VOLUME 3

HANDBOOK OF BOOLEAN ALGEBRAS

Edited by
J. Donald Monk
with *Robert Bonnet*

NORTH-HOLLAND

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Edited by

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1989

NORTH-HOLLAND

AMSTERDAM · NEW YORK · OXFORD · TOKYO

ISBN for this volume 0 444 87153 5
ISBN for the set 0 444 87291 4

Publishers:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. BOX 103
1000 AC AMSTERDAM
THE NETHERLANDS

Sole distributors for the U.S.A. and Canada:
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
655 AVENUE OF THE AMERICAS
NEW YORK, NY 10010
U.S.A.

Library of Congress Cataloging-in-Publication Data

Handbook of Boolean algebras / edited by J. Donald Monk, with the cooperation of Robert Bonnet.
p. cm.
Includes bibliographies and indexes
ISBN 0-444-70261-X (Vol. 1) / 0-444-87152-7 (Vol. 2)
0-444-87153-5 (Vol. 3) / 0-444-87291-4 (Set)
1. Algebras, Boolean--Handbooks, manuals, etc. I. Monk, J. Donald (James Donald), 1930- II. Bonnet, Robert.
QA10.3.H36 1988
511.3'24--dc 19 88-25140
CIP

PRINTED IN THE NETHERLANDS

Introduction to the Handbook

The genesis of the motion of a Boolean algebra (BA) is, of course, found in the works of George Boole; but his works are now only of historical interest – cf. HAILPERIN [1981] in the bibliography (elementary part). The notions of Boolean algebra were developed by many people in the early part of this century – Schröder, Löwenheim, etc. usually working with the concrete operations union, intersection, and complementation. But the abstract notion also appeared early, in the works of Huntington and others. Despite these early developments, the modern theory of BAs can only be considered to have started in the 1930s with works of M.H. Stone and A. Tarski. Since then there has been a steady development of the subject.

The present Handbook treats those parts of the theory of Boolean algebras of most interest to pure mathematicians: the set-theoretical abstract theory and applications and relationships to measure theory, topology, and logic. Aspects of the subject *not* treated here are discussion of axiom systems for BAs, finite Boolean algebras and switching circuits, Boolean functions, Boolean matrices, Boolean algebras with operators – including cylindric algebras and related algebraic forms of logic – and the role of BAs in ring theory and in functional analysis.

The Handbook is divided into two parts (published in three volumes). The first part (Volume 1) is a completely self-contained treatment of the fundamentals of the subject, which mathematicians in various fields may find interesting and useful. Here one will find the main results on disjointness (the Erdös–Tarski theorem), free algebras (the Gaifman–Hales, Shapirovskiĭ–Shelah, and Balcar–Franěk theorems), and the basic decidability and undecidability results for the elementary theory of BAs, as well as the systematic development of the abstract theory (ultrafilters, representation, subalgebras, ideals, topological duality, free algebras, free products, measure algebras, distributivity, interval algebras, superatomic algebras, tree algebras).

The second part of the Handbook (Volumes 2 and 3) is intended to indicate most of the frontiers of research in the subject; it consists of articles which are more or less independent of each other, although most of them assume a knowledge of at least the easier portions of Part I. The second part is arranged in four sections, with two appendices and a bibliography. Section A, Arithmetical properties of BAs, has two chapters: on distributive laws and their relationships to games on BAs, and on disjoint refinements, treating extensively this elementary notion discussed in Part I. Section B, Algebraic properties of BAs, treats subalgebras, particularly the lattice of all subalgebras and the existence of complements in this lattice; cardinal functions on Boolean spaces; the number of BAs of various sorts; endomorphisms of BAs, including the existence of endo-rigid BAs; automorphisms groups; reconstruction of BAs from their automorphism groups; embeddings and automorphisms, especially for complete rigid

BAs; rigid BAs; and homogeneous BAs. Section C is devoted to special classes of BAs: superatomic algebras, mainly thin-tall and related BAs; projective BAs; and two lengthy chapters on countable BAs, with Ketonen's theorem; and on measure algebras, giving an extensive survey of this topic which is perhaps the most important subfield of the theory of BAs for most mathematicians. Section D deals with logical questions: decidable and undecidable theories of BAs in various languages; recursive BAs; Lindenbaum–Tarski algebras; and Boolean-valued models of set theory. Two appendices, on set theory and on topology, explain some more advanced notions used in some places in the Handbook. There is a chart of topological duality. Finally, there is a comprehensive Bibliography on the aspects of the theory of Boolean algebras treated in the Handbook.

Many people contributed to the Handbook by checking manuscripts for mathematical and typographical errors; in addition to several of the authors of the Handbook, the editor is indebted to the following for help of this sort: Hajnal Andréka, Aleksander Błaszczyk, Tim Carlson, Ivo Düntsch, Francisco J. Freniche, Lutz Heindorf, Istvan Németi, Stevo Todorčević, and Petr Vojtaš. Thanks are also due to the North-Holland staff, especially Leland Pierce, for their editorial work.

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Section C

SPECIAL CLASSES OF BOOLEAN ALGEBRAS

The four chapters of this Section give an in-depth coverage of some important classes of Boolean algebras discussed in Part I.

Chapter 19, Superatomic Boolean algebras, by Judy Roitman, goes into two facets of the theory of these algebras which have been widely studied: the problem of existence of superatomic algebras with a long cardinal sequence, each term of which is small (thin–tall algebras and related concepts), and the realization of automorphism groups in superatomic algebras in a natural way.

Chapter 20, Projective Boolean algebras, by Sabine Koppelberg, gives some characterizations of these algebras, Shchepin's characterization of when a projective algebra is free, and discusses the number of isomorphism types of them.

Chapter 21, Countable Boolean algebras, by R.S. Pierce, is a comprehensive treatment of this important class of algebras. It deals with several kinds of invariants which have been induced for countable algebras, the structures induced on the isomorphism types of these algebras by the product and coproduct constructions – culminating in the theorems of Dobbertin, Ketonen, and Trnková – and a discussion of the notion of primitive Boolean algebra induced by Hanf.

Chapter 22, Measure algebras, by David H. Fremlin, gives many of the known results about this important class of algebras; Maharam's theorem, liftings, a discussion of which BAs can have measures are discussed, and other topics are treated.

CHAPTER 19

Superatomic Boolean Algebras

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HANDBOOK OF BOOLEAN ALGEBRAS
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0. Introduction

A Boolean algebra is superatomic iff every subalgebra is atomic. Equivalently, a Boolean algebra is superatomic iff every quotient algebra is atomic iff its Stone space is scattered (every non-empty subset has an isolated point). These equivalences and more will be proved in Section 1.

We abbreviate "superatomic Boolean algebra" to "sBA". Notice that, by the third equivalence above, sBAs are implicit in the early work of Cantor (the Cantor–Bendixson derivative, which predated transfinite ordinals). However, the field has been largely dormant until fairly recently.

The reader who has not worked with sBAs before might appreciate some examples.

0.1. EXAMPLE. Almost disjoint sets. Let A be an almost disjoint family on some cardinal κ so that if a, b are distinct elements of A then $a \cap b$ is finite. (Note: if $\kappa > \omega$ this is not the usual set-theoretic usage.) Let B_A be the Boolean subalgebra of $\mathcal{P}(\kappa)$ generated by finite Boolean operations on A. We show that B_A is superatomic.

Let C be a subalgebra of B_A. If C has a finite subset of κ as an element or if C is the two-element algebra, we are done. So assume that C has some element of the form $a_1 \cup \cdots \cup a_n$, where each a_i differs in at most a finite set from some element of A. Let $E = \{a_i, \ldots, a_n\}$. Say that $F \subset E$ is minimal iff $\bigcup F \in C$ and for all proper subsets $H \subset F$, $\bigcup H \not\in C$. Clearly, E has a minimal subset and if F is minimal then $\bigcup F$ is an atom of C. $\square$

0.2. EXAMPLE. Well-ordered sets. Let X be a set well-ordered under $\leq$. For the topologist: Let B_X be the algebra of clopen subsets of X. Note that B_X is generated by the clopen intervals of X, hence is called the interval algebra on X. Hence, for the reader of Part I: Let $B_X = \operatorname{Intalg} X$. We show that B_X is superatomic.

Let $\tilde{X}$ be the Stone space of B_X. Then either $\tilde{X} = X$ or $\tilde{X} = X \cup \{p\}$, where we assign $p > x$ for all $x \in X$. Hence, a non-empty subset of $\tilde{X}$ with at least two elements contains a relatively isolated point, its minimum. $\square$

Interval algebras on well-ordered sets were basic to the early study of sBAs. In 1920, Mazurkiewicz and Sierpiński showed that every countable sBA is the interval algebra on some countable ordinal. The basic characterizations of sBAs were given, and the basic questions raised, in the paper by MOSTOWSKI and TARSKI [1939] on interval algebras on ordered sets.

0.3. EXAMPLE. Trees. A tree T is a partially ordered set so that $t^< = \{s \in T: s < t\}$ is well-ordered for all $t \in T$. Given a tree T we define the interval algebra B_T to be the subalgebra of $\mathcal{P}(T)$ generated by all intervals, where an interval is a linearly ordered set $I \subset T$ so that if $s, t \in I$ and $s \leq r \leq t$, then $r \in I$. Note that if J is an interval, then $B_T \mid J = \{b \in B_T: b \leq J\}$ is an interval algebra on a well-ordered set. Note that every subalgebra of B_T which is not the two-element algebra has an element which is a finite union of intervals. Using Example 0.2 it is easily seen that B_T is an sBA. □

Interval algebras on trees can be useful for quick examples on cardinal invariants. Unfortunately, if the result is non-trivial, the trees are not constructible in ZFC alone.

Other relatively simple examples will appear in Sections 1 and 2. After that, most of the chapter will be concerned with constructing difficult examples or showing that sBAs with certain properties need not exist.

Although the essential questions were asked in 1939 by Tarski and Mostowski, the study of sBAs only began to take off in the late 1960s and early 1970s. The two main categories of results are: cardinal invariants, and automorphism groups and isomorphism types. After Section 1, it is these results we will focus on. Section 1 will present preliminary work and some miscellaneous results.

In no sense is this an exhaustive survey. The author bears sole responsibility for deciding which results to include. Results will be presented in varying degree of detail, with an eye always towards illuminating the Boolean algebraic aspect of the proof. Several of the results involve quite technical set-theoretic arguments, either forcing or combinatorial constructions or both, and these will generally be abbreviated on the grounds that the reader who knows enough set theory to follow the arguments can look up the originals, and the reader with little set theory is done no service by including them.

A word about topology. While the Stone spaces of sBAs are exactly the compact scattered spaces, the study of sBAs has been largely disjoint from topology. The reason for this is that few topological concepts translate well into Boolean algebraic terminology (try "first countable", for instance). Where results have first been proved topologically I have generally translated the proof back into Boolean algebra. While something is lost (more topological properties in the conclusion or fewer in the hypothesis) something is also gained in the combinatorial clarity of the resulting picture. And this is, after all, a handbook of Boolean algebra.

Special thanks are due to Martin Weese for enlarging my perspective and sense of the history of this area in conversations in Warsaw in 1984, and to Don Monk for first getting me interested in the subject.

1. Preliminaries

Since every sBA is atomic we will assume that each sBA is a subalgebra of some $\mathscr{P}(\kappa)$, and will use set-theoretic notation rather than Boolean algebra notation (e.g. "$a \cap b$" instead of "$a \cdot b$"). We write $S(B)$ for the Stone space of a Boolean algebra B.

Let us characterize sBAs.

1.1. THEOREM. *For a Boolean algebra B the following are equivalent*:

(a) *every quotient algebra of B is atomic*,

(b) *$S(B)$ is scattered,*

(c) *$S(B)$ has no weak subspace homeomorphic to the rationals (where a weak subspace is a subset under a topology weaker than the subspace topology),*

(d) *B has no free subalgebra on infinitely many generators, and*

(e) *every subalgebra of B is atomic.*

PROOF. Recall from Part I: quotient algebras of B correspond to closed subsets of $S(B)$; atoms of B correspond to isolated points of $S(B)$. Recall from topology: every closed subspace of a space has an isolated point, iff every subspace has an isolated point. So (a) is equivalent to (b).

Let X be a compact zero-dimensional space. The following equivalences are a homework exercise in graduate topology: X is not scattered iff X has an embedded copy of the Cantor set as a weak subspace iff X has an embedded copy of the rationals as a weak subspace iff the family of clopen subsets of X has a copy of the standard basis on the rationals, which is the unique countably generated infinite non-atomic Boolean algebra, hence is the countably generated free algebra. So (b), (c), and (d) are equivalent.

Finally, any free subalgebra on infinitely many generators is not atomic, and every non-atomic algebra has a non-atomic countable subalgebra which, by uniqueness, must be the free algebra on countably many generators. So (d) is equivalent to (e). □

Theorem 1.1 was proved in Mostowski and Tarski [1939]. In the same paper they discussed the major cardinal invariants of an sBA.

1.2. DEFINITION. Let B be a Boolean algebra.

(a) $\mathrm{At}(B)$ is the set of atoms of B.

(b) If J is an ideal in B, then J^+ is the ideal in B generated by $J \cup \{x: x/J \in \mathrm{At}(B/J)\}$.

(c) $J_0(B) = \emptyset$; $J_{\alpha+1}(B) = (J_\alpha(B))^+$; if α is a limit then $J_\alpha(B) = \bigcup_{\beta<\alpha} J_\beta(B)$.

When the context is clear, we will write J_α instead of $J_\alpha(B)$.

1.3. DEFINITION. Let B be a Boolean algebra.

(a) $\mathrm{ht}(B)$ is the least α for which $J_\alpha = J_{\alpha+1}$.

(b) $\mathrm{wd}_\alpha(B) = |\mathrm{At}(B/J_\alpha)|$

(c) $\mathrm{wd}(B) = \sup\{\mathrm{wd}_\alpha(B): \alpha < \mathrm{ht}(B)\}$

(d) the cardinal sequence of B is the function Φ_B where domain $\Phi_B = \mathrm{ht}(B)$ and each $\Phi_B(\alpha) = \mathrm{wd}_\alpha(B)$.

We call $\mathrm{ht}(B)$ the Cantor–Bendixson height of B, $\mathrm{wd}(B)$ the Cantor–Bendixson width of B.

Let us codify some basic facts. (The proof is left as an exercise.)

1.4. FACT. (a) B is superatomic iff $B = J_{\mathrm{ht}(B)}$.

(b) If B is an sBA, then $\mathrm{ht}(B)$ is a successsor ordinal, and $\mathrm{At}(B/J_{\mathrm{ht}(B)-1})$ is finite.

(c) Each J_α is closed under finite union, finite intersection, and relative complement (i.e. if a, b are elements of J_α, then so is $a - b$).
(d) If $\mathrm{wd}_\alpha(B) = \lambda$, then $|B/J_\alpha| \leq 2^\lambda$.
(e) If $\mathrm{wd}_{\alpha+1}(B) = \lambda$, then $\mathrm{wd}_{\alpha+2}(B) \leq \lambda^\omega$. □

If B is an sBA, then by 1.4(a) we define the rank function on B: rank x is the least $\alpha + 1$ with $x \in J_{\alpha+1} - J_\alpha$.

The J_α's provide a rigid structure for an sBA B. For example, any automorphism of B must carry each J_α into itself. Notice that any automorphism q of B induces an automorphism q_α of B/J_α. We say that B is almost rigid iff for any automorphism q of B some q_α is the identity, for some $\alpha \lll \mathrm{ht}(B)$ (where $\alpha \lll \beta$ iff $\alpha < \beta$ and $\alpha + 1 \neq \beta$).

1.5. DEFINITION. Let B be an sBA. $N(B) = \{q \in \mathrm{aut}(B)$: for some $\alpha \lll \mathrm{ht}(B)$, q_α is the identity$\}$.

1.6. FACT. $N(B)$ is a normal subgroup of $\mathrm{Aut}(B)$.

PROOF. Let $q \in N(B)$, q_α is the identity, $\alpha \lll \mathrm{ht}(B)$, $h \in \mathrm{Aut}(B)$, rank $x > \alpha$. Then $h^{-1}(x/J_\alpha) \in B/J_\alpha$ so $qh^{-1}(x/J_\alpha) = h^{-1}(x/J_\alpha)$ so $hqh^{-1}(x/J_\alpha) = x/J_\alpha$. □

1.7. DEFINITION. $G(B) = \mathrm{Aut}(B)/N(B)$.

The overriding issues for sBAs are: What functions can be cardinal sequences for an sBA? What groups can be $G(B)$ for an sBA B? How different can two sBAs be if they have the same cardinal sequence? If they have the same $G(B)$?

2. Odds and ends

In this section we present some easy examples and quick facts.

2.1. EXAMPLE. Partition algebras. A partitioner of an almost disjoint family $\mathcal{A}$ on ω is an infinite set $a \subset \omega$ so that for all $b \in \mathcal{A}$ either $b - a$ is finite (i.e. b is almost contained in a) or $b \cap a$ is finite (i.e. b is almost disjoint from a). The set of partitioners for a given almost disjoint family is easily seen to be a Boolean algebra, called the partition algebra of the family. Partition algebras were introduced by Weese and are explored in BAUMGARTNER and WEESE [1982]. In particular, every countable Boolean algebra is a partition algebra, under CH every Boolean algebra of size no greater than 2^ω is a partition algebra, and it is consistent that no infinite complete Boolean algebra is a partition algebra (for proofs, see BAUMGARTNER and WEESE [1982]).

When is a partition algebra an sBA? Here is a partial answer.

2.1.1. THEOREM (Baumgartner and Weese). *The partition algebra of a maximal almost disjoint family on ω which has size less than 2^ω is an sBA.*

PROOF. Let $\mathscr{A}$ be maximal almost disjoint, $|\mathscr{A}| < 2^\omega$, let B be the partition algebra for $\mathscr{A}$, and suppose B is not an sBA. Then B has an independent family $\{a_n: n < \omega\}$. For each $f \in {}^\omega\omega$ let $b_n^f = a_n$ if $f(n) = 0$, $\omega - a_n$ if $f(n) = 1$. Since $\{b_n^f: n < \omega\}$ is a countable free filter on ω there is an infinite $a_f \subset \omega$, where $a_f - b_n^f$ is finite for all n. Since $\mathscr{A}$ is maximal there is $b_f \in A$ with $b_f \cap a_f$ infinite. Since each a_n is a partitioner of $\mathscr{A}$, $b_f - a_n^f$ is also finite for all n. Hence, if $f \neq g$, b_f and b_g are distinct, contradicting $|\mathscr{A}| < 2^\omega$. □

By a theorem of Hechler, the existence of maximal almost disjoint families of size less than 2^ω is independent of ZFC.

2.2. EXAMPLE. More on trees. Let us consider the cardinal invariants of an interval algebra B_T on a tree T, as in 1.3.

Suppose T has size κ, height λ, where λ is a regular cardinal, and ρ many branches of length λ. The following are easy to verify:

(1) There is a closed unbounded set of type λ in λ, A, so that if $\mathrm{ht}(x) \in A$ for $x \in T$, $b(x) = \{y: y \leq x\}$, and $B_{b(x)}$ is as in 1.2, then rank $b(x)$ in B_T equals $\mathrm{ht}(x)$ in T, which equals the order type of $b(x)$.

(2) If b is a branch of length λ, then rank $b = \lambda + 1$ in B_T.

From (1) and (2) we can immediately conclude that $\mathrm{wd}_\lambda(B_T) = \rho$, $\mathrm{ht}(B_T) = \lambda + 2$, and $|B_T| \leq 2^\kappa$. □

2.3. EXAMPLE. S spaces. A topological space $X = \{x_\alpha: \alpha < \kappa\}$ is said to be right separated of type κ iff $\{x_\alpha: \alpha < \beta\}$ is open for each $\beta < \kappa$. Clearly, every right separated space is scattered. A regular right separated space of type ω_1 with no uncountable discrete subspaces is an example of an S space (an S space is a regular hereditarily separable space which is not hereditarily Lindelöf). Were such a space X to be locally compact its one-point compactification $\tilde{X}$ would still be scattered, hence would be the Stone space of some sBA B. Since each $\mathrm{At}(B/J_\alpha)$ corresponds to a discrete subspace of $\tilde{X}$, each $\mathrm{wd}_\alpha(B) = \omega$. Since each point x_α corresponds to an element of B with rank at most $\alpha + 1$, $\mathrm{ht}(B) = \omega_1 + 1$. An sBA with such cardinal invariants is called thin–tall.

Real thin–tall sBAs exist, but this is not the way to prove it, since locally compact right-separated S spaces exist under CH (JUHÁSZ, KUNEN and RUDIN [1976]); under $\diamondsuit$ (OSTASZEWSKI [1976]); and fail to exist under MA + $\neg$CH (SZENTMIKLÓSSY [198?]).

For the proof that real thin–tall sBA's exist, see Section 3.

Let us investigate how many non-isomorphic sBAs of a given class we can have.

2.4. THEOREM. (a) *There are at least κ^+ pairwise non-isomorphic sBAs of size κ.*

(b) *There are 2^{2^ω} pairwise non-isomorphic sBAs built out of almost disjoint families as Example* 0.1 *in Section* 0.

(Part (a) was proved in DAY [1967], part (b) in Mrówka [1977].)

Sketch of proof. For (a): if $\kappa < \alpha < \kappa^+$, then there is an sBA as in 0.2 of height $\alpha + 1$, which proves what we want.

For (b): Given $B_{\mathscr{A}}$, $B_{\mathscr{A}'}$, as in 0.1, each isomorphism π from $B_{\mathscr{A}}$ to $B_{\mathscr{A}'}$ is completely determined by a function f_π from ω to ω, and f_π gives a complete description of $B_{\mathscr{A}'}$ mod finite. So each $B_{\mathscr{A}}$ can be isomorphic to at most 2^ω many $B_{\mathscr{A}'}$'s. But there are 2^{2^ω} maximal disjoint families on ω. $\square$

We now turn our attention to some peculiar set-theoretic facts about sBAs. Suppose V is a model of set theory and B is a Boolean algebra, $B \in V$. If B is not an sBA, then B contains an infinite free subalgebra; hence any extension of V which adds reals adds points to $S(B)$. However

2.5. THEOREM. *Suppose V is a model of set theory, B is an sBA in V, and V' is a model extending V. Then*
(a) $(S(B))^V = (S(B))^{V'}$.
(b) $(S(B))^V$ *is n-compact iff* $(SB))^{V'}$ *is.*

Theorem 2.5 needs some explanation. Define $(S(B))^M = S(B) \cap M$; it is isomorphic to the collection of maximal filters on B which lie in M. So (a) says that no new maximal filters are added in any extension.

As for (b), a space is n-compact iff it has a sub-basis $\mathscr{U}$ so that any open cover via sets in $\mathscr{U}$ has a subcover of size n. (Supercompact = 2-compact.)

PROOF. Note that points in $S(B)$ correspond exactly to atoms in some B/J_α, and that both B and the rank function are absolute, i.e. do not change from model to model. Hence, B/J_α is absolute, for each α. If a new ultrafilter were added to $S(B)$, a new element would be added to some B/J_α, which cannot happen. So (a) follows.

For the proof of (b): Suppose $\mathscr{U} \in V$, where any open cover of $S(B)$ via sets in $\mathscr{U}$ has a subcover of size n, and suppose $\mathscr{A}$ is a cover of $S(B)$, $\mathscr{A} \in V'$, $\mathscr{A} \subset \mathscr{U}$. By compactness there is finite $C \subset \mathscr{A}$, C a cover of $S(B)$. But by finiteness, $C \in V$, so it must have subcover of size n. $\square$

Finally, we briefly mention three uses of sBAs to give a sense of how they can unexpectedly pop up in the strangest places.

2.6. THEOREM (Juhász, Nyikos). *If $\kappa = \kappa^{<\kappa}$, B is an sBA and $|S(B)| > \kappa$, then $S(B)$ has a closed subspace of size κ.*

The proof can be found in JUHÁSZ and NYIKOS [198?], which asks in general when large spaces are guaranteed to have closed subspaces of a given small size.

2.7. THEOREM (Day). *A Boolean algebra B has a free complete extension iff B is an sBA.*

An explanation of the terminology of 2.7: a free complete extension B^* of B is a complete algebra completely generated by B in which the following diagram commutes for all complete C:

$$\begin{tikzcd} B^* \arrow[dr, dashed] & \\ B \arrow[u] \arrow[r] & C \end{tikzcd}$$

Theorem 2.7 is proved in DAY [1956].

2.8. THEOREM (van Douwen). *ω_1 is the depth of the dynamical system (ω^*, s), where s is generated by the shift operator on ω, i.e. $s(n) = n + 1$.*

Note that sBAs are not mentioned in Theorem 2.8, but are used in the proof, which can be found in VAN DOUWEN [198?].

3. Thin–tall Boolean algebras

Recall that a thin–tall sBA has width ω and height at least ω_1. In the previous section we learned that the existence of thin–tall sBAs is consistent with the axioms of set theory. In this section we show that in fact there is always a thin–tall sBA; and, better yet, there are always two non-isomorphic thin–tall sBAs of height $\omega_1 + 1$.

A brief history. The question of whether thin–tall sBAs exist was asked by Telgarsky in 1968, although never published by him. It was answered in the affirmative, and generalized, in JUHÁSZ and WEISS [1978] – they got thin–tall sBAs of arbitrary height below ω_2 which in ZFC is the best you could hope for. Weese, around 1980, then saw a way of generalizing their construction so that the width at each level could slither back and forth between ω and ω_1, even allowing cardinals up to and including 2^ω thrown in at levels whose cofinality is countable – it is Weese's result which is presented here.

Weese [1982] showed the existence of two non-isomorphic thin–tall sBAs of height $\omega_1 + 1$ assuming CH. In late 1984, Simon and Weese showed the existence of two non-isomorphic thin–tall sBAs of height $\omega_1 + 1$ with no extra set-theoretic assumptions. We present that result in this section as well.

After this section, all results presented will be consistency results.

Both of the results presented in this section use the same machinery, introduced forthwith.

3.1. DEFINITION. A *pre-algebra* on a cardinal κ is a set $J \subset \mathbb{P}(\kappa)$ which is closed under finite union, finite intersection, and relative complement as in Fact 1.4. If J is a pre-algebra on κ we define the algebra J^* to be the closure of J under complement in κ.

Note that the rank function defined on J^* defines a rank function on J, and that if J^* is an sBA then any element of $J^* - J$ has rank strictly greater than the rank of any element of J. Hence, if J^* is an sBA, then J is the union of a sequence of ideals $\{J_\alpha : \alpha < \gamma\}$ as in Definition 1.2.

If J^* is an sBA, we say that J is a pre-sBA.

Note that if $I \subset J$ are both pre-sBAs with rank functions r_I and r_J, then for each $x \in I$, $r_I(x) \leq r_J(x)$.

3.2. DEFINITION. Suppose r is the rank function on a pre-sBA J, dom $r = F$. Let E be a set of successor ordinals of the same order type as F, where ϕ is the order isomorphism between F and E. The function $\Phi \circ r$: $J \to E$ is called a pre-ranking of J. A rank function is called a ranking.

Given a pre-sBA J with a pre-ranking s derived as in 3.2 from a ranking r and a function Φ, and an ordinal $\alpha + 1 \in$ range s, we denote by J_α the ideal more properly referred to as $J_{\Phi[(\alpha+1)]-1}$.

3.3. DEFINITION. Let J be a pre-sBA with pre-ranking s, range $s = E$. A *representation sequence for* J is a collection $\{A_\alpha : \alpha + 1 \in E\}$, where each $|A_\alpha| = |\mathrm{At}(J/J_\alpha)|$ and each A_α has exactly one element from each equivalence class x/J_α, where $s(x) = \alpha + 1$. A representation sequence is said to be *normal* iff each A_α is pairwise disjoint. Notice that if $\mathrm{wd}_\alpha = \omega$ for all $\alpha \in E$, then J has a normal representation sequence. A normal representation sequence is said to be *special* iff for all $\alpha < \beta$ if $x \in A_\alpha$ and $y \in A_\beta$, then either $x \subset y$ or $x \cap y = \emptyset$. A simple induction shows that every countable pre-sBA has a special representation sequence.

The idea is to piece together countable pre-sBAs preserving their predetermined pre-rankings. This is reasonable since we are trying to construct objects of size ω_1. (The movement to 2^ω in Weese's result is an afterthought and not part of the essential construction.) For the Simon–Weese result, straightforward induction where the pre-rankings are the actual rankings suffices. For the Weese result a more complicated induction is used, and it helps to have on hand the following countably closed partial order.

3.4. DEFINITION. The partial order $\mathbb{P}$ consists of all pairs (J, s), where J is a countable pre-sBA and s is a pre-ranking of J. For $p \in \mathbb{P}$ we write $p = (J_p, s_p)$. The order is: $p \leq q$ if $J_q \subset J_p$ and $s_p \mid J_q = s_q$. Notice that if $\{p_n : n \in \omega\}$ is a descending sequence, then $(\bigcup J_n, \bigcup s_n)$ is an element of $\mathbb{P}$ below each p_n.

3.5. THEOREM (Weese). *Suppose $\alpha < \omega_2$ and f: $\alpha + 1 \to CARDS$ is a function with the following properties*:

(1) *if β has countable cofinality, then $\omega \leq f(\beta) \leq 2^\omega$*;
(2) *if* $\mathrm{cf}(\beta) = \omega_1$, *then $\omega \leq f(\beta) \leq \omega_1$*;
(3) $1 \leq f(\alpha) < \omega$.

Then f is a cardinal sequence of some sBA on ω.

3.6. COROLLARY (Juhász and Weiss). *For each $\alpha < \omega_2$ there is a thin–tall sBA of height $\alpha + 1$.*

The proof of Theorem 3.5 proceeds by a series of lemmas.

3.7. LEMMA. *Suppose $(j, s) \in \mathbb{P}$, $\alpha < \omega_2$. Then there is $(J', s') \in \mathbb{P}$, $(J', s') \leq (J, s)$ and* $|\mathrm{At}(J'/J'\alpha) - \mathrm{At}(J/J\alpha)| \neq 0$.

PROOF. Let $E = \text{range } s$ and suppose $\{A_\beta: \beta+1 \in E\}$ is a special representation sequence for J. Without loss of generality $\alpha+1 \in E$.

Case 1: $\alpha+1$ has an immediate predecessor β in E. Let $A_\beta = \{a_n: n \in \omega\}$ and let $A_\alpha = \{b_n: n \in \omega\}$. For each n let $c_n = \{k: a_k \subset b_n\}$ and let C be a countable almost disjoint family on ω extending $\{c_n: n \in \omega\}$. For $c \in C$ let $b_c = \bigcup \{a_k: k \in c\}$. Then the pre-sBA generated by $J \cup \{b_c: c \in C\}$ extends J. If $s'(x) = s(x)$ for all $x \in J$ and $s'(b_c) = \alpha+1$ for all $c \in C$, then s' extends to a pre-ranking on J' with $(J', s') < (J, s)$.

Case 2: α is the limit in E of an increasing sequence of ordinals α_n+1, $n<\omega$. Construct, for each $\sigma \in 2^{<\omega}$, a set a_σ so that if σ has length n, then $a_\sigma \in A_{\alpha_n}$, so that if the length of σ is less than the length of τ, then $a_\sigma \cap a_\tau = \emptyset$, and so that if $a \in A_a$, then for each n there is at least one σ of length n with $a_\sigma \subset a$. If h is a branch of the binary tree of length ω, define $a_h = \bigcup a_\sigma$. For each $a \in A_\alpha$ we let H_a be the set of branches h for which $a_h \subset a$. Let H be a countable set of branches avoiding each H_a. Then letting J' be the pre-sBA generated by J together with the a_h's for $h \in H$, and assigning each a_h a pre-rank of $\alpha+1$ as in Case 1, completes the proof. □

3.8. LEMMA. *Suppose $\omega_1 \le \alpha < \omega_2$ and g is a function on α, where range $g \mid \alpha + 1 = \{\omega, \omega_1\}$ and $1 \le g(\alpha) < \omega$. Then g is a cardinal sequence for an sBA.*

PROOF. For each $\gamma < \alpha$ let $A'_\gamma = \{a'_{\beta,\gamma}: \beta < g(\gamma)\}$ be a collection of names. We must assign to each $a'_{\beta,\gamma}$ a set $a_{\beta,\gamma}$ so that if $A_\gamma = \{a_{\beta,\gamma}: \beta < g(\gamma)\}$, then the collection of all A_γ's is a representation sequence for an sBA with cardinal sequence g.

Note that $\bigcup_{\gamma<\alpha} A'_\gamma$ has size ω_1, hence we may list it as a sequence of length ω_1, $\{b_\delta: \delta < \omega_1\}$. By invoking Lemma 3.7 we construct a decreasing sequence of conditions in $\mathbb{P}$, $\{p_\delta: \delta < \omega_1\}$, where if $b_\delta = a'_{\beta,\gamma}$ then p_δ constructs $a_{\beta,\gamma}$ and gives it a pre-ranking of $\beta+1$. □

Note that Corollary 3.6 follows immediately from Lemma 3.8.

Proof of Theorem 3.5. Suppose f is as in 3.5, range $f = \alpha$. Let g be a cardinal sequence for an sBA B of height $\alpha+1$, where each $g(\beta) = \omega$. We rearrange α as $\{\beta_\delta: \delta < \omega_1\}$, and construct an increasing series of sBAs, $\{B_\delta: \delta, \omega_1\}$ with cardinal sequences g_δ where

(i) $B_0 = B$,

(ii) each $g_\delta(\beta_\delta) = f(\beta_\delta)$,

(iii) $g_\delta(\beta_{\delta'}) = \omega$ for $\delta' > \delta$,

(iv) if $\delta < \delta'$ then the rank function on $B_{\delta'}$ restricted to B_δ agrees with the rank function on B_δ.

To do this we examine the proof of Lemma 3.7 a little more closely. Note that the almost disjoint families we constructed did not have to be countable to get the desired extension. Note that we did not need a special representation of the entire pre-sBA J. Instead we needed:

(1) The level α we are working on is countable.

(2) There is a countable pre-sBA $\hat{J} \subset J$ and a special representation of $\hat{J}$

contained in the given representation of J so that we may construct A_β out of elements of $\hat{J}$ as in Lemma 3.7, for each $\beta \geq \alpha$.

Thus, by imitating the proof of 3.7, we can construct the desired sequence of B_δ's, g_δ's, and Theorem 3.5 is proved. □

We have shown that a large class of functions are cardinal sequences. In Theorem 2.4(b) we showed that short fat cardinal sequences do not determine isomorphism type. What about long skinny ones?

3.9. THEOREM (Simon and Weese). *There are two non-isomorphic thin–tall sBAs of height* $\omega_1 + 1$.

The proof of Theorem 3.9 needs yet more definitions.

3.10. DEFINITION. Suppose $\{A_\alpha: \alpha < \omega_1\}$ is a normal representation sequence of a pre-sBA, where each $A_\alpha = \{a_{\alpha,n}: n < \omega\}$. A *refinement* is an almost disjoint collection $\{b_{\alpha,n}: n < \omega, \alpha < \omega_1\}$, where each $b_{\alpha,n} \subset a_{\alpha,n}$. We write $B_\alpha = \{b_{\alpha,n}: n < \omega\}$. A refinement is *sparse* if

(a) for each $n < \omega$, $\alpha \geq 1$, if $\beta > \alpha$, then $a_{\alpha,n} \cap \bigcup B_\beta$ is finite.

A refinement is unbounded if

(b) for each $\alpha \geq 1$, n, $p < \omega$, $b_{\beta,k} \cap b_{\alpha,n}$ has size greater than p for all but finitely many $(\beta, k) \in \alpha \times n + 1$.

Theorem 3.9 will be proved by showing that there is a thin–tall pre-sBA of height ω_1 with a normal representation sequence which has a sparse, unbounded refinement; and there is a thin–tall pre-sBA of height ω_1 for which no normal representation sequence has a sparse, unbounded refinement. The proof of the second part in SIMON and WEESE [1985] is topological; we have excised the topology.

3.11. DEFINITION. A pre-sBA of height α is *split* iff there are infinite disjoint sets c, d and a representation A of J so that every $a \in A$ is either equivalent mod $J_{\operatorname{rank} a}$ to a subset of c or to a subset of d, and both c and d have subsets of arbitrarily high rank below α. We say that c, d split J.

For example, given a pre-sBA J of height α, consider the pre-sBA $J + J$ constructed by taking two disjoint copies of J. $J + J$ is split and has the same cardinal sequence as J. Thus, every cardinal sequence of an sBA is also the cardinal sequence of an sBA J^* as in Definition 3.1, where J is split.

3.12. LEMMA. *A split pre-sBA of height* ω_1 *and width* ω *has no normal representation sequence with a sparse, unbounded refinement.*

PROOF. Suppose, by contradiction, we have a pre-sBA J of height ω_1 and width ω, J is split by c, d, and $\{a_{\alpha,n}: \alpha < \omega_1, n < \omega\}$ is a normal representation sequence with the sparse, unbounded refinement $\{b_{\alpha,n}: \alpha < \omega_1, n < \omega\}$. Fix $a = a_{\alpha,n}$. Since a is an atom of J^*/J_α, exactly one of $a \cap c$, $a \cap d$ gives rise to an atom of J^*/J_α; call it a^*. Without loss of generality, $a^* = a \cap c$. Then $a \cap d$ is an element of J_α.

Let $b = b_{\alpha,n}$. If $b - a^*$ is infinite, then $b - a^*$ is an infinite subset of an element of J_α, hence some infinite subset of $b - a^*$ is a subset of some $a_{\beta,m}$, where $\beta < \alpha$. But this contradicts sparseness. So $b - a^*$ is finite. Let $b^* = b \cap a^*$.

There are fixed n, m, k, j and uncountable C, D so that if $\alpha \in C$, then $b^*_{\alpha,n} \subset c$ and $|b_{\alpha,n} - c| = k$ and if $\alpha \in D$, then $b^*_{\alpha,m} \subset d$ and $|b_{\alpha,m} - d| - j$. Without loss of generality, $m \leq n$. Fix $\alpha \in C$ so that $D \cap \alpha$ is infinite and let $E = \{\beta: b^*_{\beta,m} \cap b^*_{\alpha,n} = 0\}$. $D \cap \alpha \in E$. But then, for each $\beta \in D \cap \alpha$, $|b_{\alpha,n} \cap b_{\beta,m}| \leq k + j$, which contradicts the unboundedness of the original refinement. □

3.13. LEMMA. *There is a pre-sBA of height ω_1 and width ω which has a normal representation sequence with a sparse, unbounded refinement.*

PROOF. Suppose we have an initial segment of our eventual representation sequence $A = \{a_{\beta,n}: \beta < \alpha, n < \omega\}$, and a refinement $B = \{b_{\beta,n}: \beta < \alpha, n < \omega\}$, where A is a normal representation sequence of the pre-sBA it generates, rank $a_{\beta,n} = \beta$, and B is a sparse, unbounded refinement of A. We first construct, by countable induction, a family $\{b_n: n \in \omega\}$ so that if $a \in A$, then $a \cap \bigcup b_n$ is finite, and for each finite p and all but finitely many $(\beta, k) \in \alpha \times n + 1$, $|b_n \cap b_{\beta,k}| > p$. Let $b = \bigcup b_n$, let $a' = a - \bigcup b_n$ for each $a \in A$. Then $A' = \{a': a \in A\}$ is a normal representation sequence for a pre-sBA and we may add infinitely many disjoint elements a'_n of rank α, as in Lemma 3.7, where $\bigcup_{n<\omega} a'_n = \bigcup A'$. Let $a_n = a'_n \cup b_n$ for each n. Then $A \cup \{a_n: n \in \omega\}$ is the desired refinement. □

By Lemma 3.12, some thin–tall sBA of height $\omega_1 + 1$ has no normal representation sequence with a sparse, unbounded refinement; by Lemma 3.13, some sBA with the same cardinal invariants does. So Theorem 3.9 is proved. □

4. No big sBAs

As Example 0.1 makes clear, there is always an sBA with ω many atoms of size 2^ω, since there is always a maximal disjoint family on ω with that size. What about the general case? For arbitrary infinite κ, is there always an sBA with κ many atoms and size 2^κ? In 1983 Szentmiklóssy showed the answer was no.

Some background: notice that if $2^\omega = 2^\kappa$, then the answer is yes: just take κ many disjoint copies of the sBAs on ω gotten from almost disjoint families of size 2^ω as in 0.1. Similarly, if $2^{<\kappa} = \kappa$, then the answer is yes: take the sBA generated as in 0.3 by the binary tree of height κ. By BAUMGARTNER [1976] you can show more, e.g. if $2^\omega < \min(\mathrm{cf}(2^{\omega_1}), \aleph_{\omega_1})$, then there is a tree of size ω_1 with 2^{ω_1} many branches, hence again by 1.3 the answer is yes for ω_1. So to get Szentmiklóssy's result for $\kappa = \omega_1$ we need a model where 2^ω is at least $\aleph_{\omega_1}$ and the cofinality of 2^{ω_1} is at least $\aleph_{\omega_1}$. The simplest such model is where you add $\aleph_{\omega_1}$ many Cohen reals, and that is exactly the model that Szentmiklóssy used.

4.1. THEOREM (Szentmiklóssy). *Suppose V is a model of GCH and $\mathbb{P}$ adds $\aleph_{\omega_1}$ many Cohen reals via the usual forcing conditions (finite functions from $\aleph_{\omega_1}$ into 2). In $V^{\mathbb{P}}$, if $\omega < \kappa < \aleph_\omega$, then no sBA with κ many atoms has size 2^κ. (Note that in this model $2^\kappa = 2^{\omega_1} = \aleph_{\omega_1+1}$.)*

PROOF. The proof proceeds in two parts. The first is a forcing lemma; it is assumed here, as with subsequent forcing proofs, that the reader is familiar with the basic facts about forcing. The second part is sheer Boolean algebra, using the combinatorial fact proved in the forcing lemma.

Notice that it suffices to prove the theorem for κ a successor cardinal. So let $\kappa = \lambda^+$ be an uncountable successor cardinal no bigger than $\aleph_{\omega_1}$.

4.2. LEMMA. *In $V^{\mathbb{P}}$ let A be a family of at least κ^{++} many subsets of $\mathbb{P}(\kappa)$, $|A|$ is regular. Then there is a countable subset B of A and a $C \subset A$ with $|C| = |A|$ so $\bigcup C = \bigcup B$.*

PROOF. First let us do it for $|A| = \kappa^{++}$. For each $a \in A$ we have $\mathring{a}$ a name for a in V, and since $a \subset \kappa$, each $\mathring{a}$ has support of size κ. Since GCH holds and each range $\mathring{a} \in V$, we may assume that all $\mathring{a}$ have the same range (where the range of $\mathring{a}$ is $\{\beta$: some $p \Vdash \beta \in \mathring{a}\}$). By a Δ-system argument we may assume that their supports are pairwise disjoint. Now let B be any countably infinite subset of this reduced version of A. A density argument shows that $\bigcup B = \bigcup A$.

If $|A| < \aleph_{\omega+1}$, the above argument goes through verbatim. So suppose $|A| = \aleph_{\omega_1+1}$. The density argument in the above paragraph relied only on the fact that the ranges were equal and $\{\text{supp } \mathring{a}: a \in A - B\}$ is pairwise disjoint. So we let D be any subset of A of size κ^+ whose supp $\mathring{b}$'s are pairwise disjoint (if necessary, we can absorb the root of a Δ-system into the ground model). Then for each $a \in A$, supp $\mathring{a}$ intersects at most κ many supp $\mathring{b}$ for $b \in D$. Hence, since $2^\kappa = \kappa^+$ in the ground model there is some $C \subset A$, $|C| = \aleph_{\omega_1+1}$ and some $D' \subset D$ of size κ so that if $a \in C$ then $\text{supp } \mathring{a} \subset \bigcup \{\text{supp } \mathring{b}: b \in D'\}$. But then any countably infinite $B \subset D - D'$ suffices to finish the proof. □

4.3. LEMMA. *Suppose $2^\kappa > 2^\omega$ and κ has the following property: if A is a family of subsets of κ of size 2^κ, then there is some subset C of A of size κ and some countable $B \subset A$ with $\bigcup C = \bigcup B$. Then there is no sBA of size $(2^\omega)^+$ with only κ atoms.*

PROOF. Suppose, by contradiction, there is an sBA of size $(2^\omega)^+$ with κ atoms. We may assume this sBA is a subset of $\mathbb{P}(\kappa)$ and, by possibly moving to a smaller sBA, we may assume it is some J^*, where J is a pre-sBA and if $a \in J$, then $\{b \in J: b \subset a\}$ has size 2^ω. (You do this by noticing that without loss of generality $J = J_{(2^\omega)^+}$ and either no level of J has $(2^\omega)^+$ atoms, in which case we are done; or some J_α is the first to have more than 2^ω atoms, in which case we use J_α as our pre-sBA.) Since $|J| = (2^\omega)^+$ we have some countable $B \subset J$ with $\bigcup B = \bigcup J$. Let $B' = \{a \in J$: for some $b \in B$, $a \subset b\}$. Notice that for each a in J, $a = \bigcup \{a \cap b: b \in B\}$, hence we have a one-to-one map from J into the countable subsets of B'. But $|B'| = 2^\omega$, hence $|J| = |B'|^\omega = 2^\omega < |J|$, a contradiction. □

Lemmas 4.2 and 4.3 prove Theorem 4.1. □

A historical note: Szentmiklóssy's proof was topological and his theorem states more than the topological dual of our 4.1. Since he has not yet written up his theorem, here it is for the record:

4.4. THEOREM (Szentmiklóssy). *In the model gotten by adding $\aleph_{\omega_1}$ Cohen reals to a model of GCH the following holds: if $\omega_1 \leq \kappa < 2^\omega$ no regular, scattered, locally Lindelöf space with exactly κ many isolated points has size 2^κ.*

Note the weakening of compact to regular locally Lindelöf: not every scattered regular locally Lindelöf space has a scattered compactification.

Going from our proof of 4.1 to the topological proof of 4.4 is not difficult, and is left to the topologically oriented reader.

5. More negative results

In this section we present three more negative results on cardinal invariants: Baumgartner's theorem that "there are no ω_1-thin thick sBAs" is consistent (modulo an inaccessible cardinal); and Just's theorems that "there are no thin very thick sBAs" and "there are no thin very tall sBAs" are consistent, in fact in the same model. Just's results generalize (see ROITMAN [1984]) but for simplicity's sake we give just the proofs of Just's theorems.

First, some explanation of terminology (certainly "thin thick" is a barbarism which had better have some justification).

5.1. DEFINITION. (a) An sBA is *ω_1-thin thick* iff each $\mathrm{wd}_\alpha = \omega_1$ for $\alpha < \omega_2$ but $\mathrm{wd}(\omega_2) = \omega_2$.

(b) An sBA is *thin very tall* if its width is ω and its height is $\geq \omega_2$.

(c) An sBA is *very thin thick* iff each $\mathrm{wd}_\alpha = \omega$ for countable α, but $\mathrm{wd}(\omega_1) = \omega_2$.

Baumgartner's result is the easiest to give here, since it rests on a combinatorial principle, the proof of whose consistency we cannot hope to communicate in this paper. Thus, its difficulties remain invisible.

5.2. THEOREM (Baumgartner). *If "there is an inaccessible cardinal" is consistent with the axioms of set theory, then so is "there are no ω_1-thin thick sBAs".*

To prove Theorem 5.2 we need some more definitions.

5.3. DEFINITION. Let F be a family of uncountable subsets of ω_1. We say that F is *graded almost disjoint* iff there is partition $\{B_\alpha : \alpha < \omega_1\}$ of ω_1 where

(1) for every set $A \in F$, $\{\alpha : A \cap B_\alpha \neq \emptyset\}$ is uncountable,

(2) if $A, A' \in F$ and $A \neq A'$, then $\{\alpha : A \cap A' \cap B_\alpha \neq 0\}$ is countable. That is to say, if you think of the B_α's as demarcating levels, every set in F hits uncountably many levels, but no two sets in F have more than countably many levels in common.

5.4. DEFINITION. The principle GR says: if F is graded almost disjoint and $|F| = \omega_2$, then $|\{A \cap A'\colon A, A' \text{ are distinct elements of } F\}| = \omega_2$.

We will not prove

5.5. THEOREM. *If "there is an inaccessible cardinal" is consistent with the axioms of set theory, then so is GR.*

For the curious afficiando, GR holds in the Mitchell model for no Kurepa trees.

Proof of Theorem 5.2. We show that GR implies there is no ω_1-thin thick sBA. Suppose, by contradiction, that GR holds and in fact there is an ω_1-thin thick sBA. We take a representation sequence $\{A_\alpha\colon \alpha < \omega_2 + 2\}$ and denote $J_{\alpha+1} - J_\alpha$ by B_α. For each $x \in A_{\omega_1}$ we easily see that the B_α's make F a graded almost disjoint family, where $F = \{K_x\colon x \in A_{\omega_1}\}$ and $K_x = \{y \in J_{\omega_1}\colon y \subset x\}$. But since each $K_x \cap K_y$ is determined by an element in J_{ω_1} and $|J_{\omega_1}| = \omega_1$, there are only ω_1 many such pairwise intersections, contradicting GR. □

The inaccessible cardinal is necessary in Theorem 5.2: Weese noticed that, by Example 2.2, if there is a Canadian tree (size = height = ω_1 and with ω_2 uncountable branches), then there is an ω_1-thin thick sBA, but the consistency of the non-existence of a Canadian tree is equivalent to the consistency of an inaccessible cardinal. While large cardinals are needed to destroy ω_1-thin thick sBAs, they are not needed to construct them. In ROITMAN [1984] it is proved that MA + $\neg$CH implies the existence of ω_1-thin thick sBAs.

We now sketch the proof of Just's results. These proofs involve some technical forcing lemmas which the reader is assumed to be familiar with. Details can be found in ROITMAN [1984] (where they are given in greater generality).

5.6. THEOREM (Just). *Let $\mathbb{P}$ be the usual forcing adding ω_2 many Cohen reals to a model of ZFC + CH. Then in $V^{\mathbb{P}}$ there are no thin very tall and no very thin thick sBAs.*

Sketch of proof. Let us consider the very thin thick case first, since it is the easiest. Suppose there is a thin very thick sBA B in $V^{\mathbb{P}}$. You let $\{\mathring{x}_\alpha\colon \alpha < \omega_2\}$ name elements of B generating distinct atoms in B/J_{ω_1}, and note that without loss of generality, by CH and the Δ-system lemma for countable sets, the supports of the $\mathring{x}_\alpha$'s forms a Δ-system. Since you are adding uncountably many Cohen reals and the root of the Δ-system is countable, you can absorb the root into an extension by just countably many Cohen reals, so without loss of generality the supports of the $\mathring{x}_\alpha$'s are disjoint. Since each $\mathring{x}_\alpha$ names a subset of ω and each $\mathring{x}_\alpha$ is a set of pairs (p, n), where p is in a countable partial order and $n \in \omega$, $(p, n) \in \mathring{x}_\alpha$ iff $p \Vdash n \in \mathring{x}_\alpha$, by CH we can assume the $\mathring{x}_\alpha$'s look alike, that is, given any distinct $\mathring{x}_\alpha$, $\mathring{x}_\beta$, there is a permutation Φ of $\mathbb{P}$ moving supp $\mathring{x}_\alpha$ to supp $\mathring{x}_\beta$ and leaving everything else alone, so that for any formula Ψ, $p \Vdash \Psi(\mathring{x}_\alpha, \mathring{x}_\beta)$ iff $\Phi(p) \Vdash \Phi(\Psi)(\Phi(\mathring{x}_\alpha), \Phi(\mathring{x}_\beta))$, where $\Phi(\Psi)$ is the formula changing each parameter $\mathring{a}$ in Ψ to $\Phi(\mathring{a})$. In particular since, for some fixed countable α, $\mathbb{1} \leftrightarrow \mathring{x}_0 \cap \mathring{x}_1 \in$

J_α, $\mathbb{1} \Vdash \mathring{x}_\alpha \cap \mathring{x}_\beta \in J_\alpha$ for all distinct $\alpha, \beta < \omega_2$. But then B/J_α has ω_2 many atoms, so $|J_{\alpha+1}| = \omega_2$, a contradiction.

Now suppose we have a thin very tall sBA B. This time $\{\mathring{x}_\alpha : \alpha < \omega_2\}$ names elements from distinct levels, where we may assume that $\mathring{x}_\alpha$ generates an atom over $J_{f(\alpha)}$, where $f(\alpha)$ has cofinality ω_1. Again we get disjoint supports, in fact we can assume more. Letting $\mathring{J}_\beta$ be a name for J_β we may assume that if $\alpha < \beta$, then $\operatorname{supp} \mathring{x}_\beta$ is disjoint from $\operatorname{supp} \mathring{J}_{f(\alpha)}$. Again we get isomorphic terms and automorphisms Φ as above. Notice that, for each distinct α, β, $\mathbb{1} \Vdash \mathring{x}_\beta \cap \mathring{x}_\beta \in \mathring{J}_{f(\alpha)+1}$ and $|f(\alpha)+1| = \omega_1$. So by the pigeonhole principle, we can find some $\alpha, \gamma < \omega_2$ so that $C = \{\mathring{x}_\beta : \mathring{x}_\beta \cap \mathring{x}_\alpha \in J_\gamma\}$ has since ω_2 and γ is the least such. We can assume by ccc that γ has countable cofinality.

The proof now divides into two cases. If $\gamma = f(\alpha)+1$ we have to worry about $\mathring{x}_\alpha$ being a factor in the $\mathring{x}_\beta \cap \mathring{x}_\beta$'s, i.e. $\mathring{x}_\beta \cap \mathring{x}_\alpha = (\mathring{a}_\beta \cup \mathring{x}_\beta) - \mathring{b}_\beta$. By shrinking C down using a counting argument we get a common $\mathring{a} = \mathring{a}_\beta$ and $\mathring{b} = \mathring{b}_\beta$ for every $\mathring{x}_\beta \in C$. So we have a common intersection for any two distinct elements of C, thus forcing B/J_γ to have ω_2 many atoms, which is a contradiction.

The other case is when $\gamma < f(\alpha)$. Here we can immediately imitate the proof that there is no thin very thick sBA. □

Notice that in the proof of Theorem 5.6 we did not use the full strength of thin very tall or of very thin thick. Instead what we used is that the algebras we were destroying had only countably many atoms, and at intermediate levels the widths were no greater than ω_1. Also, if we want to generalize to larger cardinals, this is possible with either Cohen reals or with their generalizations to subsets of higher cardinals. Thus, we have the generalization:

5.7. THEOREM (Roitman). *Add at least κ^{++} many Cohen subsets of some regular $\lambda \leq \kappa$ to a model of GCH and the following hold in the forcing extension*:

(a) *there is no sBA in which* $\mathrm{wd}_0 = \kappa$, $\mathrm{wd}_\alpha \leq \kappa^+$ *for* $a < \kappa^+$, *and* $\mathrm{wd}_{\kappa^+} = \kappa^{++}$;

(b) *there is no sBA in which* $\mathrm{wd}_0 = \kappa$ *and* $\mathrm{wd}_\alpha \leq \kappa^+$ *for* $\alpha < \kappa^{++}$.

For an explication of precisely what the hypothesis of Theorem 5.7 means, see ROITMAN [1974].

6. A very thin thick sBA

An unusual aspect of Just's results presented in the previous section is that they concerned the non-existence of objects which were not at the time known to consistently exist. Spurred on by Just's negative results, however, the following theorems were proved:

6.1. THEOREM (Roitman). *The existence of a very thin thick sBA is consistent with the axioms of set theory.*

The proof of Theorem 6.1 is quite technical, so we give just a sketch. For details, see ROITMAN [1985].

6.2. THEOREM (Baumgartner and Shelah). *The existence of a thin very tall sBA is consistent with the axioms of set theory.*

The proof of Theorem 6.2 is similar to, but more technical than, the proof of Theorem 6.1, so we give just a brief sketch of how to change the sketch of the proof of 6.1 into a proof of 6.2. For details, see BAUMGARTNER and SHELAH [198?].

Sketch of proof. There is a canonical partial order for forcing an sBA with particular cardinal invariants to exist: each condition predicts a finite amount of information about finitely many elements in the representing sequence. To take the very thin thick case, conditions will be triples $p = (D_p, \leq_p, I_p)$, where D_p is a finite set of ordered pairs, $\leq_p$ is a partial order on D_p, and I_p is a function associating to each unordered pair of elements from D_p a subset of D_p. The idea is that the pair $(\alpha, \gamma) \in D_p$ represents the γth set $x_{\alpha,\gamma}$ in A_α, where $\{A_\alpha: \alpha \leq \omega_1\}$ is the representing sequence of the desired pre-sBA. If $\alpha < \omega_1$, then γ is finite; otherwise, $\gamma < \omega_2$. The order $\leq_p$ is predicting the intersection of two elements: we will have $p \Vdash x_s \subset x_t$ iff $s <_p t$. I_p is predicting the intersection of two elements: we will have $\Vdash x_s \cap x_t = \bigcup \{x_r: r \in I_p(s, t)\}$. The reader familiar with forcing will have an easy time defining the partial order to make sure that you really get an sBA (i.e. checking that the ideals satisfy the properties of definition 3.1). The reader unfamiliar with forcing will gain nothing from a half-page list of technical-looking conditions.

Now there is just one problem with this sort of forcing – it collapses cardinals like crazy (see below). You are trying to build something with ω_2 atoms at the top, but suddenly ω_2 is now ω_1 and your dreams are for naught. What has to be done is to find a proper suborder which does not collapse cardinals but which is still rich enough in dense sets to give you your desired cardinal invariants at the end.

This is the general problem with the general form of this partial order to which in general the solution is not known. Here, however, it is known. We can, in fact, find a ccc suborder which does the job.

The reason cardinals are collapsed is that there is no restriction on where any $I_p(s)$ comes from if $s = \{(\omega_1, \alpha), (\omega_1, \beta)\}$ – uncountably many choices gives us an uncountable antichain. Worse, we actually collapse ω_1 to ω: for each $n < m < \omega$ let $A_{n,m}$ be a maximal antichain where, if $p \in A_{n,m}$, then $D_p = \{(\omega_1, n), (\omega_1, m)\}$. Suppose G is the generic filter and $p_{n,m} \in G \cap A_{n,m}$. Then if $f(n) = \sup\{\sup I_{p_{k,m}}(D_{p_{k,m}}): m < k \leq n\}$, f maps l cofinally into the ω_1 of the ground model, thus collapsing ω_1. So we must restrict at the beginning the field from which we may choose a given $I_p(s)$, i.e. associate to each $s = \{(\omega_1, \alpha), (\omega_1, \beta)\}$ a countable set F_s and insist that $I_p(s) \subset F_s$. It clearly suffices if F_s is just a countable ordinal, so what we are looking for is a function f where each $f(s)$ is a countable ordinal, for $s = \{(\omega_1, \alpha), (\omega_1, \beta)\}$. This, in turn, reduces to a function from $[\omega_2]^2$ (=the set of unordered pairs of elements of ω_2) into ω_1. Not any old function will do, however. What works are the functions known as new Δ functions, to wit:

6.3. DEFINITION. Given a function $f: [\omega_2]^2 \to \omega_1$ define for each finite $a, b \subset [\omega_2]^2$, $f^*(\{a, b\}) = \inf\{f(\{\alpha, \beta\}): \alpha \in a, \beta \in b\}$. Then f is a *new Δ function* iff, for every uncountable pairwise disjoint collection A of finite subsets of ω_2, $f^* | [A]^2$ is bounded in ω_1 (where $[A]^2$ is the set of unordered pairs of elements of A).

In other words, given uncountably many conditions we can find an uncountable subset so that the lower bounds for possible pairwise intersections go as high as we like. A technical argument then shows that this indeed works.

NDP is the statement that a function with the new Δ property exists. That NDP is consistent follows from work of Galvin and Kunen.

How can we change a proof of Theorem 6.1 into a proof of 6.2? We again have the pair $(\alpha, \gamma) \in D_p$ representing the γth set $x_{\alpha,\gamma}$ in A_α, only now $\alpha < \omega_2$ and $\gamma < \omega$. Now suppose $t \in [\omega_2]^2$, $t = \{\alpha, \beta\}$. We now restrict, for all $s = \{(\alpha, n), (\beta, m)\}$, $I_p(s)$ to a countable set F_t, i.e. the intersections of arbitrary elements at two fixed levels must come from a fixed set of countable levels. But it no longer suffices to make each F_t a countable ordinal, since the F_t's must climb arbitrarily high towards ω_2. A more complicated condition must therefore hold for the F_t's, as follows:

6.4. DEFINITION. Given a function f: $[\omega_2]^2 \to [\omega_2]^{\leq\omega}$, we say that f has property Δ iff each $f(t) \subset \min t$, for $t \in [\omega_2]^2$ and, for any uncountable set $D \subset [\omega_2]^{<\omega}$, there are distinct $a, b \in D$, where if $\alpha \in a - b$, $\beta \in b - a$, $\gamma \in a \cap b$, and $t = \{\alpha, \beta\}$ then

(a) if $\alpha, \beta > \gamma$, then $\gamma \in f(t)$,
(b) if $\beta > \gamma$, then $f\{\alpha, \gamma\} \subset f(t)$,
(c) if $\alpha > \gamma$, then $f\{\beta, \gamma\} \subset f(t)$.

If, given a function f having property Δ, we define each $F_t = f(t)$, then the ensuing partial order has the countable chain condition.

DP is the statement that a function having property Δ exists. In BAUMGARTNER and SHELAH [1987] it is shown how to force the consistency of DP. In a handwritten note, Veličković has derived DP from $\square$: thus, DP not only holds in L but is difficult to destroy.

7. Any countable group can be $G(B)$

Recall the definition of $G(B)$ from Definition 1.7. The purpose of this section will be to give Koppelberg's result:

7.1. THEOREM. *Assume CH. Then any countable group can be a $G(B)$ for a thin–tall sBA.*

Some history: Weese independently proved that, under CH, there are thin–tall sBAs whose $G(B)$'s have cardinality 1 (these are the semi-rigid sBAs). Roitman has shown that the conclusion of Koppelberg's theorem is independent of CH. The proof given here is an adaptation of Roitman's proof.

Proof of 7.1. Suppose we already have a thin sBA with representation sequence $\{A_\alpha : \alpha < \rho\}$. Then without loss of generality any element of $G(B)$ permutes each A_α and is the identity on none of them. Note also that, since any group acts as is a permutation on itself, each countable group can be considered a permutation group on ω in which there are infinitely many orbits (recall that the orbit for x is $\{g(x): g \in G\}$). To do this just consider G as acting on $\omega \times G$, where $g(n, h) = (n, gh)$.

Thus, we may assume that we are given a countable group G which is a permutation group on ω with infinitely many orbits. Our task is to construct a representation sequence $\{A_\alpha: \alpha < \omega_1\}$ so that the only permutations on ω which induce non-trivial permutations of each A_α are exactly those equivalent modulo $N(B)$ to some element of G.

There are two stages to the proof: Given $\{A_\beta: \beta < \alpha\}$, at successor stages α we are just concerned about defining A_α so the element of G still permute A_α and preserve Boolean operations. At limit stages, we are also concerned with ruling out other possible permutations of ω.

It will be helpful to assume as our induction hypothesis not only that each element of G permutes each earlier A_β, but also that G acting on A_β has infinitely many orbits and if $a \in A_\beta$, then the orbit of a is pairwise disjoint, and that if x, $y \in A_\beta$ and $x = g(y)$ for some $g \in G$, then $x = h(y)$ implies $h = g$ for each $g \in G$. We say that such a G acts nicely on A_β. In particular, if G acts nicely on A_β, then no non-identity $g \in G$ fixes any element of any A_β.

Let us do the successor stage. Suppose we have $\{A_\beta: \beta < \alpha\}$, where α is a countable successor ordinal, $\alpha = \gamma + 1$, $\{A_\beta: \beta < \alpha\}$ is a normal representing sequence for a countable pre-sBA, and G acts nicely on each A_β, and each $g \in G$ is an automorphism of the Boolean algebra generated by $\{A_\beta: \beta < \alpha\}$. We pick disjoint sets $a_{k,i}$ from A_γ with pairwise disjoint orbits and let $a_i = \bigcup \{a_{k,i}: k < \omega\}$. Then defining (as we must) $g(a_i) = \bigcup \{g(a_{k,i}): k < \omega\}$ for each $g \in G$, we let $A_\alpha = \{g(a_i): g \in G, i < \omega\}$. It is easy to check that A_α is pairwise disjoint, G still acts nicely on A_α and that each $g \in G$ is an automorphism of the Boolean algebra generated by $\{A_\beta: \beta \leq \alpha\}$.

Now consider a limit state α. By CH we only have ω_1 many permutations of ω to take care of, so if we can show how to take care of one at one limit stage we will be done.

Suppose f is a permutation of ω which extends to an automorphism of the Boolean algebra generated by $\{A_\beta: \beta < \alpha\}$ and f differs infinitely often from each $g \in G$ on each A_β. (If f fails to meet these criteria, then it cannot extend to an automorphism in $G(B)$ differing, mod $N(B)$, from each $g \in G$ and therefore we need not worry about it.)

Since α is a limit and f is not the identity on any A_β/I_β, by the construction at successor stages $\{x \in \omega: f(x) \neq x\}$ is not covered by finitely many orbits at any level, and is infinitely often moved outside elements of a given level, i.e.

(*) for all $\beta < \alpha$ there is a sequence $\{a_{i,\beta}: i < \omega\} \subset A_\beta$ of disjoint sets with distinct orbits so that, for each i, there is some $x \in a_i$ with $f(x) \not\in a_{i,\beta}$.

Given such a sequence, where do such x go? There are two cases:

Case a. For all but finitely many i, if $x \in a_{i,\beta}$, then $f(x) \in G(a_{i,\beta}) = \bigcup \{g(a_{i,\beta}): g \in G\}$.

Case b. There are infinitely many i so that, for some $x \in a_{i,\beta}$, $f(x) \not\in G(a_{i,\beta})$.

Note that Case b is the negation of Case a. Note further that, by (*), Case a implies that we may assume, without loss of generality, that for infinitely many i, $a_{i,\beta}$ is split by G where a set c is split by G iff there are $x \neq y$ in c and $g \neq h$ in G with $f(x) \in g(c)$ and $f(y) \in h(c)$.

If, for some $\gamma < \alpha$, Case a holds for all β with $\gamma < \beta < \alpha$ we say we are in Case a. Otherwise, we are in Case b: Case b holds for a sequence of β's cofinal in α.

We will construct, for each $i < \omega$, two sequences: $\{a_{k,i}: k < \omega\}$ and $\{p_{k,i}: k < \omega\}$ where

(1) the family $\{a_{k,i}: i < \omega, k < \omega\}$ is pairwise disjoint with pairwise disjoint orbits;

(2) $k < n$ implies $\mathrm{rank}(a_{k,i}) < \mathrm{rank}(a_{n,i})$ for all k, n;

(3) for each i, $\sup\{\mathrm{rank}(a_{k,i}): k < \omega\} = \alpha$;

(4) $\{p_{k,i}: k < \omega\} \subset \omega$ and if $(k, i) \neq (m, j)$, then the orbit of $p_{k,i}$ is disjoint from the orbit of $p_{m,j}$;

(5) each $p_{k,i} \in a_{k,i}$;

(6a) if Case a holds, then each $a_{k,i}$ is split by G;

(6b) if Case b holds, then for each $g \in G$ and each n, i, j there are $k > n$ and m with $f(p_{k,i}) \in g(a_{m,j})$.

Note that if $f(p_{k,i}) \in g(a_{m,j})$, then, for all n and all $j' \neq j$, $f(p_{k,i}) \notin g(a_{n,j'})$.

Using countable induction, conditions (1) through (5) are easy to meet; condition (6) is met via $(*)$ and Cases a and b.

Now let $A_\alpha = \{g(a_i): i < \omega\}$. Note that by (3), each element of A_α has rank α. As in the successor case, A_α is pairwise disjoint, and each $g \in G$ is a nice permutation of A_α and is an automorphism of the Boolean algebra B generated by $\{A_\beta: \beta \leq \alpha\}$. The claim is that f is not an automorphism of B. Let us prove this claim.

Suppose the claim fails. By (1) and (6a), if Case a holds, then for each i, j, if $f(a_i) = g(a_j)$, then $j = i$; and there are distinct g, h in G so $f(a_i) = g(a_i) = h(a_i)$, a contradiction. By (1) and (6b), if Case b holds, then for each i, j there are infinitely many k with $f(p_{k,i}) \notin g(a_j)$, hence $f(a_i)$ cannot equal any $g(a_j)$. $\square$

To get the independence of "every countable G can be $G(B)$ for a thin–tall sBA B" from CH, just imitate this proof over a model of $\neg CH$ with a ccc iteration of length ω_1, substituting Cohen reals for the countable induction at limit stages.

Added in proof: In 1987 Dow and Simon showed, in ZFC, that for every countable group G there are 2^{ω_1} non-isomorphic thin-tall sBAs for which G is the non-trivial automorphism group.

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Keywords: Boolean algebra, superatomic, scattered, almost rigid, partition algebras, thin tall, thin thick, group.

MOS subject classification: primary 06E05; secondary 03G05, 20B27, 54A35.

CHAPTER 20

Projective Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

Projective Boolean algebras, defined by a property which is the category-theoretic dual of injectivity, were first studied in HALMOS [1961]; see also HALMOS [1963]. They are most conveniently described as being retracts of free algebras. In particular, every projective Boolean algebra is a subalgebra of a free one and hence its dual space is dyadic, i.e. a continuous image of a generalized Cantor space. Consequently, a great deal of information on projective Boolean algebras can be found in topological papers on dyadic spaces; see ENGELKING and PELCZYŃSKI [1968], EFIMOV [1965], ENGELKING [1965], ENGELKING and KARLOWICZ [1965], EFIMOV and ENGELKING [1965], EFIMOV [1969a], [1969b].

The most basic results due to Halmos are contained in Section 1 of this chapter: countable non-trivial Boolean algebras are projective and so are free Boolean algebras; the free product of any family of projective algebras is projective; the product of any finite non-empty family of projective algebras is projective. The section includes a number of additional elementary results on projective Boolean algebras and three examples disproving several obvious questions.

A major breakthrough in the theory of projective Boolean algebras was achieved, in the setting of functional analysis and topology, by HAYDON [1974]. Haydon was able to characterize, in purely topological terms, Dugundji spaces, a class of compact Hausdorff spaces defined by a functional-analytic property in PEŁCZYŃSKI [1968]. Haydon's characterization implies that

(1) a Boolean space is Dugundji iff its dual algebra is projective;

(2) a Boolean algebra A is projective iff it is the union of a continuous sequence $(A_\alpha)_{\alpha<\rho}$ of subalgebras such that $A_0 = 2$, A_α is relatively complete in $A_{\alpha+1}$ and $A_{\alpha+1}$ is countably generated over A_α.

The main goal of Section 2 is a proof of (2) above (Corollary 2.8). We obtain two additional characterizations:

(3) A is projective iff there is a skeleton for A over 2.

See Section 2 for this notion. A topological version of (3) for arbitrary compact Hausdorff spaces was first given in SČEPIN [1976].

(4) A is projective iff it is the union of a continuous chain $(A_\alpha)_{\alpha<\rho}$ of subalgebras such that $A_0 = 2$, A_α is relatively complete in $A_{\alpha+1}$ and $A_{\alpha+1}$ is a simple extension of A_α.

Let us call a sequence $(A_\alpha)_{\alpha<\rho}$ with the properties described in (4) a standard sequence for A. The description of projective algebras by standard sequences is an almost trivial consequence of Haydon's result (2) but turns out to be quite fruitful in Section 3. On the other hand, the description via skeletons is a versatile tool allowing the construction of standard sequences for various purposes and giving isomorphism proofs via back-and-forth arguments. We shall, as a matter of fact, prove a slightly more general theorem characterizing not projective *algebras* but projective *extensions* (C, A), where C is a subalgebra of A. These are defined in Section 2; an algebra A is then projective iff A is a projective extension of its two-element subalgebra.

It turns out in Sections 3 and 4 that projective Boolean algebras are largely determined by the characters $\chi(p)$ of their ultrafilters p and by the subspaces

$$M_\kappa(A) = \{p \in \mathrm{Ult}\, A\colon \chi(p) < \kappa\}$$

of Ult A (κ an infinite cardinal). We prove in Section 3, as a direct consequence of the characterizations obtained in Section 2, a famous theorem from ŠČEPIN [1976]: a projective Boolean algebra of infinite size κ is free iff each of its ultrafilters has character κ. We then show how to compute, for an ultrafilter p of A, $\chi(p)$ from a fixed standard sequence for A. Applying this method and making good use of skeletons (resp. standard sequences), we obtain rather easy proofs of the following results on $M_\kappa(A)$ if $\kappa = |A|$ is regular and uncountable:

(5) $M_\kappa(A)$ is a proper closed subspace of Ult A;

(6) the weight of $M_\kappa(A)$ is less than κ.

It follows from (5) and Ščepin's theorem quoted above that, for $\kappa = |A|$ regular and uncountable, A has a factor isomorphic to the free Boolean algebra on κ generators. In particular, A is not rigid in this case. It should, however, be noted that (5) and (6) were proved previously under weaker assumptions than projectivity of A; see the references given in Section 3.

Section 4 contains several new results. We define, for C a projective Boolean algebra, J an ideal of C and κ a cardinal such that $\omega \leq |C| \leq \kappa$, the notion of a κ-extension of (C, J). This is basically a projective algebra A of power κ containing C as a subalgebra such that the extension $C \subseteq A$ is projective and $M_\kappa(A) \cong \mathrm{Ult}(C/J)$. Each pair (C, J) as given above has, up to isomorphism, exactly one κ-extension. It follows immediately that there are at least $2^{<\kappa}$ non-isomorphic projective Boolean algebras of power κ. Iteration of λ-extensions for $\lambda < \kappa$ gives 2^κ non-isomorphic projective Boolean algebras of power κ for singular κ and a rigid projective algebra of power κ under the assumption that $\kappa = \aleph_\omega < 2^\omega$. Moreover, if κ is regular and uncountable, then each projective algebra of power κ is the κ-extension of some pair (C, J), where $|C| < \kappa$; hence there are exactly $2^{<\kappa}$ non-isomorphic projective Boolean algebras of power κ. Imposing some fairly technical restrictions on (C, J) even gives, for κ regular and uncountable, a one-to-one correspondence between isomorphism types of projective algebras of size κ and isomorphism types of pairs (C, J), where $|C| < \kappa$.

Let us briefly give or recall some notation. $s\colon A \to \mathrm{Clop\,Ult}\, A$ is the Stone isomorphism, sometimes denoted by s_A. 2 is the two-element Boolean algebra (resp. the two-element discrete space). $C \leq A$ says that C is a subalgebra of the Boolean algebra A. If $X \subseteq A$, then $\langle X \rangle$ is the subalgebra of A generated by X. Fr κ is the free Boolean algebra on κ independent generators. $w(X)$ is the weight (the minimal cardinality of some base) of a topological space X; if $X = \mathrm{Ult}\, A$ for an infinite Boolean algebra A, then $w(X) = |A|$. For Y a subset of a topological space X, int Y is the interior and cl Y the closure of Y in X.

1. Elementary results

A Boolean space X is said to be *injective* if it has, in the category of Boolean spaces and continuous maps, the universal property defining, in Section 5 of Part I of this Handbook, injective Boolean algebras in the category of Boolean algebras and homomorphisms. That is, X is injective if, for every embedding $e: Y \to Z$ of Boolean spaces, every continuous map $f: Y \to X$ can be "extended" to a continuous map $\bar{f}: Z \to X$ such that $\bar{f} \circ e = f$. The Boolean algebras dual to injective spaces are called *projective*. Thus, A is projective if, for every epimorphism $g: C \to B$ of Boolean algebras, every homomorphism $h: A \to B$ can be "lifted" to a homomorphism $k: A \to C$ such that $g \circ k = h$ (see Fig. 20.1).

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[ur, dashed, "k"] \arrow[r, "h"'] & B \end{tikzcd}$$

Fig. 20.1

The following results, 1.1 through 1.9, with the exception of 1.5 and 1.7, are due to Halmos (HALMOS [1961], [1963]).

1.1. PROPOSITION. (a) *Every free Boolean algebra is projective.*
(b) *The trivial (one-element) Boolean algebra is not projective.*

PROOF. (a) Let h, C, B and g be given as in Fig. 20.1 and let U be a set of free generators of A. For each $u \in U$, choose a preimage c_u of $h(u)$ under g and let $k: A \to C$ be the unique homomorphism mapping $u \in U$ onto c_u. Then $g(k(x)) = h(x)$ holds for every $x \in U$, hence for every $x \in A$.

(b) In Fig. 20.1, let B be trivial, C non-trivial and h, g the obvious homomorphisms. If A is trivial, then there is no homomorphism from A into C. □

Let us recall from Section 5 of Part I of this Handbook that a Boolean algebra A is called a *retract* of an algebra B if there are homomorphisms $e: A \to B$ and $f: B \to A$ such that $f \circ e = \mathrm{id}_A$; e is then one-to-one and f onto. Dually, a topological space X is a retract of a space Y if there are continuous maps $h: X \to Y$ and $k: Y \to X$ such that $k \circ h = \mathrm{id}_X$ (Fig. 20.2).

$$A \underset{e}{\overset{f}{\leftrightarrows}} B \qquad\qquad X \underset{k}{\overset{h}{\rightleftarrows}} Y$$

Fig. 20.2

By Stone duality, A is a retract of B iff $\mathrm{Ult}\, A$ is a retract of $\mathrm{Ult}\, B$.

The equivalence of (a) and (c) below seems to be the most useful characterization available without the more sophisticated methods of Section 2. In fact, it will turn out that projective algebras are extremely close to free ones.

1.2. THEOREM. *The following are equivalent, for every Boolean algebra A:*
(a) *A is projective,*
(b) *for every algebra C and every epimorphism $g: C \to A$, there is a monomorphism $k: A \to C$ such that $g \circ k = \mathrm{id}_A$,*
(c) *A is a retract of a free algebra.*

PROOF. (a) implies (b): in Fig. 20.1, let $B = A$ and $h = \mathrm{id}_A$.

(b) implies (c) since every Boolean algebra is a quotient of a free one.

(c) implies (a): let A be a retract of a free algebra F via $e: A \to F$ and $f: F \to A$; let B, C, h and g be given as in Fig. 20.1.

$$\begin{tikzcd}
F \arrow[r, "j"] \arrow[d, shift left, "f"] & C \arrow[d, "g"] \\
A \arrow[u, shift left, "e"] \arrow[r, "h"'] \arrow[ur, dashed, "k"] & B
\end{tikzcd}$$

Fig. 20.3

By projectivity of F, there is a homomorphism $j: F \to C$ such that $g \circ j = h \circ f$. Then $k = j \circ e$ satisfies $g \circ k = h$ (see Fig. 20.3) □

1.3. COROLLARY. *Every retract of a projective Boolean algebra is projective.*

PROOF. If A' is a retract of A and A is a retract of a free algebra F, then A' is a retract of F. □

1.4. COROLLARY. *Let A be projective. Then A is a subalgebra of a free algebra and hence*

(a) *A satisfies the countable chain condition*;

(b) *no uncountable subalgebra of A is isomorphic to an interval algebra*;

(c) *no infinite subalgebra of A has the countable separation property.*

PROOF. (a) By 1.2(c), A is a subalgebra of a free algebra, say F, and F satisfies the countable chain condition – see 9.18 in Part I of this Handbook.

(b) F and hence A have no uncountable chain by 9.17 of Part I.

(c) Assume that B is an infinite subalgebra of A with the countable separation property. It follows from 12.2 in Part I and can, in fact, easily be shown, that B is uncountable. Since B is a subalgebra of the free algebra F, there is, by 9.16 of Part I, an infinite independent subset of B, say U. The subalgebra F_0 of B generated by U is atomless, hence includes a chain Q isomorphic to the rationals, and by the countable separation property again, B includes a chain isomorphic to the reals. This contradicts (b). □

The following characterization is contained in HOFFMANN [1979] where a less trivial topological statement is proved involving arbitrary compact Hausdorff spaces.

1.5. PROPOSITION. *A Boolean algebra A is projective iff there is an embedding $e: A \to F$ into a free algebra F such that for every homomorphism $g: A \to B$ into an arbitrary Boolean algebra B there is a homomorphism $\bar{g}: F \to B$ such that $\bar{g} \circ e = g$ (Fig. 20.4).*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[dr, "g"'] & F \arrow[d, dashed, "\bar{g}"] \\ & B \end{tikzcd}$$

Fig. 20.4

PROOF. If A is projective, let A be a retract of a free algebra F via e: $A \to F$, f: $F \to A$. Then for g: $A \to B$, put $\bar{g} = g \circ f$.

Conversely, assume e: $A \to F$ has the universal property of Fig. 20.4. Letting $B = A$ and $g = \mathrm{id}_A$, we see that A is a retract of F. □

An elucidating topological proof of the following result is given in HALMOS [1961], [1963].

1.6. THEOREM. *Every non-trivial at most countable Boolean algebra is projective.*

PROOF. Assume $2 \leq |A| \leq \omega$; let F be the free Boolean algebra on ω generators and f: $F \to A$ an epimorphism. By 15.10 and 15.18 of Part I of this Handbook, F is retractive, hence A is a retract of F and projective. □

We next examine several algebraic constructions under which the class of projective Boolean algebras is closed. It is certainly not closed under the formation of homomorphic images, by 1.1. Let us mention the following result; its topological dual is a consequence of Corollary 2 in ENGELKING and PEŁCZYŃSKI [1963].

1.7. PROPOSITION. *If A is projective and I a countably generated ideal of A, then A/I is projective.*

1.8. PROPOSITION. *The free product of a family $(A_i)_{i \in I}$ of Boolean algebras is projective iff every A_i is projective.*

PROOF. Each A_i is easily seen to be a retract of the free product $\oplus_{i \in I} A_i$, hence one direction follows from 1.3. Conversely, if each A_i is projective, say a retract of some free algebra F_i, then $\oplus_{i \in I} A_i$ is a retract of the free algebra $\oplus_{i \in I} F_i$. □

1.9. PROPOSITION. *The product of a family $(A_i)_{i \in I}$ of non-trivial Boolean algebras is projective iff I is finite, non-empty and each A_i is projective. In particular, if A is projective, then so is each relative algebra $A \restriction a$ of A, where $a \in A \setminus \{0\}$.*

PROOF. First assume that I is finite, say $I = \{1, \ldots, n\}$, and each A_i is projective. Let $F = \mathrm{Fr}(\kappa)$, where κ is infinite and $\kappa \geq |A_i|$ for $i \in I$. Then each A_i is a retract of F and $A_1 \times \cdots \times A_n$ is a retract of F^n, an algebra isomorphic to F by 9.14 in Part I. So $A_1 \times \cdots \times A_n$ is projective.

Conversely, assume $A = \Pi_{i\in I} A_i$ is projective. Then I is non-empty by 1.1 and finite by 1.4(c), since ${}^{I}2$ is a subalgebra of A. We show that each A_i is a retract of A, hence projective. Let f be the projection map from A onto its ith coordinate A_i. Pick, for $j \in I$, an arbitrary homomorphism e_j: $A_i \to A_j$ such that $e_i = \mathrm{id}_{A_i}$ and define e: $A_i \to A$ by $e(x) = (e_j(x))_{j\in I}$. Clearly, $f \circ e = \mathrm{id}_{A_i}$. □

The following result seems to appear first in the paper KOPPELBERG [1973] but has been found independently by several topologists.

1.10. PROPOSITION. *The weak product of a family $(A_i)_{i\in I}$ of non-trivial Boolean algebras is projective iff I is non-empty, at most countable and each A_i is projective.*

PROOF. It is enough, by 1.9, to consider an infinite I. If $A = \Pi^{w}_{i\in I} A_i$ is projective, then I is countable by 1.4(a). Each A_i is a factor, hence a retract of A and therefore projective.

Conversely, assume that $I = \omega$ and that each A_i is projective. Let F be free and f: $F \to A$ an epimorphism; we find a monomorphism e: $A \to F$ such that $f \circ e = \mathrm{id}_A$. By induction, choose pairwise disjoint elements b_i of F such that $f(b_i)$ is the element a_i of A given by $a_i(j) = 1$ if $i = j$ and 0 otherwise. Now $A \restriction a_i \cong A_i$ is projective and $f_i = f \restriction (F \restriction b_i)$ maps $F \restriction b_i$ onto $A \restriction a_i$, so let e_i: $A \restriction a_i \to F \restriction b_i$ be a monomorphism satisfying $f_i \circ e_i = \mathrm{id}_{A \restriction a_i}$. By Sikorski's extension criterion (5.5 in Part I) and since A is generated by $\bigcup_{i\in\omega} A \restriction a_i$, there is a homomorphism e: $A \to F$ extending $\bigcup_{i\in\omega} e_i$. Clearly, $f \circ e = \mathrm{id}_A$. □

A topological space X is said to have the *Bockstein separation property* if any two disjoint open subsets of X can be separated by disjoint open F_σ sets; the classical Bockstein theorem (BOCKSTEIN [1948]) states that a product space $\Pi_{i\in I} X_i$ has the Bockstein separation property if every X_i is second countable. We say that a Boolean algebra A has the Bockstein separation property if Ult A has; this is equivalent to saying that for arbitrary disjoint ideals I, J of A (i.e. satisfying $I \cap J = \{0\}$) there are countably generated disjoint ideals $I' \supseteq I$ and $J' \supseteq J$.

1.11. LEMMA. *A topological space X has the Bockstein separation property iff each regular open subset of X is F_σ.*

PROOF. Let X have the Bockstein separation property and let $U \subseteq X$ be regular open. Then $V = \mathrm{int}(X \setminus U)$ is regular open and $U \cap V = \emptyset$; moreover, if $U' \supseteq U$ and $V' \supseteq V$ are open and disjoint, then $U' = U$ and $V' = V$. Thus, U is F_σ.

Conversely, assume that every regular open subset of X is F_σ and that U and V are open in X and disjoint. Then $U' = \mathrm{int}\,\mathrm{cl}\,U$ and $V' = \mathrm{int}\,\mathrm{cl}\,V$ are regular open and disjoint, hence F_σ; moreover, $U \subseteq U'$, $V \subseteq V'$. □

1.12. THEOREM (ENGELKING [1965]). *Every projective Boolean algebra has the Bockstein separation property.*

PROOF. For free Boolean algebras, the assertion follows from the classical Bockstein theorem; see, for example, Problem 2.7.12 in ENGELKING [1977] for a sketch of proof. Now assume that A is an arbitrary projective algebra, say a retract of a free algebra F via e and f, and that I and J are disjoint ideals of A. The ideals $\bar{I}$ (resp. $\bar{J}$) generated in F by $e[I]$ (resp. $e[J]$) can be separated by disjoint ideals generated by countable subsets $\{x_n: n \in \omega\}$ (resp. $\{y_n: n \in \omega\}$) of F. Then $x_n \cdot y_m = 0$ for $n, m \in \omega$, and the ideals I' (resp. J') of A generated by $\{f(x_n): n \in \omega\}$ (resp. $\{f(y_n): n \in \omega\}$) are disjoint. Clearly, $I \subseteq I'$ and $J \subseteq J'$. $\square$

We close this subsection with three easy examples of projective algebras providing counterexamples to several popular questions. It is convenient here to explain matters topologically. For X a topological space and p a point of X, define the *character* of p in X by

$$\chi(p, X) = \min\{|B|: B \text{ a neighbourhood base for } p \text{ in } X\} ;$$

we write $\chi(p)$ if X is understood. If $X = \mathrm{Ult}\, A$ for a Boolean algebra A and $p \in X$, then

$$\chi(p) = \min\{|M|: M \subseteq p \text{ and } p \text{ is the filter generated by } M \text{ in } A\} .$$

For X compact and Hausdorff and $p \in X$, $\chi(p)$ is the minimal cardinal κ such that $\{p\}$ is the intersection of κ open subsets of X. If X is a product space $X = \Pi_{i\in I} X_i$, I is infinite and each X_i is Hausdorff and has at least two points, then for $p = (p_i)_{i\in I}$ in X,

$$\chi(p, X) = \max(|I|, \sup_{i\in I} \chi(p_i, X_i)) .$$

For example, each point of the generalized Cantor space ${}^{\kappa}2$ has character κ, if κ is infinite.

Consider two classes $\mathbb{C}$ and $\mathbb{H}$ of Boolean algebras defined as follows: let $C \in \mathbb{C}$ iff there is a family $(A_i)_{i\in I}$ of Boolean algebras such that $2 \leq |A_i| \leq \omega$ and $C \cong \bigoplus_{i\in I} A_i$; let $H \in \mathbb{H}$ if there are $n \geq 1$ and $C_1, \ldots, C_n \in \mathbb{C}$ such that $H \cong C_1 \times \cdots \times C_n$. By 1.6, 1.8 and 1.9, each algebra in $\mathbb{H}$ is projective. The question in HALMOS [1961], [1963] whether, conversely, each projective algebra is in $\mathbb{H}$ was solved in GÖRNEMANN [1972] by a complicated counterexample, and in KOPPELBERG [1973] by an easy one, the latter one having been re-discovered by several authors.

To describe the structure of algebras in $\mathbb{H}$, consider $C = \bigoplus_{i\in I} A_i \in \mathbb{C}$, where $2 \leq |A_i| \leq \omega$. We may assume that $4 \leq |A_i|$, by 11.6(b) in Part I of this Handbook. Now if $|I| < \omega$, then C is countable; if $|I| = \omega$, then $C \cong \mathrm{Fr}\, \omega$ by 5.16 in Part I, since C is atomless; if $|I| > \omega$, then $C \cong \mathrm{Fr}\, |I|$ as follows from the case $|I| = \omega$. Thus, if $H \in \mathbb{H}$, then $\mathrm{Ult}\, H$ is the disjoint union of finitely many clopen subspaces $X_1, \ldots, X_n$, where each X_i is either second countable or homeomorphic to a Cantor space ${}^{\kappa_i}2$ for some cardinal κ_i. In particular, the subset $\{p \in \mathrm{Ult}\, H: \chi(p) \leq \omega\}$ is clopen in $\mathrm{Ult}\, H$.

1.13. EXAMPLE. Let, for $n \in \omega$, $A_n = (\mathrm{Fr}\ \omega_1) \times 2$ and put $A = \bigoplus_{n\in\omega} A$. Then A is projective but not in $\mathbb{H}$, since

$$\mathrm{Ult}\ A \cong {}^{\omega}({}^{\omega_1}2 \cup \{x\}) ,$$

where ${}^{\omega_1}2 \cup \{x\}$ is the disjoint union of the Cantor space ${}^{\omega_1}2$ and an isolated point x. The point p of Ult A defined by $p(n) = x$, for each n, is the only point of Ult A having character ω, and it is not isolated.

Another question raised in HALMOS [1961], [1963] is whether every subalgebra of a free Boolean algebra is projective. A negative answer was given in ENGELKING [1965].

1.14. EXAMPLE. Let κ be an uncountable cardinal and X_1, X_2 be two disjoint copies of the Cantor space ${}^{\kappa}2$. Fix two points $x_1 \in X_1$ and $x_2 \in X_2$ and let X be the quotient space of the disjoint union $X_1 \cup X_2$ obtained by identifying x_1 and x_2. Let π: $X_1 \cup X_2 \to X$ be the natural map and put $x^* = \pi(x_1) = \pi(x_2)$.

X is a Boolean space and, by $X_1 \cup X_2 \cong {}^{\kappa}2$, Clop X is isomorphic to a subalgebra of Fr κ. But Clop X is not projective, since we will see that X does not have the Bockstein separation property. The sets $U_1 = \pi[X_1 \setminus \{x_1\}]$ and $U_2 = \pi[X_2 \setminus \{x_2\}]$ are open and disjoint in X; since $x^* \in \mathrm{cl}(U_i) \setminus U_i$ for $i = 1, 2$, they are both regular open. If U_1 and U_2 are both F_σ, then $\{x^*\}$ is a G_δ-set in X and $\chi(x^*) \leq \omega$. But it is easily seen that $\chi(x^*, X) = \chi(x_1, X_1) = \kappa$.

We shall obtain, in Sections 3 and 4, very strong structure theorems for projective Boolean algebras of cardinality κ, if κ is a regular uncountable cardinal. For example, for such κ every projective algebra of size κ has a factor isomorphic to Fr κ and hence is not rigid. These results fail to hold for singular κ, as is shown by the following example from EFIMOV [1969b].

1.15. EXAMPLE. Let κ be singular and $\lambda = \mathrm{cf}\ \kappa$, e.g. $\kappa = \sup_{\alpha<\lambda} \kappa_\alpha$, where $\kappa_\alpha < \kappa$ for $\alpha < \lambda$. For $\alpha < \lambda$, let X_α be the disjoint union space

$$X_\alpha = {}^{\kappa_\alpha}2 \cup \{q_\alpha\} ,$$

where q_α is the only isolated point of X_α; put

$$X = \prod_{\alpha<\lambda} X_\alpha .$$

Then X is an injective Boolean space of weight κ. Define, for each cardinal μ,

$$N_\mu = \{p \in X\colon \chi(p) = \mu\} .$$

Both N_κ and, for $\nu < \kappa$, $\bigcup_{\nu\leq\mu<\kappa} N_\mu$ are dense in X. Hence, there is no clopen subset U of X such that $\chi(p) = \chi(p')$ for all $p, p' \in U$, and X has no clopen subset homeomorphic to a Cantor space.

2. Characterizations of projective algebras

The main goal of this section is Corollary 2.8, an internal characterization of projective Boolean algebras which is the key for the structure of these algebras.

A subalgebra C of A is *relatively complete* in A if, for each a in A, there is a largest element c of C satisfying $c \leq a$. c is then denoted by $\mathrm{pr}_C^A(a)$, the projection of a from A onto C. See Section 8 of Part I of this Handbook for a discussion of this situation. We will often prove relative completeness of C in A by displaying the function pr_C^A. Write $C \leq_{\mathrm{rc}} A$ if C is relatively complete in A.

2.1. LEMMA. (a) *If $D \leq_{\mathrm{rc}} C \leq_{\mathrm{rc}} A$, then $D \leq_{\mathrm{rc}} A$.*

(b) *If $D \leq C \leq A$ and $D \leq_{\mathrm{rc}} A$, then $D \leq_{\mathrm{rc}} C$.*

(c) *Let $(A_\alpha)_{\alpha<\lambda}$ be an increasing chain of Boolean algebras such that $A_\alpha \leq_{\mathrm{rc}} A_\beta$ for $\alpha < \beta < \lambda$. Then $A_\alpha \leq_{\mathrm{rc}} \bigcup_{\alpha<\lambda} A_\alpha$ for each $\alpha < \lambda$.*

(d) *If $C \leq_{\mathrm{rc}} A$ and A has the Bockstein separation property, then so has C.*

PROOF. (a) $D \leq_{\mathrm{rc}} A$ is exemplified by $\mathrm{pr}_D^A = \mathrm{pr}_D^C \circ \mathrm{pr}_C^A$.

(b) $\mathrm{pr}_D^C = \mathrm{pr}_D^A \restriction C$.

(c) Let $A = \bigcup_{\alpha<\lambda} A_\alpha$. Then $\mathrm{pr}_{A_\alpha}^A = \bigcup_{\alpha\leq\beta<\lambda} \mathrm{pr}_{A_\alpha}^{A_\beta}$.

(d) Let I and J be disjoint ideals of C and denote by K (resp. L) the ideals generated by I (resp. J) in A. If A has the Bockstein separation property, then there are elements $k_0 \leq k_1 \leq \cdots$ and $l_0 \leq l_1 \leq \cdots$ of A such that K (resp. L) are included in the ideal generated by $\{k_n\colon n \in \omega\}$ (resp. $\{l_n\colon n \in \omega\}$) in A and $k_n \cdot l_n = 0$. Let I' (resp. J') be the ideals generated by $\{\mathrm{pr}_C^A(k_n)\colon n \in \omega\}$ (resp. $\{\mathrm{pr}_C^A(l_n)\colon n \in \omega\}$) in C. Clearly, I' and J' are disjoint and $I \subseteq I'$, $J \subseteq J'$. □

Let ρ be any ordinal. An ascending chain $(A_\alpha)_{\alpha<\rho}$ of Boolean algebras is said to be *continuous* if $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ for each limit ordinal $\lambda < \rho$.

2.2. COROLLARY. *Let $(A_\alpha)_{\alpha<\rho}$ be a continuous chain of Boolean algebras such that $A_\alpha \leq_{\mathrm{rc}} A_{\alpha+1}$ for $\alpha + 1 < \rho$. Then $A_\alpha \leq_{\mathrm{rc}} A_\beta$ for $\alpha < \beta < \rho$.*

PROOF. Fix α and proceed by induction on β, using (a) and (c) of the preceding lemma. □

2.3. LEMMA. *Assume $C \leq_{\mathrm{rc}} A$ and $x_1, \ldots, x_n$ are arbitrary elements of A. Then $C(x_1 \ldots x_n) \leq_{\mathrm{rc}} A$.*

PROOF. Each element γ of $C(x_1 \ldots x_n)$ has the form:

$$\gamma = \sum_{\varepsilon\in E} p_\varepsilon c_\varepsilon \,,$$

where

$$E = {}^{\{1,\ldots,n\}}\{+1, -1\} \,,$$

$$p_\varepsilon = \prod_{i=1}^{n} \varepsilon(i) x_i \,,$$

and $c_\varepsilon \in C$ for $\varepsilon \in E$; cf. the remarks on normal forms in Section 4 of Part I. If $a \in A$ and $\gamma \in C(x_1 \dots x_n)$ is as above, then $\gamma \leq a$ iff $c_\varepsilon p_\varepsilon \leq a$ for each $\varepsilon \in E$ iff $c_\varepsilon \leq -p_\varepsilon + a$ for each $\varepsilon \in E$. Thus,

$$\mathrm{pr}^A_{C(x_1 \dots x_n)}(a) = \sum_{\varepsilon \in E} \mathrm{pr}^A_C(-p_\varepsilon + a) \cdot p_\varepsilon . \quad \square$$

It is convenient to have the following notation. We write $C \leq_{\mathrm{rcs}} A$ if $C \leq_{\mathrm{rc}} A$ and A is a simple extension of C, i.e. if $A = C(x)$ for some $x \in A$. Similarly, $C \leq_{\mathrm{rc}\omega} A$ denotes that $C \leq_{\mathrm{rc}} A$ and A is countably generated over C, i.e. $A = \langle C \cup X \rangle$ for some countable subset X of A.

2.4. COROLLARY. *Assume $C \leq_{\mathrm{rc}\omega} A$. Then there is a sequence $(A_n)_{n\in\omega}$ of subalgebras of A such that $A_0 = C$, $A_n \leq_{\mathrm{rcs}} A_{n+1}$ and $A = \bigcup_{n\in\omega} A_n$.*

PROOF. By 2.3 and 2.1(b). $\square$

For application in Section 4, we shall characterize not only projective algebras but a slightly more general situation. Let C be a subalgebra of A. We call A a *projective extension* of C and write $C \leq_{\mathrm{proj}} A$ if there are a free Boolean algebra F and homomorphisms $e\colon A \to C \oplus F$, $q\colon C \oplus F \to A$ such that $q \circ e = \mathrm{id}_A$ and $e \restriction C = q \restriction C = \mathrm{id}_C$ – as usual we assume C and F to be independent subalgebras of their free product $C \oplus F$ (Fig. 20.5).

$$\begin{tikzcd}
A \arrow[rr, shift left, "e"'] & & C \oplus F \arrow[ll, shift left, "q"'] \\
& C \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

Fig. 20.5

Note that this is a property of the *pair* (C, A) rather than of A and does not imply that A is projective in its own right – see 2.11. However, A is a projective extension of its two-element subalgebra 2 iff A is a retract of a free Boolean algebra, i.e. iff A is projective.

Our final device for characterizing or applying projective extensions is as follows. Let C be a subalgebra of A. A set $\mathcal{S}$ of subalgebras of A is called a *skeleton* for A over C if it satisfies the following conditions.

(S1) $C \in \mathcal{S}$;

(S2) $S \in \mathcal{S}$ implies $C \leq S \leq_{\mathrm{rc}} A$;

(S3) if $\mathcal{C} \subseteq \mathcal{S}$ is a non-empty chain under set-theoretical inclusion, then $\bigcup \mathcal{C} \in \mathcal{S}$;

(S4) for $S \in \mathcal{S}$ and X a countable subset of A, there is $S' \in \mathcal{S}$ such that $S \cup X \subseteq S'$ and $S \leq_{\mathrm{rc}\omega} S'$.

The proof of 2.7 essentially reduces to a couple of technical lemmas, the first of which is motivated by the following example of a skeleton. Let $A = C \oplus F$, where F is free over U. Then

$$\mathcal{S} = \{C \oplus \langle V \rangle : V \subseteq U\}$$

is a skeleton for A over C.

2.5. LEMMA. *Let A be a projective extension of C via F, e, q and let U be a set of free generators of F. For $V \subseteq U$ put*

$$F_V = C \oplus \langle V \rangle\,, \qquad A_V = q[F_V]\,;$$

call V closed if $e[A_V] \subseteq F_V$. Then

$$\mathcal{S} = \{A_V : V \subseteq U \text{ closed}\}$$

is a skeleton for A over C.

PROOF. (S1) Consider $V = \emptyset$. Then $F_V = C$, $A_V = C$ and $e[A_V] = F_V$. Thus, V is closed and $C = A_V \in \mathcal{S}$.

(S2) Clearly, $C \leq A_V \leq A$ for each $V \subseteq U$. Assume V is closed with the aim of showing $A_V \leq_{\mathrm{rc}} A$. Consider Fig. 20.6 where $q_0 = q \restriction F_V$ and $e_0 = e \restriction A_V$.

$$\begin{tikzcd}[column sep=large, row sep=large]
A \arrow[r, shift right, "e"'] \arrow[d, "k"', "\supseteq"] & C \oplus F \arrow[l, shift right, "q"'] \arrow[d, "h", "\supseteq"'] \\
A_V = q[F_V] \arrow[r, shift right, "e_0"'] & C \oplus \langle V \rangle = F_V \arrow[l, shift right, "q_0"']
\end{tikzcd}$$

Fig. 20.6

F_V is relatively complete in $C \oplus F$ since $C \oplus F$ is isomorphic to $F_V \oplus \langle U \backslash V \rangle$ over F_V; let $h = \mathrm{pr}^{C \oplus F}_{F_V}$. Then $k = q_0 \circ h \circ e$ maps A into A_V, and it is easily checked that $k = \mathrm{pr}^{A}_{A_V}$.

(S3) Assume that $V_i \subseteq U$ is closed for every $i \in I$ and that $\{A_{V_i} : i \in I\}$ is a non-empty chain of subalgebras of A. Put

$$V = \bigcup_{i \in I} V_i\,; \qquad B = \bigcup_{i \in I} A_{V_i}\,;$$

we show that V is closed and $B = A_V$; hence $B \in \mathcal{S}$.

Now $B = \bigcup_{i \in I} q[F_{V_i}]$; we claim that $B = q[F_V]$. Trivially, $B \subseteq q[F_V]$. To prove the converse, it suffices to show that $q[V] \subseteq B$ since B is a subalgebra of A, q is a homomorphism and $q[C] = C \subseteq B$. But

$$q[V] = \bigcup_{i \in I} q[V_i] \subseteq \bigcup_{i \in I} q[F_{V_i}] = B\,.$$

V is closed since

$$
\begin{aligned}
e[A_V] &= e[q[F_V]] \\
&= e\Big[\bigcup_{i\in I} q[F_{V_i}]\Big] \qquad\qquad \text{by } q[F_V]=B \\
&= \bigcup_{i\in I} e[q[F_{V_i}]] \\
&\subseteq \bigcup_{i\in I} F_{V_i} \\
&\subseteq F_V\,.
\end{aligned}
$$

Moreover, $B = q[F_V] = A_V$.

(S4) Assume $S = A_V$, where $V \subseteq U$ is closed, and that $X \subseteq A$ is countable. By induction define an increasing sequence of countable subsets X_n of A and an increasing sequence of countable subsets V_n of U: let $X_0 = X$. Given X_n, let $V_n \subseteq U$ be countable such that $e[X_n] \subseteq F_{V_n}$ and $V_0 \cup \cdots \cup V_{n-1} \subseteq V_n$. Then put $X_{n+1} = q[V_n] \cup X_0 \cup \cdots \cup X_n$.

Now (S4) is shown by letting $S' = A_W$, where $W = V \cup \bigcup_{n\in\omega} V_n$. Then $S \leq S'$, F_W is countably generated over F_V and hence, q being a homomorphism, S' is countably generated over S. S' includes X since

$$e[X] = e[X_0] \subseteq F_{V_0} \subseteq F_W\,,$$

hence

$$X = q[e[X]] \subseteq q[F_W] = A_W = S'\,.$$

Closedness of W follows, as in the proof of $q[F_V] \subseteq B$ in (S3), from

$$e\Big[q\Big[\bigcup_{n\in\omega} V_n\Big]\Big] = \bigcup_{n\in\omega} e[q[V_n]] \subseteq \bigcup_{n\in\omega} e[X_{n+1}] \subseteq \bigcup_{n\in\omega} F_{V_{n+1}} \subseteq F_W\,. \quad \square$$

2.6. LEMMA. *Let A be a projective extension of C via F, e, q. Assume $A \leq_{\text{rcs}} A(x)$ and $C \oplus F$ is canonically embedded into $C \oplus F \oplus 4$, where 4 is the four-element Boolean algebra. Then there are homomorphisms*

$$e'\colon A(x) \to C \oplus F \oplus 4\,, \qquad q'\colon C \oplus F \oplus 4 \to A(x)$$

such that $e \subseteq e'$, $q \subseteq q'$ and $q' \circ e' = \mathrm{id}_{A(x)}$ (Fig. 20.7).

$$
\begin{tikzcd}
A(x) \arrow[r, dashed, shift right, "e'"'] & C \oplus F \oplus 4 \arrow[l, dashed, shift right, "q'"'] \\
A \arrow[u, phantom, sloped, "\leq"] \arrow[r, shift right, "e"'] & C \oplus F \arrow[l, shift right, "q"'] \arrow[u, phantom, sloped, "\leq"] \\
& C \arrow[ul, phantom, sloped, "\supseteq"] \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}
$$

Fig. 20.7

PROOF. 4 is the free Boolean algebra over one generator, say u. Since 4 is independent from $C \oplus F$ in $C \oplus F \oplus 4$, there is exactly one homomorphism q': $C \oplus F \oplus 4 \to A(x)$ such that $q \subseteq q'$ and $q'(u) = x$. By relative completeness of A in $A(x)$, define

$$\alpha = \mathrm{pr}_A^{A(x)}(x)\,, \qquad \beta = \mathrm{pr}_A^{A(x)}(-x)\,, \qquad \gamma = -(\alpha + \beta)\,.$$

Then by Sikorski's extension criterion, there is exactly one homomorphism e': $A(x) \to C \oplus F \oplus 4$ such that $e \subseteq e'$ and

$$e'(x) = e(\gamma) \cdot u + e(\alpha)\,.$$

To prove $q' \circ e' = \mathrm{id}_{A(x)}$, it suffices to show that $q'(e'(x)) = x$. Note that $\alpha + \beta + \gamma = 1$ in A and thus, by definition of α, β, γ:

$$\begin{aligned} x &= \alpha x + \beta x + \gamma x \\ &= \alpha + 0 + \gamma x \\ &= \alpha + \gamma x\,. \end{aligned}$$

Now

$$\begin{aligned} q'(e'(x)) &= q(e(\gamma)) \cdot q'(u) + q(e(\alpha)) \\ &= \gamma \cdot x + \alpha \\ &= x\,. \quad \square \end{aligned}$$

It is now easy to obtain the central result of this section. The equivalence of (a) and (c) of 2.8 was first proved in HAYDON [1974] in terms of functional analysis. A topological analogue of the equivalence of (a) and (b) in 2.7 is contained in ŠČEPIN [1976].

2.7. THEOREM. *Let C be subalgebra of A. The following are equivalent*:

(a) *A is a projective extension of C*;

(b) *there is a skeleton for A over C*;

(c) *there are an ordinal ρ and a continuous chain $(B_\alpha)_{\alpha<\rho}$ of subalgebras of A such that $B_0 = C$, $\bigcup_{\alpha<\rho} B_\alpha = A$ and $B_\alpha \leq_{\mathrm{rc}\omega} B_{\alpha+1}$*;

(d) *there are an ordinal λ and a continuous chain $(A_\alpha)_{\alpha<\lambda}$ of subalgebras of A such that $A_0 = C$, $\bigcup_{\alpha<\lambda} A_\alpha = A$ and $A_\alpha \leq_{\mathrm{rcs}} A_{\alpha+1}$.*

Moreover, if κ, defined by

$$\kappa = \min\{|X| : X \subseteq A \text{ and } \langle C \cup X \rangle = A\}$$

is infinite, then we can assume that in (c) *and* (d), $\rho = \lambda = \kappa$.

PROOF. By 2.5, (a) implies (b). (b) implies (c) (with $\rho = \kappa$ if $\kappa \geq \omega$) by an obvious inductive construction. (c) implies (d) (with $\lambda = \kappa$ if $\rho = \kappa \geq \omega$) by 2.4.

Finally, assume that $(A_\alpha)_{\alpha<\lambda}$ is as in (d). For $\alpha<\lambda$, let F_α be free over $\{u_\nu: \nu<\alpha\}$. By induction on α, construct homomorphisms

$$e_\alpha: A_\alpha \to C \oplus F_\alpha\ , \qquad q_\alpha: C \oplus F_\alpha \to A_\alpha$$

such that $q_\alpha \circ e_\alpha = \mathrm{id}_{A_\alpha}$, $e_0 = q_0 = \mathrm{id}_C$, and if $\alpha<\beta<\lambda$, then $e_\alpha \subseteq e_\beta$ and $q_\alpha \subseteq q_\beta$; in successor steps use 2.6. Then $F = \bigcup_{\alpha<\lambda} F_\alpha$, $e = \bigcup_{\alpha<\lambda} e_\alpha$ and $q = \bigcup_{\alpha<\lambda} q_\alpha$ show that A is a projective extension of C. $\square$

2.8. COROLLARY. *The following are equivalent, for every Boolean algebra A.*
(a) *A is projective, i.e. a projective extension of* 2;
(b) *there is a skeleton for A over* 2;
(c) *there is a continuous chain $(B_\alpha)_{\alpha<\rho}$ of subalgebras of A such that $B_0 = 2$, $\bigcup_{\alpha<\rho} B_\alpha = A$ and $B_\alpha \leq_{\mathrm{rc}\omega} B_{\alpha+1}$;*
(d) *there is a continuous chain $(A_\alpha)_{\alpha<\lambda}$ of subalgebras of A such that $A_0 = 2$, $\bigcup_{\alpha<\lambda} A_\alpha = A$ and $A_\alpha \leq_{\mathrm{rcs}} A_{\alpha+1}$.*
Moreover, if A is infinite then we can assume that $\rho = \lambda = |A|$. $\square$

We know from 1.2 and Example 1.14 that embeddability of an algebra A into a free algebra F is strictly weaker than projectivity. Naturally, one could ask which assumptions on an embedding $e: A \to F$ guarantee that A is projective. Consider the following conditions on an embedding $e: A \to A'$, where A' is any Boolean algebra:
(A) A' is a projective extension of $e[A]$, i.e. there is a continuous chain $(A_\alpha)_{\alpha<\lambda}$ of subalgebras of A' such that $e[A] = A_0$, $A' = \bigcup_{\alpha<\lambda} A_\alpha$ and $A_\alpha \leq_{\mathrm{rcs}} A_{\alpha+1}$;
(B) there is an epimorphism $q: A' \to A$ such that $q \circ e = \mathrm{id}_A$;
(B′) for each homomorphism $g: A \to B$ from A into an arbitrary Boolean algebra B there is a homomorphism $\bar{g}: A' \to B$ such that $\bar{g} \circ e = g$.
(C) $e[A]$ is relatively complete in A'.
Much of the following material is implicit in ŠČEPIN [1976].

2.9. PROPOSITION. *Let $e: A \to A'$ be an embedding of Boolean algebras. Then* (A) *implies both* (B) *and* (C), *and* (B) *is equivalent to* (B′).

PROOF. Equivalence of (B) and (B′) is contained in the proof of 1.5.
(A) implies (B′): assume $(A_\alpha)_{\alpha<\lambda}$ is given as in (A) and $g: A \to B$ as in (B′). By induction, choose homomorphisms $g_\alpha: A_\alpha \to B$ such that $g_0 = g \circ e^{-1}$ and $g_\alpha \subseteq g_\beta$ for $\alpha<\beta<\lambda$; this is possible for successor steps by $A_\alpha \leq_{\mathrm{rcs}} A_{\alpha+1}$ and Sikorski's extension criterion. Let then $\bar{g} = \bigcup_{\alpha<\lambda} g_\alpha$.
Finally (A) implies (C) by 2.2 and 2.1(c). $\square$

It is not difficult to see that even for countable A, property (B) of an embedding does not imply (C) and hence does not imply (A). There are, however, two more results connecting the properties (A) through (C), 2.10 and 2.12. For the first of these, we apply Ščepin's theorem 3.4 whose proof, given in Section 3, requires only Corollary 2.8.

2.10. PROPOSITION. *For every Boolean algebra A, the following are equivalent:*
(a) *there is an embedding e from A into a free algebra F such that e satisfies* (A);
(b) *there is an embedding e' from A into a free algebra F such that e' satisfies* (B) – *i.e. A is projective.*

PROOF. (a) implies (b) by 2.9. Conversely, assume A is projective. Let F be a free algebra of power κ, where $\kappa \geq \max(\omega, |A|)$. Trivially $A \oplus F$ is a projective extension of A. By 1.8, $A \oplus F$ is projective; also each point in $\mathrm{Ult}(A \oplus F)$ has character κ. By 3.4, $A \oplus F$ is free over κ free generators, so $\mathrm{id}_A\colon A \to A \oplus F$ shows (A). □

Thus, projectivity of A is both equivalent to being a projective extension of 2 and to having a projective extension which is a free algebra. More generally, the following holds.

2.11. PROPOSITION. *Let A be a projective extension of C. Then A is projective iff C is.*

PROOF. If C is projective, then so is A, being a retract of $C \oplus F$ for some free algebra F. Conversely, assume A is projective. By the implication from (A) to (B) in 2.9, C is a retract of A, hence projective. □

2.12. PROPOSITION. *Assume that F is free, $A \leq_{\mathrm{rc}} F$ and $|A| \leq \omega_1$. Then A is projective.*

PROOF. We may assume that $|A| = \omega_1$ since countable algebras are projective anyway. Let U be a set of free generators of F. For $V \subseteq U$, let

$$F_V = \langle V \rangle\,, \qquad A_V = F_V \cap A\,,$$

and call V closed if $\mathrm{pr}_A^F[F_V] \subseteq F_V$. The following facts on closed sets are easily established:
(1) the union of any chain of closed subsets is closed;
(2) for every countable subset X of A there is a countable closed $V \subseteq U$ such that $X \subseteq A_V$;
(3) $A_V \leq_{\mathrm{rc}} A$ if V is closed.
In fact, (3) is proved by checking that $\mathrm{pr}_{A_V}^A = (\mathrm{pr}_A^F \circ \mathrm{pr}_{F_V}^F) \restriction A$. By (1) through (3), there is a continuous chain $(B_\alpha)_{\alpha<\omega_1}$ of countable subalgebras of A such that $A = \bigcup_{\alpha<\omega_1} B_\alpha$ and $B_\alpha \leq_{\mathrm{rc}} A$ for $\alpha < \omega_1$. This, by 2.8(c), shows the projectivity of A. □

It is shown in ŠČEPIN [1976] that the conclusion of 2.12 fails if $|A| = \omega_2$.

3. Characters of ultrafilters

This section applies the technique of the preceding one to show that a projective Boolean algebra can be described to a large extent by the characters of its ultrafilters. The best example here is Sčepin's theorem 3.4 claiming that if A is projective and each ultrafilter of A has character $\kappa \geq \omega$, then A is free over κ independent generators. By refining our technique we investigate, for A a projective algebra of size κ, the subspace of Ult A consisting of all points with character less than κ. It is convenient in the proof and necessary in Section 4 to have the following notation and lemmas. Some of these require lengthy verifications but are intuitively clear from the sheaf representation of a Boolean algebra over a subalgebra, given in Section 8 of Part I of this Handbook. No explicit use of this representation is, however, made in our proofs.

Let C be a subalgebra of A. We say that $q \in \text{Ult } C$ *splits* in A if there are distinct $p, p' \in \text{Ult } A$ such that $p \cap C = p' \cap C = q$.

In the following three lemmas, assume that $C \leq_{\text{rc}} A$. For $x \in A$, define

$$\text{indp}_C^A(x) = -(\text{pr}_C^A(x) + \text{pr}_C^A(-x))$$

(the independent part of x); thus $\{\text{pr}_C^A(x), \text{pr}_C^A(-x), \text{indp}_C^A(x)\}$ is a partition of unity in C. $\text{indp}_C^A(x)$ may be considered as being a measure of independence of x from C. For example, $\text{indp}_C^A(x) = 0$ iff $x \in C$ and $\text{indp}_C^A(x) = 1$ iff the subalgebras C and $\{0, 1, x, -x\}$ of A are independent. Let

$$J = \{\text{indp}_C^A(x) \colon x \in A\} .$$

3.1. LEMMA. *Let $q \in \text{Ult } C$ and $x \in A$.*

(a) *x is incompatible with q (i.e. $q \cup \{x\}$ does not have the finite intersection property) iff $\text{pr}_C^A(-x) \in q$.*

(b) *x and $-x$ are both compatible with q iff $\text{indp}_C^A(x) \in q$.*

(c) *Let $U = \{q \in \text{Ult } C\colon q$ splits in $A\}$. Then $U = \bigcup_{j \in J} s(j)$, where $s\colon C \to$ Clop Ult C is the Stone isomorphism; in particular, U is open in* Ult C.

PROOF. (a) is checked by a straightforward computation, and it clearly implies (b). For (c), note that $q \in U$ iff, for some $x \in A$, both x and $-x$ are compatible with q iff $j \in q$ for some $j \in J$. □

3.2. LEMMA. *J is an ideal of C, in fact, the ideal dual to $U \subseteq$* Ult C.

PROOF. $0 \in J$ since $0 = \text{indp}_C^A(0)$. If $\alpha \in C$ and $\alpha \leq \text{indp}_C^A(x)$ for some $x \in A$, then $\alpha = \text{indp}_C^A(z)$, where $z = \alpha \cdot x$. Finally, let $\alpha = \text{indp}_C^A(x)$ and $\alpha' = \text{indp}_C^A(x')$ with the aim of showing that $\alpha + \alpha' \in J$. It is sufficent to prove this for disjoint α and α'. In this case, $\alpha + \alpha' = \text{indp}_C^A(z)$, where $z = \alpha \cdot x + \alpha' \cdot x'$. The rest of the lemma holds since, by 3.1(c), U is the open subset of Ult C dual to J. □

3.3. LEMMA. *Let α, β, γ be pairwise disjoint elements of C such that $\alpha + \beta + \gamma = 1$ and $\alpha \in J$. Assume $x \in A$ and $\text{indp}_C^A(x) \leq \alpha$. Then there is some $z \in A$ such that* $\text{indp}_C^A(z) = \alpha$, $\text{pr}_C^A(z) = \beta$, $\text{pr}_C^A(-z) = \gamma$ and $x \in C(z)$.

PROOF. Define $\alpha' = \mathrm{indp}_C^A(x)$ and $\alpha'' = \alpha \cdot -\alpha'$. Pick $y \in A$ such that $\alpha = \mathrm{indp}_C^A(y)$; then put $z = \alpha' \cdot x + \alpha'' \cdot y + \beta$. x is generated by $C \cup \{z\}$ since

$$x = \alpha' \cdot x + \mathrm{pr}_C^A(x) = \alpha' \cdot z + \mathrm{pr}_C^A(x) . \quad \square$$

3.4. THEOREM (ŠČEPIN [1976]). *Let A be a projective Boolean algebra and $\kappa = |A| \geq \omega$; assume that $\chi(p) = \kappa$ for each $p \in \mathrm{Ult}\, A$. Then A is isomorphic to* $\mathrm{Fr}\, \kappa$.

PROOF. This is trivial for $\kappa = \omega$, since every countable atomless Boolean algebra is free. So let $\kappa \geq \omega_1$. By 2.8, fix a skeleton $\mathscr{S}$ for A over 2.

Consider any $C \in \mathscr{S}$ satisfying $|C| < \kappa$. Then $C \leq_{\mathrm{rc}} A$ and every ultrafilter q of C splits in A – otherwise the ultrafilter p of A generated by q would have character at most $|C| < \kappa$, a contradiction. Thus, in the terminology of the preceding lemmas, $U = \mathrm{Ult}\, C$ and $J = C$. It follows from 3.3 that for each $x \in A$ there is some $z \in A$ such that $x \in C(z)$ and $\mathrm{indp}_C^A(z) = 1$, i.e. z is independent from C.

Hence, by a simple chain argument we find a continuous chain $(C_\alpha)_{\alpha<\kappa}$ in $\mathscr{S}$ such that $C_0 = 2$, $\bigcup_{\alpha<\kappa} C_\alpha = A$ and each $C_{\alpha+1}$ is isomorphic, over C_α, to $C_\alpha \oplus \mathrm{Fr}\, \omega$. Thus, $A \cong \mathrm{Fr}\, \kappa$. $\square$

For C a subalgebra of A and $p \in \mathrm{Ult}\, A$, we define

$$\chi_C(p) = \min\{|X| : X \subseteq p \text{ and } p \text{ is the filter of } A \text{ generated by } (p \cap C) \cup X\}$$

(the character of p over C). If $C = 2$, we will stick to the notation $\chi(p)$ instead of $\chi_C(p)$. We are going to compute $\chi_C(p)$ if A is a projective extension of C. By Theorem 2.7, there is a continuous chain $(A_\alpha)_{\alpha<\rho}$ for some ordinal ρ such that $A_0 = C$, $\bigcup_{\alpha<\rho} A_\alpha = A$ and $A_\alpha \leq_{\mathrm{rcs}} A_{\alpha+1}$, say

$$A_{\alpha+1} = A_\alpha(x_\alpha) ,$$

where, without loss of generality, $x_\alpha \notin A_\alpha$. Let then

$$a_\alpha = \mathrm{indp}_{A_\alpha}^A(x_\alpha) ,$$

so $a_\alpha \in A_\alpha \setminus \{0\}$. We call $(A_\alpha)_{\alpha<\rho}$ a *standard representation* of A over C. The elements x_α are, of course, not uniquely determined by a given standard representation, but a moment's reflection shows that the elements a_α are. Let us call $(a_\alpha)_{\alpha+1<\rho}$ the sequence *attached* to $(A_\alpha)_{\alpha<\rho}$; for easier notation, we write $(a_\alpha)_{\alpha<\rho}$.

3.5. THEOREM. *Let A be a projective extension of C and $p \in \mathrm{Ult}\, A$. Given any standard representation $(A_\alpha)_{\alpha<\rho}$ of A over C with attached sequence $(a_\alpha)_{\alpha<\rho}$, define*

$$I_p = \{\alpha < \rho : a_\alpha \in p\} .$$

Then $\chi_C(p) = |I_p|$ *if* I_p *is empty or infinite.*

PROOF. Without loss of generality, we may assume that $x_\alpha \in p$ for all α, otherwise replacing x_α by $-x_\alpha$; note that this does not change a_α and I_p.

Clearly p is the filter of A generated by $p_0 \cup \{x_\nu : \nu + 1 < p\}$; we show by induction on α that

$$(1_\alpha) \qquad p \cap A_\alpha \text{ is generated, in } A_\alpha \text{, by } p_0 \cup \{x_\nu : \nu \in I_p \cap \alpha\} ,$$

thus proving $\chi_C(p) \leq |I_p|$. The only non-trivial point here is to show $(1_{\alpha+1})$ if (1_α) holds and $\alpha \not\in I_p$. But in this case, $a_\alpha \not\in p$; also $\mathrm{pr}^A_{A_\alpha}(-x_\alpha) \not\in p$ since otherwise $-x_\alpha \in p$. So $\mathrm{pr}^A_{A_\alpha}(x_\alpha)$ is an element of $p \cap A_\alpha$ and $\mathrm{pr}^A_{A_\alpha}(x_\alpha) \leq x_\alpha$, which shows $(1_{\alpha+1})$.

Assume for contradiction that I_p is infinite and $\chi_C(p) < |I_p|$. Since $a_\alpha \cdot x_\alpha \in p$ for each α in I_p, there are a fixed element x of p, an infinite subset $\{\alpha(n) : n \in \omega\}$ of I_p satisfying $\alpha(0) < \alpha(1) < \dots$, and elements c_n of p_0 such that

$$(2) \qquad c_n \cdot x \leq a_{\alpha(n)} \cdot x_{\alpha(n)} .$$

Consider

$$\alpha^* = \sup\{\alpha(n) : n \in \omega\} , \qquad a = -\mathrm{pr}^A_{A_{\alpha^*}}(-x)$$

(if $\alpha^* = p$, let $a = x$). So a is an element of $A_{\alpha^*} = \bigcup_{n\in\omega} A_{\alpha(n)}$, $a \in p$ since $x \leq a$, and by (2),

$$c_n \cdot a \leq a_{\alpha(n)} \cdot x_{\alpha(n)}$$

holds for each $n \in \omega$. Pick n large enough to guarantee that $a \in A_{\alpha(n)}$. It follows from $c_n \cdot a \leq x_{\alpha(n)}$ and $c_n \cdot a \in A_{\alpha(n)}$ that $c_n \cdot a$ is disjoint from $a_{\alpha(n)}$. But then $c_n \cdot a = 0$, which contradicts $c_n \cdot a \in p$. $\square$

In the remainder of this subsection, we study the subspaces

$$M_\kappa = M_\kappa(A) = \{p \in \mathrm{Ult}\, A : \chi(p) < \kappa\}$$

of Ult A, where A is projective and κ is an infinite cardinal.

3.6. THEOREM (EFIMOV [1969b] for successor cardinals). *Let A be projective and κ a regular uncountable cardinal. Then M_κ is a closed subspace of* Ult A.

PROOF. We show that Ult $A \backslash M_\kappa$ is open. So let $p \in \mathrm{Ult}\, A$ such that $\chi(p) \geq \kappa$; we find $a \in A$, $a \neq 0$, such that $s(a)$ is a clopen neighbourhood of p disjoint from M_κ. Note that $\kappa \leq \chi(p) \leq |A|$. By 2.8, fix a skeleton $\mathscr{S}$ for A over 2.

Choose a continuous sequence $(S_\beta)_{\beta<|A|}$ in $\mathscr{S}$ such that
(3) $S_0 = 2$ and $\bigcup_{\beta<|A|} S_\beta = A$;

(4) $S_{\beta+1}$ is countably generated, but not finitely generated, over S_β;
(5) if $\beta<\kappa$, then there exists $y_\beta \in S_{\beta+1}\backslash S_\beta$ such that $\operatorname{indp}^A_{S_\beta}(y_\beta)\in p$.

Given S_β, where $\beta<\kappa$, the existence of $S_{\beta+1}$ and y_β satisfying (5) is seen as follows. In the terminology of 3.1, $p\cap S_\beta$ splits in A since $\chi(p)\geq\kappa$ but $|S_\beta|<\kappa$. Thus, by 3.1(c), there is $y_\beta\in A$ such that $\operatorname{indp}^A_{S_\beta}(y_\beta)\in p$; then choose $S_{\beta+1}$ such that $y_\beta\in S_{\beta+1}$.

By refining the sequence $(S_\beta)_{\beta<|A|}$, we find a standard representation $(A_\alpha)_{\alpha<|A|}$ of A over 2 with attached sequence $(a_\alpha)_{\alpha<|A|}$ such that if $\alpha=\omega\cdot\beta$ is a limit ordinal then $A_\alpha=S_\beta$ and if $\alpha=\omega\cdot\beta<\kappa$, then $A_{\alpha+1}=A_\alpha(x_\alpha)$, where $x_\alpha=y_\beta$; so $a_\alpha=\operatorname{indp}^A_{A_\alpha}(y_\beta)\in p$ in this case. Put $A_\kappa=A$ if $\kappa=|A|$.

We may assume by $|A_\kappa|=\kappa$ that A_κ has κ as its underlying set. Thus,

$$S=\{\alpha<\kappa\colon \alpha \text{ limit and } A_\alpha \text{ has underlying set } \alpha\}$$

is closed unbounded in κ. Since $a_\alpha\in A_\alpha$ for $\alpha<|A|$, the function assigning a_α to $\alpha\in S$ is regressive. By Fodor's theorem, there is a stationary subset T of S and some $a\in A$ such that $a_\alpha=a$ for $\alpha\in T$. This element a is as required. In fact, $a\in p$ since $a_\alpha\in p$ for each limit ordinal $\alpha<\kappa$. Moreover, if $q\in s(a)$, then

$$T\subseteq\{\alpha<\kappa\colon a_\alpha=a\}\subseteq\{\alpha<|A|\colon a_\alpha\in q\}=I_q$$

and by $|T|=\kappa$ and 3.5, we have $\chi(q)\geq\kappa$. □

The conclusion of 3.6 fails for $\kappa=\omega$ as is shown by any countable Boolean algebra with infinitely many atoms. It also fails, by Example 1.15, for singular κ.

Let S be a subalgebra of A. We say that S *determines* an ultrafilter p of A if $p\cap S$ does not split in A, i.e. if p is the only ultrafilter of A extending $p\cap S$. Similarly, we say that S determines $Y\subseteq \operatorname{Ult} A$ if S determines each $p\in Y$.

The assertion that $w(M_\kappa)<\kappa$ in the following theorem was stated in EFIMOV [1969a] for A a subalgebra of a free algebra and κ a successor cardinal.

3.7. THEOREM. *Let A be projective and κ a regular uncountable cardinal. Then there is a subalgebra S of A determining M_κ such that $|S|<\kappa$. Hence, $w(M_\kappa)<\kappa$.*

PROOF. Fix a skeleton $\mathscr{S}$ for A over 2 and let $\mathscr{S}'=\{S\in\mathscr{S}\colon |S|<\kappa\}$. To prove the first assertion, assume that no S in $\mathscr{S}'$ determines M_κ; in particular, $\kappa\leq|A|$. Then for each $S\in\mathscr{S}'$ there is some $p\in M_\kappa$ not determined by S, i.e. by 3.1(b) for each $S\in\mathscr{S}'$ there are $p\in M_\kappa$ and $t\in A$ such that $\operatorname{indp}^A_S(t)\in p$. We shall reach a contradiction much as in the proof of 3.6.

There is a continuous chain $(S_\alpha)_{\alpha<|A|}$ in $\mathscr{S}$ and a sequence $(t_\alpha)_{\alpha<|A|}$ in A such that $S_0=2$, $S_{\alpha+1}$ is countably generated but not finitely generated over S_α, $\bigcup_{\alpha<|A|}S_\alpha=A$ and, for $\alpha<\kappa$, $\operatorname{indp}^A_{S_\alpha}(t_\alpha)\in p$ for some $t_\alpha\in S_{\alpha+1}$ and some $p\in M_\kappa$. By refining this chain we obtain a standard representation $(A_\alpha)_{\alpha<|A|}$ of A over 2 with attached sequence $(a_\alpha)_{\alpha<|A|}$ such that for each limit ordinal $\alpha<\kappa$, $a_\alpha\in p$ for some $p\in M_\kappa$. As in the proof of 3.6, there are, by Fodor's theorem, a subset T of κ and some $a\in A$ such that $|T|=\kappa$ and $\alpha\in T$ implies that α is limit and $a_\alpha=a$.

If $\alpha \in T$, then $s(a) \cap M_\kappa = s(a_\alpha) \cap M_\kappa \neq \emptyset$ by the above choice of $(A_\alpha)_{\alpha<\kappa}$, $(a_\alpha)_{\alpha<\kappa}$. Fix $p \in s(a) \cap M_\kappa$. Then by 3.5 and $T \subseteq I_p$, $\chi(p) \geq \kappa$. This contradicts $p \in M_\kappa$ and proves the first assertion.

The second assertion is seen as follows. Let $S \leq A$ determine M_κ and $\omega \leq |S| < \kappa$; denote by r the continuous map from Ult A onto Ult S given by $r(p) = p \cap S$. M_κ is, by 3.6, a compact subspace of Ult A; hence, the image Y of M_κ under r is a compact subspace of Ult S. Moreover, $r \upharpoonright M_\kappa$ is a continuous bijection from M_κ onto Y since S determines M_κ, hence a homeomorphism. Thus,

$$w(M_\kappa) = w(Y) \leq w(\text{Ult } S) = |S| < \kappa . \quad \square$$

3.8. COROLLARY. *Assume that A is projective and that $\kappa = |A|$ is regular and uncountable. Then there is some $a \in A \setminus \{0\}$ such that $A \upharpoonright a \cong \text{Fr } \kappa$.*

PROOF. M_κ is a closed subspace of Ult A by 3.6. It is a proper subspace of Ult A since $w(M_\kappa) < \kappa = w(\text{Ult } A)$ by 3.7. Thus, let $a \in A \setminus \{0\}$ be such that $s(a)$ is disjoint from M_κ.

$A \upharpoonright a$ is projective by 1.9, and it has cardinality κ. For $|A \upharpoonright a| \leq \kappa$ is trivial and $|A \upharpoonright a| \geq \kappa$ follows from

(6) $\chi(p, s(a)) = \chi(p, \text{Ult } A) = \kappa$ for $p \in s(a)$.

By (6) and Sčepin's theorem 3.4, $A \upharpoonright a \cong \text{Fr } \kappa$. $\square$

It is shown by Example 1.15 that the assertion of 3.8 does not hold for singular κ; every countable atomic Boolean algebra shows that it fails for $\kappa = \omega$.

Corollary 3.8 in particular shows that, for $\kappa = |A|$ regular and uncountable, there is some $p \in \text{Ult } A$ such that $\chi(p) = \kappa$. Similar statements can be proved under weaker assumptions on κ. The following theorem is included, for $C = 2$ and A a subalgebra of a free algebra, in ESENIN-VOLPIN [1949].

3.9. THEOREM. *Let A be a projective extension of C and assume that*

$$\kappa = \min\{|X| : X \subseteq A \text{ and } C \cup X \text{ generates } A\}$$

is uncountable. Then

(a) $\kappa = \sup\{\chi_C(p) : p \in \text{Ult } A\}$,

(b) $\kappa = \max\{\chi_C(p) : p \in \text{Ult } A\}$ *if* $\text{cf } \kappa > \omega$.

PROOF. Obviously, $\chi_C(p) \leq \kappa$ for each $p \in \text{Ult } A$. Let $(A_\alpha)_{\alpha<\kappa}$ be a standard representation of A over C with attached sequence $(a_\alpha)_{\alpha<\kappa}$.

We first prove (b). Since A is embeddable into a free Boolean algebra, it follows, e.g. from 10.9 in MONK [1983], that κ has a subset I such that $|I| = \kappa$ and $\{a_\alpha : \alpha \in I\}$ is independent (for regular κ, this was proved in Section 9 of Part I of this Handbook). Thus, there is some p in Ult A including $\{a_\alpha : \alpha \in I\}$, and $\chi_C(p) = \kappa$ follows from 3.5.

To prove (a), assume that κ is singular; we find, for each regular uncountable $\lambda < \kappa$, some $p \in \text{Ult } A$ such that $\chi_C(p) \geq \lambda$. To do this, pick as in the proof of (b) some $I \subseteq \lambda$ such that $|I| = \lambda$ and $\{a_\alpha : \alpha \in I\}$ is independent. Then for every $p \in \text{Ult } A$ including $\{a_\alpha : \alpha \in I\}$, we have $\chi_C(p) \geq \lambda$. $\square$

Again the conclusion of 3.9(b) fails if $\omega = \mathrm{cf}\, \kappa < \kappa$, for let $\kappa = \sup_{n\in\omega} \kappa_n$, where $\omega \le \kappa_n < \kappa_{n+1}$ and consider the projective algebra

$$A = \prod_{n\in\omega}^{\mathrm{w}} \mathrm{Fr}\, \kappa_n .$$

Then $|A| = \kappa$ but $\chi(p) < \kappa$ for each $p \in \mathrm{Ult}\, A$.

We can now slightly improve Sčepin's theorem inasmuch as we do not have to assume that $\chi(p) = |A|$ for each $p \in \mathrm{Ult}\, A$.

3.10. COROLLARY. *Let A be projective and κ an infinite cardinal.*
(a) *If $\chi(p) = \kappa$ for every $p \in \mathrm{Ult}\, A$, then $A \cong \mathrm{Fr}\, \kappa$.*
(b) *If A is homogeneous and $|A| = \kappa$, then $A \cong \mathrm{Fr}\, \kappa$.*

PROOF. (a) It follows from $\chi(p) = \kappa$ for $p \in \mathrm{Ult}\, A$ that $|A| \ge \kappa$. If $|A| > \kappa$, then application of 3.9(a) to $C = 2$ shows that $\chi(p) \ge \kappa^+$ for some $p \in \mathrm{Ult}\, A$. Thus, $|A| = \kappa$ and the assertion follows from 3.4.

(b) It is sufficient by 3.4 to prove that $\chi(p) = \kappa$ for every $p \in \mathrm{Ult}\, A$. This is trivial if $\kappa = \omega$, since each countably infinite homogeneous algebra is atomless.

Let κ be uncountable and regular. If $\chi(p) < \kappa$ for some $p \in \mathrm{Ult}\, A$, then M_κ is non-empty, hence dense in Ult A by homogeneity of A and closed by 3.6. Thus, $M_\kappa = \mathrm{Ult}\, A$, contradicting $w(M_\kappa) < \kappa$.

If κ is singular and $\chi(p) < \kappa$ for some $p \in \mathrm{Ult}\, A$, then there is some regular uncountable cardinal $\lambda < \kappa$ such that $\chi(p) < \lambda$. It follows as above that $M_\lambda = \mathrm{Ult}\, A$, contradicting $w(M_\lambda) < \lambda < \kappa$. □

4. The number of projective Boolean algebras

It is well known that, for every infinite cardinal κ, there are exactly 2^κ non-isomorphic Boolean algebras of power κ. We shall evaluate, in this section, the number of non-isomorphic projective algebras of cardinality κ. This is 2^ω for $\kappa = \omega$, since every countable Boolean algebra is projective; we will see that it is 2^κ for κ singular, but $2^{<\kappa}$ for κ uncountable and regular. The latter follows from the fact that if κ is regular and uncountable then each projective Boolean algebra of power κ is determined up to isomorphism by a pair (C, J), where C is a Boolean algebra of size less than κ and J an ideal of C.

Call (C, J) an *algebra-ideal pair* if C is a Boolean algebra and J an ideal of C. For such a pair, denote by U_J the open subset of Ult C dual to J. Thus,

$$U_J = \{q \in \mathrm{Ult}\, C\colon q \cap J \neq \emptyset\}$$

and the closed subspace $\mathrm{Ult}\, C \setminus U_J$ of Ult C is homeomorphic to $\mathrm{Ult}(C/J)$.

The following notion is what makes our proofs work. Let (C, J) be an algebra-ideal pair and κ an infinite cardinal. An algebra-ideal pair (A, K) is said to be a *κ-extension* of (C, J) if

(1) A is a projective extension of C;
(2) K is the ideal of A generated by J;
(3) if $p \in \mathrm{Ult}\, A$ and $p \cap C \notin U_J$, then p is determined by C, i.e. $\chi_C(p) = 0$;
(4) if $p \in \mathrm{Ult}\, A$ and $p \cap C \in U_J$, then $\chi_C(p) = \kappa$.

By abuse of language, A will also be called a κ-extension of (C, J). We will see in a moment that κ-extensions exist under rather mild assumptions and are uniquely determined. For $p \in \mathrm{Ult}\, A$, we sometimes write, for the sake of clarity, $\chi(p, A)$ instead of $\chi(p)$.

4.1. LEMMA. *Assume $D \leq_{\mathrm{proj}} C \leq_{\mathrm{proj}} A$ and $p \in \mathrm{Ult}\, A$ such that both $\chi_D(p \cap C)$ and $\chi_C(p)$ are either zero or infinite. Then*

$$\chi_D(p) = \chi_D(p \cap C) + \chi_C(p) .$$

In particular for $D = 2$, $\chi(p, A) = \chi(p \cap C, C) + \chi_C(p)$.

PROOF. Pick standard representations $(A_\alpha)_{\alpha<\rho}$ of C over D and $(B_\nu)_{\nu<\tau}$ of A over C. Then $(A_\alpha)_{\alpha<\rho+\tau}$ (ordinal addition), where $A_{\rho+\nu} = B_\nu$ for $\nu < \tau$, is a standard representation of A over D, and the lemma follows from 3.5. □

The following remark indicates how to find many non-isomorphic projective Boolean algebras.

4.2. REMARK. Assume A is a κ-extension of (C, J) and $|C| < \kappa$. If C is projective, then so is A by 2.11. The preceding lemma shows that for $p \in \mathrm{Ult}\, A$, $\chi(p, A) = \kappa$ if $p \cap C \in U_J$ and $\chi(p, A) = \chi(p \cap C, C) < \kappa$ if $p \cap C \not\in U_J$. Thus, $M_\kappa(A)$ is the closed subspace $r^{-1}[\mathrm{Ult}\, C \backslash U_J]$ of $\mathrm{Ult}\, A$, where r: $\mathrm{Ult}\, A \to \mathrm{Ult}\, C$ is the continuous map assigning $p \cap C$ to $p \in \mathrm{Ult}\, A$. The restriction of r to $M_\kappa(A)$ is a continuous bijection, hence a homeomorphism from $M_\kappa(A)$ onto $\mathrm{Ult}\, C \backslash U_J \cong \mathrm{Ult}(C/J)$. Thus, A codes the Boolean space $\mathrm{Ult}(C/J)$. For example, if C is free, then by choosing a suitable ideal J of C, $\mathrm{Ult}(C/J)$ may be any Boolean space of weight less than κ, as follows from the next theorem.

4.3. EXISTENCE THEOREM. *Let $\kappa \geq \omega$ and (C, J) be an algebra-ideal pair such that J is generated, as an ideal of C, by a subset of power at most κ (this holds, for example, if $|C| \leq \kappa$). Then there exists a κ-extension of (C, J).*

PROOF. Choose $J' \subseteq J$ such that $|J'| \leq \kappa$, $J' \neq \emptyset$ and J is generated by J'. Fix a sequence $(a_\alpha)_{\alpha<\kappa}$ in J' in which each $j \in J'$ occurs κ times. Then let $(A_\alpha)_{\alpha<\kappa}$ be a continuous chain of Boolean algebras such that $A_0 = C$, $A_\alpha \leq_{\mathrm{rc}} A_{\alpha+1}$ and $A_{\alpha+1} = A_\alpha(x_\alpha)$, where $\mathrm{indp}_{A_\alpha}^{A_{\alpha+1}}(x_\alpha)$ is the element $a_\alpha \in J'$ fixed above; this is possible by Exercise 12, Section 5 of Part I of this Handbook. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$ and let K be the ideal of A generated by J. Then for each $q \in U_J$ there is some $j \in J'$ such that $j \in q$, hence $\{\alpha < \kappa \colon a_\alpha \in q\}$ has power κ and, by 3.5, $\chi_C(p) = \kappa$. Thus, (A, K) is as required. □

Call a mapping f: $C \to C'$ an isomorphism of the algebra-ideal pairs (C, J) and (C', J') if f is a Boolean isomorphism from C onto C' mapping J onto J'.

4.4. UNIQUENESS THEOREM. *Let f be an isomorphism from (C, J) onto (C', J'). Assume that (A, K) is a κ-extension of (C, J) and (A', K') a κ-extension of (C', J'). Then there is an isomorphism from (A, K) onto (A', K') extending f.*

PROOF. If $k: A \to A'$ is an isomorphism extending f, then k maps K onto K' since K (resp. K') is generated by J (resp. J'). Thus, we can forget about K. We may also assume that $J \neq \{0\}$, for otherwise $A = C$ and $A' = C'$. Moreover,

$$\min\{|X|: X \subseteq A,\ C \cup X \text{ generates } A\} = \kappa\ .$$

For denote the left-hand side by κ'. By $J \neq \{0\}$, there exists some $p \in \text{Ult } A$ such that $p \cap C \in U_J$. If $\kappa' < \kappa$, then $\chi_C(p) \leq \kappa' < \kappa$, a contradiction. On the other hand, if $\kappa < \kappa'$, then 3.9(a) shows that, for some $p' \in \text{Ult } A$, $\chi_C(p') \geq \kappa^+$, a contradiction.

Assume first that $\kappa > \omega$. Fix skeletons $\mathcal{S}$ for A over C and $\mathcal{S}'$ for A' over C'. It is sufficient to prove the following:

Claim. Let $S \in \mathcal{S}$, $S' \in \mathcal{S}'$ and $g: S \to S'$ an isomorphism extending f. Assume that S (resp. S') is generated over C (resp. C') by less than κ elements. Let $x \in A$ be arbitrary. Then there are $T \in \mathcal{S}$, $T' \in \mathcal{S}'$ and an isomorphism $h: T \to T'$ such that $S \leq_\omega T$, $S' \leq_\omega T'$, $g \subseteq h$ and $x \in T$ (Fig. 20.8).

$$\begin{tikzcd}
T \arrow[r, dashed, "h"'] \arrow[d, dash, dashed] & T' \arrow[d, dash, dashed] \\
S \arrow[r, "g"'] \arrow[d, dash] & S' \arrow[d, dash] \\
(C, J) \arrow[r, "f"'] & (C', J')
\end{tikzcd}$$

Fig. 20.8

The claim allows us, by a back-and-forth argument, to construct a continuous chain of isomorphisms $f_\alpha: S_\alpha \to S'_\alpha$, where $\alpha < \kappa$, $S_\alpha \in \mathcal{S}$, $S'_\alpha \in \mathcal{S}'$, such that $\bigcup_{\alpha<\kappa} f_\alpha$ is an isomorphism from A onto A' extending f.

Subclaim. Let $C \leq S \leq_{\text{rc}} A$ and $C' \leq S' \leq_{\text{rc}} A'$ such that S (resp. S') is generated over C (resp. C') by less than κ elements. Let $g: S \to S'$ be an isomorphism extending f; let $x \in A$. Then there are $x' \in A'$ and an isomorphism g': $S(x) \to S'(x')$ such that $g \subseteq g'$ and $g'(x) = x'$.

Proof of subclaim. Let $\bar{J}$ (resp. $\overline{J'}$) be the ideal of S (resp. S') generated by J (resp. J'). Now S is generated over C by less than κ elements and A is a κ-extension of (C, J). It follows that for $p \in \text{Ult } A$, $\chi_S(p) = \kappa$ if $p \cap S \in U_{\bar{J}}$ and $\chi_S(p) = 0$ otherwise. Thus,

$$\bar{J} = \{\text{indp}_S^A(t): t \in A\}\ ,$$

$$\overline{J'} = \text{indp}_{S'}^{A'}(t'): t' \in A'\}\ .$$

(See Lemma 3.2.)

Moreover, g maps $\bar{J}$ onto $\overline{J'}$ since $f \subseteq g$. Hence, x' and g' exist by 3.3 and Sikorski's extension criterion.

Proof of claim. If $S \leq_{\mathrm{rc}} A$ and $x_1, \ldots, x_n \in A$, then $S(x_1 \ldots x_n) \leq_{\mathrm{rc}} A$ by 2.3; so the subclaim applies to $S(x_1 \ldots x_n)$ instead of S. Let $g\colon S \to S'$ and $x \in A$ be given as in the claim. By induction, construct, for $n \in \omega$: $x_n \in A$, $x'_n \in A'$, an isomorphism $g_n\colon S(x_0 \ldots x_n) \to S'(x'_0 \ldots x'_n)$ and $S_n \in \mathcal{S}$, $S'_n \in \mathcal{S}'$ such that

$$x_0 = x \,,$$

$$g \subseteq g_0 \,, \quad g_n \subseteq g_{n+1} \,, \quad g_n(x_i) = x'_i \quad \text{for } i \leq n \,,$$

$$S_0 = S \,, \quad S'_0 = S' \,, \quad S_n \leq_\omega S_{n+1} \,, \quad S'_n \leq_\omega S'_{n+1}$$

and

$$\bigcup_{n\in\omega} S_n = \bigcup_{n\in\omega} S(x_0 \ldots x_n) \,, \qquad \bigcup_{n\in\omega} S'_n = \bigcup_{n\in\omega} S'(x'_0 \ldots x'_n) \,.$$

This is possible by the subclaim and a bookkeeping device guaranteeing, by $S \leq_\omega S_n$, that $\bigcup_{n\in\omega} S_n \subseteq \bigcup_{n\in\omega} S(x_0 \ldots x_n)$. Let then $T = \bigcup_{n\in\omega} S_n$, $T' = \bigcup_{n\in\omega} S'_n$, and $h = \bigcup_{n\in\omega} g_n$, concluding the case $\kappa > \omega$.

Finally, consider the case that $\kappa = \omega$. Then, using the subclaim, construct a chain $(g_n)_{n\in\omega}$ of isomorphisms $g_n\colon C(x_0 \ldots x_n) \to C'(x'_0 \ldots x'_n)$ such that $f \subseteq g_0$ and $A = \bigcup_{n\in\omega} C(x_0 \ldots x_n)$, $A' = \bigcup_{n\in\omega} C'(x'_0 \ldots x'_n)$. $\square$

The uniqueness theorem generalizes Sčepin's theorem 3.4: let A be projective such that $\chi(p) = \kappa$ for each $p \in \mathrm{Ult}\, A$; let $A' = \mathrm{Fr}(\kappa)$. Then both A and A' are κ-extensions of the pair (C, J), where $2 = C = J$, thus $A \cong A'$.

By the preceding theorems, we shall henceforth denote the κ-extension of a pair (C, J), where $|C| \leq \kappa$, by $\mathrm{ext}_\kappa(C, J)$. If necessary, we may assume that $\mathrm{ext}_\kappa(C, J)$ is constructed from C and J as in the proof of the existence theorem.

4.5. COROLLARY. *Let κ be uncountable. Then for every Boolean space R of weight less than κ, there is a projective Boolean algebra A of power κ such that $M_\kappa(A) \cong R$. Hence, there are at least $2^{<\kappa}$ pairwise non-isomorphic projective Boolean algebras of power κ.*

PROOF. Let λ be an infinite cardinal such that $w(R) \leq \lambda < \kappa$; let $C = \mathrm{Fr}\, \lambda$ and J an ideal of C such that $\mathrm{Ult}(C/J) \cong R$. Put $A = \mathrm{ext}_\kappa(C, J)$. Then by 4.2, $M_\kappa(A) \cong R$.

For each infinite $\lambda < \kappa$ there are 2^λ pairwise non-isomorphic Boolean algebras of power λ, hence 2^λ pairwise non-homeomorphic Boolean spaces of weight λ. Each of these is coded by a projective algebra of power κ. $\square$

We can do even better for singular κ by iterating the construction of Corollary 4.5. For C a subalgebra of A, denote by r^A_C the continuous map dual to the inclusion map from C into A:

$$r^A_C\colon \mathrm{Ult}\, A \to \mathrm{Ult}\, C \,,$$

where $r^A_C(p) = p \cap C$.

4.6. LEMMA. *Let J be generated, as an ideal of C, by at most κ elements; let $A = \mathrm{ext}_\kappa(C, J)$. If $q \in U_J$ and Y is the preimage of q under r^A_C, then Y is homeomorphic to the generalized Cantor space ${}^\kappa 2$.*

PROOF. Pick a standard representation $(A_\alpha)_{\alpha<\kappa}$ of A over C with attached sequence $(a_\alpha)_{\alpha<\kappa}$. Let $X_\alpha = \mathrm{Ult}\, A_\alpha$ and, for $\beta \leq \alpha < \kappa$, $r^\alpha_\beta = r^{A_\alpha}_{A_\beta}$; then $X = \mathrm{Ult}\, A$ is the inverse limit of the inverse system $((X_\alpha)_{\alpha<\kappa}, (r^\alpha_\beta)_{\beta\leq\alpha<\kappa})$.

Let Y_α be the preimage of q under r^α_0 and s^α_β: $Y_\alpha \to Y_\beta$ the restriction of r^α_β to Y_α. Then $((Y_\alpha)_{\alpha<\kappa}, (s^\alpha_\beta)_{\beta\leq\alpha<\kappa})$ is an inverse system with inverse limit Y.

For $\alpha < \kappa$ we have $a_\alpha \in J \subseteq C = A_0$. If $a_\alpha \not\in q$, then $a_\alpha \not\in y$ for each $y \in Y_\alpha$ and $s^{\alpha+1}_\alpha$ is a homeomorphism. Otherwise, $a_\alpha \in y$ for each $y \in Y_\alpha$, and it is easily seen that, up to homeomorphism, $Y_{\alpha+1} = Y_\alpha \times 2$ and $s^{\alpha+1}_\alpha$ is the projection onto the first coordinate. Since $q \in U_J$ and $A = \mathrm{ext}_\kappa(C, J)$, the latter situation arises κ times. Thus, $Y \cong {}^\kappa 2$. $\square$

4.7. THEOREM. *Let κ be singular, $\lambda = \mathrm{cf}\,\kappa$ and $(\kappa_\alpha)_{\alpha<\lambda}$ a strictly increasing continuous sequence of cardinals such that $\kappa_0 = \lambda$ and $\sup_{\alpha<\lambda} \kappa_\alpha = \kappa$. For $\alpha < \lambda$ let $R_{\alpha+1}$ a Boolean space of weight at most $\kappa_{\alpha+1}$. Then there is a projective Boolean algebra A of power κ such that, for each $\alpha < \lambda$,*

$$N_{\alpha+1} = \{p \in \mathrm{Ult}\, A: \chi(p) = \kappa_{\alpha+1}\} \cong R_{\alpha+1}\,.$$

Hence, there are, up to isomorphism, exactly 2^κ projective Boolean algebras of power κ.

PROOF. We shall construct a continuous chain $(A_\alpha)_{\alpha<\lambda}$ of projective algebras satisfying $|A_\alpha| = \kappa_\alpha$ and then put $A = \bigcup_{\alpha<\lambda} A_\alpha$. The dual space of A_α will be denoted by X_α.

First, let $A_0 = \mathrm{Fr}(\lambda)$, thus $X_0 = {}^\lambda 2$. The one-point compactification of a discrete space of power λ is a Boolean space of weight λ, hence embeddable into X_0. So there are points $x_\alpha \in X_0$ for $\alpha < \lambda$ such that

$$x_\alpha \not\in \mathrm{cl}\{x_\nu : \nu < \alpha\}$$

(cl y denotes the topological closure of $Y \subseteq X_0$).

We abbreviate, for $\beta \leq \alpha < \lambda$, $r^{A_\alpha}_{A_\beta}$ by r^α_β. Along with A_α and X_α we shall construct a closed subspace Y_α of X_α such that

$$(5)_\alpha \qquad r^\alpha_0[Y_\alpha] \subseteq \mathrm{cl}\{x_\nu : \nu < \alpha\}\,.$$

Let $Y_0 = \emptyset$; $(5)_0$ is trivial. If $\alpha < \lambda$ is a limit ordinal, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $Y_\alpha = \mathrm{cl}(\bigcup_{\beta<\alpha} (r^\alpha_\beta)^{-1}[Y_\beta])$; $(5)_\alpha$ follows by induction. Given A_α, X_α and Y_α, we let

$$A_{\alpha+1} = \mathrm{ext}_{\kappa_{\alpha+1}}(A_\alpha, J_\alpha)\,,$$

where J_α is the ideal of A_α dual to $X_\alpha \backslash Y_\alpha$. To find $Y_{\alpha+1}$, define

$$Z_{\alpha+1} = (r_\alpha^{\alpha+1})^{-1}[Y_\alpha];$$

thus, $Z_{\alpha+1}$ is a closed subspace of $X_{\alpha+1}$ satisfying, by $(*)_\alpha$,

$$(r_0^{\alpha+1})[Z_{\alpha+1}] \subseteq \mathrm{cl}\{x_\nu : \nu < \alpha\}.$$

In particular, $x_\alpha \not\in r_0^{\alpha+1}[Z_{\alpha+1}]$. Fix $q \in X_\alpha$ such that $r_0^\alpha(q) = x_\alpha$. Then $q \not\in Y_\alpha$, hence $q \in U_{J_\alpha} = X_\alpha \setminus Y_\alpha$. By the preceding lemma, $Y = (r_\alpha^{\alpha+1})^{-1}[\{q\}]$ is homeomorphic to $^{K_{\alpha+1}}2$, and since $w(R_{\alpha+1}) \leq \kappa_{\alpha+1}$, we may assume that $R_{\alpha+1}$ is a closed subspace of Y. Then put

$$Y_{\alpha+1} = Z_{\alpha+1} \cup R_{\alpha+1}.$$

$(5)_{\alpha+1}$ holds since $r_\alpha^{\alpha+1}[R_{\alpha+1}] = \{x_\alpha\}$. This finishes the construction.

Now abbreviate Ult A by X and $r_{A_\alpha}^A$ by r_α. For each $p \in X$ and $\alpha < \lambda$, 4.1 and the construction of $A_{\alpha+1}$ and $Y_{\alpha+1}$ show that

$$\chi(p) \leq \kappa_\alpha \quad \text{if } r_{\alpha+1}(p) \in Z_{\alpha+1};$$

$$\chi(p) \geq \kappa_{\alpha+2} \quad \text{if } r_{\alpha+1}(p) \in X_{\alpha+1} \setminus Y_{\alpha+1};$$

$$\chi(p) = \kappa_{\alpha+1} \quad \text{if } r_{\alpha+1}(p) \in R_{\alpha+1}.$$

This implies, together with $R_{\alpha+1} \subseteq Y_{\alpha+1}$, that

$$N_{\alpha+1} = \{p \in X : r_{\alpha+1}(p) \in R_{\alpha+1}\} \cong R_{\alpha+1}.$$

The rest of the theorem follows easily: trivially, 2^κ is an upper bound on the number of isomorphism types of projective algebras of size κ; on the other hand, there are 2^κ distinct sequences $(t_\alpha)_{\alpha<\lambda}$ such that $t_{\alpha+1}$ is the homeomorphism type of a Boolean space of weight $\kappa_{\alpha+1}$. □

A Boolean algebra A is said to be rigid if id_A is the only automorphism of A. Obviously, no countably infinite Boolean algebra can be rigid, and by 3.8, no projective algebra of regular uncountable cardinality is rigid. We now show that under an additional set-theoretical hypothesis, known to be consistent with ZFC, there is a rigid projective algebra of power $\aleph_\omega$.

4.8. LEMMA. *Let C be a subalgebra of* Fr λ, *where $\lambda \leq 2^\omega$. Then* Ult C *has a countable dense subset.*

PROOF. Ult C is the image of $^\lambda 2$ under a continuous map, and by the Hewitt–Marczewski–Pondiczery theorem (see ENGELKING [1977]), $^\lambda 2$ has a countable dense subset iff $\lambda \leq 2^\omega$. □

4.9. THEOREM. *Assume that $\aleph_\omega < 2^\omega$. Then there is a rigid projective Boolean algebra of cardinality $\aleph_\omega$.*

PROOF. Choose, for $n>0$, a rigid Boolean space R_n of weight at most ω_n; e.g. R_n might be a one-point space.

As in the proof of 4.7, we shall construct a chain $(A_n)_{n\in\omega}$ of projective algebras satisfying $|A_n| = \omega_n$ and closed subspaces Y_n of $X_n = \mathrm{Ult}\, A_n$. Denote, for $k \leq n < \omega$, $r^{A_n}_{A_k}$ by r^n_k. Let $A_0 = \mathrm{Fr}\,\omega$ and $Y_0 = \emptyset$. Given A_n, X_n and Y_n, we define

$$A_{n+1} = \mathrm{ext}_{\omega_{n+1}}(A_n, J_n)\,,$$

where J_n is the ideal of A_n dual to $X_n \setminus Y_n$ and let

$$Z_{n+1} = (r^{n+1}_n)^{-1}[Y_n]\,.$$

We shall point out below how to find a subspace K_{n+1} of $X_{n+1}\setminus Z_{n+1}$, homeomorphic to R_{n+1}, such that

$$Y_{n+1} = Z_{n+1} \cup K_{n+1}$$

is a proper subspace of X_{n+1}. Then for $A = \bigcup_{n\in\omega} A_n$, the construction guarantees that N_{n+1} defined by

$$N_{n+1} = \{p \in \mathrm{Ult}\, A\colon \chi(p) = \omega_{n+1}\}$$

is homeomorphic to R_{n+1}.

We now explain how to choose the spaces K_{n+1} such that A is rigid. Since each A_k is projective and $|A_k| = \omega_k < 2^\omega$, we fix, after having constructed A_k and X_k, a countable dense subset D_k of X_k by the preceding lemma. We will see that, by a bookkeeping device, the K_{n+1} can be chosen to satisfy

(6) for each $r \in D_k$, there are $n \geq k$ and $q \in Y_{n+1}$ such that $r^{n+1}_k(q) = r$.

For assume $r \in D_k$, $n \geq k$ and A_{n+1} has been constructed; we choose K_{n+1} as follows. Fix $q \in X_{n+1}$ such that $r^{n+1}_k(q) = r$. If $q \in U = X_{n+1}\setminus Z_{n+1}$, then q has a clopen neighbourhood $V \subseteq U$ homeomorphic to $\omega_{n+1}2$; this is seen as in the proof of 4.6. We may choose V such that $V \cup Z_{n+1} \neq X_{n+1}$. Let K_{n+1} be a closed subspace of V homeomorphic to R_{n+1}. Since for any two points q and q' of $V \cong \omega_{n+1}2$ there is a homeomorphism of V mapping q' onto q, we may also assume that $q \in K_{n+1}$. – Finally, if $q \in Z_{n+1}$, we choose $V \subseteq U$ clopen in X_{n+1} such that $V \cong \omega_{n+1}2$ and $V \cup Z_{n+1} \neq X_{n+1}$ and then let $K_{n+1} \subseteq V$ be closed such that $K_{n+1} \cong R_{n+1}$.

It follows from (6) that

$$N = \bigcup_{n\in\omega} N_{n+1}$$

is dense in $X = \mathrm{Ult}\, A$. To see this, let $a \in A\setminus\{0\}$ with the aim of finding $p \in N$ such that $a \in p$. Assume $a \in A_k$; then there is some $r \in D_k$ such that $a \in r$. Pick $n \geq k$ and $q \in Y_{n+1}$ by (6) and let $p \in X$ be the unique ultrafilter of A extending q. Then $\chi(p, A) = \chi(q, A_{n+1}) \leq \omega_{n+1}$. So $p \in N$; moreover, $a \in r \subseteq q \subseteq p$.

By Stone duality, A is rigid iff id_X is the only homeomorphism from X onto itself. So assume φ is such a homeomorphism. For each $n \geq 1$, $\varphi_{n+1} = \varphi \restriction N_{n+1}$ is a homeomorphism of N_{n+1} onto itself. Now $\varphi_{n+1} = \mathrm{id}_{N_{n+1}}$ since $N_{n+1} \cong R_{n+1}$ is rigid. It follows that $\varphi \restriction N = \mathrm{id}_N$, and $\varphi = \mathrm{id}_X$ since N is dense in X. □

In the final part of this subsection, we describe, for κ regular and uncountable, projective Boolean algebras of cardinality κ by "invariants" in a set $\mathrm{Inv}(\kappa)$ to be defined below. Remember that originally, for $|C| \leq \lambda$, $\mathrm{ext}_\lambda(C, J)$ was defined to be a *pair* (A, K), where K is the ideal of A generated by J.

4.10. REMARK. If $|C| \leq \lambda \leq \kappa$, then by 4.1,

$$\mathrm{ext}_\kappa(\mathrm{ext}_\lambda(C, J)) \cong \mathrm{ext}_\kappa(C, J) .$$

Call an algebra-ideal pair (C, J) *closed* if $J \neq \{0\}$ and $\mathrm{ext}_\lambda(C, J) \cong (C, J)$, where $\lambda = |C| \geq \omega$. For example, $\mathrm{ext}_\lambda(C, J)$ is closed by the preceding remark whenever $|C| \leq \lambda$ and $J \neq \{0\}$. If (C, J) is closed, then $\chi(p) = \lambda$ for each $p \in U_J$. Call (C, J) *minimal* if $\lambda = |C| \geq \omega$ and there is no pair (D, L) such that $\omega \leq |D| < |C|$ and $(C, J) \cong \mathrm{ext}_\lambda(D, L)$. Note that for a minimal pair (C, J) there may nevertheless be some pair (D, L) such that $|D| < |C|$ and an isomorphism φ from C onto the algebra part A of $\mathrm{ext}_\lambda(D, L)$ – simply φ will not map J onto the ideal generated in A by L.

For κ regular and uncountable, define

$$\begin{aligned}\mathrm{Inv}(\kappa) = \{(C, J)\colon & (C, J) \text{ an algebra-ideal pair}, |C| < \kappa, \\ & C \text{ projective}, J \neq \{0\}, (C, J) \text{ closed and minimal}\} ,\end{aligned}$$

$$\mathrm{Pr}(\kappa) = \{A\colon A \text{ a projective Boolean algebra}, |A| = \kappa\} .$$

Modulo two lemmas to be proved below, we achieve the main goal of this section:

4.11. THEOREM. *Let κ be regular and uncountable.*

(a) *If (C, J) is in $\mathrm{Inv}(\kappa)$, then (the algebra part of) $\mathrm{ext}_\kappa(C, J)$ is in $\mathrm{Pr}(\kappa)$.*

(b) *For each A in $\mathrm{Pr}(\kappa)$ there is some $(C, J) \in \mathrm{Inv}(\kappa)$ such that $A \cong \mathrm{ext}_\kappa(C, J)$.*

(c) *Let $A \cong \mathrm{ext}_\kappa(C, J)$ and $A' = \mathrm{ext}_\kappa(C', J')$ where (C, J) and (C', J') are in $\mathrm{Inv}(\kappa)$. Then $A \cong A'$ iff $(C, J) \cong (C', J')$.*

Hence, there are, up to isomorphism, exactly $2^{<\kappa}$ projective Boolean algebras of power κ.

PROOF. (a) follows from 2.11, (b) from 4.12 and (c) from 4.13. The final assertion is a consequence of 4.5(b) and (c) and the fact that $\mathrm{Inv}(\kappa)$ has, up to isomorphism, at most $2^{<\kappa}$ elements. □

4.12. LEMMA. *Let A be projective and $\kappa = |A|$ be regular and uncountable; let K be the ideal of A dual to* $\mathrm{Ult}\, A \setminus M_\kappa(A)$. *Then $(A, K) \cong \mathrm{ext}_\kappa(C, J)$ for some pair (C, J) in* $\mathrm{Inv}(\kappa)$.

PROOF. By 3.7, there is a subalgeba D of A such that $|D| < \kappa$ and D determines $M_\kappa(A)$. Fix a skeleton $\mathcal{S}$ of A over 2 and pick $S \in \mathcal{S}$ such that $D \leq S$ and $\omega \leq |S| < \kappa$. By $|S| < \kappa$ and the properties of a skeleton, $S \leq_{\text{proj}} A$. Moreover, since S determines $M_\kappa(A)$, $(A, K) \cong \text{ext}_\kappa(S, L)$, where $L = \{\text{indp}_S^A(x): x \in A\}$. Let $\lambda = |S|$. Now $(T, M) = \text{ext}_\lambda(S, L)$ is closed and, by 4.10, $\text{ext}_\kappa(T, M) \cong (A, K)$. We have thus shown that there is a closed algebra-ideal pair (C, J) such that $\omega \leq |C| < \kappa$, C is projective and $(A, K) \cong \text{ext}_\kappa(C, J)$. Any such pair with $|C|$ minimal proves the lemma. □

4.13. LEMMA. *Let A be projective and $\kappa = |A|$ regular and uncountable. Assume that (C, J) and (C', J') are in* $\text{Inv}(\kappa)$ *such that $A \cong \text{ext}_\kappa(C, J) \cong \text{ext}_\kappa(C', J')$. Then $(C, J) \cong (C', J')$.*

PROOF. Without loss of generality, $C \leq_{\text{proj}} A$, $C' \leq_{\text{proj}} A$ and $\lambda = |C| \leq \lambda' = |C'|$. Assume for contradiction that $\lambda < \lambda'$ and consider the closed subspaces $M_\kappa(A)$ of $\text{Ult}\, A$, $M = \text{Ult}\, C \backslash U_J$ of $\text{Ult}\, C$ and $M' = \text{Ult}\, C' \backslash U_{J'}$ of $\text{Ult}\, C'$. By 4.2 and $A = \text{ext}_\kappa(C, J)$, $M_\kappa(A) \cong M$; in particular, $w(M_\kappa(A)) \leq \lambda$. Also, if $p \in M_\kappa(A)$, then $\chi(p, A) = \chi(p \cap C, C) \leq \lambda$. It follows similarly that $M_\kappa(A) \cong M'$.

Let $q \in \text{Ult}\, C'$ and choose $p \in \text{Ult}\, A$ such tht $p \cap C' = q$. If $q \in M'$, then $p \in M_\kappa(A)$ and thus

$$\chi(q, C') = \chi(p, A) \leq \lambda < \lambda' . \tag{7}$$

If $q \not\in M'$, then $\chi(q, C') = \lambda'$ since (C', J') is closed. Thus,

$$M_{\lambda'}(C') = M' = M_{\lambda^+}(C') .$$

Application of 3.7 to C' and λ^+ shows that C' has a subalgebra D' such that D' determines $M_{\lambda'}(C')$ and $|D| \leq \lambda < \lambda'$. It follows, as in the proof of 4.12, that $(C', J') \cong \text{ext}_{\lambda'}(T, M)$ for some algebra-ideal pair (T, M) such that $|T| \leq \lambda$, contradicting minimality of (C', J').

We thus have $|C| = \lambda = |C'|$. By $A = \text{ext}_\kappa(C, J)$, A is a projective extension of C, so let $\mathcal{S}$ be a skeleton of A over C and, similarly, $\mathcal{S}'$ a skeleton of A over C'. By induction choose $S_0 = C \in \mathcal{S}$, $S_0' = C' \in \mathcal{S}'$, $S_1 \in \mathcal{S}$, $S_1' \in \mathcal{S}'$, ... such that $|S_n| = |S_n'| = \lambda$ and $S_n \leq_{\text{proj}} S_{n+1}$, $S_n' \leq_{\text{proj}} S_{n+1}'$, $S_n' \leq S_{n+1} \leq S_{n+1}'$ (Fig. 20.9).

$$\begin{tikzcd}
C = S_0 \arrow[rr] & & S_1 \arrow[rr] \arrow[dr, dashed] & & S_2 \arrow[rr] \arrow[dr, dashed] & & \cdots \\
& C' = S_0' \arrow[rr] \arrow[ur, dashed] & & S_1' \arrow[rr] \arrow[ur, dashed] & & S_2' \arrow[rr] & & \cdots
\end{tikzcd}$$

Fig. 20.9

(In the figure, broken arrows represent inclusion and unbroken ones projective inclusion.) Then, for T defined by

$$T = \bigcup_{n\in\omega} S_n = \bigcup_{n\in\omega} S'_n ,$$

we have $C = S_0 \leq_{\text{proj}} T$ and $C' = S'_0 \leq_{\text{proj}} T$.

Let L (resp. L') be the ideals of T generated by J (resp. J'); let K be the ideal of A dual to Ult $A\backslash M_\kappa(A)$. Then K is generated in A by J, hence $L = K \cap T$. Similarly, $L' = K \cap T$, so $L = L'$.

By $A = \text{ext}_\kappa(C, J)$, we may choose S_1 in $\mathscr{S}$ large enough to guarantee that $(T, L) \cong \text{ext}_\lambda(C, J)$ and similarly, by choosing S'_1 in $\mathscr{S}'$ large enough, we obtain $(T, L) \cong \text{ext}_\lambda(C', J')$. Then $(C, J) \cong (C', J')$ since (C, J) and (C', J') are closed. □

We finally sketch how to translate, for $\kappa = \omega_1$, the invariants in $\text{Inv}(\omega_1)$ into similarly defined ones with a more obvious topological meaning. For let $(C, J) \in \text{Inv}(\omega_1)$. We may replace algebra-ideal pairs (C, J) by pairs (X, Y), where $X = \text{Ult}\, C$ and Y is the closed subspace $X\backslash U_J$ of X. Then $(C, J) \in \text{Inv}(\omega_1)$ iff $(X, Y) \in \mathit{Inv}'$, where

$$\mathit{Inv}' = \{(X, Y)\colon X \text{ a Boolean space, } w(X) = \omega,\ Y \subseteq X \text{ closed in } X,\ X\backslash Y \neq \emptyset, \text{ no point of } X\backslash Y \text{ is isolated}\} .$$

These invariants can still be simplified. Define

$$\mathit{Inv}'' = \{(Y, R)\colon Y \text{ a Boolean space, } w(Y) \leq \omega,\ R \text{ a closed subspace of } Y\} .$$

Let $\Phi\colon \mathit{Inv}' \to \mathit{Inv}''$ be the map assigning $(Y, \text{bd}_X Y)$ to (X, Y), where $\text{bd}_X Y$ is the boundary of Y in the space X. It can be proved that Φ is onto and, using Vaught's isomorphism theorem 5.15 in Part I of this Handbook, that for (X, Y) and (X', Y') in Inv$'$, $(X, Y) \cong (X', Y')$ iff $\Phi(X, Y) \cong \Phi(X', Y')$. Thus, the dual space X of a projective Boolean algebra of cardinality ω_1 is determined, up to homeomorphism, but its closed subspaces $Y = \{p \in X\colon \chi(p) \leq \omega\}$ and $R = \text{bd}_X Y$.

Added in proof: I am grateful to Sakaé Fuchino for pointing out an error in the original proof of Theorem 3.5.

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Keywords: Boolean algebra, projective, injective space, free, retract, Bockstein separation property, relatively complete, ultrafilter, character, invariants.

MOS subject classification: primary 06E05; secondary 03G05, 06E15, 54A25, 54D30.

CHAPTER 21

Countable Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

Boolean algebras with countable universes – countable Boolean algebras – have a richer theory than general Boolean algebras. It appears that there is not enough room in a countable Boolean algebra to support the outrageous pathology that often flourishes in uncountable structures. Also, the pervasiveness of countable Boolean algebras in logic, ring theory, and topology has made them attractive targets for investigation.

This chapter describes some important features of countable Boolean algebras. Our discussion is divided into three sections: classification; the semiring of types; and special classes of countable algebras. The first part includes descriptions of the partial and full invariants that have been introduced by several authors. Next, we examine the semiring structure that is imposed on the set of isomorphism types of countable Boolean algebras by the product and coproduct constructions. The major results treated here are associated wih the names Dobbertin, Ketonen, and Trnková. The last part deals with the primitive Boolean algebras that were introducted by Hanf, as well as two subclasses of the primitive algebras that have been studied by Hansoul and the author.

To aid the reader's digestive process, each section has been broken into short subsections featuring a main proposition or theorem, sometimes supported by a few lemmas and/or corollaries. Many of these subsections conclude with historical comments, references, and supplementary results that were not mentioned in the mainstream of our development.

The author is grateful to Dr. Lutz Heindorf, whose careful reading of a preliminary version of this chapter led to numerous additions, corrections, and improvements.

1. Invariants

1.1. Vaught's theorem revisited

Most structure theorems for countable Boolean algebras can be derived from Vaught's Theorem, Theorem 5.15 of Part I of this Handbook. Henceforth, references to results in Part I of this book will be identified by one decimal number; two decimal numbers refer to statements in this chapter. We will use a variation of Vaught's theorem which is a bit more technical. The extra complexity will be exploited later.

The class of countable Boolean algebras includes the finite algebras, whose classification problem is about as interesting as elementary arithmetic. However, the way in which finite algebras occur as subalgebras of a Boolean algebra tells a great deal, because every Boolean algebra is the directed union of its finite subalgebras. This fact is an obvious consequence of the next observation.

1.1.1. LEMMA. *If C is a finite subalgebra of the Boolean algebra A, and if $x \in A$, then the subalgebra D of A that is generated by $C \cup \{x\}$ is finite.*

Indeed, the collection of sums of the elements in the finite set $\{x \cdot a: a \in \mathrm{At}\ C\} \cup \{(-x) \cdot a: a \in \mathrm{At}\ C\}$ is a subalgebra of A that includes $C \cup \{x\}$. This subalgebra must therefore be D, so that D is finite. □

It will be convenient to give the definition of Vaught relations (V-relations) a somewhat different form than has been used in Part I. In the following statements, the expressions of the form $x \dot{+} y$ denote the Boolean algebra sum $x + y$ with the implied assumption that x and y are disjoint.

1.1.2. DEFINITION. Let A and B be Boolean algebras. A subset R of $A \times B$ is a V-relation between A and B if

(i) $1_A R 1_B$;

(ii) $xR0_B$ implies $x = 0_A$; $0_A Ry$ implies $y = 0_B$;

(iii) $xR(y_1 \dot{+} y_2)$ implies $x = x_1 \dot{+} x_2$, where x_1Ry_1 and x_2Ry_2;
$(x_1 \dot{+} x_2)Ry$ implies $y = y_1 \dot{+} y_2$, where x_1Ry_1 and x_2Ry_2.

Symmetry is built into the concept of a V-relation: if R is a V-relation between A and B, then R^{-1} is a V-relation between B and A. The composition of V-relations is a V-relation, and the identity relation Δ_A on A is a V-relation. Therefore, the V-relations between Boolean algebras induce an equivalence on the class of all Boolean algebras: A is equivalent to B if there is a V-relation between A and B. This equivalence is at least as coarse as isomorphism, since if f: $A \to B$ is an isomorphism, then $R = \{(x, f(x)): x \in A\}$ is a V-relation between A and B. By Theorem 5.15, if A and B are countable, then the existence of a V-relation between A and B implies that A is isomorphic to B.

1.1.3. VAUGHT'S THEOREM. *If R is a V-relation between the countable Boolean algebras A and B, then there is an isomorphism $f: A \to B$ such that $y = f(x)$ implies the existence of decompositions $x = x_0 \dot{+} \cdots \dot{+} x_{n-1}$, $y = y_0 \dot{+} \cdots \dot{+} y_{n-1}$ for which x_iRy_i and $f(x_i) = y_i$ for all $i < n$.*

The proof of this statement is essentially the "back and forth" argument that was used in Part I. If $A = \{x_n: n < \omega\}$ and $B = \{y_n: n < \omega\}$ are enumerations of the given Boolean algebras, then the V-relation R is used to construct sequences $\{C_n: n < \omega\}$ and $\{D_n: n < \omega\}$ of finite subalgebras of A and B, respectively, such that $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$; $D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$; $x_0, \ldots, x_{m-1} \in C_{2m}$; $y_0, \ldots, y_m \in D_{2m+1}$; and a sequence of isomorphisms f_n: $C_n \to D_n$ satisfying $f_0 \subseteq f_1 \subseteq f_2 \subseteq \cdots$ and $(a, f_n(a)) \in R$ for all $a \in \mathrm{At}\ C_n$. The construction starts with $C_0 = \{0_A, 1_A\}$, $D_0 = \{0_B, 1_B\}$. If C_i, D_i, and f_i have been constructed for $i < n = 2m + 1$, let D_n be the subalgebra of B that is generated by D_{n-1} and y_m. By the lemma, D_n is finite. Since $(a, f_{n-1}(a)) \in R$ for all $a \in \mathrm{At}\ C_{n-1}$, the definition of a V-relation implies that there is a subalgebra C_n of A whose atoms correspond bijectively via R to the atoms of D_n. That is, f_{n-1} extends to an isomorphism f_n: $C_n \to D_n$ with $(a, f_n(a)) \in R$ for all $a \in \mathrm{At}(C_n)$. When $n = 2m$, a similar construction extends C_{n-1} to C_n, a finite subalgebra of A with $x_{m-1} \in C_n$, D_{n-1} to D_n, and f_{n-1} to f_n. The required isomorphism f is the union of the f_n. □

Notes. Vaught's Theorem was a small part of Vaught's doctoral dissertation, a fact that can be confirmed in the archives of the University of California, Berkeley. The result existed as folklore (well known to Western logicians) until Hanf published the proof in 1974 (HANF [1974]). The "back and forth" argument is older, of course: CANTOR [1883] used it in his characterization of the rational numbers.

1.2. Topological matters

The special properties of the Stone spaces of countable Boolean algebras play an important part in the study of these algebras. Most of our initial discussion in this section will be in the language of topology rather than algebra. It is therefore convenient to summarize some properties of the Stone spaces of countable Boolean algebras that have been established in Part I of this Handbook.

If X is the Stone space of the Boolean algebra A, then A is countable if and only if X is metrizable, or equivalently, X has a countable open basis (Theorem 7.23). Since the atoms of A correspond to the isolated points of X, it follows from Theorems 5.16, 9.7 and 9.11 that A is free on a countably infinite set of generators if and only if X is homeomorphic to the Cantor space $\mathscr{C}$. Since every countable Boolean algebra is a homomorphic image of a free Boolean algebra on countably many generators, the Stone Duality Theorem (Theorem 8.2) implies that every metrizable Boolean space is homeomorphic to a closed subset of $\mathscr{C}$.

It will be convenient to have a topological version of Vaught's Theorem. For variety, we present the global form that was given in Section 17 of Part I. Of course, a local topological form and a global algebraic statement of Vaught's Theorem can be given. As is usual, expressions of the form $X \uplus Y$ stand for disjoint set unions.

1.2.1. DEFINITION. Let $\mathscr{G}$ be a hereditary class of Boolean spaces, that is, along with any X, $\mathscr{G}$ includes all clopen subsets of X. A binary relation R on $\mathscr{G}$ is a V-relation if:

(i) R is symmetric;
(ii) $XR\emptyset$ implies $X=\emptyset$;
(iii) $XRY_1 \uplus Y_2$ implies $X = X_1 \uplus X_2$, where X_1RY_1 and X_2RY_2.

1.2.2. VAUGHT'S THEOREM (topological version). *If R is a V-relation on a hereditary class of metrizable Boolean spaces, then XRY implies that X is homeomorphic to Y.*

To obtain this form of Vaught's Theorem from Theorem 1.1.3, let A and B be the Boolean algebras of clopen sets in X and Y, respectively. Then $R' = \{(a, b)\colon a \in \operatorname{Clop} X, b \in \operatorname{Clop} Y, aRb\}$ is a V-relation between A and B in the sense of Definition 1.1.2. Hence, $A \cong B$. It follows that X is homeomorphic to Y by Stone's Theorem. $\square$

The classical theorem that any perfect, metrizable Boolean space is homeomorphic to $\mathscr{C}$ follows directly from Theorem 1.2.2. Indeed, the class $\mathscr{G}$ of all such spaces is hereditary: if Y is a clopen subset of $X \in \mathscr{G}$, then since there are no isolated points in X (i.e. X is perfect), the same holds true for Y. It follows that $(\mathscr{G} - \{\emptyset\}) \times (\mathscr{G} - \{\emptyset\}) \cup \{(\emptyset, \emptyset)\}$ is a V-relation on $\mathscr{G}$. Of course, this topological result was implicit in the earlier remark that all countable, atomless Boolean algebras are free.

Notes. The topological characterization of $\mathscr{C}$ goes back to the work of BROUWER [1910]. The fact that every metrizable Boolean space can be embedded in $\mathscr{C}$ also appears in Brouwer's work. This topological fact can be used to give another proof of Corollary 15.10: every countable Boolean algebra is an interval algebra. In fact, if X is a closed subspace of $\mathscr{C}$, then an easy compactness argument shows that Clop X is generated by the totally ordered collection of sets $X \cap [0, r]$, where r ranges over the triadic rational numbers such that $\mathscr{C} \cap [0, r]$ is open in $\mathscr{C}$ (see Example 1.3.1 below).

1.3. *Examples*

It is often easier to distinguish two non-homeomorphic Boolean spaces than to detect structural differences between the clopen algebras of these spaces. Moreover, it is usually easier to describe a Boolean space than to present the corresponding Boolean algebra. Three examples will illustrate this phenomenon.

1.3.1. EXAMPLE. The grandfather of all Boolean spaces is the Cantor discontinuum $\mathscr{C}$. The most concrete realization of $\mathscr{C}$ is the set of numbers r in the real interval $[0, 1]$ whose triadic decimal representation has the form $r = \Sigma_{k<\omega} c_k 3^{-(k+1)}$ with all c_k equal to 0 or 2. It then follows that $\mathscr{C} = I \backslash \bigcup_{n<\omega} J_n$, where $I = [0, 1]$, and the J_n are disjoint open intervals (s_n, t_n) which could (but will not) be listed explicitly. It is useful to note that $\bigcup_{n<\omega} J_n$ is a dense open subset of I. An alternative description of the Cantor space is $\mathscr{C} = {}^{\omega}\{0, 2\}$, a countably infinite product of discrete two-point spaces. The homeomorphism between ${}^{\omega}\{0, 2\}$ and $\mathscr{C}$, viewed as a subset of $[0, 1]$, is defined by mapping $(c_0, c_1, c_2, \ldots)$ to $\Sigma_{k<\omega} c_k 3^{-(k+1)}$. Finally, thanks to Vaught's Theorem, we know that $\mathscr{C}$ can be realized by any perfect, metrizable Boolean space.

1.3.2. EXAMPLE. At the opposite extreme from the Cantor space are the spaces of countable ordinal numbers. Explicitly, let $\xi < \omega_1$ be a countable ordinal number. Denote by $\xi + 1$ the ordered space of all ordinal numbers $\eta \leq \xi$. Since $\xi + 1$ is a complete ordered set, it is a compact Hausdorff space. If $\zeta < \eta \leq \tau \leq \xi$, then $[\zeta + 1, \tau]$ is a clopen neighborhood of η. Thus, $\xi + 1$ is a Boolean space. In contrast to $\mathscr{C}$, the isolated points in $\xi + 1$ are dense. In fact, every closed subset of $\xi + 1$ shares this property. Thus Clop$(\xi + 1)$ is a superatomic Boolean algebra.

1.3.3. EXAMPLE. Here is a Boolean space that combines features of the examples above. Let $\mathscr{C} = I \backslash \bigcup_{n<\omega} J_n$ be the standard realization of the Cantor set that was given in 1.3.1, say $J_n = (s_n, t_n)$ with $s_n < t_n$. Define $X = \mathscr{C} \cup \{(1/2)(s_n + t_n): n < \omega\}$. That is, X is obtained from the Cantor set by restoring the midpoint of each omitted interval. It is easy to check that X is a closed, nowhere dense subset of I, hence a Boolean space. The midpoints $(1/2)(s_n + t_n)$ are isolated, and this set is dense in X. On the other hand, the perfect space $\mathscr{C}$ appears as a closed subspace of X. These remarks imply that the Boolean algebra $A = \operatorname{Clop} X$ is atomic but not superatomic.

1.4. Topological derivatives

One of the oldest constructions in point set topology, the Cantor–Bendixson derivative, plays an important role in the study of countable Boolean spaces. We will now describe the derivative and its transfinite iterates.

For any Hausdorff space Y, the derivative of Y is the set Y' of all accumulation points of Y. That is, Y' is the complement of the set of points that are isolated in Y. If Y is a subset of a larger space X, then the derivative Y' refers to the subspace topology of Y. Specifically, $Y \setminus Y'$ is the set of points that are isolated in Y, though not necessarily in X.

By transfinite iteration, the derivative construction produces a sequence of closed subsets of a topological space. Explicitly, define

$$X^{(0)} = X\,, \qquad X^{(\xi+1)} = (X^{(\xi)})'\,, \quad \text{and} \quad X^{(\eta)} = \bigcap_{\xi<\eta} X^{(\xi)}$$

when η is a limit ordinal. The list $(X^{(0)}, X^{(1)}, \ldots, X^{(\xi)}, \ldots)$ is called the *Cantor–Bendixson sequence* of the space X. Here are the basic properties of these derivatives.

1.4.1. PROPOSITION. *Assume that X is a Hausdorff space, and ξ, η are ordinal numbers.*

(a) *$X^{(\xi)}$ is closed in X.*
(b) *$\xi < \eta$ implies $X^{(\xi)} \supseteq X^{(\eta)}$.*
(c) *$(X^{(\xi)})^{(\eta)} = X^{(\xi+\eta)}$.*
(d) *$Y \subseteq X$ implies $Y^{(\xi)} \subseteq X^{(\xi)}$.*
(e) If *W is open in X, then $W^{(\xi)} = W \cap X^{(\xi)}$.*
(f) *If $X = \bigcup_{i\in I} W_i$ with all W_i open in X, then $X^{(\xi)} = \bigcup_{i\in I} (W_i)^{(\xi)}$.*
(g) *If ϕ: $X \to Y$ is a homeomorphism, then $\phi(X^{(\xi)}) = Y^{(\xi)}$.*
(h) *There is a smallest ordinal number η such that $X^{(\eta+1)} = X^{(\eta)}$; and if $\xi \geq \eta$, then $X^{(\xi)} = X^{(\eta)}$.*

Most of the statements in this proposition are routine consequences of the definition of the derivative, using transfinite induction. As a sample, here is how the proof of (e) goes. Since W is open, a point is isolated in W if and only if it is isolated in X. Thus, $W' = W \cap X'$, which is the case $\xi = 1$ of (e). For the induction step, just note that $W \cap X^{(\xi)}$ is open in $X^{(\xi)}$, so that $W^{(\xi+1)} = (W^{(\xi)})' = (W \cap X^{(\xi)})' = W \cap X^{(\xi)} \cap (X^{(\xi)})' = W \cap X^{(\xi+1)}$. Plainly, (f) is a corollary of (e). $\square$

1.4.2. EXAMPLE. The Cantor space $\mathscr{C}$ is perfect, that is, it has no isolated points. Thus, $\mathscr{C}^{(\xi)} = \mathscr{C}$ for all ordinals ξ.

1.4.3. EXAMPLE. For a countable ordinal number $\mu > 0$, let X be the well-ordered set $\omega^{\mu} + 1$, viewed as a Boolean space (as in 1.3.2). Then $X^{(\mu)} = \{\omega^{\mu}\}$. This important fact is proved by induction on μ. If $\mu = 1$, then the assertion is obvious. If $\beta < \omega^{\mu}$, then $\beta = \gamma + \omega^{\xi}$ for some $\xi < \mu$. The clopen subset $[0, \beta]$ of X decomposes as a disjoint union $[0, \gamma] \cup [\gamma + 1, \gamma + \omega^{\xi}]$ of open sets with $[\gamma + 1, \gamma + \omega^{\xi}]$ homeomorphic to $\omega^{\xi} + 1$. The induction hypothesis implies $[\gamma + 1, \gamma + \omega^{\xi}]^{(\mu)} = \emptyset$, so that $\beta \not\in X^{(\mu)}$. Thus, $X^{(\mu)} \subseteq \{\omega^{\mu}\}$. If $\mu = \nu + 1$, then $X = \{\omega^{\mu}\} \uplus [0, \omega^{\nu}] \uplus [\omega^{\nu} + 1, \omega^{\nu} \cdot 2] \uplus \cdots$. In this case, the induction hypothesis yields $X^{(\nu)} \supseteq \{\omega^{\nu}, \omega^{\nu} \cdot 2, \ldots\}$, and ω^{μ} is a limit point of this set. Thus, $X^{(\mu)} = \{\omega^{\mu}\}$. A similar application of the induction hypothesis shows that $\omega^{\mu} \in X^{(\mu)}$ when μ is a limit ordinal.

1.4.4. EXAMPLE. If X is the well-ordered set $\omega^{\mu} \cdot n + 1$ with $1 \leq n < \omega$ and $1 \leq \mu < \omega_1$, then X is the disjoint union of the clopen intervals $[0, \omega^{\mu}]$, $[\omega^{\mu} + 1, \omega^{\mu} \cdot 2], \ldots, [\omega^{\mu} \cdot (n-1) + 1, \omega^{\mu} \cdot n]$. Since $[\omega^{\mu} \cdot k + 1, \omega^{\mu} \cdot (k+1)]$ is homeomorphic to $\omega^{\mu} + 1$, it follows from Example 1.4.3 and the proposition that $X^{(\mu)} = \{\omega^{\mu}, \omega^{\mu} \cdot 2, \ldots, \omega^{\mu} \cdot n\}$ and $X^{(\mu+1)} = \emptyset$.

Notes. The sequence of topological derivatives were introduced by CANTOR [1883] and independently by BENDIXSON [1883]. The calculation of Cantor–Bendixson sequences of the spaces in 1.4.3 and 1.4.4 can be found in a paper of MEYER and PIERCE [1960], but they were probably known earlier.

1.5. Invariants

The topological derivatives can be used to define some natural invariants for the class of metrizable Boolean spaces. By virtue of the Stone duality, these definitions translate to invariants for countable Boolean algebras.

For each non-empty, metrizable Boolean space X, define

$$\nu(X) = \min\{\eta\colon X^{(\eta)} = X^{(\eta+1)}\}\ .$$

Since a strictly decreasing sequence of closed sets in a compact metric space can be at most countable, it follows that $\nu(X)$ is a well-defined, countable ordinal number. It is clear from Proposition 1.4.1(g) that $\nu(X)$ is an invariant of homeomorphism classes.

We will use the notation $K(X)$ for $X^{(\nu(X))}$. By definition, $K(X)$ is perfect: $K(X)' = K(X)$. Indeed, $K(X)$ is the largest perfect subspace of X; it is appropriately called the *perfect kernel* of X. By Proposition 1.4.1(a), $K(X)$ is a (metrizable) Boolean space, so that either $K(X) = \emptyset$ or $K(X) \simeq \mathscr{C}$. If $K(X) = \emptyset$, then X is called *scattered*. The scattered Boolean spaces are exactly the Stone spaces of superatomic Boolean algebras. (See 17.8 in Part I of this Handbook.) In this case, $\nu(X)$ is a non-limit ordinal. In fact, $\nu(X) = \alpha(X) + 1$, where $\alpha(X)$ is the invariant that was defined in Section 17 of Part I.

Another invariant can be defined for a Boolean space X in the following way:

$$\lambda(X) = \min\{\eta\colon X^{(\eta)}\backslash K(X) \text{ is compact}\}\,.$$

Since X is compact, the conditions "$X^{(\eta)}\backslash K(X)$ is compact" and "$X^{(\eta)}\backslash K(X)$ is closed in X" are equivalent. They will be used interchangeably.

1.5.1. PROPOSITION. *Let X be a Boolean space.*
(a) *If $X^{(\eta)}\backslash K(X)$ is compact and $\xi \geq \eta$, then $X^{(\xi)}\backslash K(X)$ is compact.*
(b) $0 \leq \lambda(X) \leq \nu(X) < \omega_1$.
(c) $\nu(X) = 0$ *if and only if* $X = \emptyset$ *or* $X \simeq \mathscr{C}$.
(d) $\lambda(X) = 0$ *if and only if* $X = K(X) \cup Y$, *where Y is compact and scattered.*
(e) *If* $\lambda(X) < \nu(X)$, *then* $\nu(X) = \mu + 1$ *for some ordinal* μ, *and* $1 \leq |X^{(\mu)}\backslash K(X)| < \omega$.

The statement (a) is a consequence of the observation that $X^{(\xi)}\backslash K(X) = X^{(\xi)} \cap (X^{(\eta)}\backslash K(X))$ is an intersection of closed subsets of the compact space $X^{(\eta)}$. The properties (b), (c), and (d) follow directly from the definitions of $\nu(X)$ and $\lambda(X)$. If $\lambda(X) < \nu(X)$, then by compactness and (a), $\bigcap_{\lambda(X)\leq\eta<\nu(X)} X^{(\eta)}\backslash K(X)$ is not empty. Thus, $\nu(X)$ cannot be a limit ordinal, say $\nu(X) = \mu + 1$. Moreover, $X^{(\mu)}\backslash K(X)$ is a compact, non-empty space such that $(X^{(\mu)}\backslash K(X))' = X^{(\nu(X))}\backslash K(X) = \emptyset$. That is, $X^{(\mu)}\backslash K(X)$ is a union of isolated points. By compactness, $X^{(\mu)}\backslash K(X)$ must be finite. □

The last part of this proposition leads to another invariant. For a Boolean space X, define

$$n(X) = |X^{(\mu)}\backslash K(X)| \quad \text{if } \nu(X) = \mu + 1 > \lambda(X)$$

and

$$n(X) = -\infty \qquad\qquad \text{if } \lambda(X) = \nu(X)\,.$$

The choice of $-\infty$ for $n(X)$ in the case that $\lambda(X) = \nu(X)$ seems arbitrary now, but it will simplify the formulation of our next proposition. If X is scattered, then $n(X)$ is the invariant that was introduced in Section 17 of Part I.

The bijective correspondence between isomorphism classes of Boolean algebras and homeomorphism classes of Boolean spaces can be used to transfer the invariants ν, λ, n from Boolean spaces to Boolean algebras: if A is a countable Boolean algebra, define $\nu(A) = \nu(X)$, $\lambda(A) = \lambda(X)$, $n(A) = n(X)$, and $K(A) = \operatorname{Clop} K(X)$, where $X = \operatorname{Ult} A$ is the Stone space of A. It is sometimes useful to have an intrinsic characterization of $\nu(A)$, $\lambda(A)$, $n(A)$, and $K(A)$. The set N of isolated points in the Stone space X of A is open in X. By Proposition 7.18, N corresponds to the ideal $\langle \mathrm{At}\, A\rangle$ that is generated by the atoms of A. The derived space $X' = X\backslash N$ can then be identified with the factor algebra $A_{(1)} = A/\langle \mathrm{At}\, A\rangle$; and the dual of the inclusion mapping $X' \to X$ is the natural projection $A \to A_{(1)}$. By iteration, this process leads to the dual of the Cantor–Bendixson sequence:

$$A = A_{(0)} \to A_{(1)} \to A_{(2)} \to \cdots \to A_{(\xi)} \to \cdots$$

in which $A_{(\eta+1)} = A_{(\eta)}/\langle \mathrm{At}\, A_{(\eta)}\rangle$, and for a limit ordinal $\eta > 0$, $A_{(\eta)}$ is the direct limit of the system $\{A_{(\xi)}: \xi < \eta\}$. With this definition in hand, the invariants ν, λ, and n are defined by

$$\nu(A) = \min\{\eta: \mathrm{At}\, A_{(\eta)} = \emptyset\}\,,$$

$$K(A) = A_{(\nu(A))}\,,$$

$$\lambda(A) = \min\{\eta: K(A) \text{ is a direct factor of } A_{(\eta)}\}\,,$$

$$n(A) = |\mathrm{At}\, A_{(\mu)}| \quad \text{if } \nu(A) = \mu + 1 > \lambda(A)\,,$$

and

$$n(A) = -\infty \quad \text{if } \nu(A) = \lambda(A)\,.$$

Notes. The invariants $\nu(X)$ and $n(X)$ were introduced for scattered spaces in a paper of MAZURKIEWICZ and SIERPIŃSKI [1920]. The ordinal number $\lambda(X)$ occurs in a work of the author (PIERCE [1970]).

1.6. *Additivity properties*

We adopt the usual conventions for sums that involve a natural number n and the symbol $-\infty$; that is, $-\infty + n = n + (-\infty) = -\infty + (-\infty) = -\infty$.

1.6.1. PROPOSITION. *Let $X = Y \uplus Z$, where Y and Z are metrizable Boolean spaces.*

(a) $\nu(X) = \max\{\nu(Y), \nu(Z)\}$.
(b) $\lambda(X) = \max\{\lambda(Y), \lambda(Z)\}$.
(c) $K(X) = K(Y) \uplus K(Z)$.
(d) *If $\nu(Y) < \nu(Z)$, then $n(X) = n(Z)$.*
(e) *If $\nu(Y) = \nu(Z)$, then $n(X) = n(Y) + n(Z)$.*

PROOF. By Proposition 1.4.1, $X^{(\xi)} = Y^{(\xi)} \uplus Z^{(\xi)}$. The statements (a) and (c) follow from this observation. Moreover, $X^{(\xi)}\backslash K(X) = (Y^{(\xi)}\backslash K(Y)) \uplus (Z^{(\xi)}\backslash K(Z))$ is compact if and only if $Y^{(\xi)}\backslash K(Y)$ and $Z^{(\xi)}\backslash K(Z)$ are compact, so that (b) is correct. If $\nu(Y) < \nu(Z)$, then $\nu(X) = \nu(Z) > \nu(Y) \geq \lambda(Y)$ by Proposition 1.5.1(b). It follows from (b) that $n(X) = -\infty$ if and only if $\max\{\lambda(Y), \lambda(Z)\} = \lambda(X) = \nu(X) = \nu(Z)$, in which case $\lambda(Z) = \nu(Z)$ and $n(Z) = -\infty$. Assume that $n(X) \neq -\infty$, so that $\max\{\lambda(Y), \lambda(Z)\} = \lambda(X) < \nu(X) = \nu(Z)$. By Proposition 1.5.1(e), $\nu(X) = \nu(Z) = \mu + 1 > \nu(Y)$. Therefore, $X^{(\mu)}\backslash K(X) = (Y^{(\mu)}\backslash K(Y)) \cup (Z^{(\mu)}\backslash K(Z)) = Z^{(\mu)}\backslash K(Z)$, and $n(X) = |X^{(\mu)}\backslash K(X)| = |Z^{(\mu)}\backslash K(Z)| = n(Z)$. Finally, suppose that $\nu(Y) = \nu(Z)$. If $n(X) = -\infty$, then $\max\{\lambda(Y), \lambda(Z)\} = \lambda(X) = \nu(X) = \nu(Y) = \nu(Z)$. In this case, $n(Y) = -\infty$ or $n(Z) = -\infty$, so that $n(Y) +$

$n(Z) = -\infty = n(X)$. If $\lambda(X) < \nu(X)$, then by Proposition 1.5.1(e), $\nu(Y) = \nu(Z) = \nu(X) = \mu + 1 > \lambda(X) = \max\{\lambda(Y), \lambda(Z)\}$. Hence, $n(X) = |X^{(\mu)} \backslash K(X)| = |Y^{(\mu)} \backslash K(Y)| + |Z^{(\mu)} \backslash K(Z)| = n(Y) + n(Z)$. □

1.7. Lifting decompositions

Topological considerations enable us to lift decompositions of $X^{(\xi)}$ to X. This simple fact is the basis of the proof of the Uniqueness Theorem 1.10.1.

1.7.1. PROPOSITION. *Let X be a Boolean space. If $X^{(\xi)} = U \uplus V$ with U and V clopen in $X^{(\xi)}$, then $X = Y \uplus Z$ with Y and Z clopen in X, and $Y \cap X^{(\xi)} = U$, $Z \cap X^{(\xi)} = V$.*

Indeed, U and V are disjoint, compact subsets of the Boolean space X, so that they can be separated by a clopen set Y: $U \subseteq Y$, $V \subseteq X \backslash Y = Z$. By Proposition 1.4.1(e), $Y \cap X^{(\xi)} = U$ and $Z \cap X^{(\xi)} = V$. □

1.7.2. COROLLARY. *If m is a natural number such that $m \leq |X^{(\xi)} \backslash X^{(\xi+1)}|$, then $X = Y \uplus Z$ with Y and Z clopen in X, and $|Y^{(\xi)}| = m$.*

PROOF. Since the points of $X^{(\xi)} \backslash X^{(\xi+1)}$ are isolated in $X^{(\xi)}$ and $m \leq |X^{(\xi)} \backslash X^{(\xi+1)}|$, there is a clopen subset U of $X^{(\xi)}$ such that $|U| = m$. The proposition therefore gives the desired decomposition of X. □

1.7.3. COROLLARY. *If m is any natural number, and either $\xi < \lambda(X)$ or $\xi + 1 < \nu(X)$, then $X = Y \uplus Z$ with Y and Z clopen in X, and Y is a scattered space such that $\nu(Y) = \xi + 1$, $n(Y) = m$.*

PROOF. By 1.7.2, it is sufficient to show that $X^{(\xi)} \backslash X^{(\xi+1)}$ is infinite. If $X^{(\xi)} \backslash X^{(\xi+1)}$ is finite, then $X^{(\xi+1)}$ is perfect. In this case, $\nu(X) \leq \xi + 1$, $X^{(\xi+1)} = K(X)$, $X^{(\xi)} \backslash K(X)$ is compact, and $\xi \geq \lambda(X)$. □

1.8. Uniform spaces

A non-empty, metrizable Boolean space X is *uniform* if $\lambda(X) = \nu(X)$. Such a space cannot be scattered by Proposition 1.5.1. A countable Boolean algebra is uniform if its Stone space is uniform.

If $X = Y \uplus Z$, where Y and Z are metrizable Boolean spaces, then by Proposition 1.6.1, X is uniform if and only if $\nu(Y) \leq \nu(Z)$ and Z is uniform, or vice versa. In particular, if Y and Z are uniform, then so is X.

1.8.1. THEOREM. *If X is a metrizable Boolean space that is not uniform, then $X = Y \uplus Z$ with Y and Z clopen in X, and*

(a) *Z is scattered, $\nu(Z) = \nu(X)$ and $n(Z) = n(X)$,*

(b) *either Y is empty, or Y is uniform, $\lambda(Y) = \lambda(X)$, and $K(Y) = K(X)$.*

PROOF. Let $\lambda = \lambda(X)$. By the definition of $\lambda(X)$, we have $X^{(\lambda)} = (X^{(\lambda)} \setminus K(X)) \uplus K(X)$, where $X^{(\lambda)} \setminus K(X)$ and $K(X)$ are closed in $X^{(\lambda)}$. An application of Proposition 1.7.1 gives a decomposition $X = Y \uplus Z$ of X into clopen subsets such that $Y^{(\lambda)} = X^{(\lambda)} \cap Y = K(X)$ and $Z^{(\lambda)} = X^{(\lambda)} \cap Z = X^{(\lambda)} \setminus K(X)$. In particular, $K(Y) = K(X)$ and $\nu(Y) \leq \lambda$. On the other hand, $X^{(\xi)} \setminus K(X) = (Y^{(\xi)} \setminus K(Y)) \uplus Z^{(\xi)}$ is not compact if $\xi < \lambda$, so that $Y^{(\xi)} \setminus K(Y)$ is not compact when $\xi < \lambda$. Hence, $\lambda(Y) \geq \lambda \geq \nu(Y) \geq \lambda(Y)$, and Y is either empty or uniform. By Proposition 1.6.1, $K(Z) = \emptyset$, $\nu(Z) = \nu(X)$, and $n(Z) = n(X)$. □

It will follow from the Uniqueness Theorem 1.10.1 that the decomposition of this theorem is unique up to homeomorphism of Y and Z.

Since the algebraic dual of the disjoint union of Boolean spaces is the product of the corresponding Boolean algebras, the algebraic version of the theorem takes the following form.

1.8.2. COROLLARY. *If A is a countable Boolean algebra that is not uniform, then $A = B \times C$, where C is superatomic, $\nu(C) = \nu(A)$ and $n(C) = n(A)$, and either B is the one element algebra, or B is uniform with $\lambda(B) = \lambda(A)$ and $K(B) = K(A)$.*

Notes. Uniform Boolean algebras were introduced by KETONEN [1978]. Theorem 1.8.1 also appears in that fundamental paper.

1.9. The rank function

The *rank function* of a metrizable Boolean space X is the mapping

$$r_X\colon K(X) \to \omega_1$$

that is defined for each $p \in K(X)$ by

$$r_X(p) = \min\{\xi\colon p \not\in (X^{(\xi)} \setminus K(X))^-\}\ .$$

In particular, if $K(X) = \emptyset$, then r_X is the empty mapping.

Properties of the rank function that we will use are easy consequences of some set theoretical observations.

1.9.1. LEMMA. *Let r be a mapping from the non-empty set K to ω_1. For each $\xi < \omega_1$, denote*

$$W_\xi = W_\xi(r) = \{p \in K\colon r(p) > \xi\}\ .$$

(a) *If $\lambda \leq \omega_1$ is the least upper bound of $\{r(p)\colon p \in K\}$, then $K \supseteq W_0 \supseteq W_1 \supseteq \cdots \supseteq W_\xi \supseteq \cdots \supseteq W_\lambda = \emptyset$ and $W_\xi \neq \emptyset$ for all $\xi < \lambda$.*

(b) *If $\eta = \xi + 1$, then $\{p \in K\colon r(p) \geq \eta\} = W_\xi$; if η is a limit ordinal, then $\{p \in K\colon r(p) \geq \eta\} = \bigcap_{\xi < \eta} W_\xi$.*

(c) *For all $p \in K$, $r(p) = \min\{\xi\colon p \not\in W_\xi\}$.*

We use the lemma with $K = K(X)$ and $r = r_X$. It is clear from the definition of r_X that

(1) $$W_\xi(r_X) = K(X) \cap (X^{(\xi)} \backslash K(X))^- .$$

In particular, $W_\xi(r_X)$ is closed in X. Therefore, by part (b) of the lemma, the function r_X is upper semicontinuous: $\{p \in K(X): r_X(p) \geq \eta\}$ is closed for all $\eta < \omega_1$.

1.9.2. PROPOSITION. *The rank function r_X is an upper semicontinuous mapping of $K(X)$ to ω_1 such that $\lambda(X) = l.u.b.\ \{r_X(p): p \in K(X)\}$. If Y is a clopen subset of X, then $r_Y = r_X \mid K(Y)$.*

PROOF. By (1) and the definition of λ, it is clear that $W_\eta(r_X) = \emptyset$ if and only if $\eta \geq \lambda(X)$. Thus, l.u.b. $\{r_X(p): p \in K(X)\} = \lambda(X)$. (Note that if $K(X) = \emptyset$, then $\lambda(X) = 0 = \text{l.u.b. } \emptyset$.) The second statement of the proposition follows from part (c) of the lemma, because $X = Y \mathbin{\dot\cup} Z$ with Y and Z clopen implies $K(X) = K(Y) \mathbin{\dot\cup} K(Z)$ and $W_\xi(X) = W_\xi(Y) \mathbin{\dot\cup} W_\xi(Z)$. □

It follows from the proposition that if X is a uniform Boolean space, then the invariants $\nu(X)$, $\lambda(X)$, and $n(X)$ are determined by r_X. At the opposite extreme, the rank function r_X is of no interest when X is scattered. Even in the mixed case, r_X determines $\lambda(X)$. However, $\lambda(X)$ has the virtue of being an invariant of the isomorphism types of X, while r_X is not.

1.10. The uniqueness theorem

Here is the first major result of this section.

1.10.1. THEOREM. *If X and Y are metrizable Boolean spaces, then X is homeomorphic to Y if and only if $\nu(X) = \nu(Y)$, $\lambda(X) = \lambda(Y)$, $n(X) = n(Y)$, and there is a homeomorphism ϕ of $K(X)$ to $K(Y)$ such that $r_X = r_Y \circ \phi$.*

PROOF. If $\chi: X \to Y$ is a homeomorphism, then $\chi(X^{(\xi)}) = Y^{(\xi)}$ for all $\xi < \omega_1$. It follows that $\chi(K(X)) = K(Y)$, $\nu(X) = \nu(Y)$, $\lambda(X) = \lambda(Y)$, and $n(X) = n(Y)$. Moreover, if ϕ is the restriction of χ to $K(X)$, then $r_X = r_Y \circ \phi$. The converse is obtained from the topological version of Vaught's Theorem. Define the relation R on the class of metrizable Boolean spaces by XRY if
(1) $\nu(X) = \nu(Y)$, $\lambda(X) = \lambda(Y)$, $n(X) = n(Y)$, and
(2) $r_X = r_Y \circ \phi$ for some homeomorphism ϕ: $K(X) \to K(Y)$.
Clearly, the conditions (1) are symmetric; so is the condition (2) because $r_X = r_Y \circ \phi$ if and only if $r_Y = r_X \circ \phi^{-1}$. Thus, R is symmetric. If $XR\emptyset$, then $\nu(X) = 0$ and $r_X = \emptyset$, so that $X = \emptyset$ by Proposition 1.5.1(c). The main thing to show is that R satisfies the third condition of Definition 1.2.1. Our strategy is to use the decomposition of Theorem 1.8.1 to split the problem into the cases of scattered and uniform spaces. This detour could be avoided, but the shortcut

would take us through some ugly case analyses. Several preliminary remarks are needed. They are consequences of the Propositions 1.5.1, 1.6.1, and 1.9.2.

(3) If X_1RY_1 and X_2RY_2, then $(X_1 \uplus X_2)R(Y_1 \uplus Y_2)$.

(4) If X and Y are scattered, then XRY if and only if $\nu(X) = \nu(Y)$ and $n(X) = n(Y)$.

(5) If X and Y are uniform, then XRY if and only if there is a homeomorphism ϕ: $K(X) \to K(Y)$ such that $r_X = r_Y \circ \phi$.

(6) If X and Y are uniform, Z and W are scattered, and $(X \uplus Z)R(Y \uplus W)$, then XRY; moreover, if $\nu(X) < \nu(Z)$, then ZRW.

Suppose that XRY and $Y = Y_1 \uplus Y_2$, where Y_1 and Y_2 are metrizable Boolean spaces. Our objective is to obtain a decomposition $X = X_1 \uplus X_2$ so that X_1RY_1 and X_2RY_2. It can be assumed that Y_1 and Y_2 are not empty. By Theorem 1.8.1, there exist decompositions $X = X_u \uplus X_s$, $Y_1 = Y_{1u} \uplus Y_{1s}$, $Y_2 = Y_{2u} \uplus Y_{2s}$ into uniform (or empty) and scattered clopen subspaces. It follows from (6) that $X_uR(Y_{1u} \uplus Y_{2u})$; and either $X_sR(Y_{1s} \uplus Y_{2s})$, or $\nu(X_s) \leq \nu(X_u)$, in which case X is uniform. By (3), the proof is reduced to three cases: (i) Y_1 and Y_2 are scattered; (ii) Y_1 and Y_2 are uniform; and (iii) X and Y_1 are uniform, and Y_2 is scattered. Assume that Y_1 and Y_2 are scattered. Then $r_X = r_Y \circ \phi^{-1} = \emptyset$ implies that X is scattered, and $\nu(X) = \nu(Y) = \max\{\nu(Y_1), \nu(Y_2)\}$ by Proposition 1.6.1(a). If $\nu(Y_1) = \nu(Y_2)$, then $n(X) = n(Y) = n(Y_1) + n(Y_2)$ with $n(Y_1) \geq 1$ and $n(Y_2) \geq 1$. It follows from 1.7.2 that $X = X_1 \uplus X_2$, where $\nu(X_1) = \nu(X_2) = \nu(X) = \nu(Y) = \nu(Y_1) = \nu(Y_2)$, $n(X_1) = n(Y_1)$, and $n(X_2) = n(Y_2)$. Hence, X_1RY_1 and X_2RY_2 by (4). Assume that $\nu(Y_1) < \nu(Y_2)$. Since $\lambda(Y_1) = 0 < \nu(Y_1)$, Proposition 1.5.1(e) shows that $\nu(Y_1)$ is not a limit ordinal. Therefore, 1.7.3 implies that there is a decomposition $X = X_1 \uplus X_2$ such that $\nu(X_1) = \nu(Y_1)$ and $n(X_1) = n(Y_1)$. It follows from Proposition 1.6.1(a) and (d) that $\nu(X_2) = \nu(Y_2)$ and $n(X_2) = n(Y_2)$. Therefore, X_1RY_1 and X_2RY_2. If $\nu(Y_2) < \nu(Y_1)$, the same argument is applicable. Thus, the proof is finished in case (i). Assume that Y_1 and Y_2 are uniform. Then Y is uniform, and therefore so is X. The hypothesis XRY guarantees the existence of a homeomorphism ϕ: $K(X) \to K(Y) = K(Y_1) \uplus K(Y_2)$ such that $r_X = r_Y \circ \phi$. By Proposition 1.7.1, there is a decomposition $X = Z_1 \uplus Z_2$ into clopen subsets such that $Z_1 \cap K(X) = \phi^{-1}(K(Y_1))$ and $Z_2 \cap K(X) = \phi^{-1}(K(Y_2))$. Hence, $r_{Z_1} = r_{Y_1} \circ \phi_1$, $r_{Z_2} = r_{Y_2} \circ \phi_2$, with $\phi_1 = \phi \mid K(Z_1)$, $\phi_2 = \phi \mid K(Z_2)$. Thus, $\lambda(Z_1) = \lambda(Y_1)$, and $\lambda(Z_2) = \lambda(Y_2)$, according to Proposition 1.9.2. If Z_1 and Z_2 are uniform, then (5) yields Z_1RY_1 and Z_2RY_2, so that $X_1 = Z_1$, $X_2 = Z_2$ will do the job. We can assume that $\nu(Y_1) \leq \nu(Y_2)$. In this case, $\lambda(Z_1) = \lambda(Y_1) \leq \lambda(Y_2) = \lambda(Z_2)$, and $\lambda(Z_2) = \lambda(X) = \nu(X) = \max\{\nu(Z_1), \nu(Z_2)\} \geq \lambda(Z_2)$. Hence, $\nu(Z_1) \leq \nu(Z_2) = \lambda(Z_2)$. By Theorem 1.8.1, $Z_1 = Z_{1u} \uplus Z_{1s}$ with Z_{1u} uniform, $K(Z_1) = K(Z_{1u})$, Z_{1s} scattered, and $\nu(Z_{1s}) = \nu(Z_1) \leq \nu(Z_2)$. It follows that $X = X_1 \uplus X_2$, where $X_1 = Z_{1u}$ and $X_2 = Z_{1s} \uplus Z_2$ are uniform; and $r_{X_1} = r_{Z_1} = r_{Y_1} \circ \phi$, $r_{X_2} = r_{Z_2} = r_{Y_2} \circ \phi$. Thus, X_1RY_1, X_2RY_2 by our previous observation. Finally, suppose that X and Y_1 are uniform, and Y_2 is scattered. In this case, $\nu(Y_2) \leq \nu(Y) = \lambda(Y) = \lambda(Y_1) = \nu(Y_1)$, and $\nu(Y_2)$ is a non-limit ordinal by Proposition 1.5.1(e). By 1.7.3, $X = X_1 \uplus X_2$, where X_2 is scattered, $\nu(X_2) = \nu(Y_2)$ and $n(X_2) = n(Y_2)$. It follows from (4) that X_2RY_2. Moreover, $\lambda(X) = \lambda(X_1) \leq \nu(X_1) \leq \nu(X) = \lambda(X)$, so that X_1 is uniform. Hence, X_1RY_1 by (6). The proof of case (iii) is therefore finished. Consequently, R is a V-relation. By Vaught's Theorem, the Uniqueness Theorem is proved. $\square$

1.10.2. COROLLARY. *Let X be a metrizable Boolean space that is not uniform and not scattered. If $X = Y_2 \uplus Z_1 = Y_2 \uplus Z_2$, where Y_1 and Y_2 are uniform and Z_1 and Z_2 are scattered, then $Y_1 \simeq Y_2$ and $Z_1 \simeq Z_2$.*

PROOF. Since $\nu(Y_1) = \lambda(Y_1) = \lambda(X) < \nu(X) = \nu(Z_1)$, it follows from (6) above that Y_1RY_2 and Z_1RZ_2. Thus, $Y_1 \simeq Y_2$ and $Z_1 \simeq Z_2$. □

1.10.3. COROLLARY. *If X is the Stone space of a non-trivial countable, superatomic Boolean algebra, then $X \simeq \omega^{\mu} \cdot n + 1$ for a unique, countable ordinal number μ, and a unique natural number n.*

PROOF. Since X is scattered and not empty, the theorem implies that X is determined up to homeomorphism by the countable ordinal μ such that $\nu(X) = \mu + 1$ and the natural number $n = n(X)$. By Example 1.4.4, $\nu(\omega^{\mu} \cdot n + 1) = \mu + 1$ and $n(\omega^{\mu} \cdot n + 1) = n$. □

Notes. The Uniqueness Theorem is a consequence of the main result in a paper by the author (PIERCE [1970]). The original proof was a nightmare of transfinite induction. It was pointed out by BREHM [1975] that a much simpler proof of the theorem could be based on Vaught's Theorem. In his paper cited earlier, Ketonen used this approach to prove the result for uniform spaces. The Corollary 1.10.3 is an old theorem due to MAZURKIEWICZ and SIERPIŃSKI [1920].

1.11. Existence theorem

The uniqueness theorem raises a natural question: What combinations of ν, λ, n, r occur as the invariants associated with a metrizable Boolean space? The Decomposition Theorem 1.8.1 essentially reduces this problem to the characterization of rank functions.

1.11.1. THEOREM. *If r is an upper semicontinuous mapping of the Cantor space $\mathscr{C}$ to ω_1, then there is a uniform Boolean space X such that $K(X) = \mathscr{C}$ and $r_X = r$.*

PROOF. Denote $V_\xi = \{p \in \mathscr{C}: r(p) \geq \xi\}$. Since r is upper semicontinuous, these sets are closed. If λ is the least upper bound of r, then $V_\lambda \neq \emptyset$. This assertion is clear if λ is not a limit ordinal; and it is a consequence of the compactness of $\mathscr{C}$ in case λ is a limit ordinal, since $V_\lambda = \bigcap_{\xi<\lambda} V_\xi$. Thus, r attains its upper bound λ, which must therefore be countable. It follows that there is a countable set $U = \{p_0, p_1, p_2, \ldots\} \subseteq \mathscr{C}$ such that $V_\xi = (U \cap V_\xi)^-$ for all $\xi \leq \lambda$. Let X be a metrizable Boolean space such that:

(1) $X = \mathscr{C} \uplus \biguplus_{n<\omega} T_n$;

(2) T_n is open in X for all $n < \omega$;

(3) there is a homeomorphism θ_n: $\omega^{r(p_n)} + 1 \to T_n \cup \{p_n\}$ such that $\theta_n(\omega^{r(p_n)}) = p_n$;

(4) if N is open in $\mathscr{C}$, then there exists M open in X satisfying $M \cap \mathscr{C} = N$ and $M \cap T_m = \emptyset$ for all $m < \omega$ such that $p_m \not\in N$.

We will later describe the construction of such a space. The theorem will be proved by showing that $\lambda(X) = \nu(X) = \lambda$ (so that X is uniform), $K(X) = \mathscr{C}$, and $r_X = r$. Since $\mathscr{C}' = \mathscr{C}$ and T_n is an open subset of X that is homeomorphic to the ordered set $\omega^{r(p_n)}$, it follows from Proposition 1.4.1(f) and Example 1.4.3 that $X^{(\xi)} = \mathscr{C} \cup \bigcup_{n<\omega} T_n^{(\xi)}$, and $T_n^{(\xi)} \neq \emptyset$ if and only if $\xi < r(p_n)$. Moreover, $V_\lambda \neq \emptyset$ and U is dense in V_λ, so that there is some $p_n \in V_\lambda$. For such an n, $p_n \in (T_n^{(\xi)})^-$, when $\xi < \lambda$ by (3). In particular, $X^{(\xi)} \supset \mathscr{C}$ if $\xi < \lambda$. On the other hand, since $r(p_m) \leq \lambda$, it follows that $T_m^{(\lambda)} = \emptyset$ for all $m < \omega$. Hence, $X^{(\lambda)} = \mathscr{C}$, $K(X) = \mathscr{C}$, and $\nu(X) = \lambda$. Suppose that $q \in \mathscr{C}$ and $r(q) = \xi$. Since r is upper semicontinuous, there is a neighborhood N of q in $\mathscr{C}$ such that $r(p) < \xi + 1$ for all $p \in N$. In particular, $r(p_n) \leq \xi$ for all $p_n \in N$. Thus, $p_n \in N$ implies $T_n^{(\xi)} = \emptyset$. By (4), there is an open set M in X such that $M \cap \mathscr{C} = N$ and $M \cap T_m = \emptyset$ whenever $p_m \not\in N$. That is, M is a neighborhood of q, and $M \cap (X^{(\xi)} \backslash K(X)) \subseteq M \cap \bigcup \{T_m\colon p_m \not\in N\} = \emptyset$. This argument shows that $q \not\in (X^{(\xi)} \backslash K(X))^-$, which yields $r_X(q) \leq \xi$ by the definition of r_X. To reverse this inequality, note that if $p_n \in V_\xi$, then $r(p_n) \geq \xi$. Since $\theta_n\colon\ \omega^{r(p_n)} + 1 \to T_n \cup \{p_n\}$ is a homeomorphism with $\theta_n(\omega^{r(p_n)}) = p_n$, it follows that $p_n \in (X^{(\eta)} \backslash K(X))^-$ for all $\eta < \xi$. Thus, $r_X(p_n) \geq \xi$. The upper semicontinuity of r_X implies that $r_X(p) \geq \xi$ for all $p \in (U \cap V_\xi)^- = V_\xi$. In particular, $r_X(q) \geq \xi$. We have therefore proved that $r_X = r$. Finally, $\lambda(X) =$ l.u.b. $\{r_X(p)\colon p \in K(X)\} =$ l.u.b. $\{r(p)\colon p \in K(X)\} = \lambda$ by Proposition 1.9.2. The proof of the theorem will be finished by describing a space X that satisfies (1)–(4). Define $S = \mathscr{C} \times \Pi_{n<\omega} (\omega^{r(p_n)} + 1)$ with the product topology. Thus, S is a metrizable Boolean space, the elements of which are the sequences $(p, \xi_0, \xi_1, \ldots)$ with $p \in \mathscr{C}$ and $\xi_n \leq \omega^{r(p_n)}$. Let $\pi\colon S \to \mathscr{C}$ and $\pi_n\colon S \to \omega^{r(p_n)} + 1$ be the projection mappings. Define $\theta\colon \mathscr{C} \to S$ by $\theta(p) = (p, \omega^{r(p_0)}, \omega^{r(p_1)}, \ldots)$, and $\theta_n\colon \omega^{r(p_n)} + 1 \to S$ by $\theta_n(\xi) = (p_n, \omega^{r(p_0)}, \ldots, \omega^{r(p_{n-1})}, \xi, \omega^{r(p_{n+1})}, \ldots)$. Plainly, θ and θ_n are continuous embeddings. Define $W_n = \{\xi\colon \xi < \omega^{r(p_n)}\}$, $T_n = \theta_n(W_n)$, and $X = \theta(\mathscr{C}) \uplus \biguplus_{n<\omega} T_n$. Since

$$S \backslash X = \bigcup_{n<m<\omega} \pi_n^{-1}(W_n) \cap \pi_m^{-1}(W_m) \cup \bigcup_{n<\omega} \pi_n^{-1}(W_n) \cap \pi^{-1}(\mathscr{C} \backslash \{p_n\})$$

is open in S, it follows that X is a closed subspace of S. Hence, X is a metrizable Boolean space. Moreover, $X \backslash T_n = X \cap \pi_n^{-1}(\{\omega^{r(p_n)}\})$ is closed, so that T_n is an open subset of X. It is clear that θ_n maps $\omega^{r(p_n)} + 1$ homeomorphically to $T_n \cup \{\theta(p_n)\}$ with $\theta_n(\omega^{r(p_n)}) = \theta(p_n)$. If N is open in $\mathscr{C}$, then $M = X \cap \pi^{-1}(N)$ is an open subset of X such that $M \cap \theta(\mathscr{C}) = \theta(N)$ and $M \cap T_m = \emptyset$ for all m such that $p_m \not\in N$. Therefore, if θ is used to identify $\mathscr{C}$ with its image in X, then conditions (1), (2), (3), and (4) are satisfied. □

1.11.2. COROLLARY. *Let $\lambda \leq \mu < \omega_1$ and $1 \leq n < \omega$. If $r\colon \mathscr{C} \to \omega_1$ is an upper semicontinuous mapping such that l.u.b. $\{r(p)\colon p \in \mathscr{C}\} = \lambda$, then there is a metrizable Boolean space Y such that $\nu(Y) = \mu + 1$, $\lambda(Y) = \lambda$, $n(Y) = n$, and $r_Y = r$.*

In fact, if X is uniform and such that $r_X = r$, then $Y = X \uplus (\omega^\mu \cdot n + 1)$ has the required invariants. □

1.11.3. COROLLARY. *There exist continuum many homeomorphism classes of uniform Boolean spaces Y such that $\lambda(Y) = 1$.*

PROOF. Let $\mathscr{C} = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ be a strictly decreasing chain of closed subsets of the Cantor space $\mathscr{C}$. Define r: $\mathscr{C} \to \omega + 1$ by $r(p) = n$ if $p \in V_n \setminus V_{n+1}$, $r(p) = \omega$ if $p \in \bigcap_{n<\omega} V_n$. Then r is upper semicontinuous and $r(\mathscr{C}) = \omega + 1$. For each strictly increasing mapping α: $\omega \to \omega$, define r_α: $\mathscr{C} \to \omega + 1$ by $r_\alpha(p) = \alpha(r(p))$ if $r(p) < \omega$ and $r_\alpha(p) = \omega$ if $r(p) = \omega$. Then r_α is upper semicontinuous with $r_\alpha(\mathscr{C}) = \alpha(\omega) \cup \{\omega\}$. Let X_α be a uniform Boolean space such that $r_{X_\alpha} = r_\alpha$. If $X_\alpha \simeq X_\beta$, then there is a homeomorphism ϕ: $\mathscr{C} \to \mathscr{C}$ such that $r_\alpha = r_\beta \circ \phi$. It follows that $\alpha(\omega) = \beta(\omega)$, and therefore $\alpha = \beta$ because these functions are strictly increasing. Since there are continuum many strictly increasing maps from ω to ω, this construction provides the required number of metrizable Boolean spaces. To achieve the same result with spaces of length 1, note that each X_α can be realized as a closed subset of $\mathscr{C}$. Let s_α be the characteristic function of X_α. Then, s_α: $\mathscr{C} \to \{0, 1\}$ is upper semicontinuous because X_α is closed in $\mathscr{C}$. Hence, there is a uniform Boolean space Y_α such that $r_{Y_\alpha} = s_\alpha$. Then $\alpha \neq \beta$ implies $Y_\alpha \not\simeq Y_\beta$; otherwise there is a homeomorphism ϕ of $\mathscr{C}$ to itself such that $s_\alpha = s_\beta \circ \phi$, and therefore $\phi(Y_\alpha) = Y_\beta$. Finally, $\lambda(Y_\alpha) = \text{l.u.b.}\ \{s_\alpha(p)\colon p \in \mathscr{C}\} = 1$. $\square$

Notes. A variant of Theorem 1.11.1 and its Corollary 1.11.2 were proved by the author in the paper that was cited in subsection 1.10. Our proof here uses a construction that was introduced by REICHBACH [1958]. The Uniqueness and Existence Theorems were used by the author to prove a conjecture of Feiner that every countable Boolean algebra has an ordered basis that is a lexicographic sum of well ordered sets, indexed by the ordered set of rational numbers (PIERCE [1973]). This result was obtained from Vaught's Theorem (independently, and at about the same time) by D. Cossack (unpublished).

1.12. Monoids and measures

The essential content of the Existence and Uniqueness Theorems is that the isomorphism classes of uniform Boolean algebras are in one-to-one correspondence with the equivalence classes of upper semicontinuous mapping of the Cantor set $\mathscr{C}$ to ω_1. Here, equivalence of two such upper semicontinuous mappings r and s means that $r = s \circ \phi$ for some homeomorphism ϕ of $\mathscr{C}$ to itself. Our next major objective is to classify the equivalence classes of upper semicontinuous mappings of $\mathscr{C}$ to ω_1. It seems to be more natural to pursue this program in the algebraic rather than the topological context. Thus, our first step is a straightforward translation of Theorems 1.10.1 and 1.11.1 into the language of Boolean algebras.

We must first determine the algebraic analog of an upper semicontinuous mapping of $\mathscr{C}$ to ω_1. This is a straightforward project. The dual of the Cantor space $\mathscr{C}$ is the free Boolean algebra $\mathscr{F}$ on a countably infinite set of free generators. The upper semicontinuous mappings of $\mathscr{C}$ to ω_1 correspond to functions on $\mathscr{F}$ that are additive in a suitable sense. The target of these functions is the set $\mathscr{W} = \omega_1 \cup \{o\}$, that is, the set of all countable ordinal numbers together with a new zero element o such that $o < \xi$ for all $\xi \in \omega_1$. In later work it will be necessary to consider mappings from $\mathscr{F}$ to more general algebraic systems.

In the rest of this section, the term *m-monoid* (which is short for measure monoid) will designate a commutative monoid M whose zero element is the unique unit. Thus, M is a set with a commutative and associative binary operation $+$ and a zero element 0 such that $a+0=a$ for all $a \in M$; and if $a+b=0$, then $a=b=0$. This last condition is equivalent to the requirement that $M^* = M \backslash \{0\}$ is a subsemigroup of M. The set $\mathscr{W}$ is an m-monoid in which $a+b=\max\{a, b\}$ and o is the zero element. Indeed, any totally ordered set with smallest element 0 can be turned into an m-monoid with the maximum defining the binary operation $+$.

The class of m-monoids becomes a category in which a morphism from M to N is defined to be a monoid homomorphism Φ such that $\Phi(M^*) \subseteq N^*$. Equivalently, Φ is a semigroup homomorphism such that $\Phi(a)=0$ if and only if $a=0$. It is clear that the collection of such mappings is closed under composition and includes all identity maps of m-monoids. That is, the conditions that define a category are fulfilled.

1.12.1. DEFINITION. Let A be a Boolean algebra, and suppose that M is an m-monoid. An M-measure on A is a mapping σ: $A \to M$ such that

(i) $\sigma(x \dot{+} y) = \sigma(x) + \sigma(y)$ for all pairs (x, y) of disjoint elements in A, and

(ii) $\sigma(x) = 0$ if and only if $x = 0$.

When there is no danger of confusion, the expression "M-measure" will be shortened to "measure".

The set of all M-measures on the countable free Boolean algebra $\mathscr{F}$ will be denoted by $\mathscr{M}(M)$. In the special case that $M = \mathscr{W}$, this notation will be shortened to $\mathscr{M}$.

Two observations need to be made; they .are obvious consequences of the definition above. First, if $\sigma \in \mathscr{M}(M)$ and Φ: $M \to N$ is a morphism of m-monoids, then $\Phi \circ \sigma \in \mathscr{M}(N)$. The importance of the second remark will be clear from the next proposition. If k is an automorphism of $\mathscr{F}$ and $\sigma \in \mathscr{M}(M)$, then $\sigma \circ k \in \mathscr{M}(M)$. It follows that there is an equivalence relation on $\mathscr{M}(M)$ defined by

$$\sigma \cong \tau \quad \text{if } \tau = \sigma \circ k \text{ for some } k \in \operatorname{Aut} \mathscr{F} .$$

We can now fulfill our promise to relate the upper semicontinuous mappings r from $\mathscr{C}$ to ω_1 with the $\mathscr{W}$-measures σ on $\mathscr{F}$. The correspondence is defined by:

$$\sigma_r(O_{\mathscr{F}}) = o , \qquad \sigma_r(x) = \text{l.u.b. } \{r(p)\colon p \in x\} \quad \text{if } x \in \mathscr{F} \backslash \{O_F\} ;$$

$$r_\sigma(p) = \min\{\sigma(x)\colon p \in x \in \mathscr{F}\} \qquad \text{if } p \in \mathscr{C} .$$

1.12.2. PROPOSITION. (a) *σ_r is a $\mathscr{W}$-measure on $\mathscr{F}$ and r_σ is an upper semicontinuous mapping of $\mathscr{C}$ to ω_1.*

(b) *The correspondences $r \mapsto \sigma_r$, $\sigma \mapsto r_\sigma$ are inverse bijections.*

(c) *If r, s: $\mathscr{C} \to \omega_1$ are upper semicontinuous mappings, then there is a homeomorphism ϕ of $\mathscr{C}$ to itself such that $s = r \circ \phi$ if and only if $\sigma_s = \sigma_r \circ k$ for some $k \in \operatorname{Aut} \mathscr{F}$, that is, $\sigma_r \cong \sigma_s$.*

The statements in this proposition are routine consequences of the definitions of σ_r and r_σ. For instance, the equality $\sigma_{r_\sigma} = \sigma$ is obtained as follows. If $\sigma(x) = \xi$ and $p \in x$, then $r_\sigma(p) \leq \xi$; hence, $\sigma_{r_\sigma}(x) \leq \xi$. If $\sigma_{r_\sigma}(x) < \xi$, then $r_\sigma(p) < \xi$ for all $p \in x$. Thus, for each $p \in x$, there exists $y \in \mathscr{F}$ such that $p \in y$ and $\sigma(y) < \xi$. The compactness of x yields a finite set $\{y_0, \ldots, y_{n-1}\} \subseteq \mathscr{F}$ such that $x = y_0 \dotplus \cdots \dotplus y_{n-1}$ and $\sigma(y_i) < \xi$ for all $i < n$. The additivity of σ then leads to the contradiction $\sigma(x) = \sigma(y_0) + \cdots + \sigma(y_{n-1}) = \max\{\sigma(y_0), \ldots, \sigma(y_{n-1})\} < \xi = \sigma(x)$. The automorphism k in statement (c) is the dual of the homeomorphism ϕ. In this context, ϕ is related to k by $k(x) = \{\phi^{-1}(p)\colon p \in x\}$ and $\phi(p) = \{k^{-1}(x)\colon p \in x \in \mathscr{F}\}$. □

1.12.3. COROLLARY. *For each $\sigma \in \mathcal{M}$ there is a uniform Boolean algebra B_σ whose Stone space X satisfies $K(X) = \mathscr{C}$ and $r_X = r_\sigma$; every uniform Boolean algebra is isomorphic to B_σ for some $\sigma \in \mathcal{M}$; if $\sigma, \tau \in \mathcal{M}$ are such that $B_\sigma \cong B_\tau$, then $\sigma \cong \tau$; conversely, $\sigma \cong \tau$ in $\mathcal{M}$ implies $B_\sigma \cong B_\tau$.*

1.13. Derived monoids

Every uniform space is determined by its rank function, and by the translation developed in the previous subsection, every uniform Boolean algebra is determined by its corresponding measure. Thus, to classify uniform Boolean algebras it is sufficient to find a complete set of invariants for the equivalence classes of $\mathscr{W}$-measures on the free Boolean algebra $\mathscr{F}$.

If $r = r_X$ is the rank function of the uniform Boolean space X, then the associated measure σ_r satisfies $\sigma_r(1) = \text{l.u.b. } \{r(p)\colon p \in K(X)\} = \lambda(X)$ by Proposition 1.4.1 and the definition of σ_r. In particular, $\sigma_r(1)$ is an invariant of the homeomorphism classes of uniform spaces. A refinement of this invariant can be obtained by using partitions of $1_{\mathscr{F}}$ in the Boolean algebra $\mathscr{F}$. For each $\sigma \in \mathcal{M}$, denote $\Delta\sigma(1) = \{(\sigma x_0, \ldots, \sigma x_{n-1})\colon x_i \in \mathscr{F},\ 1 = x_0 \dotplus \cdots \dotplus x_{n-1}\}$. Thus, $\Delta\sigma(1)$ is a countable set of finite sequences of elements in $\mathscr{W}$. If $k \in \operatorname{Aut} \mathscr{F}$, then the mapping $(x_0, \ldots, x_{n-1}) \mapsto (kx_0, \ldots, kx_{n-1})$ is a permutation of the set of all partitions of $1_{\mathscr{F}}$ in $\mathscr{F}$. Therefore, $\Delta(\sigma \circ k)(1) = \Delta\sigma(1)$, so that $\Delta\sigma(1)$ is indeed an invariant of the equivalence classes of $\mathscr{W}$-measures. These invariants still do not answer the question "when is $\sigma \cong \tau$?" but they give more information than $\sigma(1)$. For example, let X be a non-empty, proper, closed subset of $\mathscr{C}$, r the characteristic function of X, and σ the corresponding measure on $\mathscr{F}$. If X is infinite, then $\Delta\sigma(1)$ consists of all sequences $(a_0, \ldots, a_{n-1})$ with $a_i = o$, O, or 1 and $a_i = 1$ for at least one $i < n$. If $|X| = m < \omega$, then $\Delta\sigma(1)$ consists of all such sequences with at most m of the $a_i = 1$. On the other hand, $\sigma(1) = 1$ is independent of the cardinality of X. Using the homogeneity of $\mathscr{C}$, it is easy to show that $\Delta\sigma(1)$ does indeed characterize the uniform spaces whose rank function is the characteristic function of a finite set. Thus, the passage from $\sigma(1)$ to $\Delta\sigma(1)$ represents some progress. Our aim is to show that by iterating this construction (transfinitely) a complete set of invariants is obtained for uniform Boolean algebras.

If M is a set, then the collection of all finite, non-empty sequences of elements from M is denoted by $M^{<\omega}$. The sequences in $M^{<\omega}$ will be distinguished from elements of M by using boldface symbols. Typically, the sequence $(a_0, \ldots, a_{n-1}) \in M^{<\omega}$ will be denoted by $(a_i)_{i<n}$ or $\boldsymbol{a}$. In this case, n is called the length of $\boldsymbol{a}$, and it is denoted by $l(\boldsymbol{a})$. If M is a commutative monoid, then the addition operation on M extends to a partial operation on $M^{<\omega}$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are sequences of equal length n, then the sum $\boldsymbol{a}+\boldsymbol{b}$ is defined componentwise, that is $(a_0, \ldots, a_{n-1}) + (b_0, \ldots, b_{n-1}) = (a_0 + b_0, \ldots, a_{n-1} + b_{n-1})$. Of course, this partial operation satisfies the commutative and associative laws when they are meaningful. Moreover, $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$, where $\boldsymbol{0}$ denotes the sequence of length $l(\boldsymbol{a})$ that consists of zeros. If M is an m-monoid, then it is clear that $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{0}$ implies that $\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{0}$. In other words, $M^{<\omega}$ is a partial m-monoid.

If M is a commutative monoid, then there is an important mapping $T\colon M^{<\omega} \to M$ that is defined by

$$T((a_i)_{i<n}) = \sum_{i<n} a_i .$$

We will call T the *trace map*. Since M is commutative, $T(\boldsymbol{a}+\boldsymbol{b}) = T(\boldsymbol{a}) + T(\boldsymbol{b})$, whenever $l(\boldsymbol{a}) = l(\boldsymbol{b})$. Moreover, if M is an m-monoid, then $T(\boldsymbol{a}) = 0$ if and only if $\boldsymbol{a}$ is a sequence of zeros.

One more concept is needed before we can state the definition of derived monoids. Let $\boldsymbol{a} = (a_0, \ldots, a_{n-1})$ and $\boldsymbol{b} = (b_0, \ldots, b_{m-1})$ be elements of $M^{<\omega}$, where M is a commutative monoid. If there is a mapping $\lambda\colon n \to m$ such that $b_j = \Sigma\,\{a_i\colon \lambda(i) = j\}$ for all $j < m$, then $\boldsymbol{a}$ is called a *refinement* of $\boldsymbol{b}$. In this case we write $\boldsymbol{a} < \boldsymbol{b}$. Plainly, $<$ is reflexive and transitive. Two cases of the refinement concept have special importance. First, if λ is a permutation of n, then the sequence $(a_0, \ldots, a_{n-1})$ is a refinement of the rearranged list $(a_{\lambda(0)}, \ldots, a_{\lambda(n-1)})$. By symmetry, $(a_{\lambda(0)}, \ldots, a_{\lambda(n-1)}) < (a_0, \ldots, a_{n-1})$. Second, if we take λ to be the inclusion map of n to $n+1$, then it follows that $(a_0, \ldots, a_{n-1})$ refines $(a_0, \ldots, a_{n-1}, 0)$ (interpreting the empty sum as 0). Conversely, $(a_0, \ldots, a_{n-1}, 0)$ is a refinement of $(a_0, \ldots, a_{n-1})$ via the mapping λ such that $\lambda(i) = i$ for $i < n$, $\lambda(n) = n-1$. Our final remark on the concept of refinement is that if $\boldsymbol{a}$ refines $\boldsymbol{b}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same trace: $\boldsymbol{a} < \boldsymbol{b}$ implies $T(\boldsymbol{a}) = T(\boldsymbol{b})$.

1.13.1. DEFINITION. Let M be an m-monoid. The *derived* monoid ΔM of M is the set of all countably infinite subsets α of $M^{<\omega}$ that satisfy the following three conditions.

(i) *Collection property* (C.P.): if $\boldsymbol{a} \in \alpha$ and $\boldsymbol{a} < \boldsymbol{b}$, then $\boldsymbol{b} \in \alpha$.

(ii) *Refinement property* (R.P.): if $(a_i)_{i<n}$ and $(b_j)_{j<m}$ are members of α, then there exists $(c_{ij})_{i<n,j<m}$ in α such that $a_i = \Sigma_{j<m}\, c_{ij}$ for all $i<n$ and $b_j = \Sigma_{i<n}\, c_{ij}$ for all $j < m$.

(iii) *Splitting property* (S.P.): if $(a_i)_{i<n} \in \alpha$ and $a_0 \in M^*$, then there exist b, $c \in M^*$ such that $a_0 = b + c$ and $(b, c, a_1, \ldots, a_{n-1}) \in \alpha$.

If α, $\beta \in \Delta(M)$, define

$$\alpha + \beta = \{\boldsymbol{a}+\boldsymbol{b}\colon \boldsymbol{a} \in \alpha, \boldsymbol{b} \in \beta, l(\boldsymbol{a}) = l(\boldsymbol{b})\} .$$

Denote $O = \{(0), (0, 0), (0, 0, 0), \ldots\}$.

This complicated definition takes time and effort to digest. A few remarks may help the process. The collection property implies that a set $\alpha \in \Delta M$ is closed under permutations of its sequences. This observation is needed to make sense out of the statement of the refinement property. Moreover, the privileged location of a_0 in the statement of the splitting property is only a notational convenience; every non-zero a_i in $\boldsymbol{a}$ can be decomposed. The collection property also implies that zeros can be added to, or removed from, sequences in α at will. One of the important consequences of the refinement property is that all sequences in an $\alpha \in \Delta M$ have a common trace. Indeed, if $\boldsymbol{a}, \boldsymbol{b} \in \alpha$, then R.P. guarantees the existence of $\boldsymbol{c}$ such that $\boldsymbol{c} < \boldsymbol{a}$ and $\boldsymbol{c} < \boldsymbol{b}$; hence, $T(\boldsymbol{a}) = T(\boldsymbol{c}) = T(\boldsymbol{b})$. It follows that the trace map from $M^{<\omega}$ to M induces a map from ΔM to M. We will also call this induced map the trace and denote it by T. An alternative definition of T on ΔM is that $T(\alpha)$ is the unique $a \in M$ such that the one element sequence (a) is a member of α.

1.13.2. PROPOSITION. *If M is an m-monoid, then $\langle \Delta M; +, O \rangle$ is an m-monoid, and T: $\Delta M \to M$ is a morphism of m-monoids.*

PROOF. Routine calculations show that if α and β satisfy C.P., R.P., and S.P., then so does $\alpha + \beta$. Thus, ΔM is closed under addition. Plainly ΔM is commutative and associative, and O acts as a zero element. It is clear that T: $\Delta M \to M$ is a monoid homomorphism. Since M is an m-monoid, $T(\alpha) = 0$ implies that every $\boldsymbol{a} \in \alpha$ is a sequence of zeros, that is, $\alpha = O$. Consequently, if $\alpha + \beta = O$, then $T(\alpha) + T(\beta) = 0$, $T(\alpha) = T(\beta) = 0$, and $\alpha = \beta = O$. Thus, ΔM is an m-monoid and T is a morphism. □

Notes. Derived monoids were introduced by KETONEN [1978]. We have taken liberties with Ketonen's notation and terminology, but otherwise our development of invariants for uniform Boolean algebras closely follows his work.

1.14. *The derived monoid of $\mathscr{W}$*

In general, derived monoids are rather complicated objects. However, it is possible to give an alternative description of $\Delta\mathscr{W}$ that is more tractable than Definition 1.13.1. This interpretation of $\Delta\mathscr{W}$ will be needed in Section 2; for now, its only purpose is to illuminate a new concept.

For each $\alpha \in (\Delta\mathscr{W})^*$, define a mapping $\theta = \theta_\alpha$: $\omega_1 \to \omega + 1$ by $\theta(\zeta) = \text{l.u.b.}\ \{|\{i < n: a_i = \zeta\}|: (a_i)_{i<n} \in \alpha\}$. Thus, $\theta(\zeta) = 0$ if ζ does not appear in any sequence belonging to α, $\theta(\zeta) = m < \omega$ if ζ appears m times in some $\boldsymbol{a} \in \alpha$ but no more than m times in any $\boldsymbol{a} \in \alpha$, and $\theta(\zeta) = \omega$ if there is no upper bound on the number of occurrences of ζ in the sequences of α. Since $\alpha \neq O$, the trace $\eta = T(\alpha)$ is a countable ordinal. If $\boldsymbol{a} = (a_i)_{i<n} \in \alpha$, then $\eta = T(\boldsymbol{a}) = \max_{i<n} a_i$. Thus, $\theta(\zeta) = 0$ for all $\zeta > \eta$, and $\theta(\eta) \geq 1$. If $\xi = \min\{\zeta < \omega_1: \theta(\zeta) > 0\}$, then $\xi \leq \eta$, and there exists a sequence in α of the form $(\xi, a_1, \ldots, a_{n-1})$. Since $\xi \in \omega_1 = \mathscr{W}^*$, the splitting property implies that $\xi = b + c = \max\{b, c\}$, where b,

$c \in \mathscr{W}^*$ and $(b, c, a_1, \ldots, a_{n-1}) \in \alpha$. Thus, $b, c \in \omega_1$, $b \leq \xi$, $c \leq \xi$, and $\theta(b) > 0$, $\theta(c) > 0$. Since ξ is minimal with these properties, it follows that $b = c = \xi$. Therefore, $(\xi, \xi, a_1, \ldots, a_{n-1})$. By repeating this argument we obtain sequences in α with arbitrarily many occurrences of ξ. Hence, $\theta(\xi) = \omega$.

Let $\mathscr{N}^*$ be the set of all mappings θ: $\omega_1 \to \omega + 1$ for which there exist ordinals $\xi \leq \eta < \omega_1$ (with ξ, η depending on θ) so that $\theta(\zeta) = 0$ for $\zeta < \xi$ and $\eta < \zeta < \omega_1$, $\theta(\xi) = \omega$, and $\theta(\eta) \geq 1$. The previous paragraph shows that if $\alpha \in (\Delta\mathscr{W})^*$, then $\theta_\alpha \in \mathscr{N}^*$. If $\theta_1, \theta_2 \in \mathscr{N}^*$, then $\theta_1 + \theta_2 \in \mathscr{N}^*$, where $(\theta_1 + \theta_2)(\zeta) = \theta_1(\zeta) + \theta_2(\zeta)$ if these ordinals are finite, and $(\theta_1 + \theta_2)(\zeta) = \omega$ if either $\theta_1(\zeta) = \omega$ or $\theta_2(\zeta) = \omega$. It follows that $\mathscr{N} = \mathscr{N}^* \mathbin{\dot\cup} \{0\}$ is an m-monoid, with 0 acting as a zero element in the obvious way.

1.14.1. PROPOSITION. *The mapping $\alpha \mapsto \theta_\alpha$, $O \mapsto 0$ is an isomorphism of $\Delta\mathscr{W}$ to $\mathscr{N}$.*

PROOF. Let $\alpha, \beta \in (\Delta\mathscr{W})^*$. A typical element of $\alpha + \beta$ has the form $(\max\{a_0, b_0\}, \ldots, \max\{a_{n-1}, b_{n-1}\})$, where $\boldsymbol{a} = (a_i)_{i<n} \in \alpha$ and $\boldsymbol{b} = (b_i)_{i<n} \in \beta$. If ζ occurs k times in $\boldsymbol{a}$ and l times in $\boldsymbol{b}$, then there can be at most $k + l$ occurrences of ζ in $\boldsymbol{a} + \boldsymbol{b}$. Therefore, $\theta_{\alpha+\beta}(\zeta) \leq \theta_\alpha(\zeta) + \theta_\beta(\zeta)$. The opposite inequality follows from the observation that if $(a_i)_{i<n} \in \alpha$ and $(b_j)_{j<m} \in \beta$, then $(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}) = (a_0, \ldots, a_{n-1}, o, \ldots, o) + (o, \ldots, o, b_0, \ldots, b_{m-1}) \in \alpha + \beta$ by virtue of C.P. Thus, the mapping $\alpha \mapsto \theta_\alpha$ is a semigroup homomorphism from $(\Delta\mathscr{W})^*$ to $\mathscr{N}^*$. It is in fact an isomorphism whose inverse can be described as follows. Let $\theta \in \mathscr{N}^*$; say $\theta(\zeta) = 0$ for all $\zeta < \xi$ and all $\zeta > \eta$, $\theta(\xi) = \omega$, $\theta(\eta) \geq 1$. Define $\alpha = \alpha_\theta$ to be the subset of $\mathscr{W}^{<\omega}$ that consists of all sequences $\boldsymbol{a}$ such that $T(\boldsymbol{a}) = \eta$ and for all ordinals $\zeta \leq \eta$ there are at most $\theta(\zeta)$ occurrences of ζ in $\boldsymbol{a}$. In particular, both o and ξ can appear arbitrarily many times the sequences of α. It is clear from this definition that α satisfies C.P. The properties of ξ guarantee that S.P. holds in α. By appending zeros and permuting elements, it suffices to prove R.P. for sequences $(a_i)_{i<n}$ and $(b_i)_{i<n}$ such that $\eta = a_0 \geq a_1 \geq \cdots \geq a_{n-1}$ and $\eta = b_0 \geq b_1 \geq \cdots \geq b_{n-1}$. The desired conclusion is then obtained by a straightforward induction on n. Thus, $\alpha_\theta \in \Delta\mathscr{W}$. It is obvious that $\theta_{\alpha_\theta} = \theta$ for all $\theta \in \mathscr{N}^*$. Suppose that $\alpha \in \Delta\mathscr{W}$. Our definitions clearly yield $\alpha \subseteq \alpha_{\theta_\alpha}$. Let $\boldsymbol{a} \in \alpha_{\theta_\alpha}$. For the proof that $\boldsymbol{a} \in \alpha$, it can be assumed (by C.P.) that the entries of $\boldsymbol{a}$ are countable ordinals, say $\zeta_0 > \zeta_1 > \cdots > \zeta_{m-1}$ occur in $\boldsymbol{a}$ with the respective multiplicities $k_0, k_1, \ldots, k_{m-1} \geq 1$. Then $\theta_\alpha(\zeta_j) \geq k_j$ for $j < m$, and $\zeta_0 = T(\alpha)$. It follows from C.P. that there exists $\boldsymbol{b}_j \in \alpha$ that has the form $(\zeta_0, \zeta_j, \ldots, \zeta_j)$ in which ζ_j occurs k_j times. The refinement property guarantees the existence of $\boldsymbol{c} \in \alpha$ such that $\boldsymbol{c} < \boldsymbol{b}_j$ for all $j < m$. It follows that $\boldsymbol{c} < \boldsymbol{a}$, so that $\boldsymbol{a} \in \alpha$ by C.P. □

1.15. Derived measures

We are now ready to describe the basic building block that is used to construct the invariants for uniform algebras. Recall that if M is an m-monoid, then $\mathscr{M}(M)$ denotes the set of all M-measures on the free algebra $\mathscr{F}$.

1.15.1. DEFINITION. Let M be an m-monoid, and $\sigma \in \mathcal{M}(M)$. Define the mapping $\Delta\sigma$: $\mathcal{F} \to \Delta M$ by

$$\Delta\sigma(x) = \{(\sigma y_0, \ldots, \sigma y_{n-1}): x = y_0 \dot{+} \cdots \dot{+} y_{n-1} \text{ in } \mathcal{F}\}.$$

1.15.2. PROPOSITION. (a) $\Delta\sigma \in \mathcal{M}(\Delta M)$.

(b) *If $x \in \mathcal{F}$, then $T(\Delta\sigma(x)) = \sigma(x)$.*

(c) *If $z \in \mathcal{F}$ and k: $\mathcal{F} \to \mathcal{F} \upharpoonright z$ is an isomorphism, then $\Delta(\sigma \circ k) = \Delta\sigma \circ k$.*

(d) *$\sigma \cong \tau$ implies $\Delta\sigma \cong \Delta\tau$.*

PROOF. Clearly, $\Delta\sigma(0)$ is the zero of ΔM. Let $x \neq 0$. By definition, $\Delta\sigma(x)$ is a countable subset of $M^{<\omega}$. We must show that this set satisfies C.P., R.P., and S.P. The collection property follows directly from the additivity of measures: if $(\sigma y_0, \ldots, \sigma y_{n-1}) \in \Delta\sigma(x)$ with $x = y_0 \dot{+} \cdots \dot{+} y_{n-1}$, and $\lambda: n \to m$, then $(\Sigma_{\lambda(i)=0}\, \sigma y_i, \ldots, \Sigma_{\lambda(i)=m-1}\, \sigma y_i) = (\sigma z_0, \ldots, \sigma z_{m-1})$, where $z_j = \Sigma_{\lambda(i)=j}\, y_i$; thus, $(\Sigma_{\lambda(i)=0}\, \sigma y_i, \ldots, \Sigma_{\lambda(i)=m-1}\, \sigma y_i) \in \Delta\sigma(x)$, since $x = z_0 \dot{+} \cdots \dot{+} z_{m-1}$. The refinement property is a consequence of the analogous property of Boolean algebras. Specifically, if $x = y_0 \dot{+} \cdots \dot{+} y_{n-1} = z_0 \dot{+} \cdots \dot{+} z_{m-1}$, then $x = \cdot\Sigma_{i<n, j<m}\, w_{ij}$, where $w_{ij} = x_i \cdot y_j$, and $y_i = \cdot\Sigma_{j<m}\, w_{ij}$, $z_j = \cdot\Sigma_{i<n}\, w_{ij}$. Using the additivity of σ, these relations imply that $\Delta\sigma(x)$ satisfies R.P. To prove the splitting property, suppose that $(\sigma y_0, \sigma y_1, \ldots, \sigma y_{n-1}) \in \Delta\sigma(x)$ with $\sigma y_0 \neq 0$. By the definition of a measure, $y_0 \neq 0$. Since $\mathcal{F}$ has no atoms, we can write $y_0 = z \dot{+} w$ with $z \neq 0$ and $w \neq 0$. It follows that $\sigma y_0 = \sigma z + \sigma w$ with $\sigma z \neq 0$, $\sigma w \neq 0$, and $(\sigma z, \sigma w, \sigma y_1, \ldots, \sigma y_{n-1}) \in \Delta\sigma(x)$. This finishes the proof that $\Delta\sigma(x) \in \Delta M$. Next, we prove that $\Delta\sigma$ is additive. Let $x = x' \dot{+} x''$ in $\mathcal{F}$. If $\boldsymbol{a}' = (\sigma y'_0, \ldots, \sigma y'_{n-1}) \in \Delta\sigma(x')$, $\boldsymbol{a}'' = (\sigma y''_0, \ldots, \sigma y''_{n-1}) \in \Delta\sigma(x'')$, where $x' = \cdot\Sigma_{i<n}\, y'_i$, $x'' = \cdot\Sigma_{i<n}\, y''_i$, then $x = \cdot\Sigma_{i<n}\, y_i$ with $y_i = y'_i \dot{+} y''_i$ for $i < n$. Hence, $\boldsymbol{a}' + \boldsymbol{a}'' = (\sigma y_0, \ldots, \sigma y_{n-1}) \in \Delta\sigma(x)$, since $\sigma y_i = \sigma y'_i + \sigma y''_i$. This calculation shows that $\Delta\sigma(x') + \Delta\sigma(x'') \subseteq \Delta\sigma(x)$. To reverse the inclusion, suppose that $x = y_0 \dot{+} \cdots \dot{+} y_{n-1}$. Then $x' = \cdot\Sigma_{i<n}\, y'_i$, $x'' = \cdot\Sigma_{i<n}\, y''_i$, where $y'_i = x' \cdot y_i$, $y''_i = x'' \cdot y_i$, $y_i = y'_i \dot{+} y''_i$, for all $i < n$. It follows that $(\sigma y_0, \ldots, \sigma y_{n-1}) = (\sigma y'_0, \ldots, \sigma y'_{n-1}) + (\sigma y''_0, \ldots, \sigma y''_{n-1}) \in \Delta\sigma(x') + \Delta\sigma(x'')$. Since $(\sigma x) \in \Delta\sigma(x)$, it follows that $T(\Delta\sigma(x)) = \sigma(x)$. In particular, $\Delta\sigma(x) = 0$ implies $\sigma x = 0$, hence $x = 0$. If k: $\mathcal{F} \to \mathcal{F} \upharpoonright z$ is an isomorphism, then $\sigma \circ k \in \mathcal{M}(M)$ for all M-measures σ. Moreover, $\Delta(\sigma \circ k)(x) = \{(\sigma(ky_0), \ldots, \sigma(ky_{n-1})): x = y_0 \dot{+} \cdots \dot{+} y_{n-1}\} = \{(\sigma(ky_0), \ldots, \sigma(ky_{n-1})): kx = ky_0 \dot{+} \cdots \dot{+} ky_{n-1}\} = \{(\sigma z_0, \ldots, \sigma z_{n-1}): kx = z_0 \dot{+} \cdots \dot{+} z_{n-1}\} = \Delta\sigma(kx)$. Finally, (d) is a special case of (c). □

The measure $\Delta\sigma$ will be called the (first) *derivative* of σ. The proposition justifies this terminology.

1.16. Existence of measures

It follows from Proposition 1.15.2(c) that if $\sigma \cong \tau$ in $\mathcal{M}(M)$, then $\Delta\sigma(1) = \Delta\tau(1)$. (This fact was noted earlier in the case that $M = \mathcal{W}$.) In other words,

$\sigma \mapsto \Delta\sigma(1)$ is an invariant of the equivalence classes of $\mathcal{M}(M)$. Our aim now is to show that every element of ΔM has the form $\Delta\sigma(1)$ for a suitable $\sigma \in \mathcal{M}(M)$.

1.16.1. LEMMA. *Assume that M is an m-monoid. Let α be a subset of $M^{<\omega}$ that satisfies C.P. and R.P. If $\boldsymbol{a} = (a_i)_{i<n}$ and $\boldsymbol{b} = (b_j)_{j<m}$ are members of α, and $p \geq m$, then there exists $\boldsymbol{c} = (c_{ij})_{i<n,j<p} \in \alpha$ such that $\Sigma_{j<p}\, c_{ij} = a_i$ for all $i<n$, and $\boldsymbol{c} < \boldsymbol{b}$. If α also satisfies S.P. and $a_i \neq 0$ for all $i<n$, then it can be assumed that $c_{ij} \neq 0$ for all $i<n$ and $j<p$.*

PROOF. By R.P., there exists $(d_{ij})_{i<n,j<m} \in \alpha$ such that $\Sigma_{j<m}\, d_{ij} = a_i$ for $i<n$ and $\Sigma_{i<n}\, d_{ij} = b_j$ for $j<m$. The first assertion then follows by taking $c_{ij} = d_{ij}$ for $j<m$ and $c_{ij} = 0$ for $m \leq j < p$. Assume that α satisfies S.P. and $a_i \neq 0$ for all i. Then for each $i<n$, there exists $j(i)<m$ such that $d_{ij(i)} \neq 0$. Hence, $1 \leq r(i) \leq m \leq p$, where $r(i) = |\{j<m\colon d_{ij} \neq 0\}|$. Repeated use of S.P. enables us to write $d_{ij(i)} = \Sigma_{k \leq p-r(i)}\, c_{ik}$ with all $c_{ik} \neq 0$ and $\boldsymbol{d} = (d_{ij}, c_{ik})_{i<n, j<m, j\neq j(i), k\leq p-r(i)} \in \alpha$. The required sequence $\boldsymbol{c}$ is obtained from $\boldsymbol{d}$ by using C.P. to delete zeros. $\square$

It is now convenient to introduce notation that will be used frequently. The (standard) *dyadic tree* is the set $\mathcal{D}$ of all finite sequences of zeros and ones, including the empty sequence 0. Thus, $\mathcal{D} = \bigcup_{n<\omega} \mathcal{D}_n$, where $\mathcal{D}_n = \{\boldsymbol{i} \in \mathcal{D}\colon l(\boldsymbol{i}) = n\}$. In particular, $\mathcal{D}_0 = \{0\}$. If the elements of $\mathcal{D}$ are considered as functions in the sense of sets of ordered pairs, then the inclusion relation gives a partial ordering of $\mathcal{D}$. More concretely, $(i_0, \ldots, i_{n-1}) \leq (j_0, \ldots, j_{m-1})$ if and only if $n \leq m$ and $j_k = i_k$ for all $k<n$. It follows that the set of predecessors of $\boldsymbol{i} \in \mathcal{D}$ is a finite chain of length $l(\boldsymbol{i})$. Thus, $\mathcal{D}$ is a tree in the usual graph theoretical sense. A couple of items of notation will be useful. If a is a mapping from $\mathcal{D}$ to a set M and $G \subseteq \mathcal{D}_n$ for some n, we will write $a(G)$ to represent the sequence $(a(\boldsymbol{i}))_{\boldsymbol{i}\in G}$, listed according to the lexicographic order of G. In particular, denote $a_n = a(\mathcal{D}_n)$. Also, if $G \subseteq \mathcal{D}_n$ and $n \leq m$, denote $G \uparrow m = \{\boldsymbol{j} \in \mathcal{D}_m\colon \boldsymbol{i} \leq \boldsymbol{j}$ for some $\boldsymbol{i} \in G\}$. For economy, write $\boldsymbol{i} \uparrow m$ instead of $\{\boldsymbol{i}\} \uparrow m$.

1.16.2. DEFINITION. Let α be a countable subset of $M^{<\omega}$, where M is an m-monoid. An *α-tree* is a mapping $a\colon \mathcal{D} \to M$ such that

(i) $a_n \in \alpha$ for all $n<\omega$, and

(ii) if $\boldsymbol{i} \in \mathcal{D}_n$ and $n \leq m$, then $a(\boldsymbol{i}) = \Sigma_{\boldsymbol{j}\in \boldsymbol{i}\uparrow m}\, a(\boldsymbol{j})$.

An α-tree a is called *dense* if it satisfies

(iii) for all $\boldsymbol{b} \in \alpha$, there exists $n<\omega$ such that $a_n < \boldsymbol{b}$.

An α-tree a is *homogeneous* if $a(\boldsymbol{i}) \neq 0$ for all $\boldsymbol{i} \in \mathcal{D}$.

Two simple properties of α-trees follow directly from this definition: if the elements of α have a common trace $T(\alpha)$ and a is an α-tree, then $a_0 = (T(\alpha))$; and $n \leq m$ implies $a_m < a_n$.

1.16.3. LEMMA. *Let M be an m-monoid. If α is a countable subset of $M^{<\omega}$ that satisfies C.P. and R.P., then there is a dense α-tree a. If α also satisfies S.P., and $\alpha \neq 0$, then a can be chosen to be homogeneous.*

PROOF. Enumerate $\alpha = \{b_0, b_1, \ldots, b_n, \ldots\}$ so that $b_0 = (T(\alpha))$. Define a on $\mathscr{D}_0$ by $a(0) = T(\alpha)$. Assume that for some $m \geq 0$ the mapping a has been defined on $\bigcup_{k\leq m} \mathscr{D}_k$ in such a way that $a_k \in \alpha$ for $k \leq m$, $a(i) = \Sigma_{j\in i\uparrow m} a(j)$, and a_m refines each of the sequences $b_0, b_1, \ldots, b_n$. If $\alpha \neq 0$ satisfies S.P., then we add the condition $a(i) \neq 0$ for all $i \in \bigcup_{k\leq m} \mathscr{D}_k$ to these induction hypotheses. It will suffice to extend a to $\bigcup_{k\leq r} \mathscr{D}_k$ for some $r > m$ so that conditions (i) and (ii) of the definition are satisfied, $a_r < b_{n+1}$, and (if $\alpha \in \Delta M^*$) $a(i) \neq 0$ for all $i \in \bigcup_{k\leq r} \mathscr{D}_k$. By 1.16.1, there is an integer $s \geq 1$ and a sequence

(1) $c = (c_{ij})_{i\in\mathscr{D}_m, j\in\mathscr{D}_s} \in \alpha$, such that

(2) $\Sigma_{j\in\mathscr{D}_s} c_{ij} = a(i)$ for all $i \in \mathscr{D}_m$,

(3) $c < b_{n+1}$,

and $c_{ij} \neq 0$ for all i and j if $\alpha \in \Delta M^*$. Let $r = m + s$, and for $k \leq s$, define

(4) $a(i_0, \ldots, i_{m-1}, i_m, \ldots, i_{m+k-1}) = \Sigma\, \{c_{ij}\colon i = (i_0, \ldots, i_{m-1}),\ j = (i_m, \ldots, i_{m+k-1}, j_k, \ldots, j_{s-1})\}$.

By (1) and C.P., $a_k \in \alpha$ for all $k \leq r$. Condition (ii) of the definition is a consequence of (4). By (3), $a_r = c < b_{n+1}$; and $a_r < a_m < b_0, \ldots, b_n$ by (2). Finally, in the case that S.P. holds, $a(i) \neq 0$ for all $i \in \bigcup_{k<r} \mathscr{D}_k$ by (4) (since the c_{ij} are not 0 and M is an m-monoid). □

1.16.4. PROPOSITION. *If α is a non-zero element of ΔM, where M is an m-monoid, then there exists $\sigma \in \mathscr{M}(M)$ such that $\Delta\sigma(1) = \alpha$.*

PROOF. We identify the Cantor space $\mathscr{C}$ with ${}^{\omega}2$, the countable product of two-point spaces. The points of $\mathscr{C}$ are sequences $p = (i_0, i_1, i_2, \ldots)$ in which $i_k \in \{0, 1\}$. For $i \in \mathscr{D}_n$, define

(1) $x(i) = \{p \in \mathscr{C}\colon p \restriction n = i\}$.

By definition of the product topology, $x(i) \in \operatorname{Clop}\mathscr{C} = \mathscr{F}$, the free Boolean algebra on a countable set of generators. In fact, $\{x(i)\colon i \in \mathscr{D}\}$ is a clopen basis for $\mathscr{C}$. Two observations will be needed, the first of which is clear from (1):

(2) if $k \leq l < \omega$ and $i \in \mathscr{D}_k$, then $x(i) = \dot\cup\{x(j)\colon i \leq j \in \mathscr{D}_l\}$;

(3) if $1_{\mathscr{F}} = y_0 \dot{+} \cdots \dot{+} y_{n-1}$ in $\mathscr{F}$, then there exists $k < \omega$ and a partitioning $\mathscr{D}_k = G_0 \dot\cup \cdots \dot\cup G_{n-1}$ such that $y_j = \Sigma\, \{x(i)\colon i \in G_j\}$ for all $j < n$.

The assertion (3) follows from (2) and the fact that $\{x(i)\colon i \in \mathscr{D}\}$ is a basis of $\mathscr{C}$, using a standard compactness argument. The required M-measure on $\mathscr{F}$ is constructed by means of a dense, homogeneous α-tree a, the existence of which follows from 1.16.3. By (3), if $y \in \mathscr{F}$, then there exists $k < \omega$ and $G \subseteq \mathscr{D}_k$ such that $y = \cdot\Sigma_{i\in G}\, x(i)$. In this case, define

$$\sigma(y) = \sum_{i\in G} a(i)\,.$$

It follows from condition (ii) in the definition of an α-tree that for every $l \geq k$, $\Sigma_{i\in G}\, a(i) = \Sigma_{j\in G\uparrow l}\, a(j)$. This observation and property (2) above imply that the definition of σ is consistent. Moreover, σ is additive by (3). Since a is homogeneous and M is an m-monoid, if $\sigma(y) = 0$, then $y = 0$. Thus, $\sigma \in \mathscr{M}(M)$. It remains to show that $\Delta\sigma(1) = \alpha$. By (3), condition (i) in the definition of trees, and the collection property of α, $(\sigma y_0, \ldots, \sigma y_{n-1}) \in \Delta\sigma(1)$ (where $1_{\mathscr{F}} = y_0 \dot{+}$

$\cdots + y_{n-1})$ implies $(\sigma y_0, \ldots, \sigma y_{n-1}) \in \alpha$. That is, $\Delta\sigma(1) \subseteq \alpha$. To reverse this inclusion, note that for all $k < \omega$, $a_k = (a(i))_{i \in \mathscr{D}_k} \in (\Delta\sigma)(1)$ (since $1_{\mathscr{F}} = \cdot\,\Sigma_{i \in \mathscr{D}_k} x(i)$ by (1)). If $\boldsymbol{b} \in \alpha$, then $a_k < \boldsymbol{b}$ for some $k < \omega$ because a is dense. The fact that $\Delta\sigma(1)$ satisfies C.P. then yields the required conclusion $\boldsymbol{b} \in \Delta\sigma(1)$. □

1.17. *Stable measures*

If $\sigma \in \mathscr{M}(M)$ satisfies $\Delta\sigma(x) = \Delta\sigma(y)$ for some $x, y \in \mathscr{F}$, then $\sigma(x) = T(\Delta\sigma(x)) = T(\Delta\sigma(y)) = \sigma(y)$. The converse is generally false. We will call an M-measure σ *stable* if, for all $x, y \in \mathscr{F}$,

$$\sigma(x) = \sigma(y) \text{ implies } \Delta\sigma(x) = \Delta\sigma(y) .$$

Here is the basic property of stable measures.

1.17.1. PROPOSITION. *Let M be an m-monoid. If σ, $\tau \in \mathscr{M}(M)$ are stable, then*

$$\sigma \cong \tau \quad \textit{if and only if } \Delta\sigma(1) = \Delta\tau(1) .$$

PROOF. If $\sigma \cong \tau$, then $\Delta\sigma(1) = \Delta\tau(1)$ by Proposition 1.15.2. Conversely, assume that $\Delta\sigma(1) = \Delta\tau(1)$. Define the relation R on $\mathscr{F}$ by xRy if and only if $\sigma(x) = \tau(y)$. The main step of the proof is to show that R is a V-relation. The hypothesis that $\Delta\sigma(1) = \Delta\tau(1)$ implies that $\sigma(1) = \tau(1)$, so that $1R1$. If $xR0$, then $\sigma x = \tau 0 = 0$, so that $x = 0$. Similarly, $0Ry$ implies $y = 0$. Suppose that $xRy_0 \dot{+} y_1$; that is, $\sigma(x) = \tau(y_0 \dot{+} y_1)$. Since $1 = y_0 \dot{+} y_1 \dot{+} (1 - (y_0 + y_1))$, it follows that $(\tau y_0, \tau y_1, \tau(1 - (y_0 + y_1))) \in \Delta\tau(1) = \Delta\sigma(1)$. Therefore, $1 = z_0 \dot{+} z_1 \dot{+} z_2$ with $\sigma z_0 = \tau y_0$ and $\sigma z_1 = \tau y_1$. If we define $z = z_0 \dot{+} z_1$, then $\sigma z = \sigma z_1 + \sigma z_2 = \tau y_0 + \tau y_1 = \tau(y_0 \dot{+} y_1) = \sigma x$. The assumption that σ is stable implies $\Delta\sigma(x) = \Delta\sigma(z)$. In particular, there is a partition $x = x_0 \dot{+} x_1$ such that $\sigma x_0 = \sigma z_0 = \tau y_0$ and $\sigma x_1 = \tau y_1$. That is, $x_0 R y_0$ and $x_1 R y_1$. A symmetrical argument uses the hypothesis that τ is stable to prove: if $x_0 \dot{+} x_1 R y$, then there is a decomposition $y = y_0 \dot{+} y_1$ such that $x_0 R y_0$ and $x_1 R y_1$. This discussion completes the proof that R is a V-relation. The algebraic version of Vaught's Theorem (Theorem 1.1.3) yields the existence of an automorphism k of $\mathscr{F}$ such that if $y = k(x)$, then there are partitions $x = x_0 \dot{+} \cdots \dot{+} x_{n-1}$ and $y = y_0 \dot{+} \cdots \dot{+} y_{n-1}$ for which $k(x_i) = y_i$ and $x_i R y_i$, all $i < n$. Thus, by the definition of R, $\sigma x_i = \tau y_i = \tau(k x_i)$. By the additivity of measures, $\sigma x = \Sigma_{i<n} \sigma x_i = \Sigma_{i<n} \tau(k x_i) = \tau(kx)$. Hence, $\sigma = \tau \circ k$ and $\sigma \cong \tau$. □

1.17.2. EXAMPLE. Let X be a closed, proper subset of $\mathscr{C}$ that is not empty. Denote by σ the $\mathscr{W}$-measure on $\mathscr{F}$ that corresponds to the characteristic function of X. If X is not perfect and $|X| > 1$, then there are clopen sets x and y in $\mathscr{C}$ such that $|x \cap X| = 1$ and $|y \cap X| > 1$. It follows that $\sigma(x) = \sigma(y) = 1$, but $\Delta\sigma(x) \neq \Delta\sigma(y)$. In fact, $(1, 1) \in \Delta\sigma(y)$ and $(1, 1) \notin \Delta\sigma(x)$. Thus, σ is not stable. Similarly, if X contains a non-empty, perfect, clopen subset x of $\mathscr{C}$, then $\sigma(x) = \sigma(1) = 1$. However, the entries in the sequences of $\Delta\sigma(x)$ are o or 1, while $\Delta\sigma(1)$ contains sequences that include 0 because $X \subset \mathscr{C}$. Thus, $\Delta\sigma(x) \neq \Delta\sigma(1)$, and again, σ is not stable. These remarks show that if σ is stable, then either $|X|=1$ or X is perfect and nowhere dense in $\mathscr{C}$. It is easy to check that in these cases σ is indeed stable.

This example shows that the stability of a measure σ cannot be characterized by properties of $\Delta\sigma(1)$. Indeed, if σ is defined as in the example corresponding to an infinite (proper) subset X of $\mathscr{C}$, then $\Delta\sigma(1)$ consists of all sequences $(a_i)_{i<n}$ with $a_i = o$, 0, or 1, such that $a_i = 1$ for at least one $i < n$. In some of these cases, σ is stable; in others it is not. However, it is possible to characterize those $\alpha \in \Delta M$ such that $\alpha = \Delta\sigma(1)$ for *some* stable $\sigma \in \mathscr{M}(M)$, as we will now see.

1.18. *Fragments*

Throughout this subsection, M denotes a fixed m-monoid. If $\boldsymbol{b} = (b_0, \ldots, b_{n-1})$ and $\boldsymbol{c} = (c_0, \ldots, c_{m-1})$ are finite sequences of elements in M, then the sequence $(b_0, \ldots, b_{n-1}, c_0, \ldots, c_{m-1})$ will be abbreviated $(\boldsymbol{b}, \boldsymbol{c})$. Note that $T(\boldsymbol{b}, \boldsymbol{c}) = T\boldsymbol{b} + T\boldsymbol{c}$.

For a subset α ot $M^{<\omega}$ and $a \in M$, define

$$\Phi(\alpha) = \{\boldsymbol{b} \in M^{<\omega}: (\boldsymbol{b}, \boldsymbol{c}) \in \alpha \text{ for some } \boldsymbol{c} \in M^{<\omega}\}\,,$$

$$\Phi_a(\alpha) = \{\boldsymbol{b} \in \Phi(\alpha): T\boldsymbol{b} = a\}\,.$$

The sequences in $\Phi(\alpha)$ will be called *fragments* of α; in particular, $\boldsymbol{b}$ is an a-fragment if $\boldsymbol{b} \in \Phi_a(\alpha)$.

A few obvious consequences of these definitions are worth mentioning once. First, note that if α is a countable subset of $M^{<\omega}$, then $\Phi(\alpha)$ is countable; therefore $\Phi_a(\alpha)$ is countable for all $a \in M$. The map $\alpha \mapsto \Phi(\alpha)$ is a closure operator on the subsets of $M^{<\omega}$; and $\alpha \subseteq \beta$ implies $\Phi_a(\alpha) \subseteq \Phi_a(\beta)$, $\Phi_a^2(\alpha) = \Phi_a(\alpha)$ for all $a \in M$.

Most of our interest in the mappings Φ and Φ_a will concern their effect on elements $\alpha \in \Delta M$. In that case, $\alpha = \Delta\sigma(1)$ for some $\sigma \in \mathscr{M}(M)$. The following observation is therefore useful.

(1) If $\alpha = \Delta\sigma(1)$, then $\Phi(\alpha) = \bigcup_{x \in \mathscr{F}} (\Delta\sigma)(x)$ and $\Phi_a(\alpha) = \bigcup \{\Delta\sigma(x): x \in \mathscr{F}, \sigma x = a\}$.

If $\alpha \subseteq M^{<\omega}$ has the collection property, then $\Phi(\alpha)$ and $\Phi_a(\alpha)$ obviously also satisfy C.P. The same conclusion holds for the splitting property, but not for the refinement property in general. (Except in trivial cases, $\Phi(\alpha)$ could not satisfy R.P. because the traces of sequences in $\Phi(\alpha)$ need not be equal.)

1.18.1. LEMMA. *Let $\sigma \in \mathscr{M}(M)$ be stable. Denote $\Delta\sigma(1)$ by α. If $a \in M$ and $\boldsymbol{c} \in M^{<\omega}$ are such that $(a, \boldsymbol{c}) \in \alpha$, then $\Phi_a(\alpha) \in \Delta M$ and $(\boldsymbol{b}, \boldsymbol{c}) \in \alpha$ for all $\boldsymbol{b} \in \Phi_a(\alpha)$.*

PROOF. The hypothesis $(a, \boldsymbol{c}) \in \alpha = \Delta\sigma(1)$ implies that there is a partition $1_{\mathscr{F}} = x \dotplus z$ such that $\sigma x = a$ and $\boldsymbol{c} \in \Delta\sigma(z)$. If $\boldsymbol{b} \in \Phi_a(\alpha)$, then by (1) there exists $y \in \mathscr{F}$

such that $\sigma(y) = a$ and $\boldsymbol{b} \in \Delta\sigma(y)$. Since σ is stable and $\sigma(y) = a = \sigma(x)$, it follows that $\boldsymbol{b} \in \Delta\sigma(y) = \Delta\sigma(x)$. Thus, $\Phi_a(\alpha) = \Delta\sigma(x) \in \Delta M$ by (1) and Proposition 1.15.2. Moreover, if $\boldsymbol{b} \in \Phi_a(\alpha)$, then $(\boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{b}, \boldsymbol{0}) + (\boldsymbol{0}, \boldsymbol{c}) \in \Delta\sigma(x) + \Delta\sigma(z) = \Delta\sigma(1) = \alpha$. □

Our next objective is to prove the converse of this lemma; that is, if the conditions of the lemma are satisfied, then $\alpha = \Delta\sigma(1)$, where σ is a stable measure.

1.18.2. DEFINITION. Let M be an m-monoid. A countable subset α of $M^{<\omega}$ has the *local refinement property* (abbreviated L.P.) if

(i) $\Phi_a(\alpha)$ has the refinement property for all $a \in M$, and

(ii) if $(\boldsymbol{a}, \boldsymbol{c}) \in \alpha$, then $(\boldsymbol{b}, \boldsymbol{c}) \in \alpha$ for all $\boldsymbol{b} \in \Phi_a(\alpha)$.

The local refinement property does not imply the refinement property because R.P. implies that all elements of α have the same trace, whereas L.P. does not have this consequence. In the case that $\Phi_a(\alpha) = \emptyset$, we adopt the convention that (i) is vacuously satisfied.

The conclusion of 1.18.1 is that if σ is a stable M-measure, then $\Delta\sigma(1)$ satisfies L.P. Our aim is to prove that L.P. characterizes the elements α of ΔM such that $\alpha = \Delta\sigma(1)$ for a stable $\sigma \in \mathcal{M}(M)$. The argument follows the pattern that was established by the proof of Proposition 1.16.4, but some extra frills are needed.

1.18.3. LEMMA. *Assume that α is a countable subset of $M^{<\omega}$ that satisfies C.P., R.P., and L.P. Suppose that $\boldsymbol{a} = (a_i)_{i<n} \in \alpha$, $q \le n$, $a = \Sigma_{i<q} a_i$, $\boldsymbol{b} = (b_k)_{k<m} \in \Phi_a(\alpha)$, and $p \ge m$. Then there exists $\boldsymbol{c} = (c_{ij})_{i<n, j<p}$ in α such that $\Sigma_{j<p} c_{ij} = a_i$ for all $i < n$ and $(c_{ij})_{i<q, j<p} < \boldsymbol{b}$. If α also satisfies S.P. and $a_i \neq 0$ for all $i < n$, then $\boldsymbol{c}$ can be chosen so that $c_{ij} \neq 0$ for all $i < n$ and $j < p$.*

PROOF. Let $d = \Sigma_{q \le i < n} a_i$. By C.P., $(a, d) \in \alpha$; and $\Phi_a(\alpha)$, $\Phi_d(\alpha)$ have the refinement property by L.P. These sets also satisfy C.P. Thus, it follows from Lemma 1.16.1 that there exists $\boldsymbol{c}' = (c_{ij})_{i<q, j<p} \in \Phi_a(\alpha)$ such that $\Sigma_{j<p} c_{ij} = a_i$ for all $i < q$, and $\boldsymbol{c}' < \boldsymbol{b}$. Moreover, L.P. guarantees that $(\boldsymbol{c}', d) \in \alpha$. Apply Lemma 1.16.1 once more to get $\boldsymbol{c}'' = (c_{ij})_{q \le i < n, j<p} \in \Phi_d(\alpha)$ that satisfies $\Sigma_{j<p} c_{ij} = a_i$ for $q \le i < n$. Again, by L.P., $\boldsymbol{c} = (\boldsymbol{c}', \boldsymbol{c}'') \in \alpha$, so that $\boldsymbol{c}$ fulfills the claims of the lemma. If α satisfies S.P., then so do $\Phi_a(\alpha)$ and $\Phi_d(\alpha)$. In this case, 1.16.1 guarantees that $\boldsymbol{c}$ can be chosen so that all $c_{ij} \neq 0$. □

Let α be a countable subset of $M^{<\omega}$. An α-tree $\mathfrak{a}$ is *uniformly dense* if, for every $n < \omega$, $G \subseteq \mathcal{D}_n$, and $\boldsymbol{b} \in \Phi_{T(\mathfrak{a}(G))}(\alpha)$, there exists $m \ge n$ such that $\mathfrak{a}(G \uparrow m) < \boldsymbol{b}$. In this case $\mathfrak{a}$ is also dense: if $\boldsymbol{b} \in \alpha$, then $\boldsymbol{b} \in \Phi_{T\alpha}(\alpha)$, and $T\alpha = T(\mathfrak{a}(\mathcal{D}_n))$ for every $n < \omega$.

1.18.4. LEMMA. *If α is a countable subset of $M^{<\omega}$ that satisfies C.P., R.P., and L.P., then there is a uniformly dense α-tree $\mathfrak{a}$. If α also satisfies S.P. and $\alpha \neq 0$, then $\mathfrak{a}$ can be chosen to be homogeneous.*

PROOF. Let $\{\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n, \ldots\}$ be an enumeration of $\Phi(\alpha)$, beginning with $\boldsymbol{b}_0 = (T(\alpha))$. The construction starts by defining a on the empty sequence to be $a(0) = T(\alpha)$. This choice satisfies the case $n = 0$ of the following induction hypothesis: there exists $m = m(n) \geq n$ and a mapping a of $\bigcup_{r\leq m} \mathscr{D}_r$ to M such that

(1) $a_r = a(\mathscr{D}_r) \in \alpha$;

(2) if $\boldsymbol{i} \in \mathscr{D}_r$ and $r \leq s \leq m$, then $a(\boldsymbol{i}) = \Sigma_{\boldsymbol{j}\in \boldsymbol{i}\uparrow s}\, a(\boldsymbol{j})$;

(3) if α satisfies S.P. and $\alpha \neq 0$, then $a(\boldsymbol{i}) \neq 0$ for all $\boldsymbol{i} \in \bigcup_{r\leq m} \mathscr{D}_r$;

(4) if $k \leq n$ and $G \subseteq \mathscr{D}_n$ are such that $\boldsymbol{b}_k \in \Phi_{T(a(G))}(\alpha)$, then $a(G \uparrow m) < \boldsymbol{b}_k$.

For the induction step, list the finite set $\{(k, G)\colon k \leq n+1,\ G \subseteq \mathscr{D}_{n+1},\ \boldsymbol{b}_k \in \Phi_{T(a(G))}(\alpha)\}$ as $(k_0, G_0), (k_1, G_1), \ldots, (k_{l-1}, G_{l-1})$. Use 1.18.3 (as in the proof of 1.16.3) to extend a to $\bigcup_{r\leq m_0} \mathscr{D}_r$, $m_0 > m$, so that conditions (1), (2), and (3) are satisfied for all $r \leq m_0$, and $a(G_0 \uparrow m_0) < \boldsymbol{b}_{k_0}$. Repeat the construction to extend a to $\bigcup_{r\leq m_1} \mathscr{D}_r$, $m_1 \geq m_0$, so that (1), (2), and (3) hold for $r \leq m_1$ and $a(G_1 \uparrow m_1) < \boldsymbol{b}_{k_1}$. Repeat this extension process l times to obtain $m(n+1) = m_{l-1} > m(n)$, with a defined on $\bigcup_{r\leq m(n+1)} \mathscr{D}_r$, satisfying (1), (2), and (3) with $n+1$ replacing n. Moreover, if $t < l$, then $a(G_t \uparrow m(n+1)) < a(G_t \uparrow m_t) < \boldsymbol{b}_{k_t}$. Thus, (4) holds for $n+1$, which completes the induction step. The end-product a of this recursive construction is an α-tree that is homogeneous if α satisfies S.P. and is not zero. Suppose that $\boldsymbol{b} \in \Phi_a(\alpha)$, where $a = T(a(G))$ for some $G \subseteq \mathscr{D}_l$. Let $n = \max\{l, k\}$, where $k < \omega$ is such that $\boldsymbol{b} = \boldsymbol{b}_k$. It can be assumed that $G \subseteq \mathscr{D}_n$ by changing G to $G \uparrow n$ if necessary. It follows from (4) that $a(G \uparrow m) < \boldsymbol{b}$ for some $m \geq n \geq l$. Thus, a is uniformly dense. □

1.18.5. PROPOSITION. *Let $\alpha \in \Delta M$, where M is an m-monoid. There is a stable M-measure σ on $\mathscr{F}$ such that $\Delta\sigma(1) = \alpha$ if and only if α has the local refinement property.*

PROOF. If $\sigma \in \mathscr{M}(M)$ is stable, then $\Delta\sigma(1)$ satisfies L.P. by 1.18.1. Conversely, assume that $\alpha \in \Delta M^*$ satisfies L.P. By 1.18.4, there is a homogeneous, uniformly dense α-tree a. Use a to construct $\sigma \in \mathscr{M}(M)$ as in the proof of Proposition 1.16.4. Thus, if $G \subseteq \mathscr{D}_n$ and $y = \Sigma_{\boldsymbol{i}\in G}\, x(\boldsymbol{i})$, then $\sigma(y) = \Sigma_{\boldsymbol{i}\in G}\, a(\boldsymbol{i}) = T(a(G))$. The result of Proposition 1.16.4 still holds; that is, $\Delta\sigma(1) = \alpha$. Suppose that $\sigma(y) = \sigma(z)$, where $y, z \in \mathscr{F}$. Write $y = \Sigma_{\boldsymbol{i}\in G}\, x(\boldsymbol{i})$, $G \subseteq \mathscr{D}_n$, so that $\sigma(y) = T(a(G))$ as above. Let $\boldsymbol{b} \in \Delta\sigma(z)$. Then $T(\boldsymbol{b}) = \sigma(z) = \sigma(y) = T(a(G))$, that is, $\boldsymbol{b} \in \Phi_{T(a(G))}(\alpha)$. Since a is uniformly dense, there exists $m \geq n$ such that $a(G \uparrow m) < \boldsymbol{b}$. Note that $\Sigma_{\boldsymbol{j}\in G\uparrow m}\, x(\boldsymbol{j}) = \Sigma_{\boldsymbol{i}\in G}\, x(\boldsymbol{i}) = y$ implies that $a(G \uparrow m) = (\sigma(x(\boldsymbol{j})))_{\boldsymbol{j}\in G\uparrow m} \in \Delta\sigma(y)$. Thus, $\boldsymbol{b} \in \Delta\sigma(y)$ by C.P. This argument shows that $\Delta\sigma(z) \subseteq \Delta\sigma(y)$. By symmetry, $\Delta\sigma(y) \subseteq \Delta\sigma(z)$. Hence, σ is stable. □

1.19. Iterated derivatives

The construction of Definitions 1.13.1 and 1.15.1 can be iterated transfinitely, starting with $\mathscr{W}$ and $\mathscr{M}(\mathscr{W})$. We will now show that this process leads to full invariants for the equivalence classes of $\mathscr{W}$-measures on $\mathscr{F}$.

1.19.1. DEFINITION. Let M be an m-monoid. For countable ordinals $\eta \leq \zeta$, define m-monoids $\Delta^\eta M$, $\Delta^\zeta M$, and morphisms T^ζ_η: $\Delta^\zeta M \to \Delta^\eta M$ by the conditions:

(i) T^ζ_ζ = identity map on $\Delta^\zeta M$;

(ii) $\Delta^0 M = M$;

(iii) $\Delta^{\zeta+1}M = \Delta(\Delta^\zeta M)$, $T^{\zeta+1}_\eta = T^\zeta_\eta \circ T$, where T: $\Delta(\Delta^\zeta M) \to \Delta^\zeta M$ is the trace morphism;

(iv) if ζ is a limit ordinal, then $\Delta^\zeta M$ is the limit of the inverse system $(\Delta^\eta M;\ T^\eta_\xi)_{\xi \leq \eta < \zeta}$, and T^ζ_ξ is the limit of the morphisms $(T^\eta_\xi)_{\eta<\zeta}$.

To justify this definition, we have to check several things: $\xi \leq \eta \leq \zeta$ implies $T^\zeta_\xi = T^\eta_\xi \circ T^\zeta_\eta$; $\Delta^\zeta M$ is an m-monoid in the case where ζ is a limit ordinal; and the maps T^ζ_η are morphisms of m-monoids. Using the realization of the elements in the inverse limit as sequences $(a_\eta)_{\eta<\zeta}$ with $a_\eta \in \Delta^\zeta M$ and $T^\eta_\xi a_\eta = a_\xi$ for $\xi < \eta$, with the limit map given by $T^\zeta_\xi (a_\eta)_{\eta<\zeta} = a_\xi$, these verifications are routine.

1.19.2. PROPOSITION. *Let M be an m-monoid. If $\sigma \in \mathcal{M}(M)$, then for each $\zeta < \omega_1$, there exists a unique $\Delta^\zeta\sigma \in \mathcal{M}(\Delta^\zeta M)$ such that:*

(a) $\Delta^0\sigma = \sigma$;

(b) $\Delta^{\zeta+1}\sigma = \Delta(\Delta^\zeta\sigma)$;

(c) $\eta \leq \zeta$ *implies* $T^\zeta_\eta \circ \Delta^\zeta\sigma = \Delta^\eta\sigma$;

(d) *if ζ is a limit ordinal, then $\Delta^\zeta\sigma$ is the limit of the inverse system of maps* $(\Delta^\eta\sigma)_{\eta<\zeta}$.

PROOF. Only (c) requires comment. It is obtained by induction on ζ. If $\zeta = \eta$, then T^ζ_η is the identity map, as required. If (c) holds for ζ, then by (b) and property (iii) of the definition, $T^{\zeta+1}_\eta \Delta^{\zeta+1}\sigma = T^\zeta_\eta \circ T \circ \Delta(\Delta^\zeta\sigma) = T^\zeta_\eta \circ \Delta^\zeta\sigma = \Delta^\eta\sigma$ by Proposition 1.15.1(a). If ζ is a limit ordinal, then the definition of $\Delta^\zeta\sigma$ in (d) implies $T^\zeta_\eta \circ \Delta^\zeta\sigma = \Delta^\eta\sigma$. $\square$

1.20. The depth of measures

We now show that the sequence of derivatives of an M-measure must eventually stabilize.

1.20.1. LEMMA. *If M is an m-monoid and $\sigma \in \mathcal{M}(M)$, then there is a smallest ordinal number $d(\sigma) < \omega_1$ such that $\Delta^\zeta\sigma$ is stable for all $\zeta \geq d(\sigma)$.*

PROOF. For each $\zeta < \omega_1$, denote $E_\zeta = \{(x, y) \in \mathcal{F} \times \mathcal{F}\colon \Delta^\zeta\sigma(x) = \Delta^\zeta\sigma(y)\}$. Clearly, E_ζ is an equivalence relation on $\mathcal{F}$. By Proposition 1.19.2(c), $\eta \leq \zeta$ implies $E_\eta \supseteq E_\zeta$. Since $\mathcal{F}$ is countable, there exists $d(\sigma) < \omega_1$ such that $E_{d(\sigma)} = E_{d(\sigma)+1} = \cdots$. That is, $\Delta^\zeta\sigma$ is stable for all $\zeta \geq d(\sigma)$. $\square$

The ordinal number $d(\sigma)$ will be called the *depth* of σ.

1.20.2. THEOREM. *Let M be an m-monoid. If $\sigma, \tau \in \mathcal{M}(M)$, then $\sigma \cong \tau$ if and only if $\Delta^{\zeta+1}\sigma(1) = \Delta^{\zeta+1}\tau(1)$ for some countable ordinal number $\zeta \geq \max\{d(\sigma), d(\tau)\}$.*

PROOF. If there exists $k \in \operatorname{Aut} \mathcal{F}$ such that $\sigma = \tau \circ k$, then $\Delta\sigma = \Delta\tau \circ k$ by Proposition 1.15.2(c). By induction, $\Delta^{\zeta}\sigma = \Delta^{\zeta}\tau \circ k$ for all $\zeta < \omega_1$. In particular, $\Delta^{\zeta}\sigma(1) = \Delta^{\zeta}\tau(1)$ for all $\zeta < \omega_1$. Conversely, if $\Delta^{\zeta+1}\sigma(1) = \Delta^{\zeta+1}\tau(1)$ for some $\zeta \geq \max\{d(\sigma), d(\tau)\}$, then (since $\Delta^{\zeta}\sigma$ and $\Delta^{\zeta}\tau$ are stable) there exists $k \in \operatorname{Aut} \mathcal{F}$ satisfying $\Delta^{\zeta}\sigma = (\Delta^{\zeta}\tau) \circ k = \Delta^{\zeta}(\tau \circ k)$, according to Proposition 1.17.1. Hence, $\sigma = T_0^{\zeta} \circ \Delta^{\zeta}\sigma = T_0^{\zeta} \circ \Delta^{\tau}(\tau \circ k) = \tau \circ k$ by Proposition 1.19.2. □

The proof above shows more than is stated in the theorem: if $\Delta^{\zeta+1}\sigma(1) = \Delta^{\zeta+1}\tau(1)$ for some $\zeta \geq \max\{d(\sigma), d(\tau)\}$, then $\Delta^{\zeta}\sigma(1) = \Delta^{\zeta}\tau(1)$ for all $\zeta < \omega_1$.

1.21. The Boolean hierarchy

The usefulness of Theorem 1.20.2 is no greater than our ability to recognize the elements of $\Delta^{\zeta+1}\mathcal{W}$ that have the form $\Delta^{\zeta+1}\sigma(1)$ for some $\sigma \in \mathcal{M} = \mathcal{M}(\mathcal{W})$ such that $d(\sigma) \leq \zeta$. To facilitate the discussion of this problem, we introduce more notation:

$$\mathcal{K}^{\zeta} = \{\Delta^{\zeta+1}\sigma(1) \colon \sigma \in \mathcal{M},\ d(\sigma) \leq \zeta\}\,.$$

The sequence of sets $\{\mathcal{K}^{\zeta} \colon \zeta < \omega_1\}$ is called the *Boolean hierarchy*. By Theorem 1.20.2, the elements of $\mathcal{K}^{\zeta}$ correspond one-to-one with equivalence classes in $\mathcal{M}$, or equivalently, with isomorphism classes of uniform Boolean algebras. Before we discuss the characterization of $\mathcal{K}^{\zeta}$, it is useful to take a closer look at the Boolean hierarchy and its limit.

1.21.1. PROPOSITION. *For $\eta \leq \zeta < \omega_1$, there is an injective mapping $\Sigma_{\eta}^{\zeta} \colon \mathcal{K}^{\eta} \to \mathcal{K}^{\zeta}$ such that*

(a) *$\xi \leq \eta \leq \zeta < \omega_1$ implies $\Sigma_{\eta}^{\zeta} \circ \Sigma_{\xi}^{\eta} = \Sigma_{\xi}^{\zeta}$, and*
(b) *$T_{\eta+1}^{\zeta+1} \circ \Sigma_{\eta}^{\zeta}$ is the identity map on $\mathcal{K}^{\eta}$.*

PROOF. An element of $\mathcal{K}^{\eta}$ has the form $\Delta^{\eta+1}\sigma(1)$, where $\sigma \in \mathcal{M}$ is such that $\Delta^{\zeta}\sigma$ is stable for all $\zeta \geq \eta$. Thus, $\Delta^{\zeta+1}\sigma(1) \in \mathcal{K}^{\zeta}$. Define $\Sigma_{\eta}^{\zeta}(\Delta^{\eta+1}\sigma(1)) = \Delta^{\zeta+1}\sigma(1)$. This choice makes sense: if $\tau \in \mathcal{M}$, $\Delta^{\eta}\tau$ is stable, and $\Delta^{\eta+1}\tau(1) = \Delta^{\eta+1}\sigma(1)$, then $\Delta^{\zeta+1}\tau(1) = \Delta^{\zeta+1}\sigma(1)$ for all $\zeta \geq \eta$. The properties (a) and (b) are direct consequences of the definition of Σ_{η}^{ζ} and Proposition 1.19.2(c). It follows from (b) that Σ_{η}^{ζ} is injective. □

The pair $((\mathcal{K}^{\zeta})_{\zeta<\omega_1}, (\Sigma_{\eta}^{\zeta})_{\eta\leq\zeta<\omega_1})$ is a direct system of sets. The limit of this system is a pair $(\mathcal{K}, (\Sigma_{\eta})_{\eta<\omega_1})$ that consists of a set $\mathcal{K}$ together with a collection of injective mappings $\Sigma_{\eta} \colon \mathcal{K}^{\eta} \to \mathcal{K}$ that satisfy $\Sigma_{\eta} = \Sigma_{\zeta} \circ \Sigma_{\eta}^{\zeta}$ for $\eta \leq \zeta < \omega_1$. By making suitable identifications, the mapping Σ_{η} can be viewed as inclusion of $\mathcal{K}^{\eta}$ into $\mathcal{K} = \bigcup_{\zeta<\omega_1} \mathcal{K}^{\zeta}$.

The basic property of $\mathscr{K}$ is that its elements classify uniform Boolean algebras via the sequence

$$A \mapsto X \mapsto r \mapsto \sigma \mapsto \Delta^{\zeta+1}\sigma(1) \mapsto \Sigma_\zeta(\Delta^{\zeta+1}\sigma(1)) ,$$

where: A is a uniform Boolean algebra; $X = \text{Ult } A$; $r = r_X$ is the rank function of X; $\sigma = \sigma_r$ is the $\mathscr{W}$-measure that corresponds to r; $\Delta^{\zeta+1}\sigma$ is the $\zeta + 1$'st iterated derivative of σ with $\zeta = d(\sigma)$ (the depth of σ); and Σ_ζ: $\mathscr{K}^\zeta \to \mathscr{K}$ is the injective mapping that was just defined.

1.22. *The hierarchy property*

The last chapter in the story of Ketonen's classification of uniform Boolean algebras is a characterization of the sets $\mathscr{K}^\zeta = \{\Delta^{\zeta+1}\sigma(1)\colon \sigma \in \mathscr{M},\ d(\sigma) \leq \zeta\}$.

1.22.1. LEMMA. *Let $\alpha \in \mathscr{K}^\zeta \subseteq \Delta(\Delta^\zeta \mathscr{W})$, where $\zeta < \omega_1$. Suppose that $a \in \Delta^\zeta \mathscr{W}$ is such that (a) is a fragment of α. If $\eta < \zeta$ and $\boldsymbol{c} \in T^\zeta_{\eta+1}a$, then there exists $\boldsymbol{b} \in \Phi_a(\alpha)$ such that $\boldsymbol{c} = T^\zeta_\eta \boldsymbol{b}$.*

PROOF. The hypothesis that $\alpha \in \mathscr{K}^\zeta$ means that there exists $\sigma \in \mathscr{M}$ such that $\Delta^\zeta\sigma$ is stable and $\alpha = \Delta^{\zeta+1}\sigma(1)$. Since $(a) \in \Phi(\alpha)$, there is a partition $1_\mathscr{F} = x \dotplus y_0 \dotplus \cdots \dotplus y_{n-1}$ with $a = \Delta^\zeta\sigma(x)$. Thus, $\boldsymbol{c} \in T^\zeta_{\eta+1}a = T^\zeta_{\eta+1}\Delta^\zeta\sigma(x) = \Delta(\Delta^\eta\sigma)(x)$. That is, we can write $x = x_0 \dotplus x_1 \dotplus \cdots \dotplus x_{m-1}$ in $\mathscr{F}$ so that $\boldsymbol{c} = (c_j)_{j<m}$ with $c_j = \Delta^\eta\sigma(x_j)$ for all $j < m$. Define $b_j = \Delta^\zeta\sigma(x_j)$, $\boldsymbol{b} = (b_j)_{j<m}$. Then $\boldsymbol{b} \in \Phi_a(\alpha)$ and $T^\zeta_\eta \boldsymbol{b} = \boldsymbol{c}$. □

1.22.2. DEFINITION. Let M be an m-monoid and $\zeta < \omega_1$. A set α in $\Delta(\Delta^\zeta M)$ has the *hierarchy property* (H.P.) if:

($*$) for all $a \in \Delta^\zeta M$ such that $(a) \in \Phi(\alpha)$ and for all $\boldsymbol{c} \in T^\zeta_{\eta+1}a$ where $\eta < \zeta$, there exists $\boldsymbol{b} \in \Phi_a(\alpha)$ such that $\boldsymbol{c} = T^\zeta_\eta \boldsymbol{b}$.

In the case that $\zeta = 0$, the hierarchy property is satisfied vacously because there is no $\eta < \zeta$.

1.22.3. PROPOSITION. *If $\zeta < \omega_1$ and $\alpha \in \Delta^{\zeta+1}\mathscr{W}$, then $\alpha \in \mathscr{K}^\zeta$ if and only if α satisfies the local refinement and hierarchy properties.*

PROOF. By the lemma and Lemma 1.18.1, every $\alpha \in \mathscr{K}^\zeta$ satisfies H.P. and L.P. Conversely, assume that α satisfies these conditions. By Proposition 1.18.5, there is a stable measure $\tau \in M(\Delta^\zeta \mathscr{W})$ such that $\alpha = \Delta\tau(1)$. The first step of the proof consists of showing that for any $x \in \mathscr{F}$:

(1) if $\eta < \zeta$ and $\boldsymbol{c} = (c_j)_{j<m} \in (T^\zeta_{\eta+1} \circ \tau)(x)$, then $x = x_0 \dotplus \cdots \dotplus x_{m-1}$ with $c_j = (T^\zeta_\eta \circ \tau)(x_j)$ for all $j < m$.

Since $(\tau(x)) \in \Phi(\alpha)$, H.P. provides a sequence $\boldsymbol{b} = (b_j)_{j<m} \in \Phi_{\tau(x)}(\alpha)$ such that $\boldsymbol{c} = T^\zeta_\eta \boldsymbol{b}$. By using the fact that $\boldsymbol{b}$ is a fragment of α, we obtain $y \in \mathscr{F}$ and a partition $y = y_0 \dotplus \cdots \dotplus y_{m-1}$ such that $\boldsymbol{b} = (\tau y_j)_{j<m} \in \Delta\tau(y)$. In this case, $\tau(y) = T\boldsymbol{b} = \tau(x)$, because $\boldsymbol{b} \in \Phi_{\tau(x)}(\alpha)$. Thus, $\boldsymbol{b} \in \Delta\tau(x)$ by the stability of τ. Therefore, there is a partition $x = x_0 \dotplus \cdots \dotplus x_{m-1}$ such that $\boldsymbol{b} = (\tau(x_j))_{j<m}$, which yields the

assertion of (1): $c = T^\zeta_\eta b = (T^\zeta_\eta \circ \tau(x_j))_{j<m}$. To finish the proof, define $\sigma = T^\zeta_0 \circ \tau$. Since $T^\zeta_0\colon \Delta^\zeta \mathscr{W} \to \mathscr{W}$ is a morphism of m-monoids, it follows that $\sigma \in \mathscr{M}$. The proof of the proposition can be completed by showing (by induction on η) that

(2) for $0 \le \eta \le \zeta$, $T^\zeta_\eta \circ \tau = \Delta^\eta \sigma$.

If $\eta = 0$, then (2) comes directly from the defintion of σ. Assume that (2) holds for some $\eta < \zeta$. If $c = (c_j)_{j<m} \in T^\zeta_{\eta+1} \circ \tau(x)$, then by (1) $x = x_0 \dot{+} \cdots \dot{+} x_{m-1}$ with $c_j = T^\zeta_\eta \circ \tau(x_j) = \Delta^\eta \sigma(x_j)$, using the induction hypothesis. Thus, $c \in \Delta(\Delta^\eta \sigma)(x) = \Delta^{\eta+1}\sigma(x)$. Conversely, if $c = (c_j)_{j<m} \in \Delta^{\eta+1}\sigma(x) = \Delta(\Delta^\eta\sigma)(x)$, then there is a partition $x = x_0 \dot{+} \cdots \dot{+} x_{m-1}$ so that $c_j = \Delta^\eta \sigma(x_j)$. By the induction hypothesis, $c_j = T^\zeta_\eta \circ \tau(x_j) = T(T^\zeta_{\eta+1} \circ \tau)(x_j)$. Hence, $(c_j) \in (T^\zeta_{\eta+1} \circ \tau)(x_j)$ and the additivity of the measure $T^\zeta_{\eta+1} \circ \tau$ give the desired conclusion: $c = (c_0, 0, \ldots, 0) + \cdots + (0, 0, \ldots, c_{m-1}) \in (T^\zeta_{\eta+1} \circ \tau)(x_0) + \cdots + (T^\zeta_{\eta+1} \circ \tau)(x_{m-1}) = (T^\zeta_{\eta+1} \circ \tau)(x)$. Since x was any element of $\mathscr{F}$, we have proved that (2) holds for $\eta + 1$. Finally, suppose that $\xi \le \zeta$ is a limit ordinal, and (2) is satisfied for all $\eta < \xi$. Using Proposition 1.19.2, we obtain $T^\xi_\eta \circ \Delta^\xi \sigma = \Delta^\eta \sigma = T^\zeta_\eta \circ \tau = T^\xi_\eta \circ T^\zeta_\xi \circ \tau$, which gives the required result $T^\zeta_\xi \circ \tau = \Delta^\xi \sigma$. The inductive proof of (2) is therefore completed. The case $\eta = \zeta$ of (2) is the equation $\Delta^\zeta \sigma = \tau$. Therefore, $d(\sigma) \le \zeta$ because τ is stable. Moreover, $\alpha = \Delta\tau(1) = \Delta^{\zeta+1}\sigma(1) \in \mathscr{K}^\zeta$. □

Notes. The presentation in subsections 1.13 through 1.22 is based on the work of KETONEN [1978]. We have rearranged, trimmed, and sometimes expanded his exposition. However, credit for the ideas underlying the classification of countable Boolean algebras belongs almost entirely to Ketonen.

1.23. Length of the hierarchy

We conclude Section 1 with an example. It shows that for any countable ordinal μ, there are $\mathscr{W}$-measures of depth μ. Thus, the full Boolean hierarchy is needed to classify countable Boolean algebras.

1.23.1. EXAMPLE. Let X be a closed subset of the Cantor set $\mathscr{C}$ such that X is homeomorphic to the space of ordinals $\omega^\mu + 1$, where $\mu \ge 1$ is a countable ordinal number. Denote by σ the $\mathscr{W}$-measure on $\mathscr{F} = \operatorname{Clop} \mathscr{C}$ that corresponds to the characteristic function of X; that is, $\sigma(0) = o$, $\sigma(x) = 0$ if $x \in \mathscr{F}\backslash\{0\}$ and $x \cap X = \emptyset$, and $\sigma(x) = 1$ if $x \cap X \ne \emptyset$. Our objective is to prove:

(1) $d(\sigma) = \mu$.

The proof of (1) is based on two easy consequences of 1.6.1 and the Uniqueness Theorem 1.10.1:

(2) if $x = x_0 \dot{+} \cdots \dot{+} x_{r-1}$ in $\mathscr{F}$, then $\nu(x_i \cap X) \le \nu(x \cap X)$ for all $i < r$, and $\Sigma\, \{n(x_i \cap X)\colon \nu(x_i \cap X) = \nu(x \cap X)\} = n(x \cap X)$;

(3) if $(\eta_0, n_0), \ldots, (\eta_{r-1}, n_{r-1})$ satisfy $\eta_i + 1 \le \nu(x \cap X)$ and $1 \le n_i < \omega$ for all $i < r$, and if $\Sigma\, \{n_i\colon \eta_i + 1 = \nu(x \cap X)\} = n(x \cap X)$, then there is a decomposition $x = x_0 \dot{+} \cdots \dot{+} x_{r-1}$ such that $\nu(x_i \cap X) = \eta_i + 1$ and $n(x_i \cap X) = n_i$ for all $i < r$. Indeed, (2) comes directly from 1.6.1; and 1.6.1 together with 1.10.1 imply the existence of a decomposition $x \cap X = X_0 \dot{\cup} \cdots \dot{\cup} X_{r-1}$ such that $\nu(X_i) = n_i + 1$, $n(X_i) = n_i$. We can then obtain $x = x_0 \dot{+} \cdots \dot{+} x_{r-1}$ with $x_i \cap X = X_i$.

The depth computation (1) will follow from:

(4) for $x, y \in \mathscr{F}$ and $\zeta < \omega_1$,

(a) $\nu(x \cap X)$, $\nu(y \cap X) > \zeta$ implies $\Delta^{\zeta}\sigma(x) = \Delta^{\zeta}\sigma(y)$, and

(b) if $\nu(x \cap X) \leq \zeta$, then $\Delta^{\zeta}\sigma(x) = \Delta^{\zeta}\sigma(y)$ is equivalent to the conditions $x = y = 0$; $x, y \neq 0$ and $x \cap X = y \cap X = \emptyset$; or $x \cap X$, $y \cap X \neq \emptyset$ and $\nu(x \cap X) = \nu(y \cap X)$, $n(x \cap X) = n(y \cap X)$.

The assertions of (4) are obtained by induction on ζ. Note that $\nu(x \cap X) = 0$ is equivalent to $x \cap X = \emptyset$, so that the case $\zeta = 0$ of (4) follows directly from the definition of σ. If ζ is a non-zero limit ordinal, then (a) and (b) follow easily from the induction hypothesis and the observations that $\nu(x \cap X)$ is a non-limit ordinal whenever $x \cap X \neq \emptyset$ and $\Delta^{\zeta}\sigma(x) = \lim_{\eta \leq \xi < \zeta} \Delta^{\xi}\sigma(x)$ for any $\eta < \zeta$. Suppose that $\zeta = \xi + 1$ and (4) is valid if ζ is replaced by ξ. This induction hypothesis, together with (2) and (3), justifies the notation:

$\alpha(\eta, n) = \Delta^{\xi}\sigma(x)$, where $1 \leq \nu(x \cap X) = \eta \leq \xi$, $n(x \cap X) = n$;

$\beta = \Delta^{\xi}\sigma(x)$, where $\nu(x \cap X) > \xi$;

$\gamma = \Delta^{\xi}\sigma(x)$, where $x \neq 0$, $x \cap X = \emptyset$;

$\delta = \Delta^{\xi}\sigma(0)$.

Moreover, the $\alpha(\eta, n)$, β, γ, and δ are distinct elements of $\Delta^{\xi}\mathscr{W}$. It follows from (2) and (3) that if $\nu(x \cap X) > \zeta$, then $\Delta^{\zeta}\sigma(x) = \Delta(\Delta^{\xi}\sigma)(x)$ consists of all finite sequences of elements in $\{\alpha(\eta, n)\colon 1 \leq \eta \leq \xi,\ 1 \leq n < \omega\} \cup \{\beta, \gamma, \delta\}$; if $\nu(x \cap X) = \zeta$ and $n(x \cap X) = m$, then $\Delta^{\zeta}\sigma(x)$ consists of all finite sequences with 1 to m occurrences of β and any number of occurrences of elements in $\{\alpha(\eta, n)\colon 1 \leq \eta \leq \xi,\ 1 \leq n < \omega\} \cup \{\gamma, \delta\}$; if $\nu(x \cap X) = \eta' + 1 \leq \xi$, then $\Delta^{\zeta}\sigma(x)$ consists of all finite sequences with terms $\alpha(\eta', n_0), \ldots, \alpha(\eta', n_{k-1})$ such that $n_0 + \cdots + n_{k-1} = n(x \cap X)$, together with any number of occurrences of elements in $\{\alpha(\eta, n)\colon 1 \leq \eta < \eta',\ 1 \leq n < \omega\} \cup \{\gamma, \delta\}$; if $x \neq 0$ and $x \cap X = \emptyset$, then $\Delta^{\zeta}\sigma(x)$ consists of all finite sequences of elements in $\{\gamma, \delta\}$; and if $x = 0$, then $\Delta^{\zeta}\sigma(x)$ consists of all finite sequences of δ's. Statements (a) and (b) for the case $\zeta = \xi + 1$ of (4) are direct consequences of these observations.

We now prove (1). If $\zeta < \mu$, then by (2) and (3) there exist $x, y \in \mathscr{F}$ such that $\nu(x \cap X) = \nu(y \cap X) = \zeta + 1$ and $n(x \cap X) \neq n(y \cap X)$. Thus, $\Delta^{\zeta}\sigma(x) = \Delta^{\zeta}\sigma(y)$ and $\Delta^{\zeta+1}\sigma(x) \neq \Delta^{\zeta+1}\sigma(y)$. Thus, $d(\sigma) < \zeta$. On the other hand, if $\nu(x \cap X)$, $\nu(y \cap X) > \mu$, then $\nu(x \cap X) = \nu(y \cap X) = \mu + 1$ and $n(x \cap X) = n(y \cap X) = 1$. Hence, for all $x, y \in \mathscr{F}$, $\Delta^{\mu}\sigma(x) = \Delta^{\mu}\sigma(y)$ implies $\Delta^{\mu+1}\sigma(x) = \Delta^{\mu+1}\sigma(y)$ by (4). Thus, $d(\sigma) = \mu$.

The Stone space Y of B_{σ} can be realized as a closed subspace of $(\omega + 1) \times \mathscr{C}$. Let $\{p_n\colon n < \omega\}$ be an enumeration of the isolated points of X, a dense subset because X is scattered. Denote $Z = \{(m, p_n) \in \omega \times \mathscr{C}\colon n < m < \omega\}$ and $Y = Z \cup \{\omega\} \times \mathscr{C}$. A topological calculation shows that $Y^{(1)} = \{\omega\} + \mathscr{C}$ and $\bar{Z} \cap Y^{(1)} = \{\omega\} \times X$. Thus, if $\{\omega\} \times \mathscr{C}$ is identified with $\mathscr{C}$, then the rank function of Y is the characteristic function of X; that is, $Y \simeq \operatorname{Ult} B_{\sigma}$.

Notes. Credit for the example in this subsection is due to Lutz Heindorf. The author is very grateful to Dr. Heindorf for permission to include his unpublished result in this chapter.

2. Algebras of isomorphism types

2.1. The monoid of isomorphism types

Our discussion of countable Boolean algebras can be made more orderly by introducing structure on the isomorphism classes of these algebras.

For a Boolean algebra A, let $[A]$ denote the isomorphism class of A, that is, the class of all Boolean algebras B such that $B \cong A$. More generally, the same notation will be used for the isomorphism class of an object in any category. In particular, if X is a Boolean space, then $[X]$ denotes the homeomorphism class of X.

The expression $\boldsymbol{BA}$ denotes the set of all isomorphism classes of *countable* Boolean algebras.

2.1.1. PROPOSITION. *$\boldsymbol{BA}$ is an m-monoid in which $[A] + [B] = [A \times B]$, and whose zero O is the isomorphism class of one element algebras. Moreover, $\boldsymbol{BA}$ has the refinement property, that is, if $\Sigma_{i<n} a_i = \Sigma_{j<m} b_j$ in $\boldsymbol{BA}$, then there exist elements $c_{ij} \in \boldsymbol{BA}$ such that $\Sigma_{j<m} c_{ij} = a_i$ and $\Sigma_{i<n} c_{ij} = b_j$ for $i < n$, $j < m$.*

This elementary result is based on the observation that if $A \cong A'$ and $B \cong B'$, then $A \times B \cong A' \times B'$, so that $+$ is a well-defined binary operation. Moreover, $A \cong B \times C$ if and only if there exist elements x and y in A such that $1_A = x \dot{+} y$ and $B \cong A \restriction x$, $C \cong A \restriction y$. It then follows from properties of the disjoint sum operation of Boolean algebras that $+$ is commutative and associative; and $a + O = a$ for all $a \in \boldsymbol{BA}$. The refinement property in $\boldsymbol{BA}$ is obtained by the same argument that was used in the proof of Proposition 1.15.1. Finally, if $|A \times B| = 1$, then $|A| = |B| = 1$, so that $a + b = O$ implies $a = b = O$ in $\boldsymbol{BA}$. Thus, $\boldsymbol{BA}$ is an m-monoid. □

The Stone duality $A \to \mathrm{Ult}\, A$ induces a bijection between isomorphism classes of countable Boolean algebras and homeomorphism classes of metrizable Boolean spaces. If $\boldsymbol{BS}$ denotes the set of homeomorphism classes of metrizable Boolean spaces, then $[A] \mapsto [\mathrm{Ult}\, A]$ and $[X] \mapsto [\mathrm{Clop}\, X]$ are inverse bijections between $\boldsymbol{BA}$ and $\boldsymbol{BS}$. Under this correspondence, the sum in $\boldsymbol{BA}$ corresponds to the operation on $\boldsymbol{BS}$ that is defined by $[X] + [Y] = [X \uplus Y]$; and $[\emptyset]$ is the zero element of $\boldsymbol{BS}$.

The results in 1.5.1, 1.6.1, 1.8.1, 1.10.1, 1.11.2, and 1.11.3 yield the following facts about the structure of $\boldsymbol{BA}$.

2.1.2. COROLLARY. *$\boldsymbol{BA}$ has the cardinality $2^{\aleph_0}$.*

2.1.3. COROLLARY. *$\boldsymbol{BA} = \boldsymbol{SBA} \uplus \boldsymbol{UBA} \uplus \boldsymbol{MBA}$, where $\boldsymbol{SBA}$ is the set of isomorphism classes of superatomic Boolean algebras, $\boldsymbol{UBA}$ is the set of isomorphism classes of uniform Boolean algebras, and $\boldsymbol{MBA}$ is the set of mixed Boolean algebra types, the elements of which are uniquely representable as sums $b + c$ with $b \in \boldsymbol{SBA}$, $c \in \boldsymbol{UBA}$. Moreover, $\boldsymbol{SBA}$ is a submonoid of $\boldsymbol{BA}$, and $\boldsymbol{UBA}$ and $\boldsymbol{MBA}$ are subsemigroups of $\boldsymbol{BA}$.*

The invariants ν, λ, and n for metrizable Boolean spaces can be interpreted as functions on $\boldsymbol{BA}$ by defining

$$\nu[A] = \nu(\mathrm{Ult}\, A)\,, \qquad \lambda[A] = \lambda(\mathrm{Ult}\, A)\,, \qquad n[A] = n(\mathrm{Ult}\, A)\,.$$

2.1.4. COROLLARY. *Let $a \in \boldsymbol{SBA}$ and $b \in \boldsymbol{UBA}$. If $\nu(a) \leq \nu(b)$, then $a + b = b$; if $\nu(a) > \nu(b)$, then $a + b \in \boldsymbol{MBA}$.*

2.1.5. COROLLARY. *The mapping $a \mapsto (\nu(a), n(a))$ is a semigroup isomorphism of $\boldsymbol{SBA}^*$ to $\omega_1 \times \mathbb{N}$, where the operation on $\omega_1 \times \mathbb{N}$ is defined by $(\xi_1, n_1) + (\xi_2, n_2) = (\xi, n)$ with $\xi = \max\{\xi_1, \xi_2\}$, $n = n_1$ if $\xi_1 > \xi_2$, $n = n_2$ if $\xi_2 > \xi_1$, and $n = n_1 + n_2$ if $\xi_1 = \xi_2$.*

Notes. The idea of viewing the collection of isomorphism classes of Boolean algebras as a monoid is a special case of a construction that was systematically developed by Tarski in his book *Cardinal Algebras* (TARSKI [1949]).

2.2. *Refinement monoids*

The result in Proposition 2.1.1 is a foretaste of the important role that will be played by monoids in the result of this section. It is convenient to survey here some concepts concerning m-monoids that will be encountered often. Throughout this subsection, M is an m-monoid. In particular, M is commutative and has a zero element.

The *natural ordering* of M is defined by

$$a \leq b \quad \text{if } b = a + c \text{ for some } c \in M\,.$$

The existence of a zero element O guarantees that $\leq$ is reflexive, and $0 \leq a$ for all $a \in M$. The commutative and associative laws imply that $\leq$ is transitive, and addition is monotone, that is, $a \leq b$ implies $a + c \leq b + c$. In general, $\leq$ is not antisymmetric, that is, not a partial ordering. An element $a \in M$ has the *Schröder–Bernstein property* (S-B. property for short) if $a \leq b$ and $b \leq a$ imply $b = a$. Using the definition of the natural order, the S-B. property is equivalent to

$$a = a + c + d \text{ implies } a = a + c\,,$$

which of course explains the terminology "Schröder–Bernstein property". Since M is an m-monoid, it follows that 0 has the S-B. property.

If a is an element of the m-monoid M, denote

$$M \restriction a = \{b \in M \colon b \leq a\}\,.$$

We will call M *locally countable* if $M \restriction a$ is a countable set for all $a \in M$. Of course, every countable m-monoid is locally countable. For our purposes, the most interesting example of a locally countable m-monoid is $\boldsymbol{BA}$, which is uncountable. Indeed, if $b \in \boldsymbol{BA} \restriction [A]$, then $b = [A \restriction x]$ for some $x \in A$; since A is countable, so is $\boldsymbol{BA} \restriction [A]$.

If N is a submonoid of the m-monoid M, then N is called a *hereditary submonoid* of M if

$$a \in M,\ b \in N,\ a \leq b, \text{ implies } a \in N .$$

An equivalent statement is that $M \restriction a = N \restriction a$ for all $a \in N$. It is therefore clear that if M is locally countable, then so is every hereditary submonoid of M.

Most of the m-monoids M that will interest us satisfy the *refinement property*: $\Sigma_{i<n} a_i = \Sigma_{j<m} b_j$ in M implies the existence of elements $c_{ij} \in M$ such that $a_i = \Sigma_{j<m} c_{ij}$ for all $i < n$, and $b_j = \Sigma_{i<n} c_{ij}$ for all $j < m$. In this case, M is called an *r-monoid*, abbreviating *refinement monoid*. As we saw in Proposition 2.1.1, $\boldsymbol{BA}$ is an r-monoid. Moreover, it is clear that every hereditary submonoid of an r-monoid is an r-monoid. Thus, the hereditary submonoids of $\boldsymbol{BA}$ are r-monoids.

2.2.1. LEMMA. *An m-monoid M is an r-monoid if and only if M has the* $(2,2)$ *refinement property*:

for all a_0, a_1, b_0, b_1 in M, if $a_0 + a_1 = b_0 + b_1$, then there exist $c_{ij} \in M$ for $i<2$, $j<2$ such that $a_i = c_{i0} + c_{i1}$ and $b_j = c_{0j} + c_{1j}$ for $i, j<2$.

PROOF. Suppose that $\Sigma_{i<n} a_i = \Sigma_{j<m} b_j$ with $n \geq 2$, $m \geq 2$. The existence of the required refinement is proved by induction on $m+n$. We can assume that $2 \leq n \leq m$. The induction hypothesis applies to $\Sigma_{i<n} a_i = \Sigma_{j<m-2} b_j + b'_{m-2}$, where $b'_{m-2} = b_{m-2} + b_{m-1}$. Thus, we can write $a_i = \Sigma_{j<m-2} c_{ij} + c'_{i\,m-2}$ for $i<n$, $b_j = \Sigma_{i<n} c_{ij}$ for $j < m-2$, and $b_{m-2} + b_{m-1} = \Sigma_{i<n} c'_{i\,m-2}$. Another application of the induction hypothesis gives $c'_{i\,m-2} = c_{i\,m-2} + c_{i\,m-1}$ for $i<n$, and $b_{m-2} = \Sigma_{i<n} c_{i\,m-2}$, $b_{m-1} = \Sigma_{i<n} c_{i\,m-1}$, as required. □

The refinement property leads to a useful fact concerning the natural ordering of an r-monoid.

2.2.2. LEMMA. *Let M be an r-monoid. If $a \leq \Sigma_{j<m} b_j$ in M, then $a = \Sigma_{j<m} c_j$, where $c_j \leq b_j$.*

In fact, $\Sigma_{j<m} b_j = a_0 + a_1$, where $a_0 = a$. Hence, $a_i = \Sigma_{j<m} c_{ij}$ for $i = 0, 1$ and $b_j = c_{0j} + c_{1j}$. The elements $c_j = c_{0j}$ therefore satisfy $c_j \leq b_j$ and $a = \Sigma_{j<m} c_j$. □

We will say that an m-monoid M is *atomless* if for each $a \in M^*$, there exit b, $c \in M^*$ such that $a = b + c$. The monoid $\boldsymbol{BA}$ is not atomless: if A is a two-element Boolean algebra, then $[A]$ is a non-zero element of $\boldsymbol{BA}$ that cannot be written as the sum of two non-zero elements.

The following notation will be useful. Let M be an m-monoid, and $a \in M$. Denote

$$\delta(a) = \delta_M(a) = \left\{(b_i)_{i<n} \colon b_i \in M, 1 \leq n < \omega, \sum_{i<n} b_i = a\right\}.$$

2.2.3. LEMMA. *If M is a localy countable r-monoid, then $\delta(a)$ is a countable subset of $M^{<\omega}$ that satisfies the collection, refinement, and local refinement properties. If M is atomless, then $\delta(a)$ has the splitting property.*

This result is an easy consequence of our definitions.

2.2.4. PROPOSITION. *Let M be a locally countable, atomless r-monoid. If $\delta M = \{\delta(a)\colon a \in M\}$, then δM is a submonoid of ΔM, and the mapping $a \mapsto \delta(a)$ is an isomorphism of M to δM whose inverse is the trace map.*

PROOF. By 2.2.3, $\delta(a) \in \Delta M$ for all $a \in M$. It is clear from the definition of δ that $T\delta(a) = a$, and $\delta(a) + \delta(b) \subseteq \delta(a+b)$ for all $a, b \in M$. If $(c_i)_{i<n} \in \delta(a+b)$, then $\Sigma_{i<n} c_i = a + b$. By the refinement property of M, there exist $(a_i)_{i<n} \in \delta(a)$ and $(b_i)_{i<n} \in \delta(b)$ such that $(a_i)_{i<n} + (b_i)_{i<n} = (c_i)_{i<n}$. Thus, $\delta(a+b) = \delta(a) + \delta(b)$. □

2.3. *The V-radical*

The concept of a V-relation can be fruitfully carried to the context of monoids.

2.3.1. DEFINITION. Let M and N be m-monoids. A V-relation between M and N is a set $R \subseteq M \times N$ such that:

(i) if aRb, then $a = 0$ if and only if $b = 0$;

(ii) if aRb, then $a = a_0 + a_1$ in M is equivalent to $b = b_0 + b_1$ in N with a_0Rb_0 and a_1Rb_1.

By routine induction, condition (ii) in this definition can be strengthened to: if aRb, then $a = \Sigma_{i<n} a_i$ in M is equivalent to $b = \Sigma_{i<n} b_i$ in N with a_iRb_i for all $i < n$.

In two special cases it is natural to modify the terminology of the definition. If $N = M$, then a V-relation between M and N will be called a V-relation on M. In particular, if R is an equivalence or congruence relation and also a V-relation, then we refer to R as a V-equivalence or V-congruence on M. In the case that the V-relation R is a morphism from M to N, then R will be called a V-morphism. Note that a morphism $\Theta\colon M \to N$ is a V-morphism if and only if $\Theta a = b_0 + b_1$ implies $a = a_0 + a_1$ with $\Theta a_0 = b_0$ and $\Theta a_1 = b_1$. In this case, $\operatorname{Ker} \Theta = \{(a_0, a_1)\colon \Theta a_0 = \Theta a_1\}$ is a V-congruence on M. Conversely, if Θ is a surjective morphism such that $\operatorname{Ker} \Theta$ is a V-congruence, then Θ is a V-morphism. The proof of this assertion is straightforward.

2.3.2. LEMMA. *Let M, N, and P be m-monoids and suppose that $R \subseteq M \times N$ and $S \subseteq N \times P$ are V-relations. The composition $R \circ S$ is a V-relation between M and P; R^{-1} is a V-relation between N and M; and Δ_M is a V-relation on M. The union of any set of V-relations between M and N is a V-relation.*

The statements in the lemma are easy consequences of the definition of a V-relation.

For any m-monoid M, define

$$Y(M) = \bigcup \{R \subseteq M \times M : R \text{ is a V-relation}\} .$$

It follows from the lemma that $Y(M)$ is a V-equivalence on M. We will say that M is *V-simple* if $Y(M) = \Delta_M$. In this case, $R \subseteq \Delta_M$ for all V-relations R on M.

2.3.3. PROPOSITION. *If M is an r-monoid, then $Y(M)$ is a V-congruence on M, $M/Y(M)$ is a V-simple r-monoid, and the natural projection of M to $M/Y(M)$ is a V-morphism.*

PROOF. We have noted that $Y(M)$ is a V-equivalence on M. To prove that $Y(M)$ is a congruence relation, it is necessary to show that $(a_0, a_1) \in Y(M)$ and $(b_0, b_1) \in Y(M)$ imply $(a_0 + b_0, a_1 + b_1) \in Y(M)$. Denote $R = \{(a_0 + b_0, a_1 + b_1): (a_0, a_1), (b_0, b_1) \in Y(M)\}$. The refinement property and the fact that M is an m-monoid lead to the conclusion that R is a V-relation on M. Thus, $R \subseteq Y(M)$, the required conclusion. Clearly, $M/Y(M)$ is an r-monoid, and our earlier remark shows that the natural projection π: $M \to M/Y(M)$ is a V-morphism. If S is a V-relation on $M/Y(M)$, then $\pi^{-1}(S) = \pi \circ S \circ \pi^{-1}$ is a V-relation on M. Thus, $\pi^{-1}(S) \subseteq Y(M)$ and $S \subseteq \Delta_{M/Y(M)}$. In particular, $Y(M/Y(M)) = \Delta_{M/Y(M)}$, that is, $M/Y(M)$ is V-simple. $\square$

The congruence relation $Y(M)$ on the r-monoid M is called the V-radical of M.

2.3.4. COROLLARY. *If Θ: $M \to N$ is a V-morphism of r-monoids, then the image of Θ is a hereditary submonoid of N, and* Ker $\Theta \subseteq Y(M)$. *Moreover, if N is V-simple, then* Ker $\Theta = Y(M)$, *and Θ is the unique V-morphism from M to N.*

PROOF. Since Θ is a V-relation, so is Ker $\Theta = \Theta \circ \Theta^{-1}$ a V-relation on M. Hence, Ker $\Theta \subseteq Y(M)$. If $a \in M$ and $b \in N \restriction \Theta(a)$, then $\Theta(a) = b + c$ for some $c \in N$. Therefore, $a = a_0 + a_1$ with $\Theta(a_0) = b$, since Θ is a V-relation. This argument shows that $N \restriction \Theta(a) \subseteq \text{Im}\ \Theta$. That is, Im Θ is hereditary. Suppose that N is V-simple and Θ, Ψ: $M \to N$ are V-morphisms. Then $\Psi^{-1} \circ \Theta$ is a V-relation on N. Thus, $\Psi^{-1} \circ \Theta \subseteq Y(N) = \Delta_N$, that is, $\Psi = \Theta$. Moreover, $M/\text{Ker}\ \Theta \cong \text{Im}\ \Theta$ is V-simple. Indeed, since Im Θ is a hereditary submonoid of N, every V-relation on Im Θ is a V-relation on N. In view of the fact that $\Theta(Y(M))$ is a V-relation on Im Θ, it follows that $Y(M) = \text{Ker}\ \Theta$. $\square$

If M is a hereditary submonoid of N, then the inclusion map of M to N is clearly a V-morphism. Thus, if N is V-simple, then so is M.

Notes. The results in this subsection are particular consequences of Dobbertin's systematic study of r-monoids (DOBBERTIN [1982]).

2.4. Dobbertin's theorem

The major result in this subsection provides a characterization of the monoid ***BA***.

2.4.1. LEMMA. *If M is a locally countable r-monoid and $a \in M$, then there is a countable Boolean algebra A that is uniquely determined up to isomorphism, and a relation $R \subseteq A \times M \restriction a$ such that*:

(i) $1_A Ra$;

(ii) $xR0$ *implies* $x = 0$; $0Rb$ *implies* $b = 0$;

(iii) $xRb_0 + b_1$ *implies* $x = x_0 \dot{+} x_1$ *with* x_0Rb_0, x_1Rb_1; $x_0 \dot{+} x_1Rb$ *implies* $b = b_0 + b_1$ *with* x_0Rb_0, x_1Rb_1.

PROOF. If $a = 0$, then the trivial algebra and relation fulfill the requirements, so that we can assume $a \neq 0$. Denote $\delta(a) = \{\boldsymbol{b} \in M^{<\omega}: T\boldsymbol{b} = a\}$, as in subsection 2.2. By Lemma 2.2.3, $\delta(a)$ satisfies C.P., R.P. and L.P. Therefore, Lemma 1.18.4 yields a uniformly dense $\delta(a)$-tree $\mathfrak{a}$. We obtain A by a construction that starts with the description of the countable free algebra $\mathscr{F}$ that was given in the proof of Proposition 1.16.4: $\mathscr{F} = \text{Clop}\, \mathscr{C}$, where $\mathscr{C} = {}^{\omega}2$ has the clopen basis $\{x(\boldsymbol{i}): \boldsymbol{i} \in \mathscr{D}\}$, $x(\boldsymbol{i}) = \{p \in {}^{\omega}2: p \restriction n = \boldsymbol{i}\}$ for $\boldsymbol{i} \in \mathscr{D}_n$. If J is the ideal of $\mathscr{F}$ that is generated by $\{x(\boldsymbol{i}): \boldsymbol{i} \in \mathscr{D},\ \mathfrak{a}(\boldsymbol{i}) = 0\}$, then

(1) $x(\boldsymbol{i}) \in J$ if and only if $\mathfrak{a}(\boldsymbol{i}) = 0$.

(The implication $x(\boldsymbol{i}) \in J$ implies $\mathfrak{a}(\boldsymbol{i}) = 0$ is a consequence of (ii) in the Definition 1.16.2 of α-trees.) Define $A = \mathscr{F}/J$, and let

(2) $R = \{((\Sigma_{\boldsymbol{i} \in G}\, x(\boldsymbol{i}))/J,\ \Sigma_{\boldsymbol{i} \in G}\, \mathfrak{a}(\boldsymbol{i})): G \subseteq \mathscr{D}_n,\ n < \omega\}$.

Since $\Sigma_{\boldsymbol{i} \in \mathscr{D}_n} \mathfrak{a}(\boldsymbol{i}) = a$, it follows that $1_A Ra$. If xRb, then $x = 0$ if and only if $b = 0$ by (1). If $xRb_0 + b_1$, then by (2) there exists $n < \omega$ and $G \subseteq \mathscr{D}_n$ such that $x = (\Sigma_{\boldsymbol{i} \in G}\, x(\boldsymbol{i}))/J$ and $b_0 + b_1 = \Sigma_{\boldsymbol{i} \in G}\, \mathfrak{a}(\boldsymbol{i})$. The (tacit) assumption that $b_0 + b_1 \leq a$ implies that (b_0, b_1) is a fragment of $\delta(a)$, so that $\mathfrak{a}(G \uparrow m) < (b_0, b_1)$ for some $m \geq n$, because $\mathfrak{a}$ is uniformly dense. That is, $G \uparrow m = H_0 \cup H_1$ with $b_j = T(\mathfrak{a}(H_j))$ for $j = 0, 1$. Thus, $x = x_0 \dot{+} x_1$, where $x_j = (\Sigma_{\boldsymbol{i} \in H_j}\, x(\boldsymbol{i}))/J$ and x_jRb_j for $j = 0, 1$. Similarly, if $x_0 \dot{+} x_1Rb$, then it follows from (1) that we can write $x_j = (\Sigma_{\boldsymbol{i} \in G_j}\, x(\boldsymbol{i}))/J$ for $j = 0, 1$, and $b = \Sigma_{\boldsymbol{i} \in G}\, \mathfrak{a}(\boldsymbol{i})$, where $G = G_0 \cup G_1 \subseteq \mathscr{D}_n$. Consequently, $b = b_0 + b_1$ with $b_j = \Sigma_{\boldsymbol{i} \in G_j}\, \mathfrak{a}(\boldsymbol{i})$ satisfying x_jRb_j for $j = 0, 1$. The uniqueness of A follows from Vaught's Theorem and the observation that if A' is a countable Boolean algebra and $R' \subseteq A' \times M \restriction a$ satisfies (i), (ii), and (iii), then $R \circ (R')^{-1} \subseteq A \times A'$ is a V-relation. $\square$

2.4.2. THEOREM. (a) ***BA*** *is a locally countable, V-simple r-monoid.*

(b) *If M is a locally countable r-monoid, then there is a unique V-morphism Θ: $M \to \boldsymbol{BA}$; moreover,* $\operatorname{Ker} \Theta = Y(M)$ *and* $\operatorname{Im} \Theta$ *is a hereditary submonoid of* ***BA***.

(c) ***BA*** *is determined to within isomorphism by the properties* (a) *and* (b).

PROOF. We have already noted that $\boldsymbol{BA}$ is a locally countable r-monoid. Let $S = Y(\boldsymbol{BA})$. Then S is a V-congruence on $\boldsymbol{BA}$ that induces a V-relation R on the class of all countable Boolean algebras by ARB if $[A]S[B]$. If $[A]S[B]$, then ARB; hence $A \cong B$ by Vaught's Theorem; that is, $[A] = [B]$. Thus, $\boldsymbol{BA}$ is V-simple. Suppose that M is a locally countable r-monoid. For each $a \in M$, let A_a and R_a be the Boolean algebra and the relation corresponding to $M \restriction a$ as in the lemma. Since A_a is unique to isomorphism, Θ: $a \mapsto [A_a]$ is a well-defined map of M to $\boldsymbol{BA}$. By (i) and (ii) of the lemma , $\Theta(a) = 0$ if and only if $a = 0$. To prove that Θ is a homomorphism, suppose that $a = a_0 + a_1$ in M. Denote $A = A_a$,

$R = R_a$. Since $1_A R a_0 + a_1$, it follows from the lemma that there exists $x \in A$ such that xRa_0 and $-xRa_1$. A routine check shows that $R \cap (A \restriction x \times M \restriction a_0)$ satisfies the conditions of the lemma relative to $A \restriction x$ and $M \restriction a_0$. Hence, $\Theta(a_0) = [A \restriction x]$. Similarly, $\Theta(a_1) = [A \restriction (-x)]$. Thus, $\Theta(a_0) + \Theta(a_1) = [A \restriction x] + [A \restriction (-x)] = [A] = \Theta(a_0 + a_1)$. A similar argument shows that Θ is a V-morphism: if $\Theta(a) = e + f$, then there exists $x \in A_a$ such that $e = [A_a \restriction x]$, $f = [A_a \restriction (-x)]$; by part (iii) of the lemma, $a = b + c$ with $xR_a b$, $-xR_a c$, and $\Theta(b) = e$, $\Theta(c) = f$ as before. By Corollary 2.3.4, Θ is unique, Ker $\Theta = \Upsilon(M)$, and Im Θ is a hereditary submonoid of $\boldsymbol{BA}$. Finally, the fact that $\boldsymbol{BA}$ is uniquely determined to isomorphism by the properties (a) and (b) is obtained by the familiar uniqueness argument for terminal objects in a category. □

Notes. Theorem 2.4.2 is due to DOBBERTIN [1982]. Its proof is much simpler than that of the related theorem of Ketonen, which we will discuss next.

2.5. *Ketonen's theorem*

Undoubtedly the most striking result in the theory of countable Boolean algebras is a theorem that was proved by Ketonen in 1976.

2.5.1. THEOREM. *If M is a countable, commutative semigroup, then M is isomorphic to a subsemigroup of* $\boldsymbol{BA}$.

Ketonen's proof of this theorem will be given in the next 15 subsections. It uses the Boolean hierarchy in a fairly technical way. The ideas and subsidiary results in this proof will not be used in the rest of this chapter. Readers whose main interest is the general theory of countable Boolean algebras can pass directly to subsection 2.21. Plainly, Ketonen's theorem would follow from Dobbertin's result (Theorem 2.4.2) if it could be shown that countable, commutative semigroups can be embedded in V-simple r-monoids. In fact, Ketonen's theorem shows that they can, but a direct proof of this fact is elusive; indeed, none has yet been found.

The motivation for Ketonen's work seems to have been some arithmetical questions from cardinal arithmetic. They are most easily formulated as questions about $\boldsymbol{BA}$. Suppose that a, b, and c are elements of the additive monoid $\boldsymbol{BA}$.

(1) Does $a = a + b + c$ imply $a = a + b$?
(2) Does $a = a + b + b$ imply $a = a + b$?
(3) Does $a = a + a + a$ imply $a = a + a$?

The first of these questions can be restated by asking if every $a \in \boldsymbol{BA}$ has the Schröder–Bernstein property. Problems (2) and (3) are progressively weaker versions of the S-B. property. By Ketonen's Theorem $\boldsymbol{BA}$ contains a submonoid that is isomorphic to a cyclic group of order two. Thus, (3) fails in $\boldsymbol{BA}$, and therefore so do (2) and (1). An explicit (but very ingeneous) counterexample to (2) was given by Hanf in 1957. The third question, which was known as the "Tarski cube problem", was only settled with the publication of Ketonen's work.

Notes. Our reference for Theorem 2.5.1 and its proof is the work of Ketonen that was mentioned before. Hanf's result appears in HANF [1957]. A topological formulation of Hanf's theorem is presented in the book by HALMOS [1947]. The special case of problem (2) in which b is the isomorphism type of a non-trivial finite Boolean algebra has a positive solution by Theorem 1.10.1 and Proposition 1.6.1, or by Proposition 6.4.

2.6. *Products of measures*

A consequence of Proposition 1.9.1 is the fact that the rank function of the disjoint union of uniform Boolean spaces Y and Z is the union of the rank functions r_Y and r_Z. We will now translate that observation into the context of measures on the free Boolean algebra $\mathscr{F}$. As usual, M is an m-monoid.

2.6.1. DEFINITION. Let k: $\mathscr{F} \to \mathscr{F} \times \mathscr{F}$ be an isomorphism of Boolean algebras. Denote the first and second component projections of $\mathscr{F} \times \mathscr{F}$ to $\mathscr{F}$ by p_1 and p_2, that is, $p_1(x, y) = x$, $p_2(x, y) = y$. For σ, $\tau \in \mathscr{M}(M)$ and $x \in \mathscr{F}$ define

$$(\sigma \oplus_k \tau)(x) = \sigma(p_1 kx) + \tau(p_2 kx) .$$

This definition provides a mapping from $\mathscr{F}$ to M.

2.6.2. LEMMA. (a) $\sigma \oplus_k \tau \in \mathscr{M}(M)$.

(b) *If* $\sigma' \cong \sigma$, $\tau' \cong \tau$, *and* k' *is another isomorphism of* $\mathscr{F}$ *to* $\mathscr{F} \times \mathscr{F}$, *then* $\sigma' \oplus_{k'} \tau' \cong \sigma \oplus_k \tau$.

(c) *For all* $\xi < \omega_1$, $\Delta^\xi(\sigma \oplus_k \tau) = \Delta^\xi \sigma \oplus_k \Delta^\xi \tau$.

PROOF. A routine calculation shows that $\sigma \oplus_k \tau$ is additive and maps non-zero elements of $\mathscr{F}$ to M^*. If $\sigma' = \sigma \circ l_1$ and $\tau' = \tau \circ l_2$ with $l_1, l_2 \in \operatorname{Aut} \mathscr{F}$, then $\sigma' \oplus_{k'} \tau' = (\sigma \oplus_k \tau) \circ l$, where $l = k^{-1}(l_1 \times l_2)k' \in \operatorname{Aut} \mathscr{F}$. It suffices to prove (c) when $\xi = 1$; the general case will follow by transfinite induction. A computation gives $\Delta(\sigma \oplus_k \tau)(x) \subseteq (\Delta\sigma \oplus_k \Delta\tau)(x)$. Suppose that $(b_i)_{i<n} \in \Delta\sigma(p_1 kx)$ and $(c_i)_{i<n} \in \Delta\tau(p_2 kx)$, say $p_1 kx = y_0 \dot{+} \cdots \dot{+} y_{n-1}$, $p_2 kx = z_0 \dot{+} \cdots \dot{+} z_{n-1}$ with $b_i = \sigma y_i$, $c_i = \tau z_i$. Then $x = x_0 \dot{+} \cdots \dot{+} x_{n-1}$, where $x_i = k^{-1}(y_i, z_i)$, and $(\sigma \oplus_k \tau)(x_i) = \sigma y_i + \tau z_i = b_i + c_i$. Thus, $(\Delta\sigma \oplus_k \Delta\tau)(x) \subseteq \Delta(\sigma \oplus_k \tau)(x)$. □

To simplify notation, we will usually write $\sigma \oplus \tau$ instead of $\sigma \oplus_k \tau$. If one is only interested in equivalence classes of measures, this convention is justified by part (b) of the lemma. However, the omission of a reference to the isomorphism k suggests that the binary operation $\oplus$ turns $\mathscr{M}(M)$ into a semigroup. Unfortunately, $\oplus_k$ is not associative.

2.6.3. PROPOSITION. *Let* $0 < x < 1$ *in* $\mathscr{F}$, *and suppose that* k_1: $\mathscr{F} \to \mathscr{F} \restriction x$ *and* k_2: $\mathscr{F} \to \mathscr{F} \restriction (-x)$ *are isomorphisms. If* $\rho \in \mathscr{M}(M)$ *and* $\sigma = \rho \circ k_1$, $\tau = \rho \circ k_2$, *then* σ, $\tau \in \mathscr{M}(M)$ *and* $\sigma \oplus \tau \cong \rho$. *Moreover, if* ρ *is stable, then so are* σ *and* τ.

PROOF. It is clear that σ and τ are additive; they map $\mathscr{F} \backslash \{0\}$ to M^* because x and $-x$ are not $0_{\mathscr{F}}$. A calculation shows that $\rho = \sigma \oplus_k \tau$, where k is the isomorphism of $\mathscr{F}$ to $\mathscr{F} \times \mathscr{F}$ that is defined by $k(y) = (k_1^{-1}(x \cdot y), k_2^{-1}((-x) \cdot y)))$. Assume that ρ is stable and $\sigma(y) = \sigma(z)$. In this case, $\rho(k_1 y) = \rho(k_1 z)$, so that $\Delta\sigma(y) = \Delta\rho(k_1 y) = \Delta\rho(k_1 z) = \Delta\sigma(z)$ by the stability of ρ. □

2.6.4. COROLLARY. *If $\sigma, \tau \in \mathscr{M}$, then $B_\sigma \times B_\tau \cong B_{\sigma\oplus\tau}$.*

Recall that $\mathscr{M} = \mathscr{M}(\mathscr{W})$ is the set of all $\mathscr{W}$-measures. The notation $B_\sigma, B_\tau, B_{\sigma\oplus\tau}$ refers to uniform Boolean algebras that correspond to the respective measures, as in Corollary 1.12.3. To see how the corollary follows from the proposition, let $X = \text{Ult}\, B_\sigma$, $Y = \text{Ult}\, B_\tau$, so that $\text{Ult}(B_\sigma \times B_\tau) = X \uplus Y$. By Proposition 1.9.1, $K(X \uplus Y) = K(X) \uplus K(Y)$ and $r_X = r_{X\uplus Y} \restriction K(X)$, $r_Y = r_{X\uplus Y} \restriction K(Y)$. Identify $\mathscr{F}$ with Clop $K(X \uplus Y)$, so that $K(X)$ becomes an element $x \in \mathscr{F}$ and $K(Y)$ is $-x$. Let ρ be the measure corresponding to $r_{X\cup Y}$. The measures corresponding to r_X and r_Y can then be written in the form $\rho \circ k_1$ and $\rho \circ k_2$, where k_1 and k_2 are isomorphisms of $\mathscr{F}$ to $\mathscr{F} \restriction x$ and $\mathscr{F} \restriction (-x)$, respectively. Thus, $\sigma \cong \rho \circ k_1$ and $\tau \cong \rho \circ k_2$. By the proposition and the lemma, $\rho \cong \rho \circ k_1 \oplus \rho \circ k_2 \cong \sigma \oplus \tau$. Thus, $B_{\sigma\oplus\tau} \cong B_\rho \cong B_\sigma \times B_\tau$.

2.7. *The strict hierarchy property*

Let M be an arbitrary m-monoid. If ξ is a countable ordinal, then an element α in the transfinite derivative $\Delta^{\xi+1}M$ has the *strict hierarchy property* (relative to M) if:

($*$) for every $(a, \boldsymbol{b}) \in \alpha$, $\eta < \xi$, and $(e_i)_{i<n} \in T^\xi_{\eta+1}(a)$ there exists $\boldsymbol{a} = (a_i)_{i<n}$ such that $\Sigma_{i<n}\, a_i = a$, $(\boldsymbol{a}, \boldsymbol{b}) \in \alpha$, and $T^\xi_\eta a_i = e_i$ for all $i < n$.

The special case of ($*$) in which $\boldsymbol{b}$ is the empty sequence is the hierarchy property (H.P.) of Definition 1.22.2. We were able to get by with the weaker H.P. in Proposition 1.22.3 because if α satisfies both L.P. and H.P., then α clearly has the strict hierarchy property.

For $\xi < \omega_1$, denote

$$\mathscr{L}^\xi = \{\alpha \in \Delta^{\xi+1}\mathscr{W} : \alpha \text{ has the strict hierarchy property}\}\,.$$

The above remarks imply that the sets of the Boolean hierarchy satisfy $\mathscr{K}^\xi \subseteq \mathscr{L}^\xi$. However, $\mathscr{L}^\xi$ has an advantage over $\mathscr{K}^\xi$.

2.7.1. LEMMA. *For all $\xi < \omega_1$, $\mathscr{L}^\xi$ is a submonoid of $\Delta^{\xi+1}\mathscr{W}$.*

PROOF. Since the trace maps are morphisms of m-monoids, it is clear that $0 \in \mathscr{L}^\xi$. Let α, β be non-zero elements of $\mathscr{L}^\xi$. Suppose that $(a, \boldsymbol{b}) \in \alpha + \beta$. Then $a = a_0 + a_1$, $\boldsymbol{b} = \boldsymbol{b}_0 + \boldsymbol{b}_1$, where $(a_0, \boldsymbol{b}_0) \in \alpha$, $(a_1, \boldsymbol{b}_1) \in \beta$. If $\boldsymbol{e} \in T^\xi_{\eta+1}(a) = T^\xi_{\eta+1}(a_0) + T^\xi_{\eta+1}(a_1)$, then $\boldsymbol{e} = \boldsymbol{e}_0 + \boldsymbol{e}_1$ with $\boldsymbol{e}_k = (e_{ki})_{i<n} \in T^\xi_{\eta+1}(a_k)$ for $k = 0, 1$. The strict hierarchy property in α and β yield $\boldsymbol{a}_k = (a_{ki})_{i<n}$ such that $\Sigma_{i<n}\, a_{ki} = a_k$, $T^\xi_\eta a_{ki} = e_{ki}$ for $i < n$, $(\boldsymbol{a}_0, \boldsymbol{b}_0) \in \alpha$, and $(\boldsymbol{a}_1, \boldsymbol{b}_1) \in \beta$. Then $(a_i)_{i<n} = \boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{a}_1$ satisfies $\Sigma_{i<n}\, a_i = a$, $(\boldsymbol{a}, \boldsymbol{b}) \in \alpha + \beta$, and $T^\xi_\eta a_i = e_i$ for $i < n$. Hence, $\alpha + \beta \in \mathscr{L}^\xi$. □

The proof of this lemma shows somewhat more than is stated: if $\alpha, \beta \in \Delta^{\xi+1}M$ have the strict hierarchy property, then so does $\alpha + \beta$. This remark will be used later.

2.7.2. LEMMA. *Let $\xi < \omega_1$. If $\alpha \in \Delta^{\xi+2}\mathscr{W}$ is such that*
(1) $\boldsymbol{a} \in (\mathscr{L}^{\xi})^{<\omega}$ *for all* $\boldsymbol{a} \in \alpha$, *and*
(2) $(a, \boldsymbol{b}) \in \alpha$, $(e_i)_{i<n} \in a$ *implies there exists* $\boldsymbol{a} = (a_i)_{i<n} \in (\mathscr{L}^{\xi})^{<\omega}$ *such that* $\Sigma_{i<n} a_i = a$, $Ta_i = e_i$ *for all* $i < n$, *and* $(\boldsymbol{a}, \boldsymbol{b}) \in \alpha$,
then $\alpha \in \mathscr{L}^{\xi+1}$.

This result is an easy consequence of the equation $T_\eta^{\xi+1} = T_\eta^{\xi} \circ T$ in Definition 1.19.1.

2.7.3. PROPOSITION. *Let M be a locally countable submonoid of $\mathscr{L}^{\xi}$ such that*
(1) *M has the refinement property*, *and*
(2) *if $(a_i)_{i<n} \in \alpha \in M$, then there exist $\alpha_i \in M$ such that $T\alpha_i = a_i$ for all $i < n$ and $\Sigma_{i<n} \alpha_i = \alpha$.*
*Then M^* is isomorphic to a subsemigroup of **UBA**.*

PROOF. It follows from (2) and the splitting property of the elements in M that M is atomless. Therefore, by Lemma 2.2.3 and Proposition 2.2.4, $\alpha \mapsto \delta(\alpha)$ is an embedding of M into $\Delta^{\xi+2}\mathscr{W}$, and each $\delta(\alpha)$ satisfies L.P. By (2) and 2.7.2, $\delta(\alpha)$ also satisfies the strict hierarchy property, that is, $\delta(\alpha) \in \mathscr{K}^{\xi+1}$ by Proposition 1.22.3. For each $\alpha \in M$, choose $\sigma_\alpha \in \mathscr{M}$ such that $\Delta^{\xi+2}\sigma_\alpha(1) = \delta(\alpha)$ and $d(\sigma_\alpha) \leq \xi + 1$. If $\sigma_\alpha \cong \sigma_\beta$, then $\delta(\alpha) = \Delta^{\xi+2}\sigma_\alpha(1) = \Delta^{\xi+2}\sigma_\beta(1) = \delta(\beta)$, so that $\alpha = \beta$. Assume that $\alpha, \beta \in M^*$. Then $(\Delta^{\xi+1}\sigma_\alpha(1), \Delta^{\xi+1}\sigma_\beta(1)) = (\Delta^{\xi+1}\sigma_\alpha(1), 0) + (0, \Delta^{\xi+1}\sigma_\beta(1)) \in \Delta^{\xi+2}\sigma_\alpha(1) + \Delta^{\xi+2}\sigma_\beta(1) = \delta(\alpha) + \delta(\beta) = \delta(\alpha + \beta) = \Delta(\Delta^{\xi+1}\sigma_{\alpha+\beta}(1)$. Therefore, there exists $x \in \mathscr{F}$ such that $\Delta^{\xi+1}\sigma_\alpha(1) = \Delta^{\xi+1}\sigma_{\alpha+\beta}(x)$ and $\Delta^{\xi+1}\sigma_\beta(1) = \Delta^{\xi+1}\sigma_{\alpha+\beta}(-x)$. Since α and β are not zero, $0 < x < 1$. If k_1: $\mathscr{F} \to \mathscr{F} \restriction x$ and k_2: $\mathscr{F} \to \mathscr{F} \restriction (-x)$ are isomorphisms, then $\Delta^{\xi+1}(\sigma_{\alpha+\beta} \circ k_1)(1) = \Delta^{\xi+1}\sigma_\alpha(1)$ and $\Delta^{\xi+1}(\sigma_{\alpha+\beta} \circ k_2)(1) = \Delta^{\xi+1}\sigma_\beta(1)$. By Proposition 2.6.3, $\Delta^{\xi+1}\sigma_{\alpha+\beta} \circ k_1$ and $\Delta^{\xi+1}\sigma_{\alpha+\beta} \circ k_2$ are stable. Therefore, Proposition 1.17.1 implies that $\Delta^{\xi+1}\sigma_{\alpha+\beta} \circ k_1 \cong \Delta^{\xi+1}\sigma_\alpha$ and $\Delta^{\xi+1}\sigma_{\alpha+\beta} \circ k_2 \cong \Delta^{\xi+1}\sigma_\beta$. It follows from Proposition 2.6.3 and Lemma 2.6.2 that $\Delta^{\xi+1}\sigma_{\alpha+\beta} \cong \Delta^{\xi+1}\sigma_\alpha \oplus \Delta^{\xi+1}\sigma_\beta \cong \Delta^{\xi+1}(\sigma_\alpha \oplus \sigma_\beta)$; and these measures are stable. Since $\Delta^{\xi+2}\sigma_{\alpha+\beta}(1) = \delta(\alpha + \beta) = \delta(\alpha) + \delta(\alpha) = \Delta^{\xi+2}\sigma_\alpha(1) + \Delta^{\xi+2}\sigma_\beta(1) = \Delta^{\xi+2}(\sigma_\alpha \oplus \sigma_\beta)(1)$, Theorem 1.20.2 yields $\sigma_{\alpha+\beta} \cong \sigma_\alpha \oplus \sigma_\beta$. Finally, for each $\alpha \in M$, let B_α be a uniform Boolean algebra whose associated measure is equivalent to σ_α. By Corollary 2.6.3. $B_{\alpha+\beta} \cong B_\alpha \times B_\beta$. The mapping $\alpha \mapsto [B_\alpha]$ is therefore a homomorphism from M^* to **UBA**. This homomorphism is injective because $B_\alpha \cong B_\beta$ implies $\sigma_\alpha \cong \sigma_\beta$, and therefore $\alpha = \beta$. □

The proposition contains the skeleton of the proof of Ketonen's Theorem. Our program is to embed a commutative semigroup in a submonoid M of $\mathscr{L}^2$ so that conditions (1) and (2) are satisfied. The first stage of the proof consists of shifting from the sequence of derived monoids of $\mathscr{W}$ to a modified hierarchy that is easier to handle. The translated problem will also reduce the number of hierarchy levels requiring attention from four to three. The next four subsections describe this new setup.

2.8. Shifting monoids

For convenience let us recall the notation and results from subsection 1.14. Proposition 1.14.1 established an isomorphism between the derived monoid of $\mathcal{W} = \omega_1 \cup \{o\}$ and a certain additive monoid $\mathcal{N}$ of mappings from ω_1 to $\omega + 1$. If θ is such a mapping, then the *support* of θ is

$$\operatorname{Supp} \theta = \{\xi < \omega_1 \colon \theta(\xi) > 0\} .$$

The semigroup $\mathcal{N}^*$ of non-zero elements in $\mathcal{N}$ consists of all θ such that Supp θ contains a largest $\eta < \omega_1$ (hence Supp θ is countable and not empty), and $\theta(\xi) = \omega$ if $\xi = \min \operatorname{Supp} \theta$. The isomorphism Ψ from $\mathcal{N}^*$ to $(\Delta\mathcal{W})^*$ can be described conveniently in terms of new notation. For each $\boldsymbol{a} = (a_i)_{i<n} \in \mathcal{W}^{<\omega}$, define the mapping $\phi_{\boldsymbol{a}} \colon \omega_1 \to \omega$ by $\phi_{\boldsymbol{a}}(\xi) = |\{i < n \colon a_i = \xi\}|$. The proof of Proposition 1.14.1 shows that

$$\Psi(\theta) = \{\boldsymbol{a} \in \mathcal{W}^{<\omega} \colon \phi_{\boldsymbol{a}} \le \theta \text{ pointwise, and } \max \operatorname{Supp} \phi_{\boldsymbol{a}} = \max \operatorname{Supp} \theta\} .$$

The element max Supp θ is precisely $T(\Psi(\theta))$.

We will use a relatively small submonoid of $\mathcal{N}$. The functions in $\mathcal{N}^{0^*}$ will have support that is contained in $\omega + 1$ instead of ω_1, so it is convenient to consider the elements of $\mathcal{N}^{0^*}$ to be mappings from $\omega + 1$ to $\omega + 1$. Two mappings θ, χ: $\omega + 1 \to \omega + 1$ are *equal almost everywhere* if $\{k \le \omega \colon \theta(k) \ne \chi(k)\}$ is finite. In this case, we will write $\theta \equiv \chi$. Plainly, $\equiv$ is a congruence on the additive monoid of mappings from $\omega + 1$ to $\omega + 1$. The symbols 0^* and ω^* will designate constant mappings; specificaly, $0^*(k) = 0$ and $\omega^*(k) = \omega$ for all $k \le \omega$. Define $\mathcal{N}^{0^*}$ to be all the mappings from $\omega + 1$ to $\omega + 1$ that fulfill one of the conditions:

(i) $\theta \equiv 0^*$;

(ii) $\theta \equiv \omega^*$ and $\theta(\omega) = \omega$;

(iii) $\theta(k) < \omega$ for almost all k and $\lim_{k\to\omega} \theta(k) = \omega = \theta(\omega)$,

where the limit refers to the order topology of $\omega + 1$. Addition in $\mathcal{N}^{0^*}$ is defined componentwise: $(\theta + \chi)(k) = \theta(k) + \chi(k)$. To complete our description, adjoin a (new) zero 0 to $\mathcal{N}^{0^*}$ to produce $\mathcal{N}^0 = \mathcal{N}^{0^*} \cup \{0\}$. It is convenient to extend $\equiv$ to the congruence on $\mathcal{N}^0$ that is obtained by adding the relation $0 \equiv 0^*$. Then 0 is congruent to θ if and only if θ is equal to 0 almost everywhere.

2.8.1. LEMMA. *$\langle \omega + 1, + \rangle$ is an r-monoid. Moreover, if $r_0 + r_1 = s_0 + s_1$ in $\omega + 1$, and $n \ge 2$, then $r_i = \Sigma_{j<2} t_{ij}$, $s_j = \Sigma_{i<2} t_{ij}$, and $\min\{t_{ij} \colon i, j < 2\} = m_n$, where m_n is the greatest integer $\le (1/n) \min\{r_0, r_1, s_0, s_1\}$.*

PROOF. It is sufficient by Lemma 2.2.1 to prove the second statement. If $r_0 + r_1 = s_0 + s_1 = \omega$, then by adjusting subscripts, it can be assumed that $r_1 = s_1 = \omega$. In this case, $t_{00} = m_n$, $t_{01} = r_0 - m_n$, $t_{10} = s_0 - m_n$, $t_{11} = \omega$ (with $\omega - \omega =$

ω) fulfills our requirements. Suppose that r_0, r_1, s_0, s_1 are finite. Since $r_0 + r_1 = s_0 + s_1$, we can assume that $r_1 \geq s_0$. Then $t_{00} = m_n$, $t_{01} = r_0 - m_n$, $t_{10} = s_0 - m_n$, $t_{11} = r_1 - s_0 + m_n$ does the job. $\square$

2.8.2. LEMMA. *$\mathcal{N}^0$ is an r-monoid. If $\theta_0 + \theta_1 = \psi_0 + \psi_1$ in $\mathcal{N}^{0^*}$ and $\theta_0, \theta_1, \psi_0$, and ψ_1 are not congruent to 0^*, then there are uncountably many sets $\{\chi_{ij}: i, j < 2\} \subseteq \mathcal{N}^{0^*}$ such that $\theta_i = \Sigma_{j<2}\, \chi_{ij}$ and $\psi_j = \Sigma_{i<2}\, \chi_{ij}$.*

PROOF. A check of cases shows that if θ and ψ each satisfies one of the conditions (i), (ii), or (iii) (not necessarily the same one), then so does $\theta + \psi$. Hence, $\mathcal{N}^0$ is an m-monoid. If $\theta_0 + \theta_1 = \psi_0 + \psi_1$ in $\mathcal{N}^{0^*}$, then $\theta_0(k) + \theta_1(k) = \psi_0(k) + \psi_1(k)$ for all $k \leq \omega$. The use of 2.8.1 with $n = 2$ yields $\chi_{ij} \in \mathcal{N}^{0^*}$ such that $\theta_i = \Sigma_{j<2}\, \chi_{ij}$ and $\chi_j = \Sigma_{i<2}\, \chi_{ij}$. By Lemma 2.2.1, $\mathcal{N}^0$ is an r-monoid. Assume that $\theta_0, \theta_1, \psi_0, \psi_1 \not\equiv 0^*$. Denote $\mu(k) = \min\{\theta_0(k), \theta_1(k), \psi_0(k), \psi_1(k)\}$ for all $k \leq \omega$. Plainly, $\mu \in \mathcal{N}^{0^*}$ and $\mu \not\equiv 0^*$. If $V = \{k < \omega: \mu(k) = \omega\}$ is confinite, let $\chi_{01}(k) = \chi_{10}(k) = \chi_{11}(k) = \omega$ and $\chi_{00}(k) = \chi(k)$ (with $\chi \in \mathcal{N}^{0^*}$ arbitrary) for all $k \in V$; use 2.8.1 to choose the appropriate values of $\chi_{ij}(k)$ for k in the finite set $\omega \backslash V$. Suppose that $\mu \not\equiv \omega^*$. For each subset E of ω, use 2.8.1 to define functions χ^E_{ij} on $\omega + 1$ so that $\theta_i = \Sigma_{j<2}\, \chi^E_{ij}$, $\psi_j = \Sigma_{i<2}\, \chi^E_{ij}$, $\min\{\chi^E_{ij}(k): i, j < 2\} = [(1/2)\mu(k)]$ if $k \in E$, and $\min\{\chi^E_{ij}(k): i, j < 2\} = [(1/3)\mu(k)]$ if $k \in (\omega + 1)\backslash E$. (In these last formulas, $[t]$ denotes the greatest integer $\leq t$.) Plainly, this recipe produces uncountably many sets $\{\chi^E_{ij}: i, j < 2\} \subseteq \mathcal{N}^{0^*}$. $\square$

For $\theta \in \mathcal{N}^{0^*}$, define $\Omega\theta$: $\omega_1 \to \omega + 1$ by $\Omega\theta(0) = \omega$, $\Omega\theta(k) = \theta(k-1)$ if $0 < k \leq \omega$, and $\Omega\theta(k) = 0$ if $\omega < k < \omega_1$. Clearly, $\Omega\theta \in \mathcal{N}^*$ for all $\theta \in \mathcal{N}^{0^*}$. Extend Ω to $\mathcal{N}^0$ by $\Omega 0 = 0$.

2.8.3. LEMMA. *Ω is an injective morphism of m-monoids. Morever, $T\Psi\Omega 0 = o$, $T\Psi\Omega 0^* = 0$, $T\Psi\Omega\theta = 1 + \max \operatorname{Supp} \theta$ if $0, 0^* \neq \theta \equiv 0^*$, and $T\Psi\Omega\theta = \omega$ otherwise.*

These statements are clear from our previous remarks.
Extend the congruence relation $\equiv$ to $(\mathcal{N}^0)^{<\omega}$ in the obvious way:

$$(\theta_i)_{i<n} \equiv (\chi_i)_{i<n} \quad \text{if } \theta_i \equiv \chi_i \text{ for all } i < n\,.$$

2.8.4. DEFINITION. $\mathcal{N}^{1^*}$ is the set of all $\alpha \in \Delta\mathcal{N}^0$ such that:
(i) $(\theta_i)_{i<n} \equiv (\chi_i)_{i<n} \in \alpha$ and $\Sigma_{i<n}\, \theta_i = \Sigma_{i<n}\, \chi_i$ implies $(\theta_i)_{i<n} \in \alpha$;
(ii) $(\theta, \chi_1, \ldots, \chi_{n-1}) \in \alpha$, $\theta \not\equiv 0$ implies $\theta = \psi_0 + \psi_1$ in $\mathcal{N}^0$ with $\psi_0, \psi_1 \not\equiv 0$ and $(\psi_0, \psi_1, \chi_1, \ldots, \chi_{n-1}) \in \alpha$.
Define $\mathcal{N}^1 = \mathcal{N}^{1^*} \cup \{O\}$, where O is the zero of $\Delta\mathcal{N}^0$.

2.8.5. LEMMA. *$\mathcal{N}^1$ is a submonoid of $\Delta\mathcal{N}^0$.*

PROOF. Let $\alpha, \beta \in \mathcal{N}^{1^*}$, $(\theta_i)_{i<n} \in \alpha$, $(\chi_i)_{i<n} \in \beta$, $(\psi_i)_{i<n} \equiv (\theta_i + \chi_i)_{i<n}$, and $\Sigma_{i<n}\, \psi_i = \Sigma_{i<n}\, \theta_i + \chi_i$. We can assume that $\theta_i, \chi_i \neq 0$ for all i: replace occurrences of 0 by 0^*. Define $\theta'_i(k) = \theta_i(k)$, $\chi'_i(k) = \chi_i(k)$ for all $k \leq \omega$ such that $\psi_i(k) =$

$\theta_i(k) + \chi_i(k)$ for all $i < n$ (which is almost all $k \leq \omega$). Use 2.8.1 to define $\theta_i'(k)$, $\chi_i'(k)$ for the remaining i and k so that $\psi_i(k) = \theta_i'(k) + \chi_i'(k)$ and $\Sigma_{i<n} \theta_i'(k) = \Sigma_{i<n} \theta_i(k)$, $\Sigma_{i<n} \chi_i'(k) = \Sigma_{i<n} \chi_i(k)$. With these choices, $(\psi_i)_{i<n} = (\theta_i')_{i<n} + (\chi_i')_{i<n} \in \alpha + \beta$. Clearly, $\alpha + \beta$ satisfies (ii). Hence, $\alpha + \beta \in \mathcal{N}^1$. □

The morphisms Ψ and Ω give rise to a morphism from $\mathcal{N}^1$ to $\Delta^2\mathcal{W}$. Indeed, if Θ: $M \to N$ is any morphism of m-monoids, then Θ induces a morphism $\Delta\Theta$: $\Delta M \to \Delta N$ by

$$\Delta\Theta(\alpha) = \{(\Theta a_i)_{i<n} : (a_i) \in \alpha\} .$$

It is clear that if Θ is injective (an isomorphism), then so is $\Delta\Theta$. Note that this definition makes Δ functorial. In particular, $\Delta(\Psi \circ \Omega) = \Delta\Psi \circ \Delta\Omega$.

2.8.6. PROPOSITION. *$\Delta(\Psi \circ \Omega)$ is an injective morphism of $\mathcal{N}^1$ to $\mathcal{L}^1$.*

PROOF. Let $(\theta, \chi_1, \ldots, \chi_{m-1}) \in \alpha \in \mathcal{N}^1$, and suppose that $\boldsymbol{a} = (a_i)_{i<n} \in \Psi\Omega\theta$. We have to show that there exist $\theta_i \in \mathcal{N}^0$ such that $(\theta_i, \chi_j)_{i<n, j<m} \in \alpha$, $\Sigma_{i<n} \theta_i = \theta$, and $T\Psi\Omega\theta_i = a_i$. By the collection property, there is no harm in assuming that $a_0 \geq a_1 \geq \cdots \geq a_{r-1} \geq 1 > a_r = \cdots = a_{s-1} = 0 > a_s = \cdots = a_{n-1} = o$, where $r \leq s \leq n$. For $1 \leq i < r$, define $\theta_i \in \mathcal{N}^0$ by $\theta_i(k) = 1$ if $k = a_i - 1$ and $\theta_i(k) = 0$ otherwise. If $r \leq i < s$, let $\theta_i = 0^*$, and put $\theta_i = 0$ for $s \leq i < n$. Let $\theta_0 = \theta - \Sigma_{1 \leq i < r} \theta_i$, with the convention that $\omega - a = \omega$. The assumption that $\boldsymbol{a} \in \Psi\Omega\theta$ yields $|\{i < n : a_i = k\}| = \phi_{\boldsymbol{a}}(k) \leq \theta(k-1)$, which implies $\theta_0(k) \geq 0$. Moreover, $\max \operatorname{Supp} \theta_0 = a_0$. By the definition, $\Sigma_{i<n} \theta_i = \theta$, $T\Psi\Omega\theta_i = a_i$ for $i < n$ by Lemma 2.8.3, $\theta_i \equiv 0^*$ for $1 \leq i < n$, and $\theta_0 \equiv \theta$. The collection property for $\Delta\mathcal{N}^0$ implies that $(\theta, 0^*, \ldots, 0^*, \chi_1, \ldots, \chi_{m-1}) \in \alpha$. Consequently, $(\theta_0, \theta_1, \ldots, \theta_{n-1}, \chi_1, \ldots, \chi_{m-1}) \in \alpha$ by (i). □

2.9. *The monoid $\mathcal{P}$*

We can now describe the monoid that plays the central role in the proof of Ketonen's Theorem.

For $\theta \in \mathcal{N}^0$, denote

$$[\theta] = \{\chi \in \mathcal{N}^0 : \theta \equiv \chi\} .$$

Let $\mathcal{P} = \{[\theta] : \theta \in \mathcal{N}^0\}$. Since $\equiv$ is a congruence relation on $\mathcal{N}^0$, there is a unique commutative and associative operation $+$ on $\mathcal{P}$ such that the mapping $\Pi\theta = [\theta]$ is a surjective homomorphism of monoids. It is not a morphism of m-monoids, however, even though $\mathcal{P}$ is an m-monoid, as we will see. For instance, $\Pi(0^*) = \Pi(0)$ and $0^* \neq 0$ in $\mathcal{N}^0$.

As usual, we will use 0 to designate the zero of $\mathcal{P}$. Hence, $0 = [0] = [0^*]$. It will also be convenient to write w instead of $[\omega^*]$ in many situations.

2.9.1. LEMMA. *If $\theta \in \mathcal{N}^0$, then $[\theta]$ is countable.*

PROOF. If $\chi \in [\theta] \cap \mathcal{N}^{0^*}$, then $W(\chi) = \{k \leq \omega\colon \chi(k) \neq \theta(k)\}$ is a finite subset of $\omega + 1$. Consequently, the mapping $\chi \mapsto (W(\chi), \chi \mid W(\chi))$ is injective from $[\theta] \cap \mathcal{N}^{0^*}$ to a countable set. $\square$

2.9.2. PROPOSITION. (a) *$\mathcal{P}$ is an m-monoid whose natural ordering is antisymmetric (hence a partial ordering); $0 \leq a \leq w$ for all $a \in \mathcal{P}$.*

(b) *$\mathcal{P}\backslash\{w\}$ is a submonoid of $\mathcal{P}$ that is isomorphic to a submonoid of a rational vector space.*

(c) *$\mathcal{P}$ is an r-monoid; moreover, if $a_0 + a_1 = b_0 + b_1$ with $a_0, a_1, b_0, b_1 \in \mathcal{P}^*$, then there are uncountably many quadruples $(c_{00}, c_{01}, c_{10}, c_{11})$ in $\mathcal{P}^*$ such that $a_i = \Sigma_{j<2} c_{ij}$ and $b_j = \Sigma_{j<2} c_{ij}$ for $i, j < 2$.*

(d) *If $a \in \mathcal{P}^*$, then $\mathcal{P} \restriction a = \{b \in \mathcal{P}\colon b \leq a\}$ is uncountable.*

PROOF. (a) If $\theta \not\equiv 0$, then $\lim_{k\to\omega} \theta(k) = \omega$. Thus, $\theta + \psi \not\equiv 0$ for all $\psi \in \mathcal{N}^0$, and $\theta + \psi \not\equiv \psi$ if $\psi \not\equiv \omega^*$. These observations mean that if $a + b = 0$ in $\mathcal{P}$, then $a = b = 0$ (hence $\mathcal{P}$ is an m-monoid); and if $a \neq w$, $b \neq 0$, then $a + b \neq a$ (so that a has the S-B. property). However, w also has the S-B. property because $\omega^* + \theta = \omega^*$ for all $\theta \in \mathcal{N}^0$. Therefore, the natural ordering partially orders $\mathcal{P}$. Clearly, $0 \leq a \leq w$ for all $a \in \mathcal{P}$.

(b) If θ, $\psi \in \mathcal{N}^0$ and $\theta \not\equiv \omega^* \not\equiv \psi$, then $\theta(k) + \psi(k) < \omega$ for almost all $k < \omega$. Thus, $\theta + \psi \not\equiv \omega^*$. This observation shows that $\mathcal{P}\backslash\{w\}$ is a submonoid of $\mathcal{P}$. Let W be the $\mathbb{Q}$-space ${}^{\omega}\mathbb{Q}$ with pointwise operations. Denote $N' = W \cap \{\psi \restriction \omega\colon \psi \in \mathcal{N}^{0^*}\}$. It is clear that if $a \in \mathcal{P}\backslash\{w\}$, then $a = [\theta_a]$ for some $\theta_a \in \mathcal{N}^{0^*}$ such that $\theta_a \restriction \omega \in N'$. If θ', $\psi' \in W$, define $\theta' \sim \psi'$ if $\theta'(k) = \psi'(k)$ for almost all $k < \omega$. Clearly, $\sim$ is a vector space congruence on W; and if θ, $\psi \in \mathcal{N}^{0*}$ satisfy $\theta \restriction \omega$, $\psi \restriction \omega \in W$, then $\theta \equiv \psi$ if and only if $\theta \restriction \omega \sim \psi \restriction \omega$. It follows that if π: $W \to W/\!\sim$ is the natural projection, then $a \to \pi(\theta_a \restriction \omega)$ is an injective monoid homomorphism of $\mathcal{P}\backslash\{w\}$ to $W/\!\sim$.

The result (c) is a corollary of the lemma and 2.8.2; and (d) is obtained by applying the second assertion of (c) to the equation $a + w = w + w$. $\square$

2.10. *The monoid $\mathcal{P}^1$*

In order to keep our notation consistent, we will denote the derived monoid $\Delta\mathcal{P}$ by $\mathcal{P}^1$. The purpose of this subsection is to establish a lifting of the homomorphism $\Pi\colon \mathcal{N}^0 \to \mathcal{P}$.

2.10.1. LEMMA. *If θ_i, $\chi_j \in \mathcal{N}^{0^*}$ satisfy $\Sigma_{i<n} \theta_i = \Sigma_{j<m} \chi_j$, and if $\bar{\psi}_{ij} \in \mathcal{P}$ are such that $\Pi\theta_i = \Sigma_{j<m} \bar{\psi}_{ij}$ and $\Pi\chi_j = \Sigma_{i<n} \bar{\psi}_{ij}$ for $i < n$, $j < m$, then $\psi_{ij} \in \mathcal{N}^{0^*}$ exist with the properties $\Pi\psi_{ij} = \bar{\psi}_{ij}$, $\Sigma_{j<m} \psi_{ij} = \theta_i$, and $\Sigma_{i<n} \psi_{ij} = \chi_j$ for $i < n$, $j < m$.*

PROOF. Since Π is surjective, there exist $\psi'_{ij} \in \mathcal{N}^{0^*}$ satisfying $\Pi\psi'_{ij} = \bar{\psi}_{ij}$ for $i < n$, $j < m$. Then $\Pi\theta_i = \Pi(\Sigma_{j<m} \psi'_{ij})$ and $\Pi\chi_j = \Pi(\Sigma_{i<n} \psi'_{ij})$, so that there is a cofinite subset V of $\omega + 1$ with the property that $\theta_i(k) = \Sigma_{j<m} \psi'_{ij}(k)$, $\chi_j(k) = \Sigma_{i<n} \psi'_{ij}(k)$ for all $i < n$, $j < m$ and $k \in V$. Define $\psi_{ij} \in \mathcal{N}^{0^*}$ by the conditions $\psi_{ij}(k) = \psi'_{ij}(k)$ for $k \in V$; and for $k \not\in V$, $\Sigma_{j<m} \psi_{ij}(k) = \theta_i(k)$ and $\Sigma_{i<n} \psi_{ij}(k) = \chi_j(k)$ (which is

possible because $\omega+1$ is an r-monoid by Lemma 2.8.1). Clearly, $\Pi\psi_{ij}=\bar{\psi}_{ij}$, $\Sigma_{j<m}\psi_{ij}=\theta_i$, and $\Sigma_{i<n}\psi_{ij}=\chi_j$. □

The homomorphism Π: $\mathscr{N}^0 \to \mathscr{P}$ can be lifted to a homomorphism of $\mathscr{N}^1$ to $\Delta\mathscr{P}$ in the same way that morphisms of m-monoids are lifted:

$$\Pi_1(\alpha)=\{(\Pi\theta_i)_{i<n}\colon (\theta_i)_{i<n}\in\alpha\}\ .$$

2.10.2. LEMMA. *Π_1 is a monoid homomorphism of $\mathscr{N}^1$ to $\mathscr{P}^1$.*

PROOF. The only non-trivial point of this proof is the verification that if $\alpha\in\mathscr{N}^1$, then $\Pi_1(\alpha)$ has the splitting property. This result follows directly from the strong form of the splitting property in $\mathscr{N}^1$, that is, condition (ii) of Definition 2.8.4. □

2.10.3. PROPOSITION. *If $\alpha\in\Delta\mathscr{P}$ and $T\alpha=\Pi\chi$, where $\chi\in\mathscr{N}^0$, then there is a unique $\beta\in\mathscr{N}^1$ such that $\Pi_1(\beta)=\alpha$ and $T\beta=\chi$.*

PROOF. Define $\beta=\{(\theta_i)_{i<n}\in(\mathscr{N}^0)^{<\omega}\colon\ (\Pi\theta_i)_{i<n}\in\alpha,\ \Sigma_{i<n}\theta_i=\chi\}$. It suffices to show that $\beta\in\mathscr{N}^1$. The definition of β will then imply that $\Pi_1(\beta)=\alpha$ and $T\beta=\chi$. Moreover, it is clear from condition (i) of Definition 2.8.4 that β is the only element of $\mathscr{N}^1$ that satisfies these conditions. By 2.9.1, β is a countable subset of $(\mathscr{N}^0)^{<\omega}$, and β plainly satisfies C.P. With minor adjustments for the occurrence of zeros, the refinement property in β is a consequence of Lemma 2.10.1. If $\chi\neq 0$ and $(\theta_i)_{i<n}\in\beta$, then $(0^*,\theta_0,\dots,\theta_{n-1})\in\beta$ because $(\Pi 0^*,\Pi\theta_0,\dots,\Pi\theta_{n-1})=(0,\Pi\theta_0,\dots,\Pi\theta_{n-1})\in\alpha$ and $0^*+\Sigma_{i<n}\theta_i=\chi$. Hence, β satisfies S.P. Finally, it is clear from the definition of β that the condition (i) of Definition 2.8.4 is satisfied. □

Since $[\omega^*]+[\omega^*]=[\omega^*]$ and T is a morphism, the set $\mathscr{Q}^1$ of all $\alpha\in\mathscr{P}^1$ such that $T\alpha=[\omega^*]$ is a subsemigroup of $\mathscr{P}^1$.

2.10.4. COROLLARY. *There is a unique homomorphism Γ_1: $\mathscr{Q}^1\to\mathscr{N}^1$ such that $\Pi_1(\Gamma_1\alpha)=\alpha$ for all $\alpha\in\mathscr{Q}^1$ and $T(\Gamma_1\alpha)=\omega^*$.*

The existence of a unique mapping Γ_1: $\mathscr{Q}^1\to\mathscr{N}^1$ with the required properties follows from the proposition; and if $\alpha_1,\alpha_2\in\mathscr{Q}^1$, then $\Pi_1(\Gamma_1\alpha_1+\Gamma_1\alpha_2)=\alpha_1+\alpha_2$, $T(\Gamma_1\alpha_1+\Gamma_1\alpha_2)=\omega^*$ (since Γ_1 and T are homomorphisms), so that $\Gamma_1(\alpha_1+\alpha_2)=\Gamma_1\alpha_1+\Gamma_1\alpha_2$, and Γ_1 is a homomorphism.

REMARK. Since $\Pi_1\circ\Gamma_1$ is the identity on $\mathscr{Q}^1$, the homomorphism Γ_1 is injective.

2.11. One step up the hierarchy

Our discussion in the previous two subsections has produced the following commutative diagram of m-monoids and homomorphisms.

$$\begin{array}{ccccc}
\mathscr{L}^1 & \xleftarrow{\Delta\Psi\Omega} & \mathscr{N}^1 & \xrightarrow{\Pi_1} & \mathscr{P}^1 \\
T\downarrow & & T\downarrow & & T\downarrow \\
\mathscr{L}^0 & \xleftarrow[\Psi\Omega]{} & \mathscr{N}^0 & \xrightarrow[\Pi]{} & \mathscr{P}
\end{array}$$

The notation $\mathscr{P}^1$ abbreviates $\Delta\mathscr{P}$; the remaining monoids have been defined before. The morphisms $\Psi\Omega$ and $\Delta\Psi\Omega$ are injective; Π and Π_1 are surjective.

We will need a corresponding diagram that lies in one higher level of the hierarchy of derived monoids. In order to minimize the confusion in our discussion of three hierarchies at once, it is expedient to borrow the notation A, B, C, D that has been reserved for the designation of Boolean algebras. Throughout the rest of the proof of Ketonen's Theorem, these letters will denote sets at the highest level of the hierarchies under discussion. Later these letters will resume their distinguished roles as designators of algebras.

A limited version of the (strict) hierarchy property will be used in the following definition. This concept occurred in Lemma 2.7.2. It is recalled here for convenience. If M is an m-monoid and $A \in \Delta^2 M$, then A satisfies the (strict) hierarchy property (relative to M) if:

> $(\alpha, \beta_1, \ldots, \beta_{m-1}) \in A$ and $(a_0, \ldots, a_{n-1}) \in \alpha$ implies the existence of $\alpha_i \in \Delta M$ such that $(\alpha_0, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{m-1}) \in A$, $\Sigma_{i<n} \alpha_i = \alpha$, and $T\alpha_i = a_i$ for all $i < n$.

Generally, the parenthetical qualifiers "strict" and "relative to M" will be omitted from discussions that involve the hierarchy property. To verify the hierarchy property, it clearly suffices by induction to treat the case in which $n = 2$, that is, $(\alpha, \beta_1, \ldots, \beta_{m-1}) \in A$, $(a_0, a_1) \in \alpha$ implies $\alpha = \alpha_0 + \alpha_1$, $T\alpha_0 = a_0$, $T\alpha_1 = a_1$, and $(\alpha_0, \alpha_1, \beta_1, \ldots, \beta_{m-1}) \in A$.

2.11.1. DEFINITION. (a) $\mathscr{N}^2$ is the set of all $A \in \Delta\mathscr{N}^1$ that satisfy the hierarchy property and:

(i) if $(\alpha_0, \ldots, \alpha_{n-1}) \in A$, and if $\beta_i \in \mathscr{N}^1$ satisfy $\Pi_1(\beta_i) = \Pi_1(\alpha_i)$ for all $i < n$ and $\Sigma_{i<n} \beta_i = \Sigma_{i<n} \alpha_i$, then $(\beta_0, \ldots, \beta_{n-1}) \in A$;

(ii) if $(\alpha, \beta_1, \ldots, \beta_{n-1}) \in A$ and $\Pi_1\alpha \neq 0$, then $\alpha = \gamma_0 + \gamma_1$ in $\mathscr{N}^1$, where $\Pi_1\gamma_0$, $\Pi_1\gamma_1 \neq 0$ and $(\gamma_0, \gamma_1, \beta_1, \ldots, \beta_{n-1}) \in A$.

(b) $\mathscr{P}^2$ is the set of all $A \in \Delta\mathscr{P}^1$ that satisfy the hierarchy property.

Conditions (i) and (ii) in the definition of $\mathscr{N}^2$ are analogs of the corresponding properties of the elements in $\mathscr{N}^1$.

Our first lemma is an elevated version of 2.10.1.

2.11.2. LEMMA. *If $\alpha_i, \beta_j \in \mathscr{N}^1$ satisfy $\Sigma_{i<n} \alpha_i = \Sigma_{j<m} \beta_j$, and if $\bar{\gamma}_{ij} \in \mathscr{P}^1$ are such that $\Pi_1\alpha_i = \Sigma_{j<m} \bar{\gamma}_{ij}$ and $\Pi_1\beta_j = \Sigma_{i<n} \bar{\gamma}_{ij}$ for $i < n$ and $j < m$, then $\gamma_{ij} \in \mathscr{N}^1$ exist with the properties $\Pi_1\gamma_{ij} = \bar{\gamma}_{ij}$, $\Sigma_{j<m} \gamma_{ij} = \alpha_i$, $\Sigma_{i<n} \gamma_{ij} = \beta_j$ for $i < n$, $j < m$.*

PROOF. Let $\theta_i = T\alpha_i$, $\chi_j = T\beta_j$ and $\bar{\psi}_{ij} = T\bar{\gamma}_{ij}$. The hypotheses of Lemma 2.10.1 are satisfied, so that $\bar{\psi}_{ij} = \Pi\psi_{ij}$ with $\theta_i = \Sigma_{j<m}\,\psi_{ij}$ and $\chi_j = \Sigma_{i<n}\,\psi_{ij}$. Use Proposition 2.10.3 to get $\gamma_{ij} \in \mathcal{N}^1$ so that $\Pi_1\gamma_{ij} = \bar{\gamma}_{ij}$ and $T\gamma_{ij} = \psi_{ij}$. Since $\Pi_1\alpha_i = \Pi_1(\Sigma_{j<m}\,\gamma_{ij})$ and $T\alpha_i = T(\Sigma_{j<m}\,\gamma_{ij})$, the uniqueness statement of Proposition 2.10.3 implies $\alpha_i = \Sigma_{j<m}\,\gamma_{ij}$. Similarly, $\beta_j = \Sigma_{i<n}\,\gamma_{ij}$. □

2.11.3. LEMMA. *$\mathcal{N}^2$ is a submonoid of $\Delta\mathcal{N}^1$, and $\mathscr{P}^2$ is a submonoid of $\Delta\mathscr{P}^1$.*

PROOF. By the remark after the proof of Lemma 2.7.1, if A and B have the strict hierarchy property, then so does $A + B$. In particular, $\mathscr{P}^2$ is a submonoid of $\Delta^2\mathscr{P} = \Delta\mathscr{P}^1$. It remains to show that if $A, B \in \Delta\mathcal{N}^1$ satisfy (i) and (ii) of the definition, then so does $A + B$. Let $(\alpha_i)_{i<n} \in A$, $(\beta_i)_{i<n} \in B$, and suppose that the elements $\gamma_i \in \mathcal{N}^1$ are such that $\Pi_1\gamma_i = \Pi_1(\alpha_i + \beta_i)$ for all $i < n$, and $\Sigma_{i<n}\,\gamma_i = \Sigma_{i<n}(\alpha_i + \beta_i)$. It follows from 2.11.2 that there exist α'_i, β'_i in $\mathcal{N}^1$ such that $\Pi_1\alpha'_i = \Pi_1\alpha_i$, $\Pi_1\beta'_i = \Pi_1\beta_i$, $\gamma_i = \alpha'_i + \beta'_i$ for $i < n$, and $\Sigma_{i<n}\,\alpha'_i = \Sigma_{i<n}\,\alpha_i$, $\Sigma_{i<n}\,\beta'_i = \Sigma_{i<n}\,\beta_i$. Thus, $(\gamma_i)_{i<n} = (\alpha'_i)_{i<n} + (\beta'_i)_{i<n} \in A + B$. It is obvious that $A + B$ satisfies (ii). Therefore, $A + B \in \mathcal{N}^2$. □

We can now lift the homomorphism. For $A \in \mathcal{N}^2$, define

$$\Pi_2 A = \{(\Pi_1\alpha_i)_{i<n} : (\alpha_i)_{i<n} \in A\}\,.$$

2.11.4. LEMMA. *Π_2 is a monoid homomorphism of $\mathcal{N}^2$ to $\mathscr{P}^2$.*

PROOF. It is clear that if $A \in \mathcal{N}^2$, then $\Pi_2 A$ is a countable subset of $(\mathscr{P}^1)^{<\omega}$. Clearly, $\Pi_2 A$ has the collection property; the refinement property is a consequence of 2.11.2. Condition (ii) in the definition of $\mathcal{N}^2$ guarantees that $\Pi_2 A$ satisfies S.P. Thus, $\Pi_2 A \in \Delta\mathscr{P}^1$. If $(\alpha, \beta_1, \ldots, \beta_{n-1}) \in A \in \mathcal{N}^2$ and $(a_0, \ldots, a_{m-1}) \in \Pi_1(\alpha)$, then there exists $(\theta_0, \ldots, \theta_{m-1}) \in \alpha$ such that $\Pi\theta_j = a_j$ for all $j < m$. Since A has the hierarchy property, we can write $\alpha = \Sigma_{i<m}\,\alpha_i$, where $(\alpha_0, \ldots, \alpha_{m-1}, \beta_1, \ldots, \beta_{n-1}) \in A$ and $T\alpha_j = \theta_j$ for $j < m$. Consequently, $(\Pi_1\alpha_0, \ldots, \Pi_1\alpha_{m-1}, \Pi_1\beta_1, \ldots, \Pi_1\beta_{n-1}) \in \Pi_2 A$, $\Sigma_{j<m}\,\Pi_1\alpha_j = \Pi_1\alpha$, and $T\Pi_1\alpha_j = \Pi T\alpha_j = \Pi\theta_j = a_j$ for $j < m$. Thus, $\Pi_2 A \in \mathscr{P}^2$. It follows from our definition that Π_2 is a homomorphism of monoids. □

These lemmas fulfill our promise to lift the hierarchy diagram one step higher. It is clear from the definitions of Π_1 and Π_2 that the squares of the following diagram commute.

$$\begin{tikzcd}
\mathscr{L}^2 \arrow[d, "T"'] & \mathcal{N}^2 \arrow[l, "\Delta^2\Psi\Omega"'] \arrow[r, "\Pi_2"] \arrow[d, "T"'] & \mathscr{P}^2 \arrow[d, "T"'] \\
\mathscr{L}^1 & \mathcal{N}^1 \arrow[l, "\Delta\Psi\Omega"] \arrow[r, "\Pi_1"'] & \mathscr{P}^1
\end{tikzcd}$$

2.11.5. PROPOSITION. *If $A \in \mathscr{P}^2$ and $TA = \Pi_1\beta$ with $\beta \in \mathcal{N}^1$, then there is a unique $B \in \mathcal{N}^2$ such that $\Pi_2(B) = A$ and $TB = \beta$.*

PROOF. If $\gamma, \gamma' \in \mathcal{N}^1$ satisfy $\Pi_1\gamma = \Pi_1\gamma'$, then $[T\gamma] = [T\gamma']$ is a countable set. Moreover, if $\Pi_1\gamma = \Pi_1\gamma'$ and $T\gamma = T\gamma'$, then $\gamma = \gamma'$ by Proposition 2.10.3. Therefore, for each $\alpha \in \mathcal{P}^1$, $\{\gamma \in \mathcal{N}^1 : \Pi_1\gamma = \alpha\}$ is a countable set. It follows that

$$B = \left\{(\beta_i)_{i<n} \in (\mathcal{N}^1)^{<\omega} : (\Pi_1\beta_i)_{i<n} \in A,\ \sum_{i<n} \beta_i = \beta\right\}$$

is countable. Plainly, B satisfies C.P. The refinement property for B is a consequence of the refinement property in A by virtue of 2.11.2; and the fact that B satisfies S.P. is a consequence of the definition of B. Indeed, if $(\beta_0, \beta_1, \ldots, \beta_{n-1}) \in B$ with $\beta_0 \neq 0$, then $(\gamma, \beta_0, \beta_1, \ldots, \beta_{n-1}) \in B$, where $\gamma = \{(\theta_0, \ldots, \theta_{m-1}) \in (\mathcal{N}^0)^{<\omega} : \Sigma_{j<m}\, \theta_j = 0^*\}$ is a non-zero element of $\mathcal{N}^1$ such that $\Pi_1\gamma = 0$. The hierarchy property for B is obtained as follows. Let $(\alpha, \beta_1, \ldots, \beta_{n-1}) \in B$ and $(\theta_0, \ldots, \theta_{m-1}) \in \alpha$. Then $\alpha + \Sigma_{i=1}^{n} \beta_i = \beta$, $(\Pi_1\alpha, \Pi_1\beta_1, \ldots, \Pi_1\beta_{n-1}) \in A$, and $(\Pi\theta_0, \ldots, \Pi\theta_{m-1}) \in \Pi_1\alpha$. The hierarchy property in A and Proposition 2.10.3 yield elements $\alpha_j \in \mathcal{N}^1$ that satisfy $(\Pi_1\alpha_0, \ldots, \Pi_1\alpha_{m-1}, \Pi_1\beta_1, \ldots, \Pi_1\beta_{n-1}) \in A$, $\Sigma_{j<m}\, \Pi_1\alpha_j = \Pi_1\alpha$, and $T\alpha_j = \theta_j$ for all $j < m$. Since $\Pi_1(\Sigma_{j<m}\, \alpha_j) = \Pi_1\alpha$ and $T(\Sigma_{j<m}\, \alpha_j) = \Sigma_{j<m}\, \theta_j = T\alpha$, it follows that $\Sigma_{i<m}\, \alpha_i = \alpha$. Consequently, $(\alpha_0, \ldots, \alpha_{m-1}, \beta_1, \ldots, \beta_{n-1}) \in B$, which proves the hierarchy property. The construction of B is such that condition (i) in the definition of $\mathcal{N}^2$ is automatically fulfilled. If $(\alpha, \beta_1, \ldots, \beta_{n-1}) \in B$ and $\Pi_1\alpha \neq 0$, then $\alpha = \alpha_0 + \alpha_1$ with $\Pi\alpha_0, \Pi\alpha_1 \neq 0$ by S.P. in A and Proposition 2.10.3. Thus, $(\alpha_0, \alpha_1, \beta_1, \ldots, \beta_{n-1}) \in B$. Therefore, B satisfies (ii). Hence, $B \in \mathcal{N}^2$. The definition of B guarantees that $\Pi_2 B = A$ and $TB = \beta$. Finally, it follows from condition (i) in the definition that B is the only element of $\mathcal{N}^2$ with these properties. $\square$

Denote $\mathcal{Q}^2 = \{A \in \mathcal{P}^2 : TA \in \mathcal{Q}^1\}$. Since T is a morphism and $\mathcal{Q}^1$ is a subsemigroup of $\mathcal{P}^1$, it follows that $\mathcal{Q}^2$ is a subsemigroup of $\mathcal{P}^2$.

2.11.6. COROLLARY. *There is a unique semigroup homomorphism $\Gamma_2 : \mathcal{Q}^2 \to \mathcal{N}^2$ such that $\Pi_2 \circ \Gamma_2$ is the identity mapping of $\mathcal{Q}^2$ and $T\Gamma_2 = \Gamma_1 T$.*

This result follows from the proposition in the same way that Corollary 2.10.4 came from Proposition 2.10.3. Note that Γ_2 is necessarily injective.

2.12. First reduction step

We are now able to put Proposition 2.7.3 into more usable form. The three-tiered notation of the previous subsection remains in effect.

2.12.1. PROPOSITION. *Let N be a countable submonoid of $\mathcal{P}^2$ such that.*

(i) *N has the refinement property*;

(ii) *if $(\alpha_0, \alpha_1) \in A \in N$, then $A = A_0 + A_1$ in N, where $TA_0 = \alpha_0$ and $TA_1 = \alpha_1$.*

Then $N \cap \mathcal{Q}^2$ is isomorphic to a subsemigroup of ***UBA****.*

PROOF. Let $N_1 = \{B \in \mathcal{N}^2 \colon \Pi_2 B \in N\}$. By Lemma 2.9.1 and Propositions 2.10.3 and 2.11.5, N_1 is a countable submonoid of $\mathcal{N}^2$. The map Γ_2 embeds the semigroup $N \cap \mathcal{Q}^2$ into N_1 by Corollary 2.11.6. Since $\Delta^2\Psi\Omega\colon \mathcal{N}^2 \to \mathcal{L}^2$ is an injective morphism of m-monoids satisfying $T\Delta^2\Psi\Omega = \Delta\Psi\Omega T$, the proof can be completed by showing that N_1 satisfies the analogs of conditions (1) and (2) in Proposition 2.7.3. Suppose that $\Sigma_{i<n} A_i = \Sigma_{j<m} B_j$ in N_1. This equation implies that $\Sigma_{i<n} \Pi_2 A_i = \Sigma_{j<m} \Pi_2 B_j$ in N, and $\Sigma_{i<n} \alpha_i = \Sigma_{j<m} \beta_j$, where $\alpha_i = TA_i$ and $\beta_j = TB_j$. Since N satisfies (i), there exist $C_{ij} \in N$ such that $\Pi_2 A_i = \Sigma_{j<m} C_{ij}$ and $\Pi_2 B_j = \Sigma_{i<n} C_{ij}$ for $i<n$, $j<m$. Denote $\gamma_{ij} = TC_{ij} \in \mathcal{P}^1$. Then $\Pi_1 \alpha_i = T\Pi_2 A_i = \Sigma_{j<m} \gamma_{ij}$. Similarly, $\Pi_1 \beta_j = \Sigma_{i<n} \gamma_{ij}$. By Lemma 2.10.1, there are elements $\delta_{ij} \in \mathcal{N}^1$ such that $\Pi_1 \delta_{ij} = \gamma_{ij}$, $\alpha_i = \Sigma_{j<m} \delta_{ij}$, and $\beta_j = \Sigma_{i<m} \delta_{ij}$. Proposition 2.11.5 then yields $D_{ij} \in \mathcal{N}^2$ satisfying $\Pi_2 D_{ij} = C_{ij}$ and $TD_{ij} = \delta_{ij}$. Hence, $TA_i = \alpha_i = \Sigma_{j<m} \delta_{ij} = T(\Sigma_{j<m} D_{ij})$ and $\Pi_2 A_i = \Pi_2(\Sigma_{j<m} D_{ij})$, so that $A_i = \Sigma_{j<m} D_{ij}$, with $D_{ij} \in N_1$ (by the definition of N_1). Similarly, $B_j = \Sigma_{i<n} D_{ij}$ in N_1. Thus, N_1 has the refinement property. By induction, it suffices to prove the case $n=2$ of the condition 2.7.3(2). Suppose that $(\alpha_0, \alpha_1) \in A \in N_1$. Then $TA = \alpha_0 + \alpha_1$, $\Pi_2(A) \in N$, and $(\Pi_1(\alpha_0), \Pi_1(\alpha_1)) \in \Pi_2(A)$. It follows from (ii) that $\Pi_2(A) = B_0 + B_1$ in N, with $TB_0 = \Pi_1(\alpha_0)$, $TB_1 = \Pi_1(\alpha_1)$. By Proposition 2.11.5, A_0, A_1 exist in $\mathcal{N}^2$ satisfying $\Pi_2 A_0 = B_0$, $\Pi_2 A_1 = B_1$, $TA_0 = \alpha_0$, $TA_1 = \alpha_1$. Consequently, $A_0, A_1 \in N_1$; and $\Pi_1(A_0 + A_1) = \Pi_1(A)$, $T(A_0 + A_1) = T(A)$ implies $A_0 + A_1 = A$. □

2.13. *Special elements*

We will need two theorems on embeddings of monoids. Their proofs occupy this subsection and the next one. As usual, M denotes an arbitrary m-monoid. For the moment, we will designate elements of general monoids by e, f, g, saving a, b, c for the particular monoid $\mathcal{P}$.

An element $e \in M$ is *special in M* if

$$e + f = e \text{ implies } f = 0 \quad \text{for all } f \in M .$$

The set of all elements that are special in M will be designated by M_s. The monoid M will be called *special* if every element of M is a sum of special elements; that is, M_s generates M as a monoid.

For example, it is easy to see that every element except $[\omega]$ is special in the monoid $\mathcal{P}$. Thus, $\mathcal{P}_s = \mathcal{P} \backslash \{[\omega]\}$. In this case, $\mathcal{P}_s$ is a proper submonoid of $\mathcal{P}$, so that $\mathcal{P}$ is not special.

2.13.1. LEMMA. *If e is special in M, then e has the Schröder–Bernstein property: $e \le f$ and $f \le e$ implies $f = e$ for all $f \in M$.*

Indeed, if $e \le f$ and $f \le e$ in the natural order of M, then $f = e + g$ and $e = e + g + h$ for some $g, h \in M$. Since e is special, it follows that $g + h = 0$. Therefore, $g = h = 0$, and $f = e + g = e$, since M is an m-monoid. □

In general, the converse of this lemma is false. For instance, in $\mathcal{P}$, w has the S-B. property by Proposition 2.9.2; however, $w + w = w$, so that w is not special in $\mathscr{P}$. Nevertheless, it is the S-B. property of special elements that makes them important. The advantage in using the stronger property "specialness" is that this condition is easier to monitor.

If M is a submonoid of the m-monoid N, then N is called a *special extension* of M if $M_s \subseteq N_s$; that is, every element that is special in M is also special in N.

2.13.2. LEMMA. *If N is a special extension of M and P is a special extension of N, then P is a special extension of M. If $\{M_k: k<\omega\}$ is a sequence of m-monoids such that M_{k+1} is a special extension of M_k for all $k<\omega$, then $\bigcup_{k<\omega} M_k$ is a special extension of all M_l, $l<\omega$.*

These statements are clear from the definition of special extensions.

2.13.3. PROPOSITION. *If e is a non-zero element of the m-monoid M, then there is a special extension N of M such that:*

(a) *$e=e_0+e_1$, where e_0 and e_1 are not zero and special in N;*

(b) *N is generated as a monoid by $M \cup \{e_0, e_1\}$;*

(c) *if $f'=f+me_0+ne_1$ with $f', f \in M$ and $m, n \in \omega$, then $m=n$ and $f'=f+me$;*

(d) *if $\Lambda: M \to M'$ is a morphism of m-monoids, and $\Lambda e=f_0+f_1$ in M', then there is an extension of Λ' to a morphism of N to M' such that $\Lambda' e_0=f_0$ and $\Lambda' e_1=f_1$.*

PROOF. Let $P=M\times\omega\times\omega$ be the monoid product, where the sum in both copies of ω is ordinary addition. Define a congruence relation $\sim$ on P by $(f,m,n)\sim(f',m',n')$ if and only if $m-n=m'-n'$, $f+me=f'+m'e$, and $f+ne=f'+n'e$. Let $N=P/\sim$ be the factor monoid, and denote the natural projection of P to N by Π. Define $\Psi: M\to N$ by $\Psi(f)=\Pi(f,0,0)$. Clearly, Ψ embeds M in N. If $\Psi(f')=\Pi(f,m,n)$, then $(f',0,0)\sim(f,m,n)$. That is, $m-n=0$ and $f'=f+me$. In particular, $\Psi(f)=\Psi(f)+\Pi(g,m,n)$ implies $m=n$ and $f=f+g+me$. Consequently, if f is special in M, then $g=0$ and $m=n=0$. Therefore, $\Psi(f)$ is special in N. By the definition of $\sim$, we have $(e,0,0)\sim(0,1,0)+(0,0,1)$. Thus, $\Psi(e)=e_0+e_1$, where $e_0=\Pi(0,1,0)$ and $e_1=\Pi(0,0,1)$ are not zero. If $e_0+\Pi(f,m,n)=e_0$, then $(f,m+1,n)\sim(0,1,0)$, $m+1-n=1$, and $f+ne=0$. Therefore $f=0$ and $m=n=0$; that is, e_0 is special. Similarly, e_1 is special. By the usual identification process, it can be assumed that $M\subseteq N$ and Ψ is the inclusion mapping. Our discussion then shows that N is a special extension of M that is generated by $M\cup\{e_0,e_1\}$, $e=e_0+e_1$, e_0 and e_1 are special in N, and $f'=f+me_0+ne_1$ with $f, f'\in M$, $m, n\in\omega$ implies $m=n$ and $f'=f+me$. Suppose that $\Lambda: M\to M'$ is a morphism and $\Lambda e=f_0+f_1$ in M'. Plainly, Λ induces a morphism Γ of P to M' by $\Gamma(f,m,n)=\Lambda(f)+mf_0+nf_1$. If $(f,m,n)\sim(f',m',n')$ with (say) $m\geq n$, then $\Gamma(f,m,n)=\Lambda(f)+mf_0+nf_1=\Lambda(f)+n(f_0+f_1)+(m-n)f_0 = \Lambda(f)+n\Lambda(e)+(m-n)f_0 = \Lambda(f+ne)+(m-n)f_0 = \Lambda(f'+n'e)+(m'-n')f_0=\Lambda(f')+m'f_0+n'f_1=\Gamma(f',m',n')$. Thus, Γ factors through Π: there is a morphism $\Lambda': N\to M'$ such that $\Gamma=\Lambda'\circ\Pi$. It follows that $\Lambda'(\Psi(f))=\Lambda(f)$ for all $f\in M$, $\Lambda'(e_0)=f_0$, and $\Lambda'(e_1)=f_1$. Therefore, if Ψ is idenified with the inclusion mapping of M to N, then Λ' is the extension of Λ whose existence is predicted in (d). □

REMARK. If M is special, that is, generated by special elements, then N is also special by (a), (b), and the fact that N is a special extension of M.

2.13.4. COROLLARY. *If M is a countable m-monoid, then there is a special extension N of M such that N is a countable, special, atomless m-monoid.*

PROOF. Let $\{e_k: k < \omega\}$ be an enumeration of M. Use the proposition to construct a sequence $M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k \subseteq \cdots$ of special extensions of countable m-monoids such that $e_k = e_{0k} + e_{1k}$ with e_{0k} and e_{1k} non-zero, special elements of M_{k+1}. Define $M^{(1)} = \bigcup_{k<\omega} M_k$. By 2.13.2, $M^{(1)}$ is a special extension of M such that if $e \in M$, then $e = e_0 + e_1$ with e_0 and e_1 non-zero, special elements of $M^{(1)}$. Iteration of this construction gives a sequence of special extensions of countable m-monoids $M = M^{(0)} \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(k)} \subseteq \cdots$ whose union N is a special extension of M by a countable m-monoid. Moreover, every non-zero element of N is a sum of two non-zero special elements. Hence, N is special and atomless. □

2.14. *Constructing refinements*

The result in Proposition 2.13.3 gives a controlled way to split elements in an m-monoid; a corresponding result on refinements is needed.

As a notational device, we will use the monoid $M_2(\omega)$ of 2×2 matrices with entries in ω. In this subsection only, these matrices will be denoted by lower case Greek letters. The monoid operation in $M_2(\omega)$ is ordinary matrix addition. Let ε_{ij} be the matrix units in $M_2(\omega)$: ε_{ij} has 1 in the (i, j) position and 0 elsewhere. Note that $\{\varepsilon_{ij}: i, j < 2\}$ is a free basis for $M_2(\omega)$. Denote $\alpha_i = \sum_{j<2} \varepsilon_{ij}$ and $\beta_j = \sum_{i<2} \varepsilon_{ij}$ for $i, j < 2$. Define a semigroup homomorphism ϕ: $M_2(\omega) \to \mathbb{Z}$ by $\phi(\sum_{i,j<2} r_{ij} \varepsilon_{ij}) = r_{00} + r_{11} - r_{01} - r_{10}$. Clearly, $\phi(\alpha_i) = \phi(\beta_j) = 0$ for $i, j < 2$.

2.14.1. LEMMA. (a) *Every $\gamma \in M_2(\omega)$ can be represented in the form*

$$(*) \qquad \gamma = \delta + \sum_{i<2} m_i \alpha_i + \sum_{j<2} n_j \beta_j ,$$

where $\delta \in M_2(\omega)$ is either diagonal or counter-diagonal (that is, $\delta = r_0 \varepsilon_{00} + r_1 \varepsilon_{11}$ or $\delta = r_0 \varepsilon_{01} + r_1 \varepsilon_{10}$), and the $m_i, n_j \in \omega$ satisfy $n_0 n_1 = 0$ (in other words, $n_0 = 0$ or $n_1 = 0$). Call such a representation reduced.

(b) *In the reduced representation $(*)$ of γ, the matrix δ is zero if and only if $\phi(\gamma) = 0$; in this case, the reduced representation is unique.*

PROOF. If $\gamma \in M_2(\omega)$ is neither diagonal nor counter-diagonal, then for some i or $j < 2$ it is possible to write $\gamma = \gamma' + \alpha_i$ or $\gamma' + \beta_j$ with the sum of the entries in γ' less than the sum of the entries in γ. Therefore, by induction, every $\gamma \in M_2(\omega)$ has a representation of the form $(*)$. The possibility of making either n_0 or n_1 equal to zero comes about because $\beta_0 + \beta_1 = \alpha_0 + \alpha_1$. The first assertion in (b) is a consequence of the observations: $\phi(\alpha_i) = \phi(\beta_j) = 0$ for $i, j < 2$; and if δ is diagonal or counter-diagonal, then $\phi(\delta)=0$ only if $\delta=0$. To prove the second statement in (b), suppose that $\Sigma_{i<2} m_i\alpha_i + \Sigma_{j<2} n_j\beta_j = \Sigma_{i<2} m'_i\alpha_i + \Sigma_{j<2} n'_j\beta_j$, with $n_0n_1 = n'_0n'_1 = 0$. In this case, $m_i + n_j = m'_i + n'_j$ for all i and j. Hence, $m_i - m'_i = n'_j - n_j$ for $i, j<2$. Thus, there is an integer p such that $m'_i = m_i + p$ and $n'_j = n_j - p$ for $i<2$, $j<2$. Using the fact that all m_i, m'_i, n_j, n'_j are non-negative and $n_0n_1 = n'_0n'_1 = 0$, it follows that p must be 0. □

2.14.2. PROPOSITION. *If e_0, e_1, f_0, and f_1 are elements of the m-monoid M that satisfy $e_0+e_1=f_0+f_1$, then there is a special extension N of M such that*

(a) *N contains special elements $g_{00}, g_{01}, g_{10}, g_{11}$ with the properties $e_i = \Sigma_{j<2} g_{ij}$, $f_j = \Sigma_{i<2} g_{ij}$ for $i, j<2$;*

(b) *N is generated as a monoid by $M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$;*

(c) *if $g' = g + \Sigma_{i,j<2} r_{ij}g_{ij}$ with $g, g' \in M$, $r_{ij}\in\omega$, then there exist m_i and n_j in ω so that $r_{ij} = m_i + n_j$ for $i, j<2$, and $g' = g + \Sigma_{i<2} m_ie_i + \Sigma_{j<2} n_jf_j$ in M;*

(d) *if Λ: $M\to M'$ is a morphism of m-monoids, and $\Lambda e_i = \Sigma_{j<2} h_{ij}$, $\Lambda f_j = \Sigma_{i<2} h_{ij}$ $(i, j<2)$ in M', then there is an extension Λ' of Λ to a morphism of N to M' that satisfies $\Lambda'(g_{ij}) = h_{ij}$ for $i, j<2$.*

PROOF. Let P denote the monoid product $M\times M_2(\omega)$. Define a binary relation $\sim$ on P by $(g,\gamma)\sim(g',\gamma')$ if there exists $\delta\in M_2(\omega)$ and $m_i, n_j, m'_i, n'_j \in\omega$ for i, $j<2$ such that

$$\gamma = \delta + \sum_i m_i\alpha_i + \sum_j n_j\beta_j\,, \qquad \gamma' = \delta + \sum_i m'_i\alpha_i + \sum_j n'_j\beta_j\,, \tag{1}$$

and

$$g + \sum_i m_ie_i + \sum_j n_jf_j = g' + \sum_i m'_ie_i + \sum_j n'_jf_j\,. \tag{2}$$

The equations (1) can be put into reduced form using the hypothesis that $e_0+e_1=f_0+f_1$ to make corresponding modifications of (2). Note that equations (1) imply that $\phi(\gamma)=\phi(\delta)=\phi(\gamma')$. The relation $\sim$ is obviously reflexive, symmetric, and additive on P. Therefore, the transitive closure $\approx$ of $\sim$ is a congruence relation. Define $N = P/\approx$, and let Π: $P\to N$ be the natural projection map. The mapping $\Psi(g)=\Pi(g,0)$ of M to N is plainly a semigroup homomorphism. Suppose that $\Psi(g')=\Pi(g,\gamma)$. That is, there is a sequence

$$(g',0)=(g_0,\gamma_0)\sim(g_1,\gamma_1)\sim\cdots\sim(g_t,\gamma_t)=(g,\gamma)\,.$$

Our previous remark gives $\phi(\gamma)=\phi(\gamma_t)=\cdots=\phi(\gamma_1)=\phi(\gamma_0)=\phi(0)=0$. By the lemma, the reduced forms in (1) are unique. Thus, for all $s\le t$, there are relations

$$\gamma_s = \sum_i m_{si}\alpha_i + \sum_j n_{sj}\beta_j\,, \tag{3}$$

and

$$g_s + \sum_i m_{si}e_i + \sum_j m_{sj}f_j = g_{s+1} + \sum_i m_{s+1i}e_i + \sum_j n_{s+1j}f_j\,. \tag{4}$$

In particular, since $\gamma_0 = 0$, it follows from (3) and (4) that

$$(5) \qquad g' = g + \sum_i m_{ti} e_i + \sum_j n_{tj} f_j \, .$$

By specializing these calculations, we deduce from (3) and (5) that $\Psi(g') = \Psi(g)$ implies $g' = g$; and if g' is special in M, then $\Psi(g')$ is special in N. The definitions (1) and (2) imply that $(e_i, 0) \sim (0, \alpha_i)$ and $(f_j, 0) \sim (0, \beta_j)$. Hence, $\Psi(e_i) = \Pi(0, \alpha_i) = \Sigma_{j<2}\, g_{ij}$ and $\Psi(f_j) = \Sigma_{i<2}\, g_{ij}$, where $g_{ij} = \Pi(0, \varepsilon_{ij})$. As usual, it can be assumed that $M \subseteq N$ and Ψ is the inclusion mapping. When this identification is made, N becomes a special extension of M and the conditions (a), (b), and (c) of the proposition are fulfilled. Suppose that Λ: $M \to M'$ is a morphism, and $\Lambda(e_i) = \Sigma_{j<2}\, h_{ij}$, $\Lambda(f_j) = \Sigma_{i<2}\, h_{ij}$ in M' for i, $j < 2$. Define the morphism Γ: $P \to M'$ by

$$\Gamma\Big(g, \sum_{i,j<2} r_{ij}\varepsilon_{ij}\Big) = \Lambda(g) + \sum_{i,j<2} r_{ij} h_{ij} \, .$$

If $(g, \gamma) \sim (g', \gamma')$, then $\Gamma(g, \gamma) = \Gamma(g', \gamma')$. Indeed, this implication follows from (1) and (2) because $\Gamma(0, \alpha_i) = \Lambda(e_i)$ and $\Gamma(0, \beta_j) = \Lambda(f_j)$ for i, $j < 2$. By transitivity, the kernel of Γ contains $\approx$. Thus, Γ factors through Π, giving Λ': $N \to M'$, an extension of Λ that satisfies $\Lambda'(g_{ij}) = h_{ij}$ for i, $j < 2$. □

Again we note that if M is a special monoid, then so is the monoid N.

Notes. By iterating the proposition, as in the proof of Corollary 2.13.4, any m-monoid can be embedded in an r-monoid. The same conclusion could be obtained from the existence theorem for saturated structures. However, the fine structure of the extensions in Propositions 2.13.3 and 2.14.2 will be needed.

2.15. Models

The rest of the proof of Ketonen's Theorem will deal mainly with morphisms of m-monoids to $\mathcal{P}^1$. In order to gain flexibility, it is expedient to enlarge the target of these morphisms and to relax the additivity conditions. The fruit of this strategy will be called a submodel. The "good" submodels – the ones that really interest us – are called models.

Before submodels can be defined, we must describe the appropriate enlarged target monoid.

Denote by $\mathcal{R}$ the set of countable, non-empty subsets α of $\mathcal{P}^{<\omega}$ such that α has the collection property and the elements of α have a common sum: if $(a_i)_{i<n} \in \alpha$ and $(b_j)_{j<m} \in \alpha$, then $\Sigma_{i<n}\, a_i = \Sigma_{j<m}\, b_j$. We can therefore define the trace map T on $\mathcal{R}$ by putting $T\alpha = \Sigma_{i<n}\, a_i$, where $(a_i)_{i<n}$ is an element of α. Define addition in $\mathcal{R}$ by the rule that gives the sum in $\Delta\mathcal{P}$, that is, $\alpha + \beta = \{\boldsymbol{a} + \boldsymbol{b}$: $\boldsymbol{a} \in \alpha$, $\boldsymbol{b} \in \beta$, $l(\boldsymbol{a}) = l(\boldsymbol{b})\}$. With this operation, $\mathcal{R}$ becomes an m-monoid that contains $\Delta\mathcal{P}$ as a submonoid. Indeed, $\Delta\mathcal{P}$ is the set of all $\alpha \in \mathcal{R}$ that have the refinement and splitting properties. As in the case of $\Delta\mathcal{P}$, the trace T is a morphism from $\mathcal{R}$ to $\mathcal{P}$.

A simple observation about $\mathcal{R}$ is often useful: if α, $\beta \in \mathcal{R}$ and $T\alpha = T\beta$, then

$\alpha \cup \beta \in \mathcal{R}$. Indeed, C.P. is clearly preserved under arbitrary set unions; and the assumption $T\alpha = T\beta$ guarantees that the sequences in $\alpha \cup \beta$ have a common sum.

It will be important to have a supply of elements from $\mathcal{R}$. For $a \in \mathcal{P}$, denote

$$\hat{a} = \{(c_i)_{i<m} \in \mathcal{P}^{<\omega}: \text{there exists } l < m \text{ such that } c_l = a \text{ and } c_j = 0 \text{ for } j \neq l\}.$$

2.15.1. LEMMA. *If $a \in \mathcal{P}$, then $\hat{a} \in \mathcal{R}$ and $T\hat{a} = a$. Moreover, if $\alpha \in \mathcal{R}$ and $(a_i)_{i<n} \in \alpha$, then $\Sigma_{i<n} \hat{a}_i \subseteq \alpha$.*

PROOF. The first assertion is obvious from the definition of $\hat{a}$. Suppose that $(b_j)_{j<m} \in \Sigma_{i<n} \hat{a}_i$. That is, there exist $(c_{ij})_{j<m} \in \hat{a}_i$ such that $b_j = \Sigma_{i<n} c_{ij}$ for $j < m$. According to the definition of $\hat{a}_i$, there is a mapping λ: $n \to m$ such that $c_{i\lambda(i)} = a_i$ and $c_{ij} = 0$ if $j \neq \lambda(i)$. Thus, $b_j = \Sigma \{a_i: \lambda(i) = j\}$. In other words, $(a_i)_{i<n}$ is a refinement of $(b_j)_{j<m}$. By the collection property, $(b_j)_{j<m} \in \alpha$. $\square$

2.15.2. DEFINITION. Let M be a countable m-monoid. A *submodel* of M is a mapping Ψ: $M \to \mathcal{R}$ such that

(i) $\Psi(e) = 0$ if and only if $e = 0$;

(ii) $\Psi(e) + \Psi(f) \subseteq \Psi(e + f)$ for all $e, f \in M$.

A *model* of M is an m-monoid morphism of M to $\mathcal{P}^1 = \Delta\mathcal{P}$ such that

(iii) if $e \in M_s$, $f \in M$, and $e \neq f$, then $\Psi(e) \neq \Psi(f)$.

Thus, for a submodel Ψ to be a model, we require that Ψ maps M to $\mathcal{P}^1$ rather than just $\mathcal{R}$, (i) is satisfied, the strengthened form $\Psi(e) + \Psi(f) = \Psi(e + f)$ of additivity holds, and the limited injectivity property (iii) is satisfied. It will turn out to be easier to satisfy (iii) than to get models that are injective.

The condition (ii) in the definition of a submodel is considerably weaker than the corresponding equality that defines a morphism. However, it follows from (i) and (ii) that the composite $T \circ \Psi$ is a morphism from M to $\mathcal{P}$.

2.15.3. LEMMA. *If Ψ: $M \to \mathcal{R}$ is a submodel of the m-monoid M, then the mapping $e \to T(\Psi(e))$ is a morphism from M to $\mathcal{P}$.*

PROOF. By (i) of the definition, $T\Psi(e) = 0$ if and only if $e = 0$. Also, if either e or f is not zero, then $\Psi(e) + \Psi(f)$ is a non-empty subset of $\Psi(e + f)$, so that $T\Psi(e + f) = T(\Psi(e) + \Psi(f)) = T\Psi(e) + T\Psi(f)$. $\square$

Among other things, 2.15.3 shows that the hypothesis (ii) in the next result is natural.

2.15.4. PROPOSITION. *Let E be a generating set in the countable m-monoid M. Suppose that Ψ: $E \to \mathcal{R}$ is a mapping such that:*

(i) *if $e \in E$, then $\Psi(e) = 0$ if and only if $e = 0$;*

(ii) *if $\Sigma_{i<n} e_i = \Sigma_{j<m} f_j$, where $e_i, f_j \in E$ for $i < n$, $j < m$, then $\Sigma_{i<n} T\Psi(e_i) = \Sigma_{j<m} T\Psi(f_j)$.*

For $g \in M$, define

$$\Psi^+(g) = \bigcup \left\{ \sum_{i<n} \Psi e_i \colon e_i \in E, \sum_{i<n} e_i = g \right\}.$$

Then Ψ^+ *is a submodel of* M.

PROOF. Since M and the elements of $\mathcal{R}$ are countable, the definition of $\Psi^+(g)$ produces a countable subset of $\mathcal{P}^{<\omega}$. The collection property is preserved under unions, so that $\Psi^+(g)$ satisfies C.P. The hypothesis (ii) guarantees that the elements of $\Psi^+(g)$ have a common sum. Thus, $\Psi^+(g) \in \mathcal{R}$. By (i) and the assumption that M is an m-monoid, it follows that $\Psi^+(0) = 0$. On the other hand, since $\mathcal{R}$ is an m-monoid, condition (i) also implies that $\Psi^+(g) \neq 0$ if $g \neq 0$. Finally, it is obvious from the definition of Ψ^+ that $\Psi^+(g) + \Psi^+(h) \subseteq \Psi^+(g + h)$. □

The submodel Ψ^+ defined in the proposition is called the *closure* of Ψ. The definition of Ψ^+ makes sense for any mapping Ψ of E to $\mathcal{R}$, but to verify that Ψ^+ is a submodel, it is necessary to show that (i) and (ii) of the proposition are satisfied. We will often affirm that they are by saying that the closure of Ψ is a submodel.

2.15.5. COROLLARY. *If* Ψ *is a submodel of* M, *and* $\Theta\colon M \to \mathcal{R}$ *satisfies* $\Theta(e) \supseteq \Psi(e)$ *for all* $e \in M$, *then the closure of* Θ *is a submodel of* M.

This corollary is clear from the observation that $\Theta(e) \supseteq \Psi(e)$ implies $T\Theta(e) = T\Psi(e)$.

Notes. Our terminology for models differs from Ketonen's usage (KETONEN [1978]). He uses the phrase "faithful model mapping into $\mathcal{P}^1$" to describe models as we have defined them. The term "model" for him means a morphism to $\mathcal{R}$.

2.16. *Expansions of submodels*

For a subset α of $\mathcal{P}^{<\omega}$, the *domain* of α is defined by

$$\text{Dom}\,\alpha = \{a \in \mathcal{P} \colon a \text{ occurs in some } \boldsymbol{a} \in \alpha\}.$$

If α satisfies C.P., then the definition of Dom α can be simplified: $a \in \text{Dom}\,\alpha$ if and only if $(a, b) \in \alpha$ for some $b \in \mathcal{P}$. If M is a set of subsets of $\mathcal{P}^{<\omega}$, denote $\text{Dom}\,M = \bigcup \{\text{Dom}\,\alpha\colon \alpha \in M\}$. For us, the case of interest is when M is a submonoid of $\mathcal{R}$. In this situation, Dom M is clearly a submonoid of $\mathcal{P}$.

2.16.1. DEFINITION. Let M and N be m-monoids with M a submonoid of N. If Ψ is a submodel of M and Θ is a submodel of N, then Θ is an *expansion* of Ψ if

$$(*) \qquad \Theta(e) \cap (\text{Dom}\,\Psi(M))^{<\omega} = \Psi(e)$$

for all $e \in M$.

In particular, if Θ is an expansion of Ψ, then $\Psi(e) \subseteq \Theta(e)$ for all $e \in M$. Consequently, $T\Psi(e) = T\Theta(e)$ in this case. It also follows from $(*)$ that if $\Psi(e) \neq \Psi(f)$ for some $e, f \in M$, then $\Theta(e) \neq \Theta(f)$.

An important case of expansions occurs when $M = N$. Since the condition $(*)$ in the definition refers only to elements in M, it is clear that Θ is an expansion of Ψ if and only if $\Theta \restriction M$ is an expansion of Ψ. It should be emphasized that condition $(*)$ does not imply that $\Theta \restriction M = \Psi$. Of course, if this condition holds, then Θ is surely an expansion of Ψ.

2.16.2. LEMMA. *If M_0, M_1, M_2 are m-monoids, M_0 is a submonoid of M_1, M_1 is a submonoid of M_2, Θ_0, Θ_1, Θ_2 are submodels of M_0, M_1, M_2, respectively, and Θ_1 is an expansion of Θ_0, Θ_2 is an expansion of Θ_1, then Θ_2 is also an expansion of Θ_0.*

2.16.3. PROPOSITION. *Let $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k \subseteq \cdots$ be a chain of submonoids of the m-monoid M, such that $M = \bigcup_{k<\omega} M_k$. For each $k < \omega$, suppose that Ψ_k is a submodel of M_k; and Ψ_{k+1} is an expansion of Ψ_k for all $k < \omega$. If $e \in M_k$, define*

$$\Theta_k(e) = \bigcup_{n \geq k} \Psi_n(e) .$$

With this definition, Θ_k is a submodel of M_k such that:

(a) *Θ_k is an expansion of Ψ_k;*

(b) *if $k \leq n$, then $\Theta_k = \Theta_n \restriction M_k$;*

(c) *if all Ψ_n, $n \geq k$, are models, then Θ_k is a model.*

Define $\Theta_\omega : M \to \mathcal{R}$ by $\Theta_\omega(e) = \Theta_k(e)$ if $e \in M_k$. Then Θ_ω is a submodel of M, and Θ_ω is an expansion of Ψ_k for all $k < \omega$. If all Ψ_k are models, then Θ_ω is a model.

The statements of this proposition are easy consequences of the definitions of models, submodels, expansions, and the maps Θ_k.

The submodel Θ_ω that is constructed in the proposition will be called the *limit* of the sequence $\{\Psi_k : k < \omega\}$.

2.17. Second reduction step

Using the ideas that have just been introduced, we can move beyond the result in Proposition 2.12.1.

2.17.1. PROPOSITION. *Let $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k \subseteq \cdots$ be a chain of countable submonoids of the m-monoid M, and suppose that for each $k < \omega$, Ψ_k is a model of M_k, with Ψ_{k+1} an expansion of Ψ_k. Assume that:*

(i) *each M_k is special, and M_{k+1} is a special extension of M_k;*

(ii) *if $e_0 + e_1 = f_0 + f_1$ in M_k, then there exist $g_{ij} \in M_{k+1}$ such that $e_i = \Sigma_{j<2}\, g_{ij}$, $f_j = \Sigma_{i<2}\, g_{ij}$ for $i, j = 0, 1$;*

(iii) *if $e \in M_k$ and $(a_0, a_1) \in \Psi_k(e)$, then $e = e_0 + e_1$ in M_{k+1}, where $T\Psi_{k+1}(e_i) = a_i$ for $i = 0, 1$.*

If $\Psi_0 M_0^* \subseteq \mathcal{Q}^1$, then M_0^* is isomorphic to a subsemigroup of **UBA**.

PROOF. We can assume that $M = \bigcup_{k<\omega} M_k$. It then follows from (i) and (ii) that M is a countable r-monoid. Moreover, (iii) and the splitting property imply that M is atomless. By Proposition 2.16.3, the limit Θ of the sequence of Ψ_k is a model of M and an expansion of each Ψ_k. For $e \in M$, define

$$\Lambda(e) = \left\{ (\Theta f_i)_{i<n} \colon \sum_{i<n} f_i = e \right\},$$

and let $N = \{\Lambda(e) \colon e \in M\}$. By definition, Λ is the composition of the mappings $a \to \delta(a)$ from M to ΔM with $\Delta\Theta \colon \Delta M \to \Delta\mathcal{P}^1$. By Proposition 2.2.4, N is a submonoid of $\Delta\mathcal{P}^1$ and Λ is a morphism from M to N. In particular, N has the refinement property. If $(\alpha_0, \alpha_1) \in A \in N$, then $A = \Lambda(g)$, $\alpha_0 = \Theta(e_0)$, $\alpha_1 = \Theta(e_1)$, and $e_0 + e_1 = g$. It follows that $A = A_0 + A_1$ in N, where $A_0 = \Lambda(e_0)$, $A_1 = \Lambda(e_1)$, $TA_0 = \alpha_0$, and $TA_1 = \alpha_1$. Thus, N satisfies condition (ii) of Proposition 2.12.1. Next, note that if $g \in M$, then $\Lambda(g)$ has the hierarchy property. Indeed, if $(\Theta(e), \Theta(f_1), \ldots, \Theta(f_{m-1})) \in \Lambda(g)$, where $e + f_1 + \cdots + f_{m-1} = g$, and $(a_0, a_1) \in \Theta(e)$, then there exists $k < \omega$ such that $e \in M_k$ and $(a_0, a_1) \in \Psi_k(e)$. By (iii), $e = e_0 + e_1$ in M_{k+1} with $T\Psi_{k+1}(e_i) = a_i$ for $i = 0, 1$. Hence, $T\Theta(e_i) = a_i$ $(i = 0, 1)$, and $(\Theta(e_0), \Theta(e_1), \Theta(f_1), \ldots, \Theta(f_{m-1})) \in \Lambda(g)$ because $e_0 + e_1 + f_1 + \cdots + f_{m-1} = g$. This argument shows that $N \subseteq \mathcal{P}^2$. Since $\Psi_0 M_0^* \subseteq \mathcal{Q}^1$ and $T\Lambda(e) = \Theta e \supseteq \Psi_0 e$ for all $e \in M_0$, it follows that $T^2\Lambda(e) = [\omega^*]$ for all $e \in M_0^*$; that is, $\Lambda M_0^* \subseteq \mathcal{Q}^2$. Therefore, ΛM_0^* is isomorphic to a subsemigroup of $\boldsymbol{UBA}$ by Proposition 2.12.1. The proof can be completed by showing that Λ is injective. Suppose that $\Lambda(e) = \Lambda(f)$. Since M is special, there exist elements $e_i \in M_s$ such that $e = \Sigma_{i<n} e_i$. It follows that $(\Theta(e_i))_{i<n} \in \Lambda(e) = \Lambda(f)$. Hence, $f = \Sigma_{i<n} f_i$, where $\Theta(f_i) = \Theta(e_i)$. Since Θ is a model, condition (iii) of Definition 2.15.2 yields $f_i = e_i$. Thus, $f = \Sigma_{i<n} f_i = \Sigma_{i<n} e_i = e$. $\square$

2.18. Two expansions

We will translate the results in Propositions 2.13.3 and 2.14.2 into existence theorems for expansions.

2.18.1. PROPOSITION. *Let M be a countable m-monoid, and suppose that Θ is a submodel of M.*

(a) *If $e \in M^*$ and $(a, \boldsymbol{b}) \in \Theta(e)$, then there is a special extension N of M and an expansion Θ' of Θ to a submodel of N with the properties*:

(i) *N is generated by $M \cup \{e_0, e_1\}$, where e_0, e_1 are special in N and $e = e_0 + e_1$;*

(ii) *$(a) \in \Theta'(e_0)$ and $\boldsymbol{b} \in \Theta'(e_1)$;*

(iii) *$\Theta' \restriction M = \Theta$.*

(b) *If $e_0 + e_1 = f_0 + f_1$ in M^*, then there is a special extension N of M that is generated by $M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$ with the g_{ij} special elements that satisfy $e_i = \Sigma_{j<2} g_{ij}$ and $f_j = \Sigma_{i<2} g_{ij}$ for $i, j < 2$, and there is an expansion Θ' of Θ to a submodel of N.*

PROOF. (a) The existence of the monoid N satisfying (i) is guaranteed by Proposition 2.13.3. Define the mapping $\Psi \colon M \cup \{e_0, e_1\} \to \mathcal{R}$ by $\Psi(f) = \Theta(f)$ for

$f \in M$, $\Psi(e_0) = \hat{a}$, $\Psi(e_1) = \Sigma_{i=1}^{n-1} \hat{b}_i$, where $\boldsymbol{b} = (b_1, \ldots, b_{n-1})$. In this definition we are using the notation that was introduced in subsection 2.15. The mapping $T \circ \Psi$ is a morphism of M to $\mathcal{P}$ such that $T\Psi(e) = a + \Sigma_{i=1}^{n-1} b_i = T\hat{a} + T(\Sigma_{i=1}^{n-1} \hat{b}_i)$. By Proposition 2.13.3(d) this morphism extends to N. In other words, condition (ii) of Proposition 2.15.4 is satisfied. Thus, the closure Ψ^+ of Ψ is a submodel of N. Let $\Theta' = \Psi^+$. By our definitions, $(a) \in \Psi(e_0) \subseteq \Theta'(e_0)$ and $(b_1, \ldots, b_{n-1}) \subseteq \Sigma_{i=1}^{n-1} \hat{b}_i \subseteq \Theta'(e_1)$, so that (ii) is satisfied. If $f + me_0 + ne_1 = f' \in M$, then by Proposition 2.13.3(c), $m = n$. Hence, $\Psi(f) + m\Psi(e_0) + n\Psi(e_1) = \Psi(f) + m(\hat{a} + \Sigma_{i=1}^{n} \hat{b}_i) \subseteq \Psi(f) + m\Psi(e) \subseteq \Psi(f + me) = \Psi(f')$ by Lemma 2.15.1. Therefore, $\Psi^+(f') \subseteq \Psi(f') \subseteq \Psi^+(f')$ by the definitions of closure. Hence, $\Theta' \restriction M = \Theta$, which implies that Θ' is an expansion of Θ and completes the proof of (a).

(b) Let N be the special extension of M that was associated with the equation $e_0 + e_1 = f_0 + f_1$ in Proposition 2.14.2. Then N is generated by $M \cup \{g_{ij}\colon i, j < 2\}$, where the g_{ij} are special in N and $e_i = \Sigma_{j<2}\, g_{ij}$, $f_j = \Sigma_{i<2}\, g_{ij}$. The expansion of Θ to a submodel of N follows roughly the same path that we took in part (a). For i, $j = 0, 1$, denote $a_i = T\Theta e_i$ and $b_j = T\Theta f_j$. We must choose elements $c_{ij} \in \mathcal{P}$ so that

(1) $a_i = \Sigma_{j<2}\, c_{ij}$ and $b_j = \Sigma_{i<2}\, c_{ij}$ for $i, j < n$.

However, the choice requires some care. In accordance with Proposition 2.9.2, $\mathcal{P}\backslash\{w\}$ can be viewed as a submonoid of a rational vector space $\mathcal{V}$. Since M is countable, Dom ΘM is a countable subset of $\mathcal{V}$; hence, (Dom $\Theta M)\backslash\{w\}$ is contained in a countable subspace of $\mathcal{V}$. On the other hand, since e_0, e_1, f_0, f_1 are not zero, it follows that a_0, a_1, b_0, b_1 are not zero. Therefore, by Proposition 2.9.2, there are uncountably many choices of $\{c_{ij}\colon i, j < 2\} \subseteq \mathcal{P}$ that satisfy (1). Thus, by reindexing if necessary, it can be assumed that the c_{ij} are chosen to satisfy:

(2) $0 < c_{00} < w$; c_{00} is linearly independent of Dom ΘM.

Define the mapping $\Psi\colon M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\} \to \mathcal{R}$ by $\Psi(f) = \Theta(f)$ for $f \in M$, and $\Psi(g_{ij}) = \hat{c}_{ij}$ for $i, j < 2$. By the same argument that we used in part (a), the closure Ψ^+ is a submodel of N. Let $\Theta' = \Psi^+$. It remains to prove that Θ' is an expansion of Θ, that is, $\Theta'(f) \cap (\text{Dom } \Theta M)^{<\omega} = \Theta(f)$ for all $f \in M$. The inclusion of the right-hand side in the left-hand side of this equation is clear. Suppose that $\boldsymbol{d} = (d_i)_{i<n} \in \Theta'(f) = \Psi^+(f)$ with each $d_i \in \text{Dom } \Theta M$. By the definition of Ψ^+, there is a representation $f = f' + \Sigma_{i,j<2}\, r_{ij} g_{ij}$ with $f' \in M$ and $r_{ij} \in \omega$ such that $\boldsymbol{d} \in \Theta(f') + \Sigma_{i,j<2}\, r_{ij}\hat{c}_{ij}$. Moreover, by Proposition 2.14.2, there exist m_i, $n_j \in \omega$ such that

(3) $r_{ij} = m_i + n_j$ and $f = f' + \Sigma_{i<2}\, m_i e_i + \Sigma_{j<2}\, n_j f_j$.

Thus, $\boldsymbol{d} = \boldsymbol{d}' + \Sigma_{i,j<2}\, r_{ij} \boldsymbol{c}_{ij}$, where $\boldsymbol{d}' = (d'_k)_{k<n} \in \theta(f')$ and $\boldsymbol{c}_{ij} \in \hat{c}_{ij}$ has length n. By the definition of $\hat{c}_{ij}$,

(4) $d_k = d'_k + \Sigma_{i,j<2}\, s_{ijk} c_{ij}$ for $k < n$, where $\Sigma_{k<n}\, s_{ijk} = r_{ij}$.

It follows from (1) and (4) that if $k < n$, then $(s_{00k} + s_{11k} - s_{01k} - s_{10k})c_{00}$ is in the subspace of $\mathcal{V}$ that is spanned by Dom θM. Hence, $s_{00k} + s_{11k} = s_{01k} + s_{10k}$. By 2.14.1, there exist m_{ik}, $n_{jk} < \omega$ such that $s_{ijk} = m_{ik} + n_{jk}$ for $i, j < 2$, $k < n$. Using (1) and (4) with these equations gives $d_k = d'_k + \Sigma_{i<2}\, m_{ik} a_i + \Sigma_{j<2}\, n_{jk} b_j$. Since $(a_i) \in \theta(e_i)$ and $(b_j) \in \theta(f_j)$, it follows from C.P. that $\boldsymbol{d} \in \theta(f') + \Sigma_{k<n}(\Sigma_{i<2}\, m_{ik}\theta(e_i) + \Sigma_{j<2}\, n_{jk}\theta(f_j)) \subseteq \theta(f' + \Sigma_{i<2}\, m'_i e_i + \Sigma_{j<2}\, n'_j f_j)$, where $m'_i = \Sigma_{k<n}\, m_{ik}$ and $n'_j = \Sigma_{k<n}\, n_{jk}$. Thus, $m'_j + n'_j = \Sigma_{k<n}\, (m_{ik} + n_{jk}) = \Sigma_{k<n}\, s_{ijk} = r_{ij} = m_i + n_j$. The uniqueness property in 2.14.1(b) and the equation $e_0 + e_1 = f_0 + f_1$ yield $f' + \Sigma_{i<k}\, m'_i e_i + \Sigma_{j<n}\, n'_j f_j = f$. Hence, $\boldsymbol{d} \in \theta(f)$. $\square$

2.19. The Basic Lemma

Proposition 2.15.4 is a convenient device for constructing submodels of a monoid, but to use Proposition 2.17.1 we need models. The result that accomplishes the transition from submodels to models is the following *Basic Lemma*.

2.19.1. PROPOSITION. *If M is a countable, special m-monoid, and Ψ is a submodel of M, then there is an expansion of Ψ to a model of M.*

The proof of this result is preceded by several lemmas. In all of them, it is assumed that M is a countable m-monoid, Ψ is a submodel of M, and $e, f \in M$.

2.19.2. LEMMA. *If $e \not\leq f$ in M and $a = T\Psi(e)$, then $a = a_0 + a_1$ in $\mathcal{P}$ with a_0, $a_1 \in \mathcal{P}^*$, and there is an expansion Θ of Ψ such that $(a_0, a_1) \in \Theta(e)$, $(a_0, a_1) \not\in \Theta(f)$.*

PROOF. Since $e \not\leq f$, it follows that a is a non-zero element of $\mathcal{P}$. Moreover, $\operatorname{Dom} \Psi M$ is countable, so that by Proposition 2.9.2 there is a decomposition $a = a_0 + a_1$ in $\mathcal{P}$, with a_0 linearly independent of $\operatorname{Dom} \Psi M$. Define the mapping Λ: $M \to \mathcal{R}$ by $\Lambda(g) = \Psi(g)$ for $g \neq e$, $\Lambda(e) = \Psi(e) \cup \hat{a}_0 + \hat{a}_1$. Since $T(\hat{a}_0 + \hat{a}_1) = a_0 + a_1 = a = T\Psi(e)$, it follows that $\Lambda(e) \in \mathcal{R}$. By Corollary 2.15.4 the closure Θ of Λ is a submodel of M such that $(a_0, a_1) \in \hat{a}_0 + \hat{a}_1 \subseteq \Lambda(e) \subseteq \Theta(e)$. On the other hand, $e \not\leq f$, so that $\Theta(f) = \Psi(f)$ by the definition of the closure of Λ. Thus, $(a_0, a_1) \not\in \Theta(f)$ because $a_0 \not\in \operatorname{Dom} \Psi M$. It remains to show that Θ is an expansion of Ψ. Let $g \in M$ and $\boldsymbol{b} = (b_i)_{i<n} \in \Theta(g)$, where $b_i \in \operatorname{Dom} \Psi M$ for all $i < n$. By definition of the closure, there is a decomposition $g = g' + re$, $g' \in M$, $r \in \omega$, such that $\boldsymbol{b} \in \Psi(g') + r(\Psi(e) \cup \hat{a}_0 + \hat{a}_1)$. Thus, $\boldsymbol{b} = \boldsymbol{b}' + \boldsymbol{b}_1 + \cdots + \boldsymbol{b}_r$, with $\boldsymbol{b}' \in \Psi(g')$, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l \in \Psi(e)$, and $\boldsymbol{b}_{l+1}, \ldots, \boldsymbol{b}_r \in \hat{a}_0 + \hat{a}_1$. Hence, $\boldsymbol{b}_{l+1} + \cdots + \boldsymbol{b}_r = (c_0, \ldots, c_{n-1})$, where $c_k = s_{0k}a_0 + s_{1k}a_1$ for $k < n$, $s_{ik} < \omega$, and $\Sigma_{k<n} s_{0k} = \Sigma_{k<n} s_{1k} = r - l$. Since a_0 is independent of $\operatorname{Dom} \Psi M$, it follows that $s_{0k} = s_{1k}$, $c_k = s_{0k}a$, and $\boldsymbol{b}_{l+1} + \cdots + \boldsymbol{b}_r \in (r-l)\hat{a} \subseteq (r-l)\Psi(e)$. Consequently, $\boldsymbol{b} \in \Psi(g' + re) = \Psi(g)$. $\square$

2.19.3. LEMMA. *If $\boldsymbol{a} \in \Psi(e_0 + e_1)$, then there is an expansion Θ of Ψ such that $\boldsymbol{a} \in \Theta(e_0) + \Theta(e_1)$.*

PROOF. Induce on $l(\boldsymbol{a}) = n$. If $n = 1$, then $\boldsymbol{a} \in \Psi(e_0) + \Psi(e_1)$ by Lemma 2.15.3. Suppose that $\boldsymbol{a} = (a, \boldsymbol{b})$ with $l(\boldsymbol{b}) = l(\boldsymbol{a}) - 1$. By Proposition 2.18.1, there is an extension N' of M and an expansion Ψ' of Ψ to a submodel of N' such that $e_0 + e_1 = f_0 + f_1$ in N', and $a = T\Psi'(f_0)$, $\boldsymbol{b} \in \Psi'(f_1)$. By part (b) of Proposition 2.18.1, there is an extension N'' of N' and an expansion Ψ'' of Ψ' to a submodel of N'', such that $e_i = \Sigma_{j<2}\, g_{ij}$, $f_j = \Sigma_{i<2}\, g_{ij}$ in N''. Then $\boldsymbol{b} \in \Psi'(f_1) \subseteq \Psi''(f_1) = \Psi''(g_{01} + g_{11})$, so that by the induction hypothesis, $\boldsymbol{b} = \boldsymbol{b}_0 + \boldsymbol{b}_1$, where $\boldsymbol{b}_i \in \Theta'(g_{i1})$ for a suitable expansion Θ' of Ψ''. Note that $a = T\Psi'(f_0) = T\Theta'(f_0) = T\Theta'(g_{00}) + T\Theta'(g_{01}) = a_0 + a_1$ by Lemma 2.15.3. Thus, $(a_i, \boldsymbol{b}_i) = (a_i, \boldsymbol{0}) + (0, \boldsymbol{b}_i) \in \Theta'(g_{i0}) + \Theta'(g_{i1}) \subseteq \Theta'(g_{i0} + g_{i1}) = \Theta'(e_i)$ by C.P. in $\mathcal{R}$. Therefore, if $\Theta = \Theta' \restriction M$, then Θ is an expansion of Ψ such that $(a, \boldsymbol{b}) = (a_0, \boldsymbol{b}_0) + (a_1, \boldsymbol{b}_1) \in \Theta(e_0) + \Theta(e_1)$. $\square$

2.19.4. LEMMA. *If $\boldsymbol{a} = (a_i)_{i<n}$ and $\boldsymbol{b} = (b_j)_{j<m}$ are elements of $\Psi(e)$, then there is an expansion Θ of Ψ to a submodel of M such that $\Theta(e)$ contains a sequence $(c_{ij})_{i<n,j<m}$ with the properties $a_i = \Sigma_{j<m} c_{ij}$, $b_j = \Sigma_{i<n} c_{ij}$ for $i < n$ and $j < m$.*

PROOF. By Proposition 2.18.1 (extended inductively), there is an extension N of M and an expansion Ψ' of Ψ to a submodel of N such that $e = \Sigma_{i<n} e_i$ in N and $T\Psi'(e_i) = a_i$ for all $i < n$. Since $\boldsymbol{b} \in \Psi(e) \subseteq \Psi'(e)$, it follows from 2.19.3 that there is an expansion Θ' of Ψ' such that $\boldsymbol{b} \in \Sigma_{i<n} \Theta'(e_i)$, say $\boldsymbol{b} = \Sigma_{i<n} \boldsymbol{c}_i$ with $\boldsymbol{c}_i = (c_{ij})_{j<m} \in \Theta'(e_i)$. Then $\Sigma_{j<m} c_{ij} = T\Theta'(e_i) = T\Psi'(e_i) = a_i$ for all $i < n$, and $\Sigma_{i<n} c_{ij} = b_j$ because $\boldsymbol{b} = \Sigma_{i<n} \boldsymbol{c}_i$. Finally, by C.P., $(c_{ij})_{i<n,j<m} = (\boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}) = (\boldsymbol{c}_0, \boldsymbol{0}, \ldots, \boldsymbol{0}) + (\boldsymbol{0}, \boldsymbol{c}_1, \ldots, \boldsymbol{0}) + \cdots + (\boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{c}_{n-1}) \in \Sigma_{i<n} \Theta'(e_i) \subseteq \Theta'(\Sigma_{i<n} e_i) = \Theta'(e)$. Consequently, $\Theta = \Theta' \restriction M$ is an expansion of Ψ with the required properties. □

2.19.5. LEMMA. *If $(a, \boldsymbol{b}) \in \Psi(e)$ with $a \neq 0$, then there is an expansion Θ of Ψ such that $(a_0, a_1, \boldsymbol{b}) \in \Theta(e)$, where $a = a_0 + a_1$ and $a_0 \neq 0 \neq a_1$.*

PROOF. By Proposition 2.18.1, there is an extension N of M and an expansion Ψ' of Ψ to a submodel of N such that $e = e_0 + e_1$ in N and $T\Psi'(e_0) = a$, $\boldsymbol{b} \in \Psi'(e_1)$. By 2.19.2, there is an expansion Θ' of Ψ' such that $a = a_0 + a_1$ in $\mathscr{P}$, a_0, $a_1 \neq 0$, and $(a_0, a_1) \in \Theta'(e_0)$. It follows that $(a_0, a_1, \boldsymbol{b}) \in \Theta'(e_0) + \Theta'(e_1) \subseteq \Theta'(e)$. Thus, $\Theta = \Theta' \restriction M$ has the desired properties. □

The proof of the Basic Lemma can now be given. Our objective is to expand Ψ to Θ, a submodel such that:

(a) if $e \neq f$ in M and e is special, then $\Theta(e) \neq \Theta(f)$;
(b) if $(e_0, e_1) \in M \times M$, then $\Theta(e_0) + \Theta(e_1) = \Theta(e_0 + e_1)$;
(c) if $e \in M$, then $\Theta(e)$ has R.P.;
(d) if $e \in M$, then $\Theta(e)$ has S.P.

Condition (a) can be fulfilled at the beginning of our construction. Let (e_k, f_k), $k < \omega$, be an enumeration of all pairs (e, f) such that e is special and $f \neq e$. Since e_k is special, it has the S-B. property. Thus, $e_k \not\leq f_k$ or $f_k \not\leq e_k$. By 2.19.2, we can construct a sequence $\Psi = \Psi_0, \Psi_1, \Psi_2, \ldots$ of submodels of M such that Ψ_{k+1} is an expansion of Ψ_k and $\Psi_k(e_k) \neq \Psi_k(f_k)$ for all $k < \omega$. The limit Ψ_ω of this sequence is a submodel of M that expands Ψ by Proposition 2.16.3. Thus, Ψ_ω satisfies (a). It can be assumed that Ψ itself satisfies (a); every expansion of Ψ will then satisfy (a).

To get an expansion that satisfies (b), (c), and (d), a more elaborate construction is needed. We make three numerations: $\{(e_{0k}, e_{1k}, \boldsymbol{a}_k): k < \omega\}$ are all triples $(e_0, e_1, \boldsymbol{a})$ such that $e_0, e_1 \in M$, $\boldsymbol{a} \in \Psi(e_0 + e_1)$; $\{(f_k, \boldsymbol{b}_k, \boldsymbol{c}_k): k < \omega\}$ are all triples $(f, \boldsymbol{b}, \boldsymbol{c})$ such that $f \in M^*$ and $\boldsymbol{b}, \boldsymbol{c} \in \Psi(f)$; $\{(g_k, (d_k, \boldsymbol{d}_k)): k < \omega\}$ are all pairs such that $g_k \in M^*$, $(d_k, \boldsymbol{d}_k) \in \Psi(g_k)$, and $d_k \neq 0$. Use 2.19.3, 2.19.4, and 2.19.5 to construct a sequence $\Psi = \Psi_0, \Psi_1, \Psi_2, \ldots$ of submodels of M such that Ψ_{m+1} is an expansion of Ψ_m for all $m < \omega$; $\boldsymbol{a}_k \in \Psi_{m+1}(e_{0k}) + \Psi_{m+1}(e_{1k})$ if $m = 3k$; there exists $(a_{ij})_{i<n,j<m} \in \Psi_{m+1}(f_k)$ such that $b_{ki} = \Sigma_{j<m} a_{ij}$ and $c_{kj} = \Sigma_{i<n} a_{ij}$ (where $\boldsymbol{b}_k = (b_{ki})_{i<n}$, $\boldsymbol{c}_k = (c_{kj})_{j<m}$) if $m = 3k + 1$; and there exists $(d_{k0}, d_{k1}, \boldsymbol{d}_k) \in$

$\Psi_{m+1}(g_k)$ with $d_{k0}+d_{k1}=d_k$, $d_{k0}\neq 0\neq d_{k1}$ if $m=3k+2$. The limit $\Psi^{(1)}$ of the sequence $\{\Psi_m\colon m<\omega\}$ is a submodel of M such that: $\Psi(e_0+e_1)\subseteq\Psi^{(1)}(e_0)+\Psi^{(1)}(e_1)$; if $\boldsymbol{b}$, $\boldsymbol{c}\in\Psi(f)$, $\boldsymbol{b}=(b_i)_{i<n}$, $\boldsymbol{c}=(c_j)_{j<m}$, then there exists $(a_{ij})_{i<n,j<m}$ in $\Psi^{(1)}(f)$ such that $b_i=\Sigma_{j<m}a_{ij}$, $c_j=\Sigma_{i<n}a_{ij}$; and if $(d,\boldsymbol{d})\in\Psi(g)$, then there exists $(d_0,d_1,\boldsymbol{d})\in\Psi^{(1)}(g)$ such that $d=d_0+d_1$ with $d_0\neq 0\neq d_1$.

To complete the construction of the desired Θ, we iterate the process that leads from Ψ to $\Psi^{(1)}$. This produces a sequence $\Psi=\Psi^{(0)}$, $\Psi^{(1)}$, $\Psi^{(2)},\ldots$ of submodels of M such that $\Psi^{(k+1)}$ is an expansion of $\Psi^{(k)}$ satisfying the conditions above for $\Psi^{(k)}$. Finally, let Θ be the limit of the sequence of $\Psi^{(k)}$. Then Θ is an expansion of Ψ that satisfies (a), (b), (c), and (d). (Reference to the definition of the limit in Proposition 2.16.3 makes this assertion clear.) Conditions (a)–(d) imply that Θ is a model of M.

2.20. *Proof of Ketonen's Theorem*

Let P be a countable, commutative semigroup. We wish to prove that P can be embedded in ***UBA***. Our strategy is to construct a chain of countable m-monoids M_k and models Ψ_k of M_k such that the hypotheses of Proposition 2.17.1 are satisfied, and P is isomorphic to a subsemigroup of M_0^*. The construction of the M_k is inductive. The heart of the induction step is the following corollary of Proposition 2.18.1 and the Basic Lemma.

2.20.1. LEMMA. *Let M be a countable, special m-monoid, and suppose that Ψ is a model of M.*

(a) *If $e\in M$ and $(a_0,a_1)\in\Psi(e)$, then there is a special extension of M to a countable, special m-monoid N, and an expansion Θ of Ψ to a model of N such that $e=e_0+e_1$ in N and $T\Theta(e_i)=a_i$ for $i=0,1$.*

(b) *If $e_0+e_1=f_0+f_1$ in M^*, then there is a special extension of M to a countable, special m-monoid N, and an expansion Θ of Ψ to a model of N, such that $e_i=\Sigma_{j<2}g_{ij}$ and $f_j=\Sigma_{i<2}g_{ij}$ in N.*

To start our construction, we adjoin a new zero element 0 to P to obtain an m-monoid. By Corollary 2.13.4, $P\cup\{0\}$ can be embedded in a countable special m-monoid N. Define the submodel Ψ of N by $\Psi(0)=0$ and $\Psi(e)=\hat{w}$ for all $e\in N^*$. The Basic Lemma gives an expansion Ψ_0 of Ψ to a model of N. Since $w=T\Psi(e)=T\Psi_0(e)$ for all $e\in N^*$, it follows that $\Psi_0P\subseteq\Psi_0N^*\subseteq\mathscr{Q}^1$. By setting $M_0=N$, we have taken the base step in the inductive construction that will fulfill the hypotheses of Proposition 2.17.1. Suppose that for $m\geq 0$, countable, special m-monoids $M_0, M_1,\ldots,M_m$ have been obtained, together with models Ψ_k of M_k for $k\leq m$ such that if $k<m$, then M_{k+1} is a special extension of M_k, Ψ_{k+1} is an expansion of Ψ_k, and the conditions (ii) and (iii) of Proposition 2.17.1 are satisfied. Let $\{(e_k,a_{k0},a_{k1})\colon k<\omega\}$ enumerate the triples (e,a_0,a_1) such that $e\in M_m$ and $(a_0,a_1)\in\Psi_m\Psi_m(e)$; and let $\{(e_{k0},e_{k1},f_{k0},f_{k1})\colon k<\omega\}$ be a listing of the quadruples $(e_0,e_1,f_0,f_1)\in{}^4M_m$ such that $e_0+e_1=f_0+f_1$. By alternating (a) and (b) of the lemma, we produce a sequence $M_m=N^{(0)}$, $N^{(1)}$, $N^{(2)},\ldots$ of countable, special m-monoids, and a sequence $\Psi_m = \Theta^{(0)}, \Theta^{(1)}, \Theta^{(2)}, \ldots$ of models such that $N^{(l+1)}$ is a special extension of $N^{(l)}$, $\Theta^{(l+1)}$ is an expansion of $\Theta^{(l)}$ to a model of $N^{(l+1)}$, and

(1) if $l = 2k$, then $e_k = f_0 + f_1$ in $N^{(l+1)}$, where $T\Theta^{(l+1)}(f_i) = a_{ki}$, $i = 0, 1$;

(2) if $l = 2k + 1$, then $e_{ki} = \Sigma_{j<2}\, g_{ij}$, $f_{kj} = \Sigma_{i<2}\, g_{ij}$ $(i, j = 0, 1)$ in $N^{(l+1)}$.

Define $M_{m+1} = \bigcup_{l<\omega} N^{(l)}$, and let Ψ_{m+1} be the limit of the sequence of expansions $\Theta^{(l)}$. By Lemma 2.13.2, M_{m+1} is a special monoid that is a special extension of M_m. Moreover, Proposition 2.16.3 assures us that Ψ_{m+1} is an expansion of Ψ_m to a model of M_{m+1}. It follows from (1) and (2) that the conditions (ii) and (iii) of Proposition 2.17.1 are satisfied for $k = m$. The inductive construction is therefore finished, so that $N^* = M_0^*$ is isomorphic to a subsemigroup of ***UBA***. Since $P \subseteq N^*$, we have finally come to the end of our long pilgrimage.

Notes. Locally, our exposition of Theorem 2.5.1 is very different from Ketonen's; globally, the two proofs are the same. Here is a dictionary that associates the various sections in KETONEN's [1978] paper with the subsections of our exposition: {[Ketonen] 0, 1, 2} $\leftrightarrow$ {1.1 – 1.11}; {[Ketonen] 3} $\leftrightarrow$ {1.12 – 1.22}; {[Ketonen] 4} $\leftrightarrow$ {2.8 – 2.11}; {[Ketonen] 5} $\leftrightarrow$ {2.13 – 2.16}; {[Ketonen] 6} $\leftrightarrow$ {2.18, 2.19}; and {[Ketonen] 7} $\subseteq$ {2.7, 2.12, 2.17, 2.20}.

There are interesting questions about the embedding of uncountable semigroups. ADÁMEK,. KOUBEK and TRNKOVÁ [1975] have shown that every abelian group can be embedded in the additive semigroup of isomorphism types of (not necessarily countable) Boolean algebras, but there seems to be no corresponding result for uncountable commutative semigroups. Moreover, very little is known about which uncountable semigroups can be embedded in the monoid ***BA*** of countable Boolean algebra types.

2.21. *Coproducts*

For a long time, the principal subjects of this section, Boolean algebras, have been off the stage. They now return to resume the major roles in our drama.

Every Boolean algebra is an associative algebra over a two element field. Consequently, the tensor product $A \otimes B$ of the Boolean algebras A and B (over the field of two elements) is a well-defined subject. In fact, $A \otimes B$ is a Boolean algebra because it satisfies the identity $x^2 = x$. From a categorical standpoint, $A \otimes B$ is the coproduct of A and B in the category of Boolean algebras. For this reason, the notation $A \oplus B$ is generally used for the coproduct (that is, tensor product) in the literature of Boolean algebra. We will henceforth write $A \oplus B$ instead of $A \otimes B$. The dual of $A \oplus B$ in the category of Boolean spaces is the product $(\mathrm{Ult}\, A) \times (\mathrm{Ult}\, B)$ of the Stone spaces of A and B. This simple observation makes it possible to translate questions about tensor products of Boolean algebras to problems about topological products of Boolean spaces.

If X and Y are metrizable Boolean spaces, $p \in X$ and $q \in Y$, then the point (p, q) is isolated in the product space $X \times Y$ if and only if p is isolated in X and q is isolated in Y. Consequently, the topological derivative satisfies Leibnitz's rule:

$$(X \times Y)' = X' \times Y \cup X \times Y' .$$

Finite induction leads to the rule $(X\times Y)^{(n)} = \bigcup_{i+j=n} X^{(i)} \times Y^{(j)}$ for all $n<\omega$, but the generalization of this formula to transfinite derivatives is less obvious. For instance, $(X\times Y)^{(\omega)} = \bigcap_{n<\omega} \bigcup_{i\leq n} X^{(i)} \times Y^{(n-i)} = \bigcup_{\phi\leq id} (\bigcap_{n<\omega} X^{(\phi(n))} \times Y^{(n-\phi(n))}) = X\times Y^{(\omega)} \cup X^{(\omega)} \times Y$. An inductive generalization of this calculation (as in TELGÁRSKI [1968]) gives:

$$(1) \qquad (X\times Y)^{(\xi)} = \bigcup_{\eta\#\zeta=\xi} X^{(\eta)} \times Y^{(\zeta)},$$

where $\#$ denotes the natural sum of ordinal numbers. Explicitly, if $\eta = \omega^{\alpha_1}\cdot n_1 + \cdots + \omega^{\alpha_k}\cdot n_k$ and $\zeta = \omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_k}\cdot m_k$ with $\alpha_1 > \cdots > \alpha_k \geq 0$, $n_i, m_i \in \omega$, then $\eta\#\zeta = \omega^{\alpha_1}\cdot(n_1+m_1) + \cdots + \omega^{\alpha_k}\cdot(n_k+m_k)$. Note that $\eta\#\zeta$ is strictly monotone in η and ζ. Also, (1) is a finite union: $|\{(\eta,\zeta)\colon \eta\#\zeta=\xi\}|<\omega$. For large ξ, (1) becomes

$$(2) \qquad K(X\times Y) = X\times K(Y) \cup K(X)\times Y.$$

In particular, $X\times Y$ is scattered if and only if X and Y are both scattered.

If p is isolated in $X^{(\eta)}$, then $p\not\in X^{(\eta')}$ for all $\eta'>\eta$. Using a similar statement for Y and the strict monotonicity of $\#$, it follows that if (p,q) is isolated in $X^{(\eta)}\times Y^{(\zeta)}$, then $(p,q)\not\in X^{(\eta')}\times Y^{(\zeta')}$ whenever $(\eta',\zeta')\neq(\eta,\zeta)$ and $\eta'\#\zeta' = \eta\#\zeta$. Thus, (p,q) is isolated in $(X\times Y)^{(\xi)}$ if and only if (p,q) is isolated in $X^{(\eta)}\times Y^{(\zeta)}$, for some η and ζ such that $\eta\#\zeta=\xi$. Turning this statement around, we see that $(X\times Y)^{(\xi)}$ is perfect if and only if $\xi>\eta\#\zeta$ for all $\eta<\nu(X)$ and $\zeta<\nu(Y)$. For ordinal numbers μ and ν, denote

$$\mu * \nu = \min\{\xi\colon \xi > \eta\#\zeta \text{ for all } \eta<\mu,\ \zeta<\nu\}.$$

Our discussion shows that $\nu(X\times Y) = \nu(X) * \nu(Y)$.

It is useful to have an explicit formula for $\mu*\nu$. If $\mu=0$ or $\nu=0$, then $\mu*\nu=0$. Let $\mu = \omega^{\alpha_1}\cdot n_1 + \cdots + \omega^{\alpha_k}\cdot n_k$, $\nu = \omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_k}\cdot m_k$, where $\alpha_1 > \cdots > \alpha_k$ and $0 < m_i + n_i < \omega$ for $1\leq i\leq k$. If $n_k>0$ and $m_k>0$, then $\mu*\nu = \omega^{\alpha_1}\cdot(m_1+n_1) + \cdots + \omega^{\alpha_i}\cdot(m_k+n_k-1)$. If $0 = n_k = \cdots = n_{l+1}$, $n_l\neq 0$ with $l<k$, then $\mu*\nu = \omega^{\alpha_1}\cdot(m_1+n_1) + \cdots + \omega^{\alpha_l}\cdot(m_l+n_l)$. The formula for $\mu*\nu$ is similar if $m_k=0$. Note that if neither μ nor ν is a limit ordinal, then $\mu*\nu = (\mu\#\nu)-1$.

Equation (2) shows that the rank function on $X\times Y$ cannot be expressed solely in terms of r_X and r_Y. We need an extension of these functions to the entire space. Here is the appropriate definition. For any metrizable Boolean space X and point $p\in X$, define

$$s_X(p) = \min\{\xi\colon p\not\in (X^{(\xi)}\backslash K(X))^-\}.$$

Clearly, $r_X = s_X \restriction K(X)$, whereas if $p\in X\backslash K(X)$, then $s_X(p) = \min\{\xi\colon p\not\in X^{(\xi)}\}$. In particular,

$$(3) \qquad \nu(X) = \text{l.u.b. } \{s_X(p)\colon p\in X\}.$$

2.21.1. PROPOSITION. *If X and Y are metrizable Boolean spaces, $p \in X$, and $q \in Y$, then*:
(a) $\nu(X \times Y) = \nu(X) * \nu(Y)$;
(b) $\lambda(X \times Y) = \max\{\nu(X) * \lambda(Y), \lambda(X) * \nu(Y)\}$;
(c) $n(X \times Y) = n(X)n(Y)$:
(d) $s_{X \times Y}(p, q) = s_X(p) * s_Y(q)$.

PROOF. We have already proved (a). By (1) and (2), $(p, q) \not\in ((X \times Y)^{(\xi)} \backslash K(X \times Y))^- = \bigcup_{\eta \# \zeta = \xi} (X^{(\eta)} \backslash K(X))^- \times (Y^{(\zeta)} \backslash K(Y))^-$ if and only if $\eta < s_X(p)$ and $\zeta < s_Y(q)$ implies $\eta \# \zeta < \xi$. This observation proves (d). Since $\lambda(X \times Y) = \text{l.u.b.}\ \{s_{X \times Y}(p, q)\colon (p, q) \in K(X \times Y)\}$, equation (b) follows from (2) and (3). If either X or Y is uniform, then so is $X \times Y$ by (a) and (b). Thus, if either $n(X) = -\infty$ or $n(Y) = -\infty$, then $n(X \times Y) = -\infty$, which conforms to (c) if we adopt the strange convention that $(-\infty)^2 = -\infty$. If X and Y are not uniform, then $\nu(X)$ and $\nu(Y)$ are not limit ordinals, so that $\nu(X \times Y) = \nu(X) \# \nu(Y) - 1 = \mu + 1$, where $\mu = \nu(X) - 1 \# \nu(Y) - 1$. In this case $(X \times Y)^{(\mu)} \backslash K(X \times Y) = \bigcup_{\eta \# \zeta = \mu} (X^{(\eta)} \backslash K(X)) \times (Y^{(\zeta)} \backslash K(Y)) = (X^{(\nu(X)-1)} \backslash K(X)) \times (Y^{(\nu(Y)-1)} \backslash K(Y))$ is a finite set of cardinality $n(X)n(Y)$. ☐

2.21.2. COROLLARY. *If X and Y are non-empty metrizable Boolean spaces, then $X \times Y$ is uniform if and only if either X is uniform or Y is uniform.*

Notes. A proof of equation (1) can be found in a paper of TELGARSKY [1968]. The invariant $\nu(X \times Y)$ for X and Y scattered was computed in an earlier work of MAYER and PIERCE [1960].

2.22. Semiring of isomorphism classes

If A and B are countable Boolean algebras, then $A \oplus B$ is also a countable Boolean algebra. If $A \cong A'$ and $B \cong B'$, then $A \oplus B \cong A' \oplus B'$. Therefore, the tensor product induces a multiplication on $\boldsymbol{BA}$ that is defined by

$$[A] \cdot [B] = [A \oplus B].$$

2.22.1. PROPOSITION. ***BA** is a commutative semiring with identity; that is, **BA** is a commutative monoid under addition and under multiplication, and the product distributes over the sum. Moreover, **BA** satisfies the following two conditions.*
(a) *Integrality property (I.P.): $a \cdot b = 0$ if and only if $a = 0$ or $b = 0$.*
(b) *Product decomposition property (P.D.P.): if $a \cdot b = \Sigma_{k<r} c_k$, then there are decompositions $a = \Sigma_{i<n} a_i$, $b = \Sigma_{j<m} b_j$, and a partition $n \times m = G_0 \cup \cdots \cup G_{r-1}$ such that $c_k = \Sigma_{(i,j) \in G_k} a_i \cdot b_j$. The subset **SBA** is a subsemiring of **BA**, **UBA** is a semiring ideal of $\boldsymbol{BA}^*$, and **MBA** is closed under multiplication.*

PROOF. It is convenient to view the elements of $\boldsymbol{BA}$ as homeomorphism classes of metrizable Boolean spaces. From this point of view, the operations on $\boldsymbol{BA}$ are defined by $[X]+[Y]=[X \uplus Y]$ and $[X]\cdot[Y]=[X \times Y]$. The commutative and associative laws of multiplication then reflect the fact that the topological product is commutative and associative up to homeomorphism, and distributivity comes from the obvious identity $X \times (Y \uplus Z) = (X \times Y) \uplus (X \times Z)$. The identity element for multiplication is the class of one-point spaces (or dually, the class of two-element Boolean algebras). The integrality property amounts to the observation that $X \times Y = \emptyset$ if and only if $X = \emptyset$ or $Y = \emptyset$. The topological statement of the hypothesis of the product decomposition property is: $X \times Y = Z_0 \uplus \cdots \uplus Z_{r-1}$, where each Z_k is clopen in $X \times Y$. Since $X \times Y$ has a basis that consists of rectangles $U \times V$ with U clopen in X and V clopen in Y, it follows from the compactness of Z_k that there is a partition $Z_k = (U_{k0} \times V_{k0}) \uplus \cdots \uplus (U_{ks-1} \times V_{ks-1})$ of Z_k into clopen rectangles. Choose clopen partitions $X = M_0 \uplus \cdots \uplus M_{n-1}$, $Y = N_0 \uplus \cdots \uplus N_{m-1}$ so that each U_{kl} $(k < r,\ l < s = s(k))$ is a union of a subset of $\{M_i : i < n\}$ and each V_{kl} is a union of a subset of $\{N_j : j < m\}$. It is then clear that P.D.P. is satisfied by taking $a_i = [M_i]$ and $b_j = [N_j]$. The last statement of the proposition follows from Proposition 2.21.1 $\square$

2.23. *r-semirings*

Let S be a commutative semiring with unity element 1. We will call S an *m-semiring* if $\langle S, + \rangle$ is an m-monoid and S has the integrality property of Proposition 2.22.1. If $\langle S, + \rangle$ is an r-monoid and S satisfies the integrality and product decomposition properties, then S is called an *r-semiring*. Most of Proposition 2.22.1 is summarized by the statement that $\boldsymbol{BA}$ is an r-semiring. As usual, the product of two elements in a semiring will be denoted by concatenation: we write ab instead of $a \cdot b$.

It can be shown that if S is an m-semiring with the refinement property, then S is an r-semiring if and only if S has the 2-P.D.P.: if $ab = c_0 + c_1$ in S, then $a = \Sigma_{i<n} a_i$, $b = \Sigma_{j<m} b_j$, and there is a partition $n \times m = G_0 \uplus G_1$ such that $c_k = \bigcup_{(i,j)\in G_k} a_i b_j$ for $k = 0, 1$.

The proof is a straightforward induction that uses 2-P.D.P. and several applications of the refinement property. We omit the argument because the result will not be used in this chapter.

Our objective in this subsection is to sharpen Dobbertin's monoid characterization of $\boldsymbol{BA}$ to a semiring characterization. The desired conclusion follows directly from Theorem 2.4.2, Proposition 2.22.1, and the following result.

2.23.1. PROPOSITION. *Let S and T be r-semirings such that $\langle T, + \rangle$ is V-simple. If Θ: $S \to T$ is a V-morphism of additive monoids, then Θ is a semiring homomorphism.*

PROOF. Define $R \subseteq T \times T$ to consist of all pairs

$$\left(\sum_{i<r} \Theta(a_i)\Theta(b_i),\ \sum_{i<r} \Theta(a_i b_i) \right),$$

where a_i and b_i range over S. Since T is V-simple, the proof can be finished by showing that R is a V-relation. Indeed, since $(\Theta(a)\Theta(b),\ \Theta(ab)) \in R$ for all a, $b \in S$, it will then follow that $\Theta(ab) = \Theta(a)\Theta(b)$. Let $R_0 \subseteq T \times S$ consist of all pairs $(\Sigma_{i<n} \Theta(a_i)\Theta(b_i),\ \Sigma_{i<n} a_i b_i)$. Plainly, $R = R_0 \circ \Theta$. Therefore, since Θ is a V-relation, it is sufficient to show that R_0 is a V-relation. It follows from I.P. and the fact that S is an m-monoid that $(e, 0) \in R_0$ implies $e = 0$. Using the added assumption that Θ is a morphism of m-monoids, we see that $(0, d) \in R_0$ implies $d = 0$. Let $(\Sigma_{i<r} \Theta(a_i)\Theta(b_i),\ \Sigma_{i<r} a_i b_i) \in R_0$, and suppose that $\Sigma_{i<r} \Theta(a_i)\Theta(b_i) = c_0 + c_1$. The refinement property gives $\Theta(a_i)\Theta(b_i) = d_{i0} + d_{i1}$, where $c_p = \Sigma_{i<r} d_{ip}$ for $p < 2$. Use P.D.P. to obtain $\Theta(a_i) = \Sigma_{k<n} e_{ik}$, $\Theta(b_i) = \Sigma_{l<m} f_{il}$, $d_{ip} = \Sigma_{(k,l)\in G_p} e_{ik} f_{il}$, where $n \times m = G_0 \uplus G_1$. Since Θ is a V-morphism, there exist elements $a_{ik}, b_{il} \in S$ such that $e_{ik} = \Theta(a_{ik})$, $f_{il} = \Theta(b_{il})$ for all i, k, l, and $a_i = \Sigma_{k<n} a_{ik}$, $b_i = \Sigma_{l<m} b_{il}$. It follows that $c_p = \Sigma_{i<r} \Sigma_{(k,l)\in G_p} \Theta(a_{ik})\Theta(b_{il})$, $p = 0, 1$, $\Sigma_{p<2} (\Sigma_{i<r} \Sigma_{(k,l)\in G_p} a_{ik} b_{il}) = \Sigma_{i<r} a_i b_i$, and $(\Sigma_{i<r} \Sigma_{(k,l)\in G_p} \Theta(a_{ik})\Theta(b_{il}),\ \Sigma_{i<r} \Sigma_{(k,l)\in G_p} a_{ik} b_{il}) \in R_0$. This argument shows that the conditions

$$(c_0 + c_1, e) \in R_0 \text{ implies } e = e_0 + e_1 \text{ with } (c_0, e_0),\ (c_1, e_1) \in R_0$$

is satisfied. A similar proof yields the symmetrical condition. Thus, R_0 is a V-relation. □

2.23.2. COROLLARY. (a) ***BA*** *is a locally countable, V-simple r-semiring.*

(b) *If S is any locally countable, r-semiring, then there is a unique V-morphism of semirings Θ: $S \to$* ***BA***; *moreover,* $\operatorname{Ker} \theta = Y(M)$ *and* $\operatorname{Im} \Theta$ *is a hereditary submonoid of* ***BA***.

(c) ***BA*** *is determined to within semiring isomorphism by the properties* (a) *and* (b).

It is a consequence of (b) that if S is a locally countable, r-semiring, then the V-radical $Y(S)$ is a semiring congruence. In fact, it can be shown that the same result holds without the assumption that S is locally countable.

Notes. The results in this subsection are due to DOBBERTIN [1982].

2.24. *Natural ordering of r-semirings*

The natural order of a semiring refers to the natural ordering of the underlying additive monoid: $a \le b$ if $b = a + c$ for some c. As usual, we will write $a < b$ if $a \le b$ and $a \ne b$. In the absence of antisymmetry, it is possible that $a < b$ and $b < a$.

2.24.1. LEMMA. *In a semiring S, multiplication is monotone; that is, $a \le b$ and $c \le d$ implies $ac \le bd$.*

This statement is clear from the definition of the natural ordering.

If S is an m-semiring, then $0 \le a$ for all a. Moreover, if $0 < a$ and $0 < b$, then $0 < ab$. Since 0 has the S-B. property, $0 < a$ excludes $a < 0$.

The following property of elements in a monoid will be very important. An element a in the commutative monoid M is *pseudo-indecomposable* (in M) if $a = b + c$ implies $a = b$ or $a = c$. We will often abbreviate the expression "pseudo-indecomposable" by PI. By an obvious induction, if a is PI and $a = \Sigma_{i<n} b_i$, then there exists $i < n$ such that $a = b_i$.

There is a weaker property than pseudo-indecomposability that is sometimes useful. An element a in the commutative monoid M is *prime* if $a = b + c$ implies that $a \leq b$ or $a \leq c$.

2.24.2. LEMMA. *Let M be a commutative monoid, $a \in M$.*

(a) *If a is PI, then a is prime.*

(b) *If a is prime and has the S-B. property, then a is PI.*

These statements are obvious from the definitions of PI and prime elements.

2.24.3. PROPOSITION. *If M is a V-simple, r-monoid, then every prime element of M has the S-B. property. In particular, the prime elements of M are the same as the PI elements of M.*

PROOF. If a is prime, $a \leq b$ and $b \leq a$, then b is also prime. Indeed, $a \leq b = b_0 + b_1$ implies $a = a_0 + a_1$ with $a_0 \leq b_0$, $a_1 \leq b_1$ by Lemma 2.2.2. Thus, $b \leq a \leq a_0 \leq b_0$ or $b \leq a \leq a_1 \leq b_1$ since a is prime. To prove the proposition, it therefore suffices to show that

$$R = \{(a, b) \in M \times M\colon a = b, \text{ or } a \text{ and } b \text{ are prime and } a \leq b,\ b \leq a\}$$

is a V-relation. Plainly, R is an equivalence relation on M, and $aR0$ implies $a = 0$, because 0 has the S-B. property in an m-monoid. Suppose that $aRb = b_0 + b_1$. It can be assumed that a and b are prime, $a \leq b$, and $a = b + c$. In this case, $b \leq b_0$ and $b_0 \leq b$, or $b \leq b_1$ and $b_1 \leq b$; say $b \leq b_0$ and $b_0 \leq b$. Thus, b_0 is prime. Moreover, $b_0 + c \leq b + c = a \leq b \leq b_0 \leq b_0 + c$ implies that $b_0 + c$ is also prime. Since $a = b + c = (b_0 + c) + b_1$ with $b_0 + cRb_0$ and $b_1 R b_1$, it follows that R is a V-relation. □

The results in this proposition apply to ***BA***, of course. In particular, every PI (or prime) element of ***BA*** has the Schröder–Bernstein property.

*2.25. Essential elements in **BA***

In this section we describe some simple arithmetical properties of ***BA***. Some of the results are valid (with suitable definitions) in the general setting of r-semirings.

As one might expect, the element $[\mathscr{F}]$ plays a central role in the study of ***BA***. To simplify notation, this element will be designated by f.

2.25.1. LEMMA. *Let $a \in$ **BA**.*

(i) *If $0 \neq a$, then $af = f$.*

(ii) $f + f = f$.

(iii) *If* $0 \neq a \leq f$, *then* $a = f$; *in particular*, f *is PI.*
(iv) *If* $0 \neq a \neq f$, *then* $1 \leq a$.

Implication (i) follows from Proposition 2.21.1; the remaining statements are variations of the fact that every non-empty, perfect, metrizable Boolean space is homeomorphic to the Cantor set. □

A Boolean algebra A is atomic if $\mathrm{At}\, A$ is dense in A. We will transfer this terminology to $\boldsymbol{BA}$: $a = [A]$ is atomic if A is an atomic Boolean algebra. By a legalistic reading of this definition, we see that 0 is atomic.

It is useful to note that A is atomic if and only if the isolated points in $\mathrm{Ult}\, A$ are dense; equivalently, no non-empty clopen subset of $\mathrm{Ult}\, A$ is perfect. This observation more or less proves the first three statements of our next result.

2.25.2. LEMMA. *Let a and b be elements of $\boldsymbol{BA}$.*
(i) *The following conditions are equivalent*: a *is atomic*; $f \not\leq a$; $a \neq f + a$.
(ii) *If* $a \leq b$ *and* b *is atomic, then* a *is atomic.*
(iii) *If* a *and* b *are atomic, then* ab *is atomic.*
(iv) *If* a *and* b *are atomic and* $a + f = b + f$, *then* $a = b$.

PROOF. (iv) The refinement property produces $c_{ij} \in \boldsymbol{BA}$ such that $a = c_{00} + c_{01}$, $b = c_{00} + c_{10}$, and $f = c_{10} + c_{11} = c_{01} + c_{11}$. Since a and b are atomic, it follows that $c_{01} = c_{10} = 0$. Hence, $a = c_{00} = b$. □

An element $a \in \boldsymbol{BA}$ is *inessential* if $a = b + f$, where b is atomic; otherwise, a is called *essential*. In particular, f is inessential, while every atomic element is essential.

2.25.3. LEMMA. $a \in \boldsymbol{BA}$ *is essential if and only if* $a = b + f$ *implies* $a = b$.

PROOF. If a is inessential, then $a = b + f$, where b is atomic. Since $f \leq a$ implies a is not atomic, it follows that $a \neq b$. Conversely, if we can write $a = b + f$ with $b \neq a$, then b is atomic by 2.25.2. Thus, a is inessential. □

2.25.4. PROPOSITION. *If a and b are essential elements of $\boldsymbol{BA}$ such that* $ab = c_0 + c_1$, *then* $a \leq c_0$ *or* $b \leq c_1$.

PROOF. By the P.D.P. in $\boldsymbol{BA}$, we can write $a = \Sigma_{i<n} a_i$, $b = \Sigma_{j<m} b_j$, $n \times m = G_0 \cup G_1$, $c_p = \Sigma_{(i,j) \in G_p} a_i b_j$, $p = 0, 1$. It can be assumed that $a_i \neq 0, f$ and $b_j \neq 0, f$ for $i < r$, $j < s$ (where $r \leq n$, $s \leq m$), and a_i, $b_j = 0$ or f, for $i \geq r$, $j \geq s$. Since a and b are essential, it follows from 2.25.3 that $a = \Sigma_{i<r} a_i$ and $b = \Sigma_{j<s} b_j$. Moreover, if $j < s$, then $1 \leq b_j$, so that $a_i \leq a_i b_j$ for all $i < n$. Similarly, $i < r$, implies $b_j \leq a_i b_j$ for all $j < m$. If, for all $i < r$, there exists $j < s$ such that $(i, j) \in G_0$, then $a = \Sigma_{i<r} a_i \leq \Sigma_{(i,j) \in G_0} a_i b_j = c_0$. Otherwise, there exists $i < r$ such that $(i, j) \in G_1$ for all $j < s$. In this case, $b \leq c_1$. □

2.25.5. COROLLARY. *If a is essential in **BA** and $a^2 \leq a$, then a is pseudo-indecomposable, $a^2 = a$, and a has the Schröder–Bernstein property.*

The corollary follows from the proposition and Proposition 2.24.3.

2.26. Trnková's Theorem

After Ketonen's discovery that every countable, commutative semigroup can be embedded in ***BA***, it was generally assumed that the arithmetical structure of this semiring is completely intractable. It was a major surprise when Trnková showed that the multiplicative analog of the cube problem has a positive solution in ***BA***.

2.26.1. LEMMA. *If a, $b \in$ **BA** are essential, then ab is essential.*

PROOF. It can be assumed that $a \neq 0$ and $b \neq 0$, so that $a \not\leq f$ and $b \not\leq f$ by Lemma 2.25.1. Suppose that $ab = c + f$, where c is atomic. It then follows from Proposition 2.25.4 that $a \leq c$ and $b \leq c$. Consequently, a and b are atomic. Thus, ab is atomic by Lemma 2.25.2, which is inconsistent with $ab = c + f$. □

2.26.2. THEOREM. *If $a \in$ **BA** satisfies $a^m = a^n$ with $m < n$, then $a^m = a^{m+1}$.*

PROOF. If $a = 0$, then there is nothing to show. Assume that a is essential and $a \neq 0$. In particular, $1 \leq a$ by Lemma 2.25.1. The equation $a^m = a^n = a^{m+(n-m)}$ implies $a^{m+k(n-m)} = a^m$ for all $k \geq 0$. Thus, $(a^m)^2 \leq a^m$. By the lemma and Corollary 2.25.5, a^m has the S-B. property. Since $a^m \leq a^{m+1} \leq a^n = a^m$, we conclude that $a^m = a^{m+1}$. If a is inessential, say $a = b + f$ with b atomic, then $b^m + f = a^m = a^n = b^n + f$ implies $b^m = b^n$ by Lemma 2.25.2. Since atomic elements are essential, it follows that $b^m = b^{m+1}$. Therefore, $a^m = a^{m+1}$. □

Notes. The results in Propositions 2.24.3, 2.25.4, and Theorem 2.26.2 are due to TRNKOVÁ [1980]. Our exposition of this material follows DOBBERTIN [1982].

It is interesting that Trnková's Theorem fails in the semiring S of isomorphism types of Boolean algebras of cardinality $\leq 2^{\aleph_0}$. In fact, TRNKOVÁ and KOUBEK [1977] have shown that every finite abelian group can be embedded in the multiplicative semigroup of S.

3. Special classes of algebras

3.1. Pseudo-indecomposable Boolean algebras

The concept of a pseudo-indecomposable element in an m-monoid that was introduced in subsection 2.24 has a natural counterpart in the realm of Boolean algebras. A Boolean algebra A is *pseudo-indecomposable* (or PI for short) if the isomorphism class $[A]$ is a pseudo-indecomposable element of the monoid ***BA***. This means that if $A = B \times C$, then $A \cong B$ or $A \cong C$.

The zero algebra and the two element algebra are clearly pseudo-indecomposable. Indeed, these are the unique Boolean algebras that are indecomposable rings. The infinite free algebra $\mathcal{F}$ is PI by Lemma 2.25.1. By the Uniqueness and Existence Theorems (1.10.1 and 1.10.2) and Proposition 1.6.1, a superatomic Boolean algebra A is PI if and only if $n(A)=1$. Countable Boolean algebras of mixed type cannot be pseudo-indecomposable. In fact, if A is of mixed type, then A is neither superatomic nor uniform, but A is a product of a superatomic algebra with a uniform algebra. As we will see, there are many examples of uniform algebras that are PI. These can be characterized in terms of their Ketonen invariants. Indeed, it is not difficult to see that if σ is the measure associated with the uniform algebra A, and if the depth of σ is ζ, then A is pseudo-indecomposable if and only if $(a_0, \ldots, a_{n-1}) \in (\Delta^{\zeta+1}\sigma)(1)$ implies $T((\Delta^{\zeta+1}\sigma)(1)) = a_i$ for some $i<n$.

Pseudo-indecomposability can also be formulated in the language of Boolean spaces: a metrizable Boolean space X is PI if, for every clopen subset W of X, either $W \simeq X$ or $X \backslash W \simeq X$.

3.1.1. PROPOSITION. *If X is a metrizable Boolean space, then X is PI if and only if there is a point $p \in X$ such that if W is a clopen neighborhood of p, then $W \simeq X$.*

PROOF. Assume that X is PI. By Proposition 2.24.3, $[X]$ has the S-B. property in ***BA***. Let $\{W_k\colon k<\omega\}$ be an enumeration of the clopen subsets of X. Using the assumption that X is PI, construct a sequence $\{V_k\colon k<\omega\}$ of clopen subsets of X such that: $X = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k \supseteq \cdots$; $V_k \simeq X$ for all $k<\omega$; and $V_k \subseteq W_k$ or $V_k \subseteq X \backslash W_k$ for all k. It follows from these properties and the compactness of X that $\bigcap_{k<\omega} V_k = \{p\}$ for some $p \in X$. If W is a clopen neighborhood of p, then $V_k \subseteq W \subseteq X$ for some k. Thus, $[X]=[V_k] \leq [W] \leq [X]$. By the S-B. property, $W \simeq X$. The converse implication is clear. □

A point $p \in X$ such that $W \simeq X$ for all clopen neighborhoods W of p will be called a *point of homogeneity* for X. If q is another point of homogeneity for X and W is a clopen neighborhood of p such that $q \notin W$, then $W \simeq X \simeq X \backslash W$. Conversely, if such a clopen set W exists in X, then by the proposition there are two points of homogeneity for X (in fact, infinitely many).

3.1.2. COROLLARY. *If X is the Stone space of a pseudo-indecomposable Boolean algebra A, then $A \cong A \times A$ if and only if there are at least two points of homogeneity for X.*

3.2. Primitive Boolean algebras

A countable Boolean algebra A is *primitive* if every $e \in A$ admits a decomposition $e = f_0 \dot{+} \cdots \dot{+} f_{n-1}$ such that $A \restriction f_i$ is pseudo-indecomposable for all $i<n$. If X is the Stone space of a primitive Boolean algebra, then every clopen subset W of X is a disjoint union of PI clopen subsets. In particular, the PI clopen subsets of X form a basis for the topology. The converse is true, but less obvious.

3.2.1. PROPOSITION. *If X is the Stone space of a countable Boolean algebra A, then A is primitive if and only if the PI clopen subsets of X form a basis for the topology.*

PROOF. Assume that the PI clopen subsets of X form a basis. If W is clopen in X, then by compactness, $W = W_0 \cup \cdots \cup W_{n-1}$ with each W_i a PI clopen subset of X. We can assume that n is minimal. For each i, let $p_i \in W_i$ be a homogeneity point for W_i. It will be sufficient to show that $p_i \not\in W_j$ if $i \neq j$. Indeed, it will then follow that $W = V_0 \cup V_1 \cup \cdots \cup V_{n-1}$, where $V_0 = W_0$, $V_1 = W_1 \backslash W_0$, $V_2 = W_2 \backslash (W_1 \cup W_1), \ldots$, and $V_{n-1} = W_{n-1} \backslash (\bigcup_{i<n-1} W_i)$ are PI, because $p_i \in V_i \subseteq W_i$ for all $i < n$. Suppose that $p_i \in W_j$, $j \neq i$. Since p_i is a point of homogeneity for W_i, it follows that (in $\boldsymbol{BA}$) $[W_i \cap W_j] = [W_i]$, and $[W_j] = [W_j \backslash (W_i \cap W_j)] + [W_i \cap W_j] = [W_j \backslash (W_i \cap W_j)] + [W_i] = [W_i \cup W_j]$. Thus, $W_i \cup W_j$ is PI, contrary to the minimality of n. □

We will call a point $p \in X$ *homogeneous* if there is a clopen neighborhood W of p such that p is a point of homogeneity for W. In this case, W is called a *uniform neighborhood* of p. Necessarily, W is PI. Plainly, isolated points are homogeneous; so are points with a perfect neighborhood. Thus, the homogeneous points are dense in any metrizable Boolean space. If every $p \in X$ is homogeneous, then Clop X is clearly primitive, but the converse is not true, as we will see in 3.11.2 and 3.12.3.

Notes. Primitive Boolean algebras were introduced by HANF [1974]. However, Hanf imposed the extra restriction that primitive algebras must be pseudo-indecomposable. The algebras that we will call "primitive" are called "quasi-primitive" by Hanf.

3.3. Isomorphism classes of primitive algebras

The set of all isomorphism classes of primitive Boolean algebras will be denoted by $\boldsymbol{P}$. It follows from Theorems 1.10.1 and 1.11.1 and Proposition 1.6.1 that every countable, superatomic Boolean algebra is primitive. Thus, $\boldsymbol{SBA} \subseteq \boldsymbol{P}$. In particular, $0, 1 \in \boldsymbol{P}$. It is also clear that $f \in \boldsymbol{P}$. These examples suggest that $\boldsymbol{P}$ is a fairly large subset of $\boldsymbol{BA}$. On the other hand, we will see that $\boldsymbol{P}$ has nicer arithmetical properties than $\boldsymbol{BA}$.

If M is an m-monoid, then the set of all non-zero PI elements of M will be denoted by $\mathscr{D}(M)$.

3.3.1. LEMMA. *If M is an r-monoid and every element of $\mathscr{D}(M)$ has the S-B. property, then every sum of elements in $\mathscr{D}(M)$ has the S-B. property.*

PROOF. Let $a = \Sigma_{i<n} e_i$ with all $e_i \in \mathscr{D}(M)$. We use induction on n to prove that if $a \leq b$ and $b \leq a$ for some $b \in M$, then $a = b$. The case $n = 1$ is covered by the assumption that the elements in $\mathscr{D}(M)$ have the S-B. property. By the refinement property, $b \leq \Sigma_{i<n} e_i$ yields

(1) $$b = \sum_{i<n} c_i,\ c_i \leq e_i \text{ for all } i < n .$$

Thus, $e_i = c_i + d_i$ with $d_i \in M$. Since e_i is PI, either $e_i = c_i$ or $e_i = d_i$. By modifying the indexing, it can be assumed that

$$(2) \qquad c_i = e_i \quad \text{for } i < r \quad \text{and} \quad c_j < e_j = e_j + c_j \quad \text{for } r \leq j < n .$$

If $j \geq r$, then since $e_j \leq a \leq b = \Sigma_{i<n} c_i$, there exists $k < n$ such that

$$(3) \qquad e_j \leq c_k \leq e_k .$$

In fact, by the refinement property, $e_j = \Sigma_{i<n} f_i$ with $f_i \leq c_i$; hence, $e_j = f_k \leq c_k$ for some k, since e_j is PI. If $k = j$ in (3), then $c_j = c_k = e_j$ because e_j has the S-B. property. Since this is impossible for $j \geq r$, it follows that $k \neq j$. The inequality $e_j \leq e_k$ implies that either $e_j = e_k$ or $e_j + e_k = e_k$. In the second case, $a = \Sigma_{i \neq j} e_i$, so that a has the S-B. property (and $a = b$) by the induction hypothesis. If $e_j = e_k$, then it follows from (3) that $c_k = e_k$. Thus, $k < r$. Moreover, by (2), $c_j + c_k = c_j + e_j = e_j = c_k$. Since $j \geq r$ was arbitrary, $b = \Sigma_{i<n} c_i = \Sigma_{i<r} c_i = \Sigma_{i<r} e_i$. If $r = n$, then $b = a$. If $r < n$, then b has the S-B. property by induction, and $b = a$ in this case also. □

3.3.2. LEMMA. *If S is an r-semiring such that every element of $\mathscr{D}(S)$ has the S-B. property, then $\mathscr{D}(S)$ is closed under multiplication.*

PROOF. We must prove that if $e, f \in \mathscr{D}(S)$ and $a, b \in S$ satisfy $ef = a + b$, then $a = ef$ or $b = ef$. By P.D.P., $e = \Sigma_{i<n} c_i$, $f = \Sigma_{j<m} d_j$, $n \times m = G_0 \cup G_1$, $a = \Sigma_{(i,j) \in G_0} c_i d_j$, and $b = \Sigma_{(i,j) \in G_1} c_i d_j$. Since e and f are PI, it can be assumed that $c_i = e$ for $i < r$, $c_i < e$ for $r \leq i < n$, $d_j = f$ for $j < s$, and $d_j < f$ for $s \leq j < m$, where $r \geq 1$ and $s \geq 1$. Moreover, with properly chosen notation, $(0, 0) \in G_0$. If $r \leq i < n$, then $e + c_i = e$ and $ef + c_i f = ef$. Similarly, $ef + ed_j = ef$ if $s \leq j < n$. Consequently, $ef = (e + c_i)(f + d_j) = ef + c_i d_j$ if $i \geq r$ and $j \geq s$. Thus, since $(0, 0) \in G_0$, $a = kef$, where k is an integer that satisfies $1 \leq k \leq rs$. If $k = 1$, then this equation is the desired result. If, say, $r > 1$, we have $2e \leq e \leq 2e$, so that $e = 2e$ because $e \in \mathscr{D}(S)$ has the S-B. property. In this case, $ef = 2ef = \cdots = kef = a$. □

3.3.3. THEOREM. (a) ***P*** *is a hereditary subsemiring of* ***BA***.
(b) *The natural ordering of* ***P*** *is a partial order.*
(c) *$\mathscr{D}(\boldsymbol{P})$ is a submonoid of the multiplicative monoid of* ***BA***.

PROOF. The refinement property implies that $M \restriction (a + b) \subseteq M \restriction a + M \restriction b$ for any r-monoid M. The definition of primitivity guarantees that if $a \leq b \in \boldsymbol{P}$, then $a \in \boldsymbol{P}$. Thus, $\boldsymbol{P}$ is a hereditary submonoid of $\boldsymbol{BA}$. By Proposition 2.24.3, the elements of $\mathscr{D}(\boldsymbol{P})$ have the S-B. property. Since $\mathscr{D}(P)$ generates $\mathscr{P}$ as a monoid, it follows from 3.3.1 that all elements in $\boldsymbol{P}$ have the S-B. property. Moreover, $\mathscr{D}(\boldsymbol{P})$ is closed under multiplication by 3.3.2. Finally, since 1 is indecomposable, it is clear that $1 \in \mathscr{D}(\boldsymbol{P}) \subseteq \boldsymbol{P}$. □

3.3.4. COROLLARY. ***P*** *is a V-simple, r-semiring that is additively generated by $\mathscr{D}(\boldsymbol{P})$.*

Indeed, we noted at the end of subsection 2.3 that a hereditary submonoid of a V-simple r-monoid is V-simple.

3.4. *Primitive monoids*

Our objective is to get a structure theorem for $\boldsymbol{P}$. Most of this program can be accomplished for suitably restricted m-monoids. The appropriate setting is captured in the next definition.

3.4.1. DEFINITION. A *primitive monoid* is an r-monoid M such that
(i) the natural ordering of M is a partial order, and
(ii) M is generated (as a monoid) by the set $\mathscr{D}(M)$ of all non-zero elements that are pseudo-indecomposable in M.

The monoid $\boldsymbol{P}$ of isomorphism classes of primitive Boolean algebras is the prototype of primitive monoids. In fact, any hereditary submonoid of $\boldsymbol{P}$ is also a primitive monoid.

It is convenient to record a couple of simple consequences of our definitions and Lemma 2.2.2.

3.4.2. LEMMA. *Let M be a primitive monoid, and assume that $e \in \mathscr{D}(M)$.*
(a) *If $e \leq \Sigma_{i<n} a_i$, then $e \leq a_i$ for some $i \leq n$.*
(b) *If $a \leq e$, then either $a = e$ or $a + e = e$.*

The natural ordering of M restricts to a partial ordering of $\mathscr{D}(M)$. A refinement of this ordering will provide more information about the structure of $\mathscr{D}(M)$.
For elements $e, f \in \mathscr{D}(M)$, write

$$e \lhd f \quad \text{if and only if } e + f = f .$$

It follows from 3.4.2 that $e \lhd f$ if and only if either $e < f$, or $e = f$ and $e + e = e$. In particular $\lhd$ is transitive and antisymmetric, but generally not reflexive. Indeed, the set of elements of $\mathscr{D}(M)$ on which $\lhd$ is reflexive plays an important role in the theory of primitive monoids. Denote this set by $\mathscr{D}_1(M)$; that is, $\mathscr{D}_1(M) = \{e \in \mathscr{D}(M)\colon e + e = e\}$. The advantage that $\lhd$ has over $\leq$ is that it encodes $\mathscr{D}_1(M)$, whereas $\mathscr{D}_1(M)$ cannot be recovered from the knowledge of $\leq$.

3.4.3. LEMMA. *Let M be a primitive monoid. Suppose that $e \in \mathscr{D}(M)$ and $f_k \in \mathscr{D}(M)$ for $k < r$ satisfy $ne \leq me + \Sigma_{k<r} f_k$ with $n > m \geq 0$ and $e \not\leq f_k \not\leq e$ for all $k < r$. Then $e \in \mathscr{D}_1(M)$.*

PROOF. The refinement property yields $ne = \Sigma_{j<m} a_i + \Sigma_{k<r} b_k$ with all $a_i \leq e$, and $b_k \leq f_k$ for $k < r$. By 3.4.2, $f_k \not\leq e$ implies $f_k \not\leq ne$, so that $b_k \neq f_k$. Hence, $b_k + f_k = f_k$. Consequently, $ne + \Sigma_{k<r} f_k = \Sigma_{j<m} a_k + \Sigma_{k<r} f_k \leq me + \Sigma_{k<r} f_k \leq (m+1)e + \Sigma_{k<r} f_k \leq ne + \Sigma_{k<r} f_k$. By the S-B. property, $e + me + \Sigma_{k<r} f_k = me + \Sigma_{k<r} f_k$. Since $e \not\leq f_k$, it follows from 3.4.2 that $m \geq 1$. We can assume by way of induction that either $m = 1$ or the result of the lemma holds when m is

replaced by $m-1$. To simplify notation, denote $(m-1)e + \Sigma_{k<r} f_k$ by c. Thus, $e+e+c = e+c$. The proof can be completed by showing that either $e+e=e$ or $m>1$ and $e+c=c$. By the refinement property, $e = e_0 + e_1 + e_2$, $c = c_0 + c_1 + c_2$, $e = e_0 + c_0 = e_1 + c_1$, and $c = e_2 + c_2$. Since e is PI, at least one of e_0, e_1, e_2 is e. If two of these elements are equal to e, then $e \le e+e \le e$, so that $e+e=e$. If $e_2 = e \ne e_0$, e_1, then $c_0 = c_1 = e$, and $c_2 + e = c = e + e + c_2$. In this case, $e + c = c$, which implies $m - 1 \ge 1$ as we noted before. Assume that $e_0 = e \ne e_1$, e_2. Then $c_1 = e$, so that $c = c_0 + e + c_2 = e + c_2$. Also, $e_2 < e$ implies $e = e + e_2$. Thus, $c = e + c_2 = e + e_2 + c_2 = e + c$. Again, $m > 1$. The case in which $e_1 = e \ne e_0$, e_2 is similar. □

3.4.4. PROPOSITION. *If M is a primitive monoid, then every $a \in M$ has a unique representation*

$$a = \sum_{k<r} e_k ,$$

in which $e_k \in \mathscr{D}(M)$ for all $k<r$, and $e_k \not\mathrel{\triangleleft} e_l$ for $k \ne l$.

PROOF. Since $\mathscr{D}(M)$ generates M, there is a representation $a = \Sigma_{i<n} e_i$ with $e_i \in \mathscr{D}(M)$. If the e_i are listed so that $\{e_0, \ldots, e_{r-1}\}$ are maximal in the set $\{e_0, \ldots, e_{n-1}\}$ relative to $\triangleleft$, then $a = \Sigma_{k<r} e_k$, since $e_i \triangleleft e_k$ implies $e_i + e_k = e_k$. Suppose that $\Sigma_{k<r} e_k = \Sigma_{j<m} f_j$ with $f_j \in \mathscr{D}(M)$ and $f_i \not\mathrel{\triangleleft} f_j$ for $i \ne j$. By 3.3.2, there are mappings $\kappa\colon r \to m$ and $\lambda\colon m \to r$ such that $e_k \le f_{\kappa(k)}$, $f_j \le e_{\lambda(j)}$ for all $k<r$ and $j<m$. Thus, $e_k \le f_{\kappa(k)} \le e_{\lambda(\kappa(k))}$ for $k<r$. Hence, $e_k = f_{\kappa(k)} = e_{\lambda(\kappa(k))}$ by the S-B. property and the assumption that $e_k \not\mathrel{\triangleleft} e_l$ for $k \ne l$. Similarly, $f_j = e_{\lambda(j)} = f_{\kappa(\lambda(j))}$. Thus, with suitable rearrangements, $\Sigma_{k<r} e_k = \Sigma_{i<n} m_i g_i$ and $\Sigma_{j<m} f_j = \Sigma_{i<n} m_i' g_i$, where $\{g_0, \ldots, g_{n-1}\} = \{e_0, \ldots, e_{r-1}\} = \{f_0, \ldots, f_{m-1}\}$, and the m_i, m_i' are natural numbers such that $m_i = m_i' = 1$ if $g_i \in \mathscr{D}_1(M)$, and if $g_i \not\in \mathscr{D}_1(M)$, then m_i and m_i' are, respectively, the number of occurrences of g_i in the sequences $(e_0, \ldots, e_{r-1})$ and $(f_0, \ldots, f_{m-1})$. By 3.4.3, $m_i = m_i'$ for all $i<n$, so that the sequence of f's is a rearrangement of the sequence of e's. □

A representation $a = \Sigma_{k<r} e_k$ satisfying the conditions of the proposition is called the *reduced* representation of a.

3.5. Construction of primitive monoids

The next step of our program is to show that the correspondence between a primitive monoid M and the relational system $\langle \mathscr{D}(M), \triangleleft \rangle$ is bijective (up to isomorphism). We will do so by constructing a (categorical) inverse of the mapping $M \mapsto \langle \mathscr{D}(M), \triangleleft \rangle$.

Let $\triangleleft$ be a transitive, antisymmetric relation on the non-empty set D. Denote the free, commutative monoid on D by $F(D)$. As usual, the elements of $F(D)$ will be viewed as formal sums of elements from D: $x = \Sigma_{k<r} n_k e_k$, where $n_k < \omega$ and the $e_k \in D$ are distinct. The addition in $F(D)$ is componentwise, so that D is identified with a subset of $F(D)\backslash\{0\}$. Let $\equiv$ be the monoid congruence that is generated by the relations

$$e+f\equiv f \quad \text{if } e \lhd f .$$

Denote the quotient monoid $F(D)/\equiv$ by $\mathcal{M}(D)$, and let Π: $F(D)\to\mathcal{M}(D)$ be the natural projection homomorphism. An obvious remark is that if D and D' are isomorphic relational systems, then $\mathcal{M}(D)\cong\mathcal{M}(D')$.

A characterization of $\equiv$ is needed. For $x=\Sigma_{k<r} n_k e_k \in F(D)$ let $P(x)=\Sigma_{i\in J} m_i e_i$, where $J=\{i<r$: e_i is maximal among $e_0,\dots,e_{r-1}$ relative to $\lhd\}$, and $m_i=1$ if $e_i \lhd e_i$, $m_i=n_i$ if $e_i \not\lhd e_i$. It is clear from the definition of $\equiv$ that if $e_j \lhd e_i$ for some $i\neq j$, then $\Sigma_{k<r} n_k e_k \equiv \Sigma_{k<r,k\neq j} n_k e_k$. Moreover, if $e_j \lhd e_j$, then $\Sigma_{k<r} n_k e_k \equiv e_j + \Sigma_{k\neq j} n_k e_k$. Consequently, $x\equiv P(x)$ for all $x\in F(D)$.

3.5.1. LEMMA. *For x, $y\in F(D)$, $x\equiv y$ if and only if $P(x)=P(y)$.*

PROOF. If $P(x)=P(y)$, then $x\equiv y$ since $x\equiv P(x)$ and $y\equiv P(y)$. Moreover, if $e\lhd f$, then $P(e+f)=P(f)$. Thus, to prove the lemma, it is sufficient to show that $\{(x, y)$: $P(x)=P(y)\}$ is a monoid congruence on $F(D)$. By the definition of P, $P(x+z)=P(P(x)+P(z))$. In particular, if $P(x)=P(y)$ and $P(z)=P(w)$, then $P(x+z)=P(y+w)$. □

3.5.2. PROPOSITION. (a) *Let $\lhd$ be a transitive, antisymmetric relation on the non-empty set D. Then $\mathcal{M}(D)$ is a primitive monoid and Π is a bijective mapping of D to $\mathcal{D}(\mathcal{M}(D))$ such that $e\lhd f$ if and only if $\Pi e+\Pi f=\Pi f$.*

(b) *If M is a primitive monoid, then $M\cong\mathcal{M}(\mathcal{D}(M))$.*

PROOF. (a) To simplify notation, we will write $\bar{x}$ for Πx. It is a consequence of the lemma that $\mathcal{M}(D)$ is an m-monoid and every non-zero element of $\mathcal{M}(D)$ has a unique "reduced" representation: $\bar{x}=\Sigma_{k<r}\bar{e}_k$, $e_k\in D$, $e_k\not\lhd e_l$ for $k\neq l$. Hence, Π maps D bijectively to $\mathcal{D}(\mathcal{M}(D))$. Moreover, $\bar{e}\lhd\bar{f}$ if and only if $P(e+f)=P(f)$; that is $e\lhd f$. We next note that the elements of $\mathcal{D}(\mathcal{M}(D))$ have the S-B. property in $\mathcal{M}(D)$. Indeed, suppose that $e\in D$, x, $y\in F(D)$, and $\bar{e}+\bar{x}+\bar{y}=\bar{e}$. By the lemma, $P(e+x+y)=P(e)=e$. Thus, if $x=\Sigma_{i<r} n_i f_i$, $y=\Sigma_{j<s} m_j g_j$, then $f_i\lhd e$ for $i<r$ and $g_j\lhd e$ for $j<s$. Hence, $P(e+x)=e$ and $\bar{e}+\bar{x}=\bar{e}$, which establishes the S-B. property. By Lemma 3.3.1, the proof of part (a) can be completed by showing that $\mathcal{M}(D)$ has the refinement property. It suffices to prove that if $\bar{x}_0+\bar{x}_1=\bar{y}_0+\bar{y}_1$ in $\mathcal{M}(D)$, then there exist $\bar{z}_{ij}$, $i,j<2$, such that $\bar{x}_i=\Sigma_{j<2}\bar{z}_{ij}$ and $\bar{y}_j=\Sigma_{i<2}\bar{z}_{ij}$. The hypothesis $\bar{x}_0+\bar{x}_1=\bar{y}_0+\bar{y}_1$ implies $P(x_0+x_1)=P(y_0+y_1)=\Sigma_{k<r} e_k$ with all $e_k\in D$ and $e_k\not\lhd e_l$ for $k\neq l$. It follows that there are partitions $r=I(0)\cup I(1)=J(0)\cup J(1)$ and reduced representations $x_0=\Sigma_{k\in I(0)} e_k+\Sigma_{i<r(0)} f_{0i}$, $x_1=\Sigma_{k\in I(1)} e_k+\Sigma_{i<r(1)} f_{1i}$, $y_0=\Sigma_{k\in J(0)} e_k+\Sigma_{j<s(0)} g_{0j}$, $y_1=\Sigma_{k\in J(1)} e_k+\Sigma_{j<s(1)} g_{1j}$, such that: for all $i<r(0)$, there exists $k\in I(1)$ satisfying $f_{0i}\lhd e_k$; for all $i<r(1)$, there exists $k\in I(0)$ such that $f_{1i}\lhd e_k$; and similar relations hold among the g_{0j}, g_{1j} and e_k. Since $I(0)\cup I(1)=J(0)\cup J(1)$, every f_{0i} satisfies $f_{0i}\lhd e_k$ for some $k\in J(0)\cup J(1)$. Write $r(0)=K(0,0)\cup K(0,1)$, where $i<r(0)$ is in $K(0,0)$ if $f_{0i}\lhd e_k$ for some $k\in J(0)$, and $i\in K(0,1)$ otherwise

(hence $f_{0i} \lhd e_k$ for some $k \in J(1)$). Define partitions $r(1) = K(1,0) \uplus K(1,1)$, $s(0) = L(0,0) \uplus L(0,1)$, and $s(1) = L(1,0) \uplus L(1,1)$ in a similar way. Let

$$z_{00} = \sum_{k \in I(0) \cap J(0)} e_k + \sum_{i \in K(0,0)} f_{0i} + \sum_{j \in L(0,0)} g_{0j} ,$$

$$z_{01} = \sum_{k \in I(0) \cap J(1)} e_k + \sum_{i \in K(0,1)} f_{0i} + \sum_{j \in L(1,0)} g_{1j} ,$$

$$z_{10} = \sum_{k \in I(1) \cap J(0)} e_k + \sum_{i \in K(1,0)} f_{1i} + \sum_{j \in L(0,1)} g_{0j} ,$$

$$z_{11} = \sum_{k \in I(1) \cap J(1)} e_k + \sum_{i \in K(1,1)} f_{1i} + \sum_{j \in L(1,1)} g_{1j} .$$

Then $z_{00} + z_{01} = \Sigma_{k \in I(0)} e_k + \Sigma_{j < r(0)} f_{0j} + \Sigma_{j \in L(0,0)} g_{0j} + \Sigma_{j \in L(1,0)} g_{1j} \equiv x_0$, since $j \in L(0,0)$ implies $g_{0j} + e_k = e_k$ for some $k \in I(0)$ and $j \in L(1,0)$ implies $g_{1j} + e_k = e_k$ with $k \in I(0)$. Hence, $\bar{x}_0 = \bar{z}_{00} + \bar{z}_{01}$. Similar computations give $\bar{x}_1 = \bar{z}_{10} + \bar{z}_{11}$, $\bar{y}_0 = \bar{z}_{00} + \bar{z}_{10}$, and $\bar{y}_1 = \bar{z}_{01} + \bar{z}_{11}$.

(b) Define the morphism Ψ: $F(\mathscr{D}(M)) \to M$ by $\Psi(e_0 + \cdots + e_{n-1}) = e_0 * \cdots * e_{n-1}$, where $*$ denotes the sum in M. It is the content of Proposition 3.4.4 that Ψ is a surjective mapping such that $\Psi(x) = \Psi(y)$ if and only if $P(x) = P(y)$. Hence, Ψ induces an isomorphism of $\mathscr{M}(\mathscr{D}(M))$ to M. $\square$

Notes. The results in Propositions 3.4.4 and 3.5.2 are extensions of a part of Pierce's work on metrizable Boolean spaces of finite type (PIERCE [1972]). The existence of such an extension was suggested by results of WILLIAMS [1975] on primitive algebras, and large parts of the program were carried out by HANSOUL [1985].

3.6. *QO systems*

A relational system $\langle D, \lhd \rangle$ such that $\lhd$ is a transitive binary relation on the non-empty set D is called a *quasi-ordered system* or *QO system*. In general, $\lhd$ is not assumed to be antisymmetric or reflexive. However, if $\lhd$ is antisymmetric, then it can be used to construct a primitive monoid by the recipe that was described in the previous subsection. That is why we are interested in QO systems.

The relation $\lhd \cup \Delta_D$ is a preordering of D; it is a partial order if $\lhd$ is antisymmetric. We will denote this relation by $\leq$. The symbol $<$ will designate the relation $\lhd \backslash \Delta_D$. Thus, $e \leq f$ means that $e \lhd f$ or $e = f$; and $e < f$ is the statement that $e \lhd f$ and $e \neq f$. In most applications, our use of $\leq$ and $<$ is consistent with the notation that was introduced earlier in connection with the natural ordering of a monoid.

If $\langle D, \lhd \rangle$ is a QO system, denote

$$D_1 = \{e \in D : e \lhd e\} , \qquad D_2 = D \backslash D_1 .$$

It is also convenient to introduce the mapping κ: $D \to \{1, 2\}$ by $\kappa(e) = 1$ if $e \in D_1$, $\kappa(e) = 2$ if $e \in D_2$.

A subset J of a QO system D is an *ideal* of D if J is hereditary: $e \lhd f \in J$ implies $e \in J$. Plainly, the union and intersection of any set of ideals is an ideal. Because of the failure of reflexivity, there are three possible choices for the definition of principal ideals. It is convenient to have notation for each of these options. For $e \in D$, denote

$$J(e) = \{f \in D\colon f \lhd e\}\,, \qquad J_1(e) = \{f \in D\colon f \leq e\}\,,$$

$$J_2(e) = \{f \in D\colon f < e\}\,.$$

3.6.1. DEFINITION. Let D and D' be QO systems. A morphism from D to D' is a mapping $\Phi\colon D \to D'$ such that

$$\Phi(J(e)) = J(\Phi(e)) \quad \text{for all } e \in D\,.$$

It is enlightening to split the condition $\Phi(J(e)) = J(\Phi(e))$ into two inclusions:
(1) $f \lhd e$ implies $\Phi(f) \lhd \Phi(e)$;
(2) $h \lhd \Phi(e)$ implies the existence of $f \lhd e$ such that $\Phi(f) = h$.
Implication (1) asserts that Φ is monotone, relative to $\lhd$. It then follows that Φ is monotone relative to $\leq$. If Φ is also injective, then it is monotone relative to $<$.

3.6.2. LEMMA. *If Φ is a morphism from the QO system $\langle D, \lhd\rangle$ to the QO system $\langle D', \lhd'\rangle$, then $\Theta = \operatorname{Ker} \Phi$ satisfies*
(3) $\lhd \circ \Theta \subseteq \Theta \circ \lhd$.
That is, if $e \lhd f\Theta g$, then there exists $h \in D$ such that $e\Theta h \lhd g$. Conversely, if Θ is an equivalence relation on D that satisfies (3), *then there is a unique, transitive relation $\lhd'$ on D/Θ such that the natural projection of D to D/Θ is a morphism from $\langle D, \lhd\rangle$ to the QO system $\langle D/\Theta, \lhd'\rangle$.*

PROOF. The inclusion (3) is a two-line deduction from (1) and (2). The definition of $\lhd'$ is given by $\bar{e} \lhd' \bar{f}$ if there exist $e \in \bar{e}, f \in \bar{f}$ such that $e \lhd f$. The transitivity of $\lhd'$ and the fact that projection is a morphism will then follow from (3). ☐

An equivalence relation Θ on the QO system $\langle D, \lhd\rangle$ that satisfies (3) is called a congruence relation on D. If Δ_D is the only congruence relation on D, then D is called a *simple QO system*. For instance, if D is a simple QO system with more than one element, then $\lhd$ cannot be reflexive; otherwise $D \times D$ is a congruence.

The principal result of this subsection is a directory of important properties of simple QO systems.

3.6.3. PROPOSITION. *Let $\langle D, \lhd\rangle$ be a simple QO system, and $e, f \in D$.*
(a) *$\leq$ is a partial ordering of D.*
(b) *If J is an ideal of D, then J is a simple QO system.*
(c) *If I and J are ideals of D, and $\Phi\colon I \to J$ is an order isomorphism, then $I = J$ and Φ is the identity map; thus, D is rigid.*
(d) *If $J_2(e) = J_2(f)$ and $\kappa(e) = \kappa(f)$, then $e = f$.*
(e) *If $\kappa(e) = 1$, then $J(e) \neq J_2(f)$ for all $f \in D$.*
(f) *If $J(f) \neq \emptyset$ for all $f \in J_1(e)$, then $J(e) = \{e\}$.*

(g) *If D does not have the form $J(f)$, $f \in D$, then there exist simple QO systems $D^{(1)}$ and $D^{(2)}$ that contain D as a subset of the form $J_2(e_1)$ and $J_2(e_2)$, respectively, where $\kappa(e_1) = 1$ and $\kappa(e_2) = 2$.*

PROOF. (a) An easy calculation shows that $(\leq) \cap (\leq)^{-1}$ is a congruence relation on D; hence $(\leq) \cap (\leq)^{-1} = \Delta_D$. (b) If Θ is a congruence on J, then $\Theta \cup \Delta_D$ is a congruence on D. (c) Define $e\Theta f$ if $e = f$ in D, or there is an integer n (possibly negative) such that $\Phi^k(e)$ is defined for all k between 0 and n, and $\Phi^n(e) = f$. Plainly, Θ is an equivalence relation on D. Let $e \lhd f\Theta g$, say $g = \Phi^n(f)$. If $n \geq 0$, then $e \in J(f) \subseteq I$. Hence, $\Phi^k(e) \lhd \Phi^k(f) \in I$ for $0 \leq k < n$, and $e\Theta\Phi^n(e) \lhd \Phi^n(f) = g$. The case in which $n < 0$ is treated similarly. Thus, Θ is a congruence. By simplicity, $\Theta = \Delta_D$, so that Φ is the identity mapping. (d) Define $\Theta = \Delta_D \cup \{(e, f), (f, e)\}$. Clearly, Θ is an equivalence relation. If $g \lhd e\Theta f$ with $g \neq e$, then $g \in J_2(e) = J_2(f)$, and $g = g \lhd f$. If $g = e$, then $e \lhd e\Theta f$ implies $\kappa(f) = \kappa(e) = 1$, so that $g\Theta f \lhd f$. Other possible cases of $g \lhd h\Theta k$ are treated similarly, so that Θ is a congruence. By simplicity, $\Theta = \Delta_D$ and $e = f$. (e) Assume that $J(e) = J_2(f)$ for some $f \in D$. Then $\kappa(e) = 1$ implies $e \in J(e) = J_2(f)$, so that $e < f$. In particular, $e \neq f$. Define Θ as in (d). If $g \lhd e\Theta f$, then $g \in J(e) = J_2(f)$ implies $g = g \lhd f$. If $g \lhd f\Theta e$, $g \neq f$, then $g \in J_2(f) = J(e)$ implies $g = g \lhd e$. In the remaining case of interest, $f \lhd f\Theta e$ implies $f\Theta e \lhd e$. Thus, Θ is a congruence. Simplicity gives the contradiction $e = f$. Hence, $J(e) \neq J_2(f)$ for all $f \in D$. (f) $J(f) \neq \emptyset$ for all $f \leq e$ implies $J(e) \times J(e)$ is a congruence on $J(e)$. (g) Define $D^{(1)} = D \uplus \{e_1\}$, $D^{(2)} = D \uplus \{e_2\}$; and let $f \lhd g$ in $D^{(1)}$ (in $D^{(2)}$) if $f \lhd g$ in D, or $f \in D$ and $g = e_1$ ($g = e_2$), or $f = g = e_1$. Clearly, $D^{(1)}$ and $D^{(2)}$ are QO systems such that $D = J_2(e_1) = J_2(e_2)$, $\kappa(e_1) = 1$ and $\kappa(e_2) = 2$. Suppose that Θ is a congruence on $D^{(1)}$ or $D^{(2)}$. Since $\Theta \cap (D \times D)$ is a congruence on D, the simplicity of D implies that $\Theta \cap (D \times D) = \Delta_D$. Suppose that $e\Theta f$, where $e = e_1$ or e_2, and $f \in D$. By construction, $g \lhd e\Theta f$ for every $g \in D$. Since Θ is a congruence, there exists h such that $g\Theta h \lhd f$. Necessarily, $h \in D$ because D is an ideal in $D^{(1)}$ and $D^{(2)}$. Hence $g = h$ by the simplicity of D. The fact that $g \in D$ was arbitrary yields $D = J(f)$, which contradicts our hypothesis. Thus, $D^{(1)}$ and $D^{(2)}$ are simple. $\square$

3.7. *Simplicity and V-simplicity*

We can now relate the V-simplicity of a primitive monoid M to the simplicity of the associated QO system $\mathscr{D}(M)$.

3.7.1. PROPOSITION. *A primitive monoid M is V-simple if and only if $\mathscr{D}(M)$ is a simple QO system.*

PROOF. Assume that the QO system $D = \mathscr{D}(M)$ is simple. Denote $\Theta = Y \cap (D \times D)$, where Y is the V-radical of M. We will show that Θ is a congruence on D. The assumed simplicity of D then implies that $\Theta = \Delta_D$, from which it follows easily that $Y = \Delta_M$. Plainly, Θ is an equivalence relation on D. Suppose that $e \lhd f\Theta g$. Then $e + fYg$. Since Y is a V-relation, it follows that $g = a + b$ with eYa

and fYb. Write $a = h_0 + \cdots + h_{n-1}$ in reduced form, that is, $h_i \in D$ and $h_i \not\trianglelefteq h_j$ if $i \neq j$. The relation $eYh_0 + \cdots + h_{n-1}$ yields $e = c_0 + \cdots + c_{n-1}$ with $c_i Y h_i$ for $i < n$. Since e is PI, there exists $i < n$ such that $c_i = e$. It follows that $e\Theta h_i \leq a \leq g$. If $h_i < g$, then $e\Theta h_i \lhd g$, as required. Otherwise, $h_i = a = g$. In this case, $eYa = gYf$, so that $e\Theta f$. A similar argument leads from fYb to $fYk \leq b \leq g$, with $k \in D$. Thus, either $k < g$ (in which case $e\Theta f \Theta k \lhd g$, as required), or $b = g$. The equations $h_i = a = g = b$ imply $g = a + b = g + g$, hence $e\Theta g \lhd g$. This completes our proof that Θ is a congruence, hence $\Theta = \Delta_D$. Suppose that aYb. Write $a = e_0 + \cdots + e_{n-1}$ with $e_i \in D$. Since Y is a V-relation, $b = a_0 + \cdots + a_{n-1}$ with $e_i Y a_i$ for $i < n$. The argument that we used before yields $e_i \Theta f_i \leq a_i$. Hence, $e_i \leq a_i$. Thus, $a = e_0 + \cdots + e_{n-1} \leq a_0 + \cdots + a_{n-1} = b$. By symmetry and the S-B. property, $a = b$. Conversely, suppose that M is V-simple. To show that $D = \mathscr{D}(M)$ is simple, it suffices to prove that if Θ is a congruence relation on D, then there is a V-relation R on M such that $\Theta \subseteq R$. For $a, b \in M$, define aRb if $a = b = 0$ or there exist $e_i, f_i \in D$ such that $a = \Sigma_{i<n} e_i$, $b = \Sigma_{i<n} f_i$, and $e_i \Theta f_i$ for all $i < n$. Plainly, R is a reflexive, symmetric relation on M with $\Theta \subseteq R$. Moreover, R is additive: if $c_j R d_j$ for $j < m$, then $\Sigma_{j<m} c_j R\, \Sigma_{j<m} d_j$. Using this additivity, the proof can be completed by showing that if $a, b \in M$ and $e \in D$ satisfy $aRe + b$, then there is a decomposition $a = a_0 + a_1$ such that $a_0 Re$ and $a_1 Rb$. By the definition of R, $a = \Sigma_{i<n} e_i$ and $e + b = \Sigma_{i<n} f_i$ with $e_i \Theta f_i$ for $i < n$. Let $b = \Sigma_{j<m} g_j$ be the reduced representation of b. If $e \not\trianglelefteq g_j \not\trianglelefteq e$ for all $j < m$, then $e + \Sigma_{j<m} g_j$ is reduced. Thus, $e = f_i$ for some $i < n$, say $e = f_{n-1}$. Also there is an injective map λ: $m \to n - 1$ such that $g_j = f_{\lambda(j)}$; and if $i \in (n-1)\backslash\{\lambda(k)\colon k < m\}$, then either $f_i \lhd e$ or $f_i \lhd g_j$ for some $j < m$. Let $a_0 = e_{n-1} + \Sigma\,\{e_i\colon i \in (n-1)\backslash\{\lambda(k)\colon k < m\},\ f_i \lhd e\}$, $a_1 = \Sigma_{j<m} e_{\lambda(j)} + \Sigma\,\{e_i\colon i \in (n-1)\backslash\{\lambda(k)\colon k < m\},\ f_i \not\trianglelefteq e\}$. These choices give $a = a_0 + a_1$ with $a_0 Re$, $a_1 Rb$. Assume that $e \lhd g_l$ for some $l < m$. In this case, $b = e + b = \Sigma_{i<n} f_i$. Also, $e \lhd g_l = f_i \Theta e_i$ for some $i < n$. Since Θ is a congruence, $e\Theta h \lhd e_i$, where $h \in D$. Thus, $a = h + \Sigma_{i<n} e_i = h + a$, and hRe, $aRe + b = b$. Finally, assume that some of the $g_j \lhd e$, say $g_j \lhd e$ for $j \geq r$ and $g_j \not\trianglelefteq e$ for $j < r$. In this case, $e + b = e + \Sigma_{j<r} g_j$ is the reduced representation of $e + b$. It can be assumed that $f_j = g_j$ for $j < r$ and $f_r = e$. Hence, if $r < i < n$, then there exists $\lambda(i) \leq r$ such that $f_i \lhd f_{\lambda(i)}$. For $j \geq r$, we have $g_j \lhd e = f_r \Theta e_r$. Since Θ is a congruence, there exists $k_j \in D$ such that $g_j \Theta k_j \lhd e_r$. Consequently, $a = \Sigma_{i<n} e_i + \Sigma_{j\geq r} k_j$. Let $a_0 = e_r + \Sigma\,\{e_i\colon \lambda(i) = r\}$, $a_1 = \Sigma_{i<r} e_i + \Sigma\,\{e_i\colon \lambda(i) < r\} + \Sigma_{j\geq r} k_j$. It follows that $a_0 + a_1 = a$. Moreover, $a_0 Re$ and $a_1 Rb$, because $e = f_r + \Sigma\,\{f_i\colon \lambda(i) = r\}$ and $b = \Sigma_{j<m} g_j = \Sigma_{i<r} f_i + \Sigma\,\{f_i\colon \lambda(i) < r\} + \Sigma_{j\geq r} g_j$. The proof that R is a V-relation is complete. $\square$

3.8. *Diagrams*

A QO system D is *locally countable* if $J_1(e) = \{f \in D\colon f \leq e\}$ is countable for all $e \in D$. It is clear that if D is locally countable, then so is every ideal of D locally countable.

3.8.1. LEMMA. *If $\langle D, \lhd\rangle$ is a locally countable QO system such that $\lhd$ is antisymmetric, then $\mathscr{M}(D)$ is a locally countable, primitive monoid.*

PROOF. We use the notation and results of subsection 3.5. If $a \in \mathcal{M}(D)$, then $a = \Pi e_0 + \cdots + \Pi e_{n-1}$, where $e_i \in D$ for $i < n$. If $f \in D$ and $\Pi f \le a$, then $\Pi f \le \Pi e_i$ for some $i < n$ because Πf is PI. Hence, $f \in J_1(e_i)$. This argument shows that $\mathcal{M}(D) \restriction a$ is spanned by the countable set $\bigcup_{i<n} \Pi(J_1(e_i))$. □

The main result of this subsection is an analog for QO systems of Dobbertin's Theorem.

3.8.2. THEOREM. (a) *$\mathcal{D}(\boldsymbol{P})$ is a locally countable, simple QO system.*

(b) *If D is a locally countable, simple QO system, then there is a unique order isomorphism of D to an ideal of $\mathcal{D}(\boldsymbol{P})$.*

(c) *$\mathcal{D}(\boldsymbol{P})$ is characterized up to isomorphism by properties* (a) *and* (b).

PROOF. (a) Since $\boldsymbol{P}$ is a hereditary submonoid of $\boldsymbol{BA}$, it is V-simple and locally countable. Thus, by Proposition 3.7.1, $\mathcal{D}(\boldsymbol{P})$ is a simple QO system. Clearly, $\mathcal{D}(\boldsymbol{P})$ is locally countable. (b) By Propositions 3.5.2 and 3.7.1, $\mathcal{M}(D)$ is a V-simple, primitive monoid with $\mathcal{D}(\mathcal{M}(D)) \cong D$. The lemma implies that $\mathcal{M}(D)$ is locally countable. Hence, by Dobbertin's Theorem (2.4.2), $\mathcal{M}(D)$ is isomorphic to a hereditary submonoid N of $\boldsymbol{BA}$, and $N \subseteq \boldsymbol{P}$ because N is primitive. Thus, $D \cong \mathcal{D}(N)$, which is an ideal of $\mathcal{D}(\boldsymbol{P})$ because N is hereditary. The uniqueness of the isomorphism is a consequence of Proposition 3.6.3. Statement (c) is obvious. □

A QO system with a maximum element is called a *diagram*. For each primitive Boolean algebra A, denote

$$\mathcal{D}(A) = \{e \in \mathcal{D}(\boldsymbol{P}) \colon e \le [A]\} .$$

We will call $\mathcal{D}(A)$ the *diagram of* A, even though it is a diagram in the above sense only if A is pseudo-indecomposable. In that case, $\mathcal{D}(A)$ is the principal ideal $J_1([A])$ in $\mathcal{D}(\boldsymbol{P})$.

3.8.3. COROLLARY. *The mapping $A \mapsto \mathcal{D}(A)$ induces a bijection between the isomorphism classes of primitive, pseudo-indecomposable Boolean algebras and the isomorphism classes of countable, simple diagrams.*

PROOF. Clearly, $A \cong B$ if and only if $\mathcal{D}(A) = \mathcal{D}(B)$. On the other hand, by the uniqueness statement of the theorem, $\mathcal{D}(A) = \mathcal{D}(B)$ is equivalent to $\mathcal{D}(A) \cong \mathcal{D}(B)$. Finally, by the theorem, every countable, simple diagram is isomorphic to $\mathcal{D}(A)$ for some primitive, pseudo-indecomposable Boolean algebra A. □

A countable Boolean algebra A is said to be *structured* by a countable diagram D if there is a mapping Φ from D to the family of non-empty subsets of A such that the union of the range of Φ disjointly generates A, contains 1 but not 0, and: (i) $x \in \Phi(e)$, $y \in \Phi(f)$, $x \le y$ implies $e \le f$; (ii) if $x, y, z \in \Phi(e)$ and $x \dot{+} y \le z$, then $e \lhd e$; (iii) if $x = y \dot{+} z \in \Phi(e)$, then $A \restriction y \cap \Phi(e) \ne \emptyset$ or $A \restriction z \cap \Phi(e) \ne \emptyset$; (iv) if $x \in \Phi(f)$, $e \lhd f$, then $x = y \dot{+} z$ with $y \in \Phi(e)$, $z \in \Phi(f)$. It is easy to see that if A is structured by a countable diagram, then A is primitive and PI. Moreover, it can be shown, using Vaught's Theorem, that if A and B are structured by the same diagram, then $A \cong B$. The above corollary implies that every primitive Boolean algebra is structured in a canonical way by a simple diagram.

Notes. The results in this subsection are due essentially to Hanf, Williams, and Hansoul. Their work can be found in the papers that have been cited earlier. Hansoul has given an inductive construction of the universal QO system $\mathscr{D}(\boldsymbol{P})$.

3.9. *Primitive semirings*

An r-semiring that is also primitive (as an additive monoid) is called a *primitive semiring*. We are interested in the structure of $\mathscr{D}(S)$ in the case that S is a primitive semiring.

3.9.1. LEMMA. *If S is a primitive semiring, then $\mathscr{D}(S)$ is a subsemigroup of the multiplicative monoid of S such that $e \lhd f$ in $\mathscr{D}(S)$ implies $eg \lhd fg$ for all $g \in \mathscr{D}(S)$. If e, f, $g \in \mathscr{D}(S)$, then $e \lhd fg$ if and only if there exist f_0, $g_0 \in \mathscr{D}(S)$ such that $e = f_0 g_0$, and either*
(i) *$f_0 \lhd f$ and $g_0 \leq g$, or*
(ii) *$f_0 \leq f$ and $g_0 \lhd g$.*

PROOF. By Lemma 3.3.2, $\mathscr{D}(S)$ is closed under multiplication, and $e \lhd f$ implies $e + f = f$, $eg + fg = fg$, and $eg \lhd fg$. Let $e \lhd fg$, so that $fg = e + fg$. By the P.D.P., we can write $f = \Sigma_{i<n} f_i$, $g = \Sigma_{j<m} g_j$, $n \times m = G_0 \uplus G_1$, $e = \Sigma_{(i,j) \in G_0} f_i g_j$, and $fg = \Sigma_{(i,j) \in G_1} f_i g_j$. It can be assumed that all f_i and g_j are *PI* because S is primitive. Also, using the pseudo-indecomposability of e and fg, there exist pairs $(i, j) \in G_0$ and $(k, l) \in G_1$ such that $e = f_i g_j$ and $fg = f_k g_l$. With an adjustment of notation, it can be assumed that $(i, j) = (0, 0)$. Note that $f_0 \leq f$ and $g_0 \leq g$, so that if $f_0 < f$ or $g_0 < g$, then one of the conditions (i) or (ii) is fulfilled. Otherwise, $e = fg = f_k g_l$. Since $f = f + \Sigma_{1 \leq i<n} f_i$, it follows from Proposition 3.4.4 that $f_i \lhd f$ for all $i \geq 1$. Similarly, $g_j \lhd g$ for all $j \geq 1$. Since $(k, l) \neq (0, 0)$, one of $k \geq 1$ or $l \geq 1$ is satisfied. That is, (i) or (ii) holds. □

The converse of this lemma is true: if D is a QO system whose order relation is antisymmetric, and D satisfies the conclusions of the lemma, then D induces a product on $\mathscr{M}(D)$ under which $\mathscr{M}(D)$ becomes a primitive semiring. Only the P.D.P. requires attention. The proof is omitted.

3.9.2. LEMMA. *If S is a primitive semiring, and I, J are ideals in $\mathscr{D}(S)$, then $IJ = \{ef\colon e \in I, f \in J\}$ is an ideal in $\mathscr{D}(S)$.*

This result is a corollary of the last part of 3.9.1.

3.9.3. LEMMA. *Let S be a V-simple, primitive semiring. Assume that $e, f \in \mathscr{D}(S)$. Denote $I = J_1(e)J_2(f) \cup J_1(f)J_2(e)$.*

(a) *If* $I = J(g)$, *where* $g \in \mathscr{D}_1(S)$ (*that is*, $\kappa(g) = 1$), *then* $ef = g$.
(b) *If* $I \neq J(g)$ *for all* $g \in \mathscr{D}_1(S)$, *then* $I = J_2(ef)$ *and* $\kappa(ef) = \min\{\kappa(e), \kappa(f)\}$.

PROOF. By 3.9.1, $J_2(ef) \subseteq I \subseteq J(ef)$. If $I = J(g)$ for some $g \in \mathscr{D}_1(S)$, then $I \neq J_2(ef)$ by Proposition 3.6.3(e). Hence, $ef = g$. Otherwise, $I = J_2(ef)$ (since $ef \in I$ implies $\kappa(ef) = 1$). If $\kappa(e) = 1$ or $\kappa(f) = 1$, then $ef \lhd ef$ and $\kappa(ef) = 1 = \min\{\kappa(e), \kappa(f)\}$. Conversely, $\kappa(ef) = 1$ implies $ef \lhd ef$. By 3.9.1, $ef = e_0 f_0$ with $e_0 \lhd e$, $f_0 \leq f$ or $e_0 \leq e$, $f_0 \lhd f$. If $e_0 = e$, then $\kappa(e) = 1$. If $e_0 < e$, then $e_0 \in J_2(e)$, and $ef = e_0 f_0 \in I$. In this case, $\kappa(ef) = 1$. □

3.9.4. PROPOSITION. (a) *$\mathscr{D}(P)$ is a commutative semigroup with multiplicative unit* 1 *and multiplicative zero* f.
(b) $\langle \mathscr{D}(P), \lhd \rangle$ *is a simple QO system.*
(c) $\leq$ *partially orders* $\mathscr{D}(P)$; 1 *and* f *are the unique minimal elements in* $\mathscr{D}(P)$, *and* $\kappa(1) = 2$, $\kappa(f) = 1$.
(d) *If* $f \neq e \in \mathscr{D}(P)$, *then* $1 \leq e$.
(e) $e \lhd f$ *implies* $eg \lhd fg$ *for* $e, f, g \in \mathscr{D}(P)$.
(f) $e \lhd fg$ *if and only if* $e = f_0 g_0$, *where* $f_0 \lhd f$, $g_0 \leq g$, *or* $f_0 \leq f$, $g_0 \lhd g$.
(g) *If* $J \subseteq \mathscr{D}(P)$ *is an ideal such that* $J \neq J(e)$ *for all* $e \in \mathscr{D}_1(P)$, *then there are exactly two elements* e_1, e_2 *in* $\mathscr{D}(P)$ *such that* $J = J_2(e_1) = J_2(e_2)$, $\kappa(e_1) = 1$, $\kappa(e_2) = 2$.

The proposition is a consequence of 3.9.1, 3.9.3, and Proposition 3.6.3.

3.10. Well-founded QO systems

A QO system D is *well founded* if, for every non-empty set $E \subseteq D$, there is an element $e \in E$ such that $J_2(e) \cap E = \emptyset$. In other words, every non-empty subset of D contains an element that is minimal (relative to $\leq$). In particular, $\lhd$ is antisymmetric. Thus, a QO system is well founded if and only if $\leq$ is a partial ordering that satisfies the descending chain condition.

3.10.1. PROPOSITION. *A well-founded QO system D is simple if and only if*:
(i) $J_2(e) = J_2(f)$ *and* $\kappa(e) = \kappa(f)$, *then* $e = f$;
(ii) $\kappa(e) = 1$, *then* $J(e) \neq J_2(f)$ *for all* $f \in D$.

PROOF. If D is simple, then (i) and (ii) are satisfied by Proposition 3.6.3. Conversely, assume that (i) and (ii) hold. We wish to prove that if Θ is a congruence on D, then $\Theta = \Delta_D$. It is sufficient to contradict the assumption that $E = \{e \in D\colon e\Theta f \text{ for some } f \neq e\}$ is not empty. The hypothesis that D is well founded guarantees the existence of $e \in E$ such that

$$(1) \qquad J_2(e) \cap E = \emptyset .$$

Fix such an e and denote $F = \{f \in D\colon e\Theta f, f \neq e\}$. Then $\emptyset \neq F \subseteq E$. Again using the hypothesis that D is well founded, we can find $f \in F$ so that

(2) $\quad J_2(f) \cap F = \emptyset$.

Since $e \neq f$, a contradiction can be obtained from (i) by showing that $J_2(e) = J_2(f)$ and $\kappa(e) = \kappa(f)$. If $g \in J_2(e)$, then $g \lhd e\Theta f$ implies that $g\Theta h \lhd f$ for some $h \in D$. By (1), $g \not\in E$, so that $h = g$. Thus, $g \lhd f$, and $g \in J_2(f)$. This argument shows that $J_2(e) \subseteq J_2(f)$. On the other hand, if $g \in J_2(f)$, then $g\Theta h \lhd e$ for some $h \in D$, as before. If $h \neq e$, then $h \in J_2(e)$, so that $h \not\in E$ by (1). Hence, $g = h < e$, and $g \in J_2(e)$. If $h = e$, then $e \lhd e$; that is $\kappa(e) = 1$. Moreover, $g\Theta e$. Since $g \not\in F$ by (2), this relation implies that $e = g \in J_2(f)$. We conclude that either $J_2(f) = J_2(e)$, or $J_2(f) = J(e)$ and $\kappa(e) = 1$. The last option is excluded by (ii). Note that since $e \neq f$, the equality $J_2(e) = J_2(f)$ implies $e \not\in J(f)$ and $f \not\in J(e)$. If $\kappa(f) = 1$, then $\kappa(e) = 1$. In fact $f \lhd f\Theta e$ implies $f\Theta g \lhd e$ for some $g \in D$. If $g \neq e$, then $g \not\in E$ by (1). Thus, $f = g \lhd e$, which is impossible since $f \not\in J(e)$. Consequently, $e = g \lhd e$, and $\kappa(e) = 1$ as we claimed. Suppose that $\kappa(e) = 1$. In this case, $e \lhd e\Theta f$ implies the existence of $g \in D$ such that $e\Theta g \lhd f$. Since $e \not\in J(f)$, it follows that $e \neq g$. Therefore, $g \in F \cap J(f) = \{f\}$, by (2). Thus, $f = g \lhd f$, and $\kappa(f) = 1$. □

Notes. A less general version of Proposition 3.10.1 is found in PIERCE [1972]. The observation that the original proof applies to well founded QO systems was made by HANSOUL [1985].

3.11. *Well-founded Boolean algebras*

A Boolean algebra A is *well-founded* if A is primitive and $\mathscr{D}(A)$ is a well-founded QO system, where $\mathscr{D}(A)$ is the diagram of A. We will show that the Stone spaces of well-founded algebras have a nice characterization. Throughout this subsection, X denotes the Stone space of the countable Boolean algebra A.

Recall that a point p in X is homogeneous if there is a neighborhood W of p (called a uniform neighborhood) such that if $p \in V \subseteq W$ with V clopen, then $V \simeq W$. We define the *character* of the homogeneous point p to be

$$\text{Char } p = [W],$$

where W is a uniform neighborhood of p. By Proposition 3.1.1, $[W]$ is a PI element of $\boldsymbol{BA}$. Thus, Char is a surjective mapping from the set of homogeneous points in X to $\mathscr{D}(A)$. If $e \in \mathscr{D}(A)$, denote

$$V(e) = \{p \in X \colon \text{Char } p = e\}.$$

Then $V(e)$ is a non-empty subset of X such that if $e \neq f$ in $\mathscr{D}(A)$, then $V(e) \cap V(f) = \emptyset$.

For any subset U of X, let Acc U be the set of accumulation points of U in X. Thus, a point $q \in X$ belongs to Acc U if and only if every neighborhood of q contains a $p \in U$ such that $p \neq q$. It follows that Acc U consists of all $q \in U^-$ such that q is not an isolated point of U: Acc $U = U^- \backslash (U \backslash U')$. In particular, $U^- = (\text{Acc } U) \cup U$.

3.11.1. LEMMA. *Let e and f be elements of $\mathscr{D}(A)$.*
(a) *If $e \lhd f$, then $V(f) \subseteq \operatorname{Acc} V(e)$.*
(b) *If $e \not\lhd f$, then $V(f) \cap \operatorname{Acc} V(e) = \emptyset$.*

PROOF. (a) Let W be a uniform neighborhood of a point $p \in V(f)$. Then $[W] = f = e + f$ implies $W = W_1 \uplus W_2$, where W_1 and W_2 are clopen sets such that $[W_1] = e$, $[W_2] = f$ and $p \in W_2$. Since W_1 is PI, there is a homogeneous point q such that W_1 is a uniform neighborhood of q (by Proposition 3.1.1). Thus, $q \in V(e) \cap W$ and $q \neq p$. Since the uniform neighborhoods form a basis at p, it follows that $p \in \operatorname{Acc} V(e)$.

(b) Assume that there is a point $p \in V(f) \cap \operatorname{Acc} V(e)$. Let W be a uniform neighborhood of p. Then $[W] = f$ because $p \in V(f)$. Moreover, there exists $q \neq p$ such that $q \in V(e) \cap W$, since $p \in \operatorname{Acc} V(e)$. If W_1 is a uniform neighborhood of q such that $p \not\in W_1 \subseteq W$, then $f = [W_1] + [W \setminus W_1] = e + f$. Thus, $e \lhd f$. □

3.11.2. PROPOSITION. *If X is the Stone space of the countable Boolean algebra A, then A is well founded if and only if every point of X is homogeneous.*

PROOF. Assume that A is well founded and in particular primitive. By Proposition 3.2.1, the PI clopen sets in X form a basis for the topology. Thus, if $p \in X$, then $E = \{[W] \in \mathscr{D}(A)\colon p \in W\}$ is not empty. Since $\mathscr{D}(A)$ is well founded, there is a clopen neighborhood W of p such that $[W]$ is minimal in E. If $p \in V \subseteq W$ with V clopen and PI, then $[V] \leq [W]$ and $[V] \in E$. Thus, $[V] = [W]$ by the minimality of $[W]$. That is, $V \simeq W$. Since the PI clopen sets form a basis for the topology, it follows that p is homogeneous. Conversely, assume that every point of X is homogeneous. Then the PI clopen subsets of X form a basis, so that A is primitive by Proposition 3.2.1. If $\mathscr{D}(A)$ is not well founded, then there is a sequence $e_0, e_1, e_2, \ldots$ in $\mathscr{D}(A)$ such that $e_{k+1} < e_k$ for all $k < \omega$. In particular, $e_{k+1} \lhd e_k$. By the lemma, $V(e_k) \subseteq \operatorname{Acc} V(e_{k+1})$ for all $k < \omega$, so that there is a nested sequence $W_0 \supset W_1 \supset W_2 \supset \cdots$ of PI clopen sets such that $[W_k] = e_k$. It can be assumed that the diameters of the sets W_k go to zero as k increases. By the compactness of X, there is a point $p \in X$ such that $\bigcap_{k<\omega} W_k = \{p\}$. If p has a uniform neighborhood W, then $p \in W_k \subseteq W$ for all k that exceed a sufficiently large value m. In this case, $e_k = [W_k] = [W]$ for $k > m$, a contradiction. Thus, p is not homogeneous. □

3.11.3. COROLLARY. *The set $\mathbf{W}$ of isomorphism types of countable, well-founded Boolean algebras is a hereditary subsemiring of $\mathbf{P}$.*

Since the cartesian product of two primitive spaces is primitive, hence has the S-B. property, the corollary follows easily from the proposition.

3.11.4. COROLLARY. *The countable Boolean algebra A is well founded if and only if the restriction of the natural ordering of $\mathbf{BA}$ to $\delta([A])$ satisfies the descending chain condition (D.C.C.).*

PROOF. Let $X = \text{Ult}\, A$. A descending chain in $\delta([A])$ corresponds to a nested sequence $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$ of clopen subsets of S. Let $Z = \bigcap_{n<\omega} U_n$. If A is well founded, then by the proposition each $p \in Z$ is contained in a uniform neighborhood $W(p)$. The open set $W = \bigcup_{p \in Z} W(p)$ covers Z, so that by the compactness of X, there is an $n < \omega$ such that $U_n \subseteq W$. Invoke compactness again to obtain a finite set of points $\{p_i : i < r\} \subseteq Z$ and clopen sets V_i such that $p_i \in V_i \subseteq W(p_i)$ and $U_n = V_0 \uplus \cdots \uplus V_{r-1}$. Thus, V_i is a uniform neighborhood of p_i. If $m \geq n$, then $[U_m] = \Sigma_{i<r} [V_i \cap U_m] = \Sigma_{i<r} [V_i] = [U_n]$. Therefore, D.C.C. holds in $\delta([X]) = \delta([A])$. Conversely, assume that the natural order on $\delta([X])$ satisfies D.C.C. If $p \in X$, then the non-empty subset $D = \{[W] : p \in W \in \text{Clop}\, X\}$ of $\delta([X])$ includes a minimal element. Say $[U]$ is minimal in D, where $p \in U \in \text{Clop}\, X$. If V is a clopen neighborhood of p and $V \subseteq U$, then $[V] \leq [U]$ under the natural order. By the minimality of $[U]$ in D, it follows that $V \simeq U$. Thus, U is a uniform neighborhood of p. Since p was an arbitrary point in X, the proposition implies that A is well founded. □

3.12. *Primitive examples*

Many but not all primitive Boolean algebras are well founded. Here is some evidence that supports this assertion.

3.12.1. PROPOSITION. *Every countable, superatomic Boolean algebra is well founded.*

PROOF. By 1.10.3 and 3.11.2, it suffices to show that every point of a well-ordered space $X = \omega^{\mu} \cdot m + 1$ is homogeneous. Except 0 (which is isolated, hence homogeneous), every $p \in X$ has a representation $\omega^{\eta_0} \cdot m_0 + \cdots + \omega^{\eta_k} \cdot m_k$, with $\mu \geq \eta_0 > \cdots > \eta_k \geq 0$ and $1 \leq m_i < \omega$ for $i \leq k$ ($m_0 \leq m$ if $\eta_0 = \mu$, and $k = 0$ if $\eta_0 = \mu$, $m_0 = m$). Let $q = \omega^{\eta_0} \cdot m_0 + \cdots + \omega^{\eta_k} \cdot (m_k - 1) + 1$. By 1.4.3 and 1.10.1, the closed interval from q to p is a uniform, clopen neighborhood of p. □

Among the uniform Boolean algebras, there are well-founded algebras at all levels of the Boolean hierarchy.

3.12.2. EXAMPLE. Let σ be the $\mathscr{W}$-measure that was constructed in Example 1.23.1: $\sigma(1_{\mathscr{C}}) = 1$ and $d(\sigma) = \mu$. The Stone space of B_σ can be realized as a subset $Y = (\{\omega\} \times \mathscr{C}) \cup Z$ of $(\omega + 1) \times \mathscr{C}$, where $Y^{(1)} = \{\omega\} \times \mathscr{C}$, $Y^{(1)} \cap \bar{Z} = \{\omega\} \times X$, and $X \simeq \omega^{\mu} + 1$. It suffices to show that all of the points in Y are homogeneous. The points of Z are isolated, and the points of $\{\omega\} \times (\mathscr{C} \backslash X)$ have perfect neighborhoods. We have only to consider the points (ω, p) with $p \in X$. By the proposition, there exists $x \in \mathscr{F}$ such that $x \cap X$ is a uniform neighborhood of p in X. Let V be the neighborhood $Y \cap ((\omega + 1) \times x)$ of (ω, p). If $(\omega, p) \in W \subseteq V$ with W clopen, then there exists $m < \omega$ and $y \in \mathscr{F}$ with $p \in y \subseteq x$ such that $W \backslash (\{n \leq \omega : m < n\} \times y) \cap Y$ is finite. Thus, $W \simeq Y \cap (\{n \leq \omega :\ m < n\} \times y) \simeq Y \cap ((\omega + 1) \times y)$. Since $x \cap X$ is a uniform neighborhood of p, and $p \in y \cap X \subseteq x \cap X$, it follows that $y \cap X \simeq x \cap X$. Consequently $W \simeq Y \cap ((\omega + 1) \times y) \simeq Y \cap$

$((\omega+1)\times x) = V$ by 1.20.2 and the proof of 1.23.1 (or an application of Vaught's Theorem). Thus, V is a uniform neighborhood of (ω, p).

Here is an example of a primitive Boolean algebra that is not well founded. It is due to HANSOUL [1985].

3.12.3. EXAMPLE. Let $\{A_n: n<\omega\}$ be a set of well-founded, PI Boolean algebras such that $e_n \not\leq e_m$ if $n \neq m$, where $e_n = [A_n] \in \mathscr{D}(P)$. We will see in subsection 3.14 that such a collection exists. Denote $J = \bigcup_{n<\omega} J_1(e_n)$. Clearly, J is a countable ideal in $\mathscr{D}(P)$. By Proposition 3.6.3, $\langle J, \lhd\rangle$ is a simple QO system. Moreover, since $J_1(e_n) = \mathscr{D}(A_n)$, the set J is well founded. Define $D = J \uplus J'$, where $J' = \{f_n: n<\omega\}$ is an indexed set of distinct objects. Extend the relation $\lhd$ on J to D by defining

(1) $e \lhd f_m$ if $e \in J_1(e_n)$ and $n \geq m$, and

(2) $f_n \lhd f_m$ if $n>m$.

Plainly, $\langle D, \lhd\rangle$ is a QO system for which $\leq$ is a partial ordering, and f_0 is the largest element of D. In fact,

(3) $\langle D, \lhd\rangle$ is simple.

To prove (3), we first note that if Θ is a congruence on D, then $\Theta \cap (J\times J) = \Delta_J$ because J is a simple ideal in D. If the set $E = \{e \in J: e\Theta f_m$ for some $m<\omega\}$ is not empty, then there exists $e\in E$ such that $J_2(e)\cap E = \emptyset$ (because J is well founded). If e is such a minimal element with $e\Theta f_m$, then $f_n \lhd f_m \Theta e$ implies $f_n \Theta g \lhd e$ for all $n \geq m$. The minimality of e forces $g=e$, so that $e\Theta f_n$ for all $n\geq m$. By transitivity, $f_n\Theta f_m$ for all $n\geq m$. Choose $n \geq m$ so that $e_n \neq e$. For $k>n$, $e_n \lhd f_n \Theta f_k$, so that there exists $h\in D$ satisfying $e_n \Theta h \lhd f_k$. Since J is simple and $e_n \not\lhd f_k$ when $k>n$, it follows that $h=f_l$ for some $l\geq k$. This conclusion leads to the contradiction $e_n\Theta f_l \Theta e$, $e_n\neq e$. Therefore, $E=\emptyset$. If $f_n\Theta f_m$ with $m>n$, then $e_n \lhd f_n \Theta f_m$ implies $e_n \Theta g \lhd f_m$. However, as we have seen, this can happen only if $g=e_n$. In this case $e_n \lhd f_m$, which is contrary to the definition of $\lhd$ if $m>n$. Thus, $\Theta = \Delta_D$. This conclusion completes the proof of (3). Since D is a countable, simple diagram, Corollary 3.8.3 guarantees the existence of a primitive Boolean algebra A such that $\mathscr{D}(A)\cong D$. The algebra A cannot be well founded because $f_0 > f_1 > f_2 > \cdots$ in D.

3.13. Finitary Boolean algebras

A countable Boolean algebra A is *finitary* if A is primitive and the diagram $\mathscr{D}(A)$ of A is finite. Of course, every finitary algebra is well founded. Thus, every point in the Stone space X of a finitary algebra is homogeneous. Moreover, the set of characters of points in X is finite. Conversely, if X is the Stone space of a countable Boolean algebra, all points in X are homogeneous, and the image of Char is finite, then A is finitary. Unfortunately, this observation is not very useful when it comes to identifying the Stone spaces of finitary algebras. Lemma 3.11.1 suggests a different approach.

For any subset U of the Boolean space X, the set of accumulation points (in X) of U is denoted by Acc U, as before. It is clear that Acc U is a closed subset of X, and the following conditions are satisfied:

(1) $\operatorname{Acc}(U \cup V) = \operatorname{Acc} U \cup \operatorname{Acc} V$;
(2) $\operatorname{Acc}(\operatorname{Acc} U) \subseteq \operatorname{Acc} U$;
(3) $U^- = (\operatorname{Acc} U) \cup U$, $U' = (\operatorname{Acc} U) \cap U$, $\operatorname{Acc} U = U^- \backslash (U \backslash U')$.

We will view the map $U \mapsto \operatorname{Acc} U$ as a unary operation on the set $P(X)$ of all subsets of X. When $P(X)$ is viewed as a universal algebra with the usual (finite) union, intersection, complement, and the constant $\emptyset$, plus the unary operation Acc, this system is called a *topological Boolean algebra*, or TBA for short. The standard notions of homomorphism, subalgebra, and so on apply to $P(X)$, viewed as a TBA. In particular, $P(X)$ contains a smallest subalgebra, the prime topological subalgebra of $P(X)$. This prime subalgebra will be denoted by $\mathscr{P}r(X)$. The space X is of *finite type* if $\mathscr{P}r(X)$ is a finite set.

3.13.1. LEMMA. *Let C be a finite, sub-TBA of $P(X)$. For V, $W \in \operatorname{At} C$, define $V \lhd W$ if $W \subseteq \operatorname{Acc} V$.*

(a) *If $V \not\lhd W$, then $W \cap \operatorname{Acc} V = \emptyset$; in particular, $V \not\lhd V$ implies $V' = \emptyset$.*
(b) *$\langle \operatorname{At} C, \lhd \rangle$ is a QO system.*
(c) *$C = \mathscr{P}r(X)$ if and only if $\langle \operatorname{At} C, \lhd \rangle$ is simple.*

PROOF. (a) If $V \not\lhd W$, then $W \not\subseteq \operatorname{Acc} V$; thus, $W \supset W \cap \operatorname{Acc} V \in C$. Since W is an atom of C, it follows that $W \cap \operatorname{Acc} V = \emptyset$.

(b) The transitivity of $\lhd$ follows from (2).

(c) Let Θ be an equivalence relation on $\operatorname{At} C$. Denote the set of unions of equivalence classes of Θ by D. Thus, $V_1 \Theta V_2$ if and only if $V_1 \cup V_2 \subseteq W$ for some $W \in D$. Plainly, distinct sets in D are disjoint. Define B to be the sub-Boolean algebra of C that is generated by D, that is, the set of finite unions of the sets in D. Thus, $D = \operatorname{At} B$. Since B is finite, it follows from (1) that B is a sub-TBA of C if and only if

(3) $V \in D$ implies $\operatorname{Acc} V \in B$.

Straightforward application of (a) and the definition of $\lhd$ leads to the conclusion that (3) is satisfied if and only if Θ is a congruence on $\langle \operatorname{At} C, \lhd \rangle$. Therefore, $\langle \operatorname{At} C, \lhd \rangle$ is simple if and only if there are no proper sub-TBAs of C, that is, $C = \mathscr{P}r(X)$. □

3.13.2. THEOREM. *If X is the Stone space of the countable Boolean algebra A, then A is finitary if and only if X is a space of finite type. In this case, $\mathscr{D}(A)$ is isomorphic as a QO system to* $\operatorname{At}(\mathscr{P}r(X))$.

PROOF. Assume that A is finitary. For $e \in \mathscr{D}(A)$, denote $V(e) = \{p \in X\colon \operatorname{Char} p = e\}$, as in subsection 3.11. By Proposition 3.11.2, X is the disjoint union of the sets $V(e)$, each of which is non-empty. Let C be the sub-Boolean algebra of $P(X)$ that is generated by $\{V(e)\colon e \in \mathscr{D}(A)\}$. By Lemma 3.11.1, $\operatorname{Acc} V(e) = \bigcup \{V(f)\colon e \lhd f\} \in C$. Hence, C is a sub-TBA of $P(X)$. The same lemma implies that $e \mapsto V(e)$ is an isomorphism of QO systems from $\mathscr{D}(A)$ to $\operatorname{At} C$. Since $\mathscr{D}(A)$ is simple, it follows from the lemma above that $C = \mathscr{P}r(X)$. It remains to show that if X is a space of finite type, then A is primitive and $\mathscr{D}(A)$ is finite. Denote $D = \operatorname{At} \mathscr{P}r(X)$. By the lemma, $\langle D, \lhd \rangle$ is a simple QO system, where $\lhd$ is defined by $V \lhd W$ if $W \subseteq \operatorname{Acc} V$. In particular, $\leq$ is a partial ordering of D. For each

$x \in \operatorname{Clop} X$, denote by $I(x)$ the maximal elements in $\{V \in D\colon x \cap V \neq \emptyset\}$. Since D is finite and $X = \bigcup D$, it follows that $I(x) \neq \emptyset$ if $x \neq \emptyset$. Denote $I_1(x) = \{V \in I(x)\colon V \lhd V\}$ and $I_2(x) = I(x) \backslash I_1(x)$. If $V \in I_2(x)$, then $x \cap V$ is a finite set; otherwise $x \cap \operatorname{Acc} V \neq \emptyset$ by compactness, so that $x \cap W \neq \emptyset$ for some $W \in D$ that satisfies $V \lhd W$, contrary to the maximal property of V. If $p \in V \in D$, then there is a clopen neighborhood x of p with the property that $x \cap W = \emptyset$ for all $W \in D$ such that $p \notin \bar{W} = (\operatorname{Acc} W) \cup W$. Thus, $W \not\leq V$ implies $x \cap W = \emptyset$. Hence, if $p \in y \subseteq x$, then $I(y) = \{V\}$. In the case that $V \in I_2(x)$, all sufficiently small clopen neighborhoods y of p will also satisfy $y \cap V = \{p\}$. The following terminology will be useful. A u-neighborhood of the point $p \in V \in D$ is a clopen neighborhood x of p such that $I(x) = \{V\}$, and if $V \ntriangleleft V$, then $x \cap V = \{p\}$. Our discussion shows that every $p \in X$ has a u-neighborhood; and if $p \in y \subseteq x$, where x is a u-neighborhood of p, then y is also a u-neighborhood of p. Thus, to complete the proof, it is sufficient to show that if p, q are points in the same $V \in D$, and if x is a u-neighborhood of p and y is a u-neighborhood of q, then x is homeomorphic to y. Define the equivalence relation R on clop X by

$$xRy \quad \text{if } I(x) = I(y) \quad \text{and} \quad |x \cap V| = |y \cap V| \quad \text{for all } V \in I_2(x)\,.$$

By Vaught's Theorem, it will be sufficient to show that R is a V-relation. If $xR\emptyset$, then $I(x)$ is empty, which implies $x = \emptyset$, as we noted before. Suppose that $xRy_0 \uplus y_1$. Let $I(y_0) = \{V_{00}, \ldots, V_{0r-1}, \quad W_{00}, \ldots, W_{0s-1}\}$ and $I(y_1) = \{V_{10}, \ldots, V_{1t-1}, W_{10}, \ldots, W_{1u-1}\}$, where the V_{0i} and V_{1i} are in $I_1(y_0)$ and $I_1(y_1)$, while the W_{0j} and W_{1j} belong to $I_2(y_0)$ or $I_2(y_1)$. The assumption that $xRy_0 \uplus y_1$ means $I(x) = I(y_0 \uplus y_1)$ and $|x \cap W| = |(y_0 \uplus y_1) \cap W|$ for all $W \in I_2(x)$. It follows that there are distinct points $p_{0i} \in x \cap V_{0i}$ $(i < r)$, $p_{1i} \in x \cap V_{1i}$ $(i < s)$, $q_{0jk} \in x \cap W_{0j}$ $(j < t,\ k < |y_0 \cap W_{0j}|)$, $q_{1jk} \in x \cap W_{1j}$ $(j < u,\ k < |y_1 \cap W_{1j}|)$. If $V_{0i} \in I(x)$, the existence of the p_{0i} is clear; otherwise, $V_{0i} \lhd V$ for some $V \in I(x)$, and $x \cap V_{0i}$ is infinite because $V \subseteq \operatorname{Acc} V_{0i}$. Similar arguments apply to V_{1i}, W_{0j}, and W_{1j}. Choose disjoint u-neighborhoods z_{0i} of p_{0i}, z_{1i} of p_{1i}, z_{0jk} of q_{0jk}, and z_{1jk} of q_{1jk}, all of which are contained in x. Let $w_0 = \bigcup_i z_{0i} \cup \bigcup_j \bigcup_k z_{0jk}$ and $w_1 = \bigcup_i z_{li} \cup \bigcup_j \bigcup_k z_{1jk}$. By construction, w_0 and w_1 are disjoint clopen sets in x such that w_0Ry_0 and w_1Ry_1. If $p \in x \backslash (w_0 \cup w_1)$, then there exists $W \in D$ such that $p \in W \in \{V \in D\colon x \cap V \neq \emptyset\}$. It follows that $W \leq V$ for some $V \in I(x)$. In fact, $W \lhd V$ for such a V, because $V \in I_2(x)$ implies $x \cap V \subseteq w_0 \cup w_1$. Thus, $W \lhd V$, where $V \in I(w_0)$ or $V \in I(w_1)$, say $V \in I(w_0)$. In this case, $I(w_0 \cap v) = I(w_0)$ for some u-neighborhood v of p. By compactness, $x \backslash (w_0 \cup w_1) = v_{00} \uplus \cdots \uplus v_{0m-1} \uplus v_{10} \uplus \cdots \uplus v_{1n-1}$, where $x_0 = w_0 \cup v_{00} \cup \cdots \cup v_{0m-1}$ satisfies $I(x_0) = I(w_0) = I(y_0)$ and $|x_0 \cap V| = |y_0 \cap V|$ for all $V \in I_2(x_0)$; and $x_1 = w_1 \cup v_{10} \cup \cdots \cup v_{1n-1}$ satisfies $I(x_1) = I(y_1)$ and $|x_1 \cap V| = |y_1 \cap V|$ for all $V \in I_2(x_1)$. Thus, $x = x_0 \uplus x_1$, x_0Ry_0, and x_1Ry_1. Therefore, R is a V-relation, and the proof is finished. $\square$

If A is both uniform and primitive, then A is determined up to isomorphism by its corresponding $\mathscr{W}$-measure σ as well as its diagram $\mathscr{D}(A)$. It is therefore natural to look for a direct relation between σ and $\mathscr{D}(A)$. If A is finitary, then there is one obvious connection. Let $X = \mathrm{Ult}\ A$. The sets $X^{(\xi)} \setminus X^{(\xi+1)}$ constitute a chain in $\mathrm{At}(\mathscr{P}r(X))$. Consequently, $n = \nu(X)$ is a finite ordinal that is no greater than the length of the partially ordered set $\mathscr{D}(A)$. In particular, $\sigma(1) < \omega$. A deeper result can be obtained by an inductive argument that is similar to the discussion of Example 1.23.1: the depth $d(\sigma)$ of the measure σ is also bounded by the length of $\mathscr{D}(A)$. Thus, if the $\mathscr{W}$-measure σ corresponds to a finitary Boolean algebra, then $d(\sigma)$ and $\sigma(1)$ are both finite. Remarkably, the converse is true. Heindorf has proved that if σ is a $\mathscr{W}$-measure such that $d(\sigma)$ and $\sigma(1)$ are finite, then the associated uniform algebra B_σ is finitary. His proof is based on a constructive characterization of the class of finitary algebras in terms of certain product constructions.

Notes. Finitary Boolean algebras were introduced independently by PALYUTIN [1971] and PIERCE [1972]. Their descriptions of these algebras were different from each other, and they also differed from the definition above. An alternative proof of Theorem 3.13.2 is given in PIERCE [1972]. This result has been generalized to the Stone spaces of arbitrary primitive algebras, in part by MUTH [1975], and in full by HANSOUL [1985].

3.14. *Isomorphism classes of finitary algebras*

Let $\boldsymbol{F}$ denote the set of isomorphism types of finitary Boolean algebras. If A and B are primitive Boolean algebras, then $\mathscr{D}(A \times B) = \mathscr{D}(A) \cup \mathscr{D}(B)$ (by Lemma 2.2.2) and $\mathscr{D}(A \oplus B) \subseteq \mathscr{D}(A)\mathscr{D}(B)$ (by Lemma 3.9.1). Consequently, $\boldsymbol{F}$ is a hereditary subsemiring of $\boldsymbol{P}$. In particular, $\boldsymbol{F}$ is V-simple and primitive. By Lemma 3.9.1, $\mathscr{D}(\boldsymbol{F})$ is a subsemigroup of $\boldsymbol{F}$ that additively generates $\boldsymbol{F}$. The multiplication in $\boldsymbol{F}$ is therefore uniquely determined by the semigroup structure of $\mathscr{D}(\boldsymbol{F})$. Questions about the semiring structure of $\boldsymbol{F}$ can usually be translated into problems about $\mathscr{D}(\boldsymbol{F})$. Moreover, the results of the previous few subsections provide a complete description of $\mathscr{D}(\boldsymbol{F})$, both as a QO system, and as a semigroup.

Since $\boldsymbol{F}$ is a hereditary submonoid of $\boldsymbol{P}$, $\mathscr{D}(\boldsymbol{F})$ is an ideal of $\mathscr{D}(\boldsymbol{P})$, and $\langle \mathscr{D}(\boldsymbol{F}), \lhd \rangle$ is a simple QO system. Therefore, $\leq$ is a partial ordering of $\boldsymbol{F}$. By the definition of $\boldsymbol{F}$, the principal ideals in $\mathscr{D}(\boldsymbol{F})$ are finite. It follows that $\mathscr{D}(\boldsymbol{F})$ is well founded.

For a subset E of a partially ordered set, let $\max E$ denote the set of elements that are maximal in E. If E is finite and not empty, then $\max E$ is not empty. In particular, if $e \in \mathscr{D}(\boldsymbol{F})$ and $e \neq 1, f$, then $J_2(e) \neq \emptyset$ by Proposition 3.9.4. In this case, $\max J_2(e)$ is a non-empty antichain consisting of the elements in $\mathscr{D}(\boldsymbol{F})$ that are covered by e. Note that since $J_2(e)$ is an ideal of $\mathscr{D}(\boldsymbol{F})$, it can be recovered from its set of maximal elements: $J_2(e) = \{f \in \mathscr{D}(\boldsymbol{F})\colon f \leq g$ for some $g \in \max J_2(e)\}$.

If $e, f \in \mathscr{D}(\boldsymbol{F})$, then the ideal $J_2(e)J_1(f) \cup J_1(e)J_2(f)$ is finite. The following notation will be useful:

$$\Gamma(e, f) = \max(J_2(e)J_1(f) \cup J_1(e)J_2(f)) .$$

3.14.1. PROPOSITION. *$\mathscr{D}(\boldsymbol{F})$ is uniquely determined as a QO system by the following properties.*

(a) *$\leq$ is a partial ordering of $\mathscr{D}(\boldsymbol{F})$.*

(b) *All principal ideals in $\mathscr{D}(\boldsymbol{F})$ are finite.*

(c) *$\mathscr{D}(\boldsymbol{F})$ has two minimal elements 1 and f and $1 \leq e$ if $e \neq f$; moreover, $\kappa(1) = 2$, $\kappa(f) = 1$, that is $1 \not\lhd 1, f \lhd f$.*

(d) *If $e, f \in \mathscr{D}(\boldsymbol{F})$ satisfy $J_2(e) = J_2(f)$ and $\kappa(e) = \kappa(f)$, then $e = f$.*

(e) *If $\kappa(e) = 1$, then $J_1(e) \neq J_2(f)$ for all $f \in \mathscr{D}(\boldsymbol{F})$.*

(f) *If E is a finite, non-empty antichain in $\mathscr{D}(\boldsymbol{F})$, and $E \neq \{e\} \subseteq \mathscr{D}_1(\boldsymbol{F})$, then there exist elements e_1 and e_2 in $\mathscr{D}(\boldsymbol{F})$ such that $E = \max J_2(e_1) = \max J_2(e_2)$, $\kappa(e_1) = 1$, and $\kappa(e_2) = 2$.*

There is a product operation under which $\mathscr{D}(\boldsymbol{F})$ is a commutative semigroup with unity element 1 and zero f. Moreover, the following conditions are satisfied.

(g) *$e \lhd f$ implies $eg \lhd fg$.*

(h) *$e \lhd fg$ if and only if $e = f_0 g_0$, where either $f_0 \lhd f$ and $g_0 \leq g$, or $f_0 \leq f$ and $g_0 \lhd g$.*

(i) *If $\Gamma(f, g) = \{e\}$ and $\kappa(e) = 1$, then $fg = e$; if $\Gamma(f, g) = \{e\}$ and $\kappa(e) = 2$, or if $|\Gamma(f, g)| > 1$, then $\Gamma(f, g) = \max J_2(fg)$ and $\kappa(fg) = \min\{\kappa(f), \kappa(g)\}$.*

The statements of this proposition summarize our earlier remarks about $\boldsymbol{F}$, combined with the various parts of Propositions 3.6.3 and 3.9.4. Induction on $|J(e)|$ shows that the properties (a)–(f) characterize $\mathscr{D}(\boldsymbol{F})$ as an ordered set. It is clear from (d) that the product operation of $\mathscr{D}(\boldsymbol{F})$ is determined by (i). □

By virtue of part (d) in the proposition, an element e of $\mathscr{D}(\boldsymbol{F})$ is uniquely represented by the Hasse diagram of the finite ordered set $J_1(e)$ if some device is used to distinguish the elements of $\mathscr{D}_1(\boldsymbol{F})$ from those of $\mathscr{D}_2(\boldsymbol{F})$. We will make this distinction by using small circles for the elements in $\mathscr{D}_1(\boldsymbol{F})$ and small crosses for elements in $\mathscr{D}_2(\boldsymbol{F})$ (Fig. 21.1.). For instance,

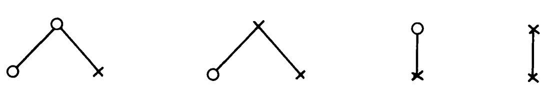

Fig. 21.1

are the diagrams of all elements of height one in $\mathscr{D}(\boldsymbol{F})$. We will often combine such diagrams to form Hasse diagrams of ideals in $\mathscr{D}(\boldsymbol{F})$.

It is clear that the partially ordered set $\mathscr{D}(\boldsymbol{F})$ is countably infinite. The width of this poset grows with height k roughly like the function $f(k)$ that is defined recursively by $f(0) = 2$, $f(k+1) = 2^{f(k)+1}$. Such rapid growth makes it possible to construct a continuum of distinct ideals in $\mathscr{D}(\boldsymbol{F})$. For example, if $(n_0, n_1, n_2, \ldots)$ is a sequence with $n_k = 1$ or 2, then there is an ideal J in $\mathscr{D}(\boldsymbol{F})$ such that the number of elements of height k in J is exactly n_k. For instance, the sequence (2, 1, 1, 2, 2, 1, 2, . . .) corresponds to an ideal whose diagram is shown in Fig. 21.2:

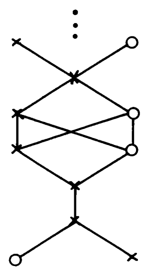

Fig. 21.2

If J is an unbounded ideal in $\mathscr{D}(\boldsymbol{F})$, then J is a simple QO system that does not have the form $J(e)$ for any $e \in J$. By Proposition 3.6.3(g), there is a simple diagram D with largest element e satisfying $J_2(e) = J$. Since D is countable, there is a primitive, PI Boolean algebra A such that $\mathscr{D}(A) \cong D$, by Corollary 3.8.3. Clearly, D is well founded, so that A is a well-founded Boolean algebra.

3.14.2. COROLLARY. *There is a set* $\{A_\xi \colon \xi < \exp \aleph_0\}$ *of well-founded Boolean algebras such that*:
(a) *if* $\xi \neq \eta$, *then* $[A_\xi] \not\leq [A_\eta]$;
(b) *if* $A_\xi = B \times C$, *then* $B \cong A_\xi$ *and* C *is finitary* (*or vice versa*).

In particular, there are continuum many isomorphism classes of well-founded (hence primitive) Boolean algebras. We remark that the corollary more than fulfills the promise that was made in Example 3.12.3 to exhibit an infinite sequence of well-founded algebras whose isomorphism types form an antichain.

3.15. Arithmetic in $\mathscr{D}(\boldsymbol{F})$

Proposition 3.14.1 is the basis of an algorithm that constructs $J_1(ef)$ from $J_1(e)$ and $J_1(f)$ for any two elements $e, f \in \mathscr{D}(\boldsymbol{F})$. The algorithm can be programmed, but if e and f have even rather modest heights (say around 10), then the length of the computation becomes enormous.

One would like to have something analogous to the fundamental theorem of arithmetic for $\mathscr{D}(\boldsymbol{F})$. Some progress in this direction can be made, using only the shallowest properties of this semigroup. We will describe the simplest arithmetical facts concerning $\mathscr{D}(\boldsymbol{F})$ in this subsection and the next one. A deeper result that requires the use of the full power of Proposition 3.14.1 will conclude this chapter.

The zero element f is an anomaly in the arithmetic of $\mathscr{D}(\boldsymbol{F})$. It is more convenient to state our main results for the subsemigroup

$$\mathscr{D}^* = \mathscr{D}(\boldsymbol{F}) - \{f\}$$

of $\mathscr{D}(\boldsymbol{F})$. Since $e \geq 1$ for all $e \in \mathscr{D}^*$, it is clear that $\mathscr{D}^*$ is indeed closed under multiplication. Of course, $\mathscr{D}(F)$ can be recovered from $\mathscr{D}^*$ because $ef=f$ for all $e \in \mathscr{D}(F)$.

An element $p \in \mathscr{D}^*$ is *multiplicatively pseudo-indecomposable* (or MPI for short) if

$$p = ef \text{ implies } p = e \text{ or } p = f .$$

Denote the set of all MPI elements in $\mathscr{D}^*$ by $\mathscr{P}$.

3.15.1. PROPOSITION. *$\mathscr{D}^*$ is a countable, partially ordered, commutative monoid that satisfies:*

(a) *$e \leq f$ implies $eg \leq fg$;*

(b) *$\mathscr{D}^*$ is positively ordered, that is, $e \leq ef$;*

(c) *$\mathscr{D}^*$ is distributive, that is, $e \leq fg$ implies $e = f_0 g_0$, where $f_0 \leq f$ and $g_0 \leq g$;*

(d) *every element of $\mathscr{D}^*$ is a product of MPI elements.*

PROOF. Statements (a) and (c) are special cases of parts of Proposition 3.14.1, and (b) follows from (a) and the fact that $1 \leq f$ for all $f \in \mathscr{D}^*$. For the proof of (d), we note that if e is not MPI, then by (b), $e = fg$ with $f < e$, $g < e$. Let E be the set of all products of MPI elements. In particular, $\mathscr{P} \subseteq E$, and E is closed under multiplication. The existence of a minimal element in $\mathscr{D}^* \backslash E$ would therefore give a contradiction. Since $\mathscr{D}^*$ is well founded, it follows that $\mathscr{D}^* = E$, which is the statement (d). □

3.15.2. COROLLARY. *If $p \in \mathscr{P}$ satisfies $p \leq e_0 \cdots e_{n-1}$, then $p \leq e_i$ for some $i < n$.*

The corollary follows from (c) and the definition of multiplicative pseudo-indecomposability.

Unfortunately, there is no uniqueness theorem for the representation of an element in $\mathscr{D}^*$ as a product of MPI elements.

3.16. Torsion in $\mathscr{D}^*$

It follows from Proposition 3.15.1 that if $e \in \mathscr{D}^*$, then $e \leq e^2 \leq e^3 \leq \cdots$. If $e^n = e^{n+1}$ for some $n < \omega$, then e is called a *torsion* element of $\mathscr{D}^*$. The set of all torsion elements of $\mathscr{D}^*$ will be denoted by $\mathscr{T}$.

3.16.1. LEMMA. *$\mathscr{T}$ is an (order) ideal and a submonoid of $\mathscr{D}^*$.*

PROOF. Clearly, $1 \in \mathscr{T}$. If $e^n = e^{n+1}$, $f^m = f^{m+1}$ and $n \leq m$, then $(ef)^m = (ef)^{m+1}$. Thus, $\mathscr{T}$ is a submonoid of $\mathscr{D}^*$. Also, $e \leq f \in \mathscr{T}$ implies the existence of $m < \omega$ such that $e^n \leq f^m$ for all $n < \omega$. Since $J(f^m)$ is finite, it follows that $e \in \mathscr{T}$. □

The idempotents in $\mathscr{D}^*$ are of course torsion elements. Denote by $\mathscr{E}$ the set of all idempotent elements in $\mathscr{D}^*$. Plainly, $\mathscr{E}$ is a submonoid of $\mathscr{T}$. However, $\mathscr{E}$ is not an order ideal in $\mathcal{T}$. Indeed, the order ideal generated by $\mathcal{E}$ is precisely $\mathcal{T}$, since $e^n = e^{n+1}$ implies $e^n \in \mathcal{E}$.

3.16.2. LEMMA. *For $e \in \mathcal{T}$, define $I(e) = e^n$, where n is the least natural number such that $e^n = e^{n+1}$. Then I is a retractive mapping from $\mathcal{T}$ to $\mathcal{E}$ that is a monotone monoid homomorphism.*

If $I(e) = e^n$, then $I(e) = e^m$ for all $m \geq n$. This observation and our earlier remarks lead easily to the lemma.

3.16.3. LEMMA. *If $f \in \mathcal{T}$ and $e \in \mathcal{E}$, then $f \leq e$ if and only if $ef = e$.*

Indeed, $f \leq e$ implies $ef \leq e^2 = e \leq ef$.

In its own right, $\mathcal{E}$ is a semilattice, that is, a commutative, idempotent semigroup. Lemma 3.16.3 shows that the usual semilattice ordering of $\mathcal{E}$ is the same as the order that $\mathcal{E}$ inherits from $\mathcal{D}^*$.

3.16.4. PROPOSITION. *$\mathcal{E}$ is a distributive lattice with the ordering $\leq$. The join operation in $\mathcal{E}$ is the semigroup product.*

PROOF. For $e \in \mathcal{E}$, denote

$$Q(e) = \{p \in \mathcal{P} \colon p \leq e\},$$

where $\mathcal{P}$ is the set of all MPI elements in $\mathcal{D}^*$. Clearly, $Q(e)$ is a finite ideal of the poset $\mathcal{P} \cap \mathcal{T}$ and $e_1 \leq e_2$ implies $Q(e_1) \subseteq Q(e_2)$. Since the family of ideals in $\mathcal{P} \cap \mathcal{T}$ is closed under set unions and intersections, the distributivity of $\mathcal{E}$ can be proved by showing that $e \mapsto Q(e)$ is an order isomorphism. Let $Q = \{p_0, \ldots, p_{n-1}\}$ be a finite ideal in $\mathcal{P} \cap \mathcal{T}$. Define $I(Q) = I(p_0) \cdots I(p_{n-1}) \in \mathcal{E}$. Clearly, $Q_1 \subseteq Q_2$ implies $I(Q_1) \leq I(Q_2)$, $I(Q(e)) \leq e$ for all $e \in \mathcal{E}$, and $Q \subseteq Q(I(Q))$. The equality $I(Q(e)) = e$ is obtained because e is a product of MPI elements that necessarily belong to $Q(e)$. If $p \in Q(I(Q))$, then $p \in \mathcal{P}$ and $p \leq I(p_0) \cdots I(p_{n-1})$. By Corollary 3.15.2, $p \leq p_i$ for some $i < n$, so that $p \in Q$. Thus, $Q(I(Q)) = Q$. It remains only to note that $I(Q_0 \cup Q_1) = I(Q_0)I(Q_1)$ for any two finite ideals Q_0, Q_1 of $\mathcal{P} \cap \mathcal{T}$. □

3.16.5. COROLLARY. *The mapping $p \mapsto I(p)$ is an order isomorphism from $\mathcal{P} \cap \mathcal{T}$ to the set of all join irreducible elements of $\mathcal{E}$.*

In fact, it is obvious that a finite ideal of $\mathcal{P} \cap \mathcal{T}$ is join irreducible if and only if it is principal.

In a distributive lattice with the minimum condition, every element has a unique representation as an irredundant join of join irreducible elements. In the present context, this fact translates to the statement that every $e \in \mathcal{E}$ is uniquely represented in the form $e = I(p_0) \cdots I(p_{m-1})$, where $p_0, \ldots, p_{m-1}$ are the distinct elements in $\max Q(e)$.

Notes. The results in subsections 3.15 and 3.16 are taken from the work of PIERCE [1975].

3.17. *The converse of Trnková's Theorem*

The theorem of Trnková (Theorem 2.26.2) states that if $a \in \boldsymbol{BA}$ satisfies $a^n = a^m$ for some $m > n$, then $a^n = a^{n+1}$. It is natural to ask if this result is optimal. That is, for which values of n are there elements $a \in \boldsymbol{BA}$ such that $a, a^2, \ldots, a^n$ are distinct, and $a^{n+1} = a^n$? The following result completely settles this question; it also sheds light on the complexity of the arithmetic in $\mathscr{D}^*$.

3.17.1. THEOREM. *If $1 \leq n < \omega$, then there exists $q \in \mathscr{P} \cap \mathscr{T}$ such that $q < q^2 < \cdots < q^n = q^{n+1}$.*

If $n = 1$, then $q = 1$ fulfills the condition of the theorem. For $n = 2$, the element q in the diagram of Fig. 21.3 has the desired property. Clearly, $\Gamma(e, e) = \{e\}$, so that $e^2 = e$. Also, $\Gamma(e, q) = \{q\}$. Hence, eq covers only q and $\kappa(eq) = 1$. It follows that $\Gamma(q, q) = \{eq\}$, so that $q^2 = eq$. Thus, $q^3 = eq^2 = e^2q = eq = q^2$. The Boolean space X such that $q = [X]$ can be described in various ways. Here is one of them: $X = Y_1 \cup Y_2$, where Y_1 is the Cantor set, embedded in $[0, 1]$, $Y_2 = Z_1 \cup Z_2$, where Z_1 is the reflection of the Cantor set through 0 and Z_2 consists of the midpoints of the gaps in Z_1 (the omitted intervals). It is easy to check that $\mathrm{At}(\mathscr{P}r(X)) = \{\{0\}, Y_1\backslash\{0\}, Z_1\backslash\{0\}, Z_2\}$ is isomorphic as a QO system to the ideal $J_1(q)$ in Fig. 21.3. The correspondence is defined by $1 \mapsto Z_2$, $f \mapsto Y_1\backslash\{0\}$, $e \mapsto Z_1\backslash\{0\}$, $q \mapsto \{0\}$. It then follows from Theorem 3.13.2 that $q = [X]$.

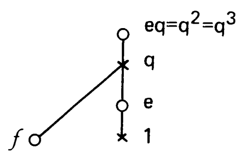

Fig. 21.3

The proof of the theorem in the general case uses two preliminary constructions.

3.17.2. LEMMA. *For each $m \geq 1$, Fig.* 21.4 *below is the diagram of the principal ideal $J_1(e_{m+1})$, where $e_{m+1} \in \mathscr{D}^*$. The elements of the diagram multiply in the following way*:

(a) $p_0^2 = p_0$, $e_0^2 = e_0$;
(b) $p_ie_0 = e_{i+1}$ *for* $i \leq m$;
(c) $p_ip_0 = e_{i+1}$ *for* $1 \leq i \leq m$;
(d) $p_ip_j = e_{j+1}$ *for* $1 \leq i \leq j \leq m$;
(e) $e_ip_j = e_{j+1}$ *for* $1 \leq i \leq j \leq m$;
(f) $e_ip_j = e_i$ *for* $0 \leq j < i \leq m+1$;
(g) $e_ie_j = e_j$ *for* $0 \leq i \leq j \leq m+1$.

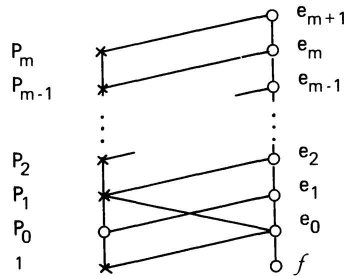

Fig. 21.4

PROOF. Examination of the diagram shows that it satisfies conditions (c), (d), and (e) of Proposition 3.14.1. Hence, it is the diagram of the principal ideal $J_1(e_{m+1})$ in $\mathscr{D}(\boldsymbol{F})$. The multiplication table of the diagram is obtained inductively, using part (i) of Proposition 3.14.1. In fact, $\Gamma(p_0, p_0) = \{p_0\}$, $\Gamma(e_0, e_0) = \{e_0\}$ imply (a). Also, $\Gamma(p_0, e_0) = \{p_0, e_0\}$, so that $p_0e_0 = e_1$. It then follows that $\Gamma(p_1, e_0) = \max\{p_0e_0, e_0^2, p_1, f\} = \{p_1, e_1\}$; thus, $p_1e_0 = e_2$. Induction on $j \geq 2$ yields $\Gamma(p_j, e_0) = \{p_j, e_j\}$, so that $p_je_0 = e_{j+1}$ for $j \leq n$. The proof of (c) is similar. Equation (d) can be proved first for $i = 1$ by induction on j, then in general for $1 < i \leq j$ by induction on $i + j$. The equations (e), (f), and (g) follow from (a)–(d). For instance, if $j > i$, the $p_ie_j = p_ip_{j-1}e_0 = e_je_0 = p_{j-1}e_0^2 = p_{j-1}e_0 = e_j$. □

The next step of the proof uses an extension of the diagram in Fig. 21.4. The diagram that we need appears in Fig. 21.5. Again, it is a routine matter to check that the conditions (c), (d), and (e) of Proposition 3.14.1 are fulfilled, so that the figure does represent an ideal in $\mathscr{D}(\boldsymbol{F})$.

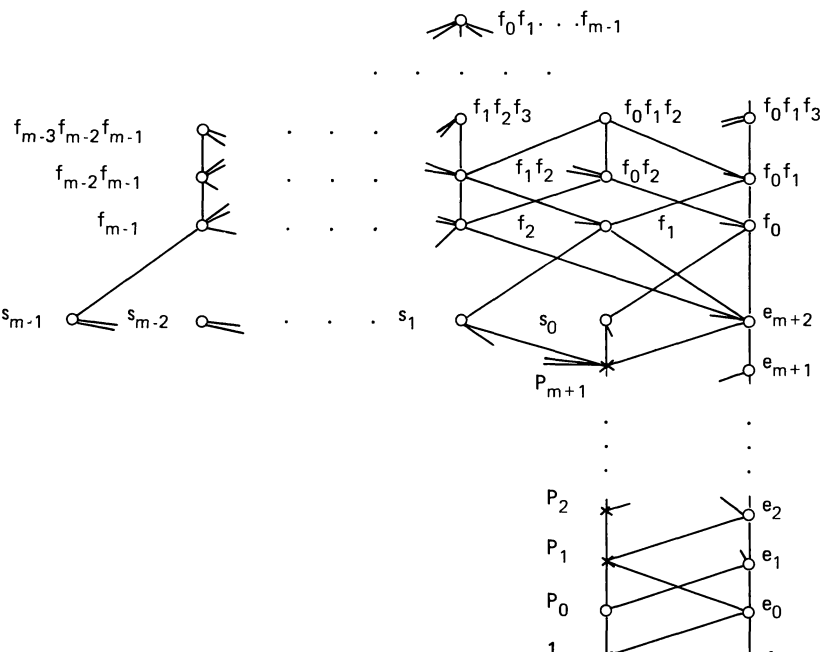

Fig. 21.5

3.17.3. LEMMA. *In Fig.* 21.5, $\{f_0, f_1, \ldots, f_{m-1}\}$ *is the set of atoms of a sub-Boolean algebra of* $\mathscr{E}$.

(*The covering relations in Fig.* 21.5 *are defined by* $\max J_2(p_j) = \{p_{j-1}\}$ *for* $2 \le j \le m+1$, $\max J_2(e_k) = \{p_{k-1}, e_{k-1}\}$ *for* $1 \le k \le m+2$, $\max J_2(s_i) = \{p_{m+1}, e_{i+1}\}$, *and* $\max J_2(f_i) = \{s_i, e_{m+1}\}$ *for* $i < m$.)

PROOF. The elements below e_{m+2} in Fig. 21.5 form an ideal that coincides with the diagram of Fig. 21.4 (replacing m by $m+1$). The multiplication formulas of 3.17.2 therefore remain valid. The same arguments that were used in the proof of that lemma give

$$f_i = s_i p_j = s_i e_k = s_i^2 = f_i p_j = f_i e_k = f_i s_i = f_i^2 ,$$

$$s_i s_j = f_i s_j = s_i f_j = f_i f_j ,$$

for $i < m$, $j \le m+1$, and $k \le m+2$. Thus, e_{m+1} and f_i belong to $\mathscr{E}$. Clearly, if $i \ne k$, then $f_i \wedge f_j = e_{m+2}$ in $\mathscr{E}$. Since $\mathscr{E}$ is distributive by Proposition 3.16.4, it follows that the mapping $G \mapsto f(G) = \Pi_{i\in G} f_i$ is an injective lattice homomorphism from the Boolean algebra of all subsets of m to $\mathscr{E}$. $\square$

This lemma already gives the converse of Trnková's Theorem. Indeed, if $a = f_0 + f_1 + \cdots + f_{n-1}$ in $\boldsymbol{F}$, then it is easy to see that $a^i \ne a^j$ if $1 \le i < j \le n$, and $a^n = f_0 \cdots f_{n-1} = a^{n+1}$. Of course, $a \notin \mathscr{D}(\boldsymbol{F})$ if $n > 1$.

Proof of the theorem. We can assume that $n \ge 3$ because of the example in Fig. 21.3. Let $m = 2(n-1)$ in Lemma 3.17.2(b). For $G \subseteq m$, define

$$f(G) = \prod_{i\in G} f_i , \qquad g(G) = f(m - G) .$$

Abbreviate $g(\{i_0, \ldots, i_k\}) = g(i_0, \ldots, i_k)$. Denote

$$E = \{g(0), g(1,2), g(1,3,4), \ldots, g(1,3, \ldots, 2k-1, 2k), \ldots,$$
$$g(1,3, \ldots, 2m-3, 2m-2), g(1,3, \ldots, 2m-3, 2m-1)\} .$$

By 3.17.3, E is antichain in $\mathscr{D}(\boldsymbol{F})$. Proposition 3.14.1(f) guarantees the existence of $q \in \mathscr{D}(\boldsymbol{F})$ such that $\max J_2(q) = E$ and $\kappa(q) = 1$. If $q = ef$ with $e < q$ and $f < q$, then there exist $h_0, h_1 \in E$ such that $e \le h_0$, $f \le h_1$. It would then follow that $q \le \Pi_{i<n} f_i$. However, the only MPI elements in $J_1(\Pi_{i<n} f_i)$ are the p_i and s_j. If q is written as a product of such elements, then the identities that were given in the proof of 3.17.3 show that $q = f(G)$ for some $G \subseteq m$. This is impossible because $\max J_1(f(G)) \ne E$ for all $G \subseteq m$. It is therefore sufficient to prove that $q < q^2 < \cdots < q^{n-1} < q^n = q^{n+1}$. The key step is a proof (by induction on k) that $\max J_2(q^{k+1}) = \{q^k g(0),\ q^k g(1,2), \ldots, q^k g(1,3, \ldots, 2m-2k-2),\ q^k g(1,3, \ldots, 2m-2k-3, 2m-2k-1)\}$. The details of this argument will not be given. $\square$

Notes. Theorem 3.17.1 is due to PIERCE [1983]. The detailed proof of the theorem can be found in that paper. Among the most interesting open questions in the theory of countable Boolean algebras is: "Which commutative semigroups can be embedded in the multiplicative monoid ***BA***?" Trnková's Theorem provides a necessary condition. Dobbertin has conjectured that any finite commutative semigroup that satisfies an identity $x^{n+1} = x^n$ for some n can be embedded in ***BA***.

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Keywords: Boolean algebra, countable, Vaught's theorem, monoids, measures, Boolean hierarchy, Dobbertin's theorem, Ketonen's theorem, primitive Boolean algebras, Trnková's theorem.

MOS subject classification: primary 06E05; secondary 03G05, 06E15.

CHAPTER 22

Measure Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

Since Lebesgue's work at the beginning of this century, theories of measure and integration, unless wilfully perverse, have been dominated by measures on σ-algebras of sets, and for many analysts such algebras have been their first introduction to Boolean algebra. To be sure, the abstract theory of Boolean algebras is hardly necessary to us so long as we are working exclusively with algebras of sets. But we quickly learn that if two sets differ only by a negligible set they are interchangeable for many of the purposes of measure theory, and it is natural to speculate on the structure that results if such equivalent sets are identified. In this way we are led to the idea of a "measure algebra" as the quotient of an algebra of measurable sets by an ideal of negligible sets. It turns out that we have here a construction which is both a leading example for the general theory of Boolean algebras (focusing attention on such concepts as completeness, the countable chain condition, weak distributivity, and homogeneity) and a natural vehicle for some of the most important results of measure theory (Maharam's theorem, the lifting theorem).

I have elsewhere (FREMLIN [1974]) attempted an exposition of measure theory which began with abstract Boolean algebras and specialized a couple of chapters later to measure theory. (The same idea is pursued in CARATHÉODORY [1963].) Here I shall not proceed by this route but by the more conventional approach which takes the idea of measure space as fundamental. My aim is to give an account of those topics which have been important in my own work in recent years, and which can be naturally discussed within the framework of measure algebras, fairly narrowly defined. Thus, finitely additive measures, and countably additive measures which are not strictly positive, will appear only as auxiliary notions; and applications in measure theory are included only when they seem to illuminate phenomena in measure algebras.

The centre of this chapter lies in Sections 3 and 4, dealing with Maharam's classification theorem for measure algebras (3.9) and the von Neumann–Maharam lifting theorem (4.4). These are preceded by sections on the "elementary" theory of measure algebras (Section 2) and on preliminaries from measure theory (Section 1); and followed by three sections on special topics (the problem of characterizing measurable algebras, in Section 5; cardinal functions, in Section 6; and real-valued-measurable cardinals, in Section 7).

I have already confessed to following personal inclination in my choice of topics, and no doubt there are many serious omissions. (There would have been more but for the timely aid of J.R. Choksi.) The most glaring gap of which I am conscious lies in the theory of automorphism groups of measure algebras, on which I offer only a few notes (3.23); a more balanced treatment would give them at least a full section. I should also make it plain that the bibliography makes no attempt at historical completeness. I have listed only the papers to which I referred while writing this chapter, and I have generally not sought out earliest versions of theorems; in particular, the frequent references to my own work should never be taken as claims to priority.

I should like to thank the editor of this Handbook, J.D. Monk, for his energetic and sympathetic assistance; N.J. Kalton, M. Talagrand and F. Topsøe for sending copies of their work; J.R. Choksi, D. Maharam and K. Prikry for suggesting suitable topics for inclusion; and M. Burke, N. Mowbray and S. Todorčević for comments on the MS.

I wrote above that I have not attempted to do justice to the history of the subject. However, there is one historical point on which I feel confident of my judgement. The classification theorem for measure algebras, as we know it, is due to Maharam. (Kolmogorov may have had some of the same ideas, but no matter.) The first published proof of the lifting theorem was given by Maharam. (Von Neumann announced the theorem, and presumably his proof was correct; but he seems never to have written it down.) The control measure problem, which outcrops all over the theories of vector measures, topological Riesz spaces and topological Boolean algebras, and here is the keystone of Section 5, was first formulated by Maharam. On a personal note, reading MAHARAM [1942] was one of the formative experiences of my apprenticeship; and since then she has become a generous and delightful friend. So it is a pleasure, as well as a duty, to declare Dorothy Maharam the chief architect of the present theory of measure algebras, and to dedicate this chapter to her.

1. Measure theory

Important parts of the theory of measure algebras are inaccessible without rather more measure theory than can usually be fitted into undergraduate courses. I therefore give a brief account, mostly without proofs, of some definitions and results I shall call upon. Do not be daunted if some of these are unfamiliar, as they may well be relevant only to parts of the work here. Specialists, on the other hand, will appreciate that I do not always give theorems in their full strengths.

1.1. DEFINITION. A *measure space* is a triple (X, B, μ), where X is a set, B is a σ-algebra of subsets of X, and $\mu: B \to [0, \infty]$ is a function such that

$$\mu(\emptyset) = 0\,,$$

$$\mu\Big(\bigcup_{n\in\omega} b_n\Big) = \sum_{n=0}^{\infty} \mu(b_n) \text{ whenever } (b_n)_{n\in\omega} \text{ is a sequence in } B \text{ such that}$$

$$b_n \cap b_m = \emptyset \text{ for all } m \neq n\,.$$

REMARK. The most important ideas of this chapter will need only measure spaces in which $\mu(X) = 1$. It seems however ungrateful to present a theory which excludes Lebesgue's original measure on $\boldsymbol{R}$ from consideration. And having once accepted the notion of infinite measure there seems to be no natural stopping place short of the definition above, even though it admits some uninstructive complications.

1.2. DEFINITIONS. Let (X, B, μ) be a measure space.

(a) Two matters of notation. For $a \subseteq X$ set $\mu^*(a) = \min\{\mu(b): a \subseteq b \in B\} \in$

$[0, \infty]$. (The minimum is attained because $\mu(\bigcap_{n\in\omega} b_n) \le \inf_{n\in\omega} \mu(b_n)$ for any sequence $(b_n)_{n\in\omega}$ in B.) Observe that $\mu^*(\bigcup_{n\in\omega} a_n) \le \Sigma_{n=0}^{\infty} \mu^*(a_n)$ for any sequence $(a_n)_{n\in\omega}$ in $P(X)$ (because $\mu(\bigcup_{n\in\omega} b_n) \le \Sigma_{n=0}^{\infty} \mu(b_n)$ for any sequence $(b_n)_{n\in\omega}$ in B). Now I write $N_\mu = \{a: a \subseteq X, \mu^*(a) = 0\}$. N_μ is a σ-complete ideal of $P(X)$; its members are called *negligible* sets. (X, B, μ) is *complete* if $N_\mu \subseteq B$.

(b) (i) (X, B, μ) is a *probability space* if $\mu(X) = 1$. (ii) (X, B, μ) is *totally finite* if $\mu(X) < \infty$. (iii) (X, B, μ) is *σ-finite* ("totally σ-finite" in HALMOS [1950]) if there is a sequence $(b_n)_{n\in\omega}$ in B such that $\mu(b_n) < \infty$ for each $n \in \omega$ and $\bigcup_{n\in\omega} b_n = X$. (iv) (X, B, μ) is *decomposable* (or "strictly localizable"; but I am not sure that all authors use this term in exactly the same sense) if there is a partition $(b_i)_{i\in I}$ of X such that

$$B = \{b: b \subseteq X, b \cap b_i \in B \quad \forall i \in I\},$$

$$\mu(b_i) < \infty \quad \forall i \in I,$$

$$\mu(b) = \sum_{i\in I} \mu(b \cap b_i) \quad \forall b \in B.$$

(v) (X, B, μ) is *semi-finite* if whenever $b \in B$ and $\mu(b) = \infty$, then there is a $c \in B$ such that $c \subseteq b$ and $0 < \mu(c) < \infty$.

(c) If $\varphi(x)$ is a formula with a free variable x I say that "$\varphi(x)$ μ-a.e.(x)" or "φ a.e. on X" or perhaps just "φ a.e." if $\{x \in X: \neg\varphi(x)\} \in N_\mu$.

REMARKS. It is easy to check that the conditions (i)–(v) of (b) are arranged in descending order of strength, so that σ-finite spaces are decomposable, and so forth. (See FREMLIN [1974] for details.) There is an easy construction of the "completion" of a measure space, replacing B by $\hat{B} = \{b \triangle n: b \in B, n \in N_\mu\}$ and μ by $\mu^* \restriction \hat{B}$; I recommend doing this as a matter of habit, so as to deal with complete measure spaces whenever possible.

1.3. LEMMA. *Let B be a σ-complete Boolean algebra and $\nu: B \to \boldsymbol{R}$ a functional which is "countably additive", i.e. $\Sigma_{n=0}^{\infty} \nu(b_n)$ exists and is equal to $\nu(\Sigma_{n\in\omega} b_n)$ whenever $(b_n)_{n\in\omega}$ is a sequence in B such that $b_n \cdot b_m = 0$ for all $m \ne n$. Then there is a $b_0 \in B$ such that $\nu(b) \ge 0$ for every $b \le b_0$ and $\nu(b) \le 0$ for every $b \le -b_0$.*

PROOF. In this form the result may be found in FREMLIN [1974, 52Ha], and CARATHÉODORY [1963, §248]. Elsewhere it generally appears as a lemma (the "Hahn decomposition") on the way to the Radon–Nikodým theorem; B is usually taken to be an algebra of sets, but this does not affect the proof; see Halmos [1950, §29, Theorem A]; BERBERIAN [1962, §49, Theorem 1]; DUNFORD and SCHWARTZ [1958, III.4.10]; HEWITT and STROMBERG [1965, 19.6]. □

1.4. THE RADON–NIKODÝM THEOREM. *Let (X, B, μ) be a σ-finite measure space and $\nu: B \to \boldsymbol{R}$ a countably additive functional such that $\nu(b) = 0$ for every $b \in B \cap N_\mu$. Then there is a B-measurable function $f: X \to \boldsymbol{R}$ such that $\int_b f\, d\mu$ exists and is equal to $\nu(b)$ for every $b \in B$. The function f is essentially unique in the sense that if $g \in {}^X\boldsymbol{R}$ has the same properties, then $f = g$* a.e.

PROOF. HALMOS [1950, §31, Theorem B]; BERBERIAN [1962, §52, Theorem 1]; DUNFORD and SCHWARTZ [1958, III.10.2]; FREMLIN [1974, 63J]; HEWITT and STROMBERG [1965, 19.23]. □

1.5. CONDITIONAL EXPECTATIONS. Let (X, B, μ) be a probability space and $C \subseteq B$ a σ-complete subalgebra. Then $(X, C, \mu \restriction C)$ is again a probability space. If $f: X \to \mathbf{R}$ is any μ-integrable function then the functional

$$c \mapsto \int_c f \, \mathrm{d}\mu : C \to \mathbf{R}$$

is countably additive. So there are C-measurable functions $g: X \to \mathbf{R}$ such that $\int_c f \, \mathrm{d}\mu = \int_c g \, \mathrm{d}\mu$ for every $c \in C$. I shall say that such functions *represent the conditional expectation of f on C*. (I regard the "conditional expectation" itself as the equivalence class of its representatives.)

A different method of constructing conditional expectations is described by IONESCU TULCEA [1969, §II.4]. See also 2.26(h) below.

1.6. CLOSED MARTINGALE THEOREM (HEWITT and STROMBERG [1965, 20.56]). *Let (X, B, μ) be a probability space and $(C_n)_{n\in\omega}$ an increasing sequence of σ-subalgebras of B. Let C be the σ-subalgebra of B generated by $\bigcup_{n\in\omega} C_n$. Let $f: X \to \mathbf{R}$ be a C-measurable μ-integrable function and for each $n \in \omega$ let g_n be a C_n-measurable function representing the conditional expectation of f on C_n. Then* $\lim_{n\to\infty} g_n(x) = f(x)$ μ-a.e. (x).

PROOF. (a) The first step is to show that f can be approximated "in the mean" by C_n-measurable functions if n is large enough. Consider

$$C' = \left\{ b \in C : \inf\left\{ \mu(b \,\triangle\, c) : c \in \bigcup_{n\in\omega} C_n \right\} = 0 \right\}.$$

Then C' is a σ-complete subalgebra of C including $\bigcup_{n\in\omega} C_n$, so is the whole of C.

Now consider the set of H of C-measurable, μ-integrable functions h such that for every $\varepsilon > 0$ there are an $n \in \omega$ and a C_n-measurable, μ-integrable function g such that $\int |h - g| \, \mathrm{d}\mu \leq \varepsilon$. Then H is a linear space of functions containing the characteristic function of any member of $C' = C$; so all C-simple functions belong to H. But for every $\varepsilon > 0$ there is a C-simple function h such that $\int |f - h| \, \mathrm{d}\mu \leq \varepsilon$; it follows at once that $f \in H$.

(b) So given $\varepsilon > 0$ we can find an $m \in \omega$ and a C_m-measurable μ-integrable function g such that $\int |f - g| \, \mathrm{d}\mu \leq \varepsilon^2$. Now, for $m \leq n \in \omega$, set

$$d(n) = \{x \in X : g_n(x) > g(x) + \varepsilon,\ g_i(x) \leq g(x) + \varepsilon \text{ if } m \leq i \leq n\}.$$

Then $d(n) \in C_n$ so

$$\varepsilon\mu(d(n)) \leq \int_{d(n)} g_n \, \mathrm{d}\mu - \int_{d(n)} g \, \mathrm{d}\mu = \int_{d(n)} f \, \mathrm{d}\mu - \int_{d(n)} g \, \mathrm{d}\mu$$
$$= \int_{d(n)} (f - g) \, \mathrm{d}\mu .$$

Observe that $d(i) \cap d(j) = \emptyset$ if $m \leq i < j \in \omega$, so that setting $d = \bigcup_{n \geq m} d(n)$ we have:

$$\varepsilon\mu(d) = \sum_{n=m}^{\infty} \varepsilon\mu(d(n)) \leq \sum_{n=m}^{\infty} \int_{d(n)} (f-g)\,\mathrm{d}\mu = \int_{d} (f-g)\,\mathrm{d}\mu \ ;$$

while also

$$d = \{x \in X\colon \sup_{n \geq m} g_n(x) > g(x) + \varepsilon\} \ .$$

Now if we write

$$d' = \{x \in X\colon g(x) > f(x) + \varepsilon\} \ ,$$
$$e = \{x \in X\colon \limsup_{n \to \infty} g_n(x) > f(x) + 2\varepsilon\}$$

we see that

$$\varepsilon\mu(d') \leq \int_{d'} (g-f)\,\mathrm{d}\mu \ , \quad e \subseteq d \cup d' \ ,$$

so that

$$\varepsilon\mu(e) \leq \int_{d} (f-g)\,\mathrm{d}\mu + \int_{d'} (g-f)\,\mathrm{d}\mu \leq \int |f-g|\,\mathrm{d}\mu \leq \varepsilon^2$$

and $\mu(e) < \varepsilon$. As ε is arbitrary, $\limsup_{n\to\infty} g_n(x) \leq f(x)$ a.e. (x). But, similarly, $\liminf_{n\to\infty} g_n(x) \geq f(x)$ a.e. (x), so $\lim_{n\to\infty} g_n(x)$ exists $= f(x)$ a.e. (x), as required. □

REMARKS. I call this the "closed martingale theorem" to distinguish it from "Doob's martingale theorem" (MEYER [1966, V.T17]), which is a good deal deeper. The proof here is for the benefit of those who want a minimal complete proof of the lifting theorem (4.4 below).

1.7. DEFINITION. A *Radon measure space* is a quadruple $(X, \mathfrak{T}, B, \mu)$ where
(i) (X, B, μ) is a complete measure space;
(ii) if $a \subseteq X$ and $a \cap b \in B$ whenever $b \in B$ and $\mu(b) < \infty$, then $a \in B$;
(iii) $\mathfrak{T}$ is a Hausdorff topology on X;
(iv) $\mathfrak{T} \subseteq B$;
(v) $\mu(b) = \sup\{\mu(c)\colon c \subseteq b,\ c \text{ is compact}\}$ for every $b \in B$;
(vi) for every $x \in X$ there is an open neighborhood u of x such that $\mu(u) < \infty$.

1.8. REMARKS. (a) There is a substantial theory of topological measure spaces, most of which we shall not need.

(b) You may prefer, on first reading, to ignore measure spaces which are not totally finite. In this case the definition simplifies usefully. A totally finite Radon measure space is a quadruple $(X, \mathfrak{T}, B, \mu)$ such that
(i) (X, B, μ) is a complete totally finite measure space;
(ii) $\mathfrak{T}$ is a Hausdorff topology on X;

(iii) $\mathfrak{T} \subseteq B$;

(iv) $\mu(b) = \sup\{\mu(c)\colon c \subseteq b, c \text{ is compact}\}\ \forall b \in B$.

(c) Many authors use the phrase "Radon measure" to mean a measure whose domain is a Borel algebra. In the language of SCHWARTZ [1973], my B is the algebra of "m-essentially measurable" sets. In the language of BOURBAKI [1969], my μ is his $\mu^{\boldsymbol{\cdot}}$. It seems a shame, however, to exclude Lebesgue's measure from the principal class of topological measure spaces, and 1.10 below becomes harder to express.

1.9. SELF-SUPPORTING SETS. Let $(X, \mathfrak{T}, B, \mu)$ be a Radon measure space.

(a) A set $a \subseteq X$ is *self-supporting* if $a \cap u \notin N_\mu$ whenever $u \subseteq X$ is open and $a \cap u \neq \emptyset$.

(b) If $a \subseteq X$ is any set there is a self-supporting set $a' \subseteq a$, relatively closed in a, such that $a \backslash a' \in N_\mu$. To see this, argue as follows. (i) Set $U = \{u \in \mathfrak{T}\colon a \cap u \in N_\mu\}$; then U is upwards-directed; set $v = \bigcup U$, $a' = a\backslash v$. If $c \subseteq v$ is compact, then there is some $u \in U$ such that $c \subseteq u$, so $c \cap a \in N_\mu$. (ii) Suppose that $b \in B$ and $\mu(b) < \infty$. Then there is a sequence $(c_n)_{n\in\omega}$ of compact subsets of $b \cap v$ such that $\sup_{n\in\omega} \mu(c_n) = \mu(b \cap v)$, by 1.7(v). Now $c_n \cap a \in N_\mu$ for each $n \in \omega$, and also $b \cap v \backslash \bigcup_{n\in\omega} c_n \in N_\mu$, so $b \cap a \cap v \in N_\mu$. (iii) Putting (i) and (ii) of 1.7 together, we see that $a \cap v \in B$; using 1.7(v) again, with (i) just above, $a \cap v \in N_\mu$. (iv) Thus, we have a relatively closed $a' \subseteq a$ with $a\backslash a' \in N_\mu$. But now observe that if $u \in \mathfrak{T}$ and $u \cap a' \neq \emptyset$ then $u \notin U$ so that $a \cap u \notin N_\mu$ and $a' \cap u \notin N_\mu$. So a' is self-supporting.

(c) In particular, if $b \in B\backslash N_\mu$, then b has a non-empty self-supporting compact subset; for there is a compact non-negligible $c \subseteq b$, and now we can take c' from (b) above.

1.10. THEOREM. *If $(X, \mathfrak{T}, B, \mu)$ is a Radon measure space, then (X, B, μ) is decomposable.*

PROOF. This is a special case of FREMLIN [1974, 72B]. The essential ideas may be found in BOURBAKI [1969, §1.8], and SCHWARTZ [1973, p. 46]. Since none of these uses exactly the language of this chapter, I repeat the argument here.

(a) It will help to begin with an elementary fact: if $a \subseteq X$ and $a \cap c \in B$ for every compact set $c \subseteq X$, then $a \in B$. For let $b \in B$ be such that $\mu(b) < \infty$. Then there is a sequence $(c_n)_{n\in\omega}$ of compact subsets of b such that $b\backslash\bigcup_{n\in\omega} c_n \in N_\mu$. So $a \cap b\backslash\bigcup_{n\in\omega} c_n \in N_\mu \subseteq B$, and

$$a \cap b = \bigcup_{n\in\omega} (a \cap c_n) \cup \left(a \cap b \Big\backslash \bigcup_{n\in\omega} c_n\right) \in B .$$

Now 1.7(ii) assures us that $a \in B$.

(b) Let D be a maximal disjoint family in B such that each member of D is a non-empty self-supporting compact set. The key fact is this: if $c \subseteq X$ is compact, then $\{d \in D\colon c \cap d \neq \emptyset\}$ is countable. To see this, observe first that $U\{u \in \mathfrak{T}\colon \mu(u) < \infty\}$ is an upwards-directed open cover of X, by 1.7(vi), so there is a $u \in U$ such that $c \subseteq u$. Now, because D is disjoint,

$$|\{d \in D: \mu(u \cap d) \geq 2^{-n}\}| \leq 2^n \mu(u)$$

for every $n \in \omega$; but also each member of D is self-supporting, so

$$\begin{aligned}\{d \in D: c \cap d \neq \emptyset\} &\subseteq \{d \in D: u \cap d \neq \emptyset\} \\ &= \{d \in D: \mu(u \cap d) > 0\}\end{aligned}$$

is countable.

(c) It follows at once that $c \cap \bigcup D \in B$ for every compact $c \subseteq X$. Consequently, by (a), $\bigcup D \in B$. Now $X \setminus \bigcup D \in N_\mu$ by the maximality of D and 1.9(c) above.

(d) So let $(b_i)_{i \in I}$ be an indexation of D (if $X = \bigcup D$) or of $D \cup \{X \setminus \bigcup D\}$ (if $X \neq \bigcup D$). This is a partition of X into measurable sets of finite measure, and $\{i \in I: c \cap b_i \neq \emptyset\}$ is countable for every compact set $c \subseteq X$.

Suppose that $a \subseteq X$ and that $a \cap b_i \in B$ for every $i \in I$. Then $a \cap c \in B$ for every compact $c \subseteq X$, so $a \in B$. Also

$$\begin{aligned}\mu(a) &= \sup\{\mu(c): c \subseteq a, c \text{ is compact}\} \\ &= \sup\left\{\sum_{i \in I} \mu(c \cap b_i): c \subseteq a, c \text{ is compact}\right\} \\ &\leq \sum_{i \in I} \mu(a \cap b_i) \leq \mu(a)\,.\end{aligned}$$

So $(b_i)_{i \in I}$ witnesses that (X, B, μ) is decomposable. □

1.11. PRODUCTS: THEOREM. *Let $((X_i, \mathfrak{T}_i, B_i, \mu_i))_{i \in I}$ be a family of compact Radon probability spaces. Set $X = \Pi_{i \in I} X_i$ and let $\mathfrak{T}$ be the product topology on X. For each $i \in I$ let pr_i: $X \to X_i$ be the canonical projection.*

(a) *There is a unique pair (B, μ) such that $(X, \mathfrak{T}, B, \mu)$ is a Radon probability space and*

$$\mu\left(\bigcap_{m \leq n} \mathrm{pr}_{i_m}^{-1}[b_m]\right) = \mu_{i_0}(b_0) \times \cdots \times \mu_{i_n}(b_n)$$

whenever $i_0, \ldots, i_n$ are distinct members of I and $b_m \in B_m$ for each $m \leq n$.

(b) *For any $c \in B$ there is a c' belonging to the σ-algebra of subsets of X generated by $\{\mathrm{pr}_i^{-1}[b]: i \in I, b \in B_i\}$ such that $c \,\triangle\, c' \in N_\mu$.*

(c) *For any $c \in B$ and any real $\varepsilon > 0$ there is a c' belonging to the algebra of subsets of X generated by $\{\mathrm{pr}_i^{-1}[b]: i \in I, b \in B_i\}$ such that $\mu(c \,\triangle\, c') \leq \varepsilon$.*

PROOF. All the required ideas are in SCHWARTZ [1973, Theorem 17, p. 63, and Theorem 22, p. 81]; see also BOURBAKI [1965, chap. III, §4, no. 6]. A very brief outline of a possible method of proof is given in FREMLIN [1984, A7E]. □

1.12. REMARK. The uniqueness claim in 1.11(a) means that the construction is fully associative; that is, if $\mathcal{J}$ is any partition of I, then the product measure on $\Pi_{i \in I} X_i$ can be identified with the repeated product measure on $\Pi_{J \in \mathcal{J}}(\Pi_{i \in J} X_i)$.

In particular, for instance, the product of a sequence $(X_n)_{n\in\omega}$ can be identified with $X_n \times \Pi_{m\neq n} X_m$ for any $n \in \omega$.

Note that the product here is the Radon measure space product; it will on occasion be essential to know that every closed subset of $\Pi_{i\in I} X_i$ is measurable.

1.13. FUBINI'S THEOREM. *Let $(X_0, \mathfrak{T}_0, B_0, \mu_0)$ and $(X_1, \mathfrak{T}_1, B_1, \mu_1)$ be a pair of compact Radon probability spaces with product $(X, \mathfrak{T}, B, \mu)$. Let $b \in B$. Set*

$$b_x = \{y: (x, y) \in b\} \quad \forall x \in X_0 ,$$

$$a = \{x \in X_0: b_x \in B_1\} .$$

Then $\mu_0(X_0 \setminus a) = 0$ and $\int_a \mu_1(b_x)\mu_0(\mathrm{d}x)$ exists $= \mu(b)$.

PROOF. BOURBAKI [1969, §2.6, Proposition 12]; SCHWARTZ [1973, p. 65]. □

1.14. DEFINITION. A Radon measure space $(X, \mathfrak{T}, B, \mu)$ is *completion regular* if

$$\mu(b) = \sup\{\mu(c): c \subseteq b, c \text{ is a zero set in } X\}$$

for every $b \in B$. (Recall that a *zero set* in a topological space is one of the form $f^{-1}[\{0\}]$ where f is a continuous real-valued function.)

1.15. THE MEASURE OF ${}^I\{0, 1\}$. Let I be any set. For $i \in I$ set $X_i = \{0, 1\}$ and $\mathfrak{T}_i = B_i = P(X_i)$. Define $\mu_i: B_i \to \{0, \frac{1}{2}, 1\}$ by setting $\mu_i(b) = \frac{1}{2}|b|$ for $b \in B_i$. (Thus, μ_i is the "uniform probability" on the finite set $\{0, 1\}$.) Evidently, $(X_i, \mathfrak{T}_i, B_i, \mu_i)$ is a compact Radon probability space. So the construction of 1.11 gives a Radon probability μ on $\Pi_{i\in I} X_i = {}^I\{0, 1\}$. I shall call this the "usual measure" on ${}^I\{0, 1\}$.

The σ-algebra B' of subsets of ${}^I\{0, 1\}$ generated by $\{\mathrm{pr}_i^{-1}[b]: i \in I, b \in B_i\}$ is also important; it is the *Baire σ-algebra* of ${}^I\{0, 1\}$, that is, the σ-algebra generated by the zero sets. (This is because ${}^I\{0, 1\}$ is zero-dimensional and B' is the σ-algebra generated by the clopen sets.)

Many authors in fact take for their "usual" measure on ${}^I\{0, 1\}$ the restriction $\mu \restriction B'$. By Theorem 1.16 below, $({}^I\{0, 1\}, B, \mu)$ is just the measure space completion of $({}^I\{0, 1\}, B', \mu \restriction B')$. (This cannot be taken for granted; see FREMLIN [1976].)

The special case $I = \omega$ is of course particularly significant. In this case we have a natural surjection $f: {}^\omega\{0, 1\} \to [0, 1]$ given by setting $f(x) = \Sigma_{n=0}^{\infty} 2^{-n-1}x(n)$. This is not quite a bijection, but the set

$$d = \{x \in {}^\omega\{0, 1\}: \exists x' \neq x, f(x') = f(x)\}$$

is countably infinite. So there are bijections $g: {}^\omega\{0, 1\} \to [0, 1]$ agreeing with f except on d. Now any such bijection gives an isomorphism between the product probability space $({}^\omega\{0, 1\}, B, \mu)$ and the probability space $([0, 1], B_L, \mu_L)$, where μ_L is the restriction of Lebesgue measure to subsets of $[0, 1]$, and B_L its domain.

1.16. THEOREM. *For any set I, the usual measure on ${}^I\{0, 1\}$ is completion regular.*

PROOF. This is due to KAKUTANI [1943]. It may be deduced from HALMOS [1950, §64, Theorem I], since the usual measure on ${}^I\{0, 1\}$ is translation-invariant. A simple proof is given in CHOKSI and FREMLIN [1979]. $\square$

1.17. DEFINITIONS. (a) Let (X, B, μ) and (Y, C, ν) be measure spaces. A function $f: X \to Y$ is *inverse-measure-preserving* if $f^{-1}[c] \in B$ and $\mu(f^{-1}[c]) = \nu(c)$ for every $c \in C$.

(b) Let (X, B, μ) be a measure space and $(Y, \mathfrak{S})$ a topological space. A function $f: X \to Y$ is *measurable* if $f^{-1}[u] \in B$ for every $u \in \mathfrak{S}$.

(c) Let $(X, \mathfrak{T}, B, \mu)$ be a Radon measure space and $(Y, \mathfrak{S})$ a topological space. A function $f: X \to Y$ is *almost continuous* if

$$\mu(b) = \sup\{\mu(c): c \subseteq b,\ c \in B,\ f \restriction c \text{ is continuous}\}$$

for every $b \in B$.

REMARKS. The functions of (c) are called "μ-measurable" functions by BOURBAKI [1969]. SCHWARTZ [1973] calls them "Lusin μ-measurable"; when (X, B, μ) is complete and decomposable, the functions of (b) are his "$(\mu, \mathfrak{S})$-measurable" functions.

1.18. PROPOSITION. *Let (X, B, μ) be a complete measure space and $(Y, \mathfrak{T}, C, \nu)$ a completion regular zero-dimensional totally finite Radon measure space. If $f: X \to Y$ is such that $\mu(f^{-1}[v])$ exists and is equal to $\nu(v)$ for every clopen set $v \subseteq Y$, then f is inverse-measure-preserving.*

PROOF. Note that $\mu(X) = \nu(Y) < \infty$. Let C' be

$$\{c \in C: \mu(f^{-1}[c]) \text{ exists} = \nu(c)\}\,.$$

Then every clopen subset of Y belongs to C'; if $c \in C'$ then $Y \setminus c \in C'$ (because $\nu(Y) < \infty$); if $(c_n)_{n\in\omega}$ is an increasing sequence in C' then $\bigcup_{n\in\omega} c_n \in C'$; if $c, d \in C'$ and $c \cap d = \emptyset$, then $c \cup d \in C'$; and if $c \subseteq d \in C' \cap N_\nu$ then $c \in C'$. I have to show that $C' = C$.

If $c \subseteq Y$ is a zero set, it is a closed G_δ set; as Y is zero-dimensional, there is an increasing sequence $(v_n)_{n\in\omega}$ of clopen sets in Y with union $Y \setminus c$; so $Y \setminus c$ and c belong to C'.

If c is any member of C, then (because ν is completion regular) there are increasing sequences $(c_n)_{n\in\omega}$, $(c'_n)_{n\in\omega}$ of zero sets such that $\bigcup_{n\in\omega} c_n \subseteq c$, $\bigcup_{n\in\omega} c'_n \subseteq Y \setminus c$, $\lim_{n\to\infty} \nu(c_n) = \nu(c)$, and $\lim_{n\to\infty} \nu(c'_n) = \nu(Y \setminus c)$. Set $d = \bigcup_{n\in\omega} c_n$, $d' = \bigcup_{n\in\omega} c'_n$; then d and d' both belong to C', and they are disjoint, so $d > d'$ and $Y \setminus (d \cup d')$ belong to C'. Since $\nu(d) = \nu(c)$ and $\nu(d') = \nu(Y \setminus c)$, $Y \setminus (d \cup d') \in N_\nu$; so $c \setminus d \in C'$, and $c = d \cup (c \setminus d) \in C'$. So we're through. $\square$

2. Measure algebras

We are now ready to begin the main work of this chapter. In this section I deal with results which are "elementary" in that they do not need either Maharam's theorem or the lifting theorem. Naturally they are somewhat miscellaneous. It may be helpful if I give a list of topics treated, as follows: definitions, 2.1 and 2.7; elementary properties, 2.2–2.3 and 2.9; representation theorems, 2.4–2.6 and 2.11–2.14; weak distributivity, 2.10; homomorphisms and their realizations, 2.15–2.18 and 2.21–2.24; the metric and topology of a measure algebra, 2.19–2.20; measure algebra of a product of measure spaces, 2.25; and function spaces over a measure algebra, 2.26.

2.1. DEFINITIONS. (a) A *measure algebra* is a pair (A, μ) where A is a σ-complete Boolean algebra and $\mu: A \to [0, \infty]$ is a function such that

$$\mu(a) = 0 \qquad \text{iff } a = 0\,,$$

$$\mu\Big(\sum_{n\in\omega} a_n\Big) = \sum_{n=0}^{\infty} \mu(a_n) \quad \text{if } (a_n)_{n\in\omega} \text{ is a sequence in } A \text{ such that } a_m \cdot a_n = 0 \text{ whenever } m \neq n\,.$$

(b) A *measurable algebra* is a Boolean algebra A such that there is some *finite-valued* μ for which (A, μ) is a measure algebra.

REMARKS. I think it best to confront immediately the dilemma of finite-valued vs. infinite-valued measures. On the one hand it is easy to see (the details are spelt out in Section 5) that nearly everything interesting happens with finite measures, and for a Boolean algebraist definition (b) is the natural one to concentrate on. On the other hand I, as a measure theorist, find definition (a) necessary to cover some questions that concern me. So I give definitions which provide a language in which both views can be readily expressed, at the cost of a somewhat unnatural relation between the phrases "measure algebra" and "measurable algebra".

I note that M. Rubin, in Chapter 15 in this Handbook, omits the requirement "$\mu(a) > 0$ whenever $a \neq 0$" from his definition of "measure algebra". You will have no difficulty in determining which of the elementary results here are valid in the wider context.

2.2. ELEMENTARY PROPERTIES OF MEASURE ALGEBRAS. Let (A, μ) be a measure algebra.

(a) If $a \leq b$ in A, then $\mu(a) \leq \mu(b)$ (taking $a_0 = a$, $a_1 = b \cdot -a$, $a_n = 0$ for $n \geq 2$ in Definition 2.1(a)).

(b) For any $a, b \in A$,

$$\mu(a + b) = \mu(a) + (b \cdot -a) \leq \mu(a) + \mu(b)\,.$$

(c) For any increasing sequence $(a_n)_{n\in\omega}$ in A,

$$\mu\Big(\sum_{n\in\omega} a_n\Big) = \mu(a_0) + \sum_{n=0}^{\infty} \mu(a_{n+1} \cdot -a_n)$$
$$= \lim_{n\to\infty} \Big(\mu(a_0) + \sum_{i=0}^{n-1} \mu(a_{i+1} \cdot -a_i)\Big) = \lim_{n\to\infty} \mu(a_n)\,.$$

(d) For any sequence $(a_n)_{n\in\omega}$ in A,

$$\mu\Big(\sum_{n\in\omega} a_n\Big) = \lim_{n\to\infty} \mu\Big(\sum_{i\le n} a_i\Big) \le \sum_{n=0}^{\infty} \mu(a_n)\,.$$

(e) If $D \subseteq A$ is upwards-directed and not empty, and $\sup_{d\in D} \mu(d) = \alpha < \infty$, then $\Sigma\, D$ exists in A and $\mu(\Sigma\, D) = \alpha$. For let $(d_n)_{n\in\omega}$ be a sequence in D such that $\lim_{n\to\infty} \mu(d_n) = \alpha$. Then $a = \Sigma_{n\in\omega} d_n$ exists in A and $\mu(a) \ge \alpha$. Also, for every $n \in \omega$, there is a $d \in D$ such that $\Sigma_{i\le n} d_i \le d$, so

$$\mu(a) = \lim_{n\to\infty} \mu\Big(\sum_{i\le n} d_i\Big) \le \alpha.$$

Thus, $\mu(a) = \alpha$.

Now take any $d \in D$, and examine $d \cdot -a$. For each $n \in \omega$,

$$\mu(d \cdot -a) \le \mu(d \cdot -d_n) = \mu(d + d_n) - \mu(d_n) \le \alpha - \mu(d_n)\,.$$

(The subtractions are valid because $\mu(d + d_n) \le \alpha < \infty$.) So

$$\mu(d \cdot -a) \le \lim_{n\to\infty} (\alpha - \mu(d_n)) = 0$$

and $d \cdot -a = 0$ (because μ is strictly positive). Thus, $d \le a$ for every $d \in D$, and $a = \Sigma\, D$.

(f) If $D \subseteq A$ is upwards-directed and not empty and $\Sigma\, D$ exists in A, then $\mu(\Sigma\, D) = \sup_{d\in D} \mu(d)$. (For (e) covers the case in which $\sup_{d\in D} \mu(d)$ is finite, and the other is trivial.)

(g) If $D \subseteq A$ is downwards-directed and not empty and $\inf_{d\in D} \mu(d) < \infty$, then $\Pi\, D$ exists in A and $\mu(\Pi\, D) = \inf_{d\in D} \mu(d)$. (Apply (e) to $\{d_0 \cdot -d\colon d \in D\}$ where d_0 is any member of D such that $\mu(d_0) < \infty$.) In particular, if $(a_n)_{n\in\omega}$ is a decreasing sequence in A and $\mu(a_0) < \infty$, then $\mu(\Pi_{n\in\omega} a_n) = \inf_{n\in\omega} \mu(a_n)$.

(h) If $a \in A$ and $\mu(a) < \infty$, then the relative algebra $A \restriction a$ satisfies the countable chain condition and is complete. (For if $X \subseteq A \restriction a$ is pairwise disjoint, then

$$|\{x \in X\colon \mu(x) \ge 2^{-n}\}| \le 2^n \mu(a)$$

for every $n \in \omega$, so X must be countable. Thus, $A \restriction a$ satisfies the countable chain condition. But as it is σ-complete it is therefore complete, by Lemma 10.2 of Part I.)

2.3. UNIQUENESS OF MEASURES. If A is a given measurable algebra we naturally ask whether different measures on A must be related. We shall find in Section 3 that the algebraic structure of A does indeed determine the measures that it can carry to a very great extent. Here I give only a pair of useful elementary facts.

PROPOSITION. *Let A be a Boolean algebra and μ, ν two finite-valued functionals such that (A, μ) and (A, ν) are measure algebras. Then*
(a) *for every real $\varepsilon > 0$ there is a $\delta > 0$ such that $\nu(a) \leq \varepsilon$ whenever $\mu(a) \leq \delta$;*
(b) *for every $\varepsilon > 0$ there are $\gamma > 0$ and $c \in A$ such that $\mu(-c) + \nu(-c) \leq \varepsilon$,*

$$\gamma^{-1}\mu(a) \leq \nu(a) \leq \gamma\mu(a) \quad \forall a \in A \restriction c .$$

PROOF. (a) Suppose, if possible, otherwise. Then there is a sequence $(a_n)_{n\in\omega}$ in A such that $\mu(a_n) \leq 2^{-n}$ and $\nu(a_n) \geq \varepsilon$ for every $n \in \omega$. Consider $a = \prod_{n\in\omega} \sum_{m\geq n} a_m$. Then

$$\mu(a) \leq \inf_{n\in\omega} \sum_{m\geq n} \mu(a_m) = 0 ,$$

$$\nu(a) = \inf_{n\in\omega} \nu\Big(\sum_{m\geq n} a_m\Big) \geq \varepsilon ,$$

which is impossible, as μ is strictly positive and $\nu(0) = 0$.

(b) Let γ be so large that $\mu(1) + \nu(1) \leq \varepsilon\gamma/2$ and $\mu(a) \leq \varepsilon/4$ whenever $\nu(a) \leq \gamma^{-1}\mu(1)$ and $\nu(a) \leq \varepsilon/4$ whenever $\mu(a) \leq \gamma^{-1}\nu(1)$ (using (a) in both directions). Now apply 1.3 to the functional $\gamma\mu - \nu$: $A \to \boldsymbol{R}$ to find a $c_0 \in A$ such that

$$(\gamma\mu - \nu)(a) \geq 0 \quad \text{if } a \leq c_0 , \qquad (\gamma\mu - \nu)(a) \leq 0 \quad \text{if } a \leq -c_0 .$$

In particular, $\gamma\mu(-c_0) \leq \nu(-c_0) \leq \nu(1)$ so $\mu(-c_0) \leq \gamma^{-1}\nu(1)$ and $\nu(-c_0) \leq \varepsilon/4$. Similarly, there is a $c_1 \in A$ such that

$$(\gamma\nu - \mu)(a) \geq 0 \quad \text{if } a \leq c_1 ,$$

$$\nu(-c_1) \leq \gamma^{-1}\mu(1) , \qquad \mu(-c_1) \leq \frac{\varepsilon}{4} .$$

So if $c = c_0 \cdot c_1$ we have

$$\mu(-c) + \nu(-c) \leq \gamma^{-1}(\nu(1) + \mu(1)) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon .$$

If $a \leq c$, then $(\gamma\mu - \nu)(a) \geq 0$ and $(\gamma\nu - \mu)(a) \geq 0$, i.e. $\gamma^{-1}\mu(a) \leq \nu(a) \leq \gamma\mu(a)$. $\square$

REMARKS. In part (a), it is not necessary to assume that μ is finite-valued.
There is a generalization of (a) in 5.5(f) below.

2.4. MEASURE ALGEBRAS FROM MEASURE SPACES: THEOREM. *Let (X, B, μ) be any measure space. Then $N_\mu \cap B$ is a σ-complete ideal of B. Let A be the Boolean algebra $B/N_\mu \cap B$ and π: $B \to A$ the canonical epimorphism. Then π is σ-*

complete, and there is a function $\tilde{\mu}$: $A \to [0, \infty]$ defined by writing $\tilde{\mu}(\pi(b)) = \mu(b)$ for every $b \in B$. Now $(A, \tilde{\mu})$ is a measure algebra.

PROOF. π is σ-complete because its kernel $N_\mu \cap B$ is a σ-complete ideal (see the remark following Lemma 5.22 of Part I). To see that $\tilde{\mu}$ is well-defined, we need to know that $\mu(b) = \mu(b')$ whenever $\pi(b) = \pi(b')$, i.e. $b \triangle b' \in N_\mu$; this is because

$$\mu(b) = \mu(b) + \mu(b' \setminus b) = \mu(b' \cup b') = \mu(b') .$$

To see that $(A, \tilde{\mu})$ is a measure algebra, note first that $\tilde{\mu}(a) = 0$ iff a is expressible as $\pi(b)$, where $\mu(b) = 0$, i.e. $b \in N_\mu$; that is, iff $a = 0$. If $(a_n)_{n\in\omega}$ is a sequence in A such that $a_m \cdot a_n = 0$ for $m \neq n$, take any $b_n \in B$ such that $\pi(b_n) = a_n$, for each $n \in \omega$. Set $b'_n = b_n \setminus \bigcup_{i<n} b_i$ for each n; then $\pi(b'_n) = a_n \cdot - \Sigma_{i<n}\, a_i = a_n$ for each n, and $\pi(\bigcup_{n\in\omega} b'_n) = \Sigma_{n\in\omega}\, a_n$ (because π is σ-complete), so

$$\tilde{\mu}\Big(\sum_{n\in\omega} a_n\Big) = \mu\Big(\bigcup_{n\in\omega} b'_n\Big) = \sum_{n=0}^{\infty} \mu(b'_n) = \sum_{n=0}^{\infty} \tilde{\mu}(a_n) .$$

Finally, A is σ-complete because B is a σ-complete Boolean algebra and π is a σ-complete epimorphism. □

2.5. DEFINITION. If (X, B, μ) is a measure space, then $(A, \tilde{\mu})$, as constructed in 2.4, is "the measure algebra of (X, B, μ)".

2.6. MEASURE SPACES FROM MEASURE ALGEBRAS: THEOREM. *Let (A, μ) be any measure algebra. Then it is isomorphic, as measure algebra, to the measure algebra of some measure space.*

PROOF. By the Loomis–Sikorski theorem (Theorem 12.7 of Part I) there are a set X, a σ-algebra B of subsets of X, and a σ-complete ideal M of B such that A is isomorphic, as Boolean algebra, to B/M. Let π: $B \to A$ be the corresponding epimorphism with kernel M; then π is σ-complete. Set $\nu = \mu \circ \pi$: $B \to [0, \infty]$. It is easy to check that (X, B, ν) is a measure space, that $M = B \cap N_\nu$, and that the isomorphism between B/M and A identifies the measure $\tilde{\nu}$ on B/M, constructed as in 2.4, with μ, so that $(A, \mu) \cong (B/M, \tilde{\nu})$. □

2.7. DEFINITIONS (compare 1.2(b)). Let (A, μ) be a measure algebra. (i) (A, μ) is a *probability algebra* if $\mu(1) = 1$. (ii) (A, μ) is *totally finite* if $\mu(1) < \infty$. (iii) (A, μ) is *σ-finite* if there is a sequence $(a_n)_{n\in\omega}$ in A such that $\Sigma_{n\in\omega}\, a_n = 1$ and $\mu(a_n) < \infty$ for every $n \in \omega$. (iv) (A, μ) is a *Maharam algebra* if it is semi-finite (see (v) following) and the Boolean algebra A is complete. (v) (A, μ) is *semi-finite* if, whenever $a \in A$ and $\mu(a) = \infty$, there is a $b \in A \restriction a$ such that $0 < \mu(b) < \infty$.

It is easy to see that a measure space (X, B, μ) is totally finite, or σ-finite, or semi-finite iff its measure algebra is. The measure algebra of (X, B, μ) is a Maharam algebra iff the natural duality between $L^1(X)$ and $L^\infty(X)$ identifies $L^\infty(X)$ with the Banach space dual of $L^1(X)$ (FREMLIN [1974], 64B).

2.8. PROPOSITION. *Let (A, μ) be a semi-finite measure algebra. Then the following are equivalent*:

(i) *(A, μ) is σ-finite*;

(ii) *A satisfies the countable chain condition*;

(iii) *A is a measurable algebra.*

PROOF. (a)(i)$\Rightarrow$(iii). Let $(a_n)_{n\in\omega}$ be a sequence in A such that $\Sigma_{n\in\omega} a_n = 1$ and $\mu(a_n) < \infty$ for every $n \in \omega$. Set $I = \{n \in \omega: \mu(a_n) > 0\}$. For $a \in A$ set

$$\nu(a) = \sum_{n\in I} 2^{-n}\mu(a \cdot a_n)/\mu(a_n) .$$

A straightforward calculation shows that (A, ν) is a totally finite measure algebra, so that A is a measurable algebra.

(b)(iii)$\Rightarrow$(ii). Immediate from 2.2(h).

(c)(ii)$\Rightarrow$(i). Let $D \subseteq A^+$ be a maximal pairwise disjoint family such that $\mu(d) < \infty$ for every $d \in D$. Then D is countable. Because A is σ-complete, $c = \Sigma\, D$ is defined in A. By the maximality of D, there is no $a \leq -c$ such that $0 < \mu(a) < \infty$; because (A, μ) is semi-finite, $\mu(-c) = 0$ and $c = 1$. So any sequence $(a_n)_{n\in\omega}$ running over $D \cup \{0\}$ will witness that (A, μ) is σ-finite. □

REMARK. The idea is that non-semi-finite measure algebras are uninteresting (see 5.1), and that this proposition identifies the measurable algebras among the rest.

2.9. PROPOSITION. *Let (A, μ) be a semi-finite measure algebra.*

(a) *If $c(A) > \omega$, then any maximal pairwise disjoint set $D \subseteq A^+$ such that $\mu(d) < \infty$ for every $d \in D$ has cardinal $c(A)$.*

(b) *There is always a pairwise disjoint $D \subseteq A^+$ such that $|D| = c(A)$.*

PROOF. (a) For let $X \subseteq A^+$ be any other pairwise disjoint set. By 2.2(h), $c(A \restriction d) \leq \omega$ for each $d \in D$, so $X_d = \{x \in X: x \cdot d \neq 0\}$ is countable for each $d \in D$. On the other hand, because D is maximal and (A, μ) is semi-finite, $\bigcup_{d\in D} X_d = X$. So $|X| \leq \max(|D|, \omega)$. As X is arbitrary, $c(A) \leq \max(|D|, \omega)$. so $|D| = c(A)$.

(b) This follows from (a), because if $c(A) \leq \omega$ it is certainly attained (see the remarks following Definition 3.8 of Part I). □

REMARKS. Thus, $c(A)$ is attained for any semi-finite measure algebra (A, μ).

In FREMLIN [1978] I discuss the relationship between properties of a semi-finite measure space and the cellularity of its measure algebra.

2.10. THEOREM. *Let (A, μ) be a semi-finite measure algebra. Then A is weakly (ω, ∞)-distributive.*

PROOF (compare Theorem 14.30 of Part I). Let $(a_{ij})_{i\in\omega, j\in J}$ be any family in A such that $a = \Pi_{i\in\omega} \Sigma_{j\in J}\, a_{ij}$, $b_h = \Pi_{i\in\omega} \Sigma_{j\in h(i)}\, a_{ij}$ exist for every $h: \omega \to [J]^{<\omega}$. I need to show that $a = \Sigma_h b_h$. Since $b_h \leq a$ for every h, I have only to show that there is no upper bound for the b_h strictly less than a.

Suppose, if possible, otherwise; that $a' < a$ and that $b_h \leq a'$ for every $h: \omega \to [J]^{<\omega}$. Because (A, μ) is semi-finite, there is a $c \leq a \cdot -a'$ such that $0 < \mu(c) < \infty$.

Now, for each $i \in \omega$, $c = \Sigma_{j \in J} c \cdot a_{ij}$. Furthermore, $A \restriction c$ satisfies the countable chain condition. So there is a countable set $K \subseteq J$ such that $c = \Sigma_{j \in K} c \cdot a_{ij}$ for every $i \in \omega$, by Lemma 10.2 of Part I.

Let $(j(n))_{n \in \omega}$ be a sequence running over K and for $i, n \in \omega$ set $d_{in} = \Sigma_{m \leq n} c \cdot a_{i, j(m)}$. Then $(d_{in})_{n \in \omega}$ is an increasing sequence and $\Sigma_{n \in \omega} d_{in} = c$; so $\mu(c) = \lim_{n \to \infty} \mu(d_{in})$ for each $i \in \omega$ (by 2.2(c)). For each $i \in \omega$ choose $g(i)$ such that

$$\mu(c) \leq \mu(d_{i, g(i)}) + 2^{-i-2}\mu(c),$$

and consider $h(i) = \{ j(m): m \leq g(i)\} \in [J]^{<\omega}$. We have $c \cdot b_h = \Pi_{i \in \omega} d_{i, g(i)}$, so

$$\begin{aligned} \mu(c \cdot -b_h) = \mu\Big(\sum_{i \in \omega} c \cdot -d_{i, g(i)}\Big) &\leq \sum_{i=0}^{\infty} \mu(c \cdot -d_{i, g(i)}) \\ &\leq \sum_{i=0}^{\infty} 2^{-i-2}\mu(c) < \mu(c), \end{aligned}$$

and $c \cdot b_h \neq 0$. But this is impossible, as $b_h \leq a' \leq -c$.

This shows that $a = \Sigma_h b_h$, as required. □

2.11. THEOREM. *Let (X, B, μ) be a decomposable measure space. Then its measure algebra $(A, \tilde{\mu})$ is a Maharam algebra.*

PROOF. I have already remarked that a decomposable measure space is semi-finite (1.2) and that a semi-finite measure space has a semi-finite measure algebra (2.7). So all I have to show is that A is complete.

Let $\pi: B \to A$ be the canonical epimorphism and let $(b_i)_{i \in I}$ be a partition of X as in 1.2(b(iv)). For each $i \in I$ set $a_i = \pi(b_i)$ so that $\tilde{\mu}(a_i) = \mu(b_i) < \infty$ and $A \restriction a_i$ is complete (2.2(h)).

Let M be any subset of A. Then $d_i = \Sigma \{m \cdot a_i: m \in M\}$ exists for each $i \in I$, and $d_i \leq a_i$. So we can choose $c_i \subseteq b_i$ such that $\pi(c_i) = d_i$ for each $i \in I$. Consider $c = \bigcup_{i \in I} c_i$. Then $c \cap b_i = c_i \in B$ for each $i \in I$, so $c \in B$. I claim that $\pi(c) = \Sigma M$ in A.

To see this, start by taking an arbitrary $m \in M$. Then $m = \pi(b)$ for some $b \in B$. For each $i \in I$,

$$\mu(b_i \cap b \setminus c) = \mu(b_i \cap b \setminus c_i) = \tilde{\mu}(a_i \cdot m \cdot -d_i) = 0.$$

So $\mu(b \setminus c) = \Sigma_{i \in I} \mu(b_i \cap b \setminus c) = 0$ and $m = \pi(b) \leq \pi(c)$. As m is arbitrary, $\pi(c)$ is an upper bound of M.

On the other hand, if a is any upper bound of M in A, we can express a as $\pi(b')$ for some $b' \in B$. For each $i \in I$,

$$\mu(b_i \cap c \setminus b') = \mu(c_i \setminus b') = \tilde{\mu}(d_i \cdot -a) = 0.$$

So $\mu(c \setminus b') = 0$ and $\pi(c) \leq a$. Thus, $\pi(c)$ is the least upper bound of M, as claimed. □

2.12. COROLLARY. *The measure algebra of any Radon measure space is a Maharam algebra.*

PROOF. Put 1.10 and 2.11 together. □

REMARK. Observe that completeness of a measure space (1.2(a)) does not have much to do with the completeness of a measure algebra, except that the proof that Radon measure spaces are decomposable does rely on the hypothesis that they are complete (as measure spaces). For further discussion of these questions see FREMLIN [1975b], [1978].

2.13. THEOREM. *Let (A, μ) be a Maharam algebra. Then it is isomorphic to the measure algebra of a Radon measure space $(Z, \mathfrak{S}, C, \nu)$; if (A, μ) is totally finite, then Z can be taken to be compact.*

PROOF. (a) Let us consider in more detail the Loomis–Sikorski theorem already quoted in 2.6. We know that $\mathrm{Ult}(A)$ is an extremally disconnected compact Hausdorff space, because A is complete (Proposition 7.21 of Part I). Also, the union of any sequence of nowhere dense sets is nowhere dense, because A is weakly (ω, ∞)-distributive (2.10 above and Exercise 9 of Section 14 in Part I); that is to say, every meager set in $\mathrm{Ult}(A)$ is nowhere dense.

For $a \in A$ let $s(a) \subseteq \mathrm{Ult}(A)$ be the corresponding clopen set. Consider

$$Z = \bigcup \{s(a) \colon a \in A,\ \mu(a) < \infty\} .$$

This is an open subset of $\mathrm{Ult}(A)$. Also, it is dense; for if $u \subseteq \mathrm{Ult}(A)$ is any non-empty open set, there is an $a \in A^+$ such that $s(a) \subseteq u$; next, because (A, μ) is semi-finite, there is an $a' \in A \restriction a$ such that $0 < \mu(a') < \infty$, and now $\emptyset \neq s(a') \subseteq Z \cap u$. Let $\mathfrak{S}$ be the subspace topology on Z, generated by $\{Z \cap s(a) \colon a \in A\}$.

(b) Let M be the ideal of nowhere dense subsets of Z. Note that a subset of Z is nowhere dense in Z iff it is nowhere dense in $\mathrm{Ult}(A)$; consequently, M is a σ-complete ideal. Let C be the algebra

$$\{c \colon c \subseteq Z, \exists a \in A \text{ such that } c \,\triangle\, (Z \cap s(a)) \in M\} .$$

Clearly, we have an epimorphism $\pi \colon C \to A$ such that $\pi(c) = a$ iff $c \,\triangle\, (Z \cap s(a)) \in M$, because Z is dense in $\mathrm{Ult}(A)$, and the kernel of π is precisely M, so that π is σ-complete. Note also that C is a σ-algebra. For if $(c_n)_{n \in \omega}$ is any sequence in C, then $a = \Sigma_{n \in \omega} \pi(c_n)$ exists in A, and $s(a) \setminus \bigcup_{n \in \omega} s(a_n)$ is nowhere dense in $\mathrm{Ult}(A)$, so $(\bigcup_{n \in \omega} c_n) \,\triangle\, (Z \cap s(a)) \in M$.

(c) Define $\nu = \mu \circ \pi \colon C \to [0, \infty]$. As in 2.6, it is easy to check that (Z, C, ν) is a measure space, that $N_\nu = M$, and that π identifies (A, μ) with the measure algebra of (Z, C, ν).

(d) We have now to check that $(Z, \mathfrak{S}, C, \nu)$ is a Radon measure space. It will help to have an alternative description of C. If $c \in C$, then $Z \cap s(\pi(c))$ is a clopen set in Z, and differs from c by a nowhere dense set; so the boundary $\partial c = \mathrm{cl}(c)\setminus\mathrm{int}(c)$ (taken in Z) is nowhere dense. On the other hand, if $c \subseteq Z$ and $\partial c \in M$, then $\mathrm{int}(c)$ is an open set in $\mathrm{Ult}(A)$, so its closure in $\mathrm{Ult}(A)$ is $s(a)$ for some $a \in A$ (this is where we really use the hypothesis that A is complete), and $c \triangle (Z \cap s(a)) \in M$, so $c \in C$.

(e) Now let us work through the conditions of Definition 1.7.

(i) Because $N_\nu = M \subseteq C$, (Z, C, ν) is complete.

(ii) Suppose that $d \subseteq Z$ and that $d \cap c \in C$ whenever $c \in C$ and $\nu(c) < \infty$. Then, in particular, $d \cap s(a) \in C$ whenever $a \in A$ and $\mu(a) < \infty$. But this means that $\partial d \cap s(a) = \partial(d \cap s(a)) \in M$ whenever $a \in A$ and $\mu(a) < \infty$. Because $\{s(a)\colon a \in A,\ \mu(a) < \infty\}$ is an open cover of Z, $\partial d \in M$, and $d \in C$.

(iii) Of course $\mathfrak{S}$ is a Hausdorff topology on Z.

(iv) If $u \in \mathfrak{S}$, then $\partial u \in M$ so $u \in C$, by (d) above.

(v) Suppose that $c \in C$. Consider

$$D = \{a\colon a \in A,\ s(a) \subseteq c\}\ .$$

Then D is an upwards-directed set in A and

$$\sup_{d \in D} \mu(d) \le \sup\{\nu(c')\colon c' \subseteq c \text{ is compact}\} \le \nu(c)\ .$$

Suppose, if possible, that $\sup_{d\in D} \mu(d) < \nu(c)$. Set $a = \Sigma\, D \in A$; then $\mu(a) = \sup_{d\in D} \mu(d) < \infty$ (2.2(e)–(f)). As $\nu(c) > \mu(a) = \nu(s(a))$, $c\setminus s(a) \notin M$ and $\mathrm{int}(c)\setminus s(a) \notin M$ and there is a $b \in A^+$ such that $s(b) \subseteq c\setminus s(a)$. But in this case $b \in D$ while $b \cdot a = 0$, which is impossible. Thus, $\sup_{d\in D} \mu(d) = \nu(c)$ and

$$\nu(c) = \sup\{\nu(c')\colon c' \subseteq c \text{ is compact}\}\ .$$

(vi) $Z = \bigcup \{s(a)\colon a \in A,\ \mu(a) < \infty\}$ is covered by open sets of finite measure.

(f) Finally, if (A, μ) is totally finite, then $Z = \mathrm{Ult}(A)$ is compact.

This completes the proof. □

2.14. REMARKS. (a) 2.12–2.13 characterize Maharam algebras as the measure algebras of Radon measure spaces, and totally finite measure algebras as the measure algebras of compact Radon measure spaces.

(b) The construction of 2.13 is in some sense canonical, and it is perhaps worth listing the special properties of the spaces Z obtained there. They are extremally disconnected locally compact Hausdorff spaces in which all meager sets are nowhere dense. The Radon measure ν on Z has the property that the ideal N_ν is precisely the ideal of nowhere dense subsets of Z. In particular, every non-empty open subset of Z has measure greater than 0; Z is self-supporting. The domain C of ν is the algebra of sets with nowhere dense boundaries; it is also the algebra of sets with the Baire property (namely those sets $w \subseteq Z$ such that $w \triangle u$ is meager for some open $u \subseteq Z$). An open subset of Z has compact closure iff it has finite measure. (For if $u \subseteq Z$ is an open set of finite measure, then $u \subseteq s(\pi(u)) \subseteq Z$.) If

e is any subset of Z, then $\nu^*(e) = \nu(\mathrm{cl}(e))$, since if $e \subseteq c \in C$ then $\nu(c) = \nu(\mathrm{cl}(c)) \geq \nu(\mathrm{cl}(e))$. Finally, $(Z, \mathfrak{S}, C, \nu)$ is completion regular, because

$$\nu(c) = \sup\{\nu(c'): c' \subseteq c \text{ is a clopen set}\}$$

for every $c \in C$ (see part (e)(v) of the proof of 2.13).

(c) I shall call a zero-dimensional locally compact Hausdorff space Z *hyperstonian* if for every non-empty open $u \subseteq Z$ there is a Radon measure λ on Z such that $\lambda(u) > 0$ and $\lambda(v) = 0$ for every nowhere dense set $v \subseteq Z$. (For Boolean spaces Z, this is Definition 3 of DIXMIER [1951].) The spaces obtained in 2.13 are of course hyperstonian in this sense. I shall therefore refer to the construction, on occasion, in such terms as: "Let (Z, ν) be the hyperstonian space of (A, μ)."

Observe that the topological space $(Z, \mathfrak{S})$, and even the algebra C, depend only on the algebra A and the ideal $\{a: \mu(a) < \infty\}$, not on any further specification of μ.

(d) For a given measure algebra (A, μ) there is a vast variety of measure spaces which have it as measure algebra. The hyperstonian space is in some sense the most complicated of the possible Radon measure spaces (see 2.17 below). In Section 3 I shall give a completely different proof of 2.13 which produces more or less the simplest Radon measure space from which A can be derived.

2.15. HOMOMORPHISMS. (a) Let (A, μ) and (B, ν) be measure algebras. A homomorphism $f: A \to B$ is *measure-preserving* if $\nu(f(a)) = \mu(a)$ for every $a \in A$. Such a homomorphism must be injective.

(b) If (A, μ) and (B, ν) are totally finite measure algebras and $f: A \to B$ is a measure-preserving homomorphism, it is complete. For if $D \subseteq A$ is any non-empty upwards-directed set, we have $f(\Sigma D) \geq \Sigma f[D]$, but also $\nu(f(\Sigma D)) = \mu(\Sigma D) = \sup_{d \in D} \mu(d) = \sup_{b \in f[D]} \nu(b) = \nu(\Sigma f[D]]$ by 2.2(e)–(f); so that $f(\Sigma D) = \Sigma f[D]$. In this case $f[A]$ is a complete subalgebra of B, isomorphic to A.

2.16. PROPOSITION. *Let (X, B, μ) and (Y, C, ν) be measure spaces, with measure algebras $(A_X, \tilde{\mu})$ and $(A_Y, \tilde{\nu})$. If $f: X \to Y$ is inverse measure-preserving there is a σ-complete measure-preserving homomorphism $g: A_Y \to A_X$ given by the formula*

$$g(\pi_Y(c)) = \pi_X(f^{-1}[c]) \quad \forall c \in C,$$

where $\pi_X: B \to A_X$ and $\pi_Y: C \to A_Y$ are the canonical epimorphisms.

PROOF. To see that g is well-defined we need note only that, for $c, c' \in C$,

$$\begin{aligned}\pi_Y(c) = \pi_Y(c') \Rightarrow c \,\triangle\, c' \in N_\nu \Rightarrow f^{-1}[c] \,\triangle\, f^{-1}[c'] \in N_\mu \\ \Rightarrow \pi_X(f^{-1}[c]) = \pi_X(f^{-1}[c']) .\end{aligned}$$

It now follows at once that g is a measure-preserving homomorphism. It is σ-complete because its kernel is

$$\{\pi_Y(c): c \in C, f^{-1}[c] \in N_\mu\}$$

and this is a σ-complete ideal of A_Y. $\square$

2.17. THEOREM. *Let $(X, \mathfrak{T}, B, \mu)$ be a Radon measure space with measure algebra $(A, \tilde{\mu})$. Let $(Z, \mathfrak{S}, C, \nu)$ be the hyperstonian space of $(A, \tilde{\mu})$ (2.13–2.14), and let $\pi_X: B \to A$ and $\pi_Z: C \to A$ be the canonical epimorphisms. Then there is an almost continuous inverse-measure-preserving function $f: Z \to X$ such that $\pi_Z(f^{-1}[b]) = \pi_X(b)$ for every $b \in B$. If X is compact then f can be taken to be continuous.*

PROOF. (a) Set

$$D = \{\pi_X(v): v \subseteq X \text{ is compact}\} \subseteq A .$$

Then D is an upwards-directed subset of A, and $D \setminus \{0\}$ is dense, by 1.7(v). If $d \in D$, then $\tilde{\mu}(d) < \infty$, so that $s(d) \subseteq Z$, where $s: A \to \mathrm{Clop}(\mathrm{Ult}(A))$ is the canonical isomorphism. Set $W = \bigcup \{s(d): d \in D\}$; then W is a dense open subset of Z.

(b) For each $z \in W$ consider

$$K_z = \{v: v \subseteq X \text{ is compact, } z \in s(\pi_X(v))\} .$$

Then K_z is a non-empty, downwards-directed family of compact subsets of X, not containing $\emptyset$. So $\bigcap K_z \neq \emptyset$. But in fact $\bigcap K_z$ is a singleton. For suppose that $x \in \bigcap K_z$ and $x' \in X \setminus \{x\}$. Then there are disjoint open sets $u, u' \subseteq X$ such that $x \in u$ and $x' \in u'$. (This is one of the few places where we use essentially the requirement that Radon measure spaces must be Hausdorff.) Fix $v_0 \in K_z$. Then $v_0 \setminus u \notin K_z$ (since $x \notin v_0 \setminus u$) so $z \notin s(\pi_X(v_0 \setminus u))$ and $z \in s(\pi_X(v_0 \cap u)) \subseteq s(\pi_X(v_0 \setminus u'))$ and $v_0 \setminus u' \in K_z$, so $x' \notin \bigcap K_z$.

(c) We can therefore define a function $g: W \to X$ by writing $\bigcap K_z = \{g(z)\}$ for each $z \in W$. In this case g is continuous. For suppose that $g(z) \in u \in \mathfrak{T}$. Then $\bigcap K_z \subseteq u$; because K_z is a downwards-directed family of compact sets, there is a $v \in K_z$ such that $v \subseteq u$. Now if z' is any member of $s(\pi_X(v))$, $v \in K_{z'}$ so $g(z') \in v \subseteq u$. Thus $s(\pi_X(v))$ is a neighbourhood of z included in $g^{-1}[u]$. As z and u are arbitrary, g is continuous.

(d) If $X = \emptyset$, then $Z = \emptyset$ and there is nothing more to be said. Otherwise, we can extend g arbitrarily to a function $f: Z \to X$. Such an f is almost continuous because $f \restriction W = g$ is continuous and W is dense in Z, so that $\nu(Z \setminus W) = 0$. (Recall that N_ν is just the ideal of nowhere dense subsets of Z.)

(e) Now suppose that $b \in B$. Set

$$D_1 = \{\pi_X(v): v \subseteq b \text{ is compact}\} , \qquad W_1 = \bigcup \{s(d): d \in D_1\} ,$$

$$D_2 = (\pi_X(v): v \subseteq X \setminus b \text{ is compact}\} , \qquad W_2 = \bigcup \{s(d): d \in D_2\} .$$

Then D_1 and D_2 are upwards-directed sets in A, $d_1 \cdot d_2 = 0$ whenever $d_1 \in D_1$ and $d_2 \in D_2$, and $(D_1 \cup D_2) \setminus \{0\}$ is dense in A^+. So W_1, W_2 are disjoint open subsets of W and $W_1 \cup W_2$ is dense in Z. Now if $z \in W_1$ there is a compact $v \subseteq b$ such that $z \in s(\pi_X(v))$, and $f(z) = g(z) \in v$; thus $f[W_1] \subseteq b$. Similarly, $f[W_2] \subseteq X \setminus b$. So

$$W_1 \subseteq f^{-1}[b] \subseteq Z \setminus W_2 .$$

But as $W_1 \cup W_2$ is dense in Z, $Z \setminus (W_1 \cup W_2) \in N_\nu$, so $f^{-1}[b] \in C$, because (Z, C, ν) is complete. Also,

$$\pi_Z(f^{-1}[b]) = \pi_Z(W_1) \geq \sum D_1 ,$$
$$\pi_Z(Z \setminus f^{-1}[b]) = \pi_Z(W_2) \geq \sum D_2 .$$

Since $(D_1 \cup D_2) \setminus \{0\}$ is dense in A^+, $\Sigma D_2 = -\Sigma D_1$ and $\pi_Z(f^{-1}[b])$ must be exactly ΣD_1. At the same time we know that $\Sigma D_1 \leq \pi_X(b)$ and $\Sigma D_2 \leq \pi_X(X \setminus b)$, so $\pi_X(b) = \Sigma D_1 = \pi_Z(f^{-1}[b])$.

(f) It follows at once that f is inverse-measure-preserving; for if $b \in B$, then

$$\nu(f^{-1}[b]) = \tilde{\mu}(\pi_Z(f^{-1}[b])) = \tilde{\mu}(\pi_X(b)) = \mu(b) .$$

(g) Finally, if X is compact, then $W = Z = \mathrm{Ult}(A)$ and $f = g$ is continuous. □

2.18. REMARK. Observe that, in the general case of 2.17, f is not just "almost continuous"; there is a conegligible open subset W of Z such that $f \restriction W$ is continuous.

When X is compact, we may reasonably call the function f of 2.17 "the canonical map" from $Z = \mathrm{Ult}(A)$ to X. When X is not compact, it would be natural to prefer g.

EXERCISE. Consider the application of the above theorem when $X = \boldsymbol{R} \setminus \boldsymbol{Q}$ and μ is the restriction of Lebesgue measure to subsets of X. Relate this to the case $X = \boldsymbol{R}$.

2.19. THE UNIFORMITY OF A MEASURE ALGEBRA. (a) Let (A, μ) be a totally finite measure algebra. For $a, b \in A$ set

$$\rho(a, b) = \mu(a \triangle b) .$$

An easy computation shows that ρ is a metric for which the Boolean operations $+, \cdot, -$ are uniformly continuous. Moreover, A is complete under this metric, for if $(a_n)_{n\in\omega}$ is a sequence in A such that $\Sigma_{n=0}^{\infty} \rho(a_n, a_{n+1}) < \infty$ then $(a_n)_{n\in\omega}$ converges for ρ to $\Pi_{n\in\omega} \Sigma_{m\geq n} a_m$. (See 5.6 below for a fuller account of a more general case.)

I shall call ρ "the" metric of the totally finite measure algebra (A, μ).

(b) If A is a measurable algebra, then two different totally finite measures on A give rise to uniformly equivalent metrics on A, by 2.3(a). So we may speak of "the" uniformity or topology of a measurable algebra.

(c) If (A, μ) is an arbitrary measure algebra then we can define pseudo-metrics ρ_d on A, for those $d \in A$ such that $\mu(d) < \infty$, by writing $\rho_d(a, b) = \mu(d \cdot (a \triangle b))$. These pseudo-metrics define a uniformity on A which is Hausdorff iff (A, μ) is semi-finite. If (A, μ) is semi-finite, then it is complete under the uniformity iff it is a Maharam algebra. (For in this case it can be identified, as uniform space, with a product of totally finite relative subalgebras.) The uniformity is metrizable iff (A, μ) is σ-finite.

If A is a Boolean algebra for which there is some μ such that (A, μ) is a semi-finite measure algebra, then all such μ give rise to the same uniformity; so once again we may speak of "the" uniformity and topology of A.

(d) If (A, μ) is any measure algebra then $(A, \triangle)$ is a topological group under the uniformity described in (a) or (c). (So the uniformity is determined by the topology.)

(e) Let (A, μ) and (B, ν) be semi-finite measure algebras and $f: A \to B$ a homomorphism. (i) If f is σ-complete, it is sequentially continuous for the topologies of (a) or (c). (ii) If f is complete, it is uniformly continuous for the uniformities of (a) or (c). (iii) Conversely, if f is (sequentially) continuous for the topologies, then it is (σ-)complete. I leave the proofs as an exercise; they are straightforward from 2.2.

2.20. COMPLETE SUBALGEBRAS. If A is a measurable algebra, then a subalgebra B of A is Boolean-complete-in-A (i.e. closed under arbitrary suprema and infima) iff it is closed for the topology $\mathfrak{T}$ of A described in 2.19(a)–(b). (For let μ be a totally finite measure on A. If B is Boolean-complete in A, then $(B, \mu \restriction B)$ is a measure algebra, so is metrically complete, and B must be a metrically-closed subset of A. While if B is metrically closed in A and D is any upwards-directed subset of B, then 2.2(f) tells us that $\Sigma\, D$ belongs to the metric closure of D, which is included in B.)

If B is any subalgebra of A, then the $\mathfrak{T}$-closure $\mathrm{cl}(B)$ of B is also a subalgebra, because the Boolean operations are continuous; so $\mathrm{cl}(B)$ is a complete subalgebra of A; evidently it is the complete subalgebra of A generated by B.

These ideas apply equally to Maharam algebras which are not σ-finite, if we delete the words "metric", "metrically" appropriately.

2.21. PROPOSITION. *Let (X, B, μ) be a measure space, with measure algebra $(A, \tilde{\mu})$, and I any set. Let μ_I be the usual measure on ${}^I\{0,1\}$ (1.15); write B_I for the domain of μ_I, and B'_I for the Baire σ-algebra of ${}^I\{0,1\}$; let $(A_I, \tilde{\mu}_I)$ be the measure algebra of $({}^I\{0,1\}, B_I, \mu_I)$. Let $\pi: B \to A$ and $\pi_I: B_I \to A_I$ be canonical epimorphisms. Now suppose that $f: A_I \to A$ is any σ-complete homomorphism. Then there is a function $g: X \to {}^I\{0,1\}$ such that $g^{-1}[b] \in B$ and $\pi(g^{-1}[b]) = f(\pi_I(b))$ for every $b \in B'_I$. If (X, B, μ) is complete, then $g^{-1}[b] \in B$ and $\pi(g^{-1}[b]) = f(\pi_I(b))$ for every $b \in B_I$. If f is measure-preserving, then g will be inverse-measure-preserving for μ, $\mu_I \restriction B'_I$ or μ, μ_I accordingly.*

PROOF. For $i \in I$ set

$$b_i = \{x \in {}^I\{0,1\}: x(i) = 1\} \in B'_I,$$

and choose $c_i \in B$ such that $\pi(c_i) = f(\pi_I(b_i))$. Define $g: X \to {}^I\{0,1\}$ by setting $g(x)(i) = 1$ if $x \in c_i$, 0 otherwise. Then we see that (because π, π_I and f are all σ-complete homomorphisms)

$$C = \{b \in B_I: g^{-1}[b] \in B \text{ and } \pi(g^{-1}[b]) = f(\pi(b))\}$$

is a σ-algebra of sets containing every b_i (because $g^{-1}[b_i] = c_i$). So $C \supseteq B'_I$.

If (X, B, μ) is complete and $b \in B_I$, there are $b', b'' \in B_I'$ such that $b' \subseteq b \subseteq b''$ and $\pi_I(b') = \pi_I(b) = \pi_I(b'')$, because $({}^I\{0,1\}, B, \mu)$ is completion regular (1.16). Now $\pi(g^{-1}[b''\setminus b']) = 0$, i.e. $g^{-1}[b'']\setminus g^{-1}[b'] \in N_\mu$; since $N_\mu \subseteq B$, it follows that $g^{-1}[b] \in B$ and that $\pi(g^{-1}[b]) = \pi(g^{-1}[b']) = \pi_I(b)$.

If f is measure-preserving, then

$$\mu(g^{-1}[b]) = \tilde{\mu}(\pi(g^{-1}[b])) = \tilde{\mu}(f(\pi_I(b))) = \tilde{\mu}_I(\pi_I(b)) = \mu_I(b)$$

for every $b \in C$, so g is inverse-measure-preserving for μ and μ_I (if (X, B, μ) is complete) and for μ and $\mu_I \restriction B_I'$ in any case. $\square$

2.22. I give two further results of this kind without proof.

PROPOSITION. *Let $(X_i)_{i\in I}$ be a family of Polish spaces, with product X; let B be the Baire σ-algebra of X. Let Z be any set, C a σ-algebra of subsets of Z, M a σ-complete ideal of C, and $\pi: C \to C/M$ the canonical epimorphism.*

Then for every σ-complete homomorphism $f: B \to C/M$ there is a function $g: Z \to X$ such that $g^{-1}[b] \in C$ and $\pi(g^{-1}[b]) = f(b)$ for every $b \in B$.

2.23. THEOREM. *Let I be a countable set and K any set. For each $i \in I$, let $(X_\xi^{(i)})_{\xi\in K}$ be a family of Polish spaces, with product X_i; let C_i be the Baire σ-algebra of X_i. Suppose that we are given a proper σ-complete ideal M_i of C_i such that*

$$\left.\begin{array}{l}\textit{for every } \xi \in K \textit{ there is an uncountable Borel set}\\ m \subseteq X_\xi^{(i)} \textit{ such that } \{x \in X_i: x(\xi) \in m\} \in M_i\,.\end{array}\right\}\quad (*)$$

Let A_i be the quotient algebra C_i/M_i and $\pi_i: C_i \to A_i$ the canonical epimorphism.

Now supose that we have for each pair i, j of members of I a countable set F_{ij} of σ-complete homomorphisms from A_i to A_j. Set $F = \bigcup_{i,j\in I} F_{ij}$. Then there is a family $(g_f)_{f\in F}$ such that

(i) *whenever $i, j \in I$ and $f \in F_{ij}$, g_f is a function from X_j to X_i such that $g_f^{-1}[b] \in C_j$ and $\pi_j(g_f^{-1}[b]) = f(\pi_i(b))$ for every $b \in C_i$;*

(ii) *whenever $i, j, k \in I$ and $f \in F_{ij}$ and $f' \in F_{jk}$ and $f' \circ f \in F_{ik}$, then $g_{f'\circ f} = g_f \circ g_{f'}: X_k \to X_i$;*

(iii) *whenever $i \in I$ and the identity homomorphism $e(i): A_i \to A_i$ belongs to F_{ii}, then $g_{e(i)}$ is the identity map from X_i to itself.*

2.24. REMARKS. (a) 2.21–2.23 all deal with the "realization" of homomorphisms. Suppose that for $i = 0, 1$ we have systems $(X_i, B_i, M_i, A_i, \pi_i)$ where X_i is a set, B_i is a subalgebra of $P(X_i)$, M_i is an ideal of B_i, A_i is the quotient B_i/M_i and $\pi_i: B_i \to A_i$ is the canonical epimorphism. If $f: A_0 \to A_1$ is a homomorphism, we can say that a function $g: X_1 \to X_0$ *realizes* f if $g^{-1}[b] \in B_1$ and $\pi_1(g^{-1}[b]) = f(\pi_0(b))$ for every $b \in B_0$.

(b) 2.21 is of course perfectly elementary; any obscurity is due entirely to the language I am using. I have taken the trouble to work through it in detail because it is a foundation for deep results in Section 4 below.

2.22 is also elementary in view of the following facts: (i) if X is any uncountable Polish space and B its Borel algebra, then (X, B) is isomorphic to $({}^{\omega}\{0,1\}, B'_{\omega})$, where B'_{ω} is the Baire σ-algebra of ${}^{\omega}\{0,1\}$; (ii) if $(X_i)_{i\in I}$ is any family of separable metric spaces, then the Baire σ-algebra of $X = \Pi_{i\in I} X_i$ is precisely the σ-algebra of subsets of X generated by

$$\{\mathrm{pr}_i^{-1}[b]: i \in I, b \text{ is a Borel subset of } X_i\}\ .$$

It follows that if X is any product of Polish spaces and C is its Baire σ-algebra, then (unless X is countable) (X, C) is isomorphic to $({}^{\kappa}\{0,1\}, B'_{\kappa})$, where κ is the weight of X.

(c) 2.23 goes much further in that it deals with the simultaneous coherent realization of many homomorphisms. An important special case is the following. If I is a singleton $\{i\}$ we can start with a single automorphism $f: A_i \to A_i$ and set $F_{ii} = \{f, e(i), f^{-1}\}$. Then we shall obtain a realization g_f of f which is bijective and such that $g_f^{-1} = g_{f^{-1}}$ realizes f^{-1}. This is in effect the case dealt with in CHOKSI [1972a], [1972b] and MAHARAM [1979], where all the ideas necessary for the general case may be found.

(d) In 2.23 the conclusion involves only the structures (X_i, C_i, M_i), and as explained in (b) the structures (X_i, C_i) are determined by $|K|$, being isomorphic to $({}^{\kappa}\{0,1\}, B'_{\kappa})$, where $\kappa = \max(|K|, \omega)$ unless $K = \emptyset$. The systems $(X_\xi^{(i)})_{\xi\in K}$ appearing in the hypotheses are used to describe a sense in which each M_i is adequately large. It is easy to see that this requirement cannot be entirely dispensed with. But, in applications, if the hypothesis $(*)$ of 2.23 is not immediately satisfied for every $i \in I$, we can seek to replace the system $(X_\xi^{(i)})_{i\in I, \xi\in K}$ by $(\Pi_{\xi\in K(\eta)} X_\xi^{(i)})_{i\in I, \eta<\lambda}$, where $(K(\eta))_{\eta<\lambda}$ is a partition of K into countable sets. This amounts to a preliminary re-organization of the factors which does not affect the products (X_i, C_i); but it evidently renders $(*)$ easier to achieve. In the (relatively simple) special case where K is countable (but not empty), for instance, we can take $\lambda = 1$, and $(*)$ becomes just "M_i contains an uncountable set". And if K is infinite then $(*)$ can be replaced by such forms as "for every countably infinite $J \subseteq K$ there is a $z \in \Pi_{\xi\in J} X_\xi^{(i)}$ such that $\{x \in X_i: x \upharpoonright J = z\} \in M_i$" (for then any partition of K into countably infinite sets will achieve $(*)$ itself). Observe that this last version of the hypothesis is satisfied whenever A_i satisfies the countable chain condition and every $X_\xi^{(i)}$ has at least two members.

(e) In measure-theoretic applications the ideals M_i of 2.23 are of course generally of the form $C_i \cap N_{\mu_i}$ for appropriate measures μ_i. Naturally, the results are most useful when the μ_i are σ-finite and completion regular, so that they can be easily reconstructed from $\mu_i \upharpoonright C_i$, as in 2.21. But I have tried to express 2.22–2.23 in a form which makes it clear that they do not really belong to measure theory; in fact their proofs seem to belong rather to descriptive set theory.

(f) The Boolean algebras A_i of 2.23 must be σ-complete, but there is no other restriction on them. Indeed, given a σ-complete Boolean algebra A, then it is isomorphic to a quotient B'_{κ}/M for any κ large enough. (The necessary technique is described, in a special case, in FREMLIN [1984], 12F. The minimal value of κ that can be used is exactly the least value of $|X|$ for any subset X of A such that A is the σ-complete subalgebra of itself generated by X. If A satisfies the countable chain condition, this is just $\tau(A)$.) The natural construction does not automatically achieve the requirement $(*)$; but if we embed ${}^{\kappa}\{0, 1\}$ in ${}^{\kappa}[0, 1]$ and replace M by

$$M' = \{b\colon b \text{ belongs to the Baire } \sigma\text{-algebra of } {}^{\kappa}[0,1] \text{ and}$$
$$b \cap {}^{\kappa}\{0,1\} \in M\},$$

then we obtain a representation of A to which 2.23 can be applied, and which may seem more accessible than the representation provided by the usual proof of the Loomis–Sikorski theorem, as given in Theorem 12.7 of Part I.

(g) Of course representations based on Stone spaces can give much stronger results on the realization of homomorphisms. If A_0 and A_1 are arbitrary σ-complete Boolean algebras, represented as B_i/M_i, where, for each i, B_i is the algebra of Borel subsets of $\mathrm{Ult}(A_i)$ and M_i is the ideal of meager Borel sets, then every σ-complete homorphism $f\colon A_0 \to A_1$ is uniquely realized by a continuous function $g_f\colon \mathrm{Ult}(A_1) \to \mathrm{Ult}(A_0)$; and thus we can obtain simultaneous realizations, satisfying (i)–(iii) of 2.23, with no restrictions on the cardinalities of I or of the F_{ij}, and with much better-behaved realizing functions g_f. (In the context of 2.23, for instance, the g_f are not in general Borel measurable.) The point of such results as 2.23 is that important algebras appear more or less directly in the form considered there.

2.25. FREE PRODUCTS OF MEASURE ALGEBRAS. (a) Let $(C_i)_{i\in I}$ be a family of Boolean algebras and for each $i \in I$ let $\nu_i\colon C_i \to \boldsymbol{R}$ be a functional which is *additive*, i.e. such that $\nu_i(a + b) = \nu_i(a) + \nu_i(b)$ whenever $a \cdot b = 0$; suppose further that $\nu_i(1) = 1$ for each $i \in I$. Let $C = \bigoplus_{i\in I} C_i$ be "the" free product of $(C_i)_{i\in I}$, with $e_i\colon C_i \to C$ the canonical homomorphisms. (See Section 11 of Part I.) Then there is a unique additive functional $\nu\colon C \to \boldsymbol{R}$ such that $\nu(\Pi_{i\in J} e_i(c_i)) = \Pi_{i\in J} \nu_i(c_i)$ whenever $J \subseteq I$ is finite and $c_i \in C_i$ for each $i \in J$. (To see this, it is perhaps easiest to begin with the case in which I and all the C_i are finite, so that C is finite and we can begin by defining ν on the atoms of C. For the general case, observe that $\bigoplus_{i\in I} C_i$ is an inductive limit of algebras of the form $\bigoplus_{i\in J} D_i$, where J is a finite subset of I and D_i is a finite subalgebra of C_i for each $i \in J$.)

(b) Now suppose that $((C_i, \nu_i))_{i\in I}$ is a family of probability algebras. Let $C = \bigoplus_{i\in I} C_i$ and let $\nu\colon C \to \boldsymbol{R}$ be the corresponding additive functional. We see that $\nu(c) > 0$ for every $c \in C^+$. (C, ν) is not a measure algebra, except in trivial cases, because C is not σ-complete (Proposition 11.9 of Part I). But we still have a metric ρ on C given by $\rho(a, b) = \nu(a \triangle b)$, as in 2.19(a); and if we complete C with respect to this metric we obtain a complete Boolean algebra $\hat{C}$ on which a strictly positive measure $\hat{\nu}$ can be defined as the unique continuous extension of ν (for details see 5.6(b) below).

(c) If $((X_i, \mathfrak{T}_i, B_i, \mu_i))_{i\in I}$ is a family of compact Radon probability spaces, consider the following constructions. (i) For each $i \in I$ let $(C_i, \tilde{\mu}_i)$ be the measure algebra of (X_i, B_i, μ_i); now let $(\hat{C}, \hat{\mu})$ be the measure algebra completion of $\bigoplus_{i\in I} C_i$, as in (b). (ii) Let $(X, \mathfrak{T}, B, \mu)$ be the Radon measure space product of

$((X_i, \mathfrak{T}_i, B_i, \mu_i))_{i\in I}$ and $(A, \tilde{\mu})$ its measure algebra. (iii) We have a natural embedding of $\bigoplus_{i\in I} B_i$ into B given by sending $e_i(b)$ to $\mathrm{pr}_i^{-1}[b]$ for $i\in I$, $b\in B_i$, where pr_i: $X\to X_i$ is the canonical projection; it is easy to check that this gives rise to an embedding of $\bigoplus_{i\in I} C_i$ in A, which respects the measures; and 1.11(c) is the fact we need to show that A can be identified with the metric space completion $\hat{C}$ of $\bigoplus_{i\in I} C_i$.

(d) For finite index sets I the construction of (a)–(b) can be attempted for arbitrary measure algebras. If $((C_i, \nu_i))_{i\in I}$ is a finite family of measure algebras, and $C=\bigoplus_{i\in I} C_i$, then there is a unique ν: $C\to[0,\infty]$ such that $\nu(\Pi_{i\in I} e_i(c_i)) = \Pi_{i\in I} \nu_i(c_i)$ whenever $c_i\in C_i$ for each $i\in I$, and $\nu(a+b)=\nu(a)+\nu(b)$ whenever $a, b\in C$ and $a\cdot b=0$. (In interpreting the product $\Pi_{i\in I}\nu_i(c_i)$ we must take $0\cdot\infty=0$ and $\alpha\cdot\infty=\infty$ for every $\alpha>0$.) If each (C_i, ν_i) is semi-finite, then $\{c\in C: 0<\nu(c)<\infty\}$ will be dense in C^+, so that we can use the approach of 2.19(c) to construct a Hausdorff uniformity on C; now once again we can complete C with respect to this uniformity to obtain a Maharam algebra $(\hat{C}, \hat{\nu})$.

If, for each $i\in I$, (C_i, ν_i) is the measure algebra of a Radon measure space $(X_i, \mathfrak{T}_i, B_i, \mu_i)$, then, just as in (c), $(\hat{C}, \hat{\nu})$ can be identified with the measure algebra of the Radon measure space product of $((X_i, \mathfrak{T}_i, B_i, \mu_i))_{i\in I}$, as constructed in BOURBAKI [1969, §2.5], or SCHWARTZ [1973, p.63].

(e) Of course it is also possible to discuss products of measure spaces which are not Radon measure spaces. In FREMLIN [1984], A6K, I describe the appropriate constructions. For finite products the best method is the (measure space completion of the) construction of BERBERIAN [1962, §39]. (HALMOS [1950] will not do, except for σ-finite measures.) For infinite products (of probability spaces) we need the (measure space completion of the) product described by DUNFORD and SCHWARTZ [1958, III.11.20], by HALMOS [1950, note 2 to §38], and by HEWITT and STROMBERG [1965, 22.8].

Having got this right, we find that for decomposable measure spaces the measure algebra of the product is the completed free product of the measure algebras, as in (c)–(d). But for non-decomposable measure spaces some new problems can arise, touched on in FREMLIN [1978].

(f) It is essential to understand that C is not the Boolean algebra completion of $\bigoplus_{i\in I} C_i$ except in elementary cases. In fact, if (C_0, ν_0) and (C_1, ν_1) are atomless probability algebras, $(C_0\oplus C_1)^+$ is never dense in $\hat{C}^+$. To see this, observe first that there is a sequence $(a_n)_{n\in\omega}$ in C_0 such that $\nu_0(\Pi_{n\in J} a_n)=\Pi_{n\in J}[(n+1)/(n+2)]$ for every finite $J\subseteq\omega$. (This can be deduced from Lemma 3.2 below. Given $B_n=\langle\{a_i: i<n\}\rangle$, use 3.2 to find $a_n\in C_0$ such that $\nu(a_n\cdot b)=[(n+1)/(n+2)]\nu(b)$ for every $b\in B_n$. Of course, this is a particularly easy special case of 3.2.) Similarly, there is a sequence $(a_n')_{n\in\omega}$ in C_1 such that $\nu_1(\Pi_{n\in J} a_n')=\Pi_{n\in J}[(n+1)/(n+2)]$ for every finite $J\subseteq\omega$. Now consider

$$a=-\sum_{n\in\omega} e_0(-a_n)\cdot e_1(-a_n')\in\hat{C}\,.$$

We have

$$\hat{\nu}(a)\geq 1-\sum_{n=0}^{\infty}\nu(e_0(-a_n)\cdot e_1(-a_n'))=1-\sum_{n=0}^{\infty}(n+2)^{-2}>0\,,$$

so $a > 0$. But suppose, if possible, that there is a $b \in C_0 \oplus C_1$ such that $0 < b \leq a$. Then there are $b_0 \in C_0^+$, $b_1 \in C_1^+$ such that $e_0(b_0) \cdot e_1(b_1) \leq a$. As $e_0(b_0) \cdot e_1(b_1) \cdot e_0(-a_n) \cdot e_1(-a'_n) = 0$ for every $n \in \omega$, $\omega = I \cup J$ where

$$I = \{n \in \omega \colon b_0 \leq a_n\}\ , \qquad J = \{n \in \omega \colon b_1 \leq a'_n\}\ .$$

Now

$$\begin{aligned} 0 < \nu_0(b_0) \cdot \nu_1(b_1) &\leq \nu_0\Big(\prod_{n \in I} a_n\Big) \cdot \nu_1\Big(\prod_{n \in J} a'_n\Big) \\ &= \prod_{n \in I} \frac{n+1}{n+2} \prod_{n \in J} \frac{n+1}{n+2} \leq \prod_{n \in \omega} \frac{n+1}{n+2} = 0 \end{aligned}$$

(because $\Sigma_{n=0}^{\infty} 1/(n+2) = \infty$). So this cannot happen.

(g) If $(C_i)_{i \in I}$ is a finite family of measurable algebras, then any choices of probabilities on the C_i give rise to uniformly equivalent metrics on $\bigoplus_{i \in I} C_i$; this is an easy consequence of 2.3(b). So we have a unique measurable algebra $\widehat{\bigoplus}_{i \in I} C_i$. Observing that $\widehat{\bigoplus}$ is distributive over simple products, we can now extend this idea to see that if $(C_i)_{i \in I}$ is a finite family of complete algebras, each of which carries a semi-finite measure, there is a unique corresponding algebra $\widehat{\bigoplus}_{i \in I} C_i$ of the same kind.

REMARK. The result of (f) is due to ERDÖS and OXTOBY [1955]. The proof here is due to R.O. Davies.

2.26. FUNCTION SPACES. In FREMLIN [1974] I set out a theory of function spaces over Boolean algebras which was designed to clarify the connections between the measure algebra of a measure space and the classical function spaces over it. I shall not attempt to describe the whole of this work here, but shall content myself with a handful of definitions which show the sort of questions it addresses.

(a) Let A be any Boolean algebra, and $\mathrm{Ult}(A)$ its Stone space. Write $L^\infty(A)$ for the Banach space of continuous real-valued functions on $\mathrm{Ult}(A)$. We have a natural map $\chi \colon A \to L^\infty(A)$ given by taking $\chi(a)$ to be the characteristic function of the clopen set $s(a) \subseteq \mathrm{Ult}(A)$ corresponding to $a \in A$. The function χ has the properties

χ is *additive*, i.e. $\chi(a+b) = \chi(a) + \chi(b)$ whenever $a \cdot b = 0$,
$\|\chi(a)\| \leq 1$ whenever $a \in A$.

Now suppose that E is any Banach space and that $\nu \colon A \to E$ is an additive function such that $\|\nu(a)\| \leq 1$ for every $a \in A$. Then there is a unique continuous linear operator $T \colon L^\infty(A) \to E$ such that $T \circ \chi = \nu$, and $\|T\| \leq 1$.

Evidently this universal mapping property specifies the Banach space $L^\infty(A)$ and the map χ up to isomorphism.

(b) The point of characterizing $L^\infty(A)$ in this way is that we can now recognize it in other contexts. If, for instance, X is a set and B is a σ-algebra of subsets of

X, then $L^\infty(B)$ can be identified with the space of bounded B-measurable functions from X to $\boldsymbol{R}$; $\chi(b)$ is just the characteristic function of $b \in B$. If M is a σ-complete ideal of B and $\pi\colon B \to B/M = A$ is the canonical epimorphism, then in $L^\infty(B)$ we have a closed linear subspace

$$F = \{f \in L^\infty(B)\colon \{x \in X\colon f(x) \neq 0\} \in M\}\ ,$$

so we can form the quotient Banach space $L^\infty(B)/F$, with the associated map $\hat{\pi}\colon L^\infty(B) \to L^\infty(B)/F$. Now we see that, for $b, b' \in B$,

$$\hat{\pi}(\chi(b)) = \hat{\pi}(\chi(b')) \quad \text{iff } \pi(b) = \pi(b')\ ,$$

so that we have a map $\chi_1\colon A \to L^\infty(B)/F$ given by

$$\chi_1(\pi(b)) = \hat{\pi}(\chi(b)) \quad \forall b \in B\ .$$

It is easy to check that χ_1 can be used to identify $L^\infty(A)$ with $L^\infty(B)/F$.

In this way, if (X, B, μ) is a measure space with measure algebra $(A, \tilde{\mu})$, the space $L^\infty(A)$ as defined here can be identified with the usual Banach space $L^\infty(X, B, \mu)$ of equivalence classes of functions.

(c) Of course, $L^\infty(A) = C(\mathrm{Ult}(A))$ is a Banach lattice and a Banach algebra as well as a Banach space. These structures are defined by the relations

$$\chi(a \cdot b) = \chi(a) \wedge \chi(b) = \chi(a) \times \chi(b)$$

for all $a, b \in A$. For instance, if E is any Banach lattice and $\nu\colon A \to E$ is an additive function such that $\nu(a \cdot b) = \nu(a) \wedge \nu(b)$ for all $a, b \in A$, there is a unique Riesz homomorphism $T\colon L^\infty(A) \to E$ such that $T \circ \chi = \nu$.

(d) Having identified spaces of (equivalence classes of) bounded measurable functions, as in (b), it is natural to seek corresponding abstract descriptions of the spaces of all measurable functions and of their equivalence classes. Here the situation is more complex because we want different constructions for different types of algebra A. Since this chapter is about measure algebras I shall describe only a construction which is adapted to σ-complete algebras. If A is a σ-complete Boolean algebra, let B be the algebra of Borel sets in $\mathrm{Ult}(A)$, and M the ideal of meagre Borel sets, so that A is isomorphic to B/M. Let E be the Riesz space (= vector lattice) of all B-measurable real-valued functions on $\mathrm{Ult}(A)$, and F the solid linear subspace (or "ideal")

$$\{f \in E\colon \{x \in \mathrm{Ult}(A)\colon f(x) \neq 0\} \in M\}.$$

Then E/F is a Riesz space; I shall call it $L^0(A)$. We have a function $\chi\colon A \to L^0(A)$ given by taking $\chi(a)$ to be the image in $L^0(A) = E/F$ of the characteristic function of $s(a)$.

The characteristic property of $L^0(A)$ is the following. Let H be any Archimedean Riesz space and $E \subseteq H$ a Riesz subspace (= vector sublattice) such that for every $h \in H$ with $h \geq 0$ there is a countable set $D \subseteq E$ such that $h = \sup D$ in H.

Let T: $E \to L^0(A)$ be a Riesz homomorphism which is sequentially order-continuous. Then T has an extension to a sequentially order-continuous Riesz homomorphism from H to $L^0(A)$. (See FREMLIN [1974], 62K; [1975a], 1.6B.) If we add the facts that (i) χ: $A \to L^0(A)$ is additive; (ii) $\chi(a) > 0$ whenever $a \in A^+$; (iii) $\chi(a \cdot b) = \chi(a) \wedge \chi(b)$ for all $a, b \in A$; and (iv) whenever $h \in L^0(A)$ and $h \geq 0$ there are sequences $(a_n)_{n\in\omega}$ in A, $(\alpha_n)_{n\in\omega}$ in $[0, \infty)$ such that $h = \sup_{n\in\omega} \alpha_n \chi(a_n)$ in $L^0(A)$, then the pair $L^0(A)$, χ is determined up to isomorphism.

What this means is that other function spaces associated with A can generally be embedded into $L^0(A)$. In the language of FREMLIN [1975a], $L^0(A)$ is "sequentially inextensible", and can be identified with the sequentially inextensible extension of $L^\infty(A)$.

An alternative description of $L^0(A)$ identifies it with the set of σ-complete homomorphisms from the Borel algebra of $\boldsymbol{R}$ to A. (This uses 2.22 above.) Or (since such a homomorphism is determined by its values on sets of the form $(-\infty, \alpha]$) we can identify $L^0(A)$ with the set of functions h: $\boldsymbol{R} \to A$ such that h is increasing and order-continuous on the right and $\Sigma_{\alpha\in\boldsymbol{R}} h(\alpha) = 1$, $\Pi_{\alpha\in\boldsymbol{R}} h(\alpha) = 0$. (This is nearly what is done by CHOKSI and PRASAD [1982].)

Any of these descriptions of $L^0(A)$ is enough to show that if (X, B, μ) is a measure space with measure algebra A, then $L^0(A)$ can be identified with the usual space of equivalence classes of measurable real-valued functions on X.

(e) Now suppose that $(A, \tilde{\mu})$ is a measure algebra. Express it as the measure algebra of a measure space (X, B, μ) in any canonical way (e.g. by the method of 2.6, taking $X = \mathrm{Ult}(A)$ and B the Borel algebra of X). Write A^f for the ideal $\{a \in A: \tilde{\mu}(a) < \infty\}$, and $L^1(A, \tilde{\mu})$ for the Banach space $L^1(X, B, \mu)$ of equivalence classes of μ-integrable real-valued functions on X. Just as in (b) above, we have a natural map χ: $A^f \to L^1(A, \tilde{\mu})$ such that $\|\chi(a)\| = \tilde{\mu}(a)$ for every $a \in A^f$. And $L^1(A, \tilde{\mu})$, χ are characterized by the universal mapping theorem: if E is any Banach space and ν: $A^f \to E$ is an additive function such that $\|\nu(a)\| \leq \tilde{\mu}(a)$ for every $a \in A^f$, there is a unique continuous linear operator T: $L^1(A, \tilde{\mu}) \to E$ such that $T \circ \chi = \nu$, and $\|T\| \leq 1$. Just as in (c) above, the Riesz space structure of $L^1(A, \tilde{\mu}) = L^1(X, B, \mu)$ can also be described in terms of the function χ. And, just as in (b) above, if (Y, C, ν) is any measure space, then any isomorphism between the measure algebra of (Y, C, ν) and $(A, \tilde{\mu})$ gives rise to an isomorphism between $L^1(Y, C, \nu)$ and $L^1(A, \tilde{\mu})$.

(f) Evidently, similar ideas can be applied to L^p-spaces, Orlicz spaces, etc. For instance, $L^2(A, \tilde{\mu})$ corresponds to additive functions ν: $A^f \to E$ such that

$$\left\|\sum_{i\leq n} a_i \nu(a_i)\right\|^2 \leq \sum_{i=0}^{n} |\alpha_i|^2 \tilde{\mu}(a_i)$$

whenever $(a_i)_{i\leq n}$ is a pairwise disjoint family in A^f and $(\alpha_i)_{i\leq n} \in \boldsymbol{R}^{n+1}$.

(g) Part of the point of doing all this work is to clarify the effect of homomorphisms on function spaces. For instance, the description of $L^\infty(A)$ above makes it clear that if A and B are Boolean algebras and f: $A \to B$ is any homomorphism, then there is a unique continuous linear operator T: $L^\infty(A) \to L^\infty(B)$ such that $T(\chi_A(a)) = \chi_B(f(a))$ for every $a \in A$. If $(A, \tilde{\mu})$ and $(B, \tilde{\nu})$ are any measure algebras and f: $A^f \to B^f$ is a measure-preserving ring homomorphism (i.e. $f(0) =$

0, $f(a \cdot a') = f(a) \cdot f(a')$, $f(a + a') = f(a) + f(a')$, $\tilde{\nu}(f(a)) = \tilde{\mu}(a)$ for all $a, a' \in A^f$), then there is a unique continuous linear operator $T: L^1(A, \tilde{\mu}) \to L^1(B, \tilde{\nu})$ such that $T(\chi_A(a)) = \chi_B(f(a))$ for every $a \in A^f$; and so on. In particular, the group of measure-preserving automorphisms of a measure algebra $(A, \tilde{\mu})$ acts on $L^\infty(A)$ and $L^0(A)$ and $L^1(A, \tilde{\mu})$; always faithfully in the first two cases, and faithfully in the last if $(A, \tilde{\mu})$ is semi-finite. Its action on $L^2(A, \tilde{\mu})$ has been especially intensively studied. In this way the general theory of linear operators can be brought to bear on the theory of automorphism groups of Boolean algebras.

(h) If $(A, \tilde{\mu})$ is a measure algebra we have a bi-additive functional $(a, a') \mapsto \mu(a \cdot a')$: $A^f \times A \to \boldsymbol{R}$. This corresponds to a duality between $L^1(A, \tilde{\mu})$ and $L^\infty(A)$ such that $(\chi(A) \mid \chi(a')) = \tilde{\mu}(a \cdot a')$ if $a \in A^f$, $a' \in A$. We find that this duality is separating iff $(A, \tilde{\mu})$ is semi-finite, in which case $L^1(A, \tilde{\mu})$ becomes identified with the order-continuous dual of $L^\infty(A)$, called $L^\infty(A)^\times$ in FREMLIN [1974]; while $L^\infty(A)$ becomes identified with the Banach space dual $L^1(A, \tilde{\mu})'$ of $L^1(A, \tilde{\mu})$ iff $(A, \tilde{\mu})$ is a Maharam algebra (FREMLIN [1974], 53B).

This duality has the following effect. If $(A, \tilde{\mu})$ and $(B, \tilde{\nu})$ are measure algebras and $f: A \to B$ is a measure-preserving homomorphism, we have corresponding linear isometries $T_1: L^1(A, \tilde{\mu}) \to L^1(B, \tilde{\nu})$ and $T_\infty: L^\infty(A) \to L^\infty(B)$. Now these have transposes $T_1': L^1(B, \tilde{\nu})' \to L^1(A, \tilde{\mu})'$ and $T_\infty': L^\infty(B)' \to L^\infty(A)'$. If $(A, \tilde{\mu})$ and $(B, \tilde{\nu})$ are semi-finite, and f is a complete homomorphism (which will necessarily be true if $(A, \tilde{\mu})$ is totally finite), then $L^1(B, \tilde{\nu})$ and $L^1(A, \tilde{\mu})$ become identified with subspaces of $L^\infty(B)'$ and $L^\infty(A)'$, and T_∞ is order-continuous; from which we can deduce that $T_\infty'[L^1(B, \tilde{\nu})] \subseteq L^1(A, \tilde{\mu})$. Now $T_\infty' \restriction L^1(B, \tilde{\nu})$ corresponds to the conditional-expectation projection touched on in 1.5. (I call it a "projection" because T_1 embeds $L^1(A, \tilde{\mu})$ as a subspace of $L^1(B, \tilde{\nu})$, and $T_\infty' \circ T_1$ is the identity on $L^1(A, \tilde{\mu})$.) If $(A, \tilde{\mu})$ and $(B, \tilde{\nu})$ are Maharam algebras, then $T_1': L^\infty(B) \to L^\infty(A)$ is a different kind of conditional-expectation operator.

3. Maharam's theorem

In this section I present the central astonishing fact about measure algebras: there are very few of them. Measure *spaces* can be exceedingly various; even such limited classes as the compact self-supporting Radon measure spaces are hard to describe adequately. But totally finite measure *algebras* (in fact, all Maharam algebras) can be reached by two constructions: (i) form the measure algebra A_κ of ${}^\kappa\{0, 1\}$ for each cardinal κ; (ii) fit them together with appropriate weights (Theorem 3.9 below). This structure theory enables us to solve, quickly and easily, a very large number of problems, some of which are dealt with here.

3.1. INTRODUCTION. Recall that for a Boolean algebra A

$$\tau(A) = \min\{|X|: X \text{ completely generates } A\}$$

(Definition 13.18 in Part I). We know that $\tau(A \restriction a) \leq \tau(A)$ for every $a \in A$ (see the remarks before 13.12 in Part I). Consequently, if A is complete, it is isomorphic to a product $\Pi_{i\in I} A \restriction a_i$, where each $A \restriction a_i$ is τ-homogeneous, that is, $\tau(A \restriction a_i) = \tau(A \restriction a)$ for every $a \in (A \restriction a_i)^+$ (Lemma 13.12 of Part I).

The essence of Maharam's theorem is the following fact: if A is a τ-homogeneous probability algebra it is actually isomorphic, as measure algebra, to the measure algebra of ${}^{\kappa}\{0, 1\}$, where $\kappa = \tau(A)$ (Corollary 3.8). In order to discuss and apply this result it will be helpful to fix some notation. For any cardinal κ I shall write $(A_\kappa, \tilde{\mu}_\kappa)$ for the measure algebra of ${}^{\kappa}\{0, 1\}$, constructed as in 2.4 from the usual measure on ${}^{\kappa}\{0, 1\}$ described in 1.11. (If $\kappa = 0$, then ${}^{0}\{0, 1\} = \{\emptyset\}$ and A_0 is a simple atom of mass 1.) A σ-finite measure space (X, B, μ) is *Maharam homogeneous* if its measure algebra A is τ-homogeneous; the *Maharam type* of (X, B, μ) is then $\tau(A)$. (I have decided that I got this wrong in FREMLIN [1984] and elsewhere.)

The basic theorems are 3.5 and 3.7 below. For the sake of other applications of the method I approach 3.5 through a couple of lemmas.

3.2. LEMMA. *Let (A, μ) be a totally finite measure algebra and $B \subseteq A$ a complete subalgebra such that $A \restriction a \neq \{a \cdot b\colon b \in B\}$ for every $a \in A^+$. Let $\nu\colon B \to \mathbf{R}$ be an additive functional (see 2.25(a)) such that $0 \le \nu(b) \le \mu(b)$ for every $b \in B$. Then there is an $a \in A$ such that $\nu(b) = \mu(a \cdot b)$ for every $b \in B$.*

PROOF. (a) Observe first that ν is actually countably additive (see 1.3); for if $(a_n)_{n\in\omega}$ is a pairwise disjoint sequence in B, then

$$\nu\Big(\sum_{i\in\omega} a_i \cdot - \sum_{i\le n} a_i\Big) \le \mu\Big(\sum_{i\in\omega} a_i \cdot - \sum_{i\le n} a_i\Big) \to 0$$

as $n \to \infty$, so that

$$\nu\Big(\sum_{i\in\omega} a_i\Big) = \lim_{n\to\infty} \nu\Big(\sum_{i\le n} a_i\Big) = \sum_{i=0}^{\infty} \nu(a_i)\,.$$

(b) For each $a \in A$ define $\nu_a\colon B \to \mathbf{R}$ by setting $\nu_a(b) = \mu(a \cdot b)$. Then ν_a is countably additive.

The key is the following observation: if $a \in A^+$ there is a $c \in (A \restriction a)^+$ such that $\nu_c \le \frac{1}{2}\nu_a$. For we know that there is a $d \in A \restriction a$ which is not of the form $a \cdot b$ for any $b \in B$. Consider the functional $\nu_a - 2\nu_d\colon B \to \mathbf{R}$. By Lemma 1.3 there is a $b_0 \in B$ such that

$$(\nu_a - 2\nu_d)(b) \ge 0 \quad \text{if } b \le b_0\,, \qquad (\nu_a - 2\nu_d)(b) \le 0 \quad \text{if } b \le -b_0\,.$$

If $d \cdot b_0 > 0$, take $c = d \cdot b_0 \in (A \restriction a)^+$; then for any $b \in B$ we have

$$\nu_c(b) = \mu(b \cdot d \cdot b_0) = \nu_d(b \cdot b_0) \le \tfrac{1}{2}\nu_a(b \cdot b_0) \le \tfrac{1}{2}\nu_a(b)\,,$$

so this c will serve.

If this fails, then $d \le a \cdot - b_0$; but d cannot be equal to $a \cdot -b_0$, so $c = a \cdot -b_0 \cdot -d > 0$. If $b \in B$, then

$$\begin{aligned}\nu_c(b) &= \mu(b \cdot a \cdot -b_0 \cdot -d) = \mu(a \cdot b \cdot -b_0) - \mu(d \cdot b \cdot -b_0)\\ &= \nu_a(b \cdot -b_0) - \nu_d(b \cdot -b_0) \leq \tfrac{1}{2}\nu_a(b \cdot -b_0) \leq \tfrac{1}{2}\nu_a(b)\,,\end{aligned}$$

and again we are done.

(c) Using (b) repeatedly, we see that whenever $a \in A^+$ and $\delta > 0$ there is a $c \in (A \restriction a)^+$ such that $\nu_c \leq \delta\nu_a$.

(d) Now let $D \subseteq A$ be a maximal upwards-directed set such that $\nu_d \leq \nu$ whenever $d \in D$. Set $a = \Sigma\, D$. Then $a \cdot b = \Sigma\, \{d \cdot b \colon d \in D\}$ for every $b \in B$, so by 2.2(f)

$$\nu_a(b) = \mu(a \cdot b) = \sup_{d \in D} \mu(d \cdot b) = \sup_{d \in B} \nu_d(b) \leq \nu(b)$$

for every $b \in B$; consequently, a must belong to D.

(e) Suppose, if possible, that $\nu_a \neq \nu$. Then there is some $b^* \in B$ such that $\nu_a(b^*) < \nu(b^*)$; as $\nu_a(-b^*) \leq \nu(-b^*)$, $\mu(a) = \nu_a(1) < \nu(1)$. set $\delta = \frac{1}{2}(\nu(1) - \mu(a))/\mu(1)$ and define $\nu' \colon B \to \boldsymbol{R}$ by

$$\nu'(b) = \nu(b) - \delta\mu(b) - \mu(a \cdot b)\,.$$

Then ν' is countably additive so there is a $b_0 \in B$ such that $\nu'(b) \geq 0$ if $b \leq b_0$ and $\nu'(b) \leq 0$ if $b \leq -b_0$. Now

$$\nu'(1) = \nu(1) - \delta\mu(1) - \mu(a) > 0\,,$$

so

$$\begin{aligned}0 < \nu'(1) &= \nu'(b_0) + \nu'(-b_0) \leq \nu'(b_0)\\ &= \nu(b_0) - \delta\mu(b_0) - \mu(a \cdot b_0) \leq \mu(b_0) - \mu(a \cdot b_0)\,,\end{aligned}$$

so $b_0 \neq a \cdot b_0$ and $e = b_0 \cdot -a > 0$.

By (c), there is a $c \in (A \restriction e)^+$ such that $\nu_c \leq \delta\nu_e$. This means that, for any $b \in B$,

$$\begin{aligned}\nu_c(b) &\leq \delta\nu_e(b) = \delta\mu(b_0 \cdot -a \cdot b)\\ &= \nu(b_0 \cdot -a \cdot b) - \mu(a \cdot b_0 \cdot -a \cdot b) - \nu'(b_0 \cdot -a \cdot b)\\ &\leq \nu(b_0 \cdot -a \cdot b)\end{aligned}$$

(because $b_0 \cdot -a \cdot b \leq b_0$, so $\nu'(b_0 \cdot -a \cdot b) \geq 0$)

$$\begin{aligned}&= \nu(b) - \nu(a \cdot b) - \nu(b \cdot -a \cdot -b_0)\\ &\leq \nu(b) - \nu_a(a \cdot b) = \nu(b) - \nu_a(b)\,.\end{aligned}$$

But also $c \cdot a = 0$, so

$$\nu_{a+c}(b) = \nu_a(b) + \nu_c(b) \leq \nu(b)$$

for every $b \in B$. This means that we ought to have put $a + c$ into D; but we did not, because $a + c > a = \Sigma D$.

This contradiction shows that $\nu_a = \nu$ and the lemma is proved. □

REMARK. I have cast this lemma in its "abstract" form, using only 1.3. But it is fair to remark that the intuition on which it is based relies essentially on representing B as the measure algebra of a measure space X, and then using the Radon–Nikodým theorem to represent the functionals ν_a by integrable functions on X.

3.3. COROLLARY. *Let (A, μ) be an atomless semi-finite measure algebra, and $a \in A$. Suppose that $0 \leq \beta \leq \mu(a)$. Then there is a $b \in A \restriction a$ such that $\mu(b) = \beta$.*

PROOF. (a) First consider the case $\mu(a) < \infty$. Apply Lemma 3.2 to $(A \restriction a, \mu \restriction (A \restriction a))$, $B = \{0, a\}$ and ν, where $\nu(0) = 0$ and $\nu(a) = \beta$.

(b) Now suppose that $\mu(a) = \infty$. If $\beta = \infty$, then take $b = a$. Otherwise consider

$$D = \{d\colon d \leq a,\ \mu(d) < \infty\}\ .$$

Then D is upwards-directed and $\Sigma D = a$, because (A, μ) is semi-finite. By 2.2(f) there is a $d \in D$ such that $\mu(d) \geq \beta$. By case (a) there is a $b \leq d$ such that $\mu(b) = \beta$, and we are done. □

REMARK. Of course this is really much easier than 3.2.

3.4. LEMMA. *Let (A, μ) and (B, ν) be totally finite measure algebras and $C \subseteq A$ a complete subalgebra. Suppose that $f\colon C \to B$ is a measure-preserving homomorphism such that $B \restriction b \neq \{b \cdot f(c)\colon c \in C\}$ for any $b \in B^+$. Then for any $a \in A$ there is a measure-preserving homomorphism from $\langle C \cup \{a\}\rangle$ to B which extends f.*

PROOF. Because f is a complete homomorphism (2.15(b)), $f[C]$ is a complete subalgebra of B; of course, f is an isomorphism between C and $f[C]$. So we can define a functional $\lambda\colon f[C] \to \boldsymbol{R}$ by setting $\lambda(d) = \mu(a \cdot f^{-1}(d))$ for $d \in f[C]$. Evidently, λ is additive, and $\lambda(d) \leq \mu(f^{-1}(d)) = \nu(d)$ for $d \in f[C]$. So Lemma 3.2 tells us that there is a $b_0 \in B$ such that $\lambda(d) = \nu(b_0 \cdot d)$ for every $d \in f[C]$.

If $c, c' \in C$ and $c \leq a \leq c'$, then

$$\nu(b_0 \cdot f(c)) = \mu(a \cdot c) = \mu(c) = \nu(f(c))\ ,$$
$$\nu(b_0 \cdot f(c')) = \mu(a \cdot c') = \mu(a \cdot 1) = \nu(b_0)\ .$$

So $b_0 \cdot f(c) = f(c)$ and $b_0 \cdot f(c') = b_0$, i.e. $f(c) \leq b_0 \leq f(c')$. By Corollary 5.8 of Part I, there is a homomorphism $g\colon \langle C \cup \{a\}\rangle \to B$, extending f, such that $g(a) = b_0$.

Now any member of $\langle C \cup \{a\}\rangle$ is expressible in the form

$$x = c \cdot a + c' \cdot -a\ ,$$

where $c, c' \in C$ (Corollary 4.7 of Part I). For such an x,

$$\begin{aligned}\nu(g(x)) &= \nu(f(c)\cdot b_0 + f(c')\cdot -b_0)\\ &= \nu(f(c)\cdot b_0) + \nu(f(c')) - \nu(f(c')\cdot b_0)\\ &= \mu(c\cdot a) + \mu(c') - \mu(c'\cdot a) = \mu(x)\,.\end{aligned}$$

So g is measure-preserving, as required. □

3.5. THEOREM. *Let (A, μ) and (B, ν) be τ-homogeneous totally finite measure algebras such that $\tau(A) = \tau(B)$ and $\mu(1) = \nu(1)$. Then there is a measure-preserving isomorphism $f\colon A \to B$.*

PROOF. (a) If $\tau(A) = \tau(B) = 0$, this is trivial. So let us take $\kappa = \tau(A) = \tau(B) \geq \omega$. Let $(a_\xi)_{\xi<\kappa}$ and $(b_\xi)_{\xi<\kappa}$ enumerate completely generating sets in A, B, respectively. I seek to define f as the last of an increasing family $(f_\xi)_{\xi\leq\kappa}$ of measure-preserving isomorphisms between subalgebras C_ξ, D_ξ of A and B. The inductive hypothesis will be that, for some families $(a'_\xi)_{\xi<\kappa}$ and $(b'_\xi)_{\xi<\kappa}$ to be determined:

C_ξ is the complete subalgebra of A generated by

$$\{a_\eta\colon \eta < \xi\} \cup \{a'_\eta\colon \eta < \xi\}\,,$$

D_ξ is the complete subalgebra of B generated by

$$\{b_\eta\colon \eta < \xi\} \cup \{b'_\eta\colon \eta < \xi\}\,,$$

$f_\xi\colon C_\xi \to D_\xi$ is a measure-preserving isomorphism,
f_ξ extends f_η whenever $\eta \leq \xi$.

(b) The induction starts with $C_0 = \{0, 1\}$, $D_0 = \{0, 1\}$, $f_0(0) = 0$ and $f_0(1) = 1$.

(c) For the inductive step to a successor ordinal $\xi + 1$, suppose that C_ξ, D_ξ and f_ξ have been defined.

(i) Consider the simple extension $C'_\xi = \langle C_\xi \cup \{a_\xi\}\rangle$ of C_ξ by a_ξ. This is a complete subalgebra of A. Now we see that, for any $b \in B^+$,

$$\tau(\{b\cdot d\colon d \in D_\xi\}) \leq \tau(D_\xi) \leq |\xi| + |\xi| < \kappa = \tau(B \restriction b)\,,$$

because B is τ-homogeneous. So Lemma 3.4 tells us that there is a measure-preserving homomorphism $f'_\xi\colon C'_\xi \to B$ extending f_ξ. Set $b'_\xi = f'_\xi(a_\xi)$, so that $f'_\xi[C'_\xi] = D'_\xi = \langle D_\xi \cup \{b'_\xi\}\rangle$.

(ii) Now $f'^{-1}_\xi\colon D'_\xi \to C'_\xi$ is a measure-preserving isomorphism and $\tau(C'_\xi) \leq |\xi| + |\xi| + 1 < \tau(A \restriction a)$ for every $a \in A^+$, so by the same argument there is a measure-preserving homomorphism $g_\xi\colon D_{\xi+1} \to A$ extending f'^{-1}_ξ, where $D_{\xi+1} = \langle D'_\xi \cup \{b_\xi\}\rangle$. Set $a'_\xi = g_\xi(b_\xi)$ and $C_{\xi+1} = \langle C'_\xi \cup \{a'_\xi\}\rangle$; then $f_{\xi+1} = g_\xi^{-1}$ is a measure-preserving isomorphism from $C_{\xi+1}$ to $D_{\xi+1}$ extending f_ξ. This achieves the inductive step to $\xi + 1$.

(d) For the inductive step to a limit ordinal $\xi > 0$, consider

$$C_\xi^* = \bigcup_{\eta<\xi} C_\eta\,, \qquad D_\xi^* = \bigcup_{\eta<\xi} D_\eta\,, \qquad f_\xi^* = \bigcup_{\eta<\xi} f_\eta\,.$$

C_ξ^* and D_ξ^* are subalgebras (not necessarily complete) of A and B, respectively, and f_ξ^* is an isomorphism; moreover, $\nu(f_\xi^*(a)) = \mu(A)$ for every $a \in C_\xi^*$. This means that f_ξ^* is an isometry for the metrics on C_ξ^*, D_ξ^* (2.19(a)). But C_ξ and D_ξ are just the closures of C_ξ^*, D_ξ^* for the metrics on A and B (2.20), and A and B are complete for these matrices; so f_ξ^* has a unique extension to an isometry $f_\xi\colon C_\xi \to D_\xi$. Because the Boolean operations are continuous, f_ξ is a Boolean isomorphism. This achieves the inductive step to ξ.

(e) The induction finishes at κ, when $C_\kappa = A$ and $D_\kappa = B$ and $f = f_\kappa\colon A \to B$ is the required measure-preserving isomorphism. □

3.6. COROLLARY. *Let A be a τ-homogeneous measurable algebra. Then A is a homogeneous Boolean algebra.*

PROOF. Let μ be such that (A, μ) is a totally finite measure algebra. Take any $a \in A^+$. Define $\nu\colon A \restriction a \to \boldsymbol{R}$ by

$$\nu(b) = \mu(b)\mu(1)/\nu(a) \quad \forall b \in A \restriction a\,.$$

Then $(A \restriction a, \nu)$ is a totally finite measure algebra. Both A and $A \restriction a$ are τ-homogeneous, $\tau(A \restriction a) = \tau(A)$, and $\nu(a) = \mu(1)$. So there is a measure-preserving isomorphism $f\colon A \to A \restriction a$, and $A \cong A \restriction a$. □

3.7. PROPOSITION. *Let κ be an infinite cardinal.*

(a) *If C is a measurable algebra and $f\colon A_\kappa \to C$ is a complete homomorphism, then $\tau(C \restriction c) \geq \kappa$ for every $c \in C^+$.*

(b) *A_κ is homogeneous and $\tau(A_\kappa) = \kappa$.*

PROOF. (a) Let μ be the usual measure on ${}^\kappa\{0,1\}$ and let B' be the Baire σ-algebra of ${}^\kappa\{0,1\}$; let $\pi\colon B' \to A$ be the canonical epimorphism (see 1.11(b), 1.15). For each $\xi < \kappa$, set

$$b_\xi = \{x \in {}^\kappa\{0,1\}\colon x(\xi) = 1\} \in B'\,,$$
$$a_\xi = \pi(b_\xi) \in A_\kappa\,.$$

Then for any countable infinite $I \subseteq \kappa$,

$$\mu\Big(\bigcap_{\xi\in I} b_\xi\Big) = 0\,, \quad \mu\Big(\bigcup_{\xi\in I} b_\xi\Big) = 1\,;$$

so that

$$\prod_{\xi\in I} a_\xi = \pi\Big(\bigcap_{\xi\in I} b_\xi\Big) = 0\,, \qquad \sum_{\xi\in I} a_\xi = \pi\Big(\bigcup_{\xi\in I} b_\xi\Big) = 1\,,$$

because π is σ-complete.

Let $c \in C^+$. Then $C_1 = C \restriction c$ is a measurable algebra, not $\{0\}$; let ν be a function such that (C_1, ν) is a totally finite measure algebra. Set $c_\xi = c \cdot f(a_\xi) \in C_1$ for $\xi < \kappa$; then, for any infinite $I \subseteq \kappa$,

$$\sum_{\xi \in I} c_\xi = c \cdot \sum_{\xi \in I} f(a_\xi) = c \cdot f\Big(\sum_{\xi \in I} a_\xi\Big) = c\,,$$

and similarly $\prod_{\xi \in I} c_\xi = 0$.

Let ρ be the metric on C_1 associated with ν (2.19). Then for every $d \in C_1$ there is a neighbourhood U of d for ρ such that $\{\xi: \xi < \kappa, c_\xi \in U\}$ is finite. For suppose, if possible, otherwise. Then we can choose inductively a sequence $(\xi(n))_{n\in\omega}$ such that, for each $n \in \omega$, $\rho(d, c_{\xi(n)}) \leq 2^{-n-2}\nu(c)$ and $\xi(n) \neq \xi(i)$ for $i < n$. In this case

$$\begin{aligned}
\nu(c) &= \nu(d) + \nu(c \cdot -d) \\
&= \nu\Big(d \cdot -\prod_{n\in\omega} c_{\xi(n)}\Big) + \nu\Big(\sum_{n\in\omega} c_{\xi(n)} \cdot -d\Big) \\
&\leq \sum_{n=0}^{\infty} \nu(d \cdot -c_{\xi(n)}) + \sum_{n=0}^{\infty} \nu({}_{\xi(n)} \cdot -d) \\
&= \sum_{n=0}^{\infty} \rho(d, c_{\xi(n)}) \leq \tfrac{1}{2}\nu(c)\,,
\end{aligned}$$

which is impossible.

This means that we have an open cover $\mathcal{U}$ of C_1 such that each member of $\mathcal{U}$ contains only finitely many of the c_ξ. So any subset of $\mathcal{U}$ covering C_1 must have cardinal at least κ. Accordingly, any base for the topology of C_1 must have cardinal at least κ; but as the topology of C_1 is metrizable, it follows that any topologically dense subset of C_1 has cardinal at least κ.

If $X \subseteq C_1$ is a completely generating set, then the subalgebra D of C_1 generated by X is topologically dense in C_1 (2.20), so $|D| \geq \kappa$. It follows at once that $|X| \geq \kappa$. As X is arbitrary, $\tau(C_1) \geq \kappa$, as claimed.

(b) Consider a_ξ, b_ξ as defined in (a). Because π is σ-complete and B' is the σ-complete subalgebra of itself generated by $\{b_\xi: \xi < \kappa\}$, $A_\kappa = \pi[B']$ must be included in the σ-complete subalgebra of A_κ generated by $\{a_\xi: \xi < \kappa\}$. So $\{a_\xi: \xi < \kappa\}$ completely generates A_κ, and $\tau(A_\kappa) \leq \kappa$.

On the other hand, applying (a) to the identity homomorphism from A to itself, we see that $\tau(A_\kappa \restriction a) \geq \kappa$ for every $a \in A_\kappa^+$. So A_κ must be τ-homogeneous, with $\tau(A_\kappa) = \kappa$ precisely. From 3.6 it follows that A_κ is actually homogeneous. □

REMARK. In the language of 3.1, (b) means that $({}^\kappa\{0,1\}, B, \mu)$ and $({}^\kappa\{0,1\}, B', \mu \restriction B')$ (where B is the domain of μ) are Maharam homogeneous, of Maharam type κ.

In fact, both these spaces are homogeneous in a very much stronger sense. If $b \in B \setminus N_\mu$, define $\nu: B \cap P(b) \to \mathbf{R}$ by $\nu(c) = \mu(c)/\mu(b)$ for $c \in B \cap P(b)$. Then $(b, B \cap P(b), \nu)$ is isomorphic, as measure space, to $({}^\kappa\{0,1\}, B, \mu)$; and if

$b \in B'$ then $(b, B' \cap P(b), \nu \restriction B' \cap P(b))$ is isomorphic to $({}^{\kappa}\{0,1\}, B', \mu \restriction B')$. But I shall not discuss isomorphisms of measure spaces in any detail in this chapter.

An alternative proof of (b) can be constructed using the technique of part (b)(i) of the proof of 3.11 below.

3.8. COROLLARY. Let (A, μ) be a homogeneous probability algebra. Then it is isomorphic, as measure algebra, to exactly one of the measure algebras $(A_\kappa, \tilde{\mu}_\kappa)$.

PROOF. For 3.5 and 3.7 show that (A, μ) is isomorphic to $(A_\kappa, \tilde{\mu}_\kappa)$ iff $\kappa = \tau(A)$. (I have not discussed the case $\kappa < \omega$. But evidently the only τ-homogeneous Boolean algebras A with $\tau(A) < \omega$ are the algebras $\{0, 1\}$ and $\{0\}$, and the latter cannot carry a probability.) □

3.9. MAHARAM'S THEOREM. *Let (A, μ) be any Maharam algebra. Then there are families $(\kappa(i))_{i\in I}$ and $(\alpha_i)_{i\in I}$ such that each $\kappa(i)$ is either an infinite cardinal or 0, each α_i is a strictly positive real number, and there is an isomorphism $f: A \to \prod_{i\in I} A_{\kappa(i)}$ such that $\mu(a) = \sum_{i\in I} \alpha_i \tilde{\mu}_{\kappa(i)}(\mathrm{pr}_i(f(a)))$ for every $a \in A$, where $\mathrm{pr}_i: \prod_{j\in I} A_{\kappa(j)} \to A_{\kappa(i)}$ is the canonical map for each $j \in J$.*

PROOF. Let $(a_i)_{i\in I}$ be a maximal pairwise disjoint family in A^+ such that $\mu(a_i) < \infty$ and $A \restriction a_i$ is τ-homogeneous for each $i \in I$. Then $\sum_{i\in I} a_i = 1$; since if $a \in A^+$ there is an $a' \in (A \restriction a)^+$ such that $\mu(a') < \infty$ (because (A, μ) is semi-finite) and now there is an $a'' \in (A \restriction a')^+$ which is τ-homogeneous, in which case $a'' \cdot a_i > 0$ for some $i \in I$.

Now, for each $i \in I$, set $\alpha_i = \mu(a_i) \in (0, \infty)$ and $\kappa(i) = \tau(A \restriction a_i)$; define $\nu_i: A \restriction a_i \to [0, 1]$ by setting $\nu_i(a) = \alpha_i^{-1}\mu(a)$ for $a \in A \restriction a_i$. Then $(A \restriction a_i, \nu_i)$ is a τ-homogeneous probability algebra and $\tau(A \restriction a_i) = \kappa(i)$. So there is a measure-preserving isomorphism $f_i: (A \restriction a_i, \nu_i) \to (A_{\kappa(i)}, \tilde{\mu}_{\kappa(i)})$, by 3.5 and 3.7. Define $f: A \to \prod_{i\in I} A_{\kappa(i)}$ by writing $f(a) = (f_i(a \cdot a_i))_{i\in I}$ for each $a \in A$. Because $\sum_{i\in I} a_i = 1$ and each f_i is a monomorphism, f is a monomorphism; because A is complete and $(a_i)_{i\in I}$ is pairwise disjoint and each f_i is an epimorphism, f is an epimorphism. Finally, if $a \in A$, then

$$\begin{aligned}\mu(a) &= \sum_{i\in I} \mu(a \cdot a_i) \quad \text{(by 2.2(f))}\\ &= \sum_{i\in I} \alpha_i \nu_i(a \cdot a_1) = \sum_{i\in I} \alpha_i \tilde{\mu}_{\kappa(i)}(f_i(a \cdot a_i))\\ &= \sum_{i\in I} \alpha_i \tilde{\mu}_{\kappa(i)}(\mathrm{pr}_i(f(a))),\end{aligned}$$

as required. □

REMARK. For the case of totally finite measure algebras (when I must be countable) this is Theorem 2 of MAHARAM [1942].

The right picture seems to be that shown in Fig. 22.1, where each a_n gives rise to a homogeneous relative algebra $A \restriction a_n$ such that the measure on that part is isomorphic, apart from a scalar multiplication of the measure, to the measure on A_κ for some κ.

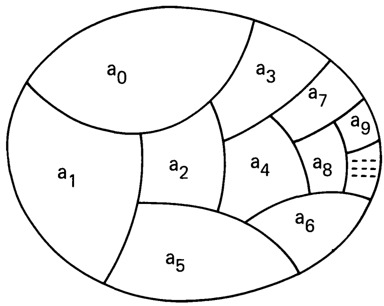

Fig. 22.1

3.10. COROLLARY. *Let (A, μ) be a Maharam algebra. Take $(\kappa(i))_{i\in I}$ and $(\alpha_i)_{i\in I}$ from 3.9. Let $(X, \mathfrak{T})$ be "the" topological disjoint union $\bigcup_{i\in I} {}^{\kappa(i)}\{0,1\}$; for each $i \in I$ let $\mu_{\kappa(i)}$ be the usual measure on ${}^{\kappa(i)}\{0,1\}$, $B_{\kappa(i)}$ its domain, and h_i: ${}^{\kappa(i)}\{0,1\} \to X$ the canonical one-to-one function. Set*

$$B = \{b : b \subseteq X,\ h_i^{-1}[b] \in B_{\kappa(i)}\ \forall i \in I\}\ ,$$

and for $b \in B$ set

$$\nu(b) = \sum_{i\in I} \alpha_i \mu_{\kappa(i)}(h_i^{-1}[b])\ .$$

Then $(X, \mathfrak{T}, B, \nu)$ is a locally compact Radon measure space and its measure algebra is isomorphic to (A, μ).

PROOF. The measure space (X, B, ν) is easy to handle because if $b_i = h_i[{}^{\kappa(i)}\{0,1\}]$ for each $i \in I$, the family $(b_i)_{i\in I}$ is a decomposition of X of the type described in 1.2(b)(iv). Now the map $b \mapsto (h_i^{-1}[b])_{i\in I}$: $B \to \prod_{i\in I} B_{\kappa(i)}$ introduces a Boolean isomorphism between the measure algebra of (X, B, ν) and $\prod_{i\in I} A_{\kappa(i)}$; it is easy to check that the weights α_i arrange that this produces a measure-preserving isomorphism between the measure algebra of (X, B, ν) and (A, μ). $\square$

REMARK. This gives a new proof of 2.13. Of course X here may fail to be compact even if (A, μ) is totally finite, since $|I|$ could be ω; but in this case we can replace X by its one-point compactification, and the obvious extension of ν will give us a compact Radon measure space with the right measure algebra.

3.11. PROPOSITION. *Let (A, μ) be a probability algebra, and C a complete subalgebra of A. For $a \in A$ set*

$$\tau_C(a) = \min\{|X| : A \upharpoonright a \text{ is completely generated by } \{a \cdot x : x \in X \cup C\}\}\ .$$

(a) *Suppose that $\kappa \geq \max(\omega, \tau_C(1))$. Let $(C \hat{\oplus} A_\kappa, \nu)$ be the completed free product of C and A_κ, as described in 2.25(b); let e: $C \to C \hat{\oplus} A_\kappa$ be the canonical* *measure-preserving homomorphism. Then there is an extension of e to a measure-preserving homomorphism* $f: A \to C \hat{\oplus} A_\kappa$.

(b) *If, in* (a), $\tau_C(a) = \kappa$ *for every* $a \in A^+$, *then f may be taken to be an isomorphism.*

PROOF. Write $B = C \hat{\oplus} A_\kappa$; let $e': A_\kappa \to B$ be the canonical measure-preserving isomorphism. Let $(a_\xi)_{\xi<\kappa}$ be the canonical independent family in A_κ, as described in the proof of 3.7, and set $b_\xi = e'(a_\xi)$ for each $\xi < \kappa$.

(a)(i) Let $(c_\xi)_{\xi<\kappa}$ be a family in A such that A is completely generated by $C \cup \{c_\xi: \xi < \kappa\}$, and for $\xi < \kappa$ let C_ξ be the complete subalgebra of A generated by $C \cup \{c_\eta: \eta < \kappa\}$. I seek to define f as the last of an increasing family of measure-preserving homomorphisms $f_\xi: C_\xi \to B$, as in the proof of 3.5.

Because $\kappa \geq \omega$, there is a disjoint family $(J_\xi)_{\xi<\kappa}$ of infinite subsets of κ. For each $\xi \leq \kappa$, let D_ξ be the complete subalgebra of B generated by $e[C] \cup \{b_\eta: \eta \in \bigcup_{\theta<\xi} J_\theta\}$.

(ii) The inductive hypothesis will be that $f_\xi[C_\xi] \subseteq D_\xi$ for each $\xi < \kappa$. The induction starts easily with $C_0 = C$, $D_0 = e[C]$ and $f_0 = e$. The hard step is that to a successor ordinal $\xi + 1$. Since $C_{\xi+1} = \langle C_\xi \cup \{a_\xi\}\rangle$ I seek to apply Lemma 3.4. Let $b \in D^+_{\xi+1}$. Since $D_{\xi+1}$ is the metric closure of the subalgebra

$$D'_{\xi+1} = \left\langle e[C] \cup \left\{b_\eta: \eta \in \bigcup_{\theta\leq\xi} J_\theta\right\}\right\rangle,$$

there is a $b' \in D'_{\xi+1}$ such that $\nu(b \triangle b') \leq \varepsilon = \frac{1}{4}\nu(b)$. Now there is some finite $J \subseteq \bigcup_{\theta\leq\xi} J_\theta$ such that $b' \in \langle e[C] \cup \{b_\eta: \eta \in J\}\rangle$; take $\zeta \in J_\xi \backslash J$.

If $a \in C_\xi$, then $f_\xi(a)$ and b' both belong to the complete subalgebra of B generated by $e[C] \cup \{b_\eta: \eta \neq \zeta\}$; so $\nu(b' \cdot f_\xi(a) \cdot b_\zeta) = \frac{1}{2}\nu(b' \cdot f_\xi(a))$. Now examine $b \cdot b_\zeta \in D_{\xi+1} \restriction b$. If $a \in C_\xi$, then

$$\begin{aligned}
\nu((b \cdot b_\zeta) \triangle (b \cdot f_\xi(a))) &= \nu(b \cdot (b_\zeta \triangle f_\xi(a))) \\
&\geq \nu(b' \cdot (b_\zeta \triangle f_\xi(a))) - \varepsilon \\
&= \nu(b' \cdot b_\zeta) + \nu(b' \cdot f_\xi(a)) - 2\nu(b' \cdot b_\zeta \cdot f_\xi(a)) - \varepsilon \\
&= \tfrac{1}{2}\nu(b') - \varepsilon \\
&\geq \tfrac{1}{2}(\nu(b) - 3\varepsilon) > 0 .
\end{aligned}$$

So $b \cdot b_\zeta$ is not of the form $b \cdot f_\xi(a)$ for any $a \in C_\xi$.

Since b is an arbitrary member of $D^+_{\xi+1}$, the condition of Lemma 3.4 is satisfied, and there is an extension of f_ξ to a measure-preserving homomorphism $f_{\xi+1}: C_{\xi+1} \to D_{\xi+1}$.

(iii) At limit ordinals $\xi > 0$ we can define f_ξ as the unique continuous extension of $\bigcup_{\eta<\xi} f_\eta$, just as in 3.5. So at the end we have $f = f_\kappa: A \to B$, as required.

(b)(i) The point is that

$$\tau_C(b) = \min\{|X|: B \restriction b \text{ is completely generated by } \{b \cdot x: x \in X \cup e[C]\}\}$$

is equal to κ for every $b \in B^+$. To see this, note first that $\{a_\xi: \xi \le \kappa\}$ completely generates A_κ (as in 3.7(b)), so that $e[C] \cup \{b_\xi: \xi < \kappa\}$ completely generates B, and $\tau_C(b) \le \tau_C(1) \le \kappa$ for every $b \in B^+$.

The next step is to see that $\tau_C(b) > 0$ for every $b \in B^+$. For if $b \in B^+$ there are a finite $J \subseteq \kappa$ and a $b' \in \langle e[C] \cup \{b_\xi: \xi \in J\}\rangle$ such that $\nu(b \triangle b') \le \frac{1}{4}\nu(b)$, and if $\zeta \in \kappa \setminus J$, then $b \cdot b_\zeta$ cannot be of the form $b \cdot e(c)$ for any $c \in C$, as in (a)(iii) above. So $B \restriction b \ne \{b \cdot e(c): c \in C\}$; but this is a complete subalgebra of $B \restriction b$, so $B \restriction b$ is not completely generated by $\{b \cdot e(c): c \in C\}$.

It follows that $\tau_C(b) \ge \omega$ for every $b \in B^+$. For if $b \in B^+$ and $X \subseteq B$ is finite, there is a $b' \in (B \restriction b)^+$ such that, for every $x \in X$, either $b' \le x$ or $b' \cdot x = 0$; there is a $b'' \le b'$ such that $b'' \ne b' \cdot e(c)$ for any $c \in C$; and now b'' cannot belong to the complete subalgebra of $B \restriction b$ generated by $\{b \cdot x: x \in X \cup e[C]\}$.

If $\kappa = \omega$ we need go no further. So suppose that $\kappa > \omega$, $b \in B^+$, $X \subseteq B$ and $|X| < \kappa$; set $\lambda = \max(|X|, \omega) < \kappa$. For each $x \in X \cup \{b\}$, there is countable set $Y_x \subseteq A$ such that x belongs to the complete subalgebra of B generated by $e[C] \cup e'[Y_x]$. Set $Y = \bigcup_{x \in X \cup \{b\}} Y_x$; then $|Y| \le \lambda$. For each $y \in Y$ there is a countable set $J_y \subseteq \kappa$ such that y belongs to the complete subalgebra of A_κ generated by $\{a_\xi: \xi \in J_y\}$. Set $J = \bigcup_{y \in Y} J_y$; then $|J| \le \lambda$, so there is a $\zeta \in \kappa \setminus J$. Let B_1 be the complete subalgebra of B generated by $e[C] \cup \{b_\xi: \xi \in J\}$. Then $X \cup \{b\} \cup e[C] \subseteq B_1$, and $\nu(d \cdot b_\zeta) = \frac{1}{2}\nu(d)$ for every $d \in B_1$. It follows at once that $b \cdot b_\zeta \notin B_1$, since $\nu(b \cdot b_\zeta) = \frac{1}{2}\nu(b) > 0$. But in this case $b \cdot b_\zeta$ cannot belong to the complete subalgebra of $B \restriction b$ generated by $\{b \cdot x: x \in e[C] \cup X\}$. As X is arbitrary, $\tau_C(b) \le \kappa$, which is what we needed to know.

(ii) We can now copy the proof of 3.5 faithfully, building f up in stages from $f_0 = e$. The inductive step will work because, for instance, D_ξ will be the complete subalgebra of B generated by $e[C] \cup \{b_\eta: \eta < \xi\} \cup \{b'_\eta: \eta < \xi\}$, so that $B \restriction b$ can never be of the form $\{b \cdot d: d \in D_\xi\}$ if $b \in B^+$. □

3.12. COROLLARY. *Let (A, μ) be a probability algebra and set $\kappa = \max(\tau(A), \omega)$. Then there is a measure-preserving homomorphism $f: A \to A_\kappa$.*

PROOF. Set $C = \{0, 1\}$ in 3.11(a), or argue directly as in 3.5–3.7. □

REMARK. The ideas of 3.11–3.12 are taken from MAHARAM [1950]. For further developments see 3.16 and 3.23–3.24 below.

3.13. EXTENSIONS OF THEOREM 3.5. *Let (A, μ) and (B, ν) be totally finite measure algebras and C a complete subalgebra of A, and let $g: C \to B$ be a measure-preserving homomorphism. Set $\kappa = \min\{\tau(B \restriction b): b \in B^+\}$.*

(a) *If $\tau(A) < \kappa$ and $\tau(C) < \kappa$, there is a measure-preserving homomorphism $f: A \to B$ extending g.*

(b) *If, moreover, A and B are both homogeneous and $\tau(A) = \kappa$, then f can be taken to be an isomorphism.*

PROOF. (a) Let $(a_\xi)_{\xi<\kappa}$ run over a set which completely generates A, and let C_ξ be the complete subalgebra of A generated by $C \cup \{a_\eta: \eta < \xi\}$. Argue as in 3.5 or

3.11 to find an increasing family of measure-preserving homomorphisms $f_\xi\colon C_\xi \to B$, with $f_0 = g$, and set $f = f_\kappa$.

(b) Copy 3.5 more closely, starting with $f_0 = g$ again. □

3.14. COROLLARY. *Let (A, μ) be a homogeneous totally finite measure algebra and a, b two members of A such that $\mu(a) = \mu(b)$. Then there is a measure-preserving automorphism $f\colon A \to A$ such that $f(a) = b$.*

PROOF. Take $C = \{0, a, -a, 1\}$ in 3.13(b). □

REMARK. The force of this corollary is that $A \restriction a$ and $A \restriction b$ are isomorphic not only as Boolean algebras but as measure algebras.

It is of course possible to have (non-homogeneous) probability algebras (A, μ) and (B, ν) such that A and B are isomorphic as Boolean algebras, but (A, μ) and (B, ν) are not isomorphic as measure algebras; indeed, this can be done with $|A| = |B| = 4$, by giving them atoms of different weights. But if (A, μ) and (B, ν) are semi-finite measure algebras, and $f\colon A \to B$ is a Boolean algebra isomorphism, then we can find a partition $(a_i)_{i\in I}$ of A such that $\mu(a_i) < \infty$, $\nu(f(a_i)) < \infty$, and $A \restriction a_i$ is homogeneous for each $i \in I$ (compare 3.9); and in this case there will be Boolean algebra isomorphisms $g_i\colon A \restriction a_i \to B \restriction f(a_i)$ such that, for suitable real $\beta_i > 0$, $\nu(g_i(a)) = \beta_i \mu(a)$ for every $a \in A \restriction a_i$. This provides a new gloss on 2.3, taking μ and ν to be different measures on $A = B$, and f the identity map.

3.15. PROPOSITION. *Let (A, μ) be an atomless Maharam algebra. Then the complete subalgebra C of elements fixed under every Boolean algebra automorphism of A is precisely the set of elements fixed under every measure-preserving automorphism of A; it is in itself an atomic Boolean algebra.*

PROOF. For each infinite cardinal κ let

$$a_\kappa = \sum \{a \in A\colon A \restriction a \cong A_\kappa \text{ as Boolean algebra}\}\,.$$

Let K be the set of cardinals κ such that $a_\kappa \neq 0$. Then $(a_\kappa)_{\kappa\in K}$ is a partition of A. Of course, every a_κ belongs to C, so C includes the closed subalgebra D generated by $\{a_\kappa\colon \kappa \in K\}$. But now take any $a \in A \setminus D$. Then there is a $\kappa \in K$ such that $a_\kappa \cdot a$ and $a_\kappa \cdot -a$ are both non-zero. Because (A, μ) is atomless and semi-finite, there are $b \leq a_\kappa \cdot a$ and $c \leq a_\kappa \cdot -a$ such that $0 < \mu(b) = \mu(c) < \infty$. Now there is a measure-preserving isomorphism $f\colon A \restriction b \to A \restriction c$, and we have a measure-preserving automorphism $g\colon A \to A$ given by

$$g(d) = f(d \cdot b) + f^{-1}(d \cdot c) + d \cdot -b \cdot -c\,.$$

Evidently $g(a) \neq a$.

Thus, the set of elements fixed under every measure-preserving automorphism of A is included in D; it is therefore exactly equal to D, and also to C. Of course D is an atomic Boolean algebra.

3.16. PROPOSITION. *Let (A, μ) be a totally finite measure algebra and G any group of measure-preserving automorphisms of A. Let C be the complete subalgebra $\{c \in A\colon g(c) = c\ \forall g \in G\}$. Then there is a partition $(c_i)_{i\in I}$ of 1 in A, with $c_i \in C$ for each $i \in I$, and a family $(\kappa(i))_{i\in I}$ of non-zero cardinals, such that A is isomorphic as measure algebra, to $\Pi_{i\in I}(C \restriction c_i)\hat{\oplus} B_i$. Here $B_i = A_{\kappa(i)}$ if $\kappa(i)$ is infinite, and $P(\kappa(i))$ if $1 \le \kappa(i) < \omega$; in the former case B_i carries the measure $\hat{\mu}_{\kappa(i)}$, and in the latter case B_i carries the uniform measure which gives mass $1/\kappa(i)$ to each atom. The isomorphism $f\colon A \to \Pi_{i\in I}(C \restriction c_i)\hat{\oplus} B_i$ can be taken such that, for $c \in C$, $f(c) = (e_i(c \cdot c_i))_{i\in I}$, where $e_i\colon C \restriction c_i \to (C \restriction c_i)\hat{\oplus} B_i$ is the canonical embedding.*

PROOF. (a) Let D be the set of those $d \in C^+$ which are candidates for membership of the family $(c_i)_{i\in I}$; i.e. such that the embedding of $C \restriction d$ in $A \restriction d$ is isomorphic to the embedding of $C \restriction d$ in $(C|d)\hat{\oplus} B$ for a probability algebra B of one of the types described. (Here $\hat{\oplus}$ is the completed free product as described in 2.25(d).)

If we can prove that D is dense in C^+ then we shall be home; for then we can take $(c_i)_{i\in I}$ to be any maximal pairwise disjoint family in D, and $(B_i)_{i\in I}$ the associated family of probability algebras. Because D is dense, $(c_i)_{i\in I}$ will be a partition of 1 in A, so that $a \mapsto (a \cdot c_i)_{i\in I}$ is an isomorphism between A and $\Pi_{i\in I} A \restriction c_i$; now we have isomorphisms $f_i\colon A \restriction c_i \to (C \restriction c_i)\hat{\oplus} B_i$ extending the canonical embeddings $e_i\colon C \restriction c_i \in (C \to c_i)\hat{\oplus} B_i$, and we set $f(a) = (f_i(a \cdot c_i))_{i\in I}$ for $a \in A$.

(b) Let us therefore fix on $c^* \in C^+$ and seek a $d \in D \cap (C \restriction c^*)$. For each $a \in A$ set

$$\tau_C(a) = \min\{|X|\colon A \restriction a \text{ is completely generated by } \{a \cdot x\colon x \in X \cup C\}\}\,.$$

(See 3.11.) Then we see that $\tau_C(a') \le \tau_C(a)$ whenever $a' \le a$, and that $\tau_C(a) = \tau_C(g(a))$ whenever $a \in A$ and $g \in G$ (because C is invariant under the automorphism g). Also

$$\tau_C\Big(\sum_{j\in J} a_j\Big) \le \sum_{j\in J} (\tau_C(a_j) + 1)$$

for any family $(a_j)_{j\in J}$ in A. Consequently, because A satisfies the countable chain condition,

$$\tau_C\Big(\sum_{g\in G} g(a)\Big) \le \max(\omega, \tau_C(a))$$

for every $a \in A$ (since there is a countable set $H \subseteq G$ such that $\Sigma_{g\in G}\, g(a) = \Sigma_{g\in H}\, g(a)$).

(c) Let $a \in (A \restriction c^*)^+$ be such that $\tau_C(a)$ is minimal. We need to deal with two cases separately.

(d) *Case 1.* Suppose that $\tau_C(a) = \kappa \ge \omega$. Set $d = \Sigma_{g\in G}\, g(a)$. Then $a \le d \le c^*$, because $g(c^*) = c^*$ for every $g \in G$. Also $g(d) = \Sigma_{h\in G}\, g(h(a)) = d$ for every

$g \in G$, so $d \in (C \restriction c^*)^+$. Finally, as remarked in (b), $\tau_C(d) \leq \max(\omega, \tau_C(a)) = \kappa$; so $\tau_C(b) = \kappa$ for every $b \in (A \restriction d)^+$, by the minimality of κ.

3.11(b) (applied to a suitable normalization of the measure on $A \restriction d$) now shows that the embedding of $C \restriction d$ in $A \restriction d$ extends to an isomorphism from $A \restriction d$ to $(C \restriction d) \hat{\oplus} A_\kappa$. So $d \in D$, as required.

(e) *Case 2.* Suppose that $\tau_C(a) < \omega$. Then $\tau_C(a) = 0$. For suppose, if possible, otherwise; let X be a set of cardinal $\tau_C(a)$ such that $A \restriction a$ is completely generated by $\{a \cdot x: x \in X \cup C\}$. Fix any $x \in X$. One of $a \cdot x$, $a \cdot -x$ is not 0; call it z; then $A \restriction z$ is completely generated by $\{y \cdot z: y \in C \cup (X \setminus \{x\})\}$ so $\tau_C(z) < \tau_C(a)$, contrary to the choice of a.

Thus, $A \restriction a$ is completely generated by $\{a \cdot c: c \in C\}$; because this is already a complete subalgebra of $A \restriction a$, it must the whole of $A \restriction a$.

(f) Let $m \in \omega$ be such that $m\mu(a) > \mu(1)$ and set $\nu(c) = m\mu(c \cdot a) - \mu(c)$ for $c \in C$. Then ν is countably additive so there is a $b_0 \in C$ such that $\nu(c) \geq 0$ for $c \leq b_0$, $\nu(c) \leq 0$ for $c \leq -b_0$. In particular,

$$m\mu(-b_0 \cdot a) \leq \mu(-b_0) \leq \mu(1) < m\mu(a)\,,$$

so $b_0 \cdot a > 0$ and $b_0 > 0$. Since $m\mu(c \cdot a) = 0 < \mu(c)$ if $c \cdot c^* = 0$ and $c \neq 0$, $b_0 \leq c^*$.

(g) Let $(a_j)_{j \in J}$ be a maximal pairwise disjoint family in A^+ such that for each $j \in J$ there is a $g \in G$ such that $a_j \leq g(a)$. Then $d_0 = \Sigma_{j \in J}\, a_j = \Sigma_{g \in G}\, g(a) \in C$ (as in (d) above); as $b_0 \cdot a > 0$, $b_0 \cdot d_0 > 0$.

For each $j \in J$, take $g_j \in G$ such that $a_j \leq g_j(a)$, and consider $g_j^{-1}(a_j) \leq a$. Because $A \restriction a = \{a \cdot c: c \in C\}$, there is a $c_j \in C$ such that $g_j^{-1}(a_j) = a \cdot c_j$, so that $a_j = g_j(a) \cdot g_j(c_j) = g_j(a) \cdot c_j$.

Now suppose that $K \subseteq J$ is such that $\Pi_{j \in K}\, c_j \cdot b_0 = c \neq 0$. Then $|K| \leq m$. For if $j \in K$ we have

$$\mu(c \cdot a_j) = \mu(c \cdot g_j(a) \cdot c_j) = \mu(c \cdot a)$$

(because $c \leq c_j$ and $g_j^{-1}(c) = c$ and g_j^{-1} is measure-preserving)

$$\geq \frac{1}{m}\,\mu(c)$$

(because $c \leq b_0$). Since $(a_j)_{j \in K}$ is disjoint, $|K| \leq m$.

(h) It follows at once (because $b_0 \cdot a_j > 0$ for some $j \in J$) that there is a non-empty maximal $K \subseteq J$ such that $d = \Pi_{j \in K}\, b_0 \cdot c_j > 0$. Set $n = |K|$. By the argument at the end of (*g*), we see that for any $c \leq d$ we have $\mu(c \cdot a_j) = \mu(c \cdot a)$ for every $j \in K$; while if $j \in J \setminus K$, $d \cdot c_j = 0$ so $d \cdot a_j = 0$. This means that $(d \cdot a_j)_{j \in K}$ is a partition of d in A, while $\mu(c \cdot a_j) = (1/n)\mu(c)$ for every $c \in C \restriction d$ and every $j \in K$. Finally, if $b \in A \restriction d$ and $j \in K$, then $b \cdot a_j \leq g_j(a)$ so $g_j^{-1}(b \cdot a_j) \leq a$ and $g_j^{-1}(b \cdot a_j)$ is expressible as $a \cdot c$ for some $c \in C$; in which case $b \cdot a_j = g_j(a) \cdot c$; since also $b \leq d \leq c_j$, $b \cdot a_j = d \cdot a_j \cdot c$.

(i) A moment's reflection shows that this means that $A \restriction d$ is isomorphic and isometric to $(C \restriction d) \oplus P(K)$, the atoms of $P(K)$ corresponding to the $d \cdot a_j$. Because $(C \restriction d) \oplus P(K)$ is already complete (both as Boolean algebra and as metric space), it is identical to $(C \upharpoonright d) \hat{\oplus} P(K)$. Thus, $d \in D$ and again we have a member of D in $C \upharpoonright c^*$, since $d \leq b_0 \leq c^*$. □

REMARK. This result is based on ideas in MAHARAM [1950].

3.17. LEMMA. *Let A be a measurable algebra and $C \subseteq A$ a complete subalgebra. Let ν: $C \to \boldsymbol{R}$ be such that (C, ν) is a totally finite measure algebra. Then there is a function μ: $A \to \boldsymbol{R}$, extending ν, such that (A, μ) is a totally finite measure algebra.*

PROOF. Let μ_0: $A \to \boldsymbol{R}$ be such that (A, μ_0) is a totally finite measure algebra. By 2.6, (A, μ_0) is isomorphic to the measure algebra of a measure space (X, B, μ_0'); let π: $B \to A$ be the corresponding epimorphism. Set $D = \pi^{-1}[C] \subseteq B$; because C is a complete subalgebra of A and π is a σ-complete homomorphism, D is a σ-complete subalgebra of B. Consider the functional $d \mapsto \nu(\pi(d))$: $D \to \boldsymbol{R}$. This is countably additive and is zero on $D \cap N_{\mu_0'}$, so by the Radon–Nikodým theorem there is a D-measurable function f: $X \to \boldsymbol{R}$ such that $\nu(\pi(d)) = \int_d f \, \mathrm{d}\mu_0'$ for every $d \in D$.

Consider $e = \{x \in X\colon f(x) < 0\}$. Then $e \in D$ and $0 \leq \nu(\pi(e)) = \int_e f \, \mathrm{d}\mu_0' \leq 0$. So $\mu_0'(e) = 0$. Define μ': $B \to \boldsymbol{R}$ by setting $\mu'(b) = \int_b f \, \mathrm{d}\mu_0'$ for every $b \in B$. Then $\mu'(b) = 0$ iff $\mu_0'(b) = 0$. So there is a function μ: $A \to \boldsymbol{R}$ such that $\mu'(b) = \mu(\pi(b))$ for every $b \in B$. It is easy to check that (A, μ) is a totally finite measure algebra and that μ extends ν. □

3.18. PROPOSITION. *Let A and B be measurable algebras and C a complete subalgebra of A, and let g: $C \to B$ be a complete homomorphism. Set $\kappa = \min\{\tau(B \upharpoonright b)\colon b \in B^+\}$.*

(a) *If $\tau(A) \leq \kappa$ and $\tau(C) < \kappa$ there is a complete homomorphism f: $A \to B$ extending g.*

(b) *If, moreover, A and B are both homogeneous and $\tau(A) = \kappa$ and g is a monomorphism, then f can be taken to be an isomorphism.*

PROOF. (a) Set $c_0 = \Sigma\, \{c \in C\colon g(c) = 0\} \in C$, $c_1 = -c_0$, $C_1 = C \upharpoonright c_1$, $g_1 = g \upharpoonright C_1$ and $A_1 = A \upharpoonright c_1$. Then A_1 is a measurable algebra, C_1 is a complete subalgebra of A_1, and g_1: $C_1 \to B$ is a complete monomorphism.

Let ν: $B \to \boldsymbol{R}$ be such that (B, ν) is a totally finite measure algebra. Then $(C_1, \nu \circ g_1)$ is a totally finite measure algebra. By Lemma 3.17 there is a μ: $A_1 \to \boldsymbol{R}$, extending $\nu \circ g_1$, such that (A_1, μ) is totally finite measure algebra. Now g_1 is measure-preserving for μ and ν. So 3.13(a) tells us that there is a measure-preserving homomorphism f_1: $A_1 \to B$ extending g_1. Define f: $A \to B$ by setting $f(a) = f_1(a \cdot c_1)$; this is a complete homomorphism extending g.

(b) In this case $C_1 = C$ and we use 3.13(b) to find a measure-preserving isomorphism $f_1 = f$ from A to B. □

REMARK. Recall that Theorem 5.13 of Part I tells us that (because B is complete) there is a homomorphism f: $A \to B$ extending g. But we need to know much more to be sure that f can be complete.

3.19. COROLLARY. *Let A be a homogeneous measure algebra. Let $C \subseteq A$ be a complete subalgebra with $\tau(C) < \tau(A)$, and $g: C \to A$ any complete monomorphism. Then g extends to an automorphism of A.*

PROOF. Take $B = A$ in 3.18(b). □

REMARK. If B is any homogeneous Boolean algebra and $C \subseteq B$ is a finite subalgebra, then any monomorphism from C to B extends to an automorphism of B. So this corollary is saying that measurable algebras, if they are (τ-)homogeneous, are homogeneous in an especially strong sense.

3.20. THEOREM. *Let A and B be measurable algebras, neither of them $\{0\}$. Set*

$$U = \{\tau(A \restriction a) : a \in A^+,\ A \restriction a \text{ is homogeneous}\}\ ,$$

$$V = \{\tau(B \restriction b) : b \in B^+,\ B \restriction b \text{ is homogeneous}\}\ .$$

Then there is a complete monomorphism from A to B iff (i) $c(A) \le c(B)$; (ii) $\min U \le \min V$; and (iii) *for every $\alpha \in U$ there is a $\beta \in V$ such that $\alpha \le \beta$.*

PROOF. Write $H_A = \{a \in A^+ : A \restriction a$ is homomogeneous$\}$, $H_B = \{b \in B^+ : B \restriction b$ is homogeneous$\}$.

(a) Suppose that there is a complete monomorphism $f: A \to B$. (i) Of course, $c(A) \le c(B)$, since $f[X]$ is pairwise disjoint whenever X is. (ii) Take $b \in H_B$. Because $\Sigma H_A = 1$, there is an $a \in H_A$ such that $b \cdot f(a) > 0$. Now $c \mapsto b \cdot f(c): A \restriction a \to B \restriction b \cdot f(a)$ is a complete homomorphism. Also, $A \restriction a \cong A_\alpha$, where $\alpha = \tau(A \restriction a) \in U$. So 3.7(a) tells us that $\alpha \le \tau(B \restriction b \cdot f(a)) = \tau(B \restriction b)$ (if $\alpha \ge \omega$; but the case $\alpha = 0$ is trivial). As b is an arbitrary member of H_B, $\min U \le \min V$. (iii) Now take $\alpha \in U$. There are an $a \in H_A$ such that $\tau(A \restriction a) = \alpha$ and a $b \in H_B$ such that $b \le f(a)$. So $c \mapsto b \cdot f(a)$ is a complete homomorphism from $A \restriction a$ to $B \restriction b$. By 3.7(a) again, $\alpha \le \tau(B \restriction b)$, which belongs to V. So (iii) is also satisfied.

(b) Suppose that conditions (i)–(iii) are satisfied. Let $(a_i)_{i \in I}$, $(b_j)_{j \in J}$ be maximal pairwise disjoint families in H_A, H_B, respectively; then they are both partitions of 1. Let $(c_k)_{k \in K}$ be a partition of 1 in B refining $(b_j)_{j \in J}$ (meaning that for every $k \in K$ there is a $j \in J$ such that $c_k \le b_j$) and such that if $j \in J$ and $B \restriction b_j$ is infinite then there are infinitely many $k \in K$ with $c_k \le b_j$. In this case

$$U = \{\tau(A \restriction a_i) : i \in I\}\ ,$$

$$V = \{\tau(B \restriction b_j) : j \in J\} = \{\tau(B \restriction c_k) : k \in K\}\ ,$$

and moreover

$$K_\beta = \{k \in K : \tau(B \restriction c_k) = \beta\}$$

is infinite for every $\beta \in V \setminus \{0\}$.

It follows that there is a surjection h from K onto I such that $\tau(A \restriction a_{h(k)}) \leq \tau(B \restriction c_k)$ for every $k \in K$. For choose $h \restriction K_\beta$, for each $\beta \in V$, such that

$$h[K_\beta] \subseteq \{i \in I\colon \tau(A \restriction a_i) \leq \beta\} = I'_\beta$$

with equality whenever $|I'_\beta| \leq |K_\beta|$; this is possible because $\min U \leq \min V$, so $I'_\beta \neq \emptyset$ whenever $K_\beta \neq \emptyset$. To see that h is surjective, observe that if $V = \{0\}$, i.e. B is atomic, then $U = \{0\}$ and $|I| = c(A) \leq c(B) = |K| = |K_0|$, so that $h[K_0] = I$; while if $V \neq \{0\}$, then for every $\alpha \in U$ there is a $\beta \in V$ such that $\max(\alpha, \omega) \leq \beta$, in which case $|I'_\beta| \leq \omega = |K_\beta|$, and $\alpha \in I'_\beta = h[K_\beta]$.

For each $k \in K$, we have a complete monomorphism $f_k\colon A \restriction a_{h(k)} \to B \restriction c_k$; for $A \restriction a_{h(k)} \cong A_\lambda$ and $B \restriction c_k \cong A_\kappa$, where $\lambda \leq \kappa$, and the restriction map $x \mapsto x \restriction \lambda\colon {}^\kappa\{0,1\} \to {}^\lambda\{0,1\}$ is an inverse-measure-preserving function which induces a measure-preserving homomorphism from A_λ to A_κ. Define $f\colon A \to B$ by writing

$$f(a) = \sum_{k \in K} f_k(a \cdot a_{h(k)}) \quad \forall a \in A\,.$$

Because $(f_k(a_{h(k)}))_{k \in K} = (c_k)_{k \in K}$ is a partition of 1 in B, f is a complete homomorphism; because $\Sigma_{k \in K}\, a_{h(k)} = \Sigma_{i \in I}\, a_i = 1$ in A, f is a monomorphism, as required. □

3.21. COROLLARY. *Let A and B be measurable algebras, neither equal to $\{0\}$. Then there is a complete homomorphism from A to B iff* $\min\{\tau(A \restriction a)\colon a \in A^+\} \leq \min\{\tau(B \restriction b)\colon b \in B^+\}$.

PROOF. There is a complete homomorphism from A to B iff there are an $a \in A^+$ and a complete monomorphism from $A \restriction a$ to B; now the conditions of 3.20 translate easily into the condition here. □

3.22. COROLLARY. *If A is an infinite measurable algebra, then there is a monomorphism from the free product $A \oplus A$ to A.*

PROOF. By 3.9 we have a countable partition $(a_i)_{i \in I}$ of 1 in A such that $A \restriction a_i$ is homogeneous for each $i \in I$; set $\kappa(i) = \tau(A \restriction a_i)$ for each i. Let B be the completed free product $A \hat{\oplus} A$ as described in 2.25, and $e_0\colon A \to B$, $e_1\colon A \to B$ the canonical homomorphisms. Then $(e_0(a_i) \cdot e_1(a_j))_{i,j \in I}$ is a partition of 1 in B. If $i, j \in I$, then

$$\begin{aligned} B \restriction e_0(a_i) \cdot e_1(a_j) &\cong (A \restriction a_i) \hat{\oplus} (A \restriction a_j) \\ &\cong A_{\kappa(i)} \hat{\oplus} A_{\kappa(j)} \cong A_{\max(\kappa(i), \kappa(j))}\,. \end{aligned}$$

If follows easily that

$$\begin{aligned} &\{\tau(A \restriction a)\colon a \in A^+,\ A \restriction a \text{ is homogeneous}\} \\ &\quad = \{\kappa(i)\colon i \in I\} \\ &\quad = \{\tau(B \restriction b)\colon b \in B^+,\ B \restriction b \text{ is homogeneous}\}\,. \end{aligned}$$

Also, $c(A) = c(B) = \omega$ because A is infinite. So there is a complete monomorphism from B to A, and there is a monomorphism from $A \oplus A$ to A. $\square$

REMARK. I give 3.18–3.22 as examples of questions which for general complete (or even complete homogeneous) Boolean algebras can be obscure, but which for measurable algebras are readily resolved using Maharam's theorem and the associated ideas.

3.23. THEOREM. *Let A be a measurable algebra, not $\{0\}$, and C a complete subalgebra of A.*

(a) *There are a non-empty countable set I, a family $(c_i)_{i\in I}$ in C^+ and a family $(\kappa(i))_{i\in I}$ of cardinals such that A is isomorphic to $\Pi_{i\in I}\,(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$. (Here the completed free products are those described in 2.25(g).) The isomorphism $f\colon A \to \Pi_{i\in I}\,(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ may be taken such that $f(c) = (e_i(c\cdot c_i))_{i\in I}$ for every $c \in C$, where $e_i\colon C \restriction c_i \to (C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ is the canonical homomorphism for each $i \in I$.*

(b) *There are an infinite countable ordinal ζ, a family $(c'_\xi)_{\xi<\zeta}$ in C, a family $(\kappa'(\xi))_{\omega\le\xi<\zeta}$ of infinite cardinals, and an isomorphism $f'\colon A \to \Pi_{n\in\omega}\,(C \restriction c'_n) \times \Pi_{\omega\le\xi<\zeta}\,(C \restriction c'_\xi)\hat{\oplus} A_{\kappa'(\xi)}$ such that $c'_{n+1} \le c'_n$ if $n<\omega$, $c'_\xi > 0$ if $\omega \le \xi < \zeta$, $\kappa'(\eta) < \kappa'(\xi)$ if $\omega \le \eta < \xi < \zeta$, and $f'(c) = ((c\cdot c_n)_{n\in\omega}, (e'_\xi(c\cdot c'_\xi))_{\omega\le\xi<\zeta})$ for every $c \in C$, where $e'_\xi\colon C \restriction c'_\xi \to (C \restriction c'_\xi)\hat{\oplus} A_{\kappa'(\xi)}$ is the canonical homomorphism for $\omega \le \xi < \zeta$. Subject to these requirements, ζ, $(c'_\xi)_{\xi<\zeta}$ and $(\kappa'(\xi))_{\omega\le\xi<\zeta}$ (but not f') are uniquely determined by A and C.*

(c) *There is a measurable algebra B such that A is isomorphic to a relative subalgebra $C\hat{\oplus} B \restriction d$ of the completed free product $C\hat{\oplus} B$. The isomorphism $g\colon A \to C\hat{\oplus} B \restriction d$ may be taken such that $g(c) = e(c)\cdot d$ for every $c\in C$, where $e\colon C \to C\hat{\oplus} B$ is the canonical homomorphism.*

PROOF. (a) For $a \in A$ set

$$\tau_C(a) = \min\{|X|\colon A \restriction a \text{ is completely generated by } \{a\cdot x\colon x \in X \cup C\}\}\,.$$

Then there is a partition $(a_i)_{i\in I}$ of 1 in A such that $\tau_C(a) = \tau_C(a_i) = \kappa(i)$ for every $a \in (A \mid a_i)^+$, $i \in I$. (Compare 3.16 above.) For each $i\in I$ set

$$c_i = \mathrm{upr}(a_i, C) = \prod\{c \in C\colon a_i \le c\}\,,$$
$$C_i = \{a_i\cdot c\colon c\in C\} \subseteq A \restriction a_i\,.$$

Then C_i is a complete subalgebra of $A \restriction a_i$ and $c \mapsto c\cdot a_i$ is an isomorphism from $C \restriction c_i$ to C_i. Now we see that

$$\tau_{C_i}(a) = \min\{|X|\colon A \restriction a \text{ is completely generated by } \{a\cdot x\colon x\in X \cup C_i\}\}$$

is equal to $\kappa(i)$ for every $a \in (A \restriction a_i)^+$. So by 3.11(b) (applied with any probability on $A \restriction a_i$), $A \restriction a_i$ is isomorphic to $C_i \hat{\oplus} A_{\kappa(i)}$, which in turn is isomorphic to $(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$; moreover, the isomorphism f_i: $A \restriction a_i \to (C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ can be taken to extend the natural embedding of C_i into $(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ derived from the isomorphism between C_i and $C \restriction c_i$.

Because $(a_i)_{i\in I}$ is a partition of 1 in A, and A is complete, the natural map $a \mapsto (a \cdot a_i)_{i-I}$: $A \to \Pi_{i\in I}(A \restriction a_i)$ is an isomorphism. Setting $f(a) = (f_i(a \cdot a_i))_{i\in I}$ for $a \in A$, we obtain a suitable isomorphism from A to $\Pi_{i\in I}(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$.

(b) For the canonical version, start from $(c_i)_{i\in I}$ and $(\kappa(i))_{i\in I}$ as in (a), and set $J = \{i \in I\colon \kappa(i) \geq \omega\}$, $K = I\setminus J$, $U = \{\kappa(i)\colon i \in J\}$. Then U is a countable well-ordered set so can be enumerated in ascending order as $(\kappa'(\xi))_{\omega\leq\xi<\zeta}$, where $\omega \leq \zeta < \omega_1$. For $\omega \leq \xi < \zeta$ set

$$a'_\xi = \sum \{a_i\colon i \in I, \kappa(i) = \kappa'(\xi)\},\ c'_\xi = \mathrm{upr}(a', C)\,.$$

Then $\tau_C(a) = \kappa'(\xi)$ whenever $0 < a \leq a'_\xi$, so $(c_i)_{i\in J}$, $(\kappa(i))_{i\in J}$ may be replaced by $(c'_\xi)_{\omega\leq\xi<\zeta}$, $(\kappa'(\xi))_{\omega\leq\xi<\zeta}$.

If $K = \emptyset$, then we can take $c'_n = 0$ for $n < \omega$. Otherwise, let $(k(n))_{n\in\omega}$ be a sequence running over K. Define $(a^*_{ai})_{n,\, i\in\omega}$, $(a'_n)_{n\in\omega}$ and $(c'_n)_{n\in\omega}$ inductively, as follows:

$$a^*_{ni} = a_{k(i)} \cdot -\Big(\sum_{j<n} a'_j\Big)\,,$$

$$a'_n = \sum_{i\in\omega}\Big(a^*_{ni} \cdot -\Big(\sum_{j<i} \mathrm{upr}(a^*_{nj}, C)\Big)\Big)\,,$$

$$c'_n = \mathrm{upr}(a'_n, C) = \sum_{i\in\omega} \mathrm{upr}(a^*_{ni}, C)\,.$$

Then we find that

$$\tau_C(a'_n) = \tau_C(a^*_{ni}) = \tau_C(a_{k(n)}) = 0 \quad \text{for } n, i \in \omega$$

(compare the first part of (e) of the proof of 3.16),

$$(a'_n)_{n\in\omega} \text{ is disjoint}\,,$$

$$a_{k(n)} \leq \sum_{j\leq n} a'_j \leq \sum_{i\in\omega} a_{k(i)}\,, \quad c'_{n+1} \leq c'_n \textit{ for } n \in \omega\,.$$

Consequently, we may replace $(a_i)_{i\in K}$, $(c_i)_{i\in K}$ by $(a'_n)_{n\in\omega}$, $(c'_n)_{n\in\omega}$, recognizing that we may now have $c'_m = a'_m = 0$ for some $m \in \omega$, so that $c'_n = 0$ for every $n \geq m$. It is now natural to use $C \restriction c'_n$ rather than $(C \mid c'_n)\hat{\oplus} A_0$, to obtain the required variant.

For the uniqueness claimed, observe that for $\omega \leq \xi < \zeta$,

$$c'_\xi = \sum \{\mathrm{upr}(a, C)\colon a \in A, \tau_C(a') = \kappa'(\xi)\ \forall a' \in (A \restriction a)^+\}\,;$$

while for $n \in \omega$,

$$c'_n = \Sigma\left\{\prod_{i\le n}\operatorname{upr}(v_i, C)\colon (v_i)_{i\le n} \text{ is a pairwise disjoint family in } A^+ \right.$$
$$\left. \text{such that } \tau_C(v_i) = 0\ \forall i \le n\right\}.$$

The arguments of 3.11 make it plain that f' cannot be unique except when $\zeta = \omega$.

(c) For the second representation, start from $(c_i)_{i\in I}$, $(\kappa(i))_{i\in I}$ and $f\colon A \to \Pi_{i\in I}(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ as in (a). Set $B = \Pi_{i\in I} A_{\kappa(i)}$; as I is countable, B is a measurable algebra (put 5.2 below together with 2.8 above). Take $e_i\colon C \restriction c_i \to (C \restriction c_i)\oplus A_{\kappa(i)}$, $e\colon C \to C\hat{\oplus} B$ and $e'\colon B \to C \hat{\oplus} B$ to be the canonical homomorphisms. In B set $b_i = (b_{ij})_{j\in I}$ for each $i \in I$, where

$$b_{ii} = 1 \in A_{\kappa(i)}\,, \qquad b_{ij} = 0 \in A_{\kappa(j)} \quad \text{if } j \in I\setminus\{i\}\,.$$

Then $(b_i)_{i\in I}$ is a partition of 1 in B. In $C\hat{\oplus} B$ set

$$d_i = e(c_i)\cdot e'(b_i) \quad \forall i \in I\,, \qquad d = \sum_{i\in I} d_i\,.$$

Since $A_{\kappa(i)}$ is naturally isomorphic to $B \restriction b_i$, the uncompleted free product $(C \restriction c_i)\oplus A_{\kappa(i)}$ can be identified with $C \oplus B \restriction d_i$; and on completing with respect to appropriate metrics, we identify $(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ with $C\hat{\oplus} B \restriction d_i$. This works for each $i \in I$. Now $(d_i)_{i\in I}$ is pairwise disjoint, so $\Pi_{i\in I}(C\hat{\oplus} B \restriction d_i)$ is isomorphic to $C\hat{\oplus} B \restriction d$. Combining these canonical isomorphisms with f, we obtain the required isomorphism $g\colon A \to C\hat{\oplus} B \restriction d$. □

3.24. REMARKS. (a) The ideas here are taken from MAHARAM [1950].

(b) If (A, μ) is a given totally finite measure algebra (not $\{0\}$) and C is a complete subalgebra of A, we can obtain metric forms of the representations above, as follows.

(i) In 3.23(a), we can make f measure-preserving by giving each $C \restriction c_i$ the measure ν_i defined by setting $\nu_i(c) = \mu(a_i \cdot c)$ for $c \in C \restriction c_i$; now give $(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ the measure $\tilde{\nu}_i$ derived from ν_i and the usual measure $\tilde{\mu}_{\kappa(i)}$ of $A_{\kappa(i)}$; and transfer these to $\Pi_{i\in I}(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ by setting

$$\nu(d) = \sum_{i\in I} \tilde{\nu}_i(d_i) \quad \text{if } d = (d_i)_{i\in I} \in \prod_{i\in I}(C \restriction c_i)\hat{\oplus} A_{\kappa(i)}\,.$$

The argument of 3.11 shows that each $f_i\colon A \restriction a_i \to (C \restriction c_i)\hat{\oplus} A_{\kappa(i)}$ can be taken to be measure-preserving for $\mu \restriction (A \restriction a_i)$ and $\tilde{\nu}_i$, in which case f will be measure-preserving for μ and ν.

(ii) In 3.23(b) we can do the same, obtaining measures ν'_ξ on $C \restriction c'_\xi$ for $\xi < \zeta$, and constructing a measure on $\Pi_{n\in\omega}(C \restriction c'_n) \times \Pi_{\omega\le\xi<\zeta}(C \restriction c'_\xi)\hat{\oplus} A_{\kappa'(\xi)}$ in the same way. For $\omega \le \xi < \zeta$, a'_ξ and ν'_ξ are uniquely determined. For $n < \omega$, a'_n and ν'_n are not quite fixed, even though c'_n is. But with a little ingenuity we can dissect and re-compose the a'_n in such a way that

$$\mu(a'_n \cdot c) \ge \mu(a'_{n+1} \cdot c) \quad \forall c \in C,\ n \in \omega\,,$$

so that

$$\nu'_{n+1}(c) \le \nu'_n(c) \quad \forall c \in C \upharpoonright c'_{n+1},\ n \in \omega .$$

Subject to this requirement the ν'_n are uniquely determined, so that the measure ν on $\Pi_{n\in\omega}(C \upharpoonright c'_n) \times \Pi_{\omega\le\xi<\zeta}(C \upharpoonright c'_\xi) \hat{\oplus} A_{\kappa'(\xi)}$ is also fixed.

(iii) In 3.23(c) it is natural to give $B = \Pi_{i\in I} A_{\kappa(i)}$ the measure $\tilde{\mu}$ defined by writing $\tilde{\mu}((b'_i)_{i\in I}) = \Sigma_{i\in I}\, \tilde{\mu}_{\kappa(i)}(b'_i)$ when $(b'_i)_{i\in I} \in B$. This is likely to make $(B, \tilde{\mu})$ a σ-finite measure algebra rather than a totally finite one; but no other difficulty arises, and g becomes measure-preserving.

(c) Note that we have already found a representation of A as a complete subalgebra of $C \hat{\oplus} A_\kappa$, where $\kappa = \max(\omega, \tau_C(1))$ (3.11). And in the special case of 3.16 there is yet another approach; the point here is that (in the language of the proof of 3.23(b)) $a'_\xi = c'_\xi \in C$ for $\omega \le \xi < \zeta$, and $\Pi_{n\in\omega} c'_n = 0$, so that $(c'_n)_{n\in\omega}$ can be replaced by $(c'_n \cdot -c'_{n+1})_{n\in L}$ for some $L \subseteq \omega$.

3.25. AUTOMORPHISM GROUPS. The fact that a measurable algebra has many relative subalgebras which are homogeneous (as Boolean algebra) is of course crucial to any understanding of its automorphism group. One of the omissions from this chapter is any thorough study of these. However, I shall give here a few hints and references, following suggestions from J.R. Choksi.

(a) If A is a Boolean algebra, I write $\mathrm{Aut}(A)$ for the group of automorphisms of A; if (A, μ) is a measure algebra, I write $\mathrm{Aut}_\mu(A)$ for the group of measure-preserving automorphisms of A.

(b) If A is a homogeneous σ-complete Boolean algebra (in particular, if (A, μ) is a homogeneous measure algebra), then $\mathrm{Aut}(A)$ is simple (Theorem 5.9b of Chapter 18, by P. Štěpánek and M. Rubin, in this Handbook). (Note that if (A, μ) is a measure algebra, and A is homogeneous, and there is some $a \in A$ such that $0 < \mu(a) < \infty$, then $A \upharpoonright a$ satisfies the countable chain condition, so that A also does; in this case A is a measurable algebra, and is isomorphic, as Boolean algebra, to some A_κ.) If (A, μ) is a totally finite homogeneous measure algebra then $\mathrm{Aut}_\mu(A)$ is simple (CHOKSI and PRASAD [1982, 6.1]).

(c) If (A, μ) is a homogeneous totally finite measure algebra, then neither $\mathrm{Aut}(A)$ nor $\mathrm{Aut}_\mu(A)$ has an outer automorphism. (See Section 6 of Chapter 15, by M. Rubin, in this Handbook, and EIGEN [198?].)

(d) If (A, μ) is any totally finite measure algebra, then there are many periodic automorphisms of A, in the following sense. If $f \in \mathrm{Aut}(A)$ and $\varepsilon > 0$, then there are $a \in A$ and $g \in \mathrm{Aut}(A)$ such that g is periodic, $g(b) = f(b)$ for $b \le a$, and $\mu(-a) \le \varepsilon$; if $f \in \mathrm{Aut}_\mu(A)$, then g can also be taken in $\mathrm{Aut}_\mu(A)$ (CHOKSI and PRASAD [1982]).

(e) I mention a very striking result from DYE [1959] (see also HAJIAN, ITO and KAKUTANI [1975]). If A is any Boolean algebra, let us say that a subgroup G of $\mathrm{Aut}(A)$ is *full* if, whenever $g \in G$, $f \in \mathrm{Aut}(A)$ and

$$\{a \in A^+ : \exists n \in \mathbf{Z},\ f(b) = g^n(b)\ \forall b \le a\}$$

is dense in A^+, then $f \in G$. Observe that if (A, μ) is a semi-finite measure algebra, then $\mathrm{Aut}_\mu(A)$ is a full subgroup of $\mathrm{Aut}(A)$. Now suppose that (A, μ) is a homogeneous σ-finite measure algebra with $\tau(A) = \omega$ (i.e. that A is isomorphic, as Boolean algebra, to A_ω), and that $f, g \in \mathrm{Aut}_\mu(A)$ are "ergodic", i.e. their only fixed points are 0 and 1. Then the full subgroups of $\mathrm{Aut}(A)$ generated by f and g are conjugate in $\mathrm{Aut}_\mu(A)$ ("Dye's Theorem").

(f) Let (A, μ) be a probability algebra. A natural and important question is to seek ways of determining when two members of $\mathrm{Aut}_\mu(A)$ are conjugate in $\mathrm{Aut}_\mu(A)$. One approach is to investigate their actions on the function spaces associated with A (2.26); in particular, the spectral theory of the associated operators on $L^2(A)$. Another is to analyse their fixed subalgebras, which must be of the form described in 3.16. A third is through the concept of "entropy" of a measure-preserving homomorphism, which can be defined as follows. (I am working from ROKHLIN [1967].) Let P be any finite partition of 1 in A. Then the *entropy* of P, $H(P)$, is

$$-\sum_{p\in P} \mu(p) \log_e \mu(p) .$$

Now if $f \in \mathrm{Aut}(A)$, and P is a finite partition of 1 in A, let $Q(P, f, n)$ be the set of atoms in the finite subalgebra $\langle\{f^k(p): p \in P, 0 \le k \le n\}\rangle$ for each $n \in \omega$, and set $h(f, P) = \inf_{n\in\omega}[1/(n+1)]H(Q(P, f, n))$. Finally, for $f \in \mathrm{Aut}_\mu(A)$, the *entropy* of f is

$$\sup\{h(f, P): P \text{ is a finite partition of 1 in } A\} .$$

It is clear that if f and g are conjugate in $\mathrm{Aut}_\mu(A)$, then they have the same entropy.

At this point I abandon this discussion before it overwhelms the rest of the chapter.

4. Liftings

From the variety and power of their consequences, two theorems stand out from the general theory of measure algebras. The first is Maharam's theorem, dealt with in Section 3; the second is the von Neumann–Maharam lifting theorem, the main result of the present section. Its most important applications lie outside the scope of this chapter, belonging rather to general probability theory; but even the limited class of corollaries that I shall offer here make it plain that it is essential to any proper understanding of the subject.

4.1. DEFINITION. Let (X, B, μ) be a measure space, with measure algebra $(A, \tilde{\mu})$. A *lifting* of (X, B, μ) is a homomorphism $f: A \to B$ such that $\pi(f(a)) = a$ for every $a \in A$, where $\pi: B \to A$ is the canonical epimorphism.

4.2. REMARKS. (a) Associated with such a lifting $f\colon A \to B$ we have a homomorphism $\theta = f \circ \pi\colon B \to B$ such that $b \triangle \theta(b) \in N_\mu$ for every $b \in B$ and $\theta(b) = \emptyset$ whenever $b \in B \cap N_\mu$. Evidently any such θ uniquely defines a lifting.

For applications in measure theory, it is the homomorphism θ that is more often used; but since this chapter is in a book on Boolean algebras I shall take the lifting f as the fundamental object.

(b) Observe that the liftings for (X, B, μ) depend only on X, B and the ideal $B \cap N_\mu$; see 4.6.

(c) Clearly, (X, B, μ) has a lifting iff there is a subalgebra C of B such that $\pi \restriction C\colon C \to A$ is bijective.

(d) What I here call a "lifting" is often called a "multiplicative lifting" to distinguish it from a "linear lifting"; see IONESCU TULCEA and IONESCU TULCEA [1969].

4.3. LEMMA. *Let (X, B, μ) be a measure space, with measure algebra $(A, \tilde{\mu})$, and $f\colon A \to B$ a lifting. Write $\mathscr{L}^\infty(B)$ for the linear space of bounded B-measurable real-valued functions on X. Then there is a linear operator $T\colon \mathscr{L}^\infty(B) \to \mathscr{L}^\infty(B)$ such that*

(i) *$T(\chi(f(a))) = \chi(f(a))$ for every $a \in A$ (where $\chi(b) \in \mathscr{L}^\infty(B)$ is the characteristic function of $b \in B$);*

(ii) *$Tg = g$ a.e. for every $g \in \mathscr{L}^\infty(B)$;*

(iii) *$Tg \geq 0$ everywhere if $g \geq 0$ a.e.;*

(iv) *$Tg = Th$ if $g = h$ a.e.*

PROOF. (a) Set $\theta = f \circ \pi\colon B \to B$, as in 4.2(a). For $x \in X$, $g \in \mathscr{L}^\infty(B)$ set

$$(Tg)(x) = \inf\{\alpha \in \boldsymbol{R}\colon x \in \theta(\{y \in X\colon g(y) \leq \alpha\})\}\,.$$

This is well-defined because g is bounded and B-measurable, so we are seeking the infimum of a non-empty bounded set. Now for any $\varepsilon > 0$,

$$\theta(\{y\colon g(y) \leq (Tg)(x) + \varepsilon\})\,, \qquad \theta(\{y\colon g(y) \leq (Tg)(x) - \varepsilon\})$$

both belong to B and only the former contains x. So

$$x \in \theta(\{y\colon |g(y) - (Tg)(x)| \leq \varepsilon\})$$

for any $\varepsilon > 0$, $g \in \mathscr{L}^\infty(B)$. It follows easily that if $g, h \in \mathscr{L}^\infty(B)$ and $\alpha \in \boldsymbol{R}$, then

$$T(g + h)(x) = (Tg)(x) + (Th)(x)\,, \qquad T(\alpha g)(x) = \alpha(Tg)(x)$$

for every $x \in X$.

(b) This defines T as a linear operator from $\mathscr{L}^\infty(B)$ to ${}^X\boldsymbol{R}$. Now fix $g \in \mathscr{L}^\infty(B)$ again and consider $Tg\colon X \to \boldsymbol{R}$. We see that, for $\alpha \in \boldsymbol{R}$,

$$\{x\colon (Tg)(x) < \alpha\} = \bigcup_{\beta \in \mathbf{Q}, \beta < \alpha} \theta(\{y\colon g(y) \le \beta\}) \in B .$$

So $Tg \in \mathcal{L}^\infty(B)$.

(c) If $b \in B$, then

$$\begin{aligned} \theta(\{y\colon (\chi b)(y) \le \alpha\}) &= X && \text{if } \alpha \ge 1 \\ &= X \setminus \theta(b) && \text{if } 0 \le \alpha < 1 \\ &= \emptyset && \text{if } \alpha < 0 . \end{aligned}$$

So $T(\chi b) = \chi(\theta(b))$. So if $a \in A$, then $T(\chi(f(a))) = \chi(\theta(f(a))) = \chi(f(a))$.

(d) If $g \in \mathcal{L}^\infty(B)$, then write $c_\alpha = \{x\colon g(x) \le \alpha\}$ for $\alpha \in \mathbf{R}$, and see that

$$\{x\colon (Tg)(x) < g(x)\} = \bigcup_{\beta \in \mathbf{Q}} (\theta(c_\beta) \setminus c_\beta) \in N_\mu$$

i.e. $g \le Tg$ a.e.; similarly, $-g \le T(-g)$ a.e., so $g = Tg$ a.e. If $g \ge 0$ a.e., then $c_\alpha \in N_\mu$ so $\theta(c_\alpha) = \emptyset$ for every $\alpha < 0$, and $Tg \ge 0$ everywhere. And the definition of T makes it plain that $Tg = Th$ whenever $g = h$ a.e. $\square$

REMARK. For an account of what is really happening here see FREMLIN [1974], 45B.

4.4. THEOREM. *Let (X, B, μ) be a complete decomposable measure space, with $\mu X > 0$. Then it has a lifting.*

PROOF. *Part A*: Assume for the time being that $\mu(X) = 1$.

(a) Enumerate A as $(a_\xi)_{\xi<\kappa}$ and for $\xi \le \kappa$ let A_ξ be the complete subalgebra of A generated by $\{a_\eta\colon \eta < \xi\}$. I seek to define f as the last of a family $(f_\xi)_{\xi \le \kappa}$, where $f_\xi\colon A_\xi \to B$ is a homomorphism for each $\xi \le \kappa$. The inductive hypothesis will be that $\pi(f_\xi(a)) = a$ whenever $a \in A_\xi$, where $\pi\colon B \to A$ is the canonical epimorphism, and that f_ξ extends f_η if $\eta \le \xi \le \kappa$.

(b) To start the induction, we have $A_0 = \{0, 1\}$; set $f_0(0) = \emptyset$, $f_0(1) = X$.

(c) *Inductive step to a successor* $\xi + 1$: Given f_ξ, then $A_{\xi+1} = \langle A_\xi \cup \{a_\xi\}\rangle$. (Compare the proof of 3.5 above.) Because A_ξ is a complete subalgebra of A,

$$d = \operatorname{lpr}(a_\xi, A_\xi) = \sum \{a \in A_\xi\colon a \le a_\xi\} ,$$

$$d' = \operatorname{upr}(a_\xi, A_\xi) = \prod \{a \in A_\xi\colon a_\xi \le a\} ,$$

both belong to A_ξ. (See Part I, Lemma 10.7.) Now $d \le a_\xi \le d'$ and $\pi(f_\xi(d)) = d$, $\pi(f_\xi(d')) = d'$. So if $b \in B$ is such that $\pi(b) = a_\xi$, set $b' = (b \cup f_\xi(d)) \cap f_\xi(d')$; we shall have $\pi(b') = a_\xi$ and, moreover,

$$f_\xi(a) \subseteq f_\xi(d) \subseteq b' \subseteq f_\xi(d') \subseteq f_\xi(a')$$

whenever $a, a' \in A$ and $a \le a_\xi \le a'$. So there is a homomorphism $f_{\xi+1}$:

$A_{\xi+1} \to B$, extending f_ξ, such that $f_{\xi+1}(a_\xi) = b'$ (Corollary 5.8 of Part I); and $\{a \in A_{\xi+1}: \pi(f_{\xi+1}(a)) = a\}$ is a subalgebra of $A_{\xi+1}$ including $A_\xi \cap \{a_\xi\}$, so is the whole of $A_{\xi+1}$. Thus, the induction proceeds.

(d) *Inductive step to a non-zero limit ordinal ξ of countable cofinality.* (i) Suppose that $0 < \xi \leq \kappa$ and that $\mathrm{cf}(\xi) = \omega$, and that f_η has been defined for $\eta < \xi$. Then for each $\eta \leq \xi$ we can set

$$B_\eta = \{b \in B: \pi(b) \in A_\eta\} .$$

Because each A_η is a complete subalgebra of A and π is σ-complete, each B_η is a σ-complete subalgebra of B. We need to know also that B_ξ is precisely the σ-algebra of sets generated by $\bigcup_{\eta<\xi} B_\eta$. For let B'_ξ be the σ-algebra of subsets of X generated by $\bigcup_{\eta<\xi} B_\eta$; then $\pi[B'_\xi]$ is a σ-complete subalgebra of A including $\bigcup_{\eta<\xi} A_\eta$; because A satisfies the countable chain condition $\pi[B'_\xi]$ is a complete subalgebra of A; and because ξ is a limit ordinal, $\pi[B'_\xi]$ contains a_η for every $\eta < \xi$, so $\pi[B'_\xi] \supseteq A_\xi$. Also, $B'_\xi \subseteq B_\xi$ so $\pi[B'_\xi] = A_\xi$. But finally $N_\mu \subseteq B_0 \subseteq B'_\xi$, so B'_ξ is precisely B_ξ.

(ii) For each $\eta < \xi$ we can apply Lemma 4.5 to $(X, B_\eta, \mu \restriction B_\eta)$ and $f_\eta: A_\eta \to B_\eta$ to find a linear map $T_\eta: \mathscr{L}^\infty(B_\eta) \to \mathscr{L}^\infty(B_\eta)$ as described there.

Define $g_\eta: A \to \mathscr{L}^\infty(B_\eta)$ as follows. Given $a \in A$ consider

$$H_a^{(\eta)} = \left\{h \in \mathscr{L}^\infty(B_\eta): \int_b h = \tilde{\mu}(a \cdot \pi(b))\ \forall b \in B\right\} .$$

By the Radon–Nikodým theorem (1.4 above), applied to the measure space $(X, B_\eta, \mu \restriction B_\eta)$ and the functional $b \mapsto \tilde{\mu}(a \cdot \pi(b))$, $H_a^{(\eta)} \neq \emptyset$; also $h = h'$ a.e. for every $h, h' \in H_a^{(\eta)}$. So we can define $g_\eta(a)$ by writing $g_\eta(a) = T_\eta(h)$ whenever $h \in H_a^{(\eta)}$. Note that as $T_\eta(h) = h$ a.e. for every $h \in \mathscr{L}^\infty(B_\eta)$, $g_\eta(a)$ is itself a member of $H_a^{(\eta)}$. We see also that if $h \in H_a^{(\eta)}$ and $b = \{x: h(x) < 0\}$, then $b \in B_\eta$ and

$$\int_b h = \tilde{\mu}(a \cdot \pi(b)) \geq 0 ,$$

so that $\mu(b) = 0$; thus $h \geq 0$ a.e. and $g_\eta(a) = T_\eta(h) \geq 0$ everywhere, by 4.3(iii).

If $a, a' \in A$ and $a \cdot a' = 0$, then for any $h \in H_a^{(\eta)}$, $h' \in H_{a'}^{(\eta)}$ we have $h + h' \in H_{a+a'}^{(\eta)}$, so

$$g_\eta(a + a') = T_\eta(h + h') = T_\eta(h) + T_\eta(h') = g_\eta(a) + g_\eta(a') .$$

If $a \in A_\eta$, then $\chi(f_\eta(a)) \in H_a^{(\eta)}$, so

$$g_\eta(a) = T_\eta(\chi(f_\eta(a))) = \chi(f_\eta(a)) .$$

In particular, $g_\eta(1) = \chi(X)$ so $g_\eta(-a) = \chi(X) - g_\eta(a)$ for any $a \in A$. Note also that

$$g_\eta(a + a') = g_\eta(a) + g_\eta(a') - g_\eta(a \cdot a') \le g_\eta(a) + g_\eta(a')$$

for all $a, a' \in A$.

(iii) All this works for every $\eta < \xi$. Now pick an increasing sequence $(\zeta(n))_{n\in\omega}$ in ξ with supremum ξ. For $x \in X$ set

$$p_x = \left\{a \in A: \lim_{n\to\infty} g_{\zeta(n)}(-a)(x) \text{ exists} = 0\right\}.$$

If $a' \ge a \in p_x$, then $-a' \le -a$, so

$$0 \le g_{\zeta(n)}(-a')(x) \le g_{\zeta(n)}(-a)(x)$$

for every $n \in \omega$, so $a' \in p_x$. If $a, a' \in p_x$, then $-(a \cdot a') = -a + -a'$, so

$$0 \le g_{\zeta(n)}(-(a \cdot a'))(x) \le g_{\zeta(n)}(-a)(x) + g_{\zeta(n)}(-a')(x)$$

for every $n \in \omega$, and $a \cdot a' \in p_x$. Finally, $1 \in p_x$ because $g_{\zeta(n)}(0)(x) = 0$ for all $n \in \omega$. Thus, p_x is a filter in A. Choose $q_x \in \mathrm{Ult}(A)$ such that $p_x \subseteq q_x$.

(iv) Define $f_\xi: A \to P(X)$ by setting

$$f_\xi(a) = \{x \in X: a \in q_x\} \quad \forall a \in A.$$

Because each q_x is an ultrafilter in A, f_ξ is a homomorphism (cf. Part I, Section 2).

At this point we must use the closed martingale theorem (1.6 above). If $a \in A_\xi$ there is a $b \in B_\xi$ such that $\pi(b) = a$. In this case $g_{\zeta(n)}(a)$ represents the conditional expectation of $\chi(b)$ on $B_{\zeta(n)}$ for each $n \in \omega$. (Read the definitions.) As B_ξ is the σ-algebra of sets generated by $\bigcup_{\eta<\xi} B_\eta = \bigcup_{n\in\omega} B_{\zeta(n)}$ ((d)(i) above), $\lim_{n\to\infty} g_{\zeta(n)}(a)(x)$ exists $= \chi(B)(x)$ a.e. (x). Set

$$b_* = \left\{x \in X: \lim_{n\to\infty} g_{\zeta(n)}(a)(x) \text{ exists} = 1\right\}$$
$$= \left\{x \in X: \lim_{n\to\infty} g_{\zeta(n)}(-a)(x) \text{ exists} = 0\right\},$$
$$b'_* = \left\{x \in X: \lim_{n\to\infty} g_{\zeta(n)}(a)(n) \text{ exists} = 0\right\}.$$

Then both b_* and b'_* belong to B, since every $g_{\zeta(n)}(a)$ is B-measurable; and $\pi(B_*) = \pi(b) = a$, $\pi(b'_*) = -a$. But we see that

$$x \in b_* \Rightarrow a \in p_x \subseteq q_x \Rightarrow x \in f_\xi(a),$$
$$x \in b'_* \Rightarrow x \in f_\xi(-a) = X \setminus f_\xi(a).$$

So $f_\xi(a) \triangle b_* \subseteq X \setminus (b_* \cup b'_*) \in N_\mu$. *Because* (X, B, μ) *is complete*, $f_\xi(a) \in B$ and $\pi(f_\xi(a)) = a$.

(v) Thus, $f_\xi: A_\xi \to B$ is a homomorphism of the right type. I have still to confirm that it extends f_η for $\eta < \xi$. But if $\eta < \xi$, there is an $m \in \omega$ such that $\eta \le \zeta(m) < \xi$. In this case, for every $a \in A$ and every $n \ge m$, $a \in A_{\zeta(n)}$ so

$g_{\zeta(n)}(a) = \chi(f_{\zeta(n)}(a)) = \chi(f_\eta(a))$. Accordingly,

$$x \in f_\eta(a) \Rightarrow a \in p_x \Rightarrow x \in f_\xi(a) ,$$
$$x \in X \setminus f_\eta(a) \Rightarrow -a \in p_x \Rightarrow x \in X \setminus f_\xi(a) ,$$

and $f_\xi(a) = f_\eta(a)$.

(e) *Inductive step to a limit ordinal of uncountable cofinality.* If $0 < \xi \leq \kappa$ and $\mathrm{cf}(\xi) > \omega$, examine $A'_\xi = \bigcup_{\eta<\xi} A_\eta$. This is a subalgebra of A. If $M \subseteq A'_\xi$, then (because A satisfies the countable chain condition) there is a countable $M' \subseteq M$ such that $\Sigma M = \Sigma M'$ in A; now (because $\mathrm{cf}(\xi) > \omega$) there is an $\eta < \xi$ such that $M' \subseteq A_\eta$, in which case $\Sigma M = \Sigma M' \in A_\eta \subseteq A'_\xi$. Thus, A'_ξ is complete, and is actually equal to A_ξ.

Acordingly we can define $f_\xi: A_\xi \to B$ to be the unique function extending f_η for every $\eta < \xi$, and the induction proceeds.

(f) The induction ends with the definition of $f_\kappa: A \to B$, which is a lifting of (X, B, μ).

Part B: The general case is now easy.

(a) Because (X, B, μ) is decomposable there is a partition $(b_i)_{i\in I}$ of X as described in 1.2(b)(iv). Since $\mu(X) > 0$, there is at least one $j \in I$ with $\mu(b_j) > 0$; now we may replace b_j by $b_j \cup \bigcup \{b_i: i \in I, \mu(b_i) = 0\}$ to obtain a partition of X into non-negligible sets with the same properties as the original partition. Thus, we may assume that $\mu(b_i) > 0$ for every $i \in I$ from the beginning.

(b) Set $B_i = B \cap P(b_i)$ and define $\mu_i: B_i \to [0, 1]$ by writing

$$\mu_i(b) = \mu(b)/\mu(b_i) \quad \forall b \in B_i .$$

Then (X, B_i, μ_i) is a complete probability space, so has a lifting (by Part A of the proof) for each $i \in I$. Let $\theta_i: B_i \to B_i$ be the corresponding homomorphisms, as described in 4.2(a). Then we have a homomorphism $\theta: B \to B$ given by

$$\theta(b) = \bigcup_{i\in I} \theta_i(b \cap b_i) \quad \forall b \in B ,$$

and θ corresponds to a lifting of (X, B, μ). □

4.5. REMARKS. (a) This theorem has a curious history. For Lebesgue measure it is due to VON NEUMANN [1931]. MAHARAM [1958b] writes that he "later (around 1942) gave an oral proof of the general theorem to S. Kakutani and the author. This proof was unfortunately forgotten beyond hope of reconstruction." (Von Neumann died in 1957.) The first published proof was given in MAHARAM [1958b]; it used Maharam's theorem to reduce the problem to the case in which X is ${}^\kappa\{0, 1\}$ with its usual measure (cf. 4.9 below), thereby (in effect) using a special case of the closed martingale theorem. The proof above follows the methods of IONESCU TULCEA and IONESCU TULCEA [1969].

(b) It is not hard to construct (non-σ-finite) measure spaces with no liftings; see IONESCU TULCEA and IONESCU TULCEA [1969, §IV.3] for a reduction of the problem (within the limited class of measure spaces considered in that book) and FREMLIN [1978] for a variety of relevant examples. A much more interesting question is: Is it relatively consistent with ZFC to assume that every probability space (whether complete or not) has a lifting? There seem to be essentially two results known in this direction, here given as 4.6 and 4.7(b).

(c) For sharper versions of 4.4, giving liftings with special properties, see IONESCU TULCEA and IONESCU TULCEA [1967], [1969, chap. VIII], and TALAGRAND [1982], [1984].

4.6. Subject to the continuum hypothesis, we can go a little way towards eliminating the hypothesis "complete" in Theorem 4.4.

THEOREM [CH]. *Let B be a σ-complete Boolean algebra and M a proper σ-ideal of B. Suppose that $A = B/M$ satisfies the countable chain condition and that $|A| \le (2^\omega)^+ = \omega_2$. Then there is a homomorphism $f: A \to B$ such that $\pi(f(a)) = a$ for every $a \in A$, where $\pi: B \to A$ is the canonical epimorphism.*

PROOF. (a) Note first that A and π are σ-complete (see the remark following 5.22 in Part I), so that A is complete, just as in 2.2(e).

Let $(a_\xi)_{\xi<\omega_2}$ run over A. For $\xi \le \omega_2$, let A_ξ be the complete subalgebra of A generated by $\{a_\eta: \eta < \xi\}$. As in 4.4, I seek to define a family $(f_\xi)_{\xi\le\omega_2}$ such that $f_\xi: A_\xi \to B$ is a homomorphism, $\pi \circ f_\xi$ is the identity on A_ξ, and f_ξ extends f_η whenever $\eta \le \xi \le \omega_2$.

(b) The start of the induction, and the inductive steps to successor ordinals and to limit ordinals of uncountable cofinality, are exactly as in 4.4 (Parts A(b), A(c) and A(e) of the proof).

(c) The step to a limit ordinal of countable cofinality is quite different. Suppose that $0 < \xi < \omega_2$, that $\mathrm{cf}(\xi) = \omega$, and that f_η has been defined for $\eta < \xi$. Because A satisfies the countable chain condition, A_ξ is in fact the σ-complete subalgebra of A generated by $\{a_\eta: \eta < \xi\}$; so $|A_\xi| \le \max(|\xi|, 2^\omega) = \omega_1$. (This is where I use the continuum hypothesis.) Let $(c_{\xi\alpha})_{\alpha<\omega_1}$ run over A_ξ. For $\alpha \le \omega_1$ let $C_{\xi\alpha}$ be the subalgebra (*not* complete subalgebra) of A generated by $\bigcup_{\eta<\xi} A_\eta \cup \{c_{\xi\beta}: \beta < \alpha\}$. I mean to define f_ξ as the last of a family $(g_{\xi\alpha})_{\alpha\le\omega_1}$, where $g_{\xi\alpha}: C_{\xi\alpha} \to B$ is a homomorphism for each $\alpha \le \omega_1$. The inductive hypothesis is, once again, that $\pi(g_{\xi\alpha}(c)) = c$ for every $c \in C_{\xi\alpha}$, and that $g_{\xi\alpha}$ extends $g_{\xi\beta}$ whenever $\beta \le \alpha \le \omega_1$.

(d) The inner induction starts with $C_{\xi 0} = \bigcup_{\eta<\xi} A_\eta$ and $g_{\xi 0} = \bigcup_{\eta<\xi} f_\eta$. For the step to a successor ordinal $\alpha+1$, we see that $C_{\xi,\alpha+1} = \langle C_{\xi\alpha} \cup \{c_{\xi\alpha}\}\rangle$. Now $C_{\xi\alpha}$ is not as a rule a complete subalgebra of A, so we cannot use the method of 4.4(A)(c) directly. But we observe that $C_{\xi\alpha}$ is expressible as the union of an increasing sequence $(D_n)_{n\in\omega}$ of complete subalgebra of A. For let $(\zeta(n))_{n\in\omega}$ be an increasing sequence in ξ with supremum ξ. If $\alpha = 0$, then $C_{\xi 0} = \bigcup_{n\in\omega} A_{\zeta(n)}$ and we can take $D_n = A_{\zeta(n)}$ for each $n \in \omega$. If $\alpha > 0$, then let $(\beta(n))_{n\in\omega}$ run over α, and take D_n to be

$$\langle A_{\zeta(n)} \cup \{c_{\xi,\beta(i)}: i \le n\}\rangle\ ;$$

it is easy to see that each D_n is complete and that $C_{\xi\alpha} = \bigcup_{n\in\omega} D_n$.

Now set

$$d_n = \sum \{a \in D_n: a \le c_{\xi\alpha}\}\ , \qquad d'_n = \prod \{a \in D_n: c_{\xi\alpha} \le a\}$$

for each $n \in \omega$. Then we see that d_n and d'_n both belong to D_n. If $m, n \in \omega$, then $d_n \leq c_{\xi\alpha} \leq d'_m$ so $g_{\xi\alpha}(d_n) \leq g_{\xi\alpha}(d'_m)$; thus $\Sigma_{n\in\omega}\, g_{\xi\alpha}(d_n) \leq \Pi_{n\in\omega}\, g_{\xi\alpha}(d'_n)$. Take any $b \in B$ such that $\pi(b) = c_{\xi\alpha}$ and try

$$b' = \Big(b + \sum_{n\in\omega} g_{\xi\alpha}(d_n)\Big) \cdot \prod_{n\in\omega} g_{\xi\alpha}(d'_n)\,.$$

Then $\pi(b') = c_{\xi\alpha}$. If $a, a' \in C_{\xi\alpha}$ and $a \leq c_{\xi\alpha} \leq a'$, then there is an $n \in \omega$ such that a, a' both belong to D_n, and now $a \leq d_n$ and $d'_n \leq a'$, so that

$$g_{\xi\alpha}(a) \leq g_{\xi\alpha}(d_n) \leq b' \leq g_{\xi\alpha}(d'_n) \leq g_{\xi\alpha}(a')\,.$$

Accordingly, there is a homomorphism $g_{\xi,\alpha+1}\colon C_{\xi,\alpha+1} \to B$, extending $g_{\xi\alpha}$, such that $g_{\xi,\alpha+1}(c_{\xi\alpha}) = b'$; in which case $\pi(g_{\xi,\alpha+1}(c)) = c$ for every $c \in C_{\xi,\alpha+1}$, and the inner induction proceeds.

(e) But since the $C_{\xi\alpha}$ are not expected to be complete, we have $C_{\xi\alpha} = \bigcup_{\beta<\alpha} C_{\xi\beta}$ for every non-zero limit ordinal $\alpha \leq \omega_1$. So for such α we can set $g_{\xi\alpha} = \bigcup_{\beta<\alpha} g_{\xi\beta}$. In this way the inner induction continues until we have $f_\xi = g_{\xi,\omega_1}\colon A_\xi \to B$.

(f) Accordingly, the outer induction proceeds uninterrupted until we have $f = f_{\omega_2}\colon A \to B$, as required. □

4.7. THEOREM. *Let B the algebra of Borel subsets of* $[0, 1]$ *and μ the restriction of Lebesgue measure to B.*

(a) *If the continuum hypothesis is true, then* $([0, 1], B, \mu)$ *has a lifting.*

(b) *It is relatively consistent with ZFC to suppose that* $([0, 1], B, \mu)$ *has no lifting.*

PROOF. (a) This is an easy special case of 4.6.

(b) See SHELAH [1983]. □

4.8. REMARKS. (a) Theorem 4.6 is due essentially to MOKOBODZKI [a, Corollary 19]. The case $|A| \leq \omega_1$, sufficient for 4.6(a), is a good deal easier; it is covered by results in VON NEUMANN and STONE [1935].

(b) Shelah's proof of 4.6(b) is an ad hoc construction, and is difficult. But it is quite unclear that such sophisticated methods are necessary. A great many models of set theory in which $2^\omega > \omega_1$ are now very well known. But as far as I am aware, the question of whether $([0, 1], B, \mu)$, as defined in 4.6, has a lifting has been resolved in only one of the standard models. CARLSON [19??] has shown that if we add just ω_2 Cohen reals to a model of ZFC + GCH we obtain a model of ZFC in which $([0, 1], B, \mu)$ has a lifting. The argument uses ideas from 4.6 and it seems that ω_2 is as far as we can go by this method.

(c) We are therefore left with the following problems, in the notation of 4.7.

(i) It is relatively consistent with ZFC to suppose that $2^\omega \geq \omega_3$ and that $([0, 1], B, \mu)$ has a lifting?

(ii) If $m = 2^\omega > \omega_1$ (i.e. Martin's axiom is true, but the continuum hypothesis is false) does $([0, 1], B, \mu)$ have a lifting?

(iii) What happens in other familiar models, e.g. random real models?

(iv) Is it relatively consistent with ZFC to suppose that there is a lifting of

$([0,1], B, \mu)$ such that the lifting takes values all lying within a Borel class of level strictly below ω_1? (I had this question from A.H. Stone.)

4.9. For general probability spaces the situation is even more open. I give an easy reduction.

PROPOSITION. *For each infinite cardinal κ let B'_κ be the Baire σ-algebra of ${}^\kappa\{0,1\}$ and μ'_κ the restriction of the usual measure on ${}^\kappa\{0,1\}$ to B'_κ. If $({}^\kappa\{0,1\}, B'_\kappa, \mu'_\kappa)$ has a lifting for every infinite κ, then every decomposable measure space has a lifting.*

PROOF. The techniques of 4.4 (Part B of the proof) and the remarks in 3.1 show that we need consider only Maharam homogeneous probability spaces; and of course spaces of finite Maharam type (with finite measure algebras) are trivial. So suppose that (X, B, μ) is a Maharam homogeneous probability space of Maharam type $\kappa \geq \omega$. Then its measure algebra $(A, \tilde{\mu})$ is isomorphic to $(A_\kappa, \tilde{\mu}_\kappa)$, the measure algebra of ${}^\kappa\{0,1\}$. Let $h: A_\kappa \to A$ be a measure-preserving isomorphism. By 2.21 there is a function $g: X \to {}^\kappa\{0,1\}$, inverse-measure-preserving for μ and μ'_κ, which realizes h^{-1}: $A_\kappa \to A$. Now let $f: A_\kappa \to B'_\kappa$ be a lifting for $({}^\kappa\{0,1\}, B'_\kappa, \mu')$ and define $f_1: A \to B$ by writing

$$f_1(a) = \dot{g}^{-1}[f(h(a))] .$$

It is straightforward to check that f_1 is a lifting for (X, B, μ). $\square$

4.10. REMARK. From Mokobodzki's theorem 4.6 again we see that the continuum hypothesis implies that $({}^\kappa\{0,1\}, B'_\kappa, \mu'_\kappa)$ has a lifting for $\kappa \leq \omega_2$. But it is an open question whether $({}^{\omega_3}\{0,1\}, B'_{\omega_3}, \mu'_{\omega_3})$ can have a lifting.

4.11. LIFTINGS AND HYPERSTONIAN SPACES. (a) Let (X, B, μ) be a measure space, $(A, \tilde{\mu})$ its measure algebra, and $\pi: B \to A$ the canonical epimorphism. If $f: A \to B$ is a lifting, we have a function $h: X \to \mathrm{Ult}(A)$ defined by

$$h(x) = \{a \in A: x \in f(a)\} \quad \forall x \in X .$$

If $s(a) \subseteq \mathrm{Ult}(A)$ is the clopen set corresponding to $a \in A$, then

$$h^{-1}[s(a)] = f(a) \quad \forall a \in A ,$$

so

$$\pi(h^{-1}[s(a)]) = a \quad \text{for every } a \in A .$$

(Evidently this reverses; if there is a function $h: X \to \mathrm{Ult}(A)$ such that $h^{-1}[s(a)] \in B$ and $\pi(h^{-1}[s(a)]) = a$ for every $a \in A$, then $a \mapsto h^{-1}[s(a)]$ is a lifting of (X, B, μ).)

(b) Now suppose that $\mu(X) < \infty$. Then we have a Radon measure ν on $\mathrm{Ult}(A)$

defined by saying that $\nu(s(a)) = \tilde{\mu}(a) = \mu(f(a))$ for every $a \in A$ (2.13–2.14 above). So the function h, if it exists, will be such that $\mu(h^{-1}[c]) = \nu(c)$ for every clopen set $c \subseteq \mathrm{Ult}(A)$. If also (X, B, μ) is complete, then h (which then surely does exist) will be inverse-measure-preserving for μ and ν, by 1.18 and 2.14(b).

(c) Specializing further, suppose that $(X, \mathfrak{T}, B, \mu)$ is a compact Radon measure space. Then we know that we have a canonical inverse-measure-preserving continuous function $g\colon \mathrm{Ult}(A) \to X$ (2.17). We can ask: When is $g \circ h$ the identity function on X? g is defined by saying that if $v \subseteq X$ is compact and $z \in s(\pi(v))$ then $g(z) \in v$. So,

$$\begin{aligned} &g(h(x)) = x \quad \forall x \in X \\ &\quad \Leftrightarrow x \in v \text{ whenever } v \subseteq X \text{ is compact and } h(x) \in s(\pi(v)) \\ &\quad \Leftrightarrow v \supseteq h^{-1}[s(\pi(v))] = f(\pi(v)) \text{ whenever } v \subseteq X \text{ is compact}\,. \end{aligned}$$

(In this case, of course, X will have to be self-supporting.) Such liftings are called *strong*; see IONESCU TULCEA and IONESCU TULCEA [1969]. They are of great importance in measure theory, but I shall not discuss them further, except to remark that not every self-supporting compact Radon measure space has a strong lifting (LOSERT [1979]).

4.12. REALIZATION OF HOMOMORPHISMS: THEOREM. Let $(X, \mathfrak{T}, B, \mu)$ be a Radon measure space and (Y, C, ν) a complete decomposable measure space; let their measure algebras be $(A_X, \tilde{\mu})$ and $(A_Y, \tilde{\nu})$, respectively, and $f\colon A_X \to A_Y$ any complete homomorphism. Then f is realized, in the sense of 2.24, by a function $\varphi\colon Y \to X$.

PROOF. (a) Write $\pi_X\colon B \to A_X$ and $\pi_Y\colon C \to A_Y$ for the canonical epimorphisms. Let $g\colon \mathrm{Ult}(A_Y) \to \mathrm{Ult}(A_X)$ be the continuous function corresponding to f (Theorem 8.2 of Part I), so that $g^{-1}[s(a)] = s(f(a))$ for every $a \in A_X$, where $s(a) \subseteq \mathrm{Ult}(A_X)$ is the clopen set corresponding to a. Let $e\colon \mathrm{Ult}(A_X) \to X$ be an inverse-measure-preserving function such that $e^{-1}[b] \triangle s(\pi_X(b))$ is negligible for every $b \in B$, as in Theorem 2.17 above. Finally, let $h\colon Y \to \mathrm{Ult}(A_Y)$ be such that $\pi_Y(h^{-1}[s(a)]) = a$ for every $a \in A_Y$, as in 4.11. Try $\varphi = e \circ g \circ h\colon Y \to X$.

(b) To see that φ realizes f, let us first consider a compact set $v \subseteq X$. Then $e(z) \in v$ for every $z \in s(\pi_X(v))$. (I am supposing that e is defined exactly by the recipe of 2.17.) So,

$$\begin{aligned} \varphi^{-1}[v] &= h^{-1}[g^{-1}[e^{-1}[v]]] \supseteq h^{-1}[g^{-1}[s(\pi_X(v))]] \\ &= h^{-1}[s(f(\pi_X(v)))] \in C\,. \end{aligned}$$

(c) Now take any $b \in B$. By 1.7(v)

$$\pi_X(b) = \textstyle\sum \{\pi_X(v) : v \subseteq b \text{ is compact}\}\,,$$

so

$$f(\pi_X(b)) = \textstyle\sum \{f(\pi_X(v)) : v \subseteq b \text{ is compact}\}$$

because f is a complete homomorphism. Let $c \in C$ be such that $\nu(c) < \infty$. Then $A_Y \restriction \pi_Y(c)$ satisfies the countable chain condition, so there is an increasing sequence $(v_n)_{n\in\omega}$ of compact subsets of b such that

$$f(\pi_X(b)) \cdot \pi_Y(c) = \sum_{n\in\omega} f(\pi_X(v_n)) \cdot \pi_Y(c) \,,$$

so that

$$\begin{aligned}\sup_{n\in\omega} \nu(c \cap h^{-1}[s(f(\pi_X(v_n)))]) \\ &= \sup_{n\in\omega} \tilde{\nu}(\pi_Y(c) \cdot f(\pi_X(v_n))) \\ &= \tilde{\nu}(\pi_Y(c) \cdot f(\pi_X(b))) \,.\end{aligned}$$

Similarly, there is an increasing sequence $(v'_n)_{n\in\omega}$ of compact subsets of $X \setminus b$ such that

$$\sup_{n\in\omega} \nu(c \cap h^{-1}[s(f(\pi_X(v'_n)))]) = \tilde{\nu}(\pi_Y(c) \cdot f(\pi_X(X \setminus b))) \,.$$

But now examine $c \cap \varphi^{-1}[b]$ and $c \cap \varphi^{-1}[X \setminus b]$. Setting

$$c_0 = c \cap \bigcup_{n\in\omega} h^{-1}[s(f(\pi_X(v_n)))] \,, \qquad c'_0 = c \cap \bigcup_{n\in\omega} h^{-1}[s(f(\pi_X(v'_n)))]$$

we see that c_0 and c'_0 both belong to C, that

$$\begin{aligned}\nu(c_0) + \nu(c'_0) &= \tilde{\nu}(\pi_Y(c) \cdot f(\pi_X(b))) + \tilde{\nu}(\pi_Y(c) \cdot f(\pi_X(X \setminus b))) \\ &= \tilde{\nu}(\pi_Y(c)) = \nu(c) \,,\end{aligned}$$

and that

$$c_0 \subseteq \bigcup_{n\in\omega} \varphi^{-1}[v_n] \,, \qquad c'_0 \subseteq \bigcup_{n\in\omega} \varphi^{-1}[v'_n]$$

(using (b) above) so that $c_0 \subseteq \varphi^{-1}[b]$ and $c' \subseteq \varphi^{-1}[X \setminus b]$. Because (Y, C, ν) is complete, it follows that $c \cap \varphi^{-1}[b] \in C$ and that

$$\pi_Y(c \cap \varphi^{-1}[b]) = \pi_Y(c_0) = \pi_Y(c) \cdot f(\pi_X(b)) \,.$$

This is true for any c of finite measure. Because (Y, C, ν) is decomposable, $\varphi^{-1}[b] \in C$; also (because (Y, C, ν) is semi-finite)

$$\begin{aligned}\pi_Y(\varphi^{-1}[b]) &= \sup\{\pi_Y(c \cap \varphi^{-1}[b]) \colon c \in C,\ \nu(C) < \infty\} \\ &= \sup\{\pi_Y(c) \cdot f(\pi_X(b)) \colon c \in C,\ \nu(C) < \infty\} \\ &= f(\pi_X(b)) \,.\end{aligned}$$

As b is arbitrary, this shows that φ realizes f. $\square$

4.13. REMARKS. (a) Versions of this result have been given by various authors; see, for example, IONESCU TULCEA and IONESCU TULCEA [1969], Chap. X, Theorem 1], or GRAF [1980, Theorem 1].

(b) It is natural to enquire whether the result will be valid for σ-complete homomorphisms f. In the most important applications, A_X satisfies the countable chain condition (equivalently, (X, B, μ) is σ-finite), so σ-complete homomorphisms with domain A_X are necessarily complete. For the general case, we need to know whether our universe contains real-valued-measurable cardinals. For suppose that $f: A \to B$ is a σ-complete homomorphism which is not complete, where (A, μ) and (B, ν) are Maharam algebras. Because (B, ν) is semi-finite, there is a $b \in B$ such that $0 < \nu(b) < \infty$ and $a \mapsto b \cdot f(a)$: $A \twoheadrightarrow B \restriction b$ is not complete. Now let $(a_i)_{i \in I}$ be a partition of 1 in A such that $\mu(a_i) < \infty$ for every $i \in I$. If $J \subseteq I$ is countable, $A \restriction \Sigma_{i \in J} a_i$ satisfies the countable chain condition and $f \restriction (A \restriction \Sigma_{i \in J} a_i)$ is complete; so $b \cdot f(\Sigma_{i \in J} a_i) = b \cdot \Sigma_{i \in J} f(a_i)$ cannot be b for any countable $J \subseteq I$. Set

$$\lambda(J) = \nu\Big(b \cdot f\Big(\sum_{i \in J} a_i\Big)\Big)$$

for $J \subseteq I$. Then $(I, P(I), \lambda)$ is a totally finite measure space and $\lambda(J) < \nu(b) = \lambda(I)$ for every countable $J \subseteq I$. But this means that there is a real-valued-measurable cardinal (Section 7 below).

Conversely, suppose that there is a real-valued-measurable cardinal κ; let λ be a probability measure with domain $P(\kappa)$ which is zero on singletons. Then the measure algebra of $(\kappa, P(\kappa), \lambda)$ is isomorphic to the measure algebra of a compact Radon measure space $(Y, \mathfrak{S}, C, \nu)$ (Theorem 2.13 above); let π: $P(\kappa) \to A_Y = C/N_\nu$ be the corresponding σ-complete epimorphism. Define μ: $P(\kappa) \to [0, \infty]$ by writing $\mu(b) = |b|$ if $b \subseteq \kappa$ is finite, ∞ if $b \subseteq \kappa$ is infinite; set $X = \kappa$, $\mathfrak{T} = B = P(\kappa)$; then $(X, \mathfrak{T}, B, \mu)$ is a Radon measure space, and $N_\mu = \{\emptyset\}$, so $A_X \cong P(\kappa)$. Define f: $A_X \to A_Y$ by setting

$$f(\pi_X(b)) = \pi(b) \quad \forall b \subseteq \kappa .$$

Then f is a σ-complete homomorphism. But it cannot be realized by any φ: $Y \to X$, because such a φ would yield a partition $\{\varphi^{-1}[\{x\}]: x \in \varphi[Y]\}$ of Y into negligible sets such that all subfamilies had measurable unions; and such partitions cannot exist for Radon measure spaces, by Theorem 1 of FREMLIN [1981] (= Theorem 6Ma of FREMLIN [a]; actually, since we can assume that either Y is a singleton or that A_Y is atomless, the hardest part of this theorem can be avoided).

Replacing Y by the disjoint union of Y and X, it is easy to make f measure-preserving in the construction above.

4.14. REALIZATION OF AUTOMORPHISMS. Now suppose that $(X, \mathfrak{T}, B, \mu)$ is a Radon measure space and that f: $A \to A$ is an automorphism, where A is the measure algebra of (X, B, μ). From 4.12 we see that f and f^{-1} can be realized by functions g, h from X to itself. The question arises: When can g and h be taken to be inverses of each other?

A partial answer is given by Choksi's theorem (CHOKSI [1972a], [1972b], generalized in 2.23 above); if $(X, \mathfrak{T})$ is a product of Polish spaces and μ is completion regular, then A can be identified with $C/C \cap N_\mu$, where C is the Baire σ-algebra of X, and now mutually inverse C-C-measurable functions can be found realizing f and f^{-1} as functions from $C/C \cap N_\mu$ to itself; and these functions will be B-B-measurable, so will realize f and f^{-1} as functions from $A = B/N_\mu$ to itself.

To see that g and h cannot always be taken to be inverses of each other, take X to be the disjoint union of $[0, 1]$, with Lebesgue measure, and its hyperstonian space Z. The canonical isomorphism between the measure algebras of $[0, 1]$ and Z is realized by functions $\varphi: [0, 1] \to Z$ and $\psi: Z \to [0, 1]$, as in 4.11 and 2.17; it is not hard to show that $\psi \circ \varphi: [0, 1] \to [0, 1]$ can be taken to be the identity on $[0, 1]$ (see 4.11(c)). But ψ is nearly uniquely defined, and cannot possibly be injective; so that $\varphi \circ \psi: Z \to Z$ cannot be the identity. Now the measure algebra A of X is (as Boolean algebra) the simple product of the measure algebras of $[0, 1]$ and Z, and the exchanging-factors automorphism of A is a measure-preserving automorphism which cannot be realized by an injective function. (This example is mentioned in MAHARAM [1975].)

5. Which algebras are measurable?

The work above (see for instance 2.10, 3.18–3.22) makes it plain that measurable algebras form a very special class of Boolean algebra. It is natural to ask whether there is any easy way to identify them in terms of their Boolean-algebra properties. No fully convincing answer to this question has been found; but there are some very interesting partial results which I present here as Theorem 5.12. I also take the opportunity to say something about chain conditions in measurable algebras (5.4).

5.1. PROPOSITION. *Let A be a Boolean algebra. Then there is a function μ such that (A, μ) is a measure algebra if, and only if, A is σ-complete.*

PROOF. By Definition 2.1(a), if (A, μ) is a measure algebra, then A is σ-complete. On the other hand, if A is σ-complete, define $\mu: A \to [0, \infty]$ by setting $\mu(0) = 0$, $\mu(a) = \infty$ for $a \in A^+$; then (A, μ) is a measure algebra. □

REMARK. This is ridiculous. Read on.

5.2. PROPOSITION. *Let A be a Boolean algebra. Then the following are equivalent*:

(i) *There is a function μ such that (A, μ) is a semi-finite measure algebra.*

(ii) *A is σ-complete and*

$$\{a \in A^+ : A \restriction a \text{ is a measurable algebra}\}$$

is dense in A^+.

(iii) *A is σ-complete and*

$$\{a \in A^+ : A \restriction a \cong A_\kappa \text{ for some cardinal } \kappa\}$$

is dense in A^+. (Here A_κ is the measure algebra of ${}^\kappa\{0, 1\}$, as in 3.1.)

PROOF. (i)$\Rightarrow$(iii). Of course A is σ-complete. Now let $c \in A^+$. Because (A, μ) is semi-finite, there is a $b \in (A \restriction c)^+$ such that $\mu(b) < \infty$. Now $(A \restriction b, \mu \restriction (A \restriction b))$ is a totally finite measure algebra, so by Maharam's theorem (3.9) there is an $a \in (A \restriction b)^+$ such that $A \restriction a$ is homogeneous, and is isomorphic to A_κ for some cardinal κ, as required.

(iii)$\Rightarrow$(ii). Trivial.

(ii)$\Rightarrow$(i). Let $(a_i)_{i \in I}$ be a maximal pairwise disjoint family in A^+ such that $A \restriction a_i$ is a measurable algebra for each $i \in I$. For each $i \in I$ let $\mu_i : A \restriction a_i \to \boldsymbol{R}$ be such that $(A \restriction a_i, \mu_i)$ is a totally finite measure algebra. Define $\mu : A \to [0, \infty]$ by writing

$$\mu(a) = \sum_{i \in I} \mu_i(a \cdot a_i) \quad \forall a \in A .$$

It is easy to check that (A, μ) is a semi-finite measure algebra. $\square$

REMARK. The point is that, on the definitions I have chosen, the question "Which Boolean algebras can be the underlying algebras of measure algebras?" is of no interest; it is disposed of summarily by 5.1. The question "Which Boolean algebras can be the underlying sets of semi-finite measure algebras?" is plainly much more to the point, and in a way is answered by the equivalence (i)$\Leftrightarrow$(iii) of 5.2. Evidently, we can give a similar answer to the question "Which Boolean algebras can be the underlying sets of Maharam algebras?", and the question "Which Boolean algebras are measurable?" is now settled in some sense by 2.8. But it is not settled very usefully, for the argument of (iii)$\Rightarrow$(i) here is altogether too transparent.

5.3. MISCELLANEOUS PROPERTIES OF MEASURABLE ALGEBRAS. If we seek non-obvious criteria for Boolean algebras to be measurable, the first step is perhaps to list the known properties of measurable algebras. Let A be a measurable algebra. Then

(a) A satisfies the countable chain condition (2.2(h)).

(b) A is complete (2.2(h)).

(c) A is weakly (ω, ∞)-distributive (2.10).

(d) If A is infinite, there is a monomorphism from $A \oplus A$ to A (3.22).

(e) A is isomorphic to a simple product of homogeneous Boolean algebras (3.9).

(f) $\gamma_\omega(A) = \gamma_\omega^+(A) = \mathrm{wdistr}(A)$ (6.15(a) below) .

Of course (d) and (f) are a rather random choice. (e) would be of some use if we had non-trivial ways of identifying homogeneous measurable algebras.

It is conceivable that these properties already characterize measurable algebras; see 5.15(b) below.

5.4. CHAIN CONDITIONS. In 2.2(h)–5.3(a) we saw that a measurable algebra satisfies the countable chain condition. In fact a great deal more can be said.

(a) A Boolean algebra A satisfies the *σ-bounded chain condition* if A^+ is expressible as $\bigcup_{n\in\omega} X_n$ where, for each $n \in \omega$, any pairwise disjoint subset of X_n has at most $n+1$ members. (See COMFORT and NEGREPONTIS [1982, p. 125].) Now evidently any measurable algebra satisfies this condition. (For if (A, μ) is a totally finite measure algebra, set $X_n = \{a \in A: n\mu(a) > \mu(1)\}$.)

(b) Many properties follow from (a). For instance, every measurable algebra A satisfies *Knaster's condition*, i.e. if $X \subseteq A^+$ is uncountable, there is an uncountable $Y \subseteq X$ such that $y \cdot y' \neq 0$ for all $y, y' \in Y$.

(c) A strengthening of Knaster's condition is the following. A Boolean algebra A has *property $K_{\kappa n}$*, where κ is an infinite cardinal and $2 \leq n \in \omega$, if whenever $(a_\xi)_{\xi<\kappa}$ is a family in A^+ there is an $I \in [\kappa]^\kappa$ such that $\Pi_{\xi\in J} a_\xi \neq 0$ for every $J \in [I]^n$. (Thus, Knaster's condition is "property $K_{\omega_1,2}$".) Now every measurable algebra has property $K_{\kappa n}$ for every cardinal κ of uncountable cofinality and every integer $n \geq 2$ (ARGYROS and KALAMIDAS [1982], repeated in COMFORT and NEGREPONTIS [1982, Corollary 6.17]).

(d) A further strengthening of Knaster's condition is this. An infinite cardinal κ is a *precaliber* of a Boolean algebra A if for every family $(a_\xi)_{\xi<\kappa}$ there is a set $I \in [\kappa]^\kappa$ such that $\Pi_{\xi\in J} a_\xi \neq 0$ for every finite $J \subseteq I$. (In this case, of course, A must have property $K_{\kappa n}$ for every n.) We now have an undecidable problem. If $\mathfrak{m} > \kappa$ (i.e. MA(κ) is true) and cf(κ) $> \omega$, then κ is a precaliber of every Boolean algebra which satisfies the countable chain condition (FREMLIN [1984], 41C; in fact, if cf(κ) $> \omega$, then $\mathfrak{m} > \kappa$ iff κ is a precaliber of every Boolean algebra satisfying the countable chain condition – see TODORČEVIĆ and VELIČKOVIĆ [1985]). In particular, it is relatively consistent with ZFC to suppose that ω_1 is a precaliber of every measurable algebra. But if the continuum hypothesis is true, then ω_1 is not a precaliber of any non-atomic measurable algebra. For if we enumerate ${}^\omega\{0,1\}$ as $(x_\xi)_{\xi<\omega_1}$ and choose a non-negligible compact set $K_\xi \subseteq {}^\omega\{0,1\} \setminus \{x_\eta: \eta \leq \xi\}$ for each $\xi < \omega_1$, then for any uncountable $I \subseteq \omega_1$ we shall have $\bigcap_{\xi\in I} K_\xi = \emptyset$, so there is a finite $J \subseteq I$ such that $\bigcap_{\xi\in J} K_\xi = \emptyset$. This means that the family of images $(\pi(K_\xi))_{\xi<\omega_1}$ in the measure algebra A_ω of ${}^\omega\{0,1\}$ witnesses that ω_1 is not a precaliber of A_ω. Now since A_ω can be embedded as a complete subalgebra of any atomless measurable algebra other than $\{0\}$ (3.20), it follows that ω_1 is not a precaliber of any non-atomic measurable algebra.

5.5. SUBMEASURES AND ADDITIVE FUNCTIONALS. I turn now to another family of ideas. Let A be a Boolean algebra.

(a) An *additive functional* (also called a "finitely additive measure", or "finitely additive signed measure") on A is a function $\nu: A \to \mathbf{R}$ such that $\nu(a+b) = \nu(a) + \nu(b)$ whenever $a \cdot b = 0$ in A. In this case $\nu(0) = 0$. ν is *positive* if $\nu(a) \geq 0$ for every $a \in A$, and *strictly positive* if $\nu(a) > 0$ for every $a \in A^+$.

(b) A *submeasure* on A is a function $\nu: A \to [0, \infty)$ such that (i) $\nu(a+b) \leq \nu(a) + \nu(b)$ for all $a, b \in A$; (ii) $\nu(0) = 0$; and (iii) $\nu(a) \leq \nu(b)$ whenever $a \leq b$.

Again, ν is *strictly positive* if $\nu(a) > 0$ for every $a \in A^+$. Note that I do not allow ∞ as a value of a submeasure. Clearly, any positive additive functional is a submeasure.

Observe that if ν is a strictly positive submeasure on A there is a corresponding metric ρ on A defined by writing

$$\rho(a, b) = \nu(a \cdot -b) + \nu(b \cdot -a)$$

(see 5.6 below, and compare 2.19).

(c) A submeasure ν: $A \to [0, \infty)$ is *exhaustive* if $\lim_{n\to\infty} \nu(a_n) = 0$ for every pairwise disjoint sequence $(a_n)_{n\in\omega}$ in A^+. It is *uniformly exhaustive* if for every $\delta > 0$ there is an $n \in \omega$ such that whenever $X \subseteq A^+$ is pairwise disjoint and $|X| \geq n$ there is an $x \in X$ such that $\nu(x) \leq \delta$. Evidently a uniformly exhaustive submeasure is exhaustive, and a positive additive functional is a uniformly exhaustive submeasure. If A carries a strictly positive exhaustive submeasure it must satisfy the countable chain condition.

(d) A submeasure ν on A is a *Maharam submeasure* (also called a "continuous outer measure") if $\lim_{n\to\infty} \nu(a_n) = 0$ whenever $(a_n)_{n\in\omega}$ is a decreasing sequence in A and $\Pi_{n\in\omega}\, a_n = 0$. A positive countably additive functional on A (see 1.3 above) is a Maharam submeasure (compare 2.2(g)). A Maharam submeasure on a σ-complete algebra is exhaustive (since if $(a_n)_{n\in\omega}$ is a pairwise disjoint sequence in A^+ and A is σ-complete, then $\Pi_{n\in\omega} \Sigma_{m\geq n}\, a_m = 0$).

(e) If ν is a Maharam submeasure on A then (i) $\nu(\Sigma_{n\in\omega}\, a_n) = \lim_{n\to\infty} \nu(a_n)$ whenever $(a_n)_{n\in\omega}$ is an increasing sequence in A for which $\Sigma_{n\in\omega}\, a_n$ exists (for $\lim_{m\to\infty} \nu(\Sigma_{n\in\omega}\, a_n \cdot -a_m) = 0$); (ii) $\nu(\Sigma_{n\in\omega}\, a_n) \leq \Sigma_{n=0}^{\infty} \nu(a_n)$ whenever $(a_n)_{n\in\omega}$ is any sequence in A such that $\Sigma_{n\in\omega}\, a_n$ exists (for $\Sigma_{n\in\omega}\, a_n = \Sigma_{n\in\omega}\, (\Sigma_{i\leq n}\, a_i)$).

(f) Suppose that A is σ-complete and that μ is a strictly positive Maharam submeasure on A. If ν is any Maharam submeasure on A, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\nu(a) \leq \varepsilon$ whenever $\mu(a) \leq \delta$. (Proof as in 2.3(a).) In particular, if μ is uniformly exhaustive, so is ν.

5.6. PROPOSITION. (a) *Let A be a σ-complete Boolean algebra and ν a strictly positive Maharam submeasure on A. Then*

(i) *A satisfies the countable chain condition and is complete (as Boolean algebra);*

(ii) *A is weakly (ω, ∞)-distributive;*

(iii) *A is complete (as metric space) under the metric ρ associated with ν.*

(b) *Let A be a Boolean algebra and ν a strictly positive exhaustive submeasure on A, with associated metric ρ. Then the completion $\hat{A}$ of A under ρ has a canonical Boolean algebra structure making A a subalgebra of $\hat{A}$, and there is a unique extension of ν to a Maharam submeasure $\hat{\nu}$ on $\hat{A}$. Moreover, $\hat{A}$ is a complete Boolean algebra.*

PROOF. (a)(i) ν is exhaustive (5.5(d)) so A satisfies the countable chain condition (5.5(c)) and must be complete. (Compare 2.2(h).)

(ii) The proof of 2.10 applies virtually unchanged, appealing to 5.7(e) instead of 2.2.

(iii) Let $(a_n)_{n\in\omega}$ be a sequence in A such that $\Sigma_{n=0}^{\infty} \rho(a_n, a_{n+1}) < \infty$. Set $b_n = \Sigma_{m\geq n} a_m$ for $n \in \omega$, $b = \Pi_{n\in\omega} b_n$. Then, for any $n \in \omega$,

$$a_n \cdot -b = \sum_{m\geq n} a_n \cdot -b_m \leq \sum_{m\geq n} a_n \cdot -a_m \leq \sum_{m\geq n} a_m \cdot -a_{m+1} ,$$
$$b \cdot -a_n \leq b_n \cdot -a_n = \sum_{m\geq n} a_m \cdot -a_n \leq \sum_{m\geq n} a_{m+1} \cdot -a_m ;$$

so

$$\begin{aligned}\rho(a_n, b) &= \nu(a_n \cdot -b) + \nu(b \cdot -a_n) \\ &\leq \sum_{m=n}^{\infty} \nu(a_m \cdot -a_{m+1}) + \sum_{m=n}^{\infty} \nu(a_{m+1} \cdot -a_m) = \sum_{m=n}^{\infty} \rho(a_{m+1}, a_m) .\end{aligned}$$

Accordingly, $\lim_{n\to\infty} \rho(a_n, b) = 0$ and $b = \lim_{n\to\infty} a_n$ for ρ. As $(a_n)_{n\in\omega}$ is arbitrary, A is complete under ρ.

(b)(i) If a, a', b, $b' \in A$ then we find easily that

$$\rho(a + b, a' + b') \leq \rho(a, a') + \rho(b, b') ,$$
$$\rho(a \cdot b, a' \cdot b') \leq \rho(a, a') + \rho(b, b') ,$$
$$\rho(-a, -b) = \rho(a, b) .$$

(The point is that, for instance,

$$(a + b) \cdot -(a' + b') \leq a \cdot -a' + b \cdot -b' ,$$

so that

$$\nu((a + b) \cdot -(a' + b')) \leq \nu(a \cdot -a') + \nu(b \cdot -b').)$$

Thus $+$, $\cdot$ and $-$ are all uniformly continuous for ρ. So they have unique continuous extensions to the metric space completion $\hat{A}$ of A. Now the identities of Definition 1.1 of Part I all extend by continuity, so $\hat{A}$ is a Boolean algebra, with A embedded as a subalgebra which is dense in the metric space sense (*not*, normally, in the Boolean algebra sense).

(ii) Next, for a, $a' \in A$,

$$\begin{aligned}|\nu(a) - \nu(a')| &\leq |\nu(a) - \nu(a \cdot a')| + |\nu(a') - \nu(a \cdot a')| \\ &= \nu(a) - \nu(a \cdot a') + \nu(a') - \nu(a \cdot a') \\ &\leq \nu(a \cdot -a') + \nu(a' \cdot -a) = \rho(a, a') .\end{aligned}$$

So $\nu: A \to \boldsymbol{R}$ is uniformly continuous and has a unique continuous extension $\hat{\nu}: \hat{A} \to \boldsymbol{R}$. The relations

$$\nu(a \cdot b) \leq \nu(a + b) \leq \nu(a) + \nu(b) ,$$
$$\rho(a, b) = \nu(a \cdot -b) + \nu(b \cdot -a) , \qquad \nu(a) = \rho(a, 0) ,$$

extend by continuity from A to $\hat{A}$ so $\hat{\nu}$ is a strictly positive submeasure, and the metric $\hat{\rho}$ on $\hat{A}$ is that associated with $\hat{\nu}$.

(iii) Suppose that $(c_n)_{n\in\omega}$ is any decreasing sequence in $\hat{A}$. Then $(c_n)_{n\in\omega}$ is $\hat{\rho}$-Cauchy. For let $\varepsilon>0$. Then we can choose for each $n\in\omega$ an $a_n\in A$ such that $\hat{\rho}(a_n, c_n)\le 2^{-n}\varepsilon$. Set $b_n=\Pi_{i\le n} a_i$ for each $n\in\omega$. Then, for each $n\in\omega$,

$$c_n\cdot -b_n=\sum_{i\le n} c_n\cdot -a_i\le \sum_{i\le n} c_i\cdot -a_i\,,$$

while $b_n\cdot -c_n\le a_n\cdot -c_n$, so that

$$\begin{aligned}\hat{\rho}(b_n, c_n)&=\hat{\nu}(b_n\cdot -c_n)+\hat{\nu}(c_n\cdot -b_n)\\&\le \hat{\nu}(a_n\cdot -c_n)+\sum_{i\le n}\hat{\nu}(c_i\cdot -a_i)\le\sum_{i\le n}\hat{\rho}(a_i, c_i)\le 2\varepsilon\,.\end{aligned}$$

On the other hand, if $(k(i))_{i\in\omega}$ is any increasing sequence in ω,

$$\lim_{n\to\infty}\rho(b_{k(n+1)}, b_{k(n)})=\lim_{n\to\infty}\nu(b_{k(n)}\cdot -b_{k(n+1)})=0\,,$$

because $(b_{k(n)}\cdot -b_{k(n+1)})\cdot(b_{k(m)}\cdot -b_{k(m+1)})=0$ whenever $m\neq n$ and ν is exhaustive. So $(b_n)_{n\in\omega}$ is a ρ-Cauchy sequence in A. Since

$$\hat{\rho}(c_m, c_n)\le\hat{\rho}(c_m, b_m)+\hat{\rho}(b_m, b_n)+\hat{\rho}(b_n, c_n)\le 4\varepsilon+\rho(b_m, b_n)$$

for all $m, n\in\omega$,

$$\limsup_{m,n\to\infty}\hat{\rho}(c_m, c_n)\le 4\varepsilon\,.$$

But ε was arbitrary, so $(c_n)_{n\in\omega}$ is a Cauchy sequence.

(iv) Continuing (iii), the decreasing sequence $(c_n)_{n\in\omega}$ in $\hat{A}$ must have a $\hat{\rho}$-limit $c\in\hat{A}$. Now, if $n\in\omega$,

$$\hat{\nu}(c\cdot -c_n)=\lim_{m\to\infty}\hat{\nu}(c_m\cdot -c_n)=0$$

by the continuity of the algebraic operations and $\hat{\nu}$; so $c\cdot -c_n=0$ for every $n\in\omega$, and c is a lower bound for $\{c_n\colon n\in\omega\}$. At the same time, if $d\in\hat{A}$ is any lower bound for $\{c_n\colon n\in\omega\}$,

$$\hat{\nu}(d\cdot -c)=\lim_{n\to\infty}\hat{\nu}(d\cdot -c_n)=0\,,$$

so $d\le c$. Thus, $c=\Pi_{n\in\omega} c_n$ in $\hat{A}$.

(v) This shows that $\hat{A}$ is σ-complete as Boolean algebra. I still have to show that $\hat{\nu}$ is a Maharam submeasure on $\hat{A}$. But suppose that $(c_n)_{n\in\omega}$ is a decreasing sequence in $\hat{A}$ such that $\Pi_{n\in\omega} c_n=0$. Then (iii)–(iv) show that 0 is the limit of $(c_n)_{n\in\omega}$ for the metric $\hat{\rho}$; so that

$$\lim_{n\to\infty}\hat{\nu}(c_n)=\lim_{n\to\infty}\hat{\rho}(c_n, 0)=0\,.$$

Thus, $\hat{\nu}$ is a Maharam submeasure. Finally, it follows from (a)(i) that $\hat{A}$ is complete (as Boolean algebra). □

5.7. PROPOSITION. *Let A be a Boolean algebra and ν a submeasure on A. For $b \in A$ set*

$$\tilde{\nu}(b) = \inf\Big\{\sup_{n\in\omega} \nu(b \cdot d_n) \colon (d_n)_{n\in\omega} \text{ is an increasing sequence in } A \text{ and } \sum_{n\in\omega} d_n = 1\Big\}.$$

Then

(a) *$\tilde{\nu}$ is a submeasure on A;*

(b) *if ν is exhaustive, then $\tilde{\nu}$ is a Maharam submeasure;*

(c) *if A is weakly (ω, ω)-distributive and ν is strictly positive, then $\tilde{\nu}$ is strictly positive;*

(d) *if ν is additive, so is $\tilde{\nu}$.*

PROOF. (a) Evidently $\tilde{\nu}(0) = 0$ and $\tilde{\nu}(a) \le \tilde{\nu}(b)$ whenever $a \le b$. Now suppose that $a, b \in A$ and that $\varepsilon > 0$. Then there are increasing sequences $(d_n)_{n\in\omega}$, $(d'_n)_{n\in\omega}$ in A such that $\Sigma_{n\in\omega}\, d_n = \Sigma_{n\in\omega}\, d'_n = 1$ and

$$\sup_{n\in\omega} \nu(a \cdot d_n) \le \tilde{\nu}(a) + \varepsilon\,, \qquad \sup_{n\in\omega} \nu(b \cdot d'_n) \le \tilde{\nu}(b) + \varepsilon\,.$$

Now $(d_n \cdot d'_n)_{n\in\omega}$ is an increasing sequence in A and $\Sigma_{n\in\omega}\, d_n \cdot d'_n = 1$ (Lemma 1.33 of Part I). So

$$\begin{aligned}\tilde{\nu}(a + b) &\le \sup_{n\in\omega} \nu((a + b) \cdot d_n \cdot d'_n) \le \sup_{n\in\omega} (\nu(a \cdot d_n) + \nu(b \cdot d'_n)) \\ &\le \tilde{\nu}(a) + \tilde{\nu}(b) + 2\varepsilon\,.\end{aligned}$$

As ε, a and b are arbitrary, $\tilde{\nu}$ is a submeasure.

(b) Suppose that ν is exhaustive and that $(a_n)_{n\in\omega}$ is a decreasing sequence in A with $\Pi_{n\in\omega}\, a_n = 0$. Let $\varepsilon > 0$. We know that

$$\lim_{n\to\infty} \nu(a_{k(n)} \cdot -a_{k(n+1)}) = 0$$

for every increasing sequence $(k(n))_{n\in\omega}$ in ω (compare part (b)(iii) of the proof of 5.6), so there is an $m \in \omega$ such that $\nu(a_m \cdot -a_n) \le \varepsilon$ for every $n \ge m$. Now $(-a_n)_{n\in\omega}$ is an increasing sequence in A and $\Sigma_{n\in\omega} - a_n = 1$, so

$$\tilde{\nu}(a_i) \le \tilde{\nu}(a_m) \le \sup_{n\in\omega} \nu(a_m \cdot -a_n) \le \varepsilon$$

for every $i \ge m$. As ε is arbitrary, $\tilde{\nu}$ is a Maharam submeasure.

(c) Now assume that A is weakly (ω, ω)-distributive and that ν is strictly positive. Suppose, if possible, that there is an $a \in A^+$ such that $\tilde{\nu}(a) = 0$. For each $n \in \omega$ there is an increasing sequence $(d_{ni})_{i\in\omega}$ in A such that $\Sigma_{i\in\omega}\, d_{ni} = 1$ and $\sup_{i\in\omega} \nu(a \cdot d_{ni}) \le 2^{-n}$. Now

$$0 < a = \prod_{n \in \omega} \sum_{i \in \omega} a \cdot d_{ni} ,$$

so, because A is weakly (ω, ω)-distributive, there is a sequence $(I(n))_{n\in\omega}$ of non-empty finite subsets of ω such that

$$\prod_{n \in \omega} \sum_{i \in I(n)} a \cdot d_{ni}$$

either does not exist or is greater than 0; that is, there is a $b > 0$ such that $b \leq \Sigma_{i\in I(n)} a \cdot d_{ni}$ for every $n \in \omega$. But now observe that each $I(n)$ must have a greatest element $m(n)$, in which case $b \leq a \cdot d_{n,m(n)}$ and $\nu(b) \leq 2^{-n}$ for every $n \in \omega$; so that $\nu(b) = 0$, which is impossible, because ν is supposed to be strictly positive.

Accordingly, $\tilde{\nu}(a) > 0$ for every $a \in A^+$, i.e. $\tilde{\nu}$ is strictly positive.

(d) Finally, suppose that ν is additive, and that $a, b \in A$ are such that $a \cdot b = 0$. Let $\varepsilon > 0$, and take an increasing sequence $(d_n)_{n\in\omega}$ in A such that $\Sigma_{n\in\omega} d_n = 1$ and

$$\sup_{n\in\omega} \nu((a+b)\cdot d_n) \leq \tilde{\nu}(a+b) + \varepsilon .$$

Then

$$\begin{aligned}
\tilde{\nu}(a) + \tilde{\nu}(b) &\leq \sup_{n\in\omega} \nu(a \cdot d_n) + \sup_{n\in\omega} \nu(b \cdot d_n) \\
&= \lim_{n\to\infty} \nu(a \cdot d_n) + \lim_{n\to\infty} \nu(b \cdot d_n) \\
&= \lim_{n\to\infty} (\nu(a \cdot d_n) + \nu(b \cdot d_n)) \\
&= \lim_{n\to\infty} \nu((a+b)\cdot d_n) \\
&= \sup_{n\in\omega} \nu((a+b)\cdot d_n) \\
&\leq \tilde{\nu}(a+b) + \varepsilon \leq \tilde{\nu}(a) + \tilde{\nu}(b) + \varepsilon .
\end{aligned}$$

As ε is arbitrary, $\tilde{\nu}(a) + \tilde{\nu}(b) = \tilde{\nu}(a+b)$; as a and b are arbitrary, $\tilde{\nu}$ is additive. □

REMARK. The last two propositions are extracts from the general theory of Boolean-algebras-with-submeasures, which should perhaps be regarded as part of a theory of topological Boolean algebras; indeed, submeasures were first introduced (MAHARAM [1947]) in order to study topologies on Boolean algebras.

Readers familiar with the theory of topological Riesz spaces will recognize most of the tricks here.

5.8. LEMMA. *Suppose that $k, l, m \in \omega$ and $3 \leq k \leq l \leq m$ and $18mk \leq l^2$. Then there is a set $R \subseteq m \times l$* (I am thinking of m, l as the sets of their predecessors) *such that* (i) *each vertical section of R has just three members*; (ii) $|R[E]| \geq |E|$ *whenever $E \in [m]^{\leq k}$*, where $R[E] = \{j : \exists i \in E, (i, j) \in R\}$.

PROOF. We need to know that $n! \geq 3^{-n}n^n$ for every $n \in \omega$; this is immediate from the inequality

$$\sum_{i=2}^{n} \log_e(i) \geq \int_1^n \log_e(x)\,\mathrm{d}x = n \log_e n - n + 1 \quad \forall n \geq 2 .$$

Let Ω be the set of those $R \subseteq m \times l$ such that each vertical section of R has just 3 members, so that

$$|\Omega| = |[l]^3|^m = \left(\frac{l!}{3!(l-3)!}\right)^m .$$

Let us regard Ω as a probability space with the uniform probability, which I shall denote Pr.

If $F \in [l]^n$, where $3 \leq n \leq k$, and $i \leq m$, then

$$\Pr(R[\{i\}] \subseteq F) = |[F]^3| / |[l]^3|$$

(because $R[\{i\}]$ is a random member of $[l]^3$)

$$= \frac{n(n-1)(n-2)}{l(l-1)(l-2)} \leq \left(\frac{n}{l}\right)^3 .$$

So if $E \in [m]^n$ and $F \in [l]^n$, then

$$\Pr(R[E] \subseteq F) = \prod_{i \in E} \Pr(R[\{i\}] \subseteq F)$$

(because the sets $R[\{i\}]$ are chosen independently)

$$\leq \left(\frac{n}{l}\right)^{3n} .$$

Accordingly,

$$\Pr(\text{there is an } E \in [m]^{\leq k} \text{ such that } |R[E]| < |E|)$$
$$\leq \Pr(\text{there is an } E \subseteq m \text{ such that } 3 \leq |E| \leq k \text{ and } |R[E]| < |E|)$$

(because $|R[E]| \geq 3$ if $E \neq \emptyset$ and $R \in \Omega$)

$$\leq \sum_{n=3}^{k} \sum_{E \in [m]^n} \sum_{F \in [l]^n} \Pr(R[E] \subseteq F)$$

$$\leq \sum_{n=3}^{k} |[m]^n| |[l]^n| \left(\frac{n}{l}\right)^{3n}$$

$$= \sum_{n=3}^{k} \frac{m!}{n!(m-n)!} \frac{l!}{n!(l-n)!} \left(\frac{n}{l}\right)^{3n}$$

$$\leq \sum_{n=3}^{k} \frac{m^n l^n n^{3n}}{(n!)^2 l^{3n}}$$

$$\leq \sum_{n=3}^{k} \frac{m^n n^n 3^{2n}}{l^{2n}} = \sum_{n=3}^{k} \left(\frac{9\,mn}{l^2}\right)^n \leq \sum_{n=3}^{k} \frac{1}{2^n} < 1 .$$

There must therefore be some $R \in \Omega$ such that $|R[E]| \geq |E|$ for every $E \in [m]^{\leq k}$. $\square$

REMARK. Of course, this argument can be widely generalized; see references in KALTON and ROBERTS [1983a].

5.9. Definition. Let A be a Boolean algebra and $C \subseteq A^+$ a non-empty set. For $c_0, \ldots, c_n \in C$ write

$$\alpha^*(c_0, \ldots, c_n) = \frac{1}{n+1} \max\Big\{|I| : I \subseteq n+1, \prod_{i \in I} c_i \neq 0\Big\}.$$

The *intersection number* of C is now

$$\alpha(C) = \inf\{\alpha^*(c_0, \ldots, c_n) : n \in \omega, c_0, \ldots, c_n \in C\}.$$

Note. In the definitions of α^*, α here the elements $c_0, \ldots, c_n$ of C are *not* assumed to be all distinct. To see that it makes a difference, consider $A = P(4)$, $C = \{\{0\}, \{1,2\}, \{1,3\}, \{2,3\}\}$; then $\alpha(C)$ is $\frac{2}{5}$, not $\frac{1}{2}$.

5.10. PROPOSITION. *Let A be a Boolean algebra and $C \subseteq A^+$ a non-empty set with intersection number α. Then there is a positive additive functional ν on A such that $\nu(1) = 1$ and $\nu(c) \geq \alpha$ for every $c \in C$.*

PROOF. (a) We may suppose that A is actually a subalgebra of $P(X)$ for some set X (using the Stone representation theorem). For each $a \in A$ let $\chi(a) : X \to \{0,1\}$ be the characteristic function of a. Let $\ell^\infty(X)$ be the Banach lattice of all bounded functions from X to $\boldsymbol{R}$. On $\ell^\infty(X)$ define a functional ρ by writing

$$\begin{aligned}\rho(f) = \inf\Big\{\gamma + (1-\alpha)\sum_{i=0}^{n} \gamma_i : \gamma \geq 0, n \in \omega, \gamma_i \geq 0\ \forall i \leq n, \\ \exists c_0, \ldots, c_n \in C \text{ such that } |f| \leq \gamma\chi(1) + \sum_{i \leq n} \gamma_i \chi(-c_i)\Big\},\end{aligned}$$

where $\alpha = \alpha(C)$. Then we see that

$$\begin{aligned}&\rho(f) \leq \rho(g) && \text{if } |f| \leq |g|, \\ &\rho(f+g) \leq \rho(f) + \rho(g) && \text{for all } f, g \in \ell^\infty(X), \\ &\rho(\gamma f) = |\gamma| \rho(f) && \text{if } \gamma \in \boldsymbol{R} \text{ and } f \in \ell^\infty(X).\end{aligned}$$

(b) The point of this construction is that $\rho(\chi(1)) = 1$. For clearly $\rho(\chi(1)) \leq 1$ (since in the defining formula for $\rho(\chi(1))$ we can take $\gamma = 1$, $n = 0$, $\gamma_0 = 0$). Now suppose, if possible, that $\rho(\chi(1)) < 1$. Then there are $n \in \omega$, $c_0, \ldots, c_n \in C$, γ, $\gamma_0, \ldots, \gamma_n \in [0,\infty)$ such that

$$\chi(1) \leq \gamma\chi(1) + \sum_{i \leq n} \gamma_i \chi(-c_i),$$

$$1 > \gamma + (1-\alpha)\sum_{i=0}^{n} \gamma_i.$$

Increasing $\gamma, \gamma_0, \ldots, \gamma_n$ slightly if need be, we can suppose that they are all rational; moving γ over to the left of each inequality and multiplying by a common denominator, we obtain integers $k, k_0, \ldots, k_n$ such that

$$k\chi(1) + \sum_{i\leq n} k_i\chi(-c_i),$$
$$k > (1-\alpha)\sum_{i=0}^{n} k_i.$$

Set $m = \sum_{i=0}^{n} k_i$ and let $b_0, \ldots, b_{m-1}$ run over $\{c_0, \ldots, c_n\}$ with the appropriate multiplicities so that

$$k\chi(1) \leq \sum_{j<m} \chi(-b_j) \tag{*}$$

while

$$k > (1-\alpha)m. \tag{**}$$

Evidently $m \geq 1$; also $\alpha = \alpha(C) \leq \alpha^*(b_0, \ldots, b_{m-1})$, so there is a $J \subseteq m$ such that $|J| \geq \alpha m$ and $b = \Pi_{j\in J} b_j \neq 0$. Now take any $x \in b$ and examine the meaning of the inequality (*) at the point x. We have

$$k\chi(1)(x) \leq \sum_{j<m} \chi(-b_j)(x), \text{ i.e. } k \leq |\{j\colon j \leq m, x \in X\setminus b_j\}|;$$

so that $k \leq |m\setminus J| \leq m(1-\alpha)$, which contradicts (**).

We conclude therefore that $\rho(\chi(1)) = 1$.

(c) The Hahn–Banach theorem (DUNFORD and SCHWARTZ [1958, II.3.10]; HEWITT and STROMBERG [1965, §14.9]) now tells us that there is a linear functional $\varphi\colon \ell^\infty(X) \to \boldsymbol{R}$ such that $\varphi(\chi(1)) = \rho(\chi(1)) = 1$ and $\varphi(f) \leq \rho(f)$ for every $f \in \ell^\infty(X)$. Define $\nu\colon A \to \boldsymbol{R}$ by setting $\nu(a) = \varphi(\chi(a))$ for $a \in A$. Then ν is additive, because $\chi(a+b) = \chi(a) + \chi(b)$ whenever $a \cdot b = 0$. We have $\nu(1) = \varphi(\chi(1)) = 1$. Also ν is positive, because

$$\begin{aligned}\nu(a) &= \nu(1) - \nu(-a) = 1 - \varphi(\chi(-a)) \geq 1 - \rho(\chi(-a))\\ &\geq 1 - \rho(\chi(1)) = 0\end{aligned}$$

for every $a \in A$.

(d) If $c \in C$, then $|\chi(-c)| \leq 0 \cdot \chi(1) + 1 \cdot \chi(-c)$, so

$$\nu(-c) = \varphi(\chi(-c)) \leq \rho(\chi(-c)) \leq 1 - \alpha,$$

and $\nu(c) = 1 - \nu(-c) \geq \alpha$. So

$$\inf_{c\in C} \nu(c) \geq \alpha = \alpha(C),$$

which is what we wanted to know. □

REMARK. This result is due to KELLEY [1959]. In fact, it characterizes intersection numbers; if ν is any positive additive functional on A such that $\nu(1)=1$, then $\inf_{c\in C}\nu(c)\leq\alpha(C)$. The role of $\ell^\infty(X)$ in the proof above could equally well be played by $L^\infty(A)$, as described in 2.26.

5.11. THEOREM. *Let A be a Boolean algebra. Then the following are equivalent:*
(i) *A carries a strictly positive additive functional;*
(ii) *A carries a strictly positive uniformly exhaustive submeasure;*
(iii) *A^+ is empty or is expressible as a countable union of sets each of which has intersection number greater than* 0.

PROOF. (a)(i)$\Rightarrow$(ii). Trivial; positive additive functionals are uniformly exhaustive submeasures.

(b)(ii)$\Rightarrow$(iii). If $A=\{0\}$, there is nothing to prove. Otherwise, let ν be a strictly positive uniformly exhaustive submeasure on A; multiplying ν by a scalar constant if need be, we can suppose that $\nu(1)=1$. For $n\in\omega$ set

$$C_n=\left\{c\in A\colon \nu(c)\geq\frac{1}{n+1}\right\}.$$

Then $A^+=\bigcup_{n\in\omega}C_n$; the aim is to show that each C_n has strictly positive intersection number.

Fix $n\in\omega$. Then there is an $r\geq 1$ such that whenever $c_0,\ldots,c_r\in A^+$ are pairwise disjoint there is an $i\leq r$ such that $\nu(c_i)\leq 1/5(n+1)$, because ν is uniformly exhaustive. Accordingly, if $a_0,\ldots,a_m\in A$ and $a_i\cdot a_j=0$ for $i\neq j$,

$$\left|\left\{i\colon i\leq m,\ \nu(a_i)>\frac{1}{5(n+1)}\right\}\right|\leq r,$$

so

$$\sum_{i=0}^{m}\nu(a_i)\leq r+\frac{m}{5(n+1)}.$$

Set $\delta=1/5r(n+1)>0$, $\varepsilon=\frac{1}{74}\delta^2$ so that $\frac{1}{18}(\delta-\varepsilon)^2\geq\frac{1}{18}(\delta^2-2\varepsilon)=4\varepsilon$.

Let m be any integer greater than or equal to $1/\varepsilon$. Then there are integers k, l such that

$$3\varepsilon\leq\frac{k}{m}\leq\frac{1}{18}(\delta-\varepsilon)^2,\qquad \delta-\varepsilon\leq\frac{l}{m}\leq\delta,$$

in which case

$$3\leq k\leq l\leq m,\qquad 18mk\leq m^2(\delta-\varepsilon)^2\leq l^2.$$

By Lemma 5.8, there is a set $R\subseteq m\times l$ such that each vertical section of R has just three members and $|R[E]|\geq|E|$ whenever $E\subseteq m$ and $|E|\leq k$. By Hall's Marriage Theorem (HALL [1935]; ANDERSON [1974, Theorem 3.3]) there is for each $E\in[m]^{\leq k}$ an injective function $f_E\colon E\to l$ such that $(i,f(i))\in R$ for every $i\in E$.

Now let $c_0, \ldots, c_{m-1}$ be any family in C_n. For $E \subseteq m$ set

$$b_E = \prod_{i \in E} c_i \cdot \prod_{i \in m \setminus E} (-c_i) .$$

Set

$$a_{ij} = \sum \{b_E : i \in E \in [m]^{\leq k},\ f_E(i) = j\}$$

for $i < m$, $j < l$. Then $a_{ij} \cdot a_{i'j} = 0$ if $i \neq i'$, because each f_E is injective and $b_E \cdot b_{E'} = 0$ if $E \neq E'$. Let

$$m_j = |\{i : i < m,\ a_{ij} \neq 0\}| \quad \forall j < l .$$

Then

$$\sum_{i=0}^{m-1} \nu(a_{ij}) \leq r + m_j / 5(n+1) \quad \forall j < l ,$$

so

$$\sum_{j=0}^{l-1} \sum_{i=0}^{m-1} \nu(a_{ij}) \leq rl + \frac{1}{5(n+1)} \sum_{j=0}^{l-1} m_j .$$

But $a_{ij} = 0$ if $(i, j) \not\in R$ so $\Sigma_{j=0}^{l-1} m_j \leq |R| = 3m$, and

$$\sum_{j=0}^{l-1} \sum_{i=0}^{m-1} \nu(a_{ij}) \leq rl + \frac{3m}{5(n+1)} .$$

On the other hand, for $i < m$,

$$\sum_{j=0}^{l-1} \nu(a_{ij}) \geq \nu\Big(\sum_{j<l} a_{ij}\Big) = \nu\Big(\sum \{b_E : i \in E \in [m]^{\leq k}\}\Big) .$$

Set

$$\begin{aligned} c'_i &= \sum \{b_E : i \in E \in [m]^{\leq k}\} \\ &= c_i \cdot \sum \{b_E : E \in [m]^{\leq k}\} = c_i \cdot \sum_{j<l} a_{ij} \end{aligned}$$

for $i < m$, so that

$$\begin{aligned} \sum_{i=0}^{m-1} \nu(c'_i) &\leq \sum_{i=0}^{m-1} \sum_{j=0}^{l-1} \nu(a_{ij}) \leq rl + \frac{3m}{5(n+1)} \leq rm\delta + \frac{3m}{5(n+1)} \\ &= \frac{4m}{5(n+1)} < \frac{m}{n+1} . \end{aligned}$$

There is therefore some $i < m$ such that $\nu(c'_i) < 1/(n+1) \leq \nu(c_i)$. But this means that $c'_i \neq c_i$ and $\Sigma \{b_E : E \in [m]^{\leq k}\} \neq 1$. And $\Sigma \{b_E : E \subseteq m\} = 1$ so there is an $E \subseteq m$ such that $|E| > k$ and $b_E \neq 0$. Accordingly, in the notation of 5.9,

$$\alpha^*(c_0, \ldots, c_{m-1}) \geq \frac{|E|}{m} \geq \frac{k+1}{m} \geq 3\varepsilon .$$

This is true for any m large enough and any $c_0, \ldots, c_{m-1} \in C_n$. But it is easy to see that

$$\alpha^*(c_0, \ldots, c_k) = \alpha^*(c_0, c_0, c_1, c_1, \ldots, c_k, c_k)$$

for any $c_0, \ldots, c_k$. So in fact $\alpha^*(c_0, \ldots, c_k) \geq 3\varepsilon$ for all $c_0, \ldots, c_k \in C_n$, i.e. $\alpha(C_n) \geq 3\varepsilon > 0$.

As this works for each $n \in \omega$, (iii) is true.

(c)(iii)$\Rightarrow$(i). If $A = \{0\}$, this is trivial. Otherwise, A^+ is expressible as $\bigcup_{n \in \omega} C_n$, where each C_n has an intersection number $\varepsilon_n > 0$. So for each $n \in \omega$ we can find a positive additive functional ν_n on A such that $\nu_n(1) = 1$ and $\nu_n(c) \geq \varepsilon_n$ for every $c \in C_n$, by 5.10. Define ν: $A \rightarrow \boldsymbol{R}$ by setting

$$\nu(a) = \sum_{n=0}^{\infty} 2^{-n} \nu_n(a) \quad \forall a \in A ,$$

and see that ν is a strictly positive additive functional on A, as required. □

REMARK. (i)$\Leftrightarrow$(iii) of this theorem is due to KELLEY [1959], (i)$\Leftrightarrow$(ii) to KALTON and ROBERTS [1983a].

5.12. THEOREM. *Let A be a σ-complete Boolean algebra. Then the following are equivalent*:

(i) *A is a measurable algebra*;

(ii) *A carries a strictly positive uniformly exhaustive Maharam submeasure*;

(iii) *A is weakly (ω, ω)-distributive and carries a strictly positive additive functional*;

(iv) *A carries a strictly positive Maharam submeasure and every countably completely generated subalgebra of A carries a strictly positive additive functional.*

PROOF. (a)(i)$\Rightarrow$(iv). Obvious.

(b)(iv)$\Rightarrow$(ii). Let μ be a strictly positive Maharam submeasure. Then μ is uniformly exhaustive. For suppose, if possible, otherwise. Then there is an $\varepsilon > 0$ such that for every $n \in \omega$ there is a pairwise disjoint family $(a_{ni})_{i \leq n}$ in A^+ with $\mu(a_{ni}) \geq \varepsilon$ for every $i \leq n$. Let B be the complete subalgebra of A generated by $\{a_{ni}\colon i \leq n \in \omega\}$. There is a strictly positive additive functional ν on B. Now $\mu \restriction B$ is a strictly positive Maharam submeasure on B, so B is weakly (ω, ∞)-distributive, by 5.6(a)(ii). So the construction of 5.7 gives us a submeasure $\tilde{\nu}$ on B which by 5.7(b)–(c) is a strictly positive Maharam submeasure. Also, $\tilde{\nu}(b) \leq \nu(b)$ for every $b \in B$, so $\tilde{\nu}$, like ν, must be uniformly exhaustive. But now we can apply 5.5(f) to see that $\mu \restriction B$ is also uniformly exhaustive; which it is not, because $(a_{ni})_{i \leq n}$ is a pairwise disjoint family in B^+ for each $n \in \omega$, with $\mu(a_{ni}) \geq \varepsilon$ for all $i \leq n$.

Thus, μ is already the required strictly positive uniformly exhaustive Maharam submeasure.

(c)(ii)$\Rightarrow$(iii). By Theorem 5.11(ii)$\Rightarrow$(i), A carries a strictly positive additive functional. By 5.6(a)(ii) again, A is weakly (ω, ω)-distributive.

(d)(iii)$\Rightarrow$(i). Let ν be a strictly positive additive functional on A. By 5.7 again, $\tilde{\nu}$ is a strictly positive additive functional on A which is also a Maharam submeasure; that is, $(A, \tilde{\nu})$ is a totally finite measure algebra, and A is a measurable algebra. □

REMARK. (iii)$\Rightarrow$(i) is due to KELLEY [1959]; (ii)$\Rightarrow$(i) is due to KALTON and ROBERTS [1983a]. (iv)$\Rightarrow$(i) answers a question in MAHARAM [1981].

5.13. THE CONTROL MEASURE PROBLEM. This famous problem amounts to asking: Can the phrase "uniformly exhaustive" be omitted from 5.12(ii)? It is a major obstacle in the theories of vector measures and topological Riesz spaces. I shall not attempt to discuss these aspects here, but I list three assertions which are equivalent in the sense that if one is true so are both the others.

CM_1: "A σ-complete Boolean algebra which carries a strictly positive Maharam submeasure must be measurable."

CM_2: "If X is a set, B is a σ-algebra of subsets of X, and $\nu: B \to [0, \infty)$ is a Maharam submeasure, there is a totally finite measure μ on X, with domain B, such that $\mu^{-1}[\{0\}] = \nu^{-1}[\{0\}]$."

CM_3: "Every exhaustive submeasure on $\mathrm{Clop}({}^{\omega}\{0, 1\})$ is uniformly exhaustive."

For the equivalence of these three statements, we can argue as follows.

(a) $\mathrm{CM}_1 \Rightarrow \mathrm{CM}_2$. Assume that CM_1 is true, and let X, B, ν be as in CM_2. Set $N = \nu^{-1}[\{0\}]$; then N is a σ-complete ideal of B, so $A = B/N$ is a σ-complete Boolean algebra, and the canonical epimorphism $\pi: B \to A$ is σ-complete. We have a functional $\tilde{\nu}: A \to \boldsymbol{R}$ defined by saying that $\tilde{\nu}(\pi(b)) = \nu(b)$ for every $b \in B$. It is easy to check that $\tilde{\nu}$ is a strictly positive Maharam submeasure on A. So CM_1 tells us that A is measurable; let $\tilde{\mu}$ be such that $(A, \tilde{\mu})$ is a totally finite measure algebra. Set $\mu = \tilde{\mu} \circ \pi: B \to \boldsymbol{R}$; then (X, B, μ) is a totally finite measure space and $\mu^{-1}[\{0\}] = N = \nu^{-1}[\{0\}]$.

(b) $\mathrm{CM}_2 \Rightarrow \mathrm{CM}_1$. Assume that CM_2 is true, and let A be a σ-complete Boolean algebra which carries a strictly positive Maharam submeasure $\tilde{\nu}$. Express A as B/M, where B is the σ-algebra of Borel subsets of $\mathrm{Ult}(A)$ and M is the ideal of meagre Borel sets. Set $\nu = \tilde{\nu} \cdot \pi: B \to \boldsymbol{R}$, where $\pi: B \to A$ is the canonical epimorphism. Then ν is a Maharam submeasure on B, with $\nu^{-1}[\{0\}] = M$. By CM_2, there is a totally finite measure μ on $\mathrm{Ult}(A)$, with domain B, such that $\mu^{-1}[\{0\}] = M$. Now we can define a function $\tilde{\mu}: A \to \boldsymbol{R}$ by setting $\tilde{\mu}(\pi(b)) = \mu(b)$ for every $b \in B$. We find that $(A, \tilde{\mu})$ is a totally finite measure algebra, so that A is a measurable algebra.

(c) $\mathrm{CM}_1 \Rightarrow \mathrm{CM}_3$. Assume CM_1, and let ν be an exhaustive submeasure on $D = \mathrm{Clop}({}^{\omega}\{0, 1\})$. Set $N = \nu^{-1}[\{0\}]$; then N is an ideal of D; let C be the quotient D/N, and $\pi: D \to C$ the canonical epimorphism.

We have a functional $\tilde{\nu}: C \to \boldsymbol{R}$ given by writing $\tilde{\nu}(\pi(d)) = \nu(d)$ for $d \in D$, and $\tilde{\nu}$ is a strictly positive submeasure on C. Moreover, because every pairwise disjoint sequence in C^+ is expressible in the form $(\pi(d_n))_{n\in\omega}$, where $(d_n)_{n\in\omega}$ is a pairwise disjoint sequence in D^+, $\tilde{\nu}$ is exhaustive. So we can apply 5.6(b) to find a metric completion $\hat{C}$ of C and an extension of $\tilde{\nu}$ to a Maharam submeasure $\hat{\nu}$ on $\hat{C}$. Now CM_1 tells us that $\hat{C}$ is a measurable algebra; let μ be such that $(\hat{C}, \mu)$ is a totally finite measure algebra. Then μ is a uniformly exhaustive Maharam submeasure on $\hat{C}$; by 5.5(f), $\hat{\nu}$ is uniformly exhaustive on $\hat{C}$. But it now follows at once that $\tilde{\nu}$ and ν are uniformly exhaustive, as required.

(d) $\mathrm{CM}_3 \Rightarrow \mathrm{CM}_1$. Assume CM_3, and let A be a σ-complete Boolean algebra with a strictly positive Maharam submeasure ν. Then A satisfies the countable chain condition (5.5(c)–(d)) and is complete. Set

$$a_0 = \sum \{a\colon a \text{ is an atom in } A\}\,.$$

Then $A \restriction a_0$ is a complete atomic Boolean algebra satisfying the countable chain condition, so is isomorphic to $P(X)$ for some countable set X, and is certainly measurable. Now consider $B = A \restriction -a_0$. This is an atomless complete Boolean algebra carrying a strictly positive Maharam submeasure $\nu \restriction B$.

Suppose, if possible, that $\nu \restriction B$ is not uniformly exhaustive. Then there is a countable subalgebra C of B such that $\nu \restriction C$ is not uniformly exhaustive, as in part (b) of the proof of 5.12. Because B is atomless, there is a countable subalgebra D of B such that $C \subseteq D$ and D is atomless. Now $\nu \restriction D$ is exhaustive (because ν is, by 5.5(d)), but not uniformly exhaustive; which contradicts CM_3, because $D \cong \mathrm{Clop}({}^{\omega}\{0,1\})$ (by Corollary 5.16 of Part I).

Thus, $\nu \restriction B$ is uniformly exhaustive. By 5.12(ii)$\Rightarrow$(i), B is measurable. It follows at once that A is measurable, as is required by CM_1. □

5.14. REMARKS. (a) The original problem, in the form "Is CM_1 true?" appeared as the penultimate sentence of MAHARAM [1947]. For other versions and further discussion, see the survey paper KALTON [1984] and the references there; also FREMLIN [n78a] and TOPSØE [1976].

(b) I conjecture that CM_1–CM_3 are all false, and that the solution involves interesting finite combinatorics. Note that CM_3 is subject to Shoenfield's Absoluteness Theorem (JECH [1978, Theorem 98]), so we can assume that the control measure problem is resolvable in ZFC.

(c) If I am right, then 5.12(ii) is, in its own terms, an efficient and economical characterization of measurable algebras, as is 5.12(iii). But both of these formulations involve assuming the existence of functionals of some kind; and our only way of escaping the mention of these is to call on 5.11(iii), which is neither very practical nor very far away from a functional. It seems to be quite unclear which of the properties of 5.3–5.4 can be assembled in an algebra which is not measurable. A great deal of work has been done on the chain conditions; for instance, GAIFMAN [1964] (repeated and refined in COMFORT and NEGREPONTIS [1982, Theorem 6.23]) gave an example of a countably generated complete Boolean algebra A which has property $K_{\omega_1,n}$ for every $n \geq 2$, and satisfies the σ-bounded chain condition, but is not measurable. Further examples are given in COMFORT and NEGREPONTIS [1982]. However, they do not address questions of distributivity. So such questions as 5.15(c) below remain open.

5.15. OTHER PROBLEMS. (a) Let A be an atomless, countably completely generated, weakly (ω, ∞)-distributive Boolean algebra, satisfying the countable chain condition, but not $\{0\}$. Is A necessarily isomorphic to the measure algebra A_ω of ${}^\omega\{0,1\}$?

I note that if I am right in supposing that the control measure problem has a negative answer, then so does this one. For if there is an exhaustive submeasure ν on $\mathrm{Clop}({}^\omega\{0,1\})$ which is not uniformly exhaustive (CM_3 of 5.13), then $\hat{C}$, as constructed in 5.13(c), will be a counter-example here.

(b) (MAHARAM [1981].) Is it relatively consistent with ZFC to suppose that every complete weakly (ω, ∞)-distributive Boolean algebra satisfying the countable chain condition is a measurable algebra?

This is what remains of Problem 163 of *The Scottish Book* (MAULDIN [1981]). We must restrain ourselves to looking for a consistency result because any ω_1-Souslin algebra (see 14.19 in Part I) provides a counter-example (MAHARAM [1947]). Moreover, a counter-example to (a) above (which I conjecture exists in ZFC) would also be a counter-example to this question.

(c) Let A be a complete weakly (ω, ∞)-distributive Boolean algebra satisfying the σ-bounded chain condition (5.4(a)). Is it necessarily a measurable algebra?

6. Cardinal functions

I now turn to a discussion of certain cardinal functions of Boolean algebras which are of interest in the context of measure algebras.

6.1. DEFINITIONS. Recall that the following cardinal functions have been defined in Part I (Definitions 3.8, 4.8, 9.19, 13.18). For a Boolean algebra A,

$$c(A) = \sup\{|X| : X \subseteq A^+ \text{ is pairwise disjoint}\}\ ,$$
$$\mathrm{sat}(A) = \sup\{|X|^+ : X \subseteq A^+ \text{ is pairwise disjoint}\}\ ,$$
$$\pi(A) = \min\{|X| : X \subseteq A^+ \text{ is dense in } A^+\}\ ,$$
$$\mathrm{ind}(A) = \sup\{|X| : X \subseteq A \text{ is independent}\}\ ,$$
$$\tau(A) = \min|X| : X \subseteq A \text{ completely generates } A\}\ .$$

Many more are examined by E.K. van Douwen in Chapter 11 of this Handbook. Most will not be important here, but a significant one is

$$d(A) = \min\{\kappa : \kappa \text{ is a cardinal and } A \text{ is isomorphic to a subalgebra of } P(\kappa)\}$$

(Theorem 5.5 of Chapter 11). In addition I should like to discuss four further functions:

$$\mathrm{wdistr}(A) = \min\{\kappa\colon A \text{ is not weakly } (\kappa, \infty)\text{-distributive}\}\,,$$

$$\gamma_\omega(A) = \min\{|X|\colon X \subseteq A^+ \text{ and there is no countable } D \subseteq A^+ \text{ which is dense in } X\}\,.$$

(Recall (Definition 10.15 of Part I) that a set $D \subseteq A^+$ is *dense in* a set $X \subseteq A^+$ if for every $x \in X$ there is a $d \in D$ such that $d \leq x$.)

$$\gamma^*_\omega(A) = \min\left\{|X|\colon X \subseteq A \text{ and there is no countable } D \subseteq A \text{ such that } x = \sum \{d \in D\colon d \leq x\}\ \forall x \in X\right\},$$

$$n(A) = \min\{|\mathscr{X}|\colon \mathscr{X} \text{ is a family of partitions of 1 in } A \text{ such that no filter in } A \text{ meets every member of } \mathscr{X}\}\,.$$

These definitions may require an interpretation of "min $\emptyset$". I shall take this to be "∞" (not, of course, to be confused with the value "∞" which can be taken by a measure), with the convention that, in this case, $\kappa < \infty$ for every cardinal κ.

Naturally enough, these functions are related to certain cardinal functions of measure spaces. I shall not attempt to discuss these thoroughly here, but the following will be useful. For any non-empty family of sets write

$$\mathrm{add}(M) = \min\left\{|E|\colon E \subseteq M,\ \bigcup E \not\subseteq m \text{ for every } m \in M\right\},$$

$$\mathrm{cf}(M) = \min\{|E|\colon E \subseteq M, \forall m \in M\ \exists e \in E \text{ such that } m \subseteq e\}\,.$$

(*Warning*: this notion of "cofinality" agrees with the ordinary notion of cofinality of a non-zero limit ordinal ξ, if we regard ξ as a family of sets; but cf(M), unlike cf(ξ), need not be a regular cardinal. Also, cf(M) here has nothing to do with cf(A) as defined in Section 12 of Chapter 11.) If (X, B, μ) is a σ-finite measure space, its *Maharam type* is $\tau(A)$, where A is its measure algebra. (See 3.1 above.)

Finally, two special notations. As in 3.1, A_κ will denote the measure algebra of ${}^\kappa\{0, 1\}$, for each cardinal κ. And in this section N will always be the ideal of negligible subsets of ${}^\omega\{0, 1\}$. As remarked in 1.15, the usual measure on ${}^\omega\{0, 1\}$ is isomorphic to Lebesgue measure on $[0, 1]$, so that the family N of sets is isomorphic, as a collection of sets, to the ideal of Lebesgue negligible subsets of $[0, 1]$; of course this is also isomorphic to the ideal of Lebesgue negligible subsets of $\boldsymbol{R}$, or indeed of $\boldsymbol{R}^n$, etc.

6.2. ELEMENTARY FACTS (I).

(a) If A is an infinite complete Boolean algebra, then $\mathrm{ind}(A) = |A| = |A|^\omega$ and $|\mathrm{Ult}(A)| = 2^{|A|}$ (13.6, 13.7 and 13.20 in Part I). Consequently, the middle section of van Douwen's Fig. 11.1 (Section 13 of Chapter 11) collapses for complete Boolean algebras.

(The Balcar–Franek theorem and Monk's theorem hardly qualify as "elementary". But for measurable algebras Maharam's theorem enables us to calculate ind(A) and $|A|$ directly without difficulty.)

(b) If A is an infinite complete Boolean algebra satisfying the countable chain condition, $|A| = \tau(A)^\omega$. (For if $X \subseteq A$ is a completely generating set of cardinal $\tau(A)$, the smallest subalgebra of A including X and closed under countable sums is A itself, and has cardinal not greater than $\tau(A)^\omega$. On the other hand, $\tau(A)^\omega \le |A|^\omega = |A|$.)

(c) If A is any Boolean algebra, and M is the ideal of nowhere dense subsets of $\mathrm{Ult}(A)$, then $\mathrm{wdistr}(A) = \mathrm{add}(M)$. (Use Exercise 14.9 of Part I.)

(d) If A is a Boolean algebra satisfying the countable chain condition, it is weakly (κ, ∞)-distributive whenever it is weakly (κ, ω)-distributive. So in this case

$$\mathrm{wdistr}(A) = \min\{\kappa\colon A \text{ is not weakly } (\kappa, \omega)\text{-distributive}\} .$$

(e) Evidently, $\omega_1 \le \gamma^*_\omega(A) \le \gamma_\omega(A)$ for every Boolean algebra A. (We shall see in 6.14 below that $\gamma^*_\omega(A) = \gamma_\omega(A)$ for every measurable algebra A. I have not been able to determine whether this is true for all Boolean algebras satisfying the countable chain condition. The best I can do is to show that if A satisfies the countable chain condition and $\gamma^*_\omega(A) > \omega_1$, then $\gamma^*_\omega(A) = \gamma_\omega(A)$.) If A does not satisfy the countable chain condition, then $\gamma_\omega(A) = \omega_1$.

(f) If A is any Boolean algebra, then $n(A)$ is the least cardinal of any cover of $\mathrm{Ult}(A)$ by nowhere dense sets. (For a set $F \subseteq \mathrm{Ult}(A)$ is nowhere dense iff there is a partition X of 1 in A such that $F \cap \bigcup_{x \in X} s(x) = \emptyset$.) Hence, or otherwise, $\mathrm{wdistr}(A) \le n(A)$.

(g) If A is a Boolean algebra which is not atomic, then $\mathrm{wdistr}(A) \le d(A)$. (For $d(A)$ is the density of the topological space $\mathrm{Ult}(A)$ (Theorem 5.5 of Chapter 11), so there is a set $w \subseteq \mathrm{Ult}(A)$ such that $|w| \le d(A)$, $\{p\}$ is nowhere dense for every $p \in w$, and w is not nowhere dense; now use (c) above.)

(h) If A is any Boolean algebra then $d(A) \le \pi(A)$ (because $d(A)$ is the density of $\mathrm{Ult}(A)$, and $\pi(A)$ is its π-weight). If $\pi(A) \le \omega$, then $\gamma^*_\omega(A) = \infty$; if $\pi(A) > \omega$, then $\gamma_\omega(A) \le \pi(A)$.

(i) Let A be a Boolean algebra and B a subalgebra of A. Then $d(B) \le d(A)$ and $c(B) \le c(A)$.

(j) Let A be a Boolean algebra and B a regular subalgebra of A (Definition 1.29 of Part I). Then $\mathrm{wdistr}(B) \le \mathrm{wdistr}(A)$ and $n(A) \le n(B)$ and $\pi(B) \le \pi(A)$ and $\gamma_\omega(A) \le \gamma_\omega(B)$ and $\gamma^*_\omega(A) \le \gamma^*_\omega(B)$.

6.3. ELEMENTARY REMARKS (II). Now let us turn to those facts about measurable algebras that are readily deducible from the work of this chapter so far (primarily, of course, Maharam's theorem).

(a) If A is an infinite measurable algebra, then

$$\mathrm{ind}(A) = |A| = \tau(A)^\omega , \qquad |\mathrm{Ult}(A)| = 2^{|A|}$$

(6.2(a)–(b) above).

(b) Let A be any measurable algebra. Then A has a well-defined topology induced by any totally finite measure on A (2.19(b)). If $X \subseteq A$ completely generates A, then the subalgebra generated by X is topologically dense in A; if

$X \subseteq A$ is topologically dense in A, then it completely generates A. (See 2.20.) So if κ is the topological density of A,

$$\max(\kappa, \omega) = \max(\tau(A), \omega) .$$

Consequently, $\tau(B) \leq \tau(A)$ for any complete subalgebra B of A. (It is for this reason also that countably completely generated σ-finite measure algebras are often called "separable".)

(c) If A is any measurable algebra, $\mathrm{wdistr}(A) \geq \omega_1$ (2.10).

(d) Let A be a measurable algebra, not $\{0\}$. By Maharam's theorem (3.9) it is isomorphic (as Boolean algebra) to a product $\Pi_{i \in I} A_{\kappa(i)}$, where I is countable and each $\kappa(i)$ is either 0 or an infinite cardinal. Now we can easily calculate the cardinal functions of A in terms of the factors.

If A is atomic, then $c(A) = d(A) = \pi(A) = |I|$, $\mathrm{wdistr}(A) = n(A) = \gamma_\omega(A) = \gamma^*_\omega(A) = \infty$.

If A is not atomic, then

$$\begin{aligned} c(A) &= \omega , \\ \tau(A) &= \sup_{i \in I} \kappa(i) , \\ \mathrm{wdistr}(A) &= \min_{i \in I} \mathrm{wdistr}(A_{\kappa(i)}) , \\ \gamma_\omega(A) &= \min_{i \in I} \gamma_\omega(A_{\kappa(i)}) , \\ \gamma^*_\omega(A) &= \min_{i \in I} \gamma^*_\omega(A_{\kappa(i)}) , \\ \pi(A) &= \sup_{i \in I} \pi(A_{\kappa(i)}) , \\ n(A) &= \sup_{i \in I} n(A_{\kappa(i)}) , \\ d(A) &= \sup_{i \in I} d(A_{\kappa(i)}) . \end{aligned}$$

(e) Accordingly, the rest of this chapter is largely devoted to the homogeneous algebras A_κ for $\kappa \geq \omega$. It turns out that A_ω is much the most difficult. Observe that if $\omega \leq \lambda \leq \kappa$, then A_λ is (isomorphic to) a complete subalgebra of A_κ (cf. 3.20), so that (reading through 6.2(i)–(j))

$$\begin{aligned} \mathrm{wdistr}(A_\lambda) &\geq \mathrm{wdistr}(A_\kappa) , \\ \gamma_\omega(A_\lambda) &\geq \gamma_\omega(A_\kappa) , \\ \gamma^*_\omega(A_\lambda) &\geq \gamma^*_\omega(A_\kappa) , \\ \pi(A_\lambda) &\leq \pi(A_\kappa) , \\ n(A_\lambda) &\geq n(A_\kappa) , \\ d(A_\lambda) &\leq d(A_\kappa) . \end{aligned}$$

Note also that if $X \subseteq A_\kappa$ and $|X| \leq \lambda \leq \kappa$, then there is a complete subalgebra B

of A_κ, including X, such that B is isomorphic to A_λ. (This is an easy consequence of 1.11(b).) It follows that if $n(A_\kappa)=\lambda$, then $n(A_\lambda)=\lambda$. (For if $\mathscr{X}$ is a family of partitions of 1 in A witnessing that $n(A_\kappa)\leq\lambda$, $|\bigcup\mathscr{X}|\leq\lambda$, so $\mathscr{X}$ witnesses that $n(A_\lambda)\leq\lambda$.)

6.4. LEMMA. *Let $(X, \mathfrak{T}, B, \mu)$ be a compact Radon probability space with Maharam type $\kappa\geq\omega$. Let $(Y, \mathfrak{S}, C, \nu)$ be the Radon probability space product of ω copies of X, as described in* 1.11. *Then (Y, C, ν) is Maharam homogeneous, with Maharam type κ.*

PROOF. (a) Let $(A_X, \tilde{\mu})$ be the measure algebra of (X, B, μ) and $(A_Y, \tilde{\nu})$ the measure algebra of (Y, C, ν). For each $i\in\omega$ we have an inverse-measure-preserving projection $\mathrm{pr}_i\colon Y\to X$ and a corresponding measure-preserving homomorphism $e_i\colon A_X\to A_Y$. (Cf. 2.25(c).)

(b) Let $D_0\subseteq A_X$ be a set of cardinal κ which completely generates A_X. Then 1.11(c) makes it plain that

$$\{e_i(d)\colon i\in\omega,\ d\in D_0\}$$

completely generates A_Y. So $\tau(A_Y)\leq\kappa$.

(c) Take any $b\in A_Y^+$, and set $\lambda=\tau(A_Y\restriction b)$. For each $i\in\omega$, set $a_i=\prod\{a\in A_X\colon e_i(a)\leq b\}$; then $e_i(a_i)\geq b$, because e_i is a complete homomorphism, and $a\mapsto b\cdot e_i(a)\colon A_X\restriction a_i\to A_Y\restriction b$ is a complete monomorphism. So $A_X\restriction a_i$ is isomorphic to a complete subalgebra of $A_Y\restriction b$ and $\tau(A_X\restriction a_i)\leq\lambda$ (6.3(b)).

Next we see that $b\leq\prod_{i\in\omega}e_i(a_i)$, so $\prod_{i=0}^{\infty}\tilde{\mu}(a_i)\geq\tilde{\nu}(b)>0$ (cf. 2.25). So $\sup_{i\in\omega}\tilde{\mu}(a_i)=1$ and $\sum_{i\in\omega}a_i=1$ in A_X. Accordingly, $\kappa=\tau(A_X)\leq\max(\omega, \sup_{i\in\omega}\tau(A_X\restriction a_i))\leq\max(\lambda,\omega)$.

(d) But we know also that $A_X\neq\{0,1\}$, so that not every a_i can be an atom, and therefore b cannot be an atom in A_Y. This means that A_Y is atomless. So, working through the argument of (c) again, $\lambda\geq\omega$ and $\kappa\leq\lambda$.

As b is arbitrary, A_Y is τ-homogeneous and $\tau(A_y)$ is exactly κ, as claimed.

6.5. LEMMA. *Let $(X, \mathfrak{T}, B, \mu)$ be a compact Radon probability space of Maharam type $\kappa\geq\omega$. Then there are functions $a\mapsto v_a\colon A_\kappa^+\to N_\mu$ and $u\mapsto c_u\colon N_\mu\to A_\kappa^+$ such that $u\subseteq v_a$ whenever $u\in N_\mu$ and $a\in A_\kappa^+$ and $a\leq c_u$.*

PROOF. (a) Let $(Y, \mathfrak{S}, C, \nu)$ be the Radon probability space product of ω copies of $(X, \mathfrak{T}, B, \mu)$, and $(A_Y, \tilde{\nu})$ the measure algebra of (Y, C, ν). Then $A_Y\cong A_\kappa$, by Lemma 6.4, so it will be enough to find functions $a\mapsto v'_a\colon A_Y^+\to N_\mu$ and $u\mapsto c'_u\colon N_\mu\to A_Y^+$ such that $u\subseteq v'_a$ whenever $a\leq c'_u$.

For $i\in\omega$ let $\mathrm{pr}_i\colon Y\to X$ be the corresponding continuous inverse-measure-preserving projection. Let $\pi_Y\colon C\to A_Y$ be the canonical epimorphism.

(b) For each $a\in A_Y^+$ choose a closed self-supporting set $h_a\subseteq Y$ such that $0<\pi_Y(h_a)\leq a$. (Use 1.9(c).) Consider the compact sets $\mathrm{pr}_i[h_a]\subseteq X$ for $i\in\omega$. We see that

$$h_a\subseteq\bigcap_{i\in\omega}\mathrm{pr}_i^{-1}[\mathrm{pr}_i[h_a]]\,,$$

so that

$$0 < \nu(h_a) \leq \prod_{i=0}^{\infty} \mu(\mathrm{pr}_i[h_a])$$

by the defining formula for ν (1.9(a)). So $\sup_{i\in\omega} \mu(\mathrm{pr}_i[h_a]) = 1$ and $v'_a = X \setminus \bigcup_{i\in\omega} \mathrm{pr}_i[h_a] \in N_\mu$.

(c) For each $u \in N_\mu$ let $(g_i(u))_{i\in\omega}$ be a sequence of closed subsets of $X \setminus u$ such that $\mu(g_i(u)) > 1 - 2^{-i}$ for each $i \in \omega$. Then

$$\nu\Big(\bigcap_{i\in\omega} \mathrm{pr}_i^{-1}[g_i(u)]\Big) = \prod_{i=0}^{\infty} \mu(g_i(u)) > 0 .$$

So $c'_u = \pi_Y(\bigcap_{i\in\omega} \mathrm{pr}_i^{-1}[g_i(u)] \in A_Y^+$.

(d) This defines the functions $a \mapsto v'_a$ and $u \mapsto c'_u$. Now suppose that $u \in N_\mu$ and that $a \in A_Y^+$ and that $a \leq c'_u$. Then

$$\pi_Y(h_a) \leq a \leq \pi_Y\Big(\bigcap_{i\in\omega} \mathrm{pr}_i^{-1}\,[g_i(u)]\Big),$$

so that $\nu(h_a \setminus \mathrm{pr}_i^{-1}[g_i(u)]) = 0$ for each $i \in \omega$. But h_a is self-supporting and $\mathrm{pr}_i^{-1}[g_i(u)]$ is closed, so $h_a \subseteq \mathrm{pr}_i^{-1}[g_i(u)]$, i.e. $\mathrm{pr}_i[h_a] \subseteq g_i(u)$ for each $i \in \omega$. This means that $u \cap \mathrm{pr}_i[h_a] = \emptyset$ for each $i \in \omega$, i.e. that $u \subseteq v'_a$, as required. □

6.6. COROLLARY. *Under the hypotheses of Lemma* 6.5,

$$\gamma_\omega(A_\kappa) \leq \mathrm{add}(N_\mu), \qquad \mathrm{cf}(N_\mu) \leq \pi(A_\kappa).$$

PROOF. (a) Suppose that $E \subseteq N_\mu$ and that $|E| < \gamma_\omega(A_\kappa)$. Then there is a countable $D \subseteq A^+$ such that for every $u \in E$ there is a $d \in D$ such that $d \leq c_u$. In this case $\bigcup E \subseteq \bigcup_{d\in D} v_d \in N_\mu$. As E is arbitrary, $\mathrm{add}(N_\mu) \geq \gamma_\omega(A_\kappa)$.

(b) Let $D \subseteq A_\kappa^+$ be a dense set of cardinal $\pi(A_\kappa)$. Then $\{v_d\colon d \in D\}$ is cofinal with N_μ, so $\mathrm{cf}(N_\mu) \leq |D| = \pi(A_\kappa)$. □

REMARK. Later, in 6.13–6.17, I shall put this corollary together with similar results to form the theorems which are the object of this section. We can observe straight away, however, that a fixed A_κ can correspond to widely varying $(X, \mathfrak{T}, B, \mu)$; e.g. X can be either ${}^\kappa\{0,1\}$ or its hyperstonian space; and in the latter case N_μ is just the ideal of nowhere dense sets in $X = \mathrm{Ult}(A_\kappa)$, so that we already have interesting relations between γ_ω, π and topological properties of Stone spaces.

6.7. LEMMA. *Let S be the set*

$$\{s\colon S \subseteq \omega \times \omega,\ |\{j\colon (i,j) \in s\}| \leq 2^{i+1}\ \forall i \in \omega\}.$$

For $f \in {}^\omega\omega$, $s \subseteq \omega \times \omega$ say that $f \subseteq^* s$ if $\{i \in \omega\colon (i, f(i)) \not\in s\}$ is finite. Now there are functions $u \mapsto s_u\colon N \to S$ and $f \mapsto v_f\colon {}^\omega\omega \to N$ such that $f \subseteq^* s_u$ whenever $f \in {}^\omega\omega$ and $v_f \subseteq u \in N$.

PROOF. (a) Let μ be the usual measure on ${}^{\omega}\{0,1\}$, so that $N = N_{\mu}$. Let $(I_{ij})_{i,j\in\omega}$ be a pairwise disjoint double sequence of subsets of ω such that $|I_{ij}| = i+1$ for every i, $j \in \omega$. Set

$$g_{ij} = \{x \in {}^{\omega}\{0,1\} : x(k) = 1 \ \forall k \in I_{ij}\}\ ;$$

then each g_{ij} is an open set in ${}^{\omega}\{0,1\}$ and

$$\mu\Big(\bigcap_{(i,\,j)\in s} g_{ij}\Big) = \prod_{(i,\,j)\in s} 2^{-i-1}$$

for every non-empty $s \subseteq \omega \times \omega$. For each $f \in {}^{\omega}\omega$ set

$$v_f = \bigcap_{n\in\omega} \bigcup_{i\geq n} g_{i,f(i)}\ .$$

Then

$$\mu(v_f) \leq \inf_{n\in\omega} \sum_{i=n}^{\infty} \mu(g_{i,f(i)}) = \inf_{n\in\omega} \sum_{i=n}^{\infty} 2^{-i-1} = 0\,,$$

so $v_f \in N$.

(b) For each $u \in N$ choose a non-empty compact self-supporting set $h_u \subseteq {}^{\omega}\{0,1\}\setminus u$. Let $(w_n(u))_{n\in\omega}$ enumerate a base for the relative topology of h_u which does not contain $\emptyset$; as h_u is self-supporting, no $w_n(u)$ belongs to N. Set

$$q(u,n,i) = \{j \in \omega : w_n(u) \cap g_{ij} = \emptyset\}$$

for $i, n \in \omega$. Then

$$0 < \mu(w_n(u)) \leq \mu\Big(\bigcap_{i\in\omega,\,j\in q(u,n,i)} {}^{\omega}\{0,1\}\setminus g_{ij}\Big)$$

$$= \prod_{i=0}^{\infty} \prod_{j\in q(u,n,i)} \mu({}^{\omega}\{0,1\}\setminus g_{ij})$$

(because the g_{ij} are stochastically independent)

$$= \prod_{i=0}^{\infty} (1 - 2^{-i-1})^{|q(u,n,i)|}\ .$$

So

$$\sum_{i=0}^{\infty} 2^{-i-1}|q(u,n,i)| < \infty\ .$$

Let $k(u,n) \in \omega$ be such that $2^{-i-1}|q(u,n,i)| \leq 2^{-n-1}$ for $i \geq k(u,n)$. Set

$$s_u = \bigcup_{n\in\omega} \{(i,j) : i \geq k(u,n),\ j \in q(u,n,i)\}\ .$$

It is easy to check that $s_u \in S$.

(c) This defines $f \mapsto v_f$ and $u \mapsto s_u$. Now suppose that $f \in {}^\omega\omega$ and that $v_f \subseteq u \in N$. Then

$$h_u \cap \bigcap_{n\in\omega} \bigcup_{i\geq n} g_{i,f(i)} = h_u \cap v_f = \emptyset .$$

But h_u, with its induced topology, is a Baire space, while v_f is a G_0 set. So there is some $n \in \omega$ such that $h_u \cap \bigcup_{i\geq n} g_{i,f(i)}$ is not dense in h_u, and an $m \in \omega$ such that $w_m(u) \cap \bigcup_{i\geq n} g_{i,f(i)} = \emptyset$; i.e. $f(i) \in q(u, m, i)$ for every $i \geq n$. So $(i, f(i)) \in s_u$ for every $i \geq \max(n, k(u, m))$, and $f \subseteq^* s_u$. □

REMARK. This result is extracted from RAISONNIER and STERN [1985]; the same ideas may be found in BARTOSZYŃSKI [1984].

6.8. COROLLARY. *In the notation of* 6.7,

(a) *if* $F \subseteq {}^\omega\omega$ *and* $|F| < \mathrm{add}(N)$ *there is an* $s \in S$ *such that* $f \subseteq^* s$ *for every* $f \in F$;

(b) *there is an* $S_0 \subseteq S$ *such that* $|S_0| \leq \mathrm{cf}(N)$ *and for every* $f \in {}^\omega\omega$ *there is an* $s \in S$ *such that* $f \subseteq^* s$.

PROOF. (a) Set $u = \bigcup_{f\in F} v_f$; then $f \subseteq^* s_u$ for every $f \in F$.

(b) Take a cofinal $E \subseteq N$ of cardinality $\mathrm{cf}(N)$ and set $S_0 = \{s_u\colon u \in E\}$. □

REMARK. As will appear from the arguments below (the details may be found in FREMLIN [1985a]) these facts characterize $\mathrm{add}(N)$ and $\mathrm{cf}(N)$ in that $\mathrm{add}(N)$ is the largest cardinal, and $\mathrm{cf}(N)$ the smallest cardinal, such that (a) and (b) above are true.

6.9. PROPOSITION. $\mathrm{add}(N) = \gamma_\omega(A_\omega) = \gamma^*_\omega(A_\omega)$ and $\mathrm{cf}(N) = \pi(A_\omega)$.

PROOF. (a) From 6.6 we know already that $\gamma^*(A_\omega) \leq \gamma_\omega(A_\omega) \leq \mathrm{add}(N)$ and that $\mathrm{cf}(N) \leq \pi(A_\omega)$.

(b) Let C be the countable algebra of clopen subsets of ${}^\omega\{0,1\}$, and let μ be the usual measure of ${}^\omega\{0,1\}$. Set

$$C_n = \{c \in C\colon \mu(c) \leq 4^{-n}\}$$

for each $n \in \omega$. Let B be the domain of μ and $\pi\colon B \to A_\omega$ the canonical epimorphism with kernel N.

(c) If $a \in A_\omega$, $k \in \omega$ there is a function $f_{ak} \in \Pi_{i\in\omega} C_i$ such that, writing $d_{ak} = \bigcup_{i\in\omega} f_{ak}(i)$, we have

$$-\pi(d_{ak}) \leq a\,, \qquad \mu(d_{ak}) \leq 1 - \tilde{\mu}(a) + 2^{-k}\,.$$

To see this, start by taking a closed set $v \subseteq {}^\omega\{0,1\}$ such that $\pi(v) \leq a$ and $\mu(v) \geq \tilde{\mu}(a) - 2^{-k}$; this is possible because μ is a Radon measure. Choose inductively a sequence $(v_n)_{n\in\omega}$ in C such that

$$v_0 = {}^{\omega}\{0,1\}\,,$$
$$v \subseteq \operatorname{int}(v_{n+1}) \subseteq \operatorname{cl}(v_{n+1}) \subseteq \operatorname{int}(v_n)\,,$$
$$\mu(v_n) \leq \mu(v) + 4^{-n} \quad \forall n \in \omega\,.$$

Set $f_{ak}(i) = v_i \setminus v_{i+1} \in C_i$ for each $i \in \omega$; then $d_{ak} = {}^{\omega}\{0,1\} \setminus v$ has the required properties.

(d) Let $\tilde{S}$ be

$$\{s\colon s \subseteq \omega \times C,\, s[\{i\}] \subseteq C_i \text{ and } |s[\{i\}]| \leq 2^{i+1}\ \forall i \in \omega\}\,,$$

where $s[\{i\}] = \{c\colon (i,c) \in s\}$. For $s \in \tilde{S}$, $m \in \omega$ and $h \in \Pi_{i<m} C_i$ set

$$e_{hs} = \bigcup_{i<m} h(i) \cup \bigcup \{c\colon \exists i \geq m,\, (i,c) \in s\} \in B\,.$$

(e) Suppose that $s \in \tilde{S}$ and $a \in A_{\omega}$ are such that $f_{ak} \subseteq^* s$ for every $k \in \omega$ (i.e. $\{i\colon (i, f_{ak}(i)) \not\in s\}$ is finite for every $k \in \omega$). Set

$$E_s = \Big\{-\pi(e_{hs})\colon h \in \bigcup_{m\in\omega} \prod_{i<m} C_i\Big\}\,.$$

Then

$$a = \Sigma\,\{e \in E_s\colon e \leq a\}\,.$$

To see this, take any $k \in \omega$ and take $m \geq k$ such that $(i, f_{ak}(i)) \in s$ for $i \geq m$. Set $h = f_{ak} \restriction m \in \Pi_{i<m} C_i$. Then $d_{ak} \subseteq e_{hs}$ so $-\pi(e_{hs}) \leq -\pi(d_{ak}) \leq a$; also

$$\mu(e_{hs}) \leq \mu(d_{ak}) + \sum_{i=m}^{\infty} 2^{i+1} 4^{-i} \leq 1 - \tilde{\mu}(a) + 2^{-k} + 4.2^{-m}\,,$$

so

$$\tilde{\mu}(-\pi(e_{hs})) \geq \tilde{\mu}(a) - 5.2^{-k}\,.$$

As k is arbitrary,

$$\tilde{\mu}(a) = \sup\{\tilde{\mu}(e)\colon e \in E_s,\, e \leq a\}\,,$$

and $a = \Sigma\,\{e \in E_s\colon e \leq a\}$.

(f) The rest is easy. If $X \subseteq A$ and $|X| < \operatorname{add}(N)$, then by 6.8(a) (moving between $(S, {}^{\omega}\omega)$ and $(\tilde{S}, \Pi_{i\in\omega} C_i)$ by means of any enumerations of the C_i) there is an $s \in \tilde{S}$ such that $f_{ak} \subseteq^* s$ for every $a \in X$, $k \in \omega$; now E_s is countable and $a = \Sigma\,\{e \in E_s\colon e \leq a\}$ for every $a \in X$; as X is arbitrary, $\gamma^*(A_{\omega}) \geq \operatorname{add}(N)$. Next, 6.8(b) tells us that there is an $\tilde{S}_0 \subseteq \tilde{S}$, of cardinal not greater than $\operatorname{cf}(N)$, such that for every $f \in \Pi_{i\in\omega} C_i$ there is an $s \in \tilde{S}_0$ such that $f \subseteq^* s$; in which case, setting $E \doteq \bigcup\{E_s\colon s \in \tilde{S}_0\} \setminus \{\emptyset\} \subseteq A_{\omega}^{+}$, we see that E is dense in A_{ω}^{+}, so that $\pi(a_{\omega}) \leq |E| \leq \operatorname{cf}(N)$. $\square$

REMARK. The result that $\mathrm{cf}(N) = \pi(A_\omega)$ is due to CICHOŃ, KAMBURELIS and PAWLIKOWSKI [1985].

6.10. LEMMA. *Let $(X, \mathfrak{T}, B, \mu)$ be a totally finite Radon measure space of Maharam type κ. Then there is a family $(e_\xi)_{\xi<\kappa}$ in N_μ such that $\{\xi : e_\xi \subseteq e\}$ is countable for every $e \in N_\mu$.*

PROOF. (a) Consider first the case in which $(X, \mathfrak{T}, B, \mu)$ is a Maharam homogeneous Radon probability space. If $\kappa \le \omega$ the result is trivial. Otherwise, by 2.21, the isomorphism between the measure algebra of (X, B, μ) and A_κ is induced by an inverse-measure-preserving function $f: X \to {}^\kappa\{0, 1\}$. For each $\xi < \kappa$ set $b_\xi = \{x \in X: f(x)(\xi) = 1\} \in B$. Then we see from 1.11(b) that for each $b \in B$ there are a countable set $I(b) \subseteq \kappa$ and a b' belonging to the σ-complete subalgebra of B generated by $\{b_\xi : \xi \in I(b)\}$ such that $b \triangle b' \in N_\mu$.

Now, for each $\xi < \kappa$, construct a sequence $(u_{\xi n})_{n\in\omega}$ of open sets in X and a sequence $(\alpha(\xi, n))_{n\in\omega}$ in κ, as follows. $\alpha(\xi, 0) = \xi$. Given $\alpha(\xi, n) < \kappa$, let $u_{\xi n}$ be an open set including $b_{\alpha(\xi,n)}$ such that $\mu(u_{\xi n} \setminus b_{\alpha(\xi,n)}) \le \frac{1}{12}3^{-n}$. Given $(u_{\xi i})_{i\le n}$ and $\alpha(\xi, n)$ let $\alpha(\xi, n+1)$ be the least ordinal strictly greater than $\alpha(\xi, n)$ and not belonging to $\bigcup_{i\le n} I(u_{\xi i})$. Observe that $\xi \le \alpha(\xi, n) < \xi + \omega_1$ for all ξ, n, so that $\{\xi < \kappa: \exists n \in \omega,\ \alpha(\xi, n) = \eta\}$ is countable for every $\eta < \kappa$.

Set $e_\xi = X \setminus \bigcup_{n\in\omega} u_{\xi n}$ for each $\xi < \kappa$. Then $e_\xi \subseteq X \setminus \bigcup_{n\in\omega} b_{\alpha(\xi,n)} \in N_\mu$ because all the $\alpha(\xi, n)$ are distinct and the b_η are stochastically independent.

Now suppose, if possible, that $J \subseteq \kappa$ is an uncountable set such that $\bigcup_{\xi\in J} e_\xi \in N_\mu$. Then there is a compact set $c \subseteq X \setminus \bigcup_{\xi\in J} e_\xi$ such that $\mu(c) \ge \frac{1}{2}$. Since

$$\{\xi : \exists n \in \omega,\ \alpha(\xi, n) \in I(c)\}$$

is countable, there is a $\zeta \in J$ such that $\alpha(\zeta, n) \notin I(c)$ for every $n \in \omega$. Set

$$c_n = c \setminus \bigcup_{i<n} u_{\zeta i}$$

for $n \in \omega$. Then there is, for each n, a c'_n in the σ-complete subalgebra of B generated by

$$\left\{ b_\xi : \xi \in I(c) \cup \bigcup_{i<n} I(u_{\zeta i}) \right\}$$

such that $c_n \triangle c'_n \in N_\mu$. So $\mu(c_n \cap b_{\alpha(\zeta,n)}) = \mu(c'_n \cap b_{\alpha(\zeta,n)}) = \frac{1}{2}\mu(c_n)$ for each $n \in \omega$, since $\alpha(\zeta, n) \notin I(c) \cup \bigcup_{i<n} I(u_{\zeta i})$. Accordingly,

$$\begin{aligned}\mu(c_{n+1}) = \mu(c_n \setminus u_{\zeta n}) &\ge \mu(c_n \setminus b_{\alpha(\zeta,n)}) - \mu(u_{\zeta n} \setminus b_{\alpha(\zeta,n)}) \\ &\ge \tfrac{1}{2}\mu(c_n) - \tfrac{1}{12}3^{-n}\end{aligned}$$

for every $n \in \omega$, and it follows by induction that $\mu(c_n) \ge \frac{1}{2}3^{-n}$ for every $n \in \omega$. In particular, no c_n can be empty; as c is compact and all the $u_{\zeta i}$ are open, there is an $x \in c \setminus \bigcup_{i\in\omega} u_{\zeta i}$. But now $x \in c \cap e_\zeta$, which is impossible.

Thus, $(e_\xi)_{\xi<\kappa}$ is a suitable family.

(b) For the general case, we can again pass over the case $\kappa \leq \omega$. By Maharam's theorem (3.9) there is a partition $(a_i)_{i\in I}$ of 1 in the measure algebra A of (X, B, μ) such that $A \restriction a_i$ is homogeneous for each $i \in I$. Since I must be countable, there is a partition $(X_i)_{i\in I}$ of X such that $\pi(X_i) = a_i$ for each $i \in I$, where $\pi: B \to A$ is the canonical map. Now setting $\mathfrak{T}_i = \{X_i \cap u: u \in \mathfrak{T}\}$, $B_i = B \cap P(X_i)$, $\mu_i = \mu \restriction B_i$, we see that $(X_i, \mathfrak{T}_i, B_i, \mu_i)$ is a Maharam homogeneous totally finite Radon measure space for each $i \in I$. Let $\kappa(i)$ be its Maharam type. By (a) (applied to normalized versions of $(X_i, \mathfrak{T}_i, B_i, \mu_i)$) there are families $(e^{(i)}_\xi)_{\xi<\kappa(i)}$ in $N_{\mu_i} = N_\mu \cap P(X_i)$ such that $\{\xi < \kappa(i): e^{(i)}_\xi \subseteq e\}$ is countable for any $e \in N_\mu$, $i \in I$. Observe now that $\kappa = \sup_{i\in I} \kappa(i)$, so that if we re-index $(e^{(i)}_\xi)_{i\in I, \xi<\kappa(i)}$ as $(e_\xi)_{\xi<\kappa}$ we shall have an appropriate family in N_μ. $\square$

6.11. COROLLARY. *Let $(X, \mathfrak{T}, B, \mu)$ be a totally finite Radon measure space of Maharam type κ. Then*

(a) $\mathrm{cf}([\kappa]^{\leq\omega}) \leq \mathrm{cf}(N_\mu)$;

(b) *if $\kappa \geq \omega_1$, then* $\mathrm{add}(N_\mu) = \omega_1$.

PROOF. Take $(e_\xi)_{\xi<\kappa}$ from 6.10. For (a), let $E \subseteq N_\mu$ be a cofinal set of cardinal $\mathrm{cf}(N_\mu)$; for $e \in E$ set $J_e = \{\xi < \kappa: e_\xi \subseteq e\} \in [\kappa]^{\leq\omega}$; then $\mathrm{cf}([\kappa]^{<\omega}) \leq |\{J_e: e \in E\}| \leq \mathrm{cf}(N_\mu)$. For (b), observe that if $\kappa \geq \omega_1$, then $\{e_\xi: \xi < \omega_1\}$ witnesses that $\mathrm{add}(N_\mu) \leq \omega_1$; and of course $\mathrm{add}(N_\mu) \geq \omega_1$ because μ is countably additive. $\square$

6.12. LEMMA. *If $(X, \mathfrak{T}, B, \mu)$ is a Maharam homogeneous Radon probability space of Maharam type $\geq\omega$, then* $\mathrm{add}(N_\mu) \leq \mathrm{add}(N)$ *and* $\mathrm{cf}(N_\mu) \geq \mathrm{cf}(N)$.

PROOF. (a) Writing A for the measure algebra of (X, B, μ) and $\kappa = \tau(A)$, there is a measure-preserving embedding of A_ω in A_κ, which is isomorphic to A; so there is a measure-preserving homomorphism from A_ω to A, which is realized by an inverse-measure-preserving function $f: X \to {}^\omega\{0, 1\}$. (Of course this is an easy consequence of 3.3.) Now f is almost continuous. To see this, set $b_n = \{x \in X: f(x)(n) = 1\}$ for each $n \in \omega$, and let $\varepsilon > 0$. For each $n \in \omega$ take closed sets $c_n \subseteq b_n$, $c'_n \subseteq X \setminus b_n$ such that $\mu(b_n \setminus c_n) \leq 2^{-n}\varepsilon$ and $\mu((X\setminus b_n)\setminus c'_n) \leq 2^{-n}\varepsilon$. Set $c = \bigcap_{n\in\omega} (c_n \cup c'_n)$. Then $f \restriction c$ is continuous. But also

$$\mu(X\setminus c) \leq \sum_{n=0}^{\infty} \mu(X\setminus(c_n \cup c'_n)) \leq 4\varepsilon .$$

As ε is arbitrary, f is almost continuous.

(b) Now suppose that $e \subseteq {}^\omega\{0, 1\}$. Then $e \in N$ iff $f^{-1}[e] \in N_\mu$. For if $e \in N$, then of course $f^{-1}[e] \in N_\mu$, since f is inverse-measure-preserving. On the other hand, if $f^{-1}[e] \in N_\mu$, there is a sequence $(d_n)_{n\in\omega}$ of compact subsets of $X \setminus f^{-1}[e]$ such that $f \restriction d_n$ is continuous for each $n \in \omega$ and $\sup_{n\in\omega} \mu(d_n) = 1$. Now $f[d_n]$ is a closed subset of ${}^\omega\{0, 1\}$ and

$$\mu_\omega(f[d_n]) = \mu(f^{-1}[f[d_n]]) \geq \mu(d_n) \quad \forall n \in \omega ,$$

where μ_ω is the measure of ${}^\omega\{0, 1\}$. So

$$e \subseteq {}^{\omega}\{0,1\} \Big\backslash \bigcup_{n\in\omega} f[d_n] \in N .$$

(c) But it follows at once that if $E \subseteq N$ is a set of cardinal $\mathrm{add}(N)$ such that $\bigcup E \not\in N$, then $\{f^{-1}[e]: e \in E\}$ witnesses that $\mathrm{add}(N_\mu) \leq \mathrm{add}(N)$. While if $E \subseteq N_\mu$ is a cofinal set of cardinal $\mathrm{cf}(N_\mu)$,

$$\{{}^{\omega}\{0,1\} \backslash f[X \backslash e]: e \in E\}$$

witnesses that $\mathrm{cf}(N) \leq \mathrm{cf}(N_\mu)$.

6.13. THEOREM. *Let $(X, \mathfrak{T}, B, \mu)$ be a Maharam homogeneous Radon probability space of Maharam type $\kappa \geq \omega$, and let A be its measure algebra. Then*

(a)(α) *If $\kappa = \omega$ then*

$$\gamma_\omega(A) = \gamma^*_\omega(A) = \mathrm{wdistr}(A) = \mathrm{add}(N_\mu) = \mathrm{add}(N) ;$$

(β) *if $\kappa \geq \omega_1$ then*

$$\gamma_\omega(A) = \gamma^*_\omega(A) = \mathrm{wdistr}(A) = \mathrm{add}(N_\mu) = \omega_1 .$$

(b) $\pi(A) = \max(\mathrm{cf}(N), \mathrm{cf}([\kappa]^\omega)) = \mathrm{cf}(N_\mu)$.
(c) $n(A) = \min\{|E|: E \subseteq N_\mu, \bigcup E = X\}$.
(d) $d(A) = \min\{|e|: e \subseteq X, \mu^*(e) = 1\}$
$= \min\{|e|: e \subseteq X, e \not\in N_\mu\}$.

PROOF. (a)(i) If $\kappa \geq \omega_1$, then $\mathrm{add}(N_\mu) = \omega_1$, by 6.11(b). From 6.6 we see that $\gamma_\omega(A) \leq \omega_1$, so that $\gamma_\omega(A) = \gamma^*_\omega(A) = \omega_1$.

(ii) If $\kappa = \omega$, then by 6.6 and 6.9 and 6.12 we have

$$\mathrm{add}(N) = \gamma_\omega(A) = \gamma^*_\omega(A) \leq \mathrm{add}(N_\mu) \leq \mathrm{add}(N) ,$$

so these are all equal.

(iii) Now let $(Z, \mathfrak{S}, C, \nu)$ be the hyperstonian space of A (2.13–2.14). Then N_ν is just the ideal of nowhere dense sets in Z so that $\mathrm{wdistr}(A) = \mathrm{add}(N_\nu)$ (6.2(c)). But from (i) or (ii) above, $\mathrm{add}(N_\nu) = \gamma_\omega(A)$. Putting these facts together we have what we need.

(b) By 6.6, $\mathrm{cf}(N_\mu) \leq \pi(A)$; by 6.9 and 6.12, $\pi(A_\omega) = \mathrm{cf}(N) \leq \mathrm{cf}(N_\mu)$; by 6.11(a), $\mathrm{cf}([\kappa]^\omega) \leq \mathrm{cf}(N_\mu)$. On the other hand, $\pi(A) = \pi(A_\kappa) \leq \max(\pi(A_\omega), \mathrm{cf}([\kappa]^\omega))$. To see this, let $\mathcal{I}$ be a cofinal subset of $[\kappa]^\omega$ of cardinality $\mathrm{cf}([\kappa]^\omega)$. For each $I \in \mathcal{I}$, let $h_I: \omega \to I$ be a bijection, and let $f_I: {}^{\kappa}\{0,1\} \to {}^{\omega}\{0,1\}$ be given by writing $f_I(x)(n) = x(h_I(n))$ for $x \in {}^{\kappa}\{0,1\}$, $n \in \omega$. Then f_I is inverse-measure-preserving; let $g_I: A_\omega \to A_\kappa$ be the associated measure-preserving homomorphism (2.16). Let $D \subseteq A^+_\omega$ be a dense set of cardinal $\pi(A_\omega)$ and set $D' = \{g_I(d): d \in D, I \in \mathcal{I}\}$. If $a \in A^+$ let $c \subseteq {}^{\kappa}\{0,1\}$ be a Baire set such that $\pi_\kappa(c) = a$, where π_κ is the canonical epimorphism onto A_κ. Let $I \in \mathcal{I}$ be such that c factors through ${}^{I}\{0,1\}$; then $c = f_I^{-1}[f_I[c]]$, so $a = g_I(a')$, where $a' = \pi_\omega(f_I[c])$.

Let $d \in D$ be such that $d \leq a'$; then $g_I(d) \in D'$ and $g_I(d) \leq a$. So D' is dense in A^+ and $\pi(A_\kappa) \leq |D'| \leq \max(\pi(A_\omega), \mathrm{cf}([\kappa]^\omega))$, as claimed.

Putting these together we have the result.

(c)–(d) These are easy if we note the following. Let (Y, C, ν) be any other probability space and $f: X \to Y$ an inverse-measure-preserving function. Then if $E \subseteq N_\nu$ and $\bigcup E = Y$, $E' = \{f^{-1}[e]: e \in E\} \subseteq N_\mu$ and $\bigcup E' = X$; while if $e \subseteq X$ and $\mu^*(e) = 1$, then $\nu^*(f[e]) = 1$, because if $f[e] \subseteq c \in C$, then $e \subseteq f^{-1}[c] \in B$ so $\nu(c) = \mu(f^{-1}[c]) = 1$. It follows at once that

$$\min\{|E|: E \subseteq N_\mu, \bigcup E = X\} \leq \min\{|E|: E \subseteq N_\nu, \bigcup E = Y\},$$

$$\min\{|e|: e \subseteq X, \mu^*(e) = 1\} \geq \min\{|e|: e \subseteq Y, \nu^*(e) = 1\}.$$

Now let $Z = \mathrm{Ult}(A)$ be the hyperstonian space of A, and write M for the ideal of nowhere dense subsets of Z, which is also the ideal of negligible subsets of Z. We have inverse-measure-preserving functions from X to Z and from Z to X (4.11(b), 2.17). So we see that

$$\min\{|E|: E \subseteq N_\mu, \bigcup E = X\} = \min\{|E|: E \subseteq M, \bigcup E = Z\} = n(A)$$

(6.2(f)). If we observe also that a subset of Z has outer measure 1 iff it is dense (since its closure must have measure 1; see 2.14(b)), then we see that

$$\min\{|e|: e \subseteq X, \mu^*(e) = 1\}$$

must be the topological density of Z, which is equal to $d(A)$.

For the last equality of (d), observe that if $e \subseteq X$ and $\mu^*(e) > 0$ there is a $b \in B$ such that $e \subseteq b$ and $\mu(b) = \mu^*(e)$. Now $(b, \mathfrak{T}_b, B \restriction b, \mu_b)$ is also a Radon probability space, where $\mathfrak{T}_b$ is the subspace topology of b, and $\mu_b(c) = \mu(c)/\mu(b)$ for $c \in B \restriction b$; its measure algebra is isomorphic to A, and $\mu_b^*(e) = 1$. So $|e| \geq d(A)$, by the work just done. □

REMARK. Part (b) of this theorem, and parts of (a), come from FREMLIN [1985b].

6.14. COROLLARY. *Let $(X, \mathfrak{T}, B, \mu)$ be a totally finite Radon measure space of Maharam type κ, and let A be its measure algebra. Then*

(a) $\mathrm{add}(N_\mu) = \mathrm{wdistr}(A)$.

(b)(α) *If A is atomic, then* $\mathrm{cf}(N_\mu) = 1$;

(β) *if A is not atomic, then* $\mathrm{cf}(N_\mu) = \max(\mathrm{cf}(N), \mathrm{cf}([\kappa]^\omega))$.

(c) $\min\{|E|: E \subseteq N_\mu, \bigcup E = X\} = n(A)$.

(d) $\min\{|e|: e \subseteq X, \mu^*(e) = \mu(X)\} = d(A)$.

PROOF. The case in which A is atomic is straightforward, because of the following fact, which is an easy consequence of 1.9(c): if $a \in A$ is an atom, there is an $x \in X$ such that $a = \pi(\{x\})$, where $\pi: B \to A$ is the canonical epimorphism. So if A is atomic, there is a countable set $D \subseteq X$ such that $N_\mu = P(X \setminus D)$.

Now suppose that A is not atomic. By Maharam's theorem (3.9), there is a partition $(a_i)_{i\in I}$ of 1 in A such that $A \restriction a_i$ is homogeneous for each $i \in I$; set $\kappa(i) = \tau(A \restriction a_i)$. Because I must be countable, there is a partition $(b_i)_{i\in I}$ of X into members of B such that $\pi(b_i) = a_i$ for each $i \in I$. Now if we write $\mathfrak{T}_i$ for the induced topology on b_i and μ_i for $\mu \restriction (B \restriction b_i)$, $(b_i, \mathfrak{T}_i, B \restriction b_i, \mu_i)$ is a Maharam homogeneous totally finite Radon measure space, of Maharam type $\kappa(i)$, with measure algebra isomorphic to $A \restriction a_i$, and with null ideal $N_{\mu_i} = N_\mu \cap P(b_i)$, for each $i \in I$. Putting 6.13 (applied to normalized versions of the μ_i) and 6.3(d) together, we easily confirm (a), (c) and (d) of this corollary.

We are left with (b)(β). If I is infinite, $\mathrm{cf}(N_\mu)$ is not instantly calculable from the $\mathrm{cf}(N_{\mu_i})$. But (supposing still that A is not atomic) we see from 6.6 and 6.13 that $\mathrm{cf}(N_\mu) \le \pi(A_\kappa) = \max(\mathrm{cf}(N), \mathrm{cf}([\kappa]^\omega))$; from 6.11 that $\mathrm{cf}([\kappa]^\omega) \le \mathrm{cf}(N_\mu)$; and from 6.12 that $\mathrm{cf}(N) \le \mathrm{cf}(N_{\nu_i}) \le \mathrm{cf}(N_{\mu_i})$ for some $i \in I$, so we are done. □

6.15. COROLLARY. *Let A be a measurable algebra, with $\tau(A) = \kappa$, and let M be the ideal of nowhere dense sets in* $\mathrm{Ult}(A)$. *Then*

(a)(α) $\gamma_\omega(A) = \gamma^*_\omega(A) = \mathrm{wdistr}(A) = \mathrm{add}(M)$;

(β) *if A is atomic, then* $\mathrm{wdistr}(A) = \infty$;

(γ) *if A is not atomic and $\kappa = \omega$, then* $\mathrm{wdistr}(A) = \mathrm{add}(N)$;

(δ) *if $\kappa \ge \omega_1$, then* $\mathrm{wdistr}(A) = \omega_1$.

(b)(α) *If A is atomic then $\pi(A) = c(A)$ and* $\mathrm{cf}(M) = 1$;

(β) *if A is not atomic then* $\mathrm{cf}(M) = \max(\mathrm{cf}(N), \mathrm{cf}([\kappa]^\omega))$;

(γ) *if A is not atomic and there is an $a \in A^+$ such that $A \restriction a$ is homogeneous and $\tau(A \restriction a) = \kappa$, then* $\pi(A) = \max(\mathrm{cf}(N), \mathrm{cf}([\kappa]^\omega))$;

(δ) *if A is not atomic and $\tau(A \restriction a) < \kappa$ whenever $a \in A^+$ and $A \restriction a$ is homogeneous, then* $\pi(A) = \max(\mathrm{cf}(N), \sup_{\lambda<\kappa} \mathrm{cf}([\lambda]^\omega))$.

(c)(α) $n(A) = \min\{|E|: E \subseteq M, \bigcup E = \mathrm{Ult}(A)\}$;

(β) *if A has an atom,* $n(A) = \infty$;

(γ) *if A is atomless and not $\{0\}$, then $n(A) = n(A_\lambda)$, where* $\lambda = \min\{\tau(A \restriction a): a \in A^+\}$.

(d)(α) *$d(A)$ is the least value of $|X|$ for any pair (X, B) such that X is a set and B is a σ-algebra of subsets of X such that there is a σ-complete epimorphism from B onto A*;

(β) *if A is atomic, then $d(A) = c(A)$*;

(γ) *if A is not atomic and there is an $a \in A^+$ such that $A \restriction a$ is homogeneous and $\tau(A \restriction a) = \kappa$, then $d(A) = d(A_\kappa)$*;

(δ) *if A is not atomic and $\tau(A \restriction a) < \kappa$ whenever $a \in A^+$ and $A \restriction a$ is homogeneous, then $d(A) = \sup_{\lambda<\kappa} d(A_\lambda)$.*

PROOF. Nearly all of this is just a matter of putting 6.13 and 6.3(d) together, with 6.14 to help with (b)(β) (because $M = N_\nu$ for a suitable measure ν on $\mathrm{Ult}(A)$, as in 2.13); 6.3(e) tells us which are the critical parts of a Maharam decomposition of A. The only part missing is (d)(α), which I now prove.

First suppose that X is a set and B is a σ-algebra of subsets of X and $\pi: B \to A$ is a σ-complete epimorphism. Set

$$N' = \{e\colon e \subseteq X, \exists b \in B \text{ such that } e \subseteq b \text{ and } \pi(b) = 0\}\ .$$

Then N' is a σ-ideal of subsets of X. Set $\hat{B} = \{b \triangle n\colon b \in B, n \in N'\}$; then $\hat{B}$ is a σ-algebra of subsets of X, and N' is a σ-complete ideal of $\hat{B}$. The inclusion map $B \to \hat{B}$ and the canonical map $\hat{B} \to \hat{B}/N'$ compose to give an epimorphism from B to $\hat{B}/N'$ with kernel $\pi^{-1}[\{0\}]$; so that A is isomorphic to $\hat{B}/N'$; let $\hat{\pi}\colon \hat{B} \to A$ be a corresponding epimorphism with kernel N'. Now let $\tilde{\mu}\colon A \to \boldsymbol{R}$ be such that $(A, \tilde{\mu})$ is a totally finite measure algebra, and set $\mu = \tilde{\mu} \circ \hat{\pi}\colon \hat{B} \to \boldsymbol{R}$. We see that $(X, \hat{B}, \mu)$ is a totally finite complete measure space and that $N_\mu = N'$. Accordingly, A is isomorphic to the measure algebra of $(X, \hat{B}, \mu)$. But now we know from 4.11(b) that there is an inverse-measure-preserving function from X to $\mathrm{Ult}(A)$, if $\mathrm{Ult}(A)$ is given the Radon measure ν corresponding to the measure $\tilde{\mu}$ on A. If $h[X] \subseteq c \in \mathrm{dom}(\nu)$, then $\nu(c) = \mu(h^{-1}[c]) = \mu(X) = \tilde{\mu}(1) = \nu(\mathrm{Ult}(A))$; so $\nu(\mathrm{cl}(h[X])) = \nu(\mathrm{Ult}(A))$ and $h[X]$ is dense in $\mathrm{Ult}(A)$. It follows at once that $d(A) \leq |h[X]| \leq |X|$, which is half of what we need to know.

For the other half of (d)(α), we know that A is isomorphic to C/M, where C is the algebra of subsets of $\mathrm{Ult}(A)$ with nowhere dense boundary and M is the ideal of nowhere dense subsets of $\mathrm{Ult}(A)$. (See 2.14.) Let $X \subseteq \mathrm{Ult}(A)$ be any topologically dense set of cardinal $d(A)$, and set $B = \{X \cap c\colon c \in C\}$, $M_1 = M \cap P(X)$. If $c \in C$, then $X \cap c \in M_1$ iff $c \in M$; so B/M_1 is isomorphic to C/M and therefore to A. But also M, and therefore M_1, are σ-complete ideals. So the canonical epimorphism from B to A is σ-complete, and (X, B) is one of the pairs we are interested in. □

6.16. REMARKS. (a) In 6.13–6.14 I have emphasized the connexions between the ideal N_μ of negligible sets in a Radon measure space and the measure algebra A. In 6.15, by way of contrast, I have collected results which can be expressed purely in terms of the Boolan algebra A and its Stone space. The point of working with arbitrary Radon measure spaces here is just that we have such a variety of different Radon measure spaces giving rise to any given measure algebra; in particular, we always have the hyperstonian space on the one hand, and the spaces based on the product measures of ${}^{\kappa}\{0, 1\}$ on the other.

(b) Most of 6.14–6.15 can readily be extended to arbitrary Radon measure spaces and to arbitrary semi-finite measure algebras, respectively. But very little of 6.14 is valid for general (totally finite) measure spaces. An exception is the fact that if (X, B, μ) is any totally finite measure space, with measure algebra A, then

$$d(A) \leq \min\{|e|\colon e \subseteq X, \mu^*(e) = \mu(X)\}$$

(compare 6.15(d)(α)). Another way of expressing this is to say that, for any infinite cardinal κ, $d(A_\kappa)$ is the smallest cardinal of any totally finite Maharam homogeneous measure space of Maharam type κ.

(c) 6.15 tells us that we can calculate the cardinal functions of any measurable algebra if we know (i) the values $\tau(A \restriction a)$ of its homogeneous relative algebras; (ii) the cardinals $\mathrm{add}(N)$, $\mathrm{cf}(N)$; and (iii) the values of $\mathrm{cf}([\kappa]^\omega)$, $n(A_\kappa)$ and $d(A_\kappa)$ for $\kappa \geq \omega$. In the next theorem I shall say what I know about these.

(d) For another version of the ideas leading to 6.13(a)–(b), see FREMLIN [6].

6.17. THEOREM. *Let λ and κ be infinite cardinals.*

(a) $\mathrm{add}(N)$ *is a regular cardinal lying between* ω_1 *and* 2^ω (*inclusive*).

(b) $\mathrm{add}(N) \le \mathrm{cf}(\mathrm{cf}(N)) \le \mathrm{cf}(N) \le 2^\omega$.

(c) (i) *If* $\lambda \le \kappa$, *then* $\mathrm{cf}([\lambda]^\omega) \le \mathrm{cf}([\kappa]^\omega)$;

(ii) *if* $\mathrm{cf}(\kappa) > \omega$, *then* $\mathrm{cf}([\kappa]^\omega) = \max(\kappa, \sup_{\omega \le \zeta < \kappa} \mathrm{cf}([\zeta]^\omega))$;

(iii) *if* $\omega = \mathrm{cf}(\kappa) < \kappa$, *then* $\mathrm{cf}([\kappa]^\omega) > \kappa$ *and* $\mathrm{cf}(\mathrm{cf}([\kappa]^\omega)) > \omega$;

(iv) *if* $\omega_1 \le \kappa < \omega_\omega$, *then* $\mathrm{cf}([\kappa]^\omega) = \kappa$;

(v) *if* $\kappa \ge 2^\omega$, *then* $\mathrm{cf}([\kappa]^\omega) = \kappa^\omega$.

(d) (i) *If* $\lambda \le \kappa$, *then* $\omega_1 \le n(A_\kappa) \le n(A_\lambda) \le \mathrm{cf}(N)$;

(ii) *if* $n(A_\kappa) \le \lambda \le \kappa$, *then* $n(A_\lambda) = n(A_\kappa)$;

(iii) *if* $\kappa \ge 2^\omega$, *then* $n(A_\kappa) = n(A_{2^\omega})$.

(e) (i) *If* $\lambda \le \kappa$, *then* $\mathrm{add}(N) \le d(A_\lambda) \le d(A_\kappa) \le \max(d(A_\omega), \mathrm{cf}([\kappa]^\omega))$;

(ii) $\mathrm{cf}(d(A_\kappa)) \ge \omega_1$;

(iv) *if* $\kappa \le 2^\lambda$, *then* $d(A_\kappa) \le \lambda^\omega$;

(v) *if* $\kappa > 2^\lambda$, *then* $d(A_k) > \lambda$;

(vi) *if* $n(A_\omega) = \mathrm{cf}(N)$, *then* $d(A_\omega) \le \mathrm{cf}(\mathrm{cf}(N))$.

PROOF. (a) For any non-empty family M of sets, $\mathrm{add}(M)$ is a regular cardinal (or perhaps ∞ or 2). Of course $\mathrm{add}(N) \ge \omega_1$ (because N is a σ-ideal of sets) and $\mathrm{add}(N) \le 2^\omega$ (because $|{}^\omega\{0,1\}| = 2^\omega$).

(b) Set $\zeta = \mathrm{cf}(\mathrm{cf}(N)$ and let $E \subseteq N$ be a cofinal set of cardinal $\mathrm{cf}(N)$; let $(E_\xi)_{\xi<\zeta}$ be a family of subsets of E such that $|E_\xi| < \mathrm{cf}(N)$ for each $\xi < \zeta$ and $E = \bigcup_{\xi<\zeta} E_\xi$. Because no E_ξ can be cofinal with N, we can choose for each $\xi < \zeta$ a set $n_\xi \in N$ such that $n_\xi \not\subseteq e$ for every $e \in E_\xi$. In this case $d = \bigcup_{\xi<\zeta} n_\xi$ cannot be included in any member of E. So $\mathrm{add}(N) \le \zeta$.

On the other hand,

$$\{e \in N\colon e \text{ is a Borel set in } {}^\omega\{0,1\}\}$$

is a cofinal subset of N of cardinal 2^ω, so $\mathrm{cf}(N) \le 2^\omega$.

(c)(i) Regarding λ as a subset of κ, then if E is a cofinal subset of $[\kappa]^\omega$, $\{e \cap \lambda\colon e \in E\}$ will be a cofinal subset of $[\lambda]^\omega$; so $\mathrm{cf}([\lambda]^\omega) \le \mathrm{cf}([\kappa]^\omega)$.

(ii) For each infinite ordinal $\xi < \kappa$, take a cofinal subset E_ξ of $[\xi]^\omega$ with $|E_\xi| = \mathrm{cf}([\xi]^\omega)$; then $\bigcup_{\xi<\kappa} E_\xi$ is cofinal with $[\kappa]^\omega$, so $\mathrm{cf}([\kappa]^\omega) \le \Sigma_{\xi<\kappa} |E_\xi| \le \max(\kappa, \sup_{\omega \le \zeta < \kappa} \mathrm{cf}([\zeta]^\kappa))$. On the other hand, if $E \subseteq [\kappa]^\omega$ is cofinal with $[\kappa]^\omega$, $\bigcup E = \kappa$, so $\kappa \le \max(\omega, |E|) = |E|$, and $\kappa \le \mathrm{cf}([\kappa]^\omega)$. Now (i) gives the rest.

(iii) As in (ii), $\mathrm{cf}([\kappa]^\omega) \ge \kappa$. But also (as in (b)) $\mathrm{cf}(\mathrm{cf}([\kappa]^\omega)) \ge \mathrm{add}([\kappa]^\omega) \ge \omega_1$. So $\mathrm{cf}([\kappa]^\omega) > \kappa$.

(iv) Immediate from (ii), inducing on κ.

(v) For if E is a cofinal subset of $[\kappa]^\omega$, of cardinal $\mathrm{cf}([\kappa]^\omega)$, $[\kappa]^\omega = \bigcup_{e \in E} P(e)$, so

$$\begin{aligned} \kappa^\omega = |[\kappa]^\omega| &\le \max(2^\omega, |E|) = \max(2^\omega, \mathrm{cf}([\kappa]^\omega)) \\ &= \max(2^\omega, \kappa, \mathrm{cf}([\kappa]^\omega)) = \mathrm{cf}([\kappa]^\omega) \\ &\le |[\kappa]^\omega| = \kappa^\omega . \end{aligned}$$

(d)(i) As already observed in 6.3(e), $n(A_\kappa) \le n(A_\lambda)$. Of course, $\omega_1 \le n(A_\kappa)$, while

$$n(A_\lambda) \le n(A_\omega) = \min\{|E|: E \subseteq N, \bigcup E = {}^\omega\{0,1\}\} \le \mathrm{cf}(N)$$

(6.13(c)).

(ii) See 6.3(e).

(iii) Follows from (ii).

(e)(i) As observed in 6.3(e), $d(A_\lambda) \le d(A_\kappa)$. Of course,

$$\mathrm{add}(N) \le \min\{|e|: e \subseteq {}^\omega\{0,1\}, \mu_\omega^*(e) = 1\} \le d(A_\omega) \le d(A_\lambda)\,,$$

using 6.13(d), where μ_ω is the usual measure of ${}^\omega\{0,1\}$. For the right-hand inequality, use the argument of 6.13(b). If $\mathcal{I}$ is a cofinal subset of $[\kappa]^\omega$, of cardinal $\mathrm{cf}([\kappa]^\omega)$, take inverse-measure-preserving functions $f_I: {}^\kappa\{0,1\} \to {}^\omega\{0,1\}$, for $I \in \mathcal{I}$, as before. If $e \subseteq {}^\omega\{0,1\}$ is such that $\mu_\omega^*(e) = 1$ and $|e| = d(A_\omega)$, then take $e' \subseteq {}^\kappa\{0,1\}$ such that $|e'| \le \max(|e|, |\mathcal{I}|)$ and $f_I[e'] \supseteq e$ for every $I \in \mathcal{I}$. Now if $e' \subseteq b \in \mathrm{dom}(\mu_\kappa)$, where μ_κ is the usual measure on ${}^\kappa\{0,1\}$, there is a set c belonging to the Baire σ-algebra of ${}^\kappa\{0,1\}$ such that $b \subseteq c$ and $\mu_\kappa(b) = \mu_\kappa(c)$; now there is an $I \in \mathcal{I}$ such that $c = f_I^{-1}[f_I[c]]$, in which case $e \subseteq f_I[c]$, so that

$$\mu_\kappa(b) = \mu_\kappa(c) = \mu_\omega(f_I[c]) = 1\,.$$

As b is arbitrary, $\mu_\kappa^*(e') = 1$, so that $d(A_\kappa) \le |e'| \le \max(d(A_\omega), \mathrm{cf}([\kappa]^\omega))$, as claimed.

(ii)–(iii) Let μ_κ be the usual measure on ${}^\kappa\{0,1\}$, and $\zeta = \mathrm{cf}(d(A_\kappa))$. Let $e \subseteq {}^\kappa\{0,1\}$ be a set of cardinal $d(A_\kappa)$ such that $\mu^*(e) = 1$. Express e as $\bigcup_{\xi<\zeta} e_\xi$, where $|e_\xi| < d(A_\kappa)$ for each $\xi < \kappa$. Then $\mu_\kappa^*(e_\xi) = 0$ for each $\xi < \kappa$, by 6.13(d). Accordingly, $\mathrm{add}(N_{\mu_\kappa}) \le \zeta$; which gives the results.

(iv) By Hausdorff's theorem, there is a topologically dense subset d of ${}^\kappa\{0,1\}$ with $|d| \le \lambda$. Let $e \subseteq {}^\kappa\{0,1\}$ be a set of cardinal λ^ω such that every sequence in d has a cluster point in e. Then every non-empty zero set in ${}^\kappa\{0,1\}$ must meet e. Because the usual measure μ_κ of ${}^\kappa\{0,1\}$ is completion regular, $\mu_\kappa^*(e) = 1$, and $d(A_\kappa) \le |e| \le \lambda^\omega$.

(v) From 6.13(d) we see that $d(A_\kappa)$ must be at least the topological density of ${}^\kappa\{0,1\}$ (since ${}^\kappa\{0,1\}$ is self-supporting) which is greater than λ (because the weight of ${}^\kappa\{0,1\}$ is $\kappa > 2^\lambda$).

(vi) Suppose that $n(A_\omega) = \mathrm{cf}(N)$ and that $\mathrm{cf}(\mathrm{cf}(N)) = \zeta$. Let $E \subseteq N$ be a cofinal subset of N with $|E| = \mathrm{cf}(N)$; express E as $\bigcup_{\xi<\zeta} E_\xi$, where $|E_\xi| < \mathrm{cf}(N) = n(A_\omega)$ for each $\xi < \zeta$. Then $\bigcup E_\xi$ cannot be equal to ${}^\omega\{0,1\}$ for any $\xi < \zeta$; choose $x_\xi \in {}^\omega\{0,1\} \setminus \bigcup E_\xi$ for each ξ. Set $e = \{x_\xi: \xi < \zeta\}$. Then e is not included in any member of E, so $e \notin N$, and $d(A_\omega) \le |e| \le \zeta$. □

6.18. Subject to the generalized continuum hypothesis, these results enable me to describe my functions exactly.

THEOREM [GCH]. *Let κ be an infinite cardinal.*

(a) $\gamma_\omega(A_\kappa) = \gamma_\omega^*(A_\kappa) = \mathrm{wdistr}(A_\kappa) = \omega_1$.

(b) (i) *If* $\mathrm{cf}(\kappa)=\omega$, $\pi(A_\kappa)=\kappa^+$;
(ii) *if* $\mathrm{cf}(\kappa)>\omega$, $\pi(A_\kappa)=\kappa$.
(c) $n(A_\kappa)=\omega_1$.
(d) (i) *If* $\mathrm{cf}(\kappa)=\omega$, $d(A_\kappa)=\kappa^+$;
(ii) *if* $\kappa=\lambda^+$ *and* $\mathrm{cf}(\lambda)>\omega$, *then* $d(A_\kappa)=\lambda$;
(iii) *otherwise*, $d(A_\kappa)=\kappa$.

PROOF. (a) Immediate from 6.13(a) and 6.17(a).

(b) An easy induction on κ, using 6.17(c), shows that $\mathrm{cf}([\kappa]^\omega)=\kappa$ if $\mathrm{cf}(\kappa)>\omega$, κ^+ if $\mathrm{cf}(\kappa)=\omega$. As $\mathrm{cf}(N)=\omega_1$, 6.13(b) gives the result.

(c) Immediate from 6.17(d)(i).

(d) If $\kappa=\omega$, then $d(A_\kappa)=\omega_1$, as required by (i). So take $\kappa>\omega$. If $\kappa=\lambda^+=2^\lambda$, then $\lambda\leq d(A_\kappa)\leq\lambda^\omega\leq\lambda^+$, by 6.17(e)(iv) and (v); so if also $\mathrm{cf}(\lambda)>\omega$, $d(A_\kappa)=\lambda$. If $\mathrm{cf}(\lambda)=\omega$, then $d(A_\kappa)\neq\lambda$, by 6.17(e)(ii), so $d(A_\kappa)$ must be λ^+.

If κ is an uncountable limit cardinal then $d(A_\kappa)\geq d(A_{\lambda^+})\geq\lambda$ for every infinite successor cardinal $\lambda<\kappa$; so $d(A_\kappa)\geq\kappa$; and $d(A_\kappa)\leq\kappa^\omega$ by 6.17(e)(i), (e)(iv). So if $\mathrm{cf}(\kappa)>\omega$, $d(A_\kappa)=\kappa$, while if $\mathrm{cf}(\kappa)=\omega$, then $d(A_\kappa)=\kappa^+$, because (as before) $\mathrm{cf}(d(A_\kappa))>\omega$. □

6.19. PROBLEMS. (a) The outstanding question seems to be: Can $n(A_\omega)$ be ω_ω? MILLER [1982] has shown that it is relatively consistent with ZFC to suppose that $n(A_\omega)=\omega_{\omega+1}$, while $n(A_{\omega_1})=\omega_\omega$. But it may be that $\mathrm{cf}(n(A_\omega))$ is always greater than ω. See BARTOSZYŃSKI [1985].

(b) Are there combinatorial characterizations of $n(A_\kappa)$, $d(A_\kappa)$ along the lines of that for $\pi(A_\kappa)$?

(c) Is there a proof of 6.15(a)(α) that does not pass through the ideas of 6.7–6.9? (The point here is that 6.5–6.11 develop parts of measure theory that are not needed, for instance, to prove 6.18. The question is whether they are really needed for the other results of this section.)

7. Envoi: Atomlessly-measurable cardinals

I conclude this chapter with a short section on some peripheral, but very interesting, questions associated with the measurable cardinal problem.

7.1. THE PROBLEMS. (a) The original problem of Banach and Ulam was the following. Suppose that (X, B, μ) is a totally finite measure space such that $B=P(X)$. Does it follow that $\mu(b)=\Sigma_{x\in b}\,\mu(\{x\})$ for every $b\subseteq X$?

(b) Using Maharam's theorem (3.9) it is natural to analyse the situation in the following way. (I ought to point out that the most important features of this analysis do not need Maharam's theorem, and were done in ULAM [1930]; they may also be found in JECH [1978, §27].) Let $(X, P(X), \mu)$ be a totally finite measure space with $\mu(X)>0$. Then there is a countable partition $(a_i)_{i\in I}$ of 1 in its measure algebra $A=P(X)/N_\mu$ such that $A\restriction a_i$ is homogeneous for each $i\in I$; this partition lifts to a partition $(b_i)_{i\in I}$ of X such that a_i is the image of b_i for each $i\in I$; and setting $\mu_i=\mu\restriction P(b_i)$ for each $i\in I$, we have Maharam homogeneous totally finite measure spaces $(b_i, P(b_i), \mu_i)$. Now if, for each $i\in I$, there is a point

$x_i \in b_i$ such that $\mu(\{x_i\}) > 0$, we shall have $\mu(b_i \setminus \{x_i\}) = 0$ for each i, and $\mu(b) = \Sigma\,\{\mu(\{x_i\})\colon i \in I,\ x_i \in b\} = \Sigma_{x\in b}\,\mu(\{x\})$ for every $b \subseteq X$. So the question becomes: Is there a totally finite Maharam homogeneous measure space $(X, P(X), \mu)$ such that $\mu(\{x\}) = 0$ for every $x \in X$, but $\mu(X) > 0$? And if such spaces exist, what Maharam types can they have?

7.2. REAL-VALUED-MEASURABLE CARDINALS. The first reduction (due to Ulam) is the following. Let $(X, P(X), \mu)$ be a totally finite measure space, with $\mu(X) > 0$ and $\mu(\{x\}) = 0$ for every $x \in X$, so that $\bigcup N_\mu = X \notin N_\mu$, and $\kappa = \mathrm{add}(N_\mu) < \infty$ (see 6.1). Now κ is *real-valued-measurable*, i.e. there is a totally finite measure ν, with domain $P(\kappa)$, such that $\nu(\kappa) > 0$ and $\nu(\{\xi\}) = 0$ for every $\xi < \kappa$ and $\mathrm{add}(N_\nu) = \kappa$. To see this, take a set $E \subseteq N_\mu$ such that $|E| = \kappa$ and $\bigcup E \notin N_\mu$; we may take E to be a pairwise disjoint family enumerated as $(e_\xi)_{\xi<\kappa}$; now there is a function $f\colon \bigcup E \to \kappa$ defined by writing $f(x) = \xi$ for $x \in e_\xi$. Set $\nu(c) = \mu(f^{-1}[c])$ for $c \subseteq \kappa$; it is easy to check that this has the required properties.

The function f just described is inverse-measure-preserving for the measures $\mu \restriction P(\bigcup E)$ and ν; it therefore induces a measure-preserving homomorphism from the measure algebra of $(\kappa, P(\kappa), \nu)$ to a relative subalgebra of the measure algebra of $(X, P(X), \mu)$ (2.16). Accordingly the Maharam type of $(\kappa, P(\kappa), \nu)$ is not greater than that of $(X, P(X), \mu)$.

7.3. ULAM'S DICHOTOMY. At this point the problem splits into atomic and atomless cases. Let κ be a real-valued-measurable cardinal, with an associated measure ν. If the measure algebra $P(\kappa)/N_\nu$ has an atom, then there must be a $b \in P(\kappa)\setminus N_\nu$ such that $N_\nu \cap P(b)$ is a maximal ideal of $P(b)$. Of course, $|b| = \kappa$, so (examining the dual filter of $N_\nu \cap P(b)$) κ is *two-valued-measurable* (or "measurable"); that is, $\kappa > \omega$ and there is a non-principal ultrafilter $\mathscr{F}$ on κ such that $\bigcap \mathscr{A} \in \mathscr{F}$ whenever $\emptyset \neq \mathscr{A} \in [\mathscr{F}]^{<\kappa}$. In this case $\kappa > 2^\omega$ (in fact, κ is strongly inaccessible; JECH [1978, 27.2]).

On the other hand, suppose that $P(\kappa)/N_\nu$ is not atomic. In this case there is a $b \in P(\kappa)\setminus N_\nu$ such that $(b, P(b), \nu_b)$ is an atomless probability space, where $\nu_b(c) = \nu(c)/\nu(b)$ for $c \subseteq b$. Accordingly, there is a measure-preserving homomorphism from the measure algebra A_ω of ${}^\omega\{0,1\}$ to the measure algebra $P(b)/N_{\nu_b}$ of b (3.12(b)), which is induced by an inverse-measure-preserving function $f\colon b \to {}^\omega\{0,1\}$ (2.21). This means that

$$E = \{f^{-1}[\{y\}]\colon y \in {}^\omega\{0,1\}\}$$

is a cover of b by not more than 2^ω negligible sets, so that $\kappa = \mathrm{add}(N_\nu) \leq 2^\omega$.

So we see that $P(\kappa)/N_\nu$ is either atomic or atomless. In the second case κ is called an *atomlessly-measurable* cardinal.

7.4. CONSISTENCY RESULTS. Ulam showed that a two-valued-measurable cardinal is strongly inaccessible, and that an atomlessly-measurable cardinal is weakly inaccessible and less than or equal to 2^ω (JECH [1978, 27.8]); it follows that it is relatively consistent with ZFC to suppose that neither exists, that is, 7.1(a) has a positive answer. It is widely supposed that it is consistent to suppose that both types of cardinal do exist. What is known (SOLOVAY [1971] is that if any one of the following three theories is consistent, so are the other two:

(i) ZFC + "there is a two-valued-measurable cardinal";
(ii) ZFC + "there is an atomlessly-measurable cardinal";
(iii) ZFC + "there is a set X and a proper σ-ideal M of subsets of X, containing all singletons, such that $P(X)/M$ satisfies the countable chain condition".

7.5. MAHARAM TYPES. Now let us return to the last question in 7.1(b). Suppose that $(X, P(X), \mu)$ is a Maharam homogeneous probability space of Maharam type $\lambda \geq \omega$. In this case we have a measure-preserving isomorphism f from A_λ (the measure algebra of ${}^\lambda\{0,1\}$) to $P(X)/N_\mu$, which by 2.21 is induced by an inverse-measure-preserving function $g: X \to {}^\lambda\{0,1\}$. If we define ν on $P({}^\lambda\{0,1\})$ by writing $\nu(c) = \mu(g^{-1}[c])$ for $c \subseteq {}^\lambda\{0,1\}$, then ν is an extension of the usual measure μ_λ of ${}^\lambda\{0,1\}$. Moreover, if $c \subseteq {}^\lambda\{0,1\}$, then the image of $g^{-1}[c]$ in A is of the form $f(a)$ for some $a \in A_\lambda$; there is some $b \in B_\lambda$ (the domain of μ_λ) such that a is the image of b in A_λ; and now $g^{-1}[b] \triangle g^{-1}[c] \in N_\mu$, i.e. $b \triangle c \in N_\nu$. What this means is that $P({}^\lambda\{0,1\}) = \{b \triangle n: b \in B_\lambda, n \in N_\nu\}$, while $B_\lambda \cap N_\nu = N_{\mu_\lambda}$; that is to say, that ν is obtained from μ_λ by adding new negligible sets.

Conversely, of course, if we can extend μ_λ to a measure ν with domain $P({}^\lambda\{0,1\})$ by adding negligible sets, then we obtain a Maharam homogeneous probability space $(X, P(X), \nu)$ with Maharam type λ.

An elementary remark will be relevant later. Suppose that μ_λ can be extended to a measure ν with domain $P({}^\lambda\{0,1\})$ by adding negligible sets. Then, in particular, any function $h: {}^\lambda\{0,1\} \to \mathbf{R}$ is ν-measurable. For each $q \in \mathbf{Q}$, there is a set $b_q \in B'_\lambda$, the Baire σ-algebra of ${}^\lambda\{0,1\}$, such that

$$b_q \triangle \{x \in {}^\lambda\{0,1\}: h(x) \leq q\} \in N_\nu .$$

Set $h'(x) = \sup\{q \in \mathbf{Q}: x \not\in b_q\}$ if this is defined in $\mathbf{R}$, 0 otherwise. Then h' is a B'_λ-measurable function and $h' = h$ ν-a.e. Consequently,

$$\mu_\lambda^*\{x \in {}^\lambda\{0,1\}: h(x) = h'(x)\} = 1 .$$

7.6. WHICH MAHARAM TYPES ARE POSSIBLE? Suppose that we write Λ for the class of those infinite cardinals λ such that there is a Maharam homogeneous probability space $(X, P(X), \mu)$ of Maharam type λ. Then it is certainly relatively consistent with ZFC to suppose that $\Lambda = \emptyset$, and (as I said in 7.4) many mathematicians have come to the conclusion that it is probably consistent to suppose that $\Lambda \neq \emptyset$. But are there (in ZFC) any restrictions on what Λ can be?

In fact I know of only one relevant argument. Suppose that $\lambda \geq \omega$ belongs to Λ. As remarked in 7.5, there must be a measure ν, defined on $P({}^\lambda\{0,1\})$, extending the usual measure μ_λ. Now take any cardinal α such that $\omega \leq \alpha \leq \lambda$. Regarding α as a subset of λ, we have a restriction map $f: {}^\lambda\{0,1\} \to {}^\alpha\{0,1\}$ which is inverse-measure-preserving for μ_λ and μ_α. Define μ on $P({}^\alpha\{0,1\})$ by setting $\mu(c) = \nu(f^{-1}[c])$ for every $c \subseteq {}^\alpha\{0,1\}$. This is a measure extending μ_α. It is not to be expected that μ can be obtained from μ_α by adding negligible sets, so we do not have an exact expression for the Maharam type β of $({}^\alpha\{0,1\}, P({}^\alpha\{0,1\}), \mu)$. But we do have bounds for β in terms of α, as follows. First,

$$\beta = \tau(P({}^{\alpha}\{0,1\})/N_{\mu}) \le |P({}^{\alpha}\{0,1\})| = 2^{2^{\alpha}}.$$

Second, the identity map on ${}^{\alpha}\{0,1\}$ is inverse-measure-preserving for μ and μ_{α}, so A_{α} is embedded as a complete subalgebra of $A = P({}^{\alpha}\{0,1\})/N_{\mu})$, and $\tau(A \restriction a) \ge \alpha$ for every $a \in A^{+}$. So if we take some $a \in A^{+}$ such that $A \restriction a$ is homogeneous, we shall have $\gamma = \tau(A \restriction a) \in \Lambda$, and $\alpha \le \gamma \le 2^{2^{\alpha}}$.

In particular, if $\Lambda \ne \emptyset$ (i.e. there is an atomlessly-measurable cardinal), then Λ contains some cardinal between ω and $2^{2^{\omega}}$ inclusive.

The argument above shows that Λ is unbounded (i.e. is a proper class) iff μ_{α} can be extended to $P({}^{\alpha}\{0,1\})$ for every cardinal α. This is called the *product measure extension axiom* (PMEA), and Kunen has given a strong plausibility argument for its consistency; see FLEISSNER [1984]. Using 4.12 it is easy to show that PMEA implies that any Radon measure on a space X can be extended to a measure defined on $P(X)$.

7.7. A FINAL PROBLEM. The chief puzzle concerning Λ is: Is it possible for ω to belong to Λ?

This has been around for some time. As with measurable cardinals in general, it may be that the difficulty is that it has a positive answer, but that there is no relative consistency proof within ZFC. Solovay's argument for his equiconsistency theorem, quoted in 7.4, does not seem to help; his methods necessarily produce Maharam types greater than 2^{ω}.

There is one thing that can be said. If $\omega \in \Lambda$, then (as remarked in 7.5) every $h: {}^{\omega}\{0,1\} \to \mathbf{R}$ is equal, on a set of μ_{ω}-outer measure 1, to a Borel function h'. As far as I am aware, this may already be inconsistent with ZFC.

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Keywords: Boolean algebra, measure algebra, measure space, Maharam's theorem, liftings, cardinal functions.

MOS subject classification: primary 28A60; secondary 06E05, 06E10, 06E15, 03G05, 46G15.

Section D

LOGICAL QUESTIONS

This Section contains five chapters treating various relationships of Boolean algebras with logic.

Chapter 23, Decidable extensions of the theory of Boolean algebras, by Martin Weese, gives survey of this topic. Some of the highlights are a short proof of Rabin's theorem on the decidability of the elementary theory of Boolean algebras with a sequence of ideals, and an exposition of applications to universal algebra.

Chapter 24, Undecidable extensions of the theory of Boolean algebras, by Martin Weese, is devoted to the opposite direction from the above chapter; in particular, it is shown that the theory of Boolean algebras with quantification over ideals is undecidable, as is the theory of Boolean algebras enriched by the Magidor–Malitz quantifier (under CH).

Chapter 25, Recursive Boolean algebras, by J.B. Remmel, discusses effective versions of several concepts in the theory of Boolean algebras, including the lattices of r.e. subalgebras and ideals, and recursive automorphisms of recursive Boolean algebras.

Chapter 26, Lindenbaum–Tarski algebras, by Dale Myers, surveys results about characterizations of the Boolean algebras of sentences for various important first-order theories.

Chapter 27, Boolean-valued models, by Thomas Jech, gives a brief discussion of the relationships between properties of a complete Boolean algebra and properties of the Boolean-valued model of set theory which it determines.

CHAPTER 23

Decidable Extensions of the Theory of Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

We denote the class of Boolean algebras by BA. In Part I of this Handbook, Section 18, it was shown that the elementary theory of Boolean algebras, Th(BA), is decidable. Thus, it is interesting to consider languages which are more expressive than the elementary language of Boolean algebras and to ask whether the arising theories are still decidable. The possible extensions can be divided into three classes:

(1) Extend the language by adding some new functions or predicates.

(2) Extend the language by adding new quantifiers.

(3) Use fragments of second-order logic.

Let us give some examples.

(1) Extend the language L of the elementary theory of Boolean algebras by a new unary predicate P. If we add to the axioms of BA a sentence saying that P defines an ideal, then the corresponding theory is decidable. If we add a sentence saying that P defines a subalgebra, then the corresponding theory is undecidable.

(2) Extend L by adding a quantifier expressing "there are infinitely many". Then the corresponding theory is decidable.

(3) Allow quantification over finite sets. Then the arising weak second-order theory is undecidable.

This chapter and the next are devoted to decidable and undecidable extensions of the theory of Boolean algebras. In this chapter we start with decidable extensions. We use the following two methods for showing decidability: (a) the method of interpretation and (b) the method of dense systems.

For proofs of the next two theorems see CHANG and KEISLER [1973], BAUDISCH ET AL. [1980] or BARWISE [1977, §8].

Let L be any language; then let Sent L denote the set of sentences of L.

The method of interpretation:

0.1. THEOREM. *Let T, T_0 be two theories with the sets* Sent L *and* Sent L_0 *of sentences, respectively. Assume that T_0 is decidable. If there is a recursive function* $*$: Sent $L \to$ Sent L_0 *such that for each $\varphi \in$ Sent L, $\varphi \in T$ iff $\varphi^* \in T_0$, then T is also decidable.*

We show that some extensions of the elementary theory of Boolean algebras can be interpreted in the monadic second-order theory of countable orders. This theory was shown to be decidable by Rabin. Moreover, using methods from universal algebra it can be shown that for every algebra in a residually small discriminator variety there is a canonical representation by sheaves and this representation can be used to construct an interpretation in $\mathrm{Th}^2(Lo_\omega)$, the monadic theory of countable orders. Thus, every residually small discriminator variety has a decidable theory.

REMARK. The method of interpretation can also be used to obtain undecidability results. This will be done in the next chapter.

The method of dense systems:

Let T be a theory in a language L, M a set of models of T. Then M is *dense* for T iff for each $\varphi \in \text{Sent } L$, if $T \cup \{\varphi\}$ is consistent, then there is $A \in M$ with $A \models \varphi$.

0.2. THEOREM. *Let T be a theory in a language L. If there exists a sequence $(A_n)_{n<\omega}$ of models of T which is dense for T and such that*

(i) $\{(\ulcorner\varphi\urcorner, n)\colon \varphi \in \text{Sent } L, A_n \models \varphi\}$ *is recursively enumerable, and*

(ii_1) *T is recursively enumerable, or*

(ii_2) *there exists a recursive function $f\colon \text{Sent } L \to \omega$ such that for each $\varphi \in \text{Sent } L$, if $T \cup \{\varphi\}$ has a model, then $A_n \models \varphi$ for some $n < f(\varphi)$.*

Then T is decidable.

This method is used, for instance, to show that extensions by generalized quantifiers remain decidable.

1. Describing the languages

We describe the languages which will be used in what follows. Let L be the language of Boolean algebras as described in Section 18 of Part I: i.e. L has the non-logical symbols $+$, $\cdot$, $-$, 0, 1. Form L and Sent L denote the set of formulas and the set of sentences of L, respectively.

By L^2 we denote the usual second-order extension of L. The atomic formulas of L^2 are all expressions of the form $s = t$, $X = Y$, $s \in X$, where s and t are terms of L and X and Y are set variables. Form L^2, the set of formulas of L^2, is the least class that contains all atomic formulas and is closed under the usual logical connectives and existential and universal quantification for both sorts of variables. Sent L^2 denotes the set of all sentences of L^2.

Let $A \in \text{BA}$, $S \subseteq P(A)$, $\varphi \in \text{Sent } L^2$. We consider (A, S) to be a two-sorted structure and define $(A, S) \models \varphi$ as in two-sorted predicate calculus. If $S = P(A)$, then $(A, S) \models \varphi$ iff φ is valid in A with respect to the usual monadic second-order logic. Taking a suitable subsystem of $P(A)$ we obtain some interesting logics.

(1) Let $S = P_{<\omega}(A)$. Then we obtain the weak second-order logic and write $A \models^{\text{ws}} \varphi$ instead of $(A, S) \models \varphi$. Formulas of the corresponding logic are called ws-formulas.

(2) Let S be the set of all ideals on A. Then, instead of $(A, S) \models \varphi$, we write $A \models^{\text{i}} \varphi$. Formulas of the corresponding logic are called i-formulas.

(3) Let S be the set of all ultrafilters on A. Then, instead of $(A, S) \models \varphi$, we write $A \models^{\text{u}} \varphi$. Formulas of the corresponding logic are called u-formulas.

We set

$$\text{Th}^{\text{ws}}(A) = \{\varphi \in \text{Sent } L^2\colon A \models^{\text{ws}} \varphi\}.$$

$\text{Th}^{\text{ws}}(A)$ is called the ws-theory of A. $\text{Th}^{\text{i}}(A)$ and $\text{Th}^{\text{u}}(A)$ are defined analogously and the corresponding theories are called the i-theory and the u-theory of A, respectively.

We write $A \equiv^{\mathrm{ws}} B$ iff $\mathrm{Th}^{\mathrm{ws}}(A) = \mathrm{Th}^{\mathrm{ws}}(B)$. $A \equiv^{\mathrm{i}} B$ and $A \equiv^{\mathrm{u}} B$ are defined analogously.

Let Q be a new n-ary quantifier. Then $L(Q)$ denotes the language which we obtain by adding the quantifier Q. The atomic formulas of $L(Q)$ are the same as those for L. Formulas of $L(Q)$ are defined as usual with the following additional clause:

$$\text{if } \varphi(x_0, \ldots, x_{n-1}, \vec{y}) \in \text{Form } L(Q)\,, \text{ then } Q\vec{x}\varphi(\vec{x}, \vec{y}) \in \text{Form } L(Q)\,.$$

Here we consider the following quantifiers:

(1) Let α be an ordinal. Then Q_α is a unary quantifier expressing "there are at least $\aleph_\alpha$". Satisfaction is defined with the following additional clause:

$$A \models Q_\alpha x\varphi(x) \quad \text{iff } |\{a \in A \colon A \models \varphi(a)\}| \geq \aleph_\alpha\,.$$

Quantifiers of this type are called *cardinality quantifiers*.

(2) Let α be an ordinal, $n \in \omega$ with $n \geq 2$. Then Q^n_α is an n-ary quantifier and satisfaction is defined with the following additional clause:

$A \models Q^n_\alpha x_0 \ldots x_{n-1}\varphi(x_0, \ldots, x_{n-1})$ iff there is a set $X \subseteq A$ with $|X| \geq \aleph_\alpha$ such that for each n-tuple $(a_0, \ldots, a_{n-1})$ of pairwise distinct elements of X, $A \models \varphi(a_0, \ldots, a_{n-1})$.

Q^n_0 is the n-ary Ramsey quantifier; for $\alpha > 0$ these quantifiers are called Magidor–Malitz quantifiers.

(3) Q_d is a unary quantifier expressing "there exist infinitely many pairwise disjoint elements". Satisfaction is defined with the following additional clause:

$A \models Q_d x\varphi(x)$ iff there is an infinite disjoint family $X \subseteq A$ such that $A \models \varphi(a)$ for each $a \in X$.

(4) For $n \in \omega$, $n \geq 2$, F_n is an n-ary quantifier, the n-ary sequence quantifier. Satisfaction is defined by the following additional clause:

$A \models F_n x_0 \ldots x_{n-1}\varphi(x_0, \ldots, x_{n-1})$ iff there is a sequence $(a_i)_{i<\omega}$ such that for each $i_0 < i_1 < \cdots < i_{n-1} < \omega$ we have $A \models \varphi(a_{i_0}, \ldots, a_{i_{n-1}})$.

Let L_1 be any language extending L. On the complexity of formulas we define the quantifier rank qr as follows:

$$\begin{aligned}
&\mathrm{qr}\ \varphi = 0 \quad \text{if } \varphi \text{ is atomic};\\
&\mathrm{qr}\ \neg\varphi = \mathrm{qr}\ \varphi\,;\\
&\mathrm{qr}\ \varphi \wedge \psi = \max\{\mathrm{qr}\ \varphi, \mathrm{qr}\ \psi\}\,;\\
&\mathrm{qr}\ \exists x\varphi(x) = \mathrm{qr}\ \varphi + 1\,;\\
&\mathrm{qr}\ \exists X\varphi(X) = \mathrm{qr}\ \varphi + 1\,;\\
&\mathrm{qr}\ Qx_0 \ldots x_{n-1}\varphi(x_0, \ldots, x_{n-1}) = \mathrm{qr}\ \varphi + 1\,.
\end{aligned}$$

Let $A \in \mathrm{BA}$, Q a generalized quantifier. Then we write $\mathrm{Th}^Q(A)$ for $\{\varphi \in \mathrm{Sent}\, L(Q)\colon A \models \varphi\}$. For $A, B \in \mathrm{BA}$, Q a generalized quantifier, we write $A \equiv^Q B$ iff $\mathrm{Th}^Q(A) = \mathrm{Th}^Q(B)$.

Let α, β be ordinals, $\varphi \in \mathrm{Form}\, L(A_\alpha)$. Then we write $\varphi^{\alpha/\beta}$ for the formula of $\mathrm{Form}\, L(Q_\beta)$ obtained from φ by replacing each occurrence of Q_α by Q_β. For $A, B \in \mathrm{BA}$, α, β ordinals, $n \in \omega$, we write $A\, {}^\alpha{\equiv}^\beta_n\, B$ iff for each $\varphi \in \mathrm{Sent}\, L(Q_\alpha)$ with $\mathrm{qr}\, \varphi \leq n$, $A \models \varphi$ iff $B \models \varphi^{\alpha/\beta}$ and we write $A\, {}^\alpha{\equiv}^\beta\, B$ iff for each $n \in \omega$, $A\, {}^\alpha{\equiv}^\beta_n\, B$.

We introduce one further logic, the so-called topological logic L^{t}. This logic was developed in order to investigate topological spaces. The atomic formulas of L^{t} are expressions like $x \in X$, $x = y$. If $\varphi, \psi \in \mathrm{Form}\, L^{\mathrm{t}}$, then $\neg\varphi$, $\varphi \wedge \psi$, $\exists x \varphi \in \mathrm{Form}\, L^{\mathrm{t}}$. If $\varphi \in \mathrm{Form}\, L^t$ and each occurrence of X is in the scope of an odd number of negation signs, then $\exists X(x \in X \wedge \varphi) \in \mathrm{Form}\, L^{\mathrm{t}}$ for each point variable x. Satisfaction is defined as follows:

> Let (P, τ) be a topological space with a basis τ for the open sets. Then $(P, \tau) \models \exists X(a \in X \wedge \varphi)$ iff there is an $X' \in \tau$ with $a \in X'$ and $(P, \tau) \models \varphi(X')$.

It is easily seen that satisfaction does not depend on the special choice of τ. Two Boolean algebras A and B are said to be t-equivalent, $A \equiv^{\mathrm{t}} B$, iff Ult A and Ult B satisfy the same L^{t}-sentences. For more information about topological logic the reader may consult FLUM and ZIEGLER [1980].

We shall use the fact that weak second-order logic satisfies the following Löwenheim–Skolem Theorem: For each ws-theory T in a countable language and each model A of T, there exists a countable structure B with $A \equiv^{\mathrm{ws}} B$.

KEISLER [1970] showed that there is an axiomatization of the logic with Q_1. This implies that $\mathrm{Th}^{Q_1}(\mathrm{BA})$ is recursively enumerable.

Approximating families, games, characteristics and configurations

It is necessary to have a good tool for checking whether two Boolean algebras A and B can be distinguished by some sentence φ with $\mathrm{qr}\, \varphi \leq n$ (for a given logic). Such tools are described in the sequel. The approximating families (introduced by HEINDORF [1981]) are closely connected with the Vaught relations (see Section 5 of Part I). Games were introduced by EHRENFEUCHT [1961] to characterize elementary equivalence (see also MONK [1976, §26]. These games were generalized by several authors for nonelementary logics (see WEESE [1980] for further information).

We start with the definition of splittings. Let $A \in \mathrm{BA}$, $a \in A$. A *splitting* is a finite sequence $(a_i)_{i<n}$ of pairwise disjoint elements with $\Sigma_{i<n}\, a_i = a$. A splitting of 1_A is said to be a splitting of the algebra. A splitting is a *binary* splitting if the sequence has length two.

(1) Approximating families for first-order logic. Let K be the class of all pairs (A, a) with $A \in \mathrm{BA}$, $a \in A$. A family $(R_n\colon n < \omega)$ of symmetric binary relations on elements of K is *e-approximating* iff

(i) $(A, a)R_0(B, b)$ implies $a = 0$ iff $b = 0$;

(ii) $(A, a)R_{n+1}(B, b)$ implies that for every binary splitting (a_0, a_1) of a there is a binary splitting (b_0, b_1) of b such that $(A, a_i)R_n(B, b_i)$ for each $i \leq 1$.

By results of FRAISSÉ [1954] and EHRENFEUCHT [1961] we have:

1.1. LEMMA. *Let $(R_n\colon n < \omega)$ be e-approximating. If, $A, B \in \mathrm{BA}$, $(a_i)_{i<k}$ a splitting of A, $(b_i)_{i<k}$ a splitting of B and $(A, a_i)R_n(B, b_i)$ for each $i < k$, then*

$$A \models \varphi(\vec{a}) \quad \textit{iff } B \models \varphi(\vec{b})$$

for every $\varphi(\vec{x}) \in \mathrm{Form}\ L$ with $\mathrm{qr}\ \varphi \leq n$.

As is easily seen, among all e-approximating families there is a largest one $(R_n^*\colon n < \omega)$ (i.e. if $(R_n\colon n < \omega)$ is any e-approximating family, then $R_n \subseteq R_n^*$ for each $n < \omega$). For each $n \in \omega$ we define an equivalence relation $\mathrm{typ}(n)$ on BA by

$$(A, B) \in \mathrm{typ}(n) \quad \text{iff } (A, 1_A)R_n^*(B, 1_B)\,.$$

Then we have

1.2. LEMMA. *For each $n \in \omega$, $A, B \in \mathrm{BA}$,*

$$(A, B) \in \mathrm{typ}(n) \quad \textit{iff } A \equiv_n B$$

and there are only finitely many equivalence classes in $\mathrm{typ}(n)$.

It is also possible to use games to check the n-elementary equivalence of structures. In the special case of Boolean algebras there exists a very simple version of these games, which we describe in the following.

Let $A, B \in \mathrm{BA}$, $n \in \omega$. Then the game $H^n(A, B)$ is played by two players and has n stages. The result before the first stage is the pair $(1_A, 1_B)$. Let (a_i, b_i) be the result before the $(i+1)$th stage $(1 \leq i < n)$. Then Player I starts and chooses one of the two structures, say B, and in this structure some $b < b_i$. Now Player II has to choose some $a < a_i$. Finally, Player I decides which of the two pairs (a, b) and $(a_i - a, b_i - b)$ is the result before the $(i+1)$th stage. The pair (a_n, b_n) is the result of the game and Player II wins if either a_n and b_n are both equal to zero or both are different from zero. Otherwise, Player I wins. We write $A \sim_n B$ if Player II can always win, otherwise we write $A \not\sim_n B$.

1.3. LEMMA. *For each $A, B \in \mathrm{BA}$, $n < \omega$,*

$$A \sim_n B \quad \textit{iff } A \equiv_n B\,.$$

(2) Approximating families for weak second-order logic. Let K be the class of all pairs (A, a) with $A \in \mathrm{BA}$, $a \in A$. A family $(R_n\colon n < \omega)$ of symmetric binary relations on elements of K is ws-approximating iff

(i) $(A, a)R_0(B, b)$ implies $a = 0$ iff $b = 0$;

(ii) $(A, a)R_{n+1}(B, b)$ implies that for every splitting $(a_i)_{i<k}$ of a there is a splitting $(b_i)_{i<k}$ of b such that $(A, a_i)R_n(B, b_i)$ for each $i<k$.

PALJUTIN [1971] showed

1.4. LEMMA. *Let $(R_n\colon n<\omega)$ be* ws-*approximating. If $A, B \in \mathrm{BA}$, $(a_i)_{i<k}$ a splitting of A, $(b_i)_{i<k}$ a splitting of B and $(A, a_i)R_n(B, b_i)$ for each $i<k$, then*

$$A \models \varphi(\vec{a}) \quad \textit{iff } B \models \varphi(\vec{b})$$

for every $\varphi(\vec{x}) \in \mathrm{Form}\ L^{\mathrm{ws}}$ with $\mathrm{qr}\ \varphi \leq n$.

There is a game for checking ws-equivalence. But as we do not need it in what follows we do not describe it here.

(3) Approximating families for Q_α. Let K be the class of all pairs (A, a) with $A \in \mathrm{BA}$, $a \in A$. A family $(R_n\colon n<\omega)$ of symmetric binary relations on elements of K is Q_α-approximating iff

(i) $(A, a)R_0(B, b)$ implies $a=0$ iff $b=0$;

(ii) $(A, a)R_{n+1}(B, b)$ implies that for every binary splitting $(a_i)_{i<2}$ of a there is a binary splitting $(b_i)_{i<2}$ of b such that $(A, a_i)R_n(B, b_i)$ for each $i<2$;

(iii) $(A, a)R_{n+1}(B, b)$ implies that for every $X \subseteq A \restriction a$ with $|X| \geq \aleph_\alpha$ there is $Y \subseteq B \restriction b$ with $|Y| \geq \aleph_\alpha$ such that for every $b_0 \in Y$ there is $a_0 \in X$ with $(A, a_0)R_n(B, b_0)$.

It was shown in WEESE [1977a] (see also BAUDISCH ET AL. [1980] or BARWISE and FEFERMAN [1985] Chapter VII) that

1.5. LEMMA. *Let $(R_n\colon n<\omega)$ be a Q_α-approximation. If $A, B \in \mathrm{BA}$, $(a_i)_{i<k}$ a splitting of B with $(A, a_i)R_n(B, b_i)$ for each $i<k$, then*

$$A \models \varphi(\vec{a}) \quad \textit{iff } B \models \varphi(\vec{b})$$

for every $\varphi(\vec{x}) \in \mathrm{Form}\ L(Q_\alpha)$ with $\mathrm{qr}\ \varphi(\vec{x}) \leq n$.

For $L(Q_\alpha)$ the downward Löwenheim–Skolem Theorem holds true, i.e. if $A \in \mathrm{BA}$, then there is a $B \in \mathrm{BA}$ with $|B| \leq \aleph_\alpha$ and $A \equiv^\alpha B$.

Among all Q_α-approximating families there is a largest one $(R_n^*\colon n<\omega)$ (i.e. if $(R_n\colon n<\omega)$ is any Q_α-approximating family, then $R_n \subseteq R_n^*$ for each $n<\omega$). For each $n<\omega$ we define an equivalence relation $\mathrm{typ}(n, \alpha)$ on BA by

$$(A, B) \in \mathrm{typ}(n, \alpha) \quad \text{iff } (A, 1_A)R_n^*(B, 1_B)\,.$$

Then we have

1.6. LEMMA. *For each ordinal α, each $n<\omega$, $A, B \in \mathrm{BA}$,*

$$(A, B) \in \mathrm{typ}(n, \alpha) \quad \textit{iff } A \equiv_n^\alpha B$$

and there are only finitely many equivalence classes in $\mathrm{typ}(n, \alpha)$.

There is a game similar to the game for the elementary case which is suitable for Q_α.

Let $A, B \in \mathrm{BA}$, $n < \omega$, α an ordinal. Then the game $H^n(A, B, \alpha)$ is played by two players and has n stages. The result before the first stage is the pair $(1_A, 1_B)$. Let (a_i, b_i) be the result before the ith stage $(1 \leq i \leq n)$. Then Player I starts and first decides whether it is an $\exists$-stage or a Q-stage. If he decides for an $\exists$-stage, then the two players play as in a stage of the elementary game $H^n(A, B)$. Now assume that Player I decided for a Q-stage. Player I chooses one of the two structures, say A, and in this structure a set $X \subseteq A \upharpoonright a_i$ with $|X| \geq \aleph_\alpha$. Now Player II has to choose a set $Y \subseteq B \upharpoonright b_i$ with $|Y| \geq \aleph_\alpha$. Now Player I chooses some $b \in Y$ and after Player I has chosen, Player II chooses some $a \in X$. At last Player I decides which of the two pairs (a, b) and $(a_i - a, b_i - b)$ is the result before the $(i+1)$th stage. The pair (a_n, b_n) is the result of the game and Player II wins if either a_n and b_n are both equal to zero or both are different from zero. Otherwise, Player I wins. We write $A \sim_n^\alpha B$ if Player II can always win, otherwise we write $A \not\sim_n^\alpha B$. We have

1.7. LEMMA. *For each ordinal α, each $n < \omega$, $A, B \in \mathrm{BA}$,*

$$A \sim_n^\alpha B \quad \textit{iff } A \equiv_n^\alpha B .$$

(4) Characteristics for Q_d. Let $A \in \mathrm{BA}$. For every $a \in A$ and every $n < \omega$ we define the n-characteristic of a, $D(n, a)$, recursively with two auxiliary functions D_0 and D_1 in the following way:

$$D(0, a) = \begin{cases} 0 & \text{if } a = 0 ; \\ 1 & \text{otherwise} ; \end{cases}$$

$$D_0(n+1, a) = \{(D(n, b), D(n, a-b)) \colon b \leq a\} ;$$

$$D_1(n+1, a) = \{(D(n, b), D(n, a-b)) \colon \text{there is an infinite disjoint family } X \subseteq A \upharpoonright a \text{ such that } b \in X \text{ and for all } c \in X, D(n, c) = D(n, b) \text{ and } D(n, a-c) = D(n, a-b)\} ;$$

$$D(n+1, a) = (D_0(n+1, a), D_1(n+1, a)) .$$

HEINDORF [1981] showed:

1.8. LEMMA. *Let $A, B \in \mathrm{BA}$. If $(a_i)_{i<k}$ is a splitting of A and $(b_i)_{i<k}$ is a splitting of B and if for each $i < k$, $D(n, a_i) = D(n, b_i)$, then*

$$A \models \varphi(\vec{a}) \quad \textit{iff } B \models \varphi(\vec{b})$$

for every Q_d-formula $\varphi(\vec{x})$ with qr $\varphi(\vec{x}) \leq n$.

It is possible to define a game which can be used to check Q_d-equivalence.

For every n-characteristic α there is a Q_d-formula $\delta_\alpha(x)$ such that for every $A \in \mathrm{BA}$ and every $a \in A$, $A \models \delta_\alpha(a)$ iff $D(n, a) = \alpha$. The construction of $\delta_\alpha(x)$ is immediately implied by the definition of $D(n, a)$. In what follows we shall treat expressions like $D(n, a) = \alpha$ themselves as Q_d-formulas.

The following two lemmas are essential for showing the decidability of $\mathrm{Th}^{Q_d}(\mathrm{BA})$.

1.9. LEMMA. *The set*

$$\{(\alpha_0, \ldots, \alpha_{k-1}; \varphi(x_0, \ldots, x_{k-1})) \colon \varphi(\vec{x}) \in \text{Form } L(Q_d),\ \text{qr } \varphi(\vec{x}) \le n, \textit{ for every } A \in \mathrm{BA} \textit{ and every splitting } (a_i)_{i<k} \textit{ of } A, \textit{ if } D(n, a_i) = \alpha_i \textit{ for each } i < k, \textit{ then } A \models \varphi(\vec{a})\}$$

is recursive.

For k a natural number, let

$$E_k = \{e \colon e \colon \{0, \ldots, k-1\} \to \{+1, -1\}\}\,.$$

1.10. LEMMA. *There is an effective procedure assigning a number m and a set of n-characteristics* $\{\alpha_i^e \colon i < m, e \in E_k\}$ *to every Q_d-formula* $\varphi(x_0, \ldots, x_{k-1})$ *with* $\text{qr } \varphi(\vec{x}) \le n$ *such that*

$$\mathrm{BA} \models \varphi(\vec{x}) \leftrightarrow \bigvee \left\{ D\Big(n, \prod_{j<k} e(j) x_j\Big) = \alpha_i^e \colon i < m, e \in E_k \right\}.$$

(5) Configurations for L^{u}. Let K be the class of all sequences $(A, p_0, \ldots, p_{k-1})$ where $A \in \mathrm{BA}$ and each p_i is either $\emptyset$ or an ultrafilter on A. A family $(R_n \colon N < \omega)$ of symmetric binary relations on K (the class of u-configurations) is called u-approximating iff

(i) $(A, p_0, \ldots, p_{k-1}) R_0 (B, q_0, \ldots, q_{k-1})$ implies: A is trivial iff B is trivial, $p_i = p_j$ iff $q_i = q_j$ and $p_i = \emptyset$ iff $q_i = \emptyset$;

(ii) $(A, p_0, \ldots, p_{k-1}) R_{n+1} (B, q_0, \ldots, q_{k-1})$ implies that (a) for every $p_k \in \mathrm{Ult}\, A$ there is a $q_k \in \mathrm{Ult}\, B$ such that $(A, p_0, \ldots, p_{k-1}, p_k) R_n (B, q_0, \ldots, q_{k-1}, q_k)$, and (b) for every $a \in A$ there is a $b \in B$ such that $(A \restriction a, p_0 \restriction a, \ldots, p_{k-1} \restriction a) R_n (B \restriction b, q_0 \restriction b, \ldots, q_{k-1} \restriction b)$ and $(A \restriction -a, p_0 \restriction -a, \ldots, p_{k-1} \restriction -a) R_n (B \restriction -b, q_0 \restriction -b, \ldots, q_{k-1} \restriction -b)$, where $p \restriction c$ denotes $\{d \le c \colon d \in p\}$.

HEINDORF [1981] showed

1.11. LEMMA. *Let $(R_n \colon n < \omega)$ be a* u-*configuration. Let $A, B \in \mathrm{BA}$, $(a_i)_{i<k}$ a splitting of A, $(b_i)_{i<k}$ a splitting of B. Let $p_i \in \mathrm{Ult}\, A$ and $q_i \in \mathrm{Ult}\, B$ for each $i < 1$. If for every $i < k$,*

$$(A \restriction a_i, p_0 \restriction a_i, \ldots, p_{l-1} \restriction a_i) R_n (B \restriction b_i, q_0 \restriction b_i, \ldots, q_{l-1} \restriction b_i),$$
then $A \models \varphi(\vec{a}, \vec{p})$ *iff* $B \models \varphi(\vec{b}, \vec{q})$

for every u-*formula* φ *with* $\text{qr } \varphi \le n$.

2. The monadic theory of countable linear orders and its application to the theory of Boolean algebras

Here we show the decidability of the monadic theory of countable orders. We need this result to show that the theory of countable Boolean algebras with quantification over ideals is decidable. On the other hand, HEINDORF [1984] showed that the theory of all Boolean algebras with quantification over ideals is undecidable (see Chapter 24 of this Handbook).

EHRENFEUCHT [1959] proved the decidability of the first-order theory of linear orders (LÄUCHLI and LEONARD [1966]). RABIN [1969] proved a very strong and difficult result, namely the decidability of the monadic theory of two successor functions. This result implies the decidability of the monadic theory of countable orders. A direct proof of this result can be found in SHELAH [1975]; there the reader can also find further information on the history of linear orders. We are obliged to S. Shelah for leaving to us a simplified version of his original proof.

FEFERMAN and VAUGHT [1959] showed that the first-order theory of sum, product and generalized products of models depends only on the first-order theories of the models. These results cannot be generalized for monadic second-order theories. We show that there is a sentence φ which is valid in the direct product of two linear orders iff the two orders are isomorphic.

Let A, B be linear orders and let M denote their direct product. For $p, q \in M$ we set

$$[p, q] = \{r \in M: p \leq r \leq q\} .$$

A set $X \subseteq M$ is called a *line segment* if the elements of X are pairwise comparable and for $p, q \in X$ with $p < q$, $[p, q] \subseteq X$. X is a line segment iff there are $p \in A$, $Y \subseteq B$ with $X = \{(p, r): r \in Y\}$ or there are $q \in B$, $Y \subseteq A$ with $X = \{(r, q): r \in Y\}$. Maximal line segments are called *axes*. Now assume that there is an isomorphism $f: A \to B$. Then $D = \{(p, f(p)): p \in A\}$ is a set of pairwise comparable elements and D intersects each axis in exactly one point. Let φ express that there is a set D of pairwise comparable elements such that D intersects each axis in exactly one point. Then it is easily seen that φ is as desired.

But, using the natural generalization of Ehrenfeucht games (see EHRENFEUCHT [1971] or WEESE [1980]) it can be shown that the monadic theory of a generalized sum depends only on the monadic theories of the summands. The results of FEFERMAN and VAUGHT [1959] can be applied directly replacing M by $(M \cup P(M), M, \in)$.

In what follows, L always denotes a first-order language with a finite number of symbols and L^2 denotes the corresponding monadic second order language. For R a symbol of L, m_R denotes its arity. For $\vec{a}$ a finite sequence of elements, $\mathrm{lh}(\vec{a})$ denotes the length of $\vec{a}$, $\vec{a} = (a_0, \ldots, a_{\mathrm{lh}(a)-1})$. We usually write $\vec{a} \in A$ instead of $\vec{a} \in A^{\mathrm{lh}(\vec{a})}$.

Let K be a class of L-models. Then we set

$$K^m = \{(M, \vec{P}): \vec{P} \in P(M), \mathrm{lh}(\vec{P}) = m\} ;$$
$$K = \bigcup_{m<\omega} K^m .$$

Let M be an L-model, $\vec{P} \in P(M)$ with $\mathrm{lh}(\vec{P}) = l$, $\vec{a} \in M$, with $\mathrm{lh}(\vec{a}) = m$, $n < \omega$, $\vec{k}$ a sequence of natural numbers with $\mathrm{lh}(\vec{k}) \geq n$, Φ a finite set of formulas of L^2. We introduce sentences of L^2 which are very similar to the Scott sentences (see, for instance, KEISLER [1971]). These sentences will be used to describe fragments of theories of linear orders. We define $\mathrm{th}^n_{\vec{k}}((M, \vec{P}, \vec{a}), \Phi)$ by induction on n:

$$\begin{aligned}
&\mathrm{th}^0_{\vec{k}}((M, \vec{P}, \vec{a}), \Phi) := \bigwedge \{\varphi(X_{i_0}, \ldots, x_{j_0}, \ldots) : \\
&\quad \varphi(X_0, \ldots, x_0, \ldots) \in \Phi \text{ and } M \models \varphi(P_{i_0}, \ldots, a_{j_0}, \ldots)\} ; \\
&\mathrm{th}^{n+1}_{\vec{k}}((M, \vec{P}, \vec{a}), \Phi) := \mathrm{th}^n_{\vec{k}}((M, \vec{P}, \vec{a}), \Phi) \\
&\quad \wedge \bigwedge \{\exists x_m \ldots x_{m+\vec{k}(n)-1}\, \mathrm{th}^n_{\vec{k}}((M, \vec{P}, \vec{a}\frown\vec{b}), \Phi) : \\
&\quad \vec{b} \in M, \mathrm{lh}(\vec{b}) = \vec{k}(n)\} \\
&\quad \wedge \forall x_m \ldots x_{m+\vec{k}(n)-1} \bigvee \{\mathrm{th}^n_{\vec{k}}((M, \vec{P}, \vec{a}\frown\vec{b}), \Phi) : \\
&\quad \vec{b} \in M, \mathrm{lh}(\vec{b}) = \vec{k}(n)\} .
\end{aligned}$$

We define $\mathrm{Th}^n_{\vec{k}}((M, \vec{P}), \Phi)$ by induction on n:

$$\begin{aligned}
&\mathrm{Th}^0_{\vec{k}}((M, \vec{P}), \Phi) := \mathrm{th}^1_{\vec{k}}((M, \vec{P}), \Phi) ; \\
&\mathrm{Th}^{n+1}_{\vec{k}}((M, \vec{P}), \Phi) := \mathrm{Th}^n_{\vec{k}}((M, \vec{P}), \Phi) \\
&\quad \wedge \bigwedge \{\exists X_l \ldots X_{l+\vec{k}(n+1)-1}\, \mathrm{Th}^n_{\vec{k}}((M, \vec{P}\frown\vec{Q}), \Phi) : \\
&\quad \vec{Q} \in P(M), \mathrm{lh}(\vec{Q}) = \vec{k}(n+1)\} \\
&\quad \wedge \forall X_l \ldots X_{l+\vec{k}(n+1)-1} \bigvee \{\mathrm{Th}^n_{\vec{k}}((M, \vec{P}\frown\vec{Q}), \Phi) : \\
&\quad \vec{Q} \in P(M), \mathrm{lh}(\vec{Q}) = \vec{k}(n+1)\} .
\end{aligned}$$

REMARKS. (1) We always assume $\vec{k}(i) \geq 1$ for any $i < \mathrm{lh}(\vec{k})$, and $\vec{k}(0) \geq m_R + 1$ for any symbol R of L.

(2) If we write $\vec{k}(i)$ for $i \geq \mathrm{lh}(\vec{k})$, then we mean 1, and if we omit $\vec{k}$ we mean $(\max m_R + 1, 1, \ldots)$.

(3) If Φ is the set of atomic formulas (which we assume to be finite in this case), then we write $\mathrm{Th}^n_{\vec{k}}(M, \vec{P})$ for $\mathrm{Th}^n_{\vec{k}}((M, \vec{P}), \Phi)$.

(4) $\mathrm{th}^N_{\vec{k}}((M, \vec{P}, \vec{a}), \Phi)$ ($\mathrm{Th}^n_{\vec{k}}((M, \vec{P}), \Phi)$, respectively) can be omitted in the definition of $\mathrm{th}^{n+1}_{\vec{k}}((M, \vec{P}, \vec{a}), \Phi)$ ($\mathrm{Th}^{n+1}_{\vec{k}}((M, \vec{P}), \Phi)$, respectively).

2.1. LEMMA. *Let L be a fixed language.*

(i) *For any $M \in \mathrm{Mod}\, L$, $\vec{P} \in P(M)$, $\varphi(\vec{X}) \in \mathrm{Form}\, L^2$ (with $\mathrm{lh}(\vec{X}) = \mathrm{lh}(\vec{P})$), there is an $n < \omega$ such that from $\mathrm{Th}^n_{\vec{k}}(M, \vec{P})$, we can effectively decide whether $M \models \varphi(\vec{P})$.*

(ii) *For every n, m, $\vec{k}$ and every finite set $\Phi \subseteq \mathrm{Form}\, L^2$ there is a finite set $\{\psi_i(\vec{X}) : i < l_0\} \subseteq \mathrm{Form}\, L^2$ ($\mathrm{lh}(\vec{X}) = m$) such that for any $M \in \mathrm{Mod}\, L$, $\vec{P} \in P(M)^m$, we have:*

$\mathrm{Th}^n_{\vec{k}}((M, \vec{P}), \Phi)$ *can be computed from* $\{i < l_0 \colon M \models \psi_i(\vec{P})\}$.

PROOF. Immediate. In (i) we can take the quantifier depth of φ for n. $\square$

REMARK. In particular (ii) implies that for any $M, N \in \mathrm{Mod}\, L$, $\vec{P} \in P(M)^m$, $\vec{Q} \in P(N)^m$, we have $\mathrm{Th}^n_{\vec{k}}((M, \vec{P}), \Phi) = \mathrm{Th}^n_{\vec{k}}((N, \vec{Q}), \Phi)$ iff $\{i < l_0 \colon M \models \psi_i(\vec{P})\} = \{i < l_0 \colon N \models \psi_i(\vec{Q})\}$.

The following two lemmas are obvious:

2.2. LEMMA. *Let L be a fixed language, $m, n < \omega$, $\vec{k} \in \omega^m$, $\vec{l} \in \omega^n$, $M \in \mathrm{Mod}\, L$, $\vec{P} \in P(M)^m$, Φ a finite set of formulas from L^2. If $\vec{k}(0) \geq \vec{l}(0)$, $1 = p_0 < p_1 < \cdots < p_n \leq m$ and for $0 < i < n$, $\vec{l}(i) \leq \Sigma_{p_{i-1} \leq j < p_i} \vec{k}(j)$, then we can effectively compute $\mathrm{Th}^n_{\vec{l}}((M, \vec{P}), \Phi)$ from $\mathrm{Th}^m_{\vec{k}}((M, \vec{P}), \Phi)$.*

2.3. LEMMA. *Let L be a fixed language. For every n, $\vec{k}$, $\vec{l}$ and every finite set $\Phi \subseteq \mathrm{Form}\, L^2$ we can compute m such that for every $M \in \mathrm{Mod}\, L$, $\vec{P} \in P(M)$, from $\mathrm{Th}^m_{\vec{l}}((M, \vec{P}), \Phi)$ we can effectively compute $\mathrm{Th}^n_{\vec{k}}((M, \vec{P}), \Phi)$.*

Let L be the first-order language of linear orders and let L_0, L_1, L_2 be extensions of L by unary predicates. A *condition* τ is a pair $(\varphi(x), \psi(x))$ with $\varphi(x) \in \mathrm{Form}\, L_0^2$, $\psi(x) \in \mathrm{Form}\, L_1^2$.

Let $N \in \mathrm{Mod}\, L_1$, and for each $i \in N$ let $M_i \in \mathrm{Mod}\, L_0$, where we assume that the M_i's are pairwise disjoint. An element $a \in \bigcup_{i \in N} M_i$, say $a \in M_{i_0}$, satisfies the condition $\tau = (\varphi(x), \psi(x))$ if $M_{i_0} \models \varphi(a)$, $N \models \psi(i_0)$.

Let σ be a function which assigns a condition to each unary predicate of L_2. Then $M = \Sigma^\sigma_{i \in N} M_i$ is an L_2-model whose universe is $\bigcup_{i \in N} M_i$, for each unary predicate $R \in L_2$,

$$R^M = \{a \in M \colon a \text{ satisfies } \sigma(R)\}$$

and

$$<^M = \{(a, b) \colon \text{if } a \in M_i,\ b \in M_j, \text{ then either } N \models i < j \text{ or otherwise } N \models i = j \text{ and } M_i \models a < b\}\,.$$

Let $\Phi(\sigma)$ ($\Psi(\sigma)$) be the set of all formulas $\varphi \in \mathrm{Form}\, L_0^2$ ($\psi \in \mathrm{Form}\, L_1^2$) that appear in the $\sigma(R)$ ($R \in L_2$). If $L_0 = L_1 = L_2 = L$, then $\sigma = \emptyset$ and $\Sigma^{\emptyset}_{i \in N} M_i$ is just the ordered sum (which we denote by $\Sigma_{i \in N} M_i$ in what follows).

2.4. LEMMA. *For any n, m, $\vec{k}$, σ, $N \in \mathrm{Mod}\, L_1$, $M^l_i \in \mathrm{Mod}\, L_0$, $\vec{P}^l_i \in P(M^l_i)^m$ (for each $i \in N$, $l \leq 1$), if $\mathrm{Th}^n_{\vec{k}}((M^0_i, \vec{P}^0_i), \Phi(\sigma)) = \mathrm{Th}^n_{\vec{k}}((M^1_i, \vec{P}^1_i), \Phi(\sigma))$ for each $i \in N$, then $\mathrm{Th}^n_{\vec{k}}(\Sigma^\sigma_{i \in N} (M^0_i, \vec{P}^0_i)) = \mathrm{Th}^n_{\vec{k}}(\Sigma^\sigma_{i \in N} (M^1_i, \vec{P}^1_i))$.*

2.5. THEOREM. *For any σ, n, m, $\vec{k}$ we can effectively find an $\vec{r}$ such that: If $M = \Sigma^\sigma_{i \in N} M_i$, $t_i = \mathrm{Th}^n_{\vec{k}}((M_i, \vec{P}_i), \Phi(\sigma))$ and $Q_t = \{i \in N \colon t_i = t\}$, $\mathrm{lh}(\vec{P}_i) = m$, then we can effectively compute $\mathrm{Th}^n_{\vec{k}}(m, \bigcup_{i \in N} \vec{P}_i)$ from $\mathrm{Th}^n_{\vec{r}}(N, \ldots, Q_t, \ldots), \psi(\sigma))$.*

For K a class of models we set

$$\mathrm{Th}^n_{\vec{k}}(K, \Phi) = \{\mathrm{Th}^n_{\vec{k}}(M, \Phi) : M \in K\} .$$

For σ, K_0, K_1 let $\mathrm{Cl}^\sigma(K_0, K_1)$ be the minimal class K such that
(i) $K_0 \subseteq K$;
(ii) if $N \in K_1$, $M_i \in K$ for each $i \in N$, then $\Sigma_{i\in N} M_i \in K$.

2.6. LEMMA. *Suppose σ, n, m, $\vec{k}$ are given, $\mathrm{lh}(\vec{k}) = n$, $L_0 = L_2$, $\Phi(\sigma)$ and $\Psi(\sigma)$ are finite sets of atomic formulas. Then there is an $\vec{r}$ such that, for every $K_0 \subseteq \mathrm{Mod}\, L_0$, $K_1 \subseteq \mathrm{Mod}\, L_1$, from $\mathrm{Th}^n_{\vec{r}}(K_1^{\vec{r}(n+1)})$, $\mathrm{Th}^n_{\vec{k}}(K_0^m)$, we can effectively compute $\mathrm{Th}^n_{\vec{h}}(K^m)$, where $K = \mathrm{Cl}^\sigma(K_0, K_1)$.*

PROOF. Choose $\vec{r}$ which relates to σ, n, m, $\vec{k}$ just as in Theorem 2.5. Let T be the set of formally possible $\mathrm{Th}^n_{\vec{k}}(M, \vec{P})$ for $M \in \mathrm{Mod}\, L_0$, $\mathrm{lh}(\vec{P}) = m$, and we set $\vec{r}(n+1) = |T|$. Let $T = \{t(0), \ldots, t(p-1)\}$ (so $p = |T| = \vec{r}(n+1)$).

Now, if $M = \Sigma^\sigma_{i\in N} (M_i, \vec{P}_i)$, $Q_l = \{i \in N\colon\ t_i = t_l\}$, $\mathrm{lh}(P_i) = m$, then from $\mathrm{Th}^n_{\vec{r}}(N, \ldots, A_l, \ldots)_{l<p}$ we can effectively compute $\mathrm{Th}^n_{\vec{k}}(M, \bigcup_{i\in N} P_i)$ and denote it by $g(t)$.

By induction on l, define $T_l \subseteq T$. Let $T_0 = T^n_{\vec{k}}(K_0^m)$ and if T_q is defined let T_{q+1} be the union of T_q with the set of $t \in T$ which satisfy the following condition:

(*) there is a $t^* \in \mathrm{Th}^n_{\vec{r}}(K_1^{\vec{r}(n+1)})$ such that $t = g(t^*)$, and if t^* implies that Q_l is not empty, then $t(l) \in T_q$.

Clearly, if $t^* = \mathrm{Th}^n_{\vec{r}}(N, \ldots, Q_l, \ldots)$, then we can compute $\mathrm{Th}^0_{\vec{r}}(N, \ldots, Q_l, \ldots)$ from t^* and hence we know whether $Q_1 \neq \emptyset$. Clearly, $T_0 \subseteq T_1 \subseteq \cdots \subseteq T$, so for some $q \leq p$ $(= |T|)$, $T_q = T_{q+1}$. We set

$$K_* = \{M \in K\colon \text{for every } \vec{P} \in P(M)^m,\ \mathrm{Th}^n_{\vec{k}}(M, \vec{P}) \in T_q\} .$$

Clearly, $\mathrm{Th}^n_{\vec{k}}(K_*^m) \subseteq T_q$ and we can effectively find T_q. Now, if $N \in K_1$, $M_i \in K_*$ for each $i \in N$, $M = \Sigma^\sigma_{i\in N} M_i$, then for any $\vec{P} \in P(M)^m$, $\mathrm{Th}^n_{\vec{k}}(M, \vec{P}) \in T_{q+1} = T_q$, and $M \in K_*$. This implies $K_* = K$. It is easily seen that $\mathrm{Th}^n_{\vec{k}}(K_*^m) \supseteq T_q$ (by induction on l we see that $\mathrm{Th}^n_{\vec{k}}(K_*^m) \supseteq T_l$ for each l. This finishes the proof. $\square$

2.7. LEMMA. *If M is a finite model, then for any n, $\vec{k}$ and finite set $\Phi \subseteq \mathrm{Form}\, L^2$, we can effectively compute $\mathrm{Th}^n_{\vec{k}}(M, \Phi)$ from M.*

Let $T(n, m, \vec{k})$ be the set of formally possible $\mathrm{Th}^n_{\vec{k}}(M, \vec{P})$, M a linear order, $\mathrm{lh}(\vec{P}) = m$.

2.8. COROLLARY. *For any n, m, $\vec{k}$, there is an $\vec{r} = \vec{r}(n, m, \vec{k})$ such that, if $P_t = \{i \in N\colon t_i = t\}$ for $t \in T(n, m, \vec{k})$, then $\Sigma_{i\in N} t_i$ can be effectively computed from $\mathrm{Th}^n_{\vec{r}}(N, \ldots, P_t, \ldots)$.*

Let I be a linear order. A *colouring* of I is a function f of unordered pairs of distinct elements of I into a finite set T of colours. We write $f(x, y)$ instead of $f(\{x, y\})$, assuming usually that $x < y$. The colouring f is additive if for $x_l < y_l < z_l \in I$ $(l \leq 1)$,

$$f(x_0, y_0) = f(x_1, y_1),\ f(y_0, z_0) = f(y_1, z_1)$$

imply

$$f(x_0, z_0) = f(x_1, z_1)\ .$$

In this case a (partial) operation $+$ is defined on T such that for $x < y < z \in I$, $f(x, z) = f(x, y) + f(y, z)$.

A set $J \subseteq I$ is *homogeneous* (for f) if there is a $t_0 \in T$ such that we have $f(x, y) = t_0$ for every $x, y \in J$, $x < y$.

2.9. LEMMA. *Let N be a linear order, $\vec{P} \in P(N)$, $n < \omega$, $\vec{k}$ a sequence of natural numbers. The colouring $f^n_{\vec{k}}$ on N is additive where*

$$f^n_{\vec{k}}(a, b) = \mathrm{Th}^n_{\vec{k}}((N, \vec{P}) \restriction [a, b))$$

and $(N, \vec{P}) \restriction [a, b)$ denotes the submodel of $(N, \vec{P})$ with the universe $[a, b)$.

PROOF. Follows from Lemma 2.4. □

Lemma 2.1(i) implies

2.10. LEMMA. *If, for any $n, \vec{k}$, we can effectively compute $\mathrm{Th}^n_{\vec{k}}(K)$, then the monadic theory of K is decidable and vice versa.*

2.11. THEOREM. *The monadic theory of the class of finite linear orders is decidable.*

PROOF. For $i \geq 1$, let K_i be the class of linear orders of cardinality i. Then K_i has (up to isomorphism) only one element. Then, by Lemma 2.7, we can compute $\mathrm{Th}^n_{\vec{k}}(K_i)$. We set $K = \mathrm{Cl}(K_1, K_2)$. Then, by Lemma 2.6, for every $n, \vec{k}$, we can compute $\mathrm{Th}^n_{\vec{k}}(K)$. But K is just the class of finite linear orders. Thus, the monadic theory of the class of finite linear orders is decidable. □

2.12. THEOREM. *The monadic theory of ω is decidable.*

PROOF. By induction on n, we show that we can compute $\{\mathrm{Th}^n_{\vec{k}}(\omega, \vec{P}) \colon \vec{P} \in P(\omega)^m\}$ for every $\vec{k}, m$ simultaneously.

For $n = 0$ this is easy (see the definition of $\mathrm{Th}^0_{\vec{k}}(\omega, \vec{P})$). Suppose now that we have done it for $n - 1$ and we shall do it for $n, m, \vec{k}$. For every $\vec{l}$ we have

$$\begin{aligned}\mathrm{Th}^n_{\vec{l}}(\omega) = \bigwedge \{\exists X_0 \ldots X_{\vec{l}(n)-1}\, \mathrm{Th}^{n-1}_{\vec{l}}(\omega, \vec{P})\colon \vec{P} \in \omega\,,\\ \mathrm{lh}(\vec{P}) = \vec{l}(n)\}\\ \wedge \forall X_0 \ldots X_{\vec{l}(n)-1} \bigvee \{\mathrm{Th}^{n-1}_{\vec{l}}(\omega, \vec{P})\colon \vec{P} \in \omega\,,\\ \mathrm{lh}(\vec{P}) = \vec{l}(n)\}\,.\end{aligned}$$

Thus, $\mathrm{Th}^a_{\vec{l}}(\omega)$ can be computed for every $\vec{l}$, especially for $\vec{r} = \vec{r}(n, m, \vec{k})$ (see Corollary 2.8).

For any $M = (\omega, P_0, \ldots, P_{m-1})$ we can find an $f^n_{\vec{k}}$-homogeneous set $\{a_i\colon i < \omega\}$ $(a_i < a_{i+1})$. We set

$$\begin{aligned}t &= \mathrm{Th}^n_{\vec{k}}((\omega, \vec{P}) \restriction [0, a_0))\,;\\ s &= \mathrm{Th}^n_{\vec{k}}((\omega, \vec{P}) \restriction [a_0, a_1))\,.\end{aligned}$$

So we have

$$\begin{aligned}\mathrm{Th}^n_{\vec{k}}(\omega, \vec{P}) &= \mathrm{Th}^n_{\vec{k}}((\omega, \vec{P}) \restriction [0, a_0)) + \sum_{i<\omega} \mathrm{Th}^n_{\vec{k}}((\omega, \vec{P}) \restriction [a_i, a_{i+1}))\\ &= t + \sum_{i<\omega} s\,.\end{aligned}$$

As $\mathrm{Th}^n_{\vec{r}}(\omega)$ is known, by Corollary 2.8 we can compute $\mathrm{Th}^n_{\vec{r}}(M, \vec{P})$ from s, t. Now, for any $s, t \in \mathrm{Th}^n_{\vec{k}}(K^m_{\mathrm{fin}})$, there is an $(\omega, \vec{P})$ such that $\mathrm{Th}^n_{\vec{k}}(\omega, \vec{P}) = t + \Sigma_{i<\omega}\, s$. As we know $\mathrm{Th}^n_{\vec{k}}(K^m_{\mathrm{fin}})$, we are finished. □

2.13. THEOREM. *The monadic theory of* $\mathbb{Q}$ (*the rationals with their order*) *is decidable.*

PROOF. By induction on n, we shall compute $\{\mathrm{Th}^n_{\vec{k}}(Q, \vec{P})\colon \vec{P} \in P(Q)^m\}$ for every $\vec{k}$, m simultaneously.

For $n = 0$ this is easy (see the definition of $\mathrm{Th}^0_{\vec{k}}(Q, \vec{P})$). Suppose now that we have done it for $n - 1$ and we shall do it for n, m, $\vec{k}$. For every $\vec{l}$ we have

$$\begin{aligned}\mathrm{Th}^n_{\vec{l}}(Q) = &\bigwedge \{\exists X_0 \ldots X_{\vec{l}(n)-1}\, \mathrm{Th}^{n-1}_{\vec{l}}(Q, \vec{P})\colon \vec{P} \in Q,\ \mathrm{lh}(\vec{P}) = \vec{l}(n)\}\\ &\wedge \forall X_0 \ldots X_{\vec{l}(n)-1} \bigvee \{\mathrm{Th}^{n-1}_{\vec{l}}(Q, \vec{P})\colon \vec{P} \in Q,\ \ \mathrm{lh}(\vec{P}) = \vec{l}(n)\}\,.\end{aligned}$$

Thus, by the induction hypothesis we can compute $\mathrm{Th}^n_{\vec{k}}(Q)$ and even $\mathrm{Th}^n_{\vec{k}}(Q, \vec{P})$ with $\vec{P} = (P_0, \ldots, P_{m-1})$, each P_i is $\emptyset$ or a dense subset of Q, the P_i's are pairwise disjoint and $\bigcup \{P_i\colon i < m\} = Q$. By the Theorem of Cantor, $(Q, P^0_0, \ldots, P^0_{m-1}) \cong (Q, P^1_0, \ldots, P^1_{m-1})$ if for each $i < m$, $j \le 1$, $P^j_i = \emptyset$ or a dense subset of Q, $P^0_i = \emptyset$ iff $P^1_i = \emptyset$, the P^0_i's (and the P^1_i's) are pairwise disjoint, $\bigcup_{i<m} P^j_i = Q$.

Let T_0 be the set of all such $\mathrm{Th}^n_{\vec{k}}(Q, \vec{P})$. So we can compute T_0. We also can compute the (finite) set of such formally possible theories $\mathrm{Th}^n_{\vec{k}}(Q, \vec{P})$.

Let m_0 denote the number of formally possible such theories. We close T_0

under the following operations (all models M have the form $(I, <, P_0, \dots, P_{m-1})$, with $(I, <) \cong (Q, <)$, $P_i \subseteq I$):

(i) if $\mathrm{Th}_k^n(M_0)$, $\mathrm{Th}_k^n(M_1) \in T$, then $\mathrm{Th}_k^n(M_0 + M_1) \in T$;

(ii) if $\mathrm{Th}_k^n(M_0)$, $\mathrm{Th}_k^n(M_1) \in T$, $N = (\{x\}, <, P_0, \dots, P_{m-1})$ ($<$ – order, $P_i \subseteq \{x\}$), then $\mathrm{Th}_k^n(M_0 + N + M_1) \in T$;

(iii) if for each $n < \omega$, $\mathrm{Th}_k^n(M_n) \in T$, $N_n = (\{a_n\}, <, P_0, \dots, P_{m-1})$, $M = \Sigma_{n<\omega} M_n$ (or $M = \Sigma_{n<\omega} M_n + N_n$), then $\mathrm{Th}_k^n(M) \in T$ and $\mathrm{Th}_k^n(M^*) \in T$ (where M^* is the order which is converse to M);

(iv) if $r < m_0$, $Q_0, \dots, Q_{r-1}$ is a partition of Q into dense sets, for $x \in Q$, $\mathrm{Th}_k^n(M_x^{\mathrm{int}}) \in T$ (where M_x^{int} is obtained from M_x by deleting the first and the last element if they exist), if $i < r$, $x, y \in Q_i$, then $\mathrm{Th}_k^n(M_x^{\mathrm{int}}) = \mathrm{Th}_k^n(M_y^{\mathrm{int}})$, M_x has a first (last) element iff M_y has a first (last) element, then $\mathrm{Th}_k^n(\Sigma_{x \in Q} M_x) \in T$.

Now each closure is computable (see Lemma 2.6) and takes no more than the number of formally possible such theories (as $\mathrm{Th}_k^n(Q, \dots, \{x\colon \mathrm{Th}_k^n(M_x) = t\}, \dots)_t$ is computable by induction hypothesis). Now it suffices to prove that $\mathrm{Th}_k^n(M) \in T$ for each M.

Let M be given. On M we define a binary relation E_0 by

$$aE_0b \quad \text{iff } a = b \text{ or } a < b \text{ and } a \le a_1 < b_1 \le b \text{ implies}$$
$$\mathrm{Th}_k^n(M \restriction (a_1, b_1)) \in T.$$

We define a binary relation E by

$$aEb \quad \text{iff } aE_0b \text{ or } bE_0a\,.$$

Then E is a convex equivalence relation. On $\{a/E\colon a \in M\}$ there is a natural ordering.

Claim 1. For each $a \in M$, $\mathrm{Th}_k^n(M \restriction (a/E)^{\mathrm{int}}) \in T$, if $(a/E)^{\mathrm{int}} \neq \emptyset$.

Proof of the claim. Let $I = \{b\colon aEb, b > a\}$. By symmetry (and (ii)) it suffices to show that $\mathrm{Th}_k^n(M \restriction I^{\mathrm{int}}) \in T$ if $I^{\mathrm{int}} \neq \emptyset$. Choose $a = a_0 < a_1 < a_2 < \cdots$, $a_i \in I$ for each $i < \omega$, $\{a_i\colon i < \omega\}$ is unbounded in I. By Ramsey's Theorem w.l.o.g. $s = \mathrm{Th}_k^n(M \restriction \{x_1\})$, $t = \mathrm{Th}_k^n(M \restriction (a_l, a_{l+1}))$ for $l = 1, 2, \dots$. Hence by (iii), $\mathrm{Th}_k^n(M \restriction \bigcup_{l \ge 1} (a_l, a_{l+1}]) \in T$ and by (ii), $\mathrm{Th}_k^n(M \restriction I^{\mathrm{int}}) \in T$.

Claim 2. For each $a, b \in M$, if $|\{c/E\colon a \le c \le b\}| \le 2$, then aEb.

This follows immediately by (i) and (ii). So the order M/E is dense or has one element; in the latter case we are finished. Otherwise, for a/E define

$$t_{a/E} = (\mathrm{Th}_k^n(M \restriction (a/E)^{\mathrm{int}}, s^0_{a/E}, s^1_{a/E})$$

where $s^0_{a/E}$ is $\emptyset$ if a/E has no first element and $\mathrm{Th}_k^n(M \restriction \{\text{first element of } a/E\})$ otherwise. $s^1_{a/E}$ similar for the last element.

There is a non-trivial interval I of M/E in which each t appears densely or not at all. So we can apply the operation in (iv) and obtain that t/E has just one element, which contradicts the assumption.

Now each countable linear order M is isomorphic to a suborder of Q and so the monadic theory of countable orders can be interpreted in the monadic theory of Q. Thus, we obtain:

2.14. THEOREM. $\mathrm{Th}^2(Lo_\omega)$ *is decidable.*

In the rest of this section we show how we can interpret $\mathrm{Th}^{\mathrm{i},Q_0}(\mathrm{BA}_\omega)$ in $\mathrm{Th}^2(Lo_\omega)$, where $\mathrm{Th}^{\mathrm{i},Q_0}(\mathrm{BA}_\omega)$ denotes the theory of countable Boolean algebras with quantification over ideals and the quantifier Q_0 (binding individual variables only).

Let M be a linear order with first element. Then Intalg M denotes the interval algebra of M (see Example 1.11 of Part I). It is known that for each countable Boolean algebra B there is a countable linear order M_B with first element such that $B \cong \mathrm{Intalg}\, M_B$ (see Corollary 15.10 of Part I). Thus, we can use the interval algebras for our interpretation.

Each element a of Intalg M is given in the form $[a_0, b_0) \cup [a_1, b_1) \cup \cdots \cup [a_{n-1}, b_{n-1})$ (or $[a_0, b_0) \cup \cdots \cup [a_{n-1}, b_{n-1}) \cup [a_n)$ (where $[a_n) = \{c \in M : a_n \leq c\}$)), with $a_0 < b_0 < a_1 < b_1 < \cdots < a_{n-1} < b_{n-1}$ (or $a_0 < b_0 < \cdots < a_{n-1} < b_{n-1} < a_n$). Thus, each $a \in \mathrm{Intalg}\, M$ can be coded by the two finite sets $\{a_0, a_1, \ldots, a_{n-1}\}$ (or $\{a_0, \ldots, a_n\}$) and $\{b_0, \ldots, b_{n-1}\}$.

A subset X of M is finite if each $Y \subseteq X$ with $Y \neq \emptyset$ has a least and a last element. Thus, we set

$$\begin{aligned}\mathrm{Fin}\, X := (\forall Y \subseteq X)[Y \neq \emptyset \to (\exists x_0 \in Y)(\exists x_1 \in Y)\\ (\forall y \in Y)(x_0 \leq y \wedge y \leq x_1)]\,.\end{aligned}$$

Let X be a subset of M. We define two sets $Z_0(X)$ and $Z_1(X)$ by

$$\begin{aligned}Z_0(X) &= \{x \in M : x \in X \text{ and } (\forall y < x)([y, x) \not\subseteq X)\}\,;\\ Z_1(X) &= \{x \in M : x \not\in X \text{ and } (\exists y < x)([y, x) \subseteq X)\}\,.\end{aligned}$$

If $a = [a_0, b_0) \cup \cdots \cup [a_{n-1}, b_{n-1})$ with $a_0 < b_0 < \cdots < a_{n-1} < b_{n-1}$, then $Z_0(a) = \{a_0, \ldots, a_{n-1}\}$ and $Z_1(a) = \{b_0, \ldots, b_{n-1}\}$. Let $X \subseteq M$; then $X \in \mathrm{Intalg}\, M$ iff it satisfies

$$\begin{aligned}\mathrm{El}(X) := (\forall x \in X)[\exists y_0[y_0 \leq x \wedge [y_0, x) \subseteq X \wedge \forall z(z < y_0 \to [z, x) \not\subseteq X]\\ \wedge [\forall z(x < z \to [x, z) \subseteq X \vee \exists y_1(x < y_1 \wedge [x, y_1) \subseteq X\\ \wedge \forall z(y_1 < z \to [x, z) \not\subseteq X))]]\\ \wedge \mathrm{Fin}\, Z_0(X) \wedge \mathrm{Fin}\, Z_1(X)\,.\end{aligned}$$

Let I be an ideal of Intalg M. Then we define a binary relation $E(I)$ by

$$a_0 E(I) a_1 \quad \text{iff } a_0 = a_1 \vee \bigvee_{i \leq 1} [a_i < a_{1-i} \wedge \exists b(a_{1-i} < b \wedge [a_i, b) \in I]\,.$$

Then it is easily seen that $E(I)$ is a convex equivalence relation on M and we can define an ideal $I(E)$ for each convex equivalence relation E on M too; the ideal which is generated by all $[a, b)$ with $a < b \wedge \forall c(a < c \wedge c < b \rightarrow aEc)$.

For each ideal I of Intalg M and for each convex equivalence relation E on M we have $I(E(I)) = I$ and $E(I(E)) = E$.

Thus, the ideals of Intalg M are in one-to-one correspondence with the convex equivalence relations. Now we show that we can quantify over convex equivalence relations in $\mathrm{Th}^2(Lo)$ and thus also in $\mathrm{Th}^2(Lo)$.

Let M be a linear order, E a convex equivalence relation on M. Then E can be coded by two subsets P, Q of M: Let P be a set of representatives for E, i.e.

$$(\forall x \in M)(\exists! y)(y \in P \wedge xEy)$$

(where $\exists! x$ means "there is exactly one x"). Clearly, such a set P exists. We set

$$Q = \{x \in M\colon (\exists y \in P)(yEx \wedge y > x)\}\,.$$

Then we have

$$\begin{aligned} x_0 E x_1 \quad \text{iff } (\exists z \in P) \bigwedge_{i \leq 1} [x_i = z \vee (x_i > z \wedge (x_i, z] \subseteq Q) \\ \vee (x_i < z \wedge [z, x_i) \cap Q = \emptyset]\,. \end{aligned}$$

On the other hand, two sets P and Q code a convex equivalence relation iff $P \cap Q = \emptyset \wedge Z_2(Q) \subseteq P$ with $Z_2(Q) = \{a \in M\colon (\forall b \in M)(a < b \rightarrow [a, b) \cap Q \neq \emptyset)\}$. Thus, we can code ideals and also have the possibility to quantify over ideals. It remains to show that also quantification with Q_0 can be translated.

Let $\varphi(x)$ be a formula in the language of $\mathrm{Th}^{\mathrm{i},Q_0}(\mathrm{BA}_\omega)$ and let D be the set of all $a \in$ Intalg M that satisfy $\varphi(x)$. Then $\bigcup \{Z_0(a)\colon a \in D\}$ and $\bigcup \{Z_1(a)\colon a \in D\}$ are definable in the language of $\mathrm{Th}^2(Lo_\omega)$ and Intalg $M \models Q_0 x \varphi(x)$ iff

$$\neg\, \mathrm{Fin}\Big(\bigcup \{Z_0(a)\colon a \in D\}\Big) \vee \neg\, \mathrm{Fin}\Big(\bigcup \{Z_1(a)\colon a \in D\}\Big)\,.$$

Thus, we have shown:

2.15. THEOREM. $\mathrm{Th}^{\mathrm{i},Q_0}(\mathrm{BA}_\omega)$ *is decidable.*

REMARK. In WEESE [1986] the decidability of $\mathrm{Th}^{\mathrm{i},Q_0}(\mathrm{BA}_\omega)$ is shown by interpreting this theory in the monadic second-order theory of two successor functions.

Theorem 2.15 implies that $\mathrm{Th}^{Q_0}(\mathrm{BA}_\omega)$ is decidable too.

The downward Löwenheim–Skolem Theorem for Q_0 implies that for each $A \in \mathrm{BA}$ there is a $B \in \mathrm{BA}$ with $|B| \leq |\aleph_0|$ such that $A \,{}^0\!\!\equiv^0 B$. Thus, we have

2.16. LEMMA. $\mathrm{Th}^{Q_0}(\mathrm{BA})$ *is decidable.*

This was shown by PINUS [1976] and WEESE [1977b] independently.

At the end of this paragraph we give an example which shows that for Boolean algebras logic with Q_0 is more expressive than elementary logic. Let

$$A = \prod_{\omega}^{<\omega} 2\,; \qquad B = A \times A\,.$$

Then using the game described in Section 1, we see that $A \equiv B$. Let

$$\varphi := \exists xy(x \cdot y = 0 \wedge Q_0 z(z \leq x) \wedge Q_0 z(z \leq y))\,.$$

Then $A \models \neg\varphi$ but $B \models \varphi$. Thus, $A \,{}^0\!\not\equiv^0 B$.

3. The theories Th$^{\mathrm{u}}$(BA) and Th^{Q_d}(BA)

Now we come to the theories Th$^{\mathrm{u}}$(BA) and Th^{Q_d}(BA). The results of this section were obtained by HEINDORF [1981]. First we show that the quantifier Q_d is more expressive than Q_0.

EXAMPLE. Let for $n < \omega$, $B_n = \operatorname{Intalg} \omega^n$. We define by induction the formulas $\varphi_n(x_n)$ as follows:

$$\varphi_0(x_0) := Q_d y(y \leq x_0)\,;$$
$$\varphi_{n+1}(x_{n+1}) := Q_d x_n \varphi_n(x_n)\,.$$

Then, using the game-theoretic equivalence for Q_0 as described in Section 1 it is easy to see

$$B_i \equiv^{Q_0} B_j \quad \text{iff } i = j = 0 \text{ or } i = j = 1 \text{ or } i, j \geq 2\,.$$

On the other hand,

$$B_n \models \varphi_k(1) \quad \text{iff } k < n$$

and thus

$$B_i \equiv^{Q_d} B_j \quad \text{iff } i = j\,.$$

While the theories Th$^{\mathrm{u}}$(BA) and Th^{Q_d}(BA) seem to be rather different it will be shown hereafter that they have the same expressive power. This is due to Stone's representation theorem. To speak about ultrafilters of a Boolean algebra is the same as to speak about points of the corresponding Boolean space.

We start with a classification of points of Boolean spaces and, having obtained invariants, we can use them to find invariants for the corresponding theories Th$^{\mathrm{u}}$(BA) and Th^{Q_d}(BA). At last these invariants can be used to show that both theories are decidable. Here we use a downward Löwenheim–Skolem theorem.

3.1. *Types of points*

We start with a classification of points of Boolean spaces. The sets T_n, $n < \omega$, of *n-types* are defined recursively as follows:

$$T_0 = \{\emptyset\}, \qquad T_{n+1} = P(T_n).$$

Let X be a Boolean space (BS denotes the class of all Boolean spaces in the following), $p \in X$. Then, for every $n < \omega$, we assign its n-type $t_n(p)$ as follows:

$$t_0(p) = \emptyset;$$
$$t_{n+1}(p) = \{\rho \in T_n\text{: every neighbourhood of } p \text{ contains a point different from } p \text{ of } n\text{-type } \rho\}.$$

Treating t_n as a function from X into T_n we have

$$t_{n+1}(p) = \bigcap \{t_n[U \setminus \{p\}]\colon U \in \operatorname{Clop} X,\ p \in U\}.$$

A neighbourhood U of p is called *n-typical* on p (or simply typical) if $t_{n+1}(p) = t_n[U\setminus\{p\}]$. It is clear that each point p has n-typical neighbourhoods for each $n < \omega$.

Let X be a fixed Boolean space. For $a \in \operatorname{Clop} X$, $n < \omega$, $\rho \in T_n$ and $m \le \omega$ we set

$$K_n(\rho, a, m) = \min\{m, |\{p \in a\colon t_n(p) = \rho\}|\}.$$

$K_n(\cdot, a, m)$ denotes the corresponding function from T_n into $m + 1$. If $m = \omega$, then we sometimes simply write $K_n(\rho, a)$ instead of $K_n(\rho, a, \omega)$.

Let $n \le k < \omega$. We define $\pi^k_n\colon T_k \to T_n$ by

$$\pi^k_0(\rho) = \emptyset;$$
$$\pi^{k+1}_{n+1}(\rho) = \{\pi^k_n(\tau)\colon \tau \in \rho\}.$$

If p has k-type ρ, then it has n-type $\pi^k_n(\rho)$. We have for $n \le k < \omega$, $\rho \in T_n$,

$$K_n(\rho, a) = \sum \{K_k(\tau, a)\colon \pi^k_n(\tau) = \rho\}.$$

Let $\rho \in T_{n+1}$, $\tau \in \rho$. Then every point of $n + 1$-type ρ is an accumulation point of points of n-type τ. Thus, $K_{n+1}(\rho, a) \ne 0$ implies $K_n(\tau, a) = \omega$.

Ziegler showed (see FLUM and ZIEGLER [1980]):

3.1. THEOREM. *Two T_3-spaces P and R satisfy the same L^t-sentences iff $K_n(\cdot, P) = K_n(\cdot, R)$ for all n.*

Remember that each Boolean space with a countable basis is regular and thus a T_3-space. A *splitting* of a topological space X is a finite set of pairwise disjoint clopen sets which is a covering of X.

Let $P \in BS$, $\alpha = \{\beta_0, \ldots, \beta_{k-1}\}$ an $n+1$-type. A splitting $P_0, \ldots, P_{k-1}$ of P is called an α-splitting iff for each $i < k$, P_i contains only finitely many points of n-type β_i.

3.2. LEMMA. *Let $P \in BS$, $\alpha \in T_{n+1}$ and $t_n[P] \subseteq \alpha$. If P has no α-splitting, then $\alpha \in t_{n+1}[P]$.*

PROOF. Assume, on the contrary, that $\alpha \not\in t_{n+1}[P]$. Let $p \in P$. Then $t_{n+1}(p) \neq \alpha$. From the assumption $t_n[P] \subseteq \alpha$ we conclude that $t_{n+1}(p) \subseteq \alpha$. Consequently, there must be a $\beta(p) \in \alpha$ with $\beta(p) \not\in t_{n+1}(p)$. Then any typical neighbourhood of p will contain at most one point of n-type $\beta(p)$ (p itself may have n-type $\beta(p)$). By compactness we can find finitely many points $p_0, \ldots, p_{l-1}$ and pairwise disjoint typical neighbourhoods $U_0, \ldots, U_{l-1}$ of these points covering P. For $i < k$ put $P_i = \bigcup \{U_m \colon \beta(p_m) = \beta_i\}$. Then $P_0, \ldots, P_{k-1}$ forms a splitting of P and P_i contains only finitely many points of n-type β_i. So P has an α-splitting, which contradicts the assumption. □

3.3. LEMMA. *Let $P \in BS$, $\alpha \in T_{n+1}$. Then the following are equivalent:*

(i) *There is a $p \in P$ with $t_{n+1}(p) = t_n[P \setminus \{p\}] = \alpha$.*

(ii) *P has no α-splitting and if $t_n[P] \setminus \alpha \neq \emptyset$, then for all $W \in \operatorname{Clop} P$, either $t_n[P \setminus W] \subseteq \alpha$ and $P \setminus W$ has an α-splitting or $t_n[W] \subseteq \alpha$ and W has an α-splitting.*

PROOF. (i)→(ii). By the previous lemma, P has no α-splitting. If there is a point $q \neq p$ with $n+1$-type α, then $t_n(p) = t_n(q) \in t_n[P \setminus \{p\}] = \alpha$. Thus, $t_n[P] \setminus \alpha = \emptyset$. Otherwise, let $W \in \operatorname{Clop} P$. If $p \in W$, then $t_n[P \setminus W] \subseteq t_n[P \setminus \{p\}] = \alpha$ and $P \setminus W$ must have an α-splitting by the above lemma. If $p \not\in W$, then $p \in P \setminus W$ and W must have an α-splitting.

(ii)→(i). Suppose P has no α-splitting. If $t_n[P] \subseteq \alpha$, then there is a $p \in P$ with $t_{n+1}(p) = \alpha$. Obviously, $t_n[P \setminus \{p\}] = \alpha$ is satisfied too.

Suppose now that $t_n[P] \setminus \alpha \neq \emptyset$ and consider the collection $F = \{W \in \operatorname{Clop} P \colon t_n[P \setminus W] \subseteq \alpha,\ P \setminus W \text{ has an } \alpha\text{-splitting}\}$. Then F is an ultrafilter on $\operatorname{Clop} P$. Hence, there is some $p \in P$ such that F is a base in p. If $q \in P \setminus \{p\}$, then there is a $W \in F$ with $q \not\in W$. Thus, $t_n(q) \in t_n[P \setminus W] \subseteq \alpha$. Thus, $t_n[P \setminus \{p\}] \subseteq \alpha$. If $t_{n+1}(p)$ were not α, then there would exist $W \in \operatorname{Clop} P$ containing only finitely many points of some n-type $\beta_i \in \alpha$. But then W and $P \setminus W$ have α-splittings and thus also $P = W \cup (P \setminus W)$ has an α-splitting. This contradicts our assumption. □

The next lemma is of technical importance. It states a sufficient condition for obtaining a partition of the whole space from a system of pairwise disjoint clopen sets such that corresponding sets have the same n-types.

3.4. LEMMA. *Let $P \in BS$, $c_0, \ldots, c_{k-1}$ be pairwise disjoint clopen sets and $m \leq \omega$. Let $F_0, \ldots, F_{k-1}$ be functions $T_{n+1} \to \omega + 1$ such that*

(i) *for all $\rho \in T_n$, $i < k$, $K_n(\rho, c_i, m) = \min\{m, \Sigma \{F_i(\tau) \colon \pi_n^{n+1}(\tau) = \rho\}\}$;*

(ii) *for all $\rho \in \tau \in T_{n+1}$, $i < k$, if $F_i(\tau) \geq 1$, then $K_n(\rho, c_i) = \omega$.*

If for each $p \in P \setminus \bigcup_{i<k} c_i$ there is an $i_0 < k$ with $F_{i_0}(t_{n+1}(p)) \geq m$, then there exists a splitting $b_0, \ldots, b_{k-1}$ of P with $K_n(\rho, b_i, m) = K_n(\rho, c_i, m)$ for all $i < k$, $\rho \in T_n$ and $c_i \subseteq b_i$ for all $i < k$.

PROOF. We set

$$D = P \Big\backslash \bigcup_{i<k} c_i \,.$$

For each $p \in D$ choose an $n+1$-typical neighbourhood U_p of p. Then there are finitely many points $p_0, \ldots, p_{l-1}$ and pairwise disjoint clopen neighbourhoods $V_0, \ldots, V_{l-1}$ of these points with $V_i \leq U_i$ $(i<l)$ and $\bigcup_{i<l} V_i = D$. For each $j<l$ choose $i(j)<k$ with $F_{i(j)}(t_{n+1}(p_j)) \geq m$. We set for $i<k$,

$$b_i = c_i \cup \bigcup \{V_j \colon j<l,\, i(j)=i\} \,.$$

This provides the desired splitting. $\square$

3.5. LEMMA. (First splitting lemma.) *Let $n \in \omega$, $P, R \in BS$ with $K_{n+1}(\rho, P) = K_{n+1}(\rho, R)$ for every $\rho \in T_{n+1}$. Then, for every splitting $b_0, \ldots, b_{k-1}$ of P, there is a splitting $c_0, \ldots, c_{k-1}$ of R such that $K_n(\tau, b_i) = K_n(\tau, c_i)$ for every $i<k$, $\tau \in T_n$.*

PROOF. For $\rho \in T_{n+1}$, $i<k$ we set

$$m(\rho, i) = \begin{cases} 1 & \text{if } K_{n+1}(\rho, c_i) = \omega \,; \\ K_{n+1}(\rho, c_i) & \text{otherwise} \,. \end{cases}$$

Then $K_{n+1}(\rho, R) = K_{n+1}(\rho, P) \geq \Sigma_{i<k}\, m(\rho, i)$. Thus, for each $\rho \in T_{n+1}$, we can find pairwise distinct points $q^{\rho}_{i,j}$ $(i<k,\ j<m(\rho, i))$ *of* $n+1$-type ρ. For these points, choose pairwise disjoint typical neighbourhoods $V^{\rho}_{i,j}$ and set

$$U_i = \bigcup \{V^{\rho}_{i,j} \colon \rho \in T_{n+1},\, j<m(\rho, i)\} \,.$$

Claim. For each $\tau \in T_n$, $K_n(\tau, b_i) = K_n(\tau, U_i)$.

Proof of the Claim. First suppose $K_n(\tau, b_i) = \omega$. b_i considered as a subspace is compact; thus there is an $n+1$-type σ containing τ with $m(\sigma, i) \geq 1$. Then $q^{\sigma}_{i,1} \in U_i$ and $q^{\sigma}_{i,1}$ has $n+1$-type σ. So $K_{n+1}(\sigma, U_i) \geq 1$ and hence $K_n(\tau, U_i) = \omega$. Now suppose $K_n(\tau, b_i) < \omega$. Then

$$\begin{aligned} K_n(\tau, b_i) &= \Sigma\, \{K_{n+1}(\sigma, b_i) \colon \pi^{n+1}_n(\sigma) = \tau\} \\ &= \Sigma\, \{m(\sigma, i) \colon \pi^{n+1}_n(\sigma) = \tau\} \,. \end{aligned}$$

Hence there are exactly $K_n(\tau, b_i)$ points of n-type τ among the $q^{\rho}_{i,j}$'s. Assume there is some point $q \in U_i$ with $t_n(q) = \tau$ which is different from the $q^{\rho}_{i,j}$'s. But then q is contained in some $V^{\rho}_{i,j} \backslash \{q^{\rho}_{i,j}\}$. As $V^{\rho}_{i,j}$ was typical and we had $\tau \in \rho$, this implies $K_n(\tau, P) = \omega$; thus the claim is shown.

If $\bigcup_{i<k} U_i = R$, then we set $c_i = U_i$ and we are done. Therefore suppose $R \backslash \bigcup_{i<k} U_i \neq \emptyset$, and consider a point s of this set. Denote $t_{n+1}(s)$ by ρ. Our assumption $K_{n+1}(\rho, P) = K_{n+1}(\rho, R)$ implies that for some $i<k$, $K_{n+1}(\rho, b_i) = \omega$. Thus we can apply Lemma 3.4. $\square$

3.6. LEMMA. (Second splitting lemma.) *Let $P, R \in BS$, $p_0, \ldots, p_{k-1}$ and $q_0, \ldots, q_{k-1}$ be pairwise distinct points of P and R, respectively. If*

$K_n(\cdot, P, 2^{m+k+1}) = K_n(\cdot, R, 2^{m+k+1})$ and $t_{n+1}(p_i) = t_{n+1}(q_i)$ for each $i < k$, then for every $S \in \mathrm{Clop}\, P$ there exists a $T \in \mathrm{Clop}\, R$ such that the following conditions are satisfied:

(i) $p_i \in S$ *iff* $q_i \in T$, *for* $i < k$;
(ii) $K_n(\cdot, S, 2^{m+k}) = K_n(\cdot, T, 2^{m+k})$;
(iii) $K_n(\cdot, P \setminus S, 2^{m+k}) = K_n(\cdot, R \setminus T, 2^{m+k})$.

PROOF. For each $\alpha \in T_{n+1}$ we can choose pairwise distinct points s_i^α ($i < K_{n+1}(\alpha, S, 2^{m+k})$) and r_i^α ($i < K_{n+1}(\alpha, P \setminus S, 2^{m+k})$) of $n+1$-type α in R. Because of $K_{n+1}(\alpha, P, 2^{m+k+1}) = K_{n+1}(\alpha, R, 2^{m+k+1})$, there are enough points of $n+1$-type α in R. The choice of the s_i^α's and r_i^α's can be done in such a way that there are those q_i^α's among the s_i^α's for which $p_i \in S$ and $t_{n+1}(p_i) = \alpha$. Among the r_i^α's we can assume the q_i^α's with $t_{n+1}(q_i) = \alpha$ for which $p_i \notin S$. Choose pairwise disjoint n-typical neighbourhoods U_i^α and V_i^α of s_i^α and r_i^α, respectively. We set

$$T_0 = \bigcup \{U_i^\alpha : \alpha \in T_{n+1}, i < K_{n+1}(\alpha, S, 2^{m+k})\,;$$

$$T_1 = \bigcup \{V_i^\alpha : \alpha \in T_{n+1}, i < K_{n+1}(\alpha, P \setminus S, 2^{m+k})\,.$$

Then $p_i \in S$ iff $q_i \in T_0$ and $p_i \notin S$ iff $q_i \notin T_1$.

As in the proof of the first splitting lemma, it is easily seen that $K_n(\cdot, S, 2^{m+k}) = K_n(\cdot, T_0, 2^{m+k})$ and $K_n(\cdot, P \setminus S, 2^{m+k}) = K_n(\cdot, T_1, 2^{m+k})$. If $T_0 \cup T_1 = R$ it remains to set $T = T_0$. Thus, let us assume that $R \setminus (T_0 \cup T_1) \neq \emptyset$. Consider a point $p \in R \setminus (T_0 \cup T_1)$ and let $t_{n+1}(p) = \alpha$. Suppose that $K_{n+1}(\alpha, S) < 2^{m+k}$ and $K_{n+1}(\alpha, P \setminus S) < 2^{m+k}$. Then $K_{n+1}(\alpha, R) \geq 1 + K_{n+1}(\alpha, T_0) + K_{n+1}(\alpha, T_1) \geq 1 + K_{n+1}(\alpha, P)$. But then $K_{n+1}(\alpha, P, 2^{m+k+1}) < K_{n+1}(\alpha, R, 2^{m+k+1})$, contradicting our assumption. So $K_{n+1}(\alpha, S) \geq 2^{m+k}$ or $K_{n+1}(\alpha, P \setminus S) \geq 2^{m+k}$ and we can apply Lemma 3.4. □

The next lemma is needed to eliminate the quantifier Q_d.

3.7. LEMMA. (Third splitting lemma.) *Let $P \in BS$, $m, k < \omega$ and $h: T_m \to k+1$, where h is not identically zero. Then the following are equivalent:*

(i) *There is an infinite family X of pairwise disjoint clopen subsets of P such that for each $\alpha \in T_m$ and each $U \in X$, $K_m(\alpha, U, k) = h(\alpha)$.*
(ii) *There is a mapping $g: T_{m+1} \to k+1$ such that*
  (a) *for each $\beta \in T_{m+1}$, if $g(\beta) > 0$, then $K_{m+1}(\beta, P) = \omega$;*
  (b) *for each $\alpha \in T_m$, $h(\alpha) = \min\{k, \Sigma\{g(\beta): \pi_m^{m+1}(\beta) = \alpha\}\}$;*
  (c) *for each $\alpha \in T_m$, if there is $\beta \in T_{m+1}$ with $\alpha \in \beta$ and $g(\beta) > 0$, then $h(\alpha) = k$.*

PROOF. (i)$\to$(ii). W.l.o.g. we can assume that $K_{m+1}(\cdot, U, k)$ does not depend on $U \subseteq X$. We take $g = K_{m+1}(\cdot, U, k)$; then (a), (b) and (c) are easily checked.

(ii)$\to$(i). We fix a function g with properties (a), (b) and (c). Let $\beta \in T_{m+1}$ with $g(\beta) > 0$. Then by (a) there are a set of points $\{p_{i,j}^\beta : \beta \in T_{m+1}, i < \omega, j < g(\beta)\}$ such that $t_{m+1}(p_{i,j}^\beta) = \beta$ and a system of pairwise disjoint clopen sets $\{V_{i,j}^\beta : \beta \in T_{m+1}, i < \omega, j < g(\beta)\}$ such that $V_{i,j}^\beta$ is m-typical for $p_{i,j}^\beta$. Put

$$U_i = \bigcup \{V^{\beta}_{i,j}\colon \beta \in T_{m+1},\ j < g(\beta)\}\ ;$$
$$X = \{U_i\colon i < \omega\}\ .$$

Then X is a system of pairwise disjoint clopen sets and it remains to show that $K_m(\cdot, U_i, k) = h$.

Suppose first that $h(\alpha) = k$. Then by (b), $\Sigma\,\{g(\beta)\colon \pi^{n+1}_n(\beta) = \alpha\} \geq k$, so U_i contains, by construction, at least k points $p^{\beta}_{i,j}$ of m-type α. Hence, $\mathrm{K_m}(\alpha, \mathrm{U_i}) \geq$ k.

Suppose now that $h(\alpha) < k$. Then all points of U_i with m-type α must be among the $p^{\beta}_{i,j}$'s. For otherwise one of them would be contained in a typical neighbourhood of some $p^{\beta}_{i,j}$. But then $\alpha \in \beta$ and $g(\beta) > 0$. So by (c), $h(\alpha) = k$, which is impossible. This implies $K_m(\alpha, U_i) = \Sigma\,\{g(\beta)\colon \pi^{m+1}_m(\beta) = \alpha\} = h(\alpha)$. $\square$

REMARKS. (1) If X is a family as stated in (i), then for each $U \in X$, $K_m(\cdot, P, k) = K_m(\cdot, P\backslash U, k)$.

(2) $K_{m+1}(\beta, P) = \omega$ iff $\Sigma\,\{K_{m+2}(\gamma, P)\colon\ \beta \in \gamma\} > 0$ iff $g(\beta) > 0$. So (ii) becomes a decidable property of h if $\{\gamma \in T_{m+2}\colon K_{m+1}(\gamma, P) > 0\}$ is given.

For $A \in \mathrm{BA}$, $a \in A$ and $p \in \mathrm{Ult}\ A$ with $a \in p$, we denote by $t_n(p)$ the n-type of p in Ult A, and by $t_n[a\backslash\{p\}]$ we denote the set $\{t_{n-1}(q)\colon\ q \in \mathrm{Ult}\ A,\ a \in q$ and $q \neq p\}$.

3.8. THEOREM. *For every n-type α there are a* u*-formula $\varphi_\alpha(X)$, a Q_d-formula $\psi_\alpha(x)$ and a* ws*-formula $\vartheta_\alpha(x)$ such that for each $A \in \mathrm{BA}$, $p \in \mathrm{Ult}\ A$, the following hold true*:

(i) $A \models \varphi_\alpha(p)$ *iff* $t_n(p) = \alpha$;

(ii) $A \models \psi_\alpha(a)$ *iff* $A \models \vartheta_\alpha(a)$ *iff there is* $q \in \mathrm{Ult}\ A$ *with* $a \in q$, $t_n(q) = \alpha$ *and if* $n > 1$, *then* $t_{n-1}[a\backslash\{q\}] = \alpha$.

PROOF. For the 0-type $\emptyset$ we take $X = X$ for $\varphi_\emptyset(X)$ and $x \neq 0$ for $\psi_\emptyset(x)$ and $\vartheta_\emptyset(x)$. Suppose that for all n-types β the formulas φ_β, ψ_β and ϑ_β have been constructed and let $\alpha = \{\beta_0, \ldots, \beta_{k-1}\}$ be an $n+1$-type. From the induction hypothesis and the definition of t_{n+1} it follows immediately that we can set

$$\varphi_\alpha(X) := \forall x\Big[x \in X \rightarrow \bigwedge_{\beta\in\alpha} \exists Y(x \in Y \wedge Y \neq X \wedge \varphi_\beta(Y))\Big]$$
$$\wedge\ \exists x\Big[x \in X \wedge \forall Y(x \in Y \rightarrow X = Y \vee \bigvee_{\beta\in\alpha} \varphi_\beta(y))\Big]\ .$$

Next we show the existence of ψ_α. Let $z_\alpha(U)$ be an abbreviation for "U has an α-splitting", formally we can set

$$z_\alpha(U) := \exists y_0 \ldots y_{k-1}\Big[x = \sum_{i<k} y_i \wedge \bigwedge_{i<j<k} y_i \cdot y_j = 0$$
$$\wedge \bigwedge_{i<k} \neg Q_d z(z \leq y_i \wedge \psi_{\beta_i}(z)\Big]\ .$$

By the induction hypothesis, $t_n(x) \subseteq \alpha$ is equivalent to

$$\bigwedge_{\beta \in T_n \setminus \{\alpha\}} \neg(\exists y \leq x)\psi_\beta(y) .$$

Now, using the equivalences given in Lemma 3.3, we can set

$$\psi_\alpha(x) := \neg z_\alpha(x) \wedge [t_n(x) \subseteq \alpha \vee (\forall y \leq x)[t_n(x-y) \subseteq \alpha \\ \wedge z_\alpha(x-y)) \vee t_n(y) \subseteq \alpha \wedge z_\alpha(y))]] .$$

To obtain the formula $\vartheta_\alpha(x)$ notice that $t_n(x) \subseteq \alpha$ is also equivalent to $\bigwedge_{\beta \in T_n \setminus \{\alpha\}} \neg(\exists y \leq x)\vartheta_\beta(y)$ and $(Q_d z \leq y_i)\psi_{\beta_i}(z)$ is equivalent to

$$\exists X[(\forall z_0 \leq y_i)[((\forall z_1 \in X)(z_0 \cdot z_1 = 0) \rightarrow \neg\vartheta_{\beta_i}(z_0)] \\ \wedge (\forall z_0 \in X)(\forall z_1 \leq z_0)[\neg\vartheta_{\beta_i}(z_1) \vee (\forall z_2 \leq z_0 - z_1)\neg\vartheta_{\beta_i}(z_2]] . \quad \square$$

3.9. COROLLARY. *For any $n < \omega$, $m \leq \omega$, and any h: $T_n \rightarrow m$, there are Q_d-, u- and ws-formulas with the only free variable x representing the predicate $K_n(\cdot, x, m) = h$.*

PROOF. Easy from Theorem 3.8. $K_n(\alpha, x) = \omega$ is equivalent to $\bigvee_{\alpha \in \beta} \times K_{n+1}(\beta, x) \geq 1$; this can be used to obtain a u-expression for $K_n(\alpha, x) = \omega$. $\square$

In what follows we will treat expressions like $K_n(\cdot, x, m) = h$, $K_n(\alpha, x, m) = i$ themselves as formulas of one of the three languages. It is easily checked that $K_n(\cdot, x, m) = h$ can also be expressed as a t-sentence. Now Theorem 3.8 together with Theorem 3.1 yields:

3.10. COROLLARY. *Let $A, B \in$ BA. Then any of the three relations $A \equiv^{\mathrm{ws}} B$, $A \equiv^{\mathrm{u}} B$ and $A \equiv^{Q_d} B$ implies $A \equiv^{\mathrm{t}} B$.*

Now the downward Löwenheim–Skolem Theorem for ws-logic implies:

3.11. COROLLARY. *For every $A \in$ BA there exists a countable $B \in$ BA with $A \equiv^{\mathrm{t}} B$.*

We define a binary relation R_n on pairs (A, a) with $A \in$ BA, $a \in A$ by

$$(A, a)R_n(B, b) \quad \text{iff } K_n(\cdot, a) = K_n(\cdot, b) .$$

Then the first splitting lemma implies that R_n, $n < \omega$ is a ws-approximating family. Theorem 3.8 implies now:

3.12. COROLLARY. *Let $A, B \in$ BA with $A \equiv^{\mathrm{t}} B$. Then $A \equiv^{\mathrm{ws}} B$.*

We define a binary relation R_n on tuples $(A, p_0, \ldots, p_{k-1})$ with $A \in$ BA, $p_0, \ldots, p_{k-1} \in \mathrm{Ult}\, A \cup \{\emptyset\}$ by

$$(A, \vec{p})R_n(B, \vec{q}) \quad \text{iff } K_n(\cdot, 1_A, 2^{n+k}) = K_n(\cdot, 1_B, 2^{n+k}) \text{ and for all } i, j < k,\ p_i = \emptyset \quad \text{iff } q_i = \emptyset,\ p_i = p_j \text{ iff } q_i = q_j, \text{ and } t_n(p_i) = t_n(q_i)\,.$$

Then the second splitting lemma implies that R_n, $n < \omega$ is a u-approximating family. Theorem 3.8 implies now:

3.13. COROLLARY. *Let $A, B \in \mathrm{BA}$, with $A \equiv^{\mathrm{t}} B$. Then $A \equiv^{\mathrm{u}} B$.*

The downward Löwenheim–Skolem Theorem for ws-logic together with Corollary 3.10 implies:

3.14. COROLLARY. *For every $A \in \mathrm{BA}$ there exists a countable $B \in \mathrm{BA}$ with $A \equiv^{\mathrm{u}} B$.*

Theorem 2.15 implies that $\mathrm{Th}^{\mathrm{u}}(\mathrm{BA}_\omega)$ is decidable. Thus, together with Corollary 3.14 we obtain:

3.15. COROLLARY. $\mathrm{Th}^{\mathrm{u}}(\mathrm{BA})$ *is decidable.*

REMARK. While $\mathrm{Th}^{\mathrm{u}}(\mathrm{BA})$ is decidable, $\mathrm{Th}^{\mathrm{i}}(\mathrm{BA})$ is undecidable (see HEINDORF [1984] or Section 7 of Chapter 24 of this Handbook).

Now we come to the theory $\mathrm{Th}^{Q_d}(\mathrm{BA})$.

3.16. LEMMA. *There is a computable function $R(n, h)$ such that $R(n, K_{2n}(\cdot, a, 2^n)) = D(n, a)$ for every $A \in \mathrm{BA}$ and $a \in A$.*

PROOF. We give a recursive definition of R with the aid of two auxiliary functions R_0 and R_1 corresponding to D_0 and D_1:

$$R(0, h) = \min\{h(\emptyset), 1\}\,;$$

$$R_0(n+1, h) = \{(R(n, h_0), R(n, h_1)): \text{the u-sentence } \forall x[K_{2n+2}(\cdot, x, 2^{n+1}) = h \to (\exists y \le x)\ [K_{2n}(\cdot, y, 2^n) = h_0 \wedge K_{2n}(\cdot, x - y, 2^n) = h_1]] \text{ holds in every algebra}\}\,;$$

$$R_1(n+1, h) = \{R(n, h_0), R(n, h_1)): h_1(\alpha) = \min\{2^n, \textstyle\sum \{h(\gamma): \pi_{2n}^{2n+1}(\gamma) = \alpha \text{ for all } \alpha \in T_{2n} \text{ and there is a function } g: T_{2n+1} \to 2^n + 1 \text{ such that}$$

$$\text{(i) } \textstyle\sum \{h(\gamma): \pi_{2n}^{2n+1}(\gamma) = \beta\} > 0 \text{ if } g(\beta) > 0\,;$$
$$\text{(ii) } h_0(\alpha) = \min\{2^n, \textstyle\sum \{g(\gamma)\ \pi_{2n}^{2n+1}(\gamma) = \alpha\}\}\,;$$
$$\text{(iii) } h_0(\alpha) = 2^n \text{ if there is a } \beta \in T_{2n+1} \text{ with } \alpha \in \beta \text{ and } g(\beta) \ne 0\}\}\,;$$

$$R(n+1, h) = R_0(n+1, h) \times \{0\} \cup R_1(n+1, h) \times \{1\}\,.$$

The condition on h_0 and h_1 in the definition of R_0 is effective because of the decidability of $\mathrm{Th}^{\mathrm{u}}(\mathrm{BA})$. As the number of possible functions g is finite, the condition in the definition of R_1 is decidable too. Consequently, R will be computable. By definition, $R(0, K_0(\cdot, a, 1)) = D(0, a)$. The splitting lemmas allow an induction to show that

$$R_0(n+1, K_{2n+2}(\cdot, a, 2^{n+1})) = D_0(n+1, a)$$

and

$$R_1(n+1, K_{2n+2}(\cdot, a, 2^{n+1})) = D_1(n+1, a)\,. \quad \square$$

Together with Theorem 3.1 the above lemma implies:

3.17. LEMMA. *Let $A, B \in \mathrm{BA}$ with $A \equiv^t B$. Then $A \equiv^{Q_d} B$.*

3.18. LEMMA. *For every n-characteristic α one can find a finite set of functions $\{h_0, \ldots, h_{l-1}\}$ in an effective way such that for every $A \in \mathrm{BA}$, $a \in A$, $D(n, a) = \alpha$ iff $A \models \bigvee_{i<l} K_{2n}(\cdot, a, 2^n) = h_i$.*

3.19. THEOREM. $\mathrm{Th}^{Q_d}(\mathrm{BA})$ *is decidable.*

PROOF. Let $\varphi \in \mathrm{Sent}\, L(Q_d)$. By Lemma 1.9 we can find an equivalent disjunction of n-characteristics for some n. Lemma 3.18 and Theorem 3.8 allow us to find an equivalent u-sentence. This u-sentence can be produced in an effective way. Now, using the decidability of $\mathrm{Th}^{\mathrm{u}}(\mathrm{BA})$ we immediately obtain the decidability of $\mathrm{Th}^{Q_d}(\mathrm{BA})$. $\square$

REMARK. HEINDORF [1984] shows that there is a simple system S of u-sentences such that for every u-sentence φ, $\mathrm{BA} \models \varphi$ iff φ can be deduced from S in the two-sorted predicate calculus. Here we consider individual variables as variables of the first sort and set variables as variables of the second sort. S consists of the following sentences:

(i) axioms expressing that the individuals form a Boolean algebra;

(ii) $\forall XY[X = Y \leftrightarrow \forall x(x \in X \leftrightarrow x \in Y)]$;

(iii) $\forall X[\neg 0 \in X \wedge \forall xy(x \in X \rightarrow x + y \in X) \wedge \forall xy(x \in X \wedge y \in X \rightarrow x \cdot y \in X) \wedge \forall x(x \in X \vee -x \in X)]$;

(iv) for each formula $\varphi(x)$, $[\neg\varphi(0) \wedge \forall xy(\varphi(x) \rightarrow \varphi(x+y)) \wedge \forall xy(\varphi(x) \wedge \varphi(y) \rightarrow \varphi(x \cdot y)] \rightarrow \exists X\, \forall x(\varphi(x) \rightarrow x \in X)$.

4. Ramsey quantifiers and sequence quantifiers

In the following we show that $\mathrm{Th}^{F^2}(\mathrm{BA})$ is decidable. This implies immediately that $\mathrm{Th}^{Q_0^2}(\mathrm{BA})$ is decidable too. We show that for each $\varphi \in \mathrm{Sent}\, L(F^2)$ we can construct in an effective way a $\varphi^* \in \mathrm{Sent}\, L(Q_d)$ such that for each Boolean algebra A, $A \models \varphi$ iff $A \models \varphi^*$. Thus, together with those results which we obtained in connection with the decidability of $\mathrm{Th}^{\mathrm{u}}(\mathrm{BA})$ and $\mathrm{Th}^{Q_d}(\mathrm{BA})$, we have that the following relations between Boolean algebras are all the same:

$$\equiv^t, \quad \equiv^{\mathrm{ws}}, \quad \equiv^{\mathrm{u}}, \quad \equiv^{Q_d}, \quad \equiv^{Q_0^2}, \quad \equiv_{F^2} .$$

MOLZAN [1981] first showed the decidability of $\mathrm{Th}^{F^2}(\mathrm{BA})$. His proof was simplified by Heindorf and we give here Heindorf's proof from HEINDORF [1984]. We start with some combinatorial lemmas.

4.1. LEMMA. *Let $(A_i)_{i<\omega}$ be a sequence of subsets of an infinite set B. Then there exists an infinite set $I \subseteq \omega$ such that*
(i) *for all $i, j \in I$, $|A_i \cap A_j| \geq \omega$* or
(ii) *for all $i, j \in I$, $|B \setminus (A_i \cup A_j)| \geq \omega$.*

PROOF. We set

$$K = \{(i, j) \colon i, j \in \omega, i < j\}$$

and define a function $f\colon K \to 3$ by

$$f(i, j) = \begin{cases} 0 & \text{if } |A_i \cap A_j| \geq \omega ; \\ 1 & \text{if } |A_i \cap A_j| < \omega \text{ and } |B \setminus (A_i \cup A_j)| \geq \omega ; \\ 2 & \text{otherwise} . \end{cases}$$

By Ramsey's Theorem, there is an infinite homogeneous set $I \subseteq \omega$. Assume that $f[K \cap I^2] = \{2\}$. Choose pairwise distinct $i, j, k \in I$. Then

$$\begin{aligned} A_i \setminus A_j &= ((A_i \cap A_k) \setminus A_j) \cup (A_i \setminus (A_k \cup A_j)) \\ &\subseteq (A_i \cap A_k) > (B \setminus (A_k \cup A_j)) . \end{aligned}$$

But $A_i \cap A_k$ and $B \setminus (A_k \cup A_j)$ are finite; thus $A_i \setminus A_j$ is finite. Permuting i and j shows that $A_j \setminus A_i$ is finite too. But then $B = (A_i \cap A_j) \cup (A_i \setminus A_j) \cup (A_j \setminus A_i) \cup (B \setminus (A_i \cup A_j))$ is the union of finite sets and thus B is finite too, which contradicts the assumption. □

4.2. LEMMA. *Let $(A_i)_{i<\omega}$ be a sequence of subsets of an infinite set B and let l be a natural number such that for all $i < j < \omega$,*

$$l = |A_i \setminus A_j| < |A_j \setminus A_i| .$$

Then, for all $m > l$, there is an infinite set $I \subseteq \omega$ such that for each $i, j \in I$, $i < j$,

$$|A_j \setminus A_i| \geq m , \qquad |A_j \cap A_i| \geq m \quad \text{and} \quad |B \setminus (A_i \cup A_j)| \geq \omega .$$

PROOF. We set

$$K = \{(i, j) \colon i, j \in \omega, i < j\}$$

and define three functions f, g, h: $K \to \omega$ by

$$f(i, j) = \begin{cases} 0 & \text{if } |B\backslash(A_i \cup A_j)| < \omega \text{ ;} \\ 1 & \text{otherwise ;} \end{cases}$$

$$g(i, j) = \min\{|A_i \cap A_j|, m\} \text{ ;}$$

$$h(i, j) = \min\{|A_j\backslash A_i|, m\} \text{ .}$$

By Ramsey's Theorem, there is an infinite set $I \subseteq \omega$ which is homogeneous for f, g and h. Let $i, j, k \in I$, $i<j<k$. Then $l = |A_i\backslash A_j| = |A_i\backslash A_k|$. Furthermore,

$$A_i\backslash A_j = (A_i\backslash(A_j \cup A_k)) \cup ((A_i \cup A_k)\backslash A_j) \text{ ,}$$
$$A_i\backslash A_k = (A_i\backslash(A_j \cup A_k)) \cup ((A_i \cap A_j)\backslash A_k)$$

and thus

$$|(A_i \cap A_k)\backslash A_j| = |(A_i \cap A_j)\backslash A_k| \text{ .}$$

Assume that $n = h(i, j) < m$. Then

$$n = |A_k\backslash A_i| = |A_k\backslash A_j| \text{ .}$$

Furthermore,

$$A_k\backslash A_i = (A_k\backslash(A_i \cup A_j)) \cup ((A_j \cap A_k)\backslash A_i) \text{ ,}$$
$$A_k\backslash A_j = (A_k\backslash(A_i \cup A_j)) \cup ((A_i \cap A_k)\backslash A_j)$$

and thus

$$|(A_i \cap A_k)\backslash A_j| = |(A_j \cap A_k)\backslash A_i| \text{ .}$$

But then

$$\begin{aligned} l = |A_j\backslash A_i| &= |(A_j\backslash(A_i \cup A_k)) \cup ((A_j \cap A_k)\backslash A_i)| \\ &= |(A_j\backslash(A_i \cup A_k)) \cup ((A_i \cap A_j)\backslash A_k)| \\ &= |A_j\backslash A_k| = n \end{aligned}$$

and consequently $l = n$, a contradiction. Thus, $f(i, j) = m$ and this implies $|A_j\backslash A_i| \geq m$.

We have

$$l = |A_i\backslash A_k| = |A_j\backslash A_k| \text{ .}$$

Furthermore,

$$A_i\backslash A_k = (A_i\backslash(A_j \cup A_k)) \cup ((A_i \cap A_j)\backslash A_k)\,,$$
$$A_j\backslash A_k = (A_j\backslash(A_i \cup A_k)) \cup ((A_i \cap A_j)\backslash A_k)$$

and thus

$$|A_i\backslash(A_j \cup A_k)| = |A_j\backslash(A_i \cup A_k)|\,.$$

Assume that $n = g(i, j) < m$. But now

$$l < |A_j\backslash A_i| = |(A_j\backslash(A_i \cup A_k)) \cup ((A_j \cap A_k)\backslash A_i)|\,;$$

hence,

$$|(A_i \cap A_j)\backslash A_k| < |(A_j \cap A_k)\backslash A_i|$$

and this implies

$$|A_i \cap A_j| < |A_j \cap A_k|\,;$$

thus

$$|A_i \cap A_j| \geq m\,.$$

This contradicts our assumption $g(i, j) < m$; thus $g(i, j) = m$ and this implies $|A_i \cap A_j| \geq m$.

Assume that, for some $i, j \in I$ with $i < j$, we have $|B\backslash(A_i \cup A_j| < \omega$. Then, for any $k \in I$ with $j < k$, we have $l = |A_i\backslash A_k| = |A_j\backslash A_k|$, and thus (as $B\backslash A_k \subseteq (A_i\backslash A_k) \cup (A_j\backslash A_k) \cup (B\backslash(A_i \cup A_j)))$,

$$|B\backslash A_k| \leq 2l + |B\backslash(A_i \cup A_j)|\,.$$

So there are $k_1, k_2 \in I$ with $j < k_1 < k_2$ such that $|B\backslash A_{k_1}| = |B\backslash A_{k_2}|$. But this implies $|A_{k_1}\backslash A_{k_2}| = |A_{k_2}\backslash A_{k_1}|$, a contradiction. Thus, $|B\backslash(A_i \cup A_j)| \geq \omega$. $\square$

4.3. LEMMA. *Let $m \in \omega$, $m > 0$, f, g, h, k: $T_n \to m + 1$ such that f or g is not constantly zero. Then for every $P \in BS$, the following conditions are equivalent*:

(i) *There is a sequence $(a_i)_{i<\omega}$ of elements of* Clop P *such that for $i < j < \omega$, $\alpha \in T_n$*,

$$K_n(\alpha, a_i, a_j, m) = h(\alpha)\,;$$
$$K_n(\alpha, a_i - a_j, m) = f(\alpha)\,;$$
$$K_n(\alpha, a_j - a_i, m) = g(\alpha)\,;$$
$$K_n(\alpha, -a_i - a_j, m) = k(\alpha)\,.$$

(ii) *There are $a \in$* Clop P *and F, G: $T_{n+1} \to m + 1$ such that for every $\alpha \in T_n$ and every $\beta \in T_{n+1}$*,

(a) $K_n(\alpha, a, m) = h(\alpha)$ *and* $K_n(\alpha, -a, m) = k(\alpha)$;
(b) $f(\alpha) = \min\{m, \Sigma\{F(\gamma)\colon \pi_n^{n+1}(\gamma) = \alpha\}\}$;
$g(\alpha) = \min\{m, \Sigma\{G(\gamma)\colon \pi_n^{n+1}(\gamma) = \alpha\}\}$;
(c) *for* $\alpha \in \beta$,
if $F(\beta) \geq 1$, *then* $f(\alpha) = m$;
if $G(\beta) \geq 1$, *then* $g(\alpha) = m$;
(d) *if* $F(\beta) + G(\beta) \geq 1$, *then* $K_{n+1}(\beta, 1) = \omega$;
(e) *if* $F(\beta) < G(\beta)$, *then* $G(\beta) = m$, $K_{n+1}(\beta, a) \geq m$ *and* $K_{n+1}(\beta, -a) = \omega$;
(f) *if* $G(\beta) < F(\beta)$, *then* $F(\beta) = m$, $K_{n+1}(\beta, -a) \geq m$ *and* $K_{n+1}(\beta, a) = \omega$.

PROOF. (i)$\to$(ii). Let $(a_i)_{i<\omega}$ be a sequence with the properties states in (i). There are only finitely many mappings $T_{n+3} \to m+1$; thus, by Ramsey's Theorem we can assume that the values of $K_n(\alpha, a_i \cdot a_j, m)$, $K_n(\alpha, a_i - a_j, m)$, $K_n(\alpha, a_j - a_i, m)$ and $K_n(\alpha, -a_i - a_j, m)$ do not depend on the special choice of i and j, if only $i < j$. We start by constructing a. We set

$$c_0 = a_0 \cdot a_1\,, \qquad c_1 = -a_0 - a_1\,.$$

Then c_0 and c_1 are disjoint. To apply Lemma 3.4 it suffices to show that for each $p \in -c_0 - c_1$,

$$K_{n+3}(t_{n+3}(p), c_0) \geq m \quad \text{or} \quad K_{n+3}(t_{n+3}(p), c_1) \geq m\,.$$

Assume that $p \in -c_0 - c_1$ and $t_{n+3}(p) = \sigma$. We set

$$A_i = \{s \in a_i\colon t_{n+3}(s) = \sigma\} \quad (i < \omega)\,;$$
$$B = \{s \in P\colon t_{n+3}(s) = \sigma\}\,.$$

$p \in -c_0 - c_1 = (a_0 - a_1) + (a_1 - a_0)$ implies that $K_{n+3}(\sigma, a_0 - a_1, m) \geq 1$ or $K_{n+3}(\sigma, a_1 - a_0, m) \geq 1$. Thus, for $i < j < \omega$, $A_i \triangle A_j \neq \emptyset$ and so B contains infinitely many disjoint subsets, i.e. $|B| \geq \omega$.

Choose $I \subseteq \omega$ with the properties stated in Lemma 4.1. For $i, j \in I$, $i \neq j$, we have

$$K_{n+3}(\sigma, c_0, m) = K_{n+3}(\sigma, a_i \cdot a_j, m) = \min\{m, |A_i \cap A_j|\}$$

and

$$K_{n+3}(\sigma, c_1, m) = K_{n+3}(\sigma, -a_i - a_j, m) = \min\{m, |B \setminus (A_i \cup A_j)|\}$$

and one of them is equal to m. Thus, we can apply Lemma 3.4 and obtain the desired element a.

Now we have for each $\gamma \in T_{n+2}$,

$$K_{n+2}(\gamma, a, m) = K_{n+2}(\gamma, c_0, m) = K_{n+2}(\gamma, a_0 \cdot a_1, m)$$

and

$$K_{n+2}(\gamma, -a, m) = K_{n+2}(\gamma, c_1, m) = K_{n+2}(\gamma, -a_0 - a_1, m) .$$

This implies that for each $\alpha \in T_n$,

$$K_n(\alpha, a, m) = K_n(\alpha, a_0 \cdot a_1, m) = h(\alpha)$$

and

$$K_n(\alpha, -a, m) = K_n(\alpha, -a_0 - a_1, m) = k(\alpha) .$$

This shows (a).

For each $\beta \in T_{n+1}$ we put

$$F(\beta) = K_{n+1}(\beta, a_0 - a_1, m) ;$$
$$G(\beta) = K_{n+1}(\beta, a_1 - a_0, m) .$$

Then, for each $i < j < \omega$, $F(\beta) = K_{n+1}(\beta, a_i - a_j, m)$ and $G(\beta) = K_{n+1}(\beta, a_j - a_i, m)$. This implies immediately (b) and (c).

Let $\beta \in T_{n+1}$. To show (d) and (e) we set

$$A_i = \{p \in a: t_{n+1}(p) = \beta\} \quad (i < \omega) ;$$
$$B = \{p \in P: t_{n+1}(p) = \beta\} .$$

Now $F(\beta) + G(\beta) \geq 1$ is equivalent to $A_i \triangle A_j \neq \emptyset$. Thus, the sets A_i $(i < \omega)$ have to be pairwise distinct and this implies $|B| \geq \omega$, i.e. $K_{n+1}(\beta, 1) = \omega$. Thus, (d) is shown. To show (e), we remark that $F(\beta) < G(\beta) \leq m$ implies

$$|A_i \backslash A_j| = F(\beta) < G(\beta) \leq |A_j \backslash A_i|$$

for any $i < j < \omega$.

By Lemma 4.2, choose an infinite set $I \subseteq \omega$ such that for all $i, j \in I$ with $i < j$, $|A_j \backslash A_i| \geq m$, $|A_i \cap A_j| \geq m$ and $|B \backslash (A_i \cup A_j)| \geq \omega$. Thus, $G(\beta) = K_{n+1}(\beta, a_j - a_i, m) = m$ and $K_{n+1}(\beta, a) = K_{n+1}(\beta, a_i \cdot a_j) \geq m$. Furthermore, $K_{n+1}(\beta, -a_i - a_j) = \min\{\omega, |B \backslash (A_i \cup A_j)|\} = \omega$. Thus, there is a $\gamma \in T_{n+2}$ with $\beta \in \gamma$, $K_{n+2}(\gamma, -a_i - a_j) \geq 1$. But then $K_{n+2}(\gamma, -a) = K_{n+2}(\gamma, -a_i - a_j) \geq 1$ and so $K_{n+1}(\beta, -a) = \omega$. (f) is similar to (e).

(ii)$\rightarrow$(i). Let a, F and G be as described in (ii).

First we show

Claim 1. Let b be a clopen set, $p \in b$ with $t_{n+2}(p) = \gamma$; then there is a family

$\{c_i^\beta: \beta \in \gamma, i < \omega\}$ of pairwise disjoint clopen subsets of b such that for each $\beta \in \gamma$, $i \in \omega$, there exists a point $p_i^\beta \in c_i^\beta$ with $t_{n+1}(p_i^\beta) = \beta$.

Proof of the Claim. Let $i < \omega$ and assume that for $j < i$ we have already constructed a family $\{c_j^\beta: j < i, \beta \in \gamma\}$ of pairwise disjoint clopen subsets of $b \backslash \{p\}$. We put

$$b_i = b - \sum \{c_j^\beta: j < i, \beta \in \gamma\} .$$

Now $p \in b_i$ implies that for each $\beta \in \gamma$ there is an ultrafilter $p_i^\beta \in b_i$ with $t_{n+1}(p_i^\beta) = \beta$. Choose pairwise disjoint clopen subsets c_i^β $(\beta \in \gamma)$ of $b \backslash \{p\}$ with $p_i^\beta \in c_i$. This proves the claim.

We set

$$M = \{\beta \in T_{n+1}(\beta, a) = \omega\} .$$

For each $\gamma \in t_{n+2}[a]$ choose $p_\gamma \in a$ with $t_{n+2}(p_\gamma) = \gamma$. Choose pairwise disjoint clopen neighbourhoods a_γ of these p_γ's which are subsets of a. Choosing for each a_γ a family as described for b in the Claim shows that there exists a family $\{b_i^\beta: \beta \in M, i \in \omega\}$ of pairwise disjoint clopen subsets of a such that for each $\beta \in M$, $i \in \omega$, there exists a $p \in b_i^\beta$ with $t_{n+1}(p) = t_n[b_i^\beta \backslash \{p\}]$. We remark that for each b_i^β,

$$K_n(\alpha, b_i^\beta) = \begin{cases} \omega & \text{if } \alpha \in \beta \text{ ;} \\ 1 & \text{if } \alpha \not\in \beta \text{ and } \pi_n^{n+1}(\beta) = \alpha \text{ ;} \\ 0 & \text{otherwise .} \end{cases}$$

We set

$$N = \{\beta \in T_{n+1}: K_{n+1}(\beta, -a) = \omega\} .$$

Similarly we can find a family $\{c_i^\beta: \beta \in N, i \in \omega\}$ of pairwise disjoint clopen subsets of $-a$ such that for each $\beta \in N$, $i \in \omega$, there exists a $p \in c_i^\beta$ with $t_{n+1}(p) = t_n[c_i^\beta \backslash \{p\}] = \beta$.

Then, also for each c_i^β,

$$K_n(\alpha, c_i^\beta) = \begin{cases} \omega & \text{if } \alpha \in \beta \text{ ;} \\ 1 & \text{if } \alpha \not\in \beta \text{ and } \pi_n^{n+1}(\beta) = \alpha \text{ ;} \\ 0 & \text{otherwise .} \end{cases}$$

It follows from (d), (e) and (f) that for each $\gamma \in T_{n+1}$ the following four cases are possible:

(1) $F(\gamma) = G(\gamma) = 0$.

Then we set

$$d_i^\gamma = e_i^\gamma = 0 .$$

(2) $0 < F(\gamma) = G(\gamma)$.

Then $K_{n+1}(\gamma, 1) = K_{n+1}(\gamma, a) + K_{n+1}(\gamma, -a) = \omega$ and we set

$$d_i = \begin{cases} \sum \{b_j^\gamma\colon F(\gamma)\cdot i \le j < F(\gamma)\cdot(i+1)\} & \text{if } K_{n+1}(\gamma, a) = \omega\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

$$e_i = \begin{cases} 0 & \text{if } K_{n+1}(\gamma, a) = \omega\,; \\ \sum \{c_j^\gamma\colon F(\gamma)\cdot i \le j < F((\gamma)\cdot(i+1)\} & \text{otherwise}\,. \end{cases}$$

Then, for each $i < j < \omega$ and $\alpha \in T_n$,

$$\begin{aligned} &K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma) \\ &\quad = K_n(\alpha, d_i^\gamma - d_j^\gamma) + K_n(\alpha, e_j^\gamma - e_i^\gamma) \\ &\quad = \begin{cases} \omega & \text{if } \alpha \in \gamma\,; \\ F(\gamma) & \text{if } \alpha \not\in \gamma \text{ and } \pi_n^{n+1}(\gamma) = \alpha\,; \\ 0 & \text{otherwise}\,. \end{cases} \end{aligned}$$

(3) $F(\gamma) < G(\gamma) = m$.
Then (e) implies that $K_{n+1}(\gamma, -a) = \omega$ and we set

$$d_i^\gamma = 0\,;$$

$$e_i^\gamma = \sum \{c_{2j}\colon j < m\cdot i\} + \sum \{c_{2j+1}\colon F(\gamma)\cdot i \le j < F(\gamma)\cdot(i+1)\}\,.$$

For $i < j < \omega$ and $\alpha \in T_n$ we have

$$\begin{aligned} &K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma) \\ &\quad = \begin{cases} \omega & \text{if } \alpha \in \gamma\,; \\ F(\gamma) & \text{if } \alpha \not\in \gamma \text{ and } \pi_n^{n+1}(\gamma) = \alpha\,; \\ 0 & \text{otherwise}\,; \end{cases} \end{aligned}$$

and

$$K_n(\alpha, d_i^\gamma - d_j^\gamma) + K_n(\alpha, e_j^\gamma - e_i^\gamma)$$

$$\begin{cases} \omega & \text{if } \alpha \in \gamma\,; \\ F(\gamma) + (j-i)\cdot m & \text{if } \alpha \not\in \gamma \text{ and } \pi_n^{n+1}(\gamma) = \alpha\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

(4) $G(\gamma) < F(\gamma) = m$.
Then (f) implies that $K_{n+1}(\gamma, a) = \omega$ and we set

$$d_i^\gamma = \sum \{b_{2j}\colon j < m\cdot i\} + \sum \{b_{2j+1}\colon G(\gamma)\cdot i \le j < G(\gamma)\cdot(i+1)\}\,;$$

$$e_i^\gamma = 0\,.$$

Then, for $i < j < \omega$ and $\alpha \in T_n$ we have

$$K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma)$$
$$= \begin{cases} \omega & \text{if } \alpha \in \gamma\,; \\ G(\gamma) + (j-i)\cdot m & \text{if } \alpha \not\subseteq \gamma \text{ and } \pi_n^{n+1}(\gamma) = \alpha\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

$$K_n(\alpha, d_i^\gamma - d_j^\gamma) + K_n(\alpha, e_j^\gamma - e_i^\gamma)$$
$$= \begin{cases} \omega & \text{if } \alpha \in \gamma\,; \\ G(\gamma) & \text{if } \alpha \not\subseteq \gamma \text{ and } \pi_n^{n+1}(\gamma) = \alpha\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

An easy calculation shows that in all four cases, for $i<j$,

$$K_{n+1}(\gamma, d_j^\gamma - d_i^\gamma) + K_{n+1}(\gamma, e_i^\gamma - e_j^\gamma) \geq F(\gamma)$$

and

$$K_{n+1}(\gamma, d_i^\gamma - d_j^\gamma) + K_{n+1}(\gamma, e_j^\gamma - e_i^\gamma) \geq G(\gamma)\,.$$

We set, for $i<\omega$,

$$a_i = a - \Sigma\,\{d_i^\gamma \colon \gamma \in T_{n+1}\} + \Sigma\,\{e_i^\gamma \colon \gamma \in T_{n+1}\}$$

and have to show that the a_i's are as desired.

First we show that for each $\alpha \in T_n$, $i<j<\omega$,

$$K_n\Big(\alpha, a - \Sigma\,\{d_i^\gamma + d_j^\gamma \colon \gamma \in T_{n+1}\}\Big) = K_n(\alpha, a)\,.$$

This is clear if $K_n(\alpha, d_i^\gamma + d_j^\gamma) = 0$ for each $\gamma \in T_{n+1}$.

Thus, assume that for some $\gamma \in T_{n+1}$, $K_n(\alpha, d_i^\gamma + d_j^\gamma) \geq 1$. But then the construction implies that there are infinitely many pairwise disjoint clopen sets

$$b_k^\gamma \subseteq a - \Sigma\,\{d_i^\delta + d_j^\delta \colon \delta \in T_{n+1}\}$$

and

$$K_n\Big(\alpha, a - \Sigma\,\{d_i^\gamma + d_j^\gamma \colon \gamma \in T_{n+1}\}\Big) = \omega = K_n(\alpha, a)\,.$$

$K_n(d_i \cdot d_j) \geq 1$ implies $d_i \cdot d_j \neq 0$ and this can happen only if $F(\gamma) < G(\gamma)$. But by (e), $K_{n+1}(\gamma, a) \geq m$ and so $K_n(\alpha, a) \geq m$. This shows that

$$\begin{aligned} K_n(\alpha, a_i \cdot a_j, m) &= \min\Big\{m, K_n\Big(\alpha, a - \Sigma\,\{d_i^\gamma + d_j^\gamma \colon \gamma \in T_{n+1}\}\Big) \\ &\qquad + K_n\Big(\alpha, \Sigma\,\{e_i^\gamma + e_j^\gamma \colon \gamma \in T_{n+1}\}\Big)\Big\} \\ &= \min\{m, K_n(\alpha, a)\} = h(\alpha) \end{aligned}$$

and thus the first equation of (i) is shown.

Moreover, we have

$$\begin{aligned}K_{n+1}(\alpha, a_i - a_j) &= \sum \{K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma) : \gamma \in T_{n+1}\} \\ &\geq \sum \{K_{n+1}(\gamma, d_j^\gamma - d_i^\gamma) \\ &\quad + K_{n+1}(\gamma, e_i^\gamma - e_j^\gamma) : \pi_n^{n+1}(\gamma) = \alpha\} \\ &\geq \sum \{F(\gamma) : \pi_n^{n+1}(\gamma)\} = \alpha \geq f(\alpha)\end{aligned}$$

by (b). This implies

$$K_n(\alpha, a_i - a_j, m) = f(\alpha)$$

if $f(\alpha) = m$.

Suppose now that $f(\alpha) < m$. Then

$$K_n(\alpha, a_i - a_j) = \sum \{K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma) : \gamma \in T_{n+1}\} .$$

Now by (c), if $\alpha \in \gamma$, then $F(\gamma) = 0$. Thus, the cases (2) and (4) of our construction are excluded. But in case (1) and in case (3) we have

$$K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma) = F(\gamma) + 0 = 0 .$$

If $\alpha \not\subseteq \gamma$ and $\pi_n^{n+1}(\gamma) \neq \alpha$, then $K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma) = 0$ in each of the four cases of the construction.

Now $f(\alpha) < m$ and (b) imply that for each $\gamma \in T_{n+1}$ with $\pi_n^{n+1}(\gamma) = \alpha$, we have $F(\gamma) < m$. Thus, for these γ case (4) of the construction is excluded and

$$K_n(\alpha, d_j^\gamma - d_i^\gamma) + K_n(\alpha, e_i^\gamma - e_j^\gamma) = F(\gamma) .$$

So we obtain:

$$K_n(\alpha, a_i - a_j) = \sum \{F(\gamma) : \alpha \not\subseteq \gamma \text{ and } \pi_n^{n+1}(\gamma) = \alpha\}$$

and using the fact that $\alpha \in \gamma$ implies $F(\gamma) = 0$, we have

$$K_n(\alpha, a_i - a_j) = \sum \{F(\gamma) : \pi_n^{n+1}(\gamma) = \alpha\} .$$

Thus, we have shown the second equation. The two remaining ones are similar.

Condition (ii) of Lemma 4.3 can be written in the form of a u-sentence, namely

$$\begin{aligned}\bigvee_{F,G \in A} \exists x \Big[& K_n(\cdot, x, m) = h \wedge K_n(\cdot, -x, m) = k \\ & \wedge \bigwedge_{\alpha \in B} K_{n+1}(\alpha, 1) = \omega \\ & \wedge \bigwedge_{\alpha \in C} (K_{n+1}(\alpha, x) \geq m \wedge K_{n+1}(\alpha, -x) = \omega) \\ & \wedge \bigwedge_{\alpha \in D} (K_{n+1}(\alpha, x) = \omega \wedge K_{n+1}(\alpha, -x) \geq m) \Big] ,\end{aligned}$$

where A is the set of all functions $F, G\colon T_{n+1} \to m+1$ which satisfy (b), (c) and

$$\text{if } F(\alpha) \neq G(\alpha), \text{ then } \max\{F(\alpha), G(\alpha)\} = m\,;$$

and B, C and D are as follows:

$$B = \{\alpha \in T_{n+1}\colon F(\alpha) + G(\alpha) \neq 0\}\,;$$
$$C = \{\alpha \in T_{n+1}\colon F(\alpha) < G(\alpha)\}\,;$$
$$D = \{\alpha \in T_{n+1}\colon G(\alpha) < F(\alpha)\}\,.$$

Remember that $K_{n+1}(\alpha, x) = \omega$ is equivalent to

$$\bigvee_{\alpha \in \beta \in T_{n+2}} K_{n+2}(\beta, x) \neq 0\,.$$

Let z be a variable. Then by $\langle h, f, g, k\rangle(z)$ we denote the relativization on z of the u-sentence just described above.

4.4. LEMMA. *There is an effective procedure providing for each F^2-formula $\varphi(z_0, \ldots, z_{k-1})$ a u-formula $\vartheta(z_0, \ldots, z_{k-1})$ such that for all $A \in \mathrm{BA}$ and all $a_0, \ldots, a_{k-1} \in A$,*

$$A \models \varphi(\vec{a}) \quad \textit{iff } A \models \vartheta(\vec{a})\,.$$

PROOF. By induction on the complexity of φ. If $\varphi(\vec{z})$ is an atomic formula of $L(F^2)$, then $\varphi(\vec{z})$ is also a u-formula.

The procedure is straightforward for the logical connections and quantifiers $\exists$ and $\forall$.

Let $\varphi(x, y, \vec{z})$ be an F^2-formula for which we have already an equivalent u-formula $\vartheta(x, y, \vec{z})$. By Lemma 1.9 and Lemma 3.18 we can assume that

$$\begin{aligned}\vartheta(x, y, \vec{z}) := \bigvee_{i<l} \bigwedge_{\varepsilon \in 2^k} \Big[& K_n\Big(\cdot, x \cdot y \cdot \prod_{j<k} \varepsilon_j \cdot z_j, 2^n\Big) = h^i_\varepsilon \\ & \wedge K_n\Big(\cdot, x \cdot (-y) \cdot \prod_{j<k} \varepsilon_j \cdot z_j, 2^n\Big) = f^i_\varepsilon \\ & \wedge K_n\Big(\cdot, (-x) \cdot y \cdot \prod_{j<k} \varepsilon_j \cdot z_j, 2^n\Big) = g^i_\varepsilon \\ & \wedge K_n\Big(\cdot, (-x) \cdot (-y) \cdot \prod_{j<k} \varepsilon_j \cdot z_j, 2^n\Big) = k^i_\varepsilon \Big]\end{aligned}$$

for suitable functions $h^i_\varepsilon, f^i_\varepsilon, g^i_\varepsilon, k^i_\varepsilon$. Now Lemma 4.3 implies that $F^2xy\vartheta(x, y, z)$ is equivalent to

$$\bigvee_{i<l} \bigwedge_{\varepsilon \in 2^k} \langle h^i_\varepsilon, f^i_\varepsilon, g^i_\varepsilon, k^i_\varepsilon\rangle \Big(\prod_{j<k} \varepsilon_j \cdot z_j\Big)\,. \quad \square$$

Now the decidability of $\mathrm{Th}^{\mathrm{u}}(\mathrm{BA})$ implies:

4.5. THEOREM. $\mathrm{Th}^{F^2}(\mathrm{BA})$ *is decidable.*

Q_0^2 can be expressed by F^2; thus

4.6. COROLLARY. $\mathrm{Th}^{Q_0^2}(\mathrm{BA})$ *is decidable.*

5. The theory of Boolean algebras with cardinality quantifiers

In Section 2 we showed that $\mathrm{Th}^{Q_0}(\mathrm{BA})$ is decidable (Corollary 2.16). We used the method of model interpretation and reduced the decidability problem of $\mathrm{Th}^{Q_0}(\mathrm{BA})$ to the decidability problem of $\mathrm{Th}^2(Lo_\omega)$. Unfortunately, this decidability proof gives little insight into the theory $\mathrm{Th}^{Q_0}(\mathrm{BA})$. Also, it is not possible to generalize this proof to $\mathrm{Th}^{Q_0}(\mathrm{BA})$ for $\alpha > 0$. The decidability of $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$ for any $\alpha > 0$ can be shown using the method of dense systems. This was done by WEESE [1977a] (see also BAUDISCH ET AL. [1980]). The full construction is rather complicated. Therefore we only explain the main ideas and show how the method of dense systems can be used to give a new proof for the decidability of Th(BA).

Let I be a linear order with first element and for each $i \in I$ let B_i be a countable Boolean algebra. For each $i \in I$ we can choose a linear order with first element, M_i, such that $B_i \cong \mathrm{Intalg}\, M_i$. We assume that the M_i's are pairwise disjoint. The following example shows that $\mathrm{Intalg}\, \Sigma_{i \in I} M_i$ can depend on the special choice of the M_i's:

EXAMPLE. Let R^+ denote the set of non-negative rational numbers with their natural ordering. For B a Boolean algebra, let id(B) be the ideal of B which is generated by the atoms and atomless elements of B; we use $B^{(1)}$ to denote $B/\mathrm{id}(B)$. For each $i \in R^+$ let $M_i = \langle \omega, < \rangle$ and $M_i^* = \langle \omega + 1, < \rangle$. Then, for each $i \in R^+$,

$$\mathrm{Intalg}\, M_i \cong \mathrm{Intalg}\, M_i^* .$$

We set

$$B = \mathrm{Intalg} \sum_{i \in R^+} M_i , \qquad B^* = \mathrm{Intalg} \sum_{i \in R^+} M_i^* .$$

Then $B^{(1)} \cong \mathrm{Intalg}\, R^+$ and thus is atomless, whereas $B^{*(1)} \cong \mathrm{Intalg}(R^+ \cdot 2)$ and thus is atomic. Thus, $B \not\cong B^*$. This leads to the following definition:

5.1. DEFINITION. Let I be a linear order with first element, for each $i \in I$ let M_i be a linear order with first element too, with $M_i \cap M_j = \emptyset$ for $i, j \in I$, $i \neq j$. For $a \in M = \Sigma_{i \in I} M_i$, say $a \in M_{i_0}$, let $[a)$ denote the segment $\{c \in M \colon c \in M_{i_0}, a \leq c\}$. Then $\mathrm{Intalg}\, {}^I M$ denotes the Boolean set algebra which is generated by all segments $[a, b)$ $(a, b \in M)$ together with all segments $[a)$ $(a \in M)$.

For the special case that each M_i is the one-element linear order we simply write $\mathrm{Intalg}_1\, M$; this is the Boolean set algebra generated by all segments $[a, b)$ $(a, b \in M)$ together with all finite subsets of M. Thus, $\mathrm{Intalg}\, M \subseteq \mathrm{Intalg}_1\, M$.

5.2. DEFINITION. (Generalized products). Let $\{B_i: i \in I\}$ be a (non-empty) family of Boolean algebras, A a subalgebra of $\mathcal{P}I$ and D an ideal of A. Then

$$\prod^{D,A} \{B_i: i \in I\}$$

denotes the subalgebra of $\Pi \{B_i: i \in I\}$ which is generated by all $a \in \Pi \{B_i; i \in I\}$ with:

> there are $d \in A$, $e \in D$ such that $a(i) = 0$ for each $i \in d$, and $a(i) = 1$ for each $i \in I - (d + e)$.

If $D = P_{<\omega}(I)$, then instead of $\Pi^{D,A} \{B_i: i \in I\}$ we simply write $\Pi^A \{B_i: i \in I\}$. The following lemma is easily seen:

5.3. LEMMA. *Let I be a linear order with first element and for each $i \in I$ let M_i be a linear order with first element too, where $M_i \cap M_j = \emptyset$ for $i, j \in I$, $i \neq j$. Then*

$$\text{Intalg} \sum_{i \in I} M_i \cong \prod^{\text{Intalg}_1 I} \{\text{Intalg}\, M_i: i \in I\}\,.$$

Next we describe a dense system for the elementary theory of Boolean algebras. Let *Term* be the term algebra generated by the three constants 1, 2 and η_0, two unary functions p_0 and p_1 and one binary function $\times$. By induction on the complexity of terms we define for each $t \in \textit{Term}$ a Boolean algebra as follows:

$$B(1) = 1\,;$$
$$B(2) = 2\,;$$
$$B(\eta_0) = \text{Intalg}\, R^+\,;$$
$$B(p_0(t_0)) = \prod_{\omega}^{<\omega} B(t_0)\,;$$
$$B(p_1(t_0)) = \prod^{\text{Intalg}_1 R^+} B(t_0)\,;$$
$$B(t_0 \times t_1) = B(t_0) \times B(t_1)\,.$$

We set

$$M = \{B(t): t \in \textit{Term}\}\,.$$

In this section L always denotes the elementary language of Boolean algebras. Next we show

5.4. LEMMA. *M is dense for* Th(BA).

PROOF. Let $A \in \text{BA}$, $\varphi \in \text{Sent}\, L$ with $A \models \varphi$. Assume that qr $\varphi = n$. We show that there is a $B \in M$ with $A \equiv_n B$; this implies that $B \models \varphi$ and thus M is dense.

We set

$$I = \{a \in A: \text{for each } b \leq a \text{ there is a } t \in \textit{Term} \text{ with } A \restriction b \equiv_n B(t)\}\,.$$

Claim. I is an ideal of A.

Proof of the Claim. It is sufficient to show: If $a, b \in I$ and $c \leq a + b$, then $c \in I$. Thus, assume that $a, b \in I$ and $c \leq a + b$. We set

$$c_0 = c \cdot a , \qquad c_1 = c \cdot (b - a) .$$

Then $c_0 \leq a$, $c_1 \leq b$, $c_0 \cdot c_1 = 0$ and $c_0 + c_1 = c$. $a \in I$ implies that there is a $t_0 \in \mathit{Term}$ with $A \restriction c_0 \equiv_n B(t_0)$ and $b \in I$ implies that there is a $t_1 \in \mathit{Term}$ with $A \restriction c_1 \equiv_n B(t_1)$. Thus,

$$A \restriction c \equiv_n B(t_0) \times B(t_1) = B(t_0 \times t_1)$$

and the Claim is shown.

Now it suffices to show that $1_A \in I$. Thus, let us assume, on the contrary, that $1_A \not\in I$. We choose a set $R \subseteq \mathit{Term}$ such that

for each $b \in I$ there is exactly one $t \in R$ with
$(A \restriction b, B(t)) \in \mathrm{typ}(n)$ and
for each $t \in R$ there is a $b \in I$ with $(A \restriction b, B(t)) \in \mathrm{typ}(n)$.

Then R is a finite set. For each $a \in A$ we set

$$R(a) = \{t \in R\text{: there is a } b \leq a \text{ with } (A \restriction b, B(t)) \in \mathrm{typ}(n)\} .$$

Then $a \leq b$ implies $R(a) \subseteq R(b)$.

We distinguish the following two cases: (i) A/I has at least one atom; (ii) A/I is atomless.

Case (i). We choose $c \in A \backslash I$ such that c/I is an atom and

(1) for each $d \leq c$, if $d \not\in I$, then $R(d) = R(c)$.

We set

$$C = \prod \{B(t)\colon t \in R(c)\} ;$$
$$B = \prod_{\omega}^{<\omega} C .$$

We show that $A \restriction c \sim_n B$. Let

$$S = \{d \leq c\colon d \not\in I\} ;$$
$$S^* = \{d \in B\colon |\{i \in \omega\colon d(i) \neq 0\}| = \aleph_0\} .$$

We show that Player II can ensure that for each $k \leq n$, if (a_k, b_k) is the result before stage $k + 1$, then

(2) $a_k \in$ iff $b_k \in S^*$ and
if $a_k \not\in S$, then $A \restriction a_k \sim_{n-k} B \restriction b_k$.

This condition is trivially satisfied for $(1_A, 1_B)$. Assume that it is satisfied for (a_i, b_i) and $i < n$. If $a_i \not\in S$, then $A \restriction a_i \sim_{n-i} B \restriction b_i$. Thus, for each $a \leq a_i$, there is a $b \leq b_i$ and for each $b \leq b_i$ there is an $a \leq a_i$ such that

$$A \restriction a \sim_{n-i-1} B \restriction b \quad \text{and} \quad A \restriction a_i - a \sim_{n-i-1} B \restriction b_i - b\,.$$

Now let us assume that $a_i \in S$. First, we consider the case that Player I chooses A and some $a \leq a_i$. W.l.o.g. $a \not\in S$ (otherwise consider $a_i - a$); then there is a $t \in R(c)$ with $A \restriction a \sim_n B(t)$. Now $b_i \in S^*$ implies that we have $b(j) = 1$ for some $j \in \omega$ and thus there is a $b < b_i$ with $b \not\in S^*$ and $B \restriction b = B(t)$. Choosing b, Player II ensures that also (a_{i+1}, b_{i+1}) satisfies (2).

Now we consider the case that Player I chooses B and some $b \leq b_i$. W.l.o.g. $b \not\in S^*$. Then $|j \in \omega\colon b(j) \neq 0\}| < \aleph_0$ and there are disjoint elements $c_0, \ldots, c_{m-1}$ and terms $s_0, \ldots, s_{m-1} \in R(c)$ with $\Sigma_{j<m} c_j = b$ and $B \restriction c_j \sim_n B(s_j)(j < m)$. Now (1) implies that there are pairwise disjoint elements $d_0, \ldots, d_{m-1} \not\in S$ with $\Sigma_{j<m} d_j < a$ and $A \restriction d_j \sim_n B \restriction c_j(j < m)$.

We set

$$a = \sum_{j<m} d_j\,.$$

Choosing a, Player II ensures that also (a_{i+1}, b_{i+1}) satisfies (2).

Case (ii). In this case A/I is atomless. Choose $c \in A \setminus I$ such that

(3) $\quad$ for each $d \leq c$, if $d \not\in I$, then $R(d) = R(c)\,.$

We set

$$C = \prod \{B(t)\colon t \in R(c)\}\,;$$
$$B = \prod\nolimits^{\mathrm{Intalg}_1 R^+} C\,.$$

We show that $A \restriction c \sim_n B$. Let

$$S = \{d \leq c\colon d \not\in I\}\,;$$
$$S^* = \{d \in B\colon |\{i \in R^+\colon b(i) \neq 0\}| = \aleph_0\}\,.$$

As in case (i), Player II can ensure that for each $k < n$, if (a_k, b_k) is the result before stage $k+1$, then

$a_k \in S$ iff $b_k \in S^*$ and
if $a_k \not\in S$, then $A \restriction a_k \sim_{n-k} B \restriction b_k\,.$

This is very similar to case (i) and thus we omit the details. □

In order to apply Theorem 0.2 we have to show that the $B(t)$'s are decidable in a uniform way. It is well known that the decidability of A and B implies the decidability of $A \times B$. The decidability of A implies also the decidability of $\Pi_\omega^{<\omega} A$

(see, for instance, MONK [1976]). Before we consider products of the form $\prod^{\mathrm{Intalg}\, R^+} A$ we give some definitions.

Let L' be some extension of L by new predicates. Let $\varphi \in \mathrm{Form}\, L'$ and z be a variable which does not occur in φ. Then we define φ^z as follows:

$$(t_0 = t_1)^z := t_0 \cdot z = t_1 \cdot z\,;$$

$R(t_0, \ldots, t_{n-1})^z := R(t_0 \cdot z, \ldots, t_{n-1} \cdot z)$, where R is an n-ary relation symbol or L';

$$(\neg\varphi)^z := \neg\varphi^z\,;$$

$$(\varphi_0 \wedge \varphi_1)^z := \varphi_0^z \wedge \varphi_1^z\,;$$

$$(\exists x\varphi)^z := \exists x(x \leq z \wedge \varphi^z)\,.$$

Let $F \subseteq \mathrm{Form}\, L'$, P a unary predicate sign which does not occur in L'. Then we define the reduction of F on P, denoted by $F \mid P$, as follows:

$$F \mid P = \{\forall z(Pz \rightarrow \varphi^z) \colon \varphi \in F,\ z \text{ does not occur in } \varphi\}\,.$$

5.5. LEMMA. *Let $B \in \mathrm{BA}$, $A = \prod^{\mathrm{Intalg}_1 R^+} B$. If $\mathrm{Th}(B)$ is decidable, then $\mathrm{Th}(A)$ is decidable too.*

PROOF. Expand the language L of Boolean algebras by adding two unary predicate signs P and S. Let T be the theory in $L \cup \{P, S\}$ which is given by the following set of axioms:

(i) The reduction of $\mathrm{Th}(B)$ on P;
(ii) $\forall x(x \neq 0 \rightarrow \exists y(Py \wedge x \cdot y \neq 0))$;
(iii) $\forall xy(Px \wedge Py \wedge x \neq y \rightarrow x \cdot y = 0)$;
(iv) $\forall x(Px \rightarrow \neg Sx)$;
(v) $\forall xy(Sx \wedge x \leq y \rightarrow Sy)$;
(vi) $\forall xy(\neg Sx \wedge \neg Sy \rightarrow \neg S(x + y))$;
(vii) $\forall x\, \exists y(Sy \wedge (y \leq x \vee x \cdot y = 0))$;
(viii) $\forall x(Sx \rightarrow \exists yz(y \cdot z = 0 \wedge y + z = x \wedge Sy \wedge Sz))$;
(ix) $\forall x\, \exists y\, \forall z[Pz \rightarrow (z \cdot x \neq 0 \rightarrow z \leq y) \wedge (z \cdot x = 0 \rightarrow z \cdot y = 0)]$;
(x) if $\varphi(x) \in \mathrm{Form}\, L$ such that z does not occur in $\varphi(x)$ and $\exists x\varphi(x) \in \mathrm{Th}(B)$, then $\forall x\, \exists y[y \leq x \wedge \forall z[Pz \rightarrow (\varphi^z(x) \rightarrow x \cdot z = y \cdot z) \wedge (\neg\varphi^z(x) \rightarrow y \cdot z = 0)]] \in T$;
(xi) if $\varphi(x)$, $\psi(x, y) \in \mathrm{Form}\, L$ such that z does not occur in $\varphi(x)$ or $\psi(x, y)$, and $\forall x[\varphi(x) \rightarrow \exists y(y \leq x \wedge \psi(x, y))] \in \mathrm{Th}(B)$, then $\forall x[\neg Sx \wedge \forall z[Pz \wedge x \cdot z \neq 0 \rightarrow \varphi^z(x)) \rightarrow \exists y(y \leq x \wedge \forall z(Pz \wedge x \cdot z \neq 0 \rightarrow \psi^z(x, y))]] \in T$.

We set

$$P^* = \{a \in A\colon \text{there is an } i \in R^+ \text{ such that } a(i) = 1 \text{ and } a(j) = 0 \text{ for } j \neq i\}\,;$$

$$S^* = \{a \in A\colon |\{i \in R^+\colon a(i) \neq 0\}| = \aleph_0\}\,.$$

Then, similarly as in the proof of Lemma 5.4, we can show that for each $D \in \operatorname{Mod} T$ and each $n \in \omega$,

$$\langle A, P^*, S^* \rangle \equiv_n D .$$

Thus, T is a complete and axiomatizable theory and hence decidable.

Now Lemma 5.5 together with the remarks preceding this lemma allow us to apply Theorem 0.2 and thus we obtain the decidability of Th(BA).

We give some examples illustrating the expressive power of Q_α for $\alpha > 0$. We set

$$\begin{aligned}
\text{at } x &:= x \neq 0 \wedge \forall y(y \cdot x = 0 \vee y \cdot x = x) ;\\
\eta x &:= x \neq 0 \wedge \neg \exists y(y \leq x \wedge \text{at } y) ;\\
\text{At } x &:= \neg \exists y(y \leq x \wedge \eta y) ;\\
\varphi_0 &:= \forall x(\eta x \rightarrow Q_1 y(y \leq x) ;\\
\varphi_1 &:= \forall x(\text{At } x \wedge Q_1 y(y \leq x) \rightarrow Q_1 y(y \leq x \wedge \text{at } y)) ;
\end{aligned}$$

5.6. EXAMPLE. Let α be an ordinal with $\alpha > 0$. H_α denotes the set of all finite sequences of ordinals less than ω_α not ending on zero. We order H_α lexicographically, i.e. $a < b$ iff a is an initial sequence of b or there is an $i < \operatorname{lh}(a)$ such that $a(i) < b(i)$ and $a(k) = b(k)$ for all $k < i$. Then H_α has a first element and for $a, b \in H_\alpha$, if $a < b$, then $|[a, b)| = \aleph_\alpha$. Then Intalg $H_\alpha \models \varphi_0^{1/\alpha}$ (for the definition of $\varphi_0^{1/\alpha}$ see Section 1). On the other hand, Intalg $\eta_0 \models \neg \varphi_0^{1/\alpha}$ and also Intalg $\omega_\alpha \cdot \eta_0 \models \neg \varphi_0^{1/\alpha}$. But it is easily seen that $\mathrm{BA} \models \varphi_0^{1/0}$.

We set

$$\begin{aligned}
\Phi(x) &:= \eta x \wedge \neg Q_1 y(y \leq x) ;\\
\Psi(x) &:= \eta x \wedge \forall y(y \neq 0 \wedge y \leq x \rightarrow Q_1 z(z \leq y)) .
\end{aligned}$$

Let $A \in \mathrm{BA}$, $a \in A$. Then a is said to be $\aleph_\alpha$*-weakly atomless* if $A \models \Phi(a)^{1/\alpha}$ and a is said to be $\aleph_\alpha$*-atomless* if $A \models \Psi(a)^{1/\alpha}$. For instance, each non-zero element of H_α is $\aleph_\alpha$-atomless.

5.7. EXAMPLE. Let α be an ordinal with $\alpha > 0$ and such that for some $\beta < \alpha$, $2^{\aleph_\beta} \geq \aleph_\alpha$. Then $\mathscr{P}\omega_\beta \models \neg \varphi_1^{1/\alpha}$. On the other hand, it is easily seen that for each ordinal β with:

$$\text{for each } \gamma < \beta, 2^{\aleph_\gamma} < \aleph_\beta ,$$

we have $\mathrm{BA} \models \varphi_1^{1/\beta}$.

These examples show that the theories $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$ (for $\alpha > 0$) are different from $\mathrm{Th}^{Q_0}(\mathrm{BA})$ and that also the theories $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$ for $\alpha > 0$ are not all the same. The algebras mentioned in the two examples already give an idea what has to be added to M in order to obtain dense sets for $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$.

Assuming GCH, we have for each limit ordinal α (>0),

(*) for each $\beta < \alpha$, $2^{\aleph_\beta} < \aleph_\alpha$,

and no successor ordinal α satisfies (*). This leads to the following notation. Let α be an ordinal greater than zero. $\aleph_\alpha$ is said to be of *limit type* if α satisfies (*); otherwise, $\aleph_\alpha$ is said to be of *successor type*.

Let *Sec* be the set of all finite 0–1 sequences and let $\wedge$ denote the empty sequence. For each ordinal $\alpha > 0$, let f_α: $Sec \to P(\omega_\alpha)$ be a function with:

(i) $f_\alpha(\wedge) = \omega_\alpha$;

(ii) for each $s \in Sec$, $f_\alpha(s0)$, $f_\alpha(s1)$ are disjoint and $f_\alpha(s0) \cup f_\alpha(s1) = f_\alpha(s)$;

(iii) for each $s \in Sec$, $|f_\alpha(s)| = \aleph_\alpha$ and $|\omega \cap f_\alpha(a)| = \aleph_0$.

Let $B(f_\alpha)$ denote the subalgebra of $\mathcal{P}\omega_\alpha$ which is generated by $P_{<\omega}(\omega_\alpha) \cup \{f_\alpha(s): s \in Sec\}$.

Let us start with the case that $\aleph_\alpha$ $(\alpha > 0)$ is of limit type. Let $Term_\omega$ be the term algebra generated by the four constants 1, 2, η, η_0, three unary functions p_0, p_1, p_2 and three binary functions p_3, p_4, $\times$.

We define a function B_α: $Term_\omega \to \mathrm{BA}$ as follows:

$$B_\alpha(1) = 1 ;$$

$$B_\alpha(2) = 2 ;$$

$$B_\alpha(\eta_0) = \mathrm{Intalg}\ R^+ ;$$

$$B_\alpha(\eta) = \mathrm{Intalg}\ H_\alpha ;$$

$$B_\alpha(t_0 \times t_1) = B_\alpha(t_0) \times B_\alpha(t_1) ;$$

$$B_\alpha(p_0(t_0)) = \prod_\omega^{<\omega} B_\alpha(t_0) ;$$

$$B_\alpha(p_1(t_0)) = \prod^{\mathrm{Intalg}\ R^+} B_\alpha(t_0) ;$$

$$B_\alpha(p_2(t_0)) = \prod^{\mathrm{Intalg}\ H_\alpha} B_\alpha(t_0) ;$$

$$B_\alpha(p_3(t_0, t_1)) = \prod^{<\omega} \{C_\beta : \beta < \omega_\alpha\}$$

with

$$C_\beta = \begin{cases} B_\alpha(t_0) & \text{for } \beta < \omega ; \\ B_\alpha(t_1) & \text{for } \omega \le \beta < \omega_\alpha ; \end{cases}$$

$$B_\alpha(p_4(t_0, t_1)) = \prod^{B(f_\alpha)} \{C_\beta : \beta < \omega_\alpha\}$$

with

$$C_\beta = \begin{cases} B_\alpha(t_0) & \text{for } \beta < \omega ; \\ B_\alpha(t_1) & \text{for } \omega \le \beta < \omega_\alpha . \end{cases}$$

Then we have

5.8. LEMMA. *Let $\aleph_\alpha$ $(\alpha > 0)$ be of limit type. Then*

$$\{B_\alpha(s): s \in Term_\omega\}$$

is dense for $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$.

PROOF. The proof is very similar to the corresponding proof of Lemma 5.5. We show that for each $A \in \mathrm{BA}$ and each $n \in \omega$ there is an $s \in Term_\omega$ such that $A\ {}^\alpha{\equiv}^\alpha_n\ B_\alpha(s)$.

Let $A \in \mathrm{BA}$, $n \in \omega$. We set

$$I = \{a \in A: \text{for each } b \leq a \text{ there is an } s \in Term_\omega \text{ with } A \upharpoonright b\ {}^\alpha{=}^\alpha_n\ B_\alpha(t)\}\,.$$

Then I is an ideal.

Assume $I \neq A$. We choose a set $R \subseteq Term_\omega$ such that

for each $b \in T$ there is exactly one $t \in R$ with $(A \upharpoonright b, B_\alpha(t)) \in \mathrm{typ}(n, \alpha)$ and for each $t \in R$ there is a $b \in I$ with $(A \upharpoonright b, B_\alpha(t)) \in \mathrm{typ}(n, \alpha)$.

Then R is a finite set and for each $a \in A$ we set

$$R_0(a) = \{t \in R: \text{there is a } b \leq a \text{ with } (A \upharpoonright b, B_\alpha(t)) \in \mathrm{typ}(n, \alpha)\}\,;$$

$$R_1(a) = \{t \in R: |\{b \leq a: (A \upharpoonright b, B_\alpha(t)) \in \mathrm{typ}(n, \alpha)\}| \geq \aleph_\alpha\}\,.$$

Then R_0 and R_1 are monotone functions on A.

Consider A/I. We have to distinguish three cases: (i) A/I contains an atom; (ii) A/I is atomless but contains an $\aleph_\alpha$-weakly atomless element; (iii) A/I is $\aleph_\alpha$-atomless.

Assume that A/I is $\aleph_\alpha$-atomless. Choose $a \in A \backslash I$ such that for $b \leq a$, $b \not\subseteq I$, we have $R_0(a) = R_0(b)$ and $R_1(a) = R_1(b)$. In particular, we have $R_1(a) \neq \emptyset$. We set

$$B_0 = \prod\{B_\alpha(t): t \in R_0(a)\}\,;$$

$$B_1 = \prod\{B_\alpha(t): t \in R_1(a)\}\,;$$

$$B = \prod\nolimits^{B(f_\alpha)}\{C_\beta: \beta < \omega_\alpha\}$$

with

$$C_\beta = \begin{cases} B_0 & \text{if } \beta < \omega\,; \\ B_1 & \text{if } \omega \leq \beta < \omega_\alpha\,. \end{cases}$$

Now it can be shown that $A \upharpoonright a \sim^\alpha_n B$. We set

$$S = \{b \leq a: b \not\subseteq I\}\,;$$

$$S^* = \{b \in B: |\{i < \omega_\alpha: b(i) \neq 0\}| \geq \aleph_\alpha\}\,.$$

Now it can be shown that Player II can ensure that for each $k \le n$, if (a_k, b_k) is the result before stage $k+1$, then

$$a_k \in S \text{ iff } b_k \in S^* \text{ and}$$
$$\text{if } a_k \notin S, \text{ then } A \restriction a_k \sim^{\alpha}_{n-k} B \restriction b_k .$$

For details see BAUDISCH ET AL. [1980]. □

Now we come to the last and most complicated case that $\aleph_\alpha$ is of successor type. We start by giving an example.

5.9. EXAMPLE. Let α be an ordinal with $\alpha > 0$ such that $\aleph_\alpha$ is of successor type. Let γ be the least ordinal with $2^{\aleph_\gamma} \ge \aleph_\alpha$. Choose a system $X \subseteq P(\omega_\gamma)$ with $|X| = \aleph_\alpha$ such that X is almost disjoint (i.e. if $a, b \in X$, $a \ne b$, then $|a \cap b| < \aleph_\gamma$). Such a system X exists, see Theorem 3.1 of the Appendix on Set Theory in this Handbook. Let A be the Boolean algebra which is generated by X together with $P_{<\aleph_\gamma}(\omega_\gamma)$. Then $|A| = \aleph_\alpha$. We set

$$\varphi_2 := \exists xy(x \cdot y = 0 \wedge Q_1 z(z \le x) \wedge Q_1 z(z \le y)) .$$

Then

$$\mathcal{P}\omega_\gamma \models \varphi_2^{1/\alpha}, \text{ but } A \models \neg\varphi_2^{1/\alpha} .$$

Let κ be a cardinal, $A, B \subseteq P(\kappa)$. A is a *refinement* of B if there is a family $\{S_a : a \in A\}$ of subsets of B with $2 \le |S_a| < \aleph_0$, the elements of S_a are pairwise disjoint, $\bigcup \{S_a : a \in A\} = B$ and $\bigcup S_a = a$ for each $a \in A$. A has property 1 iff $|A| = 2^\kappa$ and A is almost disjoint. A has property 2 iff $|A| = 2^\kappa$ and there is a sequence $(A_i)_{i<\omega}$ with $A = \bigcup \{A_i : i < \omega\}$, A_0 is almost disjoint and for each $i < \omega$, A_{i+1} is a refinement of A_i. A has property 3 iff $|A| = \aleph_0$ and there is a sequence $(A_i)_{i<\omega}$ with $A = \bigcup \{A_i : i < \omega\}$ and for each $i < \omega$, A_{i+1} is a refinement of A_i.

For each $\aleph_\alpha$ of successor type, let $\gamma(\alpha)$ be the least ordinal γ with $2^{\aleph_\gamma} \ge \aleph_\alpha$. We set

$$\varepsilon(\alpha) = \begin{cases} \omega^\omega & \text{if } \gamma(\alpha) = 0 ; \\ \omega_{\gamma(\alpha)} & \text{otherwise} \end{cases}$$

(where in ω^ω we use ordinal exponentiation).

For each ordinal $\alpha > 0$, let S_α be the set of all 0–1 sequences $\vec{a}$ of length ω_α with

$$\{i < \omega_\alpha : \vec{a}(i) = 1\} \text{ is cofinal in } \omega_\alpha$$

and let D_α be the set of all 0–1 sequences of length $<\omega_\alpha$. Let $\vec{a} \in S_\alpha$, $\vec{b} \in D_\alpha$. Then we set

$$\vec{b} < \vec{a} \quad \text{iff } \vec{b} \text{ is an initial segment of } \vec{a} .$$

We have $|S_\alpha| = 2^{\aleph_\alpha}$ and $|D\alpha| = 2^{\aleph_\alpha}$. Thus, if α satisfies (4), we have $|S_{\gamma(\alpha)}| \geq \aleph_\alpha$ and $|D_{\gamma(\alpha)}| < \aleph_\alpha$. For each ordinal $\alpha > 0$ let $g_\alpha \colon D_\alpha \to 2^{\aleph_\alpha}$ be a fixed one-to-one function from D_α onto $2^{\aleph_\alpha}$ which satisfies

for each $\vec{b} \in D_\alpha$ there are $\beta, \gamma < \omega_\alpha$ and $j < \omega$ with $\mathrm{lh}(\vec{b}) = \beta \cdot \omega + j$, $g_\alpha(\vec{b}) = \gamma \cdot \omega + j$.

For $k, l, m < \omega$ and $\vec{a} \in S_\alpha$ we set

$$D_\alpha[k, l, m] = \{\vec{b} \in D_\alpha\colon \text{there are } \beta < \omega_\alpha \text{ and } j < \omega \text{ with } \mathrm{lh}(\vec{b}) = \beta \cdot \omega^2 + l \cdot \omega + j \text{ and } j \equiv k (\mathrm{mod}\ m)\}\ ;$$

$$E_\alpha[k, l, m](\vec{a}) = \{g_\alpha(\vec{b})\colon \vec{b} \in D_\alpha[k, l, m] \text{ and } \vec{b} < \vec{a}\}\ ;$$

$$E_\alpha[k, m] = \{E_\alpha[k, l, m](\vec{b})\colon \vec{b} \in S_\alpha, l < \omega\}\ ;$$

$$F_\alpha[k, m] = \{E_\alpha[i, j, n \cdot m](\vec{b})\colon \vec{b} \in S_\alpha, i, j, n < \omega,\ i \equiv k (\mathrm{mod}\ m), n > 0\}\ ;$$

$$G_\alpha[k, m] = \{g_\alpha(\vec{b})\colon \text{there are } \beta < \omega_\alpha, i, j < \omega \text{ with } \mathrm{lh}(\vec{b}) = \beta \cdot \omega^2 + j \cdot \omega + i,\ j \equiv k (\mathrm{mod}\ 2^m)\}\ ;$$

$$G_\alpha = \{G_\alpha[i, j]\colon i, j < \omega\}\ .$$

Let S_0 be the set of all 0–1 sequences $\vec{a}$ of length ω^2 with

$\{i < \omega^2\colon \vec{a}(i) = 1\}$ is cofinal in ω^2 and has order type ω ;

and let D_0 be the set of all 0–1 sequences $\vec{a}$ with $\mathrm{lh}(\vec{a}) < \omega^2$ and $|\{i < \mathrm{lh}(\vec{a})\colon \vec{a}(i) = 1\}| < \aleph_0$ (thus $|D_0| = \aleph_0$).

Let $g_0\colon D_0 \to \omega^\omega$ be a fixed one-to-one function from D_0 onto ω^ω which satisfies:

for each $\vec{b} \in D_0$ there are $\beta, \gamma < \omega^\omega$ and $j < \omega$ with $\mathrm{lh}(\vec{b}) = \beta \cdot \omega + j$, $g_0(\vec{b}) = \gamma \cdot \omega + j$.

Let $k, l, m < \omega$ and $\vec{a} \in S_0$. We set

$$D_0[k, l, m] = \{\vec{b} \in D_0\colon \text{there are } \beta < \omega^\omega, j < \omega \text{ with } \mathrm{lh}(\vec{b}) = \beta \cdot \omega^2 + l \cdot \omega + j \text{ and } j \equiv k (\mathrm{mod}\ m)\}\ ;$$

$E_0[k, l, m](\vec{a})$, $E_0[k, m]$ and $F_0[k, m]$ are defined as above, replacing α by 0;

$$G_0[k, m] = \{g_0(\vec{b})\colon \text{there are } \beta < \omega^\omega \text{ and } i, j < \omega \text{ with } \mathrm{lh}(\vec{b}) = \beta \cdot \omega^2 + j \cdot \omega + i \text{ and } j \equiv k (\mathrm{mod}\ 2^m)\}\ ;$$

$$G_0 = \{G_0[i, j]\colon i, j < \omega\}\ .$$

Let $\alpha \geq 0$, $k, m \in \omega$. Then $E_\alpha[k, m]$ has property 1. For $n \in \omega$ we set

$$A_n = \{E_\alpha[k, l, m](\vec{a})\colon \vec{a} \in S_\alpha, i, j < \omega\ ,\ i \equiv k (\mathrm{mod}\ m)\}\ ;$$

then it is immediately seen that $F_\alpha[k, m]$ has property 2. For $n \in \omega$ we set

$$A_n^* = \{G_\alpha[k, n]: k < n\} .$$

Then it follows immediately that G_α has property 3. For $k, n, m \in \omega$, $n \leq m$ we set

$$X_\alpha[k, n, m] = \{E_\alpha[l, k+m]: k \leq l < k+n\} \cup \{F_\alpha[l, k+m]: k+n \leq l < k+m\} ;$$

$$Y_\alpha[k, n, m] = X_\alpha[k, n, m] \cup G_\alpha .$$

Let $\aleph_\alpha$ be of successor type. Then let $B_\alpha[k, n, m]$ denote the subalgebra of $\mathcal{P}\varepsilon(\alpha)$ which is generated by $X_{\gamma(\alpha)}[k, n, m]$ and let $B_\alpha^+[k, n, m]$ denote the subalgebra of $\mathcal{P}\varepsilon(\alpha)$ which is generated by $Y_{\gamma(\alpha)}[k, n, m]$. Let J be the ideal generated by $X_{\gamma(\alpha)}[k, n, m] \cup P_{<\omega}(\omega_{\gamma(\alpha)})$. Then

$$B_\alpha[k, n, m]/J \cong 2 ;$$

$$B_\alpha^+[k, n, m]/J \cong \text{Intalg } R^+ .$$

Now we describe a dense system for $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$.

Let $Term_1$ be the term algebra generated by the four constants 1, 2, η, η_0, three unary functions p_0, p_1, p_2, three binary functions p_3, p_4, $\times$ and two functions p_5, p_6 given by:

> if $m_0 > 0$, $t_0, \ldots, t_{m_0-1} \in Term_1$, $\vec{X}$ a finite sequence of elements of $P(\{t_i: i < m_0\}) \setminus \{\emptyset\}$, $n < \mathrm{lh}(\vec{X})$, then $p_5(t_0, \ldots, t_{m_0-1}, n, \vec{X})$ and $p_6(t_0, \ldots, t_{m_0-1}, n, \vec{X})$ are terms .

We define a function B_α: $Term_1 \to \mathrm{BA}$ as follows:

$$B_\alpha(1) = 1 ;$$

$$B_\alpha(2) = 2 ;$$

$$B_\alpha(\eta_0) = \text{Intalg } R^+ ;$$

$$B_\alpha(\eta) = \text{Intalg } H_\alpha ;$$

$$B_\alpha(t_0 \times t_1) = B_\alpha(t_0) \times B_\alpha(t_1) ;$$

$$B_\alpha(p_0(t_0)) = \prod_\omega^{<\omega} B_\alpha(t_0) ;$$

$$B_\alpha(p_1(t_0)) = \prod^{\text{Intalg}_1 R^+} B_\alpha(t_0) ;$$

$$B_\alpha(p_2(t_0)) = \prod^{\text{Intalg}_1 H_\alpha} B_\alpha(t_0) ;$$

$$B_\alpha(p_3(t_0, t_1)) = \prod^{<\omega} \{C_\beta: \beta < \omega_\alpha\}$$

with

$$C_\beta = \begin{cases} B_\alpha(t_0) & \text{for } \beta < \omega\,; \\ B_\alpha(t_1) & \text{for } \omega \le \beta < \omega_\alpha\,; \end{cases}$$

$$B_\alpha(p_4(t_0, t_1)) = \prod\nolimits^{B(f_\alpha)} \{C_\beta \colon \beta < \omega_\alpha\}$$

with

$$C_\beta = \begin{cases} B_\alpha(t_0) & \text{for } \beta < \omega\,; \\ B_\alpha(t_1) & \text{for } \omega \le \beta < \omega_\alpha\,; \end{cases}$$

$$B_\alpha(p_5(t_0, \ldots, t_{m_0-1}, n, \vec{X})) = \prod\nolimits^{C} \{C_\beta \colon \beta < \varepsilon(\alpha)\}$$

with

$$C = B_\alpha[m_0, n, \mathrm{lh}(\vec{X})]\,;$$

$$C_{\gamma\cdot\omega+i} = \begin{cases} B_\alpha(t_k) \text{ if } i \equiv k (\mathrm{mod}\ m_0 + \mathrm{lh}(\vec{X})) \text{ and } 0 \le k < m_0\,; \\ \prod \{B_\alpha(t);\ t \in \vec{X}(k - m_0)\} \text{ if} \\ i \equiv k (\mathrm{mod}\ m_0 + \mathrm{lh}(\vec{X})) \text{ and } m_0 \le k < m_0 + \mathrm{lh}(\vec{X})\,; \end{cases}$$

$$B_\alpha(p_6(t_0, \ldots, t_{m_0-1}, n, \vec{X})) = \prod\nolimits^{C} \{C_\beta \colon \beta < \varepsilon(\alpha)\}\,;$$

with

$$C = B_\alpha^+[m_0, n, \mathrm{lh}(\vec{X})]\,;$$

$$C_{\gamma\cdot\omega+i} = \begin{cases} B_\alpha(t_k) \text{ if } i \equiv k (\mathrm{mod}\ m_0 + \mathrm{lh}(\vec{X})) \text{ and } 0 \le k < m_0\,; \\ \prod \{B_\alpha(t) \colon t \in \vec{X}(k - m_0)\} \text{ if } i \equiv k (\mathrm{mod}\ m_0 + \mathrm{lh}(\vec{X})) \text{ and} \\ m_0 \le k < m_0 + \mathrm{lh}(\vec{X})\,. \end{cases}$$

Then we have

5.10. LEMMA. $\aleph_\alpha$ $(\alpha > 0)$ *be of successor type. Then* $\{B_\alpha(t) \colon t \in Term_1\}$ *is dense for* $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$.

PROOF. The proof is similar to the proof of Lemma 5.5 and Lemma 5.8. We show that for each $A \in \mathrm{BA}$ and each $n \in \omega$ there is an $s \in Term_1$ such that $A\ {}^{\alpha}{\equiv}_n^{\alpha}\ B_\alpha(s)$.

Let I, $R \subseteq Term_1$, R_0, R_1 and a be defined as in Lemma 5.8. We consider only the following case:

$$A/I \text{ is } \aleph_\alpha\text{-atomless, } |\{b \in A \colon A \models \mathrm{at}\ b\}| < \aleph_\alpha\,,$$

$$|\{b \in A \colon A \models \eta b \wedge \neg Q_\alpha z (z \le b)\}| < \aleph_\alpha\,.$$

We set

$$R_2(a) = \{t \in R: |B_\alpha(t)| < \aleph_\alpha; |\{b \leq a: (B_\alpha(t), A \restriction b) \in \mathrm{typ}(n, \alpha)\}| \geq \aleph_\alpha\} ;$$

$$E_0(a) = \{b \leq a: |A \restriction b| < \aleph_\alpha\} ;$$

$$E_1(a) = \{b \leq a: \text{there is a } t \in R_2(a) \text{ with } (B_\alpha(t), A \restriction b) \in \mathrm{typ}(n, \alpha)\} ;$$

$$E_2(a) = \{b \in E_1(a): \text{if } c \in E_1(a) \text{ and } c \leq b, \text{ then } (A \restriction b, A \restriction c) \in \mathrm{typ}(n, \alpha)\} .$$

Then we have $E_2(a) \subseteq E_1(a)$, $E_0(a) \cap E_1(a) = \emptyset$, $E_0(a)$ is an ideal on $A \restriction a$. Let

$$E_3(a) = \{b \in E_2(a): b/E_0(a) \text{ is an atom}\} ;$$

$$E_4(a) = \{b \in E_2(a): b/E_0(a) \text{ is atomless}\} .$$

Then $\{b/E_0(a): b \in E_3(a) \cup E_4(a)\}$ is dense in $A \restriction a/E_0(a)$. For $b \in E_3(a) \cup E_4(a)$ we set

$$R_3(b) = \{t \in R: \text{there is a } c \leq b \text{ with } c \in E_0(a) \text{ and } (A \restriction c, B_\alpha(t)) \in \mathrm{typ}(n, \alpha)\} .$$

Let $\vec{X}_0$ be any sequence exhausting $\{R_3(b): b \in E_3(a)\}$ and let $\vec{X}_1$ be any sequence exhausting $\{R_3(b): b \in E_4(a)\}$. Let $n_0 = \mathrm{lh}(\vec{X}_0)$ and let $R_0(a) = \{t_0, \ldots, t_{m-1}\}$. Similarly as in Lemma 5.8 we can show that

$$A \restriction a \;{}^\alpha\!\!\equiv_n^\alpha B(p_6(t_0, \ldots, t_{m-1}, n_0, \vec{X}_0 \frown \vec{X}_1))$$

and this leads to a contradiction. □

Using the ideas of Lemma 5.5, we can give an axiomatization of $\mathrm{Th}^{Q_1}(B_1(t))$ for each $t \in \mathit{Term}_1$. This can be done uniformly. Logic with Q_1 is axiomatizable (see KEISLER [1970]); thus $\{(t, \ulcorner\varphi\urcorner): t \in \mathit{Term}_1, \varphi \in \mathrm{Sent}\, L(Q_1), B_1(t) \models \varphi\}$ is recursively axiomatizable and thus $\mathrm{Th}^{Q_1}(\mathrm{BA})$ is decidable.

For α an ordinal we set

$$\mathit{Term}(\alpha) = \begin{cases} \mathit{Term} & \text{if } \alpha = 0 ; \\ \mathit{Term}_1 & \text{if } \alpha \text{ satisfies (4)} ; \\ \mathit{Term}_\omega & \text{otherwise} . \end{cases}$$

Using the game for logics with cardinality quantifiers as described in Section 1 it can be shown:

5.11. LEMMA. *For each ordinal α and each $t \in \mathit{Term}(\alpha)$,*

$$B_\alpha(t) \;{}^\alpha\!\!\equiv^1 B_1(t) .$$

We set

$$T_0 = \mathrm{Th}(\{B_1(t)\colon t \in \mathit{Term}\})\,;$$
$$T_\omega = \mathrm{Th}(\{B_1(t)\colon t \in \mathit{Term}_\omega\})\,;$$
$$T_1 = \mathrm{Th}(B_1(t)\colon t \in \mathit{Term}_1\})\,.$$

Then $T_1 \subseteq T_\omega \subseteq T_0$ and Examples 5.6 and 5.7 imply that these inclusions are proper. T_ω and T_0 are decidable too. Now we have the following result of WEESE [1977a] (see also BAUDISH ET AL. [1980]):

5.12. THEOREM. *For each ordinal α, $\mathrm{Th}^{Q_\alpha}(\mathrm{BA})$ is decidable and*

$$\begin{aligned} \mathrm{Th}^{Q_\alpha}(\mathrm{BA}) &= T_0^{1/\alpha} && \textit{iff } \alpha = 0\,;\\ \mathrm{Th}^{Q_\alpha}(\mathrm{BA}) &= T_\omega^{1/\alpha} && \textit{iff } \alpha > 0 \textit{ and } \aleph_\alpha \textit{ is of limit type}\,;\\ \mathrm{Th}^{Q_\alpha}(\mathrm{BA}) &= T_1^{1/\alpha} && \textit{iff } \alpha > 0 \textit{ and } \aleph_\alpha \textit{ is of successor type}\,. \end{aligned}$$

6. Residually small discriminator varieties

In this section we show that every residually small discriminator variety has a decidable theory. This was shown by Werner; see WERNER [1978] or BURRIS and WERNER [1979].

We need some facts from universal algebra. Let L be any language. Then Mod L denotes the class of all relational structures for L. If L does not contain relational symbols, then the elements of Mod L are called *algebras*.

Let L be any language, $K \subseteq \mathrm{Mod}\ L$. We define the following operators mapping classes of relational structures into classes of relational structures by:

$A \in I(K)$	iff A is isomorphic to some member of K;
$A \in S(K)$	iff A is a substructure of some member of K;
$A \in H(K)$	iff A is a homomorphic image of some member of K;
$A \in P(K)$	iff A is a direct product of a non-empty family of structures of K;
$A \in P_U(K)$	iff A is an ultraproduct of a non-empty family of structures of K.

A class K of algebras of the same type is a *variety* if it is closed under the formation of subalgebras, homomorphic images and direct products.

Let K be a class of algebras (of the same type); then $V(K)$ denotes the smallest variety including K. Tarski showed that for each class K of algebras, we have

$$V(K) = HSP(K)$$

(see BURRIS and SANKAPPANAVAR [1981] for a proof).

Let L be any language, $\Phi \subseteq \mathrm{Sent}\, L$. Then $\mathrm{Mod}\, \Phi$ denotes the class of all $A \in \mathrm{Mod}\, L$ with $A \models \varphi$ for each $\varphi \in \Phi$.

Let L be a language without relational symbols, $K \subseteq \mathrm{Mod}\, L$. Then K is a variety iff there is a set Σ of equations in L with $K = \mathrm{Mod}\, \Sigma$. This was shown by Birkhoff (see BURRIS and SANKAPPANAVAR [1981]).

Let A be a relational structure; A is a *subdirect product* of the family $\{A_i: i \in I\}$ of relational structures if

(i) $A \subseteq \prod_{i \in I} A_i$

(where $\subseteq$ denotes substructure);

(ii) $\Pi_i[A] = A_i$ for each $i \in I$

(where Π_i denotes the projection onto the ith coordinate).

A relational structure A is *subdirectly irreducible* if for every subdirect embedding

$$f: A \to \prod_{i \in I} A_i$$

there is an $i \in I$ such that

$$\pi_i \circ f: A \to A_i$$

is an isomorphism.

Let A be a subdirect product of $\{A_i: i \in I\}$, $\varphi(x_0, \ldots, x_{n-1})$ a formula in the language of A, and $a_0, \ldots, a_{n-1} \in A$. Then we define the *Boolean truth value* of φ as follows:

$$[\![\varphi(a_0, \ldots, a_{n-1})]\!] = \{i \in I: A_i \models \varphi(a_0(i), \ldots, a_{n-1}(i))\}\,.$$

Let L be a language for first-order predicate logic. Let X be a Boolean space and let $A_p \in \mathrm{Mod}\, L$ for each $p \in X$. A *Boolean product* of the family $\{A_p: p \in X\}$ is a subdirect product

$$A \subseteq \prod_{p \in X} A_p$$

such that

(i) $[\![\varphi(\vec{a})]\!] \in \mathrm{Clop}\, X$ for each atomic formula $\varphi(\vec{x})$ and each $\vec{a} \in A$;

(ii) if $a, b \in A$, $N \in \mathrm{Clop}\, X$, then $a \restriction N \cup b \restriction X \backslash N \in A$. If L does not contain relational symbols, then (i) can be replaced by

(i′) $[\![a = b]\!] \in \mathrm{Clop}\, X$ for each $a, b \in A$.

Condition (ii) is called the "patchwork property". For $K \in \mathrm{Mod}\, L$ let $\Gamma^a(K)$ denote the class of Boolean products which can be formed from subsets of K.

Boolean products are closely related to sheaves (see the definition of sheaves of L-structures given in 8.11 of Part I). Using Lemma 8.13(e) of Part I we obtain:

6.1. LEMMA. *Let L be a language without relational symbols and let $\mathscr{S} = (S, \Pi, X, (B_p)_{p\in X})$ be a Hausdorff sheaf of L-algebras with X a Boolean space. Then $\Gamma(\varphi)$ is a Boolean product.*

It is also possible to describe a Boolean product as a structure of continuous sections of some sheaf:

6.2. LEMMA. *Let A be a Boolean product of the family $\{A_p: p \in X\}$ of algebras. For $p \in X$ we set $B_p = \{a(p): a \in A\}$. Let S be the disjoint union of the B_p and define $\Pi: S \to X$ by*

$$\Pi(b) = p \quad \textit{iff } b \in B_p\,.$$

Give S the topology generated by the sets $\{\sigma_a: a \in A\}$ with

$$\sigma_a = \{a(p): p \in X\}\,.$$

Set

$$\mathscr{S} = (S, \Pi, X, (B_p)_{p\in X})\,.$$

Then $\mathscr{S}$ is a Hausdorff sheaf, with X a Boolean space and

$$A \cong \Gamma(\mathscr{S})\,.$$

The proof is straightforward and is left to the reader.

Let A be an algebra and $\{\theta_i: i \in I\}$ a set of congruences on A with $\bigcap_{i\in I} \theta_i = \Delta$. Define a function $v: A \to \Pi_{i\in I} A/\theta_i$ by $v(a)(i) = a/\theta_i$. It is immediately clear that v is a homomorphism. $\Pi_i \circ v: A \to A/\theta_i$ is onto for each $i \in I$. For $a, b \in A$ with $a \neq b$ there is an $i_0 \in I$ with $(a, b) \not\in \theta_{i_0}$, so $v(a)(i_0) \neq v(b)(i_0)$ and v is injective. Thus, v gives a subdirect embedding of A into $\Pi_{i\in I} A/\theta_i$.

6.3. LEMMA. *Let A be a non-trivial algebra. Then the following are equivalent:*
(i) *A is subdirectly irreducible;*
(ii) Con $A\backslash\{\Delta\}$ *has a minimal element.*

PROOF. (i)$\to$(ii). Assume that Con $A\backslash\{\Delta\rangle$ has no minimal element. Then $\bigcap \{\theta \in \text{Con } A: \theta \neq \Delta\} = \Delta$. Thus,

$$\alpha: A \to \prod\{A/\theta: \theta \in \text{Con } A\backslash\{\Delta\}\}$$

is a subdirect embedding and $\Pi_\theta: A/\theta$, the natural homomorphism from A onto A/θ, is not injective for $\theta \in \text{Con } A\backslash\{\Delta\}$. Thus, A is not subdirectly irreducible.

(ii)$\to$(i). Assume that Con $A\backslash\{\Delta\}$ has a minimal element θ. Choose $a, b \in A$ with $a \neq b$, $(a, b) \in \theta$; let $\alpha: A \to \Pi_{i\in I} A_i$ be any subdirect embedding. Then, for some $i_0 \in I$, $\alpha(a)(i_0) \neq \alpha(b)(i_0)$. This implies $(\Pi_{i_0} \circ \alpha)(a) \neq (\Pi_{i_0} \circ \alpha)(b)$. Thus,

$\ker(\Pi_{i_0} \circ \alpha) \subseteq \theta$ and $\ker(\Pi_{i_0} \circ \alpha) \neq \theta$. Hence, $\ker(\Pi_{i_0} \circ \alpha) = \Delta$ and so $\Pi_{i_0} \circ \alpha$: $A \to A_{i_0}$ is an isomorphism. This implies that A is subdirectly irreducible. □

The following result was obtained by Birkhoff [see BURRIS and SANKAPPANAVAR [1981]):

6.4. LEMMA. *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

PROOF. Let us assume that A is non-trivial. For $a, b \in A$ with $a \neq b$ choose a maximal congruence $\theta[a, b]$ with $(a, b) \notin \theta[a, b]$. Then $\theta_{a,b} \vee \theta[a, b]$ is a least congruence in $[\theta[a, b], \nabla] \setminus \{\theta[a, b]\}$. Thus, by Lemma 6.2, $A/\theta[a, b]$ is subdirectly irreducible. Moreover, $\bigcap \{\theta[a, b]: a, b \in A, a \neq b\} = \Delta$, thus there is a subdirect embedding

$$\alpha: A \to \prod \{A/\theta[a, b]: a, b \in A, a \neq b\} . \quad \square$$

Let L be any language for first-order predicate logic without relational sign, $K \subseteq \mathrm{Mod}\, L$. We define the operation P_S by:

$A \in P_S(K)$ iff A is a subdirect product of a non-empty family of algebras in K.

We are interested in classes of algebras which allow a good representation in the form of Boolean products. Here the lattice of congruences has a great influence. So we need some facts about this lattice and start with the necessary background from universal algebra.

Let A be an algebra. Then Con A denotes the lattice of all congruences on A. This lattice has the greatest element $\nabla_A = A \times A$ and the smallest element $\Delta_A = \mathrm{id}_A$. If A is clear from the context, then we simply write ∇ and Δ. If θ, $\phi \in \mathrm{Con}\, A$, then $\theta \wedge \phi$ coincides with the set-theoretical intersection. Thus, we also write $\theta \cap \phi$ instead of $\theta \wedge \phi$. Let $X \subseteq A$; then θ_X denotes the smallest congruence $\theta \in \mathrm{Con}\, A$ with $X \times X \subseteq \theta$. If $X = \{a_0, \ldots, a_{n-1}\}$, then we write $\theta_{a_0, \ldots, a_{n-1}}$ for θ_X.

An algebra A is said to be *trivial* if $|A| = 1$. A non-trivial algebra A is *simple* if each $\theta \in \mathrm{Con}\, A$ equals ∇_A or Δ_A.

Let A be an algebra, $\theta \in \mathrm{Con}\, A$. Then $\mathrm{Con}\, A/\theta \cong [\theta, \nabla_A]$. Thus, if θ is a maximal congruence on A, then A/θ is simple. For θ, $\phi \in \mathrm{Con}\, A$, $\theta \circ \phi$ denotes the congruence

$$\{(a, b): \text{there is a } c \in A \text{ with } (a, c) \in \theta,\ (c, b) \in \phi\} .$$

θ and ϕ *permute* if $\theta \circ \phi = \phi \circ \theta$. A is said to be *congruence-permutable* if every pair of congruences of A permutes. A is said to be *congruence-distributive* if the lattice Con A is distributive.

A variety V is *arithmetical* if each $A \in V$ is congruence-distributive and congruence-permutable.

JÓNSSON [1967] (see also BURRIS and SANKAPPANAVAR [1981]) showed the following important fact:

6.5. LEMMA. *Let $V(K)$ be a congruence-distributive variety, $A \in V(K)$. If A is subdirectly irreducible, then $A \in \mathrm{HSP}_U(K)$.*

PROOF. Suppose $A \in V(K)$ is non-trivial and subdirectly irreducible. Then there exist a direct product $\Pi_{i\in I} A_i$ of members of K, an algebra $B \subseteq \Pi_{i\in I} A_i$ and a surjective homomorphism $\alpha: B \to A$. Let $\theta = \ker \alpha$. For $J \subseteq I$ we put

$$\theta_J = \{(a, b) \in B^2: I \subseteq [\![a = b]\!]\} .$$

Then θ_J is a congruence. We put

$$W = \{J \subseteq I: \theta_J \subseteq \theta\} .$$

We show that there is an ultrafilter U contained in W. Of course, $I \in W$ and, if $J \in W$ and $J \subseteq M \subseteq I$, then $M \in W$. Furthermore, for each $J_0, J_1 \subseteq I$ we have $\theta_{J_0} \cap \theta_{J_1} = \theta_{J_0 \cup J_1}$. $J_0 \cup J_1 \in W$ implies $\theta = \theta \vee \theta_{J_0 \cup J_1}$ and by distributivity, $\theta = (\theta \vee \theta_{J_0}) \cap (\theta \vee \theta_{J_1})$. Now $\mathrm{Con}\, A \cong \mathrm{Con}\, B/\theta \cong [\theta, \nabla_B]$. Remember that A is subdirectly irreducible. Thus, $\theta = \theta \vee \theta_{J_0}$ or $\theta = \theta \vee \theta_{J_1}$, so $J_0 \in W$ or $J_1 \in W$. This implies that there exists an ultrafilter $U \subseteq W$. We set

$$\theta_U = \bigcup \{\theta_J: J \in U\} .$$

Then $\theta_U \subseteq \theta$. Let v be the natural homomorphisms from B to $\Pi_{i\in I} A_i/U$. Then $\ker v = \theta_U \subseteq \theta$ and thus

$$A \cong B/\theta \cong (B/\theta_U)/(\theta/\theta_U) ,$$

where we consider θ in θ/θ_U as a relational structure with one binary relation. Now $B/\theta_U \cong v[B] \subseteq \Pi_{i\in I} A_i/U$ and so $B/\theta_U \in \mathrm{ISP}_U(K)$. Hence $A \in \mathrm{HSP}_U(K)$. □

Arithmetical varieties are important in the following. First we show how to determine whether a variety is arithmetical.

6.6. LEMMA. *Let V be a variety. Then the following are equivalent*:

(i) *V is congruence-permutable*;

(ii) *there is a term $p(x, y, z)$ such that $V \models p(x, x, y) = y \wedge p(x, y, y) = x$.*

PROOF. (i)→(ii). Suppose V is congruence-permutable. Let $F_V(a, b, c)$ be the V-free algebra over $\{a, b, c\}$. In this algebra we have

$$(a, c) \in \theta_{a,b} \circ \theta_{b,c}$$

and as V is congruence-permutable,

$$(a, c) \in \theta_{b,c} \circ \theta_{a,b} .$$

This implies that there is a term $p(x, y, z)$ such that $(a, p(a, b, c)) \in \theta_{b,c}$ and $(p(a, b, c), c) \in \theta_{a,b}$. But then $V \models p(x, y, y) = x \wedge p(x, y, y) = y$.

(ii)$\rightarrow$(i). Let $A \in V$, $\phi, \psi \in \operatorname{Con} A$. If $(a, b) \in \phi \circ \psi$, then for some $c \in A$, $(a, c) \in \phi$ and $(c, b) \in \psi$. Then $(p(c, c, b), p(a, c, b)) \in \phi$ and $(p(a, c, b), p(a, b, b)) \in \psi$. Together with $p(c, c, b) = b$ and $p(a, b, b) = a$ this implies $(b, a) \in \phi \circ \psi$. Thus, V is congruence-permutable. □

6.7. LEMMA. *Let V be a variety. If there is a ternary term $M(x, y, z)$ such that*

$$V \models M(x, x, y) = M(x, y, x) = M(y, x, x) = x ,$$

then V is congruence-distributive.

PROOF. Let $A \in V$, $\phi, \psi, \chi \in \operatorname{Con} A$. Then $(\phi \cap \psi) \vee (\phi \cap \chi) \leq \phi \cap (\psi \vee \chi)$ and to prove distributivity it is enough to show that $\phi \cap (\psi \vee \chi) \leq (\phi \cap \psi) \vee (\phi \cap \chi)$. Let $a, b \in A$ with $(a, b) \in \phi$. Now $(a, b) \in \psi \vee \chi$ is equivalent to:

there exist $c_0, \ldots, c_{n-1} \in A$ with $(a, c_0) \in \psi$,
$(c_0, c_1) \in \chi, \ldots, (c_{n-2}, c_{n-1}) \in \psi$, $(c_{n-1}, b) \in \chi$.

For each $c, d \in A$ we have $M(a, c, a) = M(a, d, a) = a$ and together with $(a, b) \in \phi$ this implies $(M(a, c, b), M(a, d, b)) \in \phi$. Thus, we have $(M(a, a, b), M(a, c_0, b)) \in \phi \cap \psi$; $(M(a, c_0, b), M(a, c_1, b)) \in \phi \cap \chi$; $\ldots$; $(M(a, c_{n-1}, b), M(a, b, b) \in \phi \cap \chi$. This, together with $M(a, a, b) = a$ and $M(a, b, b) = b$, yields $(a, b) \in (\phi \cap \psi) \vee (\phi \cap \chi)$ and thus $\phi \cap (\psi \vee \chi) \leq (\phi \cap \psi) \vee (\phi \cap \chi)$. □

6.8. LEMMA. *Let V be a variety. Then the following are equivalent:*

(i) *V is arithmetical;*

(ii) *there are terms p and M as in Lemma* 6.6 *and Lemma* 6.7;

(iii) *there is a term $m(x, y, z)$ such that $V \models m(x, y, x) = m(x, y, y) = m(y, y, x) = x$.*

PROOF. (i)$\rightarrow$(ii). As V is congruence-permutable, there is a term p as described in Lemma 6.6. Consider $F_V(a, b, c)$, the V-free algebra over a, b, c. Then, by congruence-distributivity, as $(a, c) \in \theta_{a,c} \cap [\theta_{a,b} \vee \theta_{b,c}]$, it follows that $(a, c) \in [\theta_{a,c} \cap \theta_{a,b}] \vee [\theta_{a,c} \cap \theta_{b,c}]$ and by congruence-permutability, $(a, c) \in [\theta_{a,c} \cap \theta_{a,b}] \vee [\theta_{a,c} \cap \theta_{b,c}]$. Thus, there exists an $M(a, b, c) \in F_V(a, b, c)$ with $(a, M(a, b, c)) \in \theta_{a,c} \cap \theta_{a,b}$ and $(M(a, b, c), c) \in \theta_{a,c} \cap \theta_{b,c}$. This implies $V \models M(x, x, y) = M(x, y, x) = M(y, x, x) = x$.

(ii)$\rightarrow$(iii). We set

$$m(x, y, z) = p(x, M(x, y, z)) .$$

Then $m(x, y, z)$ is as desired.

(iii)$\rightarrow$(i). We define p and M by

$$p(x, y, z) = m(x, y, z)$$

and

$$M(x, y, z) = m(x, m(x, y, z), z) .$$

Then Lemma 6.6 and Lemma 6.7 imply that V is congruence-permutable and congruence-distributive. Thus, V is arithmetical. $\square$

EXAMPLE. Let V be the variety of Boolean algebras and put

$$m(x, y, z) = x \cdot z + x \cdot (-y) \cdot (-z) + (-x) \cdot (-y) \cdot z .$$

Then $m(x, y, z)$ is as described in Lemma 6.8(iii) and thus BA is arithmetical.

Let A be an algebra, $t(x, y, z)$ a term in the language of A, such that for every $a, b, c \in A$:

$$t(a, b, c) = \begin{cases} a & \text{if } a \neq b ; \\ c & \text{otherwise} . \end{cases}$$

Then $t(x, y, z)$ is called a *discriminator term* on A.

EXAMPLES. (1) Let V be the variety of Boolean algebras and set

$$t(x, y, z) = (-y + x \cdot z) \cdot (x + z) .$$

Then $t(x, y, z)$ is a discriminator term of the two-element Boolean algebra.

(2) A *monadic algebra* is a pair (B, c) consisting of a Boolean algebra B with a unary function c such that for each $a, b \in B$,

(i) $c(0) = 0$;

(ii) $a \leq c(a)$;

(iii) $c(a \cdot c(b)) = c(a) \cdot c(b)$.

A monadic algebra (B, c) is subdirectly irreducible iff for each $a \in B$ with $a \neq 0$ we have $c(a) = 1$. It is immediately clear that (B, c) is subdirectly irreducible if for each $a \in B \backslash \{0\}$, $c(a) = 1$. Now assume that (B, c) is a monadic algebra and $a \in B \backslash \{0\}$ with $c(a) \neq 1$. We set $b = c(a)$. Then (iii) implies that also $c(-b) = -b$ and there is a natural isomorphism between (B, c) and $(B \restriction b, c_0) \times (B \restriction -c_1)$, where c_0, c_1 are the restrictions of c to $B \restriction b$ and $B \restriction -b$. Thus, (B, c) is not subdirectly irreducible.

A discriminator term on the subdirectly irreducible monadic algebras is given by

$$t(x, y, z) = c(x \triangle y) \cdot x + (-c(x \triangle y) \cdot z .$$

(3) Let $(R, +, -, \cdot, 0, 1, g)$ be a ring with an additional unary operation

$$g(x)=\begin{cases}1 & x\neq 0\,;\\ 0 & \text{otherwise}\,;\end{cases}$$

then the following term is a discriminator:

$$t(x, y, z) = z + (x - z)\cdot g(y - x)\,.$$

Special cases of this example are the following:

(a) The finite fields $\mathrm{GF}(q)$ $(n = 0(\mathrm{mod}\ q)$ with $g(x) = x^{n-1}$.

(b) A ring R is *biregular* iff for each $x \in R$ there is a central idempotent x^+ such that x^+ generates the same principal ideal of R as x (a is central idempotent if $a \cdot a = a$ and for each $b \in R$, $a \cdot b = b \cdot a$). Obviously x^+ is uniquely determined by x, so we can consider $^+$ as a unary operation. The simple biregular rings are unitary and satisfy

$$x^+=\begin{cases}1 & \text{if } x\neq 0\\ 0 & \text{otherwise}\,.\end{cases}$$

So we can choose $g(x) = x^+$.

Let $t(x, y, z)$ be a discriminator term on A. Then we define a new term $s(x, y, u, v)$ by

$$s(x, y, u, v) = t(t(x, y, u), t(x, y, v), v)\,.$$

For any $a, b, c, d \in A$ we have

$$s(a, b, c, d)=\begin{cases}c & \text{if } a=b\,;\\ d & \text{otherwise}\,.\end{cases}$$

Each term $s'(x, y, u, v)$ with

$$s'(a, b, c, d)=\begin{cases}c & \text{if } a=b\,;\\ d & \text{otherwise}\,,\end{cases}$$

for every $a, b, c, d \in A$ is called a *switching term* (for A).

If $s(x, y, u, v)$ is a switching term, then $s(x, y, z, x)$ is a discriminator term. Thus, there exists a discriminator term iff there exists a switching term. This will be used in what follows without explicit mention.

6.9. LEMMA. *Let K be a class of algebras with a common discriminator term. Then $V(K)$ is arithmetical.*

PROOF. We have

$$K \models t(x, y, x) = t(x, y, y) = t(y, y, x) = x\,.$$

Now $V(K)$ is the least class of algebras which contains K and is closed under direct products, subalgebras and homomorphic images. Thus, it is easily seen that also

$$V(K) \models t(x, y, x) = t(x, y, y) = t(y, y, x) = x .$$

Now Lemma 6.6(iii) implies that $V(K)$ is arithmetical. □

6.10. LEMMA. *Let K be a class of algebras with a common discriminator term $t(x, y, z)$. Then, for every $A \in V(K)$ with $|A| \geq 2$, the following are equivalent:*
(i) $A \in \mathrm{ISP}_U(K)$;
(ii) *t is a discriminator term on A;*
(iii) *A is simple;*
(iv) *A is subdirectly irreducible.*

PROOF. (i)→(ii). "t is a discriminator term" is a first-order universal sentence and is thus preserved under the formation of ultraproducts and subalgebras.

(ii)→(iii). Let $\theta \in \mathrm{Con}\, A$, $(a, b) \in \theta$ and $a \neq b$. Then for each $c \in A$, we have $(a, c) = (t(a, b, c), t(a, a, c)) \in \theta$ and thus $\theta = \nabla_A$. Hence, A is simple.

(iii)→(iv) is trivial.

(iv)→(i). Let A be subdirectly irreducible. Now Lemma 6.5 implies (using the fact that $V(K)$ is distributive) that $A \in \mathrm{HSP}_U(K)$. But by (i)→(iii), all members of $\mathrm{ISP}_U(K)$ are simple. Thus, every homomorphic image of A is either isomorphic to A or trivial. □

We say that a variety V is a *discriminator variety* if $V = V(K)$ for some class of algebras and there is a term $t(x, y, z)$ which is a discriminator term for each $A \in K$ simultaneously. Discriminator varieties allow a good representation by means of sheaves.

Let $V = V(K)$ be a discriminator variety with discriminator term $t(x, y, z)$ for each $A \in K$. For $A \in V$ we set

$$\mathrm{Spec}\, A = \{\theta \in \mathrm{Con}\, A\colon (\forall \psi \in \mathrm{Con}\, A)(\theta \subseteq \psi \rightarrow \psi = \theta \vee \psi = \nabla\} ;$$

i.e. Spec A is the set of all maximal congruences on A. This set is called the *spectrum* of A. We endow this set with a topology, the *equalizer topology*, i.e. the topology which is generated by the sets

$$E(a, b) = \{\theta \in \mathrm{Spec}\, A\colon (a, b) \in \theta\}$$

and their complements

$$D(a, b) = \mathrm{Spec}\, A \backslash E(a, b) .$$

It is immediately seen that this space is Hausdorff.

6.11. LEMMA. *The sets $E(a, b)$ and $D(a, b)$ $(a, b \in A)$ form a base of clopen sets for the topology on* Spec A.

PROOF. We show that the system of sets $E(a, b)$ and $D(a, b)$ is closed with respect to intersections. Then, using the fact that the system of sets $E(a, b)$ and $D(a, b)$ is closed with respect to complements, we see that they form a Boolean algebra and thus give a basis of a topology consisting of clopen sets only.

Now we show that

(i) $E(a, b) \cap E(c, d) = E(t(a, b, c), t(b, a, d))$;

(ii) $E(a, b) \cap D(c, d) = D(t(a, b, c), t(a, b, d))$;

(iii) $D(a, b) \cap D(c, d) = D(t(a, t(a, b, c), c), t(a, t(a, b, d), d))$.

We have $\nabla \not\in D(a, b)$ and $\nabla \in E(a, b)$ for all $a, b \in A$. Let $\phi \in \operatorname{Spec} A$. Then, for $a, b, c \in A$,

$(a, b) \in \phi$ implies $(t(a, b, c), c) \in \phi$;

$(a, b) \not\in \phi$ implies $(t(a, b, c), a) \in \phi$;

thus

$(a, b) \in \phi$ and $(c, d) \in \phi$ iff $(t(a, b, c), t(b, a, d)) \in \phi$;

$(a, b) \in \phi$ and $(c, d) \not\in \phi$ iff $(t(a, b, c), t(a, b, d)) \in \phi$;

$(a, b) \not\in \phi$ and $(c, d) \not\in \phi$ iff
$(t, (a, t(a, b, c), c), t(a, t(a, b, d), d)) \not\in \phi$. □

6.12. LEMMA. *Let V be a discriminator variety and $A \in V$ a non-trivial algebra. Then* Spec A *is a Boolean space and the clopen sets of* Spec A *are the sets $E(a, b)$ and $D(a, b)$ for $a, b \in A$.*

PROOF. We have already seen that Spec A is a zero-dimensional Hausdorff space. It remains to show that Spec A is compact.

Let $\{U_i: i \in I\}$ be an open cover of Spec A. W.l.o.g. $\{U_i: i \in I\} \subseteq \{E(a, b): a, b \in A\} \cup \{D(a, b): a, b \in A\}$. Let $i_0 \in I$ be such that $\nabla \in U_{i_0}$. Then there are $a_0, b_0 \in A$ with $U_{i_0} = E(a_0, b_0)$. For each $i \in I$ we set

$$V_i = U_i \cap D(a_0, b_0).$$

Then $\{V_i: i \in I\}$ is an open cover of $D(a_0, b_0)$ and by Lemma 6.11 for each $i \in I$ there are $c_i, d_i \in A$ with $V_i = D(c_i, d_i)$. Assume that for each finite $K \subseteq I$,

$$D(a_0, b_0) \setminus \bigcup \{D(c_i, d_i): i \in K\} \neq \emptyset.$$

Consider the congruence θ which is generated by $\{(c_i, d_i): i \in I\}$. Then $(a_0, b_0) \not\in \theta$ and thus θ can be extended to a maximal congruence θ^* with $(a_0, b_0) \not\in \theta^*$. But then $\theta^* \in D(a_0, b_0) \setminus \bigcup \{V_i: i \in I\}$, which contradicts our assumption that $\{V_i: i \in I\}$ is an open cover of $D(a_0, b_0)$. □

Let A be any algebra of a discriminator variety V. For each $\phi \in \operatorname{Spec} A$ let $\rho_\phi\colon A \to A/\phi$ be the canonical map. Define

$$\rho\colon A \to \prod\{A/\phi\colon \phi \in \operatorname{Spec} A\}$$

by

$$\rho(a) = (\rho_\phi(a))_{\phi \in \operatorname{Spec} A}\,.$$

For $a, b \in A$ we have

$$\{\phi \in \operatorname{Spec} A\colon \rho_\phi(a) = \rho_\phi(b)\} = E(a, b)\,,$$

a clopen subset of $\operatorname{Spec} A$.

Let S be the disjoint union of $\{A/\phi\colon \phi \in \operatorname{Spec} A\}$. We define a map $\Pi\colon S \to \operatorname{Spec} A$ by

$$\Pi(s) = \phi \quad \text{iff } s \in A/\phi\,.$$

So we have a sheaf

$$S(A) = (S, \Pi, \operatorname{Spec} A, (A/\phi)_{\phi \in \operatorname{Spec} A}),$$

and ρ gives an embedding of A into $S(A)$. Moreover, we have

6.13. LEMMA. *Let V be a discriminator variety, $A \in V$. Then $\rho\colon A \to \Gamma(S(A))$ is an isomorphism.*

PROOF. Let $f \in S(A)$. We have to show that there is an $a \in A$ with $\rho(a) = f$.

For every $\phi \in \operatorname{Spec} A$ we have $f(\phi) \in A/\phi$, so there is a $b_\phi \in A$ with $f(\phi) = \rho(b_\phi)(\phi)$. Thus, there is neighbourhood U_ϕ of ϕ such that $\theta \in U_\phi$ implies $f(\theta) = \rho(b_\phi)(\theta)$. W.l.o.g. we assume that the U_ϕ are clopen. Now $\{U_\phi\colon \phi \in \operatorname{Spec} A\}$ is an open cover of $\operatorname{Spec} A$. By compactness there is a finite subcover $\{U_{\phi_i}\colon i < n\}$. For $i < n$ we set

$$V_i = U_{\phi_i} \Big\backslash \bigcup_{k<i} V_k\,.$$

The V_i are a clopen partition of $\operatorname{Spec} A$ and we assume that $\nabla \in V_0$. Then there are elements $c_1, \ldots, c_{n-1}, d_1, \ldots, d_{n-1} \in A$ with $V_i = D(c_i, d_i)$ for $i = 1, \ldots, n-1$. Using the switching term s we define $a_0, \ldots, a_{n-1}$ recursively as follows:

$$a_0 = c_0\,;$$

$$a_{i+1} = s(c_{i+1}, d_{i+1}, a_i, b_{\phi_i}) \quad \text{for } i < n-1\,.$$

Then a_{n-1} is as desired. $\square$

In the rest of this section we only consider sheaves which are Hausdorff, without mentioning it. Let L be any language for first-order predicate logic and let

$$\mathcal{S} = (S, \Pi, X, (B_p)_{p\in X})$$

be a sheaf of L-structures. For U an open subset of X, $\mathcal{S}|U$ denotes the restriction of $\mathcal{S}$ to U, i.e. the sheaf

$$(\Pi^{-1}[U], \Pi|\Pi^{-1}[U], U, (B_p)_{p\in U}) .$$

Let X be a topological space, $A \in \operatorname{Mod} L$. Then $\operatorname{Const}(X, A)$ denotes the sheaf

$$(X \times A, \Pi, X, (A_p)_{p\in X}) ,$$

where, in $X \times A$, A is endowed with the discrete topology and Π is the projection map onto the first component, i.e. $\Pi(p, a) = p$ for each $a \in A$, $p \in X$, and $A_p = A$ for each $p \in X$.

For $a \in A$, $e(a)$ denotes the *constant section* with value a, i.e. $e(a)(p) = a$ for each $p \in X$.

Let $\mathcal{S} = (S, \Pi, X, (B_p)_{p\in X})$ and $\mathcal{S}' = (S', \Pi', X', (B'_p)_{p\in X'})$ be sheafs. $\mathcal{S}'$ is a *subsheaf* of $\mathcal{S}$ if

(i) $S' \subseteq S$;
(ii) $X = X'$;
(iii) $\Pi' = \Pi | S'$;
(iv) $B'_p \subseteq B_p$ for each $p \in X$;

(here $S' \subseteq S$ expresses that S' is endowed with the subspace topology).

Let $\mathcal{S} = (S, \Pi, X, (B_p)_{p\in X})$ and $\varphi' = (S', \Pi', X'_{\tau}(B'_p)_{p\in X'})$ be sheafs, $f: S' \to S$, $S_0 = f[S']$. We consider S_0 as a subspace of S. Then f is an *embedding* of $\mathcal{S}'$ into $\mathcal{S}$ if f is a homeomorphism between S' and S_0, $\Pi \circ f = \Pi'$, $X' = X$ and for each $p \in X$, $f | B'_p$ is an embedding of B'_p into B_p (as L-structures).

Let $\operatorname{Const}(X, A)$ be a constant sheaf. Then $\mathcal{S} = (S, \Pi, X, (A_p)_{p\in X})$ is *embeddable into* $\operatorname{Const}(X, A)$ if for every $p \in X$ there is a neighbourhood U of p such that $\mathcal{S}|U$ is embeddable into $\operatorname{Const}(U, A)$ as a subsheaf.

Let X be a Boolean space. Then $f[X]$ is finite for each global section f of $\operatorname{Const}(X, A)$ and $f^{-1}(a) \in \operatorname{Clop} X$ for each $a \in A$. Instead of $\Gamma(\operatorname{Const}(X, A))$ we write $A[X]$ and call it the (bounded) Boolean power.

6.14. LEMMA. *Let $\mathcal{S} = (S, \Pi, X, (B_p)_{p\in X})$ be a sheaf, X a Boolean space and A a finite structure. If each B_p is isomorphic to A, then $\mathcal{S} \cong \operatorname{Const}(X, A)$.*

PROOF. Let $\{a_0, \ldots, a_{n-1}\}$ be the underlying set of A, and let D be the set of all atomic and negated atomic formulas $\varphi(\vec{x})$ with $A \models \varphi(\vec{a})$. For each $p \in X$ choose sections σ_a ($a \in A$) such that $a \mapsto \sigma_a(p)$ is an isomorphism. Put

$$V_p = \left[\!\left[(\wedge D)(\sigma_{a_0}, \ldots, \sigma_{a_{n-1}})\right]\!\right] .$$

Then $\{V_p: p \in X\}$ is a clopen cover of X. Now X is compact, thus we can find finitely many sets $V_{p_0}, \ldots, V_{p_{k-1}}$ which cover X. Choose pairwise disjoint clopen sets U_i $(i<k)$ with

$$U_i \subseteq V_{p_i} \quad (i<k);$$
$$\bigcup_{i<k} U_i = X.$$

For each $i<k$ we have a canonical isomorphism

$$f_i: \mathcal{S} \mid U_i \to \mathrm{Const}(U_i, A).$$

Then $\bigcup_{i<k} f_i$ is the desired isomorphism. □

A Boolean space has the *disjoint refinement property* if for every collection of open subsets of X, $\{U_i: i \in I\}$, there is a collection $\{V_i: i \in J\}$ of pairwise disjoint open sets such that $\bigcup_{i\in I} U_i = \bigcup_{j \in J} V_j$ and every V_j is contained in some U_i.

If X has a countable basis, then X has the disjoint refinement property.

Similar to the proof of Lemma 6.14 one can prove:

6.15. LEMMA. *Let $\mathcal{S} = (S, \Pi, X, (B_p)_{p\in X})$ be a sheaf, X a Boolean space with the disjoint refinement property, A a finite relational structure, U an open subset of X such that $\mathcal{S} \mid U$ is locally embeddable into* Const(U, A). *Then $\mathcal{S} \mid U$ is embeddable into* Const(U, A) *as a subsheaf.*

Let A and B be relational structures. Then a structure C together with an embedding $f: A \rightarrowtail C$ is said to be an *amalgamation of B over A* if for every embedding $g: A \rightarrowtail B$ there exists an embedding $g': B \rightarrowtail C$ such that $g' \circ g = f$.

6.16. LEMMA. *Let $\mathcal{S} = (S, \Pi, X, (B_p)_{p\in X})$ be a sheaf, X a Boolean space, A, B finite relational structures and $U \subseteq X$ an open set such that*

(i) *A is embeddable into B_p for each $p \in X$;*

(ii) *$\mathcal{S} \mid U$ is embeddable into* Const(U, B) *as a subsheaf;*

(iii) *for $p \in X \setminus U$, $A = B_p$.*

Furthermore, let $j: A \rightarrowtail C$ be an amalgamation of B over A with C finite. Then $\mathcal{S}$ is embeddable into Const(X, C) *as a subsheaf.*

PROOF. Similarly to the preceding lemma we choose sections $\sigma_a \in \Gamma(\varphi)$ $(a \in A)$ such that for every $p \in X$, $h_p: a \mapsto \sigma_a(p)$ is an embedding of A into B_p. Then h_p is an isomorphism for each $p \in X \setminus U$.

Choose an embedding $f: \mathcal{S} \mid U \rightarrowtail \mathrm{Const}(U, B)$. For $a \in A$ let $\tau_a = f[\sigma_a \mid U]$. Then $\tau_a \in \mathrm{Const}(U, B)$. For $p \in U$, $f \circ h_p$ is an embedding of A into B. Let $\rho \in \Gamma(\mathrm{Const}(U, B))$, $b \in B$; then $\{p \in U: \rho(p) = b\}$ is open. Thus, for every embedding $g: A \rightarrowtail B$, the set $V_g = \{p \in U: f \circ h_p = g\}$ is open and the V_g's are pairwise disjoint. For every embedding $g: A \rightarrowtail B$ there is an embedding $\bar{g}: B \rightarrowtail C$ with $j = \bar{g} \circ g$. So we have embeddings $\bar{g}_{V_g}: \mathrm{Const}(V_g, B) \rightarrowtail \mathrm{Const}(V_g, C)$ by setting

$$\bar{g}_{V_g}(b, p) = (\bar{g}(b), p) \quad (p \in V_g, b \in B).$$

Then the union of these embeddings gives an embedding

$$\bar{g}_U\colon \mathrm{Const}(U, B) \rightarrowtail \mathrm{Const}(U, C),$$

and for every $a \in A$, τ_a is mapped onto the constant section $e(j(a))$. We define a map $h\colon S \to C \times X$ by

$$h(s) = \begin{cases} (j(a), s) & \text{if } \Pi(s) \not\in U \text{ and } \sigma_a(\Pi(s)) = s; \\ \bar{g}_U \circ f(s) & \text{if } \Pi(s) \in U. \end{cases}$$

Then $h[\sigma_a] = e(j(a)) \in \mathrm{Const}(X, C)$.

In order to show that h is the desired embedding we must check the continuity of h. We have $h \restriction \mathcal{S} | U = \bar{g}_U \circ f$, so h is continuous on $\mathcal{S} | U$. Now let $s \in \mathcal{S} | (X \backslash U)$. For some $a \in A$, $s = \sigma_a(\Pi(s))$. Let N be a clopen neighbourhood of $\Pi(s)$. Then $V = \{(j(a), p)\colon p \in N\}$ is a neighbourhood of $h(s)$ and $h^{-1}(V)$ is the open set $\{\sigma_a(p)\colon p \in N\}$. □

6.17. LEMMA. *Let $X \in BS$, A a finite relational structure, $\mathcal{S}$ a subsheaf of* Const (X, A), *and for $B \subseteq A$ let*

$$X_B = \{p \in X\colon A_p \subseteq B\}.$$

Then all X_B are closed and $\Gamma(\mathcal{S})$ consists of those continuous $\sigma\colon X \to A$ that satisfy $\sigma[X_B] \subseteq B$ for each $B \subseteq A$.

PROOF. For $C \subseteq A$ we set

$$U_C = \{p \in X\colon C \subseteq A_p\}.$$

Then each U_C is open and

$$X_B = \bigcap \{X \backslash U_C\colon C \not\subseteq B\}.$$

Thus, X_B is closed. $\sigma\colon X \to A$ is a section of $\mathcal{S}$ iff σ is continuous and $x \in X_B$ implies $\sigma(x) \in B$. □

A variety V is said to be *residually small* if it has only finitely many non-isomorphic simple algebras and if these are finite.

For the rest of this section, let V be a fixed residually small discriminator variety.

Let K be a maximal set of pairwise non-isomorphic simple V-algebras (including the trivial algebra Tr), i.e. K represents all isomorphism types of simple V-algebras (including the trivial algebra). As V is residually small and has a discriminator term, K is a finite set of finite algebras and thus partially ordered by the embeddability relation $\leq$.

A system $(A, (F_i)_{i\in I})$ is a *partial algebra* if A is a set and for each $i \in I$, F_i is a partial function on A. A partial algebra $(A, (F_i)_{i\in I})$ is said to be *complete* if, for each $i \in I$, F_i is a function.

By order induction we construct for every $A \in K$ a partial algebra P_A, all complete subalgebras of which are isomorphic to members of K.

If A is maximal with respect to $\leq$, we set $P_A = A$. If A has the upper neighbours $A_0, \ldots, A_{n-1}$, let Q_A be the disjoint union of $P_{A_0}, \ldots, P_{A_{n-1}}$. Let E_A denote the set of all embeddings $f\colon A \rightarrowtail Q_A$ and let $Q_A \times E_A$ be the disjoint union of $|E_A|$ copies of Q_A. Then $Q_A \times E_A$ is a partial algebra and we define an equivalence relation $\sim$ by

$$\begin{aligned}(a, f) \sim (b, g) \text{ iff } & a = b \text{ and } f = g \text{ or} \\ & \text{for some } c \in A,\ a = f(c) \text{ and } b = g(c)\,.\end{aligned}$$

Let P_A be the quotient algebra $Q_A \times E_A/\sim$. Then it is easily seen that P_A is an amalgamation of Q_A over A. If $A_0, \ldots, A_{k-1}$ are the minimal elements of $K\backslash\{Tr\}$, let P be the disjoint union of $P_{A_0}, \ldots, P_{A_{k-1}}$. Let F denote the system of complete subalgebras of P. Then each $B \in F$ is isomorphic to some member of K.

Now we show

Claim 1. Let $R \in V$ be countable. Then $S(R)$ is embeddable as a subsheaf into $\mathrm{Const}(\mathrm{Spec}\, R, P)$.

Proof of the Claim. By Lemma 6.15 it is sufficient to show that $S(R)$ is locally embeddable into $\mathrm{Const}(\mathrm{Spec}\, R, P)$.

Let $\phi \in \mathrm{Spec}\, R$ and $R/\phi \cong A \in K$. Choose sections σ_a $(a \in A)$ such that the map $a \mapsto \sigma_a(\phi)$ is an isomorphism between A and R/ϕ. Then there is a clopen neighbourhood N of ϕ such that $a \mapsto \sigma_a(\psi)$ is an embedding of A into R/ψ for each $\psi \in N$.

By order induction on $A \in K$ we show that $S(R)|N$ is embeddable into $\mathrm{Const}(N, P_A)$.

(i) If A is maximal in K, then all mappings $a \mapsto \sigma_a(\psi)$ $(\psi \in N)$ are isomorphisms. Hence, $S(R)|N \cong \mathrm{Const}(N, A) \cong \mathrm{Const}(N, P_A)$.

(ii) Let A have the upper neighbours $A_0, \ldots, A_{n-1}$ in K. We set

$$U = \{\phi \in N\colon \text{for some } i < n,\ A_i \text{ is embeddable into } R/\phi\}\,.$$

Then U is open and by induction hypothesis, $S(R)|U$ is locally embeddable into $\mathrm{Const}(U, Q_A)$ as Q_A contains all the algebras $P_{A_0}, \ldots, P_{A_{n-1}}$. $\mathrm{Spec}\, R$ has a countable basis and thus the disjoint refinement property. So we can apply Lemma 6.15 to get a global embedding of $S(R)|U$ into $\mathrm{Const}(U, Q_A)$. P_A is an amalgamation of Q_A over A, so we can apply Lemma 6.16 to obtain an embedding of $S(R)|N$ into $\mathrm{Const}(N, P_A)$. As P contains all algebras P_A $(A \in K)$, the claim is proved.

For each countable $R \in V$ we have an embedding γ of $S(R)$ into Const $(\mathrm{Spec}\, R, P)$ which defines a closed subset X_B of $\mathrm{Spec}\, R$ for each $B \in F$.

$$X_B = \{\phi \in \mathrm{Spec}\, R\colon \text{the stalk over } \phi \text{ of } \gamma[S(R)] \text{ is included in } B\}\,.$$

Then by Lemma 6.17, R is isomorphic to the algebra of continuous sections

f: Spec $R \to P$ such that for each $B \in F$, $f[X_B] \subseteq B$. Moreover, Spec $R = \bigcup \{X_B: B \in F\}$, and for $B \in F$ the closed set X_B can be coded by the filter $\hat{X}_B$ of all clopen sets of Spec R including X_B.

Claim 2. The mapping $f \mapsto (f^{-1}(a))_{a \in P}$ is a bijection between the continuous mappings f: Spec $R \to P$ (where P is endowed with the discrete topology) and partitions $(x_a)_{a \in P}$ of Clop(Spec R). Moreover, f: Spec $R \to P$ satisfies $f[X_B] \subseteq B$ iff $\Sigma \{x_a: a \in B\} \in \hat{X}_B$.

For $\theta \in \operatorname{Con} R$ let

$$\theta^* = \{\phi \in \operatorname{Spec} R: \theta \subseteq \phi\} .$$

Then θ^* is closed and we write $\hat{\theta}$ for $\hat{\theta}^*$.

Claim 3. Let $r, s \in R$, and let $(x_a)_{a \in P}$ and $(y_a)_{a \in P}$ be the corresponding partitions. For $\theta \in \operatorname{Con} R$, we have $(r, s) \in \theta$ iff $\Sigma \{x_a \cdot y_a: a \in P\} \in \hat{\theta}$.

Let $\{c_0, \ldots, c_{n-1}\}$ be the underlying set of P. We use L to denote the language of V with quantification over congruences and with the quantifier Q_0 for individual variables. Replacing each constant c of L by a unary function f_c with constant value c we can assume that L does not contain any constants.

We are going to show that $\mathrm{Th}^{\mathrm{con}, Q_0}(V_\omega)$, the theory of countable algebras of V in the language L, is decidable.

Let L^* be the language of $\mathrm{Th}^{\mathrm{fil}, Q_0}(\mathrm{BA}_\omega)$ together with a set $\{X_B: B \in F\}$ of distinguished ideals. We define a function $*$ from Form L into Form L^* as follows:

$$(x = y)^* := \bigwedge_{a \in P} x_a = y_a ;$$

$$(t(x_0, \ldots, x_{m-1}) = y)^* := \bigwedge_{a \in P} \left[y_a = \Sigma \left\{ \prod_{i<m} x_{i, a_i}: t(a_0, \ldots, a_{m-1}) = a \right\} \right] ;$$

$$((x, y) \in \theta)^* := \Sigma \{x_a \cdot y_a: a \in P\} \in \hat{\theta}$$

(where $\hat{\theta}$ is a variable for filters);

$$(\neg \varphi)^* := \neg \varphi^* ;$$

$$(\varphi \wedge \psi)^* := \varphi^* \wedge \psi^* ;$$

we use $\operatorname{adm}(x_a)_{a \in P}$ as an abbreviation for

$$\bigwedge_{\substack{a \neq b \\ a, b \in P}} x_a \cdot x_b = 0 \wedge \Sigma \{x_a: a \in P\} = 1 \wedge \bigwedge_{B \in F} \Sigma \{x_a: a \in B\} \in \hat{X}_B ;$$

$$(\exists x \varphi)^* := (\exists x_a)_{a \in P} [\varphi^* \wedge \operatorname{adm}(x_a)_{a \in P}] ;$$

$$(\exists \theta \varphi)^* := \exists \hat{\theta} \varphi^* ;$$

$$(Q_0 x \varphi]^* := \bigvee_{c \in P} Q_0 x_c (\exists x_b)_{b \in P \setminus \{c\}} [\operatorname{adm}(x_a)_{a \in P} \wedge \varphi^*] .$$

Let $R \in V$ be a countable algebra. Then, for each sentence φ of L,

$$R \models \varphi \quad \text{iff } (\operatorname{Clop}(\operatorname{Spec} R), (\hat{X}_B)_{B \in F}) \models \varphi^* .$$

This can be easily verified by induction on the length of φ. This implies:

$$\text{for each } \varphi \in \text{Sent } L\,,$$

$$V_\omega \models \varphi \quad \text{iff } \mathrm{BA}_\omega \models (\exists X_B)_{B\in F}\Big(\forall x\Big(\bigwedge_{B\in F} x \in X_B\Big) \rightarrow x = 1\Big) \wedge \varphi^*\,.$$

Thus we have reduced the decision problem of $\mathrm{Th}^{\mathrm{con},\mathcal{Q}_0}(V_\omega)$ to that of $\mathrm{Th}^{\mathrm{fil},\mathcal{Q}_0}(\mathrm{BA}_\omega)$ and the decidability of the latter theory (see Theorem 2.15) implies

6.18. THEOREM. *Let V be a residually small discriminator variety. Then $\mathrm{Th}^{\mathrm{con},\mathcal{Q}_0}(V_\omega)$ is decidable.*

REMARK. A variety V is *modular* if for each $A \in V$, Con A is modular. BURRIS and MCKENZIE [1981] showed that every finitely generated modular variety of finite type is decidable iff it is a varietal product of a finitely generated discriminator variety and a decidable Abelain variety.

The following examples are presented to illustrate the use of Theorem 6.14. For further information see WERNER [1978].

EXAMPLES. (1) Every residually finite variety of monadic algebras has a decidable theory. This was shown by COMER [1975].

(2) A ring R is said to be an *m-ring* if $R \models \forall x(x^m = x)$. It is known that m-rings are commutative (see JACOBSON [1956, p. 217]). The subdirectly irreducible m-rings are the finite fields GF(q) with $m = 0$ (mod q). We have seen that there is a discriminator term for these fields; thus the variety of m-rings has a decidable theory (see COMER [1974]).

(3) Residually small varieties of biregular rings are decidable.

7. Boolean algebras with a distinguished finite automorphism group

For $(G \circ e)$ a group, let BA(G) denote the class of algebras $(B, (g)_{g\in G})$ such that B is a Boolean algebra and the g's are unary functions representing automorphisms of B, i.e. for $g, h \in G$, $a, b \in B$ they satisfy

(i) $g(a + b) = g(a) + g(b)$;
(ii) $g(-a) = -g(a)$;
(iii) $g(h(a)) = (g \circ h)(a)$;
(iv) $e(a) = a$.

It was shown by WOLF [1975] that BA(G) has a decidable theory for G a finite solvable group. This was generalized independently by BURRIS [1982] and JURIE [1982]:

7.1. THEOREM. *For every finite group G,* Th(BA(G)) *is decidable.*

PROOF. Let G be a fixed finite group. If $B_G \in \mathrm{BA}(G)$, then B denotes the underlying Boolean algebra. For $B_G \in \mathrm{BA}(G)$ we define a unary term $c(x)$ by

$$c(x) = \sum_{g \in G} g(x) .$$

Claim 1. (B, c) is a monadic algebra.

Proof of Claim 1. For the definition of a monadic algebra see the example before Lemma 6.9. It is immediately seen that for each $a, b \in B$,

(i) $c(0) = 0$

and

(ii) $a \leq b \rightarrow c(a) \leq c(b)$.

Furthermore, we have

$$\begin{aligned} c(a \cdot c(b)) &= \sum_{g \in G} [g(a) \cdot c(b)] = \sum_{g \in G} \left[g(a) \cdot \left(\sum_{h \in G} h(b) \right) \right] \\ &= \sum_{g \in G} \sum_{h \in G} g(a) \cdot h(b) = \sum_{g \in G} g(a) \cdot \sum_{h \in G} h(b) \\ &= c(a) \cdot c(b) . \end{aligned}$$

This shows that (B, c) is a monadic algebra.

The closed elements of (B, c) (i.e. those $a \in B$ with $c(a) = a$) are just the fixed points of G. Let θ be a congruence of B_G. Then $0/\theta$ is an ideal of B, and if $(a, 0) \in \theta$, then $(g(a), g(0)) = (g(a), 0) \in \theta$ for each $g \in G$; so $0/\theta$ is closed under g.

Conversely, let I be an ideal of B which is closed under G. Then the corresponding congruence θ on B, defined by

$$(a, b) \in \theta \quad \text{iff } a \,\triangle\, b \in I ,$$

is also a congruence on B_G: $(a, b) \in \theta$ iff $a \triangle b \in I$, so $g(a \triangle b) = g(a) \triangle g(b) \in I$, hence $(g(a), g(b)) \in \theta$ for each $g \in G$.

It is easily seen that the congruences on (B, c) correspond to the ideals on B which are closed under c. Thus, considering the definition of c, B_G and (B, c) have the same congruences. Recall that a monadic algebra (B, c) is subdirectly irreducible iff for each $a \in B \backslash \{0\}$, $c(a) = 1$.

Claim 2. If B_G is subdirectly irreducible, then $|B_G| \leq 2^{|G|}$.

Proof of Claim 2. Let $B_G \in \mathrm{BA}(G)$ and assume $|B_G| > 2^{|G|}$. If D is a finite subset of B, then the subalgebra of B_G generated by D is the subalgebra of B generated by $\{g(a): a \in D, g \in G\}$ and thus a finite subalgebra of B_G. Let A be a finite subalgebra of B_G with $|A| > 2^{|G|}$. If a is an atom of A, then $g(a)$ is also an atom for each $g \in G$. But then $c(a) < 1$ since $|\{g(a): g \in G\}| \leq |G|$ and this is less than the number of atoms of A. So B_G is not subdirectly irreducible.

Since the subdirectly irreducible members of $\mathrm{BA}(G)$ have cardinality $\leq 2^{|G|}$, it follows that $\mathrm{BA}(G)$ is generated by finitely many finite algebras. As in the case of monadic algebras, the ternary term,

$$t(x, y, z) = x \cdot c(x \triangle y) + z \cdot (-c(x \triangle y)),$$

is a discriminator term on each of the subdirectly irreducible members of BA(G).

Thus, BA(G) is a residually small discriminator variety and hence Th(BA(G)) is decidable. □

REMARKS. (1) BURRIS [1982] showed that BA($\Sigma_\omega Z_2$) is also decidable.

(2) It will be shown that for many infinite groups G, BA(G) has an undecidable theory.

HEINRICH, HENSON and MOORE [1986] investigated in more detail a subtheory of BA(Z_2). They needed this theory to obtain a complete description of a special class of Banach spaces. Here we outline their methods.

Let $B \in \mathrm{BA}$, $\sigma \in \mathrm{Aut}(B)$. σ is said to be an *involution* if σ^2 is the identity. Th(BA(Z_2)) is equivalent to the theory of the class $\{(B, \sigma)\colon B \in \mathrm{BA}, \sigma$ is an involution of $B\}$. Let us say that the involution σ is *strong*, if there is at most one non-trivial ultrafilter which is invariant under σ. If σ is strong, then let $U(\sigma)$ denote the ultrafilter which is left invariant if such an ultrafilter exists; otherwise let $U(\sigma)$ denote B.

Let BA(sin) be the class of all (B, σ) such that σ is a strong involution. We show that we can give a description of all complete extensions of Th(BA(sin)). In fact the theory Th(B, σ) for each $(B, \sigma) \in$ BA(sin) is uniquely determined by Th($B, U(\sigma)$).

In what follows, by ultrafilter we denote any element of $\mathrm{Ult}\, B \cup \{B\}$; the elements of Ult B are said to be non-trivial ultrafilters.

First we show that there is a sentence which says that an involution is strong. Let $B \in \mathrm{BA}$, σ an involution of B. For $a \in B$, $a \cdot \sigma(a)$ is left fixed under σ; we have $\sigma(a \cdot \sigma(a)) = \sigma(a) \cdot \sigma^2(a) = \sigma(a) \cdot a$. Let $p \in \mathrm{Ult}\, B \cup \{B\}$. We say that p is *invariant under* σ if $p = \{\sigma(b)\colon b \in p\}$. We say that $b \in B$ *splits* if there exists $c \in B$ with $b = c + \sigma(c)$ and $c \cdot \sigma(c) = 0$. In this case we say that b is split by c. According to this condition, 0 always splits. If b is split by c, then $\sigma(b) = \sigma(c) + \sigma^2(c) = \sigma(c) + c = b$. If $b_0 \cdot b_1 = 0$, b_0 is split by c_0 and b_1 is split by c_1, then $b_0 + b_1$ is split by $c_0 + c_1$. Assume that p is invariant under σ and $b \in p$. Then b cannot split. For if b is split by c, then $c \in p$ or $\sigma(c) \in p$ and as p is invariant under σ, both have to be in p, which is impossible.

7.2. LEMMA. *Let $B \in \mathrm{BA}$, σ an involution on B, and $b \in B$. Then there exists $p \in \mathrm{Ult}\, B$ with $b \in p$ which is left invariant under σ iff $b \cdot \sigma(b)$ does not split.*

PROOF. We have already seen that an invariant ultrafilter cannot contain an element which splits. Thus, suppose now that $b \cdot \sigma(b)$ does not split. Consider the family of all $F \subseteq B$ such that $b \in F$ and for all finite sets $\{a_0, \ldots, a_{k-1}\} \subseteq F$, $\Pi_{i<k}\, a_i \cdot \sigma(a_i)$ does not split. Choose a member F of this family which is maximal under $\subseteq$; such an F exists by Zorn's Lemma. By the above remarks F is closed under products and $a_0 \leq a_1$ and $a_0 \in F$ implies $a_1 \in F$. We show that for each $a \in B$ we have $a \in F$ or $-a \in F$. Suppose $a \notin F$ and $-a \notin F$. Then there exists

$c \in F$ such that $a \cdot c \cdot \sigma(a \cdot c)$ and $-a \cdot c \cdot \sigma(-a \cdot c)$ both split. But this implies that $c \cdot \sigma(c)$ splits, contradicting the assumption. □

The above lemma implies that σ does not leave any non-trivial ultrafilter invariant exactly when 1 splits and leaves exactly one non-trivial ultrafilter invariant exactly when $\{b \in B: b \cdot \sigma(b)$ does not split$\} \in \mathrm{Ult}\, B$. It is easily seen that there is an elementary sentence φ which states: "1 splits or the set of b such that $b \cdot \sigma(b)$ does not split is an ultrafilter."

We introduce the following modified product. Let $B \in \mathrm{BA}$ and $p \in \mathrm{Ult}\, B \cup \{B\}$. Then $B \times_p B$ is the subalgebra of $B \times B$ consisting of all $(a, b) \in B \times B$ such that either $a, b \in p$ or $a, b \not\in p$. Then $B \times_B B = B \times B$. $p \times p$ is an ultrafilter on $B \times_p B$. We define the "flip" automorphism $\tau = \tau(B, p)$ on $B \times_p B$ by $\tau(a, b) = (b, a)$. Then evidently τ is an involution which leaves $p \times p$ invariant. We show that this is the only invariant ultrafilter on $B \times_p B$: Let $q \in \mathrm{Aut}(B \times_p B)$ with $q \neq p \times p$: Then there is $(a, b) \in q$ with $a, b \not\in p$. But then $(a, 0) \in q$ or $(0, b) \in q$ and $\tau[q] = q$ implies $(0, 0) \in q$.

In the countable case this gives the typical structure of the pairs (B, σ) such that σ is a strong involution:

7.3. LEMMA. *Let B be a countable Boolean algebra and σ a strong involution on B. Then there are $C \in \mathrm{BA}$ and $p \in \mathrm{Ult}\, C \cup \{C\}$ such that*

$$(B, \sigma) \cong (C \times_p C, \tau(C, p)) .$$

PROOF. First assume that $U(\sigma) = B$, i.e. that σ does not leave any non-trivial ultrafilter invariant. Then 1 splits, say by a. We set $C = B \restriction a$, $p = C$. Then it is easily seen that

$$(B, \sigma) \cong (C \times C, \tau(C, C)) = (C \times_p C, \tau(C, C)) .$$

Now assume that $U(\sigma) \neq B$. We set $I = B \setminus U(\sigma)$. Lemma 7.2 implies that for every $a \in I$, $a \cdot \sigma(a)$ splits. Now $|B| \leq \aleph_0$ implies that there exists an increasing chain $c_0 \leq c_1 \leq \cdots$ in I such that $c_n \cdot \sigma(c_n) = 0$ for all n and for each $a \in I$ there exists n with $a \leq c_n + \sigma(c_n)$. Let J be the ideal generated by $\{c_n: n < \omega\}$. Then J and $\sigma[J]$ are disjoint and J and $\sigma[J]$ generate I. Thus, for each $a \in I$ there are $a_0 \in J$ and $a_1 \in \sigma[J]$ with $a_0 + a_1 = a$; a_0 and a_1 are uniquely determined.

Let $C = B/J$ and let $\pi: B \to C$ be the canonical homomorphism. We set $p = \pi[U(\sigma)]$. Since $J \subseteq I$ and $I \cap U(\sigma) = \emptyset$, p is a non-trivial ultrafilter on C. Moreover, for any $a \in B$, $a \in U(\sigma)$ iff $\pi(a) \in p$. We define a homomorphism $\varphi: B \to C \times_p C$ by $\varphi(a) = (\pi(a), \pi(\sigma(a)))$. If $\varphi(a) = (0, 0)$, then $\pi(a), \pi(\sigma(a)) \in J$ and thus $a = 0$, so φ is one-to-one. It remains to show that φ maps B onto $C \times_p C$. It is enough to show that φ maps I onto $(C \times_p C) \setminus (p \times p)$.

Let $(\pi(a), \pi(b)) \in (C \times_p C) \setminus (p \times p)$. Then, $a, b \in I$ and thus there are a_0, a_1, b_0, b_1 with $a = a_0 + a_1$, $b = b_0 + b_1$, $a_0, b_0 \in J$, $a_1, b_1 \in \sigma[J]$. Let $c = a_1 + \sigma(b_1)$ then $\pi(c) = \pi(a_1) = \pi(a)$ and $\pi(\sigma(c)) = \pi(b_1) = \pi(b)$. Therefore $\varphi(c) = (\pi(a), \pi(b))$ and thus φ is onto. □

In what follows we need invariants for structures (B, p) which were introduced by Eršov [1964]. They generalize the Tarski invariants for Boolean algebras as described in Part I, Section 18. We use the same notations as in that chapter. Let $U \in \mathrm{Ult}\, B \cup \{B\}$. We define $U^{(0)} = U$ and for each $i \geq 0$, $U^{(i+1)} = U^{(i)}/E(B^{(i)})$. So each $U^{(i)}$ is an ultrafilter on $B^{(i)}$. We set

$$r = \begin{cases} \text{the least } i < \omega \text{ such that } U^{(i+1)} = B^{(i+1)} \text{ if such an } i \text{ exists}; \\ \omega \text{ otherwise}; \end{cases}$$

$$s = \begin{cases} 0 & \text{if } r \in \{-1, \omega\} \text{ or } U^{(r)} \text{ contains an atom in } B^{(r)}; \\ 1 & \text{if } r < \omega \text{ and } U^{(r)} \text{ contains an element which} \\ & \text{is atomless in } B^{(r)}; \\ 2 & \text{if } r < \omega \text{ and } U^{(r)} \text{ contains an element which is atomic in} \\ & B^{(r)} \text{ but } U^{(r)} \text{ contains no atom of } B^{(r)}. \end{cases}$$

REMARK. If $U = B$, then $r = -1$.

Without proof we remark here that the game $H^n(A, B)$ can be modified for structures (B, U) where $B \in \mathrm{BA}$ and $U \in \mathrm{Ult}\, B \cup \{B\}$; we only have to demand, in addition to the other rules, that either both players choose elements from the ultrafilter or both players choose elements not in the ultrafilter. This modified game can be used to show the following theorem (Eršov [1964]):

7.4. THEOREM. *Let $B_0, B_1 \in \mathrm{BA}$, $U_0 \in \mathrm{Ult}\, B_0 \cup \{B_0\}$, $U_1 \in \mathrm{Ult}\, B_1 \cup \{B_1\}$, with invariants k_0, l_0, m_0, r_0, s_0 and k_1, l_1, m_1, r_1, s_1, respectively. Then $(B_0, U_0) \equiv (B_1, U_1)$ iff $(k_0, l_0, m_0, r_0, s_0) = (k_1, l_1, m_1, r_1, s_1)$.*

Next we have:

7.5. LEMMA. *Let $B \in \mathrm{BA}$, $U \in \mathrm{Ult}\, B \cup \{B\}$. Then the* Eršov–Tarski *invariants for (B, U) are determined by those for $(B \times_U B, U \times U)$ and conversely.*

PROOF. It is easily seen that the invariants of $B \times B$ can be calculated from the invariants of B and conversely. Thus for $U = B$ the lemma is immediately seen. Thus, let us assume that $U \neq B$. We start with some fact.

Fact 1. The atoms of $B \times_U B$ are exactly the elements $(a, 0)$ and $(0, a)$ with a an atom of B if B is not generated by an atom. If B is generated by an atom e, then the atoms of $B \times_U B$ are the elements $(a, 0)$ and $(0, a)$ with a an atom of B and $a \neq e$ as well as the element (e, e).

Fact 2. (a, b) is atomic in $B \times_U B$ iff a and b are atomic.

Fact 3. (a, b) is atomless in $B \times_U B$ iff a and b are atomless. From Fact 2 and Fact 3 it follows that

Fact 4. $E(B \times_U B) = (E(B) \times E(B)) \cap (B \times_U B)$.

From Fact 4 it follows that for each $i < \omega$ there is a natural isomorphism between $(B \times_U B)^{(i)}$ and $B^{(i)} \times_{U^{(i)}} B^{(i)}$. This shows that we have:

(i) $k(B \times_U B) = k(B)$;

(ii) $l(B \times_U B) = l(B)$;
(iii) suppose that $m(B) = k < \omega$; then

$m(B \times_U B) = 2 \cdot m(B)$ if $U^{(k)} = B^{(k)}$ or $U^{(k)} \neq B^{(k)}$ and $U^{(k)}$ contains no atom of $B^{(k)}$;

$m(B \times_U B) = 2 \cdot m(B) - 1$ if $U^{(k)} \neq B^{(k)}$ and $U^{(k)}$ contains an atom of $B^{(k)}$;

(iv) $r(B \times_U B, U \times U) = r(B, U)$;
(v) $s(B \times_U B, U \times U) = s(B, U)$.
This concludes the proof. □

Now we are prepared to show:

7.6. THEOREM. *Let $A_0, A_1 \in \mathrm{BA}$, σ_0, σ_1 strong involutions on A_0 and A_1, respectively. Then*

$$(A_0, \sigma_0) \equiv (A_1, \sigma_1) \quad \text{iff } (A_0, U(\sigma_0)) \equiv (A_1, U(\sigma_1)).$$

PROOF. ($\rightarrow$). This follows immediately from the fact that $U(\sigma)$ is uniformly definable in (A, σ).

($\leftarrow$). Let A_0, A_1, σ_0, σ_1 be as in the statement to be proved and assume $(A_0, U(\sigma_0)) \equiv (A_1, U(\sigma_1))$. Choose countable structures (A_0', σ_0') and (A_0', σ_1') with $(A_0', \sigma_0') \equiv (A_0, \sigma_0)$ and $(A_1', \sigma_1') \equiv (A_1, \sigma_1)$. Then each σ_i' $(i \leq 1)$ is a strong involution. Lemma 7.3 implies that there exist Boolean algebras B_0, B_1 and $U_0 \in \mathrm{Ult}\, B_0 \cup \{B_0\}$, $U_1 \in \mathrm{Ult}\, B_1 \cup \{B_1\}$ such that $(A_i', \sigma_i') \equiv (B_i \times_{U_i} B_i, \tau(B_i, U_i))$ $(i \leq 1)$ with $U(\sigma_i)$ corresponding to $U_i \times U_i$. Hence, we have $(B_0 \times_{U_0} B_0, U_0 \times U_0) \equiv (B_1 \times_{U_1} B_1, U_1 \times U_1)$. Lemma 7.5 implies that $(B_0, U_0) \equiv (B_1, U_1)$.

It is easily seen that $(B \times_U B, \tau(B, U))$ is definable from (B, U) in a uniform way using first-order definitions. Hence, every sentence referring to $(B \times_U B, \tau(B, U))$ can be translated to a sentence referring to (B, U) and this implies $(B_0 \times_{U_0} B_0, \tau(B_0, U_0)) \equiv (B_1 \times_{U_1} B_1, \tau(B_1, U_1))$. □

8. Boolean pairs

It was shown by Rubin (see Theorem 19.5 of Part I) that the class of all Boolean pairs, BP, has an undecidable theory. The reason is that there is much freedom for a Boolean algebra to be placed above a subalgebra. But what happens if we restrict this freedom? Here we show a decidability result which was obtained by KOPPELBERG [1982] for a restricted class of Boolean algebras.

Let BP^c be the class of all Boolean pairs (B, A) such that B and A are complete Boolean algebras and A is relatively complete in B, i.e. for each $b \in B$ there is a greatest element $a \in A$ with $a \leq b$. In this section we show that $\mathrm{Th}(BP^c)$ is decidable. The main idea of the proof is as follows: We show that sheaves allow a good representation of the Boolean pairs of BP^c. Then we use Feferman–Vaught sequences to transfer statements about BP^c to statements about Boolean algebras.

We start with the Feferman–Vaught sequences for Boolean products. Let $A \subseteq \Pi_{p\in X} A_p$ be a Boolean product. Then A satisfies the *maximality property* if for each formula $\varphi(x, x_0, \ldots, x_{n-1})$ in the language L of A and $a_0, \ldots, a_{n-1} \in A$, there exists a $b \in A$ with $[\![\exists x\varphi(x, \vec{a})]\!] = [\![\varphi(b, \vec{a})]\!]$. An easy induction shows:

> If A satisfies the maximality property, then the set $[\![\varphi(a_0, \ldots, a_{n-1}]\!]$ is clopen for each formula $\varphi(x_0, \ldots, x_{n-1}) \in \text{Form } L$ and each $a_0, \ldots, a_{n-1} \in A$.

By induction on the complexity of formulas, we define for each $\varphi \in \text{Form } L$ a *determining sequence* $(\Phi, \varphi_0, \ldots, \varphi_{k-1})$ with Φ a formula in the language of Boolean algebras and $\varphi_0, \ldots, \varphi_{k-1} \in \text{Form } L$.

We make the following convention: Let $\varphi \in \text{Form } L$; then φ^{-1} denotes $\neg\varphi$ and φ^1 denotes φ. For k a natural number let

$$E_n = \{e\colon e\colon \{0, 1, \ldots, n-1\} \to \{-1, 1\}\}\,,$$

say $E_n = \{e_0, \ldots, e_{2^k-1}\}$ with e_0 the function with $e_0(i) = -1$ for each $i < k$.

Now we can define the determining sequences.

(i) If φ is atomic, then its determining sequence is $(v_0 = 1, \varphi)$;

(ii) if φ is determined by $(\Phi(v_0, \ldots, v_{k-1}), \varphi_0, \ldots, \varphi_{k-1})$, then $\neg\varphi$ is determined by $(\neg\Phi(v_0, \ldots, v_{k-1}), \varphi_0, \ldots, \varphi_{k-1})$;

(iii) if φ is determined by $(\Phi(v_0, \ldots, v_{k-1}), \varphi_0, \ldots, \varphi_{k-1})$ and ψ is determined by $(\Psi(v_0, \ldots, v_{l-1}), \psi_0, \ldots, \psi_{l-1})$, then $\varphi \wedge \psi$ is determined by $(\Phi(v_0, \ldots, v_{k-1}) \wedge \Psi(v_k, \ldots, v_{k+l-1}), \varphi_0, \ldots, \varphi_{k-1}, \psi_0, \ldots, \psi_{l-1})$;

(iv) if $\varphi(x, \vec{y})$ is determined by the sequence $(\Phi, \varphi_0, \ldots, \varphi_{k-1})$, then $\exists x\varphi(x, \vec{y})$ is determined by the sequence $(\Psi, \psi_0, \ldots, \psi_{2^k-1})$ with

$$\psi_j := \exists x \bigwedge_{i<k} \varphi_i^{e_j(i)}$$

and

$$\Psi(v_0, \ldots, v_{2^k-1}) := \exists w_0 \ldots w_{2^k-1}\Big[\bigwedge_{i<2^k} w_i \le v_i$$

$$\wedge \bigwedge_{i<j<k} w_i \cdot w_j = 0 \wedge \sum_{i<2^k} w_i = 1$$

$$\wedge \exists u_0 \ldots u_{k-1}\Big[\bigwedge_{i<k}\Big(u_i = \sum_{e_j(i)=1} w_j\Big)\Phi(u_0, \ldots, u_{k-1})\Big]\Big]\,.$$

Then we have:

8.1. THEOREM. *Let $\varphi(\vec{x})$ be a formula with a determining sequence $(\Phi(v_0, \ldots, v_{k-1}), \varphi_0(\vec{x}), \ldots, \varphi_{k-1}(\vec{x}))$ and $A \subseteq \Pi_{p\in X} A_p$ a Boolean product* *which satisfies the maximality property and $\vec{a}$ a sequence of elements of A such that $\vec{x}$ and $\vec{a}$ have the same length. Then*

$$A \models \varphi(\vec{a}) \quad \textit{iff}\ \operatorname{Clop} X \models \Phi([\![\varphi_0(\vec{a})]\!], \ldots, [\![\varphi_{k-1}(\vec{a})]\!]) .$$

PROOF. By induction on the complexity of φ. For φ atomic it is just the definition of $\models$. The induction steps for $\neg$ and $\wedge$ are straightforward. Thus, we have to consider only the existential quantifier.

Let us assume that the theorem holds true for $\varphi(y, \vec{x})$ with the determining sequence $(\Phi, \varphi_0, \ldots, \varphi_{k-1})$ and let $(\Psi, \psi_0, \ldots, \psi_{2^k-1})$ be as in (iv).

First assume that $A \models \exists y\varphi(y, \vec{a})$. Choose $b \in A$ with $A \models \varphi(b, \vec{a})$. Then, obviously for some $j < 2^k$, we have

$$\left[\!\!\left[\bigwedge_{i<k} \varphi_i^{e_j(i)}(b, \vec{a}) \right]\!\!\right] \subseteq [\![\psi_j(\vec{a})]\!] ;$$

for $j < l < 2^k$ we have

$$\left[\!\!\left[\bigwedge_{i<k} \varphi_i^{e_j(i)}(b, \vec{a}) \right]\!\!\right] \cap \left[\!\!\left[\bigwedge_{i<k} \varphi_i^{e_l(i)}(b, \vec{a}) \right]\!\!\right] = \emptyset$$

and

$$\operatorname{Clop} X \models \Psi([\![\psi_0(\vec{a})]\!], \ldots, [\![\psi_{2^k-1}(\vec{a})]\!]) .$$

Conversely, assume that

$$\operatorname{Clop} X \models \Psi([\![\psi_0(\vec{a})]\!], \ldots, [\![\psi_{2^k-1}(\vec{a})]\!]) .$$

By the maximality property we can find $b_0, \ldots, b_{2^k-1} \in A$ such that for each $j < 2^k$,

$$\left[\!\!\left[\exists x \left(\bigwedge_{i<k} \varphi_i^{e_j(i)}(x, a) \right) \right]\!\!\right] = \left[\!\!\left[\bigwedge_{i<k} \varphi_i^{e_j(i)}(b_j, \vec{a}) \right]\!\!\right] .$$

Let $w_0, \ldots, w_{2^k-1}$ be a partition of Clop X with $w_i \leq [\![\psi_i(\vec{a})]\!]$ for each $i < 2^k$ and Clop $X \models \Psi(w_0, \ldots, w_{2^k-1})$. Let

$$b = \sum_{i<2^k} b_i \cdot w_i .$$

Then b is as desired. $\square$

We say that a determining sequence $(\Phi, \varphi_0, \ldots, \varphi_{k-1})$ for φ is a *special determining sequence* if the formulas $\bigvee_{i<k} \varphi_i$ and $\neg(\varphi_i \wedge \varphi_j)$ for $i < j < k$ are tautologies. Let $(\Phi, \varphi_0, \ldots, \varphi_{k-1})$ be a determining sequence. For $j < 2^k$ let

$$\psi_j = \bigwedge_{i<k} \varphi_i^{e_j(i)} .$$

Then it is a matter of routine to write down a sequence $(\Psi', \psi_0, \ldots, \psi_{2^k-1})$ which is a special determining sequence for φ. We remark that special determining sequences can be constructed in an effective way.

Let $(B, A) \in BP^c$. Sheaves give us a good tool to describe how B lies above A. We use the notation of 8.16 and the results 8.17 and 8.20 of Part I. Especially, let $\mathscr{S}$ be the sheaf associated with the pair (B, A). Then, by Lemma 8.17 of Part I, we can identify A with $\Gamma(\mathscr{S})$.

We set $X = \text{Ult } A$. In order to simplify notation we do not distinguish between elements of A and clopen sets of Ult X. It will be clear from the context if we are working in A or in Ult X.

In the rest of this section, L denotes the language of Boolean algebras and L_P denotes L expanded by adding a unary predicate sign P. L_P is suitable for speaking about Boolean pairs.

First we show that if c $(=[\![\varphi(\vec{b})]\!])$ is a clopen subset of X for each $\vec{b} \in B$ and each $\varphi(\vec{x}) \in \text{Form } L_P$, then f_c is first-order definable in (B, A) from $\vec{b}$ (for the definition of f_c see Definition 8.16 of Part I).

8.2. LEMMA. *There is an effective procedure assigning to each formula $\varphi(\vec{x}) \in$ Form L_P a formula $s_\varphi(y, \vec{x}) \in$ Form L_P (where y is a variable not occurring in φ) such that for each $(B, A) \in BP$,* (i) *and* (ii) *are equivalent and each of them implies* (iii):

(i) *for every $\varphi(\vec{x}) \in$ Form L_P and $\vec{b} \in B$ with $\text{lh}(\vec{b}) = \text{lh}(\vec{x})$, $[\![\varphi(\vec{b})]\!]$ is clopen*;

(ii) *for every $\varphi(\vec{x}) \in$ Form L_P, $(B, A) \models \forall \vec{x} \exists y s_\varphi(y, \vec{x})$*;

(iii) *for every $\varphi(\vec{x}) \in$ Form L_P and every $\vec{b} \in B$ with $\text{lh}(\vec{b}) = \text{lh}(\vec{x})$, if $Y = [\![\varphi(\vec{b})]\!]$ and $c = f_Y$, then c is the unique element of B with $(B, A) \models s_\varphi(c, \vec{b})$.*

PROOF. The inductive definition of s_φ will show that (i) and (ii) are equivalent and (i) implies (iii). The interesting cases are φ atomic or φ existential. In these cases the fact that $[\![\varphi(\vec{x})]\!]$ is clopen will be expressed by stating that "there is a largest element a of A with $a \leq [\![\varphi(\vec{x})]\!]$". We define φ as follows:

If

$$\varphi := t_0(\vec{x}) = t_1(\vec{x}), \text{ then}$$

$$s_\varphi := P(y) \wedge y \cdot t_0(\vec{x}) = y \cdot t_1(\vec{x}) \wedge \forall z[P(z) \wedge z \cdot t_0(\vec{x}) = z \cdot t_1(\vec{x}) \rightarrow z \leq y];$$

if

$$\varphi := P(t(\vec{x})), \text{ then}$$

$$s_\varphi := \exists y_0 y_1 [y = y_0 + y_1 \wedge s_{t(\vec{x})=1}(y_0, \vec{x}) \wedge s_{t(\vec{x})=0}(y_1, \vec{x})];$$

if

$$\varphi := \psi(\vec{x}) \wedge \chi(\vec{x}), \text{ then}$$

$$s_\varphi := \exists y_0 y_1 [y = y_0 \cdot y_1 \wedge s_\psi(y_0, \vec{x}) \wedge s_\chi(y_i, \vec{x})];$$

if

$$\varphi := \neg\psi(\vec{x}), \text{ then}$$
$$s_\varphi := \exists y_0[y = -y_0 \wedge s_\psi(y_0, \vec{x})] ;$$

if

$$\varphi := \exists y_0 \psi(y_0, \vec{x}), \text{ then}$$

$$s_\varphi := \exists y_0 s_\psi(y, y_0, \vec{x}) \wedge \forall z z_0[s_\psi(z, z_0, \vec{x}) \rightarrow z \leq y] .$$

This lemma together with 8.20 and 8.13(e) of Part I implies:

If $(B, A) \in BP$ and A is relatively complete in B, then $[\![\varphi(\vec{b})]\!]$ is definable for each $\varphi(\vec{x}) \in \text{Form } L_P$ and each $\vec{b} \in B$ with $\text{lh}(\vec{b}) = \text{lh}(\vec{x})$.

Let

$$\text{at } x := x \neq 0 \wedge \forall y(y \leq x \rightarrow y = 0 \vee y = x) ;$$

then at x states that x is an atom. Let

$$\eta x := x \neq 0 \wedge \forall y(y \leq x \rightarrow \neg \text{ at } y) ;$$
$$\text{at } x := \neg\exists y(y \leq x \wedge \eta y) .$$

ηx states that x is atomless and At x states that x is atomic.

Let

$$\sigma := \exists x(\text{At } x \wedge \eta(-x)) ,$$

i.e. σ states that the supremum of all atoms exist. σ^P denotes the relativization of σ to the one-place predicate P. Recall that a Boolean algebra is called separated if it satisfies σ. Let

$$T = \text{Th}(BP) \cup \{\forall \vec{x}\, \exists y s_\varphi(y, \vec{x}) \colon \varphi(\vec{x}) \in \text{Form } L_P\}$$
$$\cup \{\sigma^P, s_\sigma(1)\} .$$

8.3. LEMMA. *If* $(B, A) \in BP^c$, *then* $(B, A) \in \text{Mod } T$.

PROOF. Let $(B, A) \in BP^c$. Then $[\![\varphi(\vec{b})]\!]$ is clopen for every atomic formula $\varphi(\vec{x}) \in \text{Form } L_P$ and arbitrary $\vec{b} \in B$. If $[\![\varphi(\vec{b})]\!]$ and $[\![\psi(\vec{b})]\!]$ are clopen subsets of X ($= \text{Ult } A$), so are $[\![\neg\varphi(\vec{b})]\!]$ and $[\![\varphi(\vec{b})]\!] \wedge [\![\psi(\vec{b})]\!]$. Assume that

$$\varphi := \exists y \psi(y, \vec{x})$$

and that $[\![\psi(c, \vec{b})]\!]$ is clopen for fixed $\vec{b} \in B$ and arbitrary $c \in B$. Let

$$U = \bigcup \{[\![\psi(c, \vec{b})]\!] \colon c \in B\} .$$

Then U is an open subset of X. Choose a maximal family

$$F = \{(c_i, V_i) : i \in I\}$$

such that $c_i \in B$, $V_i \in \operatorname{Clop} X$ and $V_i \subseteq [\![\psi(c_i, \vec{b})]\!]$, the V_i are pairwise disjoint (the c_i need not be pairwise distinct). Let V be the closure of $\bigcup \{V_i : i \in I\}$. Then $U \subseteq V$. Now A is complete; hence, X is extremally disconnected (see the remark following Theorem 7.9 of Part I) and V is clopen. By the completeness of B, there is some $c \in B$ with $c \cdot V_i = c_i \cdot V_i$ for each $i \in I$. Thus, for $i \in I$, $V_i \subseteq [\![\psi(c, \vec{b})]\!]$. So, for $a \in A$, $[\![\psi(a, \vec{b})]\!] \subseteq U \subseteq V \subseteq [\![\psi(c, \vec{b})]\!] = [\![\exists y \psi(y, \vec{b})]\!]$.

Now we show that B_p is separated for each $p \in X$. Let

$$\varphi(x) := \text{at}\, x \vee x = 0\,;$$

$$\psi(x) := \forall y(\text{at}\, y \rightarrow y \leq x)\,.$$

Put

$$M = \{a \in B : [\![\varphi(a)]\!] = 1\}$$

and let b be the supremum of M in B. We show that $b(p)$ is the supremum of atoms of B_p for each $p \in X$.

Assume that $s \in B_p$ is an atom of B_p. Then there is an $a \in M$ with $a(p) = s$. So $a \leq b$ and $s = a(p) \leq b(p)$.

Now assume that $t \in B_p$ and $s \leq t$ for every atom s of B_p. We choose $c \in B$ with $c(p) = t$. Then $p \in [\![\psi(c)]\!]$. We set

$$V = [\![\psi(c)]\!]\,.$$

For $q \in V$, $a \in M$, $a(p)$ is zero or an atom of B_p and thus $V \cdot a \leq c_1$. This implies $b(p) \leq c(p) = t$. Thus, $b(p)$ is the union of the atoms of B_p. $\square$

Let

$$T_\sigma = \text{Th(BA)} \cup \{\sigma\}\,;$$

i.e. T_σ is the theory of separated Boolean algebras. We define the following formulas: Let $(=n)$ and $(\geq n)$ state that there are exactly n atoms and that there are at least n atoms, respectively. Let (η) state that there is an atomless element. Define, for $n \in \omega + 1$, $i \in 2$, an L-theory T_{ni} by:

$$T_{n0} = T_\sigma \cup \{(=n), \neg(\eta)\}\,;$$

$$T_{n1} = T_\sigma \cup \{(=n), (\eta)\}\,;$$

for $n < \omega$, and

$$T_{\omega 0} = T_\sigma \cup \{(\geq n) : n \in \omega\} \cup \{\neg(\eta)\}\,;$$

$$T_{\omega 1} = T_\sigma \cup \{(\geq n) : n \in \omega\} \cup \{(\eta)\}\,.$$

Then for each separated Boolean algebra A, there exist $n \in \omega + 1$ and $i \in 2$ with $\mathrm{Th}(A) = T_{ni}$, and thus T_{ni} is complete. Moreover, let cBA denote the class of complete Boolean algebras. Then

$$\mathrm{Th}(c\mathrm{BA}) = T_\sigma .$$

We set

$$S = (\omega + 1) \times 2 .$$

8.4. LEMMA. *Let $s, t \in S$. Then there exist $(B, A) \in BP^{\mathrm{c}}$ such that $A \in \mathrm{Mod}\, T_s$ and $B_p \in \mathrm{Mod}\, T_t$ for each $p \in X$.*

PROOF. We choose A, $F \in c\mathrm{BA}$ with $A \in \mathrm{Mod}\, T_s$, $F \in \mathrm{Mod}\, T_t$. Then, for the free product of A and F we have

$$A * F \cong \mathrm{Clop}(\mathrm{Ult}\, A \times \mathrm{Ult}\, F)$$

(cf. Section 11 of Part I). We have a natural embedding $A \to A * F$ given by $a \mapsto a \times \mathrm{Ult}\, F$ ($F \to A * F$ given by $f \mapsto \mathrm{Ult}\, A \times f$). Thus, we can consider A and F as subalgebras of $A * F$. A is relatively complete in $A * F$ and for each $p \in \mathrm{Ult}\, A$ $(=X)$, $(A * F)_p \cong F$. But $A * F$ is complete only if A or F is finite. Thus, we set

$$B = (\overline{A * B}) ,$$

the completion of $A * F$. $A * F$ is dense in B and $(B, A) \in BP^{\mathrm{c}}$, since the inclusion maps from A into $A * F$ and from $A * F$ into B are complete. Let $p \in X$; in general B_p is a proper extension of $(A * F)_p$. We show that B_p and F are elementarily equivalent. This is done by the following four claims.

Claim 1. For each atom $f \in F$, $f(p)$ is an atom of B_p (hence, if F has at least n atoms where $n \in \omega$, then B_p has at least n atoms).

Proof of Claim 1. Assume not. Then there are $g_0, g_1 \in B$ with $g_0(p) > 0$, $g_1(p) > 0$, $g_0 \cdot g_1 = 0$, $g_0 + g_1 \leq f$. Thus, there are $a_0, a_1 \in A$ and $f_0, f_1 \in F$ with $a_0 \cdot f_0 \leq g_0$, $a_1 \cdot f_1 \leq g_1$. But then f_0, f_1 splits f, contradicting the fact that f is an atom.

Claim 2. If B_p has at least n atoms (where $1 \leq n < \omega$), then F has at least n atoms.

Proof of Claim 2. Let $b_0, \ldots, b_{n-1} \in B_p$ be atoms. For each $i < n$ choose $g_i \in B$ with $g_i(p) = b_i$. Then there are $a \in A$ and $f_i \in F$ with $a(p) = 1$, $0 < a \cdot f_i \leq g_i$. This implies $0 < f_i(p) \leq g_i(p)$. As the b_i's are atoms, we have $f_i(p) = g_i(p)$ and the f_i are pairwise distinct atoms of F.

Claim 3. If F has a non-zero atomless element f, then each B_p has a non-zero atomless element b.

Proof of Claim 3. We set $c = f(p)$. Then $f(p) \neq 0$. Assume there is an atom $b_0 \in f(p)$. Then, similar as in the proof of Claim 2, we can show that b_0 is an atom of F, which contradicts the assumption.

Claim 4. If B_p has a non-zero atomless element b, then F has a non-zero atomless element f.

Proof of Claim 4. Choose $g \in B$ with $g(p) = b$. There are $a \in A$, $f \in F$ with $0 < a \cdot f \leq g$ and $a(p) = 1$. Then $0 < f(p) \leq g(p)$ and f is an atomless element of F. $\square$

8.5. LEMMA. *Let* $\{(B_i, A_i): i \in I\} \subseteq BP^c$, $B = \Pi_{i \in I} B_i$, $A = \Pi_{i \in I} A_i$, $\varphi(x_0, \ldots, x_{n-1}) \in \text{Form } L_P$, $b_0, \ldots, b_{n-1} \in B$, $b_j = (b_{ij})_{i \in I}$. *Put*

$$a_i = [\![\varphi(b_{i0}, \ldots, b_{in-1})]\!] .$$

Then

$$[\![\varphi(b_0, \ldots, b_{n-1})]\!] = (a_i)_{i \in I} .$$

PROOF. By induction on the complexity of φ. $\square$

Let

$$T_{\sigma 2} = T_\sigma \cup \{\forall x(P(x) \leftrightarrow x = 0 \vee x = 1\} .$$

T_σ and $T_{\sigma 2}$ are decidable because Th(BA) is decidable.

8.6. LEMMA. *There is an effective procedure deciding for every sentence* $\varphi \in$ Form L_P *whether* $T \vdash \varphi$. *Moreover,* $T \vdash \varphi$ *iff for each* $(B, A) \in BP^c$, $(B, A) \models \varphi$.

PROOF. Let φ be given. Construct a special determining sequence $(\Phi(v_0, \ldots, v_{k-1}), \varphi_0, \ldots, \varphi_{k-1})$ as described in Lemma 8.1. For every $i < k$ decide whether $T_{\sigma 2} \vdash \neg \varphi_i$. W.l.o.g. assume that $T_{\sigma 2} \vdash \neg \varphi_i$ iff $r \leq i < k$. We set

$$\Phi'(v_0, \ldots, v_{r-1}) := \Big(\bigwedge_{r \leq i < k} v_i = 0 \Big) \rightarrow \Phi(v_0, \ldots, v_{k-1}) .$$

In order to prove the lemma it is sufficient to show the equivalence of:
(i) $T \vdash \varphi$;
(ii) $(B, A) \models \varphi$ for every $(B, A) \in \text{Mod } T$;
(iii) $T_\sigma \vdash \forall \vec{v} \Phi'(\vec{v})$.

By Lemma 8.3, (i)$\rightarrow$(ii). To prove (iii)$\rightarrow$(i) assume that there is a $(B, A) \in$ Mod T with $(B, A) \models \neg \varphi$. Put $a_i = [\![\varphi_i]\!]$. Then $A \models \neg \Phi(a_0, \ldots, a_{k-1})$ and $a_r = a_{r+1} = \cdots = a_{k-1} = 0$. Thus, $A \models \neg \Phi'(a_0, \ldots, a_{r-1})$ and (iii) is false. To prove (ii)$\rightarrow$(iii) assume that (iii) is false. Choose a separated Boolean algebra A' and $a'_0, \ldots, a'_{k-1} \in A'$ with $a'_r = \cdots = a'_{k-1} = 0$ such that $A' \models \neg \Phi(a'_0, \ldots, a'_{k-1})$. W.l.o.g., $a'_i \neq 0$ for $i < r$. There are $t_0, \ldots, t_{r-1} \in S$ with $T_{t_i} \models \varphi_i$ for $i < r$. For $i < r$ choose $s_i \in S$ with $A' \restriction a'_s \in \text{Mod } T_{s_i}$. By Lemma 8.5, there are $(B, A) \in BP^c$ and $a_0, \ldots, a_{r-1} \in A$ with $\Sigma_{i<r} a_i = 1$, $a_i \cdot a_j = 0$ for $i < j < r$, $A \restriction a_i \in$ Mod T_{s_i} and $(B \restriction a_i)_p \in \text{Mod } T_{t_i}$ for those $p \in X$ satisfying $a_i(p) = 1$. So $[\![\varphi_i]\!] = a_i$. Put $a_r = \cdots = a_{k-1} = 0$. Then $A \models \neg \Phi(a_0, \ldots, a_{k-1})$ and $(B, A) \models \neg \varphi$. $\square$

Lemma 8.6 immediately implies:

8.7. THEOREM. $\mathrm{Th}(BP^c)$ *is decidable.*

In the rest of this chapter we describe the possible completions of T.

We call the sentences $(=n) \wedge (\eta)$, $(=n) \wedge \neg(\eta)$, $(\geq n) \wedge (\eta)$, $(\geq n) \wedge \neg(\eta)$ *basic sentences*. For each sentence $\delta \in \mathrm{Form}\, L$, there are basic sentences $\beta_0, \ldots, \beta_{n-1}$ such that

$$T_\sigma \vdash \Big(\delta \leftrightarrow \bigvee_{i<n} \beta_i\Big) \wedge \bigwedge_{i<j<n} \neg(\beta_i \wedge \beta_j)\,.$$

The same holds true for $T_{\sigma 2}$. Replacing each atomic formula $P(t)$ by "$t = 0 \vee t = 1$", we see that for each sentence $\delta \in \mathrm{Form}\, L_P$ there are basic sentences $\beta_0, \ldots, \beta_{n-1}$ such that

$$T_{\sigma 2} \vdash \Big(\delta \leftrightarrow \bigvee_{i<n} \beta_i\Big) \wedge \bigwedge_{i<j<n} \neg(\beta_i \wedge \beta_j)\,.$$

For γ a basic sentence, let $\Delta_\gamma(y)$ be the result of relativizing the quantifier $\exists x\varphi \ldots$ in γ to $\exists x(P(x) \wedge x \leq y \wedge \varphi^y \ldots)$.

For β, γ basic sentences, let $\Sigma\,(\beta, \gamma)$ be the formula

$$\exists y(\Delta_\gamma(y) \wedge s_\beta(y))\,,$$

where $s_\beta(y)$ is the formula assigned to β as in Lemma 8.2. If $(B, A) \in \mathrm{Mod}\, T$, then $(B, A)|Ab \models \Sigma\,(\beta, \gamma)$ iff $A \restriction [\![\beta]\!] \models \gamma$.

8.8. LEMMA. *Let* (B, A), $(B', A') \in \mathrm{Mod}\, T$. *Then* $(B, A) \equiv (B', A')$ *iff for any basic sentence* β, γ,

$$(B, A) \models \Sigma\,(\beta, \gamma) \quad \text{iff } (B', A') \models \Sigma\,(\beta, \gamma)\,.$$

PROOF. $(\leftarrow)$ is clear.

$(\rightarrow)$ Suppose (B, A) and (B', A') satisfy the same sentences $\Sigma(\beta, \gamma)$. Let $\varphi \in \mathrm{Form}\, L_P$ be a sentence with $(B, A) \models \varphi$. We show that also $(B', A') \models \varphi$.

Let $(\Phi(v_0, \ldots, v_{k-1}), \varphi_0, \ldots, \varphi_{k-1})$ be a special determining sequence for φ. Put

$$a_i = [\![\varphi_i]\!]^{(B,A)}\,;$$

then $\{a_0, \ldots, a_{k-1}\}$ is a partition of A and $A \models \Phi(a_0, \ldots, a_{k-1})$. Let

$$a_i' = [\![\varphi_i]\!]^{(B',A')}\,.$$

Then $\{a_0', \ldots, a_{k-1}'\}$ is a partition of A' and it suffices to show that

$$(A, a_0, \ldots, a_{k-1}) \equiv (A', a_0', \ldots, a_{k-1}')\,.$$

But this implies $B' \models \Phi(a_0', \ldots, a_{k-1}')$ and thus $(B', A') \models \varphi$. For each $i < k$ choose basic sentences $\beta_{i0}, \ldots, \beta_{in_i-1}$ with

$$T_{\sigma 2} \vdash \Big(\varphi_i \leftrightarrow \bigvee_{j<n_i} \beta_{ij}\Big) \wedge \bigwedge_{j<l<n_i} \neg(\beta_{ij} \wedge \beta_{il}) .$$

Put

$$a_{ij} = [\![\beta_{ij}]\!]^{(B,A)} ,$$
$$a'_{ij} = [\![\beta_{ij}]\!]^{(B',A')} .$$

Then a_i is the sum of the a_{ij} ($j < n_i$) and a'_i is the sum of the a'_{ij} (where the a_{ij} are pairwise disjoint for fixed i and also the a'_{ij} are pairwise disjoint). Now we show that

$$A \restriction a_{ij} \equiv A' \restriction a'_{ij} .$$

Let γ be a basic sentence and $A \restriction a_{ij} \models \gamma$. Then $(B, A) \models \Sigma(\beta_{ij}, \gamma)$ and thus $(B', A') \models \Sigma(\beta_{ij}, \gamma)$. But then $A' \restriction a'_{ij} \models \gamma$ and thus $A \restriction a_{ij} = A' \restriction a'_{ij}$. This implies

$$(A, a_0, \ldots, a_{k-1}) \equiv (A', a'_0, \ldots, a'_{k-1}) . \quad \square$$

Using the last lemma, we can describe the completions of T.

Let $(B, A) \in BP^c$, $n \in \omega$. Then $[\![(\geq n) \wedge \neg(\eta)]\!]$ is the union of the disjoint elements $[\![(=n) \wedge \neg(\eta)]\!]$ and $[\![(\geq n+1) \wedge \neg(\eta)]\!]$ (and $[\![(\geq n) \wedge (\eta)]\!]$ is the union of the disjoint elements $[\![(=n) \wedge (\eta)]\!]$ and $[\![(\geq n+1) \wedge (\eta)]\!]$). The type of a Boolean pair $(B, A) \in BP^c$ is completely determined by the two functions

$$\alpha, \rho \colon \omega \times 2 \to (\omega + 1) \times 2 ,$$

where α describes the types of the elements $[\![(=n) \wedge \neg(\eta)]\!]$ and $[\![(=n) \wedge (\eta)]\!]$ and ρ describes the types of the elements $[\![(\geq n) \wedge \neg(\eta)]\!]$ and $[\![(\geq n) \wedge (\eta)]\!]$.

For $m, m' \in \omega + 1$ and $j, j' \in 2$, define

$$(m, j) + (m', j') = (m'', j'') ,$$

where m'' is the cardinal sum of m and m' and j'' is the maximum of j and j' (this addition describes the type of the union of two disjoint elements of a model of T_σ). Let

$$Q = \{(\alpha, \rho) \colon \alpha \text{ and } \rho \text{ are functions from } \omega \times 2 \text{ into } (\omega+1) \times 2 \text{ and for } (n, i) \in \omega \times 2,\ \rho(n, i) = \rho(n+1, i) + \alpha(n, i)\} .$$

For $(\alpha, \rho) \in Q$ let

$$\begin{aligned} T_{\alpha\rho} = T &\cup \{\exists x(s_{(=n)\wedge\neg(\eta)}(x) \wedge \gamma^x \colon n \in \omega, \gamma \in T_{\alpha(n,0)}\} \\ &\cup \{\exists x(s_{(\geq n)\wedge\neg(\eta)}(x) \wedge \gamma^x \colon n \in \omega, \gamma \in T_{\rho(n,0)}\} \\ &\cup \{\exists x(s_{(=n)\wedge(\eta)}(x) \wedge \gamma^x \colon n \in \omega, \gamma \in T_{\alpha(n,1)}\} \\ &\cup \{\exists x(s_{(\geq n)\wedge(\eta)}(x) \wedge \gamma^x \colon n \in \omega, \gamma \in T_{\rho(n,1)}\} . \end{aligned}$$

Lemma 8.4 and Lemma 8.5 now imply:

8.9. THEOREM. *$\{T_{\alpha\rho}: (\alpha, \rho) \in Q\}$ is the set of completions of T.*

REMARK. Let BP_0 be the class of Boolean pairs (B, A) such that B is complete. HEINDORF [1984] shows that $\mathrm{Th}(BP_0)$ is undecidable.

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Keywords: Boolean algebra, decidable, dense system, approximating families, games, characteristics, monadic, linear order, Ramsay quantifier, sequence quantifier, cardinality quantifier, residually small, discriminator variety, automorphism group, Boolean pair.

MOS subject classification: primary 03G05; secondary 06E05, 08B15, 03C99.

CHAPTER 24

Undecidable Extensions of the Theory Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

In order to show that a theory is undecidable, we use the method of model interpretation (see TARSKI ET AL., [1953], RABIN [1965] and ERSHOV ET AL. [1965]).

Let L be any language. Then Form L (Sent L) denotes the set of all formulas (sentences) of L. Let A be a structure for L, $c_0, \ldots, c_{n-1} \in A$, $\varphi(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{n-1}) \in \text{Form } L$. Then we put

$$\varphi^A(c_0, \ldots, c_{n-1}) = \{(a_0, \ldots, a_{k-1}) \in A^k : A \models \varphi(a_0, \ldots, a_{k-1}, c_0, \ldots, c_{n-1})\}.$$

In particular we write $\varphi^A(\vec{c})$ instead of $\varphi^A(c_0, \ldots, c_{n-1})$.

Let L, L' be languages and let $\boldsymbol{A}$ be a structure for L; for simplicity let us assume that $\boldsymbol{A} = (A, R, F)$ with R a binary relation and F a binary function, and let $\boldsymbol{B}$ be a structure for L'. Then we say that $\boldsymbol{A}$ can be *simply semantically embedded* into $\boldsymbol{B}$ ($\boldsymbol{A} \xrightarrow{\text{sse}} \boldsymbol{B}$) if there are formulas $\varphi_0(x)$, $\varphi_1(x, y)$, $\varphi_2(x, y, z) \in$ Form L' such that $(\varphi_0^{\boldsymbol{B}}, \varphi_1^{\boldsymbol{B}}, \varphi_2^{\boldsymbol{B}}) \cong \boldsymbol{A}$ (thus we have in particular $\varphi_1^{\boldsymbol{B}} \subseteq (\varphi_0^{\boldsymbol{B}})^2$, $\varphi_2^{\boldsymbol{B}} \subseteq (\varphi_0^{\boldsymbol{B}})^3$).

In this case each element of A is represented by a unique element of B. For many purposes this notion of interpretability is not suitable. Therefore we often use a more general method.

Let $\boldsymbol{A} = (A, (R_i)_{i\in I}, (F_i)_{i\in J})$ be a structure, R_i of arity n_i for each $i \in I$, F_i of arity m_i for each $i \in J$. A set $\theta \subseteq A^2$ is a *congruence* on $\boldsymbol{A}$ if

(i) for any $i \in I$, $a_0, \ldots, a_{n_i-1}, b_0, \ldots, b_{n_i-1} \in A$, if $(a_k, b_k) \in \theta$ for each $k < n_i$, then $R_i(\vec{a})$ iff $R_i(\vec{b})$;

(ii) for any $i \in J$, $a_0, \ldots, a_{m_i-1}, b_0, \ldots, b_{m_i-1} \in A$, if $(a_k, b_k) \in \theta$ for each $k < m_i$, then $(F_i(\vec{a}), F_i(\vec{b})) \in \theta$.

We say that $\boldsymbol{A}$ can be *semantically embedded* into $\boldsymbol{B}$ ($\boldsymbol{A} \xrightarrow{\text{se}} \boldsymbol{B}$) if there are formulas $\psi(x, y, z_0, \ldots, z_{n-1})$, $\varphi_0(x, z_0, \ldots, z_{n-1})$, $\varphi_1(x, y, z_0, \ldots, z_{n-1})$, $\varphi_2(x, y, z, z_0, \ldots, z_{n-1}) \in \text{Form } L'$ and elements $c_0, \ldots, c_{n-1} \in B$ such that $\varphi^{\boldsymbol{B}}(c_0, \ldots, c_{n-1})$ is a congruence on $(\varphi_0^{\boldsymbol{B}}(\vec{c}), \varphi_1^{\boldsymbol{B}}(\vec{c}), \varphi_2^{\boldsymbol{B}}(\vec{c}))$ and $(\varphi_0^{\boldsymbol{B}}(\vec{c}), \varphi_1^{\boldsymbol{B}}(\vec{c}), \varphi_2^{\boldsymbol{B}}(\vec{c})/\psi^{\boldsymbol{B}}(\vec{c}) \cong \boldsymbol{A}$. To indicate the language L', we write $\boldsymbol{A} \xrightarrow{\text{see}} (L')\boldsymbol{B}$ ($\boldsymbol{A} \xrightarrow{\text{se}} (L')\boldsymbol{B}$, respectively) if necessary.

Let H be a class of structures for L, $\boldsymbol{A} \in H$ and let K be a class of structures for L'. We say that $\boldsymbol{A}$ can be semantically embedded into K ($\boldsymbol{A} \xrightarrow{\text{se}} K$) if there is some $\boldsymbol{B} \in K$ with $\boldsymbol{A} \xrightarrow{\text{se}} \boldsymbol{B}$; H can be semantically embedded into K if for each $\boldsymbol{A} \in H$ there is $\boldsymbol{B} \in K$ such that $\boldsymbol{A} \xrightarrow{\text{se}} \boldsymbol{B}$ (using the same formulas $\psi(x, y, \vec{z})$, $\varphi_0(x, \vec{z})$, $\varphi_1(x, y, \vec{z})$, $\varphi_2(x, y, \vec{z})$ for every $\boldsymbol{A} \in H$). Of course, the relation $\xrightarrow{\text{se}}$ is transitive: if $G \xrightarrow{\text{se}} H$ and $H \xrightarrow{\text{se}} K$, then $G \xrightarrow{\text{se}} K$.

Let N be the set of natural numbers and $\boldsymbol{N} = (N, +, \cdot)$. The following theorem was shown in TARSKI ET AL. [1953]:

0.1. THEOREM. *Let K be a class of L-structures. If $\boldsymbol{N} \xrightarrow{\text{se}} K$, then* $\text{Th}^L(K)$ *is undecidable.*

Let G_{fin} denote the class of finite graphs, i.e. of all structures (G, R) with G a finite set and R an irreflexive symmetric relation on R. It was shown in RABIN [1965] (see also BURRIS and SANKAPPANAVAR [1980]) that

0.2. THEOREM. *Let K be a class of structures for some language L. If $G_{\text{fin}} \xrightarrow{\text{se}} K$, then $\text{Th}^L(K)$ is undecidable.*

In order to show that some extension of the elementary theory of Boolean algebras is undecidable we use Theorem 0.1 and Theorem 0.2.

Let F_2 be the function on N given by $F_2(n) = \binom{n}{2}$. Then we have:

0.3. LEMMA. $(N, +, \cdot) \xrightarrow{\text{sse}} (N, +, F_2)$.

PROOF. Let

$$\varphi_2(x, y, z) :\leftrightarrow F_2(x) + F_2(y) + z = F_2(x + y).$$

Then $(N, +, F_2) \models \varphi_2(a, b, c)$ iff $a \cdot b = c$. Thus,

$$(N, +, \varphi_2^N) \cong (N, +, \cdot)$$

(where we consider $\cdot$ as a ternary relation).

Thus, in order to show the undecidability of some theory it is enough to show that $(N, +, F_2)$ can be semantically embedded.

Let $A \in BA$. We use $\text{At}(A)$ to denote the set of atoms of A. In what follows we use at x as an abbreviation for $x \neq 0 \wedge \forall y(y \leq x \wedge y \neq x \rightarrow y = 0)$. Thus, $A \models \text{at } a$ iff $a \in \text{At}(A)$.

1. Boolean algebras in weak second-order logic and second-order logic

Let $\text{Th}^{\text{ws}}(\text{BA})$ denote the weak second-order theory of Boolean algebras, i.e. the elementary language is extended by allowing quantification over finite sets. In the case of weak second-order logic the theory of Boolean algebras is undecidable (see PALJUTIN [1971]). This will be shown in the sequel. On the other hand, if A and B are Boolean algebras which cannot be distinguished in the logic with the quantifier Q_0, then they also cannot be distinguished in weak second-order logic. But $\text{Th}^{Q_0}(\text{BA})$ is decidable, as was shown in Section 5 of Chapter 23. We now come to the theorem of Paljutin:

1.1. THEOREM. *$\text{Th}^{\text{ws}}(\text{BA})$ is undecidable.*

PROOF. We show that $(N, +, F_2) \xrightarrow{\text{se}} (L^{\text{ws}})$ BA. Let $A \in \text{BA}$ with $|\text{At}(A)| \geq \aleph_0$, $a \in A$. We say that a codes the natural number n if $|\{b \leq a\colon b \in \text{At}(A)\}| = n$. Thus, two elements of the Boolean algebra are "equal" if they are atomic and code the same number. Thus, we set

$$\varphi_0(x) := \exists X\, \forall y(y \leq x \rightarrow y \in X)\,;$$

$$\begin{aligned}\varphi(x, y) := \varphi_0(x) \wedge \varphi_0(y) \wedge \exists X(\forall uv \in X)[(u = v \\ \vee\, u \cdot v = 0) \wedge u \leq x \triangle y \wedge \mathrm{at}\, u \cdot x \wedge \mathrm{at}\, u \cdot y \\ \wedge\, \forall u[\mathrm{at}\, u \wedge u \leq x \triangle y \rightarrow (\exists v \in X)(u \leq v)]]\,.\end{aligned}$$

Thus, $A \models \psi(a, b)$ iff a and b are finite and there is a one-to-one correspondence between the atoms of $a - b$ and $b - a$.

To code addition we proceed as follows. Let a and b code n and m, respectively. Then choose c_0, c_1 which are disjoint and such that they code n and m, respectively. Then $c_0 + c_1$ codes $n + m$. More formally, we set

$$\varphi_1(x, y, z) := \exists uv(u + v = z \wedge u \cdot v = 0 \wedge \psi(x, u) \wedge \psi(y, v))\,.$$

To code F_2 we proceed as follows. Let a code n. Then for each pair a_0, a_1 of distinct atoms with a_0, $a_1 \leq a$ we choose some atom b. Then the sum of all these b's codes $\binom{n}{2}$. More formally, we set

$$\begin{aligned}\varphi_2(x, y) := \exists z\{\psi(y, z) \wedge x \cdot z = 0 \wedge \exists X[(\forall uv \in X) \\ \times [u \leq x + z \wedge \mathrm{at}\, u \cdot z \wedge \exists u_0 u_1(\mathrm{at}\, u_0 \wedge \mathrm{at}\, u_1 \\ \wedge\, u_0 \neq u_1 \wedge u \cdot x = u_0 + u_1 \wedge (u \cdot x = v \cdot x \leftrightarrow u \cdot z = v \cdot z)) \\ \wedge\, \forall u_0 u_1[\mathrm{at}\, u_0 \wedge \mathrm{at}\, u_1 \wedge u_0 \neq u_1 \wedge u_0 + u_1 \leq x \\ \rightarrow (\exists v \in X)(x \cdot v = u_0 + u_1)] \wedge [\forall t(t \leq z \wedge \mathrm{at}\, t \\ \rightarrow (\exists u \in X)(u \cdot z = t)]]\}\,.\end{aligned}$$

Then

$$(\varphi_0^A, \varphi_1^A, \varphi_2^A)/\psi^A \cong (N, +, F_2)$$

and thus $\mathrm{Th}^{\mathrm{ws}}(\mathrm{BA})$ is undecidable. □

Let $\mathrm{Th}^{\mathrm{s}}(\mathrm{BA})$ denote the second order theory of Boolean algebras, i.e. the elementary language is extended by allowing quantification over arbitrary sets.

1.2. THEOREM. *$\mathrm{Th}^{\mathrm{s}}(\mathrm{BA})$ is undecidable.*

PROOF. Similar to the proof of the previous theorem we show that $(N, +, F_2) \xrightarrow{\mathrm{se}} (L^{\mathrm{s}})$ BA.

Let $A \in \mathrm{BA}$ with $|\mathrm{At}(A)| \geq \aleph_0$. We want to express that an alement a contains only a finite number of atoms. In fact, a contains only a finite number of atoms iff for each ultrafilter U on $A \restriction a$ there is some $b \in U$ which is an atom. Thus, we set

$$\varphi_0^*(x) := \forall X(\mathrm{Ult}\, X \wedge x \in X \rightarrow \exists z(\mathrm{at}\, z \wedge z \in X))\,,$$

where Ult X is an abbreviation for

$$\forall u(u \in X \vee -u \in X) \wedge (\forall uv \in X)(u \cdot v \in X) \wedge \forall u(u \in X \rightarrow \neg(-u \in X)) .$$

Let $\psi^*(x, y)$, $\varphi_1^*(x, y, z)$, $\varphi_2^*(x, y)$ be the formulas which are obtained from $\psi(x, y)$, $\varphi_1(x, y, z)$, $\varphi_2(x, y)$ in the proof of the above theorem by replacing $\varphi_0(x)$ by $\varphi_0^*(x)$. Then

$$(\varphi_0^{*A}, \varphi_1^{*A}, \varphi_2^{*A})/\psi^{*A} \cong (N, +, F_2)$$

and thus $\mathrm{Th}^{\mathrm{s}}(\mathrm{BA})$ is undecidable. □

2. Boolean algebras in a logic with the Härtig quantifier

The Härtig quantifier I is a binary quantifier. If $\varphi(x)$, $\psi(x)$ are formulas, then $Ix\varphi(x)\psi(x)$ is a formula too and its interpretation is given by

$$A \models Ix\varphi(x)\psi(x) \quad \text{iff } |\varphi^A| = |\psi^A| .$$

The quantifier Q_0 can be expressed with I as follows:

$$A \models Q_0 x\varphi(x) \quad \text{iff } A \models \exists x[\varphi(x) \wedge Iy\varphi(y)(\varphi(y) \wedge x \neq y)] .$$

It was shown in WEESE [1976] that

2.1. THEOREM. $\mathrm{Th}^I(\mathrm{BA})$ *is undecidable.*

PROOF. Similar to the case of $\mathrm{Th}^{\mathrm{ws}}(\mathrm{BA})$ we show that $(N, +, F_s) \xrightarrow{\mathrm{se}} (L^I)$ BA.

Let A be a Boolean algebra with an infinite number of atoms. We say that a codes the natural number n if a contains exactly n atoms. Then we set

$$\varphi_0(x) := \neg Q_0 y(\mathrm{at}(y) \wedge y \leq x) ;$$

$$\psi(x, y) := Iz(z \leq x \wedge \mathrm{at}(z))(z \leq y \wedge \mathrm{at}(z)) .$$

Also similar to the case of weak second-order logic we define φ_1 and φ_2:

$$\varphi_1(x, y, z) := \exists uv(u \cdot v = 0 \wedge u + v = z \wedge \psi(x, u) \wedge \psi(y, v)) ;$$

$$\varphi_2(x, y) := Iz(z \leq x \wedge \exists uv(u \cdot v = 0 \wedge \mathrm{at}(u) \wedge \mathrm{at}(v) \wedge u + v = z)(z \leq y \wedge \mathrm{at}(z)) .$$

Then

$$(\varphi_0^A, \varphi_1^A, \varphi_2^A)/\psi^A \cong (N, +, F_2)$$

and thus $\mathrm{Th}^I(\mathrm{BA})$ is undecidable. □

We have shown that not only the class BA but also the class BA^1 of atomic Boolean algebras has an undecidable I-theory. Also, BA^2, the class of atomless Boolean algebras, has an undecidable theory in the language with Härtig quantifier (see WEESE [1976]):

2.2. THEOREM. *$\mathrm{Th}^I(BA^2)$ is undecidable.*

PROOF. Let $A \in \mathrm{BA}^2$, $a \in A$ with $a \neq 0$. The cardinal κ is *determined* by a if there is $b \leq a$ with $|A \upharpoonright b| = \kappa$. We say that a is *one-cardinal-like* if there is exactly one infinite cardinal which is determined by a. Formally, we set

$$c_1(x) := x \neq 0 \wedge \forall y(y \leq x \wedge y \neq 0 \rightarrow Iz(z \leq x)(z \leq y)).$$

Thus, $A \models c_1(a)$ iff a is one-cardinal-like. We set

$$\begin{aligned} c_2(x) := \exists uv[&u \leq x \wedge v \leq x \wedge c_1(u) \wedge c_1(v) \\ &\wedge \forall y[y \leq x \wedge y \neq 0 \rightarrow Iz(z \leq y)(z \leq u) \\ &\vee Iz(z \leq y)(z \leq v)]] \\ &\wedge \neg \exists uv[x = u + v \wedge c_1(u) \wedge c_1(v)]. \end{aligned}$$

Thus, $A \models c_2(a)$ if there are exactly two infinite cardinals which are determined by a and a is not the union of two one-cardinal-like elements.

Now we show that $G_{\mathrm{fin}} \xrightarrow{\mathrm{se}} (L^I)\ \mathrm{BA}^2$. For i a natural number, let η_i^* be the set of all finite sequences of ordinals less than ω_i not ending on zero. We order η_i^* lexicographically, i.e. $a \leq b$ iff a is an initial segment of b or there is an $i < \mathrm{lh}(a)$ such that $a(i) < b(i)$ and $a(k) = b(k)$ for all $k < i$. Let $a, b \in \eta_i^*$ with $a < b$. Then $|\{c \in \eta_i^* : a < c < b\}| = \aleph_i$, i.e. η_i^* is $\aleph_i$-dense. Thus, $\mathrm{Intalg}(\eta_i^*)$ is atomless and $\aleph_i$ is the only cardinal which is determined by the elements of $\mathrm{Intalg}(\eta_i^*)$.

If $i, k < \omega$, $i \neq k$, then

$$\mathrm{Intalg}((\eta_i^* + \eta_k^*) \cdot \omega) \models c_2(1).$$

Let $(G, R) \in G_{\mathrm{fin}}$ with $G \subseteq \omega$. Then we set

$$B^* = \prod_{i \in G} \mathrm{Intalg}(\eta_i^*) \times \prod_{(i,k) \in R} \mathrm{Intalg}((\eta_i^* + \eta_k^*) \cdot \omega).$$

We define:

$$\varphi_0(x) := c_1(x);$$

$$\psi(x, y) := c_1(x) \wedge c_1(y) \wedge Iz(z \leq x)(z \leq y);$$

$$\begin{aligned} \varphi_1(x, y) := {}& c_1(x) \wedge c_1(y) \wedge \neg \psi(x, y) \\ &\wedge \exists z[c_2(z) \wedge \exists uv(u \leq z \wedge v \leq z \wedge \psi(u, x) \wedge \psi(v, y))]. \end{aligned}$$

Then

$$(\varphi_0^{B^*}, \varphi_1^{B^*})/\psi^{B^*} \cong (G, R);$$

and thus $G_{\mathrm{fin}} \xrightarrow{\mathrm{se}} (L^I)$ BA^2. But that means that $\mathrm{Th}^I(\mathrm{BA}^2)$ is undecidable. □

3. Boolean algebras in a logic with the Malitz quantifier

It was shown in the Chapter 23 that $\mathrm{Th}^{Q_1}(\mathrm{BA})$ and $\mathrm{Th}^{Q_0^2}(\mathrm{BA})$ are decidable. Here we show that the theory of Boolean algebras becomes undecidable if we add a quantifier which is a "little bit" more expressive than Q_1: The Malitz quantifier Q_1^2 is a quantifier binding two variables and defined by

$$A \models Q_1^2 xy\varphi(x, y) \quad \text{iff there is a set } X \subseteq A \text{ with } |X| = \aleph_1 \text{ such that for any } a, b \in X \text{ with } a \neq b,\ A \models \varphi(a, b).$$

It is immediately clear that Q_1 can be expressed using Q_1^2. Assuming CH, RUBIN [1983] showed that $\mathrm{Th}^{Q_1^2}(\mathrm{BA})$ is undecidable.

Let A be a Boolean algebra, $a \in A$. We say that a satisfies the ccc if A has the ccc, i.e. each system X of pairwise disjoint elements of $A \restriction a$ is at most countable. This can be expressed with the help of Q_1^2. Let

$$c(x) := \neg Q_1^2 yz(y \leq x \wedge y \cdot z = 0).$$

Then $A \models c(a)$ if a has the ccc.

Let M be a linear ordering with first element. Then it is immediately seen that M has the ccc iff Intalg M has the ccc.

SIERPIŃSKI [1950] proved the following lemma (see also BONNET [1981] or Corollary 2.2 of the Appendix on Set Theory in this Handbook):

3.1. LEMMA (CH). *Let I be the unit interval. Then there exists a family $\{P_i : i < \omega\}$ with $|P_i| = \aleph_1$, $P_i \subseteq I$ for each $i < \omega$ such that:*

> *if $X, Y \subseteq \omega$ with $X \cap Y = \emptyset$ and $f: \bigcup_{i \in X} P_i \to \bigcup_{i \in Y} P_i$ is a one-to-one increasing or decreasing function, then $|\mathrm{dom}\, f| \leq \aleph_0$.*

We set

$$\rho(u, v) := Q_1^2 xy[(x < y \vee y < x) \wedge (x < y \to x \cdot u < y \cdot u \wedge x \cdot v < y \cdot v)].$$

Thus, $A \models \rho(a, b)$ iff there are $X \subseteq A \restriction a$ and $Y \subseteq A \restriction b$ such that $|X| \geq \aleph_1$ and $(X, < \cap X^2) \cong (Y, < \cap Y^2)$.

3.2. LEMMA. *Let M be a linear ordering with first element e_0 and let $e_1 \in M$. Then*

Intalg $M \models \rho([e_0, e_1), [e_1, \infty))$ *iff there is an increasing or decreasing one-to-one function* f *with* $\operatorname{dom} f \subseteq [e_0, e_1)$, $\operatorname{rng} f \subseteq [e_1, \infty)$ *and* $|\operatorname{dom} f| = \aleph_1$.

PROOF. Assume that there is a one-to-one increasing (decreasing) function f with $\operatorname{dom} f \subseteq [e_0, e_1)$, $\operatorname{rng} f \subseteq [e_1, \infty)$ and $|\operatorname{dom} f| = \aleph_1$. Then let

$$X = \{[e_0, x) + [e_1, f(x)) \colon x \in \operatorname{dom} f\} ;$$

$$(X = \{[x, e_1) + [e_1, f(x)) \colon x \in \operatorname{dom} f\}) ;$$

and thus Intalg $M \models \rho([e_0, e_1), [e_1, \infty))$.

Let Intalg $M \models \rho([e_0, e_1), [e_1, \infty))$. Then there is a chain $X \subseteq$ Intalg M such that $|X| = \aleph_1$, if $a, b \in X$ with $a \neq b$, then $a \cdot [e_0, e_1) \neq b \cdot [e_0, e_1)$ and $a \cdot [e_1, \infty) \neq b \cdot [e_1, \infty)$. Let $X = \{x_i \colon i < \omega_1\}$. By thinning out the set X we can assume that there are $n < \omega$ and $k < 2n$ such that

$$x_i = [s_0^i, s_1^i) + [s_2^i, s_3^i) + \cdots + [s_{2n-2}^i, s_{2n-1}^i) ,$$

with $s_0^i < s_1^i < \cdots < s_{2n-2}^i$, $s_j^i \in [e_0, e_1)$ for $i < k$, $s_j^i \in [e_1, \infty)$ for $k \leq j < 2n-2$, $s_{2n-1}^i \in [e_1, \infty) \cup \{\infty\}$ and $s_{2n-2}^i < s_{2n-1}^i$ if $s_{2n-1}^i \neq \infty$.

Thinning out the set X again we can assume that for each $m < k$, either $s_m^i = s_m^j$ for any $i < j < \omega_1$ or $s_m^i \neq s_m^j$ for any $i < j < \omega_1$. Let m_0 be the least and m_1 be the last element $m < k$ such that $s_m^i \neq s_m^j$ for any $i < j < \omega_1$. Then for any $i < \omega_1$, $s_{m_0}^i < e_1$, $s_{m_1}^i > e_1$. We define $g_0, g_1 \colon X \to M$ by

$$g_0(x_i) = s_{m_0}^i , \qquad g_1(x_i) = s_{m_1}^i .$$

Let $f \colon \operatorname{rng} g_0 \to M$ be given by

$$f(a) = g_1(g_0^{-1}(a)) .$$

Then f is as desired. □

Now we are prepared to show

3.3. THEOREM (CH). $\mathrm{Th}^{Q_1^2}(\mathrm{BA})$ *is undecidable.*

PROOF. We show that $G_{\mathrm{fin}} \xrightarrow{\mathrm{se}} (L^{Q_1^2})$ BA.

Let $\{P_i \colon i < \omega\}$ be a family of subsets of I as described in Lemma 3.1, and such that each P_i has a first element. Let $(G, R) \in G_{\mathrm{fin}}$ with $G \subseteq \omega$. We set

$$B = \prod_{i \in G} \text{Intalg } P_i \times \prod_{(i,k) \in R} \text{Intalg } P_i \cup P_k .$$

Let $a \in B$ be such that $B \restriction a \cong \prod_{i \in G}$ Intalg P_i. We set

$$\varphi^*(x) := Q_1 y(y \leq x) \wedge \neg \exists uv(u \cdot v = 0 \wedge u + v = x \wedge Q_1 y(y \leq u) \wedge Q_1 y(y \leq v))\,;$$

$$\varphi_0(x, a) := x \leq a \wedge \varphi^*(x)\,;$$

$$\psi(x, y, a) := \varphi_0(x) \wedge \varphi_0(y) \wedge \varphi_0(x \cdot y)\,;$$

$$\varphi_1(x, y, a) := \varphi_0(x) \wedge \varphi_0(y) \wedge \neg \varphi_0(x \cdot y) \wedge \exists z(\varphi^*(z) \wedge \rho(x, z) \wedge \rho(y, z))\,.$$

Then

$$(\varphi_0^B, \varphi_1^B)/\psi^B \cong (G, R)$$

(here we used a as a parameter). Consequently, $G_{\text{fin}} \xrightarrow{\text{se}} (L^{Q_1^2})$ BA. □

4. Boolean algebras in stationary logic

Stationary logic was introduced in BARWISE ET AL. [1978]. It extends the logic with Q_1. Similar as in the logic with the Malitz quantifier, the theory of Boolean algebras becomes undecidable in stationary logic. But, whereas in the case of Malitz quantifiers we had to use the continuum hypothesis, we do not need any assumption from set theory in the case of stationary logic.

Let L be an elementary language, M a structure for L. Let $P_{\leq\omega}(M)$ denote the set of all countable subsets of M. A set $X \subseteq P_{\leq\omega}(M)$ is *unbounded*, if each $s_0 \in P_{\leq\omega}(M)$ is a subset of some $s \in X$. X is *closed* if the union of each increasing countable sequence $s_0 \subseteq s_1 \subseteq \cdots \subseteq s_n \subseteq \cdots$ of elements of X is again an element of X. A set X is *cub*, if it is closed and unbounded.

Let $\mathrm{Cub}(M)$ denote the filter on $P_{\leq\omega}(M)$ which is generated by all cub sets. Then $\mathrm{Cub}(M)$ is countably closed. Let $L(aa)$ be the language L with an additional second-order quantifier aa defined by

$$M \models aa\ s\ \varphi(x) \quad \text{iff } \{s \in P_{<\omega}(M)\colon M \models \varphi(s)\} \in \mathrm{Cub}(M)\,.$$

In what follows we write $s(x)$ for $x \in s$. If $M \models \neg aa\ s \neg \varphi(s)$ (we assume $|M| \geq \aleph_0$), then $\{s \in P_{\leq\omega}(M)\colon M \models \varphi(s)\}$ is stationary; thus it is convenient to write $\mathrm{stat}\, s\ \varphi(s)$ as an abbreviation for $\neg aa\ s \neg \varphi(s)$.

The quantifier Q_1 is definable in $L(aa)$. In fact we have

$$M \models Q_1 x \varphi(x) \quad \text{iff } M \models \mathrm{stat}\, s \exists x(\varphi(x) \wedge \neg s(x))\,.$$

It is known that there exists a family $\{S_\alpha\colon \alpha < \omega_1\}$ of pairwise disjoint stationary subsets of ω_1. (See Theorem 5.9 of The Appendix on Set Theory in this Handbook).

Let η (η_0 respectively) denote the set of positive (non-negative) rational numbers with its natural ordering. A linear ordering M with least element 0 is called a *$\aleph_1$-like dense linear ordering* if $|M| = \aleph_1$ and for each $a \in M$ with $a > 0$, $|\{b \in M\colon b < a\}|$ is isomorphic to η_0. For $A \subseteq \omega_1$ with $0 \in A$, let $\mathrm{Lo}(A)$ denote the set $(\omega_1 \times \eta) \cup (A \times \{0\})$, ordered lexicographically. Then $\mathrm{Lo}(A)$ is an $\aleph_1$-like dense linear ordering.

Conway (see HUTCHINSON [1976]) gave the following classification of $\aleph_1$-like dense linear orderings:

4.1. LEMMA. (i) *For any $\aleph_1$-like dense linear ordering M there is $A \subseteq \omega_1$ with $0 \in A$ such that* $\mathrm{Lo}(A) \cong M$.

(ii) *Let A, $B \subseteq \omega_1$ with $0 \in A \cap B$. Then* $\mathrm{Lo}(A) \cong \mathrm{Lo}(B)$ *iff there is a closed unbounded set $C \subseteq \omega_1$ with $A \cap C = B \cap C$.*

Similarly, it can be shown:

4.2. LEMMA. *Let A, $B \subseteq \omega_1$ with $0 \in A \cap B$. Then* $\mathrm{Lo}(A)$ *can be embedded into* $\mathrm{Lo}(B)$ *iff there is a closed unbounded set $C \subseteq \omega_1$ with $A \cap C \subseteq B \cap C$.*

Furthermore, we have

4.3. LEMMA. *Let A, $B \subseteq \omega_1$ with $0 \in A \cap B$. Then* $\mathrm{Intalg}(\mathrm{Lo}(A))$ *can be embedded into* $\mathrm{Intalg}(\mathrm{Lo}(B))$ *iff* $\mathrm{Lo}(A)$ *can be embedded into* $\mathrm{Lo}(B)$.

PROOF. First assume that there is an embedding $f\colon \mathrm{Lo}(A) \rightarrowtail \mathrm{Lo}(B)$. Then let f^* be the embedding of $\mathrm{Intalg}(\mathrm{Lo}(A))$ into $\mathrm{Intalg}(\mathrm{Lo}(B))$ satisfying

$$f^*([0, A)) = [0, f(a))\,.$$

It is immediately seen that f^* is the desired embedding. Now let us assume that there is an embedding $g\colon \mathrm{Intalg}(\mathrm{Lo}(A)) \rightarrowtail \mathrm{Intalg}(\mathrm{Lo}(B))$. Let $\pi_A\colon \mathrm{Intalg} \times (\mathrm{Lo}(A)) \rightarrowtail \omega_1$ ($\pi_B\colon \mathrm{Intalg}(\mathrm{Lo}(B)) \rightarrowtail \omega_1$, respectively) be defined by

$$\pi_A(a) = \sup\{\alpha < \omega_1\colon \text{for some } x,\ (\alpha, x) \in a\}\,.$$

We define two functions h_0, $h_1\colon \omega_1 \to \omega_1$ by

$$h_0(\alpha) = \sup\{\beta + 1\colon \text{there is } a \in \mathrm{Intalg}(\mathrm{Lo}(A)) \text{ with } \pi_A(a) < \alpha \text{ and } \pi_B(g(a)) = \beta\}\,;$$

$$h_1(\alpha) = \sup\{\beta + 1\colon \text{there is } a \in \mathrm{Intalg}(\mathrm{Lo}(A)) \text{ with } \pi_A(a) = \beta \text{ and } \pi_B(g(a)) < \alpha\}\,.$$

Let $h\colon \omega_1 \to \omega_1$ be given by

$$h(\alpha) = \max\{h_0(\alpha), h_1(\alpha)\}\,.$$

Then h is unbounded and continuous; thus

$$C = \{\alpha < \omega_1 \colon h(\alpha) = \alpha\}$$

is closed and unbounded.

Now let us assume that $\alpha \in A \cap C$. Then

$$g([(0,0), (\alpha, 0))) = \mathrm{Lo}(B) \setminus \{(\beta, x) \colon \beta < \alpha, x \in \eta_0\}$$

and thus $\mathrm{Lo}(B) \setminus \{(\beta, x) \colon \beta < \alpha, x \in \eta_0\}$ has a first element. Thus, $(\alpha, 0) \in \mathrm{Lo}(B)$. This implies $\alpha \in B \cap C$ and thus $A \cap C \subseteq B \cap C$. $\square$

Let A be a Boolean algebra, $a \in A$. a is called an *$\aleph_1$-atom* iff $|A \restriction a| = \aleph_1$ and for $b, c \in A$ with $b \cdot c = 0$, $b + c = a$, $|A \restriction b| < \aleph_1$ or $|A \restriction c| < \aleph_1$. More formally, let

$$\varphi^*(x) := Q_1 y(y \leq x) \wedge \neg \exists uv(u \cdot v = 0 \wedge u + v = x \\ \wedge Q_1 y(y \leq u) \wedge Q_1 y(y \leq v)) .$$

Then a is an $\aleph_1$-atom iff $A \models \varphi^*(a)$. Let

$$\rho^*(s, x, y) := \varphi^*(x) \wedge \forall z(s(z) \wedge z < x \wedge \neg \varphi^*(z) \rightarrow z \leq y) \\ \wedge \forall z[\forall u(s(u) \wedge u < x \wedge \neg \varphi^*(u) \rightarrow u \leq z) \rightarrow y \leq z] .$$

Thus, if $A \models \rho^*(s, a, b)$, then a is an $\aleph_1$-atom and b is the supremum of all elements of s which are less than a, and are not $\aleph_1$-atoms. We set

$$\rho(s, x) := \exists y \rho^*(s, x, y) .$$

Now we can show (see SEESE ET AL. [1982]):

4.4. THEOREM. $\mathrm{Th}^{aa}(\mathrm{BA})$ *is undecidable.*

PROOF. We show that $G_{\mathrm{fin}} \xrightarrow{\mathrm{se}} (L^{aa})$ BA.

Let $(G, R) \in G_{\mathrm{fin}}$ with $G \subseteq \omega$. Let $\{S_i \colon i < \omega\}$ be a system of stationary subsets of ω_1 such that for $i < k < \omega$, we have $S_i \cap S_k = \{0\}$. We set

$$B = \prod_{i \in G} \mathrm{Intalg}(\mathrm{Lo}(S_i)) \times \prod_{(i,k) \in R} \mathrm{Intalg}(\mathrm{Lo}(S_i \cup S_k) .$$

Let $a \in B$ be such that

$$B \restriction a = \prod_{i \in G} \mathrm{Intalg}(\mathrm{Lo}(S_i)) .$$

We set

$$\varphi_0(x, a) := x \le a \wedge \varphi^*(x) ;$$

$$\psi(x, y, a) := \varphi_0(x) \wedge \varphi_0(y) \wedge \varphi_0(x \cdot y) ;$$

$$\varphi_1(x, y, a) := \varphi_0(x) \wedge \varphi_0(y) \wedge \exists z(\varphi^*(z) \wedge aa\, s(\rho(s, x) \to \rho(x, z)) \wedge aa\, s(\rho(s, y) \to \rho(s, z))) .$$

The elements of G are coded by $\aleph_1$-atoms (in fact by those $\aleph_1$-atoms, which are less than a). This is expressed by $\varphi_0(x, a)$. Two $\aleph_1$-atoms c_0, c_1 code the same element of G if they are nearly the same, i.e. if also their intersection $c_0 \cap c_1$ is an $\aleph_1$-atom. This is expressed by $\psi(x, y, a)$. The $\aleph_1$-atoms $\le -a$ are used to code the edges of G. The vertices c_0, c_1 (i.e. $\aleph_1$-atoms $\le a$) are connected if there is some $\aleph_1$-atom $d \le -a$ such that for some $\aleph_1$-chains M_0 in $B \restriction c_0$ and M_1 in $B \restriction c_1$ we can find isomorphic copies in $B \restriction d$. This is expressed by $\varphi_1(x, y, a)$. Lemmas 4.2 and 4.3 ensure that this coding is correct. Thus,

$$(\varphi_0^B, \varphi_1^B)/\psi^B \cong (G, R)$$

(here we used a parameter for the interpretation). Thus, $\mathrm{Th}^{aa}(\mathrm{BA})$ is undecidable. □

5. Boolean algebras with a distinguished group of automorphisms

Let $G = (G, \circ, e)$ be a group. Then $\mathrm{BA}(G)$ denotes the class of algebras

$$B_G = (B, +, \cdot, -, 0, 1, (g)_{g \in G})$$

which satisfy, for $g, h \in G$,

(i) $(B, +, \cdot, -, 0, 1)$ is a Boolean algebra;

(ii) $g(x + y) = g(x) + g(y)$;
$g(-x) = -g(x)$;
$g(h(x)) = (g \circ h)(x)$;
$e(x) = x$.

WOLF [1975] showed that $\mathrm{Th}(\mathrm{BA}(G))$ is decidable if G is a finite solvable group. BURRIS [1982] and JURIE [1982] extended this result by showing that for any finite group G, $\mathrm{Th}(\mathrm{BA}(G))$ is decidable (see Section 7 of Chapter 23). On the other hand, BURRIS [1982] could show that for many groups G, $\mathrm{Th}(\mathrm{BA}(G))$ is undecidable. This was done by interpreting BP in $\mathrm{BA}(G)$.

In what follows we need the notion of Boolean powers and an extension of this notion. Let A be any algebra, B a Boolean algebra. Then we set

$$A^* = \prod_{i \in \mathrm{Ult}\, B} A_i ,$$

with $A_i = A$ for each $i \in \mathrm{Ult}\, B$. The *Boolean power* $A[B]^*$ is the substructure of A^* with universe $\{f \in A^*: f^{-1}(a) \in \mathrm{Clop}\ \mathrm{Ult}\, B \text{ for each } a \in A\}$. In what follows we usually identify Clop Ult B with B. By compactness, for each $f \in A[B]^*$, the set $\{a \in A: f^{-1}(a) \neq \emptyset\}$ is finite.

Let θ be a congruence on A, B_0 a subalgebra of B. Then the *modified Boolean power* $A[B, B_0, \theta]^*$ is the substructure of $A[B]^*$ with universe $\{f \in A[B]^*: f^{-1}[\{b \in A: (a, b) \in \theta\}] \in B_0$ for each $a \in A\}$.

Remember that a group is locally finite if, for each finite set $X \subseteq G$, the subgroup generated by X is finite.

5.1. THEOREM. *Let G be a group which is not locally finite. Then* Th(BA(G)) *is undecidable.*

PROOF. Let $B = P(G)$, the power set algebra of G. For $a \in B$, $g \in G$ let

$$g(a) = \{g \circ h: h \in a\}.$$

This gives an algebra $B_G \in \mathrm{BA}(G)$. Now, as G is not locally finite, we can choose elements $g_0, \ldots, g_{n-1} \in G$ such that $\{g_0, \ldots, g_{n-1}\}$ generates an infinite subgroup H of G.

Let $a = \{g_0\}$, $b = H$. Then, in B, $0 < a < b$, a is an atom of B and b is an infinite subset of G.

Let θ be the congruence on B_G determined by the ideal $P_{<\omega}(G)$ (it is immediately seen that the ideal is closed under the action of G).

Let $(F, F_0, \leq)$ be a Boolean pair. We show that there is some algebra in BA(G) in which we can interpret this Boolean pair. We start with

$$B^* = B_G[F, F_0, \theta]^*.$$

For $c \in B$ let c^*: Ult $F \to B$ be the function with $c^*(i) = c$ for each $i \in \mathrm{Ult}\ F$. a is an atom; thus the interval $F^* = [0, a^*]$ in B^* is order-isomorphic to F under the map $f \mapsto f^{-1}(a)$. The only elements of $B \restriction b$ that are fixed by $g_0, \ldots, g_{n-1}$ are 0 and b. Let

$$\tilde{F}_0 = \{x \in [0, b^*]: g_i(x) = x \text{ for each } i < n\}.$$

Then $\tilde{F}_0 \cong F_0$ under the map $f \mapsto f^{-1}(b)$. Thus, F_0 is isomorphic to

$$F_0^* = \{a^* \cdot x: x \in [0, b^*],\ g_i(x) = x \text{ for each } i < n\},$$

and so

$$(F, F_0, \leq) \cong (F^*, F_0^*, \leq).$$

Now we set

$$\varphi_0(x) := x \leq a^*;$$

$$\varphi_1(x) := \exists y \Big(y \leq b^* \wedge x = a^* \cdot y \wedge \bigwedge_{i<n} g_i(y) = y \Big) \wedge \varphi_0(x);$$

$$\varphi_2(x, y) := x \leq y \wedge \varphi_0(x) \wedge \varphi_0(y).$$

Then

$$(\varphi_0^{B^*}, \varphi_1^{B^*}, \varphi_2^{B^*}) \cong (F, F_0, \leq)$$

and thus $BP \xrightarrow{\text{se}} \mathrm{BA}(G)$.

REMARKS. (1) MARTYJANOV [1982] showed that the theory of Boolean algebras with a distinguished automorphism is undecidable. More precisely, let BA(aut) be the class of all structures (B, α), where B is a Boolean algebra and α is an automorphism. Then Th(BA(aut)) is undecidable. This result can easily be deduced from the above theorem: Let $\mathbb{Z}$ be the group of integers with addition. Let the integer 1 correspond to α. Then it is immediately seen that $\mathrm{BA}(Z) \xrightarrow{\text{se}} \mathrm{BA(aut)}$ and thus Th(BA(aut)) is undecidable.

(2) DULATOVA [1984] showed that the theory of atomic Boolean algebras with a distinguished automorphism is undecidable. This result is also an immediate consequence of the above theorem.

6. Single Boolean algebras with a distinguished ideal

It is known that all completions of the elementary theory of Boolean algebras are decidable (see Section 18 of Part I of this Handbook). Further on, it was shown by Rabin (see Section 2 of Chapter 23) that also the theory of Boolean algebras in the language augmented by a sequence of unary predicates varying on ideals is decidable. As we shall see now, this is not true for completions of the theory of Boolean algebras if the elementary language is augmented by one unary predicate defining an ideal (MOROZOV [1982]). Here we follow an idea of MOLZAN [1981].

By induction, we define linear orderings σ_n, τ_n, $\sigma_{m,n}$, $\tau_{m,n}$ and ideals related to each Boolean algebra A, $i_n = i_n(A)$, $j_m = j_m(A)$, $(m, n < \omega)$ as follows:

$$\sigma_0 = \omega\ ; \qquad \tau_0 = (1+1)\eta_0\ ;$$

$$\sigma_{n+1} = (\sigma_n + \tau_n)\omega\ ; \qquad \tau_{n+1} = (\sigma_n + \tau_n)\eta_0\ ;$$

$$\sigma_{0,n} = \sigma_n\ ; \qquad \tau_{0,n} = \tau_n + \eta_0\ ;$$

$$\sigma_{m+1,n} = (\sigma_{m,n} + \tau_{m,n})\omega\ ; \qquad \tau_{m+1,n} = (\sigma_{m,n} + \tau_{m,n})\eta_0\ ;$$

$$i_0(A) = \{a \in A\colon a \text{ contains only finitely many atoms}\}\ ;$$

$$j_0(A) = \{a \in A\colon a \text{ is the union of an atomic and an atomless element of } A\}\ ;$$

$$i_{n+1}(A) = \{a \in A\colon a/i_n \in j_0(A/i_n)\}\ ;$$

$$j_{n+1}(A) = \{a \in A\colon a/j_n \in j_0(A/j_n)\}\ .$$

By simultaneous induction we can prove:

6.1. LEMMA.

(i) $\mathrm{Intalg}(\sigma_n)/i_n \cong 2$;
$\mathrm{Intalg}(\tau_n)/i_n \cong \mathrm{Intalg}(\eta_0)$;

(ii) $\mathrm{Intalg}(\sigma_{m,n})/i_{m+n} \cong 2$;
$\mathrm{Intalg}(\tau_{m,n})/i_{m+n} \cong \mathrm{Intalg}(\eta_0)$;

(iii) $\mathrm{Intalg}(\sigma_{m+1,n})/j_m \cong 2$;
$\mathrm{Intalg}(\tau_{m+1,n})/j_m \cong \mathrm{Intalg}(\eta_0)$;

(iv) *for all* $a \in \mathrm{Intalg}(\sigma_{m+1,n})$: *if* $a \in i_{m+n+1}$, *then* $a \in j_m$.

Notice that the ideals j_n are elementary definable (without parameters), and that i_n, $n > 0$, are definable using a formula defining i_0.

For X an infinite subset of ω, let

$$\sigma_{m+1,X} = \sum_{n \in X} \sigma_{m+1,n} .$$

6.2. LEMMA. *For all* m, $n \in \omega$,

$$\begin{aligned}(\mathrm{Intalg}(\sigma_{m+1,X}), i_0) \models \exists x[&\text{“}x/i_{m+n+1} \textit{ is an atom}\text{”} \\ &\wedge \text{“}x/j_m \textit{ is an atom}\text{”} \\ &\wedge \forall y(\text{“}(x \cdot y) \in i_{m+n+1}\text{”} \rightarrow \text{“}x \cdot y \in j_m\text{”})]\end{aligned}$$

if $n \in X$.

PROOF. If $n \in X$, then $\mathrm{Intalg}(\sigma_{m+1,X})$ contains an element which is an interval of the type $\sigma_{m+1,n}$. According to Lemma 6.1(iv) this element satisfies the left-hand condition in the statement of the lemma. Next we suppose that $n \not\in X$, but we assume that a/i_{m+n+1} is an atom, a/j_m is an atom and for all $b \in \mathrm{Intalg}(\sigma_{m+1,X})$, $a \cdot b \in i_{m+n+1}$ implies $a \cdot b \in j_m$. If a is a bounded element of $\mathrm{Intalg}(\sigma_{m+1,X})$ (i.e. a is contained in some interval $[\alpha, \beta)$), then a intersects only finitely many elements b_k, which are the intervals of the type $\sigma_{m+1,k}$, where $k \in X$.

For $k < n$ we conclude that $a \cdot b_k/i_{m+k+1}$ is either 0 or an atom, hence $a \cdot b_k \in i_{m+k+2} \subseteq i_{m+n+1}$ and therefore $a \cdot b_k \in j_m$ by our assumption. From $k > n$ it follows that $a \cdot b_k \in i_{m+k+1}$. Otherwise a/i_{m+n+1} would contain more than one atom. Hence, again $a \cdot b_k \in j_m$. We conclude that, for a bounded, $a \in j_m$, a contradiction. If a is not a bounded element of $\mathrm{Intalg}(\sigma_{m+1,X})$, then a contains an interval of type $\sigma_{m+1,k}$ for some $k \in X$ with $k > n$. Hence, a/i_{m+n+1} cannot be an atom, as in this case $a \in i_{m+n+2} \subseteq i_{m+k+1}$. □

6.3. THEOREM. *For all* $m < \omega$, *there exist a Boolean algebra B and an ideal* $i \subseteq B$ *such that*

(i) B/j_m *is atomic*;

(ii) $\mathrm{Th}((B, i))$ *is undecidable*;

(iii) $\mathrm{Th}^{\mathcal{Q}_0}(B)$ *is undecidable.*

In the case $m = \omega$ *there are a Boolean algebra B and an ideal* $i \subseteq B$ *such that*

(iv) *for all* $n < \omega$, $B/j_n \neq 0$, *and*

(v) $\mathrm{Th}((B, i))$ *is undecidable.*

PROOF. Take a non-recursive set $X \subseteq \omega$. For the first part of the theorem we set

$$B = \text{Intalg}(\sigma_{n+1,X}) , \quad i = i_0(B) .$$

Then (i) follows from Lemma 6.1(iii). By Lemma 6.2, X is recursive in $\text{Th}((B, i))$, thus (ii) holds. Assertion (iii) holds as i_0 is definable by means of the quantifier Q_0.

To prove the second part we set

$$\sigma_X = \sum_{n \in X} \sigma_{n+1,0} .$$

Let k be the ideal of $\text{Intalg}(\sigma_X)$ which is generated by the ideals $j_n(\text{Intalg}(\sigma_{n+1,0}))$ with $n \in X$. Analogous to the first part, X is recursively encoded in $\text{Th}((\text{Intalg}(\sigma_X), k))$ by

$$\begin{aligned} n \in X \quad \text{iff}\ (\text{Intalg}(\sigma_X), k) \models \exists x[&\text{“}x/k \text{ is an atom”} \\ &\wedge \text{“}x/j_n \text{ is an atom”} \\ &\wedge \forall y(\text{“}x \cdot y \in j_n\text{”} \rightarrow \text{“}x \cdot y \in k\text{”})] . \quad \square \end{aligned}$$

REMARK. It follows from the proof of Theorem 6.3 that there are $2^{\aleph_0}$ complete $L(Q_0)$-theories of Boolean algebras. A more general result is contained in MOLZAN [1981]: there is a family $\{B_{X,Y}: X, Y \subseteq \omega, \text{infinite}\}$ of Boolean algebras, all having the same elementary theory but

$$B_{X_1,Y_1} \equiv (L^{Q_0}) B_{X_2,Y_2} \quad \text{iff } X_1 = X_2 ;$$

$$B_{X_1,Y_1} \equiv (L^{Q_0^2}) B_{X_2,Y_2} \quad \text{iff } X_1 = X_2 \text{ and } Y_1 = Y_2 .$$

7. Boolean algebras in a logic with quantification over ideals

Let L_i be the language of Boolean algebras which is extended by allowing quantification over ideals. For $K \subseteq \text{BA}$ let $\text{Th}^i(K)$ denote the theory of K in the language L_i.

It was shown by RABIN [1965] that $\text{Th}^i(\text{BA}_\omega)$ is decidable (where BA_ω denotes the class of countable Boolean algebras, see Section 2 of Chapter 23). Here we show a result of HEINDORF [1984] that $\text{Th}^i(\text{sBA})$ is undecidable (where sBA denotes the class of superatomic Boolean algebras). This implies immediately that also $\text{Th}^i(\text{BA})$ is undecidable.

We use some properties which can conveniently be formulated using topological notions. Therefore we work with Boolean spaces and a language L_{top} for describing topological properties. (*Warning*: L_{top} is different from L_t, that language which is usually used in topological model theory.)

Each formula $\varphi \in \text{Form } L_{\text{top}}$ can be translated into a formula $\Phi \in \text{Form } L_i$ such that for each $X \in BS$ (the class of Boolean spaces), $X \models \varphi$ iff $\text{Clop } X \models \Phi$. Thus, the undecidability of $\text{Th}^{\text{top}}(BS)$ (the theory of Boolean spaces in the language L_{top}) implies the undecidability of $\text{Th}^i(\text{BA})$.

Let X be a topological space, $Y \subseteq X$. $\mathrm{cl}(Y)$ denotes the closure of Y. A point $p \in X$ is said to be an *ω_1-point* if, for each countable $Y \subseteq X \setminus \{p\}$, $p \not\in \mathrm{cl}(Y)$. Let S, $T \subseteq X$. We say that S and T *can be separated* if there are disjoint open sets U, V with $S \subseteq U$, $T \subseteq V$. Remember that X is *normal* if every two disjoint closed sets F, $G \subseteq X$ can be separated. If X is the Stone space of a countable Boolean algebra, then X is metrizable. Hence, also each subspace of X is metrizable and thus normal. But in general not every subspace of a normal space has to be normal.

Let X be a normal space. A point $p \in X$ is said to be a *hinge point* if $X \setminus \{p\}$ (with the subspace topology) is not normal. The following example was given by TYCHONOFF [1930]: Let S_0 be $\omega_1 + 1$ with the order topology and let S_1 be $\omega + 1$ with the order topology. Let

$$T = S_0 \times S_1 .$$

S_0 and S_1 are Boolean spaces and so is T.

REMARK. Clop T and Intalg$(\omega \cdot \omega_1)$ have the same cardinal sequences but are not isomorphic.

We put

$$L = \{(\alpha, \omega) \colon \alpha \leq \omega_1\} ;$$

$$S = \{(\omega_1, n) \colon n \leq \omega\} ;$$

$$a = (\omega_1, \omega) .$$

L is called the *long edge* and S is called the *short edge*. We have

7.1. LEMMA. (i) *If $F \subseteq L \setminus \{a\}$, $G \subseteq S \setminus \{a\}$ and $a \in \mathrm{cl}(F)$, $a \in \mathrm{cl}(G)$, then F and G cannot be separated.*

(ii) *a is a hinge point of T and there are no other hinge points.*

(iii) *$S \setminus \{a\}$ is the set of ω_1-points of T and a is the only accumulation point of $K \setminus \{a\}$.*

(iv) *Let F be closed and assume that F does not contain ω_1-points. Then a is an accumulation point of F iff a is an accumulation point of $F \cap L$.*

(v) *Let F and G be closed and assume that neither F nor G contain accumulation points. If a is an accumulation point of F and G, then a is also an accumulation poin of $F \cap G$.*

PROOF. (i) Let U and V be open sets with $F \subseteq U$, $G \subseteq V$. We have to show that $(U \cap V) \setminus \{a\} \neq \emptyset$.

For each $(\omega_1, m) \in G \subseteq V$ there exists $\alpha_m < \omega_1$ with $(\alpha_m, \omega_1] \times \{m\} \subseteq V$. Put

$$\beta = \sup\{\alpha_m \colon (\omega_1, m) \in G\} .$$

Now a is an accumulation point of F, hence we can find γ with $\beta < \gamma < \omega_1$ and $(\gamma, \omega) \in F$. Thus, for some $m < \omega$, $(\gamma, m) \in U$ and also $(\gamma, m) \in V$.

(ii) By (i), a is a hinge point. Let $p \neq a$.

Case 1. $p \in K$, i.e. $p = (\omega_1, m)$ for some $m < \omega$. We set

$$U = (\omega_1 + 1) \times \{m\} .$$

Then $U \setminus \{p\}$ is normal (since it is an order topology) and $T \setminus U$ is compact and hence normal too. But then

$$T \setminus \{p\} = (T \setminus U) \cup (U \setminus \{p\}) ;$$

Thus $T \setminus \{p\}$ is the union of two normal spaces and also normal. So p is no hinge point.

Case 2. $p \not\in K$, i.e. $p = (\alpha, m)$ with $\alpha < \omega_1$, $m \leq \omega$. Then $U = (\alpha + 1) \times (\omega + 1)$ is metrizable, so $U \setminus \{p\}$ is normal. $T \setminus U$ is compact and thus normal. So again $T \setminus \{p\} = (T \setminus U) \cup (U \setminus \{p\})$ is normal.

(iii) Is obvious.

(iv) If a is an accumulation point of $F \cap L$, then a is also an accumulation point of F. Now assume that a is an accumulation point of F. We put

$$F^* = \{\beta < \omega_1 \colon (\beta, \omega) \in F\} .$$

Then F^* is closed. We have to show that F^* is unbounded. Let $\alpha < \omega_1$. For $i \in \omega$ we set

$$U_i = (\alpha, \omega_1] \times (i, \omega] .$$

For each $i \in \omega$ choose $(\beta_i, n_i) \in F \cap U_i$. If, for some β, the set $\{i \in \omega \colon \beta_i = \beta\}$ is infinite, then $\beta \in F^*$ and $\beta > \alpha$. Thus, assume that for each β, $\{i \in \omega \colon \beta_i = \beta\}$ is finite. Choose an increasing subsequence $(\gamma_i)_{i<\omega}$ of $(\beta_i)_{i<\omega}$. Put

$$\gamma = \lim(\gamma_i)_{i<\omega} .$$

But then $\gamma \in F^*$. Thus, F^* is unbounded.

(v) Using (iv) and the fact that the intersection of two closed unbounded subsets of ω_1 is again closed and unbounded. □

Now we generalize the example given by Tychonoff. For $n \in \omega$ let D_n be n with the discrete topology, and let X_n be the product of the topological spaces T and D_n. We define an equivalence relation θ on X_n by

$$\theta = \{((p, i), (p, j)) \colon i, j < n,\ p \in L\} .$$

Then let T_n be the quotient space X_n/θ. T_n can be described as follows. Take the disjoint union of n copies of T and identify the long edges. The long edge of T_n, denoted by L_n, is $L \times D_n/\theta$.

We show in what follows that each finite irreflexive connected graph can be coded in a suitable Boolean space. This is enough since each irreflexive finite graph can be coded by a Finite irreflexive connected graph.

Let $G=(G,R)$ be a finite irreflexive connected graph with $|G|\geq 2$. For each $g\in G$ we set

$$v(g)=|\{f\in G\colon (g,f)\in R\}|\,,$$

and choose a one-to-one function $\pi_g\colon v(g)\to\{f\in G\colon (g,f)\in R\}$. We construct a topological space T_G as follows. Take the disjoint union of $\{T_{v(g)}\colon g\in G\}$ and identify the short edges of $T_{v(g)}$ and $T_{v(f)}$ iff $(g,f)\in R$. More formally, put

$$X_G=\bigcup\{T_{v(g)}\times\{g\}\colon g\in G\}\,;$$

$$\theta_G=\{((p,i,g),(p,j,f))\colon p\in S,(g,f)\in R,\ \pi_g(i)=f,\ \pi_f(j)=g\}\,.$$

Then we set

$$T_G=X_G/\theta_G\,.$$

T_G has exactly one hinge point. We denote this point by a. Now we have to show that G is coded in T_G and can be defined in the suitable chosen language L_{top}.

Let L_{top} be the three-sorted language which is defined as follows:

- L_{top} has a constant a of the first sort;
- variables of the first sort are $p,q,\ldots$ varying over points;
- variables or the second sort are $U,V,\ldots$ varying over open sets;
- variables of the third sort are $F,G,\ldots$ varying over closed sets.

Atomic formulas are expressions like $p=q$, $U=V$, $p\in U$ and $p\in F$. Formulas are built as usual by atomic formulas. We put

$$\mathrm{ac}(p,F):=\forall U(p\in U\to\exists q(q\in U\wedge q\in F)\wedge p\not\in F)\,;$$

i.e. $\mathrm{ac}(p,F)$ expresses that p is an accumulation point of F. In compact spaces the ω_1-points can be characterized as follows: p is an ω_1-point iff each closed set F which has p as accumulation point, has an accumulation point q which is different from p. More formally, we set

$$P_{\omega_1}(p):=\forall F(\mathrm{ac}(p,F)\to\exists q(q\neq p)\wedge\mathrm{ac}(q,F))\,.$$

We like to code the edges by the long edges of the corresponding topological space. We set

$$\begin{aligned}\varphi_0(F):=\neg\exists p(p\in F\wedge P_{\omega_1}(p)\wedge\mathrm{ac}(a,F)\\ &\wedge\forall G_0G_1(G_0\subseteq F\wedge G_1\subseteq F\wedge\mathrm{ac}(a,G_0)\wedge\mathrm{ac}(a,G_1)\\ &\to\exists q(q\neq a\wedge\mathrm{ac}(q,G_0)\wedge\mathrm{ac}(q,G_1)))\,;\end{aligned}$$

i.e. $\varphi_0(F)$ iff F is closed, does not contain ω_1-points, a is an accumulation point of F and there do not exist two closed subsets G_0, G_1 of F with $G_0\cap G_1=\{a\}$. Then the long edges satisfy φ_0. Lemma 7.1(iv) implies that for each closed set F which satisfies φ_0 there exists a long edge which is not very different from F. Thus, we set

$$\psi(F, H) := \varphi_0(F) \wedge \varphi_0(H) \wedge \mathrm{ac}(a, F \cap H)\,.$$

Lemma 7.1(v) implies that ψ^{T_G} is an eqivalence relation on $\varphi_0^{T_G}$ and each equivalence class contains just one long edge.

A closed set is called *ω_1-set* if a is an accumulation point of F and every point $q \in F$ which is different from a, is an ω_1-point. Formally,

$$S_{\omega_1}(F) := \forall p(p \in F \wedge p \neq a \rightarrow P_{\omega_1}(p)) \wedge \mathrm{ac}(a, F)\,.$$

Two ω_1-sets F and H are said to be *connected* if, for every two open sets U, V with $U \supseteq F \backslash \{a\}$, $V \supseteq H \backslash \{a\}$, we can find $q \in U \cap V$ with $q \neq a$:

$$\mathrm{con}(F, H) := S_{\omega_1}(F) \wedge S_{\omega_1}(H) \wedge \forall UV(F\backslash\{a\} \subseteq U \\ \wedge H\backslash\{a\} \subseteq V \rightarrow V \cap H \not\subseteq \{a\})\,.$$

We set

$$\varphi_1(F, H) := \exists G_0(S_{\omega_1}(G_0) \wedge \forall F_0 H_0 G_1[\psi(F, F_0) \wedge \psi(H, H_0) \\ \wedge G_1 \subseteq G_0 \rightarrow \mathrm{con}(F_0, G_1) \wedge \mathrm{con}(H_0, G_1)]\,.$$

We have to convince ourselves that $T_G \models \varphi_1(L_g, L_f)$ iff $(f, g) \in R$. But this is immediately clear by Lemma 7.1(i). Thus,

$$(\varphi_0^{T_G}, \varphi_1^{T_G})/\psi^{T_G} \cong \boldsymbol{G}$$

and hence

7.2. THEOREM. $\mathrm{Th}^{\mathrm{top}}(BS)$ *is undecidable.*

L_{top} can be translated into L_i, the language of Boolean algebras where it is allowed to quantify over ideals. For each open set U of a Boolean space X let $I(U)$ denote the ideal of Clop X which consists of all clopen sets Y with $U \subseteq Y$. For each $p \in X$, let $I(p)$ denote the prime ideals of Clop X which consists of all clopen sets Y with $p \not\in Y$ and for each closed set $F \subseteq X$ let $J(F)$ be the ideal of all clopen sets which are disjoint from F. Then the following lemma is immediately seen:

7.3. LEMMA. *For each $\varphi(\vec{p}, \vec{F}, \vec{U}) \in \mathrm{Form}\, L_{\mathrm{top}}$ there is $\Phi(\vec{X}, \vec{Y}, \vec{Z}) \in \mathrm{Form}\, L_i$ such that for each Boolean space X, $\vec{p} \in X$, $\vec{F} \subseteq X$, $\vec{U} \subseteq X$ we have*

$$X \models \varphi(\vec{p}, \vec{F}, \vec{U}) \quad \text{iff } \mathrm{Clop}\, X \models \Phi(I(\vec{p}), J(\vec{F}), I(\vec{U}))$$

(where $I(\vec{p})$ denoted the sequence of ideals $(I(p_0), \ldots, I(p_{k-1}))$ and similarly for $J(\vec{F})$, $I(\vec{U})$).

Using this lemma we have:

7.4. COROLLARY. $\mathrm{Th}^i(\mathrm{BA})$ *is undecidable.*

REMARK. In the proof we used only scattered Boolean spaces. Thus, we have shown that also $\mathrm{Th}^i(\mathrm{sBA})$ is undecidable, where sBA denoted the class of superatomic Boolean algebras.

8. Some applications

In this section we present two undecidability results which are closely connected with undecidable theories of Boolean algebras.

BURRIS and LAWRENCE [1982] used modified Boolean powers to give a new proof of the following result of ZAMJATIN [1976]:

8.1. THEOREM. *A variety of rings with unity has a decidable theory iff it is generated by finitely many finite fields.*

The direction ($\leftarrow$) follows from Theorem 6.18 of Chapter 23 and the example given before Lemma 6.9 of the same chapter. Thus, we only have to show the converse. We assume in the following that the reader is familiar with the basic facts of varieties as given in Section 6 of Chapter 23. Here we need some more facts from universal algebra.

An algebra A is *semisimple* if it is isomorphic to a subdirect product of simple algebras. A variety V is said to be semisimple if every member of V is semisimple. We have the following:

8.2. LEMMA. *A variety V is semisimple iff every subdirectly irreducible member of V is simple.*

PROOF. ($\rightarrow$) Let $A \in V$ be subdirectly irreducible. Then A can be subdirectly embedded into a product of simple algebras, say by

$$\alpha: A \rightarrow \prod_{i \in I} S_i \,.$$

Since A is subdirectly irreducible, there is a projection

$$\pi_i: \prod_{i \in I} S_i \rightarrow S_i$$

such that $\pi_i \circ \alpha$ is an isomorphism; thus $A = S_i$ and this implies that A is simple.

($\leftarrow$) By a result of Birkhoff (see Chapter 23, Lemma 6.4), every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. Thus, by assumption, every $A \in V$ is isomorphic to a subdirect product of simple algebras. □

Let R be a ring, $a \in R$. Then a is *central* if, for any $b \in R$, $a \cdot b = b \cdot a$; a is *idempotent* if $a^2 = a$. Of course 0 is central and idempotent. If R has a unity 1, then also 1 is central and idempotent.

Let R be a ring with unity and let $e \in R \setminus \{0, 1\}$ be central and idempotent. Then, for any $b \in R$, we have $b = e \cdot b + (1 - e) \cdot b$. Now assume that $b \in e \cdot R$ and $b \in (1 - e) \cdot R$. Then $b = e \cdot b_0 = (1 - e) \cdot b_1$ for some $b_0, b_1 \in R$. This implies $e \cdot b = e \cdot (1 - e) \cdot b_1 = 0$ and also $(1 - e) \cdot b = (1 - e) \cdot e \cdot b_0 = 0$ and thus $b = 0$. This implies that R is the direct sum of the ideals $e \cdot R$ and $(1 - e) \cdot R$. Thus, if R is a directly indecomposable ring with unity, then the only central-idempotents are 0 and 1.

In Part I of this Handbook, Theorem 19.2, it was shown that the theory BP^2 of Boolean pairs (A, B) such that A and B have the same set of atoms is hereditarily undecidable. Let BP^D be the class of Boolean pairs (A, B) such that B is dense in A, i.e. $(A, B) \models \forall x(x \neq 0 \rightarrow \exists y(y \neq 0 \wedge U(y) \wedge y \leq x))$. Then $BP^2 \subseteq BP^D$, and thus BP^D is undecidable too.

For the following lemma recall the definition of modified Boolean power as given in Section 5.

8.3. LEMMA. *Let R be a directly indecomposable non-simple ring and let θ be a congruence of R with $\Delta < \theta < \nabla$. Then BP^D can be interpreted into $\{R[B, B_0, \theta]^*\colon (B, B_0) \in BP^D\}$.*

PROOF. Let the Boolean pairs be given in the form of structures $(B, B_0, \leq)$, where $\leq$ is the partial ordering of the Boolean algebra B. We set:

$$\varphi_0(x) := x = x\ ;$$

$$\varphi_1(x) := x^2 = x \wedge \forall y(x \cdot y = y \cdot x)\ ;$$

i.e. $\varphi_1(x)$ states that x is central and idempotent;

$$\varphi_2(x, y) := \forall z[\varphi_1(z) \rightarrow (y \cdot z = y \rightarrow x \cdot z = x)]\ ;$$

$$\psi(x, y) := \varphi_2(x, y) \wedge \varphi_2(y, x)\ .$$

Let $(B, B_0, \leq) \in BP^D$ with $B \subseteq P(I)$, $R_1 = R[B, B_0, \theta]^*$. For $f \in R_1$ we set

$$\alpha(f) = [\![f \neq 0]\!]\ .$$

Then $\{\alpha(f)\colon f \in R_1\} \subseteq B$. We show that also the inverse inclusion holds true. Choose $a \in R$ with $a \neq 0$ and $(a, 0) \in \theta$. For $b \in B$ we set

$$f_b(i) = \begin{cases} 0 & \text{for } i \in b\ ; \\ a & \text{otherwise}\ . \end{cases}$$

Then $f_b \in R_1$ and $\alpha(f_b) = b$.

CLAIM. $B_0 = \{\alpha(f)\colon f \in R_1 \text{ and } R_1 \models \varphi_1(f)\}$.

Proof of the claim. It is immediately clear that $B_0 \subseteq \{\alpha(f)\colon f \in R_1$ and $R_1 \models \varphi_1(f)\}$. Let $f \in R_1$ with $R_1 \models \varphi_1(f)$. Then, for each $i \in I$, $f(i)$ is zero or the unity of R. Now these two elements cannot be θ-equivalent and thus $f^{-1}(0) = f^{-1}(0/\theta) = \alpha(f) \in B_0$. This shows the claim.

For all $f, g \in R_1$,

$$R_1 \models \varphi_2(f, g) \quad \text{iff } \alpha(f) \subseteq \alpha(g) .$$

This holds because B_0 is dense in B. Consequently, we have

$$(\varphi_0^{R_1}, \varphi_1^{R_1}, \varphi_2^{R_1})/\psi^{R_1} \cong (B, B_0, \theta) . \quad \square$$

8.4. LEMMA. *A semisimple variety V is generated by finitely many finite fields.*

PROOF. First we show that $F_V(\emptyset)$ (the V-free ring generated by the unity 1) has to be finite. For otherwise, $F_V(\emptyset) = \mathbb{Z} \in V$. But $\mathbb{Z}_4$ is a homomorphic image of $\mathbb{Z}$ and thus $\mathbb{Z}_4 \in V$. But $\mathbb{Z}_4$ is a non-simple, subdirectly irreducible algebra; this contradicts the fact that V is semisimple. Thus, there are only finitely many p such that there is a field of characteristic p in V. For any prim p the polynomial ring $\mathbb{Z}_p[x]$ is not in V as $\mathbb{Z}_p[x]/\langle x^2 \rangle$ is subdirectly irreducible and not simple.

Let F be a field in V which has characteristic p. We show that F must be finite. Otherwise, there is a transcendental element $a \in F$ or there are elements $a_n \in F$ for $n < \omega$ such that degree $(a_n) \geq n$. In the first case $Z_p[x]$ can be embedded into F and in the second case $\mathbb{Z}_p[x]$ can be embedded into F^ω/U for each non-principal ultrafilter U on ω. Thus, V contains, up to isomorphism, only finitely many fields, and they are all finite.

Now consider $F_V(\{a\})$, the V-free algebra generated by any constant a. $F_V(\{a\})$ is commutative and, since V is semisimple, $F_V(\{a\})$ must be a subdirect product of fields. Now there are only finitely many fields in V and they are all finite; thus $a^n = a$ for some n. But then $V \models \forall x(x^n = x)$ and thus V is generated by finitely many fields. (Compare with the example following Theorem 6.18 of Chapter 23.) $\square$

Now Lemma 8.3 and Lemma 8.4 imply the directions $(\rightarrow)$ of the theorem.

REMARK. ERŠOV [1974] showed that each variety of groups with a finite non-abelian member has an undecidable theory. In BURRIS and LAWRENCE [1982] a proof is given using the undecidability of Boolean pairs.

PINUS [1985] showed that many undecidability results for Boolean algebras can be translated for congruence–distributive varieties if the main congruences are elementarily definable. We need some more facts from universal algebra.

Let A be an algebra, $a, b \in A$. Then $\theta_{a,b}$ denotes the least congruence θ with $a, b \in \theta$. Congruences of this form are called *main congruences*. Let $\mathrm{Con_m}\, A$ denote the partially ordered set of main congruences of A.

A congruence $\theta \in \mathrm{Con}\, A$ is *meet irreducible* if for all $\theta_0, \theta_1 \in \mathrm{Con}\, A$, $\theta = \theta_0 \cap \theta_1$ implies $\theta = \theta_0$ or $\theta = \theta_1$. We need the following:

8.5. LEMMA. *Each $\theta \in \operatorname{Con} A$ is the intersection of meet irreducible congruences.*

PROOF. Let $a, b \in A$ with $(a, b) \not\in \theta$. Then for $\theta_0, \theta_1 \in \operatorname{Con} A$, $\theta_0 \cap \theta_1 = \theta$ implies $(a, b) \not\in \theta_0$ or $(a, b) \not\in \theta_1$. Let

$$X = \{\varphi \in \operatorname{Con} A\colon \text{for some } \psi \in \operatorname{Con} A,\ \theta = \varphi \cap \psi\}\ .$$

Choose a maximal chain $C_{a,b} \subseteq X$ with $(a, b) \not\in \psi$ for each $\psi \in C_{a,b}$. Then $\psi[a, b] = \bigcap C_{a,b}$ is meet irreducible and $(a, b) \not\in \psi[a, b]$. Thus, we have

$$\theta = \bigcap \{\psi[a, b]\colon a, b \in A,\ (a, b) \not\in \theta\}\ .$$

Let $\theta \in \operatorname{Con} A$, $X \in BS$ and $U \subseteq X$. We set $B = \operatorname{Clop} X$. θ_U denotes the congruence on the Boolean power $A[B]^*$ (compare with Section 5), which is given by

$$(f, g) \in \theta_U \quad \text{iff } (f(p), g(p)) \in \theta \text{ for each } p \in U\ .$$

For $p \in X$ we also write $\theta(p)$ instead of $\theta_{\{p\}}$. Especially we have

$$(f, g) \in \Delta_U \quad \text{iff for each } p \in U,\ f(p) = g(p)\ . \quad \square$$

PINUS [1981] showed:

8.6. LEMMA. *Let V be a congruence–distributive variety, $A \in V$, $X \in BS$, $B = \operatorname{Clop} X$. Then each $\theta \in \operatorname{Con} A[B]^*$ is the intersection of congruences of the form $\theta(p)$ with $\theta \in \operatorname{Con} A$, $p \in X$.*

PROOF. First we show that for each meet irreducible congruence $\theta \in \operatorname{Con} A[B]^*$ there exists a $p \in X$ with $\Delta(p) \leq \theta$. W.l.o.g. we assume that $\theta \neq \Delta$. We set

$$F = \{b \in B\colon \Delta_b \leq \theta\}\ ,$$

and show that F is an ultrafilter on B. Of course, if $b \leq b_0$ and $b \in F$, then $b_0 \in F$. Let $b, b_0 \in F$. Then $\Delta_{b \cdot b_0} \leq \Delta_b \vee \Delta_{b_0} \leq \theta$ and thus $b \cdot b_0 \in F$. Let $b \in B$; we show now that $b \in F$ or $-b \in F$. We have $\Delta_b \cap \Delta_{-b} = \Delta$ and thus

$$\theta = \theta \vee \Delta = \theta \vee (\Delta_b \cap \Delta_{-b}) = (\theta \vee \Delta_b) \cap (\theta \vee \Delta_{-b})$$

and as θ is meet irreducible, $\theta = \theta \vee \Delta_b$ or $\theta = \theta \vee \Delta_{-b}$ and thus $b \in F$ or $-b \in F$. So we have shown that F is an ultrafilter on B.

If $f, g \in A[B]^*$ and $(f, g) \in \Delta(F)$, then for some $b \in F$, $(f, g) \in \Delta_b$ and so $(f, g) \in \theta$. This implies $\Delta(F) \leq \theta$.

Now, to prove the statement of the lemma it is enough to show that each meet irreducible congruence $\psi \in \operatorname{Con} A[B]^*$ has the form $\theta(p)$ for some $\theta \in \operatorname{Con} A$ and some $p \in X$.

Let $\theta \in \operatorname{Con} A[B]^*$ which is meet irreducible. Choose $p \in X$ with $\Delta(p) \leq \theta$. We set

$$\psi = \{(a, b)\colon a, b \in A \text{ and there are } f, g \in A[B]^* \text{ with } (f, g) \in \theta, f(p) = a, g(p) = b\}\,.$$

We show that ψ is as desired. Of course, we have $\theta \leq \psi(p)$. Now let $(f, g) \in \psi(p)$. Then $(f(p), g(p)) \in \psi$ and thus there are $h, k \in A[B]^*$ with $(h, k) \in \theta$, $h(p) = f(p)$, $k(p) = g(p)$. Now $\Delta(p) \leq \theta$ implies $(h, f) \in \theta$ and $(k, g) \in \theta$. Thus, we have $(f, g) \in \theta$ and $\psi(p) \leq \theta$, i.e. we also have $\theta = \psi(p)$. □

8.7. LEMMA. *Let V be a congruence-distributive variety, $A \in V$, $X \in BS$, $B = \operatorname{Clop} X$. Then the partially ordered sets* $\operatorname{Con}_{\mathrm{m}} A[B]^*$ and $(\operatorname{Con}_{\mathrm{m}} A)[B]^*$ *are isomorphic.*

PROOF. Let $f, g \in A[B]^*$. Then

$$(f, g) \in \bigcap_{p \in X} \theta_{f(p), g(p)}(p)$$

and thus

$$\theta_{f,g} \leq \bigcap_{p \in X} \theta_{f(p), g(p)}(p)\,.$$

Now we show that also

$$\bigcap_{p \in X} \theta_{f(p), g(p)}(p) \leq \theta_{f,g}\,.$$

By the preceding lemma there exists a family of congruences $\{\theta_j(p_j)\colon j \in J\}$ with

$$\theta_{f,g} = \bigcap_{j \in J} \theta_j(p_j)\,.$$

Then $(f(p_j), g(p_j)) \in \theta_j$ for each $j \in J$ and so

$$\theta_j \geq \theta_{f(p_j), g(p_j)}$$

for each $j \in J$. This implies:

$$\theta_j(p_j) \geq \theta_{f(p_j), g(p_j)}(p_j)$$

and thus

$$\begin{aligned}\theta_{f,g} = \bigcap_{j \in J} \theta_j(p_j) &\geq \bigcap_{j \in J} \theta_{f(p_j), g(p_j)}(p_j) \\ &\geq \bigcap_{p \in X} \theta_{f(p), g(p)}(p)\,.\end{aligned}$$

For each $f, g \in A[B]^*$ there exists a finite partition of B, $\{b_0, \ldots, b_{n-1}\}$ such that for each $i < n$, f and g are constant on b_i. We define

$$\varphi\colon \mathrm{Con_m}\, A[B]^* \to (\mathrm{Con_m}\, A)[B]^*$$

by

$$\varphi(\theta_{f,g})(p) = \theta_{f(p),g(p)} .$$

Then it is easily seen that φ is the desired isomorphism. □

Magari (see BURRIS and SANKAPPANAVAR [1981]) showed:

8.8. THEOREM. *Let V be a variety with a non-trivial algebra. Then there is a non-trivial simple algebra in V.*

PROOF. Let L be the underlying language for V. For Y a set of variables let Term Y denote the set of all terms of L that can be built in the variables of Y. We can consider Term Y as a set or as an L-algebra. Let $X=\{x, y\}$ and let $F_V(\{x, y\})$ denote the free algebra in V generated by x and y. For $s, t \in \mathrm{Term}\, X$ we set $(s, t) \in \psi_0$ iff $V \models s=t$. Then $\mathrm{Term}\, X/\psi_0 \cong F_V(X)$. Let $s \in \mathrm{Term}\, X$. Then we write $\bar{s}$ for s/ψ_0. We set

$$S = \mathrm{Term}\{x\}/\psi_0 .$$

First suppose that $\theta_S \neq \nabla$ in $\mathrm{Con}\, F_V(X)$ (where θ_S denotes the least congruence from $\mathrm{Con}\, F_V(X)$ containing S). Then by Zorn's lemma there is a maximal element θ_0 in $[\theta_S, \nabla]\backslash\{\nabla\}$ (use the fact that for $\theta \in [\theta_S, \nabla]$, $\theta = \nabla$ iff $(\bar{x}, \bar{y}) \in \theta$). Then $F_V(X)/\psi_0$ is a simple algebra in V.

Now assume that $\theta_S = \nabla$. But then, for some finite $S_0 \subseteq S$, $(\bar{x}, \bar{y}) \in \theta_{S_0}$. Now V is non-trivial and thus $\bar{x} \neq \bar{y}$ in $F_V(X)$ and, since $(\bar{x}, \bar{y}) \in \theta_S$, it follows that S is non-trivial.

CLAIM. $\nabla_S = \theta_{S_0}$.

Proof of the claim. Let $p(\bar{x}) \in S$. We define a homomorphism $\alpha\colon F_V(X) \to S$ by

$$\alpha(\bar{x}) = \bar{x}\ ; \qquad \alpha(\bar{y}) = p(\bar{x}) .$$

Since $(\bar{x}, \bar{y}) \in \theta_{S_0}$ in $F_V(X)$, it follows that $(\bar{x}, p(\bar{x})) \in \theta_{S_0}$ in S as $\alpha[S_0] = S_0$.

∇_S is finitely generated; hence, using Zorn's lemma, we can find a maximal congruence θ on S. Thus, S/θ is a simple algebra in V. □

Let K be a class of algebras. Then K_f denotes the class $\{(A, f)\colon A \in K, f \in \mathrm{Aut}\, A\}$, and K_S denotes the class $\{(A, P)\colon A \in K, P$ is a subalgebra of $A\}$. $\mathrm{Th}^I(K)$ denotes the theory of K in the language with the added quantifier I and similar for other extensions of the elementary theory. Now we can state the theorem of Pinus:

8.9. THEOREM. *Let V be a non-trivial congruence–distributive variety such that the main congruences are elementarily definable. Then the following theories are undecidable*: $\mathrm{Th}(V_f)$, $\mathrm{Th}(V_S)$, $\mathrm{Th}^I(V)$, $\mathrm{Th}^{ws}(V)$, $\mathrm{Th}^{aa}(V)$, $\mathrm{Th}^{Q_1^2}(V)$ (*the last assuming CH*).

PROOF. By Lemma 8.8, we can choose an $A \in V$ which is simple. Let $X \in BS$, $B = \mathrm{Clop}\, X$. Then by Lemma 8.7, $\mathrm{Con}_m\, A[B]^* \cong (\mathrm{Con}_m\, \mathrm{A})[\mathrm{B}]^* \cong 2[\mathrm{B}]^* \cong \mathrm{B}$. Let $\psi(u, v, x, y)$ be an elementary formula which defines the main congruences generated by u and v for the algebras in the variety V. Thus, for $c_0, c_1 \in A[B]^*$, $\{(a, b) \in (A[B]^*)^2 \colon A[B]^* \models \psi(c_0, c_1, a, b)\}$ is a congruence. Let $a_0, a_1 \in A[B]^*$ be such that $A[B]^* \models \forall xy\psi(a_0, a_1, x, y)$, i.e. $\{p \in X\colon a_0(p) \neq a_1(p)\} = X$. let $b \in A[B]^*$ be such that

$$\theta_{a_0,b} \vee \theta_{a_1,b} = \nabla \quad \text{and} \quad \theta_{a_0,b} \cap \theta_{a_1,b} = \Delta\,.$$

This property can be expressed by the elementary formula

$$\begin{aligned}\varphi_0(a_0, a_1, x) := \forall y_0 y_1 \exists z[\psi(a_0, x, y_0, z) \wedge \psi(a_1, x, z, y_1)\\ \wedge \forall y_0 y_1[\psi(a_0, x, y_0, y_1) \wedge \psi(a_1, x, y_0, y_1) \rightarrow y_0 = y_1]]\,.\end{aligned}$$

If $A[B]^* \models \varphi_0(a_0, a_1, b)$, then for each $p \in X$, $b(p) \in \{a_0(p), a_1(p)\}$. For $b \in A[B]^*$ with $A[B]^* \models \varphi_0(a_0, a_1, b)$ we set

$$b^* = \{p \in X; b(p) = a_0(p)\}\,.$$

Then $b^* \in B$ and we say that b^* is coded by b. We define

$$\begin{aligned}\varphi_1(a_0, a_1, x, y) := \varphi_0(a_0, a_1, x) \wedge \varphi_0(a_0, a_1, y)\\ \wedge \forall z_0 z_1[\psi(a_0, x, z_0, z_1) \rightarrow \psi(a_0, y, z_0, z_1)]\,.\end{aligned}$$

Then, for each $a_0, a_1 \in A[B]^*$ with $A[B]^* \models \forall xy\psi(a_0, a_1, x, y)$, we have

$$(\varphi_0^{A[B]^*}, \varphi_1^{A[B]^*}) \cong B$$

(here we consider B as a partially ordered set).

Having coded the Boolean algebras in V it is a matter of routine to show the undecidability of $\mathrm{Th}^I(V)$, $\mathrm{Th}^{ws}(V)$, $\mathrm{Th}^{aa}(V)$ and $\mathrm{Th}^{Q_1^2}(V)$. We only have to care about $\mathrm{Th}(V_f)$ and $\mathrm{Th}(V_S)$.

Let $f \in \mathrm{Aut}\, B$. We define $f_1 \in \mathrm{Aut}\, A[B]^*$ as follows. Let $\tilde{f}$ be the homeomorphism of X induced by f. We set

$$f_1(a)(p) = a(\tilde{f}^{-1}(p))\,.$$

This shows that the theory of Boolean algebras with one additional function varying on automorphisms can be interpreted in $\mathrm{Th}(V_f)$.

Let (B, B_1) be a Boolean pair. We define

$$A_1 = \{f \in A[B]^*: f^{-1}(a) \in B_1 \text{ for each } a \in A\}.$$

Then

$$(\varphi_0^{A[B]^*}, \varphi_0^{A_1}, \varphi_1^{A[B]^*}) \cong (B, B_1, \leq),$$

and thus the theory of Boolean pairs can be coded in $\mathrm{Th}(V_S)$. □

EXAMPLE. For each variety of commutative rings with unity the main congruences are definable. Thus, for the variety V of m-rings $(m > 1)$ all of the theories $\mathrm{Th}(V_f)$, $\mathrm{Th}(V_S)$, $\mathrm{Th}^I(V)$, $\mathrm{Th}^{\mathrm{ws}}(V)$, $\mathrm{Th}^{aa}(V)$ and $\mathrm{Th}^{Q_1^2}(V)$ (the last assuming CH) are undecidable.

REMARK. BURRIS and MCKENZIE [1981] proved hereditary undecidability of the class $P_S(F)$ of subdirect powers of a finite algebra F under different assumptions about F. They used the undecidability of the theory of Boolean pairs.

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Keywords: Boolean algebra, undecidable, semantic embedding, weak second order logic, second order logic, Härtig quantifier, Malitz quantifier, stationary logic, automorphism, ideal, variety, ring, congruence–distributive.

MOS subject classification: primary 03G05; secondary 06E05, 03D35, 08B99, 08B10.

CHAPTER 25

Recursive Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

This chapter is devoted to the study of what we might call the effective content of the theory of Boolean algebras. That is, we shall apply the tools of modern recursion theory to analyze various theorems and constructions in the theory of Boolean algebras with an eye toward discovering which theorems and constructions are "constructive" or "effective" in the recursion theoretic sense and to pursue some of the interesting questions which naturally arise out of such an endeavor. To better illustrate these notions, let us take two basic theorems about countable BAs and give their effective counterparts. We say a Boolean algebra $B = (B, +, \cdot, -)$ is *recursive* if the universe of B is a recursive subset of the natural numbers N and the basic operations of B, $+$, $\cdot$, and $-$, are partial recursive. We say that two recursive BAs A and B are *recursively isomorphic*, written $B \approx_{\mathrm{r}} C$, if there is a partial recursive function f with domain B such that $f\colon B \to C$ is an isomorphism. This given, the effective version of Cantor's theorem that any two countable atomless BAs are isomorphic would be the statement that any two recursive atomless BAs are recursively isomorphic. Our effective version of Cantor's theorem is true since as one can easily see, Cantor's back-and-forth argument is effective. In contrast, consider the effective version of VAUGHT's [1954] result that any two countable infinite atomic BAs B_1 and B_2 such that $B_i/\langle \mathrm{At}(B_i)\rangle^{\mathrm{id}} \approx 2$ for $i = 1, 2$ (i.e. B_1 and B_2 are 1-superatomic) are isomorphic. The effective version of Vaught's results is not quite so straightforward. For example, consider the three possible effective versions below.

I. If B_1 and B_2 are infinite recursive atomic BAs such that for $i = 1, 2$, $B_i/\langle \mathrm{At}(B_i)\rangle^{\mathrm{id}} \approx 2$, then $B_1 \approx_{\mathrm{r}} B_2$.

II. If B_1 and B_2 are infinite recursive atomic BAs such that for $i = 1, 2$, $\langle \mathrm{At}(B_i)\rangle^{\mathrm{id}}$ is recursive (so that $B_i/\langle \mathrm{At}(B_i)\rangle^{\mathrm{id}}$ is a recursive BA) and $B_i/\langle \mathrm{At}(B_i)\rangle^{\mathrm{id}} \approx 2$, then $B_1 \approx_{\mathrm{r}} B_2$.

III. If B_1 and B_2 are infinite recursive atomic BAs, such that for $i = 1, 2$, $\mathrm{At}(B_i)$ is recursive (which implies that $\langle \mathrm{At}(B_i)\rangle^{\mathrm{id}}$ is also recursive) and $B_i/\langle \mathrm{At}(B_i)\rangle^{\mathrm{id}} \approx 2$, then $B_1 \approx_{\mathrm{r}} B_2$.

Note that it is not obvious that there are any real differences between versions I, II, and III. That is to say, to know that these versions are different we must know whether there exist recursive 1-superatomic Boolean algebras C and D such that $\langle \mathrm{At}(C)\rangle^{\mathrm{id}}$ is not recursive and $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ is recursive while $\mathrm{At}(D)$ is not recursive. In fact such recursive BAs do exist; see Theorems 5.2 and 4.1. However, the proof of the existence of C and D requires the use of a finite injury priority argument. (Note that it is quite easy to construct a recursive 1-superatomic BA A such that $\mathrm{At}(A)$ is recursive.) It thus follows that versions I and II are false. Also, it is not difficult to see that version III is true so that we can gain some insight into the lack of effectiveness of Vaught's result. Moreover, in light of such results, one is led to a number of interesting questions. For example: What are the possibilities for the Turing degrees of $\mathrm{At}(B)$ and $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ in recursive 1-superatomic BAs? More generally: What can we say about the recursion theoretic relationships between $\mathrm{At}(B)$ and $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ in recursive BAs, e.g. is every recursive BA B isomorphic to a recursive BA D where $\mathrm{At}(D)$ is recursive or possibly both $\mathrm{At}(D)$ and $\langle \mathrm{At}(D) \rangle^{\mathrm{id}}$ are recursive? Or one could define a recursive BA B to be *recursively categorical* if every recursive BA D isomorphic to B is recursively isomorphic to B and then ask for a classification of the recursively categorical BAs. A very pleasant answer to this last question has been provided independently by Goncharov and La Roche; see Corollary 5.4.

The various results on the effective content of the theory of Boolean algebras which we shall survey in this chapter require a mixture of recursion theoretic and algebraic techniques. We shall assume a familiarity with the basics of recursion theory, e.g. Turing machines and Turing machines with oracles, partial recursive functions, recursive predicates, recursive versus recursively enumerable sets, the enumeration Theorem, Turing reducibility and Turing degrees, etc. A good general reference for all the recursion theory we shall need is ROGERS [1967]. However, we shall for the most part provide definitions for all but the most basic concepts of recursion theory so that an introduction to recursion theory on the level of the treatment in MONK'S [1976] graduate logic text should be sufficient background for one to read and understand this chapter.

The outline of this chapter is as follows. In Section 1 we deal with preliminary definitions and notations. In particular, we shall formally define recursive, r.e., decidable, and arithmetic BAs. Also in Section 1 we shall introduce four specific recursive BAs, $\tilde{Q}$, $\tilde{N}$, $\tilde{C}$, and $\tilde{G}$ and their properties which will play an important role in the rest of the sections since every infinite recursive BA has a factor isomorphic to one of these four BAs.

In Section 2 we give several equivalent characterizations of recursive, r.e., and arithmetic BAs. For example, we shall show every recursive BA is recursively isomorphic to some recursive subalgebra of the recursive atomless BA $\tilde{Q}$ and is also recursively isomorphic to $\tilde{Q}/I$ for some recursive ideal I of $\tilde{Q}$.

In Section 3, we consider some classes of recursive BAs B whose presentation is highly effective in the sense that B at the very least has a decidable complete diagram. In particular, we shall prove that an α-superatomic recursive BA exists iff α is a recursive ordinal and show that for all recursive ordinals α, there exists α-superatomic recursive BAs B where the entire sequence of Frechet ideals $\{F_\gamma(B)\}$ is effective. Moreover, we show that every countably prime, countably homogeneous, and countably saturated BA is isomorphic to a decidable BA.

In contrast to Section 3, Section 4 deals with families of Rec. BAs whose presentations have very few effective properties beyond those required to be a recursive BA. For example, for each recursive ordinal α, we sketch how to construct α-superatomic recursive BAs so that none of the ideals $F_\gamma(B)$ for $1 \le \gamma \le \alpha$ is recursive. Moreover, we outline some important coding techniques due to FEINER [1970a] which was used by FEINER [70a] to produce r.e. BAs which are not isomorphic to any recursive BA, by GONCHAROV [1975a] to produce recursive BAs not isomorphic to any decidable BA, and by REMMEL [1981b] to produce decidable BAs B such that $B/\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ is not isomorphic to any recursive BA.

In Section 5 we survey some of the results about the nature of the recursive isomorphism types that are contained within a classical isomorphism type of a recursive BA. More specifically, we survey several results about the possible recursion theoretic properties of the set of atoms $\mathrm{At}(B)$ and the ideal generated by the atoms $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ for recursive BAs B which lie in the classical isomorphism type of some recursive BA with infinitely many atoms. We shall then indicate how such results lead to the solution to the recursive categoricity problem for various classes of recursive BAs. That is, we say that for any given class $\mathbb{C}$ of BAs, a recursive BA $B \in \mathbb{C}$ is *recursively categorical with respect to* $\mathbb{C}$ if for every recursive BA $D \in \mathbb{C}$ such that D is isomorphic to B, D is in fact recursively isomorphic to B. We then shall classify the recursive BAs B which are recursively categorical with respect to $\mathbb{R}$, the class of all recursive BAs, with respect to $\mathbb{R}$ At, the class of all recursive BAs where the predicate "x is atom" is recursive, and with respect to $\mathbb{R}$AtAl, the class of all recursive BAs where both the predicates "x is atom" and "x is atomless" are recursive.

In Section 6 we give a brief survey of various results on the lattice $\mathscr{L}(B)$ of r.e. subalgebras and the lattice $\mathscr{L}I(B)$ of r.e. ideals of a Rec. BA B. For the lattice $\mathscr{L}(B)$, we shall show that there is a lattice theoretic characterization of the recursive subalgebras of B but such a result requires some nontrivial facts about the nature of complementation in the full lattice of subalgebras. Moreover, for any infinite recursive BA D, $\mathscr{L}(D)$ always has nonrecursive elements and the theory of $\mathscr{L}(D)$ is always undecidable. In contrast, we shall show that there are infinite recursive BAs B where $\mathscr{L}I(B)$ has no nonrecursive ideals and where the theory of $\mathscr{L}I(B)$ is decidable.

Finally, in Section 7 we survey some results on the group of recursive automorphisms of a recursive BA B as well as some results on the number of automorphisms for the lattices $\mathscr{L}(B)$ and $\mathscr{L}I(B)$.

1. Preliminaries

Let φ_e denote the partial recursive function computed by the eth Turing machine and φ_e^A denote the A-partial recursive function computed by the eth oracle machine with oracle A. We shall write $\varphi_{e,s}(x)\downarrow$ ($\varphi_{e,s}^A(x)\downarrow$) if the eth Turing machine (eth oracle machine with oracle A) gives an output in s or fewer steps when started on input x. We write $\varphi_e(x)\downarrow$ if there exists an s such that $\varphi_{e,s}(x)\downarrow$ and refer to $W_e = \{x \mid \varphi_e(x)\downarrow\}$ as the eth r.e. set. Similarly, we write $\varphi_e^A(x)\downarrow$ if there is an s such that $\varphi_{e,s}^A(x)\downarrow$ and refer to $W_e^A = \{x \mid \varphi_e^A(x)\downarrow\}$ as the eth r.e. set in A. We also let $u(A, x, e, s)$ denote the maximum element about which the A-oracle is questioned in the course of the computation $\varphi_{e,s}^A(x)$ if $\varphi_{e,s}^A(x)\downarrow$ and let $u(A, x, e, s)$ denote 0 otherwise. We shall assume that our computations are such that $u(A, x, e, s) \le s$ for all A, e, x, and s. Given a set $B \subseteq N$, we write

$$\varphi_e^A = B \quad \text{if for all } x \in N\,, \qquad \varphi_e^A(x) = B(x) = \begin{cases} 0 & \text{if } x \notin B\,, \\ 1 & \text{if } x \in B\,. \end{cases}$$

Given subsets A and B of N, we say B is *Turing reducible* to A, written $B \le_{\mathrm{T}} A$, if $B = \varphi_e^A$ for some e. We write $B \equiv_{\mathrm{T}} A$ if both $A \le_{\mathrm{T}} B$ and $B \le_{\mathrm{T}} A$ and let $\deg(A)$ denote the *Turing degree* of A, i.e. $\deg(A) = \{B \mid B \equiv_{\mathrm{T}} A\}$.

We shall fix a recursive pairing function $\langle, \ldots, \rangle$: $N^{<\omega} \to N$, where $N^{<\omega}$ denotes the set of all finite sequence from N. Let k_0 be some fixed total recursive function such that the range of $k_0 = \{\langle x, y\rangle \mid x \in W_y\}$. Then for any e and s, $W_e^s = \{x \mid \exists y \leq s\ (k_0(y) = \langle x, e\rangle)\}$. Let $K = \{e \mid e \in W_e\} = \{e \mid \varphi_e(e) \downarrow\}$. Then it is well known that the degree of K is the highest possible for an r.e. set, i.e. $W_e \leq_T K$ for all e. More generally, $K^A = \{e \mid e \in W_e^A\}$ has the property that $W_e^A \leq_T K^A$ for all e and the degree of K^A denoted by A' is called the jump of A. Thus, $0' = \deg(K)$ is the *jump* of any recursive set.

When we talk about an *r.e. index e* of a set A or a *partial recursive index k* of a function f, we simply mean that $A = W_e$ and $f = \varphi_k$. By the *canonical index e* of a finite set X, we mean $e = 0$ if $W = \emptyset$ and $e = x^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$ if $X = \{x_1 < \cdots < x_n\}$. We let D_k denote the finite set whose canonical index is k. We say an n-ary relation R is *arithmetical* if $R = \{(x_1, \ldots, x_n) \mid (Q_1 y_1) \ldots (Q_m y_m) S(x_1, \ldots, x_n, y_1, \ldots, y_m)\}$ for some $n+m$-ary recursive predicate S and Q_i is either $\exists$ or $\forall$ for each $i \leq m$. We classify the arithmetic predicates according to the number of alternation of quantifiers inductively as follows:

I. $\Pi_0^0 = \Sigma_0^0 = \Delta_0^0 =$ sets of all recursive predicates.

II. Having defined Π_n^0, Σ_n^0, Δ_n^0 predicates, we say an n-ary arithmetic predicate R is in

(a) Π_{n+1}^0 if $R = \{(x_1, \ldots, x_n) \mid \forall y\ S(x_1, \ldots, x_n, y)\}$ where S is in Σ_n^0,

(b) Σ_{n+1}^0 if $R = \{(x_1, \ldots, x_n) \mid \exists y\ T(x_1, \ldots, x_n, y)\}$ where T is in Π_n^0, and

(c) Δ_{n+1}^0 if R is in both Π_{n+1}^0 and Σ_{n+1}^0.

These definitions produce a hierarchy for the arithmetic predicates which is pictured in Fig. 25.1, with the lines indicating proper inclusions.

One can define a similar hierarchy, called the analytic hierarchy, if one allows quantifiers over functions f: $N \to N$ as well as quantifiers over numbers. At one point we shall need to refer to the first level of the analytic hierarchy. Thus, rather than defining the entire analytic hierarchy, we will only define the concepts we need. We say a n-ary predicate R is in Σ_1^1 if $R = \{(x_1, \ldots, x_n) \mid \exists f\ S(x_1, \ldots, x_n, f)\}$, where $\exists f$ is a quantifier over functions f: $N \to N$ and $S(x_1, \ldots, x_n, f)$ is an arithmetic predicate. Similarly, R is in Π_1^1 if $R = \{(x_1, \ldots, x_n) \mid \forall f\ S(x_1, \ldots, x_n, f)\}$, where $\forall f$ is a quantifier over functions f: $N \to N$ and $S(x_1, \ldots, x_n, f)$ is an arithmetic predicate. We refer the reader to ROGERS [1967] for further details on the analytic hierarchy.

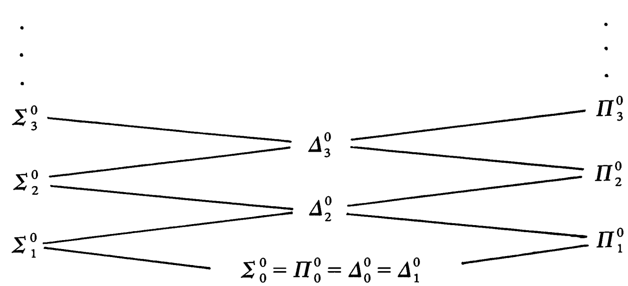

Fig. 25.1

Finally, we say a set A is *Σ_n^i-complete* (Π_n^i-complete) if for any set B in Σ_n^i (Π_n^i) there is a recursive function f such that for all $x \in N$, $x \in B$ iff $f(x) \in A$.

All the BAs in this chapter will be countable so that we shall adopt the convention that the universe of any BA A is assumed to be a subset of N unless explicitly stated otherwise. We should note that such a convention requires that some of the BAs which we mention such as the Intalg(ω) or Intal($1+\eta$) are identified via some implicit Gödel numbering with a subset of N but this should not cause any confusion. We shall also make the convention that $\leq$ or $<$ will denote the usual ordering on the natural numbers, while for any BA A, we shall denote the less than or equal and less than relations by $\leq_A$ and $<_A$, respectively. Similarly, we shall denote the zero and one of any BA A by 0_A and 1_A so as to distinguish them from 0 and 1 in the natural numbers. This given, we are now in position to make the following definitions

1.1. DEFINITION.* Given a countable BA $A = (A, +, \cdot, -, 0_A, 1_A)$ we say:

(i) A is a *recursive* BA (Rec. BA) if A is a recursive subset of N and the operations $+$, $\cdot$, and $-$ are partial recursive.

(ii) A is a *decidable BA* (Dec. BA) if A is a recursive subset of N and Th(A, $+$, $\cdot$, $-$, 0_A, 1_A, $\{a\}_{a\in A}$) is decidable.

(iii) A is a *recursively enumerable BA* (Re BA) if A is a recursive subset of N, the operations $+$, $\cdot$, and $-$ are partial recursive, and the equality relation on A, $=_A$, is r.e. (i.e. there is an r.e. congruence relation $=_A$ with respect to $+$, $\cdot$, and $-$ such that $A/{=_A}$ is a BA). (Note for Rec. and Dec. BAs A we assume the equality relation for A is just the equality on N.)

(iv) A is an *arithmetic BA* (Arith. BA) if A is an arithmetic subset of N, the operations $+$, $\cdot$, and $-$ are arithmetic, and the equality relation on A is arithmetic.

Since we shall often need to refer to interval algebras of various countable linear orderings, we must also make the following definitions.

1.2. DEFINITION. Given a partial ordering $L = (L, <_L)$, we say

(i) L is a *recursive linear ordering* (Rec. LO) if L is a recursive subset of N, and $<_L$ is recursive.

(ii) L is a *decidable linear ordering* (Dec. LO) if L is a recursive subset of N and Th(L, $<_L$, $\{l\}_{l\in L}$) is decidable.

(iii) L is a *recursively enumerable linear ordering* (R.e. LO) if L is a recursive subset of N, $<_L$ is recursive, and the equality relation on L, $=_L$, is r.e., i.e. $=_L$ is an r.e. congruence relation with respect to $<_L$. (Again for Rec. and Dec. LOs, we assume that the equality relation on L is just the equality relation on N.)

(iv) L is an *arithmetical linear ordering* (Arith. LO) if L is an arithmetic subset of N, $<_L$ is arithmetic, and the equality relation for L is arithmetic.

We should emphasize here that whether a Boolean algebra or linear ordering is recursive, decidable, etc. is dependent upon its presentation and not just its isomorphism type. This may conflict with conventions that have occurred in the

* We note that in the Russian literature the terms *constructive* and *strongly constructive* are used for our *recursive* and *decidable* BAs and LOs, respectively.

literature where a BA B is said to be decidable iff B is isomorphic to a Dec. BA in the sense of Definition 1.1. Another word about notation is relevant here. Namely, if B is a Rec. BA and we say C is a *recursive* (*r.e. arithmetic*) *subalgebra* of B we mean that the universe of C is a recursive (r.e. arithmetic) subset of the universe of B as opposed to meaning that C is a subalgebra of B which happens to be a Rec. BA, (R.e. BA, Arithm. BA) in the sense of Definition 1.1. (See the discussion following Theorem 2.1 for a distinction between these two notions.) A similar convention applies to recursive (r.e., arithmetic) subordering of a Rec. LO.

Let ω, η, and $\bar{n}$ denote the order types of the natural numbers N, the rational numbers Q, and the numbers $\{0, \ldots, n-1\}$ under the usual ordering. Recall that given two linear orderings $L_1 = (L_1, <_1)$ and $L_2 = (L_2, <_2)$, we define the product ordering $L_1 \times L_2 = (L, <)$ by setting $L = L_1 \times L_2$ and defining $\langle x_1, y_1 \rangle < \langle x_2, y_2 \rangle$ iff $y_1 <_2 y_2$ or $y_1 = y_2$ and $x_1 <_1 x_2$. Note that if L_1 and L_2 are Rec. LOs, then it is easy to see that we can consider $L_1 \times L_2$ as a recursive linear ordering by simply identifying L with $\{\langle x, y \rangle \mid x \in L_1 \text{ and } y \in L_2\}$ where $\langle\ ,\ \rangle$ is our recursive pairing function. Similarly, the product $B_1 \times B_2$ of two Rec. BAs B_1 and B_2 and the product $\Pi_{i \in N} B_i$ of an r.e. sequence of Rec. BAs $B_0, B_1, \ldots$ can be considered as recursive BAs by using our recursive pairing function. Note also that if a is an element of a Rec. BA A, then $A \mid a$ is also a Rec. BA since $\leq_A$ is also a recursive predicate given that $\cdot$ is partial recursive, i.e. $x \leq_A y$ iff $x \cdot y = x$. Thus, under the conventions above, A is not the same as $(A \mid a) \times (A \mid -a)$ since A and $(A \mid a) \times (A \mid -a)$ may have different universes but certainly if A is a Rec. BA, then A and $(A \mid a) \times (A \mid -a)$ are recursively isomorphic. In a similar vein, if B is a Rec. BA and I is a recursive ideal of B, then we consider B/I as a Rec. BA by identifying each equivalence class $[x]_I$ of B/I with its least element. That is to say, it is easy to see that $B_I = \{z \mid \exists b(b \in B \ \&\ z = \mu x(x \in [b]_i))\}$ is a recursive subset of B and if we define $+$, $\cdot$, and $-$ on B_I by $z_1 + z_2 = \mu x\ (x \in [z_1 + z_2]_I)$, etc. then the resulting Rec. BA is isomorphic to B/I. In contrast, if I is an r.e. ideal of B, then B/I may be viewed directly as the R.e. BA $(B, +, \cdot, -, =_B)$ where $x =_B y$ iff $(x - y) + (y - x) \in I$.

Next we shall introduce several special recursive BAs which will be very useful in later sections and give characterizations of their recursive isomorphism types.

(1.1) $\tilde{N}$ denotes a recursive presentation of the BA of finite and cofinite subsets of N such that $\mathrm{At}(N)$ is recursive. Thus, $\tilde{N}$ is isomorphic to $\mathrm{Intalg}(\omega)$.

(1.2) $\tilde{Q}$ denotes a recursive presentation of the BA generated by the left-closed right-open intervals of the rationals Q. Thus, $\tilde{Q} \approx \mathrm{Intalg}(1 + \eta)$.

(1.3) $\tilde{C}$ denotes a recursive presentation of the BA generated by the closed intervals of Q, or equivalently by the BA generated by $\tilde{Q}$ together with $\{\{q\} \mid q \in Q\}$ where $\mathrm{At}(\tilde{C})$, $\langle \mathrm{At}(C) \rangle^{\mathrm{id}}$, and $\tilde{Q}$ as it sits inside $\tilde{C}$ are recursive. It is easy to see that $\tilde{C} \approx \mathrm{Intalg}(1 + \omega \times \eta)$. (Here $\tilde{C}$ stands for "closed interval algebra.")

(1.4) $\tilde{H}$ denotes the recursive subalgebra of $\tilde{C}$ generated by $\tilde{Q}$ (as it sits inside $\tilde{C}$) together with $\{\{q\} \mid q \in N\}$. Thus, $\tilde{H} \approx \mathrm{Intalg}(\omega + \eta)$.

(1.5) $\tilde{G}$ denotes the recursive subalgebra of $\tilde{C}$ generated by $\tilde{Q} \cup A$, where $A = \{\{q\} \mid 0 \leq q \leq 1,\ q \in Q,\ \&\ q$ has a ternary expansion $a_0 a_1 \ldots,$

where $a_i \in \{0, 2\}$ for all $i\}$. Note that another description of A is the set of singletons $\{q\}$, where q is an endpoint of one of the "middle-thirds" eliminated in the usual constrution of the Cantor set on $[0, 1]$. It is not difficult to show that $\tilde{G} \approx \mathrm{Intalg}(1 + (\omega + \eta) \times \eta)$.

We shall not give the explicit Gödel numberings for $\tilde{N}$, $\tilde{Q}$, and $\tilde{C}$. However, from the descriptions above it should not be difficult to convince oneself that such Gödel numberings for $\tilde{N}$, $\tilde{Q}$, and $\tilde{C}$ exist so that $\tilde{H}$, $\mathrm{At}(\tilde{H})$, $\mathrm{Atl}(\tilde{H})$, $\langle \mathrm{At}(\tilde{H})\rangle^{\mathrm{id}}$, $\{x \in \tilde{H} \mid \mathrm{At}(\tilde{H} \mid x)$ is infinite$\}$, $\tilde{G}$, $\mathrm{At}(\tilde{G})$, $\mathrm{Atl}(\tilde{G})$, $\langle \mathrm{At}(\tilde{G})\rangle^{\mathrm{id}}$, and $\{x \in \tilde{G} \mid \mathrm{At}(\tilde{G} \mid x)$ is infinite$\}$ are all recursive, see REMMEL [1978], [1981a] for details. Moreover, we can easily characterize the recursive isomorphism types of these BAs. To this end, we must first characterize the classical isomorphism types. Recall that the Frechet sequence of ideals indexed by countable ordinals is defined for any countable BA B as follows.

(1.6)
$$F_0(B) = \{0_B\},$$
$$F_{\alpha+1}(B) = \{x \in B \mid x \bmod F_\alpha(B) \in \langle \mathrm{At}(B/F_\alpha(B))\rangle^{\mathrm{id}}\},$$
$$F_\lambda(B) = \bigcup_{\alpha<\lambda} F_\alpha(B) \text{ for } \lambda \text{ a limit ordinal.}$$

B is said to be *α-atomic* if $B/F_\beta(B)$ is atomic for all $\beta < \alpha$ and B is said to be *α-superatomic* if B is α-atomic and $B/F_\alpha(B) \approx 2$. Then Vaught proved the following.

1.3. THEOREM (VAUGHT [1954]). (a) *Let B and D be countable infinite atomic BAs. Then $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}} \approx D/\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ iff $B \approx D$.*

(b) *More generally, for any countable ordinal α, if B and C are countable infinite α-atomic BAs and $B/F_\alpha(B)$ and $C/F_\alpha(C)$ are nonzero, then $B/F_\alpha(B) \approx C/F_\alpha(C)$ iff $B \approx C$.*

For our purposes, the most useful characterizations of the classical isomorphism types of $\tilde{Q}$, $\tilde{N}$, $\tilde{C}$, $\tilde{H}$, and $\tilde{G}$ are given by our next proposition

1.4. PROPOSITION. *Suppose B is an infinite countable BA.*

(i) *If B is atomless, then $B \approx \tilde{Q}$.*

(ii) *If B is atomic and $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}} \approx 2$, and $B \approx \tilde{N}$.*

(iii) *If B is atomic and $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}} \approx \tilde{Q}$, then $B \approx \tilde{C}$.*

(iv) *If $\mathrm{At}(B)$ is infinite, B has no infinite atomic elements, and for all $z \in B$ either $\mathrm{At}(B \mid z)$ or $\mathrm{At}(B \mid -z)$ is finite, then $B \approx \tilde{H}$.*

(v) *If $\mathrm{At}(B)$ is infinite, B has no infinite atomic elements, and for all $z \in B$ such that $\mathrm{At}(B \mid z)$ is infinite, there exists an $x \leq_B z$ such that both $\mathrm{At}(B \mid x)$ and $\mathrm{At}(B \mid z - x)$ are infinite, then $B \approx \tilde{G}$.*

PROOF. Clearly, $\tilde{Q}$ is atomless so that (i) is just Cantor's theorem (see Chapter 5 of this Handbook). Parts (ii) and (iii) follow immediately from Theorem 1.3(a). Parts (iv) and (v) are easily established by straightforward back-and-forth arguments; see our next proposition for an example. □

Now the effective counterpart of Proposition 1.4 is the following.

1.5. PROPOSITION (REMMEL [1979], [1981a], [1981b]). *Suppose B is an infinite recursive BA. Then*

(i) *If B is atomless, then $B \approx_{\mathrm{r}} \tilde{Q}$.*

(ii) *If B is atomic, $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}} \approx 2$, and $\mathrm{At}(B)$ is recursive, then $B \approx_{\mathrm{r}} \tilde{N}$.*

(iii) *If B is atomic, $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}} \approx \tilde{Q}$, and both $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ and $\mathrm{At}(B)$ are recursive, then $B \approx_{\mathrm{r}} \tilde{C}$.*

(iv) *If B satisfies the hypothesis of Proposition 1.4(iv) and $\mathrm{At}(B)$ and $\mathrm{Atl}(B)$ are recursive, then $B \approx_{\mathrm{r}} \tilde{H}$.*

(v) *If B satisfies the hypothesis of Proposition 1.4(v) and $\mathrm{At}(B)$, $\mathrm{Atl}(B)$, $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$, and $\{x \in B \mid \mathrm{At}(B \mid x)$ is infinite$\}$ are recursive, then $B \approx_{\mathrm{r}} \tilde{G}$.*

PROOF. All the parts of this proposition can be established by effective back-and-forth arguments. As an example of this type of argument, we shall give a sketch of part (iv). So assume that B is a recursive BA satisfying the hypothesis of part (iv) and let $b_0, b_1, \ldots$ be an effective listing of the elements of B. Similarly, let $h_0, h_1, \ldots$ be an effective listing of the elements of $\tilde{H}$. Clearly, there is no loss in generality in assuming that $b_0 = 0_B$, $b_1 = 1_B$, $h_0 = 0_{\tilde{H}}$, and $h_1 = 1_{\tilde{H}}$. We shall define a partial recursive isomorphism f: $B \to \tilde{H}$ in stages. At each stage s, we shall specify finite subalgebras B_s and H_s of B and $\tilde{H}$, respectively, and an isomorphism f_s: $B_s \to H_s$ so that for all s, $B_s \subseteq B_{s+1}$, $H_s \subseteq H_{s+1}$, and $f_s \subseteq f_{s+1}$ and $f = \bigcup_s f_s$ is the desired partial recursive isomorphism.

Stage 0. Let $B_0 = \{b_0, b_1\}$, $H_0 = \{h_0, h_1\}$ and $f_0 = \{\langle b_0, h_0\rangle, \langle b_1, h_1\rangle\}$.

Stage $s+1$. Assume we have defined B_s, H_s, and f_s so that $\langle\{b_0, \ldots, b_s\}\rangle \subseteq B_s$, $\langle\{h_0, \ldots, h_s\rangle\} \subseteq H_s$, and if $x_0, \ldots, x_k$ and $y_0, \ldots, y_k$ are lists of $\mathrm{At}(B_s)$ and $\mathrm{At}(H_s)$, respectively, so that $f_s(x_i) = y_i$, then (a) $|\mathrm{At}(B \mid x_i)| = |\mathrm{At}(\tilde{H} \mid y_i)|$ and $x_i \in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ iff $y_i \in \langle \mathrm{At}(\tilde{H})\rangle^{\mathrm{id}}$. As usual, we shall divide stage s into two substages where we first add b_{s+1} to B^s and attempt to find an $h \in H$ to add to H_s so that we can extend the isomorphism f_s to an isomorphism $f_{s,0}$: $B_{s,0} \to H_{s,0}$, where $B_{s,0} = \langle B_s \cup \{b_{s+1}\}\rangle$ and $H_{s,0} = \langle H_s \cup \{h\}\rangle$. At the second substage, we add h_{s+1} to $H_{s,0}$ to get H_{s+1} and the attempt to find a $b \in B$ to add to $B_{s,0}$ so that we can extend the isomorphism $f_{s,0}$ to an isomorphism f_{s+1}: $B_{s+1} \to H_{s+1}$, where $B_{s+1} = \langle B_{s,0} \cup \{b\}\rangle$. The only thing we need to check that is different from the usual back-and-forth type argument is that our assumptions allow us to find the required h and b using recursive search procedures. Of course the hypothesis of part (iv) reflects exactly what is needed to ensure our searches are recursive. Let us consider substage s, 0.

Substage s, 0. If $b_{s+1} \in B_s$, then we let $B_{s,0} = B_s$, $H_{s,0} = H_s$ and $f_{s,0} = f_s$. If $b_{s+1} \not\in B_s$, consider each atom x_i of B_s in turn. If $b_{s+1} \cdot x_i = 0_B$ or $b_{s+1} \cdot x_i = x_i$, then let $x_{i,0} = x_i$, $x_{i,1} = 0_B$, and define $f_{s,0}(x_{i,0}) = y_{i,0} = y_i$ and $f_{s,0}(x_{i,1}) = y_{i,1} = 0_H$. Otherwise, let $x_{i,0} = b_{s+1} \cdot x_i$ and $x_{i,1} = x_i - b_{s+1}$. Then we are in one of three possible cases.

Case 1. $x_i \in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$;

Case 2. $x_i \not\in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ and $|\mathrm{At}(B \mid x_i)|$ is finite; and

Case 3. $x_i \not\in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ and $|\mathrm{At}(B \mid x_i)|$ is infinite.

Note that our assumptions on B ensure that we can effectively distinguish between these three cases. That is, we claim that since both $\mathrm{At}(B)$ and $\mathrm{Atl}(B)$ are recursive, then automatically $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ and $I_B = \{x \in B \mid \mathrm{At}(B \mid x) \text{ is infinite}\}$ are also recursive due to the isomorphism type of B. For suppose $x \in B$, then by assumption either $\mathrm{At}(B \mid x)$ is finite and $\mathrm{At}(B \mid -x)$ is infinite or $\mathrm{At}(B \mid x)$ is infinite and $\mathrm{At}(B \mid -x)$ is finite. To distinguish between these two cases, note that we can search the atoms of B until we find $a_1, \dots, a_n \in \mathrm{At}(B)$ such that either: (a) $x = \Sigma_{i=1}^n a_i$ in which case $x \in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ and $-x \in I_B$; (b) $(x - \Sigma_{i=1}^n a_i) \in \mathrm{Atl}(B)$ in which case $x \notin \langle \mathrm{At}(B)\rangle^{\mathrm{id}} \cup I_B$, and $-x \in I_B$; (c) $-x = \Sigma_{i=1}^n a_i$ in which case $-x \in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ and $x \in I_B$; or (d) $-x - \Sigma_{i=1}^n a_i \in \mathrm{Atl}(B)$ in which case $-x \notin \langle \mathrm{At}(B)\rangle^{\mathrm{id}} \cup I_B$ and $x \in I_B$.

Now in Case 1, note that since $\mathrm{At}(B)$ is recursive, we can simply search through $b_0, b_1, \dots$ until we find $n+m$ atoms of $B, a_1, \dots, a_{n+m}$, such that $x_i = \Sigma_{i=1}^{n+m} a_i$, $x_{i,0} = \Sigma_{i=1}^n a_i$, and $x_{i,1} = \Sigma_{i=n+1}^{n+m} a_i$. Now by assumption, $y_i \in \langle \mathrm{AT}(\tilde{H})\rangle^{\mathrm{id}}$ and $|\mathrm{At}(\tilde{H} \mid y_i)| = n+m$ so that since $\mathrm{At}(\tilde{H})$ is recursive, we can search through the atoms of H until we find $g_1 < \cdots < g_{n+m}$ such that $y_i = \Sigma_{i=1}^{n+m} g_i$. Then we let $f_{s,0}(x_{i,0}) = \Sigma_{i=1}^n g_i$ and $f_{s,0}(x_{i,1}) = \Sigma_{i=n+1}^{n+m} g_i$.

In Case 2, first we search through the atoms of B until we find $n+m$ atoms $a_1, \dots, a_{n+m}$ such that $x_i - \Sigma_{j=1}^{n+m} a_j$ is atomless and $\Sigma_{j=1}^n a_j \le_B x_{i,0}$ and $\Sigma_{j=n+1}^{n+m} a_j \le_B x_{i,1}$ for some $n, m \ge 0$. Note that since $\mathrm{At}(B)$ and $\mathrm{Atl}(B)$ are recursive, we can effectively find $a_1, \dots, a_{n+m}$. Then by assumption $y_i \notin \langle \mathrm{At}(\tilde{H})\rangle^{\mathrm{id}}$ and $|\mathrm{At}(\tilde{H} \mid y_i)| = n+m$ so we can also effectively find $g_1 < \cdots < g_{n+m}$ in $\mathrm{At}(\tilde{H})$ such that $y_i - \Sigma_{j=1}^{n+m} g_j$ is atomless. Then we have three possibilities to consider, namely: (a) $x_{i,0} = \Sigma_{j=1}^n a_j$ in which case we set $y_{i,0} = \Sigma_{i=0}^n g_i$ and $y_{i,1} = y_i - y_{i,0}$; (b) $x_{i,1} = \Sigma_{j=n+1}^{n+m} a_j$ in which case we set $y_{i,1} = \Sigma_{j=n+1}^{n+1} g_j$ and $y_{i,0} = y_i - y_{i,1}$; or (c) neither (a) nor (b) hold in which case we let $v = \mu z(v <_B y_i - \Sigma_{j=1}^{n+m} g_j$ and $v \neq 0_{\tilde{H}})$, and then set $y_{i,0} = v + \Sigma_{j=1}^n g_j$ and $y_{i,1} = (y_i - v - \Sigma_{j=1}^n g_j)$. In all three cases, we define $f_{s,0}(x_{i,0}) = y_{i,0}$ and $f_{s,0}(x_{i,1}) = y_{i,1}$.

For Case 3, note that by assumption either $|\mathrm{At}(x_{i,0})|$ or $|\mathrm{At}(x_{i,1})|$ must be finite and we have already argued that we can effectively decide which is which. Say $|\mathrm{At}(x_{i,0})| < \omega$. Then we can effectively find $a_1, \dots, a_n$ in $\mathrm{At}(B)$ such that either $x_{i,0} = \Sigma_{j=1}^n a_j$ or $x_{i,0} - \Sigma_{j=1}^n a_j$ is atomless. Since by assumption $|\mathrm{At}(\tilde{H} \mid y_i)|$ is infinite, we can effectively find that first n elements of $\mathrm{At}(\tilde{H} \mid y_i)$, $g_1 < \cdots < g_n$. Then if $x_{i,0} = \Sigma_{j=1}^n a_j$, we set $y_{i,0} = \Sigma_{j=1}^n g_j$ and $y_{i,1} = y_i - y_{i,0}$. Similarly, if $x_{i,0} - \Sigma_{j=1}^n a_j$ is atomless, we find the least $v <_{\tilde{H}} y_i$ such that $v \in \mathrm{Atl}(\tilde{H})$ and set $y_{i,0} = v + \Sigma_{j=1}^n g_j$ and $y_{i,1} = y_i - y_{i,0}$. Note that such a v must exist since $\tilde{H}$ has no infinite atomic elements. Under either circumstance, we define $f_{s,0}(x_{i,0}) = y_{i,0}$ and $f_{s,0}(x_{i,1}) = y_{i,1}$. The case when $|\mathrm{At}(B \mid x_{i,1})|$ is finite is symmetric.

Now it is easy to check that $B_{s,0} = \langle \{x_{i,0}, x_{i,1} \mid i = 1, \dots, k\}\rangle = \langle B_s \cup \{b_{s+1}\}\rangle$. Moreover, if we let $H_{s,0} = \langle \{y_{i,0}, y_{i,1} \mid i = 1, \dots, k\}\rangle$, then we have defined $f_{s,0}$ on the atoms of $B_{s,0}$ so that $f_{s,0}$ extends to a unique isomorphism from $B_{s,0}$ to $H_{s,0}$ and for all atoms x of $B_{s,0}$, $|\mathrm{At}(B \mid x)| = |\mathrm{At}(\tilde{H} \mid f_{s,0}(x))|$ and $x \in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ iff $f_{s,0}(x) \in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$.

It should now be clear that at substage $s.1$, we can reverse the roles of B and $\tilde{H}$ and extend $B_{s,0}$, $H_{s,0}$, and $f_{s,0}$ to B_{s+1}, H_{s+1}, and f_{s+1}, respectively, so that $h_{s+1} \in H_{s+1}$ and $f_{s+1}\colon B_{s+1} \to H_{s+1}$ is an isomorphism such that for all atoms

$x \in B_{s+1}$, $|\mathrm{At}(B \mid x)| = |\mathrm{At}(\tilde{H} \mid f_{s+1}(x))|$ and $x \in \langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ iff $f_{s+1}(x) \in \langle \mathrm{At}(\tilde{H}) \rangle^{\mathrm{id}}$. Moreover, it should be clear from our discussion that the entire process is effective so that $f = \bigcup_s f_s$ will be a partial recursive isomorphism from B to $\tilde{H}$. □

We end this section with two rather simple theorems which should give some indication why we have considered these five specific BAs.

1.6. THEOREM. *If B is a countably infinite BA, then there exists an $a \in B$ such that $B \mid a$ is isomorphic to one of $\tilde{N}$, $\tilde{Q}$, or $\tilde{C}$.*

PROOF. This is an easy consequence of Proposition 1.4. That is, if B has an atomless element a, then clearly $B \mid a \approx \tilde{Q}$. Otherwise, B is atomic and we consider $B/\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$. Now if there is an atom x in $B/\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$, then for any $a \in B$ such that $x = a \bmod \langle \mathrm{At}(B) \rangle^{\mathrm{id}}$, it is the case that $B \mid a$ is atomic and $B \mid a / \langle \mathrm{At}(B \mid a) \rangle^{\mathrm{id}} \approx 2$ so that $B \mid a \approx \tilde{N}$. The only other case is that $B/\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ is atomless in which case $B \approx \tilde{C}$ by Proposition 1.4(c). □

1.7. THEOREM (REMMEL [1981b]). *If B is a countably infinite BA and $|\mathrm{At}(B)| = \omega$, then there exists an $a \in B$ such that $B \mid a$ is isomorphic to one of $\tilde{N}$, $\tilde{C}$, $\tilde{H}$, or $\tilde{G}$.*

PROOF. By the proof of Theorem 1.6, it follows that if B has an inifinte atomic element b, then for some $a <_B b$, $B \mid a$ is isomorphic to one of $\tilde{N}$ or $\tilde{C}$. Thus, we may assume that B has no infinite atomic elements. Now if there is a $b \in B$ such that $|\mathrm{At}(B \mid b)| = \omega$ and for all $z \leq_B b$, either $\mathrm{At}(B \mid z)$ or $\mathrm{At}(B \mid b - z)$ is finite, then clearly by Proposition 1.4(iv), $B \mid b \approx \tilde{H}$. If there is no such $b \in B$, then clearly B satisfies the hypothesis of Proposition 1.4(v) and hence $B \approx \tilde{G}$. □

We should note that the effective versions of both Theorems 1.6 and 1.7 are false. For example, it is not the case that for every infinite recursive BA B there is an a such that $B \mid a$ is recursively isomorphic to one of $\tilde{N}$, $\tilde{Q}$, or $\tilde{C}$.

2. Equivalent characterizations of recursive, r.e., and arithmetic BAs

In this section we present several equivalent ways to look at recursive BAs which result from the effectivizations of various characterizations of countable BAs. For example, the effective version that the countable atomless BA is universal for countable BAs is the fact that every recursive BA is recursively isomorphic to a recursive subalgebra of $\tilde{Q}$. While this result is quite simple, we shall present the argument given in REMMEL [1979] in some detail since it will allow us to describe a general set up for constructing recursive subalgebras of $\tilde{Q}$, which is used in many of the results to follow.

2.1. THEOREM. *Every Rec. BA B is recursively isomorphic to a recursive subalgebra of $\tilde{Q}$.*

PROOF. We shall only consider the case where B is infinite. Let $1_B = d_0, d_1, \ldots$ be an effective enumeration of B. First, we use this enumeration to effectively enumerate a sequence $b_0, b_1, \ldots$ of elements of B which we call an *r.e. generating* sequence of B. We enumerate $b_0, b_1, \ldots$ in stages. At stage 0, we let $b_0 = d_0 = 1_B$. Assume we have enumerated $b_0, \ldots, b_{k_n}$ by stage n so that $D_n = \langle\{d_0, \ldots, d_n\}\rangle = \langle\{b_0, \ldots, b_{k_n}\}\rangle$. At stage $n+1$, we consider d_{n+1}. If $d_{n+1} \in D_n$, we go on to stage $n+2$. Otherwise, we let $e_1, \ldots, e_j$ be the atoms of D_n in increasing order of magnitude such that both $e_i \cdot d_{n+1}$ and $e_i - d_{n+1}$ are nonzero. Then we let $b_{k_n+i} = e_i \cdot d_{n+1}$ for $i = 1, \ldots, j$ and $k_{n+1} = k_n + j$. It is clear from our description of the stages that $b_0, b_1, \ldots$ is an effective sequence. Moreover, if $B_n = \langle\{b_0, \ldots, b_n\}\rangle$, then the following two properties are also easily established:

(i) $B = \bigcup_n B_n$.

(ii) b_{n+1} splits exactly one atom of B_n, that is, there is one $a \in \mathrm{At}(B_n)$ such that $b_{n+1} \leq_B a$ and both b_{n+1} and $a - b_{n+1}$ are nonzero.

Given $b_0, b_1, \ldots$, we then build the desired recursive subalgebra C of $\tilde{Q}$ and a partial recursive isomorphism $f\colon B \to C$ in stages. At each stage s, we will define a finite subalgebra C_s of $\tilde{Q}$ and a finite isomorphism $f_s\colon B_s \to C_s$.

Stage 0. $B_0 = \{0_B, 1_B\}$ so let $C_0 = \{0_{\tilde{Q}}, 1_{\tilde{Q}}\}$, $f_0(0_B) = 0_{\tilde{Q}}$, and $f_0(1_B) = 1_{\tilde{Q}}$.

Stage $s+1$. By construction B_s has $s+1$ atoms so let $a_1, \ldots, a_{s+1}$ be the atoms of B_s. Let a_i be the atom of B_s which b_{s+1} splits. By induction, $f(a_0), \ldots, f(a_{s+1})$ are the atoms of C_{s+1}. Then $f(a_i)$ is an atomless element in $\tilde{Q}$ so that there are infinitely many $c <_{\tilde{Q}} f_s(a_i)$. Moreover, if $c_1, c_2 <_{\tilde{Q}} f_s(a_i)$ and $c_1 \notin \{c_2, f_s(a_i) - c_2\}$, then $\langle C_s \cup \{c_1\}\rangle \cap \langle C_s \cup \{c_2\}\rangle = C_s$. It easily follows that given any finite $A \subset \tilde{Q}$ such that $A \cap C_s = \emptyset$, there is a nonzero $x <_{\tilde{Q}} f_s(a_i)$ such that $\langle C_s \cup \{x\}\rangle \cap A = \emptyset$. Thus, there exists a nonzero $c <_{\tilde{Q}} f_s(a_i)$ such that $C_s \cap \{0, \ldots, s\} = \langle C_s \cup \{c\}\rangle \cap \{0, \ldots, s\}$. Let c_{s+1} be the least such c. We define f_{s+1} by defining it on the atoms of B_{s+1} and extending it to be a homomorphism. Thus, set $f_{s+1}(a_j) = f_s(a_j)$ for $j \neq i$, $f_{s+1}(b_{s+1}) = c_{s+1}$, and $f_{s+1}(a_i - b_{s+1}) = f_s(a_i) - c_{s+1}$. Finally, we let $C_{s+1} = \langle C_s \cup \{c_{s+1}\}\rangle$.

This completes the description of the stages. It is clear that for each s, $f_s \subseteq f_{s+1}$, $C_s \subseteq C_{s+1}$, and $f_{s+1}\colon B_{s+1} \to C_{s+1}$ is an isomorphism. Thus, if $f = \bigcup_s C_s$, then $f\colon B \to C$ is a partial recursive isomorphism. Note that C is a recursive subalgebra of $\tilde{Q}$ since our construction ensures $s \in C$ if $s \in C_s$. $\square$

It will be useful to state explicitly as a lemma the result of our argument used in the middle of stage $s+1$ of the above construction since it is a basic fact which will be used repeatedly.

2.2. LEMMA. *If B is a finite subalgebra of $\tilde{Q}$, F is a finite subset of N such that $F \cap B = \emptyset$, and a is an atom of B, then there exists $0 <_{\tilde{Q}} x <_{\tilde{Q}} a$ such that $\langle B \cup \{x\}\rangle \cap F = \emptyset$.*

Next we pause to make a couple of remarks about possible extensions of Theorem 2.1 to R.e. BAs and Arith. BAs. We note that it is not true that every R.e. BA is recursively isomorphic to an r.e. subalgebra of $\tilde{Q}$, since in Section 4 we shall show the existence of R.e. BAs which are not even classically isomorphic to a Rec. BA while clearly the argument of Theorem 2.1 proves the following.

2.3. COROLLARY. *Every r.e. subalgebra of $\tilde{Q}$ is recursively isomorphic to a recursive subalgebra of $\tilde{Q}$.*

Similarly, even if we start with an Arith. BA $B = (B, +, \cdot, -, =_B)$, where $+$, $\cdot$, and $-$ are the restrictions of partial recursive operations to an arithmetic subset B of N and $=_B$ is just usual equality, REMMEL [1981d] has shown that it does not follow that B is an arithmetic subalgebra of some Rec. BA.

It is also well known that every countable BA B is isomorphic to a quotient of the atomless BA and this fact has the following effective counterpart.

2.4. THEOREM. (i) *If B is a Rec. BA, then B is recursively isomorphic to $\tilde{Q}/I$ for some recursive ideal I of $\tilde{Q}$.*

(ii) *If B is an R.e. BA, then B is recursively isomorphic to $\tilde{Q}/I$ for some r.e. ideal I of $\tilde{Q}$.*

PROOF. Let $B = (B, +_B, \cdot_B, -_B, =_B, 0_B, 1_B)$ be an R.e. BA. We shall prove only (ii) since our proof of (ii) will also give a proof (i) in the case where B is actually a Rec. BA. When we say that B is recursively isomorphic to $\tilde{Q}/I$ for some r.e. ideal I we mean there is a partial recursive surjection f: $\tilde{Q} \to B$ such that $f(x+y) =_B f(x) +_B f(y)$, $f(x \cdot y) =_B f(x) \cdot_B f(y)$, and $f(-x) =_B -_B f(x)$ for all x and y in $\tilde{Q}$ and $I = \{x \in \tilde{Q} \mid f(x) =_B 0_B)$. We shall construct the desired f in stages but we note that the construction requires some care due to the fact B becomes a BA only modulo the r.e. equivalence relation $=_B$. For example, we require only that $b -_B b =_B 0_B$ but not that $b -_B b = 0_B$. Let $b_0 = 0_B$, $b_1 = 1_B$, $b_2, b_3, \ldots$ and $r_0 = 0_{\tilde{Q}}$, $r_1 = 1_{\tilde{Q}}$, $r_2, r_3, \ldots$ be effective listings of B and $\tilde{Q}$ without repetitions. At each stage s, we shall specify a finite subalgebra A_s of $\tilde{Q}$ containing $\{r_0, \ldots, r_s\}$, a finite subset B_s of B containing $\{b_0, \ldots, b_s\}$ such that $B_s/{=_B}$ is a subalgebra of $B/{=_B}$, and a surjection f_s: $A_s \to B_s$ which induces a homomorphism from A_s to $B_s/{=_B}$. In addition, we shall define $\mathrm{At}(B_s) = \{f_s(x) \mid x \in \mathrm{At}(A_s)\ \&\ \langle f_s(x), 0_B\rangle \not\in {=^s_B}\}$ where $\langle x, y\rangle \in {=^s_B}$ if either $x = y$ or $\langle x, y\rangle$ is in the symmetric transitive closure of the first s elements in the same r.e. listing of $=_B$. That is, $\mathrm{At}(B_s)$ is the set of images of those atoms of A that currently look as if they map to nonzero elements in B.

Stage 0. Let $B_0 = \{0_B, 1_B\}$, $A_0 = \{0_{\tilde{Q}}, 1_{\tilde{Q}}\}$, $f_0 = \{\langle 0_{\tilde{Q}}, 0_B\rangle, \langle 1_{\tilde{Q}}, 1_B\rangle\}$, and $\mathrm{At}(B_0) = \{1_{\tilde{Q}}\}$.

Stage s + 1. Assume we have defined A_s, B_s, f_s, and $\mathrm{At}(B_s)$ with the properties given above. We shall divide this stage into two substages.

Substage 0. Consider r_{s+1}. If $r_{s+1} \in A_s$, then set $A_{s,0} = A_s$, $B_{s,0} = B_s$, $f_{s,0} = f_s$, and $\mathrm{At}(B_{s,0}) = \mathrm{At}(B_s)$. If $r_{s+1} \not\in A_s$, then let $a_1, \ldots, a_n, a_{n+1}, \ldots, a_k$ be a listing of the atoms of A_s, where for $i \leq n$, both $a_i \cdot r_{s+1}$ and $a_i - r_{s+1}$ are nonzero and for $i>n$ one of $a_i \cdot r_{s+1}$ or $a_i - r_{s+1}$ is zero. Then for each $i \leq n$, define $f_{s,0}(a_i \cdot r_{s+1}) = 0_B$ and $f_{s,0}(a_i - r_{s+1}) = f_s(a_i)$ and for $i>n$, define $f_{s,0}(a_i) = f_s(a_i)$. Set $A_{s,0} = \langle A_s \cup \{r_{s+1}\}\rangle$ and extend f_s to $f_{s,0}$ by setting $f_{s,0}(x) = \Sigma\ f_{s,0}(b)$ where the sum ranges over all $b \in \mathrm{At}(A_{s,0})$ with $b \leq_{\tilde{Q}} x$ if $x \in A_{s,0} - A_s$ and setting $f_{s,0}(x) = f_s(x)$ if $x \in A_s$. Finally, we set $B_{s,0}$ equal to the range of $f_{s,0}$ and $\mathrm{At}(B_{s,0}) = \mathrm{At}(B_s)$.

Substage 1. Consider b_{s+1}. If $b_{s+1} \in B_{s,0}$, then set $A_{s+1} = A_{s,0}$, $B_{s+1} = B_{s,0}$, and $f_{s+1} = f_{s,0}$. Otherwise, let $x_1, \ldots, x_n$ be the elements of $\mathrm{At}(B_{s,0})$ and $y_1, \ldots, y_n$ be the corresponding atoms in $A_{s,0}$ such that $f_{s,0}(y_i) = x_i$ for $i = 1, \ldots, n$. Then for each $i \leq n$, let u_i be the least element of $\tilde{Q}$ such that $u_i <_{\tilde{Q}} y_i$ and $v_i = y_i - u_i$. We let $A_{s+1} = \langle A_{s,0} \cup \{u_1, \ldots, u_n\}\rangle$. Let $u_1, v_1, \ldots, u_n, v_n, a_1, \ldots, a_p$ be the atoms of A_{s+1}. First we define f_{s+1} on $\mathrm{At}(A_{s+1})$ by setting $f_{s+1}(u_i) = x_i \cdot b_{s+1}$ and $f_{s+1}(v_i) = x_i - b_{s+1}$ for $i = 1, \ldots, n$ and setting $f_{s+1}(a_i) = f_{s,0}(a_i)$ for $i = 1, \ldots, p$. We then extend f_{s+1} to A_{s+1} by setting $f_{s+1}(x) = f_{s,0}(x)$ if $x \in A_{s,0}$, $f_{s+1}(x) = \Sigma\ f_{s+1}(b)$, where the sum runs over all $b \in \mathrm{At}(A_{s+1})$ with $b \leq_{\tilde{Q}} x$ if $x \in A_{s+1} - (A_{s,0} \cup \{\Sigma_{i=1}^n u_i\})$, and $f_{s+1}(x) = b_{s+1}$ if $x = \Sigma_{i=1}^n u_i$. Finally, we set B_{s+1} equal to the range of f_{s+1}.

It is then easy to check that our definitions ensure that for all s that f_{s+1} induces a homomorphism from A_{s+1} onto $B_{s+1}/{=_B}$ and $f_s \subseteq f_{s+1}$. Hence, $f = \bigcup_s f_s$ will be a partial recursive surjection from $\tilde{Q}$ onto B such that f induces a homomorphism from $\tilde{Q}$ onto $B/{=_B}$. Thus, if we set $I = \{x \in \tilde{Q} \mid f(x) =_B 0_B\}$, I will be an r.e. ideal and f will be a partial recursive isomorphism from $\tilde{Q}/I$ onto $B/{=_B}$. We note that in the case B is a recursive BA, I is actually a recursive ideal since to decide if $x \in I$, one simply waits until $x \in A_s$ and then $x \in I$ iff $f_s(x) = 0_B$. □

A third way of looking at Rec. BAs follows from the observation that the proof of the fact that every countable BA is isomorphic to an interval algebra (see Part I, Section 15 of this Handbook) is completely effective and so we have the following.

2.5. THEOREM. *Every recursive BA is recursively isomorphic to the interval algebra of some recursive linear ordering.*

We end this section with one final way of looking at Rec. BAs which will be useful in later sections.

2.6. THEOREM. *Every recursive BA is recursively isomorphic to $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ for some decidable atomic BA B.*

PROOF. First we must observe the fact that an atomic Rec. BA B is decidable iff $\mathrm{At}(B)$ is recursive. That is to say, Tarski (see Chapter 15 of the Handbook, showed that for the theory of atomic Boolean algebras, there is an effective elimination of quantifiers in terms of the predicates $A_1(x)$, $A_2(x), \ldots$, where $A_n(x)$ holds iff x is a union of exactly n atoms. Now suppose B is an atomic Rec. BA such that $\mathrm{At}(B)$ is recursive. Then $\{x \in B \mid B \models A_n(x)\}$ is recursive for all n. That is, to decide if $B \models A_n(b)$, we simply effectively list $\mathrm{At}(B)$, $a_0, a_1, \ldots.$

Then since B is atomic, we will either find k atoms $a_{i_1}, \ldots, a_{i_k}$ such that $b = \Sigma_{j=1}^{k} a_{i_j}$ for some $k \leq n$ or we will find $n+1$ atoms $a_{j_1}, \ldots, a_{j_{n+1}}$ such that $\Sigma_{i=1}^{n+1} a_{j_i} \leq_B b$. In the first case, $B \models A_n(b)$ iff $k = n$ and in the second case $B \not\models A_n(b)$. Thus, B is decidable. On the other hand, if B is decidable, then $\mathrm{At}(B)$ is recursive.

Now suppose A is a Rec. BA. Then by Theorem 2.1, A is recursively isomorphic to a recursive subalgebra D of $\tilde{Q}$. If we consider $\tilde{Q}$ as it sits inside $\tilde{C}$, then we claim that if $B = \langle D \cup \mathrm{At}(\tilde{C}) \rangle$, then B is a decidable BA such that A is recursively isomorphic to $B/\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$. First, B is a recursive subalgebra of $\tilde{C}$ since given an $x \in \tilde{C}$, $x = q_x + \Sigma_{a \in E_x} a - \Sigma_{a \in F_x} a$, where E_x and F are finite sets of atoms and $q_x \in \tilde{Q}$. Because $\mathrm{At}(\tilde{C})$, $\langle \mathrm{At}(\tilde{C}) \rangle^{\mathrm{id}}$, and $\tilde{Q}$ as it sits inside $\tilde{C}$ are all recursive, it easily follows that we can effectively find q_x, E_x, and F_x for any x. Thus, $x \in B$ if $q_x \in D$ so that B is recursive. But then $\mathrm{At}(B) = \mathrm{At}(\tilde{C})$ so that by our observation B is decidable. Moreover, $\langle \mathrm{At}(B) \rangle^{\mathrm{id}} = \langle \mathrm{At}(\tilde{C}) \rangle^{\mathrm{id}}$ so that $B/\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ is a Rec. BA which is clearly recursively isomorphic to A by our construction. □

3. Recursive Boolean algebras with highly effective presentations

In this section we survey several general results which show that some large classes of countable BAs B have "nice" effective presentations in the sense that at the very least the classical isomorphism type of B contains a decidable BA. For example, results of MEAD [1979] and MOROZOV [1982a] show that all countable prime BAs B and all countable ω-saturated BAs B are isomorphic to Dec. BAs. We shall also sketch the proof of the well-known result that an α-superatomic BA B is isomorphic to a Dec. BA iff α is a recursive ordinal. We note that the results of this section are in sharp contrast with the results of the next section where we construct many examples of Rec. BAs D such that there are no Dec. BAs in the classical isomorphism type of D.

We begin this section with some results on recursive superatomic BAs. However, before we can proceed with our presentation, we need to recall the definitions of recursive and constructive ordinals. The definition of a recursive ordinal is the obvious one.

3.1. DEFINITION. An ordinal α is said to be *recursive* if and only if α is order isomorphic to a recursive linear ordering.

The definition of a constructive ordinal arises from the theory of systems of notations for ordinal numbers developed by Church and Kleene; see ROGERS [1967] for references.

3.2. DEFINITION. A *system of notation* S is a mapping ν_S from a set of integers D_S called the set of notations of S onto a segment of the ordinal numbers such that

(i) there exists a partial recursive function k_S such that

$$\nu_S(x) = 0 \Rightarrow k_s(x) = 0\,,$$

$$\nu_S(x) \text{ is a successor} \Rightarrow k_S(x) = 1\,,$$

$$\nu_S(x) \text{ is a limit} \Rightarrow k_S(x) = 2\,;$$

(ii) there exists a partial recursive function p_S such that

$$\nu_S(x) \text{ a successor} \Rightarrow [p_S(x)\downarrow \text{ and } \nu_S(p_S(x)) + 1 = \nu_S(x)]\,;$$

(iii) there exists a partial recursive function q_S such that

$$\nu_S(x) \text{ a limit} \Rightarrow [q_S(x)\downarrow\,,\ \varphi_{q_S(x)} \text{ is total, and}$$
$$\{\nu_S(\varphi_{q_S(x)}(n))\}_{n\geq 0} \text{ is an increasing sequence with } \nu_S(x) \text{ as limit}]\,.$$

3.3a. DEFINITION. An ordinal α is said to be *constructive* if there is a system of notation S which assigns at least one notation to α.

3.3b. DEFINITION. A system of notation S is:
(a) *univalent* if ν_S is one-to-one;
(b) *recursive* if D_S is recursive;
(c) *recursively related* if $R_S = \{\langle x, y\rangle \mid x, y \in D_S \;\&\; \nu_S(x) \leq \nu_S(y)\}$ is recursive;
(d) *maximal* if S gives a notation to every constructive ordinal;
(e) *universal* if for any system of notation S' there is a partial recursive function φ mapping $D_{S'}$ into D_S such that $x \in D_{S'} \Rightarrow \nu_{S'}(x) \leq \nu_S(\varphi(x))$.

Next we list some basic facts about recursive and constructive ordinals without proof. We refer the reader to Chapters 11 and 16 of ROGERS [1967] for proofs and references.

3.4. THEOREM (Kleene, Markwald, Spector). *An ordinal α is recursive iff it is constructive.*

Given Theorem 3.4 and the definitions above, it is obvious that there are only countably many recursive or constructive ordinals so there is a least countable ordinal called ω_1^{CK} which is not constructive. Now it is easy to see that every predecessor of a recursive ordinal is recursive, the successor of recursive ordinal is recursive, and the limit of r.e. increasing sequence of recursive ordinals is recursive. Thus, ω_1^{CK} is a limit ordinal and there is no r.e. increasing sequence of recursive ordinals cofinal in ω_1^{CK}.

Next we present the standard maximal and universal system of notation due to Kleene.

3.5. DEFINITION. "Kleene's system $\mathcal{O}$". We define the functions $\nu_{\mathcal{O}}$, $k_{\mathcal{O}}$, $p_{\mathcal{O}}$, $q_{\mathcal{O}}$ and a partial ordering $<_{\mathcal{O}}$ on $D_{\mathcal{O}}$ as follows. 0 receives notation 1.

Assume all ordinals $<\gamma$ have received their notations and assume $<_{\mathcal{O}}$ has been defined on these notations.

(i) If $\gamma = \beta + 1$, then for each x such that $\nu_{\mathcal{O}}(x) = \beta$, we define $\nu_{\mathcal{O}}(2^x) \doteq \gamma$ and add the pairs $\langle z, 2^x \rangle$ to $<_{\mathcal{O}}$ for all z such that either $z = x$ or $\langle z, x \rangle \in <_{\mathcal{O}}$ already.

(ii) If γ is a limit, then for each y such that $\{\varphi_y(n)\}_{n=0}^{n=\infty}$ are notations for an increasing sequence of ordinals with limit γ and such that $\forall i \forall j$ $(i<j \Rightarrow \langle \varphi_y(i), \varphi_y(j) \rangle \in <_{\mathcal{O}}$ already), we define $\nu_{\mathcal{O}}(3 \cdot 5^y) = \gamma$ and add the ordered pairs $\langle z, 3 \cdot 5^y \rangle$ to $<_{\mathcal{O}}$ for all z such that $\exists n$ $(<z, \varphi_y(n)\rangle \in <_{\mathcal{O}}$ already).

(iii) $k_{\mathcal{O}}, p_{\mathcal{O}}, q_{\mathcal{O}}$ are defined by $k_{\mathcal{O}}(1) = 0$, $k_{\mathcal{O}}(2^x) = 1$, $k_{\mathcal{O}}(3 \cdot 5^y) = 2$, $p_{\mathcal{O}}(2^x) = x$, and $q_{\mathcal{O}}(3 \cdot 5^y) = y$.

Our next theorem gives the basic properties of Kleene's system $\mathcal{O}$.

3.6. THEOREM (Kleene). (i) *$\mathcal{O}$ is a maximal universal system of notation.*

(ii) *Given any $y \in \mathcal{O}$, $\{x \mid x <_{\mathcal{O}} t\}$ constitutes a univalent system of notations and, moreover, $\{x \mid x <_{\mathcal{O}} y\}$ is r.e. uniformly in y, i.e. there is a recursive function f such that $\omega_{f(y)} = \{x \mid x <_{\mathcal{O}} y\}$.*

(iii) *$D_{\mathcal{O}}$ is a complete Π_1^1-set.*

One last fact which we shall need about recursive or constructive ordinals is the following.

3.7. THEOREM (Tanaka, Kreisel). *Suppose A is Σ_1^1 and $\{\langle x, y \rangle \mid \langle x, y \rangle \in A\}$ is the ordering relation of a well-ordering $<$ of order type α; then α is a recursive ordinal.*

Given the machinery of Kleene's system $\mathcal{O}$, we can easily construct recursive BAs whose Frechet sequence of ideals is completely effective. Given a countable BA B and a countable ordinal α, we say that a BA D is *α-atomic of type B* if for all $\gamma < \alpha$, $D/F_\gamma(D)$ is atomic and $D/F_\alpha(D) \approx B$. For example, an α-superatomic BA is α-atomic of type 2.

3.8. DEFINITION. Given a Rec. BA B and an ordinal $\alpha \geq 1$, we say that D is a *super-recursive α-atomic BA of type B* if D is recursive and α-atomic of type B. $D/F_\alpha(D)$ is recursively isomorphic to B, $\mathrm{At}(D)$ is recursive, and the sequence of ideals $\{F_\beta(D)\}_{\beta \leq \alpha}$ is a recursive uniformly, that is, there is a notation $y \in \mathcal{O}$ and a partial recursive function f such that $\nu_{\mathcal{O}}(y) = \alpha$ and for all $z \leq_{\mathcal{O}} y$ if $\nu_{\mathcal{O}}(z) = \beta$, then $\varphi_{f(z)}$ is the characteristic function of $F_\beta(D)$.

Our next series of results will show that for any Rec. BA B, there exists a super-recursive α-atomic BA of type B iff α is a recursive ordinal. We should also note that if $\alpha > 1$, then the condition of being a super-recursive α-atomic BA of type B is much stronger than being decidable. For if D is an atomic BA, then D is a Dec. BA iff D is a Rec. BA and $\mathrm{At}(D)$ is recursive; see Theorem 2.6. However, it will follow from our results of the next section that if α is recursive, $\alpha \geq 1$, and B is any Rec. BA, then there exist Dec. BAs D which are α-superatomic of type B but have $F_\beta(D)$ nonrecursive for all $1 \leq \beta \leq \alpha$.

3.9. THEOREM. *If α is a recursive ordinal and B is a Rec. BA, then there exists a super-recursive α-atomic BA of type B.*

PROOF. We proceed by induction on α to construct for each notation $y \in D_{\mathcal{O}}$ such that $\nu_{\mathcal{O}}(y) = \alpha$ and each recursive BA B, a Rec. BA $D(y, B)$ which is a super-recursive α-atomic BA of type B.

The case $\alpha = 1$, follows from Theorem 2.4. That is, if x is notation for the ordinal 1, then let $D(x, B)$ be the Dec. BA D constructed as in Theorem 2.4 such that $D/\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ is recursively isomorphic to B. If $\alpha = \beta + 1 > 1$, then assume that for each x such that $\nu_{\mathcal{O}}(x) = \beta$, we have constructed a super-recursive β-atomic BA, $D(x, B)$, of type B for each recursive BA B. In particular, if $B = 2$, we have constructed $D(x, 2)$ which is a super-recursive β-superatomic BA. An informal description of the construction of $D(2^x, B)$ is as follows. Recall our descriptions of $\tilde{Q}$ and $\tilde{C}$ in Section 1. We thought of $\tilde{Q}$ as the BA generated by the left-closed right-open intervals of the rationals Q and $\tilde{C}$ as the BA generated by $\tilde{Q}$ together with the set of all singletons. Our recursive presentation of $\tilde{C}$ is such that $\mathrm{At}(\tilde{C}) = \{\{q\} \mid q \in Q\}$, $\langle \mathrm{At}(\tilde{C})\rangle^{\mathrm{id}}$, and $\tilde{Q}$ as it sits inside $\tilde{C}$ are all recursive so that we can effectively express each $x \in \tilde{C}$ in the form $x = I + \Sigma_{q\in S_1} q - \Sigma_{q\in S_2} q$, where $I \in \tilde{Q}$ and S_1 and S_2 are finite disjoint subsets of Q. Now by Theorem 2.1, $\tilde{C}$ is recursively isomorphic to a recursive subalgebra of a recursive atomless BA so that we shall think of $\tilde{C}$ as a subalgebra of some fixed recursive atomless BA A. This given, the BA $A \mid q$ which results from the ideal generated by q for each atom of $\tilde{C}$ is again a copy of the recursive atomless BA. Now for each $q \in \mathrm{At}(\tilde{C})$, let $D(x, 2)_q$ be a recursive subalgebra of $A \mid q$ recursively isomorphic to $D(x, 2)$ and let $\tilde{B}$ be a recursive subalgebra of $\tilde{Q}$ as it sits inside $\tilde{C}$ which is recursively isomorphic to B. Then our desired recursive BA $D(2^x, B)$ is the subalgebra of A that is generated by $\tilde{B}$ together with the set of all $\{a \in D(x, 2)_q \mid q \in \mathrm{At}(\tilde{C})\}$. Of course, one must be a little careful since the description above would only yield that $D(2^x, B)$ is an r.e. subalgebra of A rather than a recursive subalgebra but it is not difficult to see that we can carry out a simultaneous recursive embedding of $\tilde{C}$ and all the $D(x, 2)_q$ into A in stages as in the construction of Theorem 2.1 so that $D(2^x, B)$ will be a recursive subalgebra of A and for each $d \in D(2^x, B)$ we can effectively express d in the form $d = b + \Sigma_{q\in S_1} a_q - \Sigma_{q\in S_2} a_q$, where $b \in \tilde{B}$, S_1 and S_2 are disjoint finite subsets of $\mathrm{At}(\tilde{C})$, and $a_q \in D(x, 2)_q$ for each $q \in S_1 \cup S_2$. By our induction hypothesis, each $D(x, 2)_q$ is a super-recursive β-superatomic BA from which it easily follows that $D(2^x, B)$ is α-atomic of type B. Moreover, the fact that for each d we can effectively express d in the form $d = b + \Sigma_{q\in S_1} a_q - \Sigma_{q\in S_2} a_q$ as above and the fact that each $F_\gamma(D(x, q))$ is recursive uniformly for $\gamma \le \alpha$ ensures that $F_\gamma(D(2^x, B))$ is recursive uniformly for $\gamma \le \alpha$. That is, if $\gamma < \alpha$, then $d \in F_\gamma(D(2^x, B))$ iff d is of the form $d = \Sigma_{q\in S} a_q$, where S is a finite subset of $\mathrm{At}(\tilde{C})$ and $a_q \in F_\gamma(D(x, 2)_q)$ for all $q \in S$ and if $\gamma = \alpha$, then $d \in F_\gamma(D(2^x, B))$ iff $d = \Sigma_{q\in S} a_q$, where S is a finite subset of $\mathrm{At}(\tilde{C})$ and $a_q \in D(x, 2)_q$. For the same reason, it should be clear that $D(2^x, B)/F_\alpha(D(2^x, B))$ is recursively isomorphic to $\tilde{B}$ and hence to B. Note also that $x \in \mathrm{At}(D(2^x, B))$ iff $x = a_q$, where $a_q \in D(x, 2)_q$ for some q and $a_q \in \mathrm{At}(D(x, 2)_q)$ so that $\mathrm{At}(D(2^x, B))$ is recursive.

Finally, if α is a limit and $\nu_{\mathcal{O}}(3\cdot 5^y)=\alpha$, then we can assume we have constructed super-recursive $\nu_{\mathcal{O}}(\varphi_y(n))$-atomic BAs, $D(\varphi_y(n), B)$, of type B for each $n\geq 0$ and for each Rec. BA B. The construction of $D(3\cdot 5^y, B)$ is very similar to our construction at successor ordinals. The only difference is that first we must partition the rationals Q into an r.e. sequence $T_0, T_1, \ldots$ of pairwise disjoint dense recursive subsets. Thus, identifying the atoms of $\tilde{C}$ with Q, we see that each $q\in \mathrm{At}(\tilde{C})$ falls in some T_n. If $q\in T_n$, then we let $D(\varphi_y(n),2)_q$ be a recursive subalgebra of $A \mid q$ which is recursively isomorphic to $D(\varphi_y(n),2)$. Then $D(3\cdot 5^y, B)$ is constructed just as $D(2^x, B)$ except $D(x,2)_q$ is replaced by $D(\varphi_y(n),2)_q$ if $q\in T_n$. The fact that $D(3\cdot 5^y, B)$ has the desired properties follows from the same type of argument as in the successor case. □

Before giving the converse of Theorem 3.9, we pause to note two results in the literature which easily follow from Theorem 3.9. First, GONCHAROV [1972] noted that if B is recursive ω-atomic BA, then B is isomorphic to a Dec. BA. That is, if B is ω-atomic, then B is clearly isomorphic to a 1-superatomic BA of type B. Since B is recursive, then by Theorem 3.9 there exists a Rec. BA D which is super-recursive 1-atomic of type B. But, by definition, D is atomic and $\mathrm{At}(D)$ is recursive so that by our remarks in Theorem 2.5, D is a Dec. BA. In fact, a similar argument can be used to prove the following strengthening of Goncharov's result. Namely, let "Qx" denote the quantifier "there exists infinitely many".

3.10. THEOREM (PINUS [1976]). *If B is a recursive ω-atomic BA, then there is an $L(Q)$-decidable BA D which is isomorphic to B.*

PROOF. PINUS [1976] showed that the $L(Q)$-theory of atomic BAs effectively admits the elimination of quantifiers in terms of the following four predicates:

$$\phi_1(x)\Leftrightarrow B \mid x \text{ is atomic and } \mathrm{At}(B\mid x) \text{ is infinite}\,;$$

$$\phi_2(x)\Leftrightarrow B \mid x \text{ is isomorphic to } Q\times \mathcal{F} \text{ for some finite BA } \mathcal{F}\,;$$

$$\phi_2(x)\Leftrightarrow B \mid x \text{ is a finite BA, i.e. iff } x\in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}; \text{ and}$$

$$\phi_4(x)\Leftrightarrow B \mid x \text{ is an atomless BA}\,.$$

It is easy to see that any recursive atomic BA B, where $\mathrm{At}(B)$ and $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ are recursive, the four predicates $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, and $\phi_4(x)$ are recursive predicates and hence B is $L(Q)$-decidable. Thus, by our remarks preceding the statement of the theorem, every recursive ω-atomic BA B is isomorphic to a recursive BA D, where $\mathrm{At}(D)$ and $\langle \mathrm{AT}(D)\rangle^{\mathrm{id}}$ are recursive and hence to a $L(Q)$-decidable BA. □

The key to the converse of Theorem 3.9 is the following.

3.11. THEOREM. *If α is a countable ordinal such that $\alpha\geq \omega_1^{\mathrm{CK}}$, then there does not exist an arithmetic α-superatomic BA.*

PROOF. It is enough to prove the result in the case where $\alpha = \omega_1^{\mathrm{CK}}$ since if $\alpha > \omega_1^{\mathrm{CK}}$ and B is an arithmetic α-superatomic BA, then $B \mid x$ is an arithmetic ω_1^{CK}-superatomic BA for some $x \in F_{\omega_1^{\mathrm{CF}}+1}(B) - F_{\omega_1^{\mathrm{CK}}}(B)$.

We shall show that if there exists an arithmetic ω_1^{CK}-superatomic BA B, then we can construct a Σ_1^1 linear ordering $<_L$ on N of order type ω_1^{CK} which would contradict Theorem 3.7. For each countable ordinal α, let S^α be an α-superatomic BA and let $S^{\alpha,n}$ denote the n-fold product of S^α for each positive integer n. Since B is ω_1^{CK}-superatomic, it follows that for each $x \in B$, $B \mid x$ is isomorphic to $S^{\alpha,n}$ for some $\alpha < \omega_1^{\mathrm{CK}}$ and $1 \le n < \omega$ or to $S^{\omega_1^{\mathrm{CK},1}}$. Thus, for each $x \in B$, we define the characteristic of x, $\mathrm{ch}(x)$, to be (α, n), where $B \mid x \approx S^{\alpha,n}$. If we order the characteristics lexicographically, then we can define the linear ordering $<_L$ as follows. Given $e, f \in N$, we say $e <_L f$ iff

$$\begin{aligned}&(e \notin B \;\&\; f \notin B \;\&\; e < f) \vee (e \notin B \;\&\; f \in B)\\ &\vee (e \in B \;\&\; f \in B \;\&\; \mathrm{ch}(e) < \mathrm{ch}(f)) \vee (e \in B \;\&\; f \in B \;\&\\ &\exists G(G\colon B \mid e \approx B \mid f) \;\&\; e < f)\,.\end{aligned}$$

where $G\colon B \mid e \approx B \mid f$ is to mean G is an isomorphism from $B \mid e$ onto $B \mid f$. Depending on whether $|N - B|$ is finite or infinite, it is easy to see that $<_L$ is an ordering of N of order type either $\bar{n} + \omega \cdot \omega \cdot \omega_1^{\mathrm{CK}} = \omega_1^{\mathrm{CK}}$ for some n or $\omega + \omega \cdot \omega \cdot \omega_1^{\mathrm{CK}} = \omega_1^{\mathrm{CK}}$. Thus, we need only check that $<_L$ is indeed a Σ_1^1 predicate.

Since B is an Arith. BA, the first two disjuncts in the definition of $<_L$ are clearly arithmetical. For the third disjunct in the definition of $<_L$, first observe that the uniformity of the construction of Theorem 3.9 implies that there exists a partial recursive function h such that for each ordinal notation $x \in D_{\mathcal{O}}$, $h(x)$ is an r.e. index of a Rec. BA $D_{h(x)}$ which is isomorphic to S^α if $\nu_{\mathcal{O}}(x) = \alpha$. That is, $h(x)$ is a six-tuple $\langle h_1(x), h_2(x), h_3(x), h_4(x); h_5(x), h_6(x)\rangle$, where $h_1(x)$ is an r.e. index of the universe of $D_{h(x)}$, i.e. $W_{h_1(x)} = \{x \mid x \in D_{h(x)}\}$, and $h_2(x)$, $h_3(x)$, and $h_4(x)$ are the indices of the partial recursive function for meet, join, and complement in $D_{h(x)}$, and $h_5(x)$ and $h_6(x)$ are the zero and one of $D_{h(x)}$, respectively. Similarly, there is a partial recursive function g of two variables such that for each ordinal notation $x \in D_{\mathcal{O}}$ and positive integer n, $g(x, n)$ is an r.e. index of a recursive BA which is isomorphic to the n-fold product of $D_{h(x)}$. To make this point a bit clearer, we note that if x is an ordinal notation of the form 2^y, then our construction of Theorem 3.9 provides a uniform way to define $h(x)$ in terms of $h(y)$ and similarly if $x = 3 \cdot 5^y$, then our construction provides a way to define $h(x)$ in terms of the r.e. sequence $\{h(\varphi_y(n))\}_{n\in\omega}$. Thus, we can prove by induction on the ordinals that for all $x \in D_{\mathcal{O}}$, $D_{h(x)}$ is a Rec. BA isomorphic to S^α, where $\nu_{\mathcal{O}}(x) = \alpha$. Of course, if $x \notin D_{\mathcal{O}}$, $h(x)$ may or may not be defined. For example, x might be of the form $3 \cdot 5^y$, where $\varphi_y(n)$ are all ordinal notations but $\nu_{\mathcal{O}}(\varphi_y(0))$, $\nu_{\mathcal{O}}(\varphi_y(1)), \ldots$ may not be an increasing sequence of ordinals with a limit. Thus, $x \notin D_{\mathcal{O}}$ but our construction will still allow us to construct a recursive BA $D_{h(x)}$ in terms of the recursive BAs $D_{h(\varphi_y(0))}, D_{h(\varphi_y(1))}, \ldots$. Of course, it may be that if $x \notin D_{\mathcal{O}}$, $h(x)$ is not defined or $h(x)$ is defined but is not an r.e. index of a Rec. BA. Nevertheless, we can use the partial recursive functions h and g to show that $\mathrm{ch}(e) < \mathrm{ch}(f)$ is Σ_1^1. That is, if $e \in B$ and $f \in B$, then $\mathrm{ch}(e) < \mathrm{ch}(f)$ iff

$$\begin{aligned}&\exists F_1 \exists F_2 \exists F_3 \exists x \exists n \exists k \exists l\ (k \in B \\ &\quad \& \ F_1\colon B \mid k \approx D_{h(x)} \ \& \ n \geq 1 \ \& \ F_2\colon B \mid e \approx D_{g(x,n)} \\ &\quad \& \ l \in B \ \& \ l \leq_B f \ \& \ F_3\colon B \mid l \approx D_{g(x,n+1)}) \,.\end{aligned}$$

The implication from left to right is immediate since we may choose x and n such that $\mathrm{ch}(e) = (\alpha, n)$, where $\nu_{\mathcal{O}}(x) = \alpha$. For the converse, note that the first two conjuncts imply that $D_{h(x)}$ has a characteristic since $k \in B$ implies $B \mid k$ has a characteristic. Letting $\mathrm{ch}(k) = (\alpha, m)$, the next two conjuncts show that $\mathrm{ch}(e) = (\alpha, m \cdot n)$, while the final three conjuncts show that

$$\mathrm{ch}(f) \geq \mathrm{ch}(l) = (\alpha, m \cdot n + m) > (\alpha, m) = \mathrm{ch}(e)\,.$$

Thus, we need only remark that simply writing out what it means for F to be an isomorphism from $B \mid z$ onto D, where $z \in B$ and D is a Rec. BA, will show that the predicates $F_1\colon B \mid k \approx D_{h(x)}$, $F_2\colon B \mid e \approx D_{g(x,n)}$, and $F_3\colon B \mid l \approx D_{g(x,n+1)}$ are arithmetic so that $\mathrm{ch}(e) < \mathrm{ch}(f)$ and hence the entire third disjunct in the definition of $<_L$ is in Σ^1_1. Similarly, it is easy to see that $G\colon B \mid e \approx B \mid f$ is an arithmetic predicate so that the fourth disjunct is also in Σ^1_1. Thus there can be no such Arith. BA B since otherwise, by Theorem 3.7, ω_1^{CK} is a recursive ordinal. $\square$

Theorems 3.9 and 3.10 combine to prove the following result which has been proved independently by FEINER [1967], GONCHAROV [1972], and ALTON [1974].

3.12. THEOREM. *There exists a recursive α-superatomic BA iff α is a recursive ordinal.*

We also can easily prove the following.

3.13. THEOREM. *For any Rec. BA B, there exists a super-recursive α-atomic BA of type B iff α is a recursive ordinal.*

PROOF. The only if part of the theorem is just Theorem 3.9. For the other direction suppose B is a Rec. BA and D is a super-recursive α-atomic BA of type B. Now it cannot be the case that $\alpha > \omega_1^{\mathrm{CK}}$ since otherwise $D \mid x$ is a recursive ω_1^{CK}-superatomic BA for some $x \in F_{\omega_1^{\mathrm{CK}}+1}(D) - F_{\omega_1^{\mathrm{CK}}}(D)$. If $\alpha = \omega_1^{\mathrm{CK}}$, then B cannot have an atom since once again there would exist $x \in D$ such that $D \mid x$ is a recursive ω_1^{CK}-superatomic BA. Thus, the only possibility when $\alpha = \omega_1^{\mathrm{CK}}$ is that D is ω_1^{CK}-atomic of type $\tilde{Q}$. However, this case is also impossible. That is, by definition $F_{\omega_1^{\mathrm{CK}}}(D)$ is recursive so that $\{D \mid x \mid x \in F_{\omega_1^{\mathrm{CK}}}(D)\}$ is an r.e. set of Rec. BAs which contains α-superatomic BA for all $\alpha < \omega_1^{\mathrm{CK}}$. Now going back to the construction of Theorem 3.9 where we had a recursive copy of $\tilde{C}$ sitting inside a recursive atomless BA A, we let $\tilde{F}$ be a copy of the two element BA as it sets inside the copy of $\tilde{Q}$ in our copy of $\tilde{C}$. Let f be the one-to-one partial recursive function which maps the recursive set $F_{\omega_1^{\mathrm{CK}}}(D)$ onto the recursive set $\{\{q\} \in$

$\tilde{C} \mid q \in Q\}$. Since for each $x \in F_{\omega_1^{\mathrm{CK}}}(D)$, $A \mid f(x)$ is a recursive atomless BA, we can uniformly find recursive subalgebras $E_{f(x)}$ of $A \mid f(x)$ which is isomorphic to $D \mid x$ for all $x \in F_{\omega_1^{\mathrm{CK}}}(D)$. It then follows that since for each $x \in F_{\omega_1^{\mathrm{CK}}}$, $D \mid x \approx S^{\alpha,n}$ for some $\alpha < \omega_1^{\mathrm{CK}}$ and $1 \leq n < \omega$, the r.e. subalgebra E generated by $\tilde{F}$ together with $E_{f(x)}$ for $x \in F_{\omega_1^{\mathrm{CK}}}(D)$ is isomorphic to $S^{\omega_1^{\mathrm{CK}}}$. Now by Corollary 2.3, E is recursively isomorphic to a recursive subalgebra A which would mean there exists a recursive ω_1^{CK}-superatomic BA. Since such recursive BAs are ruled out by Theorem 3.11, it follows that if D is a super-recursive α-atomic BA of type B, then $\alpha < \omega_1^{\mathrm{CK}}$. $\square$

The concept of countably prime, countably saturated, and countably homogeneous models are basic and natural notions which arise in model theory. Now it is the case that the first-order theory of BAs is well enough understood so that all the countably prime, countably saturated, and countably homogeneous BAs have been classified. As a result, we can classify which of these BAs have decidable presentations. The rest of this section will be devoted to this classification.

Recall that a model A is said to be *countably prime* iff A is countable and A is elementary embeddable in every countable model of $\mathrm{Th}(A)$. A is said to be *countably saturated* if A is countable and ω-saturated (see Part I, Section 18 of this Handbook). A is said to be *countably homogeneous* if A is countable and whenever $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are finite subsets of A such that $(A, x_1, \ldots, x_n) \equiv (A, y_1, \ldots, y_n)$, then for all $a \in A$, there exists a $b \in A$ such that $(A, x_1, \ldots, x_n, a) \equiv (A, y_1, \ldots, y_n, b)$. Now it is well known that countably prime and countably saturated models are unique up to isomorphism. That is, we have the following (see CHANG and KEISLER [1973]).

3.14. THEOREM. (i) *If A and B are countably prime models and $A \equiv B$, then A is isomorphic to B.*

(ii) *If A and B are countably saturated models and $A \equiv B$, then A is isomorphic to B.*

It follows from Theorem 3.14 and the results of Section 18, Part I, of this Handbook that there is exactly one countably prime and exactly one countably saturated BA for each invariant type as introduced in Section 18. That is, recall that if A is a BA, then $E(A)$ is the ideal generated by the atomless and atomic elements of A. Then, by induction, we can define a sequence of ideals $E_i(A)$, $i < \omega$, by $E_0(A) = \{0_A\}$, $E_{i+1} = \Pi_i^{-1}[E(A^{(i)})]$, where $A^{(i)} = A/E_i(A)$ and Π_i: $A \rightarrow A^{(i)}$ is the canonical map. Finally, $E_\omega(B) = \bigcup_{i \in \omega} E_i(B)$. This given, we now introduce the following predicates by induction.

3.15. DEFINITION. (i) $\mathscr{E}_0(x)$ is the formula $x = 0$;

(ii) $\mathrm{at}_{n+1}(x)$ is the formula for "$\Pi_n(x)$ is atom in A/E_n", i.e. $\urcorner \mathscr{E}_n(x) \wedge \forall y\, [y \leq x \rightarrow \mathscr{E}_n(y) \vee \mathscr{E}_n((x-y)+(y-x))]$;

(iii) $\mathrm{atl}_{n+1}(x)$ is the formula for "$\Pi_n(x)$ is an atomless element in A/E_n", i.e. $\urcorner \mathscr{E}_n(x) \wedge \urcorner \exists y\, (\mathrm{at}_{n+1}(y)\ \&\ y \leq x)$;

(iv) $\mathrm{atc}_{n+1}(x)$ is the formula for "$\Pi_n(x)$ is an atomic element of A/E_n", i.e. $\urcorner \mathscr{E}_n(x) \wedge \forall z\, (z \leq x \wedge \urcorner \mathscr{E}_n(z) \rightarrow \exists y\, (y \leq z \wedge \mathrm{at}_{n+1}(y)))$;

(v) $\mathscr{E}_{n+1}(x)$ is the formula for "$x \in E_{n+1}$", i.e. $\mathscr{E}_n(x) \vee \mathrm{atl}_{n+1}(x) \vee \mathrm{atc}_{n+1}(x) \vee \exists y \exists z\ (x = y + z\ \&\ \mathrm{atl}_{n+1}(y)\ \&\ \mathrm{atc}_{n+1}(z))$.

By the results of Section 18, it follows that a Rec. BA B is decidable iff all the predicates $\mathscr{E}_n(x)$, $\mathrm{at}_{n+1}(x)$, $\mathrm{atl}_{n+1}(x)$, and $\mathrm{atc}_{n+1}(x)$ are recursive for all n. We shall call those elements satisfying $\mathrm{at}_{n+1}(x)$, $\mathrm{atl}_{n+1}(x)$, $\mathrm{atc}_{n+1}(x)$, and $\mathscr{E}_{n+1}(x) - \mathscr{E}_n(x)$ the *n-atoms*, *n-atomless elements*, *n-atomic elements*, and *n-elements* of B, respectively. For any BA B, we define $\mathrm{inv}(B) = \langle \mathrm{inv}_1(B), \mathrm{inv}_2(B), \mathrm{inv}_3(B)\rangle$, where

$$\mathrm{inv}_1(B) = \begin{cases} \min\{r \mid E_{r+1}(B) \text{ is trivial}\} & \text{if it exists}, \\ \omega & \text{otherwise}; \end{cases}$$

$$\mathrm{inv}_2(B) = \begin{cases} \sup\{r \mid B/E_{\mathrm{inv}_1(B)} \text{ has at least } r \text{ atoms}\} & \text{if } \mathrm{inv}_1(B) < \omega, \\ 0 & \text{otherwise}; \end{cases}$$

$$\mathrm{inv}_3(B) = \begin{cases} 1 & \text{if } B/E_{\mathrm{inv}_1(B)} \text{ has an atomless element or } \mathrm{inv}_1(B) = \omega, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, if $a \in B$, we write $\mathrm{inv}(a) = \langle \mathrm{inv}_1(a), \mathrm{inv}_2(a), \mathrm{inv}_3(a)\rangle$ for $\mathrm{inv}(B \mid a)$. We let $\mathrm{Th}_{\langle n,m,l\rangle}$ be the theory of all BAs B such that $\mathrm{inv}(B) = \langle n, m, l\rangle$ so that the collection of $\mathrm{Th}_{\langle n,m,l\rangle}$ forms the set of all complete extensions of the first-order theory of Boolean algebras.

3.16. DEFINITION. Let B be a model of $\mathrm{Th}_{\langle n,m,l\rangle}$.
(a) If $n < \omega$, then we say B is *finite-atomic* iff
  (i) for every $k < n$, every atomic element in $B/E_k(B)$ is a finite union of atoms of $B/E_k(B)$, and
  (ii) for every atomic x in $B/E_n(B)$ either x or $-x$ has at most finitely many atoms of $B/E_n(B)$ below it.
(b) If $n = \omega$, then we say B is *finite-atomic* iff B satisfies (i) and
  (iii) if $B/E_\omega(B)$ is the two element BA.

The significance of the *finite-atomic* BAs is given by the following theorem.

3.17. THEOREM (MEAD [1979]). *A BA B is a prime model for* $\mathrm{Th}_{\langle n,m,l\rangle}$ *iff* $\mathrm{inv}(B) = \langle n, m, l\rangle$ *and B is finite-atomic.*

Following MEAD [1979], we next construct a sequence of Rec. LOs $L_{\langle n,m,l\rangle}$ such that $B_{\langle n,m,l\rangle} = \mathrm{Intalg}(L_{\langle n,m,l\rangle})$ is a decidable finite-atomic BA with $\mathrm{inv}(B_{\langle n,m,l\rangle}) = \langle n,m,l\rangle$ and hence show that all countably prime BAs are isomorphic to Dec. BAs.

Recall that η denotes the order type of the rationals, ω order type of the natural numbers, ω^* the order type of the negative integers, and $\bar{n}$ the order type of the integers $\{1, \ldots, n\}$. Assume that we have fixed some decidable presentation of these orderings. First, we construct the linear orderings for those invariants whose first invariant is finite:

$$L_{\langle 0,m,0\rangle} = \overline{m+1}\,, \qquad L_{\langle 0,m,1\rangle} = \overline{m+1} + \eta\,,$$

$$L_{\langle 0,\omega,0\rangle} = \omega\,, \qquad L_{\langle 0,\omega,1\rangle} = \omega + \bar{1} + \eta\,.$$

To raise the first invariant, we modify our linear orders as follows. For each interval $[a, b) = \{a\}$, i.e. those intervals which represent atoms in the interval algebra, add the order $(\bar{1} + \eta + \bar{1}) \cdot \omega^*$ between a and b so that now the interval $[a, b)$ represents a 1-atom in the interval algebra. In general, this process will turn an n-atom into an $n+1$-atom. Next for each a that is an element of a copy of η, replace that element by a copy of $(\bar{2} + \eta) \cdot \omega$ which will have the effect of making every interval $[a, b)$ inside that copy of η isomorphic to $\bar{2} + \eta + ((\bar{2} + \eta) \cdot \omega) \cdot \eta$ which represents a 1-atomless element in the interval algebra. In general this process will turn n-atomless elements into $n+1$-atomless elements. Thus, for example, we let

$$L_{\langle 1,m,0\rangle} = \sum_{i=1}^{m+1} \bar{1} + (\bar{1} + \eta + \bar{1}) \cdot \omega^*\,,$$

$$L_{\langle 1,\omega,0\rangle} = \sum_{i=1}^{\omega} \bar{1} + (\bar{1} + \eta + \bar{1}) \cdot \omega^* = (\bar{1} + (\bar{1} + \eta + \bar{1}) \cdot \omega^*) \cdot \omega\,,$$

$$L_{\langle 1,m,1\rangle} = \left(\sum_{i=1}^{m+1} \bar{1} + (\bar{1} + \eta + \bar{1}) \cdot \omega^*\right) + ((\bar{2} + \eta) \cdot \omega) \cdot \eta\,,$$

$$L_{\langle 1,\omega,1\rangle} = (\bar{1} + (\bar{1} + \eta + \bar{1}) \cdot \omega^*) \cdot \omega + ((\bar{2} + \eta) \cdot \omega) \cdot \eta\,.$$

Note that all the linear orderings corresponding to those invariants whose first invariants are of level 1 have the following two properties:

(i) if $a < b$ and b is not an immediate successor of a, then $\exists c, d\ (a < c < d < b$ & $[c, d)$ is a dense linear ordering);

(ii) the first element does not have an immediate successor and the last element (if it exists) does not have an immediate predecessor.

Given any linear ordering L satisfying (i) and (ii), define L^* to be the linear ordering which is the lexicographic ordering on

$$\{(a, x), (b, y), (c, c) \mid a \in \Delta_L^1 \ \&\ x \in \bar{1} + (\bar{1} + \eta + \bar{1}) \cdot \omega^*\,, \quad b \in \Delta_L^2 \ \&\ x \in (\bar{2} + \eta) \cdot \omega,\ c \in \Delta_L^3\}\,,$$

where

$\Delta_L^1 = \{a \in L \mid a$ has an immediate successor or an immediate predecessor and a is not the first or last element$\}$,
$\Delta_L^2 = \{a \in L \mid a \notin \Delta_L^1$ and a is not the first or last element$\}$, and
$\Delta_L^3 = \{a \in L \mid a$ is first or last element of $L\}$.

It is then not difficult to check that if L satisfies (i) and (ii), the Intalg(L) is finite-atomic, and $\operatorname{inv}(L) = \langle n, m, l\rangle$, then Intalg$(L^*)$ is finite-atomic and $\operatorname{inv}(L) = \langle 1+n, m, l\rangle$. Thus, we can define by induction for $n \geq 1$, $L_{\langle n+1,m,l\rangle} =$

$(L_{\langle n,m,l\rangle})^*$. Finally, $L_{\langle \omega,0,0\rangle} = \Sigma_{n\in\omega} L_{\langle n,1,0\rangle}$. It is easy to see that this whole process can be carried out in an effective manner so that in each $B_{\langle n,m,l\rangle} = \mathrm{Intalg}(L_{\langle n,m,l\rangle})$, the predicates $\mathrm{at}_{k+1}(x)$, $\mathrm{atl}_{k+1}(x)$, $\mathrm{atc}_{k+1}(x)$, and $\mathscr{E}_k(x)$ are recursive for all k and hence each $B_{\langle n,m,l\rangle}$ is a Dec. BA. Thus, we have the following.

3.18. THEOREM (MEAD [1979]). *Every countably prime BA is isomorphic to a Dec. BA.*

Next we turn to the countably saturated BAs. Countably saturated models are in some sense dual to the prime models in that they are countably universal where a countable model A of a complete theory T is *countably universal* if every countable model of T is elementary embeddable in B. In fact, it follows from general model theoretic results that for BAs, a BA B is countably saturated iff B is countably universal and countably homogeneous. MOROZOV [1982a] proved the following characterization of countably saturated BAs.

3.19. THEOREM (MOROZOV [1982a]). *For any countable BA B, B is countably saturated iff all $a \in B$ satisfy*

(i) $\forall r < \mathrm{inv}_1(a) \exists a' \leq_B a\ (\mathrm{inv}(a') = \langle r, \omega, 1\rangle)$,
(ii) $\mathrm{inv}_2(a) = \omega \rightarrow \exists a' \leq a\ (\mathrm{inv}_2(a') = \mathrm{inv}_2(a - a') = \omega)$, *and*
(iii) $\mathrm{inv}_1(a) = \omega \rightarrow \exists a' \leq a\ (\mathrm{inv}_1(a') = \mathrm{inv}_1(a - a') = \omega)$.

We can construct the countably saturated BAs by a process very similar to the way we constructed the countably prime BAs. That is, we shall construct a Rec. LO $S_{\langle n,m,l\rangle}$ for each invariant type so that $C_{\langle n,m,l\rangle} = \mathrm{Intalg}(S_{\langle n,m,l\rangle})$ is a decidable saturated BA with $\mathrm{inv}(C_{\langle n,m,l\rangle}) = \langle n, m, l\rangle$.

Now it is easy to see from Theorem 3.19 that we can define

$$S_{\langle 0,m,0\rangle} = \overline{m+1}\,, \qquad S_{\langle 0,m,1\rangle} = \overline{m+1} + \eta\,,$$

$$S_{\langle 0,\omega,0\rangle} = \bar{1} + \omega \cdot \eta\,, \qquad S_{\langle 0,\omega,1\rangle} = \bar{1} + \eta + \bar{1} + \omega \cdot \eta\,.$$

Next the process of raising the first characteristic by 1 is simpler than before since we need only replace each point in a linear order L by $(\bar{1} + \eta + \bar{1} + \omega \cdot \eta) \cdot \omega = \Sigma_{n\in\omega} S_{\langle 0,\omega,1\rangle}$. That is, we can define by induction on $n \geq 0$,

$$S_{\langle n+1,m,l\rangle} = (\bar{1} + \eta + \bar{1} + \omega \cdot \eta) \cdot \omega \cdot S_{\langle n,m,l\rangle}\,.$$

Finally, we can construct $S_{\langle \omega,0,0\rangle}$ as follows. Start with the rationals Q and partition Q into countably many pairwise disjoint dense subsets $T_0, T_1, T_2, \ldots$ and form a new recursive linear ordering S by replacing each $x \in T_j$ by a copy of $S_{\langle j,\omega,1\rangle}$. Then we can let $S_{\langle \omega,0,0\rangle} = \bar{1} + S$. Again it is not difficult to verify, using Theorem 3.19, that for all $\langle n, m, l\rangle$, $C_{\langle n,m,l\rangle}$ is a saturated BA whose invariance type is $\langle n, m, l\rangle$ and that the construction of the Rec. LOs $S_{\langle n,m,l\rangle}$ can be carried out in a uniformly effective manner so as to guarantee that each $C_{\langle n,m,l\rangle}$ is a Dec. BA. Thus, we have the following.

3.20. THEOREM (MOROZOV [1982a]). *Every countably saturated BA is isomorphic to Dec. BA.*

Finally, we end this section by considering the countably homogeneous BAs following the work of MOROZOV [1982b]. The following lemma is rather straightforward to verify.

3.21. LEMMA. (i) *B is countable homogeneous BA iff*

$$\forall a\, \forall b[\mathrm{inv}(a) = \mathrm{inv}(b)\ \&\ \mathrm{inv}(-a) = \mathrm{inv}(-b) \rightarrow$$
$$\forall a_0 \leq a \exists b_0 \leq b(\mathrm{inv}(a_0) = \mathrm{inv}(b_0)\ \&\ \mathrm{inv}(a - a_0) = \mathrm{inv}(b - b_0))] .$$

(ii) *If B is countably homogeneous, then $B/E(B)$ and $B \mid b$ are countably homogeneous for all $b \in B$.*

It easily follows from Lemma 3.21 that the following is a complete list of all countably homogeneous BAs B such that $\mathrm{inv}_1(B) = 0$.

$\mathrm{inv}(B)$	
$\langle 0, m, 0\rangle$	$\mathrm{Intalg}(\overline{m+1})$
$\langle 0, m, 1\rangle$	$\mathrm{Intalg}(\overline{m+1} + \eta)$
$\langle 0, \omega, 0\rangle$	$\mathrm{Intalg}(\omega)$, $\mathrm{Intalg}(\omega + \omega)$, $\mathrm{Intalg}(\bar{1} + \omega \cdot \eta)$
$\langle 0, \omega, 1\rangle$	$\mathrm{Intalg}(\bar{1} + \eta + \omega)$, $\mathrm{Intalg}(\bar{1} + \eta + \omega + \omega)$, $\mathrm{Intalg}(\bar{1} + \eta + \bar{1} + \omega \cdot \eta)$

That is, if B is a countably homogeneous BA of type $\langle 0, \omega, 0\rangle$, then either (i) there is no $b \in B$ such that both $|\mathrm{At}(B \mid b)|$ and $|\mathrm{At}(B \mid -b)|$ are infinite in which case B is isomorphic to the BA of finite–cofinite sets of N, i.e. to $\mathrm{Intalg}(\omega)$; (ii) there is a $b \in B$ such that $|\mathrm{At}(B \mid b)| = |\mathrm{At}(B \mid -b)| = \omega$ but there is no $c \leq_B b$ such that $|\mathrm{At}(B \mid c)| = |\mathrm{At}(B \mid b - c)| = \omega$, in which case $B \approx \mathrm{Intalg}(\omega + \omega)$; or finally, (iii) there is a $b \in B$ and $c <_B b$ such that $|\mathrm{At}(B \mid b)| = |\mathrm{At}(B \mid b)| = |\mathrm{At}(B \mid c)| = |\mathrm{At}(B \mid b - c)| = \omega$, in which case the fact that B is homogeneous will show that $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ is atomless and hence by Theorem 1.3(a), $B \approx \tilde{C} \approx \mathrm{Intalg}(\bar{1} + \omega \cdot \eta)$. A similar argument applies if B is of type $\langle 0, \omega, 1\rangle$. On the other hand, it is routine to verify that all the BAs above are countably homogeneous.

Next consider the countably homogeneous BAs B, where $1 \leq \mathrm{inv}_1(B)$. We then have the following lemma.

3.22. LEMMA (MOROZOV [1982b]). *If B is a countably homogeneous BA and $\mathrm{inv}_1(B) \geq 1$, then*

(i) *if there exists an $a \in B$ such that* $\mathrm{inv}(a) = \langle 0, \omega, 0\rangle$, *then $B \mid a$ is isomorphic to either* $\mathrm{Intalg}(\omega)$ *or* $\mathrm{Intalg}(1 + \omega \cdot \eta)$,

(ii) *if there exists $a \in B/E_n(B)$ such that $B/E_n(B) \mid a$ is isomorphic to* $\mathrm{Intalg}(\omega)$, *then $B/E_{n+1}(B) \approx 2$.*

PROOF. For (i), suppose $a, b \in B$ and $\mathrm{inv}(a) = \mathrm{inv}(b) = \langle 0, \omega, 0\rangle$. Then $\mathrm{inv}(-a) = \mathrm{inv}(-b) = \mathrm{inv}(B)$ so that it follows from Lemma 3.21(i) that $B \mid a \approx$

$B \mid b$. Moreover, since $B \mid a$ is countably homogeneous, $B \mid a$ can only be one of the three BAs in our table. However, $B \mid a \approx \text{Intalg}(\omega + \omega)$ is ruled out since $\exists x \leq_B a\ (B \mid x \approx \text{Intalg}(\omega) \not\approx B \mid a\ \&\ \text{inv}(x) = \text{inv}(a) = \langle 0, \omega, 0 \rangle)$.

For (ii), note that since $B/E_n(B)$ is countably homogeneous, it is enough to show that if B is countably homogeneous and $B \mid a \approx \text{Intalg}(\omega)$ for some a, then $B/E(B) \approx 2$. Now if $B/E(B) \not\approx 2$, then let $b \in B$ be such that both $\pi(b)$ and $\pi(-b)$ are nonzero in $B/E(B)$, where π: $B \to B/E(B)$ is the canonical map. But now either $B \mid (b \cdot a)$ or $B \mid (-b \cdot a)$ is isomorphic to $B \mid a$, say $B \mid b \cdot a \approx B \mid a$. Then since $\text{inv}_1(b)$, $\text{inv}_1(-b) \geq 1$, $\text{inv}(b - a) = \text{inv}(b)$ and $\text{inv}(-(b - a)) = \text{inv}(-b)$. Thus, by homogeneity, there exists $c \leq b - a$ such that $\text{inv}(c) = \text{inv}(a)$. But then $B \mid c + b \cdot a$ is isomorphic to $\text{Intalg}(\omega + \omega)$ violating (i). $\square$

Now suppose that B is a countably homogeneous BA of type $\langle n, m, l \rangle$, where $1 \leq n < \omega$. Let $\rho(B) = (\rho_0, \ldots, \rho_{n-1})$ where

$$\rho_i = \begin{cases} 0 & \text{if } \exists a \in B/E_i(B)((B/E_i(B)) \mid a \approx \text{Intalg}(\omega))\,, \\ 1 & \text{if } \exists a \in B/E_i(B)((B/E_i(B)) \mid a \approx \text{Intalg}(1 + \omega \cdot \eta))\,, \\ 2 & \text{if } \urcorner \exists a \in B/E_i(B)(\text{inv}(a) = \langle 0, \omega, 0 \rangle)\,. \end{cases}$$

By Lemma 3.22, $\rho(B)$ is well defined. In fact, $\rho(B)$ and $\text{inv}(B)$ form a complete set of invariants for countably homogeneous BAs B with $1 \leq \text{inv}_1(B) < \omega$.

3.23. THEOREM (MOROZOV [1982b]). *If B_1 and B_2 are countably homogeneous BAs such that* $\text{inv}(B_1) = \text{inv}(B_2)$ *and* $1 \leq \text{inv}_1(B_i) < \omega$, *then* $\rho(B_1) = \rho(B_2)$ *iff* $B_1 \approx B_2$.

Moreover, it easily follows from Lemma 3.22 that the possible countably homogeneous BAs B, where $1 \leq \text{inv}_1(B) < \omega$, are as shown in Table 25.1.

Using the following theorem, we can inductively construct decidable countably homogeneous BAs for all the isomorphism types corresponding to the entries in Table 25.1 much as we did for the countably prime and countably saturated BAs.

Table 25.1

$\text{inv}(B) = \langle n, m, l \rangle$, $n \geq 1$	$\rho(B) = (\rho_0, \ldots, \rho_{n-1})$	B/E_n	Total number
$\langle n, 1, 0 \rangle$	$\rho_i \in \{1, 2\} \forall i \leq n-2$ $\rho_{n-1} \in \{0, 1, 2\}$	2	$2^{n-1} \cdot 3$
$\langle n, 1, 1 \rangle$	$\rho_i \in \{1, 2\} \forall i$	$\text{Intalg}(\bar{2} + \eta)$	2^n
$\langle n, \omega, 0 \rangle$	$\rho_i \in \{1, 2\} \forall i$	$\text{Intalg}(\omega)$, $\text{Intalg}(\omega + \omega)$ $\text{Intalg}(\bar{1} + \omega \cdot \eta)$	$2^n \cdot 3$
$\langle n, \omega, 1 \rangle$	$\rho_i \in \{1, 2\} \forall i$	$\text{Intalg}(\bar{1} + \eta + \omega)$ $\text{Intalg}(\bar{1} + \eta + \omega + \omega)$ $\text{Intalg}(\bar{1} + \eta + \bar{1} + \omega \cdot \eta)$	$2^n \cdot 3$
$\langle n, m, 0 \rangle$ $1 < m < \omega$	$\rho_i \in \{1, 2\} \forall i$	$\text{Intalg}(m + 1)$	2^n
$\langle n, m, 1 \rangle$ $1 < m < \omega$	$\rho_i \in \{1, 2\} \forall i$	$\text{Intalg}(\overline{m + 1} + \eta)$	2^n

3.24. THEOREM (MOROZOV [1982b]). *Suppose L is a linear ordering such that* $\mathrm{Intalg}(L) = B_L$ *is a countable homogeneous,* $\mathrm{inv}(B_L) = \langle n, m, l \rangle$, *where* $n < \omega$, *and* $\rho(B_L) = (\rho_0, \ldots, \rho_{n-1})$. *Then*

(i) *if* $L_1 = (\bar{1} + \omega \cdot \eta) \cdot L$ *and* $B_{L_1} = \mathrm{Intalg}(L_1)$, *then* B_{L_1} *is countably homogeneous,* $\mathrm{inv}(B_{L_1}) = \langle n+1, m, l \rangle$, *and* $\rho(B_{L_1}) = \langle 1, \rho_0, \ldots, \rho_{n-1} \rangle$;

(ii) *if* $L_2 = (\omega + \eta) \cdot L$ *and* $B_{L_2} = \mathrm{Intalg}(L_2)$, *then* B_{L_2} *is countably homogeneous,* $\mathrm{inv}(B_{L_2}) = \langle n+1, m, l \rangle$, *and* $\rho(B_{L_2}) = \langle 2, \rho_0, \ldots, \rho_{n-1} \rangle$;

(iii) $B = \mathrm{Intalg}(\omega + \omega + \eta)$ *is countably homogeneous,* $\mathrm{inv}(B) = \langle 1, 1, 0 \rangle$, *and* $\rho(B) = \langle 0 \rangle$.

3.25. COROLLARY (MOROZOV [1982b]). *Every countably homogeneous BA B such that* $\mathrm{inv}_1(B) < \omega$ *is isomorphic to Dec. BA.*

Finally, we need only consider countably homogeneous BAs B of type $\langle \omega, 0, 0 \rangle$. For such B, we need one more invariant besides $\mathrm{inv}(B)$ and $\rho(B) = (\rho_0, \rho_1, \rho_2, \ldots)$, where ρ_i is defined as above to determine the isomorphism type of B. Namely, define

$$\tau(B) = \begin{cases} 0 & \text{if } \forall b \in B\ (\mathrm{inv}_1(b) < \omega \vee \mathrm{inv}_1(-b) < \omega)\,, \\ 1 & \text{if } \exists b \in B\ (\mathrm{inv}_1(b) = \mathrm{inv}_1(-b) = \omega\ \&\ \forall c \leq b\ (\mathrm{inv}_1(c) < \omega \\ & \qquad \vee\ \mathrm{inv}_1(b - c) < \omega))\,, \\ 2 & \text{if } \exists b \in B\ \exists c \leq b \\ & (\mathrm{inv}_1(b) = \mathrm{inv}_1(-b) = \mathrm{inv}_1(c) = \mathrm{inv}_1(b - c) = \omega)\,. \end{cases}$$

We then have the following.

3.26. THEOREM (MOROZOV [1982b]). *Suppose B is a countably homogeneous BA with* $\mathrm{inv}(B) = \langle \omega, 0, 0 \rangle$. *Then*

(i) *if B' is a countably homogeneous BA with* $\mathrm{inv}(B') = \langle \omega, 0, 0 \rangle$, *then* $B \approx B'$ *iff* $\rho(B) = \rho(B')$ & $\tau(B) = \tau(B')$.

(ii) *B is isomorphic to a Dec. BA iff* $\{i \mid \rho_i = 2\} \in \Pi^0_2$.

It is not difficult to show that the techniques we have outlined above will allow one to construct countably homogeneous BAs B of type $\langle \omega, 0, 0 \rangle$ where $\rho(B) = (\rho_0, \rho_1, \ldots)$ is any given sequence such that $\rho_i \in \{1, 2\}$ for all i and $\tau(B)$ is any element from $\{0, 1, 2\}$. Thus, there are 2^ω such BAs and hence they cannot all be isomorphic to Dec. BAs. Thus, some criterion such as given in part (ii) of Theorem 3.26 is the best we can hope for as a characterization of when countably homogeneous BAs of type $\langle \omega, 0, 0 \rangle$ are isomorphic to Dec. BAs.

4. Recursive Boolean algebras with minimally effective presentations

The results of this section will be in sharp contrast to the results of the previous section in that we will give various constructions of BAs whose presentation have very few effective properties beyond those required to have the BA be recursive.

For example, we begin this section by constructing via a finite injury priority argument a Rec. BA $\tilde{M}$ isomorphic to $\tilde{N}$ such that $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ is immune. Here a subset I of N is *immune* if I is infinite and I contains no infinite r.e. set. Moreover, we shall show that the construction of $\tilde{M}$ can easily be modified to produce recursive BAs M^α isomorphic to the α-superatomic BA S^α such that $\mathscr{F}_\beta(M^\alpha)$ is not recursive for all $1 \le \beta \le \alpha$. We shall then state some general recursive model theoretic results of Ash and Nerode which can be used to produce many examples of Rec. BAs B, where the basic predicates $\mathrm{at}_{n+1}(x)$, $\mathrm{atl}_{n+1}(x)$, $\mathrm{atc}_{n+1}(x)$, and $\epsilon_n(x)$ are nonrecursive for various n. Finally, we shall end this section with a presentation of some very important coding techniques due to FEINER [1970a] which he used to give an example of an R.e. BA B which is not isomorphic to any Rec. BA. We shall actually present a modification of Feiner's construction due to REMMEL [1981b] of a recursive atomic BA B such that $\mathrm{At}(B)$ is recursive but B is not isomorphic to any Rec. BA D, where both $\mathrm{At}(D)$ and $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ are recursive. It thus follows that B is a Dec. BA such that $B/\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ is not even isomorphic to a Rec. BA. A similar modification of Feiner's construction due to GONCHAROV [1975a] produces an example of a Rec. BA R which is not isomorphic to any Dec. BA.

We start this section with our first application of the priority method to produce Rec. BAs with interesting recursion theoretic properties. Actually, the priority method has been widely used to study the effective content of various theorems throughout mathematics since it was first used for such purposes by METAKIDES and NERODE [1975], [1977]. Our next result requires a rather simple priority argument and at this point follows from more general results, but it will provide a simple enough setting in which to introduce the uses of priority arguments in the theory of Rec. BAs.

4.1. THEOREM (REMMEL [1986a]). *There exists a Rec. BA $\tilde{M}$ which is isomorphic to the BA of finite and cofinite subsets of the natural numbers such that $\langle \mathrm{At}(\tilde{M})\rangle^{\mathrm{id}}$ is immune.*

PROOF. We shall construct a recursive subalgebra $\tilde{M}$ of $\tilde{Q}$ in stages so that we meet the following set of requirements.

R_e: If W_e is infinite, then $W_e \not\subseteq \langle \mathrm{At}(\tilde{M})\rangle^{\mathrm{id}}$.

A simple minded way to construct a recursive subalgebra of $\tilde{Q}$ in stages which is isomorphic to $\tilde{N}$ is the following. At each stage s, we specify a finite subalgebra B_s of $\tilde{Q}$ and a finite sequence $b_0^s, \ldots, b_s^s$ which is the set of atoms of B_s. Then at stage $s+1$, we split b_s^s into two nonzero pieces b_s^{s+1} and b_{s+1}^{s+1} in such a way so that if we define $b_i^{s+1} = b_i^s$ for $i < s$ and $B_{s+1} = \langle \{b_j^{s+1} \mid 1 \le j \le s+1\}\rangle$, then $B_{s+1} \cap \{0, \ldots, s\} = B_s \cap \{0, \ldots, s\}$. This is possible since there are infinitely many splittings of b_s^s. It is then easy to see that $B = \bigcup_s B_s$ is a recursive subalgebra of $\tilde{Q}$ isomorphic, in fact, recursively isomorphic to $\tilde{N}$. Our idea is to modify the above construction via a finite injury priority so that we meet the requirements R_e. The basic change from the above construction is that instead of always splitting the last atom in the sequence $b_0^s, \ldots, b_s^s$ at each stage $s+1$, we will allow ourselves first to reorder the atoms and then split the last atom. We shall see that this process will still ensure that $\bigcup_s B_s$ is isomorphic to $\tilde{N}$ as long as we ensure that $\lim_s b_i^s = b_i$ exists for each i. To meet a single requirement R_e we proceed as follows. We assume that for the sake of those requirements of higher priority than R_e, i.e. $R_0, \ldots, R_{e-1}$, there is an integer $r(e-1)$ such that we must have $b_i^s = b_i^{s+1}$ for $i \leq r(e-1)$ for all sufficiently large s. Now if W_e is infinite, then for large enough s, there will exist $x \in W_e^s - \langle \{b_0^s, \ldots, b_{r(e-1)}^s\} \rangle$. Then there are two possibilities: either $x \not\in B_s$, in which case we will ensure x is never put into B_t for any $t > s$, or $x \in B^s$. In the latter case, it follows that $b_j^s \leq_{\tilde{Q}} x$ for some $j > r(e-1)$. Then at stage $s+1$, we will reorder the atoms of B_s so that b_j^s is the last atom and hence b_j^s will be split into two nonzero pieces b_s^{s+1} and b_{s+1}^{s+1} at stage s. Moreover, we will allow requirement R_e to impose the restraint that $b_i^t = b_i^{t+1}$ for all $i < s$ and $t > s$. This action will ensure that as long as we respect the restraint imposed by R_e at all later stages, then no matter how we reorder the atoms at a stage $t > s$, some atom less than b_j^s will be split at stage t. Thus, $b_j^s \not\in \langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ and since $x \geq_{\tilde{Q}} b_j^s$, x will witness that $W_e \not\subseteq \langle \mathrm{At}(B) \rangle^{\mathrm{id}}$. As usual with finite injury priority arguments, our construction will ensure that the restraint imposed for a requirement R_e can be violated only for requirements of higher priority and hence in the end, requirement R_e will be "injured" at most finitely often so that we will be guaranteed that the above strategy for R_e will eventually produce the desired witness to the fact $W_e \not\subseteq \langle \mathrm{At}(B) \rangle^{\mathrm{id}}$.

More formally, at each stage s of our construction, we will specify a finite subalgebra B_s of $\tilde{Q}$, an ordering $b_0^s, b_1^s, \ldots, b_s^s$ of the atoms of B_s, and a finite set F_s such that $F_s \cap B_s = \emptyset$. F_s will be called the set of *forbidden elements* at stage s. Moreover, for each e and $s \geq 0$, we will define an integer $r(e, s) \geq -1$ called the *restraint imposed by R_e at stage s*. (When $r(e, s) = -1$, R_e will impose no restraint at stage s.) We shall say requirement R_e is *satisfied* at stage s if either $W_e^s \cap F^s \neq \emptyset$ or $r(e, s) \geq 0$. We say that requirement R is *injured* at stage $s+1$ if for some $i \leq r(e, s)$, $b_i^s \neq b_i^{s+1}$.

Construction.

Stage 0. Let $B_0 = \{1_{\tilde{Q}}, 0_{\tilde{Q}}\}$, $b_0^0 = 1_{\tilde{Q}}$, and $F_0 = \emptyset$. For all e, let $r(e, 0) = -1$.

Stage s + 1. Look for the least requirement R_e which is not satisfied and has an $x \leq s$ that $x \in W_e^s$ and either (i) $x \not\in B_s$ or (ii) $x \in B_s$ and $b_j^s \leq_{\tilde{Q}} x$ for some j such that $j > e$ and $j > r(i, s)$ for all $i < e$. If there is no such e, let $F_{s+1} = (\{0, \ldots, s+1\} - B_s) \cup F_s$. Then let $b_i^s = b_i^{s+1}$ for $i < s$ and pick the least y such that $0_{\tilde{Q}} <_{\tilde{Q}} y <_{\tilde{Q}} b_s^s$ and $\langle B_s \cup \{y\} \rangle \cap F_{s+1} = \emptyset$. (Such a y exists by Lemma 2.2.) Then let $b_s^{s+1} = y$, $b_{s+1}^{s+1} = b_s^s - y$, and $B_{s+1} = \langle \{b_0^{s+1}, \ldots, b_{s+1}^{s+1}\} \rangle$. Finally, let $r(e, s+1) = r(e, s)$ for all e.

If there is such an e, let $e(s+1) = e$ be the least such e and let $x(s+1) = x$ be the least x corresponding to e. If $x \not\in B_s$, then proceed exactly as above except that we let $F_{s+1} = (\{0, \ldots, s+1\} - B_s) \cup F_s \cup \{x\}$. If $x \in B_s$, then let $j(s+1) = j$ be the largest i such that $b_i^s \leq_{\tilde{Q}} x$. By definition $j > r(k, s)$ for all $k < e$. Then let $F_{s+1} = (\{0, \ldots, s+1\} - B_s \cup F_s$, $b_i^{s+1} = b_i^s$ if $i < j$, and $b_i^{s+1} = b_{i+1}^s$ if $j \leq i \leq s$. Pick the least y such that $0_{\tilde{Q}} <_{\tilde{Q}} y <_{\tilde{Q}} b_j^s$ and $\langle B_s \cup \{y\} \rangle \cap F_{s+1} = \emptyset$. Then let $b_s^{s+1} = y$, $b_{s+1}^{s+1} = b_j^s - y$, and $B_{s+1} = \langle \{b_0^{s+1}, \ldots, b_{s+1}^{s+1}\} \rangle$. Finally, let $r(e, s+1) =$

s and for all $f \neq e$, let $r(f, s+1) = r(f, s)$ if $r(f, s) < j$ and $r(f, s+1) = -1$, otherwise.

This completes our construction. We let $\tilde{M} = \bigcup_s B_s$ and now prove a series of simple lemmas to verify that $\tilde{M}$ has the desired properties.

4.1.1. LEMMA. *For all e, $\lim_s b_e^s = b_e$ and $\lim_s r(e, s) = r(e)$ exist.*

PROOF. We proceed by induction on e. Fix e and assume that there is a stage $s_0 \geq e$ large enough such that $b_i^t = b_i^{s_0}$ for all $i < e$. By our construction, $b_e^{s+1} \neq b_e^s$ for some $s > s_0$ only we take an action to satisfy some requirement R_j with $j < e$ at stage $s+1$. But note that we can take action for requirement R_0 at only one stage, at most, since we are never allowed to injure the restraint imposed by R_0. Once R_0 has ceased to act, we can only take action for R_1 at most one more time since only actions for R_0 can injure R_1 restraint, etc. It thus easily follows by induction that $b_e^{s+1} \neq b_e^s$ for $s > s_0$ at most finitely many times so that $\lim_s b_e^s$ exists. The same argument will establish that $\lim r(e, s)$ exists for all e. $\square$

4.1.2. LEMMA. *$\tilde{M}$ is a recursive BA isomorphic to $\tilde{N}$.*

PROOF. Note that $\tilde{M}$ is recursive since the fact that for all s, $\{0, \ldots, s+1\} - B_s \subseteq \tilde{F}_{s+1}$ implies that $s \in B_{t+1} - B_t$ only if $t < s$. Thus, $s \in \tilde{M}$ iff $s \in B_s$. To see that $\tilde{M}$ is isomorphic to the Boolean algebra of finite and cofinite sets, it is clearly enough to show that for any $x \in \tilde{M}$ either x or $-x$ is a finite union of atoms. Now fix $x \in \tilde{M}$ and suppose that s is a stage large enough so that $x \in B_s$ and at stage $s+1$ we take an action to satisfy some requirement R_k at stage $s+1$. Note since there are infinitely many indices e, where $W_e = N$, it follows by our construction that there are infinitely many stages t at which we take an action to satisfy some requirement at stage t. Then at stage $s+1$, $r(k, s+1) = s$ and either x or $-x$ is less than $\Sigma_{i=0}^{r(k,s+1)} b_i^{s+1}$. Now if there is a first stage $t > s$ such that $b_i^{t+1} \neq b_i^t$ for some $i \leq r(k, s+1)$, then by construction, we must take an action to satisfy some requirement R_l, where $l < k$. Then as before $r(l, t+1) = t$ and either x or $-x$ is less than $\Sigma_{i=0}^{r(l,t+1)} b_i^{t+1}$. It then follows by induction for all $s' \geq s$, either x or $-x$ is less than $\Sigma_{i=1}^{r(j,s')} b_i^{s'}$, where $j \leq k$. But since $\lim_s r(e, s)$ exists for all e, it follows that either x or $-x$ is a finite union of atoms. $\square$

4.1.3. LEMMA. *All requirements R_e are met.*

PROOF. Again we proceed by induction e. We can assume by induction that all requirements R_i with $i < e$ are met. Then let t_0 be a stage large enough so that $r(i, t) = r(i, t_0)$ for all $i < e$. Note that it follows that we never take action for any R_i with $i < e$ at any stage $t > t_0$. For a contradiction, assume W_e is infinite and $W_e \subseteq \langle \mathrm{At}(\tilde{M}) \rangle^{\mathrm{id}}$. It follows that there must be some stage $s > t_0$ and $x \in W_e^s$ such that $b_j^s \leq_{\tilde{Q}} x$ for some $j > \max\{r(i, t) \mid i < e\}$. Then at stage $s+1$, x is a candidate to satisfy requirement R_e. Now our choice of $s > t_0$ ensures that either we satisfy R_e at stage $s+1$ or R_e is already satisfied at stage s. Since R_e can never be injured after stage $s+1$, it then easily follows by our remarks preceding the construction that $W_e \not\subseteq \langle \mathrm{At}(\tilde{M}) \rangle^{\mathrm{id}}$ so that R_e is met. $\square$

We pause to make a few remarks about some of the recursion theoretic properties of $\tilde{M}$. First of all we note that even though $\tilde{M}$ is an infinite Rec. BA, it is impossible to find an infinite r.e. sequence of pairwise disjoint elements in $\tilde{M}$ or an infinite r.e. increasing sequence in $\tilde{M}$ since both such sequences would have to be entirely made up of elements of $\langle \mathrm{At}(\tilde{M})\rangle^{\mathrm{id}}$ and hence would violate the immunity of $\langle \mathrm{At}(\tilde{M})\rangle^{\mathrm{id}}$. Similarly, there are no infinite r.e. decreasing sequences in $\tilde{M}$ since all such elements must be in $\tilde{M} - \langle \mathrm{At}(\tilde{M})\rangle^{\mathrm{id}}$ so that the set of complements of such a sequence would be an infinte r.e. sequence in $\langle \mathrm{At}(\tilde{M})\rangle^{\mathrm{id}}$. Next observe that since the only nonprincipal ideals of the BA of finite and cofinite subsets of N are entirely contained in the ideal generated by the atoms, it follows that the only r.e. ideals of $\tilde{M}$ are principal. Thus, if we consider $\tilde{M}$ as a Boolean ring, then every r.e. ideal is finitely generated. Recall a ring R is *Noetherian* if every ideal of R is finitely generated. We say that a ring R is *recursive* if R is a recursive set and the operators $\cdot_R$ and $+_R$ are partial recursive. We say a recursive ring R is *recursively Noetherian* if every r.e. ideal is finitely generated. Thus, by our remarks above, we have the following.

4.2. COROLLARY. *There exists a recursive ring R which is recursively Noetherian but not Noetherian.*

Next we shall modify our construction of $\tilde{M}$ to prove the following result which shows that there exist recursive α-superatomic BAs at the opposite end of scale from super-recursive α-superatomic BAs with respect to the recursion theoretic properties of its sequence of Frechet ideals.

4.3. THEOREM (REMMEL [1986a]). *For each recursive ordinal α, there exists a recursive α-superatomic BA M such that for all ordinals B with $1 \le \beta \le \alpha$, $F_\beta(M^\alpha)$ is not r.e. and $F_1(M^\alpha) = \langle \mathrm{At}(M^\alpha)\rangle^{\mathrm{id}}$ is immune.*

PROOF. We shall only give an outline of the proof. Our idea is to produce for each $y \in D_{\mathcal{O}}$, a Rec. BA $M(y)$ such that if $\nu_{\mathcal{O}}(y) = \alpha$, the $M(y)$ has the properties of M^α listed above. Now if $y = 2^x$, then we can assume by induction that $M(x)$ exists and we let $D_0, D_1, \ldots$ be an r.e. sequence of Rec. BAs all of which are recursively isomorphic to $M(x)$. If $y = 3 \cdot 5^z$, then we can assume by induction that we have constructed a Rec. BA $M(\varphi_z(j))$ for each $j \in \omega$ with the desired properties and we let $D_0, D_1, \ldots$ be an r.e. sequence of Rec. BAs such that D_j is recursively isomorphic to $M(\varphi_z(j))$ for all j. This given, let us follow the same outline as in the proof of Theorem 4.1 by first describing how we construct a recursive subalgebra of $\tilde{Q}$ isomorphic to an α-superatomic BA. That is, at each stage s, we specify a finite sequence $b_0^s, \ldots, b_s^s$ of pairwise disjoint elements whose union is $1_{\tilde{Q}}$. Moreover we assume that for each D_k, we have an r.e. generating sequence $d_{k,0}, d_{k,1}, \ldots$ for D_k and that $c_{k,0}, \ldots, c_{k,s+1}$ is a list of the atoms of $D_k^s = \langle\{d_{k,0}, \ldots, d_{k,s}\}\rangle$. The idea is then to build a recursive embedding of D_k into $\tilde{Q} \mid b_k^s$ for all k as we go along. Thus, at each stage s and for each $k < s$, assume that we have defined nonzero pairwise disjoint elements $a_{k,0}^a, \ldots, a_{k,s}^s$ whose union is b_k^s and a map f_k^s from D_k^s into $\tilde{Q} \mid b_k^s$ such that $f_k^s(c_{k,i}) = a_{k,i}^s$. Then at stage $s+1$, for each $k < s$ we can extend f_k^s to f_k^{s+1} so that if $c_{k,j}$ is the unique atom of D_k^s which is nontrivially split by $d_{k,s+1}$, then we split $a_{k,j}^s$ into two nonzero elements x and y and define $f_k^{s+1}(d_{k,s+1}) = x$, $f_k^{s+1}(c_{k,j} - d_{k,s+1}) = y$, and $f_k^{s+1}(c_{k,i}) = a_{k,i}^s$ if $i \neq j$. Next we split b_s^s into nonzero elements b_s^{s+1} and b_{s+1}^{s+1} and further split b_s^{s+1} into $s+2$ nonzero elements $a_{a,0}^{s+1}, \ldots, a_{s,s+1}^{s+1}$ and define $f_s^{s+1}\colon D_s^{s+1} \to \tilde{Q} \mid b_s^{s+1}$ by defining $f_s^{s+1}(c_{s,i}) = a_{s,i}^{s+1}$ for $i \leq s+1$. Then the only thing we have to do is to ensure that if we define $B_s = \langle \{b_i^s, a_{i,j}^s \mid i < s, j \leq s\} \rangle$ for all s, then $B_s \cap \{0, \ldots, s\} = B_{s+1} \cap \{0, \ldots, s\}$. But this last condition is easily ensured by repeated use of Lemma 2.2. It will then follow that if $B = \bigcup_s B_s$, then B will be a recursive BA and for each i, $b_i = \lim_s b_i^s$ exists and $B \mid b_i \approx D_i$ so that B will be α-superatomic.

As in the construction of Theorem 4.1, the basic modification of the above construction needed to produce a recursive α-superatomic BA with the desired properties is to no longer insist that b_s^s is the unique element in the sequence $b_0^s, \ldots, b_s^s$ which is not in $F_\alpha(B)$. That is, we have two sets of requirements to meet in this case.

R_e: $F_\alpha(M(y))$ is not r.e.

S_e: If W_e is infinite, then $W_e \not\subseteq \langle \mathrm{At}(M(y)) \rangle^{\mathrm{id}}$.

We rank the requirements in decreasing order of priority as $R_0, S_0, R_1, S_1, \ldots$. The strategy to meet a single requirement R_e is as follows. Assume that there is a stage s_0 large enough so that we are no longer required to act for any requirement of higher priority, i.e. the requirements $R_0, S_0, \ldots, R_{e-1}, S_{e-1}$, and that these requirements impose some restraint $r(e-1)$ which is to mean that we are required to have $b_i^{s+1} = b_i^s$ for all $i \leq r(e-1)$ and $s \geq s_0$. Then much as in the construction of Theorem 4.1, we wait until we find at stage $t > s_0$ such that there is an $x \in W_e^t$ such that either $x \not\in B_t$ or $x \in B_t$ and $x \cdot b_j^t \neq 0_{\tilde{Q}}$ for some j with $r(e-1) < j < s$. If $x \not\in B_t$, then we can arrange that $x \not\in B_s$ for all $s \geq t$ as in Theorem 4.1. Otherwise assume $x \in B_t$; such an x is bound to exist if it were the case that $W_e = F_\alpha(M(y))$. Then at stage $t+1$, we take the least element of the form $a_{j,i}^t \leq_{\tilde{Q}} x \cdot b_j^t$ and split it into two nonzero elements b_t^{t+1} and b_{t+1}^{t+1}. Then for $k \not\in \{j, t\}$, we define $b_k^{t+1} = b_k^t$ and extend the embeddings $f_k^t\colon D_k^t \to \tilde{Q} \mid b_k^t$ to embeddings $f_k^{t+1}\colon D_k^{t+1} \to \tilde{Q} \mid b_k^{t+1}$ as described above. We next define $b_j^{t+1} = (b_j^t - a_{j,i}^t) + b_t^t$ so that b_j^{t+1} is still a union of $t+1$ atoms in B_t and then we will have to define a new embedding ${}^*f_j^t\colon D_j^t \to b_j^{t+1}$ by letting ${}^*f_j^t(c_{j,l}) = a_{j,l}^t$ if $l \neq i$ and ${}^*f_j^t(c_{j,i}) = b_t^t$. We can then extend ${}^*f_j^t$ to an embedding $f_j^{t+1}\colon D_j^{t+1} \to \tilde{Q} \mid b_j^{t+1}$ in the usual manner. Finally, we impose restraint $r(e) = t$ for R_e which will ensure that $b_k^{s+1} = b_k^s$ for all $k \leq t$ and $s > t+1$. Then it will be the case that as long as the restraint for R_e is not injured after $t+1$, b_{t+1}^{t+1} will not be in $F_\alpha(M(y))$. But then $b_{t+1}^{t+1} \leq_{\tilde{Q}} a_{j,i}^t \leq_{\tilde{Q}} x \in W_e$ so that x will witness that $W_e \neq F_\alpha(M(y))$.

After requirement R_e no longer imposes any further restraint the strategy to meet requirement S_e is essentially the same. The point is that we assume by induction that $\langle \mathrm{At}(D_k) \rangle^{\mathrm{id}}$ are immune for each k. Since we are constructing a recursive embedding $f_k = \bigcup_{s>t} f_k^s$ from D_k into $\tilde{Q} \mid b_k^{t+1}$ for all $k \leq r(e)$, it will follow that $\langle \mathrm{At}(M(y)) \rangle^{\mathrm{id}} \cap \tilde{Q} \mid b_k^{t+1}$ is immune for each $k \leq r(e)$. Thus, if W_e is an infinite r.e. set contained in $\langle \mathrm{At}(M(y))\rangle^{\mathrm{id}}$, then $W_e \cap \tilde{Q} \mid b_k^{t+1}$ must be finite for each $k \le r(e)$. Hence, eventually, we would be able to find an $s > t+1$ and a $j > r(e)$ such that $a_{j,i}^s \in W_e^s$ for some i. But then at stage $s+1$ we can proceed exactly as we did for requirement R_e and ensure that $a_{j,i}^s \not\in F_\alpha(M(y))$ assuming no further injuries occur after stage $s+1$ to the restraint imposed by S_e at stage $s+1$.

It should be clear at this point that the strategies for meeting the present requirements are so similar to the strategies used in Theorem 4.1 that one can put them together via the *a* finite injury priority argument to construct $M(y) = \bigcup_s B_s$ and meet all the requirements. Moreover, it should be clear that for each i, $\lim b_i^s = b_i$ exists and $M(y) \mid b_i$ is recursively isomorphic to D_i. It then follows that $F_\beta(M(y))$ is not r.e. for any $1 < \beta < \alpha$ since otherwise $F_\beta(M(y)) \cap M(y) \mid b_i = F_\beta(M(y) \mid b_i)$ would be r.e. which would violate the fact that $F_\beta(D_k)$ is assumed not to be r.e. for all sufficiently large k. Of course, meeting all the requirements R_e ensures $F_\alpha(M(y))$ is not r.e. and meeting all the requirements S_e ensures $F_1(M(y))$ is immune so that $M(y)$ will have the properties required by the theorem. □

We saw in Theorems 4.1 and 4.3 how to use finite injury priority arguments to construct BAs with various recursion theoretic properties. There are a number of general recursive model theoretic theorems which can be applied to get the same type of results. The first result of this type is due to NURTAZIN [1974] and for other results of this type we refer the reader to ASH and NERODE [1981], ASH [1986], DZGOEV and GONCHAROV [1980], GONCHAROV [1975b], [1977], [1980], HIRD [1984], and MOSES [1983]. We shall present two results of ASH and NERODE [1981] which will give the flavor of such results and which are most suitable for our purposes.

4.4. DEFINITION. Let A be a model for a first-order language L whose universe is a recursive set. Then we say

(i) A is a *decidable* model if the satisfaction predicate for A is recursive.

(ii) A is a *recursive* model if the satisfaction predicate for A restricted to atomic formulas is recursive.

(iii) A is a *1-recursive* model if the satisfaction predicate for A restricted to existential formulas is recursive.

4.5. DEFINITION. Let A be a recursive model over a first-order language L and let R be a recursive relation on the universe of A. (Here R is not generally a relation of L, e.g. R may be the relation "x is an atom" for a recursive BA.) Then we say

(i) R is *intrinsically r.e.* if for every recursive model B and map f: $B \to A$ such that f is an isomorphism from B to A (over L), $f^{-1}(R)$ is r.e.

(ii) R is *intrinsically recursive* if for every recursive model B and map f: $B \to A$ as above, $f^{-1}(R)$ is recursive.

(iii) $R(x_1, \ldots, x_n)$ is *formally r.e.* on A if for some $c_1, \ldots, c_k \in A$ and some r.e. sequence $\{\phi_n\}$ of existential formulas, the following equivalence holds in A:

$$R(x_1, \ldots, x_n) \leftrightarrow \bigvee_n \phi_n(x_1, \ldots, x_n, c_1, \ldots, c_k).$$

(iv) R is *formally recursive* on A iff both R and its complement are formally r.e. on A.

It is easy to see that if a relation R is formally r.e. (recursive) on a recursive model A, then R is intrinsically r.e. (recursive) for A. The main point of ASH and NERODE [1981] is to show that under certain circumstances we have a converse of this result.

4.6. DEFINITION. Give a recursive model A over a first-order language A and a recursive relation R on the universe of A we say that A is *1-recursive with respect to R* if there is a recursive procedure for determining, given an existential formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $a_1, \ldots, a_m$ in A, whether the implication

$$\phi(x_1, \ldots, x_n, a_1, \ldots, a_m) \rightarrow R(x_1, \ldots, x_n)$$

holds in A.

Note that if A is 1-recursive with respect to R, then A is certainly 1-recursive and R is a recursive relation on the universe of A. We then have the following results.

4.7. THEOREM (ASH and NERODE [1981]). *Let $\{R_n\}_{n \in N}$ be family of relations on the universe of recursive model A such that A is 1-recursive with respect to R_n uniformly in n. Then there exists a recursive model B and an isomorphism f: $B \rightarrow A$ such that for each n, $f^{-1}(R_n)$ is not* r.e. *iff R_n is not formally* r.e.

4.7a. COROLLARY (ASH and NERODE [1981]). *Let R be a relation on the universe of a recursive model such that A is 1-recursive with respect to both R and its complement, then R is intrinsically recursive iff R is formally recursive on A.*

4.8. THEOREM (ASH and NERODE [1981]). *Given the hypothesis of Theorem* 4.7, *there exists a recursive model B and an isomorphism f: $B \rightarrow A$ such that, whenever R_n is a k-ary relation and S is a k-ary r.e. relation on B with $f(S) \subseteq R_n$, then there is a formally r.e. k-ary relation F on A such that $f(S) \subseteq F \subseteq R_n$.*

4.9. COROLLARY (ASH and NERODE [1981]). *Let R be a relation on the universe of a recursive model A such that A is 1-recursive with respect to R. Then there exists a recursive model B and an isomorphism f: $B \rightarrow A$ for which $f^{-1}(R)$ is isolated iff R has no infinite subrelation which is formally r.e.*

Here we say a set $I \subseteq N$ is *isolated* if I is either finite or immune and a relation $R(x_1, \ldots, x_n)$ on A is isolated if $\{\langle a_1, \ldots, a_n\rangle \mid A \models R(a_1, \ldots, a_n)\}$ is isolated.

In the case of Boolean algebras, it is quite easy to apply Theorems 4.7 and 4.8 and their corollaries. Suppose, for example, we want to prove Theorem 4.1. Then we let $A = \tilde{N}$ and R be the relation "$x \in \langle \mathrm{At}(\tilde{N})\rangle^{\mathrm{id}}$". Now suppose $\phi(x, x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a quantifier-free formula, $c_1, \ldots, c_m$ are in $\tilde{N}$, and

$$(*) \qquad \tilde{N} \models \exists x_1, \ldots, x_n \phi(x, x_1, \ldots, x_n, c_1, \ldots, c_m) \rightarrow R(x) .$$

Then consider the subalgebra of $\tilde{N}$, $C = \langle\{c_1, \ldots, c_m\}\rangle$, and let $a_0, \ldots, a_p$ be the atoms of C. Now suppose $N \models \phi(d_0, d_1, \ldots, d_n, c_1, \ldots, c_m)$, $D = \langle\{d_0, d_1, \ldots, d_n, c_1, \ldots, c_m\}\rangle$, and for each j, $a_{j,1}, \ldots, a_{j,l_j}$ is a list of the atoms of D under a_j for each $j \le p$. Then for each $k \le n$ there is a certain subset S_i of $\{\langle j, i\rangle \mid j \le p \;\&\; i \le l_k\} = S$ such that $d_k = \Sigma_{\langle j,i\rangle \in S_k} a_{j,i}$. Since ϕ is quantifier-free, it is easy to see that if $\{e_{i,j} \mid \langle i, j\rangle \in S\}$ is any sequence of pairwise disjoint nonzero elements such that $a_j = \Sigma_{i=1}^{l_j} e_{j,i}$ for all j and we define $e_k = \Sigma_{\langle j,i\rangle \in S_k} e_{j,i}$ for $k \le n$, then $\tilde{N} \models \phi(e_0, e_1, \ldots, e_n, c_1, \ldots, c_m)$. Because we are in $\tilde{N}$ there is exactly one atom of C, say a_p, which has infinitely many atoms under it. Thus, if d_0 is such that $d_0 \cdot a_p \neq 0_{\tilde{Q}}$, then it is clear that we can arrange our choice of the $e_{j,i}$ so that $e_0 \cdot a_p$ has infinitely many atoms under it. It follows that $(*)$ can hold only if the formula ϕ asserts $d_0 \cdot a_p = 0_{\tilde{Q}}$ or $\tilde{N} \not\models \exists x_1 \cdots x_n \phi(x, x_1, \ldots, x_n)$. Thus, since $\tilde{N}$ is decidable and $\langle \mathrm{At}(\tilde{N})\rangle^{\mathrm{id}}$ is recursive, $\tilde{N}$ is 1-recursive with respect to R. Moreover, our argument shows that for fixed $c_1, \ldots, c_m$, the only elements of $\langle \mathrm{At}(\tilde{N})\rangle^{\mathrm{id}}$ for which $(*)$ can hold with "$\to$" replaced by "$\leftrightarrow$" for any ϕ are those elements of $\tilde{N} \mid -a_p$ and hence R contains no infinite formally r.e. subsets. Thus, we can apply Corollary 4.9 to prove Theorem 4.1. (Of course one can also view Theorem 4.1 as proving in light of Corollary 4.9 that $\langle \mathrm{At}(\tilde{N})\rangle^{\mathrm{id}}$ has no infinite formally r.e. subsets.)

Clearly the above argument works for any Dec. BA B with infinitely many atoms such that $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ is recursive. Hence, we have the following.

4.10. THEOREM. *Let B be a Dec. BA such that $|\mathrm{At}(B)| = \omega$ and $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ is recursive, then there exists a Rec. BA D isomorphic to D such that $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ is immune.*

Note that the Ash–Nerode type theorems transfer the work required to produce Rec. BAs with certain noneffective properties from techniques involving priority arguments to model theoretic or definability arguments. Of course, it should be noted that the Ash–Nerode theorems themselves are proved via finite injury priority arguments. However, there is a major problem with these theorems with respect to the theory of Rec. BAs. That is, to produce such counterexamples we must start out with a Rec. BA which at the very least is 1-decidable. But note that if B is an atomic BA and B is 1-decidable, then B is in fact a Dec. BA. That is, the predicate "$x \not\in \mathrm{At}(B)$" is clearly defined by an existential formula so that $\mathrm{At}(B)$ must be recursive in a 1-decidable BA, and hence by our remarks in Theorem 2.6, B is decidable. Thus, if we want to use Theorem 4.10 to show that for any infinite atomic Rec. BA B, there is a Rec. BA D isomorphic to B such that $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ is immune, then we have to prove that every atomic Rec. BA is isomorphic to a Dec. BA, where the ideal generated by the atoms is recursive. Unfortunately, it is just not true that every atomic Rec. BA is isomorphic to such a Dec. BA as our final results of this section will show.

We now turn to the problem of showing that there exist classical isomorphism types in which every BA lacks certain effective properties. This problem is in some sense harder than the type of results seen previously in this section since we can no longer employ some effective diagonalization type argument but must code information directly into the isomorphism type. The basic techniques to achieve such codings are due to FEINER [1970a] and have later been modified by GONCHAROV [1975a] and REMMEL [1981b] to produce the results mentioned in the Introduction.

Our first step is to define a certain recursion theoretic hierarchy for functions recursive in the ωth jump of the recursive sets. That is, let $X \subseteq N$. Then let $X' = \{e \mid \varphi_e^X(e)\downarrow\}$ denote the jump of X. We then define the nth jump of X (written $X^{(n)}$) by induction on n as $X^{(0)} = X$ and $X^{(n+1)} = (X^{(n)})'$. Finally, we define the ωth jump of X, $X^{(\omega)}$, as $\{\langle m, n\rangle \mid m \in X^{(n)}\}$. Following FEINER [1970a], we define a hierarchy for functions and sets recursive in $\phi^{(\omega)}$.

4.11. DEFINITION (of Feiner hierarchy). (i) Let e, a, and b be elements of N. We say *e is of type* (a, b) (written $e \sim (a, b)$) if (a) $\varphi_e^{\phi^{(\omega)}}$ is total, and (b) $(\forall n)$ $(\forall m > a + nb)\forall q$.

(The question "Is $\langle q, m\rangle \in \phi^{(\omega)}$?" is never asked of the oracle during the computation $\varphi_e^{\phi^{(\omega)}}(n)$.)

(ii) If $X \subseteq N$ and there is an $e \in N$ such that the characteristic function of $X = \varphi_e^{\phi^{(\omega)}}$ and $e \sim (a, b)$, then we say *X is of the type* (a, b) (written $X \sim (a, b)$). (*Note*: We write $e \not\sim (a, b)$ and $X \not\sim (a, b)$ for when e and X is not of type (a, b).)

The following basic facts about this hierarchy are found in FEINER [1970a]. It is clear from the definition that for any $X \subseteq N$, $X \sim (a, b)$ if and only if $N - X \sim (a, b)$.

4.12. PROPOSITION. *If f is a recursive function and* $X = \{n \mid \exists z_1 \forall z_2 \exists z_2 \cdots (f(\langle z_1, \ldots, z_{a+nb}\rangle) = 1\}$, *then* $X \sim (a, b)$.

Let $A_n = \{\langle x, m\rangle \mid x \in \phi^{(m)}\ \&\ m \leq n\}$. Thus, $A_n \subseteq \phi^{(\omega)}$ and if $e \sim (a, b)$, then by definition $\varphi_e^{A_{a+nb}}(n) = \varphi_e^{\phi^{(\omega)}}(n)$ for all n. Let $X(a, b) = \{n \mid n$ is even and $\phi_{n/2}^{A_{a+nb}}(n) = 0\}$.

4.13. PROPOSITION. *If $Y \subseteq N$ and $\{n \mid n$ is even & $n \in Y\} = X(a, b)$, then $Y \not\sim (a, b)$.*

4.14. PROPOSITION. *For all a, $b \in N$, there is a recursive function $\rho_{(a,b)}$: $N \to N$ such that $\forall n(n \in X(a, b) \leftrightarrow \exists z_1 \forall z_2 \cdots (\rho_{(a,b)}(\langle z_1, \ldots, z_{a+nb+1}\rangle) = 1)$.*

Next we state a technical lemma which will be useful for a coding argument to follow. Let $\exists^{\aleph_0} z$ be the quantifier which asserts that there exist infinitely many z. Using the Kreisel, Shoenfield, and Wang Theorem (ROGERS [1967, p. 329]) that every Π_{2n+1}^0 predicate is equivalent to a $U^{(n)}\forall$ predicate, where $U = \exists^{\aleph_0} z$ and

$$U^{(n)} = \underbrace{U \cdots U}_{n \text{ times}}$$

and Proposition 4.14, it is easy to prove the following.

4.15. LEMMA. *For every $k \geq 1$, there is a recursive function r_k: $N \to N$ such that for all m,*

$$2m \not\in X(2k, 2) \leftrightarrow \forall i \exists h \exists^{\aleph_0} z_1 \cdots \exists^{\aleph_0} z_{2m+k-1} \forall z_{2m+k} \times (r_k(\langle i, h, z_1, \ldots, z_{2m+k}\rangle) = 1) .$$

Next in order to relate the Feiner hierarchy to the structure of BAs, we introduce the following predicates for a given BA B and n, $k \in N$:

(1) $\alpha_n(x) \leftrightarrow x \in F_n(B)$.
(2) $\lambda_n(x) \leftrightarrow x \not\in F_n(B)$.
(3) $\text{Atomistic}_n(x) \leftrightarrow x/F_n(B)$ is an atomic element of $B/F_n(B)$.
(4) $\gamma_n^k(x) \leftrightarrow$ (i) $x \not\in F_{n+k}(B)$, (ii) $y \leq_B x \to y/F_k(B)$ is a nonatomic element in $B/F_k(B)$ or $y \in F_{n+k}(B)$, and (iii) there exists infinitely many pairwise disjoint $z \leq_B x$ such that $z \in F_{n+k}(B) - F_{n+k-1}(B)$.
(5) $\Phi_n^k \leftrightarrow \exists x(\gamma_n^k(x))$.

The following examples should be instructive. First note that for any linear order L with a first element $B_L \approx B_{\omega^k \times L}/F_k(B_{\omega^k \times L})$ where in what follows we shall write B_L for $\text{Intalg}(L)$.

4.16. LEMMA. (a) $B_{\omega^n + \eta} \models \Phi_m^0$ *iff* $m = n$.
(b) $B_{\omega^k \times (\omega^n + \eta)} = B_{\omega^{k+n} + \omega^k \times \eta} \models \Phi_m^k$ *iff* $m = n$.
(c) *If* $X \subseteq N$, $B_{\sum_{m \in X} (\omega^{k+m} + \omega^k \times \eta)} \models \Phi_n^k$ *iff* $n \in X$.

PROOF. For (a) let $B = B_{\omega^n + \eta}$ and consider $x = 1_B$. Now $B/F_n(B) = B_{1+\eta}$ so that $x \not\in F_n(B)$. Moreover, it is clear that if $z \leq_B x$, either $z/F_n(B)$ is nonatomic or $z/F_n(B)$ equal zero in $B/F_n(B)$ since $B/F_n(B)$ is atomless. Thus, conditions (i) and (ii) hold of $\gamma_n^0(x)$. Note that $\omega^n = \omega^{n-1} \times \omega$ so that the ω-copies of ω^{n-1} provide the witnesses that condition (iii) of $\gamma_n^0(x)$ holds. Thus, $B \models \Phi_n^0$. It is clear that if $m > n$, $B \not\models \Phi_m^0$ because there is no $z \in B$ such that $z \in F_m(B) - F_{m-1}(B)$. So assume $m < n$ and $B \models \gamma_m^0(y)$ for some $y \in B$. Now y cannot contain an interval which includes a cofinal sequence of the ω^n part of the order because then there is an interval z of type ω^m under y. But then condition (ii) of $\gamma_m^0(y)$ is violated since $z/F_m(B)$ is an atomic element of $B/F_m(B)$ and $z \not\in F_m(B)$. It follows that y is contained in finite union of intervals whose order type $\Sigma\, \omega^{n-1} + 1 + \eta$, where $\Sigma\, \omega^{n-1}$ is some finite sum of copies of ω^{n-1}. However, the only way there can be infinitely many pairwise disjoint $z \in F_m(B) - F_{m-1}(B)$ under y if y contains an interval of type ω^m, which is impossible. Thus, there is no such y and $B \not\models \Phi_m^0$.

(b) Follows by a similar argument and (c) easily follows from (b). □

Our next lemma is easily verified.

4.17. LEMMA. *If B and D are BAs such that $B \approx D$, then*
(a) $B \models \Phi_n^k$ *iff* $D \models \Phi_n^k$ *for all n and k,*
(b) $\alpha_0(x) \leftrightarrow x = 0_B$

$$\alpha_{n+1}(x) \leftrightarrow (\exists m \in N)(\exists x_1 \cdots x_m)\Big[x = \bigvee_{i=1}^{m} x_i \ \& \ \underset{i \neq j}{\overset{m}{\&}} (x_i \wedge x_j = 0_B) \ \&$$
$$\underset{i=1}{\overset{m}{\&}} \forall y(y \leq_B x_i \rightarrow \alpha_n(y) \vee \alpha_n(x_i - y))\Big],$$

(c) $\lambda_n(x) \leftrightarrow \neg \alpha_n(x)$,

(d) $\text{Atomistic}_n(x) \leftrightarrow \forall y[y \leq_B x \rightarrow \alpha_n(y) \vee \exists z(z \leq_B y \ \& \ \forall a(a \leq_B z \rightarrow \alpha_n(a) \vee \alpha_n(z-a)) \ \& \ \neg \alpha_n(z))]$, and

(e) $\gamma_n^k(x) \leftrightarrow \lambda_{n+k}(x) \ \& \ \forall y(y \leq_B x \rightarrow \neg \text{Atomistic}_k(y) \vee \alpha_{n+k}(y)) \ \& \ (\forall m \in N)(\exists x_1 \cdots x_m)[\&_{i \neq j} (x_i \wedge x_j = 0_B) \ \& \ x_i \leq_B x \ \& \ \alpha_{n+k}(x_i) \ \& \ \lambda_{n+k-1}(x_i)]$.

Using Lemma 4.17, we can now relate those predicates to the Feiner hierarchy.

4.18. PROPOSITION. *Let B be a BA and $Z_k(B) = \{n \mid B \models \Phi_n^k\}$.*

(a) *If B is a Rec. BA such that* $\text{At}(B)$ *is recursive, then $Z_k(B) \sim (2k+1, 2)$ for $k \geq 0$.*

(b) *If B is a Rec. BA such that* $\langle \text{At}(B) \rangle^{\text{id}}$ *is recursive, then $Z_k(B) \sim (2k, 2)$.*

PROOF. Note that if B is a Rec. BA and $\text{At}(B)$ is recursive, then $\langle \text{At}(B) \rangle^{\text{id}}$ is r.e. so that the predicate $\alpha_1(x)$ is Σ_1^0. It easily follows by induction and the forms of the definitions in Lemma 4.17 that $\alpha_{n+1}(x)$ is Σ_{2n+1}^0, $\lambda_{n+1}(x)$ is Π_{2n+1}^0, $\text{Atomistic}_n(x)$ is Π_{2n+2}^0, and $\gamma_n^k(x)$ is Π_{2n+2k}^0. Thus, Φ_n^k is $\Sigma_{2n+2k+1}^0$ and hence is recursive in $A_{2n+2k+1}$. It easily follows that $Z_k(B) \sim (2k+1, 2)$.

For (b), note that if $\langle \text{At}(B) \rangle^{\text{id}}$ is recursive so that $\alpha_1(x)$ is recursive, then Φ_n^k will be Σ_{2k+2n}^0 predicate and hence $Z_k(B) \sim (2k, 2)$. □

As mentioned in the Introduction, the basic machinery above was developed by FEINER [1970a] to prove the existence of an R.e. BA which is not isomorphic to any Rec. BA. GONCHAROV [1975a] later modified Feiner's technique to produce examples of n-atomic but not $n+1$-atomic Rec. BAs C which are not isomorphic to any Rec. BA D, where $\text{At}(D)$ is also recursive for all $n \geq 1$. Note such C are not isomorphic to any Dec. BAs. In what follows we shall show how this machinery can be used to produce for each $k \geq 1$ examples of k-atomic but not $k+1$-atomic Rec. BA B such that $\text{At}(B)$ is recursive but B is not isomorphic to any Rec. BA D such that $\langle \text{At}(D) \rangle^{\text{id}}$ is recursive. In light of Proposition 4.18, it is enough to produce such B so that $Z_k(B) \not\sim (2k, 2)$. This is precisely what we shall do. Thus, for each $k \geq 0$, we will construct such B of the form:

$$^B\sum_m \sum_i \sum_h \Delta(m, i, h) + 1 + \omega^k \times \eta,$$

where $\omega^{2m+k-1} \leq \Delta(m, i, h) \leq \omega^{2m+k}$. The main fact we need about such BAs has a proof similar to Lemma 4.16. A detailed proof can be found in GONCHAROV [1975a].

4.19. PROPOSITION. *Let CO denote the collection of countable ordinals and let Δ: $N^3 \rightarrow CO$ be such that $\omega^{2m+k-1} \leq \Delta(m, i, h) \leq \omega^{2m+k}$ for all (m, i, h). Then*

$$B_{\sum_m \sum_i \sum_h \Delta(m,i,h)+1+\omega^k \times \eta} \models \phi^k_{2m} \text{ iff } \exists i \neg \exists^{\aleph_0} h(\Delta(m,i,h) = \omega^{2m+k}).$$

4.20. THEOREM (REMMEL [1981b]). *For every $k \geq 1$, there exists a k-atomic but not $k+1$-atomic BA B_k such that* $\mathrm{At}(B_k)$ *is recursive but there is no Rec. BA D such that* $\langle \mathrm{At}(D) \rangle^{\mathrm{id}}$ *is recursive and D is isomorphic to B.*

PROOF. Fix $k \geq 1$. First we construct a recursive linear order $\mathscr{L}_k = (L_k, \leq_k)$. Let $(N, \subset_k)$ be a decidable linear order of type $1 + \omega^k \times \eta$. Let L_k be the set of all $(m, i, h, z_1, \ldots, z_{2m+k}, l, g)$ such that

(a) $l = 0 \rightarrow g = -1$ & $(z_1, \ldots, z_{2m+k}) \in N^{2m+k}$,

(b) $l = 1 \rightarrow g \in N$ & $(z_1, \ldots, z_{2m+k}) = (-1, \ldots, -1)$, and

(c) $(m, i, h) \in N^3$ and $l \in \{0, 1\}$. Let f: $N \rightarrow N$ be an infinite to one, onto recursive function. Then we say $(m, i, h, z_1, \ldots, z_{2m+k}, l, g) \leq_k (\hat{m}, \hat{\imath}, \hat{h}, \hat{z}_1, \ldots, \hat{z}_{2m+k}, \hat{l}, \hat{g})$ iff

(A) $(m, i, h, l) <_{\mathrm{lex}} (\hat{m}, \hat{\imath}, \hat{h}, \hat{l})$ or

(B) $(m, i, h, l) = (\hat{m}, \hat{\imath}, \hat{h}, \hat{l})$ & $l = 1$ & $g \subset_k \hat{g}$ or

(C) $(m, i, h, l) = (\hat{m}, \hat{\imath}, \hat{h}, \hat{l})$ & $l = 0$ & either

(i) $(z_1, \ldots, z_{2m+k}) \leq_{\mathrm{lex}} (\hat{z}_1, \ldots, \hat{z}_{2m+k})$ or

(ii) $\displaystyle\mathop{\&}_{i=1}^{2m+k-1} z_i = \hat{z}_i$ & $\hat{z}_{2m+k} < z_{2m+k}$ &

$\exists n \leq \hat{z}_{2m+k}(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k-1}, n\rangle \neq 1)$

(where recall r_k is the recursive function of Lemma 4.15).

It is clear that $\mathscr{L}_k$ is a recursive linear order so that $B_k = B_{\mathscr{L}_k}$ is a recursive BA.

4.20.1. LEMMA. $\mathrm{At}(B)$ *is recursive.*

PROOF. The atoms of B arise from intervals $[a, b) = \{x \in L_k \mid a \leq_k x <_k b\}$, where b is an immediate successor of a (written $a \rightarrow b$). Thus, we need only show that we can effectively decide if $a \rightarrow b$ for any $a, b \in L_k$. Let $a = (m, i, h, z_1, \ldots, z_{2m+k}, l, g)$ and $b = (\hat{m}, \hat{\imath}, \hat{h}, \hat{z}_1, \ldots, \hat{z}_{2m+k}, \hat{l}, \hat{g})$. If $a \rightarrow b$, then clearly $(m, i, h) = (\hat{m}, \hat{\imath}, \hat{h})$. Now if $l = \hat{l} = 1$, then $a \rightarrow b$ iff $g \rightarrow \hat{g}$ in $(N, \subset_k)$. However, we can effectively decide if $g \rightarrow \hat{g}$ in $(N, \subset_k)$ since $(N, \subset_k)$ is decidable.

Next assume $l = 0$ and $\hat{l} = 1$. It is easy to see that if $k \geq 2$ or $m \geq 1$, then $a \not\rightarrow b$. So assume $m = 0$ and $k = 1$. If $\hat{g}$ is not the first element of $(N, \subset_k)$, then again $a \not\rightarrow b$. If $\hat{g}$ is the first element of $(N, \subset_k)$, then $a \rightarrow b$ iff $\exists n \leq z_{2m+k}(r_k(\langle i, f(h), n\rangle) \neq 1)$ so we can decide if $a \rightarrow b$ effectively.

Finally, we are reduced to the case where $l = \hat{l} = 0$. Again it is easy to see that if $a \rightarrow b$, then $z_i = \hat{z}_i$ for $1 \leq i \leq 2m + k - 2$. We consider two cases.

Case 1. $z_{2m+k-1} = \hat{z}_{2m+k-1}$ or $m = 0$ & $k = 1$. By clause (C) of our definition of $\leq_k$, $a \rightarrow b$ iff $\neg \exists n \leq z_{2m+k}(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k-1}, h\rangle) \neq 1)$ and either $\hat{z}_{2m+k} = z_{2m+k} + 1$ or $z_{2m+k} + 1 < \hat{z}_{2m+k}$ and $r_k(\langle i, f(h), z_1, \ldots, z_{2m+k-1}, z_{2m+k} + 1\rangle) \neq 1$.

Case 2. $z_{2m+k-1} \neq \hat{z}_{2m+k-1}$. In this case $a \rightarrow b$ iff $z_{2m+k-1} + 1 = \hat{z}_{2m+k-1}$, $a =_k (m, i, h, z_1, \ldots, z_{2m+k-1}, z_{2m+k} + 1, l, g)$, and $b =_k (m, i, h,$

$z_1, \ldots, z_{2m+k-1}+1, 0, l, g)$. Thus we can effectively decide if $a \to b$ in both Cases 1 and 2. $\square$

We let $\Delta(m, i, h) = (\{(m, i, h, z_1, \ldots, z_{2m+k}, 0, -1) \mid z_i \in N\}, \leq_k)$. Thus,

$$\mathscr{L}_k = \sum_m \sum_i \sum_h \Delta(m, i, h) + 1 + \omega^k \times \eta .$$

4.20.2. LEMMA. (a) $\omega^{2m+k-1} \leq \Delta(m, i, h) \leq \omega^{2m+k}$ *for all* m, i, h.

(b) $(\forall i, \hat{\imath}, h, \hat{h})(f(h) = f(\hat{h}) \to \Delta(m, i, h) = \Delta(m, \hat{\imath}, \hat{h}))$.

(c) $\Delta(m, i, h) = \omega^{2m+k} \leftrightarrow \exists^{\aleph_0} z_1 \cdots \exists^{\aleph_0} z_{2m+k-1} \forall z_{2m+k}(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k}\rangle) = 1)$.

PROOF. (a) and (b) are obvious by construction. For (c), let m, i, and h be fixed. First we must define a sequence of linear orders by induction:

$$\mu_1(z_1, \ldots, z_{2m+k-1}) = (\{(m, i, h, z_1, \ldots, z_{2m+k-1}, z, 0, -1) \mid z \in N\}, \leq_k) .$$

$$\mu_r(z_1, \ldots, z_{2m+k-r}) = \sum_{z \in N} \mu_{r-1}(z_1, \ldots, z_{2m+k-r}, z) \quad \text{for } 2 \leq r \leq 2m+k .$$

Similarly, we define a sequence of predicates P_r for $1 \leq r \leq 2m+k$.

$$P_1(z_1, \ldots, z_{2m+k-1}) \leftrightarrow \forall z_{2m+k}(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k}\rangle) = 1) .$$

$$P_r(z_1, \ldots, z_{2m+k-r}) \leftrightarrow \exists^{\aleph_0} z_{2m+k-r+1} \cdots \exists^{\aleph_0} z_{2m+k-1} \forall z_{2m+k} \times (r_k(\langle i, f(h), z_1, \ldots, z_{2m+k}\rangle) = 1) .$$

SUBLEMMA. *For all* $r \geq 1$ *and all* $z_1, \ldots, z_{2m+k-r}$,

$$P_r(z_1, \ldots, z_{2m+k-r}) \to u_r(z_1, \ldots, z_{2m+k-r}) = \omega^r$$

and

$$\neg P_r(z_1, \ldots, z_{2m+k-r}) \to u_r(z_1, \ldots, z_{2m+k-r}) < \omega^r .$$

PROOF. We proceed by induction on r. For $r = 1$, our definition of $\leq_k$ ensures $\mu(z_1, \ldots, z_{2m+k-1}) = \omega$ if $P_1(z_1, \ldots, z_{2m+k-1})$ holds and $\mu_r(z_1, \ldots, z_{2m+k-1})$ if finite if $\neg P_1(z_1, \ldots, z_{2m+k-1})$ holds. In fact, in the latter case, $\mu(z_1, \ldots, z_{2m+k-1})$ is of length $n+1$ where $n = \mu j(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k-1}, j\rangle) \neq 1)$.

Next assume $r > 1$ and the lemma holds for $r-1$. Since $\mu_r(z_1, \ldots, z_{2m+k-r}) = \Sigma_z\, \mu_{r-1}(z_1, \ldots, z_{2m+k-r}, z)$, where by induction for each $z \in N$, $\omega^{r-2} \leq u_{r-1}(z_1, \ldots, z_{2m+k-r}, z) \leq \omega^{r-1}$, it follows that $\mu_r(z_1, \ldots, z_{2m+k-r}) = \omega^r$ only if $\exists^{\aleph_0} z(\mu_{r-1}(z_1, \ldots, z_{2m+k-r}, z) = \omega^{r-1})$. But by induction, $\exists^{\aleph_0} z(\mu_{r-1}(z_1, \ldots, z_{2m+k-r}, z) = \omega^{r-1})$ iff $\exists^{\aleph_0} z \exists^{\aleph_0} z_{2m+k-r+1} \cdots \exists^{\aleph_0} z_{2m+k-1}$

$$\forall z_{2m+k}(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k-r}, z, \ldots, z_{2m+k}\rangle) = 1)\,.$$

The result now easily follows. $\square$

Note that $\mu_{2m+k} = \Delta(m, i, h)$ so that the sublemma asserts $\Delta(m, i, h) = \omega^{2m+k}$ $\leftrightarrow$ P_{2m+k} $\leftrightarrow$ $\exists^{\aleph_0} z_1 \cdots \exists^{\aleph_0} z_{2m+k-1} \forall z_{2m+k}(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k}\rangle) = 1)$. $\square$

4.20.3. LEMMA. *For all* $m \geq 0$, $B_k \models \Phi^k_{2m}$ *iff*

$$\neg\forall i \exists h \exists^{\aleph_0} z_1 \cdots \exists^{\aleph_0} z_{2m+k-1} \forall z_{2m+k}(r_k(\langle i, h, z_1, \ldots, z_{2m+k}\rangle) = 1)\,.$$

PROOF. Suppose

$$\forall i \exists h \exists^{\aleph_0} z_1 \cdots \exists^{\aleph_0} z_{2m+k-1} \forall z_{2m+k}(r_k(\langle i, h, z_1, \ldots, z_{2m+k}\rangle) = 1)\,.$$

For each i, let h_i be such that

$$\exists^{\aleph_0} z_1 \cdots \exists^{\aleph_0} z_{2m+1} \forall z_{2m+k}(r_k(\langle i, h_i, \ldots, z_{2m+k}\rangle) = 1)\,.$$

Then by Lemma 4.20.2 for all h such that $f(h) = h_i$, $\Delta(m, i, h) = \omega^{2m+k}$. Since f is infinite to one, it follows $\forall i \exists^{\aleph_0} h(\Delta(m, i, h) = \omega^{2m+k})$ and hence by Proposition 4.19, $B_k \not\models \Phi^k_{2m}$.

However, if

$$\exists i \forall j \neg \exists^{\aleph_0} z_1 \cdots \exists^{\aleph_0} z_{2m+k-1} \forall z_{2m+k}(r_k(\langle i, f(h), z_1, \ldots, z_{2m+k}\rangle) = 1)\,,$$

then by Lemma 4.20.2, 2, $\exists i \forall h(\Delta(m, i, h) < \omega^{2m+k})$. Thus, $\exists i \neg \exists^{\aleph_0} h(\Delta(m, i, h) = \omega^{2m+k})$ so that by Proposition 4.19, $B_k \models \Phi^k_{2m}$. $\square$

4.20.4. LEMMA. $Z_k(B_k) \not\sim (2k, 2)$.

PROOF. By Lemmas 4.15 and 4.20.3, it follows that for all m, $2m \not\in Z_k(B_k)$ iff $\forall i \exists h \exists^{\aleph_0} z_{2m+k-1} {}^{\aleph_0} z_{2m+k}(r_k(\langle i, h, z_1, \ldots, z_{2m+k}\rangle) = 1)$ iff $2m \not\in X(2k, 2)$. Thus, by Proposition 4.13, $N - Z_k(B) \not\sim (2k, 2)$ and hence $Z_k(B) \not\sim (2k, 2)$. $\square$

As a corollary of Theorem 4.20, we get the following result which is a slight strengthening of FEINER's [1970a] result.

4.21. COROLLARY (REMMEL [1981b]). *For every* $n \geq 1$, *there is an* n*-atomic but not* $n+1$*-atomic R.e. BA* D_n *such that there is no Rec. BA isomorphic to* D_n.

PROOF. Fix n and let $D_n = B_k / \langle \mathrm{At}(B_k)\rangle^{\mathrm{id}}$, where $k = n + 1$ and B_k is the Rec. BA constructed in Theorem 4.20. Since $\mathrm{At}(B_k)$ is recursive, $\langle \mathrm{At}(B_k)\rangle^{\mathrm{id}}$ is r.e. and hence D_n is an R.e. BA. However, if D_n were isomorphic to a Rec. BA D, then we can apply the construction of Theorem 2.6 to produce an atomic Dec. BA B such that $B / \langle \mathrm{At}(B)\rangle^{\mathrm{id}} \approx D$. One can observe that our construction of Theorem 2.6 in fact shows that $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ is also recursive. But then we can apply Vaught's

Theorem (Theorem 1.3) and conclude that $B_k \approx B$ which violates the fact that B_k is not isomorphic to any Rec. BA B, where $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ is recursive. □

We should note that Corollary 4.21 has important implications for recursion theory. That is, the lattice of r.e. sets $\mathscr{E}$ and the lattice of r.e. sets modulo finite sets $\mathscr{E}^*$ have been studied extensively in modern recursion theory (see SOARE [1978] for a survey). A fundamental result due to LACHLAN [1968] states that a BA B is realized as the lattice of all r.e. supersets modulo finite sets for some r.e. set $A \subset N$ if and only if B is an $\exists - \forall - \exists$ BA. All we need to know about $\exists - \forall - \exists$ BAs is that they are a certain restricted class of Arith. BAs which include all R.e. BAs, see LACHLAN [1968] for a formal definition. FEINER [1970] combined Lachlan's result with his somewhat simpler version of Corollary 4.21 to prove that it is impossible that either $\mathscr{E}$ or $\mathscr{E}^*$ are isomorphic to recursive lattice lest every R.e. BA be isomorphic to a Rec. BA. In a similar vein, Lachlan's result was combined with Theorem 3.9 to give some important results in isol theory, see MANASTER and REMMEL [1980].

We should note that Theorem 4.20 can be combined with the results of Section 3 to produce a number of results concerning the possibilities for the recursion theoretic properties of the sequence $\{F_\gamma(B) \mid \gamma \in CO\}$ for Rec. BA. For example, Theorem 4.20 is easily combined with Theorem 3.9 to produce our next theorem (see REMMEL [1981b] for further results of this type).

4.22. THEOREM (REMMEL [1981b]). *Let α be any recursive ordinal. There is an α-atomic Rec. BA B_α such that $F_\gamma(B_\alpha)$ is recursive for all $\gamma \le \alpha$, $F_{\alpha+1}(B_\alpha)$ is r.e., but there is no Rec. BA D isomorphic to B_α such that $F_{\alpha+1}(D)$ is recursive.*

5. Recursive isomorphism types of Rec. BAs

The goal of this section is to survey some results about the nature of the recursive isomorphism types that are contained within the classical isomorphism type of a Rec. BA. Ideally one would like to produce some neat list of invariants which would determine the recursive isomorphism type of a Rec. BA. However, no such general result exists at this point nor is it clear that such a general classification result should be expected. Instead, the results surveyed in this section will tend to show the existence of a wide variety of recursive isomorphism types at least for those Rec. BAs B, where $\mathrm{At}(B)$ is infinite. Of course, if $\mathrm{At}(B)$ is finite, it easily follow from Proposition 4.1(i) that any Rec. BA D isomorphic to B is recursively isomorphic to B so that there is a unique recursive isomorphism type in the classical isomorphism type of B. Now most of the results of this section involve studying the possibilities for $\mathrm{At}(B)$ and $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ for a Rec. BA B with respect to such recursion theoretic properties as immunity, Turing degree, and recursive equivalence type. In particular, such a study will allow us to solve what we term the recursive categoricity problem for various classes of Rec. BAs. That is, in general we say a recursive model A of theory T is *recursively categorical**

* We note that in the Russian literature the word *autostable* is used instead of *recursively categorical*.

(over a first order language L) if every recursive model B of T which is isomorphic to A (over L) is recursively isomorphic to A. The recursive categoricity problem is then to classify the isomorphism types of the recursively categorical models of a given theory. In the Introduction we promised that we would classify the recursively categorical BAs with respect to the three classes of BAs, namely $\mathbb{R}$, the class of all Rec. BAs, $\mathbb{R}$At, the class of all Rec. BAs whose set of atoms is recursive, and $\mathbb{R}$AtAl, the class of a Rec. BAs whose set of atoms and set of atomless elements are recursive. That is, we shall classify the recursively categorical models of the theory of BAs over the languages $\mathscr{L}_1 = (+, \cdot, -, 0, 1)$, $\mathscr{L}_2 = (+, \cdot, -, 0, 1, \phi_1(x))$, and $\mathscr{L}_2 = (+, \cdot, -, 0, 1, \phi_1(x), \phi_2(x))$, where $\phi_1(x)$ and $\phi_2(x)$ are the predicates "x is atom" and "x is atomless", respectively.

We begin this section with a study of the possible recursion theoretic properties of At(B) in a Rec. BA. A natural first question might be to ask whether for any Rec. BA B there exists a Rec. BA D isomorphic to B such that At(D) is recursive. Of course, due to Goncharov's result mentioned in the previous section, we know that the answer to this question is no. That is, there exist Rec. BAs B which are not isomorphic to any Rec. BA D, where At(D) is recursive. Our next question might be to ask whether for any Rec. BA B there exists a Rec. BA D isomorphic to B such that At(D) is immune. The answer to our second question is yes as long as we have the clearly necessary condition that At(B) is infinite. We should note that due to the negative answer to our first question, there is no hope to use one of the Ash–Nerode theorems quoted in Section 4 to prove such a result. Instead, we require a direct construction. That is, we must start with a Rec. BA B such that At(B) is infinite and construct a Rec. BA D isomorphic to B such that At(D) is immune. The construction of such a D involves the second finite injury priority argument which we present in detail. However, before proceeding we need to quote a result about countable BAs B such that At(B) is infinite to the effect that if we construct a new BA D from B by splitting each of the atoms of B into finitely many pieces, then D remains isomorphic to B. More formally, we must work with subalgebras of the countable atomless BA $\tilde{Q}$.

5.1. THEOREM (REMMEL [1981a]). *Let B be a subalgebra of $\tilde{Q}$ such that* At(B) *is infinite. Let* At$(B) = \{b_0, b_1, \ldots\}$ *and assume that for each i, $e^i_1, \ldots, e^i_{k_i}$ are nonzero pairwise disjoint elements of $\tilde{Q}$ such that $d_i = \Sigma_{j=1}^{k_i} e^i_j$. Finally, let $C = \langle B \cup \{e^i_j \mid i \geq 0 \ \& \ 1 \leq j \leq k_i\}\rangle$. Then $B \approx C$.*

We should note that if the B of Theorem 5.1 is atomic, then the result easily follows from Vaught's Theorem (see Theorem 1.3). The general result follows from straightforward back-and-forth type construction. Given Theorem 5.1, we can now prove the following.

5.2. THEOREM (REMMEL [1981a]). *Let D be a Rec. BA such that* $|\mathrm{At}(D)| = \omega$, *then there exists a Rec. BA C isomorphic to D such that* At(C) *is immune.*

PROOF. Let $d_0, d_1, \ldots$ be an r.e. generating sequence for D and let $D^s = \langle \{d_0, \ldots, d_s\} \rangle$. We shall build the desired recursive BA C in stages. C will be a recursive subalgebra of $\tilde{Q}$. At each stage s of our construction, we will specify two finite subalgebras B^s and C^s of $\tilde{Q}$ and an isomorphism $f^s \colon D^s \to B^s$. We will ensure that for all s, $B^s \subseteq C^s$, $B^s \subseteq B^{s+1}$, $C^s \subseteq C^{s+1}$, and $f^s \subseteq f^{s+1}$. At the end our construction $B = \bigcup_s B^s$ will be an r.e. subalgebra of $\tilde{Q}$, $C = \bigcup_s C^s$ will be a recursive subalgebra of $\tilde{Q}$, and $f = \bigcup_s f^s$ will be a partial recursive isomorphism from D onto B. C will be in relation to B as in Theorem 5.1, that is, if $\mathrm{At}(B) = \{b_0, b_1, \ldots\}$, then for each i, there will be finitely many pairwise disjoint nonzero elements of $\tilde{Q}$, $e_1^i, \ldots, e_{k_i}^i$, such that $b_i = \Sigma_{j=1}^{k_i} e_j^i$ and $C = \langle B \cup \{e_j^i \mid i \geq 0 \;\&\; i \leq j \leq k_i\} \rangle$. Thus, f will ensure $D \approx B$ and, by Theorem 5.1, $B \approx C$ so that $D \approx C$.

To ensure that $\mathrm{At}(C)$ is immune, we shall meet the following set of requirements for $e = 0, 1, \ldots$.

R_e: If $W_e \cap C$ is infinite, then $W_e \cap (C - \mathrm{At}(C)) \neq \emptyset$.

We say requirement R_e is *satisfied* at stage s if $W_e^s \cap (C^s - \mathrm{At}(C^s)) \neq \emptyset$.

Our basic strategy in the construction is as follows. Suppose at stage s, requirement R_e is not satisfied and there is an $x \in \mathrm{At}(C^s)$ such that $x \in W_e^{s+1}$. Then at stage $s+1$, we let x_1 and x_2 be two nonzero disjoint elements of $\tilde{Q}$ such that $x = x_1 + x_2$ and let $C^{s+1} = \langle C^s \cup \{x_1, x_2\} \rangle$. Thus $x \notin \mathrm{At}(C^{s+1})$ so that $x \notin \mathrm{At}(C)$ and x will witness that R_e is satisfied. However, this procedure of satisfying a requirement R_e may conflict with ensuring that C is isomorphic D. That is, if we blindly follow such a procedure to satisfy the requirements, at some later stages we may split x_1 and x_2 into nonzero disjoint elements so that $x_1 = y_1 + y_2$ and $x_2 = z_1 + z_2$ for the sake of other requirements R_i and R_j, and then at even later stages split each of y_1, y_2, z_1, and z_2, etc. In this way, x may end up to be an atomless element in C even though D is atomic. Our idea to control the isomorphism type of C is to build an isomorphic copy of D, B, in $\tilde{Q}$ and to ensure that C only differs from B as described in the above paragraph. In particular, we must ensure that if $x \in \mathrm{At}(B)$, then x is a union of finitely many atoms in C. We shall priority rank our requirements as $R_0, R_1, \ldots$, that is, R_0 has highest priority, R_1 has next highest priority, etc. We shall use a set of movable markers in the construction. We imagine we have a potentially infinite set of markers Γ_e for each requirement R_e. When we split an $x \in \mathrm{At}(C^s)$ at stage $s+1$ into two nonzero disjoint elements x_1 and x_2 for the sake of requirement R_e as described above, then we will place a Γ_e marker on each of x_1 and x_2. As long as a Γ_e remains on x_1 and x_2, we will allow x_1 and x_2 to be split at some later stage only for the sake of some requirement R_j which has a higher priority than R_e, i.e. only if $j < e$. It will follow that if x_1 and x_2 have Γ_e markers on them at the end of the construction, then x will be a union of at most 2^{e+1} atoms of C. However, there is one situation where we could remove the Γ_e markers from x_1 and x_2. Namely, it may be that at some stage u, $x = f^u(a)$, where $a \in \mathrm{At}(D^u)$. Then at stage $u+1$, $d_{u+1} \cdot a$ and $a - d_{u+1}$ are nonzero so that we must split x for the sake of building B to be isomorphic to D. In such a situation, we will let $B^{u+1} = \langle B^u \cup \{x_1, x_2\} \rangle$

and let $f^{u+1}(a \cdot d_{u+1}) = x_1$ and $f^{u+1}(a - d_{u+1}) = x_2$ and remove the Γ_e markers from x_1 and x_2. In this way, if a turned out to be an atomless element in D, we will be free to make x an atomless element of C.

We now proceed to give the formal description of the construction. Each stage $s > 0$ is divided into two substages; the first substage will be used to ensure $B \approx D$ and the second substage will be used to ensure that we meet all the requirements $R_0, R_1, \ldots$.

CONSTRUCTION.

Stage 0. Let $B^0 = C^0 = \{1_{\tilde{Q}}, 0_{\tilde{Q}}\}$ and let $f^0(1_D) = 1_{\tilde{Q}}$ and $f^0(0_D) = 0_{\tilde{Q}}$.

Stage s + 1. Assume B^s, C^s, and f^s have been defined so that $f^s\colon D^s \to B^s$ is an isomorphism and $B^s \subseteq C^s$.

Substage i. Let a be the atom of D^s such that $d_{s+1} <_D a$ and let $x = f^s(a)$. If $x \in \mathrm{At}(C^s)$, then let x_1 be the least nonzero element of $\tilde{Q}$ such that $x_1 <_{\tilde{Q}} x$ and $\langle C^s \cup \{x_1\}\rangle \cap \{0, \ldots, s\} = C^s \cap \{0, \ldots, s\}$. If $x \not\in \mathrm{At}(C^s)$, let t be the least stage such that $x \in \mathrm{At}(C^{t-1})$ but $x \not\in \mathrm{At}(C^t)$. Then x is a union of two atoms in C^t so let x_1 be the least of these two atoms. Let $B^{s+1} = \langle B^s \cup \{x_1\}\rangle$. Next we define f^{s+1} by defining it on the atoms of D^{s+1} and extending it to be a homomorphism. If $d \in \mathrm{At}(D^{s+1})$ and $d \not\in \{d_{s+1},\ a - d_{s+1}\}$, then let $f^{s+1}(d) = f^s(d)$ and let $f^{s+1}(d_{s+1}) = x_1$ and $f^{s+1}(a - d_{s+1}) = x - x_1$. It is clear that $f^s \subseteq f^{s+1}$ and f^{s+1} is an isomorphism from d^{s+1} onto B^{s+1}. Also, if we are in the case where $x \not\in \mathrm{At}(C^s)$, remove any markers on x_1 and $x - x_1$.

Substage ii. Look for an $e \leq s + 1$ such that R_e is not satisfied at stage s and there is a $y \in \mathrm{At}(C^s)$ such that $y \in W_e^{s+1}$ and y has no Γ_j marker on it with $j < e$. If there is no such e, let $C^{s+1} = \langle C^s \cup B^{s+1}\rangle$ and go onto the next stage. If there is such an e let $e(s + 1)$ be the least such e and $y(s + 1)$ be the least y corresponding to $e(s + 1)$. Now if $y(s + 1) = x$ where x was chosen at substage i, let $C^{s+1} = <$ $C^s \cup B^{s+1}\rangle$ and go onto the next stage. Otherwise, let y_1 be the least nonzero z such that $z <_{\tilde{Q}} y(s + 1)$ and $\langle C^s \cup B^{s+1} \cup \{z\}\rangle \cap \{0, \ldots, s\} = C^s \cap \{0, \ldots, s\}$. We then let $C^{s+1} = \langle C^s \cup B^{s+1} \cup \{y_1\}\rangle$ and place $\Gamma_{e(s+1)}$ markers on y_i and $y(s + 1) - y_1$.

This completes the description of the stages. It is clear that each stage is completely effective. We let $B = \bigcup_s B^s$, $C = \bigcup_s C^s$, and $f = \bigcup_s f^s$. It is clear that $f\colon D \to B$ is an isomorphism and that C is a recursive subalgebra of $\tilde{Q}$ since we have ensured that $s \in C$ if and only if $s \in C^s$. We shall now prove two lemmas to prove that C has the properties claimed in the theorem.

5.2.1. LEMMA. *All the requirements R_e are met.*

PROOF. Suppose the lemma is false. Let e be the least requirement R_e which fails to hold. Thus, W_e is infinite and $W_e \subseteq \mathrm{At}(C)$. Note that once a requirement R_j is satisfied at stage t, it remains satisfied at all stages $s \geq t$. Moreover, the only stage t where we introduce a Γ_j marker is if $e(t) = j$ and we satisfy requirement R_j at stage t. It follows that only finitely many atoms of C have Γ_j markers on them with $j < e$. Thus, there is an atom of C, a, such that $a \in W_e$ and a never has a Γ_j marker on it with $j < e$. Let s be a stage large enough so that $a \in C^s \cap W_e^s$ and for all

$j < e$, if R_j is ever satisfied at any stage t, it is satisfied by stage s. Then at stage $s+1$, requirement R_e must be the least requirement not satisfied such that there is a $y \in \mathrm{At}(C^s)$, namely a, such that $y \in W_e^{s+1}$ and there is no Γ_j marker on y with $j < e$. But then at substage ii of stage $s+1$, we would have $e(s+1) = e$ and hence requirement R_e would be satisfied at stage $s+1$ and $W_e \cap (C - \mathrm{At}(C)) \neq \emptyset$. Thus, there can be no such e and hence all the requirements are met. □

It now follows that $\mathrm{At}(C)$ is immune. Thus to complete the proof of the theorem, we need only show $D \approx C$.

5.2.2. LEMMA. $D \approx C$.

PROOF. Our construction ensures that $D \approx B$. Thus, we show $B \approx C$. Let $\mathrm{At}(B) = \{a_0, a_1, \ldots\}$. First we shall show that for each a_i there exists $e_1^i, \ldots, e_{k_i}^i$ elements of $\mathrm{At}(C)$ such that $a_i = \Sigma_{j=1}^{k_1} e_j^i$ and then we shall show that $C = \langle B \cup \{e_j^i \mid i \geq 0 \;\&\; 1 \leq j \leq k_i\}\rangle$. Thus, it will follow from Theorem 5.1 that $B \approx C$.

Given i, let t be the first stage such that $a_i \in B^t$. Since $a_i \in \mathrm{At}(B)$ there is an atom d of D such that $f^t(d) = a_i$. It follows that there is no stage $s > t$ such that at substage i of stage s we introduce a nonzero $x \in B^s$ such that $x <_{\tilde{Q}} a_i$. Now if a_i is not an atom of C, then there is a stage s such that the $y(s)$ chosen at substage ii of stage s is a_i. Thus, there exists an $x \in C^s$ such that x is nonzero and $x <_{\tilde{Q}} a_i$ and both x and $a_i - x$ have $\Gamma_{e(s)}$ markers on them. Since a_i is never split in substage i at any stage u, it follows that the $\Gamma_{e(s)}$ markers are never removed from x and $x - a_i$. Our construction now ensures that if there is a stage u and a nonzero y such that $y <_{\tilde{Q}} x$ or $y <_{\tilde{Q}} x - a_i$ and $y \in \mathrm{At}(C^u) - \mathrm{At}(C^{u-1})$, then y must have a Γ_j marker on it for some $j < e(s)$ and this Γ_j marker will never be removed from y. Since once a requirement R_j is satisfied at stage u, it remains satisfied for all $s \geq u$, it follows that there are only finitely many $y \in C$ with Γ_j markers on them with $j < e(s)$. Thus, both x and $a_i - x$ will be a union of finitely many atoms of C.

Next, suppose z is an arbitrary nonzero element of C. Let s be the stage where $z \in C^s - C^{s+1}$. Thus, we can express z as a finite union of atoms of C^s, $z = \Sigma_{i=1}^k z_i$. It is an easy finite induction to show that if $z' \in \mathrm{At}(C^s)$, then either $z' \in B^s$ or z' has a Γ_j marker on it for some j. Moreover, our construction ensures that if a Γ_j marker is ever removed from any x at some stage t, then $x \in B^t$. Now let u be a stage large enough so that if $x \in C^s$ and x has a Γ_j marker on it at stage s, then either x no longer has a Γ_j marker on it at stage u, in which case $x \in B^u$, or the Γ_j marker remains on x at all stages $t \geq u$. Consider the subalgebra $E = B^u \cap C^s$. For each z_i as above, either $z_i \in B^u$ or z_i has a Γ_j marker on it at stage u. In the latter case, let a be the atom of E such that $z_i <_{\tilde{Q}} a$. Consider the first stage w where $a \in \mathrm{At}(C^{w+1}) - \mathrm{At}(C^w)$. Then at stage $w+1$, there are x_1 and x_2 in $\mathrm{At}(C^{w+1})$ such that $a = x_1 + x_2$. Since $a \in C^s$, $z_i \in C^s$, and $z_i <_{\tilde{Q}} a$, it follows that $w+1 \leq s$ and x_1 and x_2 are not in $B^u \cap C^s$. Moreover, we must have split a at substage ii of stage $w+1$ and hence at stage $w+1$, x_1 and x_2 have Γ_n markers on them for some n. Since x_1 and x_2 are not in B^u, it follows that the Γ_n markers on x_1 and x_2 were not removed by stage u. Thus, x_1 and x_2 have Γ_n markers on them at stage s and hence by our choice of u, x_1 and x_2 have Γ_n markers on them at all stages $t \geq s$. It now follows that a must be an atom of B for if there is a stage t such that $a \in \mathrm{At}(B^{t+1}) - \mathrm{At}(B^t)$, then we split a at stage i of stage $t+1$ and our construction would force us to remove the Γ_n markers on x_1 and x_2. Thus, each z_i is either in B or $z_i <_Q a_j$ where a_j is some atom of B in which case z_i is a finite union of some of $e_1^j, \ldots, e_{k_j}^j$, where $e_1^j, \ldots, e_{k_j}^j$ are the atoms of C under a_j. Thus, we can conclude that $C = \langle B \cup \{e_j^i \mid i \geq 0 \ \& \ 1 \leq j \leq k_i\}\rangle$. □

In contrast to Theorem 5.2, we also have the following which is an immediate consequence of Theorem 1.7.

5.3. THEOREM (REMMEL [1981a]). *Let D be a Rec. BA such that* $|\mathrm{At}(D)| = \omega$, *then there exists a Rec. BA B isomorphic to D such that* $\mathrm{At}(B)$ *is not immune.*

PROOF. By Theorem 1.7, there exists an $a \in D$ such that $D \mid a$ is isomorphic to one of $\tilde{N}$, $\tilde{H}$, $\tilde{C}$, or $\tilde{G}$. Thus, D is isomorphic to one of $\tilde{N} \times D \mid -a$, $\tilde{H} \times D \mid -a$, $\tilde{C} \times D \mid -a$, or $\tilde{G} \times D \mid -a$ each of which is a Rec. BA whose atoms contain an infinite recursive set. □

Note that since a recursive isomorphism must preserve the immunity or nonimmunity of the set of atoms, an immediate corollary of Theorems 5.2 and 5.3 is our first recursive categoricity result which is due independently to GONCHAROV [1975a] and LAROCHE [1977].

5.4. COROLLARY (GONCHAROV [1975a], LAROCHE [1977]). *A Rec. BA B is recursively categorical with respect to* $\mathbb{R}$, *the set of all Rec. BAs iff B has only finitely many atoms.*

In fact, Theorem 5.2 allows us to say quite a bit more about the number of recursive isomorphism types within the classical recursive isomorphism type of a Rec. BA D such that $|\mathrm{At}(D)| = \omega$. But before proceeding, we need to recall a couple of facts about *recursive equivalence types* (RETs).

5.5. DEFINITION. (i) Given two sets A, B contained in N, we say A is *recursively equivalent* to B, written $A \sim B$, if there is a one-to-one partial recursive function f whose domain contains A and $f \mid A$ is a bijection from A onto B.

(ii) The *recursive equivalence type* of A, $\mathrm{RET}(A)$, equals $\{B \mid A \sim B\}$.

Now RETs were introduced by DEKKER and MYHILL [1960] as a way to study effective cardinality. In particular, the RET of an isolated set A corresponds to an effectively Dedekind finite cardinal due to the following lemma of DEKKER [1960].

5.6. LEMMA. *If A is isolated, then A is not recursively equivalent to any proper subset of itself.*

PROOF. Suppose f is a one-to-one partial recursive function such that $\mathrm{dom}(f) \supset A$ and range of $f \mid A \subset A$. Now let $a \in A - \mathrm{range}(f \mid A)$. Then it is easy to see that

$a, f(a), f^2(a), f^3(a), \ldots$ is an infinite r.e. set of elements in A so that A is not immune. $\square$

The significance of these notions for our purposes is that if $f: A \to B$ is a recursive isomorphism between Rec. BAs A and B, then f witnesses that $\mathrm{RET}(\mathrm{At}(A)) = \mathrm{RET}(\mathrm{At}(B))$ and $\mathrm{RET}(\langle \mathrm{At}(A) \rangle^{\mathrm{id}}) = \mathrm{RET}(\langle \mathrm{At}(B) \rangle^{\mathrm{id}})$. But now suppose B is a Rec. BA such that $\mathrm{At}(B)$ is immune and $z_1 <_B z_2 <_B z_3 <_B \cdots$ is a sequence of elements of B such that for each i, z_i is a union of exactly i atoms in B. Now it follows from the proof of Theorem 5.1 that if $z \in \langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ and $|\mathrm{At}(B)| = \omega$, then B is isomorphic to $B \mid -z$ (see REMMEL [1981a] for proof). Thus, $B \mid -z_0, B \mid -z_1, \ldots$ are all Rec. BAs isomorphic to B but by Lemma 5.6, $\mathrm{RET}(\mathrm{At}(B \mid -z_i)) \neq \mathrm{RET}(\mathrm{At}(B \mid -z_j)$ if $i < j$. Thus, $B, B \mid -z_1, B \mid -z_2, \ldots$ all lie in distinct recursive isomorphism types. Thus, we have the following:

5.7. THEOREM (GONCHAROV [1975a], LAROCHE [1977]). *If B is a Rec. BA, then there are either* 1 *or ω recursive isomorphism types within the classical isomorphism type of B.*

We should note that Theorems 5.2 and 5.3 can be sharpened significantly. First of all, in Theorem 5.2 one can modify the construction of C so that $\mathrm{At}(C)$ turns out to be hyperimmune rather than just immune. Here an infinite set $H \subseteq N$ is *hyperimmune* if there is no total recursive function f such that for all n, $D_{f(n)} \cap H \neq \emptyset$ and for all n and m, $D_{f(n)} \cap D_{f(m)} = \emptyset$ if $m \neq n$. (Recall D_x is the finite set with canonical index x, see Section 1.) Moreover, we can control the Turing degree of the atoms as well in Theorems 5.2 and 5.3. For Rec. BAs D such that $\mathrm{At}(D)$ is recursive and infinite, we can completely control the Turing degrees.

5.8. THEOREM (REMMEL [1981a]). *Let D be a Rec. BA such that* $\mathrm{At}(D)$ *is an infinite recursive set and let E be any nonrecursive r.e. set. Then there exist Rec. BAs $C_0, C_1, \ldots$ each isomorphic to D such that for all $e \geq 0$,* $\mathrm{At}(C_{2e})$ *is hyperimmune,* $\mathrm{At}(C_{2e+1})$ *is not immune, and* $\mathrm{At}(C_e) \equiv_T E$ *and for all $i \neq j$, $C_i \not\cong_r C_j$.*

Of course, by Goncharov's examples of Rec. BAs which are not isomorphic to any Rec. BAs with a recursive set of atoms, we know that Theorem 5.8 does not apply to all isomorphism types of Rec. BAs. Thus, for arbitrary BAs, we have the following.

5.9. THEOREM. *Let D be a Rec. BA such that* $|\mathrm{At}(D)| = \omega$.

(i) (REMMEL [1981a]). *There exist Rec. BAs $C_0, C_1, \ldots$ each isomorphic to D such that for all i,* $\mathrm{At}(C_i)$ *is hyperimmune and of degree* $0'$ *and $C_i \not\cong_r C_j$ for $i \neq j$.*

(ii) (REMMEL [∞]). *For any r.e. set E with* $\mathrm{At}(D) <_T E$, *there exist Rec. BAs $E_0, E_1, \ldots$ each isomorphic to D such that for all* $\mathrm{At}(C_i)$ *is hyperimmune and* $E \equiv_T \mathrm{At}(E_i)$ *and $E_i \not\cong_r E_j$ for $i \neq j$.*

(iii) (REMMEL [1981a]). *For any r.e. set F with* $\mathrm{At}(D) \leq_T F$, *there exist Rec. BAs $B_0, B_1, \ldots$ each isomorphic to D such that for all i,* $\mathrm{At}(B_i)$ *is not immune and* $\mathrm{At}(B_i) \equiv_T F$ *and $B_i \not\cong_r B_j$ for $i \neq j$.*

We note that even though Theorem 5.9(ii) is unpublished, its proof is essentially identical with the proof of Theorem 5.8. An interesting open problem at this point is whether every Rec. BA D with $|\mathrm{At}(D)| = \omega$ is isomorphic to a Rec. BA B whose atoms are incomplete, i.e. $\mathrm{At}(B) <_T 0'$. Such a result has been announced in an addendum to REMMEL [1981a] but the proposed proof was in error. It should be noted, however, that Goncharov's example alluded to above actually shows that there exist Rec. BAs D which are not isomorphic to any Rec. BA B, where $\mathrm{At}(B)$ is of low degree, i.e. where $\mathrm{At}(B)' \equiv_T 0'$.

One final remark about Theorems 5.2 and 5.3 is in order here. Namely it is easy to check that if $\langle \mathrm{At}(D) \rangle^{\mathrm{id}}$ started out recursive, then C and B were constructed so that $\langle \mathrm{At}(C) \rangle^{\mathrm{id}}$ and $\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ are also recursive. Thus, even though the ideal generated by the atoms in a Rec. BA is well behaved from a recursion theoretic point of view, it is possible for the atoms themselves to be quite pathological. The same remark applied to Theorems 5.8 and 5.9 which we formalize.

5.10. REMARK. If the recursive BA D in Theorems 5.8 and 5.9 is such that $\langle \mathrm{At}(D) \rangle^{\mathrm{id}}$ is recursive, then all the Rec. BAs mentioned in those theorems have the property that their ideal generated by the atoms is recursive.

In light of these last remarks, there are two natural questions which arise. That is, suppose B is a Rec. BA such that $|\mathrm{At}(B)| = \omega$ and $\langle \mathrm{At}(B) \rangle$ is recursive. Does there exist a Rec. BA D isomorphic to B such that both $\mathrm{At}(D)$ and $\langle \mathrm{At}(D) \rangle^{\mathrm{id}}$ are nonrecursive? Does there exist a Rec. BA E isomorphic to B such that both $\mathrm{At}(E)$ and $\langle \mathrm{At}(E) \rangle^{\mathrm{id}}$ are recursive? As it turns out the answer to both questions is yes as our next results show.

5.11. THEOREM. *Let B be a Rec. BA such that $\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ is infinite and recursive. Then there exists a Rec. BA D isomorphic to B such that both* $\mathrm{At}(D)$ *and* $\langle \mathrm{At}(D) \rangle^{\mathrm{id}}$ *are nonrecursive.*

PROOF. By Theorem 1.7, there exists an $a \in B$ such that $B \mid a$ is isomorphic to one $\tilde{N}$, $\tilde{H}$, $\tilde{C}$, or $\tilde{G}$. As each of the BAs $\tilde{N}$, $\tilde{H}$, $\tilde{C}$, and $\tilde{G}$ are Dec. BAs whose ideal generated by the atoms is recursive, we can apply Theorem 4.10 to show that there exist Rec. BAs $\tilde{N}_1$, $\tilde{H}_1$, $\tilde{C}_1$, and $\tilde{G}_1$ isomorphic to $\tilde{N}$, $\tilde{H}$, $\tilde{C}$, and $\tilde{G}$, respectively, such that the ideal generated by the atoms in each of the BAs are immune. It then follows that the set of atoms in $\tilde{N}_1$, $\tilde{H}_1$, $\tilde{C}_1$, and $\tilde{G}_1$ are immune as well. But then $\tilde{N}_1 \times (B \mid -a)$, $\tilde{H}_1 \times (B \mid -a)$, $\tilde{C}_1 \times (B \mid -a)$, and $\tilde{G}_1 \times (B \mid -a)$ all have the property that both their ideal generated by the atoms and their set of atoms are not recursive. As B is isomorphic to one of these BAs, the result follows. □

5.12. THEOREM (REMMEL [1981a]). *Let B be a Rec. BA such that $\langle \mathrm{At}(B) \rangle^{\mathrm{id}}$ is infinite and recursive. Then there exists a Rec. BA E isomorphic to B such that both* $\mathrm{At}(E)$ *and* $\langle \mathrm{At}(E) \rangle^{\mathrm{id}}$ *are recursive.*

PROOF. The proof is another application of Theorem 5.1. We shall enumerate a recursive subalgebra E of D such that D comes from E by splitting some of the atoms of E into finitely many elements in D. Let $d_0, d_1, \ldots$ be an r.e. generating sequence for D and recall $D^s = \langle\{d_0, \ldots, d_s\}\rangle$. Our basic idea is that when an x in $D^s - \langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ is split by d_{s+1} and say $d_{s+1} \in \langle \mathrm{At}(D)\rangle$, then we can simply let d_{s+1} be an atom of E since splitting d_{s+1} finitely many more times does not have an effect on the isomorphism type of E by Theorem 5.1. A similar statement holds if $x - d_{s+1} \in \langle \mathrm{At}(D)\rangle^{\mathrm{id}}$. In such a situation, we will simply put a marker Γ on d_{s+1} and/or $x - d_{s+1}$ depending on their membership in $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$. The set of elements with markers on them will be recursive and will be the atoms of E.

CONSTRUCTION.

Stage 0. Let $E^0 = D^0$.

Stage s + 1. Assume $E^s \subseteq D^s$ and if $b \in \mathrm{At}(E^s)$, then b has a Γ marker on it iff $b \in \langle \mathrm{At}(D)\rangle^{\mathrm{id}}$. Consider d_{s+1}. There is still a unique atom of E^s, a, such that $d_{s+1} < a$. If a has a Γ marker on it, let $E^{s+1} = E^s$ and go onto the next stage. Otherwise, let $E^{s+1} = \langle E^s \cup \{d_{s+1}\}\rangle$ and put Γ markers on those elements of $\{d_{s+1}, a - d_{s+1}\}$ which are in $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$.

This completes the description of the construction. It easily follows from Theorem 5.1 and our remarks before the construction that $E \approx D$. E is recursive since $x \not\in E$ iff either $x \not\in D$ or there is an s such that $x \in D^s - E^s$. Now $x \in \langle \mathrm{At}(E)\rangle^{\mathrm{id}}$ iff $x \in E$ and $x \in \langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ so that $\langle \mathrm{At}(E)\rangle^{\mathrm{id}}$ is recursive. Finally $x \in \mathrm{At}(E)$ iff $x \in E$ and there is a stage s such that $x \in E^s$ and x has a Γ marker on it so that $\mathrm{At}(E)$ is recursive. $\square$

Our results in this section up to this point have shown that the set of atoms of Rec. BAs within the same classical isomorphism type can have widely varying properties. Nevertheless, there are restrictions imposed by the classical isomorphism type on the recursion theoretic properties of the set of atoms. In fact, placing certain recursion theoretic restrictions on the set of atoms of a Rec. BA completely determines the classical isomorphism. For example, a set $A \subseteq N$ is *cohesive* if A is infinite and there is no r.e. set W such that both $W \cap A$ and $A - W$ are infinite. We then have the following.

5.13. THEOREM (REMMEL [1981a]). *Suppose B is a Rec. BA such that* $\mathrm{At}(B)$ *is cohesive, then B is isomorphic to* $\mathrm{Intalg}(\omega + \eta)$.

PROOF. By Theorem 1.7, we know that there exits an $a \in B$ such that $B \mid a$ is isomorphic to one of $\tilde{N}$, $\tilde{H}$, $\tilde{C}$, or $\tilde{G}$. The possibilities $\tilde{C}$ and $\tilde{G}$ are immediately ruled out since otherwise there is an $x <_B a$ such that both $\mathrm{At}(B \mid x)$ and $\mathrm{At}(B \mid a - x)$ are infinite and hence $W = \{b \in B \mid b \leq_B x\}$ is an r.e. set which violates the cohesiveness of $\mathrm{At}(B)$. Our next lemma will show that $\tilde{N}$ is also ruled out so that the only possibility is $B \mid a \approx \tilde{H} \approx \mathrm{Intalg}(\omega + \eta)$. But then it cannot be that $\mathrm{At}(B \mid -a)$ is infinite so that $B \mid -a$ is isomorphic to F_n or $F_n \times \tilde{Q}$ for some finite BA F_n. But since $\tilde{H} \approx \tilde{H} \times F_n \approx \tilde{H} \times F_n \times \tilde{Q}$, our theorem easily follows once we have proved the following lemma.

5.14. LEMMA. *If B is a Rec. BA and $B \approx \tilde{N}$, then* $\mathrm{At}(B)$ *is not cohesive.*

PROOF. In fact we shall show that $\mathrm{At}(B)$ is not even hyper-hyperimmune (h-h-immune) where a set $A \subseteq N$ is *h-h-immune* if there is no recursive function f such that
(i) $\forall n(W_{f(n)}$ is finite & $W_{f(n)} \cap A \neq \emptyset)$ and
(ii) $\forall n \forall m(m \neq n \rightarrow W_{f(n)} \cap W_{f(m)} = \emptyset)$.

We shall enumerate in stages an effective sequence of r.e. finite sets $U_0, U_1, \ldots$ which will witness the non-h-h-immunity of $\mathrm{At}(B)$. For each i, we let U_i^s denote the set of elements enumerated into U_i by the end of stage s. Let $b_0, b_1, \ldots$ be an r.e. generating sequence for B and let $B^s = \langle\{b_0, \ldots, b_s\}\rangle$. Our idea is very simple. At stage 0, we let $U_0^0 = \{1_B\}$ and $U_i^0 = \emptyset$ for all $i > 0$. At any stage $s > 0$, the fact that $b_0, b_1, \ldots$ is an r.e. generating sequence for B implies, there are exactly $s+1$ atoms of B^s. We shall ensure that $|\mathrm{At}(B^s) \cap U_i^s| = 1$ for all $i \leq s$. Thus, at stage $s+1$, we let a be the atom of B^s such that $b_{s+1} <_B a$. Now $a \in U_i^s$ for some $i \leq s$. Then we let $U_i^{s+1} = U_i^s \cup \{b_{s+1}\}$, $U_{s+1}^{s+1} = \{a - b_{s+1}\}$, and $U_j^{s+1} = U_j^s$ for $j \not\in \{i, s+1\}$. For each j, let $U_j = \bigcup_s U_j^s$. Clearly, $U_0, U_1, \ldots$ is an effective sequence of nonempty r.e. sets. Moreover, for each j, if $u_j^0, u_j^1, \ldots$ is the enumeration of U_j which comes from our stages then we have (a) $u_j^0 >_B u_j^1 >_B \cdots$ and (b) if u_j^k is not an atom of B, then there is a $u_j^{k+1} <_B u_j^k$ in U_j. It easily follows that if $u_j^k \in \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ for any k, then U_j is finite and U_j contains an atom of B. Since $B \approx \tilde{N}$, at any stage s exactly one of the atoms of B^s is not in $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$. It follows that for at most one n, U_n is infinite and in that case $U_n \subseteq B - \langle \mathrm{At}(B)\rangle^{\mathrm{id}}$. Now if there is such an n, then there is a recursive function g such that $W_{g(i)} = U_i$ for $i < n$ and $W_{g(i)} = U_{i+1}$ for $1 \geq n$ in which case g witnesses the non-h-h-immunity of $\mathrm{At}(B)$. Otherwise, there is a recursive function f such that $W_{f(i)} = U_i$ for all i and f witnesses the non-h-h-immunity of $\mathrm{At}(B)$. □

Now there are Rec. BAs B isomorphic to $\tilde{H}$ such that $\mathrm{At}(B)$ is a cohesive set. In fact, other than the obvious restriction that the set of atoms are co-r.e., the classical isomorphism type of $\mathrm{Intalg}(\omega + \eta)$ is the only isomorphism type of Rec. BA where there are no restrictions on the RET of the set of atoms.

5.15. THEOREM. *For any co-r.e. set $A \subseteq N$, there exists a Rec. BA B isomorphic to* $\mathrm{Intalg}(\omega + \eta)$ *such that* $\mathrm{RET}(A) = \mathrm{RET}(\mathrm{At}(B))$.

PROOF. Recall that we thought of $\tilde{H}$ as the subalgebra of $\tilde{C}$ generated by the set of left-closed right-open intervals of the rationals Q plus $\{\{q\} \mid q \in N\}$. Now by Theorem 2.1, $\tilde{H}$ is isomorphic to some recursive subalgebra D of $\tilde{Q}$. Then let g be any one-to-one recursive function from N onto the recursive set $\mathrm{At}(D)$. Thus, if A is a co-r.e. subset of N, then $g(A)$ is a co-r.e. subset of $\mathrm{At}(D)$ such that $\mathrm{RET}(A) = \mathrm{RET}(g(A))$. But from our presentation of $\tilde{H}$, it is clear that $C = \langle B \cup \{x \mid x \leq_{\tilde{Q}} a \ \& \ a \in \mathrm{At}(D) - g(A)\}\rangle$ is an r.e. subalgebra of $\tilde{Q}$ such that $\mathrm{At}(C) = g(A)$ and $C \approx \tilde{H}$. But by Corollary 2.3, C is recursively isomorphic to a recursive subalgebra B of $\tilde{Q}$. It then easily follows that $B \approx \tilde{H}$ and $\mathrm{RET}(\mathrm{At}(B)) = \mathrm{RET}(A)$. □

Next we turn our attention to the class $\mathbb{R}$At of recursive BAs B where $\mathrm{At}(B)$ is recursive. Much as we did for arbitrary Rec. BAs D with respect to $\mathrm{At}(D)$, it is natural to study the possible recursion theoretic properties of $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ for BAs B in $\mathbb{R}$At. Now if $\mathrm{At}(B)$ is recursive, then $\langle \mathrm{At}(B)\rangle^{\mathrm{id}}$ must be r.e. We know from Theorem 4.20 that there exist BAs $B \in \mathbb{R}\mathrm{At}$ such that B is not isomorphic to any Rec. BA D such that $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ is recursive. Thus, the obvious question to ask is whether for a $B \in \mathbb{R}\mathrm{At}$, there always exists a Rec. BA $C \in \mathbb{R}\mathrm{At}$ such that $B \approx C$ and $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ is nonrecursive. The answer to such a question leads us to our second recursive categoricity result.

5.16. THEOREM (REMMEL [1981b]). *Let $\mathcal{M}$ denote the smallest class of BAs which contains the atomless BA $\tilde{Q}$, the BA of finite–cofinite sets of N, $\tilde{N}$, plus all finite BAs and is closed under finite products. Then*

(i) *a BA D is recursively categorical with respect to $\mathbb{R}$At iff $D \in \mathcal{M}$, and*

(ii) *a BA $D \in \mathbb{R}\mathrm{At}$ has the property that every BA $E \in \mathbb{R}\mathrm{At}$ which is isomorphic to D has $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ recursive iff $D \in \mathcal{M}$.*

We shall not give the proof of Theorem 5.16 since it is much too long. Nevertheless we will outline the key steps. Now $\mathcal{M}$ consists of BAs of the form F_n, $F_n \times \tilde{Q}$, $\tilde{N}^k$, and $\tilde{N}^k \times Q$ for some $n, k < \omega$, where F_n is the finite BA with n-atoms. We have already observed that in $\tilde{N}$, the ideal generated by the atoms is recursive and that if B is a Rec. BA such that $B \approx \tilde{N}$ and $\mathrm{At}(B)$ is recursive, then $B \approx_r \tilde{N}$ (see Proposition 1.5). The "only if" parts of both (i) and (ii) then easily follow from these observations. The "if" parts use the following proposition which is proved by a finite injury priority argument.

5.17. PROPOSITION (REMMEL [1981b]). *Suppose B is a Rec. BA such that $\mathrm{At}(B)$ is recursive and there exists an $a \in B$ and Rec. BAs $B_0, B_1, B_2, \ldots$ such that for each i, $\mathrm{At}(B)_i$ is infinite and recursive and $B \mid a \approx \Pi_{i=0}^{\omega} B_i \approx \Pi_{i=0}^{\omega} D_i$, where $D_i \approx B_i$ if i is odd and D_i can be either finite or isomorphic to B_i if i is even. Then there exist Rec. BAs $E_0, E_1, \ldots$ each isomorphic to B such that for all i, $\mathrm{At}(E_i)$ is recursive and $\langle \mathrm{At}(E_i)\rangle^{\mathrm{id}} \equiv_T 0'$ and for all $i \neq j$, $E_i \not\approx_r E_j$.*

Once we have Proposition 5.17, the "iff" parts of (i) and (ii) of Theorem 5.16 then consist of a rather lengthy analysis to show that if $B \in \mathbb{R}\mathrm{At} - \mathcal{M}$, then B has a factor as described in Proposition 5.17.

A similar type argument establishes our third recursive categoricity result.

5.18. THEOREM (REMMEL [1986b]). *let $\mathcal{N}$ be the smallest class of BAs such that $\mathcal{N}$ contains $\tilde{Q} = \mathrm{Intalg}(1+\eta)$, $\tilde{N} = \mathrm{Intalg}(\omega)$, $\tilde{H} = \mathrm{Intalg}(\omega+\eta)$, and all finite BAs and is closed under finite products. Then*

(i) *a BA D is recursively categorical with respect to $\mathbb{R}$AtAl iff $D \in \mathcal{N}$, and*

(ii) *a BA $D \in \mathbb{R}\mathrm{AtAl}$ has the property that every BA $E \in \mathbb{R}\mathrm{AtAl}$ which is isomorphic to D has $\langle \mathrm{At}(D)\rangle^{\mathrm{id}}$ recursive iff $D \in \mathcal{M}$.*

Finally, we should remark that there are analogues of our first two recursive categoricity results for linear orders. That is, corresponding to Corollary 5.4 we have

5.19. THEOREM (Dzgoev (see GONCHAROV and DZGOEV [1980]) and REMMEL [1981c]). *A linear order L is recursively categorical with respect to the set of all Rec. LOs iff L has only finitely successivities where a pair $(a, b) \in L$ is a successivity iff b is an immediate successor of a in L.*

5.20. THEOREM (MOSES [1984]). *A linear L is recursively categorical with respect to the set of all Rec. LOs with recursive successivities iff the order type of L is of the form $k_1 + g_1 + k_2 + \cdots + g_{n-1} + k_n$, where each k_i is a finite order type, k_i is nonempty for $i \in \{2, \ldots, n-1\}$, and each g_i is an order type from among $\{\omega, \omega^*, \omega + \omega^*\} \cup \{k \cdot \eta \mid k < \omega\}$.*

As of 1987 there is no analogue of Theorem 5.18 for linear orders. We also refer the reader to GONCHAROV and DZGOEV [1980] for a common recursive model theoretic framework which allows one to prove a general result which implies both Corollary 5.4 and Theorem 5.19.

6. The lattices of r.e. subalgebras and r.e. ideals of a Rec. BA

The lattice of r.e. subsets of the natural numbers has been extensively studied by recursion theorists. See SOARE [1978], [1986] for a general survey. Thus, it is quite natural for recursion theorists to study the lattice of r.e. substructures of other recursive models. In this section we shall very briefly survey some of the results on the lattice $\mathscr{L}(B)$ of r.e. subalgebras and the lattice $\mathscr{L}I(B)$ of r.e. ideals of a Rec. BA B. However, our survey will be by no means complete. Instead, we will concentrate on what we feel are the most salient aspects of these lattices and refer the reader to a general survey of the lattices of r.e. substructures of recursive models by NERODE and REMMEL [1985] for a more complete picture.

We begin by studying the lattice $\mathscr{L}(B)$ for a Rec. BA B. In many ways, the theory of the lattice of r.e. subalgebras for a Rec. BA B is much cleaner than the corresponding theory for the lattices of r.e. ideals. We shall see that not only is it the case that there are natural analogues for many of the basic concepts that arise in $\mathscr{E}$ but also that we can say quite a bit about $\mathscr{L}(B)$ for general Rec. BAs. For example, Carroll has shown that the theory of $\mathscr{L}(B)$ is undecidable for every Rec. BA B. In contrast, we shall see that there are no such general results possible for $\mathscr{L}I(B)$. However, we should note that the study of r.e. ideals of a Rec. BA B is of course equivalent to the study of the lattice of r.e. filters of B. It is well known that an r.e. filter in the atomless BA $\tilde{Q}$ can be identified with an r.e. theory in a propositional logic based on infinitely many propositional letters $P_0, P_1, \ldots$ so that the study of $\mathscr{L}I(\tilde{Q})$ is of independent interest to logicians.

We begin our study of $\mathscr{L}(B)$ by considering one of the simplest theorems in $\mathscr{E}$ and asking for its analogue in $\mathscr{L}(B)$. Namely, it is a well known observation of Kleene that the recursive sets in $\mathscr{E}$ are the complemented elements in $\mathscr{E}$, i.e. an r.e. set A is recursive iff both A and its complement $\bar{A}$ are r.e. There is a complete analogue of this result for $\mathscr{L}(B)$ in any rec. BA B but it is by no means trivial and leads to some interesting results of general interest. To this end, we make the following definitions.

6.1. DEFINITION. Let B be any BA and let U and W be subalgebras of B. Then we say

(i) U is a *pseudo-complement* of W if $U \cap W = \{0_B, 1_B\}$ and for any $x \not\subseteq U$, $\langle U \cup \{x\} \rangle \cap W \neq \{0_B, 1_B\}$, i.e. U is a maximal element among those subalgebras of B which intersect W trivially;

(ii) U is a *bi-pseudo-complement* of W if U is a pseudo-complement of W and W is pseudo-complement of U;

(iii) U is a *complement* of W if U is a bi-pseudo-complement of W and $B = \langle W \cup U \rangle$.

It is easy to see that every recursive subalgebra W of a Rec. BA B has an r.e. pseudo-complement. Moreover, it is the case that if U is an r.e. subalgebra and U is an r.e. pseudo-complement of an r.e. subalgebra W, then U is a recursive subalgebra since $B - U$ is r.e., i.e. $B - U = \{x \in B \mid \langle U \cup \{x\} \rangle \cap W \neq \{0_B, 1_B\}\}$. Note however we cannot conclude that W itself is recursive in these circumstances unless we know that U being a psuedo-complement of W forces W to be a pseudo-complement of U. However, it is not clear that this property of psuedo-complements holds for arbitrary BAs. In fact, it was precisely in trying to answer this question that the author was led to prove the following.

6.2. THEOREM (REMMEL [1980a]). *Let S be any set and $B(S)$ denote the BA of finite and cofinite subsets of S. Then for all subalgebras U and W of $B(S)$, U is pseudo-complement of W iff U is a complement of W.*

6.3. THEOREM (REMMEL [1980a]). *If B is a BA and B is not isomorphic to $B(S)$ for some set S, then there exist subalgebras U and W of B such that U is a pseudo-complement of W but W is not a pseudo-complement of U and $\langle U \cup W \rangle \neq B$.*

We emphasize that the set S and BA B in Theorems 6.2 and 6.3 are not restricted to being countable but are completely arbitrary. Nevertheless, despite the failure of our desired property of pseudo-complements to hold in general, Theorem 6.2 does allow us to prove our desired lattice theoretic characterization of recursive subalgebras in $\mathscr{L}(B)$.

6.4. THEOREM (REMMEL [1981a]). *Let B be any Rec. BA, then a subalgebra W of B is recursive iff W is r.e. and has an r.e. complement.*

PROOF. By our observations above, if W is an r.e. subalgebra of B and U is an r.e. complement of B, then both W and U are recursive. For the other direction, suppose W is a recursive subalgebra of B, $b_0, b_1, \ldots$ is an r.e. generating sequence of B, and we let $B_s = \langle \{b_0, \ldots, b_s\} \rangle$ and $W_s = W \cap B_s$ for all s. Next, we define by induction an increasing sequence $U_0 \subseteq U_1 \subseteq \cdots$ of finite subalgebras of B. Let $U_0 = B_0 = \{0_B, 1_B\}$. For $s > 0$, assume we have defined $U_{s-1} \subseteq B_{s-1}$ such that U_{s-1} is a complement of W_{s-1} relative to the BA B_{s-1}. Then note that $W_s \cap U_{s-1} = B_{s-1} \cap W_s \cap U_{s-1} = W_{s-1} \cap U_{s-1} = \{0_B, 1_B\}$. We can thus easily extend U_{s-1} to a pseudo-complement of W_s relative to B_s by simply considering each of the finitely many elements of B_s in order and adding an $x \in B_s$ to our extension only if x together with U_{s-1} and all those elements added previously generate subalgebra of B_s which still intersects W_s trivially. In this way we clearly construct a subalgebra $U_s \supset U_{s-1}$ such that U_s is a pseudo-complement of W_s relative to B_s. But since B_s is a finite BA, Theorem 6.2 applies and hence U_s is in fact a complement of W_s relative to B_s. It then easily follows that $U = \bigcup U_s$ is a complement of W relative to B. Moreover, since W is recursive, we can explicitly calculate a canonical index for W_s at each stage s and hence explicitly calculate a canonical index for U_s at each stage s. Thus $U = \bigcup_s U_s$ is an r.e. complement of W as desired. □

We note the same proof where we use Zorn's lemma to construct our extensions at each stage if necessary will apply to any BA B which is a directed limit of BAs of finite and cofinite sets. That is, we have the following.

6.5. THEOREM (REMMEL [1980a]). *Let B be a BA which is a directed limit of BAs of finite and cofinite sets. (In particular B can be any countable BA.) Then the lattice of all subalgebras of B is complemented.*

Next we shall consider three basic types of r.e. sets defined in $\mathscr{E}$ and see how to define the natural analogues of these elements in $\mathscr{L}(B)$.

6.6. DEFINITION. (i) An r.e. set $S \subseteq N$ is *simple* if $\bar{S}$ is infinite and for any infinite r.e. set W_e, $W_e \cap S \neq \emptyset$.

(ii) An r.e. set $R \subseteq N$ is *r-maximal* if $\bar{R}$ is infinite and for any recursive set A either $N - (A \cup R)$ or $N - (A \cup \bar{R})$ is finite.

(iii) An r.e. set M is *maximal* if $\bar{M}$ is infinite and for any r.e. set $W_e \supseteq M$ either $N - W_e$ or $W_e - M$ is finite.

It is rather easy to see that the following implications hold

$$\text{maximal} \Rightarrow r\text{-maximal} \Rightarrow \text{simple}\,,$$

and it is known that none of the reverse implications hold.

Now the first thing one needs to do to define the analogues of these definitions for $\mathscr{L}(B)$ is to define the analogue of being coinfinite. To this end, let us fix a Rec. BA B. Then for subalgebras U and W of B, we write $U = {}^*W$ iff there exists finite sets $E_1, E_2 \subseteq B$ such that $\langle E_1 \cup U \rangle = \langle E_2 \cup W \rangle$. It is easy to see that $=^*$ is an equivalence relation but it is not a congruence relation with respect to both the $\vee$ (join) and $\wedge$ (meet) operations of our lattice $\mathscr{L}(B)$. That is, $U \vee W = \langle U \cup W \rangle$ and it is clear that if $U_1 =^* U_2$ and $W_1 =^* W_2$, then $U_1 \vee W_1 =^* U_2 \vee W_2$. However, it is not always the case that $U_1 \wedge W_1 =^* U_2 \wedge W_2$ under such circumstances (see REMMEL [1978] for counterexamples). Nevertheless, a natural analogue of U being coinfinite is to say $U \neq^* B$. This given, the following are natural analogues of the definitions in 6.6.

6.7. DEFINITION. Let B be a Rec. BA and U be an r.e. subalgebra of B. Then we say

(i) U is a *simple subalgebra* of B if $U \neq^* B$ and for any infinite r.e. subalgebra W of B, $W \cap U \neq \{0_B, 1_B\}$;

(ii) U is an *r-maximal subalgebra* of B if $U \neq^* B$ and for any pair of r.e. subalgebras R_1 and R_2 of B such that $\langle R_1 \cup R_2 \rangle = B$, either $\langle U \cup R_1 \rangle =^* B$ or $\langle U \cup R_2 \rangle =^* B$;

(iii) U is a *maximal subalgebra* of B if $U \neq^* B$ and for any r.e. subalgebra $W \supseteq U$ either $W =^* B$ or $W =^* U$.

The widely varying effective properties of Rec. BAs B make it difficult to have many general results which apply to all lattices $\mathscr{L}(B)$. In fact, we shall show that for the case of the lattices $\mathscr{L}I(B)$, there are no constructions which always produce simple, r-maximal, or maximal elements. There is, however, one important existence result for the lattices $\mathscr{L}(B)$. Namely, NERODE and REMMEL [1985] have a general existence result for r.e. generic substructures of a recursive model and when their general result is applied to Rec. BAs, it produces simple subalgebras.

6.8. THEOREM (NERODE and REMMEL [1985]). *Let B be any Rec. BA, then there exists a simple subalgebra of B.*

In fact, Remmel [∞] has shown there exist simple subalgebras of B in every nonzero r.e. degree. Note that if U is a simple subalgebra, then U certainly has no r.e. complements and hence U must be nonrecursive by Theorem 6.4. An important consequence of this last observation is that the finite subalgebras of B are definable in $\mathscr{L}(B)$. That is, it follows that a subalgebra $U \in \mathscr{L}(B)$ is finite if and only if U is complemented in $\mathscr{L}(B)$ and every subalgebra $W \subseteq U$ is complemented in $\mathscr{L}(B)$. The fact that the finite elements of $\mathscr{L}(B)$ are definable in $\mathscr{L}(B)$ is the key step that allowed CARROLL [1986] to show that the proof of the undecidability of the theory of the full lattice of subalgebras for any infinite BA B due to BURRIS and SANKAPPANAVAR [1975] can be effectivized so as to apply to $\mathscr{L}(B)$. Thus, Carroll was able to prove the following.

6.9. THEOREM (CARROLL [1986]). *If B is an infinite Rec. BA, then* $\mathrm{Th}(\langle \mathscr{L}(B), \wedge, \vee, 0, 1\rangle)$ *is undecidable.*

In contrast to Theorem 6.4, it is not known whether there exists r-maximal or maximal subalgebras in every Rec. BA. However, due to the fact that every Rec. BA D is isomorphic to a Rec. BA of the form $\tilde{N} \times B$, $\tilde{Q} \times B$, or $\tilde{C}$ by Theorem 1.6, our next results show that for every Rec. BA D, there is at least a Rec. BA C isomorphic to D such that r-maximal and maximal subalgebras exist in $\mathscr{L}(C)$. In fact our results show that we can transfer results about $\mathscr{E}$ for simple sets due to

POST [1944] and DEKKER [1954], maximal sets due to FRIEDBURG [1958] and MARTIN [1966], and r-maximal sets due to LACHLAN [1968] and ROBINSON [1967] to $\mathscr{L}(\tilde{N} \times B)$, $\mathscr{L}(\tilde{Q} \times B)$, or $\mathscr{L}(\tilde{C})$ for any Rec. BA B.

6.10. THEOREM (REMMEL [1978]). *Let B be any Rec. BA.*

(i) *If S is a maximal subset of* $\mathrm{At}(\tilde{N})$, *then* $\langle S\rangle$ *is a maximal subalgebra of* $\tilde{N}$ *and* $\langle S\rangle \times B$ *is a maximal subalgebra of* $\tilde{N} \times B$.

(ii) *If S is a hypersimple subset of* $\mathrm{At}(\tilde{N})$, *then* $\langle S\rangle$ *is a simple subalgebra of* $\tilde{N}$ *and* $\langle S\rangle \times B$ *is a simple subalgebra of* $\tilde{N} \times B$.

(iii) *There exists* $S_1, S_2 \subseteq \mathrm{At}(\tilde{N})$ *such that for* $i = 1, 2$, $\langle S_i\rangle$ *and* $\langle S_i\rangle \times B$ *are r-maximal but not maximal subalgebras of* $\tilde{N}$ *and* $\tilde{N} \times B$, *respectively,* $\langle S_1\rangle$ *and* $\langle S_1\rangle \times B$ *are not contained in any maximal subalgebras of* $\tilde{N}$ *and* $\tilde{N} \times B$, *respectively, and* $\langle S_2\rangle$ *and* $\langle S_2\rangle \times B$ *are contained in maximal subalgebras of* $\tilde{N}$ *and* $\tilde{N} \times B$, *respectively.*

Recall that $\tilde{Q}$ is the Boolean algebra generated by the left-closed right-open intervals of the rationals $\tilde{Q}$. Let $\tilde{R} = \{[i, i+1) \mid i = 0, 1, 2, \ldots\}$. Then $\tilde{R}$ is a recursive subset of $\tilde{Q}$. Let $\mathscr{H}$ denote the subalgebra of $\tilde{Q}$ generated by all intervals $[a, b)$ such that either (i) $[a, b) \subseteq (-\infty, 0)$ or (ii) for some n, $[a, b) \subseteq [n, n+1)$ and $b < n+1$. Now $\mathscr{H}$ is a recursive subalgebra of $\tilde{Q}$. For any set $S \subseteq \tilde{R}$, let $S_{\mathscr{H}}$ denote $\langle S \cup \mathscr{H}\rangle$.

6.11. THEOREM (REMMEL [1978]). *Let B be any recursive Boolean algebra.*

(i) *If S is a maximal subset of $\tilde{R}$, then $S_{\mathscr{H}}$ is a maximal subalgebra of $\tilde{Q}$ and $S_{\mathscr{H}} \times B$ is a maximal subalgebra of $\tilde{Q} \times B$.*

(ii) *If S is a hypersimple subset of $\tilde{R}$, then $S_{\mathscr{H}}$ is a simple subalgebra of $\tilde{Q}$ and $S_{\mathscr{H}} \times B$ is a simple subalgebra of $\tilde{Q} \times B$.*

(iii) *There exist $S^1, S^2 \subseteq \tilde{R}$ such that for $i = 1, 2$, $S^i_{\mathscr{H}}$ and $S^i_{\mathscr{H}} \times B$ are r-maximal but not maximal subalgebras of $\tilde{Q}$ and $\tilde{Q} \times B$, respectively, $S^1_{\mathscr{H}}$ and $S^1_{\mathscr{H}} \times B$ are not contained in any maximal subalgebras of $\tilde{Q}$ and $\tilde{Q} \times B$, respectively, and $S^2_{\mathscr{H}}$ and $S^2_{\mathscr{H}} \times B$ are contained in maximal subalgebras of $\tilde{Q}$ and $\tilde{Q} \times B$, respectively.*

Recall that $\tilde{C}$ is the Boolean algebra generated by $\tilde{Q}$ and all the singletons $\{q\}$ for $q \in Q$. Thus, $\tilde{Q}$ can be thought of as a recursive subalgebra of $\tilde{C}$.

6.12. THEOREM (REMMEL [1978], [1979]). *Let S be an r.e. subalgebra of $\tilde{Q}$.*

(i) *If S is a maximal subalgebra of $\tilde{Q}$,* $\langle S \cup \mathrm{At}(\tilde{C})\rangle$ *is a maximal subalgebra of* $\tilde{C}$.

(ii) *If S is a simple subalgebra of $\tilde{Q}$,* $\langle S \cup \mathrm{At}(\tilde{C})\rangle$ *is a simple subalgebra of* $\tilde{C}$.

(iii) *If S is an r-maximal subalgebra of $\tilde{Q}$,* $\langle S \cup \mathrm{At}(\tilde{C})\rangle$ *is an r-maximal element of* $\tilde{C}$.

(iv) *If S is an r.e. subalgebra of $\tilde{Q}$ not contained in any maximal element of $\tilde{Q}$,* $\langle S \cup \mathrm{At}(\tilde{C})\rangle$ *is not contained in any maximal element of* $\tilde{C}$.

We should note also that for all Rec. BAs B if $S \subseteq \mathrm{At}(\tilde{N})$, then $S \equiv_T \langle S\rangle \equiv_T \langle S\rangle \times B$ and if $S \subseteq \tilde{R}$, then $S \equiv_T S_{\mathscr{H}} \equiv_T S_{\mathscr{H}} \times B$. Thus, all degree theoretic results about simple, maximal, and r-maximal sets transfer from $\mathscr{E}$ to $\mathscr{L}(\tilde{N} \times B)$, $\mathscr{L}(\tilde{Q} \times B)$, and $\mathscr{L}(\tilde{C})$ as well. In fact, the lattices $\mathscr{L}(\tilde{N} \times B)$, $\mathscr{L}(\tilde{Q} \times B)$, and $\mathscr{L}(\tilde{C})$ are at least as rich as $\mathscr{E}$ because based on work of REMMEL [1978a], [1979], DOWNEY [1986] has shown that there exist subalgebras U and W in such lattices such that the interval $[U, W] = \{V \mid V$ is r.e. subalgebra and $U \subseteq V \subseteq W\}$ is isomorphic to $\mathscr{E}^*$ modulo $=^*$.

Before leaving the lattices of r.e. subalgebras, we should note that there are genuinely new phenomena in such lattices. For example, we pointed out that in $\mathscr{E}$ maximal implies simple. Our next result will show that the analogous implication does not always hold in the lattice of r.e. subalgebras of a Rec. BA B. We refer the reader to NERODE and REMMEL [1985] for a survey of other differences between $\mathscr{E}$ and the lattices $\mathscr{L}(B)$.

6.13. THEOREM (REMMEL [1978]).
(i) *Every maximal subalgebra of $\tilde{N}$ is a simple subalgebra.*
(ii) *For any infinite Rec. BA D which is not isomorphic to $\tilde{N}$, there exists a Rec. BA B isomorphic to D such that there exist maximal subalgebras of B which are not simple subalgebras.*

Next let us consider the lattice of r.e. ideals of a Rec. BA. In contrast to Theorem 6.4, it is not the case that the lattice $\mathscr{L}I(B)$ is always undecidable. For example, consider the Rec. BA $\tilde{M}$ constructed in Theorem 4.1. Now $\tilde{M} \approx \tilde{N}$, but since the ideal generated by the atoms of $\tilde{M}$ is immune, it easily follows that the only r.e. ideals of $\tilde{M}$ are principal. Thus, the lattice $\langle \mathscr{L}I(\tilde{M}), \vee, \wedge, 0, 1\rangle$ is isomorphic to $\langle \tilde{N}, +, \cdot, 0, 1\rangle$ and hence has a decidable theory. A similar remark applies to all the superatomic BAs constructed in Theorem 4.3 so that the following holds.

6.14. THEOREM (REMMEL [1986a]). *Suppose B is a superatomic Rec. BA, then B is isomorphic to a Rec. BA D such that theory of $\mathscr{L}I(D)$ is decidable.*

On the other hand, it is not difficult to show that there are lots of Rec. BAs B such that $\mathscr{L}I(B)$ is undecidable. For example, consider the BA $\tilde{N}$. Now since both $\mathrm{At}(\tilde{N})$ and $\langle \mathrm{At}(\tilde{N})\rangle^{\mathrm{id}}$ are recursive, it is easy to see that the interval $[0, \langle \mathrm{At}(\tilde{N})\rangle^{\mathrm{id}}]$ in $\mathscr{L}I(B)$ is isomorphic to the lattice of r.e. sets $\mathscr{E}$ since every $U \in \mathscr{L}I(\tilde{N})$ such that $U \subseteq \langle \mathrm{At}(N)\rangle^{\mathrm{id}}$ is completely determined by the r.e. set of atoms $S_U = \mathrm{At}(N) \cap U$. Recently HARRINGTON [1982] and HERMANN [1984] have shown that the theory of $\mathscr{E}$ is undecidable. Although they use different codings, Harrington and Hermann both show how to code all Boolean pairs $(A, B, \leq)$, where A is a Rec. BA, B is a recursive subalgebra of A, and $\leq$ is the ordering relation on A. Now RUBIN [1976] showed that the theory of BAs with a distinguished subalgebra is undecidable. In fact if one considers the proof of Rubin's result given by BURRIS and MCKENZIE [1981], it is shown that the recursive Boolean pairs $(A, B, \leq)$ are enough to code up any finite graph $G = (V, E)$, where a graph is just an irreflexive symmetric relation E on the vertex set V. But LAVROV [1965] proved that the theory of graphs is finitely inseparable, i.e. there is no recursive set R which contains the Gödel numbers of all sentences true in all graphs but does not contain the Gödel number of any sentence refutable in some finite graph. It follows that any lattice $\mathscr{L}$ which contains an interval isomorphic to $\mathscr{E}$ has an undecidable theory since otherwise by combining either the Harrington or Hermann coding with the Burris and McKenzie coding, one could use $\mathrm{Th}(\mathscr{L})$ to build a recursive set R which separates the Gödel numbers of sentences true in all graphs from the Gödel numbers of sentence refutable in some finite graph. Thus, $\mathrm{Th}(\mathscr{L}I(\tilde{N}), \vee, \wedge, 0, 1)$ is undecidable. But note that $\mathscr{L}I(\tilde{Q})$ also has an interval isomorphic to $\mathscr{E}$. That is, we know by Theorem 2.4 that there is a recursive ideal $I \subseteq \tilde{Q}$ such that $\tilde{Q}/I$ is recursively isomorphic to $\tilde{N}$. Thus, if J is the inverse image of $\langle \mathrm{At}(\tilde{N})\rangle^{\mathrm{id}}$ under the canonical map from $\tilde{Q}$ to $\tilde{Q}/I$, then the interval $[I, J]$ in $\mathscr{L}I(\tilde{Q})$ is also isomorphic to $\mathscr{E}$. Similarly, the interval $[0, \langle \mathrm{At}(\tilde{C})\rangle^{\mathrm{id}}]$ in $\mathscr{L}I(\tilde{C})$ is isomorphic to $\mathscr{E}$. It thus follows that there is an interval isomorphic to $\mathscr{E}$ in $\mathscr{L}I(\tilde{N} \times B)$, $\mathscr{L}I(\tilde{Q} \times B)$, and $\mathscr{L}I(\tilde{C})$ for any Rec. BA B and hence each of these lattices has an undecidable theory. Combining these remarks with the fact that every infinite Rec. BA is isomorphic to a Rec. BA of the form $\tilde{N} \times B$, $\tilde{Q} \times B$, or $\tilde{C}$ for some Rec. BA B, we have the following.

6.15. THEOREM. *For every infinite Rec. BA D there exists a Rec. BA B isomorphic to D such that the theory of $\mathscr{L}I(B)$ is undecidable.*

Next let us consider the possibility of defining analogues of simple, maximal, and r-maximal elements in the lattice $\mathscr{L}I(B)$. Once again our first problem is to define a proper analogue of coinfinite in $\mathscr{L}I(B)$, but the solution in this case is not so clear. That is, one could try to imitate what we did for subalgebras by defining for any BA B and any ideals $I, J \subseteq B$, $I =^* J$ iff there exists finite sets $E_1, E_2 \subseteq B$ such that $\langle I \cup E_1\rangle^{\mathrm{id}} = \langle J \cup E_2\rangle^{\mathrm{id}}$. The problem with this approach is that $B = \langle\{1_B\}\rangle^{\mathrm{id}}$ so that every ideal $I \subseteq B$ satisfies $I =^* B$. It is possible to restrict ourselves to study only ideals I contained in some recursive nonprincipal ideal K so that in the interval $[0, K]$ of $\mathscr{L}I(B)$, $I \neq^* K$, is a reasonable definition of coinfinite again. The lattice ideals of such intervals have not been explored to any great extent but there are some general results of REMMEL [1980b] and DOWNEY [1986] which apply to such lattices. Also, DOWNEY [1987] has explored some notions of maximal in such lattices.

Another approach to defining the analogue of being coinfinite in the lattice $\mathscr{L}I(B)$ is to simply say I is coinfinite if B/I is an infinite BA. This approach also has some problems. For example, suppose we just let $B = \tilde{Q}$. Then we might say I is a *simple ideal* of $\tilde{Q}$ if I is r.e., $\tilde{Q}/I$ is infinite, and for any infinite r.e. ideal J, $I \cap J \neq \{0_{\tilde{Q}}\}$. But note if J is an infinite ideal in $\tilde{Q}$ such that $I \cap J = \{0_{\tilde{Q}}\}$, then any nonzero element $x \in J$ maps to an atomless element of $\tilde{Q}/I$ under the canonical mapping so that under such a definition, the "simplicity" of I is guaranteed by the property that $\tilde{Q}/I$ be an infinite atomic BA. But there are certainly many recursive ideals I such that $\tilde{Q}/I$ is an atomic BA so that such an approach leads to recursive simple elements. This is not necessarily a problem as such phenomena occur in other lattices of r.e. substructures (see NERODE and REMMEL [1985]). Of course, no matter how we define the concepts of simple, maximal, or r-maximal ideals, we have the problem of Rec. BAs like $\tilde{M}$ as produced in Theorem 4.1 where every r.e. ideal is principal. That is, we cannot expect uniform constructions of simple, maximal, and r-maximal elements in the lattices $\mathscr{L}I(B)$. Despite such problems, there are many interesting phenomena to study in the lattices of r.e. ideals of a Rec. BA and we end this section with a few examples.

Of course, it is always the case that the lattice of r.e. ideals of Rec. BA B is equivalent to the lattice of r.e. filters, $\mathscr{L}F(B)$. Now for the atomless BA $\tilde{Q}$, each element $F \in \mathscr{L}F(\tilde{Q})$ can be identified with an r.e. theory in a propositional logic L based on ω propositional letters $P_0, P_1, \ldots$ since the set of formulas of L modulo logical equivalence is isomorphic to $\tilde{Q}$. Thus, the following result of MARTIN and POUR-EL [1970] can be viewed as a result about $\mathscr{L}F(\tilde{Q})$.

6.16. THEOREM (MARTIN–POUR-EL [1970]). *There exists an r.e. theory T in L such that T is the theory generated by $\{P_i \mid i \in A\} \cup \{\neg P_i \mid i \in B\}$ for a pair of disjoint r.e. sets $A, B \subseteq \omega$, where* (i) *T has no decidable extension and* (ii) *every r.e. extension of T is principal over T.*

Now DOWNEY [1987] called theories T with the properties described in Theorem 6.16, *Martin–Pour-el* theories. Somewhat surprisingly, Martin–Pour-el theories cannot live in all nonzero r.e. degrees.

6.17. THEOREM (DOWNEY [1987]). (i) *There exists a Martin–Pour-el theory of degree* $0'$.

(ii) *There exists a Martin–Pour-el theory T such that the degree of T is low, i.e.* $\deg(T)' = 0'$.

(iii) *There exists a nonzero r.e. degree δ such that there is no Martin–Pour-el theory T with $T \leq_T \delta$.*

We can also get interesting results in $\mathscr{L}I(\tilde{Q})$ by ideals I generated by $\{P_i \mid i \in A\} \cup \{\neg P_i \mid i \in B\}$ for certain pairs of disjoint r.e. sets A and B. For example, suppose A and B are effectively inseparable r.e. sets, that is, $A \cap B = \emptyset$ and there is a partial recursive function of two variables $f(x, y)$ such that if W_i and W_j are disjoint r.e. sets such that $W_i \supset A$ and $W_j \supset B$, then $f(i, j)\downarrow$ and $f(i, j) \not\in W_i \cup W_j$. Thus, f effectively witnesses that A and B cannot be separated by a recursive set. Now if $I = \langle \{P_i \mid i \in A\} \cup \{\neg P_j \mid j \in B\} \rangle^{\mathrm{id}}$, where A and B are effectively inseparable r.e. sets, then Q/I is an effectively universal and homogeneous R.e. BA. More formally, given two R.e. BAs B_1 and B_2, we say that $\vec{x} = (x_1, \ldots, x_n)$ from B_1 has the *same type* as $\vec{y} = (y_1, \ldots, y_n)$ from B_2, written $\vec{x} \sim \vec{y}$, if the map $x_i \rightarrow y_i$ for $i = 1, \ldots, n$ can be extended to an isomorphism between the subalgebras $\langle \{x_1, \ldots, x_n\} \rangle$ and $\langle \{y_1, \ldots, y_n\} \rangle$ in B_1 and B_2, respectively. By an index k of an R.e. BA, $B = \langle B, +, \cdot, -, \equiv \rangle$, we mean that k codes r.e. indices for B and $\equiv$ and partial recursive indices for $\cdot$, $+$, and $-$.

6.18. DEFINITION. (i) An R.e. BA B is *uniformly effective homogeneous* (EUH) if there is a uniform effective procedure which given the index k of an R.e. BA B_k, $(x_1, \ldots, x_n)$ from B_k, $(y_1, \ldots, y_n)$ from B such that $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$, and $x \in B_k$ produces $y \in B_k$ such that $(x_1, \ldots, x_n, x) \sim (y_1, \ldots, y_n, y)$.

(ii) If B is an EUH BA and B is recursively isomorphic to Q/I for some r.e. ideal I of $\tilde{Q}$, then I is called an *EUH kernel.*

6.19. THEOREM (METAKIDES–NERODE [∞]). (i) *If* $I = \langle \{P_i \mid i \not\in A\} \cup \{\neg P_j \mid j \in B\} \rangle^{\mathrm{id}}$, *where A and B are effectively inseparable r.e. sets, then $\tilde{Q}/I$ is an EUH BA.*

(ii) *Any two EUH BAs are recursively isomorphic.*

(iii) *Any two EUH kernels in $\mathscr{L}I(\tilde{Q})$ differ by a recursive automorphism of $\tilde{Q}$.*

There are also some interesting results concerning the possibility of extending r.e. ideals to recursive maximal ideals I in a Rec. BA B. Here by a maximal ideal, we do not mean some analogue of maximal sets but the usual definition of a maximal ideal, namely, an ideal I is *maximal* iff $1_B \not\in I$ and for all $x \in B$ either $x \in I$ or $-x \in I$.

6.20. THEOREM (REMMEL [1986a]). (i) *If I is a recursive ideal in a Rec. BA B, then there exists a recursive maximal ideal M such that $I \subset M$.*

(ii) *There is no uniform effective procedure which given an index for a Rec. BA B and an r.e. index for a recursive ideal I of B produces an r.e. index of a recursive maximal ideal M such that $I \subseteq M$.*

(iii) *A Rec. BA B is not isomorphic to a Rec. BA D where there exists an r.e. ideal $I \in \mathscr{L}I(D)$ such that I is not contained in a recursive maximal ideal if and only if B is finite product of superatomic BAs.*

7. Recursive automorphisms of Rec. BAs

In this section we survey two different types of results about the automorphisms of a Rec. BA. In the first part of this section we survey results of MOROZOV [1983] and REMMEL [1986c] on the group of recursive automorphisms of a Rec. BA. Then in the last part of this section we survey results of GUICHARD [1983], REMMEL [1986c], and RUBIN [∞] on the number of automorphisms of the lattices $\mathscr{L}(B)$ and $\mathscr{L}I(B)$.

Now for countable BAs B, the group of automorphisms of B, $\mathrm{Aut}(B)$, tells us quite a bit about the structure of B as our next two results show.

7.1. THEOREM (RUBIN [1980]). *If B and C are countable BAs,* $|\mathrm{At}(B)| \geq 2$, *and the groups* $\mathrm{Aut}(B)$ *and* $\mathrm{Aut}(C)$ *are elementary equivalent, then B and C are elementary equivalent.*

7.2. THEOREM (MCKENZIE [1977]). *If B and C are countable BAs,* $|\mathrm{At}(B)| \geq 2$, *B has a maximal atomic element, and* $\mathrm{Aut}(B)$ *is isomorphic to* $\mathrm{Aut}(C)$, *then B is isomorphic to C.*

We shall see that the effective versions of both of these theorems are false. For a Rec. BA B, it is easy to see that the set of all recursive automorphisms of B is a group which we denote by $\mathrm{Aut}_r(B)$. Now we say a group $G = (G, \circ)$ is *recursive* if G is a recursive set and the product $\circ$ on G is a partial recursive function. We note that for any Rec. BA B, the set of partial recursive indices of recursive automorphisms of B, $\{x \mid \varphi_x \in \mathrm{Aut}_r(B)\}$, is not a recursive set since by RICE's [1953] Theorem, the only collections C of partial recursive functions of one variable such that $\{x \mid \varphi_x \in C\}$ is recursive are the empty set or the set of all partial recursive functions. Nevertheless, one might ask whether $\mathrm{Aut}_r(B)$ is at least isomorphic to a recursive group. Our next result shows that the answer to this question is resoundingly no.

7.3. THEOREM (MOROZOV [1983]). *Let B be an infinite Rec. BA.*

(i) *If B contains an atomless element, then* $\mathrm{Aut}_r(B)$ *is not isomorphic to a recursive group.*

(ii) *If B is an atomic BA such that* $\mathrm{At}(B)$ *is recursive, then* $\mathrm{Aut}_r(B)$ *is not isomorphic to a recursive group.*

The hypothesis in part (ii) of Theorem 7.3 that the atomic BA B have $\mathrm{At}(B)$ be recursive which is equivalent to saying B is a Dec. BA is necessary since Morozov showed that it is possible to have the set of all finite permutations of ω as the group of recursive automorphisms of a Rec. BA. That is, it is quite easy to see that if B is a Rec. BA and $a_1, \ldots, a_n$ is a finite set of atoms, then for any permutation φ of $a_1, \ldots, a_n$, the map $\hat{\varphi}$: $B \to B$ defined by $\varphi(x) = z \cdot x + \Sigma_{i=1}^{k} \varphi(a_{i_j})$, where $z = -(\Sigma_{i=1}^{n} a_i)$ and $x \cdot (-z) = \Sigma_{i=1}^{k} a_{i_j}$ is a recursive automorphism of B. Thus, the set of automorphisms induced by finite permutations of the atoms is always contained in $\mathrm{Aut}_r(B)$. The next result shows that for certain Rec. BAs B, $\mathrm{Aut}_r(B)$ consists entirely of automorphisms induced by finite permutations of the atoms since it is easy to see that for atomic BAs any automorphism is completely determined by its action on the atoms.

7.4. THEOREM (MOROZOV [1983]). *Let B be an infinite atomic Dec. BA. Then there exists a Rec. BA D isomorphic to B such that every recursive automorphism of B moves only finitely many atoms.*

We know by the results of Section 4 that there are many atomic Rec. BAs which are not isomorphic to any Dec. BA so that Theorem 7.4 does not cover all recursive BAs. However, by using an infinite injury priority argument plus algebraic results similar to those used in the proof of Theorem 5.2. Remmel has proved the following.

7.5. THEOREM (REMMEL [1987]). *Let B be any Rec. BA. Then there exists a Rec. BA D isomorphic to B such that every automorphism of D moves only finitely many atoms.*

As an immediate corollary of either Theorem 7.4 or Theorem 7.5, we get recursive counterexamples to an effective version of Theorem 7.2. For example, as a corollary to Theorem 7.5, we have the following.

7.6. COROLLARY (REMMEL [1987]). *If B_1 and B_2 are any two infinite atomic Rec. BAs, then there exist Rec. BAs D_1 and D_2 isomorphic to B_1 and B_2, respectively, such that* $\mathrm{Aut}_r(D_1) \approx \mathrm{Aut}_r(D_2)$.

Another finite injury priority argument can be used to show that the effective version of Theorem 7.1 is also false. That is, note that $\tilde{H}$ has elementary invariant $\langle 1, 1, 0\rangle$ and hence is not elementary equivalent to $\tilde{H} \times \tilde{H}$ which has elementary invariant $\langle 1, 2, 0\rangle$. Thus, the following theorem provides counterexamples to the effective version of Theorem 7.1.

7.7. THEOREM (REMMEL [∞]). *There exist Rec. BAs B_1 and B_2 isomorphic to $\tilde{H}$ and $\tilde{H} \times \tilde{H}$, respectively, such that* $\mathrm{Aut}_r(B_1) \approx \mathrm{Aut}_r(B_2)$. *Thus,* $\mathrm{Aut}_r(B_1) \approx \mathrm{Aut}_r(B_2)$ *but $B_1 \not\equiv B_2$.*

We note that all the counterexamples that are constructed to prove Theorems 7.4–7.7 have the property that the set of atoms are not recursive. The next results show that this must necessarily be the case since if a Rec. BA B has a sufficiently effective presentation, then $\mathrm{Aut}_r(B)$ tells us quite a bit about B.

7.8. THEOREM (MOROZOV [1983]). (i) *Suppose B is a Rec. BA such that* $\mathrm{At}(B)$ *and* $\mathrm{Atl}(B)$ *are recursive and C is a Rec. BA such that* $\mathrm{Aut}_r(B) \approx \mathrm{Aut}_r(C)$, *then* $\mathrm{At}(C)$ *and* $\mathrm{Atl}(C)$ *must also be recursive.*

(ii) *Suppose B is an atomic Dec. BA and C is a Rec. BA such that* $\mathrm{Aut}_r(B) \approx \mathrm{Aut}_r(C)$, *then B is recursively isomorphic to C.*

We should note that the hypothesis in part (ii) of Theorem 7.8 that B be atomic is necessary.

7.9. THEOREM (MOROZOV [1983]). *There exist Dec. BAs B_0 and B_1 such that* $\mathrm{Aut}_r(B_0) \approx \mathrm{Aut}_r(B_1)$ *but $B_0 \not\cong B_1$.*

Finally, we should note that the full automorphism group of BA does not determine whether or not it has a decidable presentation.

7.10. THEOREM (MOROZOV [1983]). *There exists BAs B_0 and B_1 such that* $\mathrm{Aut}(B_0) \approx \mathrm{Aut}(B_1)$ *while B_0 is a Dec. BA and B_1 is not isomorphic to any Dec. BA.*

Next we consider results about the number of automorphisms of the lattice $\mathscr{L}(B)$ of r.e. subalgebras and the lattice $\mathscr{L}I(B)$ of r.e. ideals of a Rec. BA B. Automorphisms of $\mathscr{L}(B)$ and $\mathscr{L}I(B)$ are intimately tied to automorphisms of B since it is a theorem of SACHS [1962] that every automorphism of the lattice of all subalgebras of a BA B is induced by an automorphism of B. GUICHARD [1983] noticed that Sachs' result also holds for $\mathscr{L}(B)$ if B is a Rec. BA.

7.11. THEOREM (SACHS–GUICHARD [1983]). *Let B be a Rec. BA, then every automorphism of $\mathscr{L}(B)$ is induced by an automorphism of B.*

It is also easy to see that every automorphism of $\mathscr{L}I(B)$ is induced by an automorphism of B since every automorphism of $\mathscr{L}I(B)$ is determined by its action on the principal ideals. Now there is great contrast between the number of automorphisms of $\mathscr{L}(B)$ and $\mathscr{L}I(B)$ for a Rec. BA. For example, just considering the atomless BA $\tilde{Q}$ and our two contrasting recursive presentations, the BAs of finite and cofinite sets $\tilde{N}$ and $\tilde{M}$ (see Theorem 4.1), we have the following.

7.12. THEOREM. (i) (GUICHARD [1983]). *Every automorphism of $\mathscr{L}(\tilde{Q})$ is induced by a recursive automorphism of $\tilde{Q}$.*

(ii) (REMMEL [1986c]). *There are $2^{\aleph_0}$ automorphisms of $\mathscr{L}I(\tilde{Q})$.*

(iii) (KALANTARI–REMMEL [∞]). *Every automorphism of $\mathscr{L}(\tilde{N})$ is induced by a recursive permutation of* $\mathrm{At}(\tilde{N})$.

(iv) (REMMEL [1986c]). *There are $2^{\aleph_0}$ automorphisms of $\mathscr{L}I(\tilde{N})$.*

(v) *Every automorphism of M induces an automorphism of $\mathscr{L}I(\tilde{M})$.*

Note that part (v) of Theorem 7.12 follows immediately from the fact that every automorphism of $\tilde{M}$ induces an automorphism of the principal ideals of $\tilde{M}$ and the fact that every r.e. ideal of $\tilde{M}$ is principal.

Now the best results about the number of automorphism of the lattices $\mathscr{L}(B)$ and $\mathscr{L}I(B)$ are the following.

7.13. THEOREM. *Let B be a Rec. BA.*

(i) (RUBIN [∞]). *If B has an atomless element, then there are only countably many automorphisms of B.*

(ii) (REMMEL [1986c]). *If B is an atomic Dec. BA, then every automorphism of B is induced by a recursive permutation of* $\mathrm{At}(B)$ *and hence there are only countably many automorphisms of $\mathscr{L}(B)$.*

7.14. THEOREM (REMMEL [1986c]). *Let B be an infinite Rec. BA, then B is isomorphic to a Rec. BA D such that there are $2^{\aleph_0}$ automorphisms of $\mathscr{L}I(D)$.*

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Keywords: Boolean algebra, recursive, r.e., arithmetic, subalgebra, lattice, ideal, automorphism, presentation.

MOS subject classification: primary 06E05; secondary 03G05, 03D80, 03F65.

CHAPTER 26

Lindenbaum–Tarski Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

1. Introduction

Every countable Boolean algebra is isomorphic to an interval algebra, i.e. a Boolean algebra generated by the left-closed right-open intervals of some linear order (see MOSTOWSKI and TARSKI [1939] or HANF [1974]; also Part I of this Handbook, Corollary 15.10). A countable Boolean algebra will usually be characterized by exhibiting an isomorphic interval algebra.

Our objective is to characterize the sentence algebras of countable theories. Every countable Boolean algebra is isomorphic to such a sentence algebra since the sentence algebra of a language with ω propositional symbols is the free Boolean algebra on ω generators, and a quotient of this algebra modulo an ideal is the sentence algebra whose theory is the corresponding dual ideal. Thus, classifying countable sentence algebras amounts to classifying arbitrary countable Boolean algebras.

First, we present techniques for constructing an isomorphism between two sentence algebras. A Cantor–Bernstein type theorem of Hanf constructs such an isomorphism from a suitable pair of elementary embeddings between the two model spaces. Second, we introduce measures and the rank diagram technique for constructing isomorphisms between sentence algebras and interval algebras. Rank diagrams are duals of Hanf's structure diagrams (HANF [1971]).

2. History

Alfred Tarski studied sentence algebras in the 1940s (TARSKI [1949b], [1949c]) and characterized the very simple sentence algebras for the theories of real-closed and algebraically-closed fields and of Boolean algebras (TARSKI [1949d], [1949e]). Mostowski and Tarski incorrectly calculated the sentence algebra for well-orderings (MOSTOWSKI and TARSKI [1949]) and Szmielew established that the sentence algebra for Abelian groups had no well-ordered base (SZMIELEW [1949]).

Little further progress was made unitl the 1960s when Willian Hanf developed the structure diagram technique and used it to characterize the first complicated sentence algebra, the algebra for the theory of an equivalence relation (HANF [1974]). Using the same technique, Roger Simons calculated the sentence algebra for unary functions (SIMONS [1971]) and the author calculated the sentence algebras for Abelian groups and well-orders (MYERS [1974]). Using rank diagrams, a model-theoretic dual of structure diagrams, the author calculated the sentence algebra for linear orders (MYERS [1980]). Similar diagrams were independently discovered by R.S. Pierce (PIERCE [1970], [1972] and Chapter 21 of this Handbook). Hanf's proof that a diagram characterizes a Boolean algebra up to isomorphism used a back-and-forth construction of Vaught (VAUGHT [1954]).

The diagram techniques succeeded admirably with decidable theories, but not with undecidable theories such as the theory of a binary relation. In the early 1960s Hanf proved a Cantor–Bernstein type theorem which he used to show that the sentence algebra of any finite language with an undecidable theory was isomorphic to the sentence algebra for a binary relation (HANF [1962]). In the mid-1970s, the author discovered that elementarily definable model maps between model spaces were the natural duals of homomorphisms between sentence algebras. The duality theory was used in HANF and MYERS [1983] to simplify Hanf's unpublished proof.

The most significant sentence algebra result is Hanf's characterization (HANF [1975]) of the algebra of the theory of a binary relation (see also PERETYAT'KIN [1982]). The proof uses a complicated and ingenious encoding of Minsky machine calculations using Robinson's nonperiodic tilings (ROBINSON [1971]). A corollary of the characterization is that every r.e. Boolean algebra (where $+$, $\cdot$ and $-$ are recursive and $=$ is an r.e. equivalence relation) is isomorphic to the sentence algebra of some finitely axiomatizable theory.

Open problems.

(a) Characterize the sentence algebras of group theory and of the theory of distributive or modular lattices.

(b) Modulo the axiom that there are at least two points, all sentence algebras of infinite languages are isomorphic. Which ones are recursively isomorphic?

3. Sentence algebras and model spaces

All sentences and theories will be in the finitary first-order logic $L_{\omega\omega}$ with equality. A *theory* is a *consistent* deductively closed set of sentences.

Let L be any countable language (i.e. similarity type or signature), T be any theory, and $\mathfrak{A}$ be any structure for L.

Let $\mathrm{Sent}(L)$ = the set of sentences of L, $\mathrm{Struct}(L)$ = the set of structures of similarity type L, $\mathrm{Th}(\mathfrak{A})$ = the theory of $\mathfrak{A}$, and $\mathrm{mod}(T)$ = the set of models of T. (To avoid foundational difficulties, assume all structures lie in some subuniverse which is a set in the full universe.)

Let $\mathrm{Sent}(L)/T$ = the set of equivalence classes of sentences mod $\equiv_T$, where

$$\varphi \equiv_T \psi \text{ iff } T \models \varphi \Leftrightarrow \psi ;$$

$$[\varphi] = \text{the equivalence class of } \varphi .$$

We identify L with the axiomless theory whose only theorems are the logical validities. An *elementary class* is a set $\mathscr{X}$ of structures of the form $\mathrm{mod}(T)$ for some set T of sentences. An *EC subclass* of $\mathscr{X}$ is a set of the form $\mathrm{mod}_T(\sigma) = \mathrm{mod}(T \cup \{\sigma\})$.

3.1. SENTENCE ALGEBRA DEFINITION. The *sentence algebra* of T is the Boolean algebra $\mathrm{Sentalg}(T) = \langle \mathrm{Sent}(L)/T, \wedge, \vee, \neg \rangle$, where the operation $\wedge$ sends $[\varphi]$ and $[\psi]$ to $[\varphi \wedge \psi]$ and likewise for $\vee$ and $\neg$. This algebra is isomorphic to the Boolean set algebra $\langle$The *EC* subclasses of $\mathrm{mod}(T)$, $\cup, \cap, \sim\rangle$, where complementation is relative to $\mathrm{mod}(T)$. We prefer "sentence algebra" or "Lindenbaum–Tarski algebra" to "Lindenbaum algebra" (see HENKIN, MONK and TARSKI [1971, p. 169]).

3.2. MODEL SPACE DEFINITION. The *model space* of T is the Boolean space $\mathrm{Mod}(T) = \langle \mathrm{mod}(T), \equiv,$ the topology whose closed sets are the elementary subclasses$\rangle$ where equality is to be interpreted as the elementary equivalence relation $\equiv$. The EC classes are the clopen sets. $\mathrm{Mod}(T)$ is the canonical dual of $\mathrm{Sentalg}(T)$.

Eq EXAMPLE. Let **Eq** be the elementary class of equality structures, that is, structures (sets essentially) for the empty similarity type. Let **1**, **2**, . . . , be the EC subclasses of one element, two element, . . . equality structures. Since elementarily equivalent structures are identified, Mod(**Eq**) is isomorphic to the one point compactification of ω.

3.3. HOMOMORPHISM LEMMA. *For any theories S and T and homomorphism h: $\mathrm{Sentalg}(S) \to \mathrm{Sentalg}(T)$, if T' is a complete theory extending T, then $h^{-1}[T']$ is a complete theory extending S.*

PROOF. Given the hypothesis on S, T and h, T' is a complete theory extending T iff T' is a complete theory closed under $\equiv_T$ iff T' is a prime dual ideal in $\mathrm{Sentalg}(T)$ only if $h^{-1}(T')$ is a prime dual ideal in $\mathrm{Sentalg}(S)$ iff $h^{-1}(T')$ is a complete theory extending S. $\square$

4. Model maps

For any structures $\mathfrak{A} = \langle A, ---\rangle$ and $\mathfrak{B} = \langle B, ***\rangle$ of possibly different similarity types, any C including A and B and any additional relations or operations . . . , let $\langle C, \mathfrak{A}, \mathfrak{B}, \ldots\rangle$ be the structure $\mathfrak{C} = \langle C, A, ---, B, ***, \ldots\rangle$ of a language with symbols to name the unary relations A and B and the relations and operations in the lists $---$, $***$ and . . . . The operations of $\mathfrak{A}$ and $\mathfrak{B}$ will be undefined on points not in A or B. For any sentence σ about $\mathfrak{A}$, let σ^A be the corresponding sentence for $\mathfrak{C}$ obtained by relativizing quantifiers to A and replacing the symbols for relations and operations of $\mathfrak{A}$ by the symbols which name the corresponding relations and operations of $\mathfrak{C}$. Similarly for sentences about $\mathfrak{B}$.

4.1. MAP DEFINITION. H is an *elementary map* between elementary classes $\mathscr{X}$ and $\mathscr{Y}$, written $H: \mathscr{X} \to \mathscr{Y}$, iff H is an elementary class of models of the form $\langle C, \mathfrak{A}, \mathfrak{B}, \ldots\rangle$ such that $\mathfrak{A} \in \mathscr{X}$, $\mathfrak{B} \in \mathscr{Y}$ and $(\forall \mathfrak{A} \in \mathscr{X})$ ($\exists \mathfrak{B} \in \mathscr{Y}$ which is unique up to elementary equivalence) $(\exists C, \ldots)$ $(\langle C, \mathfrak{A}, \mathfrak{B}, \ldots\rangle \in H)$.

All maps we construct will satisfy the stronger but easy-to-verify condition that $\mathfrak{B}$ be unique up to isomorphism. Our maps are functions in the usual sense if equality is interpreted as elementary equivalence.

Note: $\{\langle A, \langle A, ---\rangle, ***\rangle : \ldots\} = \{\langle C, \langle A, ---\rangle, ***\rangle : \forall x \;\; x \in A$ and $\ldots\}$.

EXAMPLE. Let **BoolAlg** be the class of Boolean algebras and let **Derivative: BoolAlg→BoolAlg** be the map

$$\{\langle B, \langle B,+,\cdot,-,0,1\rangle, \langle A,+',\cdot',-',0',1'\rangle, h\rangle : h: \langle B,+,\cdot,-,0,1\rangle \to \langle A,+',\cdot',-',0',1'\rangle \text{ is an epimorphism whose kernel is the ideal of atomic elements}\}\,.$$

Then **BoolAlg** is an elementary class and **Derivative** is an elementary map. Here $\langle B, \langle B,+,\cdot,-,0,1\rangle, \langle A,+',\cdot',-',0',1'\rangle, h\rangle = \langle B, B,+,\cdot,-,0,1, A,+',\cdot',-',0',1',h\rangle$, the extra $\langle\,\rangle$'s enclose the argument and value structures. The condition that h is an epimorphism is equivalent to the elementary condition $(\forall a\in A)(\exists b\in B)(h(b)=a)$ & $(\forall b,c\in B)(h(b\cdot c)=h(b)\cdot' h(c)$ & $h(b+c)=h(b)+' h(c)$ & $h(-b)=-' h(b)$ & $h(0)=0'$ & $h(1)=1')$. Similarly for the kernal condition.

We will always omit the routine verification that the classes described are elementary and determine their values up to isomorphism or elementary equivalence. For other examples, see GAIFMAN [1974], MYERS [1976] and PILLAY [1977].

EXAMPLE. Let **Succ: Eq→Eq** be the map

$$\{\langle C, \langle A\rangle, \langle B\rangle, f\rangle : f: A \xrightarrow{1-1} B \text{ and } (\exists! b\in B)(b \not\in \text{image of } f)\}\,.$$

Thus **Succ(1) = 2**, **Succ(2) = 3**, ...

For any model spaces $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ and any model maps H and J:

- Let $\mathrm{id}|\mathscr{X}: \mathscr{X}\to\mathscr{X}$ be the map $\{\langle A,\mathfrak{A},\mathfrak{A}\rangle : \mathfrak{A}=\langle A,---\rangle\in\mathscr{X}\}$.
- If $H: \mathscr{X}\to\mathscr{Y}$ and $J: \mathscr{Y}\to\mathscr{Z}$, let $J\circ H: \mathscr{X}\to\mathscr{Z}$ be $\{\langle F,\mathfrak{A},\mathfrak{C},D,\mathfrak{B},---,E,***\rangle : D,E\subseteq F$ and $\langle D,\mathfrak{A},\mathfrak{B},---\rangle\in H$ and $\langle E,\mathfrak{B},\mathfrak{C},***\rangle\in J\}$.
- $H(\mathfrak{A})=\mathfrak{B}$ means $(\exists C,---)(\langle C,\mathfrak{A},\mathfrak{B},---\rangle\in H)$.
- $H: \mathscr{X}\to\mathscr{Y}$ and $J: \mathscr{X}\to\mathscr{Y}$ are *equivalent*, written $H\equiv J$, iff $(\forall\mathfrak{A},\mathfrak{B},\mathfrak{B}')(H(\mathfrak{A})=\mathfrak{B}$ and $J(\mathfrak{A})=\mathfrak{B}'$ implies $\mathfrak{B}\equiv\mathfrak{B}')$.
- For $\mathscr{U}\subseteq\mathscr{Y}$ let $H^{-1}(\mathscr{U})=\{\mathfrak{A}\in\mathscr{X}: H(\mathfrak{A})=\mathfrak{B}$ for some $\mathfrak{B}\in\mathscr{U}\}$.
- H is an *elementary isomorphism* from $\mathscr{X}$ to $\mathscr{Y}$, written $H:\mathscr{X}\simeq\mathscr{Y}$, iff $H:\mathscr{X}\to\mathscr{Y}$ and for some $G:\mathscr{Y}\to\mathscr{X}$, $G\circ H\equiv\mathrm{id}|\mathscr{X}$ and $H\circ G\equiv\mathrm{id}|\mathscr{Y}$.
- $\mathscr{X}\simeq\mathscr{Y}$ iff $H:\mathscr{X}\simeq\mathscr{Y}$ for some H.

In the example above **Succ**: **Eq** $\simeq$ (**Eq****1**), where $X\backslash Y = X\cap(\sim Y)$.

4.2. ELEMENTARY EQUIVALENCE LEMMA. *Elementary maps between elementary classes preserve elementary equivalence.*

PROOF. Let $H: \mathscr{X}\to\mathscr{Y}$ be an elementary map between elementary classes of models. Suppose $\mathfrak{C}=\langle C,\mathfrak{A},\mathfrak{B},---\rangle$ and $\mathfrak{C}'=\langle C',\mathfrak{A}',\mathfrak{B}',---'\rangle$ are in H.

Claim. $\mathfrak{A}\simeq\mathfrak{A}'$ implies $\mathfrak{B}\equiv\mathfrak{B}'$. If $\mathfrak{A}\simeq\mathfrak{A}'$, then for some $C'',\mathfrak{B}'',---''$, $\langle C,\mathfrak{A},\mathfrak{B},---\rangle\simeq\langle C'',\mathfrak{A}',\mathfrak{B}'',---''\rangle\in H$ and, by the elementary uniqueness of values of elementary maps $\mathfrak{B}'\equiv\mathfrak{B}''\simeq\mathfrak{B}$.

Now we show: $\mathfrak{A}\equiv\mathfrak{A}'$ implies $\mathfrak{B}\equiv\mathfrak{B}'$. Let $\langle C_0\cup C_0',\ C_0,\mathfrak{A}_0,\mathfrak{B}_0,---_0,$

$C_0', \mathfrak{A}_0', \mathfrak{B}_0', ---_0'\rangle$ be a countable recursively saturated structure elementarily equivalent to $\langle C \cup C', C, \mathfrak{A}, \mathfrak{B}, ---, C', \mathfrak{A}', \mathfrak{B}', ---'\rangle$. By pseudo-uniqueness (see BARWISE and SCHLIPF [1976]), $\mathfrak{A} \equiv \mathfrak{A}'$ implies $\mathfrak{A}_0 \simeq \mathfrak{A}_0'$. Since H is closed under elementary equivalence, $\langle C_0, \mathfrak{A}_0, \mathfrak{B}_0, ---_0\rangle$ and $\langle C_0', \mathfrak{A}_0', \mathfrak{B}_0', ---_0'\rangle$ are in H and, by the claim, $\mathfrak{B}_0 \equiv \mathfrak{B}_0'$. Hence, $\mathfrak{B} \equiv \mathfrak{B}_0 \equiv \mathfrak{B}_0' \equiv \mathfrak{B}'$. □

Note: The Lemma implies that if $\mathcal{U} \subseteq \mathcal{Y}$ is closed under elementary equivalence, then so is $H^{-1}(\mathcal{U})$ and $H^{-1}(\sim\mathcal{U}) = \sim H^{-1}(\mathcal{U})$.

Between Boolean spaces, a map is continuous iff the inverse image of every closed set is closed iff the inverse image of every clopen set is clopen. Since in model spaces closed equals elementary and clopen equals EC subclass, the following theorem asserts that elementary maps are continuous.

4.3. CONTINUITY LEMMA. *For any elementary map between elementary classes, the inverse image of an elementary subclass is an elementary subclass and the inverse image of an EC subclass is an EC subclass.*

PROOF. Suppose $H: \mathcal{X} \to \mathcal{Y}$ is an elementary map and $\mathcal{U}$ is an elementary subclass of $\mathcal{Y}$. By the above note, $H^{-1}(\mathcal{U})$ is closed under elementary equivalence and hence it is elementary if it is closed under ultraproducts (BELL and SLOMSON [1971, p. 153]). Suppose D is an ultrafilter on a set I and $\mathfrak{A}_k \in H^{-1}(\mathcal{U})$ for $k \in I$. Then for each k there is a $\mathfrak{C}_k = \langle C_k, \mathfrak{A}_k, \mathfrak{B}_k, ---\rangle \in H$ with $\mathfrak{B}_k \in \mathcal{U}$. Since H and $\mathcal{U}$ are elementary, $\Pi_D C_k = \langle \Pi_D C_k, \Pi_D \mathfrak{A}_k, \Pi_D \mathfrak{B}_k, ---\rangle \in H$ and $\Pi_D \mathfrak{B}_k \in \mathcal{U}$. Hence, $\Pi_D \mathfrak{A}_k \in H^{-1}(\mathcal{U})$.

If $\mathcal{U}$ is EC, then $\mathcal{U}$ and $\sim\mathcal{U}$ are elementary. Hence, so are $H^{-1}(\mathcal{U})$ and $\sim H^{-1}(\mathcal{U}) = H^{-1}(\sim\mathcal{U})$. Hence, $H^{-1}(\mathcal{U})$ is also EC. □

5. Duality

By Stone's theorem, the Boolean algebra category is dual to the Boolean space category, i.e. they are isomorphic via a contravariant functor. The logic version is:

5.1. DUALITY THEOREM. *The category of sentence algebras and homomorphisms is dual to the category of elementary classes and elementary maps.*

5.2. DEFINITION OF THE DUALITY FUNCTORS $*$. For any theories S and T, any homomorphism h: Sentalg$(S) \to$ Sentalg(T), and elementary map H: mod$(T) \to$ mod(S),

$$\begin{array}{ccc}
\text{Sentalg}(S) & \longrightarrow & \text{Sentalg}(T) \\
\sigma & \xrightarrow{h} & \gamma \\
\text{mod}(T) & \longrightarrow & \text{mod}(S) \\
\mathfrak{B} & \xmapsto{H} & \mathfrak{A}\,.
\end{array}$$

Let

$$\begin{aligned}
&\mathrm{Sentalg}(T)^* = \mathrm{mod}(T)\,,\\
&\mathrm{mod}(T)^* = \mathrm{Sentalg}(T)\,,\\
&h^* = \{C, \mathfrak{A}, \mathfrak{B}\rangle : \mathfrak{A} \in \mathrm{mod}(T), \mathfrak{B} \in \mathrm{mod}(S)\\
&\qquad \text{and } (\forall\langle[\sigma],[\tau]\rangle \in h)(\mathfrak{B} \models \sigma \text{ iff } \mathfrak{A} \models \tau)\}\\
&\quad = \{\langle C, \mathfrak{A}, \mathfrak{B}\rangle : \mathfrak{A} \in \mathrm{mod}(T), \mathfrak{B} \in \mathrm{mod}(S)\\
&\qquad \text{and } \langle C, \mathfrak{A}, \mathfrak{B}\rangle \models \{\sigma^A \Leftrightarrow \tau^B : \langle[\sigma],[\tau]\rangle \in h\}\}
\end{aligned}$$

where $[\sigma]$ is σ's equivalence class in Sentalg(S), and let

$$\begin{aligned}
H^* = \{\langle[\sigma],[\tau]\rangle &\in \mathrm{Sentalg}(S) \times \mathrm{Sentalg}(T):\\
&(\forall\langle C, \mathfrak{A}, \mathfrak{B}, ---\rangle \in H)(\mathfrak{A} \models \tau \text{ iff } \mathfrak{B} \models \sigma)\}\,.
\end{aligned}$$

Note: $h^*(\mathfrak{A}) = \mathfrak{B}$ iff $\mathrm{Th}(\mathfrak{B}) = h^{-1}(\mathrm{Th}(\mathfrak{A}))$ for $\mathfrak{A} \in \mathrm{mod}(T)$ and $\mathfrak{B} \in \mathrm{mod}(S)$. $H^*([\sigma]) = [\tau]$ iff $\mathrm{mod}_T(\tau) = H^{-1}(\mathrm{mod}_S(\sigma))$.

Proof of Theorem 5.1. The duality of the sentence and model categories via * follows from the facts below:

- * is well defined. From Sentalg(T) one can determine T and hence mod(T). Similarly, mod(T) determines T and hence Sentalg(T). Likewise for h^* and H^*.
- h: Sentalg(S)$\to$Sentalg(T) implies h^*: mod(T)$\to$mod(S). By the second equality in its definition, h^* is an elementary class. For any $\mathfrak{A} \in \mathrm{mod}(T)$, $\mathrm{Th}(\mathfrak{A})$ is a complete theory extending T and, hence, by the Homomorphism Lemma, $h^{-1}(\mathrm{Th}(\mathfrak{A}))$ is a complete theory extending S. Thus, there is a $\mathfrak{B} \in \mathrm{mod}(S)$ which is unique up to elementary equivalence such that $\mathrm{Th}(\mathfrak{B}) = h^{-1}(\mathrm{Th}(\mathfrak{A}))$, i.e. by the note, $h^*(\mathfrak{A}) = \mathfrak{B}$.
- H: mod(T)$\to$mod(S) implies H^*: Sentalg(S)$\to$Sentalg(T). For any $[\sigma] \in$ Sentalg(S), $\mathrm{mod}_S(\sigma)$ is an EC subclass and, hence, by the Continuity Lemma, so is $H^{-1}(\mathrm{mod}_S(\sigma))$. Thus, there is a sentence τ, unique up to $\equiv_\tau$, such that $\mathrm{mod}_T(\tau) = H^{-1}(\mathrm{mod}_S(\sigma))$, i.e. by the note, such that $H^*([\sigma]) = [\tau]$. Finally, if $\langle[\sigma],[\tau]\rangle$ and $\langle[\sigma'],[\gamma']\rangle \in H^*$, then $\langle[\neg\sigma],[\neg\gamma]\rangle$, $\langle[\sigma \wedge \sigma'],[\tau \wedge \tau']\rangle$ and $\langle[\sigma \vee \sigma'],[\gamma \vee \gamma']\rangle$ also satisfy the condition defining H^*.
- $\mathrm{Sentalg}(T)^{**} = \mathrm{Sentalg}(T)$ and $\mathrm{mod}(T)^{**} = \mathrm{mod}(T)$. Clear.
- h: Sentalg(S)$\to$Sentalg(T) implies $h \equiv h^{**}$. Suppose $\langle\sigma, \tau\rangle \in h$ and $\langle\sigma, \tau'\rangle \in h^{**}$ and $\mathfrak{A} \in \mathrm{mod}(T)$. Then $\langle C, \mathfrak{A}, \mathfrak{B}, ---\rangle \in h^*$ for some $C, \mathfrak{B}, ---$ and $\mathfrak{A} \models \gamma$ iff (definition of h^*) $\mathfrak{B} \models \sigma$ iff (definition of h^{**}) $\mathfrak{A} \models \tau'$. Hence $\gamma \equiv_T \tau'$ and $h \equiv h^{**}$.
- H: mod(T)$\to$mod(S) implies $H \equiv H^{**}$. Similar argument.
- Finally, the transitivity and reflexivity of "iff" guarantees that, up to equivalence, * contravariantly preserves composition and sends identities to identities. □

5.3. COROLLARY. Sentalg$(S) \simeq$ Sentalg(T) *iff* mod$(S) \simeq$ mod(T).

PROOF. Isomorphism is defined in terms of composition and identities and hence is preserved by *. □

6. Repetition and Cantor–Bernstein

For sets of points, $A \uplus B = (A \times \{1\}) \cup (B \times \{0\})$, the disjoint union of A and B. For classes of models, we will be more elaborate. Let 1 and 0 be the true and false truth values. We write $=$ instead of $\Leftrightarrow$ between truth values.

6.1. DEFINITION. Suppose $\mathscr{X}$ and $\mathscr{Y}$ are elementary classes of the same language and $H: \mathscr{X} \to \mathscr{U}$ and $J: \mathscr{Y} \to \mathscr{V}$ are elementary maps where $\mathscr{U}$ and $\mathscr{V}$ also have the same language, and suppose p and q are the first propositional symbols not in the given languages.

Let $\mathscr{X} \uplus \mathscr{Y} = \{\langle \mathfrak{A}, p\rangle : (\mathfrak{A} \in \mathscr{X} \;\&\; p = 1) \vee (\mathfrak{A} \in \mathscr{Y} \;\&\; p = 0)\}$ where $\langle\langle A, ---\rangle, p\rangle = \{A, ---, p\rangle$.

Let $H \uplus J: \mathscr{X} \uplus \mathscr{Y} \to \mathscr{U} \uplus \mathscr{V}$ be the map

$$\begin{aligned} H \uplus J = \{\langle C, \langle \mathfrak{A}, p\rangle, \langle \mathfrak{B}, q\rangle, D, ---, E, ***\rangle :\\ (\langle D, \mathfrak{A}, \mathfrak{B}, ---\rangle \in H \;\&\; p = q = 1) \text{ or}\\ (\langle E, \mathfrak{A}, \mathfrak{B}, ***\rangle \in J \;\&\; p = q = 0)\}\,. \end{aligned}$$

6.2. LEMMA. *If $\mathscr{X}$ and $\mathscr{Y}$ are elementary classes of the same language as are $\mathscr{U}$ and $\mathscr{V}$, then*

(a) $\mathscr{X} \uplus \mathscr{Y} \simeq \mathscr{U} \uplus \mathscr{V}$ *if* $\mathscr{X} \simeq \mathscr{U}$ *and* $\mathscr{Y} \simeq \mathscr{V}$.

(b) $\mathscr{X} \uplus \mathscr{Y} \simeq \mathscr{Y} \uplus \mathscr{X}$.

(c) $\mathscr{X} \simeq \mathscr{U} \uplus \mathscr{V}$ *if* $\mathscr{U}$ *and* $\mathscr{V}$ *are complementary elementary subclasses of* $\mathscr{X}$.

PROOF. (a) $H \uplus J: \mathscr{X} \uplus \mathscr{Y} \simeq \mathscr{U} \uplus \mathscr{V}$, if $H: \mathscr{X} \simeq \mathscr{U}$ and $J: \mathscr{Y} \simeq \mathscr{V}$.

(b) Let $H: \mathscr{X} \uplus \mathscr{Y} \simeq \mathscr{Y} \uplus \mathscr{X}$ be $\{\langle C, \langle \mathfrak{A}, p\rangle, \langle \mathfrak{A}, q\rangle\rangle : (\mathfrak{A} \in \mathscr{X} \;\&\; p = 1 \;\&\; q = 0) \text{ or } (\mathfrak{A} \in \mathscr{Y} \;\&\; p = 0 \;\&\; q = 1)\}$.

(c) Let $H: \mathscr{X} \to \mathscr{U} \uplus \mathscr{V}$ be $\{\langle A, \mathfrak{A}, \langle \mathfrak{A}, p\rangle\rangle : (\mathfrak{A} \in \mathscr{U} \;\&\; p = 1) \text{ or } (\mathfrak{A} \in \mathscr{V} \;\&\; p = 0)\}$. □

If $C = A \uplus B$, we will write $\langle A \uplus B, A, ---, B, ***\rangle$ for $\langle C, A, ---, B, ***\rangle$. Note that $\langle A \uplus B, \mathfrak{A}, \mathfrak{B}\rangle \equiv \langle A' \uplus B', \mathfrak{A}', \mathfrak{B}'\rangle$ iff $\mathfrak{A} \equiv \mathfrak{A}'$ and $\mathfrak{B} \equiv \mathfrak{B}'$.

6.3. DEFINITION. Given elementary classes $\mathscr{X}$ and $\mathscr{Y}$ and elementary maps $H: \mathscr{X} \to \mathscr{U}$ and $J: \mathscr{Y} \to \mathscr{V}$ let $\mathscr{X} \times \mathscr{Y} = \{\langle A \uplus B, \mathfrak{A}, \mathfrak{B}\rangle : \mathfrak{A} = \langle A, ---\rangle \in \mathscr{X}, \mathfrak{B} = \langle B, ***\rangle \in \mathscr{Y}\}$, and let $H \times J: \mathscr{X} \times \mathscr{Y} \to \mathscr{U} \times \mathscr{V}$ be the map

$$\begin{aligned} H \times J = \{\langle E, \langle A \uplus B, \mathfrak{A}, \mathfrak{B}\rangle, \langle C \uplus D, \mathfrak{C}, \mathfrak{D}\rangle, F, ---, G, ***\rangle :\\ \langle F, \mathfrak{A}, \mathfrak{C}, ---\rangle \in H \text{ and } \langle G, \mathfrak{B}, \mathfrak{D}, ***\rangle \in J\}\,. \end{aligned}$$

6.4. LEMMA. *For any elementary classes $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{U}$, and $\mathscr{V}$, $\mathscr{X}\times\mathscr{Y}\simeq\mathscr{U}\times\mathscr{V}$ if $\mathscr{X}\simeq\mathscr{U}$ and $\mathscr{Y}\simeq\mathscr{V}$.*

PROOF. $H\times J\colon \mathscr{X}\times\mathscr{Y}\simeq\mathscr{U}\times\mathscr{V}$, if $H\colon\mathscr{X}\simeq\mathscr{U}$ and $J\colon\mathscr{Y}\simeq\mathscr{V}$. $\square$

6.5. DEFINITION. For any elementary classes $\mathscr{X}$ and $\mathscr{Y}$,
- $\mathscr{X}\le\mathscr{Y}$ iff $\mathscr{X}\simeq\mathscr{Z}\subseteq\mathscr{Y}$ for some EC subclass $\mathscr{Z}$ of $\mathscr{Y}$,
- $\mathscr{X}$ *repeats* in $\mathscr{Y}$ iff $\mathscr{X}\times\mathbf{Eq}\le\mathscr{Y}$, and
- $\mathscr{X}$ is *repetitive* iff $\mathscr{X}$ repeats in itself.

6.6. LEMMA. *For any elementary classes $\mathscr{X}$ and $\mathscr{Y}$,*
(a) *$\le$ is transitive,*
(b) *if $\mathscr{X}\le\mathscr{Y}$, $\mathscr{Y}$ repeats in $\mathscr{U}$, and $\mathscr{U}\le\mathscr{V}$, then $\mathscr{X}$ repeats in $\mathscr{V}$,*
(c) *if $\mathscr{X}$ repeats in $\mathscr{Y}$, then $\mathscr{Y}\simeq\mathscr{X}\uplus\mathscr{Y}$.*

PROOF. (a) Clear.
(b) $\mathscr{X}\times\mathbf{Eq}\le\mathscr{Y}\times\mathbf{Eq}\le\mathscr{U}\le\mathscr{V}$.
(c) Suppose $\mathscr{X}$ repeats in $\mathscr{Y}$, then $H\colon\mathscr{X}\times\mathbf{Eq}\simeq\mathscr{X}_\omega$ for some H and some EC subclass $\mathscr{X}_\omega$ of $\mathscr{Y}$. Let $\mathscr{X}_1$ and $\mathscr{X}_{\omega-1}$ be the images under H of the EC subclasses $\mathscr{X}\times 1$ and $\mathscr{X}\times(\mathbf{Eq}\backslash 1)$. Clearly, $\mathscr{X}\simeq\mathscr{X}\times 1\simeq\mathscr{X}_1$ and, since $\mathbf{Eq}\simeq(\mathbf{Eq}\backslash 1)$, $\mathscr{X}_\omega\simeq\mathscr{X}\times\mathbf{Eq}\simeq\mathscr{X}\times(\mathbf{Eq}\backslash 1)\simeq\mathscr{X}_{\omega-1}$. Thus

$$\begin{aligned}\mathscr{Y}&\simeq\mathscr{X}_\omega\uplus(\sim\mathscr{X}_\omega)\simeq\mathscr{X}_1\uplus\mathscr{X}_{\omega-1}\uplus(\sim\mathscr{X}_\omega)\\&\simeq\mathscr{X}\uplus\mathscr{X}_\omega\uplus(\sim\mathscr{X}_\omega)\simeq\mathscr{X}\uplus\mathscr{Y}\,.\quad\square\end{aligned}$$

6.7. MODEL-SPACE CANTOR–BERNSTEIN THEOREM. *For any elementary classes $\mathscr{X}$ and $\mathscr{Y}$ such that either $\mathscr{X}$ or $\mathscr{Y}$ is repetitive, $\mathscr{X}\le\mathscr{Y}$ and $\mathscr{Y}\le\mathscr{X}$ implies $\mathscr{X}\simeq\mathscr{Y}$.*

PROOF. Suppose $\mathscr{X}$ repeats in itself, $\mathscr{X}\le\mathscr{Y}$, and $\mathscr{Y}\le\mathscr{X}$.
$\mathscr{Y}\le\mathscr{X}$ and $\mathscr{X}$ repeats in $\mathscr{X}\Rightarrow$, by Lemma 6.6(b), $\mathscr{Y}$ repeats in $\mathscr{X}$
$\Rightarrow$, by Lemma 6.6(c), $\mathscr{X}\simeq\mathscr{Y}\uplus\mathscr{X}$.
$\mathscr{X}$ repeats in $\mathscr{X}$ and $\mathscr{X}\le\mathscr{Y}\Rightarrow$, by Lemma 6.6(b), $\mathscr{X}$ repeats in $\mathscr{Y}$
$\Rightarrow$, by Lemma 6.6(c), $\mathscr{Y}\simeq\mathscr{X}\uplus\mathscr{Y}$.
Thus $\mathscr{X}\simeq\mathscr{Y}\uplus\mathscr{X}\simeq\mathscr{X}\uplus\mathscr{Y}\simeq\mathscr{Y}$. $\square$

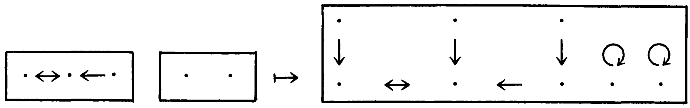
Without the repetitive condition, the theorem fails (HANF [1957, p. 208]).

7. Language isomorphisms

Let L_R be the language whose only symbol is the binary relation symbol R and let L_S be the language consisting of the ternary relation symbol S.
Let $H\colon\mathscr{X}\le\mathscr{Y}$ mean $H\colon\mathscr{X}\simeq\mathscr{Z}$ for some EC subclass $\mathscr{Z}$ of $\mathscr{Y}$.

7.1. LEMMA. *L_R is repetitive.*

PROOF. Let H: Struct$(L_R) \times \mathbf{Eq} \leq$ Struct(L_R) be the map (see Fig. 26.1) $\{\langle D, A \uplus B, \langle A, R\rangle, \langle B\rangle, \langle C, R'\rangle\rangle$: $A \uplus B \subseteq C$, $(\forall x \in B)(xR'x)$, $(\forall y \in A)(\exists! x \notin A \uplus B)(xR'y)$, $(\forall x \notin A \uplus B)(\exists! y \in A)(xR'y)$, $(\forall x, y \in A)(xR'y \Leftrightarrow xRy)$, and R' relates no other pairs of points$\}$. The range is the EC subclass $\{\langle C, R'\rangle$: if $U =$ the complement of the range of R' and $V =$ the image of U under R', then R' is one-to-one on U and $R' =$ the identity outside of $U \cup V\}$.



$\langle A, R\rangle$ $\langle B\rangle$ $\langle C, R'\rangle$ □

Fig. 26.1

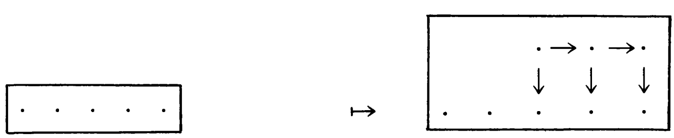
7.2. THEOREM. $L_R \simeq L_S$.

PROOF. By Lemma 7.1 and the Model-space Cantor–Bernstein Theorem, it suffices to show that $L_R \leq L_B$ and $L_S \leq L_R$.

Let H: $L_R \leq L_S$ be the map $\{\langle A, \langle A, R\rangle, \langle B, S\rangle\rangle$: $B = A$, $(\forall x, y, z)(S(x, y, z) \Leftrightarrow R(x, y))\}$. The range is the EC subclass $\{\langle B, S\rangle$: $(\forall x, y, z, z')(S(x, y, z) \Leftrightarrow S(x, y, z'))\}$.

In a structure $\langle A, R\rangle$, an *S-unit* is a 6-tuple $\langle x', y', z', x, y, z\rangle$ such that $x'Ry'Rz'$, $x'Rx$, $y'Ry$, $z'Rz$ and $x', \ldots, z$ are not R-related in any other way. The S-unit is said to *point out* the triple $\langle x, y, z\rangle$.

Let G: $L_S \leq L_R$ be the map $\{\langle A, \langle B, S\rangle, \langle A, R\rangle\rangle$: $(A \backslash B) =$ domain of R, every element of $A \backslash B$ is in a unique S-unit, if $\langle x, y, z\rangle$ is pointed out by an S-unit then $x, y, z \in B$ and $S(x, y, z)$, if $S(x, y, z)$ then $\langle x, y, z\rangle$ is pointed out by a unique S-unit, R relates no pairs other than those required above$\}$. The range is the EC subclass $\{\langle A, R\rangle$: each element in R's domain is in a unique S-unit and R relates no pairs other than those in S-units$\}$ (Fig. 26.2).



$\langle B, S\rangle = \langle\{1, 2, 3, 4, 5\}, \{\langle 3, 4, 5\rangle\}\rangle$ $\langle A, R\rangle$ □

Fig. 26.2

7.3. DEFINITION. A language is

finite	iff it has only finitely many symbols,
undecidable	iff it has ≥ 1 relation or operation symbols of ≥ 2 places or ≥ 2 unary operation symbols,
functional	iff it has exactly one unary operation symbol and all other symbols are unary relations, constants, or propositions,
monadic	iff it has only unary relation, constant, and propositional symbols.

The above proof is a typical case of the proof of:

7.4. HANF'S LANGUAGE ISOMORPHISM THEOREM. *Any two finite undecidable languages have isomorphic sentence algebras. Any two finite functional languages have isomorphic sentence algebras.*

PROOF. See HANF [1962] and HANF and MYERS [1983]. □

8. Measures

The measures below (MYERS [1977]) are a modification, appropriate for countable Boolean algebras, of the classical σ-algebra measures. The main result is that Boolean algebras with the same measure are isomorphic. The result is a generalization of Hanf's structure diagram result; its proof is a Vaught-type back-and-forth argument (VAUGHT [1954]); its statement is similar to a theorem of Caratheodory for countably additive measures (ROYDEN [1963]). The previous sections constructed isomorphisms between sentence algebras; this section constructs isomorphisms for arbitrary countable Boolean algebras.

A *pointed monoid* is a structure $\mathcal{M} = \langle M, +, 0, m\rangle$ such that $+$ is associative, 0 is a two sided identity, and m is any element of M.

In a Boolean algebra, $c = a \dot{+} b$ means c is the disjoint union of a and b.

8.1. MEASURE DEFINITION. A Boolean algebra $B = \langle B, +, \cdot, -, 0, 1\rangle$ is *measured* by a pointed monoid $\mathcal{M} = \langle M, +, 0, 1\rangle$ iff there is a function $\mu\colon B \to M$, called a *measure*, such that for any $a, b \in B$ and $m, n \in M$:

$$\mu(a) = 0 \quad \text{iff } a = 0\,,$$

$$\mu(1) = 1\,,$$

$$\mu(a \dot{+} b) = \mu(a) + (b)\,,$$

and

$$\mu(a) = m + n \text{ implies } a = b \dot{+} c \text{ for some } b, c \text{ with } \mu(b) = m \text{ and } \mu(c) = n\,.$$

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The Boolean algebra of measurable subsets of the reals modulo sets of measure zero is measured by the monoid $\langle [0, \infty], +, 0, \infty\rangle$ with μ being Lebesgue measure.

The Boolean algebra of measurable subsets of $[0, 1]$ modulo sets of measure zero is measured, via Lebesgue measure, by $\langle [0, \infty), +, 0, 1\rangle$.

The Boolean algebra of finite and cofinite subsets of ω is measured by $\langle \omega + 1, +, 0, \omega\rangle$ via the measure which sends a set to its cardinality.

An atomless Boolean algebra is measured by $\langle \{0, \infty\}, +, 0, \infty\rangle$ via the measure which sends 0 to 0 and everything else to ∞.

See also the canonical measure of Definition 12.1 and the free measure following Theorem 12.2.

8.2. DEFINITION. If A and B are both measured by $\mathcal{M}$ via μ_A and μ_B, respectively, then a map $f: A \to B$ is *measure preserving* iff $\mu_A(a) = \mu_B(f(a))$ for all a in A.

8.3. LEMMA. *If A and B are measured by $\mathcal{M}$ via μ_A and μ_B, if $f: C \to B$ is a measure preserving monomorphism from a finite subalgebra C of A, and if $\langle C \cup \{a\} \rangle$ is the subalgebra generated by C and an element a of A, then f can be extended to a measure preserving monomorphism $g: \langle C \cup \{a\} \rangle \to B$.*

PROOF. Assume the hypothesis. Each atom of $\langle C \cup \{a\} \rangle$ is of the form $c \cdot a$ or $c \cdot -a$ for some atom c of C. For each atom c of C, $\mu_B(f(c)) = \mu_A(c) = \mu_A((c \cdot a) \dotplus (c \cdot -a)) = \mu_A(c \cdot a) + \mu_A((c \cdot -a)$. Hence, by the Measure Definition, $f(c) = b_{c \cdot a} \dotplus b_{c \cdot -a}$ for some $b_{c \cdot a}$ and $b_{c \cdot -a}$ in B with $\mu_B(b_{c \cdot a}) = \mu_A((c \cdot a)$ and $\mu_B(b_{c \cdot -a} = \mu_A(c \cdot -a)$. Let g be the unique extension of f to $\langle C \cup \{a\} \rangle$ with $g(c \cdot a) = b_{c \cdot a}$ and $g(c \cdot -a) = b_{c \cdot -a}$ for each atom c of C. By choice of $b_{c \cdot a}$ and $b_{c \cdot -a}$ and additivity, g is measure preserving. $\square$

8.4. MEASURE ISOMORPHISM THEOREM. *Countable Boolean algebras measured by the same pointed monoid are isomorphic.*

PROOF. Suppose A and B are countable Boolean algebras measured by the pointed monoid $\mathcal{M}$ via μ_A and μ_B. By the first two conditions of the measure definition, if either A or B is a one element Boolean algebra, so is the other. Hence, we may assume $0 \neq 1$ in both A and B.

Let $a_1, a_3, a_5, \ldots$ and $b_2, b_4, b_6, \ldots$ be enumerations of A and B, respectively. Let f_0 be the measure preserving isomorphism between the subalgebras $\{0, 1\}$ of A and of B. Suppose f_k is a measure preserving isomorphism from a finite subalgebra of A onto a finite subalgebra of B. If k is even (odd), use the above lemma applied to f_k (to f_k^{-1}) to extend f_k to a measure preserving isomorphism f_{k+1} whose finite domain includes a_{k+1} (whose finite range includes b_{k+1}). Let $f = \bigcup_k f_k$. Then $f: A \cong B$. $\square$

9. Rank diagrams

Rank diagrams (MYERS [1978]) are a generalization of Cantor–Bendixson rank provided we regard ordinals as partial orders. They are duals of Hanf's structure diagrams (HANF [1974]) and are closely related to Pierce's diagrams (PIERCE [1970], [1972], Chapter 21 in this Handbook, and MUTH [1975]). See also FLUM and ZIEGLER [1980, pp. 78–113].

A *diagram* is a transitive antisymmetric structure $\mathcal{R} = \langle R, < \rangle$ with a greatest element. Let $\leq$ and $\lneq$ be the associated reflexive and irreflexive partial orders. Elements of a diagram are called *ranks*. A rank r is *reflexive* (*irreflexive*) iff $r < r$ ($r \not< r$); it is *separated* iff it is not the infimum of a decreasing chain of ranks strictly above it. Ranks p and q are *incomparable* iff $p \not< q$ and $q \not< p$ and *strictly incomparable* if, in addition, $p \neq q$. A sequence $q_0, q_1, \ldots$ is *strictly increasing* iff $q_k \lneq q_{k+1}$ for each k; likewise for strictly decreasing.

9.1. RANK DIAGRAM DEFINITION. For any Boolean space $\mathscr{X}$ with a countable base and any diagram $\mathscr{R} = \langle R, < \rangle$, $\mathscr{R}$ is a *rank diagram* for $\mathscr{X}$ iff there is an onto function $f: \mathscr{X} \to R$ such that for all $p, q \in R$, $x \in \mathscr{X}$:

(1) if $p > q$ and $f(x) = p$ then $x = \lim x_k$ for some $x_0, x_1, \ldots$ with $x_k \neq x$ and $f(x_k) = q$ for each k;

(2) for any $x_0, x_1, \ldots$, if $x = \lim x_k$ and $x_k \neq x$ and $f(x_k) = q$ for each k, then $f(x) > q$;

(3) if q is separated, $f(\lim x_k) \leq q$, and the $f(x_k)$'s are pairwise strictly incomparable or strictly decreasing, then $f(x_k) \leq q$ for all but finitely many k;

(4) if $f(x)$ is not separated, then $x = \lim x_k$ for some $x_0, x_1, \ldots$ with $f(x_k) \gneq f(x)$ for each k;

(5) if q is the greatest rank and $q \not< q$, then $(\exists! x \in \mathscr{X})\ f(x) = q$.

If f is as above, $\mathscr{R}$ is said to rank $\mathscr{X}$ *via* f. For $x \in \mathscr{X}$, $f(x)$ is called the *rank* of x.

Condition (2) is equivalent to:

(2′) If $f(x) \not> q$, then x is isolated from all rank q points distinct from itself.

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The one-point compactification of ω is ranked by $\langle \{0, 1\}, \langle\ \rangle$ where the rank of an isolated point is 0 and the rank of the limit point is 1.

The model space of the theory of an equivalence relation with, for example, 13 equivalence classes is ranked by $\langle \{0, 1, \ldots, 13\}, \langle\ \rangle$ where an equivalence relation structure with exactly n infinite equivalence classes has rank n.

See Theorems 10.3 and 11.1 for more examples.

See Definition 15.2 for the canonical rank diagram.

ASSUMPTION. For the rest of this section assume $\mathscr{X}$ is a Boolean space with a countable base which is ranked via f by a diagram $\mathscr{R}$ whose greatest rank is 1.

For $A \subseteq \mathscr{X}$ and $x \in A$: x is a *max point* of A and $f(x)$ is a *max rank* of A iff there is no $y \in A$ such that $f(x) \lneq f(y)$, and x is a *greatest point* iff $f(x) > f(y)$ for all $y \in A$ not equal to x.

9.2. LEMMA. (a) *The rank of any point in a clopen set is $\leq$ a max rank of the set.*

(b) *A max point of a clopen set has a separated rank.*

(c) *A point of separated rank is a greatest point of some clopen set and, if x is of separated rank and $f(x') < f(x)$, then x is a greatest point of some clopen set containing x'.*

(d) *A clopen set has only finitely many max ranks.*

(e) *A clopen set has only finitely many points of a given irreflexive max rank.*

PROOF. (a) By Zorn's Lemma, it suffices to show that every chain U of ranks of points of a clopen set A has an upper bound. We may assume U has no largest rank. Let $B_0, B_1, \ldots$ be a listing of $\mathscr{X}$'s clopen sets. Let $A_0 = A$. B_n splits A_n into two parts, let A_{n+1} be a part whose points have cofinally many ranks in U. Then $\bigcap_n A_n = \{x\}$ for some $x \in A$.

Claim: $f(x)$ is an upper bound for U. For each n, q is $<$ the rank of some $y \in A_n$, hence, by (1) of the definition, y is a limit of rank q points, almost all of which must be in A_n. Thus, each A_n has points of rank q. Thus, x is a limit of such points and, by (2), $f(x) \succ q$.

(b) By (4) a point in a clopen set of nonseparated rank is not maximal.

(c) Suppose x has a separated rank and $f(x') \prec f(x)$. Let $Y = \{y \in \mathscr{X}\colon y \neq x$ and $f(y) \not\prec f(x)\}$. We show that $x \not\in$ the closure of Y. Suppose $x = \lim y_k$ for some $y_k \in Y$. By Ramsey's Theorem (RAMSEY [1930]) and possibly taking a subsequence, we may assume that the $f(y_k)$'s are either equal, strictly increasing, strictly incomparable, or strictly decreasing. If they are equal or strictly increasing, x is a limit of distinct points of rank $f(y_0)$ (the strictly increasing case needs condition (1) of the definition which implies that each y_k and hence x is a limit of rank $f(y_0)$ points). By (2), $f(y_0) \prec f(x)$, contradicting $y_0 \in Y$. If the $f(y_k)$'s are strictly incomparable or strictly decreasing, condition (3) requires that $f(y_k) \prec f(x)$ for almost all k contradicting $y_k \in Y$. The same argument shows $x' \not\in$ the closure of Y since $f(y_0)$ or $f(y_k) \prec f(x')$ and $f(x') \prec f(x)$ implies $f(y_0)$ or $f(y_k) \prec f(x)$. Hence, x and x' are separated from Y by some clopen set A. But then x is a greatest point of A.

(d) Suppose a clopen set A has infinitely many max ranks $q_0, q_1, \ldots$. Maximality implies the q_k's are strictly incomparable. For each q_k, pick a point $x_k \in A$ with rank q_k. By compactness and possibly taking a subsequence, we may suppose $\lim x_k$ exists. By (a), $f(\lim x_k) \preceq$ some max rank q of A which, by (b), is separated. By (3) of the definition, $q_k = f(x_k) \preceq q$ for almost all k contradicting the maximality of the q_k.

(e) Suppose a clopen set A has infinitely many max points $x_0, x_1, \ldots$ with an irreflexive rank q. By possibly taking a subsequence, we may suppose $\lim x_k$ exists. By (2), $q \prec f(\lim x_k)$. Since $q \not\prec q$, $q \precneqq f(\lim x_k)$ contradicting the maximality of q. $\square$

Let $\mathcal{M} = \langle M, +, 0, 1\rangle$ be the pointed monoid where 1 is the greatest rank and $\langle M, +, 0\rangle$ is the commutative monoid freely generated by the separated ranks of $\mathscr{R}$ subject to the absorbtivity condition:

$$r + s = r \quad \text{if } s \prec r\,.$$

A sum $r_1 + \cdots + r_n$ of ranks is *reduced* iff the r_k's are incomparable. A rank is *maximal* in a sum $r_1 + \cdots + r_n$ iff it is not $\precneqq$ another rank in the sum.

9.3. LEMMA. (a) *Two sums of ranks are equal in $\mathcal{M} \Leftrightarrow$ every rank which occurs maximally in one occurs in the other and maximal irreflexive ranks occur the same number of times in each sum.*

(b) *Every element of $\mathcal{M}$ is equal to a reduced sum which is unique up to a permutation of its ranks.*

PROOF. (a) $\Rightarrow$: No application of the absorbtivity condition can add a new or delete an old maximal rank or change the number of occurrences of an irreflexive maximal rank. $\Leftarrow$: Two sums which satisfy the right-hand side of the $\Leftrightarrow$ can differ only with respect to occurrences of nonmaximal ranks and the number of occurrences of reflexive ranks but the absorbtivity condition can add or delete any occurrence of a nonmaximal rank or change a nonzero number of occurrences of a maximal reflexive rank.

(b) A sum is reduced iff all ranks are maximal and no reflexive rank (which is comparable with itself) occurs more than once. A sum of ranks may be reduced by omitting all nonmaximal ranks in the sum and then omitting all but one occurrence of each maximal reflexive rank. A reduced sum is characterized up to a permutation by listing its maximal ranks and, for each irreflexive rank, the number of times it occurs. By (a), reduced sums which differ by more than a permutation cannot be equal in $\mathcal{M}$. □

Let $\mathrm{Clop}(\mathscr{X}) = \langle$clopen subsets of $\mathscr{X}$, $\cup$, $\cap$, $\sim$, $\emptyset$, $X\rangle$ be the Boolean algebra of clopen subsets of $\mathscr{X}$. Let $\mu\colon \mathrm{Clop}(\mathscr{X}) \to \mathcal{M}$ be the function such that $\mu(\emptyset) = 0$ and for any nonempty clopen set A, $\mu(A) = \Sigma\, n_k r_k$ where $\{r_0, r_1, \ldots\}$ is the finite set of max (and hence separated) ranks of A and where $n_k =$ the number of points in A of rank r_k if r_k is irreflexive and $n_k = 1$ if r_k is reflexive. As defined, $\mu(A)$ is a reduced sum.

9.4. LEMMA. (a) *$\mu(A) = 0$ iff $A = \emptyset$.*

(b) *$\mu(A) = f(x)$ if x is a greatest point of A. Hence $\mu(X) = 1$.*

(c) *$\mu(A \uplus B) = \mu(A) + \mu(B)$.*

PROOF. (a) By definition, $\mu(\emptyset) = 0$. By Lemma 9.2(a), $A \neq \emptyset$ implies it has a max rank and so $\mu(A) \neq 0$.

(b) Clear.

(c) We verify the condition of Lemma 9.3(a). Any maximal rank of $\mu(A \uplus B)$ is the rank of a max point of $A \uplus B$ and hence a max point of A or of B and hence occurs in $\mu(A) + \mu(B)$. Any maximal rank of $\mu(A) + \mu(B)$ is the rank of max point of A or B which is a max rank of $A \uplus B$ and hence occurs in $\mu(A \uplus B)$. The number of occurrences of an irreflexive maximal rank in $\mu(A \uplus B)$ equals the number of max points of $A \uplus B$ of that rank which equals the number of such points in A plus the number in B which equals the number of occurrences of that rank in $\mu(A) + \mu(B)$. □

9.5. LEMMA. *For any clopen set A, if $\mu(A) = r_1 + \cdots + r_m$, where the r_k are separated ranks, then for some $A_1, \ldots, A_m$, $A = A_1 \uplus \cdots \uplus A_m$ and $\mu(A_k) = r_k$.*

PROOF. Reduced case. Assume $\mu(A) = r_1 + \cdots + r_m$ and that $r_1 + \cdots + r_m$ is reduced. Suppose $x \in A$. By Lemma 9.2(a), $f(x) \leq f(y)$ for some max point $y \in A$. By Lemma 9.2(b, c), $f(y)$ is separated and y is a greatest point of some clopen set B containing x. Hence, every point in A is in a clopen subset, $B \cap A$ for example, of A with a greatest point which is a max point of A. By compactness $A = B_1 \cup \cdots \cup B_n$ for some clopen sets $B_1, \ldots, B_n$ with greatest points $x_1, \ldots, x_n$, respectively, which are max points of A. If $f(x_j) < f(x_k)$, then x_k is a greatest point of $B_j \cup B_k$ and hence we may omit B_j and replace B_k by $B_k \cup B_j$. Repeating this process we obtain $A = B_1 \cup \cdots \cup B_n$, where $f(x_k)$ and $f(x_j)$ are incomparable for $k \neq j$. Let $A_k \subseteq B_k$ be clopen subsets such that $A = A_1 \uplus \cdots \uplus A_n$. Since $x_k \not\in B_j$ for $k \neq j$, $x_k \in A_k$ and hence x_k is a greatest point of A_k as well as a max point of A. By Lemma 9.4(c) and 9.4(b), $\mu(A) = \mu(A_1) + \cdots + \mu(A_n) = f(x_1) + \cdots + f(x_n)$ which is reduced since the $f(x_k)$'s are incomparable. Since reduced sums are unique, we have $m = n$ and, by possibly permuting the A_k's, we have $\mu(A_k) = r_k$.

Nonreduced case. By the proof of Lemma 9.3 and by possibly permuting the terms we may suppose that $r_1 + \cdots + r_n = r_1 + \cdots + r_k$, where $k < n$ and $r_1 + \cdots + r_k$ is reduced. By the above there are clopen sets $A_1, \ldots, A_k$ with $A = A_1 \uplus \cdots \uplus A_k$ and $\mu(A_k) = r_k$. Consider r_{k+1}. For some $j \leq k$, $r_{k+1} < r_j$. Let $x \in A_j$ be a point of rank r_j. By (1) of the rank definition, x is a limit of distinct points of rank r_{k+1}. Hence, there is a $y \neq x$ in A_j of rank r_{k+1}. By Lemma 9.2(c) there is a clopen subset A_{k+1} of A_j containing y as a greatest point. Hence, $\mu(A_{k+1}) = r_{k+1}$. We may assume $x \not\in A_{k+1}$. Hence, $x \in A_k \backslash A_{k+1}$, and $\mu(A_j \backslash A_{k+1}) = r_j$. Now replace A_j by $A_j \backslash A_{k+1}$. Then $A = A_1 \uplus \cdots \uplus A_k \uplus A_{k+1}$ and $\mu(A_j) = r_j$. Repeating the above process for $k+2$, $k+3, \ldots, n$ we get $A = A_1 \uplus \cdots \uplus A_n$ with $\mu(A_j) = r_j$ for each j. □

9.6. THEOREM. *$\mathcal{M}$ measures* Clop($\mathscr{X}$) *via μ.*

PROOF. Lemma 9.4 takes care of the first two conditions of the measure definition. To verify the last condition, suppose $\mu(A) = p + q$. Let $p = p_1 + \cdots + p_m$ and $q = q_1 + \cdots + q_n$, where p_k and q_k are separated ranks. Then $\mu(A) = p_1 + \cdots + p_m + q_1 + \cdots + q_n$ and, by Lemma 9.5, $A = B_1 \uplus \cdots \uplus B_m \uplus C_1 \uplus \cdots \uplus C_n$ for some clopen sets such that $\mu(B_k) = p_k$ and $\mu(C_k) = q_k$. Let $B = B_1 \uplus \cdots \uplus B_m$ and $C = C_1 \uplus \cdots \uplus C_n$. Then $A = B \uplus C$ and $\mu(B) = p$ and $\mu(C) = q$. □

9.7. RANK ISOMORPHISM THEOREM. *If $\mathscr{X}$ and $\mathscr{Y}$ are Boolean spaces with countable bases which have a common rank diagram, then $\mathscr{X} \simeq \mathscr{Y}$.*

PROOF. By Theorem 9.6, if $\mathscr{X}$ and $\mathscr{Y}$ have a common rank diagram, their dual algebras are measured by a common monoid and, by the Measure Isomorphism Theorem, are isomorphic. □

10. Interval algebras and cut spaces

10.1. INTERVAL ALGEBRA DEFINITION. For any linear order type γ with first element, the *interval algebra* of γ is the Boolean set algebra generated by the left-closed right-open (including $[x, \infty)$) intervals of some linear order of type γ (MOSTOWSKI and TARSKI [1939]; see also Section 15, Part I, of this Handbook).

A cut is a splitting of γ (more precisely, of some given order of type γ) into a nonempty initial segment and a terminal segment. *A* cut is *in* an interval $[a, b)$ if a is in the initial segment and b is in the terminal segment (∞ is considered to be in every terminal segment). Thus, each interval determines an interval of cuts.

10.2. CUT SPACE DEFINITION. Let $\mathscr{X}(\gamma)$, the *cut space* of γ, be the Boolean space whose universe is the set of cuts in γ and whose topology is generated by the above intervals of cuts.

The clopen sets are the finite unions of left-closed right-open intervals. Cut spaces are the cannonical duals of interval algebras.

Let η, η_0, and ω be the order types of the rationals, the non-negative rationals, and the natural numbers, respectively.

10.3. CUT SPACE THEOREM. *The cut space of each order listed in Table* 26.1 *is ranked by the transitive closure of the diagram on its left.*

Table 26.1

Diagram with largest element	Linear order
$\overset{\downarrow}{\cdot}$	1
$\cdot \to \cdot$	ω
$\cdot \to \cdot \to \cdot$	ω^2
$\overset{\circ}{\cdot}$	η_0
$\cdot \to \overset{\circ}{\cdot}$	$\omega\eta_0$
$\cdot \to \cdot \leftarrow \overset{\circ}{\cdot}$	$(\omega + \eta)$
$\cdot \to \overset{\circ}{\cdot} \leftarrow \overset{\circ}{\cdot}$	$(1 + \eta_0)\eta_0$
$\cdot \to \cdot \to \cdot \to \cdot \ \dots \ \cdot$	ω^{ω}
$\cdot \to \cdot \to \cdot \to \cdot \ \dots \ \overset{\circ}{\cdot}$	$\omega^{\omega}\eta_0$
$\cdot \to \cdot \to \cdot \to \cdot \ \dots \ \overset{\circ}{\cdot} \leftarrow \overset{\circ}{\cdot}$	$(\eta_0 + \omega^{\omega})\eta_0$
$\underset{0}{\cdot} \to \underset{1}{\cdot} \to \underset{2}{\cdot} \to \underset{3}{\cdot} \ \dots \ \underset{\omega}{\overset{\circ}{\cdot}} \to \underset{\infty}{\overset{\circ}{\cdot}} \leftarrow \underset{\eta}{\overset{\circ}{\cdot}}$	$(\eta_0 + \omega^{\omega}\eta_0)\eta_0$

PROOF. We prove only the last example. Suppose $\mathcal{R} = \langle \{0, 1, 2, 3, \dots, \omega, \infty, \eta\}, < \rangle$, where $<$ is the transitive closure of the above diagram. Cuts in $\mathcal{X}((\eta_0 + \omega^{\omega}\eta_0)\eta_0)$ are ranked in $\mathcal{R}$ as follows: $(\eta_0 + \omega^{\omega}\eta_0)\eta_0$ is a dense union of segments of type $(\eta_0 + \omega^{\omega}\eta_0)$. A cut lying between these segments has rank ∞. Otherwise the cut lies within such a segment. If it lies within the η_0 subsegment or between the η_0 and $\omega^{\omega}\eta_0$ subsegments, it has rank η. Otherwise it lies within the $\omega^{\omega}\eta_0$ segment. Each $\omega^{\omega}\eta_0$ segment is a dense union of ω^{ω} segments. If the cut lies between ω^{ω} segments, it has rank ω. If the cut lies within an ω^{ω} segment, its lower half has a final segment of type ω^{n} for some n and the cut has rank n.

To verify the conditions of the rank definition, note that:

(1) For any $r > s$ each cut of rank r is a limit of cuts of rank s.

(2) For any r and cut x of rank r if $r \not> s$ then x lies in an interval which isolates it from all cuts of ranks other than itself.

(3), (4), and (5) are trivial since there is no set of more than two incomparable ranks, there is no infinite decreasing sequence of ranks, there are no nonseparated ranks, and $\infty < \infty$. $\square$

11. Finite monadic languages

Using the terminology of Definition 7.3, every language is undecidable, functional, or monadic. By the Language Isomorphism Theorem, the sentence algebras of a countable undecidable language and of the language of a binary relation symbol are isomorphic and, by HANF [1975], the latter is the universal r.e. Boolean algebra, i.e. the unique r.e. Boolean algebra such that every r.e. Boolean algebra is isomorphic to one of its factors. Similarly, the sentence algebras of a countable functional language and of the language of a single unary operation are isomorphic and, by SIMONS [1971], the latter is the interval algebra of $(\eta_0 + \omega^{\omega}\eta_0)\eta_0$. To complete the classification of finite languages, we must characterize the sentence algebras of monadic languages which have only propositional, unary predicate and individual constant symbols.

11.1. MONADIC LANGUAGE THEOREM. *The model space of a language consisting of n unary predicates is isomorphic to* $\mathscr{X}(\omega^{2n})$.

PROOF. Let $\mathrm{Mod}(L)$ be the model space of a language L consisting of n unary predicates. Then $\mathrm{Mod}(L)$ and $\mathscr{X}(\omega^{2n})$ are both ranked by the diagram $\mathscr{R} = \langle\{0, 1, \ldots, 2^n\}, <\rangle$. The rank of a cut in $\mathscr{X}(\omega^{2n})$ is the unique k such that the lower half of the cut has a final segment of type ω^k. The conditions of the Rank diagram Definition are easily verified.

Finally, $\mathrm{Mod}(L)$ is also ranked by $\mathscr{R}$. Two structures are $\equiv_{\mathrm{m}}$-equivalent iff they satisfy the same sentences of quantifier depth m. A *component* of a structure of $\mathrm{Mod}(L)$ is an indiscernibility equivalence class where two points are indiscernible iff they satisfy exactly the same unary predicates. The rank of a structure is the number of its components with infinitely many points. Recall that if X and Y have $\geq m$ elements, then the equality structures $\langle X\rangle$ and $\langle X\rangle$ are $\equiv_m$-equivalent since player II can win an m-step Ehrenfeucht–Fraïssé game between them. The following observations verify the corresponding conditions of the Rank Diagram Definition:

(1) For any ranks $i > j$, if $\mathfrak{A}$ is a structure of rank i, then for any m a structure $\equiv_m$-equivalent to $\mathfrak{A}$ but of rank j may be constructed by replacing all but j of $\mathfrak{A}$'s infinite components by m-element components.

(2′) If $\mathfrak{A}$ has rank i and $i \leq j$, then $2^n - i$ components of $\mathfrak{A}$ are finite and $\mathfrak{A}$ can be isolated from all structures of rank j which are elementarily distinct from $\mathfrak{A}$ by the sentence which specifies the cardinality of each of these finite components.

(3) $\mathscr{R}$ has no infinite incomparable or strictly decreasing sequences.

(4) $\mathscr{R}$ has no nonseparated ranks.

(5) All components of a structure of rank 2^n are infinite and hence any two such structures are elementarily equivalent and thus equal in $\mathrm{Mod}(L)$. □

For any language L, symbol s, and sentence θ, $L \cup \{s\}$ is the result of adding the symbol s to L and $L + \theta$ is the theory with language L and axiom θ. If $C = \{c_1, \ldots, c_n\}$ is a set of constants, let "universe $= C$" be the sentence $(\forall x)(x = c_1 \vee x = c_2 \vee \cdots \vee x = c_n)$.

11.2. LEMMA. (a) $\mathrm{Mod}(L \cup \{p\}) \simeq \mathrm{Mod}(L) \uplus \mathrm{Mod}(L)$ *if p is a propositional symbol not in L.*

(b) $\mathrm{Mod}(L \cup C) \simeq \mathscr{X}(k) \uplus (\mathrm{Mod}(L) \times \mathscr{X}(k))$ *if L has no constants, C is a finite set of constants and $k =$ the number of isomorphism types of models of "universe $= C$".*

(c) $\mathscr{X}(\alpha) \uplus \mathscr{X}(\beta) \simeq \mathscr{X}(\alpha + \beta)$, *where α and β are order types with initial elements.*

(d) $\mathscr{X}(\alpha) \times \mathscr{X}(k) \simeq \mathscr{X}(\alpha \cdot k)$, *where α is an order type with initial element and $k < \omega$.*

PROOF. (a) $\mathrm{Mod}(L \cup \{p\}) \simeq \mathrm{Mod}(L \cup \{p\} + p) \uplus \mathrm{Mod}(L \cup \{p\} + \neg p) \simeq \mathrm{Mod}(L) \uplus \mathrm{Mod}(L)$.

(b) Suppose C, k, and L are as hypothesized and that $C = \{c_1, \ldots, c_n\}$. Each $L \cup C$-structure $\mathfrak{A} = \langle A, \ldots, a_1, \ldots, a_n\rangle$ can be split into two disjoint structures: the $L \cup C$ substructure $\mathfrak{A}_C$ with universe $\{a_1, \ldots, a_n\}$ and the L structure $\mathfrak{A}_{\sim C}$ with universe $A \backslash \{a_1, \ldots, a_n\}$. Then $\mathfrak{A}_C \models$ "universe $= C$" and $(\mathfrak{A} \mapsto \langle A, \mathfrak{A}_C, \mathfrak{A}_{\sim C}\rangle)$: $\mathrm{Mod}(L \cup C +$ "universe $\neq C$") $\simeq \mathrm{Mod}(L \cup C +$ "universe $= C$") $\times \mathrm{Mod}(L)$. Thus,

$$\begin{aligned}\mathrm{Mod}(L \cup C) &\simeq \mathrm{Mod}(L \cup C + \text{“universe} = C\text{”}) \uplus \mathrm{Mod}(L \cup C \\ &\qquad + \text{“universe} \neq C\text{”}) \\ &\simeq \mathrm{Mod}(L \cup C + \text{“universe} = C\text{”}) \uplus (\mathrm{Mod}(L \cup C \\ &\qquad + \text{“universe} = C\text{”}) \times \mathrm{Mod}(L)) \\ &\simeq \mathscr{X}(k) \uplus (\mathscr{X}(k) \times \mathrm{Mod}(L))\end{aligned}$$

(c) and (d) are clear. □

The model space of any finite monadic language may be computed using the above theorem and lemma.

MONADIC LANGUAGE EXAMPLE. $\mathrm{Mod}(\{U, V, c, d, p\}) \simeq \mathscr{X}(\omega^4 \cdot 40)$, where $\mathrm{Mod}(\{U, V, c, d, p\})$ is the model space of the language with unary predicates U and V, constants c and d, and propositional symbol p.

PROOF. By Theorem 11.1, $\mathrm{Mod}(\{U, V\}) \simeq \mathscr{X}(\omega^4)$. $\mathrm{Mod}(\{U, V, c, d\} +$ "universe $= \{c, d\}$") has 20 isomorphism types: 4 with a 1-element universe and 16 with a 2-element universe. By the above lemma,

$$\begin{aligned}\mathrm{Mod}(\{U, V, c, d\}) &\simeq \mathscr{X}(20) \uplus (\mathrm{Mod}(\{U, V\}) \times \mathscr{X}(20)) \\ &\simeq \mathscr{X}(20) \uplus (\mathscr{X}(\omega^4) \times \mathscr{X}(20)) \\ &\simeq \mathscr{X}(20) \uplus (\mathscr{X}(\omega^4 \cdot 20)) \\ &\simeq \mathscr{X}(20 + \omega^4 \cdot 20) \simeq \mathscr{X}(\omega^4 \cdot 20)\,.\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{Mod}(\{U, V, c, d, p\}) &\simeq \mathrm{Mod}(\{U, V, c, d\}) \uplus \mathrm{Mod}(\{U, V, c, d\}) \\ &\simeq \mathscr{X}(\omega^4 \cdot 20) \uplus \mathscr{X}(\omega^4 \cdot 20) \simeq \mathscr{X}(\omega^4 \cdot 40)\,.\end{aligned}$$

12. Factor measures

We show that every Boolean algebra has a cannonical measure.

B is a *factor* of a Boolean algebra A iff $A \simeq B \times C$ for some C. For any $a \in A$, $A \upharpoonright a$ is the quotient of A modulo the dual ideal generated by a. B is a factor of A iff it is isomorphic to $A \upharpoonright a$ for some $a \in A$.

12.1. DEFINITION. For any Boolean algebra A, the *factor monoid* for A is the pointed monoid ⟨the set of isomorphism types of factors of A and their finite cartesian products, the cartesian product operation on isomorphism types, the 1-element Boolean algebra's isomorphism type, A's isomorphism type⟩.

For more about factor monoids, see DOBBERTIN [1982–1984].

12.2. THEOREM. *Every Boolean algebra is measured by its factor monoid.*

PROOF. A Boolean algebra A is measured by its factor monoid via the measure which sends an element a of A to the isomorphism type of $A \upharpoonright a$. □

There is also a "free" measure naturally associated with each Boolean algebra. A Boolean algebra $\langle B, +, \cdot, -, 0, 1\rangle$ is measured by the pointed monoid freely generated by the pointed partial monoid $\langle B, \dot{+}, 0, 1\rangle$ via the natural injection.

13. Measure monoids

We give sufficient conditions for a countable pointed monoid to be a measure monoid for some Boolean algebra.

13.1. DEFINITION. A pointed monoid $\mathcal{M} = \langle M, +, 0, 1\rangle$ is a *refinement monoid* iff it is commutative and positive ($m + n = 0$ iff $m = n = 0$) and has the following refinement property:

for any finite sums $\Sigma_j m_j$ and $\Sigma_k n_k$ of elements of M, if $\Sigma_j m_j = \Sigma_k n_k$ then for some finite array p_{jk} of elements of M, $\Sigma_k p_{jk} = m_j$ and $\Sigma_j p_{jk} = n_k$ for each j and k.

These monoids are a generalization due to HANF [1975] of Tarski's refinement algebras (TARSKI [1949a, Definition 11.26, conditions I–IV]). See also DOBBERTIN [1983 & 1984].

13.2. THEOREM (Hanf). *Every countable refinement monoid measures some countable Boolean algebra.*

PROOF. For any Boolean algebra $B = \langle B, \ldots \rangle$ and any refinement monoid $\mathcal{M} = \langle M, +, 0, 1 \rangle$, a map $\mu: B \to M$ is a *premeasure* iff it satisfies the first three conditions of the Measure Definition 8.1.

Suppose B is finite, $\mu: B \to M$ is a premeasure, $a \in B$, and $\mu(a) = m + n$. Then there are finite extensions $B' \supseteq B$ and $\mu' \supseteq \mu$ such that $a = b \dotplus c$ for some b and c in B' with $\mu'(b) = m$ and $\mu'(c) = n$. *Proof.* Let $a_1, \ldots, a_k$ be the atoms of B below a. Then $\Sigma_j\, \mu(a_j) = \mu(a) = m + n$. By the refinement property, there is an array $p_{01}, \ldots, p_{0k}, p_{11}, \ldots, p_{1k}$ such that $p_{0j} + p_{1j} = \mu(a_j)$, $\Sigma_j\, p_{0j} = m$, and $\Sigma_j\, p_{1j} = n$. Let B' be the Boolean algebra freely generated by B and two new generators b and c subject to the relations $a = b \dotplus c$, $b \cdot a_j = 0$ if $p_{0j} = 0$, and $c \cdot a_j = 0$ if $p_{1j} = 0$. Let μ': $B' \to M$ be the unique premeasure extending μ for which $\mu'(b \cdot a_j) = p_{0j}$ and $\mu'(c \cdot a_j) = p_{1j}$. Then B' and μ' are the desired extensions.

Let $\mathcal{M}$ be a countable refinement monoid. If $0 = 1$ in $\mathcal{M}$, then $\mathcal{M}$ measures the 1-element Boolean algebra. Suppose $0 \neq 1$ in $\mathcal{M}$. Hence, $0 \neq 1$ in any Boolean algebra measured by $\mathcal{M}$. Let B_0 be the 2-element Boolean algebra and μ_0 the unique $\mathcal{M}$-valued premeasure on B_0. By the previous paragraph and the countability of $\mathcal{M}$ one can construct an increasing sequence $B_0 \subseteq B_1 \subseteq \cdots$ of finite Boolean algebras and an increasing sequence $\mu_0 \subseteq \mu_1 \subseteq \cdots$ of $\mathcal{M}$-valued premeasures on the B_j such that for any j, any $a \in B_j$, and any $m, n \in M$, if $\mu_j(a) = m + n$ then for some k and some $b, c \in B_k$, $a = b \dotplus c$ and $\mu_k(b) = m$ and $\mu_k(c) = n$. Let $B = \bigcup_j B_j$ and $\mu = \bigcup_j \mu_j$. Since the first three conditions of the Measure Definition 8.1 are preserved under directed unions and since the last condition is satisfied by construction, μ is an $\mathcal{M}$-valued measure on B. □

14. Orbits

For a Boolean space with a countable base, we give necessary and sufficient conditions for two points of the space to be in the same orbit, i.e. to have an automorphism sending one to the other (MYERS [1977]). In this section, "isomorphism" will be synonymous with "homeomorphism".

14.1. COROLLARY. (a) *In a Boolean space with countable base, two points are in the same orbit iff every clopen neighborhood of one includes a clopen neighborhood which is isomorphic to a clopen neighborhood of the other.*

(b) *In a ranked Boolean space, points of the same rank are in the same orbit.*

PROOF. This follows from Lemma 14.4 below.

Here's a counterexample due to Harold Reiter for arbitrary spaces. In the space $[0, 1] \cup [2, 3)$, every neighborhood of 0.5 is isomorphic to a neighborhood of 2.5 and vice versa but 0.5 and 2.5 are not in the same orbit.

In what follows, $\mathscr{X}$ and $\mathscr{Y}$ will be Boolean spaces with countable bases and with points a and b. $A, B, C, \ldots$ will be clopen sets. We shall also regard clopen sets as Boolean spaces with the subspace topology and $A \simeq B$ will mean A and B are isomorphic as spaces. As usual, $A \backslash B = A \cap (\sim B)$.

14.2. DEFINITION. $\mathrm{Nhd}(a, A) =$ the class of clopen neighborhoods of a which are included in A.

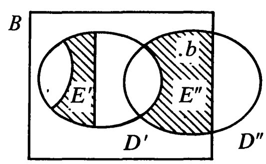
$\langle \mathscr{X}, a\rangle \simeq \langle \mathscr{Y}, b\rangle$ iff some isomorphism from $\mathscr{X}$ to $\mathscr{Y}$ sends a to b.

$\langle \mathscr{X}, a\rangle \approx \langle \mathscr{Y}, b\rangle$ iff $\mathscr{X} \simeq \mathscr{Y}$ and every clopen neighborhood of a includes a clopen neighborhood which is isomorphic to a clopen neighborhood of b (the isomorphism need not send a to b) and vice versa.

14.3. LEMMA. (a) *If* $\langle \mathscr{X}, a\rangle \approx \langle \mathscr{Y}, b\rangle$, $A \in \mathrm{Nhd}(a, \mathscr{X})$ *and* $\mathscr{B} \in \mathrm{Nhd}(b, \mathscr{Y})$ *and* $A \simeq B$, *then* $\langle A, a\rangle \approx \langle B, b\rangle$.

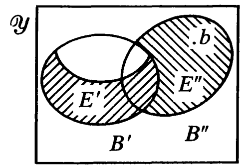
(b) *If* $\langle \mathscr{X}, a\rangle \approx \langle \mathscr{Y}, b\rangle$ *and* $A_0 \in \mathrm{Nhd}(a, \mathscr{X})$, *then there are* $A \in \mathrm{Nhd}(a, A_0)$ *and* $\mathscr{B} \in \mathrm{Nhd}(b, \mathscr{Y})$ *with* $\langle A, a\rangle \approx \langle \mathscr{B}, b\rangle$ *and* $\mathscr{X} \backslash A \simeq A\mathscr{Y} \backslash \mathscr{B}$.

PROOF. (a) Assume the hypothesis and suppose $C_0 \in \mathrm{Nhd}(a, A)$ (the vice versa case is symmetrical). We want a $C \in \mathrm{Nhd}(a, C_0)$ and a $D \in \mathrm{Nhd}(b, B)$ with $C \simeq D$.



If the image of C_0 under the isomorphism from A to B contains b, let $C = C_0$ and let D be this image. Otherwise, pick $C \in \mathrm{Nhd}(a, C_0)$ and $D'' \in \mathrm{Nhd}(b, \mathscr{Y})$ such that $C \simeq D''$ and let D' be the image of C under the isomorphism from A to B. Since D' and D'' are isomorphic to C, $D'' \simeq D'$. Let $E'' = D'' \cap B \cap (\sim D')$ and let E' be the image of E'' under the isomorphism from D'' to D'. Then $D = (D' \backslash E') \cup E''$ is the desired neighborhood of b in B. $C \simeq D$ since $C \simeq D'$ and $D' \simeq D$ by the map which maps E' to E'' and $D' \backslash E'$ to itself.

(b) Assume the hypothesis. Hence, $\mathscr{X} \simeq \mathscr{Y}$. We want an $A \in \mathrm{Nhd}(a, A_0)$ and a $B \in \mathrm{Nhd}(b, \mathscr{Y})$ with $\langle A, a\rangle \approx \langle B, b\rangle$ and $\mathscr{X} \backslash A \simeq \mathscr{Y} \backslash B$.



If the image of A_0 under the isomorphism from $\mathscr{X}$ to $\mathscr{Y}$ contains b, let $A = A_0$ and let B be this image. Otherwise, pick $A \in \mathrm{Nhd}(a, A_0)$ and $B'' \in \mathrm{Nhd}(b, \mathscr{Y})$

such that $A \simeq B''$ and let B' be the image of A under the isomorphism from $\mathscr{X}$ to $\mathscr{Y}$. Since B'' and B' are isomorphic to A, $B'' \simeq B'$. Let $E'' = B''\backslash B'$ and let E' be the image of E'' under the isomorphism from B'' to B'. Let h be the automorphism of $\mathscr{Y}$ which maps E' to E'' and E'' to E' and leaves everything else fixed. Then $B = (B'\backslash E') \cup E'' =$ the image of B' under h is the desired neighborhood of b. $A \simeq B$ since $A \simeq B'$ and $B' \simeq B$. By part (a), $\langle A, a\rangle \approx \langle B, b\rangle$. Since $\mathscr{X}\backslash A \simeq \mathscr{Y}\backslash B'$ and h maps $\mathscr{Y}\backslash B' \simeq \mathscr{Y}\backslash B$, $\mathscr{X}\backslash A \simeq \mathscr{Y}\backslash B$. □

14.4. LEMMA. (a) *If* $\langle \mathscr{X}, a\rangle \approx \langle \mathscr{Y}, b\rangle$ *then* $\langle \mathscr{X}, a\rangle \simeq \langle \mathscr{Y}, b\rangle$.

(b) *If* $\mathscr{X}$ *and* $\mathscr{Y}$ *are ranked by a diagram* $\mathscr{R}$ *via* f *and* g, *respectively, and* $f(a) = g(b)$, *then* $\langle \mathscr{X}, a\rangle \simeq \langle \mathscr{Y}, b\rangle$.

(c) *If* $a_1, \ldots, a_n$ *are distinct points of* $\mathscr{X}$ *and* $b_1, \ldots, b_n$ *distinct points of* $\mathscr{Y}$ *and* $\langle \mathscr{X}, a_k\rangle \simeq \langle \mathscr{Y}, b_k\rangle$ *for* $1 \leq k \leq n$, *then* $\langle \mathscr{X}, a_1, \ldots, a_n\rangle \simeq \langle \mathscr{Y}, b_1, \ldots, b_n\rangle$.

PROOF. (a) Since a Boolean space with a countable base is metrizable, we can pick metrics for $\mathscr{X}$ and $\mathscr{Y}$. Given $\langle \mathscr{X}, a\rangle \approx \langle \mathscr{Y}, b\rangle$, we construct a sequence $A_1 \supseteq A_2 \supseteq \cdots$ of clopen neighborhoods of a whose diameters go to 0, a similar sequence $B_1 \supseteq B_2 \supseteq \cdots$ of neighborhoods of b, and a sequence $h_1 \subseteq h_2 \subseteq \cdots$ of isomorphisms such that $h_k\colon \mathscr{X}\backslash A_k \simeq \mathscr{Y}\backslash B_k$ and $\langle A_k, a\rangle \approx \langle B_k, b\rangle$. Let $A_1 = \mathscr{X}$, $B_1 = \mathscr{Y}$, and $h_1 = \emptyset$. Suppose $h_n\colon \mathscr{X}\backslash A_n \simeq \mathscr{Y}\backslash B_n$, $\langle A_n, a\rangle \approx \langle B_n, b\rangle$ and n is odd. Pick a $C \in \mathrm{Nhd}(a, A_n)$ with diameter $\leq 1/(n+1)$ (for n even we chose $C \in \mathrm{Nhd}(b, B_n)$). By Lemma 14.3(b), there is an $A_{n+1} \in \mathrm{Nhd}(a, C)$ and $B_{n+1} \in \mathrm{Nhd}(b, \mathscr{Y})$ such that $A_n\backslash A_{n+1} \simeq B_n\backslash B_{n+1}$ by some isomorphism g and $\langle A_{n+1}, a\rangle \approx \langle B_{n+1}, b\rangle$. Let $h_{n+1} = h_n \cup g$. Finally, $\langle \mathscr{X}, a\rangle \simeq \langle \mathscr{Y}, b\rangle$ via the isomorphism $(\bigcup_n h_n) \cup \{\langle a, b\rangle\}$.

(b) Assume the hypothesis. By part (a) it suffices to show that $\langle \mathscr{X}, a\rangle \approx \langle \mathscr{Y}, b\rangle$. By the Rank Isomorphism Theorem, $\mathscr{X} \simeq \mathscr{Y}$. Suppose $A_0 \in \mathrm{Nhd}(a, \mathscr{X})$ (the vice versa case is symmetrical). We want an $A \in \mathrm{Nhd}(a, A_0)$ and a $B \in \mathrm{Nhd}(b, \mathscr{Y})$ with $A \simeq B$. By Lemma 9.5, there is an $A \in \mathrm{Nhd}(a, A_0)$ with only one max rank $r \geq f(a) = g(b)$ and, if r is irreflexive, only one point of that rank. Since g is onto, there is a point c of $\mathscr{Y}$ with rank r and, by Lemma 9.2(b, c), c is the greatest point of some $B \in \mathrm{Nhd}(b, \mathscr{Y})$. Let $\mathscr{R}_r$ be the subdiagram of $\mathscr{R}$ consisting of points of rank $\leq r$. Then A and B are both ranked by $\mathscr{R}_r$ and, by the Rank Isomorphism Theorem, $A \simeq B$.

(c) Assume the hypothesis. Since $\langle \mathscr{X}, a_1\rangle \simeq \langle \mathscr{Y}, b_1\rangle$, there is a neighborhood A of a_1 which separates it from $a_2, \ldots, a_n$ and a similar neighborhood B of b_1 such that $\langle A, a_1\rangle \simeq \langle B, b_1\rangle$ and $\mathscr{X}\backslash A \simeq \mathscr{Y}\backslash B$. For $k > 1$, $\langle \mathscr{X}, a_k\rangle \simeq \langle \mathscr{Y}, b_k\rangle$ implies $\langle \mathscr{X}, a_k\rangle \approx \langle \mathscr{Y}, b_k\rangle$ implies, by Lemma 14.3(a), $\langle \mathscr{X}\backslash A, a_k\rangle \approx \langle \mathscr{Y}\backslash B, b_k\rangle$ implies, by part (a), $\langle \mathscr{X}\backslash A, a_k\rangle \simeq \langle \mathscr{Y}\backslash B, b_k\rangle$ implies, by induction, $\langle \mathscr{X}\backslash A, a_2, \ldots, a_n\rangle \simeq \langle \mathscr{Y}\backslash B, b_2, \ldots, b_n\rangle$. Taking the union of the two isomorphisms gives $\langle \mathscr{X}, a_1, \ldots, a_n\rangle \simeq \langle \mathscr{Y}, b_1, \ldots, b_n\rangle$. □

15. Primitive spaces and orbit diagrams

The Boolean spaces which can be ranked by a rank diagram are exactly the primitive Boolean spaces defined below and each such space is canonically ranked by its orbit diagram.

15.1. DEFINITION. For any Boolean space $\mathscr{X}$ with clopen sets A and B,

- A is *pseudo-indecomposable* iff ($\forall$ clopen B, C)($A = B \cup C \Rightarrow A \simeq B$ or $A \simeq C$),
- $\mathscr{X}$ is *pseudo-indecomposable* iff its universe is,
- $\mathscr{X}$ is *primitive* iff it has a countable base, it is pseudo-indecomposable and each of its clopen sets is a disjoint union of pseudo-indecomposable clopen sets.

Of the naturally occurring theories whose model spaces are known, most are primitive and those that are not can be reduced to a finite disjoint union of primitive spaces.

15.2. DEFINITION. Let $\mathscr{X}$ be a primitive Boolean space and let R be the set of orbits $\bar{x}$ of points x in $\mathscr{X}$ acted on by homeomorphisms. Let R be ordered by $\bar{x} < \bar{y}$ iff every point in $\bar{y}$ is a limit of distinct points from $\bar{x}$ iff some point in $\bar{y}$ is a limit of distinct points from $\bar{x}$. $\langle R, < \rangle$ is the *orbit diagram* for $\mathscr{X}$.

15.3. THEOREM. (a) *A Boolean space with a countable base has a rank diagram iff it is primitive.*

(b) *Every primitive space is ranked by its orbit diagram via* $f(x) = \bar{x}$.

PROOF. This theorem is the dual of a theorem of Hanf for Boolean algebras and structure diagrams (HANF [1971]). □

16. Miscellaneous

Sentence algebras are not the only Boolean algebras of first-order logic. The set of all formulas of a theory is also a Boolean algebra, but this is essentially the sentence algebra of the theory with countably many new constants added. The algebra of formulas can be split into two factors by the sentence "there is exactly one element". If the theory has finite similarity type, the factor determined by this sentence is a finite algebra whose number of atoms is the number of one element models of the theory. The factor determined by the negation of the sentence is the countable atomless algebra since a given formula is split by the formula "$x = y$" when x and y do not occur in the given formula.

The isomorphisms given by Hanf's Language Isomorphism Theorem for finite-similarity-type languages are all recursive. *Problem*: classify languages with infinite recursive similarity types up to recursive isomorphism. In the following theories, the only axiom will be "there are at least two elements", an integer n will indicate an n-ary relation, $\underline{n}$ will indicate ω n-ary relations, $n.5$ will indicate an n-ary operation. Let, for example, $L\langle 2, 2.5, 0, \underline{0.5}\rangle$ be the theory with one binary relation symbol, one binary operation symbol, one propositional symbol, and ω constants plus the axiom "there are at least two elements".

Two theories are *recursively isomorphic* iff their sentence algebras are. Hanf has shown $L\langle 2, \underline{0.5}\rangle$ and $L\langle 2, \underline{0}\rangle$ are recursively isomorphic. By HANF [1965], $L\langle 2\rangle$ has finitely axiomatizable extensions of every intermediate degree of unsolvability and hence so do $L\langle 2, \underline{0.5}\rangle$ and $L\langle 2, \underline{0}\rangle$. $L\langle \underline{2}\rangle$ does not and thus is not recursively isomorphic to $L\langle 2, \underline{0.5}\rangle$ and $L\langle 2, \underline{0}\rangle$. *Problem*: does $L\langle 2, \underline{1}\rangle$

Table 26.2

Theory	Order type	Rank diagram	Investigator	Date
Any complete theory	1			
Any essentially undecidable theory	η_0	$\overset{\circlearrowleft}{\cdot}$		
Any countable language with infinitely many symbols	(the number of 1-element models) $+ \eta_0$			
Equality	ω	$\cdot \to \cdot$		
Algebraically-closed fields			Alfred Tarski	[1949e]
Boolean algebras	ω^2	$\cdot \to \cdot \to \cdot$	Alfred Tarski	[1949d]
A language of n unary predicates	ω^{2n}			
Equivalence relations,	$\omega^{\omega}\eta_0$	$\cdot \to \cdot \to \cdot \to \cdot \ldots \overset{\circlearrowleft}{\cdot}$	William Hanf	[1969]
Permutations,			Roger Simons	[1971]
Well-orders,			Dale Myers	[1974]
Linear orders			Dale Myers	[1980]
Unary operations, Any countable functional language	$(\eta_0 + \omega^{\omega}\eta_0)\eta_0$	$\cdot \to \cdot \to \cdot \to \cdot \ldots \overset{\circlearrowleft}{\cdot} \to \overset{\circlearrowleft}{\cdot} \leftarrow \overset{\circlearrowleft}{\cdot}$	Roger Simons	[1971]
Abelian groups	$(\eta_0 + \omega^{\omega})\eta_0$	$\cdot \to \cdot \to \cdot \to \cdot \ldots \overset{\circlearrowleft}{\cdot} \leftarrow \overset{\circlearrowleft}{\cdot}$	Dale Myers	[1974]
Binary relations, Semigroups, Lattices, Any countable undecidable language	the "universal" r.e. Boolean algebra, i.e. the unique r.e. Boolean algebra such that every r.e. Boolean algebra is isomorphic to one of its factors.		William Hanf	[1975]
Groups	?	?		
Distributive lattices	?	?		
?	ω^{ω}	$\cdot \to \cdot \to \cdot \to \cdot \ldots \cdot$		
?	$(\omega + \eta)$	$\cdot \to \cdot \leftarrow \overset{\circlearrowleft}{\cdot}$		

have such intermediate finite extensions? By FAUST, HANF and MYERS [1977], $L\langle 2, \underline{0.5}\rangle$ and $L\langle 3, \underline{0.5}\rangle$ are recursively isomorphic. By FAUST [1982], $L\langle 2, \underline{1}\rangle$ and $L\langle 2, 2, \underline{1}\rangle$ are recursively isomorphic. *Problem*: Are $L\langle 2, \underline{1}\rangle$ and $L\langle 3, \underline{1}\rangle$ recursively isomorphic? By MYERS [1976] $L\langle \underline{3}\rangle$ and $L\langle 2, \underline{3}\rangle$ are recursively isomorphic. *Problem*: Are any of $L\langle \underline{2}\rangle$, $L\langle \underline{3}\rangle$ and $L\langle 3, \underline{2}\rangle$ recursively isomorphic?

The class of *definable* (*parametrically definable*) subsets of a structure – subsets of the form $\{x \in A: \mathfrak{A} \models \varphi(x)\}$ (of the form $\{x \in A: \mathfrak{A} \models \varphi(x, a_1, \ldots, a_n)\}$) where φ is a first-order formula (and $a_1, \ldots, a_n \in A$) – is a Boolean algebra. Hanf has shown that every countable Boolean algebra is isomorphic to the definable subset algebra of some structure of finite similarity type. By an argument of MORLEY [1965], every countable atomic Boolean algebra is isomorphic to the parametrically definable subset algebra of some structure of countable similarity type. *Problem*: Can "countable similarity type" be replaced by "finite similarity type"? The author has shown that the answer is yes for primitive atomic Boolean algebras.

17. Table of sentence algebras

Table 26.2 lists various axiomatizable theories, the linear order for an isomorphic interval algebra, the rank diagram (take the transitive closure of the pictured relation), and the discoverer. A question mark in the first column means there is no known "natural" example of a finitely axiomatizable theory with the given type. η_0 and ω are the order types of the non-negative rationals and the natural numbers, respectively. A language is identified with the theory consisting of its valid sentences.

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Keywords: Boolean algebra, sentence algebra, Lindenbaum–Tarksi algebra, model maps, language isomorphisms, measures, rank diagrams, interval algebras, measure monoids, primitive spaces.

MOS subject classification: primary 03G05; secondary 06E05, 03C10, 03C65.

CHAPTER 27

Boolean-Valued Models

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HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

The similarity between Boolean operations and the rules of (classical) logic can be exploited by generalizing the notion of a model. Let $\mathscr{L}$ be a first-order language and let B be a complete Boolean algebra. We introduce the notion of *B-valued model* for $\mathscr{L}$.

Let M be a set (the *universe* of the model). We consider a function from $M \times M$ into B: for each $x \in M$ and $y \in M$, its value $\|x = y\|_B$, or simply

$$\|x = y\| ,$$

is an element of B. The function $\|x = y\|$ has to satisfy the following:

$$\begin{aligned} &\|x = x\| = 1 , \\ &\|x = y\| = \|y = x\| , \\ &\|x = y\| \cdot \|y = z\| \leq \|x = z\| . \end{aligned} \tag{1}$$

For every predicate $R(x_1, \ldots, x_n)$, of $\mathscr{L}$, let

$$\|R(x_1, \ldots, x_n)\|$$

be an n-ary function from M into B that satisfies, for each $i = 1, \ldots, n$,

$$\|x_i = y_i\| \cdot \|R(\ldots x_i \ldots)\| \leq \|R(\ldots y_i \ldots)\| . \tag{2}$$

With every n-ary operation F of $\mathscr{L}$ we associate a function F: $M^n \to M$ such that

$$\|x_i = y_i\| \leq \|F(\ldots x_i \ldots) = F(\ldots y_i \ldots)\| . \tag{3}$$

It follows from (1)–(3) that the binary relation

$$\|x = y\| = 1$$

is an equivalence relation on M, in fact a congruence relation with respect to the functions $\|\mathrm{R}\|$ and F of the model. Thus, we postulate:

$$\text{if } \|x = y\| = 1, \text{ then } x = y . \tag{4}$$

DEFINITION. *A Boolean-valued model* for $\mathscr{L}$ is

$$M = \langle M, \|x = y\|, \|R(x_1, \ldots, x_n)\|, \ldots, F, \ldots \rangle ,$$

satisfying (1)–(4).

Satisfaction in a Boolean-valued model is defined as follows. For every formula

φ of $\mathscr{L}$ and $x_1, \ldots, x_n \in M$, we define the *Boolean value* of $\varphi(x_1, \ldots, x_n)$, an element of B

$$\|\varphi(x_1, \ldots, x_n)\| .$$

If φ is an atomic formula, then its value is

$$\|x = y\| \quad \text{or} \quad \|R(x_1, \ldots, x_n)\| ,$$

whatever the case may be. The Boolean value of logical connectives is defined by

$$\begin{aligned} &\|\neg\varphi\| = \|\varphi\| , \\ &\|\varphi \wedge \psi\| = \|\varphi\| \cdot \|\psi\| , \\ &\|\varphi \vee \psi\| = \|\varphi\| + \|\psi\| . \end{aligned} \tag{5}$$

As for the quantifiers, we let

$$\begin{aligned} &\|\exists x\varphi\| = \sum_{x \in M} \|\varphi(x)\| , \\ &\|\forall x\varphi\| = \prod_{x \in M} \|\varphi(x)\| . \end{aligned} \tag{6}$$

We also introduce the notation,

$$M \models \varphi , \tag{7}$$

to mean $\|\varphi\| = 1$, and say that in that case *M satisfies* φ.

All axioms of predicate calculus are satisfied in every Boolean-valued model, and satisfaction is preserved by deductive rules. In particular, the axioms on $=$ are satisfied: properties (1)–(3) say exactly that the following are satisfied in M:

$$\begin{aligned} &x = x , \\ &x = y \leftrightarrow y = x , \\ &(x = y \wedge y = z) \rightarrow x = z , \\ &(x_i = y_i \wedge R(\ldots x_i \ldots)) \rightarrow R(\ldots y_i \ldots) , \\ &x_i = y_i \rightarrow F(\ldots x_i \ldots) = F(\ldots y_i \ldots) . \end{aligned}$$

Boolean-valued models are a generalization of ordinary (two-valued) models. If B is the algebra $\{0, 1\}$ then, under the interpretation of predicates $R(x_1, \ldots, x_n)$ as $\|R(x_1, \ldots, x_n)\| = 1$, satisfaction in M is just ordinary satisfaction.

We now present two nontrivial examples of Boolean-valued models of the language of ring theory.

EXAMPLE 1. Let B be a complete Boolean algebra, and let $\langle R, +, \cdot, 0, 1\rangle$ be the ring of all real numbers. We define a B-valued model M, the *Boolean power* of R. (And one can define the Boolean power of any structure, using the same construction.)

The universe M consists of all functions f such that

$$\begin{aligned} &\text{(i)} \quad \operatorname{dom}(f) \text{ is a partition of } B\,, \\ &\text{(ii)} \quad f(a) \in R \text{ for all } a \in \operatorname{dom}(f)\,. \end{aligned} \tag{8}$$

(Rather, M is the quotient of this set by the congruence $\|f = g\| = 1$.)

To define $\|f = g\|$, we first extend each f as follows: if $0 \neq a \leq b \in \operatorname{dom}(f)$, we define $f(a)$ by $f(a) = f(b)$. For any $f, g \in M$, let W be some common refinement of the partitions $\operatorname{dom}(f)$ and $\operatorname{dom}(g)$. We let

$$\|f = g\| = \sum \{w \in W\colon f(w) = g(w)\}\,. \tag{9}$$

The function $\|f = g\|$ does not depend on the choice of W, and satisfies (1). To interpret $+$ and $\cdot$ in M, let f and g be such that $\operatorname{dom}(f) = \operatorname{dom}(g) = W$ (we may assume that or else use a common refinement). For each $a \in W$, let

$$\begin{aligned} &(f - g)(a) = f(a) - g(a)\,, \\ &(f \cdot g)(a) = f(a) \cdot g(a)\,. \end{aligned} \tag{10}$$

Finally, 0 and 1 are interpreted as the functions defined on $\{1_B\}$ with the respective values 0_R and 1_R. It is easily seen that (3) is satisfied; hence $\langle M, +, \cdot, 0, 1\rangle$ is a Boolean-valued model.

EXAMPLE 2. Let $\mathscr{L}$ be the set of all real valued (Lebesgue) measurable functions on the interval $[0, 1]$, and let B be the algebra of all measurable subsets of $[0, 1]$, modulo null sets. Let M be the quotient of $\mathscr{L}$ by the ideal of null sets. For f and g in M, we let

$$\|f = g\| = \{x \in [0, 1]\colon f(x) = g(x)\}\,. \tag{11}$$

The operations $+$ and $\cdot$ are the pointwise addition and multiplication, and 0 and 1 are the constant functions with values 0 and 1, respectively. Properties (1), (3) and (4) are easily verified.

Note that the Boolean-valued model in Example 2 contains the B-power of R as a submodel: namely the set of all step functions, that is all $f \in M$ such that for some partition W of B, f is constant on each $w \in W$.

The Boolean-valued models in the examples above have the following property:

DEFINITION. A B-valued model M is *full* if, for every partition W of B and every function $a \mapsto x_a$ from W into B, there exists an $x \in M$ such that

$$a \leq \|x = x_a\| \tag{12}$$

for all $a \in W$.

It is clear that such an $x \in M$ is unique (because it is unique up to $\|x = x'\| = 1$). We shall use thc notation

$$x = \sum_{a \in W} x_a \cdot a\,. \tag{13}$$

If M is full and $S \subseteq M$, we denote

$$\hat{S} \tag{14}$$

the set of all $\Sigma_w\, x_a \cdot a$, where W is a partition and $x_a \in S$.

An important property of full models is the following:

PROPOSITION. *If M is full, then for any formula $\varphi(z, x_1, \ldots, x_n)$ and every $x_1, \ldots, x_n \in M$ there exists some $x \in M$ such that*

$$\|\varphi(x, x_1, \ldots, x_n)\| = \|\exists z \varphi(z, x_1, \ldots, x_n)\|\,. \tag{15}$$

PROOF. Let W be a maximal set of pairwise disjoint $a \in B$ such that for some $x_a \in M$,

$$a \leq \|\varphi(x_a, x_1, \ldots, x_n)\|\,.$$

W is a partition of $\|\exists z \varphi\|$, and $x = \Sigma_{a \in W}\, x_a \cdot a$ will do. □

PROPOSITION. *Every Boolean-valued model M can be embedded in a full B-valued model $\hat{M}$ such that for any φ and all $x_1, \ldots, x_n \in M$,*

$$\|\varphi(x_1, \ldots, x_n)\|_M = \|\varphi(x_1, \ldots, x_n\|_{\hat{M}}\,. \tag{16}$$

PROOF. Let $\hat{M}$ consist of all (formal expressions)

$$x = \sum_{a \in W} x_a \cdot a\,,$$

where W is a partition of B, and $x_a \in M$ for all $a \in W$. The structure of $\hat{M}$ is defined by

$$\left\|R\Big(\sum_{a \in W} x_a \cdot a\Big)\right\|_{\hat{M}} = \sum_{a \in W} \|R(x_a)\|_M \cdot a\,. \tag{17}$$

The formula (16) is proved by induction on φ. □

Let M be a Boolean-valued model and let G be an ultrafilter on B. The *quotient* M/G, a two-valued model, is obtained in the following way: The universe is the quotient of M by the equivalence relation

$$\|x = y\| \in G \tag{18}$$

and the predicates are interpreted by

$$R(x_1, \ldots, x_n) \quad \text{iff } \|R(x_1, \ldots, x_n)\| \in G\,. \tag{19}$$

When M is full, then satisfaction in M/G and the Boolean values $\|\varphi\|$ are related as follows:

THEOREM. *Let M be a full B-valued model and let G be an ultrafilter on B. For every formula φ and every $x_1, \ldots, x_n \in M$,*

$$M/G \models \varphi(x_1/G, \ldots, x_n/G) \quad \textit{iff } \|\varphi(x_1, \ldots, x_n)\| \in G\,. \tag{20}$$

PROOF. By (18) and (19), the equivalence (20) holds for atomic formulas. For logical connectives, use (5) and the basic properties of ultrafilters. As for the quantifiers, we have, by (6) and (15):

$$\begin{aligned} M/G \models \exists x\varphi \quad & \text{iff } \exists x \in M\ M/G \models \varphi(x)\,, \\ & \text{iff } \exists x \in M\ \|\varphi(x)\| \in G\,, \\ & \text{iff } \|\exists z\varphi\| \in G\,. \quad \square \end{aligned}$$

The relation between M/G and M can also be expressed as follows:

THEOREM. *Let M be a full B-valued model. Then for every nonzero $a \in B$, every formula φ and all $x_1, \ldots, x_n \in M$,*

$$\begin{aligned} a \le \|\varphi(x_1, \ldots, x_n)\| \quad & \textit{iff } M/G \models \varphi(x_1/G, \ldots, x_n/G) \\ & \textit{for every ultrafilter } G \ni a\,. \end{aligned} \tag{21}$$

PROOF. The implication from left to right follows directly from (20). For the converse, assume that $a \nleq \|\varphi\|$. Then there is a nonzero $b \le a$ such that $b \le -\|\varphi\| = \|\neg\varphi\|$. Let $G \ni b$ be an ultrafilter; then $M/G \models \neg\varphi$, and so not every M/G satisfies φ. $\square$

COROLLARY. *If M is full, then for every sentence σ,*

$$M \models \sigma \quad \textit{iff for all ultrafilters } G \textit{ on } B,\ M/G \models \sigma\,.$$

Now we turn our attention to Boolean-valued models of set theory. Below we describe the model V^B, the *B-valued universe*.

When dealing with Boolean-valued models of set theory, we allow M to be a proper class; the situation is similar to the two-valued case.

A Boolean-valued model M of set theory has two B-valued functions, $\|x = y\|$ and $\|x \in y\|$; in addition to (1) and (4), M satisfies the formula:

$$(x \in y \wedge x' = x \wedge y' = y) \rightarrow x' \in y' . \tag{22}$$

DEFINITION. M is *extensional* if it satisfies the axiom of extensionality:

$$\forall u(u \in x \leftrightarrow u \in y) \rightarrow x = y . \tag{23}$$

The following property states that even if M is a proper class, each $x \in M$ has only a set of B-elements:

$$\text{for each } x \in M \text{ there is a set } S \text{ such that for all } u \in M,$$
$$\|u \in x\| \leq \sum_{z \in S} \|u = z\| . \tag{24}$$

DEFINITION. M is *universal* if it has property (24), and if for every set $S \subset M$ and every function $x \mapsto a_x$ from S into B there exists $A \in M$ such that

$$\|z \in A\| = \sum_{x \in S} \|z = x\| \cdot a_x \quad (\text{all } z \in M) . \tag{25}$$

If M is extensional, then the A in (25) is unique. (The condition (25) states roughly that whenever possible we can collect any set of $x \in M$ into $A \in M$ so that $\|x \in A\| = a_x$.) Notice that if M is extensional and universal, then it is full.

The following property is the B-valued analog of well-foundedness:

DEFINITION. A Boolean-valued model M of set theory is *well-founded* if, for every nonempty $S \subseteq M$, there exists $x \in \hat{S}$ that is $\in$*-minimal*, i.e. $\forall z \in S \|z \in x\| = 0$.

We define V^B as the unique well-founded, extensional, universal B-valued model of set theory. An *isomorphism* of two B-valued models is a bijection π that satisfies

$$\|\pi x = \pi y\| = \|x = y\| \quad \text{and} \quad \|\pi x \in \pi y\| = \|x \in y\| . \tag{26}$$

THEOREM. *There is a unique (up to isomorphism) well-founded extensional universal B-valued model V^B. V^B satisfies all axioms of ZFC.*

The uniqueness of V^B is more or less obvious: if x is an $\in$-minimal element of the model for which π is not defined, we use universality and (26) to define $\pi(x)$, and extensionality to verify that π still satisfies (26).

The actual construction of V^B can be found in set theory textbooks; it is the Scott–Solovay modification of Cohen's method of forcing.

The two-valued universe V is embedded in V^B as follows: There exists a (unique) function $x \mapsto \check{x}$ of V into V^B such that

$$\begin{aligned} &\check{\emptyset} = \emptyset , \\ &\check{x}(\check{u}) = 1 \quad \text{iff } u \in x . \end{aligned} \tag{27}$$

In applications of forcing it is easier to work with the *generic extension* $V[G]$ than with the Boolean-valued universe V^B.

DEFINITION. An ultrafilter G on B is *V-generic* if, for any set $A \subset B$, $A \in V$,

$$\sum_{a\in A} a \in G \text{ implies } \exists a \in A \ a \in G . \tag{28}$$

If B is atomless, then no ultrafilter in V is V-generic; (28) leads easily to a contradiction. One can, however, postulate the existence of a generic ultrafilter G, residing in some larger universe outside V. If we assume that V is countable, then generic ultrafilters are easily constructed:

PROPOSITION. *Assume that V is a countable model of set theory and let B be, in V, a complete Boolean algebra. Then*

for every nonzero $a \in B$ there is a V-generic ultrafilter G such that $a \in G$.

PROOF. The ultrafilter G only has to satisfy countably many requirements (28). Thus, we can construct a sequence $a_0 \geq a_1 \geq \cdots \geq a_n \geq \cdots$ such that for each n we satisfy the nth requirement (28) by either choosing a_n incompatible with ΣA, or choosing some $a \in A$ and $a_n \leq a$. The a_n's generate a filter, which extends to a generic ultrafilter. □

Let $S \subset V^B$. We recall that $\hat{S}$ is the set of all $x \in V^B$ such that $\Sigma_{s\in S} \|x = s\| = 1$.

LEMMA. *Let G be a V-generic ultrafilter on B. If S is a subset of V^B, then*

$$(\forall x \in \hat{S})(\exists s \in S)\ x/G = s/G . \tag{30}$$

PROOF. We have $\Sigma_{s\in S} \|x = s\| = 1$. By genericity, there is an $s \in S$ such that $\|x = s\| \in G$. □

Let G be a V-generic ultrafilter on B, and consider V^B/G. Combining the fact that V^B is a well-founded Boolean-valued model (see the definition above) with property (30), it follows that V^B/G is an extensional well-founded (two-valued) model. Hence, V^B/G is isomorphic to a transitive class, which we denote $V[G]$. Under this isomorphism, each $\check{x} \in V^B$ becomes x.

The model $V[G]$ is a transitive model extending V; in fact, a smallest transitive model of ZFC extending V and containing G, thus the notation $V[G]$.

The B-valued universe has a *canonical name* for a generic ultrafilter, namely $\mathring{G} \in V^B$ with the property:

$$\|\check{a} \in \mathring{G}\| = a \qquad (\text{all } a \in B) . \tag{31}$$

V^B satisfies that $\mathring{G}$ is a generic ultrafilter on $\check{B}$, and whenever G is generic,

$$\mathring{G}/G = G .$$

In the context of generic extensions, we have:

THEOREM. *For every formula φ and every $x_1, \ldots, x_n \in V^B$,*

$$V[G] \models \varphi(x_1/G, \ldots, x_n/G) \quad \textit{iff} \ \|\varphi(x_1, \ldots, x_n)\| \in G . \tag{32}$$

Also, if we postulate (29), then we have:

THEOREM. *For every nonzero* $a \in B$, *every formula* φ *and every* $x_1, \ldots, x_n \in V^B$.

$$a \leq \|\varphi(x_1, \ldots, x_n)\| \quad \textit{iff } V[G] \models \varphi(x_1/G, \ldots, x_n/G) \quad \textit{for every V-generic ultrafilter } G \ni a . \tag{33}$$

The formulas (32) and (33) describe truth in the generic extension $V[G]$ in terms of Boolean values of formulas. In practice it means that properties of $V[G]$ can be described inside the ground model V in terms of properties of the Boolean algebra B. We present here three such examples. As the first example, we recall:

THEOREM. *B is (κ, λ)-distributive if and only if every function $f: \kappa \to \lambda$ in the generic extension is in the ground model.*

As the second example, we prove:

PROPOSITION. *B is atomless if and only if the generic ultrafilter is not in the ground model.*

PROOF. If a is an atom of B, then the ultrafilter $\{b \in B: b \geq a\}$ is generic and is in V. Conversely, let B be atomless, and assume that $G \in V$. Hence $A = B - G \in V$, and for the contradiction it suffices to show that $\Sigma\, A = 1$. But since B is atomless and G is an ultrafilter, we have $\forall a \neq 0 \ \exists b \leq a$ such that $b \notin G$. Hence $\Sigma\, A = 1$. $\square$

As a third example, we state (without proof) the following:

THEOREM. *If B is rigid, then G (and therefore every set in $V[G]$) is definable in $V[G]$ with parameters from V. Conversely, if G is definable in $V[G]$, then there is a partition W of B such that for each $a \in W$, B_a is rigid.*

Let A be some structure in the Boolean-valued universe V^B (and I do not just mean a structure for a first-order logic, but in a more general sense). Since V^B is a model of set theory, A has (in V^B) any property that can be proved about such structures. But we can also look at A "from the outside", that is as a B-valued structure, and investigate its properties as such. Although this subject can be treated in abstract generality, I shall rather describe it by giving several examples.

First, let A be just a set in V^B, that is $A \in V^B$. We generalize (14) by letting

$$\hat{A} = \{x \in V^B: \|x \in A\| = 1\} . \tag{34}$$

The B-valued equality $\|x = y\|$ for $x, y \in A$ makes $\hat{A}$ a B-valued model; it is full because V^B is full.

If A has an additional structure in V^B, then $\hat{A}$ can be endowed with a corresponding structure as follows. If $R \in V^B$ is such that

$$\|R(x_1, \ldots, x_n) \text{ is a relation on } A\| = 1 ,$$

then we let, for all $x_1, \ldots, x_n \in \hat{A}$,

$$\hat{R}(x_1, \ldots, x_n) \quad \text{iff } \|R(x_1, \ldots, x_n)\| = 1 . \tag{35}$$

If $f \in V^B$ is such that

$$\|f \text{ is an operation on } A\| = 1 ,$$

then we let, for all $x_1, \ldots, x_n, y \in \hat{A}$,

$$\hat{f}(x_1, \ldots, x_n) = y \quad \text{iff } \|f(x_1, \ldots, x_n) = y\| = 1 . \tag{36}$$

The y in (36) is unique.

We denote the structure so obtained

$$B * A . \tag{37}$$

EXAMPLE 3. Let $(A, \leq)$ be a partially ordered set in V^B. Using (35), we obtain a binary relation $\hat{\leq}$ on $\hat{A}$; let us denote it $\leq$. The set $(\hat{A}, \leq)$ is also a partial ordering. For example, $(\hat{A}, \leq)$ has the property

$$\forall xyz(x \leq y \wedge y \leq z \rightarrow x \leq z) \tag{38}$$

because (38) is satisfied in V^B (if $\|x \leq y\| = 1$ and $\|y \leq z\| = 1$, then because $\|(38)\| = 1$ we have $\|x \leq z\| = 1$).

The argument just given uses the particular logical structure of the axioms of partial order, and does not apply in general. For instance, if $\leq$ is a linear order in V^B, $(\hat{A}, \leq)$ need not be linearly ordered: consider the case when $x, y \in \hat{A}$ are such that for some $a \in B$, $0 \neq a \neq 1$,

$$\|x < y\| = a , \qquad \|y < x\| = -a .$$

EXAMPLE 4. Let $(A, \cdot)$ be a group in V^B. We define multiplication $\cdot$ on $\hat{A}$ using (36). $(\hat{A}, \cdot)$ satisfies the axioms for a group. For example, if $x, y \in \hat{A}$, then

$$\|\exists z\, x \cdot z = y\| = 1 ,$$

and because $\hat{A}$ is full, it follows that there exists a $z \in \hat{A}$ such that $x \cdot z = y$. Hence, $B * A$ is a group.

EXAMPLE 5. Let A be a ring in V^B. Then $B * A$ is also a ring, with ring operations defined by (36). If A is a field in V^B, $B * A$ is not necessarily a field. This is because if $x \in \hat{A}$ is such that

$$0 \neq \|x = 0\| \neq 1,$$

then x is not the zero of $B * A$ (because $\|x = 0\| \neq 1$), but does not have an inverse in $B * A$ (because $\|x = 0\| \neq 0$).

EXAMPLE 6. *Boolean power*. Let $A = (A, R, \ldots, f, \ldots)$ be a structure (in V). Then $\check{A}$ is a structure in V^B, and we can form a structure $B * \check{A} = (A^{\check{}\hat{}}, R^{\check{}\hat{}}, \ldots, f^{\check{}\hat{}}, \ldots)$ using (34)–(36). The structure $B * \check{A}$ is the *Boolean power* of A by B. It is easy to see that the present definition agrees with the standard definition of Boolean power, as in (8)–(10) (Example 1).

EXAMPLE 7. *B-valued reals*. Let $R^B \in V^B$ be the set of all real numbers in V^B:

$$\|R^B \text{ is the set of all real numbers}\| = 1 .$$

When we equip R^B with its natural ordering, and addition and multiplication, $\hat{R}^B$ becomes a partially ordered ring.

In V^B, $\hat{R}^B$ has a dense set $\check{R}$ (where R is the set of real numbers in V). It follows that every $x \in \hat{R}^B$ can be identified with the function:

$$\lambda \mapsto \|x \leq \lambda\| = a_\lambda \quad (\lambda \in R) . \tag{39}$$

The function (39) from R into B is monotone and satisfies

$$a_\lambda = \prod_{\varepsilon > 0} a_{\lambda + \varepsilon} ; \tag{40}$$

note the similarity between (39) and spectral resolutions.

EXAMPLE 8. (Example 2 revisited). Let B be a measure algebra and (Ω, μ) the corresponding measure space. Let R^B be the set of all real numbers in V^B. The ring $B * R^B$ has the following representation. With each $r \in \hat{R}^B$ one can associate a measurable function f_r: $\Omega \to R$ such that for each $\lambda \in R$,

$$\|r \leq \lambda\| = \{x \in \Omega: f_r(x) \leq \lambda\} . \tag{41}$$

Under this correspondence, $B * R^B$ is isomorphic to the ring of all measurable functions on $\Omega(\text{mod} = \mu - \text{a.e.})$.

EXAMPLE 9. *Iteration*. Let A be a Boolean algebra in V^B. The structure

$$B * A ,$$

whose universe is $\hat{A}$ and whose operations $+$ and $\cdot$ are defined by (36), is a Boolean algebra, called the *iteration* of B and A.

The algebra B embeds in $B * A$ by the complete embedding e defined as follows. For each $b \in B$, let $e(b)$ be the unique element of $\hat{A}$ such that

$$\|e(b) = 1_A\| = b\,, \qquad \|e(b) = 0_A\| = -b\,. \tag{42}$$

If A is a Boolean algebra (in V), then $B * \check{A}$ is the Boolean power of A by B; thus, iteration is a generalization of Boolean power.

The operation of iteration is often used in applications of forcing. Assume that in V^B, A is a complete Boolean algebra, i.e.

$$\|A \text{ is complete}\| = 1\,.$$

It follows that $B * A$ is complete; indeed, if $X \subseteq \hat{A}$, then $\Sigma\, X$ is the unique $a \in \hat{A}$ such that

$$\|\Sigma\, X = a\| = 1\,. \tag{43}$$

(The X in (43) is not exactly the $X \subseteq \hat{A}$, but rather the corresponding $X \in V^B$.)

A generic extension of V by a generic ultrafilter on $B * A$ amounts to an iterated extension $V[G][H]$, where G is generic on B (over V) and H is generic on H (over $V[G]$).

Iteration preserves various properties of Boolean algebras. For instance:

THEOREM. (a) *If B has the countable chain condition and if $\|A$ has the countable chain condition$\| = 1$, then $B * A$ has the countable chain condition.*

(b) *If B is $\aleph_0$-distributive and $\|A$ is $\aleph_0$-distributive$\| = 1$, then $B * A$ is $\aleph_0$-distributive.*

(c) *If B has a dense $\aleph_0$-closed subset and if $\|A$ has a dense $\aleph_0$-closed subset$\| = 1$, then $B * A$ has a dense $\aleph_0$-closed subset.*

(d) *If B is game-closed and $\|A$ is game-closed$\| = 1$, then $B * A$ is game-closed.*

(e) *If B is proper and $\|A$ is proper$\| = 1$, then $B * A$ is proper.* □

Let $\bar{A}$ denote the completion of a Boolean algebra A, and let $B[A]$ be the Boolean power of A by a complete B. The following proposition summarizes the relationship between products, Boolean powers and iterations:

PROPOSITION. *Let A and B be the Bollean algebras. The following Boolean algebras all have isomorphic completions: $A \times B$, $B \times A$, $\bar{A} \times B$, $\bar{A} \times \bar{B}$, $\bar{A}[B]$, $\bar{A}[\bar{B}]$, $\bar{A} * \check{B}$, $\bar{A} * \check{\bar{B}}$, $\bar{B} * \check{A}$.* □

EXAMPLE 10. Iteration of measure algebras. Let B be a measure algebra with measure μ, and let A be, in V^B, a measure algebra with measure m. Then $B * A$ is a measure algebra, with measure ν defined as follows. The measure $m \in V^B$ defines a function $\hat{m}\colon \hat{A} \to \hat{R}^B$, and $\hat{R}^B$ is represented, as in Example 8, by measurable functions. Thus, for $x \in \hat{A}$, $\hat{m}(x)$ is a measurable function with values in $[0, 1]$, and we let

$$\nu(x) = \int \hat{m}(x)\,\mathrm{d}\mu\,. \tag{44}$$

Note that, under the embedding $e: B \to B * A(42)$, ν extends the measure μ, i.e. $\nu(e(b)) = \int_b \mathrm{d}\mu = \mu(b)$ for all $b \in B$.

EXAMPLE 11. The operation $C : B$. Let C be a complete Boolean algebra and let B be a complete subalgebra of C. The operation $C : B$ is the converse of iteration. There is a complete Boolean algebra A in V^B such that

$$B * A \cong C\,.$$

A is, in V^B, the quotient of the Boolean algebra $\check{C}$ by the ideal $I \in V^B$ defined as follows: for $c \in C$, let

$$\|\check{c} \in I\| = \sum \{a \in B;\, a \cdot c = 0\}\,. \tag{45}$$

The ideal I is (in V^B) the dual of the canonical generic ultrafilter $\mathring{G}$ on B (31).

EXAMPLE 12. Let C be a measure algebra, with measure ν, and let B be a complete subalgebra of C; B is a measure algebra with measure $\mu = \nu \restriction B$. Let $A = C : B$.

In V^B, A is a measure algebra. To verify that, let for each nonzero $c \in C$, ν_c be the following measure on B:

$$\nu_c(b) = \frac{\nu(b \cdot c)}{\nu(c)} \quad (b \in B)\,. \tag{46}$$

By the Radon–Nikodym theorem, there is a measurable function $f^c = \mathrm{d}\nu_c/\mathrm{d}\mu$ such that

$$\nu_c(b) = \int f^c\,\mathrm{d}\mu \tag{47}$$

for all $b \in B$. As in Example 8, there is some B-valued real $r \in R^B$ such that $f^c = f_r(\mu - a \cdot e)$. Let I be the ideal (45). We define a measure $\mu \in V^B$ on $A = \check{C}/I$ by

$$m(\check{c}/I) = r\,. \tag{48}$$

EXAMPLE 13. Let W be a vector space in V^B. In addition to the ring structure, W has a multiplication by scalars, i.e. B-valued reals.

A vector space $B * W$ can be defined as follows. The universe is $\hat{W}$, and the ring operations are defined as in Example 5. As for the scalar multiplication, if $x \in \hat{W}$ and $\lambda \in R$, $\lambda \cdot x$ is the unique $y \in \hat{W}$ such that

$$\|\check{\lambda} \cdot x = y\| = 1\,. \tag{49}$$

EXAMPLE 14 (Takeuti, Ozawa). Let B be a measure algebra (with a measure space $(\Omega; \mu)$) and let H be a Hilbert space in V^B. We define a Hilbert space $K = B * H$ as follows. Consider the set $\hat{H}$ endowed with the vector space structure, as in Example 13. For $x \in \hat{H}$, the norm $|x|$ or x in H is a real number in V^B; thus, a measurable function on Ω. We let

$$K = \{x \in \hat{H}: |x| \in \mathscr{L}_2(\Omega)\}\ , \tag{50}$$

and define the inner product on K by

$$(x, y) = \int (x, y)_H \, \mathrm{d}\mu\ . \tag{51}$$

K is a Hilbert space. Although $B * H = K$ is not all of $\hat{H}$ as in the other examples, it can be shown that $\hat{K} = \hat{H}$.

If the Hilbert space H in V^B is the space l_2, then $B * H$ is isomorphic to $\mathscr{L}_2(\Omega) \otimes l_2$.

The algebra B embeds naturally into the lattice of projections of $B * H$. If $a \in B$, let P_a be the following projection operator on K. For $x \in K$, let $P_a(x) = y$, where

$$a \leq \|y = x\| \quad \text{and} \quad -a \leq \|y = 0\|\ .$$

The operation $B * H$ has a converse. Let K be a Hilbert space and let B be a complete Boolean algebra of projections of H; let us also assume that B is a measure algebra. We construct a Hilbert space H in V^B so that K is isomorphic to $B * H$.

We define an ideal on $\check{K}$ in V^B as in (45): for $x \in K$, let

$$\|\check{x} \in I\| = \sum \{E \in B: E(x) = 0\}\ . \tag{52}$$

To define the inner product, let (Ω, μ) be a measure space associated with B. For each x and each y in K,

$$\nu_{x,y}(E) = (Ex, y) \quad (E \in B)$$

is a measure on B. By the Radon–Nikodým theorem

$$f^{x,y} = \frac{\mathrm{d}\nu_{x,y}}{\mathrm{d}\mu}$$

is a measurable function on Ω, thus a complex number in V^B. We let this B-valued complex number be the inner product (in V^b) of $\check{x}/I$ and $\check{y}/I$.

$H = \check{K}/I$ is a Hilbert space in V^B, and $K \simeq B * H$.

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Keywords: Boolean algebra, complete, Boolean-valued model, full, distributive, rigid.

MOS subject classification: primary 03E40; secondary 06E05, 03G05.

Appendix on Set Theory

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

The purpose of this appendix is not to develop set theory ab initio, even in outline form, but rather to give complete proofs for some advanced set-theoretic results used in this Handbook. Thus, we assume that the reader has a good working knowledge of set theory. We do not develop the more extensive portions of set theory concerned with forcing, although this does now play a role in the theory of BAs. We treat: some advanced cardinal arithmetic; two lemmas on the unit interval; almost disjoint sets; independent sets; stationary sets including diamond; delta-systems; the partition calculus; and Hajnal's free set theorem.

1. Cardinal arithmetic

1.1. THEOREM. *Let α be an ordinal of the form $\lambda \cdot \beta$, where λ is an infinite cardinal, β is a non-zero ordinal $<\lambda^+$, and $\cdot$ denotes ordinal multiplication. Suppose that $\langle \kappa_i : i < \alpha \rangle$ is a non-decreasing sequence of non-zero cardinals. Then $\prod_{i<\alpha} \kappa_i = (\sup_{i<\alpha} \kappa_i)^\lambda$.*

PROOF. Let $\lambda = \bigcup_{i<\lambda} A_i$ be a partition of λ into λ disjoint subsets of power λ. Let $\mu = \sup_{i<\alpha} \kappa_i$. Then

$$\begin{aligned}\mu^\lambda &= \prod_{j<\lambda} \mu \leqq \prod_{j<\lambda} \left(\prod \{\kappa_{\lambda\cdot\gamma+i} : \gamma < \beta, i \in A_j\}\right) \\ &= \prod_{i<\alpha} \kappa_i \leqq \mu^\lambda ,\end{aligned}$$

so the desired result follows. □

1.2. THEOREM. *If κ is a limit cardinal and $\lambda \geqq \mathrm{cf}\,\kappa$, then $\kappa^\lambda = (\sup_{\mu<\kappa} \mu^\lambda)^{\mathrm{cf}\,\kappa}$.*

PROOF. Let $\langle \kappa_i : i < \mathrm{cf}\,\kappa \rangle$ be an increasing sequence of non-zero cardinals with supremum κ. Then

$$\begin{aligned}\kappa^\lambda &\leqq \left(\prod_{i<\mathrm{cf}\,\kappa} \kappa_i\right)^\lambda = \prod_{i<\mathrm{cf}\,\kappa} \kappa_i^\lambda \\ &\leqq \prod_{i<\mathrm{cf}\,\kappa} (\sup_{\mu<\kappa} \mu^\lambda) = (\sup_{\mu<\kappa} \mu^\lambda)^{\mathrm{cf}\,\kappa} \\ &\leqq (\kappa^\lambda)^{\mathrm{cf}\,\kappa} = \kappa^\lambda ,\end{aligned}$$

and the theorem follows. □

1.3. THEOREM (The Hausdorff formula). *For infinite cardinals κ, λ we have $(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$.*

PROOF. If $\kappa^+ \leqq \lambda$, then both sides are equal to 2^λ. If $\lambda < \kappa^+$, then

$$\begin{aligned}(\kappa^+)^\lambda = |{}^\lambda\kappa^+| = \Big|\bigcup_{\alpha<\kappa^+} {}^\lambda\alpha\Big| \\ \leqq \sum_{\alpha<\kappa^+} |\alpha|^\lambda \leqq \kappa^\lambda \cdot \kappa^+ \leqq (\kappa^+)^\lambda ,\end{aligned}$$

as desired. □

The following theorem tells how to compute cardinal exponentiation – reducing κ^λ to 2^λ, κ, μ^λ for some $\mu < \kappa$, or $\kappa^{\mathrm{cf}\,\kappa}$:

1.4. THEOREM. *Let κ and λ be infinite cardinals.*
(i) *If $\kappa \leqq \lambda$, then $\kappa^\lambda = 2^\lambda$.*
(ii) *If there is a $\mu < \kappa$ such that $\mu^\lambda \geqq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(iii) *Assume that $\mu^\lambda < \kappa$ for all $\mu < \kappa$. Then $\lambda < \kappa$ and*:
(a) *if* $\mathrm{cf}\,\kappa > \lambda$, *then* $\kappa^\lambda = \kappa$;
(b) *if* $\mathrm{cf}\,\kappa \leqq \lambda$, *then* $\kappa^\lambda = \kappa^{\mathrm{cf}\,\kappa}$.

PROOF. We take (i) as known. For (ii):

$$\kappa^\lambda \leqq (\mu^\lambda)^\lambda = \mu^\lambda \leqq \kappa^\lambda ,$$

as desired. Assume the hypothesis of (iii). Then $2^\lambda < \kappa$, so $\lambda < \kappa$. Suppose that $\mathrm{cf}\,\kappa > \lambda$. If κ is a successor cardinal μ^+, then

$$\kappa^\lambda = (\mu^+)^\lambda = \mu^\lambda \cdot \mu^+ = \kappa$$

using 1.3 and the hypothesis of (iii). If κ is a limit cardinal, then

$$\begin{aligned}\kappa^\lambda = |{}^\lambda\kappa| = \Big|\bigcup_{\alpha<\kappa} {}^\lambda\alpha\Big| \quad (\text{since } \lambda < \mathrm{cf}\,\kappa) \\ \leqq \sum_{\alpha<\kappa} |\alpha|^\lambda \leqq \kappa\end{aligned}$$

by the hypothesis of (iii), so $\kappa^\lambda = \kappa$.

Finally, assume that $\mathrm{cf}\,\kappa \leqq \lambda$, so that κ is a singular cardinal. Then by 1.2,

$$\kappa^\lambda = (\sup_{\mu<\kappa} \mu^\lambda)^{\mathrm{cf}\,\kappa} \leqq \kappa^{\mathrm{cf}\,\kappa} \leqq \kappa^\lambda$$

using the hypothesis of (iii), as desired. □

Supplements to the above facts can be found in JECH [1978, pp. 42–52].

1.5. THEOREM. *Suppose that χ and σ are infinite cardinals such that $2^{<\sigma} \leqq \chi$. Then $\{\chi^\kappa : \kappa < \sigma\}$ is finite.*

PROOF. Suppose not: say $\langle \chi^{\kappa(n)}: n<\omega \rangle$ is a one-to-one sequence with $\kappa(n)<\kappa(n+1)<\sigma$ for all $n\in\omega$. Let $\chi_n=\min\{\mu: \mu^{\kappa(n)}\geqq\chi\}$ for all $n\in\omega$. Then

$$(1)\qquad \chi_n\geqq\chi_{n+1}\,,$$

$$(2)\qquad \chi_n\leqq\chi\,,$$

$$(3)\qquad \chi_n^{\kappa(n)}=\chi^{\kappa(n)}\,,$$

each holding for all $n\in\omega$. By (1) we may assume that $\langle \chi_n: n\in\omega\rangle$ is constant; call the constant value θ. Since the $\chi^{\kappa(n)}$ are distinct, by (3) the $\theta^{\kappa(n)}$ are distinct. If $\theta=2$, then $\chi\leqq 2^{\kappa(n)}\leqq\chi$ hence $\chi^{\kappa(n)}=\chi$ for each n, a contradiction. So $\theta\geqq\omega$. Now for all $\mu<\chi_n$ we have $\mu^{\kappa(n)}<\chi_n$, that is, for all $\mu<\theta$ we have $\mu^{\kappa(n)}<\theta$. Hence, by Theorem 1.4(iii), $\langle \theta^{\kappa(n)}: n\in\omega\rangle$ is eventually constant. This contradicts the distinctness of the $\theta^{\kappa(n)}$. □

Our final theorem on cardinal arithmetic is a simplified form of a theorem in SHELAH [1984].

1.6. THEOREM. *Let $\langle \mu_i: i\in I\rangle$ be a system of cardinals >1, with $I\neq 0$, and set $\mu=\Pi_{i\in I}\,\mu_i$.*

(i) *If κ and λ are cardinals, λ limit, $\kappa<\lambda$, and for every cardinal ν with $\kappa<\nu<\lambda$ we have $|\{i: \kappa<\mu_i\leqq\lambda\}|=|\{i: \nu<\mu_i\leqq\lambda\}|$, then $\Pi\{\mu_i: \kappa<\mu_i\leqq\lambda\}=\lambda^\rho$, where $\rho=|\{i: \kappa<\mu_i\leqq\lambda\}|$.*

(ii) *There exist an $n<\omega$ and cardinals χ_l, κ_l, $l<n$, such that $\chi_l\leqq\sup_{i\in I}\mu_i$ for each $l<n$, $\Sigma_{l<n}\,\kappa_l=|I|$, and $\mu=\Pi_{l<n}\,\chi_l^{\kappa_l}$.*

(iii) *If μ is infinite and $\mu>\mu_i$ for each $i\in I$, then $\mu^\omega=\mu$; in fact, $\mu=\chi_l^{\kappa_l\cdot\omega}$ for some $l<n$.*

(iv) *If χ and σ are infinite cardinals $\chi\geqq 2^{<\sigma}$, then $\{\Pi_{j\in J}\,\nu_j: |J|<\sigma,\ \nu_j\leqq\chi$ for all $j\in J$, $\Pi_{j\in J}\,\nu_j>\chi\}$ is finite.*

PROOF. Let σ be a one-to-one function from some ordinal α onto $\{i: \kappa<\mu_i\leqq\lambda\}$, chosen so that $\langle \mu_{\sigma i}: i<\alpha\rangle$ is non-decreasing. Write $\alpha=\rho\cdot\beta+\gamma$ with $\gamma<\rho$ (ordinal $\cdot$ and $+$). By the hypothesis of (i), $\gamma=0$. Hence, Theorem 1.1 applies to give the desired conclusion (the case $\rho=0$ being trivial).

(ii) We define $\chi_0>\chi_1>\cdots>\chi_{n-1}$ by induction. Let $\chi_0=\sup_{i\in I}\mu_i$. If χ_l has been defined, we consider three possibilities. If $\chi_l=1$, the construction stops, with $n=l+1$. If χ_l is a successor cardinal >1, write $\chi_l=\chi_{l+1}^{+}$. Finally, suppose that χ_l is a limit cardinal. Now

$$\langle |\{i\in I: \nu \text{ is a cardinal, } \nu<\mu_i\leqq\chi_l\}|: \nu<\chi_l\rangle$$

is a non-increasing sequence of cardinals, so we can choose $\chi_{l+1}<\chi_l$ so that $\chi_{l+1}>1$ and

$$|\{i\in I: \chi_{l+1}<\mu_i\leqq\chi_l\}|=\{i\in I: \kappa<\mu_i\leqq\chi_l\}|$$

for all κ such that $\chi_{l+1} < \kappa < \chi_l$. This finishes the construction. Then, setting $\chi_n = 0$,

$$\mu = \prod_{l<n} \prod \{\mu_i : \chi_{l+1} < \mu_i \leqq \chi_l\} = \prod_{l<n} \chi_l^{\kappa l},$$

where $\kappa_l = |\{i \in I : \chi_{l+1} < \mu_i \leqq \chi_l\}|$, by (i).

(iii) We continue with the notation introduced in the proof of (ii). If $\mu > \chi_0$, then necessarily $\mu = \chi_l^{\kappa l}$ for some $l < n$, where $\kappa_l \geqq \omega$ since $\chi_l \leqq \sup_{i\in I} \mu_i < \mu$. The only other possibility is that $\mu = \chi_0$. Then the hypothesis of (iii) implies that μ is a limit cardinal, hence $\kappa_0 \geqq \omega$. By (ii), $\mu = \chi_l^{\kappa l}$ for some $l < n$, so $\mu \geqq \chi_0^{\kappa 0} \geqq \chi_l$ implies that $\mu \geqq \chi_l^{\omega}$ hence $\mu \geqq \chi_l^{\kappa l \cdot \omega} \geqq \mu$, as desired.

(iv) By (iii), if $|J| < \sigma$, $\nu_j \leqq \chi$ for all $j \in J$, and $\prod_{j\in J} \nu_j > \chi$, then $\prod_{j\in J} \nu_j$ has the form δ^κ for some $\delta \leqq \chi$ and $\kappa \geqq \omega$, with $\kappa \leqq |J|$. Now

$$\delta^\kappa \leqq \chi^\kappa \leqq \left(\prod_{j\in J} \nu_j\right)^\kappa = \delta^\kappa .$$

Thus $\prod_{j\in J} \nu_j = \chi^\kappa$ for some $\kappa \leqq |J|$. The desired result now follows by Lemma 1.5. □

2. Two lemmas on the unit interval

These lemmas, Theorems 2.1 and 2.4 here, are taken from BONNET [1980]; they generalize a construction of SIERPIŃSKI [1950].

2.1. THEOREM. *Let I be the unit interval. Then there is a set $P \subseteq I$ of cardinality 2^ω such that if $P' \subseteq P$ and $f: P' \to P$ is either strictly increasing or strictly decreasing, then $|\{x \in P' : fx \neq x\}| < 2^\omega$, and such that if $0 \leqq a < b \leqq 1$, then $|P \cap (a, b)| = 2^\omega$.*

PROOF. For the empty set 0 we let $\sup 0 = 0$, $\inf 0 = 1$. For any subset Q of I we let $\mathrm{cl}\, Q$ be its topological closure in I, and we let $C_1 Q = \{f : f : Q \to I$, and f is either strictly increasing or strictly decreasing$\}$. For $Q \subseteq I$ and $f \in C_1 Q$ we define $f_{\mathrm{cl}} : \mathrm{cl}\, Q \to I$ by

$$f_{\mathrm{cl}} x = \begin{cases} fx & \text{if } x \in Q , \\ \sup\{fy : x > y \in Q\} & \text{if } x \notin Q \text{ and } x = \sup\{y \in Q : y < x\} \\ \inf\{fy : x < y \in Q\} & \text{if } x \notin Q \text{ and } x \neq \sup\{y \in Q : y < x\} . \end{cases}$$

Note that if $x, y \in \mathrm{cl}\, Q \backslash Q$, $x < y$, and $f_{\mathrm{cl}} x = f_{\mathrm{cl}} y$, then $x = \sup\{z : x > z \in Q\}$ and $y \neq \sup\{z : y > z \in Q\}$, so $y = \inf\{z : y < z \in Q\}$. Hence, for each $y \in I$, $f_{\mathrm{cl}}^{-1}\{y\}$ has at most three elements. Furthermore, f strictly increasing (strictly decreasing) implies that f_{cl} is increasing (decreasing).

For any $Q \subseteq I$ let $C_2 Q$ be the set of all functions $f : Q \to I$ such that

(1) f is either increasing or decreasing ,

(2) $\qquad f^{-1}\{y\}$ is finite for all $y \in I$,

(3) $\qquad |\{x \in Q\colon fx \neq x\}| = 2^\omega$.

Given $f \in C_2Q$, say f increasing, let F_1 be a countable dense subset of Q, and let

$$F_2 = \{x \in Q\colon \sup\{fy\colon x \geqq y \in F_1\} < \inf\{fy\colon x \leqq y \in F_1\}\}\,.$$

If $x \in F_2$, then $x \not\in F_1$. Hence, if $x, y \in F_2$ and $x < y$, then there is a $z \in F_1$ with $x < z < y$. It follows that the sup and inf above determine an interval U_x so that $U_x \cap U_y = 0$ for $x \neq y$. So F_2 is countable. Note that f is determined by its restriction to $F_1 \cup F_2$. From these considerations it follows that $|C_2Q| \leqq 2^\omega$. Also recall that there are just 2^ω closed sets, since every closed set is the closure of a countable dense subset. Hence, the set

$$C = \bigcup\,\{C_2F\colon F \subseteq I,\ F \text{ closed}\}$$

has cardinality $\leqq 2^\omega$. Let $\langle f_\alpha\colon \alpha < 2^\omega\rangle$ be an enumeration of C, with each element repeated 2^ω times. For each $\alpha < 2^\omega$ let

$$N_\alpha = \{x \in \operatorname{dom}(f_\alpha)\colon f_\alpha x \neq x\}\,.$$

We construct a sequence $\langle x_\alpha\colon \alpha < 2^\omega\rangle$ of distinct elements of I such that, for each $\alpha < 2^\omega$,

(C_α) $\qquad \{x_\beta\colon \beta \leqq \alpha\}$ and $\{f_\gamma x_\beta\colon \gamma \leqq \beta \leqq \alpha,\ x_\beta \in N_\gamma\}$ are disjoint.

Let $\alpha < 2^\omega$ and suppose that x_β has been defined for each $\beta < \alpha$ so that (C_β) holds. To satisfy (C_α) it suffices to find $x_\alpha \in N_\alpha$ such that

(C_α^1) $\qquad x_\alpha \not\in \{f_\gamma x_\beta\colon \gamma \leqq \beta < \alpha,\ x_\beta \in N_\gamma\} \cup \{x_\beta\colon \beta < \alpha\}$

and

(C_α^2) $\qquad$ for each $\gamma \leqq \alpha$ for which $x_\alpha \in N_\gamma$, $f_\gamma x_\alpha \not\in \{x_\beta\colon \beta < \alpha\}$.

Set

$$P_\alpha = \{x_\beta\colon \beta < \alpha\}\,,$$

$$Q_\alpha = \{f_\gamma x_\beta\colon \gamma \leqq \beta < \alpha,\ x_\beta \in N_\gamma\} \cup \{f_\gamma^{-1} P_\alpha\colon \gamma \leqq \alpha\}\,.$$

Then P_α and Q_α have power $< 2^\omega$, so we can choose $x_\alpha \in N_\alpha \setminus (P_\alpha \cup Q_\alpha)$. Clearly, (C_α^1) and (C_α^2) hold, so the construction is finished.

Let $P = \{x_\alpha\colon \alpha < 2^\omega\}$. Suppose that $P' \subseteq P$, $f\colon P' \to P$ is strictly increasing or strictly decreasing, but $\{x \in P'\colon fx \neq x\}$ has power 2^ω. Then there is an $\alpha < 2^\omega$ such that $f_{\mathrm{cl}} = f_\alpha$. Choose $\beta > \alpha$ so that $fx_\beta \neq x_\beta$. Thus, $f_\alpha x_\beta \neq x_\beta$, so $x_\beta \in N_\alpha$. Say $fx_\beta = x_\gamma$. This contradicts (C_δ), $\delta = \max\{\beta, \gamma\}$.

It remains only to check the final cardinality condition of the theorem. Suppose that $0 \leqq a < b \leqq 1$. Let f be a strictly increasing function from (a, b) into I which has no fixed points. Then $f \in C$ and hence f appears 2^ω times in the sequence $\langle f_\alpha: \alpha < 2^\omega \rangle$. It follows that $|P \cap (a, b)| = 2^\omega$. $\square$

2.2. COROLLARY (CH). *Let I be the unit interval. Then there exists a system $\langle P_i: i < \omega \rangle$ of pairwise disjoint uncountable subsets of I such that if X and Y are disjoint subsets of ω and f: $P' \to \bigcup_{i \in Y} P_i$, $P' \subseteq \bigcup_{i \in X} P_i$, is strictly increasing or strictly decreasing, then P' is countable.* $\square$

Theorem 2.1 could also be stated for the real line $\mathbf{R}$ rather than I; one can just take an order-isomorphism of $(0, 1)$ with $\mathbf{R}$. Now we want to give a variant of Theorem 2.1. It depends to a small extent on the following well-known lemma, which is of independent interest. A subset S of $\mathbf{R}$ is called *κ-dense* provided that $|S| = \kappa$ and between any two members of S there are κ members of S.

2.3. LEMMA. *If $S \subseteq \mathbf{R}$ and $|S| = \kappa$ where $\mathrm{cf}\, \kappa > \omega$, then there is a κ-dense $S' \subseteq S$ with $|S'| = \kappa$.*

PROOF. Define $x \equiv y$ iff $x, y \in S$ and $x = y$, or $x \neq y$ and if, say, $x < y$, then $|(x, y) \cap S| < \kappa$. Then $\equiv$ is an equivalence relation on S. Let S' have one element from each $\equiv$-class. Note that each $\equiv$-class is an interval, and hence there are only countably many $\equiv$-classes with more than one element. Moreover, since $\mathrm{cf}\, \kappa > \omega$, it follows that each $\equiv$-class has $< \kappa$ elements. So $|S \setminus S'| < \kappa$, hence $|S'| = \kappa$. Between two elements of S' there are κ elements of S, hence also κ elements of S', as desired. $\square$

2.4. THEOREM. *Let I be the unit interval. Then there is a set $P \subseteq I$ of cardinality $\mathrm{cf}\, 2^\omega$ such that if $P' \subseteq P$ and f: $P' \to P$ is either strictly increasing or strictly decreasing, then $\{x \in P': fx \neq x\}$ has power $< \mathrm{cf}\, 2^\omega$, and such that if $0 \leqq a < b \leqq 1$ then $|P \cap (a, b)| = \mathrm{cf}\, 2^\omega$.*

PROOF. We continue the proof of Theorem 2.1. For all rational a, b in I with $a < b$ choose $Q_{ab} \subseteq P \cap (a, b)$ of the form $\{x_\alpha: \alpha \in A\}$, where A is a cofinal subset of 2^ω of order type $\mathrm{cf}\, 2^\omega$. Let $P' = \bigcup \{Q_{ab}: a, b \in I,\ a, b \text{ rational},\ a < b\}$. Note that for each $\beta < 2^\omega$, the set $\{x_\alpha: \alpha < \beta,\ x_\alpha \in P\}$ is of power $< \mathrm{cf}\, 2^\omega$. Hence, if Q is any subset of P of power $\mathrm{cf}\, 2^\omega$, and if $\beta < 2^\omega$ is arbitrary, then there is an $\alpha > \beta$ such that $x_\alpha \in Q$. This is important at the very end of this proof. We claim that P' is as desired. Clearly, $|P'| = \mathrm{cf}\, 2^\omega$ and the last cardinality condition in the formulation of the theorem is true. Now suppose that $R \subseteq P'$, f: $R \to P'$ is strictly increasing (the case strictly decreasing is similar), but $N \overset{\mathrm{def}}{=} \{x \in R: fx \neq x\}$ has power $\mathrm{cf}\, 2^\omega$.

We claim that $f_{\mathrm{cl}} \in C$. To show this we need to show that $T \overset{\mathrm{def}}{=} \{x \in \mathrm{dom}\, f_{\mathrm{cl}}: f_{\mathrm{cl}} x \neq x\}$ has cardinality 2^ω. By Lemma 2.3, choose an $\aleph_1$-dense $N' \subseteq N$. Let

$$L = \{x \in N': \sup\{y \in N': y < x\} < \inf\{y \in N': x < y\}\} .$$

By $\aleph_1$-denseness (really, just by denseness) we infer that L is countable. Now suppose, to get a contradiction, that T has power $<2^\omega$. Let $x \in N' \setminus L$. Thus, $\sup\{y \in N' : y < x\} = \inf\{y \in N' : x < y\} = x$. For any $y \in N'$ with $y < x$ we have $(y, x) \cap N'$ uncountable, so $[x, y] \cap \mathrm{cl}\, R$ is closed and uncountable, hence of cardinality 2^ω. Hence, choose $z \in (y, x) \cap \mathrm{cl}\, R$ with $z \not\in T$. Then $z = f_{\mathrm{cl}} z \leqq f_{\mathrm{cl}} x$. Since y is arbitrary, it follows that $x \leqq f_{\mathrm{cl}} x$. Similarly, $f_{\mathrm{cl}} x \leqq x$, so $f_{\mathrm{cl}} x = x \neq fx$, a contradiction.

Hence, $f_{\mathrm{cl}} \in C$. We now obtain a contradiction just as at the end of the proof of Theorem 2.1, using the remark above following the definition of P'. □

3. Almost-disjoint sets

Sets A and B are *almost-disjoint* if $|A \cap B| < |A|, |B|$.

3.1. THEOREM (Sierpiński, Tarski). *Suppose κ and λ are cardinals with $2 \leqq \lambda \leqq \kappa \leqq \omega$. Let μ be the least cardinal such that $\kappa < \lambda^\mu$. Then there is an $\mathscr{A} \subseteq \mathscr{P}\kappa$ such that $|\mathscr{A}| = \lambda^\mu$, $|X| = \mu$ for all $X \in \mathscr{A}$, and the members of $\mathscr{A}$ are almost disjoint.*

PROOF. Let $T = \bigcup_{\alpha<\mu} {}^\alpha\lambda$. Thus, $|T| \leqq \kappa$. For each $f \in {}^\mu\lambda$ let $X_f = \{t \in T : t \subseteq f\}$. Then set $\mathscr{A} = \{X_f : f \in {}^\mu\lambda\}$. □

3.2. COROLLARY. *There is a family of 2^ω almost-disjoint infinite subsets of ω.* □

4. Independent sets

Let κ be an infinite cardinal. For $\Gamma \subseteq \kappa$ we define $(+1) \cdot \Gamma = \Gamma$, $(-1) \cdot \Gamma = \kappa \setminus \Gamma$. A family $\mathscr{A} \subseteq \mathscr{P}\kappa$ of subsets of κ is *independent* if, for any finite sequence $\Gamma_0, \ldots, \Gamma_m$ of distinct elements of $\mathscr{A}$ and any sequence $\varepsilon_0, \ldots, \varepsilon_m$ of $+1$'s and -1's, we have $\varepsilon_0 \cdot \Gamma_0 \cap \cdots \cap \varepsilon_m \cdot \Gamma_m \neq 0$. We call $\mathscr{A}$ *strongly independent* if the conclusion of the preceding statement is replaced by $|\varepsilon_0 \cdot \Gamma_0 \cap \cdots \cap \varepsilon_m \cdot \Gamma_m| = \kappa$.

4.1. THEOREM (Fichtenholz, Kantorovitch, Hausdorff). *Let $\kappa \geqq \omega$. Then there is a family $\mathscr{A} \subseteq \mathscr{P}\kappa$ of strongly independent subsets of κ with $|\mathscr{A}| = 2^\kappa$.*

PROOF. Let $\mathscr{F}$ be the set of all finite subsets of κ, and Φ the set of all finite subsets of $\mathscr{F}$. Thus, $|\mathscr{F}| = |\Phi| = |\mathscr{F} \times \Phi| = \kappa$. We construct a family $\mathscr{A} \subseteq \mathscr{P}(\mathscr{F} \times \Phi)$ of strongly independent subsets of $\mathscr{F} \times \Phi$ with $|\mathscr{A}| = 2^\kappa$. With each subset Γ of κ we associate $b_\Gamma \subseteq \mathscr{F} \times \Phi$ as follows:

$$b_\Gamma = \{(\Delta, \varphi) \in \mathscr{F} \times \Phi : \Delta \cap \Gamma \in \varphi\} .$$

Now let $\Gamma_0, \ldots, \Gamma_m$ be distinct subsets of κ, and let $0 \leqq i \leqq m$. We shall show that the set

$$(*) \qquad b_{\Gamma 0} \cap \cdots \cap b_{\Gamma i} \cap ((\mathscr{F} \times \Phi) \setminus b_{\Gamma(i+1)}) \cap \cdots \cap ((\mathscr{F} \times \Phi) \setminus b_{\Gamma m})$$

has κ members, thereby proving the theorem with $\mathscr{A} = \{b_\Gamma: \Gamma \subseteq \kappa\}$. For $0 \leqq k < j \leqq m$ pick $\alpha_{kj} \in \Gamma_k \Delta \Gamma_j$, and let $\Delta = \{\alpha_{kj}: 0 \leqq k < j \leqq m\}$, $\varphi = \{\Delta \cap \Gamma_0, \ldots, \Delta \cap \Gamma_i\}$. Now we claim that if $\varphi \subseteq \psi \in \Phi$ but $\Delta \cap \Gamma_j \not\in \psi$ for $i + l \leqq j \leqq m$, then (Δ, ψ) is in the set $(*)$; since $\Delta \cap \Gamma_j \not\in \varphi$ for $i+1 \leqq j \leqq m$, this will show that $(*)$ has κ members. Let $0 \leqq j \leqq i$. Then $\Delta \cap \Gamma_j \in \varphi \subseteq \psi$, so $(\Delta, \psi) \in b_{\Gamma j}$. Now let $i+1 \leqq j \leqq m$, and suppose that $(\Delta, \psi) \in b_{\Gamma j}$. Then $\Delta \cap \Gamma_j \in \psi$, a contradiction. This completes the proof. □

For a Boolean-algebraic proof of Theorem 4.1 see Part I of this Handbook, Example 9.21, and also Corollary 13.15.

5. Stationary sets

Let α be an ordinal. A subset $\Gamma \subseteq \alpha$ is *unbounded in* α if $\bigcup \Gamma = \alpha$. It is *closed in* α if $\lambda \in \Gamma$ whenever λ is a limit ordinal $< \alpha$ such that $\Gamma \cap \lambda$ is unbounded in λ. We use "club" as an abbreviation for "closed and unbounded".

5.1. THEOREM. *If κ is a regular uncountable cardinal and $\mathscr{A}$ is a collection of clubs in κ with $|\mathscr{A}| < \kappa$, then $\bigcap \mathscr{A}$ is club in κ.*

PROOF. Clearly, $\bigcap \mathscr{A}$ is closed in κ. Now suppose that $\alpha < \kappa$; we must find β such that $\alpha < \beta < \kappa$ and $\beta \in \bigcap \mathscr{A}$. Let $\lambda = \omega \cup |\mathscr{A}|$, and write $\lambda = \bigcup_{\Gamma \in \mathscr{A}} \Delta_\Gamma$, where $|\Delta_\Gamma| = \lambda$ for all $\Gamma \in \mathscr{A}$ and $\Delta_\Gamma \cap \Delta_\Theta = 0$ for distinct $\Gamma, \Theta \in \mathscr{A}$. Then we define $\beta: \lambda \to \kappa$ by recursion, setting for $\gamma < \lambda$

$$\beta_\gamma = \min\{\delta \in \Gamma: \alpha < \delta \text{ and } \beta_\varepsilon < \delta \text{ for all } \varepsilon < \gamma\},$$

where Γ is chosen so that $\gamma \in \Delta_\Gamma$. This is possible since Γ is unbounded in κ and $\sup_{\varepsilon<\gamma} \beta_\varepsilon < \kappa$ (because κ is regular and uncountable). Thus, β is strictly increasing, so $\mu \stackrel{\text{def}}{=} \sup_{\gamma<\lambda} \beta_\gamma$ is a limit ordinal, which is $<\kappa$ since κ is regular and $\lambda < \kappa$. Now for all $\Gamma \in \mathscr{A}$, $\Gamma \cap \mu$ is unbounded in μ: if $\gamma < \mu$, then there is a $\nu \in \Delta_\Gamma$ with $\gamma < \nu$, and hence $\gamma < \nu \leqq \beta_\nu \in \Gamma$. Thus, $\mu \in \Gamma$, so $\mu \in \bigcap \mathscr{A}$ and $\alpha < \mu$, as desired. □

Let α be an ordinal. A subset S of α is *stationary in* α if $S \cap C \neq 0$ for every club C in α.

5.2. COROLLARY. *Let κ be uncountable and regular. If C is club in κ, then C is stationary in κ; moreover, if S is stationary in κ, then $S \cap C$ is also stationary in κ.* □

5.3. COROLLARY. *If κ is uncountable and regular, S is stationary in κ, $S = \bigcup_{i \in I} T_i$, and $|I| < \kappa$, then there is an $i \in I$ such that T_i is stationary in κ.*

PROOF. Assume otherwise. For each $i \in I$ let C_i be a club in κ such that $T_i \cap C_i = 0$. Then $S \cap \bigcap_{i \in I} C_i = 0$, contradicting Theorem 5.1. □

Let Γ be a set of ordinals, f a function from Γ into a set of ordinals. We call f *regressive* if $f\alpha < \alpha$ for all $\alpha \in \Gamma \backslash \{0\}$. The following fundamental theorem about this notion is also called the *pressing-down* lemma.

5.4. THEOREM (Fodor). *Let κ be an uncountable regular cardinal, S a stationary set in κ, and f: $S \to \kappa$ regressive. Then there is a stationary set $S' \subseteq S$ in κ such that $|f[S']| = 1$.*

PROOF. It suffices to show

(1) there is a $\beta < \kappa$ such that $f^{-1}[\{\beta\}]$ is stationary in κ .

Suppose that (1) is false. For all $\beta < \kappa$ let C_β be club in κ such that $f^{-1}[\{\beta\}] \cap C_\beta = 0$. Now we define β: $\kappa \to \kappa$ by recursion. Let $\beta 0 = 0$. Having defined β_γ, choose $\beta_{\gamma+1} \in \bigcap_{\varepsilon < \beta\gamma} C_\varepsilon$ such that $\beta_{\gamma+1} > \beta_\gamma$, using Theorem 5.1. For limit $\lambda < \kappa$ let $\beta_\lambda = \sup_{\gamma<\lambda} \beta_\gamma$. This defines β. Let $B = \{\beta_\lambda$: λ limit $<\kappa\}$. Clearly, B is club in κ. Choose $\gamma \in B \cap S$. We claim

(2) $\gamma \in C_{f\gamma}$.

For, γ is a limit ordinal since $\gamma \in B$, so it suffices to show that $C_{f\gamma} \cap \gamma$ is unbounded in γ. Le $\nu < \gamma$. Say $\gamma = \beta_\lambda$, λ limit. Choose $\rho < \lambda$ such that $f\gamma < \beta_\rho$ and $\nu < \beta_\rho$. Then by construction $\beta_{\rho+1} \in C_{f\gamma} \cap \gamma$, as desired. So (2) holds.

But then $\gamma \in f^{-1}[\{f\gamma\}] \cap C_{f\gamma}$, a contradiction. □

5.5. THEOREM. *Let κ be an uncountable regular cardinal, S a stationary subset of κ, and $\langle S_i$: $i \in I\rangle$ a partition of S into non-stationary subsets. Then there is a stationary subset S' of S such that* (a) *$S \backslash S'$ is non-stationary*, (b) $|S' \cap S_i| = 1$ *for all $i \in I$.*

PROOF. By Corollary 5.3 we know that $|I| = \kappa$, so we may assume that $I = \kappa$. For each $\alpha < \kappa$ let β_α be the least element of S_α. Set $S' = \{\beta_\alpha$: $\alpha < \kappa\}$. Thus, $|S' \cap S_\alpha| = 1$ for all $\alpha < \kappa$. By Corollary 5.3 again, it suffices now to show that $S \backslash S'$ is non-stationary.

Assume that $S \backslash S'$ is stationary. For each $\alpha \in S \backslash S'$ choose $\gamma < \kappa$ so that $\alpha \in S_\gamma$, and let $f\alpha = \beta_\gamma$. Thus, f is regressive on $S \backslash S'$, so by Fodor's theorem there is a stationary $T \subseteq S \backslash S'$ on which f is constant. Clearly, $T \subseteq S_\alpha$ for some α, so S_α is stationary, a contradiction. □

5.6. THEOREM. *Let κ be an uncountable regular cardinal, S a stationary subset of κ, and f a function mapping S into κ. Then there is a stationary subset S' of S such that one of the following conditions holds*:

(i) *$f \restriction S'$ is constant.*
(ii) *$f \restriction S'$ is the identity.*
(iii) *$f \restriction S'$ is strictly increasing and $f\alpha > \alpha$ for all $\alpha \in S'$.*

Moreover, (iii) *implies that $f[S']$ is non-stationary in κ.*

PROOF. Let S_1 (resp. S_2, S_3) be the set of all $\alpha \in S$ such that $f\alpha < \alpha$ (resp. $f\alpha = \alpha$, $f\alpha > \alpha$). By Corollary 5.3, one of S_1, S_2, S_3 is stationary. If S_1 is stationary, Fodor's theorem yields a stationary $S' \subseteq S_1$ satisfying (i). If S_2 is stationary, $S' = S_2$ satisfies (ii). Suppose that S_3 is stationary. Define $\gamma 0 = 0$, $\gamma(\alpha + 1) = (\bigcup \{f\gamma\alpha\colon \text{if } \gamma\alpha \in S_3\} \cup \gamma\alpha) + 1$, $\gamma\lambda = \bigcup_{\alpha<\lambda} \gamma\alpha$ for λ a limit ordinal $<\kappa$. Then ran γ is a club. Let $S' = S_3 \cap \operatorname{ran} \gamma$. Then S' is a stationary subset of S, and $f\alpha > \alpha$ for all $\alpha \in S'$. Suppose that $\alpha, \beta \in S'$, $\alpha < \beta$. Say $\alpha = \gamma\alpha'$, $\beta = \gamma\beta'$. Then $f\beta > \beta = \gamma\beta' \geqq \gamma(\alpha' + 1) > f\gamma\alpha' = f\alpha$; thus f is strictly increasing.

Finally, assume the conditions in (iii). For any $\alpha \in S'$ let $gf\alpha = \alpha$; so $gf\alpha < f\alpha$, hence g is regressive on $f[S']$. Since g is one-to-one, $f[S']$ is non-stationary by Fodor's theorem. □

If $\Gamma \subseteq \alpha$ and $f\colon \Gamma \to \alpha$, we call f *divergent over* α if, for every $\beta < \alpha$, $f^{-1}[\beta]$ is bounded in α.

5.7. LEMMA. *Let α be a limit ordinal, $\Gamma \subseteq \alpha$, and Γ non-stationary in α. Then there is a divergent regressive function $f\colon \Gamma \to \alpha$ over α. If* cf $\alpha = \omega$, *then such an f exists whether or not Γ is non-stationary.*

PROOF. Let C be club in α such that $\Gamma \cap C = 0$. For each $\gamma \in \Gamma$ let $f\gamma = \bigcup (\gamma \cap C)$. Thus, for γ non-limit with $\gamma \neq 0$ we have $f\gamma = \bigcup (\gamma \cap C) \subseteq \bigcup \gamma < \gamma$. For γ limit, $\gamma \cap C$ is bounded in γ (otherwise $\gamma \in C \cap \Gamma$, a contradiction), so $f\gamma < \gamma$. Thus, f is regressive.

Now let $\beta < \alpha$. Choose $\gamma \in C$ such that $\beta < \gamma < \alpha$. For any $\delta > \gamma$ with $\delta \in \Gamma$ we then have $f\delta = \bigcup (\delta \cap C) \geqq \gamma$, so $\delta \notin f^{-1}[\beta]$. Thus, $\gamma + 1$ is a bound for $f^{-1}[\beta]$, as desired.

If cf $\alpha = \omega$, say $\sup_{n\in\omega} \beta_n = \alpha$ with $\langle \beta_n\colon n \in \omega \rangle$ strictly increasing and $\beta_0 = 0$. For each $\gamma \in \Gamma \backslash \{0\}$ let $f\gamma = \beta_n$, where $\beta_n < \gamma \leqq \beta_{n+1}$. Clearly, f is as desired. □

5.8. LEMMA. *Let β be a limit ordinal with* cf $\beta > \omega$, *and let C be club in β. Define C' to be the set of all limit points of C, i.e. $C' = \{\alpha < \beta\colon \alpha$ is a limit ordinal and $C \cap \alpha$ is unbounded in $\alpha\}$. Then C' is club in β.*

Furthermore, if $\alpha \in C'$ and cf $\alpha > \omega$, *then $C' \cap \alpha$ is club in α.*

PROOF. Note that $C' \subseteq C$. If $\alpha < \beta$ is a limit ordinal and $C' \cap \alpha$ is unbounded in α, then $C \cap \alpha$ is unbounded in α and so $\alpha \in C'$. Hence, C' is closed in β. Now let $\alpha < \beta$. By the unboundedness of C in β one can construct $\gamma_0, \gamma_1, \ldots \in C$ such that $\alpha < \gamma_0 < \gamma < \cdots$. Then, since cf $\beta > \omega$, we have $\bigcup_{i\in\omega} \gamma_i < \beta$, so $\bigcup_{i\in\omega} \gamma_i \in C$ and therefore $\bigcup_{i\in\omega} \gamma_i \in C'$. So C' is unbounded in β. We have shown that C' is club in β.

Finally, let $\alpha \in C'$ with cf $\alpha > \omega$. Then $C \cap \alpha$ is unbounded in α, and the above argument yields that $C' \cap \alpha$ is unbounded in α. Since $C' \cap \alpha$ is closed in α, because it is closed in β, $C' \cap \alpha$ is club in α. □

5.9. THEOREM (Solovay). *Let κ be an uncountable regular cardinal, and let S be stationary in κ. Then S can be split into κ pairwise disjoint stationary subsets.*

PROOF. Let

$$N = \{\alpha < \kappa\colon \alpha \text{ is a limit ordinal, and either cf } \alpha = \omega \text{ or else } S \cap \alpha \text{ is non-stationary in } \alpha\} .$$

We claim:

(1) $N \cap S$ is stationary in κ .

For, let C be club in κ; we need to find $\alpha \in C \cap N \cap S$. Let C' be the set of limit points of C. By Lemma 5.8, C' is club in κ, so $S \cap C' \neq 0$; let α be the least element of $S \cap C'$. We claim that $\alpha \in N$. To show this, assume that cf $\alpha > \omega$. Now $C' \cap \alpha$ is club in α by Lemma 5.8, and $(C' \cap \alpha) \cap (S \cap \alpha) = 0$ by choice of α, so $S \cap \alpha$ is non-stationary in α – thus $\alpha \in N$, proving (1).

By (1) we may assume that $N \cap S = S$; thus for every $\alpha \in S$, α is a limit ordinal and either cf $\alpha = \omega$ or else $S \cap \alpha$ is non-stationary in α. Hence, by Lemma 5.7, for every $\alpha \in S$ let $f_\alpha\colon S \cap \alpha \to \alpha$ be divergent and regressive. Then

(2) for all $\beta \in S$ there is a $\delta < \beta$ such that $\{\alpha \in S\colon f_\alpha \beta = \delta\}$ is stationary.

In fact, let $\beta \in S$. Then $S \setminus (\beta + 1) = \bigcup_{\delta < \beta} \{\alpha \in S\colon \alpha \geqq \beta + 1 \text{ and } f_\alpha \beta = \delta\}$, and $S \setminus (\beta + 1)$ is stationary, so by Corollary 5.3 we get the conclusion of (2).

For each $\beta \in S$ let $g\beta$ be a δ as in (2). Thus g is regressive, so by Fodor's theorem pick $\gamma < \kappa$ so that $g^{-1}[\{\gamma\}]$ is stationary in κ. Now for each $\alpha \in S$ with $\alpha > \gamma$ the set $f_\alpha^{-1}[\gamma + 1]$ is bounded in α (since f_α is divergent), so we can choose $h\alpha < \alpha$ such that for all ρ, if $h\alpha < \rho \in S \cap \alpha$ then $\gamma < f_\alpha \rho$.

The theorem now follows from

(3) $\{\sigma < \kappa\colon h^{-1}[\{\sigma\}]$ is stationary in $\kappa\}$ is unbounded in κ .

To prove (3), let $\delta < \kappa$. Since $g^{-1}[\{\gamma\}]$ is stationary in κ and hence unbounded in κ, choose $\beta \in S$ with $\delta < \beta$ so that $g\beta = \gamma$. By the definition of g, the set $T \overset{\text{def}}{=} \{\alpha \in S\colon \alpha > \gamma, f_\alpha \beta = \gamma\}$ is stationary. For each $\alpha \in T$ with $\beta < \alpha$ we have $h\alpha \geqq \beta$ (otherwise $h\alpha < \beta \in S \cap \alpha$ and hence $\gamma < f_\alpha \beta$, a contradiction); furthermore, h is regressive on T. So, by Fodor's theorem, there is a σ with $h^{-1}[\{\sigma\}] \cap T$ stationary in κ. This implies that $\sigma \geqq \beta > \delta$, so (3) holds, as desired. □

We now discuss the set-theoretical principle *diamond*. $\diamond$ is the following statement: there is a system $\langle \Gamma_\alpha\colon \alpha < \omega_1 \rangle$ of sets such that $\Gamma_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$, and for every $\Delta \subseteq \omega_1$, $\{\alpha < \omega_1\colon \Delta \cap \alpha = \Gamma_\alpha\}$ is stationary in ω_1. $\langle \Gamma_\alpha\colon \alpha < \omega_1 \rangle$ is called a *diamond sequence*. This is a very useful principle, which goes beyond the axioms of ZFC but is consistent relative to ZFC. We derive the two most important consequences of it: the continuum hypothesis (CH), and the existence of Suslin trees.

5.10. THEOREM. $\diamond$ *implies CH.*

PROOF. Let $\langle \Gamma_\alpha \colon \alpha < \omega_1 \rangle$ be a diamond sequence. Then for every subset Δ of ω the set $\{\alpha < \omega_1 \colon \Delta \cap \alpha = \Gamma_\alpha\}$ is stationary in ω_1, hence has a member $\alpha \geqq \omega$. Thus, $\Delta = \Gamma_\alpha$. So $\mathscr{P}\omega \subseteq \{\Gamma_\alpha \colon \alpha < \omega_1\}$, and CH follows. $\square$

To show that $\diamond$ implies the existence of a Suslin tree, we need to introduce some of the (mostly) standard terminology about trees. A *tree* is a partially ordered set $\langle T, \leqq \rangle$ such that for all $t \in T$, $\{x \colon x < t\}$ is well-ordered by $<$; the order type of this set is called the *level* of t. The *height* of T is $\sup\{\text{level } t + 1 \colon t \in T\}$. A *branch* of T is a maximal simply ordered subset. Elements of level 0 are called *roots*.

Let α be a limit ordinal. A *normal α-tree* is a tree $\langle T, \leqq \rangle$ of height α with the following properties:

(1) T has only one root.

(2) For any limit ordinal $\beta < \alpha$ and any elements x, y of level β, if $\{z \colon z < x\} = \{z \colon z < y\}$ then $x = y$.

(3) If $\beta < \gamma < \alpha$ and $x \in T$ is of level β, then there is a $y \in T$ of level γ with $x < y$.

(4) Every $x \in T$ has ω immediate successors.

Finally, a Suslin tree is a tree of height ω_1 with no uncountable chain and no uncountable set of pairwise incomparable elements. It is a *normal Suslin tree* if, in addition, it is a normal ω_1-tree. It is not very hard to show that if there is a Suslin tree, then there is a normal Suslin tree; but we will not use this fact.

5.11. THEOREM. *$\diamond$ implies that there is a normal Suslin tree.*

PROOF. We construct a sequence $\langle L_\alpha \colon \alpha < \omega_1 \rangle$ by recursion, and a tree ordering $\preccurlyeq_\alpha$ on $\bigcup_{\beta \leq \alpha} L_\beta$; L_α will be the αth level of the desired tree. Let $\langle S_\alpha \colon \alpha < \omega_1 \rangle$ be a diamond sequence. We will have $L_\alpha \subseteq \omega_1$, and the desired tree $(T, \preccurlyeq \rangle$ will have $T = \omega_1$.

Let $L_\alpha = \{0\}$. If L_α has been defined, with $|L_\alpha| \leqq \omega$, we let $L_{\alpha+1}$ be obtained by giving each $x \in L_\alpha$ ω immediate successors (using the first ordinals in $\omega_1 \backslash \bigcup_{\beta \leq \alpha} L_\beta$).

Now suppose α is a limit ordinal $< \omega_1$ and L_β has been defined for all $\beta < \alpha$ so that $T_\alpha \stackrel{\text{def}}{=} \bigcup_{\beta < \alpha} L_\beta$ is a countable normal α-tree. Then

There is a system $\langle b_x \colon x \in T_\alpha \rangle$ of branches of T_α, each of order type α, such that

(a) $x \in b_x$, for all $x \in T_\alpha$;

(b) if $x \neq y$ then $b_x \neq b_y$, for all $x, y \in T_\alpha$;

(c) if $S_\alpha \subseteq T_\alpha$ and S_α is a maximal set of pairwise incomparable elements of T_α, and if $x \in T_\alpha$, then $b_x \cap S_\alpha \neq 0$.

To see this, let $T_\alpha = \{x_n \colon n \in \omega\}$. Suppose b_{xm} has been defined for all $m < n$ so that (a)–(c) hold. If $S_\alpha \subseteq T_\alpha$ and S_α is a maximal set of pairwise incomparable elements of T_α, choose $y \in S_\alpha$ so that x_n and y are comparable, and let $z_0 = x_n$ if $y \leqq x_n$, y if $x_n < y$. If S_α is not as indicated, let $z_0 = x_n$. Let z_1 be an immediate successor of z_0 with $z_1 \not\in b_{x0} \cup \cdots \cup b_{x(n-1)}$. Then, using $\operatorname{cf} \alpha = \omega$ and the normality of T_α it is easy to construct a branch b_{xn} of T_α with $z_1 \in b_{xn}$. Clearly, (a)–(c) then hold. Hence, the definition of the b_{xn} is complete, and (1) holds.

Let L_α be defined by putting an element above each branch b_x (again using the first ordinals in $\omega_1 \backslash T_\alpha$).

This defines our tree $\langle \omega_1, \preccurlyeq \rangle$, with $\gamma \preccurlyeq \delta$ iff $\gamma \in L_\alpha$, $\delta \in L_\beta$, $\alpha \leqq \beta$, and $\gamma \preccurlyeq \delta$ by the definition of L_β. Clearly, $\langle \omega_1, \preccurlyeq \rangle$ is a normal ω_1-tree. Now if X is a branch of $\langle \omega_1, \preccurlyeq \rangle$ of power ω_1, we can construct from X an uncountable set of pairwise incomparable elements by choosing for each $x \in X$ a successor y_x of x with $y_x \not\in X$; then $\{y_x: x \in X\}$ is the indicated set. So, it suffices to show that if X is a maximal set of pairwise incomparable elements then X is countable. Let

$$C = \{\alpha < \omega_1 \colon \alpha \text{ is a limit ordinal, and } X \cap \alpha \text{ is a maximal set of pairwise incomparable elements of } T_\alpha\}\,.$$

We claim that C is club in ω_1. To show that C is closed, suppose that α is a limit ordinal $<\omega_1$ and $C \cap \alpha$ is unbounded in α. If $x \in X \cap \alpha$, choose $y \in C \cap \alpha$ with $x < y$. Since $y \in C$, $X \cap y$ is a maximal pairwise incomparable family in T_y. Since $x \in X \cap y$, we have $x \in T_y \subseteq T_\alpha$. Thus, $X \cap \alpha \subseteq T_\alpha$. Now let $z \in T_\alpha$; we want to show that z is $\preccurlyeq$-comparable with some member of $X \cap \alpha$. Say $z \in T_\beta$ with $\beta < \alpha$, since α is limit. Choose $\gamma \in C \cap \alpha$ with $\beta < \gamma$. Then $z \in T_\gamma$ and $X \cap \gamma$ is a maximal pairwise incomparable family in T_γ, so there is a $u \in X \cap \gamma$ such that z and u are $\preccurlyeq$-comparable. Since $\gamma < \alpha$, we have $u \in X \cap \alpha$, as desired. Thus, $\alpha \in C$, and C is closed.

To show that C is unbounded, let $\beta < \omega_1$; we are to find $\alpha \in C$ with $\beta < \alpha$. Let $\gamma_0 = \beta$. Choose $\gamma_1 > \gamma_0$ with $X \cap \gamma_0 \subseteq T_{\gamma 1}$. Choose $\gamma_2 > \gamma_1$ so that each member of $T_{\gamma 1}$ is comparable with some member of $X \cap \gamma_2$; etc. Let $\alpha = \bigcup_{n \in \omega} \gamma_n$. Clearly, $\beta < \alpha \in C$, as desired. So C is a club.

By the $\diamond$ property, choose $\alpha \in C$ such that $X \cap \alpha = S_\alpha$. By our construction, every element of L_α lies above (in the tree sense) a member of S_α. Hence, $X = X \cap \alpha$, and X is countable. $\square$

6. Δ-systems

A system of sets $\langle A_i \colon i \in I \rangle$ is a *Δ-system with kernel B* if for all distinct $i, j \in I$ we have $A_i \cap A_j = B$.

6.1. THEOREM (Marczewski, Erdös, Hajnal). *Suppose that $\omega \leqq \kappa < \lambda$, κ and λ are regular, and for all $\mu < \lambda$, $\mu^{<\kappa} < \lambda$. Assume that $\langle A_\alpha \colon \alpha < \lambda \rangle$ is a system of sets such that for all $\alpha < \lambda$, $|A_\alpha| < \kappa$. Then there is a $\Gamma \in [\lambda]^\lambda$ such that $\langle A_\alpha \colon \alpha \in \Gamma \rangle$ is a Δ-system.*

PROOF. Since $|\bigcup_{\alpha<\lambda} A_\alpha| \leqq \lambda$, we may assume that $A_\alpha \subseteq \lambda$ for each $\alpha < \lambda$. Let $S = \{\alpha < \lambda \colon \mathrm{cf}\, \alpha = \kappa\}$. Clearly, S is stationary in λ. For each $\alpha \in S$ choose $g\alpha$ so that $\bigcup (A_\alpha \cap \alpha) < g\alpha < \alpha$. Thus, g is regressive, so by Fodor's theorem there is a stationary $S' \subseteq S$ and a $\beta < \lambda$ such that $g[S'] = \{\beta\}$. For all $\alpha \in S'$ let $F\alpha = A_\alpha \cap \alpha$. Thus, $F\alpha \in [\beta]^{<\kappa}$ and $|[\beta]^{<\kappa}| < \lambda$, so there is an $S'' \in [S']^\lambda$ and a $B \in [\beta]^{<\kappa}$ such that $A_\alpha \cap \alpha = B$ for all $\alpha \in S''$.

Now we define $\alpha \in {}^{\lambda}S''$ by recursion: for any $\xi < \lambda$, α_ξ is a member of S'' such that $\alpha_\xi > \alpha_\eta$ for all $\eta < \xi$, and $\alpha_\xi > \delta$ for all $\delta \in \bigcup_{\eta<\xi} A_{\alpha\eta}$. It suffices now to show

(*) If $\eta < \xi < \lambda$, then $A_{\alpha\eta} \cap A_{\alpha\xi} = B$.

To do this, note that $\supseteq$ is obvious. Now let $\delta \in A_{\alpha\eta} \cap A_{\alpha\xi}$. Then $\delta < \alpha_\xi$ by construction, so $\delta \in A_{\alpha\xi} \cap \alpha_\xi = \beta$, as desired. □

6.2. COROLLARY. *If λ is a regular uncountable cardinal and $\langle A_\alpha \colon \alpha < \lambda \rangle$ is a system of finite sets, then there is a $\Gamma \in [\lambda]^\lambda$ such that $\langle A_\alpha \colon \alpha \in \Gamma \rangle$ is a Δ-system.* □

7. The partition calculus

We introduce three special cases of the arrow notation in the partition calculus. Let κ, λ, μ, ν be non-zero cardinals. We write

$$\kappa \to (\lambda)^\nu_\mu$$

provided that whenever $f \colon [\kappa]^\nu \to \mu$ there is an $H \subseteq \kappa$ such that $|H| = \lambda$ and $|f[H]| = 1$; then we call H a *homogeneous* set for f. We write

$$\kappa \to (\kappa, \lambda)^2$$

provided that whenever $f \colon [\kappa]^2 \to 2$ there is $H \subseteq \kappa$ such that $|f[H]| = 1$ and either $|H| = \kappa$ or $|H| = \lambda$. Again, H is called *homogeneous* for f. Our third arrow notation is of an auxiliary nature. Suppose that κ, λ, μ are non-zero cardinals and n is a positive integer. We write

$$\kappa \underset{\text{pre}}{\to} (\lambda)^n_\mu$$

provided that whenever $f \colon [\kappa]^n \to \mu$ there is an $H \subseteq \kappa$ such that $|H| = \lambda$ and $f(F \cup \{\alpha\}) = f(F \cup \{\beta\})$ whenever $F \cup \{\alpha, \beta\} \subseteq H$, $|F| = n - 1$, and $F < \alpha$, $F < \beta$; H is called *pre-homogeneous* for f.

7.1. LEMMA. *Let λ and κ be infinite cardinals. Suppose that $\mu^{<\lambda} < \kappa$. Then $\kappa \underset{\text{pre}}{\to} (\lambda)^n_\mu$ for every integer $n \geq 2$.*

PROOF. Let $f \colon [\kappa]^n \to \mu$ be given. Set

$$T = \bigcup_{n-2<\xi<\lambda} A_\mu, \quad \text{where } A = [\xi]^{n-1}.$$

For each $s \in T$ let ls be the $\xi < \lambda$ such that s maps $[\xi]^{n-1}$ into μ. We now define $A \colon T \to \mathscr{P}\kappa$ by induction on ls, for $s \in T$; for each A_s which has been defined we let $h(ls, s)$ be the least element of A_s, or $h(ls, s) = 0$ if $A_s = 0$. In addition, we let

$h(m, 0) = m$ for $m < n-1$. Note that if $s \in T$ and $\eta < n-1$, then $s \upharpoonright [\eta]^{n-1} = 0$. We shall define A_s so that the following condition holds:

$$\text{(1)} \qquad \text{If } \eta < ls \text{ and } A_s \neq 0, \text{ then } h(\eta, s \upharpoonright [\eta]^{n-1}) < h(ls, s)\,.$$

To start with, for $ls = n-1$ we set $A_s = \kappa \backslash n$. Clearly, (1) holds. Now suppose that $ls = \xi + 1 \geqq n$. We set $A_2 = 0$ if $A_{s \upharpoonright B} = 0$, where $B = [\xi]^{n-1}$, and if $A_{s \upharpoonright B} \neq 0$ then we set

$$A_s = A_{s \upharpoonright B} \cap \{\gamma \in \kappa\colon h(\xi, s \upharpoonright [\xi]^{n-1}) < \gamma \text{ and for all } F \in [\xi + 1]^{n-1},$$
$$f(\{h(\eta, s \upharpoonright [\eta]^{n-1})\colon \eta \in F\} \cup \{\gamma\}) = sF\}\,.$$

Clearly, (1) still holds. For ls limit we set

$$A_s = \bigcap_{n-2<\xi<ls} A_{s \upharpoonright B}\,, \quad \text{where } B = [\xi]^{n-1}\,.$$

Obviously, (1) still holds. The definition of A is complete. Now

$$\text{(2)} \qquad |T| < \kappa\,.$$

For, $|T| \leqq \Sigma_{\xi<\lambda}\, \mu^{|\xi|} < \kappa$ by hypothesis.

(3) If $s \in T$, $ls = \xi \geqq n$, $F \in [\xi]^{n-1}$, and $h(\eta, s \upharpoonright [\eta]^{n-1}) < \gamma$ for all $\eta \in F$, with $\gamma \in A_s$, then $f(\{h(\eta, s \upharpoonright [\eta]^{n-1})\colon \eta \in F\} \cup \{\gamma\}) = sF$.

For, (3) is clear if ξ is a successor ordinal. If ξ is a limit ordinal, then $F \subseteq \nu + 1$ for some $\nu < \xi$, and $A_s \subseteq A_{s \upharpoonright B}$, with $B = [\nu + 1]^{n-1}$, so it follows again.

$$\text{(4)} \qquad \text{If } s \in T \text{ and } A_s \neq 0, \text{ then } A_s = \{h(ls, s)\} \cup \bigcup \{A_t\colon s \subseteq t \text{ and } lt = ls + 1\}\,.$$

For, $\supseteq$ is clear. Now suppose $\gamma \in A_s$ and $\gamma \neq h(ls, s)$. Let $ls = \xi$. For any $F \in [\xi + 1]^{n-1}$ set

$$tF = f(\{h(\eta, s \upharpoonright [\eta]^{n-1})\colon \eta \in F\} \cup \{\gamma\})\,.$$

By (1) we have $h(\eta, s \upharpoonright [\eta]^{n-1}) < \gamma$ for all $\eta \in F$. Hence, if $F \in [\xi]^{n-1}$ we get $sF = tF$ by (3). Thus, $s \subseteq t$. Clearly, $\gamma \in A_t$, proving $\subseteq$. So (4) holds.

Now choose $\gamma \in \{\kappa \backslash (\{h(ls, s)\colon s \in T\} \cup n)$ using (2). We now define $t\colon \lambda \backslash (n - 1) \to T$ by recursion. Let t_{n-1} be any function mapping $[n-1]^{n-1}$ into μ. Suppose that t_ξ has been defined so that $lt_\xi = \xi$ and $\gamma \in A_{t\xi}$. By (4), choose $t_{\xi+1} \in T$ so that $lt_{\xi+1} = \xi + 1$, $t_\xi \subseteq t_{\xi+1}$, and $\gamma \in A_{t(\xi+1)}$. For ξ limit $<\lambda$ let $t_\xi = \bigcup_{\eta<\xi} t_\eta$. Clearly, then

$$\text{(5)} \qquad \text{if } \xi < \lambda, \text{ then } lt_\xi = \xi \text{ and } \gamma \in A_{t\xi}\,;$$

$$\text{(6)} \qquad \text{if } \xi < \eta < \lambda, \text{ then } t_\xi \subseteq t_\eta\,.$$

Let $H = \{h(\xi, t_\xi)\colon n-1 \leqq \xi < \lambda\}$. We claim that H is as desired in the definition of $\underset{\text{pre}}{\to}$. To prove this, suppose that $F \cup \{\alpha, \beta\} \subseteq H$, $|F| = n-1$, $F < \alpha$, $F < \beta$. Say $\alpha = h(\eta, t_\eta)$, $\beta = h(\nu, t_\nu)$, $F \subseteq A_{t\xi}$, $\xi < \eta, \nu$. Then $h(\eta, t_\eta), h(\nu, t_\nu) \in A_{t(\xi+1)}$ and hence

$$f(F \cup \{h(\eta, t_\eta)\}) = t_{\xi+1}F = f(F \cup \{h(\nu, t_\nu)\}),$$

as desired. □

Now for any cardinal κ we define

$$2_0^\kappa = \kappa,$$

$$2_{n+1}^\kappa = 2^\lambda, \quad \text{with } \lambda = 2_n^\kappa, \text{ for all } n \in \omega.$$

7.2. THEOREM (Erdős, Rado). *$(2_{n-1}^\kappa)^+ \to (\kappa^+)_\kappa^n$ for any cardinal $\kappa \geqq \omega$ and any positive integer n.*

PROOF. By induction on n. $n = 1$ says $\kappa^+ \to (\kappa^+)_\kappa^1$, which is obvious. Now assume the theorem for n, and let $f\colon [(2_n^\kappa)^+]^{n+1} \to \kappa$. Now

$$\kappa^{<\lambda^+} = \kappa^\lambda = 2_n^\kappa < (2_n^\kappa)^+, \quad \text{with } \lambda = 2_{n-1}^\kappa,$$

so we can apply Lemma 7.1 to get $H \subseteq (2_n^\kappa)^+$ such that $|H| = (2_{n-1}^\kappa)^+$, H has no last element, with H pre-homogeneous for f. Now define $f'\colon [H]^n \to \kappa$ by setting, for each $F \in [H]^n$, $f'F = f(F \cup \{\gamma\})$, where γ is any member of H such that $F < \gamma$; this is unambiguous by pre-homogeneity. Now by the induction hypothesis let $H' \subseteq H$ be homogeneous for f' with $|H'| = \kappa^+$. We claim that H' is homogeneous for f. Let $G \subseteq H'$, $|G| = n+1$, $G' \subseteq H'$, $|G'| = n+1$. Say $G = F \cup \{\gamma\}$, $F < \gamma$, $G' = F' \cup \{\gamma'\}$, $F' < \gamma'$. Then

$$fG = f(F \cup \{\gamma\}) = f'F = f'F' = f(F' \cup \{\gamma'\}) = fG',$$

as desired. □

We prove one more theorem in the partition calculus:

7.3. THEOREM. *If κ is an infinite regular cardinal, then $\kappa \to (\kappa, \omega)^2$. (The theorem also holds for singular κ, but we shall not consider this case.)*

PROOF. Let $f\colon [\kappa]^2 \to 2$. Assume that

(1) for all $A \subseteq \kappa$, if $f[[A]^2] \subseteq \{0\}$, then $|A| < \kappa$.

We want to find an infinite $B \subseteq \kappa$ such that $f[[B]^2] \subseteq \{1\}$. To do this, we define x_n, A_n, S_n for all $n \in \omega$.

Let $S_0 = \kappa$. Suppose S_n has been defined so that $|S_n| = \kappa$; we define S_{n+1}, x_n,

A_n. Let A_n be a maximal subset of S_n with $f[[A_n]^2] \subseteq \{0\}$. Thus, $|A_n| < \kappa$. Now by maximality

$$S_n \backslash A_n = \bigcup_{y \in An} \{z \in S_n \backslash A_n : f\{y, z\} = 1\} ,$$

so, since $|S_n \backslash A_n| = \kappa$, there is $x_n \in A_n$ such that the set $S_{n+1} \overset{\text{def}}{=} \{z \in S_n \backslash A_n : f\{x_n, z\} = 1\}$ has κ elements. This completes the definition.

Note the following facts about the sequences constructed:

(2) $\quad S_0 \supset S_1 \supset \cdots ;$

(3) $\quad x_n \in A_n \subseteq S_n ;$

(4) $\quad$ for all $z \in S_{n+1}$ we have $f\{x_n, z\} = 1$.

Thus, $f[[\{x_0, x_1, \ldots\}]^2] \subseteq \{1\}$, as desired. $\square$

8. Hajnal's free set theorem

Let F: $X \to \mathscr{P}(X)$. Then we call a subset M of X *free under* F if, for all distinct $x, y \in M$, we have $x \notin Fy$.

8.1. THEOREM (Hajnal). *Let κ and λ be infinite cardinals, $\kappa < \lambda$, $|X| = \lambda$, F: $X \to \mathscr{P}(X)$, and assume that for all $x \in X$ we have $|fx| < \kappa$. Then X has a free subset for F of power λ.*

PROOF. Clearly, we may assume that $x \notin Fx$ for all $x \in X$.

For we prove the theorem in the special case $\kappa < \operatorname{cf} \lambda$. Suppose that $|M| < \lambda$ for every free $M \subseteq X$; we shall get a contradiction. First we define by recursion a sequence $\langle S_\nu : \nu < \kappa \rangle$ of subsets of X, by letting S_ν be a maximal free subset of $X \backslash \bigcup_{\eta < \nu} S_\eta$. Thus, $|S_\nu| < \lambda$ for all $\nu < \kappa$, so the assumption $\kappa < \operatorname{cf} \lambda$ yields: the set $S^* \overset{\text{def}}{=} \bigcup_{\nu < \kappa} S_\nu$ has $<\lambda$ elements. Let $S^{**} = S^* \cup \bigcup_{x \in S^*} Fx$. Then

$$|S^{**}| \leqq |S^*| + \sum_{x \in S^*} |Fx| < \lambda ,$$

since $|S^*| < \lambda$ and $|Fx| < \kappa$ for all $x \in S^*$.

Choose $y \in X \backslash S^{**}$. By the maximality of the S_ν's, for all $\nu < \kappa$ the set $S_\nu \cup \{y\}$ is not free, so for some $x_\nu \in S_\nu$ we have $x_\nu \in Fy$ or $y \in Fx_\nu$. Since $Fx_\nu \subseteq S^{**}$, it follows that $y \notin Fx_\nu$. So $x_\nu \in Fy$, hence $S_\nu \cap Fy \neq 0$ for all $\nu < \kappa$. But $\langle S_\nu : \nu < \kappa \rangle$ is a system of pairwise disjoint sets, hence $|Fy| \geqq \kappa$, a contradiction. This takes care of the special case $\kappa < \operatorname{cf} \lambda$.

Now assume that $\operatorname{cf} \lambda \leqq \kappa$. Hence, λ is singular; say $\mu_\xi \uparrow \lambda$ for $\xi < \operatorname{cf} \lambda$, with $\kappa^+ < \mu_0$ and each μ_ξ regular. It is now straightforward to construct $\langle A_\xi : \xi < \operatorname{cf} \lambda \rangle$ so that the following conditions hold:

(1) $\quad$ if $\xi < \eta < \operatorname{cf} \lambda$, then $A_\xi \subseteq A_\eta$;

(2) if $\xi < \mathrm{cf}\,\lambda$, then $|A_\xi| = \mu_\xi$;

(3) if $\xi < \mathrm{cf}\,\lambda$ and $x \in A_\xi$, then $Fx \subseteq A_\xi$.

Now we apply our special case to each A_ξ (and $F \restriction A_\xi$) to get a free subset B_ξ of A_ξ of power μ_ξ (for all $\xi < \mathrm{cf}\,\lambda$). We now define by recursion $B'_\xi \subseteq B_\xi$ (for all $\xi < \mathrm{cf}\,\lambda$) so that

(4_η) if $\eta < \mathrm{cf}\,\lambda$, then $|B'_\eta| = \mu_\mu$;

(5_η) if $\xi \leqq \eta < \mathrm{cf}\,\lambda$ and $x \in B'_\xi$, then $Fx \cap B'_\eta = 0$.

Suppose that $\xi < \mathrm{cf}\,\lambda$ and we have already defined B'_η for all $\eta < \xi$ so that (4_η) and (5_η) hold. Let

$$H = \bigcup_{\eta<\xi} B'_\eta .$$

Thus, $|H| \leqq \Sigma_{\eta<\xi} |B'_\eta| = \Sigma_{\eta<\xi} \mu_\eta < \mu_\xi$, since μ_ξ is regular and $\xi < \mathrm{cf}\,\lambda \leqq \kappa < \mu_0 \leqq \mu_\xi$. So

$$\left|\bigcup_{x\in H} Fx\right| \leqq \kappa \cdot |H| < \mu_\xi .$$

Thus, if we set $B'_\xi = B_\xi \setminus \bigcup_{x\in H} Fx$, then ($4_\xi$) is guaranteed. ($5_\xi$) holds since B_ξ is free. So, the construction is finished.

We make one last transfinite construction, of $\langle D_\xi \colon \xi < \mathrm{cf}\,\lambda\rangle$ and $\langle D_{\xi\rho} \colon \xi < \mathrm{cf}\,\lambda,\ \rho < \kappa^+\rangle$. We suppose that $\xi < \mathrm{cf}\,\lambda$, and D_η and $D_{\eta\rho}$ have been constructed for all $\eta < \xi$ and $\rho < \kappa^+$ so that the following conditions hold:

(6_η) $D_{\eta\rho} \cap D_{\eta\sigma} = 0$ for distinct $\rho, \sigma < \kappa^+$, and $\bigcup_{\rho<\kappa^+} D_{\eta\rho} = D_\eta \subseteq B'_\eta$;

(7_η) for any $\rho < \kappa^+$ we have $|D_{\eta\rho}| = |D_\eta| = \mu_\eta$;

(8_η) there is a $\rho_\eta < \kappa^+$ such that for all $x \in D_\eta$, all $\nu < \eta$, and all $\rho \in (\rho_\eta, \kappa^+)$ we have $Fx \cap D_{\nu\rho} = 0$.

Now $|Fx| < \kappa$ for all $x \in B'_\xi$, and $\kappa^+ < \mu_0$. Hence, for all $x \in B'_\xi$ and all $\eta < \xi$ we can choose $\rho_{x\eta} < \kappa^+$ such that $Fx \cap D_{\eta\rho} = 0$ for all $\rho \in (\rho_{x\eta}, \kappa^+)$. Since $\xi < \mathrm{cf}\,\lambda \leqq \kappa$, for $\rho_x \overset{\mathrm{def}}{=} \bigcup_{\eta<\xi} \rho_{x\eta}$ we have $\rho_x < \kappa^+$ and $Fx \cap D_{\eta\rho} = 0$ for all $\eta < \xi$ and $\rho \in (\rho_x, \kappa^+)$. Now $\rho\colon B'_\xi \to \kappa^+$ and $\kappa^+ < \mu_\xi = |B'_\xi|$, and μ_ξ is regular, so there is $\rho_\xi < \kappa^+$ such that the set

$$D_\xi \overset{\mathrm{def}}{=} \{x \in B'_\xi \colon \rho_x = \rho_\xi\}$$

has power μ_ξ. Thus, (8_ξ) holds. We let $\langle D_{\xi\rho} \colon \rho < \kappa^+\rangle$ be any partition of D_ξ into sets of power μ_ξ. Thus, (6_ξ) = (8_ξ) hold. Hence, the construction is finished and (6_η)–(8_η) hold for all $\eta < \mathrm{cf}\,\lambda$.

Finally, we consider the ordinals ρ_η given in (8_η). Let $\sigma = \bigcup_{\eta<\mathrm{cf}\,\lambda}(\rho_\eta + 1) < \kappa^+$ and set

$$M = \bigcup_{\eta<\mathrm{cf}\,\lambda} D_{\eta\sigma}\,.$$

We claim that this is the desired set. By (7_η) it is clear that $|M| = \lambda$. Let $x \in M$; we show that $Fx \cap M = 0$, finishing the proof. Say $\eta < \mathrm{cf}\,\lambda$ and $x \in D_{\eta\sigma}$. Pick any $\xi < \mathrm{cf}\,\lambda$; we want to show that $Fx \cap D_{\xi\sigma} = 0$. If $\eta \leqq \xi$, then $x \in B'_\eta$ by (6_η), $D_{\xi\sigma} \subseteq B'_\xi$ by (6_ξ), and so $Fx \cap D_{\xi\sigma} = 0$ by (5_ξ). If $\xi < \eta$, then $x \in D_\eta$ by (6_η), so $Fx \cap D_{\xi\sigma} = 0$ by (8_ξ). □

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Keywords: cardinal, unit interval, almost disjoint set, independent set, stationary set, diamond, Suslin tree, Δ-system, partition calculus, Hajnal's free set theorem.

MOS subject classification: primary 04A20; secondary 03E05.

Chart of Topological Duality

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HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

According to the Stone duality (Part I, Chapter 3) we shall use the following notations: A, B Boolean algebras; X, Y Boolean spaces, i.e. compact Hausdorff totally disconnected spaces; φ a homomorphism; f a continuous mapping. Assume that the dual relations are as follows:

$A = \mathrm{Clop}(X)$	$X = \mathrm{Ult}(A)$
$B = \mathrm{Clop}(Y)$	$Y = \mathrm{Ult}(B)$
$\varphi = A \to B$	$f\colon Y \to X$
$\varphi = f^d$	$f = \varphi^d$

Boolean algebra(s)	Boolean space(s)
Ultrafilter	Point
Atom, principal ultrafilter	Isolated point
Filter	Closed set
Filter F with $\prod F = 0$	Closed nowhere dense set
Ideal	Open set
Ideal with a countable set of generators	Cozero set
Dense set	π-base
Family $\mathscr{F}$ of ultrafilters with $\bigcup \mathscr{F} = A^+$	Dense set
Rasiowa–Sikorksi lemma	Baire category theorem
Homomorphism	Continuous mapping
embedding	onto
onto	one-to-one (embedding)
regular embedding	semi-open and onto
embedding onto a dense subalgebra	irreducible and onto
$\varphi\colon B/I \to B$ is lifting	$f\colon Y \to X$ is retraction
Algebra	Space
complete	extremally disconnected
σ-complete	basically disconnected
countable separation property	F-space
superatomic	scattered
$\lvert A\rvert$ for A infinite	weight of X for X infinite
$\pi(A)$-minimal size of a dense subset	π-weight of X
Minimal size of a set generating ultrafilter	$\chi(x, X)$-character of a point x in X
$\mathrm{Ind}(A)$	$\sup\{\tau\colon X$ maps continuously onto ${}^{\tau}\{0,1\}\}$

Let us recall the definitions not mentioned in Part I or in the Appendix on General Topology.

A family $\mathscr{B}$ of open sets in a topological space X is called a π-base if each non-void open set contains some non-void member of $\mathscr{B}$. The smallest cardinality of a π-base is called a π-weight of X. A continuous mapping $f\colon X \to Y$ is called semi-open provided that $\mathrm{int}_Y f[U] \neq \emptyset$ whenever $\mathrm{int}_X U \neq \emptyset$. A space X is called basically disconnected if the closure of any cozero set is open. If $Y \subseteq X$, $f\colon X \to Y$ is continuous, then f is called a retraction provided that $f \circ f = \mathrm{id}_Y$. Let $I \subseteq A$ be an ideal, then a homomorphism $\varphi\colon A/I \to A$ is called a lifting, if $\varphi[a] \in [a]$ for each $a \in A$; [ ] denotes the I-equivalence class.

Appendix on General Topology

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
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0. Introduction

This appendix surveys several basic facts from general topology. Our aim is not to present here another textbook or monograph, but rather to convince the reader that topological spaces and methods are useful for Boolean algebras and vice versa. Until the 1930s, both disciplines developed independently. Topology offered interesting algebras of sets naturally described, e.g. Borel sets, clopen sets, and regular open sets, but set theory did the same, too. Then Stone duality tied Boolean algebras and topology together. Recently, the mutual influence is getting stronger and stronger.

However, not all of general topology can be reduced to Boolean spaces and not all of the theory of Boolean algebras deals with the study of properties interesting mainly from the topological point of view. The tightest connection between Boolean algebras and topology lies in the problems concerning compact spaces where connectedness does not play an important role. Plenty of statements valid for Boolean spaces hold for much wider classes, usually for all compact spaces and sometimes even for the completely regular ones. And it may happen that the most transparent proof of the theorem in one theory is to prove the dual statement in the other and then use Stone duality. There is no rule where to start, whether in topology or in Boolean algebras; it depends on the case in question. Anyway, it is good to know both of them; this explains why the authors of the present section have to work jointly.

1. Basics

1.1. TOPOLOGICAL SPACE. A topological space is a pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a family of subsets of X, the so-called open subsets of a topological space. The axioms are simple:

(i) $X \in \mathcal{O}$, $\emptyset \in \mathcal{O}$;

(ii) $\mathcal{O}$ is closed under arbitrary unions;

(iii) $\mathcal{O}$ is closed under finite intersections.

A few examples: $(X, \mathcal{P}(X))$ is a topological space which is called discrete. $(X, \{\emptyset, X\})$ is a topological space which is called indiscrete. If (X, d) is a metric space, then $U \subseteq X$ is open in metric topology iff, for each $x \in U$, there is some $\varepsilon > 0$ such that the open ball with center x and radius ε, $B(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$ is contained in U.

If the topology $\mathcal{O}$ is clear from the context, we shall speak about a topological space X instead of $(X, \mathcal{O})$.

1.2. SUBSPACE. If $(X, \mathcal{O})$ is a topological space, $Z \subseteq X$, then the family $\{U \cap Z: U \in \mathcal{O}\}$ is a topology on Z. Equipped with this topology, Z is called a subspace of X.

1.3. BASE, WEIGHT. The last example shows that the topology can be fully described by declaring only a part of all open subsets of X. A base of a topological space is a subfamily $\mathcal{B} \subseteq \mathcal{O}$ such that for each $U \in \mathcal{O}$ there is some $\mathcal{B}' \subseteq \mathcal{B}$ with $\bigcup \mathcal{B}' = U$. Any base $\mathcal{B}$ of a topology satisfies:

(i) $\bigcup \mathcal{B} = X$;

(ii) whenever $x \in X$, $B_0, B_1 \in \mathcal{B}$ and $x \in B_0 \cap B_1$, then there is some $B \in \mathcal{B}$ with $x \in B \subseteq B_0 \cap B_1$.

If some family $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfies (i) and (ii), then there is a unique family $\mathcal{O}$ such that $\mathcal{B}$ is a base of the topological space $(X, \mathcal{O})$.

The weight of a topological space X is defined as

$$w(X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a base of } X\} .$$

EXAMPLES. Let K be a set and consider the set of all functions from ω to K. The generalized Baire space is the set ${}^{\omega}K$ with the base consisting of all sets of the form $[\varphi] = \{f \in {}^{\omega}K: f \supseteq \varphi\}$, where φ is a function, dom φ is a finite subset of ω, rng $\varphi \subseteq K$. Observe that if one defines for $f, g \in {}^{\omega}K$,

$$d(f, g) = \begin{cases} 0 & \text{if } f = g , \\ 1/(n+1) & \text{if } f(n) \neq g(n) \text{ and } n \text{ is the first such ,} \end{cases}$$

then d is the metric inducing the topology just described.

In the special case when $K = \{0, 1\}$, the space ${}^{\omega}\{0, 1\}$ is called the Cantor space.

The Stone space topology on the set Ult(B) of all ultrafilters on a Boolean algebra B is determined also using a base. The base is the set of all $s(b)$ with b ranging in B, where

$$s(b) = \{\mathcal{U} \in \mathrm{Ult}(B): b \in \mathcal{U}\} .$$

1.4. CLOSURE, NEIGHBORHOOD AND FURTHER SETS. For a topological space X, a set $F \subseteq X$ is closed iff $X - F$ is open. By the de Morgan laws, the family of all closed sets contains $\emptyset$, X and is closed under finite unions and arbitrary intersections. If $M \subseteq X$, then the closure cl M (also denoted by $\mathrm{cl}_X M$, if X should be emphasized) is the smallest closed set containing M, i.e. $\mathrm{cl}\, M = \bigcap \{F: F \subseteq X \text{ is closed } \& \ M \subseteq F\}$.

A neighbourhood of a point x (of a set M) is a set V containing x (containing M) such that for some open U, $x \in U \subseteq V$ ($M \subseteq U \subseteq V$). It is easy to see that for $x \in X$ and $M \subseteq X$ we have $x \in \mathrm{cl}\, M$ iff for each neighborhood V of x, $V \cap M \neq \emptyset$.

The family of all neighborhoods of a point x is a filter on X; a local base or a neighborhood base at x is the family $\mathcal{V}$ of neighborhoods of x such that for each neighborhood U there is some $V \in \mathcal{V}$ with $x \in V \subseteq U$. If one describes a neighborhood base at each point, one knows the topology.

The interior of a set M, int M, is the largest open subset of M, i.e. $\mathrm{int}\, M = \bigcup \{U: U \subseteq X \text{ is open } \& \ U \subseteq M\}$. A set M is regular open, if $M = \mathrm{int}\,\mathrm{cl}\, M$. The family of all regular open subsets of X, RO(X), is a complete Boolean algebra if the Boolean operations are defined by $M \cdot N = M \cap N$, $M + N = \mathrm{int}\,\mathrm{cl}(M \cup$

$N)$, $-M = \mathrm{int}(X - M)$. A set is called clopen if it is simultaneously closed and open in X. In our examples of generalized Baire space and the Stone space of a Boolean algebra, the bases consisted of clopen subsets. For each topological space X the family $\mathrm{Clop}(X)$ of all clopen subsets is a Boolean subalgebra of the power set algebra $\mathscr{P}(X)$.

A set M is called dense if $\mathrm{cl}\, M = X$. A set is called nowhere dense in X, if its closure has empty interior.

An easy argument gives the following characterization of nowhere dense sets. A set M is nowhere dense iff for each non-empty open set V there is a non-empty open set $U \subseteq V$ such that $U \cap M = \emptyset$.

The union of finitely many nowhere dense sets is nowhere dense, too, therefore all nowhere dense subsets of a space X constitute an ideal. Assuming $X \neq \emptyset$, then the whole space X is not nowhere dense.

A very important notion is the notion of a meager set. A set is called meager (or first category) if it can be represented as a countable union of nowhere dense sets. The family of all meager sets is a σ-complete ideal in $\mathscr{P}(X)$.

Let us say that a set $M \subseteq X$ has the Baire property, if there is an open set $U \subseteq X$ such that both $U - M$ and $M - U$ are meager.

Recall that the family of all sets which have the Baire property is a σ-algebra in $\mathscr{P}(X)$ and is denoted by $\mathrm{Baire}(X)$ (see Part I of this Handbook, 1.31). Moreover, $\mathrm{Borel}(X) \subseteq \mathrm{Baire}(X)$, where $\mathrm{Borel}(X)$ denotes the smallest σ-algebra of subsets of X which contains all open sets.

1.5. EXAMPLES. (a) Consider the space $\mathbb{Q}$ of rational numbers with the usual metric topology. $\mathbb{Q}$ is countable and each singleton is a nowhere dense set, hence the whole space $\mathbb{Q}$ is meager. Therefore $\mathscr{P}(\mathbb{Q}) = \mathrm{Baire}(\mathbb{Q}) = \mathrm{Borel}(\mathbb{Q})$.

(b) The Baire category theorem. Call a space X a Baire space if no non-void open set in X is meager. Equivalently, every intersection of countably many dense open subsets of X is dense in X.

The proof of the following Baire category theorem can be found in any textbook of general topology or functional analysis.

THEOREM. *Every locally compact Hausdorff space is a Baire space* (*for the unexplained notions, see 2, 4.8*).

REMARK. The class of all Baire spaces is wider than stated here, e.g. each generalized Baire space (see 1.3) is a Baire space.

(c) The algebra $\mathrm{Borel}(\mathbb{R})/\mathrm{meager}$ of all Borel subsets of reals modulo the ideal of all Borel meager sets is complete.

Since the real line is a locally compact metric space, hence a Baire space, this statement is a consequence of the next one.

If X is a non-empty Baire space, then

$$\mathrm{Baire}\ (X)/\mathrm{meager} \cong \mathrm{Borel}\ (X)/\mathrm{meager} \cong \mathrm{RO}(X)\,,$$

where $\mathrm{RO}(X)$ is the complete Boolean algebra of all regular open sets in X.

PROOF. We shall find a correspondence between sets with the Baire property and regular open sets in X. Let $A = (U \cup M_1) - M_2$, where U is open, M_1, M_2 are meager. Let us define $f(A) = \operatorname{int} \operatorname{cl}(U)$. It is obvious that $f(A) \in \mathrm{RO}(X)$. Since $U \subseteq f(A)$ and $f(A) - U$ is nowhere dense, the symmetric difference $A \triangle f(A)$ is meager. If U and V are regular open sets and $U - V \neq \emptyset$, then there is a non-empty regular open set $W \subseteq U$ such that $W \cap V = \emptyset$. That is a consequence of the fact that the inclusion $\subseteq$ is a canonical order in the algebra $\mathrm{RO}(X)$. Since X is the Baire space, $U \triangle V$ is not meager, whenever $U \triangle V \neq \emptyset$.

We have shown that the mapping f maps $\mathrm{Baire}(X)$ onto $\mathrm{RO}(X)$ and preserves inclusion; moreover, for $U \in \mathrm{RO}(X)$ the set $f^{-1}[U] = \{A \in \mathrm{Baire}(X)\colon A \triangle U$ is meager$\}$. Thus, $\mathrm{Baire}(X)/\text{meager} \cong \mathrm{RO}(X)$. Since all regular open sets are Borel, and $\mathrm{Borel}(X) \subseteq \mathrm{Baire}(X)$, the same argument gives $\mathrm{Borel}(X)/\text{meager} \cong \mathrm{RO}(X)$. □

(d) The topology of the Cantor space ${}^{\omega}\{0,1\}$ gives a natural topology for $\mathscr{P}(\omega)$: identify each $X \in \mathscr{P}(\omega)$ with its characteristic function $\chi_X \in {}^{\omega}2$. Therefore, each ideal $\mathscr{I}$ on ω can be viewed as a subset of ${}^{\omega}2$.

PROPOSITION (M. Talagrand). *An ideal $\mathscr{I}$ on ω is meager in the Cantor space if and only if there is a sequence $\langle a_n\colon n \in \omega\rangle$ of non-empty pairwise disjoint finite subsets of ω such that no union of infinitely many a_n's belongs to $\mathscr{I}$.*

PROOF. Assume $\mathscr{I}$ is meager. Then there are nowhere dense sets $\{R_n\colon n \in \omega\}$ such that $\mathscr{I} \subseteq \bigcup_{n\in\omega} R_n$, $\chi_\emptyset \in R_0$ and $R_n \subseteq R_{n+1}$ for all $n \in \omega$. We shall construct an increasing sequence $\{t_n\colon n \in \omega\}$ of natural numbers and a sequence of finite functions $\{\psi_n\colon n \in \omega\}$. Choose $t_0 > 0$ arbitrarily. Consider all functions $\{f_i\colon i < 2^{t_0}\}$ from $[0, t_0)$ to $\{0,1\}$. Since R_0 is nowhere dense, we can choose finite functions φ_i $(i < 2^{t_0})$ with $\operatorname{dom} \varphi_i \subseteq \omega - t_0$ such that $\varphi_i \subseteq \varphi_j$ for $i < j < 2^{t_0}$ and $[f_i \cup \varphi_i] \cap R_0 = \emptyset$ for all $i < 2^{t_0}$. Let $t_1 > \max \operatorname{dom} \varphi_{2^{t_0}-1}$ and pick $\psi_0\colon [t_0, t_1) \to \{0,1\}$ such that $\psi_0 \supseteq \varphi_{2^{t_0}-1}$. The construction guarantees that for each $f\colon [0, t_0) \to \{0,1\}$ we have $[f \cup \psi_0] \cap R_0 = \emptyset$. Similarly, we obtain $t_2 > t_1$ and $\psi_1\colon [t_1, t_2) \to \{0,1\}$ such that for all $f\colon [0, t_1) \to \{0,1\}$, $[f \cup \psi_1] \cap R_1 = \emptyset$, and so on.

For $n \in \omega$, let $a_n = \{i \in [t_n, t_{n+1})\colon \psi_n(i) = 1\}$. Since $\chi_\emptyset \in R_0$, each a_n is non-empty.

Let $A = \bigcup_{n\in Z} a_n$, where $|Z| = \omega$. If $A \in R_k$ for some $k \in \omega$ and if $n \geq k$, then a_n cannot be a subset of A since $[\psi_n] \cap R_k = \emptyset$. This contradiction shows that $A \notin \bigcup_{k\in\omega} R_k \supseteq \mathscr{I}$.

In the opposite direction, let $\{a_n\colon n \in \omega\}$ be a sequence of finite sets with the property as stated. If we set $R_k = \{A \subseteq \omega\colon (\forall n \geq k) a_n - A \neq \emptyset\}$ for $k \in \omega$, then all R_k's are nowhere dense and $\mathscr{I} \subseteq \bigcup_{k\in\omega} R_k$. Hence $\mathscr{I}$ is meager. □

(e) Suppose M is a subset of a generalized Baire space ${}^{\omega}K$, where $1 < |K| < \omega$. Then M is nowhere dense if and only if there is a family $\{\varphi_n\colon n \in \omega\}$ of finite functions from ω to K such that $\operatorname{dom} \varphi_n \cap \operatorname{dom} \varphi_m = \emptyset$ for $n \neq m$ and $M \cap [\varphi_n] = \emptyset$ for all $n \in \omega$.

The proof uses similar ideas as explained in (d). It could be noticed that this characterization of nowhere dense subsets in ${}^{\omega}K$ fails if $|K| \geq \omega$.

1.6. CONTINUOUS MAPPINGS. If $(X, \mathcal{O})$, $(Y, \mathcal{V})$ are topological spaces, $f: X \to Y$ a mapping, then f is called continuous if, for each $V \in \mathcal{V}$, $f^{-1}[V] \in \mathcal{O}$. Another easy exercise: $f: X \to Y$ is continuous iff for each $x \in X$ and for each neighborhood V of $f(x)$ there is some neighborhood U of x with $f[U] \subseteq V$. If $f: X \to Y$ is a one-to-one onto mapping and both f and f^{-1} are continuous, then f is called a homeomorphism. If $f: X \to Y$ is one-to-one and $f: X \to f[X]$ is a homeomorphism, then f is an embedding.

1.7. BASIC CONSTRUCTIONS. Assume $\{X_a: a \in A\}$ to be a family of topological spaces.

The topological sum of this family is a topological space $(Y, \mathcal{O})$, where $Y = \{\langle a, x\rangle: a \in A, x \in X_a\}$ and U is open in Y iff for each $a \in A$ the set $\{x \in X_a: \langle a, x\rangle \in U\}$ is open in X_a.

The topological product of a family $\{X_a: a \in A\}$ is the space $Z = \Pi\{X_a: a \in A\}$ and the basis of the topology consists of all sets $\Pi\{T_a: a \in A\}$, where $T_a = X_a$ for all but finitely many a's and for each $a \in A$, T_a is open in X_a.

Thus a generalized Baire space is the product of countably many copies of a discrete space K.

If X is a topological space and $f: X \to Q$ is an onto mapping, then there is the largest topology $\mathcal{O}$ on Q, namely $\mathcal{O} = \{A \subseteq Q: f^{-1}[A]$ is open in $X\}$, such that f is continuous; this topology is called a quotient topology. With the quotient topology, Q is denoted by X/f. Suppose we are given some equivalence relation $\sim$ on the space X. Let Q be the set of all $\sim$-equivalence classes and $\pi: X \to Q$ the projection induced by $\sim$. Then the quotient space X/π is usually denoted by $X/\sim$.

If $f_a: X \to X_a$ $(a \in A)$ is a mapping for each $a \in A$, then $\triangle_{a\in A} f_a$ is the mapping from X into $\Pi_{a\in A} X_a$ defined by $\triangle_{a\in A} f_a(x) = \langle f_a(x): a \in A\rangle$.

If each f_a is continuous, then $\triangle_{a\in A} f_a$ is continuous as well.

A family $\{f_a: a \in A\}$ of continuous mappings defined on X separates points and closed sets, if for each $x \neq y$ there is some $a \in A$ with $f_a(x) \neq f_a(y)$ and if for each $x \in X$ and closed $C \subseteq X - \{x\}$ there is some $a \in A$ with $f_a(x) \not\in \operatorname{cl} f_a[C]$.

1.8. EMBEDDING LEMMA. *If a family $\{f_a: a \in A\}$ separates points and closed sets, then $\triangle_{a\in A} f_a$ is an embedding.*

2. Separation axioms

Call two points x, y in a space X separated by sets A, B, if $x \in A \subseteq X$, $y \in B \subseteq X$ and $A \cap B = \emptyset$. Separation of sets is defined analogously.

A space X is called T_1 if each one-point set is closed.

A space X is called T_2 or Hausdorff, if any two distinct points can be separated by open sets.

A T_1-space X is called regular, if for each point $x \in X$ and closed set $C \subseteq X$ with $x \not\in C$, x and C can be separated by open sets.

A T_1-space X is called normal, if any two disjoint closed sets can be separated by open sets.

A T_1-space X is called completely regular, if for each $x \in X$ and a closed C with $x \notin C$ there is a continuous mapping $f: X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in C$.

Clearly, each completely regular space is regular. There are completely regular spaces which are not normal.

Let us mention a few important properties satisfied under a suitable separation axiom.

2.1. PROPOSITION. *Let Y be a Hausdorff space, f, g: $X \to Y$ continuous mappings. If there is a dense subset D of X such that $f \restriction D = g \restriction D$, then $f = g$.*

PROOF. Assuming the contrary, choose $x \in X$ with $f(x) \neq g(x)$. Since Y is Hausdorff, there are disjoint open neighborhoods U and V of $f(x)$ and $g(x)$, respectively. By the continuity of both mappings, $f^{-1}[U] \cap g^{-1}[V]$ is an open neighborhood of x. The set D is dense in X, thus there is some $d \in D \cap f^{-1}[U] \cap g^{-1}[V]$. For this d, $f(d) \in U$ and $g(d) \in V$, therefore $f \restriction D \neq g \restriction D$. □

2.2. PROPOSITION. *Every infinite Hausdorff space X contains an infinite discrete subspace.*

PROOF. There is nothing to prove if X is discrete. So assume X is not discrete. Then there is a point $x \in X$ such that each neighborhood of x is infinite.

Denote $V_0 = X$, pick a point $y_0 \in V_0 - \{x\}$ and proceed by induction as follows. If $k < \omega$ and $y_0, \ldots, y_k$, $V_1, \ldots, V_k$ have been defined, then by Hausdorffness, there are disjoint open neighborhoods G_k of the set $\{y_0, \ldots, y_k\}$ and H_k of the point x. Let $V_{k+1} = V_0 \cap \cdots \cap V_k \cap H_k$ and choose a point $y_{k+1} \in V_{k+1} - \{x\}$.

The subspace $Y = \{y_k : k \in \omega\}$ is discrete. Clearly, for each $k \in \omega$ the set $G_k \cap V_k$ is an open neighborhood of y_k which does not contain any y_j, $j \neq k$. □

2.3. PROPOSITION. (i) *A T_1 space X is regular iff for each $x \in X$ and a neighborhood U of x there is an open neighborhood V of x such that $x \in V \subseteq \mathrm{cl}\, V \subseteq U$.*

(ii) *In a regular space X, $\mathrm{RO}(X)$ is a base of the topology.*

PROOF. Suppose X is regular. Since U is a neighborhood of x, $x \notin \mathrm{cl}(X - U)$. Let open sets V and W separate x and $\mathrm{cl}(X - U)$. Then V is a neighborhood of x and $\mathrm{cl}\, V \cap W = \emptyset$, hence $\mathrm{cl}\, V \subseteq U$.

To prove the opposite implication, let C be a closed subset of X, $x \notin C$. Then $X - C$ is a neighborhood of x. If V is as stated in (i), then V, $X - \mathrm{cl}\, V$ separate x and C.

(ii) Follows immediately by the fact that if $x \in \mathrm{int}\, V$, then $x \in \mathrm{int}\, \mathrm{cl}\, V$, too, and by (i). □

2.4. URYSOHN LEMMA. *A T_1 space X is normal iff for each two disjoint closed sets C, D there is a continuous mapping f: $X \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in D$. In particular, every normal space is completely regular.*

PROOF. Denote $U(1) = X - D$. The set $U(1)$ is open and $C \subseteq U(1)$. By normality, there is an open set $U(0)$ such that $C \subseteq U(0) \subseteq \mathrm{cl}\, U(0) \subseteq U(1)$. Let $n \in \omega$ and suppose that we have open sets $U(p/2^n)$ $(0 \leq p \leq 2^n)$ such that $U(p/2^n) \subseteq \mathrm{cl} U(p/2^n) \subseteq U((p+1)/2^n)$ for all $p < 2^n$. By normality, there is some open set $U((2p+1)/2^{n+1})$ satisfying $\mathrm{cl}\, U(p/2^n) \subseteq U((2p + 1/2^{n+1}) \subseteq \mathrm{cl}\, U((2p+1)/2^{n+1}) \subseteq U((p+1)/2^n)$. Let $f(x) = 1$ for $x \in X - U(1) = D$, otherwise let $f(x) = \inf\{q\colon q$ is dyadic rational, $x \in U(q)\}$. That's that. The rest is obvious. □

The most widely used consequence of the Urysohn lemma is Tietze's theorem. We shall omit the standard proof.

2.5. TIETZE'S THEOREM. *In a normal space X, each bounded continuous real-valued function defined on a closed subspace has a continuous extension to the whole of X.*

3. Compactness

3.1. DEFINITION. An open cover of a space X is a family $\mathcal{V}$ of open sets such that $\bigcup \mathcal{V} = X$. A topological space X is compact if each open cover of X contains a finite subcover. A set $Z \subseteq X$ is compact if the subspace Z is compact.

3.2. THEOREM. *A topological space X is compact iff for each ultrafilter $\mathcal{U}$ on X there is a point $x \in X$ such that $V \in \mathcal{U}$ for each neighborhood V of x (let us say that $\mathcal{U}$ converges to x).*

PROOF. Indeed, let X be compact. If $\mathcal{U}$ is an ultrafilter which does not converge, then each point $x \in X$ has a neighborhood V_x with $X - V_x \in \mathcal{U}$. Then the cover $\{V_x\colon x \in X\}$ has no finite subcover, a contradiction.

Suppose each ultrafilter to converge and let $\mathcal{V}$ be an open cover without a finite subcover. Then the system $\{X - \bigcup \mathcal{V}'\colon \mathcal{V}' \in [\mathcal{V}]^{<\omega}\}$ is closed under finite intersections and consists of non-void closed subsets of X. Since its intersection is empty, no ultrafilter extending it can converge, a contradiction. □

3.3. COROLLARY. *In a Hausdorff compact space, each ultrafilter converges to precisely one point.*

3.4. PROPOSITION. *Every closed subspace of a compact space is compact, too. Every compact subset of a Hausdorff space is closed.*

PROOF. If $\mathcal{V}$ is an open cover of a closed subspace $C \subseteq X$, then for some collection $\mathcal{W}$ of open subsets of X, $\mathcal{V} = \{W \cap C\colon W \in \mathcal{W}\}$. Pick a finite subcover from the cover $\mathcal{W} \cup \{X - C\}$.

Let X be Hausdorff, Z a compact subset of X and $x \in \mathrm{cl}\, Z$. Choose an arbitrary ultrafilter $\mathcal{U}$ containing the filter $\{V \cap Z\colon V$ is a neighborhood of $x\}$. By the compactness of Z, $\mathcal{U}$ converges to some $z \in Z$, but clearly $\mathcal{U}$ converges to x, too. Since X is Hausdorff, $x = z$. □

3.5. PROPOSITION. *Each compact Hausdorff space is regular.*

PROOF. If C is closed in X and $x \notin C$, for each $y \in C$ choose open $U_y \ni y$, $V_y \ni x$, $U_y \cap V_y = \emptyset$. If $\{U_{y_1}, \ldots, U_{y_n}\}$ is a finite subcover of a compact subspace C, then $\bigcup \{U_{y_i}: i = 1, \ldots, n\}$ and $\bigcap \{V_{y_i}: i = 1, \ldots, n\}$ are disjoint neighborhoods of C and x. $\square$

Applying the same trick once more, we obtain

3.6. PROPOSITION. *Each compact Hausdorff space is normal.*

3.7. PROPOSITION. *Let X be a compact space, $f: X \to Y$ a continuous mapping. Then $f[X]$ is compact, too. If, moreover, f is one-to-one and Y is Hausdorff, then f is an embedding.*

PROOF. Let $\mathcal{U}$ be an open cover of $f[X]$. By the compactness of X, there is a finite subcover $\{V_1, \ldots, V_n\} \subseteq \{f^{-1}[U]: U \in \mathcal{U}\}$. Clearly, $\{f[V_1], \ldots, f[V_n]\} \subseteq \mathcal{U}$ is the desired finite cover of $f[X]$.

For the second half of the proposition, choose arbitrary open $U \subseteq X$. By 3.4, $X - U$ is compact, hence by the previous, $f[X - U]$ is compact, too. Since Y is Hausdorff, $f[X - U]$ is closed. Therefore $f[X] - f[X - U]$ is open in $f[X]$. Since f is one-to-one, $f[X] - f[X - U] = f[U]$. We have proved that $f[U]$ is open; equivalently, the mapping f^{-1}: $f[X] \to X$ is continuous. $\square$

Our aim now is to discuss the conditions under which a given compact Hausdorff space is the Stone space of some Boolean algebra. We need several new notions.

3.8. DEFINITION. A zero set in a topological space X is the preimage of zero under a real-valued continuous mapping defined on X; a cozero set is a complement of zero set. Notice that the preimage of any closed subset of reals under a continuous mapping is a zero set.

3.9. OBSERVATION. If Z_0, Z_1 are disjoint zero sets in a topological space X, then there is a continuous mapping $g: X \to \mathbb{R}$ with $Z_0 = g^{-1}\{0\}$, $Z_1 = g^{-1}\{1\}$.

PROOF. Let $Z_0 = f^{-1}\{0\}$, $Z_1 = h^{-1}\{0\}$. We may and shall assume that f and h are non-negative and bounded, for if f is continuous, then $\min\{|f|, 1\}$ is continuous, too, and has the same preimage of zero. Define

$$g(x) = \frac{f(x)}{f(x) + h(x)} \, . \quad \square$$

3.10. DEFINITION. A space X is called connected, if there is no pair of disjoint non-void open sets covering X. Notice that the space X is connected iff the only clopen subsets of X are $\emptyset$ and X.

A space X is called totally disconnected, if each connected subspace contains at most one point.

A space X is called zero-dimensional, if it has a base consisting of clopen subsets.

A space C is called strongly zero-dimensional, if any two disjoint zero sets are separated by clopen sets.

In the class of completely regular spaces, strongly zero-dimensional implies zero-dimensional implies totally disconnected. All three notions coincide in compact Hausdorff spaces.

3.11. THEOREM. *A compact Hausdorff space is strongly zero-dimensional iff it is totally disconnected.*

PROOF. We shall show only that a compact Hausdorff totally disconnected space is zero-dimensional, since the rest is trivial.

Assume X to be totally disconnected. Choose a point $x \in X$ and define $C = \bigcap \{U \subseteq X: x \in U \;\&\; U \text{ is clopen}\}$.

Let us show first that the set C is connected. If not, then $C = C_0 \cup C_1$ with $C_0 \cap C_1 = \emptyset$ and both C_i closed. By the normality of X, there are open $V_i \supseteq C_i$ such that $\mathrm{cl}\, V_0 \cap \mathrm{cl}\, V_1 = \emptyset$. There is some clopen $U \supseteq C$ such that $V_0 \cup V_1 \supseteq U$. If not, then the family $\{U - (V_0 \cup V_1): U \text{ clopen in } X,\ U \supseteq C\}$ is a centered system of non-void closed sets, thus by compactness, some $z \in X$ belongs to its intersection. Now $z \in C - (V_0 \cup V_1)$, which contradicts our assumption that $V_0 \cup V_1$ covers C. So we found clopen U and disjoint open V_0, V_1 with $V_0 \cup V_1 \supseteq U$. The closures $\mathrm{cl}\, V_0, \mathrm{cl}\, V_1$ are disjoint, hence both $V_0 \cap U$, $V_1 \cap U$ are clopen in X and both of them meet C. But then, if, say, $x \in V_0$, the set $U \cap V_0$ is clopen and contains x, which contradicts our choice of C. Thus, C is connected.

Since X is totally disconnected, $C = \{x\}$. Therefore, again by compactness, for each neighborhood W of x there is some clopen U with $x \in U \subseteq W$. □

3.12. DEFINITION. A totally disconnected compact Hausdorff space is called a Boolean space.

Thus, Boolean spaces are Stone spaces of Boolean algebras.

3.13. TYCHONOFF THEOREM. *An arbitrary product of compact spaces is compact.*

PROOF. Let $X = \Pi\{X_a: a \in A\}$, let $\mathcal{U}$ be an ultrafilter on X. For each $a \in A$, $\pi_a[\mathcal{U}] = \mathcal{U}_a$ is an ultrafilter on X_a converging to x_a; π_a denoting the ath projection. We claim that $\mathcal{U}$ converges to $x = \langle x_a: a \in A\rangle$. Suppose V to be a basic open neighborhood of x, $V = \Pi_{a \in F} V_a \times \Pi_{a \in A-F} X_a$ for some finite $F \subseteq A$ and open $V_a \ni x_a$ for $a \in F$. Then for each $b \in F$, the set $W_b = V_b \times \Pi_{a \in A-\{b\}} X_a$ belongs to $\mathcal{U}$: if not, then $U \cap W_b = \emptyset$ for some $U \in \mathcal{U}$, therefore $\pi_b[U] \cap V_b = \emptyset$, but this is impossible by the choice of x_b. Since $\mathcal{U}$ is centered,

$$\prod_{a \in F} V_a \times \prod_{a \in A-F} X_a = \bigcap_{a \in F} W_a \in \mathcal{U}\,. \quad \square$$

3.14. COROLLARY. *An arbitrary product of Boolean spaces is Boolean.*

4. The Čech–Stone compactification

Let X be a completely regular space, denote by $\mathscr{C}(X, I)$ the set of all continuous mappings from X to the unit interval I. By complete regularity, the embedding lemma 1.8 ensures that the mapping $\varphi = \triangle_{f\in\mathscr{C}(X,I)} f\colon X \to I^{\mathscr{C}(X,I)}$ is an embedding. Using the Tychonoff theorem we conclude that each completely regular space is (homeomorphic to) a subspace of a compact Hausdorff space.

Let us investigate for a moment the space $\operatorname{cl}\varphi[X] \subseteq I^{\mathscr{C}(X,I)}$. This space is compact, $\varphi[X]$ is dense in it and, what is important, each continuous mapping $g\colon \varphi[X] \to I$ can be continuously extended to $\operatorname{cl}\varphi[X]$. To see this, consider $\tilde{g} = \pi_{g\circ\varphi} \restriction \operatorname{cl}\varphi[X]$. Being a restriction of a continuous mapping $\pi_{g\circ\varphi}$, it is continuous, and for $y \in \varphi[X]$, $\tilde{g}(y) = g(y)$. Indeed, $y = \varphi(x)$ for some $x \in X$ and $\tilde{g}(y) = \pi_{g\circ\varphi}(\varphi(x)) = g\circ\varphi(x) = g(y)$.

Thus, we have shown:

4.1. Theorem. *For each completely regular space X there is a compact Hausdorff space, the so called Čech–Stone compactification and denoted by βX, such that X is dense in βX and each continuous mapping from X to a unit interval has a continuous extension to βX.*

We gave a proof of the existence theorem concerning βX; the proof of the characterization theorem below will be omitted.

4.2. Theorem. *Let X, Y be completely regular spaces. If*

(i) *Y is compact Hausdorff, $X \subseteq Y$,*

(ii) *X is dense in Y,*

(iii) *each continuous mapping from X to I can be continuously extended to Y, then Y is homeomorphic to βX.*

Condition (iii) can be replaced by each of the following:

(iii)′ Each bounded real-valued continuous mapping from X extends to Y,

(iii)″ for each compact space Z and for each continuous mapping $f\colon X \to Z$, f extends to Y.

4.3. Ultrafilter limits. From the variety of descriptions of βX, let us mention without proof the following one.

If $\mathscr{U}$ is an ultrafilter on a set X, Y a topological space, $y \in Y$ and $f\colon X \to Y$ a mapping, then the point y is called a $\mathscr{U}$-limit of f, in short, $y = \mathscr{U} - \lim_{x\in X} f(x)$, provided that for each neighborhood V of y, $f^{-1}[V] \in \mathscr{U}$.

The basic feature of $\mathscr{U}$-lim is that whenever Y is compact Hausdorff, then $\mathscr{U} - \lim_{x\in X} f(x)$ always exists and is unique.

If X is a discrete (infinite) topological space, then βX is the Stone space of the Boolean algebra $\mathscr{P}(X)$ and for $\mathscr{U} \in \operatorname{Ult}(P(X))$ and $f\colon X \to I$, $f(\mathscr{U})$ equals $\mathscr{U} - \lim_{x\in X} f(x)$.

If X is an arbitrary completely regular space, consider the space Y to be the set X with the discrete topology. If $\mathscr{U}$, $\mathscr{V} \in \beta Y$, denote $\mathscr{U} \sim \mathscr{V}$ if for each continuous mapping $f: X \to I$, $\mathcal{U} - \lim_{x \in X} f(x) = \mathcal{V} - \lim_{x \in X} f(x)$. The quotient space $\beta Y/\sim$ is nothing else than βX.

Let us remark that limits over an ultrafilter may serve in many instances where the common approach needs the Hahn–Banach theorem. Consider, for example, the function $\mu: \mathcal{P}(\omega) \to I$ defined by

$$\mu(Z) = \mathcal{U} - \lim_{n \in \omega} \frac{|Z \cap [0, n)|}{n},$$

where $\mathcal{U}$ is some fixed uniform ultrafilter on ω. A moment's reflection suffices to check that μ is a finitely additive probabilistic measure with domain $\mathcal{P}(\omega)$.

4.4. MORE ABOUT βX. Our aim now is to deduce several properties of βX from the properties of X. This is trivial if X is compact; for $\beta X = X$, then, the general case needs some care. We are interested mainly under what conditions βX or $\beta X - X$ are Boolean spaces.

4.5. THEOREM. *For a completely regular space X, βX is Boolean iff X is strongly zero-dimensional.*

PROOF. Suppose βX to be zero-dimensional and take Z_0, Z_1 disjoint zero sets in X. By 3.9, there is some continuous $g: X \to I$ with $Z_0 = g^{-1}\{0\}$, $Z_1 = g^{-1}\{1\}$, therefore $\mathrm{cl}_{\beta X} Z_0$ is disjoint with $\mathrm{cl}_{\beta X} Z_1$. The compactness of βX implies that they can be separated by clopen sets in βX, hence in X, too. Thus, X is strongly zero-dimensional.

Let X be strongly zero-dimensional, $p \in \beta X$, U an open neighborhood of p. Choose a continuous mapping $f: \beta X \to I$ with $f(p) = 0$, $f(y) = 1$ for all $y \in \beta X - U$ and consider $Z_0 = X \cap f^{-1}[0, 1/3]$, $Z_1 = X \cap f^{-1}[2/3, 1]$. The sets Z_0, Z_1 are disjoint zero sets in X and Z_0 is non-void, since X is dense in βX. The space X is strongly zero-dimensional, hence there is some clopen $C \subseteq X$, $Z_0 \subseteq C$, $Z_1 \subseteq X - C$. Let g be a mapping such that $g(x) = 0$ for $x \in C$, $g(x) = 1$ for $x \in X - C$. Clearly, $g: X \to I$ is continuous and can be continuously extended to $\beta g: \beta X \to I$. Since X is dense in βX, βg admits two values only, i.e. 0 and 1. Thus, $(\beta g)^{-1}\{0\}$ is a clopen subset of βX, $p \in (\beta g)^{-1}\{0\} \subseteq U$. $\square$

REMARK. In the above proof, once we separated zero-sets, we no longer used the strong zero-dimensionality of X. So as a byproduct, we have:

4.6. COROLLARY OF PROOF. *For each completely regular space X, if U is clopen in X, then $\mathrm{cl}_{\beta X} U$ is clopen in βX and $\mathrm{cl}_{\beta X} U \cap \mathrm{cl}_{\beta X}(X - U) = \emptyset$.*

4.7. PROPOSITION. *Let X be a completely regular space, Z_0, Z_1 zero sets in X. Then $(\mathrm{cl}_{\beta X} Z_0 \cap \mathrm{cl}_{\beta X} Z_1) - X = \emptyset$ iff $Z_0 \cap Z_1$ is compact.*

PROOF. If $\mathrm{cl}_{\beta X} Z_0 \cap \mathrm{cl}_{\beta X} Z_1$ is a subset of X, then it is a compact set containing a closed subset $Z_0 \cap Z_1$. Therefore $Z_0 \cap Z_1$ is compact.

Assume $Z_0 \cap Z_1$ to be compact and let $p \in \beta X - X$ be arbitrary. Since $Z_0 \cap Z_1$

is compact, it is closed in βX and does not contain p, so there is some mapping $f: \beta X \to I$ with $f(p)=0$ and $f(x)=1$ for all $x \in Z_0 \cap Z_1$. Denote $F=f^{-1}[0,1/2] \cap X$, $G=f^{-1}(1/2,1] \cap X$. Since X is dense in βX, $p \in \mathrm{cl}_{\beta X}F$, the other possibility being ruled out by $\mathrm{cl}_{\beta X}G \subseteq f^{-1}[1/2,1] \not\ni p$. Now $F \cap Z_0$ and $F \cap Z_1$ are disjoint zero sets in X, hence $\mathrm{cl}_{\beta X}(F \cap Z_0)$ and $\mathrm{cl}_{\beta X}(F \cap Z_1)$ are disjoint. But this means that the point p belongs to at most one of them. □

4.8. PROPOSITION. *Let X be completely regular. Then X is open in βX iff X is locally compact, i.e. each point has a compact neighborhood.*

PROOF. Indeed, each point has a neighborhood which misses $\beta X - X$, namely the compact one, provided that X is locally compact. If X is open in βX, then by the regularity, for each $x \in X$ there is a neighborhood V of x in βX such that $\mathrm{cl}_{\beta X}V \cap (\beta X - X) = \emptyset$. But $\mathrm{cl}_{\beta X}V$ is a compact neighborhood of x, $\mathrm{cl}_{\beta X}V \subseteq X$. □

Summarizing, we obtain the following useful information.

4.9. THEOREM. *Let X be a completely regular space. Then*

(i) *βX is Boolean if and only if X is strongly zero-dimensional. In this case, βX is the Stone space of the algebra* Clop(X) *of all clopen subsets of X.*

(ii) *$\beta X - X$ is Boolean if X is locally compact and strongly zero-dimensional. In this case, $\beta X - X$ is the Stone space of the quotient algebra* Clop(X)$/\mathscr{C}$, *where $\mathscr{C}$ is the ideal of all compact members of* Clop(X).

PROOF. The first part of (i) has been already proved in 4.5. If βX is Boolean, then the mapping h: Clop(βX) $\to$ Clop(X) defined by $h(U) = U \cap X$ for each U clopen in βX is an embedding of a Boolean algebra Clop(βX) into Clop(X). But if V is clopen in X, then there is a continuous mapping $f: X \to I$ with $f(x)=0$ for $x \in V$, $f(x)=1$ for $x \in X - V$. Let βf be the continuous extension of f to βX. Then the set $W = (\beta f)^{-1}\{0\}$ is clopen in βX and $h(W) = U$. Thus, h is onto. The rest follows by the Stone duality.

To show (ii), in view of 4.8 and 4.5, we need to verify only the second part. We assume now that X is locally compact and strongly zero-dimensional. For U clopen in X, denote $[U] = \{U' \in \mathrm{Clop}(X)\colon U' \Delta U \text{ is compact}\}$.

Our aim is to find an isomorphism h between Clop $(\beta X - X)$ and $\{[U]\colon U \in \mathrm{Clop}(X)\}$. First, we shall show that for each clopen subset of $\beta X - X$ there is a clopen subset $U(V)$ of X such that $\mathrm{cl}_{\beta X}U(V) \cap (\beta X - X) = V$. Since V is clopen in $\beta X - X$, the mapping $f: \beta X - X \to I$ defined by $f(x)=0$ if $x \in V$, $f(x)=1$ if $x \in \beta X - X - V$, is continuous. Since $\beta X - X$ is closed in βX, there is a continuous extension $f \subseteq g: \beta X \to I$ by the Tietze theorem. By the strong zero-dimensionality of βX, there is some clopen $U \supseteq g^{-1}\{0\}$, U disjoint with $g^{-1}\{1\}$. Let $U(V) = U \cap X$. Since the mapping g admits only two values on $\beta X - X$, we have $V = U \cap (\beta X - X)$. Trivially, $U \cap (\beta X - X) \subseteq U$. Since U is open, $U \cap X$ is dense in U. Thus, $\mathrm{cl}_{\beta X}(U \cap X) = \mathrm{cl}_{\beta X}U = U$, because U is closed. So we have proved $V = (\beta X - X) \cap \mathrm{cl}_{\beta X}U(V)$.

Define $h(V) = [U(V)]$.

If U, U' are clopen in X and $\mathrm{cl}_{\beta X} U \cap (\beta X - X) = \mathrm{cl}_{\beta X} U' \cap (\beta X - X)$, then $U \triangle U'$ is compact by 4.7. Therefore our definition of h does not depend on the actual choice of $U(V)$. A similar argument using 4.7, shows that h is a one-to-one order preserving mapping, thus h is an embedding of $\mathrm{Clop}(\beta X - X)$ into $\mathrm{Clop}(X)/\mathscr{C}$. Finally, if $U \subseteq X$ is clopen and non-compact, then $\mathrm{cl}_{\beta X} U \cap (\beta X - X) \neq \emptyset$, so h is onto. □

4.10. EXAMPLES. (i) For rational numbers $\mathbb{Q}$ with the usual topology, $\beta\mathbb{Q}$ is homeomorphic to $\mathrm{Ult}(\mathrm{Clop}(\mathbb{Q}))$.

In order to apply 4.9, we shall show that $\mathbb{Q}$ is strongly zero-dimensional. Let Z_0, Z_1 be two disjoint zero sets in $\mathbb{Q}$. By 3.9, there is a continuous mapping $f: \mathbb{Q} \to I$ such that $Z_0 \subseteq f^{-1}\{0\}$, $Z_1 \subseteq f^{-1}\{1\}$. The space $\mathbb{Q}$ is countable, so there is some real number t, $0 < t < 1$, such that $t \not\subseteq f[\mathbb{Q}]$. Clearly, the sets $f^{-1}[0, t)$, $f^{-1}(t, 1]$ are clopen in $\mathbb{Q}$ and separate Z_0 and Z_1.

(ii) Consider the set ω with the discrete topology. Then the Stone space of $\mathscr{P}(\omega)/\mathit{fin}$ is homeomorphic to $\beta\omega - \omega$.

Notice that ω is locally compact and that the only compact subsets of ω are the finite ones. Now use 4.9(ii).

5. Extremally disconnected and Gleason spaces

From the categorical point of view, the Čech–Stone compactification is a reflection from the category of completely regular spaces into the category of compact Hausdorff spaces. Let us turn now to the projective objects in the category of compact Hausdorff spaces.

5.1. DEFINITION. A topological space is called extremally disconnected, if the closure of every open subset is open.

An easy exercise on the Stone duality gives immediately the following theorem.

5.2. THEOREM. *A compact Hausdorff space is extremally disconnected iff it is a Stone space of a complete Boolean algebra.*

Every dense subspace of an extremally disconnected space is extremally disconnected, too. That is an immediate consequence of the definition.

5.3. THEOREM. *Let X be a completely regular space. Then βX is extremally disconnected iff X is.*

PROOF. Suppose X is extremally disconnected, let $U \subseteq \beta X$ be open. Then $U \cap X$ is open in X and, by the extremal disconnectedness of X, $\mathrm{cl}_X(U \cap X)$ is open, too. Consider the mapping $f: X \to I$ defined by $f(x) = 0$ if $x \in \mathrm{cl}_X(U \cap X)$, $f(x) = 1$ otherwise. Since $\mathrm{cl}_X(U \cap X)$ is clopen, f is continuous. The mapping $\beta f: \beta X \to I$, a continuous extension of f, admits only two values, since X is dense in βX. Therefore $(\beta f)^{-1}\{0\}$ is a clopen subset of βX. It suffices to check that

$\mathrm{cl}_{\beta X}U = (\beta f)^{-1}\{0\}$. If $x \in \mathrm{cl}_{\beta X}U$, then each neighborhood V of x meets U and by the density of X in βX, $V \cap U \cap X$ is non-void. Thus, $(\beta f)(x) = 0$. If $x \not\in \mathrm{cl}_{\beta X}U$, then there is a neighborhood V of x disjoint with U. Thus, $f(y) = 1$ for each $y \in V \cap X$, therefore $(\beta f)(x) = 1$.

If βX is extremally disconnected, then so is X, because X is dense in βX. □

5.4. DEFINITION. Let $f: X \to Y$ be a continuous onto mapping. Then f is called irreducible if $f[F] \neq Y$ for each proper closed subset $F \subsetneq X$.

5.5. PROPOSITION. *A continuous onto mapping $f: X \to Y$ between two compact Hausdorff spaces is irreducible iff for each non-void open $U \subseteq X$ there is some non-void open $V \subseteq Y$ with $f^{-1}[V] \subseteq U$.*

PROOF. Let f be irreducible. Choose non-void open $U \subseteq X$. Then $X - U$ is closed, $X - U \subsetneq X$, therefore $f[X - U] \neq Y$. Since X is compact and Y Hausdorff, $V = Y - f[X - U]$ is as required.

Suppose the condition holds, let F be a proper closed subset of X. By the condition, there is some non-void open $V \subseteq Y$ with $f^{-1}[V] \cap F = \emptyset$. Thus, $f[F] \subseteq Y - V \neq Y$. □

5.6. PROPOSITION. *Let X, Y be Boolean spaces, $f: X \to Y$ a continuous onto mapping, $\mathscr{B}(f)$: Clop$(Y) \to$ Clop(X) the Stone dual of f, i.e. $\mathscr{B}(f)(U) = f^{-1}[U]$ for each clopen $U \subseteq Y$.*

Then f is irreducible iff $\mathscr{B}(f)$ embeds the algebra Clop(Y) *onto a dense subalgebra of the algebra* Clop(X).

PROOF. If f is irreducible and $U \in$ Clop(X), then $f[X - U] \neq Y$ whenever $U \neq \emptyset$. Since Y is Hausdorff and f continuous, $Y - f[X - U]$ is open. Choose some non-void clopen $V \subseteq Y - f[X - U]$. Then $f^{-1}[V]$ is clopen, too, and $f^{-1}[V] \subseteq U$. Since U was chosen arbitrarily, this shows that $\mathscr{B}(f)$[Clop(Y)] is dense in Clop(X). Since f is onto, $\mathscr{B}(f)$ is one-to-one by the Stone duality.

Conversely, suppose $\mathscr{B}(f)$[Clop(Y)] is dense in Clop(X) and let F be a proper closed subset of X. Then there is some non-void clopen $V \subseteq Y$ with $f^{-1}[V] \subseteq X - F$. Therefore $f[F] \subseteq Y - V \neq Y$, so f is irreducible. □

As an important corollary, we obtain:

5.7. THEOREM. *Let X be Boolean, Y an extremally disconnected compact space, $f: X \to Y$ an irreducible mapping. Then f is a homeomorphism.*

PROOF. We shall show that the dual mapping $\mathscr{B}(f)$: Clop$(Y) \to$ Clop(X) is an isomorphism. By the previous proposition, $\mathscr{B}(f)$[Clop(Y)] is a dense subalgebra of Clop(X), therefore by the completeness of Clop(Y), $\mathscr{B}(f)$ is onto. Moreover, $\mathscr{B}(f)$ is an embedding. Thence $\mathscr{B}(f)$ is an isomorphism. □

The complete Boolean algebras are injective objects in the category of all Boolean algebras (see Part I, Theorem 5.13). Similarly, the extremally disconnected compact Hausdorff spaces are projective in the category of compact Hausdorff spaces. This is the content of the Gleason theorem below.

5.8. GLEASON THEOREM. *For each compact Hausdorff space X there is an extremally disconnected compact Hausdorff space GX and an irreducible mapping $p\colon GX \to X$ such that for each compact Hausdorff Y and for each continuous onto mapping $f\colon Y \to X$ there is a continuous mapping $g\colon GX \to Y$ with $p = f \circ g$.*

The space GX, the so-called absolute or projective cover or Gleason space of X, is unique up to homeomorphism.

PROOF. The validity of the theorem is clear for Boolean spaces by Stone duality and the Sikorski theorem. In the general case, GX is the Stone space of the algebra $\mathrm{RO}(X)$ and the mapping p sends each ultrafilter in $\mathrm{RO}(X)$ to the (unique) point in the intersection of the closures of its members. It is clear that GX is extremally disconnected as well as that p is irreducible.

If $f\colon Y \to X$ is an onto mapping, then there is a closed subset $Y_1 \subseteq Y$ such that $f[Y_1] = X$ and $f \upharpoonright Y_1$ is irreducible. Indeed, let Y_1 be the minimal element in the family of all compact subspaces of Y which are mapped onto X by f. Thus, from now on we are allowed to assume that $f\colon Y \to X$ is irreducible.

Under the assumption of irreducibility of f we shall show that for each ultrafilter $\mathcal{U}$ in $\mathrm{RO}(X)$ (thus the point of GX), the family $\{\mathrm{cl}\, f^{-1}[U]\colon U \in \mathcal{U}\}$ has a unique point in its intersection.

To see this, if x, y are distinct points in Y, choose an open neighborhood V of x such that $y \not\in \mathrm{cl}\, V$. We want to apply Proposition 5.5. To this end, let $\mathcal{G} = \{U\colon U$ open in $X, f^{-1}[U] \subseteq V\}$, $\mathcal{H} = \{U\colon U$ open in $X, f^{-1}[U] \cap \mathrm{cl}_Y V = \emptyset\}$, $U_0 = \mathrm{int\,cl} \bigcup \mathcal{G}$, $U_1 = \mathrm{int\,cl} \bigcup \mathcal{H}$. Since U_0, U_1 are regular open, disjoint and $U_0 \cup U_1$ is dense in X, just one of them belongs to $\mathcal{U}$. Furthermore, $x \in \mathrm{cl}_Y f^{-1}[U_0]$ and $y \in \mathrm{cl}_Y f^{-1}[U_1]$, because $\mathrm{cl}_Y f^{-1}[U_0] = \mathrm{cl}_Y V$ and $\mathrm{cl}_Y f^{-1}[U_1] = Y - V$, by Proposition 5.5.

The value of $g\colon GX \to Y$ in $\mathcal{U}$ is, as can be expected, the point in $\bigcap \{\mathrm{cl}_Y f^{-1}[U]\colon U \in \mathcal{U}\}$.

It remains to prove the unicity of GX. Consider any other extremally disconnected compact space Z with the properties of GX. Denote by p_Z the irreducible mapping $Z \to X$. Then there is a continuous mapping $g\colon Z \to GX$ such that $p_Z = p \circ g$. Since both p and p_Z are irreducible, g is irreducible too. Since GX is extremally disconnected and Z is Boolean, g is a homeomorphism. □

5.9. DEFINITION. Let X, Y be completely regular spaces. Then X and Y are called coabsolute, if the algebras $\mathrm{RO}(X)$ and $\mathrm{RO}(Y)$ are isomorphic.

5.10. PROPOSITION. (i) *Each completely regular space X is coabsolute to βX.*

(ii) *Two compact Hausdorff spaces X, Y are coabsolute iff GX and GY are homeomorphic.*

PROOF. (ii) is clear, while (i) follows immediately from the fact that $\mathrm{RO}(X)$ is isomorphic to $\mathrm{RO}(Y)$ whenever X is dense in Y. □

5.11. EXAMPLE. All separable metric spaces without isolated points are co-absolute.

PROOF. Indeed, for X separable metrizable let $\mathscr{C} \subseteq \mathrm{RO}(X)$ be the algebra generated by all $B(x, 1/n)$, where n ranges in $\omega - \{0\}$ and x in the countable dense subset of X. The algebra $\mathscr{C}$ is countable and dense in $\mathrm{RO}(X)$ and if X has no isolated point, then $\mathscr{C}$ is atomless. Thus, by Part I, Corollary 5.16, $\mathrm{RO}(X) \approx \mathrm{RO}({}^{\omega}2)$. $\square$

Sometimes it is useful to reduce the general case to the zero-dimensional one, and it may happen that the absolute cannot help. The next proposition is one of the possible solutions.

5.12. PROPOSITION. *Let X be a compact Hausdorff space, $\mathscr{F}$ a family of continuous mappings from X onto X. Then there is a compact space X^0, a family $\mathscr{F}^0 = \{f^0 \colon f \in \mathscr{F}\}$ of continuous mappings from X^0 onto X^0 and a continuous onto mapping $p \colon X^0 \to X$ such that*
 (i) *X^0 is Boolean,*
 (ii) *$w(X^0) \le w(X) \cdot |\mathscr{F}|$,*
 (iii) *for each $f \in \mathscr{F}$, $f \circ p = p \circ f^0$,*
 (iv) *if $f, g \in \mathscr{F}$ commute, then so do f^0, g^0.*

PROOF. The theorem is trivial if X is finite. In an infinite case, choose an open base $\mathscr{B}$ for X with $|\mathscr{B}| = w(X)$. Let $\mathscr{A} \subseteq \mathscr{P}(X)$ be the smallest Boolean subalgebra with $\mathscr{B} \subseteq \mathscr{A}$ and $f^{-1}[A] \in \mathscr{A}$ for all $A \in \mathscr{A}, f \in \mathscr{F}$. We shall define X^0 to be the Stone space of $\mathscr{A}$. Immediately from the definition, X^0 is Boolean and $w(X^0) \le |\mathscr{A}| \le |\mathscr{B}| \cdot |\mathscr{F}| = w(X) \cdot |\mathscr{F}|$.

If $\mathscr{U}$ is an ultrafilter in $\mathscr{A}$, then there is a unique $x \in X$ with $x \in \bigcap \{\mathrm{cl}\, U \colon U \in \mathscr{U}\}$. Indeed, if $x_0 \neq x_1$, then there are $B_0, B_1 \in \mathscr{B}$ with $x_i \in B_i$ and $\mathrm{cl}\, B_0 \cap \mathrm{cl}\, B_1 = \emptyset$. It is impossible to have $B_0 \in \mathscr{U}$, $B_1 \in \mathscr{U}$ simultaneously. For $\mathscr{U} \in X^0$, let $p(\mathscr{U})$ be the unique point in $\bigcap \{\mathrm{cl}\, U \colon U \in \mathscr{U}\}$; this defines $p \colon X^0 \to X$. The mapping p is obviously onto. If $p(\mathscr{U}) = x$ and $B \in \mathscr{B}$ is a neighborhood of x, choose a $B_1 \in \mathscr{B}$ with $x \in B_1 \subseteq \mathrm{cl}\, B_1 \subseteq B$. Then $B_1 \in \mathscr{U}$, too, and for each $\mathscr{V} \in X^0$, if $\mathscr{V} \in s(B_1)$, then $p(\mathscr{V}) \in B$. Thus, p is continuous.

For $f \in \mathscr{F}$ and $\mathscr{U} \in X^0$, let $f^0(\mathscr{U})$ be the ultrafilter $\mathscr{V} \in X^0$ such that for each $V \in \mathscr{V}$, $f^{-1}[V] \in \mathscr{U}$. Clearly, f^0 is continuous.

It remains to prove (iii) and (iv). Let $\mathscr{U} \in X^0$, $\mathscr{V} = f^0(\mathscr{U})$, $x = p(\mathscr{U})$, $y = p(\mathscr{V})$. We need to show that $f(x) = y$. Aiming for a contradiction, assume $f(x) \neq y$ and choose $B_0, B_1 \in \mathscr{B}$ with $\mathrm{cl}\, B_0 \cap \mathrm{cl}\, B_1 = \emptyset$, $f(x) \in B_0$, $y \in B_1$. Since $\mathrm{cl}\, B_0 \cap \mathrm{cl}\, B_1 = \emptyset$, we have $f^{-1}[\mathrm{cl}\, B_0] \cap f^{-1}[\mathrm{cl}\, B_1] = \emptyset$; $f(x) \in B_0$ implies $x \in f^{-1}[\mathrm{cl}\, B_0]$. Thus, for the desired contradiction we need to show that $x \in f^{-1}[\mathrm{cl}\, B_1]$. Since $y \in B_1$, $B_1 \in \mathscr{V}$ and having $\mathscr{V} = f^0(\mathscr{U})$ we have $f^{-1}[B_1] \in \mathscr{U}$; finally, $f^{-1}[B_1] \in \mathscr{U}$ implies $x \in \mathrm{cl}\, f^{-1}[B_1] \subseteq f^{-1}[\mathrm{cl}\, B_1]$.

For (iv), assume $f \circ g = g \circ f$ and let $\mathscr{U} \in X^0$, $\mathscr{V} = f^0 \circ g^0(\mathscr{U})$. Then $V \in \mathscr{V}$ iff $g^{-1}[f^{-1}[V]] \in \mathscr{U}$, but from the commutativity of f, g, $g^{-1}[f^{-1}[V]] = f^{-1}[g^{-1}[V]]$. So $V \in f^0 \circ g^0(\mathscr{U})$ iff $V \in g^0 \circ f^0(\mathscr{U})$, which shows $f^0 \circ g^0 = g^0 \circ f^0$. $\square$

5.13. ALEXANDROFF THEOREM. *Each compact metric space is a continuous image of* ${}^{\omega}2$.

PROOF. Apply 5.12 to a compact metric space X and $\mathscr{F} = \{id\}$. Since $w(X) \le \omega$, X^0 is Boolean and $w(X^0) \le \omega$, the same being true for $X^0 \times {}^{\omega}2$. $X^0 \times {}^{\omega}2$ is homeomorphic to ${}^{\omega}2$ for it is compact, metrizable, zero-dimensional without isolated points. The projection from $X^0 \times {}^{\omega}2$ onto X^0 composed with p is the mapping we need. □

6. κ-Parovičenko spaces

It was proved in Part I that any countable Boolean algebra embeds into the countable infinite atomless Boolean algebra Clop(${}^{\omega}2$), and that any Boolean algebra of size $\le \omega_1$ embeds into $\mathscr{P}(\omega)/fin$. In this section we give a generalization to higher cardinals.

6.1. DEFINITION. Let κ be an infinite cardinal. A Boolean space X is called κ-Parovičenko, if for any two families A, C of clopen subsets of X such that $|A| < \kappa$, $|C| < \kappa$ and $\bigcup A_1 \subsetneqq C_1$ whenever $A_1 \in [A]^{<\omega}$, $C_1 \in [C]^{<\omega}$, there is some clopen $U \subseteq X$ with $V \subsetneqq U \subsetneqq W$ for each $V \in A$, $W \in C$.

6.2. LEMMA. *Each non-void open subset of a* κ*-Parovičenko space* X *contains at least* κ *disjoint non-void clopen subsets.*

PROOF. Let $W \subseteq X$ be clopen and non-void. Consider $A = \{\emptyset\}$, $C = \{W\}$ in the definition of a κ-Parovičenko space. There is some non-void clopen U with $U \subsetneqq W$. This shows that the κ-Parovičenko space has no isolated points; in particular, there is some infinite family A of disjoint clopen subsets of W.

It suffices to prove that $|A| < \kappa$ implies A is not maximal. Indeed, for $C = \{W\}$ and A, there is some clopen $U \subsetneqq W$ with $U \supsetneqq V$ for all $V \in A$. Obviously $W - U$ is a non-void clopen subset of W disjoint with all $V \in A$. □

6.3. COROLLARY. *If* κ *is an infinite cardinal, then the weight of a* κ*-Parovičenko space* X *is at least* $\kappa^{<\kappa}$.

PROOF. By 6.2, each clopen subset contains a disjoint family consisting of κ non-void clopen sets. By the fact that X is κ-Parovičenko, for each decreasing chain of $<\kappa$ non-void clopen subsets of X there is a non-void clopen set in its intersection. Using a standard branching argument, we are therefore able to find for each $\tau < \kappa$ and for each $\varphi \in {}^{\tau}\kappa$ a clopen set U_φ such that U_φ, U_ψ are disjoint for distinct $\varphi, \psi \in {}^{\tau}\kappa$. Since we have found κ^τ disjoint non-void clopen sets in X, $w(X) \ge \kappa^\tau$, for each $\tau < \kappa$. □

6.4. THEOREM. *Every compact space* X *of weight* $\le\kappa$ *is a continuous image of a* κ*-Parovičenko space* Y.

PROOF. By 5.12, we may assume that X is Boolean (take $\mathcal{F} = \{id\}$). Using Stone duality, it suffices to show that Clop(X) can be embedded into Clop(Y). We shall proceed by a transfinite induction similarly as in Part I, 5.29. Enumerate $\text{Clop}(X) = \{b_\alpha : \alpha < \kappa\}$ and denote by $\mathcal{B}_\alpha$ the subalgebra generated by $\{b_\beta : \beta < \alpha\}$, for $\alpha < \kappa$. Let $\varphi_0 : \mathcal{B}_0 \to \text{Clop}(Y)$ be defined by $\varphi_0(\emptyset) = \emptyset$, $\varphi_0(X) = Y$.

Assume an embedding $\varphi_\alpha : \mathcal{B}_\alpha \to \text{Clop}(Y)$ has been defined and for $\beta < \gamma \leq \alpha$, $\varphi_\beta \subseteq \varphi_\gamma$. If $b_\alpha \in \mathcal{B}_\alpha$, then $\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha$ and $\varphi_{\alpha+1} = \varphi_\alpha$. If $b_\alpha \not\in \mathcal{B}_\alpha$, let us denote $C_0 = \{\varphi_\alpha(b) : b \in \mathcal{B}_\alpha,\ b > b_\alpha\}$, $A_0 = \{\varphi_\alpha(b) : b \in \mathcal{B}_\alpha,\ b < b_\alpha\}$, $D = \{\varphi_\alpha(b) : b \in \mathcal{B}_\alpha,\ b \cap b_\alpha \neq \emptyset \neq b - b_\alpha\}$.

For each $d \in D$, define $C^+(d) = \{d \cap c : c \in C_0\}$, $A^+(d) = \{\emptyset\}$, $C^-(d) = \{Y\}$, $A^-(d) = \{d - a : a \in A_0\rangle$. Using the fact that Y is κ-Parovičenko, we shall find two members $d^+, d^- \in \text{Clop}(Y)$ for each $d \in D$ such that $\emptyset \neq d^+ \subsetneqq d \cap c$ for all $c \in C_0$ and $d - a \subsetneqq d^- \subsetneqq Y$ for all $a \in A_0$.

The family $\{d^+, d^- : d \in D\}$ is of size $< \kappa$; hence by 6.2 and the Balcar–Vojtáš theorem (Part I, 3.14), there is a disjoint family $\{d_1, d_2 : d \in D\}$ such that for each $d \in D$, $d_1 \subseteq d^+$, $d_2 \subseteq d^-$.

Let $A = A_0 \cup \{d_1 : d \in D\}$, $C = C_0 \cup \{Y - d_2 : d \in D\}$. It is easy to check that for each $A' \in [A]^{<\omega}$ and each $C' \in [C]^{<\omega}$, $\bigcup A' \subsetneqq \bigcap C'$. Let $u \in \text{Clop}(Y)$ separate A, C.

If $x \in \mathcal{B}_{\alpha+1}$, then for some $b, c \in \mathcal{B}_\alpha$, $x = (b \cap b_\alpha) \cup (c - b_\alpha)$. Define then $\varphi_{\alpha+1}(x) = (\varphi_\alpha(b) \cap u) \cup (\varphi_\alpha(c) - u)$. By our choice of u, the definition does not depend on the choice of b and c for given x. Since φ_α is a homomorphism, $\varphi_{\alpha+1}$ is a homomorphism too. In order to show that $\varphi_{\alpha+1}$ is one-to-one, suppose $b \cap b_\alpha \neq \emptyset$ for some $b \in \mathcal{B}_\alpha$. If $b \subseteqq b_\alpha$, then $\varphi_\alpha(b) \subsetneqq u$, so $\varphi_{\alpha+1}(b \cap b_\alpha) = \varphi_\alpha(b) \neq \emptyset$. If $b - b_\alpha \neq \emptyset$, then $\varphi_\alpha(b) = d \in D$, thus $\varphi_\alpha(b) \supsetneqq d_1$. Therefore $\varphi_{\alpha+1}(b \cap b_\alpha) = u \cap \varphi_\alpha(b) \supseteq u \cap d_1 = d_1 \neq \emptyset$. A symmetrical argument shows that $\varphi_{\alpha+1}(b - b_\alpha) \neq \emptyset$ whenever $b - b_\alpha \neq \emptyset$.

For α limit, $\varphi_\alpha = \bigcup_{\beta<\alpha} \varphi_\beta$ of course. It is clear now that φ_κ is the required embedding. □

6.5. REMARK. Let κ be an infinite cardinal and suppose that there is a κ-Parovičenko space X of weight κ.

Then by Corollary 6.3, $\kappa^{<\kappa} = \kappa$ and κ is a regular cardinal. X is up to a homeomorphism a unique κ-Parovičenko space of weight κ, and if Z is a compact Hausdorff space of weight $< \kappa$ and $f, g : X \to Z$ are two continuous onto mappings, then there is a homeomorphism $h : X \to X$ such that $f \circ h = g$.

In both cases it is necessary to find a Boolean isomorphism from Clop(X) onto Clop(X). This is done similarly as in the proof of 6.4, with an obligatory back-and-forth modification. The interested reader may find all details in COMFORT and NEGREPONTIS [1974, §6].

6.6. THEOREM. *For each infinite regular κ there is a κ-Parovičenko space of weight $\kappa^{<\kappa}$.*

There is nothing to prove if $\kappa = \omega$ – the Cantor space ${}^{\omega}2$ is such.

Let $\kappa > \omega$; we shall show first the existence of a κ^+-Parovičenko space. The main trick, using an ultraproduct, stems from the model theory.

Choose a κ^+-good uniform ultrafilter $\mathcal{U}$ on κ. (Let us recall what a good ultrafilter is. If $\varphi: [\kappa]^{<\omega} \to \mathcal{P}(\kappa)$ is a mapping, call φ monotone if $\varphi(K) \supseteq \varphi(H)$ whenever $K \subseteq H$, $K, H \in [\kappa]^{<\omega}$, and call it multiplicative if $\varphi(H \cup K) = \varphi(H) \cap \varphi(K)$ for $K, H \in [\kappa]^{<\omega}$. An ultrafilter $\mathcal{U}$ on κ is called κ^+-good provided that for each monotone mapping $\varphi: [\kappa]^{<\omega} \to \mathcal{U}$ there is a multiplicative mapping $\psi: [\kappa]^{<\omega} \to \mathcal{U}$ with $\psi(H) \subseteq \varphi(H)$ for all $H \in [\kappa]^{<\omega}$ and there is a decreasing sequence $\{U_n: n \in \omega\} \subseteq \mathcal{U}$ with empty intersection. There are 2^{2^κ}-many κ^+-good uniform ultrafilters on κ as proved by K. Kunen. See KUNEN [1972] or COMFORT and NEGREPONTIS [1974, 10.6].)

Take $\kappa > \omega$ with the discrete topology, let $\pi: \kappa \times {}^\omega 2 \to \kappa$ be the natural projection. Then π extends to $\beta\pi: \beta(\kappa \times {}^\omega 2) \to \beta\kappa$; denote $Y = (\beta\pi)^{-1}\{\mathcal{U}\}$. Y is a closed subspace of $\beta(\kappa \times {}^\omega 2)$; we shall show that Y is a κ^+-Parovičenko space.

Obviously, $\kappa \times {}^\omega 2$ is locally compact and strongly zero-dimensional, thus $\beta(\kappa \times {}^\omega 2) - (\kappa \times {}^\omega 2)$ is Boolean; the same applies for its closed subset Y.

For the sake of brevity, denote $Z = \kappa \times {}^\omega 2$ and $Z_\xi = \{\xi\} \times {}^\omega 2$ for $\xi \in \kappa$. Since Z is a topological sum of $\{Z_\xi: \xi \in \kappa\}$, each Z_ξ is clopen in Z. Before giving the proof that Y satisfies the desired separation property, let us state an easy claim.

Claim. Let D_ξ, E_ξ be clopen subsets of Z_ξ. Then

(i) $\quad Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} D_\xi \neq \emptyset \quad$ iff $\{\xi \in \kappa: D_\xi \neq \emptyset\} \in \mathcal{U}$;

(ii) $\quad Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} D_\xi \subseteq Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} E_\xi \quad$ iff $\{\xi \in \kappa: D_\xi \subseteq E_\xi\} \in \mathcal{U}$;

(iii) $\quad Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} D_\xi = Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} E_\xi \quad$ iff $\{\xi \in \kappa: D_\xi = E_\xi\} \in \mathcal{U}$;

(iv) $\quad Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} D_\xi \subsetneqq Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} E_\xi \quad$ iff $\{\xi \in \kappa: D_\xi \subsetneqq E_\xi\} \in \mathcal{U}$.

We show (i). Denote $T = \{\xi \in \kappa: D_\xi \neq \emptyset\}$. If $T \not\in \mathcal{U}$, then $\mathrm{cl}_{\beta\kappa}(\kappa - T)$ is an open neighborhood of $\mathcal{U}$, therefore $(\beta\pi)^{-1}[\mathrm{cl}_{\beta\kappa}(\kappa - T)]$ is an open neighborhood of Y, disjoint with $\bigcup_{\xi \in \kappa} D_\xi$. Thus, $Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} D_\xi = \emptyset$.

If $T \in \mathcal{U}$, for each $\xi \in T$ pick some $x_\xi \in D_\xi$. By the compactness of βZ, $\mathcal{U} - \lim x_\xi = x$ exists and clearly $x \in Y$. Thus, $Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} D_\xi \neq \emptyset$ in this case.

Now, (ii) follows from (i) and (iii) follows from (ii); finally, (iv) is a consequence of (ii) and (iii). The claim is proved.

Let $\{A_\alpha: \alpha \in \kappa\}$, $\{C_\alpha: \alpha \in \kappa\}$ be families of clopen subsets of Y such that for each $H \in [\kappa]^{<\omega}$, $\bigcup_{\alpha \in H} A_\alpha \subsetneqq \bigcap_{\alpha \in H} C_\alpha$.

There are families $\{A_{\xi,\alpha}: \alpha \in \kappa\}$ and $\{C_{\xi,\alpha}: a \in \kappa\}$ of clopen subsets of Z_ξ such that for each $\alpha \in \kappa$, $A_\alpha = Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} A_{\xi,\alpha}$, $C_\alpha = Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} C_{\xi,\alpha}$.

For $H \in [\kappa]^{<\omega}$, let $\varphi(H) = \{\xi \in \kappa: \bigcup_{\alpha \in H} A_{\xi,\alpha} \subsetneqq \bigcap_{\alpha \in H} C_{\xi,\alpha}\}$. By the claim, for each $H \in [\kappa]^{<\omega}$ we have $\varphi(H) \in \mathcal{U}$. Moreover, φ is obviously a monotone mapping. For each $H \in [\kappa]^{<\omega}$ and $\xi \in \varphi(H)$ choose a clopen set $B_{\xi,H} \subseteq Z_\xi$ with $\bigcup_{\alpha \in H} A_{\xi,\alpha} \subsetneqq B_{\xi,H} \subsetneqq \bigcap_{\alpha \in H} C_{\xi,\alpha}$.

There is a family $\kappa = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n \supseteq \cdots$ of members of $\mathcal{U}$ with $\bigcap_{n \in \omega} U_n = \emptyset$. Let $\varphi': [\kappa]^{<\omega} \to \mathcal{U}$ be defined by $\varphi'(H) = \varphi(H) \cap U_{|H|}$. The mapping φ' is monotone, since φ is. By the κ^+-goodness of $\mathcal{U}$, there is a multiplicative mapping $\psi: [\kappa]^{<\omega} \to \mathcal{U}$, $\psi(H) \subseteq \varphi'(H)$ for all $H \in [\kappa]^{<\omega}$.

For $\xi \in \kappa$, consider the set $H_\xi = \{\alpha \in \kappa: \xi \in \psi(\{\alpha\})\}$. There is some $i \in \omega$

with $\xi \not\in U_i$; we show that then $|H_\xi| \leq i$. If not, then there is some $H \in [H_\xi]^{i+1}$ and

$$\xi \in \bigcap_{\alpha \in H} \psi(\{\alpha\}) = \psi(H) \subseteq \varphi'(H) \subseteq U_{|H|} \subseteq U_i ,$$

which contradicts our choice of i. Thus, for each $\xi \in \kappa$ the set H_ξ is finite and we are justified to define B as $Y \cap \mathrm{cl}_{\beta Z} \bigcup_{\xi \in \kappa} B_{\xi, H_\xi}$.

The set B is a clopen subset of Y. If $\alpha \in \kappa$ and $\xi \in \psi(\{\alpha\})$, then $\alpha \in H_\xi$ and $\xi \in \psi(H_\xi) \subseteq \varphi(H_\xi)$. Therefore we have for each $\xi \in \psi(\{\alpha\})$,

$$A_{\xi,\alpha} \subseteq \bigcup_{\beta \in H_\xi} A_{\xi,\beta} \subsetneqq B_{\xi,H_\xi} \subsetneqq \bigcap_{\beta \in H_\xi} C_{\xi,\beta} \subseteq C_{\xi,\alpha} .$$

By the Claim, $A_\alpha \leq B \subseteq C_\alpha$ since $\psi(\{\alpha\}) \in \mathcal{U}$.

So we have found a κ^+-Parovičenko space Y. The theorem follows by an easy trick, which can be explained best in Boolean terms.

Take a subalgebra $\mathcal{B}_0 \subseteq \mathrm{Clop}(Y)$ with $|\mathcal{B}_0| \leq \kappa^{<\kappa}$. Proceeding by induction, suppose $\mathcal{B}_\alpha \subseteq \mathrm{Clop}(Y)$ has been found with $|\mathcal{B}_\alpha| \leq \kappa^{<\kappa}$; choose $\mathcal{C}_\alpha \subseteq \mathrm{Clop}(Y)$ of cardinality $\kappa^{<\kappa}$ such that any two subfamilies of $\mathcal{B}_\alpha$ of size $<\kappa$ which are still not separated by a member of $\mathcal{B}_\alpha$ can be separated by a member of $\mathcal{C}_\alpha$ and define $\mathcal{B}_{\alpha+1}$ to be the smallest subalgebra of $\mathrm{Clop}(Y)$ containing $\mathcal{B}_\alpha \cup \mathcal{C}_\alpha$. At limit stages take unions of all the $\mathcal{B}_\alpha$'s already known.

Let $\mathcal{B} = \bigcup_{\alpha \in \kappa} \mathcal{B}_\alpha$. Since κ is regular, $|\mathcal{B}| = \kappa^{<\kappa}$ and the Stone space of $\mathcal{B}$ is a κ-Parovičenko space of weight $\kappa^{<\kappa}$. □

There is an easy description of how to pass from a κ-Parovičenko space to a κ^+-Parovičenko space.

6.7. THEOREM (A. Dow). *Let $\kappa > \omega$ be a regular cardinal and let $\{X_\alpha : \alpha < \kappa\}$ be a strictly increasing family of non-empty clopen subsets of a topological space X such that each X_α is a κ-Parovičenko space. Then for $Y = \bigcup \{X_\alpha : \alpha < \kappa\}$, the Čech–Stone remainder $\beta Y - Y$ is a κ^+-Parovičenko space.*

6.8. COROLLARY (A. Dow). *The space $\beta\, \mathrm{SU}(\omega_1) - \mathrm{SU}(\omega_1)$ is an ω_2-Parovičenko space. Here $\mathrm{SU}(\omega_1)$ is the space of all subuniform ultrafilters on ω_1, i.e. all non-trivial ultrafilters on ω_1 which contain a countable set.*

PROOF. The corollary is an immediate consequence of the theorem: put $X = \beta\omega_1 - \omega_1$, $X_\alpha = \beta\alpha - \alpha$ for $\alpha < \omega_1$.

In order to prove the theorem, let us show first that Y is strongly zero-dimensional, hence $\beta Y - Y$ is Boolean. Let Z_0, Z_1 be two disjoint zero sets in Y. The proof goes by induction. Choose a clopen subset $C_0 \subseteq X_0$ such that $Z_0 \cap X_0 \subseteq C_0$, $Z_1 \cap X_0 \subseteq X_0 - C_0$. Suppose $\beta < \kappa$ and that for $\alpha < \beta$ the sets $C_\alpha \subseteq X_\alpha$ have been found such that: each C_α is clopen in X_α, $C_\alpha \supseteq Z_0 \cap X_\alpha$, $Z_1 \cap X_\alpha \subseteq X_\alpha - C_\alpha$ and if $\alpha < \gamma < \beta$, then $C_\gamma \cap X_\alpha = C_\alpha$. Since X_β is κ-Parovičenko and $|\beta| < \kappa$, there is some D_β clopen in X_β such that for all $\alpha < \beta$, $C_\alpha \subsetneqq D_\beta \subsetneqq C_\alpha \cup (X_\beta - X_\alpha)$.

But we cannot be sure that D_β contains $Z_0 \cap X_\beta$ and is disjoint from $Z_1 \cap X_\beta$. However, $Z_0 \cap X_\beta - D_\beta$ is closed and disjoint from $\mathrm{cl} \bigcup_{\alpha<\beta} X_\alpha$, for if a point x belongs to $Z_0 \cap \mathrm{cl} \bigcup_{\alpha<\beta} X_\alpha$, then x belongs to $\mathrm{cl} \bigcup_{\alpha<\beta} Z_0 \cap X_\alpha \subseteq \mathrm{cl} \bigcup_{\alpha<\beta} C_\alpha \subseteq D_\beta$. Thus, there is a clopen set $E_\beta \subseteq X_\beta$ such that $E_\beta \supseteq Z_0 \cap X_\beta - D_\beta$, $E_\beta \cap \bigcup_{\alpha<\beta} X_\alpha = \emptyset$. A symmetrical argument produces a clopen set $F_\beta \subseteq X_\beta$ with $F_\beta \supseteq Z_1 \cap X_\beta \cap (D_\beta \cup E_\beta)$, $F_\beta \cap (Z_0 \cap \bigcup_{\alpha<\beta} X_\alpha) = \emptyset$. It remains now to set $C_\beta = (D_\beta \cup E_\beta) - F_\beta$.

The set $C = \bigcup_{\alpha<\kappa} C_\alpha$ is evidently clopen in Y and separates Z_0 from Z_1, thus Y is strongly zero-dimensional.

It remains to check the separating property. Let $\{A_\xi : \xi < \kappa\}$ and $\{C_\xi : \xi < \kappa\}$ be families of clopen subsets of $\beta Y - Y$ such that for each $H \in [\kappa]^{<\kappa}$, $\bigcup_{\xi \in H} A_\xi \subsetneqq \bigcap_{\xi \in H} C_\xi$. Then there are families of clopen subsets of Y, $\{U_\xi : \xi \in \kappa\}$ and $\{V_\xi : \xi \in \kappa\}$ such that for each $\xi < \kappa$, $A_\xi = \mathrm{cl}_{\beta Y} U_\xi - Y$, $C_\xi = \mathrm{cl}_{\beta Y} V_\xi - Y$. Our plan is to find more convenient representation of the A_ξ's and C_ξ's and then to use the separation property in the X_α's.

Using 4.9 we know that for each $H \in [\kappa]^{<\omega}$, the inequality $\bigcup_{\xi \in H} A_\xi \subseteq \bigcap_{\xi \in H} C_\xi$ holds iff $\bigcup_{\xi \in H} U_\xi - \bigcap_{\xi \in H} V_\xi$ is compact in Y. But each compact subset of Y is contained in some X_α. Thus, by an easy induction we can find a strictly increasing sequence $\{\alpha(\xi) : \xi \in \kappa\}$ such that for each $\xi \in \kappa$ and for each $H \in [\xi]^{<\omega}$,

$$\bigcup_{\eta \in H} U_\eta - \bigcap_{\eta \in H} V_\eta \subseteq X_{\alpha(\xi)}$$

and

$$\bigcup_{\eta \in H} U_\eta \cap (X_{\alpha(\xi+1)} - X_{\alpha(\xi)}) \subsetneqq \bigcap_{\eta \in H} V_\eta \cap (X_{\alpha(\xi+1)} - X_{\alpha(\xi)}) .$$

Define now $\tilde{U}_\xi = U_\xi - X_{\alpha(\xi)}$, $\tilde{V}_\xi = V_\xi \cup X_{\alpha(\xi)}$. By 4.9, $\mathrm{cl}_{\beta Y} \tilde{U}_\xi - Y = A_\xi$ and $\mathrm{cl}_{\beta Y} \tilde{V}_\xi - Y = C_\xi$, too.

The rest of the proof resembles how we showed the strong zero-dimensionality.

Suppose $\xi < \kappa$ and for each $\eta < \xi$ we have found W_η clopen in $X_{\alpha(\eta)}$ such that the following holds. For all $H \in [\eta]^{<\omega}$, $\bigcup_{\zeta \in H} \tilde{U}_\zeta \cap X_{\alpha(\eta)} \subsetneqq W_\eta \subsetneqq \bigcap_{\zeta \in H} \tilde{V}_\zeta \cap X_{\alpha(\eta)}$ and if $\zeta < \eta < \xi$, then $W_\zeta = X_{\alpha(\zeta)} \cap W_\eta$. Knowing that $X_{\alpha(\xi)}$ is κ-Parovičenko and using the fact that $|\xi| < \kappa$, we are able to find W_ξ such that for each $\eta < \xi$, $W_\eta \subsetneqq W_\xi \subsetneqq W_\eta \cup (X_{\alpha(\xi)} - X_{\alpha(\eta)})$ and for each $H \in [\xi]^{<\omega}$, $\bigcup_{\eta \in H} \tilde{U}_\eta \cap X_{\alpha(\xi)} \subsetneqq W_\xi \subsetneqq \bigcap_{\eta \in H} \tilde{V}_\eta \cap X_{\alpha(\xi)}$.

It remains to define $W = \bigcup_{\xi \in \kappa} W_\xi$ and $B = \mathrm{cl}_{\beta Y} W - Y$. If $\xi \in \kappa$, then from $\tilde{U}_\xi - W \subseteq X_{\alpha(\xi)}$ we have $A_\xi \subseteq B$, from $\tilde{U}_\xi \cap X_{\alpha(\zeta)} \subsetneqq W \cap X_{\alpha(\zeta)}$ whenever $\xi < \zeta < \kappa$ we have $A_\xi \subsetneqq B$. Similarly for $B \subsetneqq C_\xi$. □

7. *F*-spaces

A completely regular space X is called an F-space, if for any pair of disjoint cozero sets C_0, C_1 there is a continuous real-valued function f on X with $f(x) = i$ whenever $x \in C_i$, $i = 0, 1$.

7.1. REMARK. A completely regular space X is an F-space whenever each continuous bounded real-valued mapping defined on a cozero set in X extends continuously to the whole of X. The proof of this fact is an analogy to the proof of Tietze's theorem from Urysohn lemma, and can be found, for example, in GILLMAN and JERISON [1960].

For a Boolean space X, X is an F-space if and only if it is a Stone space of an algebra with the countable separation property (Part I, 12.1). Hence, all extremally disconnected compact spaces as well as ω_1-Parovičenko spaces are examples of F-spaces. However, there are also compact F-spaces which are not Boolean. We shall show this in 7.11.

Our goal of this section will be the Frolík's theorem on non-homogeneity. We shall start with two lemmas.

7.2. LEMMA. *A countable union of cozero sets in a topological space is a cozero set too.*

PROOF. Let $f_n: X \to I$ be continuous, $X - C_n = f_n^{-1}\{0\}$. Then for $h = \Sigma_{n=0}^{\infty} 2^{-n} \cdot f_n$, $h^{-1}\{0\} = X - \bigcup_{n=0}^{\infty} C_n$. □

7.3. LEMMA. *If D is a countable discrete subspace of a compact Hausdorff F-space X, then* cl D *is homeomorphic to* $\beta\omega$.

PROOF. Let $D = \{d_n : n \in \omega\}$. The mapping $f: \omega \to D$ defined by $f(n) = d_n$ has a continuous extension βf from $\beta\omega$ onto cl D. Let us recall that $\beta f(\mathcal{U}) = \mathcal{U} -$ lim $f(n)$ for each $\mathcal{U} \in \beta\omega$. Therefore to show that βf is a homeomorphism we need only to verify that f is one-to-one (cf. 3.7). If $\mathcal{U} \neq \mathcal{V}$, $\mathcal{U}$, $\mathcal{V} \in \beta\omega$, then there are $U \in \mathcal{U}$ and $V \in \mathcal{V}$ with $U \cap V = \emptyset$.

Let us prove that any two disjoint subsets of D have disjoint closures. Since D is discrete, for each $d_n \in D$ we have $d_n \notin \mathrm{cl}_X(D - \{d_n\})$. Thus, there is a continuous function $\varphi_n: X \to I$ such that $\varphi_n(d_n) = 1$, $\varphi_n(x) = 0$ for all $x \in \mathrm{cl}_X(D - \{d_n\})$. Let $\psi = \Sigma_{n=0}^{\infty} 2^{-n} \cdot \varphi_n$. Denote $C_n = \psi^{-1}(2^{-n} - 2^{n-2}, 2^{-n} + 2^{-n-2})$. Then each C_n is a cozero neighborhood of d_n and C_n is disjoint with C_k for $n \neq k$.

If A, B are disjoint subsets of D, then $C_A = \bigcup_{d_n \in A} C_n$ and $C_B = \bigcup_{d_n \in B} C_n$ are disjoint cozero subsets of X by 7.2. Hence, there is a continuous mapping $g: X \to I$ such that $g^{-1}\{0\} \supseteq C_A$, $g^{-1}\{1\} \supseteq C_B$. Both sets $g^{-1}\{0\}$, $g^{-1}\{1\}$ are closed and disjoint, thence $\mathrm{cl}_X A \cap \mathrm{cl}_X B = \emptyset$.

The rest is clear now. By the continuity of βf, $\beta f(\mathcal{U}) \in \mathrm{cl}_X\{d_n : n \in U\}$, $\beta f(\mathcal{V}) \in \mathrm{cl}_X\{d_n : n \in V\}$. So $\beta f(\mathcal{U}) \neq \beta f(\mathcal{V})$. □

7.4. DEFINITION. A topological space X is called homogeneous, if for any two points $x, y \in X$ there is a homeomorphism $h: X \to X$ such that $h(x) = y$.

7.5. FROLÍK'S THEOREM. *No infinite compact F-space is homogeneous.*

The proof will be given in 7.9.

7.6. EXAMPLE. The Cantor space ${}^{\omega}2$ is homogeneous, while its Gleason space is not. Notice also that $\mathrm{RO}({}^{\omega}2) \approx \mathrm{Clop}(G\,{}^{\omega}2)$ is a homogeneous Boolean algebra.

7.7. LEMMA. *If D_0, D_1 are two discrete countable subspaces of a compact Hausdorff F-space, then $D_0 \cap \operatorname{cl} D_1 = \emptyset = D_1 \cap \operatorname{cl} D_0$ implies $\operatorname{cl} D_0 \cap \operatorname{cl} D_1 = \emptyset$.*

PROOF. We claim that $D_0 \cup D_1$ is discrete. Indeed, if $d \in D_0$, then $d \notin \operatorname{cl} D_1$ by the assumption and $d \notin \operatorname{cl}(D_0 - \{d\})$ since D_0 is discrete. Thus, $d \notin \operatorname{cl}(D_0 - \{d\}) \cup \operatorname{cl} D_1 = \operatorname{cl}(D_0 \cup D_1 - \{d\})$. Similarly for $d \in D_1$. Thus, $D_0 \cup D_1$ is discrete and, by 7.3, $\operatorname{cl} D_0 \cap \operatorname{cl} D_1 = \emptyset$ because $D_0 \cap D_1 = \emptyset$. □

7.8. THEOREM (KUNEN [1978]). *There are 2^{2^ω} weak P-points in $\beta\omega - \omega$. Recall that a point x is called a weak P-point in X if there is no countable set $S \subseteq X$ with $x \in \operatorname{cl}(S - \{x\})$.*

7.9. PROOF OF FROLÍK'S THEOREM. Since X is infinite, by 2.2 and 7.3, one can choose two disjoint discrete countable subspaces D, E with disjoint closures. We know that $\operatorname{cl} D$ as well as $\operatorname{cl} E$ are copies of $\beta\omega$, hence there are embeddings $\varphi, \psi\colon \beta\omega \to X$ such that $\operatorname{cl} D = \varphi[\beta\omega]$, $\operatorname{cl} E = \psi[\beta\omega]$. Choose two distinct weak P-points p, q in $\beta\omega - \omega$ such that for each permutation g of ω, $q \neq \beta g(p)$. This is clearly possible, for there are 2^{2^ω}-many weak P-points in $\beta\omega - \omega$, but only 2^ω-many permutations.

Let $x = \varphi(p)$, $y = \psi(q)$. Then x and y are as required.

Suppose not; let $h\colon X \to X$ be a homeomorphism with $h(x) = y$. Aiming for a contradiction, let us examine what happens with $h[D] = B$. Since h is a homeomorphism and D is discrete, B is discrete too. Since x belongs to $\operatorname{cl} D - D$, its image y belongs to $\operatorname{cl} B - B$.

The set B splits into three parts as follows: $B_0 = B \cap (\operatorname{cl} E - E)$, $B_1 = B \cap E$, $B_2 = B - \operatorname{cl} E$. Since B is countable discrete, $\operatorname{cl} B$ is homeomorphic to $\beta\omega$, hence y is in the closure of just one of B_i's. We shall check all three possibilities.

Suppose $y \in \operatorname{cl} B_0 - B_0$. Since $y \notin B_0$, B_0 must be infinite. So B_0 is a countable subset of $\operatorname{cl} E - E$ with $y \in \operatorname{cl} B_0 - \{y\}$, but $y = \psi(q)$ and q is a weak P-point of $\beta\omega - \omega$, a contradiction.

Suppose $y \in \operatorname{cl} B_2$ and denote $E_1 = E \cap \operatorname{cl} B_2$, $E_2 = E - \operatorname{cl} B_2$. Since we have $B_2 \cap \operatorname{cl} E = \emptyset$, we obtain from 7.7 that $\operatorname{cl} B_2 \cap \operatorname{cl} E_2 = \emptyset$ too. Since $y \in \operatorname{cl} B_2 \cap \operatorname{cl} E = \operatorname{cl} B_2 \cap (\operatorname{cl} E_1 \cup \operatorname{cl} E_2)$, again by 7.7, we have $E_1 \subseteq \operatorname{cl} B_2 - B_2$ and $y \in \operatorname{cl} E_1$. Hence, E_1 is infinite and $y \in \operatorname{cl}(E_1 - \{y\})$. Now, $h^{-1}[E_1]$ is a countable subset of $\operatorname{cl} D - D$ and $x = h^{-1}(y) \in \operatorname{cl}(h^{-1}[E_1] - \{x\})$. But this contradicts the assumption that p is a weak P-point in $\beta\omega - \omega$.

Finally, assume $y \in \operatorname{cl} B_1 - B_1$. Consider the sets $\psi^{-1}[B_1] \subseteq \omega$ and $\varphi^{-1}[h^{-1}[B_1]] \subseteq \omega$. Both are infinite and the homeomorphism h induces a one-to-one onto mapping between them, namely $g(n) = \psi^{-1} \circ h \circ \varphi(n)$. Now it is clear that $\beta g(p) = q$. Since g is one-to-one, it can be substituted by a permutation π of ω onto itself with $\beta\pi(p) = q$, but this contradicts our choice of p, q. To find π, choose an infinite set $V \in p$, $V \subseteq \operatorname{dom} g$ with infinite complement, such that $g[V]$ has an infinite complement, too, and let π be an arbitrary permutation which coincides with g on V. □

7.10. REMARK. Notice that in the above proof we never used the full strength of h being a homeomorphism. In fact, the same proof shows that in every infinite compact F-space X there are two points x, y such that no continuous one-to-one mapping from X to X sends x to y or y to x.

7.11. EXAMPLE. Let $\mathbb{H}=[0,+\infty)\subseteq\mathbb{R}$. Then $\beta\mathbb{H}-\mathbb{H}$ is a connected compact Hausdorff F-space.

PROOF. Clearly, $\beta\mathbb{H}-\mathbb{H}$ is closed in the compact space $\beta\mathbb{H}$, since $\mathbb{H}$ is locally compact.

$\beta\mathbb{H}-\mathbb{H}$ is connected. Assume the contrary. Let U, V be disjoint non-void open in $\beta\mathbb{H}-\mathbb{H}$, $U\cup V=\beta\mathbb{H}-\mathbb{H}$. Define $f\colon \beta\mathbb{H}-\mathbb{H}\to I$ by the rule $f(p)=0$ for $p\in U$, $f(p)=1$ for $p\in V$. Then f is continuous and by the Tietze theorem, there is a continuous extension $\tilde{f}\supseteq f$, $\tilde{f}\colon \beta\mathbb{H}\to I$. Denote $G=\tilde{f}^{-1}[0,1/3)$, $H=\tilde{f}^{-1}(2/3,1]$. The set G is open in $\beta\mathbb{H}$ and $\mathbb{H}$ is dense in $\beta\mathbb{H}$, hence $G\cap\mathbb{H}$ is non-empty.

Moreover, $G\cap\mathbb{H}$ is unbounded in $\mathbb{H}$ since otherwise $\mathrm{cl}_{\mathbb{H}}(G\cap\mathbb{H})$ is compact, thus $\mathrm{cl}_{\beta\mathbb{H}}(G\cap\mathbb{H})\subseteq\mathbb{H}$, but $\mathrm{cl}_{\beta\mathbb{H}}(G\cap\mathbb{H})\supseteq U$.

Similarly, $H\cap\mathbb{H}$ is unbounded in $\mathbb{H}$ and obviously $G\cap H=\emptyset$. Thus, we can choose for each $n\in\omega$ points a_n, $b_n\in\mathbb{H}$ such that $n<a_n<b_n<a_{n+1}$, $a_n\in G\cap\mathbb{H}$, $b_n\in H\cap\mathbb{H}$.

Notice that $\tilde{f}(a_n)<1/3$, $\tilde{f}(b_n)>2/3$, by the continuity of $\tilde{f}$ there is some $c_n\in\mathbb{H}$, $a_n<c_n<b_n$ with $\tilde{f}(c_n)=1/2$. The set $\{c_n\colon n\in\omega\}$ is unbounded and closed in $\mathbb{H}$, so if $p\in\mathrm{cl}_{\beta\mathbb{H}}\{c_n\colon n\in\omega\}-\{c_n\colon n\in\omega\}$, then $p\in\beta\mathbb{H}-\mathbb{H}$.

Since $\tilde{f}$ is continuous, $\tilde{f}(p)=1/2$, but $\tilde{f}(p)=f(p)$, which contradicts our assumption $U\cup V=\beta\mathbb{H}-\mathbb{H}$.

It remains to prove that $\beta\mathbb{H}-\mathbb{H}$ is an F-space. Let us postpone it until after the proof of the forthcoming claim. Have in mind that in the case of the space $\mathbb{H}$, "zero" and "closed" are the same.

Claim. (i) If $F\subseteq C\subseteq\beta\mathbb{H}-\mathbb{H}$, F is a zero set and C is a cozero set in $\beta\mathbb{H}-\mathbb{H}$, then there is a closed set $Z\subseteq\mathbb{H}$ with $F\subseteq\mathrm{cl}_{\beta\mathbb{H}}Z-\mathbb{H}\subseteq C$.

(ii) If M, N are zero sets in $\mathbb{H}$, then $\mathrm{cl}_{\beta\mathbb{H}}N-\mathbb{H}\subseteq\mathrm{cl}_{\beta\mathbb{H}}M$ if and only if $\mathrm{cl}_{\mathbb{H}}(N-M)$ is compact.

For the first half of the claim there is some $f\colon \beta\mathbb{H}-\mathbb{H}\to I$ with $f(x)=0$ for $x\in F$, $f(x)=1$ for $x\in\beta\mathbb{H}-\mathbb{H}-C$ by 3.9. Using the Tietze theorem, take an extension $\tilde{f}\supseteq f$ with $\operatorname{dom}\tilde{f}=\beta\mathbb{H}$. It suffices to set $Z=\tilde{f}^{-1}[0,1/2]\cap\mathbb{H}$.

For the second half of the claim suppose first $\mathrm{cl}_{\mathbb{H}}(N-M)$ is non-compact, i.e. $N-M$ is unbounded. For each $n\in\omega$ pick an $x_n\in N-M$, $x_n>n$. Thus, $\{x_n\colon n\in\omega\}$ is closed and disjoint with M. Then take an arbitrary continuous $f\colon\mathbb{H}\to I$ satisfying $f(x_n)=0$ for all $n\in\omega$, $f(y)=1$ for all $y\in M$. By compactness, there is some $x\in\mathrm{cl}_{\beta\mathbb{H}}\{x_n\colon n\in\omega\}-\mathbb{H}$ and clearly $\beta f(x)=0$ and $x\in\mathrm{cl}_{\beta\mathbb{H}}N-\mathbb{H}$, but $\beta f(y)=1$ for all $y\in\mathrm{cl}_{\beta\mathbb{H}}M$. So $\mathrm{cl}_{\beta\mathbb{H}}N-\mathbb{H}$ is not a subset of $\mathrm{cl}_{\beta\mathbb{H}}M$.

Next, let $N-M$ be bounded. Then $\mathrm{cl}_{\mathbb{H}}(N-M)$ is compact and therefore $\mathrm{cl}_{\mathbb{H}}(N-M)=\mathrm{cl}_{\beta\mathbb{H}}(N-M)\subseteq\mathbb{H}$. Therefore if $p\in\mathrm{cl}_{\beta\mathbb{H}}N-\mathbb{H}$, then $p\notin\mathrm{cl}_{\beta\mathbb{H}}(N-M)$, so $p\in\mathrm{cl}_{\beta\mathbb{H}}(N-\mathrm{cl}_{\beta\mathbb{H}}(N-M))$. Now the obvious inclusion $N-\mathrm{cl}_{\mathbb{H}}(N-M)\subseteq M$ implies $p\in\mathrm{cl}_{\beta\mathbb{H}}M$. The claim is proved.

In order to show that $\beta\mathbb{H}-\mathbb{H}$ is an F-space, let C_0, C_1 be two non-void disjoint cozero sets in $\beta\mathbb{H}-\mathbb{H}$. Our aim is to find two disjoint zero sets $Z_0, Z_1\subseteq\mathbb{H}$ such that $\mathrm{cl}_{\beta\mathbb{H}}Z_0\supseteq C_0$, $\mathrm{cl}_{\beta\mathbb{H}}Z_1\supseteq C_1$. By 3.9, that suffices.

Any cozero set in an arbitrary space is a union of a countable family of zero sets, since $I - \{0\} = \bigcup_{n=1}^{\infty} [1/n, 1]$. So there are zero sets F_n^i in $\beta\mathbb{H} - \mathbb{H}$, $F_n^i \subseteq C_i$ and $\bigcup_{n\in\omega} F_n^i = C_i$. By point (i) of the claim, there are $M_n^i \subseteq \mathbb{H}$ closed in $\mathbb{H}$ satisfying $F_n^i \subseteq \mathrm{cl}_{\beta\mathbb{H}} M_n^i - \mathbb{H} \subseteq C_i$, $i = 0, 1$. Since both C_i are non-void, we are allowed to assume that all M_n^i are unbounded.

By the disjointness of C_0, C_1 and by 4.7, for each $n, m \in \omega$, $M_n^0 \cap M_m^1$ is compact. To find Z_0, Z_1 we can proceed by induction as follows.

There is some $x_0 \in \mathbb{H}$ such that $x_0 > 0$, $x_0 \notin M_0^0 \cup M_0^1$ and $M_0^0 \cap M_0^1 \subseteq [0, x_0)$. If $n < \omega$ and if x_k have been defined for all $k < n$, choose $x_n > x_{n-1}$, $x_n > n$ so that $x_n \notin \bigcup_{k=0}^n M_k^0 \cup \bigcup_{k=0}^n M_k^1$, $\bigcup_{k=0}^n M_k^0 \cap \bigcup_{k=0}^n M_k^1 \subseteq [0, x_n)$. Notice that for some $y \in \mathbb{H}$, $\bigcup_{k=0}^n M_k^0 \cap \bigcup_{k=0}^n M_k^1 \subseteq [0, y)$. Since for arbitrary $t \geq y$, $[t, +\infty)$ is a connected space, it cannot be covered by disjoint closed sets $\bigcup_{k=0}^n M_k^0 \cap [t, +\infty)$, $\bigcup_{k=0}^n M_k^1 \cap [t, +\infty)$. Therefore, x_n can be found.

Now, it remains to define $Z_i = \bigcup \{[x_n, x_{n+1}) \cap \bigcup_{k=0}^n M_k^i \colon n \in \omega\}$ for $i = 0, 1$. We have Z_0, Z_1 closed in $\mathbb{H}$, $Z_0 \cap Z_1 = \emptyset$ and for all $n \in \omega$, $M_n^i - Z_i \subseteq [0, x_n)$, $i = 0, 1$.

Thus, by (ii) of the claim, for each $n \in \omega$ and $i = 0, 1$, we have $\mathrm{cl}_{\beta\mathbb{H}} Z_i \supseteq \mathrm{cl}_{\beta\mathbb{H}} M_n^i - \mathbb{H} \supseteq F_n^i$, consequently $\mathrm{cl}_{\beta\mathbb{H}} Z_i \supseteq C_i$, which was to be proved. □

8. Cardinal invariants

We have already mentioned the weight of a topological space X. From the variety of cardinal invariants, let us briefly discuss the following two. The character of a point x in the topological space X is $\chi(x, X) = \min\{|\mathcal{V}| \colon \mathcal{V}$ is a neighborhood base at $x\}$. The character of a set $Y \subseteq X$ is defined analogously. The tightness of a point x in the space X is defined by

$$t\{x, X) = \min\{\tau\colon \text{for each } A \subseteq X, \text{ if } x \in \mathrm{cl}\, A, \text{ then there is some } B \in [A]^{\leq\tau} \text{ with } x \in \mathrm{cl}\, B\}\,.$$

8.1. ČECH–POSPÍŠIL THEOREM. *Let X be a compact Hausdorff space, let $\omega \leq \tau \leq \chi(x, X)$ for each $x \in X$. Then $|X| \geq 2^\tau$.*

PROOF. Notice that in every normal space, any two distinct non-isolated points can be separated by infinite closed G_δ-sets. Indeed, if U, V are closed disjoint neighborhoods of x and y, then $U \subseteq \bigcap_{n=1}^{\infty} f^{-1}[0, 1/n)$, $V \subseteq \bigcap_{n=1}^{\infty} f^{-1}(1 - 1/n, 1]$, where $f\colon X \to I$ is a separating function.

Denote by $X_\emptyset$ an arbitrary infinite closed G_δ-subset of X. Suppose $\alpha < \tau$ and let X_φ be defined for each $\beta < \alpha$ and $\varphi \in {}^\beta 2$. If $\alpha = \beta + 1$, $\varphi \in {}^\beta 2$, choose distinct $x, y \in X_\varphi$ and separate them by infinite closed G_δ-sets C, D. Then $X_{\varphi\frown\{0\}} = X_\varphi \cap C$, $X_{\varphi\frown\{1\}} = X_\varphi \cap D$.

If $\alpha < \tau$ is limit and $\varphi \in {}^\alpha 2$, let $X_\varphi = \bigcap_{\beta<\alpha} X_{\varphi \restriction \beta}$. Whenever $\alpha < \tau$, α limit and $\varphi \in {}^\alpha 2$, then $\chi(X_\varphi) \leq |\alpha| \cdot \omega < \tau$, thus $|X_\varphi| > 1$ and we may continue. Since for each $f \in {}^\tau 2$ the intersection $\bigcap_{\alpha<\tau} X_{f\restriction\alpha}$ is non-empty by the compactness of X, we have $|X| \geq 2^\tau$. □

8.2. ARHANGEL'SKIĬ THEOREM. *Let X be a compact Hausdorff space, let $\tau \geq \chi(x, X)$ for each $x \in X$. Then $|X| \leq 2^\tau$.*

PROOF. Notice that for each $x \in X$, $t(x, X) \leq \tau$. If $x \in \text{cl } A$, choose a local base $\mathcal{V}$ at x, $|\mathcal{V}| \leq \tau$ and pick a point $t_V \in V \cap A$ for each $V \in \mathcal{V}$. Clearly, for $T = \{t_V: V \in \mathcal{V}\}$ we have $T \in [A]^{\leq\tau}$ and $x \in \text{cl } T$.

Furthermore, if $T \subseteq X$ and $|T| \leq \tau$, then $|\text{cl } T| \leq 2^\tau$. For $x \in \text{cl } T$, let $\mathcal{K}_x = \{V \cap T: V \in \mathcal{V}_x\}$, where $\mathcal{V}_x$ is a local base at x of cardinality at most τ. Then $\mathcal{K}_x \in [\mathcal{P}(T)]^{\leq\tau}$ and if $x \neq y$, then $\mathcal{K}_x \neq \mathcal{K}_y$ because the space X is Hausdorff. Thus, $|\text{cl } T| \leq |[\mathcal{P}(T)]^{\leq\tau}| \leq 2^\tau$.

After these preliminaries, let us proceed by induction as follows. For each $x \in X$ fix a local base $\mathcal{V}_x$ of it of cardinality $\leq\tau$. Choose an arbitrary one-point subset $Y_0 \subseteq X$. For $a < \tau^+$, assume Y_α has been defined and $|Y_\alpha| \leq 2^\tau$. Since $t(x, X) \leq \tau$, $\text{cl } Y_\alpha = \bigcup \{\text{cl } T\colon\ T \in [Y_\alpha]^{\leq\tau}\}$. But if $|T| \leq \tau$, then $|\text{cl } T| \leq 2^\tau$, so $|\text{cl } Y_\alpha| \leq (2^\tau)^\tau \cdot 2^\tau = 2^\tau$. Let $\mathcal{H}_\alpha = \bigcup \{\mathcal{V}_x: x \in \text{cl } Y_\alpha\}$. Then clearly $|\mathcal{H}_\alpha| \leq 2^\tau$ too. For each finite open cover $\mathcal{G}$ of $\text{cl } Y_\alpha$ consisting of members of $\mathcal{H}_\alpha$ choose – if possible – a point $x(\mathcal{G}) \in X - \bigcup \mathcal{G}$ and let $Y_{\alpha+1} = \text{cl } Y_\alpha \cup \{x(\mathcal{G})\colon\ \mathcal{G}$ is a finite cover of $\text{cl } Y_\alpha$, $\mathcal{G} \subseteq \mathcal{H}_\alpha\}$. Since $|\mathcal{H}_\alpha| \leq 2^\tau$, $|[\mathcal{H}_\alpha]^{<\omega}| \leq 2^\tau$ too, hence $|Y_{\alpha+1}| \leq 2^\tau$.

For $\alpha < \tau^+$, α limit, let $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$.

Let $Y = \bigcup_{\alpha<\tau^+} Y_\alpha$; then $|Y| \leq \tau^+ \cdot 2^\tau = 2^\tau$. We claim that $Y = X$, which will complete the proof.

Notice that Y is closed. If $x \in \text{cl } Y$, then there is some $T \in [Y]^{\leq\tau}$ with $x \in \text{cl } T$. But since $|T| \leq \tau$, there is some $\alpha < \tau^+$ with $T \subseteq Y_\alpha$, so $x \in \text{cl } Y_\alpha \subseteq Y_{\alpha+1} \subseteq Y$.

Assume on the contrary that there is some $p \in X - Y$. For each $x \in Y$ choose $V_x \in \mathcal{V}_x$ with $p \notin V_x$ and let $\mathcal{G}$ be a finite cover of Y with $\mathcal{G} \subseteq \{V_x: x \in Y\}$. Then for some $\alpha < \tau^+$, $\mathcal{G} \subseteq \mathcal{H}_\alpha$ and $X - \bigcup \mathcal{G}$ is non-void, since $p \in X - \bigcup \mathcal{G}$. Now $x(\mathcal{G})$ could have been defined, yet $x(\mathcal{G}) \in Y_{\alpha+1} \subseteq Y \subseteq \bigcup \mathcal{G}$, a contradiction. □

8.3. THEOREM. *Each point in $\beta\omega - \omega$ has uncountable tightness.*

PROOF. Let $p \in \beta\omega - \omega$; we have to find a set A such that $p \in \text{cl } A$ but $p \notin \text{cl } C$ whenever C is a countable subset of A.

If p is a P-point, i.e. whenever $\{U_n: n \in \omega\}$ is a family of neighborhoods of p, then p belongs to $\text{int} \bigcap_{n\in\omega} U_n$, choose a point $x(U, V) \in U - V$ for each pair U, V of neighborhoods of p with $U - V \neq \emptyset$. Let $A = \{x(U, V)\colon U, V$ neighborhoods of p, $U - V \neq \emptyset\}$. Then $p \in \text{cl } A$, but if $C \in [A]^\omega$, say $C = \{x(U_n, V_n)\colon n \in \omega\}$, then $p \notin \text{cl } C$ since $p \in \text{int} \bigcap_{n\in\omega} V_n$.

If p is not a P-point, then there is countable family of clopen neighborhoods of p, say $\{U_n: n \in \omega\}$, such that $p \notin \text{int} \bigcap_{n\in\omega} U_n$. Let $A = \text{int} \bigcap_{n\in\omega} U_n$ in this case. Since $\beta\omega - \omega$ is an ω_1-Parovičenko space, $p \in \text{cl } A$, but if $C = \{c_n: n \in \omega\} \subseteq A$, we can choose a clopen set V_n such that $c_n \in V_n \subseteq A$. Then by ω_1-Parovičenko-ness, there is some clopen set $W \subseteq \beta\omega - \omega$ with $V_n \subseteq W \subseteq U_n$ for all $n \in \omega$. Obviously, $\beta\omega - \omega - W$ is a neighborhood of p disjoint from $\{c_n: n \in \omega\}$. □

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Keywords: topology, complete, Boolean algebra, compact, Boolean space, Čech–Stone compactification, strongly zero-dimensional, extremally disconnected, Gleason space, κ-Parovičenko space, F-space, cardinal invariant.

MOS subject classification: primary 54D30; secondary 06E05, 54G05.

Bibliography

Because of the large size of the bibliography, it has been divided into nine parts: (1) a **general** part, containing advanced technical papers on the subject, (2) an **elementary** and historical part, with articles suitable for beginners in the subject, (3) papers involving connections with **functional analysis**, (4) a part with papers about connections with **logic** (Lindenbaum–Tarski algebras, decidability and undecidability questions, independence of some BA results from various axioms of set theory, etc.), (5) **measure algebras**, (6) papers on **recursive BAs**, (7) papers on **set theory** which border on the area of Boolean algebra, (8) papers in **topology** which border on the BA area, and (9) papers on **topological BAs** (BAs with a topology under which the operations are continuous).

We have striven for completeness only in the general bibliography, the one for logic, for recursive BAs, and the topological BA area. Reviews are indicated where available; Mathematical Reviews and Zentralblatt für Mathematik were checked, the former through issue 87e. The bibliography was compiled by the editor with the assistance of Sabine Koppelberg and Karl Schlechta. Mr. Schlechta's work on it was supported by a grant from the Freie Universität Berlin. In addition, the Omega Group bibliography of logic was used, and thanks are due to their generosity in making available a preliminary version of it.

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Index of Notation, Volume 3

This index lists the notation in the order that it is introduced. The page on which the notation is defined is given, and in most cases a brief definition of it is also supplied. Notation which is only in force for a page or two is omitted.

Chapter 19: Superatomic Boolean Algebras

sBA	superatomic BA, 721
$S(B)$	Stone space of B, 722
$\mathrm{At}(B)$	set of atoms of B, 723
J^+	ideal associated with ideal J, 723
$J_\alpha(B)$	αth ideal of B, 723
$\mathrm{ht}(B)$	least α such that $J_\alpha = J_{\alpha+1}$, 723
$\mathrm{wd}_\alpha(B)$	$\lvert\mathrm{At}(B/J_\alpha)\rvert$, 723
$\mathrm{wd}(B)$	$\sup\{\mathrm{wd}_\alpha(B)\colon \alpha < \mathrm{ht}(B)\}$, 723
Φ_B	cardinal sequence of B, 723
$N(B)$	a certain normal subgroup of $\mathrm{Aut}(B)$, 724
$G(B)$	$\mathrm{Aut}(B)/N(B)$, 724
J^*	closure of J under complementation, 727
$\mathbb{P}$	a certain partial order, 728
GR	a certain principle, 734
NDP	a certain principle, 737

Chapter 20: Projective Boolean Algebras

$\chi(p, X)$	character of p in X, 749
$\chi(p)$	character of p, 749
$\mathrm{pr}_c^A(a)$	projection of a from A onto C, 751
$C \leq_{\mathrm{rc}} A$	C is relatively complete in A, 751
$\mathrm{indp}_C^A(x)$	$-(\mathrm{pr}_C^A(x) + \mathrm{pr}_C^A(-x))$, 758
$M_\kappa(A)$	$\{p \in \mathrm{Ult}A\colon \chi(p) < \kappa\}$, 760
U_J	$\{q \in \mathrm{Ult}C\colon q \cap J \neq \emptyset\}$, 763
$\mathrm{ext}_\kappa(C, J)$	κ-extension of (C, J), 766

Chapter 21: Countable Boolean Algebras

$x \dotplus y$	sum of x and y, assumed disjoint, 778
$X \mathbin{\dot\cup} Y$	disjoint union, 779
Y'	set of all accumulation points of Y, 781

Chapter 22: Measure Algebras

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