Notes for Operations Research & More

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Part I

Nonlinear Programming

Chapter 1

Convex Analysis

1.1 Convex Sets

Definition 1.1.1. A set $S \in \mathbb{R}^n$ is said to be convex if $\forall x_1, x_2 \in S, \lambda \in (0,1) \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S$

The following are some familes of convex sets.

Example. Empty set is by convention considered as convex.

Example. Polyhedrons are convex sets.

Example. Let $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^\top \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ where $\mathbf{A} \in \mathbb{S}_+^{n\times n}$ and $\mathbf{b} \in \mathbb{R}_+$. The set P is a convex subset of $\mathbb{R}^n$.

Example. Let $\|.\|$ be any norm in $\mathbb{R}^n$. Then, the unit ball $B = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\| \leq b, b > 0\}$ is convex.

Let S_1, S_2 be convex set, then:

- $S_1 \cap S_2$ is convex set
- $S_1 \oplus S_2$ (Minkowski addition) is convex set, where

$$S_1 \oplus S_2 = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{x_1} + \mathbf{x_2}, \mathbf{x_1} \in S_1, \mathbf{x_2} \in S_2\} \tag{1.1}$$

- $S_1 \ominus S_2$ is convex set, where

$$S_1 \oplus S_2 = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{x_1} - \mathbf{x_2}, \mathbf{x_1} \in S_1, \mathbf{x_2} \in S_2\} \tag{1.2}$$

- $f(S_1)$ is convex iff $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$

Theorem 1.1 (Carathéodory's Theorem)**.** *Let $S \subseteq \mathbb{R}^n$. Then $\forall \mathbf{x} \in conv(S)$, there exists $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^p \in S$, where $p \leq n+1$ such that $\mathbf{x} \in conv\{\mathbf{x}^1, \mathbf{x}^2, ...\mathbf{x}^p\}$.*

Notice: This theorem means, any point $\mathbf{x} \in \mathbb{R}^n$ in a convex hull of S, i.e., $conv(S)$, can be included in a convex subset $S' \subseteq conv(S)$ that has $n+1$ extreme points.

Theorem 1.2. *Let S be a convex set with nonempty interior. Let $\mathbf{x}_1 \in cl(S)$ and $\mathbf{x}_2 \in int(S)$, then $\mathbf{y} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in int(S), \forall \lambda \in (0,1)$*

1.2 Convex Functions

Definition 1.2.1. Let $C \in \mathbb{R}^n$ be a convex set. A function $f : C \to \mathbf{R}$ is (resp. strictly) convex if

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) \tag{1.3}$$
$$\forall \mathbf{x}_1, \mathbf{x}_2 \in C, \forall \lambda \in (0,1) \tag{1.4}$$

(resp.)

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) < \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) \tag{1.5}$$
$$\forall \mathbf{x_1} \neq \mathbf{x}_2 \in C, \forall \lambda \in (0,1) \tag{1.6}$$

Notice: When calling a function convex, we imply that its domain is convex.

Example. Given any norm $\|.\|$ on $\mathbb{R}^n$, the function $f(x) = \|x\|$ is convex over $\mathbb{R}^n$.

Definition 1.2.2. Let S be a nonempty convex subset of $\mathbb{R}^n$, $f : S \to \mathbb{R}$ is (resp. strictly) **concave** if $-f(x)$ is (resp. strictly) convex.

Notice: A function may be neither convex nor concave.

Theorem 1.3. *Consider $f : \mathbb{R}^n \to \mathbb{R}$. $\forall \bar{\mathbf{x}} \in \mathbb{R}^n$ and a nonzero direction $\mathbf{d} \in \mathbb{R}^n$. Define $F_{\bar{\mathbf{x}},d}(\lambda) = f(\bar{\mathbf{x}} + \lambda \mathbf{d}$. Then f is (resp. strictly) convex iff $F_{\bar{\mathbf{x}},d}(\lambda)$ is (resp. strictly) convex for all $\bar{\mathbf{x}} \in \mathbb{R}^n, \forall \mathbf{d} \in \mathbb{R}^n \setminus \{0\}$.*

Definition 1.2.3 (Level-set)**.** Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a scalar $\alpha \in \mathbb{R}$, we refer to the set $S_\alpha = \{\mathbf{x} \in S | f(\mathbf{x}) \leq \alpha\} \subseteq \mathbb{R}^n$ as the **α-level-set** of f.

Lemma 1.4. *Let S be a nonempty convex set in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}^n$ be a convex function, then the **α-level-set** of f is a convex set for each value of $\alpha \in \mathbb{R}$.*

Notice: The converse is not necessarily true.

Definition 1.2.4 (Epigraphs, Hypographs)**.** Let $S \in \mathbb{R}^n$ be such that $S \neq \emptyset$. The **epigraph** of f, denoted by $epi(f)$ is

$$epi(f) = \{(\mathbf{x}, y) \in S | \mathbf{x} \in S, y \in \mathbb{R}, y \geq f(x)\} \in \mathbb{R}^{n+1} \tag{1.7}$$

The **hypograph** of f, denoted by $hypo(f)$ is

$$hypo(f) = \{(\mathbf{x}, y) \in S | \mathbf{x} \in S, y \in \mathbb{R}, y \leq f(x)\} \in \mathbb{R}^{n+1} \tag{1.8}$$

Theorem 1.5. *Let S be a nonempty convex subset in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$. Then f is convex iff $epi(f)$ is convex.*

Theorem 1.6. *Let S be a nonempty convex subset in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$ be a convex function on S. Then f is continuous in $int(S)$.*

1.3 Subgradients and Subdifferentials

Definition 1.3.1 (Subgradient)**.** Let S be a nonempty convex set in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$ be a convex function, then ξ is a **subgradient** of f at $\bar{\mathbf{x}}$ if

$$f(\mathbf{x}) \geq f(\bar{\mathbf{x}}) + \xi^\top(\mathbf{x} - \bar{\mathbf{x}}), \forall \mathbf{x} \in S \tag{1.9}$$

Definition 1.3.2 (Subdifferential)**.** The set of all subgradients of f at $\bar{\mathbf{x}}$ is called **subdifferential** of f at $\bar{\mathbf{x}}$, denoted as $\partial f(\bar{\mathbf{x}})$

Theorem 1.7. *Let S be a nonempty convex set in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$ be a convex function. Then for $\bar{\mathbf{x}} \in int(S)$, there exists a vector ξ such that*

$$f(\mathbf{x}) \geq f(\bar{\mathbf{x}}) + \xi^\top(\mathbf{x} - \bar{\mathbf{x}}), \forall \mathbf{x} \in S \tag{1.10}$$

In particular, the hyperplane

$$\mathcal{H} = \{(\mathbf{x}, y) | y = f(\bar{\mathbf{x}}) + \xi^\top(\mathbf{x} - \bar{\mathbf{x}})\} \tag{1.11}$$

is a supporting plane of $epi(f)$ at $(\bar{\mathbf{x}}, f(\bar{\mathbf{x}}))$

Theorem 1.8. *Let S be a nonempty convex set in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$ be a convex function. Suppose that for each $\bar{\mathbf{x}} \in S$, there exists ξ such that*

$$f(\mathbf{x}) \geq f(\bar{\mathbf{x}}) + \xi^\top(\mathbf{x} - \bar{\mathbf{x}}), \forall \mathbf{x} \in S \tag{1.12}$$

Then f is convex on $int(S)$

Notice: Not all convex functions are continuous, it has to be continuous in its interior, but it may not be continuous at the boundary.

1.4 Differentiable Functions

Definition 1.4.1 (Differentiable Functions)**.** Let S be a nonempty subset of $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$.Then f is said to be **differentiable** at $\bar{\mathbf{x}} \in int(S)$ if there exists a vector $\nabla f(\bar{\mathbf{x}})$ and a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(\mathbf{x}) = f(\bar{\mathbf{x}}) + \nabla f(\bar{\mathbf{x}})^\top(\mathbf{x} - \bar{\mathbf{x}}) + \alpha(\bar{\mathbf{x}}, \mathbf{x} - \bar{\mathbf{x}})\|\mathbf{x} - \bar{\mathbf{x}}\| \tag{1.13}$$

for all $\mathbf{x} \in S$ where $\lim_{\mathbf{x} - \bar{\mathbf{x}}} \alpha(\bar{\mathbf{x}}, \mathbf{x} - \bar{\mathbf{x}}) = 0$

Remark. If function f is differentiable, then $\nabla f(\bar{\mathbf{x}}) = (\frac{\partial f(\bar{\mathbf{x}})}{\partial \mathbf{x}_1}, \frac{\partial f(\bar{\mathbf{x}})}{\partial \mathbf{x}_2}, \cdots, \frac{\partial f(\bar{\mathbf{x}})}{\partial \mathbf{x}_n})$, and the gradient is unique.

Chapter 2

KKT Optimality Conditions

Chapter 3

Lagrangian Duality

Chapter 4

Unconstrained Optimization

Chapter 5

Penalty and Barrier Functions