Step 1 : let's define our prior as a Gamma distribution

Please note that we have two way to define a Gamma distribution

$$f\left(\lambda|\alpha,\beta\right)=\frac{\lambda^{\alpha-1}e^{-\beta\lambda}\beta^{\alpha}}{\Gamma\left(\alpha\right)}\text{ for }x>0\text{ and }\alpha>0,\beta>0$$

$$f\left(\lambda|k,\theta\right)=\frac{\lambda^{k-1}e^{-\lambda/\theta}}{\theta^{k}\Gamma(k)},\text{ for }x>0\text{ and }k,\theta>0$$

Note that

$$k=\alpha\text{ and }\theta=\beta$$

Mathematica automatically assumes the (k, θ) methods.

However, in Bayesian statistics, we prefer the (α, β) way.

Hence, the parameters will be

$$k=\alpha=7$$

$$\theta=1/\beta=1/5$$

based on the parameter values, we define the prior as a function of λ

In[275]:=

```
prior[λ_] := PDF[GammaDistribution[7, 1 / 5], λ]
```

let' s print its analytic form to see whether it works

In[276]:=

```
prior[λ]
```

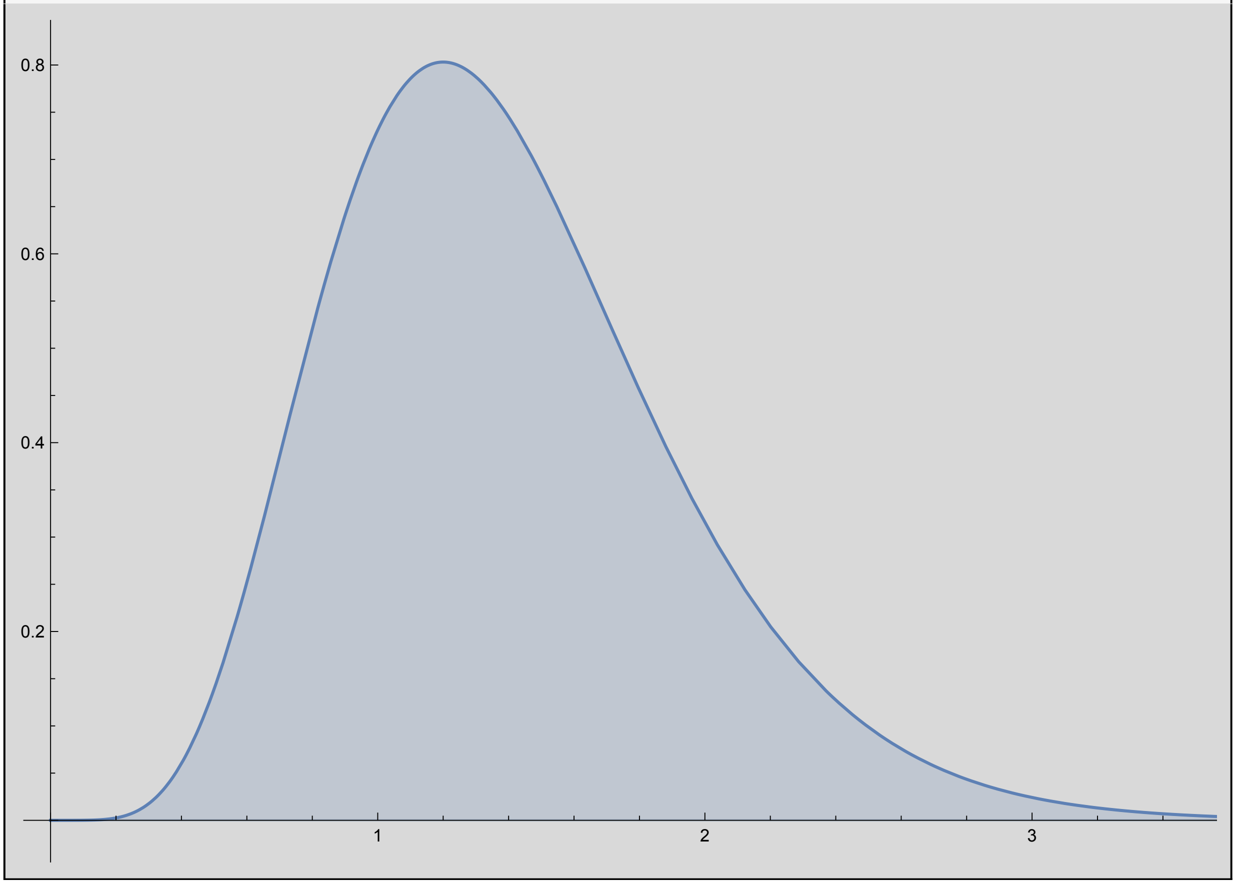
Out[276]=

$$\begin{cases} \frac{15\,625}{144}\, e^{-5\lambda}\,\lambda^{6} & \lambda>0 \\ 0 & \text{True} \end{cases}$$

let' s plot it as well

In[277]:=
```
Plot[prior[λ], {λ, 0, 4}, Filling → Axis]
```

Out[277]=



Step 2 : let’s define our likelihood function of each data point as an Exponential Distribution

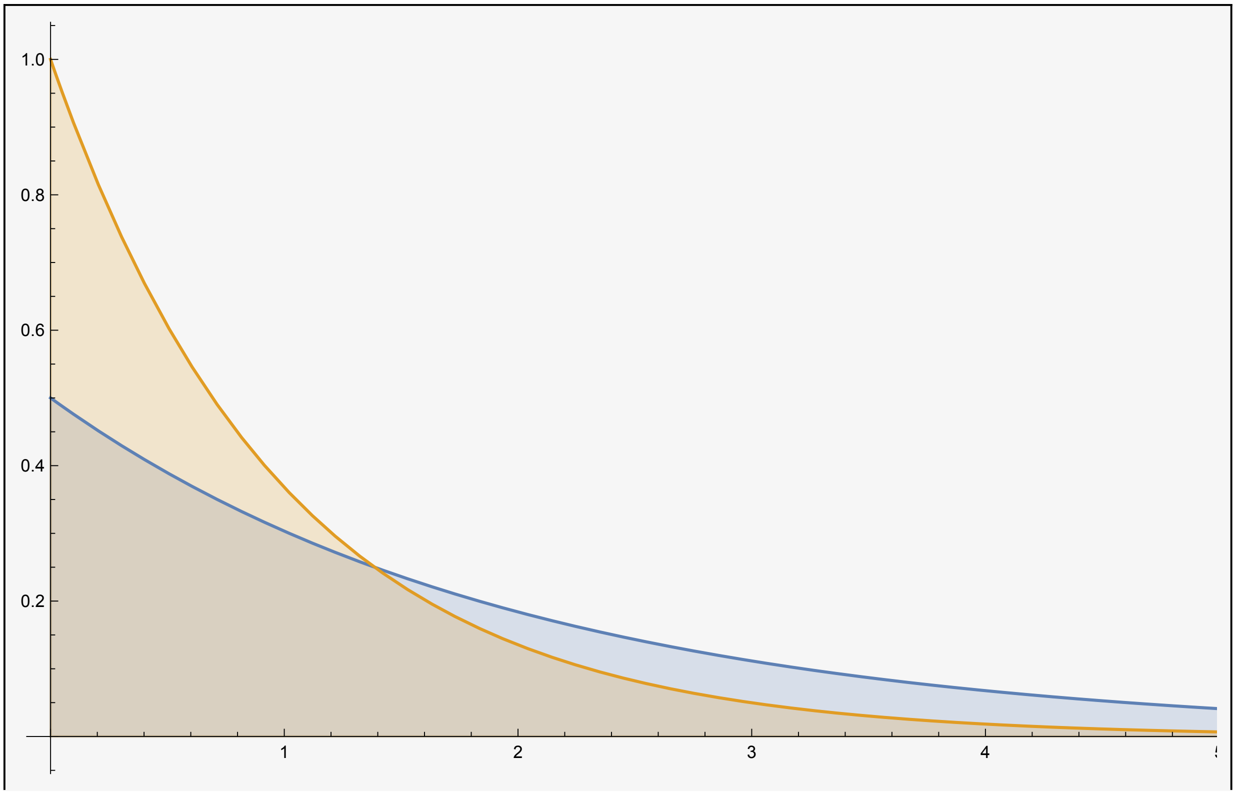
```
lf[x_] := PDF[ExponentialDistribution[λ], x]
```

```
lf[x]
```

Out[279]=

$$\begin{cases} e^{-x\lambda}\lambda & x \geq 0 \\ 0 & \text{True} \end{cases}$$

```
Plot[Table[lf[x], {λ, {0.5, 1}}] // Evaluate,
 {x, 0, 5}, Filling → Axis]
```



Step 3 : let’s define our likelihood function of all data points as the product of Exponential PDFs

In[281]:=
```
likelihood[f_, data_] := Product[PDF[f, x], {x, data}]
```

In[282]:=
```
likelihood[ExponentialDistribution[λ], {a₁, a₂, a₃, a₄, a₅}]
```

Out[282]=

$$\left(\begin{cases} e^{-\lambda a_1}\lambda & a_1 \geq 0 \\ 0 & \text{True} \end{cases}\right)\left(\begin{cases} e^{-\lambda a_2}\lambda & a_2 \geq 0 \\ 0 & \text{True} \end{cases}\right)\left(\begin{cases} e^{-\lambda a_3}\lambda & a_3 \geq 0 \\ 0 & \text{True} \end{cases}\right)\left(\begin{cases} e^{-\lambda a_4}\lambda & a_4 \geq 0 \\ 0 & \text{True} \end{cases}\right)\left(\begin{cases} e^{-\lambda a_5}\lambda & a_5 \geq 0 \\ 0 & \text{True} \end{cases}\right)$$

In[283]:=
```
l = Simplify[likelihood[
   ExponentialDistribution[λ], {a₁, a₂, a₃, a₄, a₅}]]
```

Out[283]=

$$\begin{cases} e^{-\lambda (a_1+a_2+a_3+a_4+a_5)}\lambda^5 & a_1 \geq 0 \,\&\&\, a_2 \geq 0 \,\&\&\, a_3 \geq 0 \,\&\&\, a_4 \geq 0 \,\&\&\, a_5 \geq 0 \\ 0 & \text{True} \end{cases}$$

In[285]:=
```
likelihood[ExponentialDistribution[λ], {1, 2, 3, 4, 5}]
```
Out[285]=

$$e^{-15\lambda}\lambda^5$$

Step 4 : let's calculate our posterior using Bayes Rule

The Bayes Rule suggests that

$$\pi(\lambda|x) = \frac{L(x|\lambda)p(\lambda)}{\int L(x|\lambda)p(\lambda)\,d\lambda}$$

Since the denominator is just a constant, we just need to focus on the numerator calculation as follows

In[286]:=
```
num = Simplify[l * prior[λ]]
```
Out[286]=

$$\begin{cases} \frac{15625}{144} e^{-\lambda(5+a_1+a_2+a_3+a_4+a_5)} \lambda^{11} & a_1 \geq 0 \,\&\&\, a_2 \geq 0 \,\&\&\, a_3 \geq 0 \,\&\&\, a_4 \geq 0 \,\&\&\, a_5 \geq 0 \,\&\&\, \lambda > 0 \\ 0 & \text{True} \end{cases}$$

It looks like a Gamma distribution, which is

In[339]:=

$$f\,(\lambda|\alpha,\beta) = \frac{\lambda^{\alpha-1}e^{-\beta\lambda}\beta^{\alpha}}{\Gamma\,(\alpha)} \text{ for } x > 0 \text{ and } \alpha > 0, \beta > 0$$

Let's vary our sample points (add one extra sample points) to find the pattern of Gamma paratmeter.

In[287]:=
```
l = Simplify[likelihood[
    ExponentialDistribution[λ], {a₁, a₂, a₃, a₄, a₅, a₆}]];
num = Simplify[l * prior[λ]]
```
Out[288]=

$$\begin{cases} \frac{15625}{144} e^{-\lambda(5+a_1+a_2+a_3+a_4+a_5+a_6)} \lambda^{12} & \lambda > 0 \,\&\&\, a_1 \geq 0 \,\&\&\, a_2 \geq 0 \,\&\&\, a_3 \geq 0 \,\&\&\, a_4 \geq 0 \,\&\&\, a_5 \geq 0 \,\&\&\, a_6 \geq 0 \\ 0 & \text{True} \end{cases}$$

Still a Gamma distribution

Again, let's vary our sample points (add one extra sample points) to find the pattern of Gamma paratmeter.

In[289]:=

```
l = Simplify[likelihood[ExponentialDistribution[λ],
    {a1, a2, a3, a4, a5, a6, a7}]];
num = Simplify[l * prior[λ]]
```

Out[290]=

$$\begin{cases} \frac{15625}{144} e^{-\lambda (5+a_1+a_2+a_3+a_4+a_5+a_6+a_7)} \lambda^{13} & \lambda > 0 \&\& a_1 \geq 0 \&\& a_2 \geq 0 \&\& a_3 \geq 0 \&\& a_4 \geq 0 \&\& a_5 \geq 0 \&\& a_6 \geq 0 \&\& a_7 \geq 0 \\ 0 & \text{True} \end{cases}$$

Now we can find the pattern of Gamma paratmeter:

$$\beta = 5 + \sum_i x_i$$

$\alpha = 7 + n$ where n is sample size.