CITM
Final Exam, January 2020
MATVJII

SOLUTIONS

First part: Short questions

Exercise 1:

Given the rotation vector $\boldsymbol{r}$,

$$\mathbf{R} = \mathbf{I} + \frac{\sin(\|r\|)}{\|r\|}\left[\boldsymbol{r}\right]_\times + \frac{(1-\cos(\|r\|))}{\|r\|^2}\left[\boldsymbol{r}\right]_\times^2$$

defines the rotation matrix that rotates vectors about $\boldsymbol{r}$ an amount of $\|\boldsymbol{r}\|$ rad.

15 Point

(a) Give the expression that defines its transpose and hence, its inverse as a function of $[\boldsymbol{r}]_\times$ and $\|\boldsymbol{r}\|$.

Solution:

Looking at the terms of the given equation:

- $\mathbf{I}$ is symmetric
- $[\boldsymbol{r}]_\times$ is anti-symmetric
- $[\boldsymbol{r}]_\times^2$ is also symmetric

Since the inverse of a rotation matrix equals its transpose, and

$$\mathbf{S}^T = \mathbf{S}; \mathbf{A}^T = -\mathbf{A}$$

being $\mathbf{S}$ a symmetric matrix and $\mathbf{A}$ and anti-symmetric one, then

$$\mathbf{R}^{-1} = \mathbf{I} - \frac{\sin(\|r\|)}{\|r\|}\left[\boldsymbol{r}\right]_\times + \frac{(1-\cos(\|r\|))}{\|r\|^2}\left[\boldsymbol{r}\right]_\times^2$$

Exercise 2:

A frame $\{B\}$ can be achieved by rotating an initial frame $\{A\}$ 20 degs about the direction of its z-axis.

A third frame, frame $\{C\}$, is achieved by rotating the frame $\{B\}$ 50 degs about its y-axis.

15 Point

(a) Which is the rotation matrix that allows to express a vector defined in $\{C\}$ into the frame $\{A\}$?

Solution:

$${}^B\mathbf{R}_A = \begin{pmatrix} \cos(20) & \sin(20) & 0 \\ -\sin(20) & \cos(20) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad {}^C\mathbf{R}_B = \begin{pmatrix} \cos(50) & 0 & -\sin(50) \\ 0 & 1 & 0 \\ \sin(50) & 0 & \cos(50) \end{pmatrix}$$

$${}^A\mathbf{R}_C = {}^B\mathbf{R}_A{}^\intercal\, {}^C\mathbf{R}_B{}^\intercal = \begin{pmatrix} 0.6040 & -0.3420 & 0.7198 \\ 0.2198 & 0.9397 & 0.2620 \\ -0.7660 & 0 & 0.6428 \end{pmatrix}$$

Exercise 3:

The quaternion

$$\mathring{q}_1 = (0.8660,\ 0,\ 0.4472,\ -0.2236)^\intercal$$

allows to transform vectors from a frame $\{A\}$ to a frame $\{C\}$.

In a similar way the quaternion

$$\mathring{q}_2 = (0.9659,\ 0.1830,\ 0,\ 0.1830)^\intercal$$

allows to transform vectors from a frame $\{B\}$ to the frame $\{C\}$.

15 Point (a) Which is the quaternion $\mathring{q}_3$ that transforms vectors defined in the frame $\{B\}$ to the frame $\{A\}$?

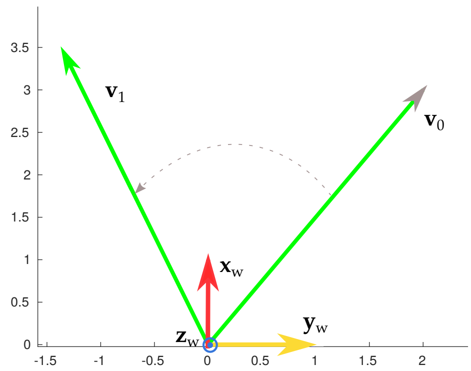
Solution:

$$\mathring{q}_3 = \Big(0.7956,\ 0.0766,\ -0.3911,\ 0.4563\Big)^\intercal$$

Exercise 4:

Take vector $\boldsymbol{v_0} = (2, 3, 0)^\intercal$ at time $t = 0$. At time $t = 1$ a certain rotation is applied to this vector obtaining $\boldsymbol{v_1} = (-1.31, 3.35, 0)^\intercal$ as shown in the figure below.

15 Point (a) Find the rotation matrix $\mathbf{R}$ that was employed to rotate this vector such that $\boldsymbol{v_1} = \mathbf{R}\boldsymbol{v_0}$.



Solution:

$$\mathbf{R} = \begin{pmatrix} 0.5736 & -0.8192 & 0 \\ 0.8192 & 0.5736 & 0 \\ 0 & 0 & 1.0000 \end{pmatrix}$$

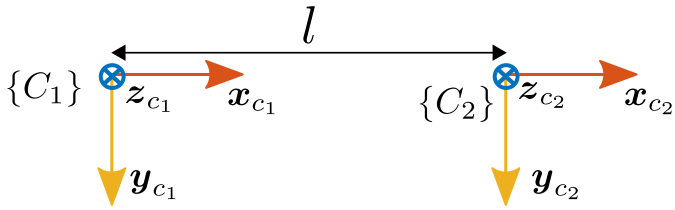


Figure 1: Rear view of the stereocam setup

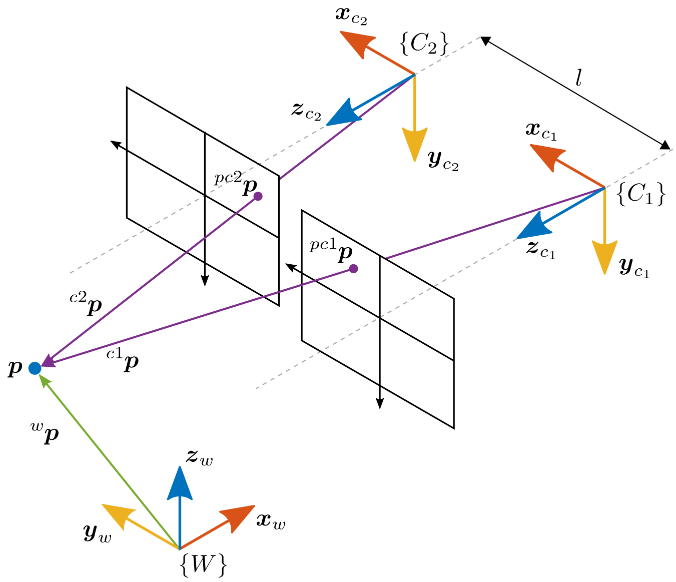


Figure 2: Scene **conceptual** view, not real scenario. The stereo pair is observing a generic point $\boldsymbol{p}$

Exercise 5

A point $\boldsymbol{p}$ is observed from the stereo-cameras pair depicted in Fig. 1 and Fig. 2.

The baseline that separates the camera centers (distance in x-direction) is

$$l = 0.5\,\mathrm{m}$$

Provide the coordinates of a point $\boldsymbol{p}$ as seen in the $\{C_1\}$ frame, i.e., ${}^{C_1}\boldsymbol{p}$ and the focal distance of the camera 2 if it is known that the focal distance of the camera 1 is $f_1 = 1/35\,\mathrm{m}$ and that the same point is observed at the both camera planes with coordinates

$${}^{pc1}\boldsymbol{p} = (0.0017,\, 0.0096)^{\intercal} \qquad {}^{pc2}\boldsymbol{p} = (0.0118,\, 0.0134)^{\intercal}$$

To this end:

10 Point (a) Find the affine transformation matrix between the cameras frames $\{C_1\}$ and $\{C_2\}$

Solution:

$${}^{C_2}\mathbf{A}_{C_1} = \begin{pmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

30 Point (b) Provide

(**Hint:**) Write the equations that relate the generic coordinates of ${}^{C_1}\boldsymbol{p}$, e.g., ${}^{C_1}\boldsymbol{p} = (a, b, c)^T$ and apply the projection and transform relations needed to find its projection on the camera planes of both cameras

Solution:

Take the unknown coordinates of the point ${}^{C_1}\boldsymbol{p}$, as, ${}^{C_1}\boldsymbol{p} = (a, b, c)^T$ then its projection on the camera plane of the camera 1 is:

$$
{}^{pc_1}\boldsymbol{p} = \begin{pmatrix} \frac{af_1}{c} \\ \frac{bf_1}{c} \end{pmatrix} = \begin{pmatrix} 0.0017 \\ 0.0096 \end{pmatrix} \tag{1}
$$

In a similar fashion, the point as seen from the camera 2 can be first converted to the frame $\{C_2\}$ using the matrix provided in section a) and then projected to the camera frame of the camera 2, then

$$
\begin{pmatrix} {}^{C_2}\boldsymbol{p} \\ 1 \end{pmatrix} = {}^{C_2}\mathbf{A}_{C_1} \begin{pmatrix} {}^{C_1}\boldsymbol{p} \\ 1 \end{pmatrix} \rightarrow {}^{C_2}\boldsymbol{p} = \begin{pmatrix} a - l \\ b \\ c \end{pmatrix}
$$

hence its projection over the camera frame 2 is:

$$
{}^{pc_2}\boldsymbol{p} = \begin{pmatrix} \frac{(a-l)f_2}{c} \\ \frac{bf_2}{c} \end{pmatrix} = \begin{pmatrix} 0.0118 \\ 0.0134 \end{pmatrix} \tag{2}
$$

with l and f_1 known, Eqs.(1 and 2), forms a system of 4 unknowns and 4 equations. Dividing the lower component of Eq(1)by the lower component of Eq.(2), the relation $\frac{f_1}{f_2} = \frac{0.0096}{0.0134}$ can be obtained from there,

$$
f_2 = \frac{0.0134 f_1}{0.0096} = 0.039 \approx \frac{1}{25}\ \mathrm{m}
$$

with f_1 calculated the same thing can be done with the upper components to extract a, i.e.,

$$
\frac{af_1}{(a-l)f_2} = \frac{0.0017}{0.0118} \rightarrow a \approx 0.1259\,\mathrm{m}
$$

having a and f_1 both eqs. can be used to find first c, and later b. Using, e.g, Eq(1):

$$
c = \frac{af_1}{0.0017} \approx 0.7107\,\mathrm{m}
$$

$$
b = \frac{0.0096c}{f_1} \approx 2.1152\,\mathrm{m}
$$

Resulting in:

$$
{}^{C_1}\boldsymbol{p} = (0.1259, 0.7107, 2.1152)^T
$$

Exercise 6

A frame attached to a video-game character is rotating. It is known that at the initial instant the orientation of the frame can be represented by the quaternion

$$\mathring{q}_1 = (0.9962,\, 0.0616,\, 0,\, 0.0616)^\intercal .$$

One second later, the orientation of the frame is given by another quaternion

$$\mathring{q}_2 = (0.9659,\, 0.1494,\, -0.1494,\, -0.1494)^\intercal$$

15 Point

(a) Using linear interpolation (LERP) determine the orientation quaternion at times $t_1 = 0.1\,\text{s}$, $t_2 = 0.5\,\text{s}$ and $t_3 = 0.8\,\text{s}$.

Solution:

$$\mathring{q}_{t1} = \begin{pmatrix} 0.9966 & 0.0706 & -0.0150 & 0.0407 \end{pmatrix}^\intercal$$
$$\mathring{q}_{t2} = \begin{pmatrix} 0.9905 & 0.1065 & -0.0754 & -0.0443 \end{pmatrix}^\intercal$$
$$\mathring{q}_{t3} = \begin{pmatrix} 0.9779 & 0.1327 & -0.1203 & -0.1079 \end{pmatrix}^\intercal$$

25 Point

(b) Using spherical interpolation (SLERP) determine the orientation quaternion at times $t_1 = 0.1\,\text{s}$, $t_2 = 0.5\,\text{s}$ and $t_3 = 0.8\,\text{s}$.

Solution:

$$\mathring{q}_{t1} = \begin{pmatrix} 0.9966 & 0.0707 & -0.0151 & 0.0405 \end{pmatrix}^\intercal$$
$$\mathring{q}_{t2} = \begin{pmatrix} 0.9905 & 0.1065 & -0.0754 & -0.0443 \end{pmatrix}^\intercal$$
$$\mathring{q}_{t3} = \begin{pmatrix} 0.9780 & 0.1326 & -0.1201 & -0.1076 \end{pmatrix}^\intercal$$

The exam continues on next page . . .

Some help

- Euler principal axis and angle to rotation matrix

$$\mathbf{R}_{\boldsymbol{u},\phi} = \mathbf{I}\cos(\phi) + (1-\cos(\phi))\left(\boldsymbol{u}\boldsymbol{u}^{\intercal}\right) + \sin(\phi)\left[\boldsymbol{u}\right]_{\times}$$

- Rotation matrix to Euler principal axis and angle

$$\phi = \arccos\left(\frac{\operatorname{trace}(\mathbf{R})-1}{2}\right); \quad \left[\boldsymbol{u}\right]_{\times} = \frac{\mathbf{R}-\mathbf{R}^{\intercal}}{2\sin(\phi)}$$

- Projection of the vector $\boldsymbol{p}$ over the direction of $\boldsymbol{u}$

$$\boldsymbol{p}' = \left(\boldsymbol{p}^{\intercal}\boldsymbol{u}\right)\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}$$

- Euler angles to rotation matrix

$$\mathbf{R}(\psi,\theta,\phi) = \begin{pmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\psi c_\phi s_\theta + s_\psi s_\phi \\ c_\theta s_\psi & s_\psi s_\theta s_\phi + c_\phi c_\psi & c_\phi s_\psi s_\theta - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{pmatrix}$$

- Quaternion multiplication

$$\mathring{r}\mathring{s} = \begin{pmatrix} r_0 s_0 - \boldsymbol{r}^{\intercal}\boldsymbol{s} \\ r_0\boldsymbol{s} + s_0\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{s} \end{pmatrix}$$

- Quaternion exponentiation

$$\mathring{q}^h = \exp\left(\ln(\mathring{q})h\right)$$

- Quaternion exponential

$$\exp\left(\mathring{q}\right) = e^{q_0}\begin{pmatrix} \cos(\|\boldsymbol{q}_v\|) \\ \sin(\|\boldsymbol{q}_v\|)\,\frac{\boldsymbol{q}_v}{\|\boldsymbol{q}_v\|} \end{pmatrix}$$

- Quaternion logarithm

$$\ln\left(\mathring{q}\right) = \begin{pmatrix} \ln\left(\|\mathring{q}\|\right) \\ \frac{q_0}{\|\mathring{q}\|}\,\frac{\boldsymbol{q}_v}{\|\boldsymbol{q}_v\|} \end{pmatrix}$$