Notes for Operations Research & More

Lan Peng, PhD Student

Department of Industrial and Systems Engineering
University at Buffalo, SUNY
lanpeng@buffalo.edu

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4.2.1 Absolute Value

Description: Consider the following model statement:

$$
\begin{aligned}
\min \quad & \sum_{j \in J} c_j |x_j|, \quad c_j > 0 \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& x_j \quad \text{unrestricted}, \quad \forall j \in J
\end{aligned}
$$

Modeling:

$$
\begin{aligned}
\min \quad & \sum_{j \in J} c_j (x_j^+ + x_j^-), \quad c_j > 0 \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \gtreqless b_i, \quad \forall i \in I \\
& x_j^+, x_j^- \geq 0, \quad \forall j \in J
\end{aligned}
$$

4.2.2 A Minimax Objective

Description: Consider the following model statement:

$$
\begin{aligned}
\min \quad & \max_{k \in K} \sum_{j \in J} c_{kj} x_j \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}
$$

Modeling:

$$
\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& \sum_{j \in J} c_{kj} x_j \leq z, \quad \forall k \in K \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}
$$

4.2.3 A fractional Objective

Description: Consider the following model statement:

$$\begin{aligned}
\min \quad & \frac{\sum_{j \in J} c_j x_j + \alpha}{\sum_{j \in J} d_j x_j + \beta} \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}$$

Modeling:

$$\begin{aligned}
\min \quad & \sum_{j \in J} c_j x_j t + \alpha t \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\
& \sum_{j \in J} d_j x_j t + \beta t = 1 \\
& t > 0 \\
& x_j \geq 0, \quad \forall j \in J \\
& (t = \frac{1}{\sum_{j \in J} d_j x_j + \beta})
\end{aligned}$$

4.2.4 A range Constraint

Description: Consider the following model statement:

$$\begin{aligned}
\min \quad & \sum_{j \in J} c_j x_j \\
\text{s.t.} \quad & d_i \leq \sum_{j \in J} a_{ij} x_j \leq e_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J
\end{aligned}$$

Modeling:

$$\begin{aligned}
\min \quad & \sum_{j \in J} c_j x_j, \quad c_j > 0 \\
\text{s.t.} \quad & u_i + \sum_{j \in J} a_{ij} x_j = e_i, \quad \forall i \in I \\
& x_j \geq 0, \quad \forall j \in J \\
& 0 \leq u_i \leq e_i - d_i, \quad \forall i \in I
\end{aligned}$$

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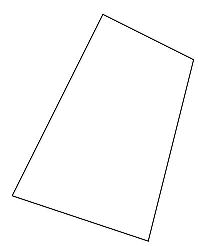
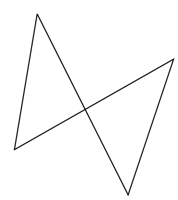
Polygon Triangulation

18.1 Types of Polygons

Def: A **simple polygon** is a closed polygonal curve without self-intersection.



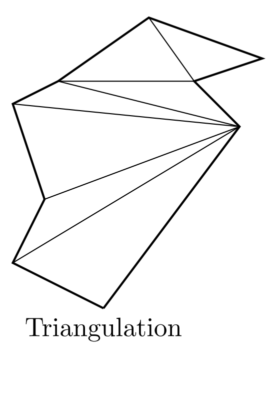


Simple Polygon Non-simple Polygon

Polygons are basic building blocks in most geometric applications. It can model arbitrarily complex shapes, and apply simple algorithms and algebraic representation/manipulation.

18.2 Triangulation

Def: **Triangulation** is to partition polygon P into non-overlapping triangles using diagonals only. It reduces complex shapes to collection of simpler shapes. Every simple n-gon admits a triangulation which has $n-2$ triangles.



Triangulation

Lemma: Every polygon with more than three vertices has a diagonal.

Proof: (by Meisters, 1975) Let P be a polygon with more than three vertices. Every vertex of a P is either *convex* or *concave*. W.L.O.G.(any polygon must has convex vertex) Assume p is a convex vertex. Denote the neighbors of p as q and r. If $\bar{q}r$ is a diagonal, done, and we call $\triangle pqr$ is an *ear*. If $\triangle pqr$ is not an ear, it means at least one vertex is inside $\triangle pqr$, assume among those vertexes inside $\triangle pqr$, s is a vertex closest to p, then $\bar{p}s$ is a diagonal.

18.3 Art Gallery Theorem

Problem: The floor plan of an art gallery modeled as a simple polygon with n vertices, there are guards which is stationed at fixed positions with 360 degree vision but cannot see through the walls. How many guards does the art gallery need for the security? (Fun fact: This problem was posted to Vasek Chvatal by Victor Klee in 1973)

Theorem: Every n-gon can be guarded with $\lfloor \frac{n}{3} \rfloor$ vertex guards

Lemma: Triangulation graph can be 3-colored.

Proof:

- P plus triangulation is a planar graph
- 3-coloring means there exist a 3-partition for vertices that no edge or diagonal has both endpoints within the same set of vertices.
- Proof by Induction: - Remove an ear (there will always exist ear)
  - Inductively 3-color the rest
  - Put ear back, coloring new vertex with the label not used by the boundary diagonal.

18.4 Triangulation Algorithms

Part IV

Integer and Combinatorial Programming

Chapter 19

Formulation

19.1 Typical Problems

19.2 Integer Programming Formulation Skills

19.2.1 A Variable Taking Discontinuous Values

Description: In algebraic notation:

$$x = 0, \quad \text{or} \quad l \le x \le u$$

Modeling:

$$\begin{aligned} & x \le uy \\ & x \ge ly \\ & y \in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } l \le x \le u \end{cases}$$

19.2.2 Fixed Costs

Description: In algebraic notation:

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

Modeling:

$$\begin{aligned} & C^*(x, y) = ky + cx \\ & x \le My \\ & x \ge 0 \\ & y \in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \ge 0 \end{cases}$$

19.2.3 Either-or Constraints

Description: In algebraic notation:

$$\sum_{j \in J} a_{1j} x_j \le b_1 \text{ or } \sum_{j \in J} a_{2j} x_j \le b_2$$

Modeling:

$$
\begin{aligned}
&\sum_{j\in J} a_{1j}x_j \le b_1 + M_1 y\\
&\sum_{j\in J} a_{2j}x_j \le b_2 + M_1(1-y)\\
&y \in \{0,1\}
\end{aligned}
$$

where

$$
y = \begin{cases} 0, & \text{if } \sum_{j\in J} a_{1j}x_j \le b_1 \\ 1, & \text{if } \sum_{j\in J} a_{2j}x_j \le b_2 \end{cases}
$$

Notice that the sign before M is determined by the inequality $\ge$ or $\le$, if it is "$\ge$", use "$-$", if it "$\le$", use "$+$".

19.2.4 Conditional Constraints

Description: If constraint A is satisfied, then constraint B must also be satisfied

$$
\text{If } \sum_{j\in J} a_{1j}x_j \le b_1 \text{ then } \sum_{j\in J} a_{2j}x_j \le b_2
$$

The key part is to find the opposite of the first condition. We are using $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$
Therefore it is equivalent to

$$
\sum_{j\in J} a_{1j}x_j > b_1 \text{ or } \sum_{j\in J} a_{2j}x_j \le b_2
$$

Furthermore, it is equivalent to

$$
\sum_{j\in J} a_{1j}x_j \ge b_1 + \epsilon \text{ or } \sum_{j\in J} a_{2j}x_j \le b_2
$$

Where ϵ is a very small positive number.

Modeling:

$$
\begin{aligned}
&\sum_{j\in J} a_{1j}x_j \ge b_1 + \epsilon - M_2 y\\
&\sum_{j\in J} a_{2j}x_j \le b_2 + M_2(1-y)\\
&y \in \{0,1\}
\end{aligned}
$$

19.2.5 Special Ordered Sets

SOS1 Description Out of a set of yes-no decisions, at most one decision variable can be yes.

$$
\begin{gathered}
x_1 = 1, x_2 = x_3 = \cdots = x_n = 0\\
\text{or}\\
x_2 = 1, x_1 = x_3 = \cdots = x_n = 0\\
\text{or ...}
\end{gathered}
$$

Modeling:

$$
\sum_i x_i = 1, \quad i \in N
$$

SOS2 Description 1 Out of a set of binary variables, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list.

Modeling: If $x_1, x_2, ..., x_n$ is a SOS2, then

$$
\begin{aligned}
&\sum_{i=1}^{n} x_i \le 2\\
&x_i + x_j \le 1, \forall i \in \{1,2,...,n\}, j \in \{i+2, i+3, ..., n\}\\
&x_i \in \{0,1\}
\end{aligned}
$$

SOS2 Description 2 There is another type of definition, that is out of a set of nonnegative variables **not binary here**, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list. All variables summing to 1.
This definition of SOS2 is used in the following section *Piecewise Linear Formulations*

19.2.6 Piecewise Linear Formulations

Description: The objective function is a sequence of line segments, e.g. $y = f(x)$, consists $k-1$ linear segments going through k given points $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$.
Denote

$$d_i = \begin{cases} 1, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Then the objective function is

$$\sum_{i \in \{1,2,...,k-1\}} y = d_i f_i(x)$$

Modeling: Given that objective function as a piecewise linear formulation, we can have these constraints

$$\begin{aligned}
&\sum_{i \in \{1,2,...,k-1\}} d_i = 1 \\
&d_i \in \{0,1\}, i \in \{1,2,...,k-1\} \\
&x = \sum_{i \in \{1,2,...,k\}} w_i x_i \\
&y = \sum_{i \in \{1,2,...,k\}} w_i y_i \\
&w_1 \le d_1 \\
&w_i \le d_{i-1} + di, i \in \{2,3,...,k-1\} \\
&w_k \le d_{k-1}
\end{aligned}$$

In this case, $w_i \in SOS2$ (second definition)

19.2.7 Conditional Binary Variables

Description: Choose at most n binary variable to be 1 out of $x_1, x_2, ...x_m, m \ge n$. If $n = 1$ then it is SOS1.
Modeling:

$$\sum_{k \in \{1,2,...,m\}} x_k \le n$$

Description: Choose exactly n binary variable to be 1 out of $x_1, x_2, ...x_m, m \ge n$
Modeling:

$$\sum_{k \in \{1,2,...,m\}} x_k = n$$

Description: Choose x_j only if $x_k = 1$
Modeling:

$$x_j = x_k$$

Description: "and" condition, iff $x_1, x_2, ..., x_m = 1$ then $y = 1$
Modeling:

$$\begin{aligned}
&y \le x_i, i \in \{1,2,...,m\} \\
&y \ge \sum_{i \in \{1,2,...,m\}} x_i - (m-1)
\end{aligned}$$

19.2.8 Elimination of Products of Variables

Description: For variables x_1 and x_2,

$$y = x_1 x_2$$

Modeling: If x_1, x_2 are binary, it is the same as "and" condition of binary variables.
If x_1 is binary, while x_2 is continuous and $0 \leq x_2 \leq u$, then

$$\begin{aligned} & y \leq u x_1 \\ & y \leq x_2 \\ & y \geq x_2 - u(1 - x_1) \\ & y \geq 0 \end{aligned}$$

If both x_1 and x_2 are continuous, it is non-linear, we can use Piecewise linear formulation to simulate.

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39.1 Relationship between Some Random Variables

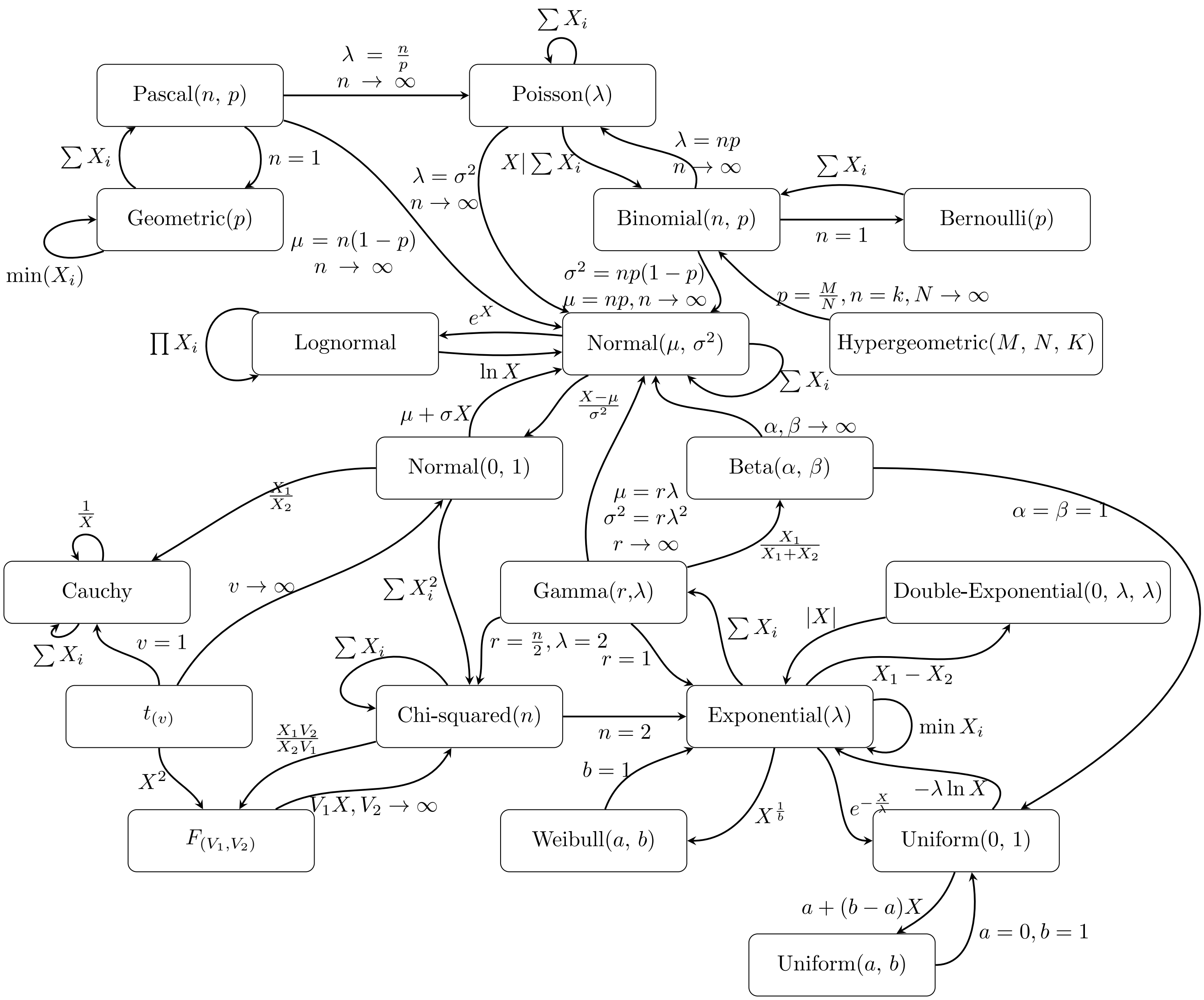

Figure 39.1: Relationship between Some Random Variables

39.2 Discrete Random Variables

Table 39.1: Discrete Random Variables

Distribution	PMF	CDF	Expectation	Variance	MGF
Discrete Uniform(a, b)	$f(x) = \frac{1}{b-a+1}$ $x = a, a+1, ..., b$	$F(x) = \frac{x-a+1}{b-a+1}$ $x = a, a+1, ..., b$	$E[X] = \frac{b-a}{2}$	$D[X] = \frac{(b-a+1)^2-1}{12}$	$M(t) = \frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$ $t \in \mathbb{R}$
Bernoulli(p)	$f(x) = p^x(1-p)^{1-x}$ $x \in \{0, 1\}$	$F(x) = \begin{cases} 0, & x < 0 \\ 1-p, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$	$E[X] = p$	$D[X] = p(1-p)$	$M(t) = 1 - p + pe^t$ $t \in \mathbb{R}$
Binomial(n, p)	$f(x) = \binom{n}{x} p^x(1-p)^{n-x}$ $x = 0, 1, ..., n$	$F(x) = \sum_{k=0}^{x} \binom{n}{k} p^k(1-p)^{n-k}$ $x = 0, 1, ..., n$	$E[X] = np$	$D[X] = np(1-p)$	$M(t) = (1-p+pe^t)^n$ $t \in \mathbb{R}$
Poisson(μ)	$f(x) = \frac{\mu^x e^{\mu}}{x!}$ $x = 0, 1, ..., n, ...$	$f(x) = \frac{\Gamma(x+1,\mu)}{\Gamma(x+1)}$ $x = 0, 1, ..., n, ...$	$E[X] = \mu$	$D[X] = \mu$	$M(t) = e^{\mu(e^t-1)}$ $t \in \mathbb{R}$
Geometric(p)	$f(x) = p(1-p)^x$ $x = 0, 1, ..., n, ...$	$F(x) = 1 - (1-p)^{x+1}$ $x = 0, 1, ..., n, ...$	$E[X] = \frac{1-p}{p}$	$D[X] = \frac{1-p}{p^2}$	$M(t) = \frac{p}{1-(1-p)e^t}$ $t < -\ln(1-p)$
Pascal(n, p)	$f(x) = \binom{n-1+x}{x} p^n(1-p)^x$ $x = 0, 1, 2, ..., n, ...$	$F(x) = 1 - I_p(k+1, n)$ $x = 0, 1, 2, ..., n, ...$	$E[X] = \frac{n(1-p)}{p}$	$D[X] = \frac{n(1-p)}{p^2}$	$M(t) = (\frac{p}{1-(1-p)e^t})^n$ $t < -\ln(1-p)$

39.3 Continuous Random Variables

Table 39.2: Continuous Random Variables

Distribution	PDF	CDF	Expectation	Variance	MGF
Uniform(a, b)	$f(x) = \frac{1}{b-a}$ $x = [a, b]$	$F(x) = \frac{x-a}{b-a}$ $x = [a, b]$	$E[X] = \frac{b-a}{2}$	$D[X] = \frac{(b-a)^2}{12}$	$M(t) = \begin{cases} 1, & t = 0 \\ \frac{e^{bt}-e^{at}}{t(b-a)}, & t \neq 0 \end{cases}$
Normal(μ, σ)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x \in \mathbb{R}$	$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x \in \mathbb{R}$	$E[X] = \mu$	$D[X] = \sigma^2$	$e^{\frac{t(t\sigma^2+2\mu)}{2}}$ $t \in \mathbb{R}$
Exponential$\lambda)$	$f(x) = \lambda e^{-\lambda x}$ $x > 0$	$F(x) = 1 - e^{\lambda x}$ $x > 0$	$E[X] = \frac{1}{\lambda}$	$D[X] = \frac{1}{\lambda^2}$	$\frac{1}{1-\frac{t}{\lambda}}$ $t < \lambda$
Erlang(n, λ)	$f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}$ $x > 0$	$F(x) = 1 - \sum_{i=0}^{n-1} \frac{\lambda^n x^n e^{-\lambda x}}{n!}$ $x > 0$	$E[X] = \frac{n}{\lambda}$	$D[X] = \frac{n}{\lambda^2}$	$\frac{1}{(1-\frac{t}{\lambda})^n}$ $t < \lambda$

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