# Notes for Operations Research & More

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# Contents

# Part I

# Preliminary Topics

# Chapter 1

# Introduction to Optimization

## 1.1 Optimization Model

The following is the basic forms of terminology:

$$\begin{aligned}
\text{(P)} \quad \min \quad & f(x) && && (1.1)\\
\text{s.t.} \quad & g_i(x) \leq 0, && i = 1, 2, ..., m && (1.2)\\
& h_j(x) = 0, && j = 1, 2, ..., l && (1.3)\\
& x \in X && && (1.4)
\end{aligned}$$

We have

- - $x \in R^n \to X \subseteq R^m$
- $g_i(x)$ are called inequality constraints
- $h_j(x)$ are called equality constraints
- $X$ is the domain of the variables (e.g. cone, polygon, $\{0,1\}^n$, etc.)
- Let $F$ be the feasible region of $(P)$:
  - $x^0$ is a feasible solution iff $x^0 \in F$
  - $x^*$ is an optimized solution iff $x^* \in F$ and $f(x^*) \leq f(x^0), \forall x^0 \in F$ (for minimized problem)

**Notice:** Not every $(P)$ has a feasible region, we can have $F = \emptyset$. Even if $F \neq \emptyset$, there might not be an solution to $P$, e.g. unbounded. If $(P)$ has optimized solution(s), it could be 1) Unique 2) Infinite number of solution 3) Finite number of solution

Types of Optimization Problem

- $m = l = 0$, $x \in R^n$, unconstrained problem
- $m + l > 0$, constrained problem
- $f(x), g_i(x), h_j(x)$ are linear, Linear Optimization
  - If $X = R^n$, Linear Programming
  - If $X$ is discrete, Discrete Optimization
  - If $X \subseteq Z^n$, Integer Programming
  - If $X \in \{0,1\}^n$, Binary Programming
  - If $X \in Z^n \times R^m$, Mixed Integer Programming

# Chapter 2

# Review of Linear Algebra

## 2.1 Field

**Definition 2.1.1** (Field)**.** Let $F$ denote either the set of real numbers or the set of complex numbers.

- Addition is commutative: $x + y = y + x, \forall x, y \in F$
- Addition is associative: $x + (y + z) = (x + y) + z, \forall x, y, z \in F$
- Element 0 exists and unique: $\exists 0, x + 0 = x, \forall x \in F$
- To each $x \in F$ there corresponds a unique element $(-x) \in F$ such that $x + (-x) = 0$
- Multiplication is commutative: $xy = yx, \forall x, y \in F$
- Multiplication is associative: $x(yz) = (xy)z, \forall x, y, z \in F$
- Element 1 exists and unique: $\exists 1, x1 = x, \forall x \in F$
- To each $x \neq 0 \in F$ there corresponds a unique element $x^{-1} \in F$ that $xx^{-1} = 1$
- Multiplication distributes over addition: $x(y + z) = xy + xz, \forall x, y, z \in F$

Suppose one has a set $F$ of objects $x, y, z, ...$ and two operations on the elements of $F$ as follows. The first operation, called addition, associates with each pair of elements $x, y \in F$ an element $(x + y) \in F$; the second operation, called multiplication, associates with each pair $x, y$ an element $xy \in F$; and these two operations satisfy all conditions above. The set $F$, together with these two operations, is then called a **field**.

**Definition 2.1.2** (Subfield)**.** A **subfield** of the field $C$ is a set $F$ of complex numbers which itself is a field.

**Example.** The set of integers is not a field.

**Example.** The set of rational numbers is a field.

**Example.** The set of all complex numbers of the form $x + y\sqrt{2}$ where $x$ and $y$ are rational, is a subfield of $\mathbb{C}$.

**Notice:** In this note, we (...Lan) assume that the field involved is a subfield of the complex numbers $\mathbb{C}$. More generally, if $F$ is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0, which does not happen in the subfield of $\mathbb{C}$. If it does happen in $F$, the least $n$ such that the sum of $n$ 1's is 0 is called **characteristic** of the field $F$. If it does not happen, then $F$ is called a field of **characteristic zero**.

## 2.2 Real Vector Spaces

## 2.3 Linear, Conic, Affine, and Convex Combinations

## 2.4 Determinants

## 2.5 Inner Products

**Definition 2.5.1** (Inner Product)**.** Let $F$ be the field of real numbers or the field of complex numbers, and $V$ a vector space over $F$. An **inner product** on $V$ is a function which assigns to each ordered pair of vectors $\alpha$, $\beta$ in $V$ a scalar $< \alpha|\beta >$ in $F$ in such a way that $\forall \alpha, \beta, \gamma \in V, c \in \mathbb{R}$ that

- $< \alpha + \beta|\gamma > = < \alpha|\gamma > + < \beta|\gamma >$
- $< c\alpha|\beta > = c < \alpha|\beta >$
- $< \alpha|\beta > = \overline{< \beta|\alpha >}$
- $< \alpha|\alpha > \geq 0$, $< \alpha|\alpha > = 0$ iff $\alpha = \mathbf{0}$

Furthermore, the above properties imply that

- $< \alpha|c\beta + \gamma > = \bar{c} < \alpha|\beta > + < \alpha|\gamma >$

**Definition 2.5.2.** On $F^n$ there is an inner product which we call the **standard inner product**. It is defined on $\alpha = (x_1, x_2, ..., x_n)$ and $\beta = (y_1, y_2, ..., y_n)$ by

$$< \alpha|\beta > = \sum_j x_j \bar{y_j} \tag{2.1}$$

For $F = \mathbb{R}^n$

$$< \alpha|\beta >= \sum_j x_j y_j \tag{2.2}$$

In the real case, the standard inner product is often called the dot product and denoted by $\alpha \cdot \beta$

**Example.** For $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in $\mathbb{R}^2$, the following is an inner product.

$$< \alpha|\beta >= x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2 \tag{2.3}$$

**Example.** For $\mathbb{C}^{n\times n}$,

$$< \mathbf{A}|\mathbf{B} >= trace(\mathbf{B}^* \mathbf{A}) \tag{2.4}$$

is an inner product, where

$$\mathbf{A}^*_{ij} = \bar{\mathbf{A}}_{ji} \quad (\textbf{conjugate transpose}) \tag{2.5}$$

For $\mathbb{R}^{n\times n}$,

$$< \mathbf{A}|\mathbf{B} >= trace(\mathbf{B}^T \mathbf{A}) = \sum_j (AB^T)_{jj} = \sum_j \sum_k A_{jk} B_{jk} \tag{2.6}$$

## 2.6 Norms

## 2.7 Eigenvectors and Eigenvalues

**Definition 2.7.1.** If $\mathbf{A}$ is an $n\times n$ matrix, then a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ is called an **eigenvector** of $\mathbf{A}$ if $\mathbf{Ax}$ is a scalar multiple of $\mathbf{x}$, i.e.

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{2.7}$$

for some scalar $\lambda$. The scalar $\lambda$ is called **eigenvalue** of $\mathbf{A}$ and the vector $\mathbf{x}$ is said to be an **eigenvector corresponding to** $\lambda$

**Theorem 2.1** (Characteristic Equation)**.** *If* $\mathbf{A}$ *is an* $n \times n$ *matrix, then* $\lambda$ *is an eigenvalue of* $\mathbf{A}$ *iff*

$$\det(\lambda I - A) = 0 \tag{2.8}$$

**Corollary 2.1.1.**

$$\sum \lambda_A = tr(\mathbf{A}) \tag{2.9}$$

**Corollary 2.1.2.**

$$\prod \lambda_A = \det(\mathbf{A}) \tag{2.10}$$

**Notice:** Gaussian elimination changes the eigenvalues.

## 2.8 Decompositions

# Chapter 3

# Review of Real Analysis

## 3.1 Sequences and Series

# Chapter 4

# Review of Topology

## 4.1 Open Sets and Closed Sets

**Definition 4.1.1** (Metric space)**.** A **metric space** is a set $X$ where we have a notion of distance. That is, if $x, y \in X$, then $d(x, y)$ is the distance between $x$ and $y$. The particular distance function must satisfy the following conditions:

- $d(x, y) > 0, \forall x, y \in X$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

**Definition 4.1.2** (Ball)**.** Let $X$ be a metric space. A **ball** $B$ of radius $r$ around a point $x \in X$ is

$$B = \{y \in X | d(x, y) < r\} \tag{4.1}$$

**Definition 4.1.3** (Open set)**.** A subset $O \subseteq X$ is **open** if $\forall x \in O, \exists r, B = \{x \in X | d(x, y) < r\} \subseteq O$

**Theorem 4.1.** *The union of any collection if open sets is open.*

*Proof.* Sets $S_1, S_2, ..., S_n$ are open sets, let $S = \cup_{i=1}^{n} S_i$, then $\forall i, S_i \subseteq S$. $\forall x \in S, \exists i, x \in S_i$. Given that $S_i$ is an open set, then for $x$, $\exists r$ that $B = \{x \in S_i | d(x, y) < r\} \subseteq S_i \subseteq S$, therefore $S$ is an open set. □

**Theorem 4.2.** *The intersection of any finite number of open sets is open.*

*Proof.* Sets $S_1, S_2, ..., S_n$ are open sets, let $S = \cap_{i=1}^{n} S_i$, then $\forall i, S \subseteq S_i$. $\forall x \in S, x \in S_i$. For any $i$, we can define an $r_i$, such that $B_i = \{x \in S_i | d(x, y) < r_i\} \subseteq S_i$. Let $r = \min_i \{r_i\}$. Noticed that $\forall i, B' = \{x \in S_i | d(x, y) < r\} \subseteq B_i \subseteq S_i$. Therefore $S$ is an open set. □

*Remark.* The intersection of infinite number of open sets is not necessarily open.

Here we find an example that the intersection of infinite number of open sets can be closed.

**Example.** Let $A_n \in \mathbb{R}$ and $B_n \in \mathbb{R}$ be two infinite series, with the following properties.

- $\forall n, A_n < a, \lim A_n = a$
- $\forall n, B_n > b, \lim B_n = b$
- $a < b$

Then we define infinite number of sets $S_i$, the $i$th set is defined as

$$S_i = (A_i, B_i) \subset \mathbb{R} \tag{4.2}$$

Then

$$S = \cap_{i=1}^{\infty} S_i = [a, b] \subset \mathbb{R} \tag{4.3}$$

and $S$ is a closed set.

**Definition 4.1.4** (Limit point)**.** A point $z$ is a **limit point** for a set $A$ if every open set $U$ that $z \in U$ intersects $A$ in a point other than $z$.

**Notice:** $z$ is not necessarily in $A$.

**Definition 4.1.5** (Closed set)**.** A set $C$ is **closed** iff it contains all of its limit points.

**Theorem 4.3.** $S \in \mathbb{R}^n$ *is closed* $\iff \forall \{x_k\}_{k=1}^{\infty} \in S, \lim_{k \to \infty} \{x_k\}_{k=1}^{\infty} \in S$

**Theorem 4.4.** *Every intersection of closed sets is closed.*

**Theorem 4.5.** *Every finite union of closed sets is closed.*

*Remark.* The union of infinite number of closed sets is not necessarily closed.

**Theorem 4.6.** *A set $C$ is a closed set if $X \setminus C$ is open*

*Proof.* Let $S$ be an open set, $x \notin S$, for any open set $S_i$ that $x \in S_i$, we can find a correspond $r_i > 0$, such that $B_i = \{x \in S_i | d(x, y) < r_i\}$. Take $r = \min_{\forall i} \{r_i\}$, set $B = \{x \notin S | d(x, y) < r\} \neq \emptyset$. Which means for any $x \notin S$, we can find at least one point $x' \in B$ that for all open set $S_i$, $x' \in S_i$, which makes $x$ a limit point of the complement of the open set. Notice that $x$ is arbitrary, then the collection of $x$, i.e., the complement of $S$ is a closed set. □

*Remark.* The empty set is open and closed, the whole space $X$ is open and closed.

# Part II

# Linear Programming

# Chapter 5

# Formulation

## 5.1 Linear Programming Problem Manipulation

### 5.1.1 Inequalities and Equalities

An inequality can be transformed into an equation by adding or subtracting the nonnegative slack or surplus variable

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j - x_{n+1} = b_i \tag{5.1}$$

or

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j + x_{n+1} = b_i \tag{5.2}$$

Although it is not the practice, equality can be transformed into inequality too

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Rightarrow \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \leq b_i \\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i \end{cases} \tag{5.3}$$

Also, in linear programming, we only care about close set, so we will not have $<, >$ in the formulation, we can use the following

$$\sum_{j=1}^{n} a_{ij}x_j > b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j \geq b_i + \epsilon \tag{5.4}$$

where $\epsilon$ is a small number.

### 5.1.2 Minimization and Maximization

To convert a minimization problem into a maximization problem, we can use the following to define a new objective function

$$\min \sum_{j=1}^{n} c_j x_j = -\max \sum_{j=1}^{n} c_j x_j \tag{5.5}$$

### 5.1.3 Standard Form and Canonical Form

Standard Form

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j && (5.6)\\ \text{s.t.} \quad & Ax = b && (5.7)\\ & x \geq 0 && (5.8) \end{aligned}$$

Canonical Form

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j && (5.9)\\ \text{s.t.} \quad & Ax \leq b && (5.10)\\ & x \geq 0 && (5.11) \end{aligned}$$

## 5.2 Typical Problems

## 5.3 Formulation Skills

### 5.3.1 Absolute Value

Consider the following model statement:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j |x_j|, \quad c_j > 0 && (5.12)\\ \text{s.t.} \quad & \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I && (5.13)\\ & x_j \quad \text{unrestricted}, \quad \forall j \in J && (5.14) \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j (x_j^+ + x_j^-), \quad c_j > 0 && (5.15)\\ \text{s.t.} \quad & \sum_{j\in J} a_{ij}(x_j^+ - x_j^-) \gtreqless b_i, \quad \forall i \in I && (5.16)\\ & x_j^+, x_j^- \geq 0, \quad \forall j \in J && (5.17) \end{aligned}$$

### 5.3.2 A Minimax Objective

Consider the following model statement:

$$\begin{aligned}
\min \quad & \max_{k\in K} \sum_{j\in J} c_{kj}x_j && (5.18)\\
\text{s.t.} \quad & \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I && (5.19)\\
& x_j \geq 0, \quad \forall j \in J && (5.20)
\end{aligned}$$

Modeling:

$$\begin{aligned}
\min \quad & z && (5.21)\\
\text{s.t.} \quad & \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I && (5.22)\\
& \sum_{j\in J} c_{kj}x_j \leq z, \quad \forall k \in K && (5.23)\\
& x_j \geq 0, \quad \forall j \in J && (5.24)
\end{aligned}$$

### 5.3.3 A fractional Objective

Consider the following model statement:

$$\begin{aligned}
\min \quad & \frac{\sum_{j\in J} c_j x_j + \alpha}{\sum_{j\in J} d_j x_j + \beta} && (5.25)\\
\text{s.t.} \quad & \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I && (5.26)\\
& x_j \geq 0, \quad \forall j \in J && (5.27)
\end{aligned}$$

Modeling:

$$\begin{aligned}
\min \quad & \sum_{j\in J} c_j x_j t + \alpha t && (5.28)\\
\text{s.t.} \quad & \sum_{j\in J} a_{ij}x_j \gtreqless b_i, \quad \forall i \in I && (5.29)\\
& \sum_{j\in J} d_j x_j t + \beta t = 1 && (5.30)\\
& t > 0 && (5.31)\\
& x_j \geq 0, \quad \forall j \in J && (5.32)\\
& (t = \frac{1}{\sum_{j\in J} d_j x_j + \beta}) && (5.33)
\end{aligned}$$

### 5.3.4 A range Constraint

Consider the following model statement:

$$\begin{aligned}
\min \quad & \sum_{j\in J} c_j x_j && (5.34)\\
\text{s.t.} \quad & d_i \leq \sum_{j\in J} a_{ij}x_j \leq e_i, \quad \forall i \in I && (5.35)\\
& x_j \geq 0, \quad \forall j \in J && (5.36)
\end{aligned}$$

Modeling:

$$\begin{aligned}
\min \quad & \sum_{j\in J} c_j x_j, \quad c_j > 0 && (5.37)\\
\text{s.t.} \quad & u_i + \sum_{j\in J} a_{ij}x_j = e_i, \quad \forall i \in I && (5.38)\\
& x_j \geq 0, \quad \forall j \in J && (5.39)\\
& 0 \leq u_i \leq e_i - d_i, \quad \forall i \in I && (5.40)
\end{aligned}$$

# Chapter 6

# Simplex Method

## 6.1 Basic Feasible Solutions and Extreme Points

**Definition 6.1.1.** Consider the system $\{A_{m\times n}x = b_m, x \geq 0\}$, suppose $rank(A, b) = rank(A) = m$, we can arrange $A$ and have a partition of $A$. Let $A = [B, N]$ where $B$ is $m\times m$ invertible matrix, and $N$ is a $m\times(n-m)$ matrix. The solution $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ to the equation $Ax = b$, where

$$x_B = B^{-1}b \tag{6.1}$$

and

$$x_N = 0 \tag{6.2}$$

is called **basic solution** of system. If $x_B \geq 0$, it is called **basic feasible solution**. If $x_B > 0$ it is called **non-degenerate basic feasible solution**. For $x_B \geq 0$, if some $x_j = 0$, those components are called **degenerated basic feasible solution**.
$B$ is called the **basic matrix**, $N$ is called **nonbasic matrix**

**Theorem 6.1.** *$x$ is an E.P. $\iff$ $x$ is a B.F.S*

*Proof.* First, Let $x$ be a B.F.S., Suppose $x = \lambda u + (1-\lambda)v$, for $u, v \in S, \lambda \in (0, 1)$
Let $I = \{i : x_i > 0\}$ Then
- if $i \notin I$ then $x_i = 0$, which implies $u_i = v_i = 0$ - $\because Au = Av = b$, $\therefore A(u-v) = 0 \Rightarrow \sum_{i=1}^{n}(u_i - v_i)a_i = 0$, $\because u_i = v_i = 0$, for $i \notin I$, it implies $u_i = v_i$ for $i \in I$, Hence $u = v$, $x$ is E.P.
Second, suppose $x$ is not B.F.S., i.e. $\{a_i : i \in I\}$ are linearly dependent.
Then there $\exists u \neq 0, u_i = 0, i \notin I$ such that $Au = 0$.
Hence, for a small $\epsilon$, $x = \frac{1}{2}(x + \epsilon u) + \frac{1}{2}(x - \epsilon u)$, $x$ is not E.P. □

## 6.2 Simplex Method

### 6.2.1 Key to Simplex Method

**Cost Coefficient**

The cost coefficient can be derived from the following

$$\begin{aligned} z &= cx && (6.3)\\ &= c_Bx_B + c_Nx_N && (6.4)\\ &= c_B(B^{-1}b - B^{-1}Nx_N) + c_Nx_N && (6.5)\\ &= c_BB^{-1}b - \sum_{j\in N}(c_BB^{-1}a_j - c_j)x_j && (6.6)\\ &= c_BB^{-1}b - \sum_{j\in N}(z_j - c_j)x_j && (6.7) \end{aligned}$$

We denote $z_0 = c_BB^{-1}b$, $z_j = c_B^{-1}a_j$, $\bar{b} = B^{-1}b$ and $y_j = B^{-1}a_j$ for all nonbasic variables.
The formulation can be transformed into

$$\begin{aligned} \min \quad & z = z_0 - \sum_{j\in N}(z_j - c_j)x_j && (6.8)\\ \text{s.t.} \quad & \sum_{j\in N} y_jx_j + x_B = \bar{b} && (6.9)\\ & x_j \geq 0, j \in N && (6.10)\\ & x_B \geq 0 && (6.11) \end{aligned}$$

In the above formulation, $z_j - c_j$ is the cost coefficient. If $\exists j$ and $z_j - c_j > 0$, it means the objective function can still be optimized. (If $\forall j$, $z_j - c_j \leq 0$, then $z \geq z_0$ for any feasible solution, $z$ is the optimal solution)

**Pivot**

After finding the most violated $z_j - c_j$, we find a variable, say $x_k$, where $z_k - c_k = \min\{z_j - c_j\}$ to be the variable leaving the basis.
If there are degenerated variables, we can perform different method to choose variable to enter basis.

**Minimum Ratio**

$$x_{B_i} = \bar{b}_i - y_{ik}x_k \geq 0 \tag{6.12}$$

Therefore we have the minimum ratio rule

$$x_k = \min_{i \in B}\{\frac{\bar{b}_i}{y_{ik}}, y_{ik} > 0\} \tag{6.13}$$

If for the that column all $y_{ik} \leq 0$, unbounded.

### 6.2.2 Simplex Method Algorithm

The pseudo-code of Simplex Method is given as following:

**Algorithm 1** Simplex Method

**Require:** Given a basic feasible solution with basis $B$
**Ensure:** Optimal objective value $\min z = cx$

Set $\mathbf{B}$ for basic variables, $\mathbf{N}$ for nonbasic variables
$\mathbf{B} \leftarrow$ all slack variables
$\mathbf{N} \leftarrow$ all variables excepts slack variables
**for** $\forall j$ **do**
  $z_j = c_B B^{-1} a_j = 0$
**end for**
**while** $\exists z_j - c_j > 0$ **do**
  $z_j = wa_j - c_j = c_B B^{-1} a_j - c_j$
  $z_k - c_k = \max_{j \in \mathbf{N}}\{z_j - c_j\}$
  $y_k = B^{-1} a_k$
  **if** $\exists y_{ik} > 0$ **then**
    $\theta_r = \min_{i \in \mathbf{B}}\{\theta_i = \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0\}$
    $\mathbf{B} \leftarrow \mathbf{B} \backslash \{k\}$
    $\mathbf{N} \leftarrow \mathbf{N} \cup \{k\}$
    $\mathbf{B} \leftarrow \mathbf{B} \cup \{r\}$
    $\mathbf{N} \leftarrow \mathbf{N} \backslash \{r\}$
  **else**
    Unbounded
  **end if**
**end while**
$x_B^* = B^{-1} b = \bar{b}$
$x_N = 0$
$z^* = c_B B^{-1} b = c_B \bar{b} \mathbf{a_{B_k}}$

## 6.3 Find Elements in Tableau

Initial tableau:

| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $z$ | 1 | 1 | 3 | 0 | 0 | 0 | 0 |
| $x_3$ | 0 | 1 | -2 | 1 | 0 | 0 | 0 |
| $x_4$ | 0 | -2 | $\boxed{1}$ | 0 | 1 | 0 | 4 |
| $x_5$ | 0 | 5 | 3 | 0 | 0 | 1 | 15 |

(6.14)

Last tableau:

| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $z$ | 1 | 0 | 0 | 0 | $-\frac{12}{11}$ | $-\frac{7}{11}$ | $-\frac{153}{11}$ |
| $x_3$ | 0 | 0 | 1 | 1 | $\frac{13}{11}$ | $\frac{3}{11}$ | $\frac{97}{11}$ |
| $x_2$ | 0 | 1 | 0 | 0 | $\frac{5}{11}$ | $\frac{2}{11}$ | $\frac{50}{11}$ |
| $x_1$ | 1 | 0 | 0 | 0 | $-\frac{3}{11}$ | $\frac{1}{11}$ | $\frac{3}{11}$ |

(6.15)

- The optimal basic variables are $x_3$, $x_2$, $x_1$. The optimal basis is the columns in the initial tableau with correspond columns

$$B = \begin{pmatrix} \frac{13}{11} & \frac{3}{11} & \frac{97}{11} \\ \frac{5}{11} & \frac{2}{11} & \frac{50}{11} \\ -\frac{3}{11} & \frac{1}{11} & \frac{3}{11} \end{pmatrix} \tag{6.16}$$

- From the initial tableau, we can see the initial basis is built from slack variables $x_3$, $x_4$, $x_5$. The $B^{-1}$ is the correspond columns in final tableau.

$$B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 3 & 5 \end{pmatrix} \tag{6.17}$$

- The optimal basic variables are $x_3$, $x_2$, $x_1$. Find $c_B$ in the initial tableau.

$$c_B = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \tag{6.18}$$

- Find $w = c_B B^{-1}$ from the final tableau, correspond to the slack variable.

$$w = c_B B^{-1} = \begin{pmatrix} 0 \\ -\frac{12}{11} \\ -\frac{7}{11} \end{pmatrix} \tag{6.19}$$

## 6.4 Artificial Variable

If some of the constraint is not in $\sum_{i=1}^{n} a_i x_i \leq 0$ form, we cannot add a positive slack variable. In this case, we add an artificial variable other than slack variable.

$$\sum_{i=1}^{n} a_i x_i \geq (or =) 0 \Rightarrow \sum_{j=1}^{n} a_i x_i + x_a = 0 \tag{6.20}$$

Notice that in an optimal solution, $x_a = 0$, otherwise it is not valid.
Artificial variables are only a tool to get the simplex method started.

### 6.4.1 Two-Phase Method

**Two-Phase Method**

For **Phase I:**
Solve the following program start with a basic feasible

solution $x = 0, x_a = b$, i.e., the artificial variable forms the basis.

$$\min \quad 1x_a \tag{6.21}$$
$$\text{s.t.} \quad Ax + x_a = b \tag{6.22}$$
$$x \geq 0 \tag{6.23}$$
$$x_a \geq 0 \tag{6.24}$$

If the optimal $1x_a \neq 0$, infeasible, stop. Otherwise proceed Phase II. For **Phase II:**
Remove the columns of artificial variables, replace the objective function with the original objective function, proceed to solve simplex method.

**Discussion**

**Case A:** $x_a \neq 0$
Infeasible.
**Case B.1:** $x_a = 0$ and all artificial variables are out of the basis
At the end of Phase I, we derive

| $x_0$ | $x_B$ | $x_N$ | $x_a$ | RHS |
|---|---|---|---|---|
| 1 | 0 | 0 | -1 | 0 |
| 0 | $I$ | $B^{-1}N$ | $B^{-1}$ | $B^{-1}b$ |

(6.25)

We can discard $x_a$ columns, (or we can leave it because it keeps track of $B^{-1}$), and then we do the Phase II

| $z$ | $x_B$ | $x_N$ | $RHS$ |
|---|---|---|---|
| 1 | 0 | $c_B B^{-1}N - c_N$ | $c_B B^{-1}b$ |
| 0 | $I$ | $B^{-1}N$ | $B^{-1}b$ |

(6.26)

**Case B.2:** Some artificial variables are in the basis at zero values
This is because of degeneracy. We pivot on those artificial variables, once they leave the basis, eliminate them.

### 6.4.2 Big M Method

### 6.4.3 Single Artificial Variable

## 6.5 Revised Simplex Method

### 6.5.1 Key to Revised Simplex Method

The procedure of Simplex Method is (almost) exactly the same as original simplex method. However, notice that we don't need to use $N$ so for the revised simplex method, we don't calculate any matrix related to $N$
The original matrix:

| $z$ | $x_B$ | $x_N$ | $RHS$ |
|---|---|---|---|
| 1 | 0 | $c_B B^{-1}N - c_N$ | $c_B B^{-1}b$ |
| 0 | $I$ | $B^{-1}N$ | $B^{-1}b$ |

(6.27)

The revised matrix:

| Basic Inverse | RHS |
|---|---|
| $w = c_B B^{-1}$ | $c_B\bar{b} = c_B B^{-1}b$ |
| $B^{-1}$ | $\bar{b} = B^{-1}b$ |

(6.28)

For each pivot iteration, calculate $z_j - c_j = wa_j - c_j = c_B B^{-1}a_j - c_j, \forall j \in N$, pivot rules are the same as simplex method, each time find a variable $x_k$ to enter basis

| $B^{-1}$ | RHS | $x_k$ |
|---|---|---|
| $w$ | $c_B b$ | $z_k - c_k$ |
| $B^{-1}$ | $\bar{b}$ | $y_k$ |

(6.29)

Do the minimum ratio rule to find the variable $x_r$ to leave the basis

| $B^{-1}$ | RHS | $x_k$ |
|---|---|---|
| $w$ | $c_B b$ | $z_k - c_k$ |
| | $\bar{b}_1$ | $y_{1k}$ |
| | $\bar{b}_2$ | $y_{2k}$ |
| | ... | ... |
| $B^{-1}$ | $\bar{b}_r$ | $y_{rk}$(pivot at here) |
| | ... | ... |
| | $\bar{b}_m$ | $y_{mk}$ |

(6.30)

### 6.5.2 Comparison between Simplex and Revised Simplex

**Advantage of Revised Simplex**

- Save storage memory
- Don't need to calculate N (including $B^{-1}N$ and $c_B B^{-1}N$)
- More accurate because round up errors will not be accumulated

**Disadvantage of Revised Simplex**

- Need to calculate $wa_j$ for all $j \in N$ (in fact don't need to calculated it for the variable just left the basis)

**Computation Complexity**

| Method | Type | Operations |
|---|---|---|
| Simplex | $\times$ | $(m+1)(n-m+1)$ |
| | $+$ | $m(n-m+1)$ |
| Revised Simplex | $\times$ | $(m+1)^2 + m(n-m)$ |
| | $+$ | $m(m+1) + m(n-m)$ |

(6.31)

**When to use?**

- When $m >> n$, do revise simplex method on the dual problem
- When $m \simeq n$, revise simplex method is not as good as simplex method
- When $m << n$ perfect for revise simplex method.

### 6.5.3 Decomposition of B inverse

Let $B = \{a_{B_1}, a_{B_2}, ..., a_{B_r}, ..., a_{B_m}\}$ and $B^{-1}$ is known. If $a_{B_r}$ is replaced by $a_{B_k}$, then $B$ becomes $\bar{B}$. Which means $a_{B_r}$ enters the basis and $a_{B_k}$ leaves the basis.

Then $\bar{B}^{-1}$ can be represent by $B^{-1}$. Noting that $a_k = By_k$ and $a_{B_i} = Be_i$, then

$$\begin{aligned}\bar{B} &= (a_{B_1}, a_{B_2}, ..., a_{B_{r-1}}, a_k, a_{B_{r+1}}, a_m) && (6.32)\\ &= (Be_1, Be_2, ..., Be_{r-1}, By_k, Be_{r+1}, ..., Be_m) && (6.33)\\ &= BT && (6.34)\end{aligned}$$

where $T$ is

$$T = \begin{bmatrix} 1 & 0 & ... & 0 & y_{1k} & 0 & ... & 0 \\ 0 & 1 & ... & 0 & y_{2k} & 0 & ... & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 1 & y_{r-1,k} & 0 & ... & 0 \\ 0 & 0 & ... & 0 & y_{rk} & 0 & ... & 0 \\ 0 & 0 & ... & 0 & y_{r+1,k} & 1 & ... & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 0 & y_{mk} & 0 & ... & 1 \end{bmatrix} \tag{6.35}$$

and

$$E = T^{-1} = \begin{bmatrix} 1 & 0 & ... & 0 & \frac{-y_{1k}}{y_{rk}} & 0 & ... & 0 \\ 0 & 1 & ... & 0 & \frac{-y_{2k}}{y_{rk}} & 0 & ... & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 1 & \frac{-y_{r-1,k}}{y_{rk}} & 0 & ... & 0 \\ 0 & 0 & ... & 0 & \frac{1}{y_{rk}} & 0 & ... & 0 \\ 0 & 0 & ... & 0 & \frac{-y_{r+1,k}}{y_{rk}} & 1 & ... & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 0 & \frac{-y_{mk}}{y_{rk}} & 0 & ... & 1 \end{bmatrix} \tag{6.36}$$

For each iteration, i.e. one variable enters the basis and one leaves the basis, $\bar{B}^{-1} = T^{-1}B^{-1} = EB^{-1}$. Given that the first iteration starts from slack variables, the first basis $B_1$ is $I$, then we have

$$B_t^{-1} = E_{t-1}E_{t-2} \cdots E_2E_1I \tag{6.37}$$

Using $E$ in calculation can simplify the product of matrix where

$$\begin{aligned} cE &= c_1, c_2, ..., c_m \begin{bmatrix} 1 & 0 & ... & g_1 & ... & 0 \\ 0 & 1 & ... & g_2 & ... & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & ... & g_m & ... & 1 \end{bmatrix} && (6.38)\\ &= (c_1, c_2, ..., c_{r-1}, cg, c_{r+1}, ..., c_m) && (6.39)\end{aligned}$$

and

$$\begin{aligned} Ea &= \begin{bmatrix} 1 & 0 & ... & g_1 & ... & 0 \\ 0 & 1 & ... & g_2 & ... & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & ... & g_m & ... & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} && (6.40)\\ &= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ 0 \\ a_{r+1} \\ \vdots \\ a_m \end{bmatrix} + a_r \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{r-1} \\ g_r \\ g_{r+1} \\ \vdots \\ g_m \end{bmatrix} && (6.41)\\ &= \bar{a} + a_r g && (6.42)\end{aligned}$$

Then we can calculate $w$, $y_k$ and $\bar{b}$

$$\begin{aligned} w &= c_B B^{-1} = c_B E_{t-1}E_{t-2}...E_2E_1 && (6.43)\\ y_k &= B^{-1}a_k = E_{t-1}E_{t-2}...E_2E_1a_k && (6.44)\\ \bar{b} &= B_{t+1}^{-1}b = E_tE_{t-1}E_{t-2}...E_2E_1b && (6.45)\end{aligned}$$

## 6.6 Simplex for Bounded Variables

### 6.6.1 Bounded Variable Formulation

$$\begin{aligned} \min \quad & cx && (6.46)\\ \text{s.t.} \quad & Ax = b && (6.47)\\ & l \le x \le b && (6.48)\end{aligned}$$

Reason why we don't the following formulation

$$\begin{aligned} \min \quad & cx && (6.49)\\ \text{s.t.} \quad & Ax = b && (6.50)\\ & x - Ix_l = l && (6.51)\\ & x + Ix_u = u && (6.52)\\ & x \ge 0 && (6.53)\\ & x_l \ge 0 && (6.54)\\ & x_u \ge 0 && (6.55)\end{aligned}$$

is that this formulation increase the number of variable from $n$ to $3n$, and the number of constraint from $m$ to $m + 2n$, the size in increase significantly.

### 6.6.2 Basic Feasible Solution

Consider the system $Ax = b$ and $l \le x \le b$, where $A$ is a $m \times n$ matrix of rank $m$, the solution $\bar{x}$ is a **basic feasible solution** if $A$ can be partition into $[B, N_l, N_u]$ where the solution $x$ can be partition into $x = (x_B, x_{N_l}, x_{N_u})$, in which $\bar{x}_{N_l} = l_{N_l}$ and $\bar{x}_{N_u} = u_{N_u}$, therefore

$$\bar{x}_B = B^{-1}b - B^{-1}N_lx_{N_l} - B^{-1}N_ux_{N_u} \tag{6.56}$$

Furthermore, similar to definition of nonnegative variables, if $l_B \le x_B \le u_B$, $x_B$ is a basic feasible solution, if $l_B < x_B < u_B$, $x_B$ is a non-degenerate basic feasible solution.

### 6.6.3 Improving Basic Feasible Solution

The basic variables and the objective function can be derived as following:

$$
\begin{aligned}
x_B &= B^{-1}b - B^{-1}N_l x_{N_l} - B^{-1}N_u x_{N_u} && (6.57)\\
z &= c_B x_B + c_{N_l} x_{N_l} + c_{N_u} x_{N_u} && (6.58)\\
&= c_B(B^{-1}b - B^{-1}N_l x_{N_l} - B^{-1}N_u x_{N_u}) && (6.59)\\
&\quad + c_B x_B + c_{N_l} x_{N_l} + c_{N_u} x_{N_u} && (6.60)\\
&= c_B B^{-1}b + (c_{N_l} - c_B B^{-1}N_l)x_{N_l} && (6.61)\\
&\quad + (c_{N_u} - c_B B^{-1}N_u)x_{N_u} && (6.62)\\
&= c_B B^{-1}b - \sum_{j\in J_1}(z_j - c_j)x_j - \sum_{j\in J_2}(z_j - c_j)x_j && (6.63)
\end{aligned}
$$

$J_1$ is the set of variables at lower bound, $J_2$ is the set of the variables at upper bound.
Notice that the right-hand-side no longer provide $c_B B^{-1}b$ and $B^{-1}b$. For the variable entering the basis, find the variable with

$$\max\{\max_{j\in J_1}\{z_j - c_j\}, \max_{j\in J_2}\{c_j - z_j\}\} \quad (6.64)$$

to enter the basis
**Tip:** "Most violated rule"
The minimum ratio rule is revised for bounded simplex

$$\Delta = \min\{\gamma_1, \gamma_2, u_k - l_k\} \quad (6.65)$$

$$\gamma_1 = \begin{cases} \min_{r\in J_1}\{\frac{\bar{b}_r - l_{B_r}}{y_{rk}} : y_{rk} > 0\} \\ \min_{r\in J_2}\{\frac{\bar{b}_r - l_{B_r}}{-y_{rk}} : y_{rk} < 0\} \\ \infty \end{cases} \quad (6.66)$$

$$\gamma_2 = \begin{cases} \min_{r\in J_1}\{\frac{u_{B_r} - \bar{b}_r}{-y_{rk}} : y_{rk} < 0\} \\ \min_{r\in J_2}\{\frac{u_{B_r} - \bar{b}_r}{y_{rk}} : y_{rk} > 0\} \\ \infty \end{cases} \quad (6.67)$$

**Tip:**
Use $l \le x + \Delta \le u$ to test the range of $\delta$, if it hits lower bound, it is called $\gamma_1$, if it hits upper bound, it is called $\gamma_2$.

## 6.7 Degeneracy and Cycling

### 6.7.1 Degeneracy

#### Degeneracy in Simplex Method

If the basic variable $x_B$ is not strictly $> 0$, i.e. if some basic variable equals to 0, we call it degenerate.

#### Degeneracy for Bounded Variables

If some basic variables are at their upper bound or lower bound, we call it degenerate.

### 6.7.2 Cycling

In the degenerate case, pivoting by the simplex rule does not always give a strict decrease in the objective function value, because it may have $b_r = 0$. It is possible that the tableau may repeat if we use the simplex rule.
Geometrically speaking, it means that at the same point - extreme point - it corresponds to more than one feasible solutions, so when we are pivoting, we stays at the same place.
In computer algorithm, we rarely care about cycling because the data in computer is not precise, it is very hard to get into cycling.

### 6.7.3 Cycling Prevent

#### Lexicographic Rule

- For entering variable, same as simplex rule
- For leaving variable, if there is a tie, choose the variable with the smallest $\frac{y_{r1}}{y_{rk}}$.

#### Bland's Rule

- For entering variable, choose the variable with smallest index where $z_j - c_j \le 0$
- For leaving variable, if there is a tie, choose the variable with smallest index.

#### Successive Ratio Rule

- Select the pivot column as any column $k$ where $z_k - c_k \le 0$
- Given $k$, select the pivot row $r$ as the minimum successive ratio row associated with column $k$.
In other words, for pivot columns where there is no tie in the usual minimum ratio, the successive ratio rule reduces to the simplex rule

## 6.8 Dual Simplex Method

Maintain dual feasibility, i.e. primal optimality, and complementary slackness and work towards primal feasibility. **Tip:** The RHS become new $z_j - c_j$, the old $z_j - c_j$ become new RHS. We are actually solving the dual problem.

## 6.9 As a Search Algorithm

### 6.9.1 Improving Search Algorithm

A simplex method is a search algorithm, for each iteration it finds a not-worse solution, which can be represented as:

$$x^t = x^{t-1} + \lambda_{t-1} d^{t-1} \tag{6.68}$$

Where
- $x^t$ is the solution of the $t$th iteration
- $\lambda_t$ is the step length of $t$th iteration
- $d^t$ is the direction of the $t$th iteration

For each iteration, it contains three steps:
- Optimality test
- Find direction
- Find the step length

### 6.9.2 Optimality Test

$$\begin{aligned}
z &= cx && (6.69)\\
&= \begin{bmatrix} c_B & c_N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} && (6.70)\\
&= c_B x_B + c_N x_N && (6.71)\\
\text{and } \because Ax &= b && (6.72)\\
\therefore B x_B + N x_N &= b, x_B \geq 0, x_N \geq 0 && (6.73)\\
\therefore x_B &= B^{-1}b - B^{-1} N x_N && (6.74)\\
z &= c_B B^{-1} b - c_B B^{-1} N x_N + c_N x_N && (6.75)
\end{aligned}$$

for current solution $\hat{x} = \begin{bmatrix} \hat{x_B} \\ 0 \end{bmatrix}$, $\hat{z} = c_B B^{-1} b$, then

$$z - \hat{z} = \begin{bmatrix} 0 & c_N - c_B B^{-1} N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} \tag{6.76}$$

The $c_N - c_B B^{-1} N$ is the reduced cost, for a minimized problem, if $c_N - c_B B^{-1} N > 0$ means $z - \hat{z} \geq 0$, it reaches the optimality because we cannot find a solution less than $\hat{z}$.

### 6.9.3 Find Direction

Suppose we choose $x_k$ as a candidate to pivot into Basis

$$x = \begin{bmatrix} B^{-1}b - B^{-1} a_k x_k \\ 0 + e_k x_k \end{bmatrix} = \begin{bmatrix} B^{-1} b \\ 0 \end{bmatrix} + \begin{bmatrix} -B^{-1} a_k \\ e_k \end{bmatrix} x_k \tag{6.77}$$

In this form, we can see: $x$ is the result after $t$th iteration, $\begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$ is the result after $(t-1)$th iteration. $\begin{bmatrix} -B^{-1} a_k \\ e_k \end{bmatrix}$ is the iteration direction, $x_k$ is the step length.
The only requirement of $x_k$ is $r_k < 0$ where $r_k = c_k - z_k$ is reduce cost, which is the $k$th entry of $c_N - c_B B^{-1} N$.
Generally speaking, we usually take $r_k = \min\{c_j - z_j\}$ (which in fact can not guarantee the efficient of the algorithm.)

### 6.9.4 Find the Step Length

We need to guarantee the non-negativity, so for each iteration, we need to make sure $x \geq 0$. Which means

$$B^{-1} b - B^{-1} a_k x_k \geq 0 \tag{6.78}$$

Denote $B^{-1}b$ as $\bar{b}$, denote $B^{-1} a_k$ as $y_k$
If $y_k \leq 0$, we can have $x_k$ as large as infinite, which means unboundedness.
If $y_k > 0$ now we can use the minimum ratio to guarantee non-negativity.
**Remember** hit the bound, basic variable leave the basis and become non-basic variable.

### 6.9.5 Simplex Method Algorithm

### 6.9.6 Simplex Method Tableau

### 6.9.7 Simplex Method as a Search Algorithm

# Chapter 7

# Duality, Sensitivity and Relaxation

## 7.1 Duality

### 7.1.1 Dual Formulation

For any prime problem

$$\begin{aligned} \min \quad & cx && (7.1)\\ \text{s.t.} \quad & Ax \geq b && (7.2)\\ & x \geq 0 && (7.3) \end{aligned}$$

we can have a dual problem

$$\begin{aligned} \max \quad & wb && (7.4)\\ \text{s.t.} \quad & wA \leq c && (7.5)\\ & w \geq 0 && (7.6) \end{aligned}$$

### 7.1.2 Mixed Forms of Duality

For the following prime problem

$$\begin{aligned} \text{P(or D)} \quad \min \quad & c_1x_1 + c_2x_2 + c_3x_3 && (7.7)\\ \text{s.t.} \quad & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \geq b_1 && (7.8)\\ & A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \leq b_2 && (7.9)\\ & A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3 && (7.10)\\ & x_1 \geq 0 && (7.11)\\ & x_2 \leq 0 && (7.12)\\ & x_3 \quad \text{unrestricted} && (7.13) \end{aligned}$$

The dual of the problem

$$\begin{aligned} \text{D(or P)} \quad \max \quad & w_1b_1 + w_2b_2 + w_3b3 && (7.14)\\ \text{s.t.} \quad & w_1A_{11} + w_2A_{21} + w_3A_{31} \leq c_1 && (7.15)\\ & w_1A_{12} + w_2A_{22} + w_3A_{32} \geq c_2 && (7.16)\\ & w_1A_{13} + w_2A_{23} + w_3A_{33} = c_3 && (7.17)\\ & w_1 \geq 0 && (7.18)\\ & w_2 \leq 0 && (7.19)\\ & w_3 \quad \text{unrestricted} && (7.20) \end{aligned}$$

In sum, the relation between primal and dual problems are listed as following

| | Minimization | | Maximization | |
|---|---|---|---|---|
| Var | $\geq 0$ | $\longleftrightarrow$ | $\leq 0$ | Cons |
| | $\leq 0$ | $\longleftrightarrow$ | $\geq 0$ | |
| | Unrestricted | $\longleftrightarrow$ | $=$ | |
| Cons | $\geq 0$ | $\longleftrightarrow$ | $\geq 0$ | Var |
| | $\leq 0$ | $\longleftrightarrow$ | $\leq 0$ | |
| | $=$ | $\longleftrightarrow$ | Unrestricted | |

### 7.1.3 Dual of the Dual is the Primal

For a primal problem (P)

$$\begin{aligned} (P) \quad \min \quad & cx && (7.21)\\ \text{s.t.} \quad & Ax \geq b && (7.22)\\ & x \geq 0 && (7.23) \end{aligned}$$

The dual problem (D) is

$$\begin{aligned} (D) \quad \max \quad & wb && (7.24)\\ \text{s.t.} \quad & wA \leq c && (7.25)\\ & w \geq 0 && (7.26) \end{aligned}$$

Rewrite the dual

$$\begin{aligned} \min \quad & -b^Tw^T && (7.27)\\ \text{s.t.} \quad & -A^Tw^T \geq -c^T && (7.28)\\ & w^T \geq 0 && (7.29) \end{aligned}$$

Find the dual of this problem

$$\begin{aligned} \max \quad & x^T(-c^T) && (7.30)\\ \text{s.t.} \quad & x^T(-A^T) \leq (-b^T) && (7.31)\\ & x^T \geq 0 && (7.32)\\ & && (7.33) \end{aligned}$$

Rewrite the dual of the dual

$$\begin{aligned} (P) \quad \min \quad & cx && (7.34)\\ \text{s.t.} \quad & Ax \geq b && (7.35)\\ & x \geq 0 && (7.36) \end{aligned}$$

### 7.1.4 Primal-Dual Relationships

#### Weak Duality Property

Let $x_0$ be any feasible solution of a primal minimization problem,

$$Ax_0 \geq b, \quad x_0 \geq 0 \tag{7.37}$$

Let $x_0$ be any feasible solution of a dual maximization problem,

$$w_0 A \leq c, \quad w_0 \geq 0 \tag{7.38}$$

Therefore, we have

$$cx_0 \geq w_0 A x_0 \geq w_0 b \tag{7.39}$$

which is called the weak duality property. This property is for any feasible solution in the primal and dual problem.
Therefore, any feasible solution in the maximization problem gives the lower bound of its dual problem, which is a minimization problem, vice versa. We use this to give the bounds in using linear relaxation to solve IP problem.

#### Fundamental Theorem of Duality

With regard to the primal and dual LP problems, one and only one of the following can be true.
- Both primal and dual has optimal solution $x^*$ and $w^*$, where $cx^* = w^*b$
- One problem has an unbounded optimal objective value, the other problem must be infeasible
- Both problems are infeasible.

#### Strong Duality Property

From KKT condition, we know that in order to make $x^*$ the optimal solution, the following condition should be met.
- Primal Optimal: $Ax^* \geq b$, $x^* \geq 0$
- Dual Optimal: $w^*A \leq c$, $w^* \geq 0$
- Complementary Slackness:

$$\begin{cases} w^*(Ax^* - b) = 0 \\ (c - w^*A)x^* = 0 \end{cases} \tag{7.40}$$

The first condition means the primal has an optimal solution, the second condition means the dual has an optimal solution. The third condition means $cx^* = w^*b$, which is also called **strong duality property**

$\boxed{\textbf{Tip:}}$ $w$ in the dual problem is the same as the $w = c_B B^{-1}$ in primal problem.

#### Complementary Slackness Theorem

Let $\boldsymbol{x^*}$ and $\boldsymbol{w^*}$ be any feasible solutions, they are optimal iff

$$(c_j - \boldsymbol{w^* a_j})x_j^* = 0, \quad j = 1, ..., n \tag{7.41}$$

$$w_i^*(\boldsymbol{a^i x^*} - b_i) = 0, \quad i = 1, ..., m \tag{7.42}$$

In particular

$$x_j^* > 0 \Rightarrow \boldsymbol{w^* a_j} = c_j \tag{7.43}$$

$$\boldsymbol{w^* a_j} < c_j \Rightarrow x_j^* = 0 \tag{7.44}$$

$$w_i^* > 0 \Rightarrow \boldsymbol{a^i x^*} = b_i \tag{7.45}$$

$$\boldsymbol{a^i x^*} > b_i \Rightarrow w_i^* = 0 \tag{7.46}$$

It means, if in optimal solution a variable is positive (has to be in the basis), the correspond constraint in the other problem is tight. If the constraint in one problem is not tight, the correspond variable in the other problem is zero.

#### Use Dual to Solve the Primal

in the dual problem, we solved some $w$ which is positive, we can know that the correspond constraint in primal is tight, furthermore we can solve the basic variables from those tight constraints, which becomes equality and we can solve it using Gaussian-Elimination.

### 7.1.5 Shadow Price

#### Shadow Price under Non-degeneracy

Let $B$ be an optimal basis for primal problem and the optimal solution $x^*$ is non-degenerated.

$$z = c_B B^{-1} b - \sum_{j \in N} (z_j - c_j) x_j = w^* b - \sum_{j \in N} (z_j - c_j) x_j \tag{7.47}$$

therefore

$$\frac{\partial z^*}{\partial b_i} = c_B B_i^{-1} = w_i^* \tag{7.48}$$

$w^*$ is the shadow prices for the right-hand-side vectors. We can also regard it as the **incremental cost** of producing one more unit of the $i$th product. Or $w^*$ is the **fair price** we would pay to have an extra unit of the $i$th product.

#### Shadow Price under Degeneracy

For shadow price under degeneracy, the $w^*$ may not be the true shadow price, for it may not be the right basis. In this case, the partial differentiation may not be valid, for component $b_i$, if $x_i = 0$ and $x_i$ is a basic variable, we can't find the differentiation.

## 7.2 Sensitivity

### 7.2.1 Change in the Cost Vector

#### Case 1: Nonbasic Variable

$c_B$ is not affected, $z_j = c_B B^{-1} a_j$ is not changed, say nonbasic variable cost coefficient $c_k$ changed into $c_k^{'}$. For now $z_k - c_k \leq 0$, if $z_k - c_k^{'}$ is positive, $x_k$ must into the basis, the optimal value changed.Otherwise stays at the same.

**Case 2: Basic Variable**

If $c_{B_t}$ is replaced by $c'_{B_t}$, then $z'_j - c_j$ is

$$z'_j - c_j = c'_B B^{-1} a_j - c_j = (z_j - c_j) - (c'_{B_t} - c_{B_t}) B^{-1} a_{B_t} \tag{7.49}$$

for $j = k$, it is a basic variable, therefore original $z_k - c_k = 0$, $B^{-1}a_k = 1$. Hence $z'_k - c_k = c'_k - c_k \Rightarrow z'_k - c'_k = 0$. The basis stays the same. The optimal solution updated as $c'_B B^{-1} b = c_B B^{-1} b + (c'_{B_t} - c_{B_t}) B^{-1} b_{B_t}$.

### 7.2.2 Change in the Right-Hand-Side

If $b$ is replaced by $b'$, then $B^{-1}b$ is replaced by $B^{-1}b'$. If $B^{-1}b' \geq 0$, the basis remains optimal. Otherwise, we perform dual simplex method to continue.

### 7.2.3 Change in the Matrix

**Case 1: Changes in Activity(Variable) Vectors for Nonbasic Columns**

If a nonbasic column $a_j$ is replaced by $a'_j$, then $z_j = c_B B^{-1} a_j$ is replaced by $z'_j = c_B B^{-1} a'_j$, if new $z'_j - c_j \leq 0$, the basis stays optimal basis, the optimal value is the same because $c_B$ stays the same.

**Case 2: Changes in Activity(Variable) Vectors for Basic Columns**

If a basic columns changed, it means $B$ and $B^{-1}$ changed, and every column changed. We can do this in two steps:
- step I: add a new column with $a'_j$
- step II: remove the original column $a_j$

If in step I the new variable can enter basis, i.e. $z'_j - c_j \leq 0$, let it enter the basis and eliminate the original column directly (because at this time the original column leave the basis the nonbasic variable is 0); otherwise, if the new column can not form a new basis, treat $x_j$, the original variable as an artificial variable.

**Add a New Activity(Variable)**

Suppose we add a new variable $x_{n+1}$ and $c_{n+1}$ and $a_{n+1}$ respectively. Calculate $z_{n+1} - c_{n+1}$ to determine if the new variable enters the basis, if not, remains the same optimal solution, otherwise, continue on to find a new optimal solution.

**Add a New Constraint**

This is the basic of Branch-and-Cut/Bound, also, we can perform dual simplex method after we add a new constraint(cut)

## 7.3 Relaxation

### 7.3.1 Why Rounding Can be Bad - IP Example

Rounding can be bad because the optimal of IP can be far away from optimal of LP. For example,

$$\begin{aligned}
\max \quad & z = x_1 + 0.64x_2 && (7.50)\\
\text{s.t.} \quad & 50x_1 + 31x_2 \leq 250 && (7.51)\\
& 3x_1 - 2x_2 \geq -4 && (7.52)\\
& x_1, x_2 \geq 0 \quad \text{(for LP)} && (7.53)\\
& x_1, x_2 \in Z^+ \quad \text{(for IP)} && (7.54)
\end{aligned}$$

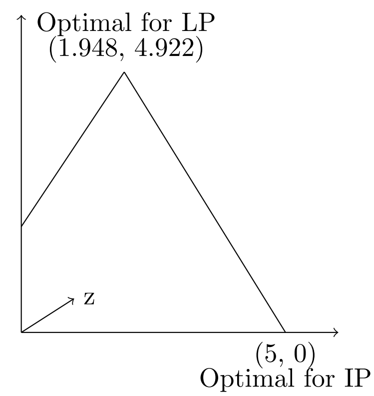



Figure 7.1: Optimal solution for LP / IP

### 7.3.2 Why Rounding Can be Bad - QAP example

Rounding can make the LP useless. For example, for QAP problem, the IP model is

$$\min \quad z = \sum_{i \in D} \sum_{s \in O} c_i s x_i s + \sum_{i \in D} \sum_{j \in D} \sum_{s \in O} \sum_{t \in O} w_{ij}^{st} y_{ij}^{st} \tag{7.55}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{i \in D} x_{is} = 1, \quad s \in D && (7.56)\\
& \sum_{s \in O} x_{is} = 1, \quad i \in D && (7.57)\\
& x_{is} \in \{0, 1\}, \quad i \in D, s \in O && (7.58)\\
& y_{ij}^{st} \geq x_{is} + x_{jt} - 1, \quad i \in D, j \in D, s \in O, t \in O && (7.59)\\
& y_{ij}^{st} \geq 0, \quad i \in D, j \in D, s \in O, t \in O && (7.60)\\
& y_{ij}^{st} \leq x_{is}, \quad i \in D, j \in D, s \in O, t \in O && (7.61)\\
& y_{ij}^{st} \leq x_{jt}, \quad i \in D, j \in D, s \in O, t \in O && (7.62)
\end{aligned}$$

We can get the optimal solution for LP supposing $\forall i, s \quad x_{is} \in [0,1]$

$$x_{is} = \frac{1}{|D|}, \quad i \in D, s \in O \tag{7.63}$$

$$y_{ij}^{st} = 0, \quad i \in D, j \in D, s \in O, t \in O \tag{7.64}$$

### 7.3.3 IP and Convex Hull

For IP problem

$$Z_{IP} \quad \max \quad z = cx \tag{7.65}$$

$$\text{s.t.} Ax \leq b \tag{7.66}$$

$$x \in Z^n \tag{7.67}$$

In feasible region $S = \{x \in Z^n, Ax \leq b\}$ , the optimal solution $Z_{IP} = \max\{cx : x \in S\}$.
Denote $conv(S)$ as the convex hull of $S$ then

$$Z_{IP}(S) = Z_{IP}(conv(S)) \tag{7.68}$$

### 7.3.4 Local Optimal and Global Optimal

Let

$$Z_s = \min\{f(x) : x \in S\} \tag{7.69}$$

$$Z_t = \min\{f(x) : x \in T\} \tag{7.70}$$

$$S \subset T \tag{7.71}$$

then

$$Z_t \leq Z_s \tag{7.72}$$

**Notice** that if $x_T^* \in S$ then $x_S^* = x_T^*$, to generalized it, We have

$$\begin{cases} x_T^* \in \arg\min\{f(x) : x \in T\} \\ x_T^* \in S \end{cases} \tag{7.73}$$

$$\Rightarrow x_T^* \in \arg\min\{f(x) : x \in S\} \tag{7.74}$$

Especially for IP, we can take the LP relaxation as $T$ and the original feasible region of IP as $S$, therefore, if we find an optimal solution from LP relaxation $T$ which is also a feasible solution of $S$, then it is the optimal solution for IP $(S)$

### 7.3.5 LP Relaxation

To perform the LP relaxation, we expand the feasible region

$$x \in \{0,1\} \to x \in [0,1] \tag{7.75}$$

$$y \in Z^+ \to y \geq 0 \tag{7.76}$$

If we have $Z_LP(s) = conv(s)$ then

$$LP(s) : x \in R_+^n : Ax \leq b \tag{7.77}$$

The closer $LP(s)$ is to $conv(s)$ the better. Interestingly, we only need to know the convex in the direction of $c$.
There are several formulation problem have the property of $Z_{LP}(s) = conv(s)$, such as:
- Assignment Problem
- Spawning Tree Problem
- Max Flow Problem
- Matching Problem

## Chapter 8

# Decomposition Principle

# Chapter 9

# Ellipsoid Algorithm

# Chapter 10

# Projective Algorithm

# Chapter 11

# Interior-Point Algorithm

# Part III

# Graph and Network Theory

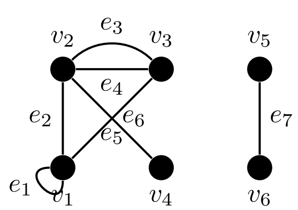
# Chapter 12

# Graphs and Subgraphs

## 12.1 Graph

**Definition 12.1.1** (Graph)**.** A **graph** G consists of a finite set $V(G)$ on vertices, a finite set $E(G)$ on edges and an **incident relation** than associates with any edge $e \in E(G)$ an unordered pair of vertices not necessarily distinct called **ends**.

**Example.** The following graph




can be represented as

$$V = V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\} \tag{12.1}$$
$$E = E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \tag{12.2}$$
$$e_1 = v_1v_2, \quad e_2 = v_2v_4, \quad \ldots \tag{12.3}$$

**Definition 12.1.2** (Loop, Parallel, Simple Graph)**.** An edge with identical ends is called a **loop**, Two edges having the same ends are said to be **parallel**, A graph without loops or parallel edges is called **simple graph**

**Definition 12.1.3** (Adjacent)**.** Two edges of a graph are **adjacent** if they have a common end, two vertices are **adjacent** if they are jointed by an edge.

## 12.2 Graph Isomorphism

## 12.3 The Adjacency and Incidence Matrices

## 12.4 Subgraph

**Definition 12.4.1** (Subgraph)**.** Given two graphs $G$ and $H$, $H$ is a **subgraph** of $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and an edge has the same ends in $H$ as it does in $G$. Furthermore, if $E(H) \neq E(G)$ then $H$ is a proper subgraph.

**Definition 12.4.2** (Spanning)**.** A subgraph $H$ on $G$ is **spanning** if $V(H) = V(G)$

**Definition 12.4.3** (Vertex-induced, Edge-induced)**.** For a subset $V^{'} \subseteq V(G)$ we define an **vertex-induced** subgraph $G[V^{'}]$ to be the subgraph with vertices $V^{'}$ and those edges of $G$ having both ends in $V^{'}$. The **edge-induced** subgraph $G[E^{'}]$ has edges $E^{'}$ and those vertices of $G$ that are ends to edges in $E^{'}$.

**Notice:** If we combine node-induced or edge-induced subgraphs $G(V^{'})$ and $G(V - V^{'})$, we cannot always get the entire graph.

## 12.5 Degree

**Definition 12.5.1** (Degree)**.** Let $v \in V(G)$, then the **degree** of $v \in V(G)$ denote by $d_G(v)$ is defines to be the number of edges incident of $v$. Loops counted twice.

**Theorem 12.1.** *For any graph $G = (V, E)$*

$$\sum_{v \in V} d(v) = 2|E| \tag{12.4}$$

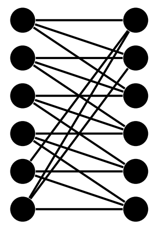
*Proof.* $\forall$ edge $e = uv$ with $u \neq v$, $e$ is counted once for $u$ and once for $v$, a total of two altogether. If $e = uu$, a loop, then it is counted twice for $u$. $\square$

**Problem 12.1.** Explain clearly, what is the largest possible number of vertices in a graph with 19 edges and all vertices of degree at least 3. Explain why this is the maximum value.

**Solution.** The maximum number is 12.

*Proof.* First we prove 12 vertices is possible, then we prove 13 vertices is not possible

- The following graph contains 12 vertices and 18 edges, each vertex has a degree of 3.



- For 13 vertices and each vertex has a degree of at least 3 will require at least

$$2|E| = \sum_{v \in V} d(v) \geq 3 \times |N| = 3 \times 13 \Rightarrow |E| \geq 19.5 > 19 \tag{12.5}$$

edges, i.e., 13 vertices is not possible.

□

**Corollary 12.1.1.** *Every graph has an even number of odd degree vertices.*

*Proof.*

$$V = V_E \cup V_O \Rightarrow \sum_{v \in V} d(v) = \sum_{v \in V_E} d(v) + \sum_{v \in V_O} d(v) = 2|E| \tag{12.6}$$

□

## 12.6 Special Graphs

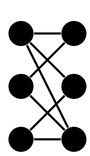
**Definition 12.6.1** (Complete Graph)**.** A **complete** graph $K_n (n \geq 1)$ is a simple graph with $n$ vertices and with exactly one edge between each pair of distinct vertices.

**Definition 12.6.2** (Cycle)**.** A **cycle** graph $C_n (n \geq 3)$ consists of $n$ vertices $v_1, ... v_n$ and $n$ edges $\{v_1, v_2\}, \{v_2, v_3\}, ... \{v_{n-1}, v_n\}$

**Definition 12.6.3** (Wheel)**.** A **wheel** graph $W_n (n \geq 3)$ is a simple graph obtains by adding one vertex to the cycle graph $C_n$, and connecting this new vertex to all vertices of $C_n$

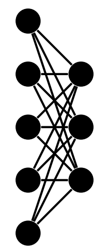
**Definition 12.6.4** (Bipartite Graph)**.** A simple graph is said to be **bipartite** if the vertex set can be expressed as the union of two disjoint non-empty subsets $V_1$ and $V_2$ such that every edges has one end in $V_1$ and another end in $V_2$

**Example.** Here is an example for bipartite graph



**Definition 12.6.5** (Complete Bipartite Graph)**.** The **complete bipartite graph** $K_{mn}$ is the bipartite graph $V_1$ containing $m$ vertices and $V_2$ containing $n$ vertices such that each vertiex in $V_1$ is adjacent to every vertex in $V_2$

**Example.** Here is an example for $K_{53}$



**Theorem 12.2.** *(König Theorem) A graph $G$ is bipartite iff every cycle is even.*

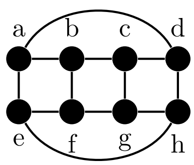
*Proof.* Hereby we prove the ⇒ and ⇐

- (⇒) If the graph $G$ is bipartite, by definition, the vertices of graph can be partition into two groups, that within the group there is no connection between vertices. Therefore, for each cycle, the odd index of vertices and even index of vertices has to be choose alternatively from each groups. Therefore the cycle has to be even.
- (⇐) Prove by contradiction. A graph can be connected or not connected.
  - If $G$ is connected and has at least two vertices, for an arbitrary vertex $v \in V(G)$, we can calculate the minimum number of edges between the other vertices $v'$ and $v$ (i.e., length, denoted by $l(v', v)$), for all the vertices that has odd length to $v$, assign them to set $V_1$, for the rest of vertices (and $v$), assign to set $V_2$. Assume that $G$ is not bipartite, which means there are at least one edge between distinct vertices in set $V_1$ or set $V_2$, without lost of generality, assume that edge is $uw$, $u, w \in V_1$. For all vertices in $V_1$ there is an odd length of path between the vertex and $v$, therefore, there exists an odd $l(u, v)$, and an odd $l(w - v)$. The length of cycle $l(u, w, v) = 1 + l(u, v) + l(w, v)$, which is an odd number, it contradict with the prerequisite that all cycles are even, which means the assumption that $G$ is not bipartite is incorrect, $G$ should be bipartite.
  - If $G$ is not connected. Then $G$ can be partition into a set of disjointed subgraphs which are connected with at least two vertices or contains only one vertex. For the subgraph that has more that one vertices, we already proved that it has to be bipartite. For the subgraph

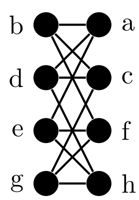
$G_i \subset G, i = 1, 2, ..., n$, the vertices can be partition into $V_{i1} \in V(G_i)$ and $V_{i2} \in V(G_i)$, where $V_{i1} \cap V_{i2} = \emptyset$, the union of those subgraphs are bipartite too because $V_1 = \cup_{i=1}^{n} V_{i1} \in V(G)$ and $V_2 = \cup_{i=1}^{n} V_{i2} \in V(G)$ satisfied the condition of bipartite. For the subgraph that has one one vertices, those vertices can be assigned into either $V_1$ or $V_2$.

□

**Example.** The following graph is bipartite, it only contains even cycles.

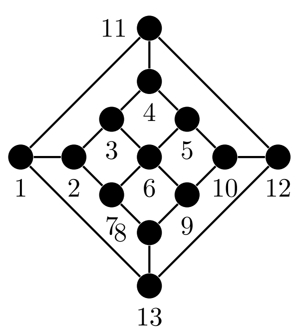



We can rearrange the graph to be more clear as following




The vertices of graph $G$ can be partition into two sets, $\{a, c, f, h\}$ and $\{b, d, e, g\}$

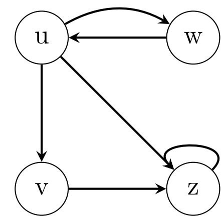
**Example.** The following graph is not bipartite




The cycle $c = v_1v_11v_4v_3v_2$ have odd number of vertices.

## 12.7 Directed Graph

**Definition 12.7.1.** A graph $G = (V, E)$ is called directed if for each edge $e \in E$, there is a **head** $h(e) \in V$ and a **tail** $t(e) \in V$ and the edges of $e$ are precisely $h(e)$ and $t(e)$, denoted $e = (t(e), h(e))$




**Definition 12.7.2.** We call directed graphs **digraphs**, we call edges in a digraph are called **arcs**, and vertices in a digraph **nodes**

**Definition 12.7.3.** Similar as in the undirected case we have walks, traces, paths and cycles in digraphs.

**Definition 12.7.4.** A vertex $v \in V$ is **reachable** from a vertex $u \in V$ if there is a $(u, v)$-dipath. If at the same time $u$ is reachable from $v$, they are **strongly connected**

**Definition 12.7.5.** A digraph is strongly connected if every pair of vertices are strongly connected.

**Definition 12.7.6.** A digraph is **strict** if it has no loops and whenever $e$ and $f$ are parallel, $h(e) = t(f)$

**Definition 12.7.7.** For a vertex $v$ in a digraph $D$, the **indegree** of $v$ in $D$, denoted by $d^+(v)$ is the number of arcs of $D$ having head $V$. The **outdegree** of $v$ is denoted by $d^-(v)$ is the number of arcs of $D$ having tail $v$.

Let $D = (V, A)$ be a digraph with no loops a vertex-arc **incident matrix** for $D$ is a $(0, 1, -1)$ matrix $N$ with rows indexed by $V = \{v_1, ..., v_n\}$ and column indexed by $A = \{e_1, ..., e_m\}$ and where entry $(i, j)$ in the matrix $n_{ij}$ is

$$n_{ij} = \begin{cases} 1, & \text{if } v_i = h(e_j) \\ -1, & \text{if } v_i = t(e_j) \\ 0, & \text{otherwise} \end{cases} \tag{12.7}$$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \tag{12.8}$$

## 12.8 Sperner's Lemma

# Chapter 13

# Paths, Trees, and Cycles

## 13.1 Walk

**Definition 13.1.1** (walk)**.** A **walk** in a graph $G$ is a finite sequence $w = v_0e_1v_1e_2...e_kv_k$, where for each $e_i = v_{i-1}v_i$ the edge and its ends exists in $G$. We say that walk $v_0$ to $v_k$ on $(v_0, v_k)$-walk.

**Example.**

$$w = v_2e_4v_3e_4v_2e_5v_3 \tag{13.1}$$

is a walk, or $(v_2, v_3)$-walk

**Definition 13.1.2** (origin, terminal, internal, length)**.** For $(v_0, v_k)$-walk, The vertices $v_0$ and $v_k$ are called the **origin** and the **terminal** of the walk w, $v_1..v_{k-1}$ are called **internal** vertices. The integer $k$ is the **length** of the walk. Length of $w$ equals to the number of edges.

We can create a reverse walk $w^{-1}$ by reversing $w$.

$$w^{-1} = v_ke_kv_{k-1}e_{k-1}...e_2v_1 \tag{13.2}$$

(The reverse walk is guaranteed to exist because it is an undirected graph)
Given two walks $w$ and $w'$ we can create a third walk denoted by $ww'$ by concating $w$ and $w'$. The new walk's origin is the same as terminal.

## 13.2 Path and Cycle

**Definition 13.2.1** (trail)**.** A **trail** is a walk with no repeating edges. e.g., $v_3e_4v_2e_5v_3$

**Definition 13.2.2** (path)**.** A **path** is a trail with no repeating vertices. e.g., $v_3e_4v_2$

**Notice:** Paths $\subseteq$ Trails $\subseteq$ Walks

**Definition 13.2.3** (closed, cycle)**.** A path is **closed** if it has positive length and its origin and terminal are the same. e.g., $v_1e_2v_2e_4v_3e_3v_1$. A closed trail where origin and internal vertices are distinct is called a **cycle** (The only time a vertex is repeated is the origin and terminal)

**Definition 13.2.4** (even/odd cycle)**.** A cycle is **even** if it has a even number of edges otherwise it is **odd**.

**Problem 13.1.** Prove that if $C_1$ and $C_2$ are cycles of a graph, then there exists cycles $K_1, K_2, ..., K_m$ such that $E(C_1)\Delta E(C_2) = E(K_1) \cup E(K_2) \cup ... \cup E(K_m)$ and $E(K_i) \cap E(K_j) = \emptyset, \forall i \neq j$. (For set $X$ and $Y$, $X\Delta Y = (X - Y) \cup (Y - X)$, and is called the symmetric difference of $X$ and $Y$)

*Proof.* Proof by constructing $K_1, K_2, ...K_m$. Denote

$$C_1 = v_{11}e_{11}v_{12}e_{12}v_{13}e_{13}...v_{1n}e_{1n}v_{11} \tag{13.3}$$
$$C_2 = v_{21}e_{21}v_{22}e_{22}v_{23}e_{23}...v_{2k}e_{2k}v_{21} \tag{13.4}$$

Assume both cycle start at the same vertice, $v_{11} = v_{12}$. (If there is no intersected vertex for $C_1$ and $C_2$, just simply set $K_1 = C_1$ and $K_2 = C_2$)
The following algorithm can give us all $K_j, j = 1, 2, ..., m$ by constructing $E(C_1)\Delta E(C_2)$. Also, the complexity is $O(mn)$, which makes the proof doable.

**Algorithm 2** Find $K_1, K_2, ...K_m$ by constructing $E(C_1)\Delta E(C_2)$

**Require:** Graph $G$, cycle $C_1$ and $C_2$
**Ensure:** $K_1, K_2, ...K_m$

Initial, $K \leftarrow \emptyset$, $j = 1$
Set temporary storage units, $v_o \leftarrow v_{11}$, $v_t \leftarrow \emptyset$
**for** $i = 1, 2, ..., n$ **do**
  **if** $e_{1i} \in C_2$ **then**
    **if** $v_o \neq v_{1i}$ **then**
      $v_t \leftarrow v_{1i}$
      concate $(v_o, v_t)$-path $\subset C_1$ and $(v_o, v_t)$-path $\subset C_2$ to create a new $K_j$
      Append $K$ with $K_j$, $K \leftarrow K \cup K_j$
      Reset temporary storage unit. $v_o \leftarrow v_{1(i+1)}$ (or $v_{11}$ if $i = n$), $v_t \leftarrow \emptyset$
    **else**
      $v_o \leftarrow v_{1(i+1)}$ (or $v_{11}$ if $i = n$)
    **end if**
  **end if**
**end for**

Now we prove that $K_i \cap K_j = \emptyset, \forall i \neq j$. For each $K_j$, it is defined by two $(v_o, v_t)$-paths in the algorithm. From the algorithm we know that all the edges in $(v_o, v_t)$-path in $C_1$ are not intersecting with $C_2$, because if the edge in $C_1$ is intersected with $C_2$, either we closed the cycle $K_j$ before the edge, or we updated $v_o$ after the edge (start a new $K_j$ after that edge). By definition of cycle, all the $(v_o, v_t)$-path that are subset of $C_1$ are not intersecting with each other, as well as all the $(v_o, v_t)$-path that are subset of $C_2$. Therefore, $K_i \cap K_j = \emptyset, \forall i \neq j$. □

**Definition 13.2.5** (connected vertices)**.** Two vertices $u$ and $v$ in a graph are said to be **connected** if there is a path between $u$ and $v$.

**Definition 13.2.6** (component)**.** Connectivity between vertices is an equivalence relation on $V(G)$, if $V_1, ...V_k$ are the corresponding equivalent classes then $G[V_1]...G[V_k]$ are **components** of G. If graph has only one component, then we say the graph is connected. A graph is connected iff every pair of vertices in G are connected, i.e., there exists a path between every pair of vertices.

**Problem 13.2.** If $G$ is a simple graph with at least two vertices, prove that $G$ has two vertices with the same degree.

*Proof.* A simple graph can only be connected or not connected.

- If $G$ is connected, i.e., for all vertices, the degree is greater than 0. Also the graph is simple, for a graph with $|N|$ vertices, the degree of each vertex is less or equal to $|N|-1$ (cannot have loop or parallel edge). For $|N|$ vertices, to make sure there is no two vertices that has same degree, it will need $|N|$ options for degrees, however, we only have $|N|-1$ option. According to pigeon in holes principle, there has to be at least two vertices with the same degree.

- If $G$ is not connected, i.e., the graph has more than one component. One of the following situation will happen:
  - For all components, each component contains only one vertex. Since we have at least two vertices, which means there are at least two component that has only one vertex. For those vertices, at least two vertices has the same degree as 0.
  - At least one component has more than one vertices. In this situation, we can find a component that has more than one vertices as a subgraph $G'$ of the graph $G$. That $G'$ is a connected simple graph by definition. We have already proved that a connected simple graph has two vertices with the same degree, which means $G$ has two vertices with the same degree.

□

## 13.3 Tree and forest

**Definition 13.3.1** (acyclic graph)**.** A graph is called **acyclic** if it has no cycles

**Definition 13.3.2** (forest, tree)**.** A acyclic graph is called a **forest**. A connected forest is called a **tree**.

**Theorem 13.1.** *Prove that $T$ is a tree, if $T$ has exactly one more vertex than it has edges.*

*Proof.*

1. First we prove for any tree $T$ that has at least two vertices, there has to be at least one leaf, i.e., now we prove that we can find $u$ with degree of 1. Proof by constructing algorithm. (In fact we can prove that there are at least two leaves.)

   **Algorithm 3** Find one leaf in a tree

   **Require:** $d(u) = 1$
   **Ensure:** A tree $T$ has at least one vertex

   ```
   Let u and v be any distinct vertex in a tree T
   Let p be the path between u and v
   while d(u) ≠ 1 do
       if d(u) > 1 then
           Let n(u) be the set of neighboring vertices of u
           In n(u), find a u' that the edge between u and
           u', denoted by e, e ∉ p
           u ← u'
           p ← p ∪ e
       end if
   end while
   ```

   The above algorithm is guaranteed to have an end because a tree is acyclic by definition

2. Then, if we remove one leaf in the tree, i.e., we remove an edge and a vertex, where that vertex only connects to the edge we removed. One of the following situations will happen:
   (a) Situation 1: The remaining of $T$ is one vertex. In this case, $T$ has two vertices an one edge. (Exactly one more vertex than it has edges)
   (b) Situation 2: The remaining of $T$ is another tree $T^{'}$ (removal of edges will not change acyclic and connectivity), where $|V(T)| = |V(T^{'})| + 1$ and $|E(T)| = |E(V^{'}| + 1$. (one edge and one vertex has been removed)

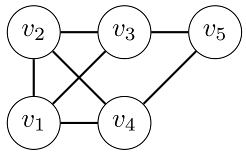
3. Do the leaf removal process recursively to $T^{'}$ if Situation 2 happens until Situation 1 happens.

□

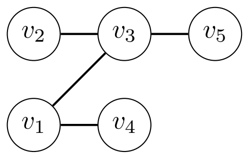
## 13.4 Spanning tree

**Definition 13.4.1** (spanning tree)**.** A subgraph T of G is a **spanning tree** if it is spanning ($V(T) = V(G)$) and it is a tree.

**Example.** In the following graph




This is a spanning tree




**Problem 13.3.** Prove that if $T_1$ and $T_2$ are spanning trees of $G$ and $e \in E(T_1)$, then there exists a $f \in E(T_2)$, such that $T_1 - e + f$ and $T_2 + e - f$ are both spanning trees of $G$.

*Proof.* One of the following situation has to happen:

1. If for given $e \in E(T_1)$, $\exists f = e \in E(T_2)$, then $T_1 - e + f = T_1$, $T_2 + e - f = T_2$ are both spanning trees of $G$

2. If for given $e \in E(T_1)$, $e \notin E(T_2)$, the following will find an edge $f$ that $T_1 - e + f$ and $T_2 + e - f$ are both spanning trees of $G$.

   (a) $T_1$ is a spanning tree, removal of $e \in E(T_1)$ will disconnect the spanning tree into two components (by definition of spanning tree), denoted by $G_1 \subset G$ and $G_2 \subset G$, by definition, $V(G_1)$ and $V(G_2)$ is a partition of $V(G)$.

   (b) Add $e$ into $T_2$. We can proof that by adding an edge into a tree will create exactly one cycle, denoted by $C$, $e \in E(C)$.

   (c) For $C$, since it is a cycle and one end of $e$ is in $V(G_1)$, the other end of $e$ is in $V(G_2)$, there has to be at least two edges (can be more) that has one end in $V(G_1)$ and the other end in $V(G_2)$, denote the set of those edges as $E \subset E(C)$, one of those edges is $e \in E$

   (d) Choose any $f \in E$ and $f \neq e$, for that $f$, $T_1 - e + f$ and $T_2 + e - f$ are both spanning trees of $G$.

   (e) Prove that $T_1 - e + f$ is a spanning tree

      i. $T_1 - e + f$ have the same set of vertices as $T_1$, therefore it is spanning.

      ii. It is connected both within $G_1$ and $G_2$, for $f$, one end is in $V(G_1)$, the other end is in $V(G_2)$ therefore $T_1 - e + f$ is connected.

      iii. $T_1 - e + f$ have the same number of edges as $T_1$, which is $|T_1| - 1$, therefore $T_1 - e + f$ is a tree. (We have proven the connectivity in the previous step.)

      iv. $T_1 - e + f$ is spanning, connected, a tree, therefore it is a spanning tree.

   (f) Prove that $T_2 + e - f$ is a spanning tree

      i. $T_2 + e - f$ have the same set of vertices as $T_2$, therefore it is spanning.

      ii. $T_2$ is connected, adding an edge will not break connectivity, therefore $T_2 + e$ is connected, removing an edge in a cycle will not break connectivity, therefore $T_2 + e - f$ is connected.

      iii. $T_2 - e + f$ have the same number of edges as $T_2$, which is $|T_2| - 1$, therefore $T_2 + e - f$ is a tree. (We have proven the connectivity in the previous step.)

      iv. $T_2 - e + f$ is spanning, connected, a tree, therefore it is a spanning tree.

□

**Theorem 13.2.** *Every connected graph has a spanning tree.*

*Proof.* Prove by constructing algorithm: □

**Algorithm 4** Find a spanning tree for connected graph (Prim's Algorithm in unweighted graph)

**Require:** a connected graph G and an enumeration $e_1, ...e_m$ of the edges of G
**Ensure:** a spanning tree T of G

```
Let T be the spanning subgraph of G with V(T) = V(G) and E(T) = ∅
i ← 1
while i ≤ |E| do
    if T + e_i is acyclic then
        T ← T + e_i
        i ← i + 1
    end if
end while
```

**Notice:** This algorithm can be improved, one idea is to make summation of edges in spanning subgraph less or equation to $|V| - 1$

For the complexity of spanning tree algorithm:

1. Space complexity, $2|E|$, which is $O(|E|)$

2. Time complexity

   (a) How to check for acyclic?

      i. At every stage $T$ has certain components $V_1, ...V_t$, (every time we add an edge, the number of components minus 1)

ii. So at the beginning $t = |V|$ with $|V_i| = 1 \forall i$ and at the end, $t = 1$.

(b) Count the amount of work for the algorithm.

i. Need to check for acyclic for each edge, which costs $O(|E|)$

ii. Need to flip the pointer for each vertex, for each vertex, at most will be flipped $\log |V|$ times, altogether $|V| \log |V|$ times.

iii. The time complexity is $O(|E| + |V| \log |V|)$

3. First we need to input the data, create an array such that the first and the second entries are the ends of $e_1$, third and fourth are the ends of $e_2$, and so on.

4. The amount of storage needs in $2|E|$, which is $O(|E|)$

5. The main work involved in the algorithm is for each edges $e_i$ and the current $T$, to determine if $T + e_i$ creates a cycle.

6. suppose we keep each component $V_i$ by keeping for each vertex a pointer from the vertex to the name of the component containing it. Thus if $\mu \in V_3$, there will be a pointer from $\mu$ to integer 3.

7. Then when edge $e_i = \mu v$ is encountered in Step 2, we see that $T + e_i$ contains a cycle if and only if $\mu$ and $v$ point to same integer which means they are in the same component

8. If they are not in the same component, we want to add the edge which means then I have to update the pointers.

To prove algorithm we need to show the output is a spanning tree, which means three properties must hold:

- spanning (Step I)
- acyclic (We never add an edge that create a cycle)
- connected (Proof by contradiction)

So it is sufficient to show that the output will be connected.

*Proof.* (Proof by Contradiction) Suppose the output graph $T$ of the algorithm is NOT connected. Let $T_1$ be a component of $T$, let $x \in T_1$ and $y \notin T_1$. But $G$ is a connected graph (given from the beginning), so there must be a path in $G$ that connects $x$ and $y$. Let such a path in $G$ be $p = xe_1v_1e_2, ..v_{k-1}e_ky$. Clearly, $p \notin T_1$. So there must be a first vertex in $P$ that not in $T_1$. So $e_i \notin E(T)$, the only way this can happen when applying the algorithm is if $T + e_i$ creates a cycle $C$, i.e., $e_i \in C$, so $C - e_i$ is a path connecting $v_{i-1}$ and $v_i$. So $c - e_i \in T$, so $v_{i-1}$ is connected to $v_i \in T$. Contradiction. $\square$

## 13.5 Cayley's Formula

## 13.6 Connectivity

## 13.7 Blocks

# Chapter 14

# Euler Tours and Hamilton Cycles

## 14.1 Euler Tours

## 14.2 Hamilton Cycles

# Chapter 15

# Planarity

## 15.1 Plane and Planar Graphs

## 15.2 Dual Graphs

## 15.3 Euler's Formula

## 15.4 Bridges

## 15.5 Kuratowski's Theorem

## 15.6 Four-Color Theorem

## 15.7 Graphs on other surfaces

# Chapter 16

# Minimum Spanning Tree Problem

## 16.1 Basic Concepts

**Example.** A company wants to build a communication network for their offices. For a link between office $v$ and office $w$, there is a cost $c_{vw}$. If an office is connected to another office, then they are connected to with all its neighbors. Company wants to minimize the cost of communication networks.

**Definition 16.1.1** (Cut vertex)**.** A vertex $v$ of a connected graph $G$ is a **cut vertex** if $G \setminus v$ is not connected.

**Definition 16.1.2** (Connection problem)**.** Given a connected graph $G$ and a positive cost $C_e$ for each $e \in E$, find a minimum-cost spanning connnected subgraph of $G$. (Cycles all allowed)

**Lemma 16.1.** *An edge $e = uv \in G$ is an edge of a cycle of $G$ iff there is a path $G \setminus e$ from $u$ to $v$.*

**Definition 16.1.3** (Minumum spanning tree problem)**.** Given a connected graph graph $G$, and a cost $C_e, \forall e \in E$, find a minimum cost spanning tree of $G$

The only way a connection problem will be different than MSP is if we relax the restriction on $C_e > 0$ in the connection problem.

## 16.2 Kroskal's Algorithm

**Algorithm 5** Kroskal's Algorithm, $O(m \log m)$

**Require:** A connected graph
**Ensure:** A MST
  Keep a spanning forest $H = (V, F)$ of $G$, with $F = \emptyset$
  **while** $|F| < |V| - 1$ **do**
    add to $F$ a least-cost edge $e \notin F$ such that $H$ remains a forest.
  **end while**

## 16.3 Prim's Algorithm

**Algorithm 6** Prim's Algorithm, $O(nm)$

**Require:** A connected graph
**Ensure:** A MST
  Keep $H = (V(H), T)$ with $V(H) = \{v\}$, where $r \in V(G)$ and $T = \emptyset$
  **while** $|V(T)| < |V|$ **do**
    Add to $T$ a least-cost edge $e \notin T$ such that $H$ remains a tree.
  **end while**

## 16.4 Comparison between Kroskal's and Prim's Algorithm

- Kroskal start with a forest that contains all vertices, Prim start with a tree that only contain one vertex.
- Kroskal cannot gurantee every step it is a tree but can gurantee it is spanning, Prim can gurantee every step it is a tree but cannot gurantee spanning.

## 16.5 Extensible MST

**Definition 16.5.1** (cut)**.** For a graph $G = (V, E)$ and $A \subseteq V$ we denote $\delta(A) = \{e \in E : e \text{ has an end in } A \text{ and an end in } V \setminus A\}$. A set of the form $\delta(A)$ for some $A$ is called a **cut** of $G$.

**Definition 16.5.2.** We also define $\gamma(A) = \{e \in E : \text{both ends of } e \text{ are in } A\}$

**Theorem 16.2.** *A graph $G = (V, E)$ is connected iff there is no $A \subseteq V$ such that $\emptyset \neq A \neq V$ with $\delta(A) = \emptyset$*

**Definition 16.5.3.** Let us call a subset $A \in E$ **extensible** to a minimum spanning tree problem if $A$ is contained in the edge set of some MST of $G$

**Theorem 16.3.** *Suppose $B \subseteq E$, that $B$ is extensible to an MST and that $e$ is a minimum cost edge of some cut $D$ satisfying $D \cap B = \emptyset$, then $B \cup \{e\}$ is extensible to an MST.*

## 16.6 Solve MST in LP

Given a connected graph $G = (V, E)$ and a cost on the edges $C_e$ for all $e \in E$, Then we can formulate the following LP

$$X_e = \begin{cases} 1, \text{if edge } e \text{ is in the optimal solution} \\ 0, \text{otherwise} \end{cases} \tag{16.1}$$

The formulation is as following

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e && (16.2)\\
\text{s.t.} \quad & \sum_{e \in E} x_e = |V| - 1 && (16.3)\\
& x_e \geq 0 && (16.4)\\
& e \in E && (16.5)\\
& \sum_{e \in E(S)} x_e = |S| - 1, \forall S \subseteq V, S \neq \emptyset && (16.6)\\
& && (16.7)
\end{aligned}$$

# Chapter 17

# Shortest-Path Problem

## 17.1 Basic Concepts

All Shortest-Path methods are based on the same concept, suppose we know there exists a dipath from $r$ to $v$ of a cost $y_v$. For each vertex $v \in V$ and we find an arc $(v, w) \in E$ satisfying $y_v + v_{vw} < y_w$. Since appending $(v, w)$ to the dipath to $v$ takes a cheaper dipath to $w$ then we can update $y_w$ to a lower cost dipath.

**Definition 17.1.1** (feasible potential)**.** We call $y = (y_v : v \in V)$ a **feasible potential** if it satisfies

$$y_v + c_{vw} \geq y_w \quad \forall (v, w) \in E \tag{17.1}$$

and $y_r = 0$

**Proposition 1.** *Feasible potential provides lower bound for the shortest path cost.*

*Proof.* Suppose that you have a dipath $P = v_0 e_1 v_1, ..., e_k v_k$ where $v_0 = r$ and $v_k = v$, then

$$C(P) = \sum_{i=1}^{k} C_{e_i} \geq \sum_{i=1}^{k} (y_{v_i} - y_{v_{i-1}}) = y_{v_k} - y_{v_0} \tag{17.2}$$

□

## 17.2 Breadth-First Search Algorithm

## 17.3 Ford's Method

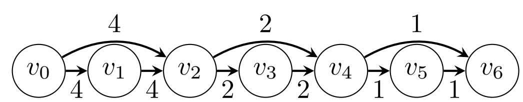
Define a predecessor function $P(w), \forall w \in V$ and set $P(w)$ to $v$ whenever $y_w$ is set to $y_v + c_{vw}$

**Notice:** Technically speaking, this is not an algorithm, for the following reasons: 1) We did not specify how to pick $e$, 2) This procedure might not stop given some situations, e.g., if there is a cycle with minus total weight

**Notice:** This method can be really bad. Here is another example that could take $O(2^n)$ to solve.

**Algorithm 7** Ford's Method

**Ensure:** Shortest Paths from $r$ to all other nodes in $V$
**Require:** A digraph with arc costs,starting node $r$
  Initialize, $y_r = 0$ and $y_v = \infty, v \in V \setminus r$
  Initialize, $P(r) = 0, P(v) = -1, \forall v \in V \setminus r$
  **while y** is not a feasible potential **do**
    Let $e = (v, w) \in E$ (this could be problematic)
    **if** $y_v + c_{vw} < y_w$ (incorrect) **then**
      $y_w \leftarrow y_v + c_{vw}$ (correct it)
      $P(w) = v$ (set $v$ as predecessor)
    **end if**
  **end while**




## 17.4 Ford-Bellman Algorithm

**Notice:** Only correct the node that comes from a node that has been corrected.

A usual representation of a digraph is to store all the arcs having tail $v$ in a list $L_v$ to **scan** $v$ means the following:

- For $(v, w) \in L_v$, if $(v, w)$ is incorrect, then correct $(v, w)$

For Bellman, will either terminate with shortest path from $r$ to all $v \in V \setminus r$ or it will terminate stating that there is a negative cycle. In $O(mn)$

In the algorithm if $i = n$ and there exists a feasible potential, the problem has a negative cycle.

Suppose that the nodes of $G$ can be ordered from left to right so that all arcs go from left to right. That is suppose there is an ordering $v_1, v_2, ..., v_n \in V$ so that $(v_i, v_j) \in V$ implies $i < j$. We call such an ordering **topological** sort.

If we order $E$ in the sequence that $v_i v_j$ precedes $v_k v_i$ if $i < k$ based on topological order then ford algorithm will terminate in one pass.

**Algorithm 8** Ford-Bellman Algorithm

**Ensure:** Shortest Paths from $r$ to all other nodes in $V$
**Require:** A digraph with arc costs,starting node $r$
  Initialize $y$ and $p$
  **for** $i = 0; i < N; i++$ **do**
    **for** $\forall e = (v, w) \in E$ **do**
      **if** $y_v + c_{vw} < y_w$ (incorrect) **then**
        $y_w \leftarrow y_v + c_{vw}$ (correct it)
        $P(w) = v$ (set $v$ as predecessor)
      **end if**
    **end for**
  **end for**
  **for** $\forall e = (v, w) \in E$ **do**
    **if** $y_v + c_{vw} < y_w$ (incorrect) **then**
      Return error, negative cycle
    **end if**
  **end for**

## 17.5 SPFA Algorithm

## 17.6 Dijkstra Algorithm

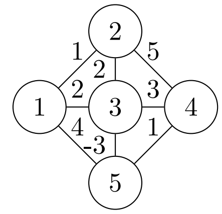
**Algorithm 9** Dijkstra Algorithm

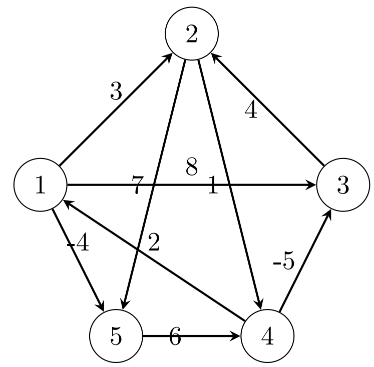
**Ensure:** Shortest Paths from $r$ to all other nodes in $V$
**Require:** A digraph with arc costs,starting node $r$
  Initialize $y$ and $p$
  $S \leftarrow V$
  **while** $S \neq \emptyset$ **do**
    Choose $v \in S$ with minimum $y_v$
    $S \leftarrow S \setminus v$
    **for** $\forall w, (v, w) \in E$ **do**
      **if** $y_v + c_{vw} < y_w$ (incorrect) **then**
        $y_w \leftarrow y_v + c_{vw}$ (correct it)
        $P(w) = v$ (set $v$ as predecessor)
      **end if**
    **end for**
  **end while**

## 17.7 A* Algorithm

## 17.8 Floyd-Warshall Algorithm

If all weights/distances in the graph are nonnegative then we could use Dijkstra within starting nodes being any one of the vertices of the graph. This method will take $O(n^3)$
If weight/distances are arbitrary and we would like to find shortest path between all pairs of vertices or detect a negative cycle we could use Bellman-Ford Algorithm with $O(n^4)$
We would like an algorithm to find shortest path between any two pairs in a graph for arbitrary weights (determined, negative, cycles) in $O(n^3)$




Let $d_{ij}^k$ denote the length of the shortest path from $i$ to $j$ such that all intermediate vertices are contained in the set $\{1, ..., k\}$
In this case $d_{14}^5 = 5$
If the vertex $k$ is not an intermediate vertex on $p$, then $d_{ij}^k = d_{ij}^{k-1}$, notice that $d_{15}^4 = -1$, node 4 is not intermediate, so $d_{15}^3 = -1$
If the vertex $k$ is an intermediate on $p$, then $d_{ij}^k = d_{ik}^{k-1} + d_{kj}^{k-1}$, $d_{14}^5 = 0$ ($p = 1 \to 3 \to 5 \to 4$), i.e., $d_{14}^5 = d_{15}^4 + d_{54}^4 = 0$
Therefore $d_{ij}^k = \min\{d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}\}$
Input: graph $G = (V, E)$ with weight on edges Output: Shortest path between all pairs of vertices on existence of a negative cycle Step 1: Initialize

$$d_{ij}^0 = \begin{cases} c_{ij} & \text{distance from } i \text{ to } j \text{ if } (i,j) \in E \\ 0 & \text{if } i = j \\ \infty & \text{if } (i,j) \notin E \end{cases} \tag{17.3}$$

Step: For k = 1 to n For i = 1 to n For j = 1 to n $d_{ij}^k = \min\{d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}\}$ Next j Next i Next k Between optimal matrix $D^n$

$$D^0 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \tag{17.4}$$

$$\Pi^0 = \begin{bmatrix} & 1 & 1 & & 1 \\ & & & 2 & 2 \\ & 3 & & & \\ 4 & & 4 & & \\ & & & 5 & \end{bmatrix} \tag{17.5}$$

$$D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \mathbf{5} & -5 & 0 & -\mathbf{2} \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \tag{17.6}$$

$$\Pi^1 = \begin{bmatrix} & 1 & 1 & & 1 \\ & & & 2 & 2 \\ & 3 & & & \\ 4 & \mathbf{1} & 4 & & \mathbf{1} \\ & & & 5 & \end{bmatrix} \tag{17.7}$$

$$D^2 = \begin{bmatrix} 0 & 3 & 8 & \mathbf{4} & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \mathbf{5} & \mathbf{11} \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \tag{17.8}$$

$$\Pi^2 = \begin{bmatrix} & 1 & 1 & \mathbf{2} & 1 \\ & & & 2 & 2 \\ & 3 & & \mathbf{2} & \mathbf{2} \\ 4 & 1 & 4 & & 1 \\ & & & 5 & \end{bmatrix} \tag{17.9}$$

$$D^3 = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -\mathbf{1} & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \tag{17.10}$$

$$\Pi^3 = \begin{bmatrix} & 1 & 1 & 2 & 1 \\ & & & 2 & 2 \\ & 3 & & 2 & 2 \\ 4 & \mathbf{3} & 4 & & 1 \\ & & & 5 & \end{bmatrix} \tag{17.11}$$

$$D^4 = \begin{bmatrix} 0 & 3 & -\mathbf{1} & 4 & -4 \\ \mathbf{3} & 0 & -\mathbf{4} & 1 & -\mathbf{1} \\ \mathbf{7} & 4 & 0 & 5 & \mathbf{3} \\ 2 & -1 & -5 & 0 & -2 \\ \mathbf{8} & \mathbf{5} & \mathbf{1} & 6 & 0 \end{bmatrix} \tag{17.12}$$

$$\Pi^4 = \begin{bmatrix} & 1 & \mathbf{4} & 2 & 1 \\ \mathbf{4} & & \mathbf{4} & 2 & \mathbf{1} \\ \mathbf{4} & 3 & & 2 & \mathbf{1} \\ 4 & 3 & 4 & & 1 \\ \mathbf{4} & \mathbf{3} & \mathbf{4} & 5 & \end{bmatrix} \tag{17.13}$$

$$D^5 = \begin{bmatrix} 0 & \mathbf{1} & -\mathbf{3} & \mathbf{2} & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \tag{17.14}$$

$$\Pi^5 = \begin{bmatrix} & \mathbf{3} & 4 & \mathbf{5} & 1 \\ 4 & & 4 & 2 & 1 \\ 4 & 3 & & 2 & 1 \\ 4 & 3 & 4 & & 1 \\ 4 & 3 & 4 & 5 & \end{bmatrix} \tag{17.15}$$

Time complexity $O(n^3)$
If during the previous processes, there exist an element of negative value in the diagonal, it means there exists negative cycle.

## 17.9 Johnson's Algorithm

# Chapter 18

# Maximum Flow Problem

## 18.1 Basic Concept

Let $D = (V, A)$ be a strict diagraph with distinguished vertices $s$ and $t$. We call $s$ the source and $t$ the sink, let $u = \{u_e : e \in A\}$ be a nonnegative integer-valued capacity function defined on the arcs of $D$. The maximum flow problem on $(D, s, t, u)$ is the following Linear program.

$$\max \quad v \tag{18.1}$$

$$\text{s.t.} \quad \sum_{h(e)=i} x_e - \sum_{t(e)=i} x_e = \begin{cases} -v, & \text{if } i = s \\ v, & \text{if } i = t \\ 0, & \text{otherwise} \end{cases} \tag{18.2}$$

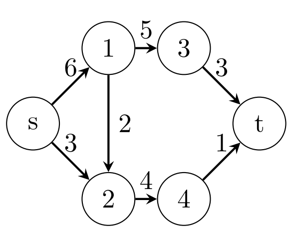
$$0 \leq x_e \leq u_e, \quad \forall e \in A \tag{18.3}$$

We think of $x_e$ as being the flow on arc $e$. Constraint says that for $i \neq s, t$ the flow into a vertex has to be equal to the flow out of vertex. That is, flow is conceded at vertex $i$ for $i = s$ and for $i = t$ the net flow in the entire digraph must be equal to $v$. A $\mathbf{x_e}$ that satisfied the above constraints is an $(s,t)$-flow of value $v$. If in addition it satisfies the bounding constraints, then it is a feasible $(s,t)$-flow. A feasible $(s,t)$-flow that has maximum $v$ is optimal on maximum.

## 18.2 Prime and Dual Problem

**Theorem 18.1.** *For $S \subseteq V$ we define $(S, \bar{S})$ to be a $(s,t)$-cut if $s \in S$ and $t \in \bar{S} = V - S$, the capacity of the cut, denoted $u(S, \bar{S})$ as $\sum\{u_e : e \in \delta^-(S)\}$ where $\delta^-(S) = \{e \in A : t(e) \in S \text{ and } h(e) \in \bar{S}\}$*

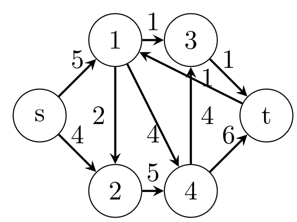
**Example.** For the following graph:



Let $S = \{1, 2, 3, s\}$, $\bar{S} = \{4, t\}$
then $\delta^-(S) = \{(2,4), (3,t)\} \Rightarrow u(S, \bar{S}) = 7$

**Definition 18.2.1.** If $(S, \bar{S})$ has minimum capacity of all $(s,t)$-cuts, then it is called **minimum cut**.

**Definition 18.2.2.** Let $\delta^+(S) = \delta^-(V - S)$



**Example.** Let $S = \{s, 1, 2, 3\}$, $\bar{S} = \{4, t\}$, $u(S, \bar{S}) = u_{14} + u_{24} + u_{3t} = 10$, $\delta^-(S) = \{(1,4), (2,4), (3,t)\}$, $\delta^+ S = \{(4,3), (t,1)\}$

**Lemma 18.2.** *If $x$ is a $(s,t)$ flow of value $v$ and $(S, \bar{S})$ is a $(s,t)$-cut, then*

$$v = \sum_{e \in \delta^-(S)} x_e - \sum_{e \in \delta^+(S)} x_e \tag{18.4}$$

*Proof.* Summing the first set of constraints over the vertices of $S$,

$$\sum_{i \in S} \left( \sum_{h(e)=i} x_e - \sum_{t(e)=i} x_e \right) = -v \tag{18.5}$$

Now for an arc $e$ with both ends in $S$, $x_e$ will occur twice once with a positive and once with negative so they cancel and the above sum is reduced to

$$\sum_{e \in \delta^+(S)} x_e - \sum_{e \in \delta^-(S)} x_e = -v \tag{18.6}$$

$\square$

**Notice:** Flow is the prime variable, capacity is the dual variable.

**Corollary 18.2.1.** *If $x$ is a feasible flow of value $v$, and $(S, \bar{S})$ is an $(s,t)$-cut, then*

$$v \leq u(S, \bar{S}) \quad \textit{(Weak duality)} \tag{18.7}$$

**Definition 18.2.3.** Define an arc $e$ to be **saturated** if $x_e = u_e$, and to be **flowless** if $x_e = 0$

**Corollary 18.2.2.** *Let $x$ be a feasible flow and $(S, \bar{S})$ be a $(s,t)$-cut, if $\forall e \in \delta^-(S)$ is saturated, and $\forall e \in \delta^+(S)$ is flowless, then $x$ is a maximum flow and $(S, \bar{S})$ is a minimum cut. (Strong duality)*

*Proof.* If every arc of $\delta^-(S)$ is saturated then

$$\sum_{e\in\delta^-(S)} x_e = \sum_{e\in\delta^-(S)} u_e \tag{18.8}$$

If every arc of $\delta^+(S)$ is flowless then

$$\sum_{e\in\delta^+(S)} x_e = 0 \tag{18.9}$$

$\Rightarrow$ $x$ is as large as it can get when as $u(S, \bar{S})$ is as small as it can get. □

The LP of maximum flow can be modeled as following

$$\max \quad f = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ f \end{bmatrix} \tag{18.10}$$

$$\text{s.t.} \quad \begin{bmatrix} \mathbf{A} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ f \end{bmatrix} = \mathbf{0} \tag{18.11}$$

$$\mathbf{Ix} \leq \mathbf{u} \tag{18.12}$$

$$\begin{bmatrix} \mathbf{x} \\ f \end{bmatrix} \geq 0 \tag{18.13}$$

In which $\mathbf{A}$ is the vertex-arc incident matrix and $\mathbf{F}$ is a column vector where the first row is 1, last row is -1 and all other rows are 0s. $\mathbf{u}$ is the column vector of upper bound of each arcs.

$$\mathbf{A} = \mathbf{A}_{|E|\times|V|} = [a_{ij}], \text{ where } a_{ij} = \begin{cases} 1, & \text{if } v_i = h(e_j) \\ -1, & \text{if } v_i = t(e_j) \\ 0, & \text{otherwise} \end{cases} \tag{18.14}$$

$$\mathbf{F} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & -1 \end{bmatrix}^\top \tag{18.15}$$

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_{|E|} \end{bmatrix}^\top \tag{18.16}$$

Then, we take the dual of LP

$$\min \quad \mathbf{uw_E} \tag{18.17}$$

$$\text{s.t.} \quad \begin{bmatrix} \mathbf{w_V} & \mathbf{w_E} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \geq 0 \tag{18.18}$$

$$\mathbf{w_V} \quad \text{unrestricted} \tag{18.19}$$

$$\mathbf{w_E} \geq \mathbf{0} \tag{18.20}$$

In which $\mathbf{w_V}$ is the capacity variable of each vertex, $\mathbf{w_E}$ is the capacity of each arc. $\mathbf{u}, \mathbf{E}, \mathbf{F}$ have the same meaning as in prime.

$$\mathbf{w_V} = \begin{bmatrix} w_1 & w_2 & \cdots & w_{|V|} \end{bmatrix}^\top \tag{18.21}$$

$$\mathbf{w_E} = \begin{bmatrix} w_{|V|+1} & w_{|V|+2} & \cdots & w_{|V|+|E|} \end{bmatrix}^\top \tag{18.22}$$

## 18.3 Maximum Flow Minimum Cut Theorem

**Definition 18.3.1.** Let $P$ be a path, (not necessarily a dipath), $P$ is called **unsaturated** if every **forward** arc is unsaturated ($x_e < u_e$) and ever **reverse** arc has positive flow ($x_e > 0$). If in addition $P$ is an $(s,t)$-path, then $P$ is called an **x-augmenting path**

**Theorem 18.3.** *A feasible flow $x$ in a digraph $D$ is maximum iff $D$ has no augmenting paths.*

*Proof.* (Prove by contradiction)
($\Rightarrow$) Let $x$ be a maximum flow of value $v$ and suppose $D$ has an augmenting path. Define in $P$ (augmenting path):

$$D_1 = \min\{u_e - x_e : e \text{ forward in } P\} \tag{18.23}$$

$$D_2 = \min\{x_e : e \text{ backward in } P\} \tag{18.24}$$

$$D = \min\{D_1, D_2\} \tag{18.25}$$

Since $P$ is augmenting, then $D > 0$, let

$$\hat{x_e} = \begin{cases} x_e + D & \text{If } e \text{ is forward in } P \\ x_e - D & \text{If } e \text{ is backward in } P \\ x_e & \textit{otherwise} \end{cases} \tag{18.26}$$

It is easy to see that $\hat{x}$ is feasible flow and that the value is $V + D$, a contradiction.
($\Leftarrow$) Suppose $D$ admits no x-augmenting path, Let $S$ be the set of vertices reachable from $s$ by x-unsaturated path clearly $s \in S$ and $t \notin S$ (because otherwise there would be an augmenting path). Thus, $(S, \bar{S})$ is a $(s,t)$-cut.
Let $e \in \delta^-(S)$ then $e$ must be saturated. For otherwise we could add the $h(e)$ to $S$
Let $e \in \delta^+(S)$ then $e$ must be flow less. For otherwise we could add the $t(e)$ to $S$.
According to previous corollary, that $x$ is maximum. □

**Theorem 18.4.** *(Max-flow = Minimum-cut) For any digraph, the value of a maximum $(s,t)$-flow is equal to the capacity of a minimum $(s,t)$-cut*

## 18.4 Ford-Fulkerson Method

Finding augmenting paths is the key of max-flow algorithm, we need to describe two functions, labeling and scanning a vertex.
A vertex is first labeled if we can find x-unsaturated path from $s$, i.e., $(s,v)$-unsaturated path.
The vertex $v$ is scanned after we attempted to extend the x-unsaturated path.
This algorithm is incomplete/incorrect, needs to be fixed FIXME

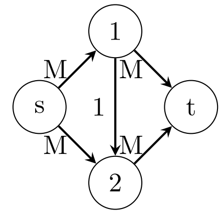
Labeling algorithm can be exponential, the following is an example

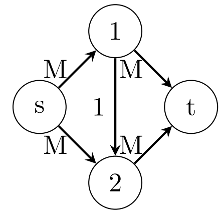
**Algorithm 10** Labeling algorithm

**Ensure:** Max-flow $x$ with value $v$
**Require:** Digraph with source $s$ and sink $t$, a capacity function $u$ and a feasible flow (could be $x_e = 0$)
  Initialize, $v \leftarrow x$
  Designate all vertices as unlabeled and unscanned
  Label $s$
  **while** There exists vertex unlabeled or unscanned **do**
    Let $i$ be such a vertex, for each arc $e$ with $t(e) = i, x_e < u_e$ and $h(e)$ unlabeled, label $h(e)$
    For each arc $e$ with $h(e) = i, x_e > 0$ and $t(e)$ unlabeled, label $t(e)$, designate $i$ as scanned.
    If $t$ is not label
  **end while**
  $x$ is the maximum.

**Algorithm 11** Ford-Fulkerson algorithm

**Ensure:** Max-flow $x$ with value $v$
**Require:** Digraph with source $s$ and sink $t$, a capacity function $u$ and a feasible flow (could be $x_e = 0$)
  Initialize, $v \leftarrow x$
  Designate all vertices as unlabeled and unscanned
  Label $s$
  **while** There exists vertex unlabeled or unscanned **do**
    Let $i$ be such a vertex, for each arc $e$ with $t(e) = i, x_e < u_e$ and $h(e)$ unlabeled, label $h(e)$
    For each arc $e$ with $h(e) = i, x_e > 0$ and $t(e)$ unlabeled, label $t(e)$, designate $i$ as scanned.
    If $t$ is not label
  **end while**
  $x$ is the maximum.




## 18.5 Polynomial Algorithm for max flow

Let $(D, s, t, u)$ be a max flow problem and let $x$ be a feasible flow for $D$, the **x-layers** of $D$ are define be the following algorithm
Layer algorithm (Dinic 1977) Input: A network $(D, s, t, u)$ and a feasible flow $x$ Output: The **x-layers** $V_0, V_1, ..., V_l$ where $V_i \cap V_j = \emptyset \forall i \neq j$
Step 1: Set $V_0 = \{s\}, i \leftarrow 0 and l(x) = 0$ Step 2: Let $R$ be the set of vertices $w$ such that there is an arc $e$ with either:

- $t(e) \in V_i, h(e) = w, x_e < u_e$ or
- $h(e) \in V_j, t(e) = w, x_e > 0$

Step 3: If $t \in R$, set $V_{i+1} = \{t\}$, $l(t) = i + 1$ and stop. Set $V_{i+1} \leftarrow R \setminus \cup_{0 \leq j \leq i} V_j$, $l \leftarrow i + 1, l(x) = i$, goto Step 2. If $V_{i+1} = \emptyset$, set $l(x) = i$ and Stop.

**Example.** For the following graph

$$\text{Second iteration} \tag{18.27}$$
$$V_0 = \{s\}, i = 0, l(x) = 0 \tag{18.28}$$
$$R = \{1, 2\} \tag{18.29}$$
$$V_1 \leftarrow \{1, 2\}, i = 1, l(x) = 1 \tag{18.30}$$
$$R = \{3, 4, 5\} \tag{18.31}$$
$$V_2 \leftarrow \{3, 4\}, i = 2, l(x) = 2 \tag{18.32}$$
$$R = \{1, 5, 6, 3\} \tag{18.33}$$
$$V_3 \leftarrow \{5, 6\}, i = 3, l(x) = 3 \tag{18.34}$$
$$R = \{4, t\} \tag{18.35}$$
$$V_4 = \{t\} \tag{18.36}$$
$$A_1 = \{(s, 1), (s, 2)\} \tag{18.37}$$
$$A_2 = \{(1, 3), (2, 4)\} \tag{18.38}$$
$$A_3 = \{(3, 5), (4, 6)\} \tag{18.39}$$
$$A_4 = \{(5, t), (6, t)\} \tag{18.40}$$

The layer network $D_x$ is defined by $V(D_x) = V_0 \cup V_1 \cup V_2 \cdots \cup V_{l(x)}$

Suppose we have computed the layers of $D$ and $t \in V_l(x)$, the last layer (last layer I am goin to $V_e$)
For each $i, 1 \leq i \leq l$, define a set of arcs $A_i$ and a function $\hat{u}$ on $A_i$ as following. For each $e \in A(D)$

- If $t(e) \in V_{i-1}, h(e) \in V_i$ and $x_e < u_e$ then add arc $e$ to $A_i$ and define $\hat{u}_e = u_e - x_e$

- If $h(e) \leftarrow V_{i-1}, t(e) \in V_i$ and $x_e > 0$ then add arc $e' = (h(e), t(e))$ to $A_i$ with $\hat{u}_e - x_e$

Let $\hat{u}$ be the capacity function on $D_x$ and let the source and sink of $D_x$ be $s$ and $t$
We can think of $D_x$ as being make of arc shortest (in terms of numbers of arcs) x-augmenting paths.
A feasible flow in a network is said to be maximal (does not means maximum) if every $(s,t)$-directed path contains at least one saturated arc.
For layered algorithm $V_0, V_1, ..., V_L$
Arcs:

- If $t(e) \in V_{i-1}$, $h(e) \in V_i$ and $x_e < u_e$, then add $e$ to $A_i$ with $\hat{u_e} = u_e - x_e$

- If $h(e) \in V_{i-1}$, $t(e) \in V_i$ and $x_e > 0$, then add arc $e' = (h(e), t(e))$ to $A_i$ and define $\hat{u_e} = x_e$

Maximal Flow: If every directed $(s,t)$-path has at least one saturated arc.
Computing maximal flow is easier than computing maximum flow, since we never need to consider canceling flows on reverse arcs,
Let $\hat{x}$ be a maximal flow on the layered network $D_x$, we can define new flows in $D(x')$ by

$$x'_e = x_e + \hat{x_e}, \quad \text{If } t(e) \in V_{i-1}, h(e) \in V_i \tag{18.41}$$
$$x'_e = x_e - \hat{x_e}, \quad \text{If } h(e) \in V_{i-1}, t(e) \in V_i \tag{18.42}$$

## 18.6 Dinic Algorithm

Input: A layered network $(D_x, s, t, \hat{u})$ and a feasible flow x Output: A maximal flow $\hat{x}$ from $D_x$
Step 1: Set $H \leftarrow D_x$ and $i \leftarrow S$ Step 2: If there is no arc $e$ with $t(e) = i$, goto Step 4, otherwise let $e$ be such an arc Step 3: Set $T(h(e)) \leftarrow i$ and $i \leftarrow h(e)$, if $i = t$ goto Step 5, otherwise goto Step 2. Step 4: If $i = s$, Stop, Otherwise delete $i$ and all incident arcs with $H$, set $i \leftarrow T(i)$ and goto Step 2 Step 5: Construct the directed path, $s = i_0 e_1 i_1 e_2 ... e_k i_k = t$ where $i_{j-1} = T(i_j), 1 \leq j \leq k$. Set $D = \min\{\hat{u_{e_j}} - \hat{x_{e_j}} : i \leq j \leq k\}$, set $\hat{x_{e_j}} \leftarrow \hat{x_{e_j}} + D, i \leq j \leq k$. Delete from $H$ all saturated arcs on this path, set $i \leftarrow 1$ and goto Step 2.

**Theorem 18.5.** *Dinic algorithm runs in $O(|E||V|^2)$*

*Proof.* Step 1 is $O(|E||V|)$ Step 2 runs Step 1 for $O(|V|)$ times □

# Chapter 19

# Minimum Weight Flow Problem

## 19.1 Transshipment Problem

Transshipment Problem $(D, b, w)$ is a linear program of the form

$$\min \quad wx \tag{19.1}$$
$$\text{s.t.} \quad Nx = b \tag{19.2}$$
$$x \geq 0 \tag{19.3}$$

Where $N$ is a vertex-arc incident matrix. For a feasible solution to LP to exist, the sum of all $b$s must be zero. Since the summation of rows of $N$ is zero. The interpretation of the LP is as follows.
The variables are defined on the edges of the digraph and that $x_e$ denote the amount of flow of some commodity from the tail of $e$ to the head of $e$
Each constraints

$$\sum_{h(e)=i} x_e - \sum_{t(e)=i} x_e = b_i \tag{19.4}$$

represents consequential of flow of all edges into k vertex that have a demand of $b_i > 0$, or a supply of $b_i < 0$. If $b_i = 0$ we call that vertex a transshipment vertex.

## 19.2 Network Simplex Method

**Notice:** Similar to Simplex Method in LP, even though in worst case may be inefficient. In most cases it is simple and empirically efficient.

**Lemma 19.1.** *Let $C_1$ and $C_2$ be distinct cycles in a graph $G$ and let $e \in C_1 \cup C_2$. Then $(C_1 \cup C_2) \setminus e$ contains a cycle.*

*Proof.* Case 1: $C_1 \cap C_2 = \emptyset$. Trivia.
Case 2: $C_1 \cap C_2 \neq \emptyset$. Let $e \in C_2$ and $f = uv \in C_1 \setminus C_2$. Starting at $v$ traverse $C_1$ in the direction away from $u$ until the first vertex of $C_2$, say $x$. Denote the $(v, x)$-path as $P$. Starting at $u$ traverse $C_1$ in the direction away from $v$ until the first vertex of $C_2$, say $y$. Denote the $(u, y)$-path as $Q$. $C_2$ is a cycle, there are two $(x, y)$-path in $C_2$. Denote the $(x, y)$-path without $e$ as $R$. Then $vPxRyQ^{-1}uf$ is a cycle. □

**Theorem 19.2.** *Let $T$ be a spanning tree of $G$. And let $e \in E \setminus T$ then $T + e$ contains a unique cycle $C$ and for any edge $f \in C$, $T + e - f$ is a spanning tree of $G$*

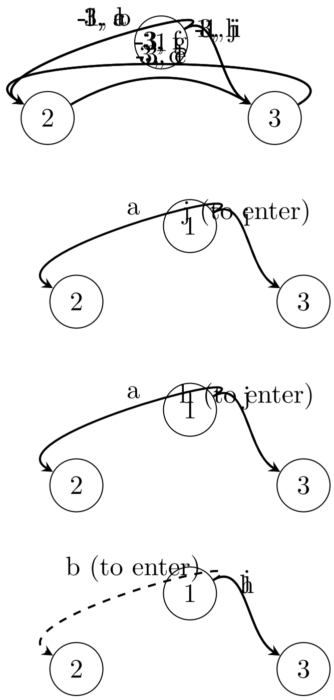
Let $(D, b, w)$ be a transshipment problem. A feasible solutions $x$ is a **feasible tree solution** if there is a spanning tree $T$ such that $||x|| = \{e \in A, x_e \neq 0\} \subseteq T$.
For any tree $T$ of $D$ and for $e \in A \setminus T$, it follows from above theorem that $T+e$ contains a unique cycle. Denote that cycle $C(T, e)$ and orient it in the direction of $e$, define $w(T, e) = \sum\{w_e : e \text{ forward in } C(T, e)\} - \sum\{w_e : e \text{ reverse in } C(T, e)\}$.
We think of $w(T, e)$ as the weight of $C(T, e)$.

**Algorithm 12** Network Simplex Method Algorithm

**Ensure:** An optimal solution or the conclusion that $(D, b, w)$ is unbounded
**Require:** A transshipment problem $(D, b, w)$ and a feasible tree solution $x$ containing to a spanning tree $T$

**while** $\exists e \in A \setminus T, w(T, e) < 0$ **do**
  let $e \in A \setminus T$ be such that $w(T, e) < 0$.
  **if** $C(T, e)$ has no reverse arcs **then**
    Return unboundedness
  **else**
    Set $\theta = \min\{x_f : f \text{ reverse in } C(T, e)\}$ and set $f = \{f \in C(T, e) : f \text{ reverse in } C(T, e), x_f = \theta\}$
    **if** $f$ forward in $C(T, e)$ **then**
      $x_f \leftarrow x_f + \theta$
    **else**
      $x_f \leftarrow x_f - \theta$
    **end if**
    Let $f \in F$ and $T \leftarrow T + e - f$
  **end if**
**end while**
Return $x$ as optimal

The following is an example of cycling
Then for
$w(T, j) = w_j - w_i = -3 - 3 = -6$
$w(T, h) = w_h + w_j - w_i = 3 - 3 - 1 = -1$
Keep going and we will find cycling.
To Avoid cycling we will introduce the modified network Simplex Method. Let $T$ be a **rooted** spanning tree. Let

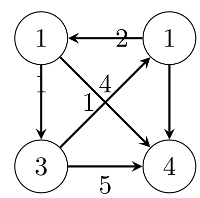



$f$ be an arc in $T$, we say $f$ is **away** from the root $r$ if $t(f)$ is the component of $T - f$. Otherwise we say $f$ is **towards** $r$.
Let $x$ be a feasible tree solution associated with $T$, then we say $T$ is a **strong feasible tree** if for every arc $f \in T$ with $x_f = 0$ then $f$ is away from $r \in T$.
Modification to NSM:

- The algorithm is initialed with a strong feasible tree.
- $f$ in pivot phase is chosen to be the first reverse arc of $C(T, e)$ having $x_f = \theta$. By "first", we mean the first arc encountered in traversing $C(T, e)$ in the direction of $e$, starting at the vertex $i$ of $C(T, e)$ that minimizes the number of arcs in the unique $(r, i)$-path in $T$.

**Notice:** In the second rule above, $r$ could also be in the cycle, in that case, $i$ is $r$.

Finding initial strong feasible tree.
Pick a vertex in $D$ to be root $r$. The tree $T$ has an arc $e$ with the $t(e) = r$ and $h(e) = v$. For each $v \in V \setminus r$ with $b_v \geq 0$ and has an arc $e$ with $h(e) = r$ and $t(e) = v$ for each $v \in V \setminus r$ for which $b_v < 0$. Wherever possible the arcs of $T$ are chosen from $A$, where an appropriate arc doesn't exist. We create an **artificial arc** and give its weight $|V|(\max\{w_e : e \in A\} + 1$. This is similar to Big-M method and if optimal solution contains artificial arcs ongoing arc problem is infeasible.




**Notice:** This algorithm can be really bad, its mimic of Simplex Method of LP, which means we can run into exponential operations

# Chapter 20

# Matchings

## 20.1 Hall's "Marriage" Theorem

## 20.2 Transversal Theory

## 20.3 Menger's Theorem

## 20.4 The Hungarian Algorithm

# Chapter 21

# Colorings

## 21.1 Edge Chromatic Number

## 21.2 Vizing's Theorem

## 21.3 The Timetabling Problem

## 21.4 Vertex Chromatic Number

## 21.5 Brooks' Theorem

## 21.6 Hajós' Theorem

## 21.7 Chromatic Polynomials

## 21.8 Girth and Chromatic Number

# Chapter 22

# Independent Sets and Cliques

## 22.1 Independent Sets

## 22.2 Ramsey's Theorem

## 22.3 Turán's Theorem

## 22.4 Schur's Theorem

# Chapter 23

# Matroids