Notes for Operations Research & More

Lan Peng, PhD Student

Department of Industrial and Systems Engineering
University at Buffalo, SUNY
lanpeng@buffalo.edu

July 12, 2019

Contents

I Preliminary Topics **7**

1 Review of Linear Algebra **9**

2 Convex Sets **11**

3 Convex Functions and Generalizations **13**

II Linear Programming **15**

4 Formulation **17**
4.1 Typical Problems . . . 17
4.2 Formulation Skills . . . 17
4.2.1 Absolute Value . . . 17
4.2.2 A Minimax Objective . . . 17
4.2.3 A fractional Objective . . . 18
4.2.4 A range Constraint . . . 18

5 Simplex Method **19**
5.1 Basic Feasible Solutions and Extreme Points . . . 19
5.2 Simplex Method . . . 19
5.2.1 Simplex Method Algorithm . . . 19
5.2.2 Simplex Method Tableau . . . 19
5.2.3 Simplex Method as a Search Algorithm . . . 19
5.3 Revised Simplex Method . . . 19
5.4 Simplex Method with Bounded Variables . . . 19
5.5 Artificial Variable . . . 19
5.5.1 Two-Phase Method . . . 19
5.5.2 Big-M Method . . . 19
5.5.3 Single Artificial Variable Technique . . . 19
5.6 Degeneracy and Cycling . . . 19
5.6.1 Degeneracy . . . 19
5.6.2 Cycling . . . 19
5.6.3 Cycling Prevention Rules . . . 19

6 Duality Theory and Sensitivity Analysis **21**

7 Decomposition Principle **23**

8 Ellipsoid Algorithm **25**

9 Projective Algorithm **27**

10 Interior-Point Algorithm **29**

III Geometry, Graph and Network Theory **31**

11 Paths, Trees, and Cycles **33**

12 Shortest-Path Problem **35**

13 Minimum Spanning Tree Problem **37**

14 Maximum Flow Problem **39**

15 Minimum Cost Flow Problem **41**

16 Assignment and Matching Problem **43**

17 Graph Algorithms **45**

18 Polygon Triangulation **47**
18.1 Types of Polygons . . . 47
18.2 Triangulation . . . 47
18.3 Art Gallery Theorem . . . 47

IV Integer and Combinatorial Programming **49**

19 Formulation **51**
19.1 Typical Problems . . . 51
19.2 Integer Programming Formulation Skills . . . 51
19.2.1 A Variable Taking Discontinuous Values . . . 51
19.2.2 Fixed Costs . . . 51
19.2.3 Either-or Constraints . . . 51
19.2.4 Conditional Constraints . . . 52
19.2.5 Special Ordered Sets . . . 52
19.2.6 Piecewise Linear Formulations . . . 53
19.2.7 Conditional Binary Variables . . . 53
19.2.8 Elimination of Products of Variables . . . 54

20 Branch and Bound **55**

21 Branch and Cut **57**

22 Packing and Matching **59**

23 Traveling Salesman Problem **61**

24 Knapsack Problem **63**

V Nonlinear Programming **65**

25 KKT Optimality Conditions **67**

26 Lagrangian Duality **69**

27 Unconstrained Optimization **71**

28 Penalty and Barrier Functions **73**

VI Algorithms and Computational Complexity **75**

29 Computational Complexity **77**

30 Sorting **79**
30.1 Elementary Sorting Algorithms . . . 79
30.2 Heap-sort . . . 79
30.3 Quick-sort . . . 79
30.4 Sorting in Linear Time . . . 79
30.5 Medians and Order Statistics . . . 79

31 Data Structures **81**
31.1 Elementary Data Structures . . . 81
31.2 Hash Tables . . . 81
31.3 Binary Search Trees . . . 81
31.4 Red-Black Trees . . . 81
31.5 B-Trees . . . 81
31.6 Fibonacci Heaps . . . 81
31.7 van Emde Boas Trees . . . 81

32 Design and Analysis Techniques **83**
32.1 Dynamic Programming . . . 83
32.2 Greedy Algorithms . . . 83
32.3 Amortized Analysis . . . 83
32.4 Multi-threaded Algorithms . . . 83
32.5 Matrix Operations . . . 83
32.6 Polynomials and the FFT . . . 83
32.7 Number-Theoretic Algorithms . . . 83
32.8 String Matching . . . 83
32.9 Computational Geometry . . . 83
32.10NP-Completeness . . . 83
32.11Approximation Algorithms . . . 83

VII Heuristics ans Meta-heuristics **85**

VIII Game Theory **87**

33 Games with Ordinal Payoffs **89**
33.1 Ordinal Games in Strategic Form . . . 89
33.2 Perfect-information Games . . . 89
33.3 General Dynamic Games . . . 89

34 Games with Cardinal Payoffs **91**
34.1 Expected Utility Theory . . . 91
34.2 Strategic-form Games . . . 91
34.3 Extensive-form Games . . . 91

35 Knowledge, Common Knowledge, Beliefs **93**
35.1 Common Knowledge . . . 93
35.2 Adding Beliefs to Knowledge . . . 93
35.3 Rationality . . . 93

36 Refinements of Subgame-perfect Equilibrium **95**
36.1 Weak Sequential Equilibrium . . . 95
36.2 Sequential Equilibrium . . . 95
36.3 Perfect Bayesian Equilibrium . . . 95

37 Incomplete Information **97**
37.1 Static Games . . . 97
37.2 Dynamic Games . . . 97
37.3 The Type-Space Approach . . . 97

IX Decision Analysis **99**

X Probability, Stochastic Processes and Markov Chains **101**

38 Probability **103**

39 Random Variables **105**
39.1 Relationship between Some Random Variables . . . 105
39.2 Discrete Random Variables . . . 106
39.3 Continuous Random Variables . . . 106

40 Limit Theorems **107**

41 The Bernoulli and Poisson Process **109**

42 Discrete-Time Markov Chains **111**

43 Continuous-Time Markov Chains **113**

XI Queuing Theory **115**

44 Queuing Model **117**

45 Birth-and-Death Queuing Models **119**

46 Multidimensional Birth-and-Death Queuing Models **121**

47 Phase-Type Queue **123**

48 Bulk Queue **125**

49 Imbedded-Markov-Chain Queuing Models **127**

50 Queuing Network **129**

XII Inventory Theory **131**

XIII Reliability Theory **133**

51 Maintenance Optimization **135**

XIV Bayesian Statistic **137**

XV Classical Statistic **139**

XVI Simulation **141**

Part I

Preliminary Topics

Chapter 1

Review of Linear Algebra

Chapter 2

Convex Sets

Chapter 3

Convex Functions and Generalizations

Part II

Linear Programming

Chapter 4

Formulation

4.1 Typical Problems

4.2 Formulation Skills

4.2.1 Absolute Value

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j |x_j|, \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & x_j \quad \text{unrestricted}, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j (x_j^+ + x_j^-), \quad c_j > 0 \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \gtreqless b_i, \quad \forall i \in I \\ & x_j^+, x_j^- \geq 0, \quad \forall j \in J \end{aligned}$$

4.2.2 A Minimax Objective

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \max_{k \in K} \sum_{j \in J} c_{kj} x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & \sum_{j \in J} c_{kj} x_j \leq z, \quad \forall k \in K \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

4.2.3 A fractional Objective

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \frac{\sum_{j\in J} c_j x_j + \alpha}{\sum_{j\in J} d_j x_j + \beta} \\ \text{s.t.} \quad & \sum_{j\in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j x_j t + \alpha t \\ \text{s.t.} \quad & \sum_{j\in J} a_{ij} x_j \gtreqless b_i, \quad \forall i \in I \\ & \sum_{j\in J} d_j x_j t + \beta t = 1 \\ & t > 0 \\ & x_j \geq 0, \quad \forall j \in J \\ & (t = \frac{1}{\sum_{j\in J} d_j x_j + \beta}) \end{aligned}$$

4.2.4 A range Constraint

Description: Consider the following model statement:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j x_j \\ \text{s.t.} \quad & d_i \leq \sum_{j\in J} a_{ij} x_j \leq e_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

Modeling:

$$\begin{aligned} \min \quad & \sum_{j\in J} c_j x_j, \quad c_j > 0 \\ \text{s.t.} \quad & u_i + \sum_{j\in J} a_{ij} x_j = e_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \\ & 0 \leq u_i \leq e_i - d_i, \quad \forall i \in I \end{aligned}$$

Chapter 5

Simplex Method

5.1 Basic Feasible Solutions and Extreme Points

5.2 Simplex Method

5.2.1 Simplex Method Algorithm

5.2.2 Simplex Method Tableau

5.2.3 Simplex Method as a Search Algorithm

5.3 Revised Simplex Method

5.4 Simplex Method with Bounded Variables

5.5 Artificial Variable

5.5.1 Two-Phase Method

5.5.2 Big-M Method

5.5.3 Single Artificial Variable Technique

5.6 Degeneracy and Cycling

5.6.1 Degeneracy

5.6.2 Cycling

5.6.3 Cycling Prevention Rules

Lexicographic Rule

Bland's Rule

Successive Ratio Rule

Chapter 6

Duality Theory and Sensitivity Analysis

Chapter 7

Decomposition Principle

Chapter 8

Ellipsoid Algorithm

Chapter 9

Projective Algorithm

Chapter 10

Interior-Point Algorithm

Part III

Geometry, Graph and Network Theory

Chapter 11

Paths, Trees, and Cycles

Chapter 12

Shortest-Path Problem

Chapter 13

Minimum Spanning Tree Problem

Chapter 14

Maximum Flow Problem

Chapter 15

Minimum Cost Flow Problem

Chapter 16

Assignment and Matching Problem

Chapter 17

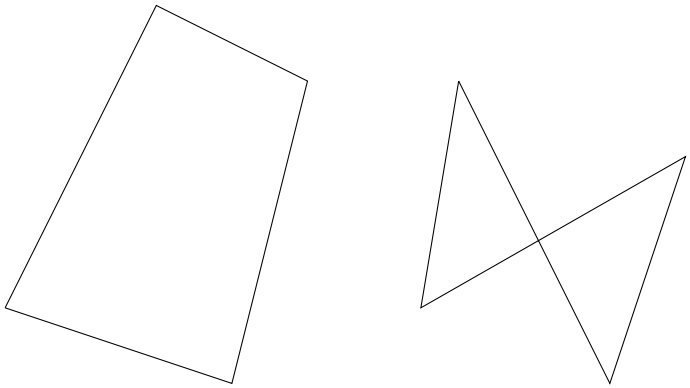
Graph Algorithms

Chapter 18

Polygon Triangulation

18.1 Types of Polygons

Def: A **simple polygon** is a closed polygonal curve without self-intersection. Polygons are basic building blocks



Simple Polygon Non-simple Polygon

in most geometric applications. It can model arbitrarily complex shapes, and apply simple algorithms and algebraic representation/manipulation.

18.2 Triangulation

Def: **Triangulation** is to partition polygon P into non-overlapping triangles using diagonals only. It reduces complex shapes to collection of simpler shapes. Every simple n-gon admits a triangulation which has $n-2$ triangles.

18.3 Art Gallery Theorem

Scenario: The floor plan of an art gallery modeled as a simple polygon with n vertices, there are guards which is stationed at fixed positions with 360 degree vision but cannot see through the walls. How many guards does the art gallery need for the security? (Fun fact: This problem was posted to Vasek Chvatal by Victor Klee in 1973)

Part IV

Integer and Combinatorial Programming

Chapter 19

Formulation

19.1 Typical Problems

19.2 Integer Programming Formulation Skills

19.2.1 A Variable Taking Discontinuous Values

Description: In algebraic notation:

$$x = 0, \quad \text{or} \quad l \leq x \leq u$$

Modeling:

$$\begin{aligned} & x \leq uy \\ & x \geq ly \\ & y \in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } l \leq x \leq u \end{cases}$$

19.2.2 Fixed Costs

Description: In algebraic notation:

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

Modeling:

$$\begin{aligned} & C^*(x, y) = ky + cx \\ & x \leq My \\ & x \geq 0 \\ & y \in \{0, 1\} \end{aligned}$$

where

$$y = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

19.2.3 Either-or Constraints

Description: In algebraic notation:

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \text{ or } \sum_{j \in J} a_{2j} x_j \leq b_2$$

Modeling:

$$\sum_{j\in J} a_{1j}x_j \le b_1 + M_1 y$$
$$\sum_{j\in J} a_{2j}x_j \le b_2 + M_1(1-y)$$
$$y \in \{0,1\}$$

where

$$y = \begin{cases} 0, & \text{if } \sum_{j\in J} a_{1j}x_j \le b_1 \\ 1, & \text{if } \sum_{j\in J} a_{2j}x_j \le b_2 \end{cases}$$

Notice that the sign before M is determined by the inequality $\ge$ or $\le$, if it is "$\ge$", use "$-$", if it "$\le$", use "$+$".

19.2.4 Conditional Constraints

Description: If constraint A is satisfied, then constraint B must also be satisfied

$$\text{If} \quad \sum_{j\in J} a_{1j}x_j \le b_1 \text{ then } \sum_{j\in J} a_{2j}x_j \le b_2$$

The key part is to find the opposite of the first condition. We are using $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$
Therefore it is equivalent to

$$\sum_{j\in J} a_{1j}x_j > b_1 \text{ or } \sum_{j\in J} a_{2j}x_j \le b_2$$

Furthermore, it is equivalent to

$$\sum_{j\in J} a_{1j}x_j \ge b_1 + \epsilon \text{ or } \sum_{j\in J} a_{2j}x_j \le b_2$$

Where ϵ is a very small positive number.
Modeling:

$$\sum_{j\in J} a_{1j}x_j \ge b_1 + \epsilon - M_2 y$$
$$\sum_{j\in J} a_{2j}x_j \le b_2 + M_2(1-y)$$
$$y \in \{0,1\}$$

19.2.5 Special Ordered Sets

SOS1 Description Out of a set of yes-no decisions, at most one decision variable can be yes.

$$x_1 = 1, x_2 = x_3 = \cdots = x_n = 0$$
$$\text{or}$$
$$x_2 = 1, x_1 = x_3 = \cdots = x_n = 0$$
$$\text{or ...}$$

Modeling:

$$\sum_i x_i = 1, \quad i \in N$$

SOS2 Description 1 Out of a set of binary variables, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list.
Modeling: If $x_1, x_2, ..., x_n$ is a SOS2, then

$$\sum_{i=1}^{n} x_i \le 2$$
$$x_i + x_j \le 1, \forall i \in \{1,2,...,n\}, j \in \{i+2, i+3, ..., n\}$$
$$x_i \in \{0,1\}$$

SOS2 Description 2 There is another type of definition, that is out of a set of nonnegative variables **not binary here**, at most two variables can be nonzero. In addition, the two variables must be adjacent to each other in a fixed order list. All variables summing to 1.
This definition of SOS2 is used in the following section *Piecewise Linear Formulations*

19.2.6 Piecewise Linear Formulations

Description: The objective function is a sequence of line segments, e.g. $y = f(x)$, consists $k-1$ linear segments going through k given points $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$.
Denote

$$d_i = \begin{cases} 1, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Then the objective function is

$$\sum_{i \in \{1,2,...,k-1\}} y = d_i f_i(x)$$

Modeling: Given that objective function as a piecewise linear formulation, we can have these constraints

$$\sum_{i \in \{1,2,...,k-1\}} d_i = 1$$
$$d_i \in \{0,1\}, i \in \{1,2,...,k-1\}$$
$$x = \sum_{i \in \{1,2,...,k\}} w_i x_i$$
$$y = \sum_{i \in \{1,2,...,k\}} w_i y_i$$
$$w_1 \le d_1$$
$$w_i \le d_{i-1} + di, i \in \{2,3,...,k-1\}$$
$$w_k \le d_{k-1}$$

In this case, $w_i \in SOS2$ (second definition)

19.2.7 Conditional Binary Variables

Description: Choose at most n binary variable to be 1 out of $x_1, x_2, ...x_m, m \ge n$. If $n = 1$ then it is SOS1.
Modeling:

$$\sum_{k \in \{1,2,...,m\}} x_k \le n$$

Description: Choose exactly n binary variable to be 1 out of $x_1, x_2, ...x_m, m \ge n$
Modeling:

$$\sum_{k \in \{1,2,...,m\}} x_k = n$$

Description: Choose x_j only if $x_k = 1$
Modeling:

$$x_j = x_k$$

Description: "and" condition, iff $x_1, x_2, ..., x_m = 1$ then $y = 1$
Modeling:

$$y \le x_i, i \in \{1,2,...,m\}$$
$$y \ge \sum_{i \in \{1,2,...,m\}} x_i - (m-1)$$

19.2.8 Elimination of Products of Variables

Description: For variables x_1 and x_2,

$$y = x_1 x_2$$

Modeling: If x_1, x_2 are binary, it is the same as "and" condition of binary variables.
If x_1 is binary, while x_2 is continuous and $0 \leq x_2 \leq u$, then

$$\begin{aligned} y &\leq u x_1 \\ y &\leq x_2 \\ y &\geq x_2 - u(1 - x_1) \\ y &\geq 0 \end{aligned}$$

If both x_1 and x_2 are continuous, it is non-linear, we can use Piecewise linear formulation to simulate.

Chapter 20

Branch and Bound

Chapter 21

Branch and Cut

Chapter 22

Packing and Matching

Chapter 23

Traveling Salesman Problem

Chapter 24

Knapsack Problem

Part V

Nonlinear Programming

Chapter 25

KKT Optimality Conditions

Chapter 26

Lagrangian Duality

Chapter 27

Unconstrained Optimization

Chapter 28

Penalty and Barrier Functions

Part VI

Algorithms and Computational Complexity

Chapter 29

Computational Complexity

Chapter 30

Sorting

30.1 Elementary Sorting Algorithms

30.2 Heap-sort

30.3 Quick-sort

30.4 Sorting in Linear Time

30.5 Medians and Order Statistics

Chapter 31

Data Structures

31.1 Elementary Data Structures

31.2 Hash Tables

31.3 Binary Search Trees

31.4 Red-Black Trees

31.5 B-Trees

31.6 Fibonacci Heaps

31.7 van Emde Boas Trees

Chapter 32

Design and Analysis Techniques

32.1 Dynamic Programming

32.2 Greedy Algorithms

32.3 Amortized Analysis

32.4 Multi-threaded Algorithms

32.5 Matrix Operations

32.6 Polynomials and the FFT

32.7 Number-Theoretic Algorithms

32.8 String Matching

32.9 Computational Geometry

32.10 NP-Completeness

32.11 Approximation Algorithms

Part VII

Heuristics ans Meta-heuristics

Part VIII

Game Theory

Chapter 33

Games with Ordinal Payoffs

33.1 Ordinal Games in Strategic Form

33.2 Perfect-information Games

33.3 General Dynamic Games

Chapter 34

Games with Cardinal Payoffs

34.1 Expected Utility Theory

34.2 Strategic-form Games

34.3 Extensive-form Games

Chapter 35

Knowledge, Common Knowledge, Beliefs

35.1 Common Knowledge

35.2 Adding Beliefs to Knowledge

35.3 Rationality

Chapter 36

Refinements of Subgame-perfect Equilibrium

36.1 Weak Sequential Equilibrium

36.2 Sequential Equilibrium

36.3 Perfect Bayesian Equilibrium

Chapter 37

Incomplete Information

37.1 Static Games

37.2 Dynamic Games

37.3 The Type-Space Approach

Part IX

Decision Analysis

Part X

Probability, Stochastic Processes and Markov Chains

Chapter 38

Probability

Chapter 39

Random Variables

39.1 Relationship between Some Random Variables

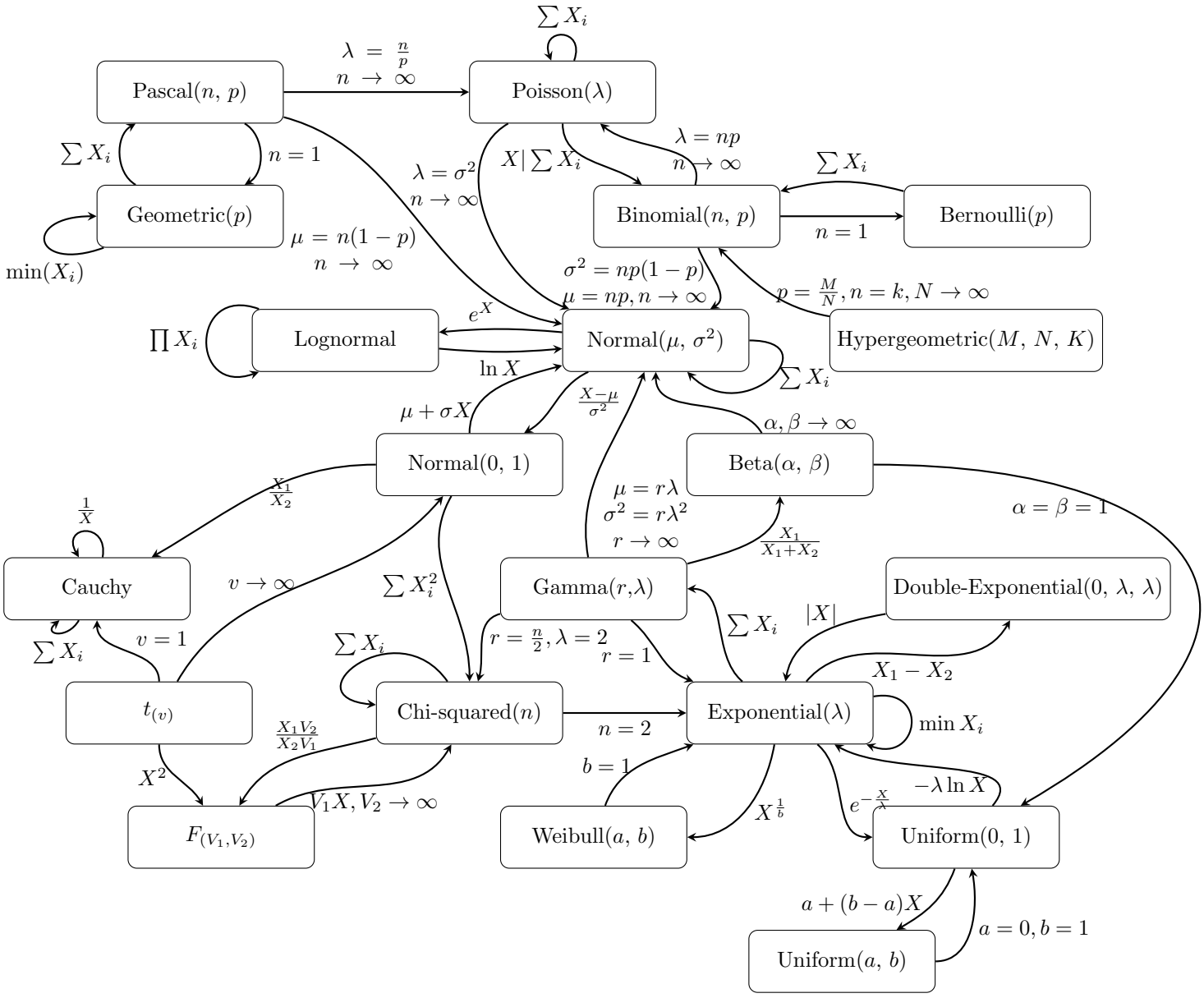


Figure 39.1: Relationship between Some Random Variables

39.2 Discrete Random Variables

Table 39.1: Discrete Random Variables

Distribution	PMF	CDF	Expectation	Variance	MGF
Discrete Uniform(a,b)	$f(x)=\frac{1}{b-a+1}$ $x=a,a+1,...,b$	$F(x)=\frac{x-a+1}{b-a+1}$ $x=a,a+1,...,b$	$E[X]=\frac{b-a}{2}$	$D[X]=\frac{(b-a+1)^2-1}{12}$	$M(t)=\frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$ $t\in\mathbb{R}$
Bernoulli(p)	$f(x)=p^x(1-p)^{1-x}$ $x\in\{0,1\}$	$F(x)=\begin{cases}0, & x<0\\ 1-p, & 0\le x\le 1\\ 1, & x>1\end{cases}$	$E[X]=p$	$D[X]=p(1-p)$	$M(t)=1-p+pe^t$ $t\in\mathbb{R}$
Binomial(n,p)	$f(x)=\binom{n}{x}p^x(1-p)^{n-x}$ $x=0,1,...,n$	$F(x)=\sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k}$ $x=0,1,...,n$	$E[X]=np$	$D[X]=np(1-p)$	$M(t)=(1-p+pe^t)^n$ $t\in\mathbb{R}$
Poisson(μ)	$f(x)=\frac{\mu^x e^{\mu}}{x!}$ $x=0,1,...,n,...$	$f(x)=\frac{\Gamma(x+1,\mu)}{\Gamma(x+1)}$ $x=0,1,...,n,...$	$E[X]=\mu$	$D[X]=\mu$	$M(t)=e^{\mu(e^t-1)}$ $t\in\mathbb{R}$
Geometric(p)	$f(x)=p(1-p)^x$ $x=0,1,...,n,...$	$F(x)=1-(1-p)^{x+1}$ $x=0,1,...,n,...$	$E[X]=\frac{1-p}{p}$	$D[X]=\frac{1-p}{p^2}$	$M(t)=\frac{p}{1-(1-p)e^t}$ $t<-\ln(1-p)$
Pascal(n,p)	$f(x)=\binom{n-1+x}{x}p^n(1-p)^x$ $x=0,1,2,...,n,...$	$F(x)=1-I_p(k+1,n)$ $x=0,1,2,...,n,...$	$E[X]=\frac{n(1-p)}{p}$	$D[X]=\frac{n(1-p)}{p^2}$	$M(t)=(\frac{p}{1-(1-p)e^t})^n$ $t<-\ln(1-p)$

39.3 Continuous Random Variables

Table 39.2: Continuous Random Variables

Distribution	PDF	CDF	Expectation	Variance	MGF
Uniform(a,b)	$f(x)=\frac{1}{b-a}$ $x=[a,b]$	$F(x)=\frac{x-a}{b-a}$ $x=[a,b]$	$E[X]=\frac{b-a}{2}$	$D[X]=\frac{(b-a)^2}{12}$	$M(t)=\begin{cases}1, & t=0\\ \frac{e^{bt}-e^{at}}{t(b-a)}, & t\neq 0\end{cases}$
Normal(μ,σ)	$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in\mathbb{R}$	$F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in\mathbb{R}$	$E[X]=\mu$	$D[X]=\sigma^2$	$e^{\frac{t(t\sigma^2+2\mu)}{2}}$ $t\in\mathbb{R}$
Exponential$\lambda)$	$f(x)=\lambda e^{-\lambda x}$ $x>0$	$F(x)=1-e^{\lambda x}$ $x>0$	$E[X]=\frac{1}{\lambda}$	$D[X]=\frac{1}{\lambda^2}$	$\frac{1}{1-\frac{t}{\lambda}}$ $t<\lambda$
Erlang(n,λ)	$f(x)=\frac{\lambda^n x^{n-1}e^{-\lambda x}}{(n-1)!}$ $x>0$	$F(x)=1-\sum_{i=0}^{n-1}\frac{\lambda^n x^n e^{-\lambda x}}{n!}$ $x>0$	$E[X]=\frac{n}{\lambda}$	$D[X]=\frac{n}{\lambda^2}$	$\frac{1}{(1-\frac{t}{\lambda})^n}$ $t<\lambda$

Chapter 40

Limit Theorems

Chapter 41

The Bernoulli and Poisson Process

Chapter 42

Discrete-Time Markov Chains

Chapter 43

Continuous-Time Markov Chains

Part XI

Queuing Theory

Chapter 44

Queuing Model

Chapter 45

Birth-and-Death Queuing Models

Chapter 46

Multidimensional Birth-and-Death Queuing Models

Chapter 47

Phase-Type Queue

Chapter 48

Bulk Queue

Chapter 49

Imbedded-Markov-Chain Queuing Models

Chapter 50

Queuing Network

Part XII

Inventory Theory

Part XIII

Reliability Theory

Chapter 51

Maintenance Optimization

Part XIV

Bayesian Statistic

Part XV

Classical Statistic

Part XVI

Simulation