Algebra Universalis, **9** (1979) 207–210 Birkhäuser Verlag, Basel

# The number of rigid Boolean algebras

J. D. MONK AND W. RASSBACH

We prove here three theorems concerning rigid Boolean algebras.

THEOREM 1. *Let $\kappa$ be an uncountable regular cardinal. Then there is a system $\langle A_\alpha : \alpha < 2^\kappa \rangle$ of Boolean algebras with the following properties:*

(i) *$A_\alpha$ satisfies the countable chain condition;*
(ii) $|A_\alpha| = \kappa$;
(iii) *the completion of $A_\alpha$, $A_\alpha^c$, is rigid;*
(iv) *$A_\alpha^c$ is nonisomorphic to $A_\beta^c$ for $\alpha \neq \beta$;*
(v) *if $0 \neq a \in A_\alpha$, then $|A_\alpha \restriction a| = \kappa$.*

The construction used to prove Theorem 1 is an easy modification, due to Monk, of that used in the proof of Theorem 1 of Balcar, Štěpánek [1], the basic idea of which is due to Shelah [4]. We follow the notation of [1], and only indicate the modifications from [1] needed to prove Theorem 1.

As an immediate corollary of Theorem 1 we obtain

THEOREM 2. *If $\kappa$ is an uncountable regular cardinal and $\kappa^{\aleph_0} = \kappa$, then there are exactly $2^\kappa$ isomorphism types of rigid complete Boolean algebras of power $\kappa$.*

This theorem partially extends the result of Monk, Solovay [3] concerning the number of complete Boolean algebras of a given cardinality. We do not know whether the methods here, or other methods, can be used to construct rigid complete Boolean algebras of all singular cardinalities $\kappa$ with $\kappa^{\aleph_0} = \kappa$.

As concerns rigid (not necessarily complete) Boolean algebras of singular cardinalities, a construction due to Rassbach gives the maximum possible number, and so we have

Presented by B. Jónsson. Received May 24, 1977. Accepted for publication in final form November 22, 1977.

THEOREM 3. *For any uncountable cardinal $\kappa$ there are exactly $2^\kappa$ isomorphism types of rigid Boolean algebras of power $\kappa$.*

This theorem solves problems 8, 9 of McKenzie, Monk [2].

*Proof of Theorem 1.* Let

(1) $W=\{\alpha<\kappa:\exists\beta(cf\beta=\omega \quad \text{and} \quad \alpha=|\alpha|\cdot\beta\neq 0)\}.$

Clearly $W$ is stationary in $\kappa$. Let $\langle W_\alpha:\alpha<\kappa\rangle$ be a partition of $W$ into stationary subsets. Let $\mathfrak{H}$ be a collection of $2^\kappa$ subsets of $\kappa$ each of power $\kappa$. For each $\Gamma\in\mathfrak{H}$ let $\Gamma'=\{\alpha\cdot 2:\alpha\in\Gamma\}\cup\{\alpha\cdot 2+1:\alpha\in\Gamma\}$, and set $\mathscr{K}=\{\Gamma':\Gamma\in\mathfrak{H}\}$. For each $\Delta\in\mathscr{K}$ let $\nu_\Delta$ be the strictly increasing function mapping $\kappa$ onto $\Delta$. Then

(2) If $\Delta_1,\Delta_2\in\mathscr{K}$ and $\Delta_1\neq\Delta_2$, then there is an odd ordinal $\alpha<\kappa$ such that $\nu_{\Delta_1}\alpha\notin\Delta_2$ or $\nu_{\Delta_2}\alpha\notin\Delta_1$.

For each $\alpha<\kappa$ and each $\Delta\in\mathscr{K}$ let $W'_{\Delta\alpha}=W_{\nu_\Delta\alpha}\sim(\alpha+1)$. Let $W''_\Delta=\bigcup_{\alpha<\kappa}W'_{\Delta\alpha}$.

For each $\Gamma\in\mathscr{K}$ let $B_\Gamma$ be a free Boolean algebra with free generators $b_{\Gamma\alpha}$, $\alpha<\kappa$. For each $s\subseteq\kappa$ let $B_\Gamma s$ be the subalgebra of $B_\Gamma$ generated by $\{b_{\Gamma\alpha}:a\in s\}$. Let $\varphi_\Gamma$ be a mapping from $\kappa$ onto $B_\Gamma$ such that $\varphi''_\Gamma\delta=B_\Gamma\delta$ for all $\delta\in W$, with $\varphi_\Gamma\alpha=b_{\Gamma\mu}$ for some non-limit $\mu$ whenever $\alpha$ is odd. For each $x\in B_\Gamma$ there is a unique smallest finite $s(\Gamma,x)\subseteq\kappa$ such that $x\in B_\Gamma s(\Gamma,x)$. For each $\alpha<\kappa$ and $\delta\in W'_{\Gamma\alpha}$ choose a strictly increasing sequence $\langle\xi(\Gamma,\delta,n):n\in\omega\rangle$ of non-limit ordinals converging to $\delta$, with $\beta<\xi^\Gamma(\delta,0)$ for each $\beta\in s(\Gamma,\varphi_\Gamma\alpha)$.

Let $I_\Gamma$ be the ideal in $B_\Gamma$ generated by all elements

$$b_{\Gamma\delta}\wedge-\varphi_{\Gamma\alpha} \quad \text{for} \quad \delta\in W'_{\Gamma\alpha},$$
$$b_{\Gamma\delta}\wedge-b_{\Gamma\xi(\Gamma,\delta,n)} \quad \text{for} \quad \delta\in W''_\Gamma, \qquad n\in\omega.$$

We set $A_\Gamma=B_\Gamma/I_\Gamma$. Now it is clear that $[b_{\Gamma\delta}]\neq 0$ whenever $\delta\notin W$. Hence we have

(3) $|A_\Gamma\upharpoonright x|=\kappa$ whenever $0\neq x\in A_\Gamma$

(4) $[\varphi_{\Gamma\alpha}]\neq 0$ for every odd ordinal $\alpha$.

The following is also easily established.

(5) If $[\varphi_{\Gamma\alpha}]\neq 0$ and $\delta\in W'_{\Gamma\alpha}$, then $[b_{\Gamma\delta}]\neq 0$.

The following are established just as in [1]:

(6) $A_\Gamma$ satisfies the countable chain condition.

(7) If $\delta\in W'_{\Gamma\alpha}$, then $[b_{\Gamma\delta}]=\bigwedge_{n\in\omega}[b_{\Gamma\xi(\Gamma,\delta,n)}]$.

(8) If $E$ is an infinite subset of $W'_{\Gamma\alpha}$, then $\bigwedge_{\delta\in E}[b_{\Gamma\delta}]=0$.

Now for each $\beta<\kappa$ let

$$C_{\Gamma\beta}=\left\{x\in A_\Gamma^c:\exists E\subseteq B_{\Gamma\beta}\left(x=\bigvee_{e\in E}[e]\right)\right\}.$$

Then, just as in [1],

(9) if $\Gamma\in\mathcal{K}$, $\delta\in W''_\Gamma$, $E\subseteq C_{\Gamma\delta}$ and $\bigwedge E\wedge[b_{\Gamma\delta}]=0$, then $\bigwedge E\in C_{\Gamma\delta}$.

The same proof gives

(10) If $\Gamma\in\mathcal{K}$, $\delta\in W$, $\delta\notin W''_\Gamma$, and $E\subseteq C_{\Gamma\delta}$, then $\bigwedge E\in C_{\Gamma\delta}$.

Now based on (7), (8), (9), one proves just as in [1] that $A_\Gamma^c$ is rigid.

We finish the proof by showing that $A_\Gamma^c$ is non-isomorphic to $A_\Delta^c$ when $\Gamma\neq\Delta$. Suppose, to the contrary, that $G$ is an isomorphism of $A_\Gamma^c$ onto $A_\Delta^c$, and $\alpha\in\Gamma\sim\Delta$. Let $\beta=\nu_\Gamma^{-1}\alpha$. By (2) we may assume that $\beta$ is odd. Hence by (4), $[\varphi_{\Gamma\beta}]\neq 0$. Now let

$$X=\{\alpha<\kappa:G''\{[x]:x\in B_\Gamma\}\subseteq C_{\Gamma\alpha}\}.$$

Clearly $X$ is stationary in $\kappa$. Thus $Y=X\cap W'_{\Gamma\beta}$ is also stationary. Note that $W''_{\Gamma\beta}\subseteq W_\alpha$, so $W''_{\Gamma\beta}\cap W''_\Delta=0$. Take any $\delta\in Y$. Then $\delta\in W'_{\Gamma\beta}$, so $0\neq[b_{\Gamma\delta}]\leq[\varphi_{\Gamma\beta}]$ by (5). Since $\delta\notin W''_\Delta$, we infer from (7), (10) and $\delta\in X$ that $G[b_{\Gamma\delta}]\in C_{\Delta\delta}$. Thus if we let $h\delta$ be the least $\gamma$ such that $0\neq[\varphi_{\Delta\gamma}]\leq G[b_{\Gamma\delta}]$, for each $\delta\in Y$, we define a regressive function $h$ on the stationary set $Y$. Hence $h$ is constant on a stationary subset of $Y$, contradicting (8).

*Proof of Theorem 3.* The proof is based upon a lemma which is of independent interest. If $I$ is an ideal in a Boolean algebra $A$ we let $-I=\{-a:a\in I\}$. As is well-known and easy to check, $I\cup -I$ is a subalgebra of $A$.

LEMMA. *If $A$ is a rigid Boolean algebra and $I$ is a non-principal ideal in $A$, then $I\cup -I$ is rigid.*

*Proof.* Suppose, to the contrary, that $F$ is a non-trivial automorphism of $I\cup -I$. Then there is a non-zero $x\in I\cup -I$ with $x\wedge Fx=0$. If $x\in I$ and $Fx\in I$, then $F$ induces an automorphism from $A\restriction x$ onto $A\restriction Fx$, contradicting $A$ rigid. Suppose that $x\in I$ and $Fx\in -I$. Since $I$ is non-principal, there is a $y\in I$ with $-Fx<y$. Thus $y\wedge Fx\neq 0$, $y\wedge Fx\in I$, $F^{-1}(y\wedge Fx)\in I$, which reduces to the case already considered. The other two cases are similar.

Now we turn to the proof of Theorem 3. The case $\kappa$ regular is given by Theorem 1, so we assume henceforth that $\kappa$ is singular. Let $\langle\lambda_\alpha:\alpha<cf\kappa\rangle$ be a

strictly increasing sequence of uncountable regular cardinals with supremum $\kappa$. For each $\alpha < cf\kappa$ let $\langle A_{\alpha\beta} : \beta < 2^{\lambda_\alpha} \rangle$ be a system of non-isomorphic rigid Boolean algebras of power $\lambda_\alpha$, with the property that $|A_{\alpha\beta} \restriction x| = \lambda_\alpha$ whenever $0 \neq x \in A_{\alpha\beta}$. For each $f \in P_{\alpha < cf\kappa} 2^{\lambda_\alpha}$ let $B_f = P_{\alpha < cf\kappa} A_{\alpha f\alpha}$. Then $B_f$ is rigid (cf. [2]). Let $I_f = \{x \in B_f : \{\alpha : x_\alpha \neq 0\}$ is finite$\}$. Clearly $I_f$ is a non-principal ideal in $B_f$, so by the lemma $C_f = I_f \cup -I_f$ is rigid. Clearly $|C_f| = \kappa$. Note that $|P_{\alpha < cf\kappa} 2^{\lambda_\alpha}| = 2^\kappa$. Thus it suffices to show that $C_f$ is not isomorphic to $C_g$ whenever $f \neq g$. Suppose, to the contrary, that $f\alpha \neq g\alpha$ but that $l$ is an isomorphism of $C_f$ onto $C_g$. Let $x$ be the element of $C_f$ which is 1 at $\alpha$ and 0 otherwise. Now $x$ is the largest element of $C_f$ such that $0 \neq y \leq x$ implies $|C_f \restriction y| = \lambda_\alpha$. Hence the same applies to $lx$. Thus $(lx)_\alpha = 1$, while $(lx)_\beta = 0$ for $\beta \neq \alpha$. So $l$ induces an isomorphism from $A_{\alpha f\alpha}$ onto $A_{\alpha g\alpha}$, contradiction.

## REFERENCES


[1] P. ŠTĚPÁNEK and B. BALCAR, *Embedding theorems for Boolean algebras and consistency results on ordinal definable sets*, J. Symb. Logic 42(1977), 64–76.

[2] R. MCKENZIE and J. D. MONK, *On automorphism groups of Boolean algebras, Infinite and finite sets*, Erdös Symposium, 1973, 951–988.

[3] J. D. MONK and R. M. SOLOWAY, *On the number of complete Boolean algebras*, Alg. Universalis 2 (1972), 365–368.

[4] S. SHELAH, *Why there are many nonisomorphic models for unsuperstable theories*, Int. Congress Math. Vancouver (1974), 259–264.



*University of Colorado*
*Boulder, Colorado*
*U.S.A.*