{0, 1\} \tag{1.56}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } l \leq x \leq u \end{cases} \tag{1.57}$$

1.6.2 Fixed Costs

In algebraic notation:

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases} \tag{1.58}$$

Modeling:

$$C^*(x, y) = ky + cx \tag{1.59}$$
$$x \leq My \tag{1.60}$$
$$x \geq 0 \tag{1.61}$$
$$y \in \{0, 1\} \tag{1.62}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \geq 0 \end{cases} \tag{1.63}$$

1.6.3 Either-or Constraints

In algebraic notation:

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \text{ or } \sum_{j \in J} a_{2j} x_j \leq b_2 \tag{1.64}$$

Modeling:

$$\sum_{j \in J} a_{1j} x_j \leq b_1 + M_1 y \tag{1.65}$$
$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_1 (1 - y) \tag{1.66}$$
$$y \in \{0, 1\} \tag{1.67}$$

where

$$y = \begin{cases} 0, & \text{if } \sum_{j \in J} a_{1j} x_j \leq b_1 \\ 1, & \text{if } \sum_{j \in J} a_{2j} x_j \leq b_2 \end{cases} \tag{1.68}$$

Notice that the sign before M is determined by the inequality $\geq$ or $\leq$, if it is "$\geq$", use "$-$", if it "$\leq$", use "$+$".

1.6.4 Conditional Constraints

If constraint A is satisfied, then constraint B must also be satisfied

$$\text{If} \quad \sum_{j \in J} a_{1j} x_j \leq b_1 \text{ then } \sum_{j \in J} a_{2j} x_j \leq b_2 \tag{1.69}$$

The key part is to find the opposite of the first condition. We are using $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$
Therefore it is equivalent to

$$\sum_{j \in J} a_{1j} x_j > b_1 \text{ or } \sum_{j \in J} a_{2j} x_j \leq b_2 \tag{1.70}$$

Furthermore, it is equivalent to

$$\sum_{j \in J} a_{1j} x_j \geq b_1 + \epsilon \text{ or } \sum_{j \in J} a_{2j} x_j \leq b_2 \tag{1.71}$$

Where ϵ is a very small positive number.
Modeling:

$$\sum_{j \in J} a_{1j} x_j \geq b_1 + \epsilon - M_2 y \tag{1.72}$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_2(1-y) \tag{1.73}$$

$$y \in \{0,1\} \tag{1.74}$$

1.6.5 Special Ordered Sets

Out of a set of yes-no decisions, at most one decision variable can be yes. Also known as SOS1.

$$x_1 = 1, x_2 = x_3 = \cdots = x_n = 0 \tag{1.75}$$

$$\text{or} \tag{1.76}$$

$$x_2 = 1, x_1 = x_3 = \cdots = x_n = 0 \tag{1.77}$$

$$\text{or ...} \tag{1.78}$$

Modeling:

$$\sum_i x_i = 1, \quad i \in N \tag{1.79}$$

Out of a set of binary variables, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list. Also known as SOS2. Modeling: If $x_1, x_2, ..., x_n$ is a SOS2, then

$$\sum_{i=1}^{n} x_i \leq 2 \tag{1.80}$$

$$x_i + x_j \leq 1, \forall i \in \{1,2,...,n\}, j \in \{i+2, i+3, ..., n\} \tag{1.81}$$

$$x_i \in \{0,1\} \tag{1.82}$$

There is another type of definition, that is out of a set of nonnegative variables **not binary here**, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list. All variables summing to 1.
This definition of SOS2 is used in Piecewise Linear Formulations.

1.6.6 Piecewise Linear Formulations

The objective function is a sequence of line segments, e.g. $y = f(x)$, consists $k-1$ linear segments going through k given points $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$.
Denote

$$d_i = \begin{cases} 1, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases} \tag{1.83}$$

Then the objective function is

$$\sum_{i \in \{1,2,...,k-1\}} y = d_i f_i(x) \tag{1.84}$$

Modeling: Given that objective function as a piecewise linear formulation, we can have these constraints

$$\sum_{i\in\{1,2,...,k-1\}} d_i = 1 \tag{1.85}$$
$$d_i \in \{0,1\}, i \in \{1,2,...,k-1\} \tag{1.86}$$
$$x = \sum_{i\in\{1,2,...,k\}} w_i x_i \tag{1.87}$$
$$y = \sum_{i\in\{1,2,...,k\}} w_i y_i \tag{1.88}$$
$$w_1 \leq d_1 \tag{1.89}$$
$$w_i \leq d_{i-1} + di, i \in \{2,3,...,k-1\} \tag{1.90}$$
$$w_k \leq d_{k-1} \tag{1.91}$$

In this case, $w_i \in SOS2$ (second definition)

1.6.7 Conditional Binary Variables

Choose at most n binary variable to be 1 out of $x_1, x_2, ...x_m, m \geq n$. If $n = 1$ then it is SOS1.
Modeling:

$$\sum_{k\in\{1,2,...,m\}} x_k \leq n \tag{1.92}$$

Choose exactly n binary variable to be 1 out of $x_1, x_2, ...x_m, m \geq n$
Modeling:

$$\sum_{k\in\{1,2,...,m\}} x_k = n \tag{1.93}$$

Choose x_j only if $x_k = 1$
Modeling:

$$x_j = x_k \tag{1.94}$$

"and" condition, iff $x_1, x_2, ..., x_m = 1$ then $y = 1$
Modeling:

$$y \leq x_i, i \in \{1,2,...,m\} \tag{1.95}$$
$$y \geq \sum_{i\in\{1,2,...,m\}} x_i - (m-1) \tag{1.96}$$

1.6.8 Elimination of Products of Variables

For variables x_1 and x_2,

$$y = x_1 x_2 \tag{1.97}$$

Modeling: If x_1, x_2 are binary, it is the same as "and" condition of binary variables.
If x_1 is binary, while x_2 is continuous and $0 \leq x_2 \leq u$, then

$$y \leq u x_1 \tag{1.98}$$
$$y \leq x_2 \tag{1.99}$$
$$y \geq x_2 - u(1 - x_1) \tag{1.100}$$
$$y \geq 0 \tag{1.101}$$

If both x_1 and x_2 are continuous, it is non-linear, we can use Piecewise linear formulation to simulate.

1.7 Typical Nonlinear Programming Problems

1.8 Nonlinear Programming Formulation Skills

Chapter 2

Review of Linear Algebra

2.1 Field

Definition 2.1.1 (Field)**.** Let F denote either the set of real numbers or the set of complex numbers.

- Addition is commutative: $x + y = y + x, \forall x, y \in F$
- Addition is associative: $x + (y + z) = (x + y) + z, \forall x, y, z \in F$
- Element 0 exists and unique: $\exists 0, x + 0 = x, \forall x \in F$
- To each $x \in F$ there corresponds a unique element $(-x) \in F$ such that $x + (-x) = 0$
- Multiplication is commutative: $xy = yx, \forall x, y \in F$
- Multiplication is associative: $x(yz) = (xy)z, \forall x, y, z \in F$
- Element 1 exists and unique: $\exists 1, x1 = x, \forall x \in F$
- To each $x \neq 0 \in F$ there corresponds a unique element $x^{-1} \in F$ that $xx^{-1} = 1$
- Multiplication distributes over addition: $x(y + z) = xy + xz, \forall x, y, z \in F$

Suppose one has a set F of objects $x, y, z, ...$ and two operations on the elements of F as follows. The first operation, called addition, associates with each pair of elements $x, y \in F$ an element $(x + y) \in F$; the second operation, called multiplication, associates with each pair x, y an element $xy \in F$; and these two operations satisfy all conditions above. The set F, together with these two operations, is then called a **field**.

Definition 2.1.2 (Subfield)**.** A **subfield** of the field C is a set F of complex numbers which itself is a field.

Example. The set of integers is not a field.

Example. The set of rational numbers is a field.

Example. The set of all complex numbers of the form $x + y\sqrt{2}$ where x and y are rational, is a subfield of $\mathbb{C}$.

Notice: In this note, we (...Lan) assume that the field involved is a subfield of the complex numbers $\mathbb{C}$. More generally, if F is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0, which does not happen in the subfield of $\mathbb{C}$. If it does happen in F, the least n such that the sum of n 1's is 0 is called **characteristic** of the field F. If it does not happen, then F is called a field of **characteristic zero**.

2.2 Real Vector Spaces

2.3 Linear, Conic, Affine, and Convex Combinations

2.4 Determinants

2.5 Inner Products

Definition 2.5.1 (Inner Product)**.** Let F be the field of real numbers or the field of complex numbers, and V a vector space over F. An **inner product** on V is a function which assigns to each ordered pair of vectors α, β in V a scalar $< \alpha|\beta >$ in F in such a way that $\forall \alpha, \beta, \gamma \in V, c \in \mathbb{R}$ that

- $< \alpha + \beta|\gamma > = < \alpha|\gamma > + < \beta|\gamma >$
- $< c\alpha|\beta > = c < \alpha|\beta >$
- $< \alpha|\beta > = \overline{< \beta|\alpha >}$
- $< \alpha|\alpha > \geq 0$, $< \alpha|\alpha > = 0$ iff $\alpha = \mathbf{0}$

Furthermore, the above properties imply that

- $< \alpha|c\beta + \gamma > = \bar{c} < \alpha|\beta > + < \alpha|\gamma >$

Definition 2.5.2. On F^n there is an inner product which we call the **standard inner product**. It is defined on $\alpha = (x_1, x_2, ..., x_n)$ and $\beta = (y_1, y_2, ..., y_n)$ by

$$< \alpha|\beta > = \sum_j x_j \bar{y_j} \tag{2.1}$$

For $F = \mathbb{R}^n$

$$< \alpha|\beta > = \sum_j x_j y_j \tag{2.2}$$

In the real case, the standard inner product is often called the dot product and denoted by $\alpha \cdot \beta$

Example. For $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in $\mathbb{R}^2$, the following is an inner product.

$$< \alpha|\beta > = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2 \tag{2.3}$$

Example. For $\mathbb{C}^{n\times n}$,

$$< \mathbf{A}|\mathbf{B} > = trace(\mathbf{B}^*\mathbf{A}) \tag{2.4}$$

is an inner product, where

$$\mathbf{A}^*_{ij} = \bar{\mathbf{A}}_{ji} \quad \textbf{(conjugate transpose)} \tag{2.5}$$

For $\mathbb{R}^{n\times n}$,

$$< \mathbf{A}|\mathbf{B} > = trace(\mathbf{B}^T\mathbf{A}) = \sum_j (AB^T)_{jj} = \sum_j \sum_k A_{jk}B_{jk} \tag{2.6}$$

2.6 Norms

Definition 2.6.1 (Norms)**.** A **norm** on a vector space $\mathcal{V}$ is a function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ for which the following three properties hold for all point $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and scalars $\lambda \in \mathbb{R}$

- (Absolute homogeneity) $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$
- (Triangle inequality) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
- (Positivity) Equality $\|\mathbf{x}\| = 0$ holds iff $\mathbf{x} = 0$

Definition 2.6.2 (L_p-norms)**.** Let $p \geq 1$ be a real number. We define the p-norm of vector $\mathbf{v} \in \mathbb{R}^n$ as:

$$\|\mathbf{x}\|_p = (\sum_{i=1}^{n} |v_i|^p)^{\frac{1}{p}} \tag{2.7}$$

Particularly

$$\|\mathbf{v}\|_1 = \sum_{i=1}^{n} |v_i| \tag{2.8}$$

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{2.9}$$

$$\|\mathbf{v}\|_\infty = \max_{i=1}^{n} |v_i| \tag{2.10}$$

Definition 2.6.3 (Frobenius norm)**.** $\mathbf{X} \in \mathbb{R}^{m\times n}$, the **Frobenius norm** is defined as

$$\|\mathbf{X}\|_F = \sqrt{trace(\mathbf{X}^\top \mathbf{X})} \tag{2.11}$$

Definition 2.6.4 (Dual norm)**.** For an arbitrary norm $\|\cdot\|$ on Euclidean space $\mathbf{E}$, the **dual norm** $\|\cdot\|^*$ on $\mathbf{E}$ is defined by

$$\|\mathbf{v}\|^* = \max\{< \mathbf{v}|\mathbf{x} > | \|\mathbf{x}\| \leq 1\} \tag{2.12}$$

For $p, q \in [1, \infty]$, the l_p and l_q norms on $\mathbb{R}^n$ are dual to each other whenever $\frac{1}{p} + \frac{1}{q} = 1$.

2.7 Eigenvectors and Eigenvalues

Definition 2.7.1. If $\mathbf{A}$ is an $n \times n$ matrix, then a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ is called an **eigenvector** of $\mathbf{A}$ if $\mathbf{Ax}$ is a scalar multiple of $\mathbf{x}$, i.e.

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{2.13}$$

for some scalar λ. The scalar λ is called **eigenvalue** of $\mathbf{A}$ and the vector $\mathbf{x}$ is said to be an **eigenvector corresponding to** λ

Theorem 2.1 (Characteristic Equation)**.** *If* $\mathbf{A}$ *is an* $n \times n$ *matrix, then* λ *is an eigenvalue of* $\mathbf{A}$ *iff*

$$\det(\lambda I - A) = 0 \tag{2.14}$$

Corollary 2.1.1.

$$\sum \lambda_A = tr(\mathbf{A}) \tag{2.15}$$

Corollary 2.1.2.

$$\prod \lambda_A = \det(\mathbf{A}) \tag{2.16}$$

Notice: Gaussian elimination changes the eigenvalues.

2.8 Decompositions

Chapter 3

Review of Real Analysis

3.1 The Real Number System

3.2 Open Sets and Closed Sets

Definition 3.2.1 (Metric space)**.** A **metric space** is a set X where we have a notion of distance. That is, if $x, y \in X$, then $d(x, y)$ is the distance between x and y. The particular distance function must satisfy the following conditions:

- $d(x, y) > 0, \forall x, y \in X$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Definition 3.2.2 (Ball)**.** Let X be a metric space. A **ball** B of radius r around a point $x \in X$ is

$$B = \{y \in X | d(x, y) < r\} \tag{3.1}$$

Definition 3.2.3 (Open set)**.** A subset $O \subseteq X$ is **open** if $\forall x \in O, \exists r, B = \{x \in X | d(x, y) < r\} \subseteq O$

Theorem 3.1. *The union of any collection if open sets is open.*

Proof. Sets $S_1, S_2, ..., S_n$ are open sets, let $S = \cup_{i=1}^{n} S_i$, then $\forall i, S_i \subseteq S$. $\forall x \in S, \exists i, x \in S_i$. Given that S_i is an open set, then for x, $\exists r$ that $B = \{x \in S_i | d(x, y) < r\} \subseteq S_i \subseteq S$, therefore S is an open set. □

Theorem 3.2. *The intersection of any finite number of open sets is open.*

Proof. Sets $S_1, S_2, ..., S_n$ are open sets, let $S = \cap_{i=1}^{n} S_i$, then $\forall i, S \subseteq S_i$. $\forall x \in S, x \in S_i$. For any i, we can define an r_i, such that $B_i = \{x \in S_i | d(x, y) < r_i\} \subseteq S_i$. Let $r = \min_i \{r_i\}$. Noticed that $\forall i, B' = \{x \in S_i | d(x, y) < r\} \subseteq B_i \subseteq S_i$. Therefore S is an open set. □

Remark. The intersection of infinite number of open sets is not necessarily open.

Here we find an example that the intersection of infinite number of open sets can be closed.

Example. Let $A_n \in \mathbb{R}$ and $B_n \in \mathbb{R}$ be two infinite series, with the following properties.

- $\forall n, A_n < a, \lim A_n = a$
- $\forall n, B_n > b, \lim B_n = b$
- $a < b$

Then we define infinite number of sets S_i, the ith set is defined as

$$S_i = (A_i, B_i) \subset \mathbb{R} \tag{3.2}$$

Then

$$S = \cap_{i=1}^{\infty} S_i = [a, b] \subset \mathbb{R} \tag{3.3}$$

and S is a closed set.

Definition 3.2.4 (Limit point)**.** A point z is a **limit point** for a set A if every open set U that $z \in U$ intersects A in a point other than z.

Notice: z is not necessarily in A.

Definition 3.2.5 (Closed set)**.** A set C is **closed** iff it contains all of its limit points.

Theorem 3.3. *$S \in \mathbb{R}^n$ is closed $\iff \forall \{x_k\}_{k=1}^{\infty} \in S, \lim_{k\to\infty}\{x_k\}_{k=1}^{\infty} \in S$*

Theorem 3.4. *Every intersection of closed sets is closed.*

Theorem 3.5. *Every finite union of closed sets is closed.*

Remark. The union of infinite number of closed sets is not necessarily closed.

Theorem 3.6. *A set C is a closed set if $X \setminus C$ is open*

Proof. Let S be an open set, $x \notin S$, for any open set S_i that $x \in S_i$, we can find a correspond $r_i > 0$, such that $B_i = \{x \in S_i | d(x, y) < r_i\}$. Take $r = \min_{\forall i}\{r_i\}$, set $B = \{x \notin S | d(x, y) < r\} \neq \emptyset$. Which means for any $x \notin S$, we can find at least one point $x' \in B$ that for all open set S_i, $x' \in S_i$, which makes x a limit point of the complement of the open set. Notice that x is arbitrary, then the collection of x, i.e., the complement of S is a closed set. $\square$

Remark. The empty set is open and closed, the whole space X is open and closed.

3.3 Functions, Sequences, Limits and Continuity

3.4 Differentiation

3.5 Integration

3.6 Infinite Series of Constants

3.7 Power Series

3.8 Uniform Convergence

3.9 Arcs and Curves

3.10 Partial Differentiation

3.11 Multiple Integrals

3.12 Improper Integrals

3.13 Fourier Series

Chapter 4

Review of Probability Theory

4.1 Relationship between Some Random Variables

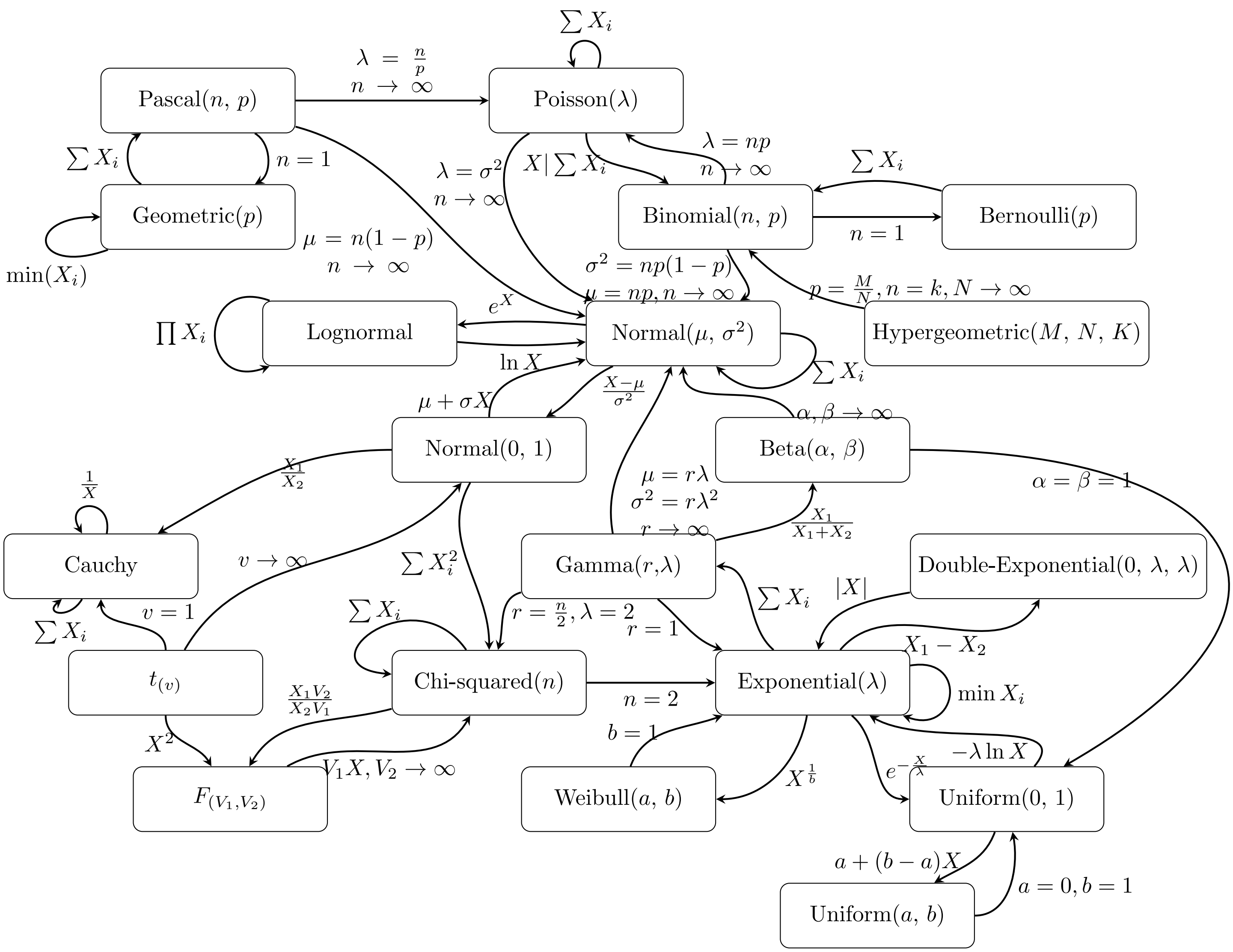

Figure 4.1: Relationship between Some Random Variables

4.2 Discrete Random Variables

Distribution	PMF	CDF	Exp.	Var.	MGF
Uniform(a,b)	$\frac{1}{b-a+1}$ $x=a,a+1,...,b$	$\frac{x-a+1}{b-a+1}$ $x=a,a+1,...,b$	$\frac{b-a}{2}$	$\frac{(b-a+1)^2-1}{12}$	$\frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$ $t\in\mathbb{R}$
Bernoulli(p)	$p^x(1-p)^{1-x}$ $x\in\{0,1\}$	$\begin{cases}0, & x<0\\ 1-p, & 0\le x\le 1\\ 1, & x>1\end{cases}$	p	$p(1-p)$	$1-p+pe^t$ $t\in\mathbb{R}$
Binomial(n,p)	$\binom{n}{x}p^x(1-p)^{n-x}$ $x=0,1,...,n$	$\sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k}$ $x=0,1,...,n$	np	$np(1-p)$	$(1-p+pe^t)^n$ $t\in\mathbb{R}$
Poisson(μ)	$\frac{\mu^x e^{\mu}}{x!}$ $x=0,1,...,n,...$	$\frac{\Gamma(x+1,\mu)}{\Gamma(x+1)}$ $x=0,1,...,n,...$	μ	μ	$e^{\mu(e^t-1)}$ $t\in\mathbb{R}$
Geometric(p)	$p(1-p)^x$ $x=0,1,...,n,...$	$1-(1-p)^{x+1}$ $x=0,1,...,n,...$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$ $t<-\ln(1-p)$
Pascal(n,p)	$\binom{n-1+x}{x}p^n(1-p)^x$ $x=0,1,2,...,n,...$	$1-I_p(k+1,n)$ $x=0,1,2,...,n,...$	$\frac{n(1-p)}{p}$	$\frac{n(1-p)}{p^2}$	$(\frac{p}{1-(1-p)e^t})^n$ $t<-\ln(1-p)$

(4.1)

4.3 Continuous Random Variables

Distribution	PDF	CDF	Exp.	Var.	MGF
Uniform(a,b)	$\frac{1}{b-a}$ $x=[a,b]$	$\frac{x-a}{b-a}$ $x=[a,b]$	$\frac{b-a}{2}$	$\frac{(b-a)^2}{12}$	$\begin{cases}1, & t=0\\ \frac{e^{bt}-e^{at}}{t(b-a)}, & t\neq 0\end{cases}$
Normal(μ,σ)	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in\mathbb{R}$	$\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in\mathbb{R}$	μ	σ^2	$e^{\frac{t(t\sigma^2+2\mu)}{2}}$ $t\in\mathbb{R}$
Exponential(λ)	$\lambda e^{-\lambda x}$ $x>0$	$1-e^{\lambda x}$ $x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{1}{1-\frac{t}{\lambda}}$ $t<\lambda$
Erlang(n,λ)	$\frac{\lambda^n x^{n-1}e^{-\lambda x}}{(n-1)!}$ $x>0$	$1-\sum_{i=0}^{n-1}\frac{\lambda^n x^n e^{-\lambda x}}{n!}$ $x>0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$	$\frac{1}{(1-\frac{t}{\lambda})^n}$ $t<\lambda$

(4.2)

Chapter 5

Review of Statistics

5.1 Classical Statistics

5.2 Bayesian Statistics