CHAPTER 13

Endomorphisms of Boolean Algebras

J. Donald MONK
University of Colorado

Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet
© Elsevier Science Publishers B.V., 1989

0. Introduction

We describe in this chapter the main results concerning endomorphisms of Boolean algebras, giving proofs for some of them. We divide the discussion into four parts.

(1) Reconstruction: the very satisfying result here is that if A and B have isomorphic endomorphism semigroups, then A and B are isomorphic.

(2) Number of endomorphisms: it is clear that $|\mathrm{Ult}\, A| \le |\mathrm{End}\, A|$, where End A is the semigroup of endomorphisms of A. The main question here seems to be to construct BAs A with $|A| = |\mathrm{End}\, A|$ of various sizes or with additional properties. We give some easy results, and some questions, along these lines.

(3) Endo-rigid BAs: A BA A is endo-rigid if, roughly speaking, it has no endomorphisms except the obvious ones. We show that there *is* one of power continuum.

(4) Hopfian BAs: a BA A is *hopfian* if every onto endomorphism of A is one-to-one. We show that there is an atomic BA of power continuum, and give some results concerning hopfian BAs of other powers.

1. Reconstruction

The main theorem here, mentioned in the introduction to this article, was proved by SCHEIN [1970]. MAGILL [1970] proved an analogous theorem for Boolean rings, where the endomorphisms preserve Δ and $\cdot$ but not necessarily 1. His proof is topological. MAXSON [1972] proved Magill's theorem in a direct algebraic way. We follow Schein's proof. His result applies to rather general ordered sets, and for BAs gives a result, Theorem 1.2, which is stronger than mere reconstruction. In fact, reconstruction is possible just from two- and four-valued endomorphisms.

A subsemigroup S of End A is *sufficient* if the following conditions hold:

(1) For any two distinct $a, b \in A$ there is a two-valued $f \in \mathrm{End}\, A$ such that $fa \neq fb$.

(2) If $f_1, f_2, f_3, f_4 \in S$ and $f_1 \neq f_2$, and all are two-valued, then there is a $g \in S$ such that $f_1 \circ g = f_3$ and $f_2 \circ g = f_4$.

(3) For each $a \in A$ there is a four-valued $f \in S$ such that f takes on the value a.

Now we first note that End A itself is sufficient:

1.1. LEMMA. *If* $|A| > 2$, *then* End A *is sufficient.*

PROOF. Condition (1) is obvious. Recall that if I and J are distinct maximal ideals on A then $|A/(I \cap J)| = 4$. Hence (3) is clear. Now assume the hypothesis of (2). If $f_3 = f_4$, then we can take $g = f_3$, since $f_1 \circ f_3 = f_3 = f_2 \circ f_3$. So, assume that $f_3 \neq f_4$. Let the kernels of f_1, f_2, f_3, and f_4 be I_1, I_2, I_3, and I_4, respectively. Choose $a \in I_1 \backslash I_2$ and $b \in I_3 \backslash I_4$. Since $|A/(I_3 \cap I_4)| = 4$, there is an endomorphism g of A with range $\{0, a, -a, 1\}$ and kernel $I_3 \cap I_4$ such that $gb = a$. Note that $x \in I_3 \cap I_4 \Rightarrow gx = 0$, $x \in I_3 \backslash I_4 \Rightarrow gx = a$, $x \in I_4 \backslash I_3 \Rightarrow gx = -a$, and $x \in A \backslash (I_3 \cup I_4) \Rightarrow gx = 1$. Hence, it is clear that $f_1 \circ g = f_3$ and $f_2 \circ g = f_4$, as desired. □

1.2. THEOREM. *Let A and B be BAs, and S and T sufficient semigroups of endomorphisms of A and B, respectively. Suppose that p is a homomorphism of S onto T. Then either $|B| \leqq 2$, or $|B| > 2$ and p is an isomorphism induced by an isomorphism of A onto B.*

PROOF. Let S_2 and S_4 be the set of all two-valued and four-valued members of S, respectively, and similarly define T_2 and T_4. Assume that $|B| > 2$. Recall that an element u of a semigroup U is a *right zero* if $v \cdot u = u$ for all $v \in U$. Now we claim

(4) S_2 is the set of all right zeros of S.

For, clearly every member of S_2 is a right zero. Now let f be a right zero of S. Let $g \in S$ be two-valued; there is such a g by (1). Then $g \circ f = f$ and $g \circ f$ is two-valued, so $f \in S_2$. Thus, (4) holds. Similarly

(5) T_2 is the set of all right zeros of T.

Next we claim

(6) $p[S_2] = T_2$.

In fact, since p maps onto T it is clear that pf is a right zero of T for each $f \in S_2$, using (4). Thus $\subseteq$ holds. Now let $g \in T_2$; say $pf = g$. Choose $h \in S_2$, using (1). Then $p(h \circ f) = ph \circ pf = ph \circ g$; $ph \circ g = g$ since g is two-valued, and $h \circ f$ is two-valued, so $g \in p[S_2]$, as desired.

Next, by (1) and $|B| > 2$ we have

(7) $|T_2| \geqq 2$.

(8) $p \restriction S_2$ is one-to-one:

For, suppose f_1, f_2 are distinct members of S_2, and $pf_1 = pf_2$. Let $f_3, f_4 \in S_2$; we show that $pf_3 = pf_4$; by (6) and (7), this is not possible. By (2), choose $g \in S$ so that $f_1 \circ g = f_3$ and $f_2 \circ g = f_4$. Applying p, we do then obtain $pf_3 = pf_4$, as claimed.

(9) $h \in S_4$ iff $|S_2 \circ h| = 2$, for any $h \in S$.

For, let $h \in S_4$; say $\operatorname{ran}(h) = \{0, a, -a, 1\}$. By (1) there exist $f_1, f_2 \in S_2$ with

$f_1a \neq 0$, $f_2a \neq 1$, hence $f_1a \neq f_2a$. Thus, $f_1 \circ h \neq f_2 \circ h$. Clearly, $|S_2 \circ h| \leqq 2$, so $|S_2 \circ h| = 2$. Conversely, suppose $h \in S$ and $|S_2 \circ h| = 2$. Since $S_2 \circ k = \{k\}$ for any two-valued k, it follows that h is not two-valued. Suppose $|\text{ran}(h)| > 4$. Then there exist $0 < a_1 < a_2 < 1$ in $\text{ran}(h)$. By (1), choose two-valued f_1, f_2, and f_3 such that $f_1a_1 \neq f_1 0$, $f_2a_1 \neq f_2a_2$, and $f_3a_2 \neq f_3 1$. Thus $f_1a_1 = 1$, $f_2a_1 = 0$, $f_2a_2 = 1$, and $f_3a_2 = 0$. Hence, $f_1 \circ h$, $f_2 \circ h$, and $f_3 \circ h$ are three distinct members of $S_2 \circ h$, a contradiction. So $h \in S_4$, as desired.

Similarly,

(10) $\quad h \in T_3$ iff $|T_2 \circ h| = 2$, for any $h \in T$.

From (6)–(10), using also $S_2 \circ h \subseteq S_2$ and $T_2 \circ h \subseteq T_2$, we obtain:

(11) $\quad p[S_4] = T_4$,

(12) $\quad p \restriction S_4$ is one-to-one.

For, suppose $f_1, f_2 \in S_4$ and $pf_1 = pf_2$. Then for any $g \in S_2$ we have $p(g \circ f_1) = pg \circ pf_1 = pg \circ pf_2 = p(g \circ f_2)$, hence $g \circ f_1 = g \circ f_2$ by (8). Then from (1) it follows that $f_1 = f_2$.

(13) $\quad$ If $a_1, a_2 \in A$, $0 < a_1 < 1$, $a_2 \notin \{0, a_1, -a_1, 1\}$, then there exist $g_1, g_2 \in S_2$ such that $g_1a_1 = g_2a_1$ and $g_1a_2 \neq g_2a_2$.

To prove (13) we consider three cases.

Case 1. $a_1 < a_2$. Then $a_1 + -a_2 \neq 1$. Choose $g_1, g_2 \in S_2$ so that $g_1a_2 = 0$, $g_2(a_1 + -a_2) = 0$.

Case 2. $a_1 < -a_2$. Similarly.

Case 3. $a_1 \cdot -a_2 \neq 0 \neq a_1 \cdot a_2$. Choose $g_1, g_2 \in S_2$ so that $g_1(a_1 \cdot -a_2) = 1 = g_2(a_1 \cdot a_2)$.

Similarly we have

(14) $\quad$ If $b_1, b_2 \in B$, $0 < b_1 < 1$, $b_2 \notin \{0, b_1, -b_1, 1\}$, then there exist $g_1, g_2 \in T_2$ such that $g_1b_1 = g_2b_1$ and $g_1b_2 \neq g_2b_2$.

(15) $\quad$ If $g \in S_4$, $\text{ran}(g) = \{0, a, -a, 1\}$, $\text{ran}(pg) = \{0, b, -b, 1\}$, and $f_1, f_2 \in S_2$, then $f_1a = f_2a$ iff $(pf_1)b = (pf_2)b$.

For, using (8) we have $f_1a = f_2a$ iff $f_1 \circ g = f_2 \circ g$ iff $p(f_2 \circ g) = p(f_2 \circ g)$ iff $pf_1 \circ pg = pf_2 \circ pg$ iff $(pf_1)b = (pf_2)b$.

(16) $\quad$ If $g_1, g_2 \in S_4$, then $\text{ran}(g_1) = \text{ran}(g_2)$ iff $\text{ran}(pg_1) = \text{ran}(pg_2)$.

For $\Rightarrow$, suppose $\text{ran}(g_1) = \text{ran}(g_2) = \{0, a, -a, 1\}$, $\text{ran}(pg_1) = \{0, b, -b, 1\}$, and $c \in \text{ran}(pg_2)$ with $c \notin \{0, b, -b, 1\}$. By (14), choose $f'_1, f'_2 \in T_2$ such that $f'_1b = f'_2b$ and $f'_1c \neq f'_2c$. Say $pf_1 = f'_1$, $pf_2 = f'_2$, with $f_1, f_2 \in S_2$. Then $f_1a = f_2a$ by (15), and then $f'_1c = f'_2c$ by (15), a contradiction. The direction $\Leftarrow$ is similar.

Now we are ready to define the desired isomorphism l from A onto B. Of course, we let $l0 = 0$ and $l1 = 1$. Now suppose $0 < a < 1$ in A. By (3) there is a four-valued $h \in S_4$ such that $a \in \operatorname{ran}(h)$. By (11) say $\operatorname{ran}(ph) = \{0, b, -b, 1\}$. Let $f \in S_2$ be such that $fa \neq 0$. By (6) we have $(pf)b = 0$ or $(pf)b = 1$, and we set $la = -b$ or $la = b$ in these respective cases. By (15) this definition does not depend on the choice of f, and by (16) it does not depend on the choice of h.

(17) $\quad l(-a) = -la$ for any $a \in A$.

In fact, (17) is obvious for $a \in \{0, 1\}$. Now suppose that $0 < a < 1$. Let $h \in S_4$ with $a \in \operatorname{ran}(h)$, and say $\operatorname{ran}(ph) = \{0, b, -b, 1\}$. Let $f_1, f_2 \in S_2$ with $f_1 a \neq 0$, $f_2(-a) \neq 0$. Then by (15) we have $(pf_1)b \neq (pf_2)b$. It follows that $l(-a) = -la$.

Now l maps onto B: suppose $0 < b < 1$ in B. Choose $f' \in T_4$ such that $b \in \operatorname{ran}(f')$, and choose $f \in S_4$ such that $pf = f'$. Say $\operatorname{ran} f = \{0, a, -a, 1\}$. Then using (17), we have $la = b$ or $l(-a) = b$.

By (16) and (17) it is clear that l is one-to-one.

Now we show that $a_1 \leqq a_2$ iff $la_1 \leqq la_2$, for any $a_1, a_2 \in A$. First suppose that $a_1 \cdot -a_2 \neq 0$. Choose $f \in S_2$ such that $f(a_1 \cdot -a_2) = 1$. Then by the definition of l, $(pf)(la_1) = 1$ and $(pf)(l(-a_2)) = 1$, i.e. by (17) $(pf)(la_2) = 0$. Hence, $la_1 \cdot -la_2 \neq 0$. Conversely, suppose $la_1 \cdot -la_2 \neq 0$. Say $f' \in T_2$ and $f'(la_1 \cdot -la_2) = 1$. Say $f' = pf$ with $f \in S_2$. If $fa_1 = 0$, then $f(-a_1) = 1$ and $(pf)(la_1) = f'la_1 = 1$, so $l(-a_1) = la_1$, contradicting (17). Thus, $fa_1 = 1$. Similarly, $fa_2 = 0$, so $a_1 \cdot -a_2 \neq 0$.

Thus, l is an isomorphism from A onto B. It remains only to show that l induces p, i.e.

(18) $\quad$ for any $f \in S$, and any $a \in A$ we have $(pf)(la) = lfa$.

For, if $a \in \{0, 1\}$, then the conclusion of (18) is clear. Assume that $0 < a < 1$. Let $h \in S_4$ with $a \in \operatorname{ran}(h)$. Suppose $fa = 1$, but $(pf)(la) = 0$. Let $g' \in T_2$; say $g' = pg$ with $g \in S_2$. Then $(p(g \circ f))la = 0$, and $(g \circ f)a = 1$, so by the definition of l, $la = -la$, a contradiction. Thus, $fa = 1$ implies $(pf)(la) = 1$. If $fa = 0$, then $f(-a) = 1$, hence $(pf)(l(-a)) = 1$ and $(pf)(la) = 0$. Now suppose $0 < fa < 1$. Suppose $(pf)(la) \neq lfa$. Then there is a $g \in S_2$ with $(pg)(pf)(la) \neq (pg)(lfa)$.

Case 1. $gfa = 1$. Then by the definition of la, $(p(g \circ f))(la) = 1$, i.e. $(pg)((pf)(la)) = 1$. By the definition of lfa, $(pg)(lfa) = 1$, a contradiction.

Case 2. $gfa = 0$. Then, as in case 1, $(pg)((pf)(l(-a))) = (pg)(lf(-a))$, a contradiction. □

1.3. Corollary. *If A and B are non-trivial and* End $A \cong$ End B, *then* $A \cong B$.

Proof. For any BA C, $|C| \leqq 2$ iff $|\operatorname{End} C| = 1$, so the corollary follows from 1.1 and 1.2. □

Reconstruction does not work for automorphism groups, since, for example, there are rigid BAs of all uncountable cardinalities. Perhaps there is some natural kind of endomorphisms other than the entire semigroup End A or the two- and four-valued endomorphisms which still yields reconstruction. Note that the class of all one-to-one endomorphisms is ruled out by the existence of many mono-rigid BAs; similarly, the class of all onto endomorphisms is ruled out by the existence of many onto-rigid BAs.

2. Number of endomorphisms

Since $|A| \leqq |\mathrm{Ult}\, A| \leqq |\mathrm{End}\, A|$, one of the most natural questions about the number of endomorphisms is: In which cardinalities κ does there exist a BA A with $|A| = |\mathrm{End}\, A| = \kappa$? A complete answer to this question is not known, but we present some simple results concerning it.

2.1. THEOREM. *Suppose L is a complete dense linear ordering of power $\lambda \geqq \omega$, and D is a dense subset of L of power κ, where $\lambda^\kappa = \lambda$. Let A be the interval algebra on L. Then $|A| = |\mathrm{End}\, A| = \lambda$.*

PROOF. Ult A is a linearly-ordered space with a dense subspace of power κ, while $|\mathrm{Ult}\, A| = \lambda$. Hence, there are at most $\lambda^\kappa = \lambda$ continuous functions from Ult A into Ult A, as desired. □

2.2. COROLLARY. *If A is the interval algebra on $\boldsymbol{R}$, then $|A| = |\mathrm{End}\, A| = 2^\omega$.* □

Now recall that if μ is any infinite cardinal and ν is minimum such that $\mu^\nu > \mu$, then there is a complete dense linear ordering L of power μ^ν with a dense subset D of power μ. Namely, we can take

$$L = {}^\nu(\mu+1) \setminus \{f \in {}^\nu(\mu+1) \colon \exists \beta < \nu(f_\beta < \mu \text{ and } \forall \gamma > \beta(f_\gamma = \mu))\}$$

and

$$D = \{f \in {}^\nu(\mu+1) \colon \exists \beta < \nu(f_\beta = \mu \text{ and } \forall \gamma > \beta(f_\gamma = 0))\}$$

under the lexicographic ordering. Hence, we obtain:

2.3. COROLLARY. *If μ is an infinite cardinal and $\forall \nu < \mu(\mu^\nu = \mu)$, then there is a BA A such that $|A| = |\mathrm{End}\, A| = 2^\mu$.* □

2.4. COROLLARY (GCH). *If κ is infinite and regular, then there is a BA A such that $|A| = |\mathrm{End}\, A| = \kappa^+$.* □

We do not know, even under GCH, the situation for κ singular or the successor of a singular cardinal.

The following simple result is relevant when CH is not assumed.

2.5. THEOREM. *If $|A| = \omega$, then $|\mathrm{End}\, A| \geqq 2^\omega$.*

PROOF. If A has an atomless subalgebra, clearly $|\mathrm{End}\, A| \geqq |\mathrm{Ult}\, A| \geqq 2^{\omega}$. Suppose A is superatomic. Then there is a homomorphism f from A onto B, the finite–cofinite algebra on ω: if a is an atom of $A/\langle \mathrm{At}\, A\rangle^{\mathrm{id}}$, then f can be taken to be the composition of the natural mappings

$$A \twoheadrightarrow A \restriction a \rightarrowtail\!\!\!\!\twoheadrightarrow C \twoheadrightarrow B\,,$$

where C is the finite–cofinite algebra on ω or ω_1. ("$\twoheadrightarrow$" means "onto", and "$\rightarrowtail\!\!\!\!\twoheadrightarrow$" means "one-to-one and onto".) There is an isomorphism g of B into A. If X is any subset of ω with $\omega \backslash X$ infinite, then $B/\langle\{i\}: i \in X\rangle^{\mathrm{id}}$ is isomorphic to B, and so there is an endomorphism k_X of B with kernel $\langle\{i\}: i \in X\rangle^{\mathrm{id}}$. Clearly, the endomorphisms $g \circ k_X \circ f$ of A are distinct for distinct X's. □

2.6. COROLLARY ($\omega_1 < 2^{\omega}$). *There is no BA A with* $|A| = |\mathrm{End}\, A| = \omega_1$. □

Another unresolved question concerning $|\mathrm{End}\, A|$ is whether $|\mathrm{End}\, A| \leqq |\mathrm{Sub}\, A|$ for all infinite A, where $\mathrm{Sub}\, A$ is the set of all subalgebras of A.

3. Endo-rigid algebras

Recall that a BA is *rigid* if it has no automorphisms except the identity. In general algebra a stronger rigidity has been extensively studied: an algebra is *strongly rigid* if it has no endomorphism except the identity. Now a BA with more than two elements is not strongly rigid in this sense. For, if $|A| > 2$ and F is an ultrafilter on A, then $|A/F| = 2$, and there is a natural embedding of A/F into A. Then $A \to A/F \to A$ gives a non-trivial endomorphism of A. Using this simple fact, one can build more complicated endomorphisms. For example, if $\langle a, b, c\rangle$ is a partition of unity in A, F is an ultrafilter on $A \restriction b$, and G is an ultrafilter on $A \restriction c$, then the composition of natural maps,

$$\begin{aligned} A &\to (A \restriction a) \times (A \restriction b) \times (A \restriction c) \\ &\to (A \restriction a) \times [(A \restriction b)/F] \times [(A \restriction c)/G] \\ &\to (A \restriction a) \times [(A \restriction c)/F] \times [(A \restriction b)/G] \\ &\to (A \restriction a) \times (A \restriction b) \times (A \restriction c) \\ &\to A\,, \end{aligned}$$

gives an endomorphism f of A defined by

$$fx = \begin{cases} x \cdot a + b + c \text{ if } x \cdot b \in F \text{ and } x \cdot c \in G\,, \\ x \cdot a + b \text{ if } x \cdot b \notin F \text{ and } x \cdot c \in G\,, \\ x \cdot a + c \text{ if } x \cdot b \in F \text{ and } x \cdot c \notin G\,, \\ x \cdot a \text{ if } x \cdot b \notin F \text{ and } x \cdot c \notin G\,. \end{cases}$$

It is possible to describe the most general kind of endomorphism constructible from ultrafilters in the above way; we do this below. A BA A is called *endo-rigid* if it has only endomorphisms of this sort. One can give a definition of endo-rigid which is not so complicated, and we now do this. Only this simpler definition is actually used later.

If f is an endomorphism of A, then the *extended kernel of* f is

$$\operatorname{exker} f = \{a + b : fa = 0 \text{ and } fx = x \text{ for all } x \leqq b\} .$$

Clearly exker f is an ideal of A.

Two ideals I and J of A are *complementary* if $I \cap J = \{0\}$, while $\langle I \cap J\rangle^{\mathrm{id}}$ is a maximal ideal of A.

We call A *endo-rigid if* it satisfies the following three conditions:

(19) A is atomless ,

(20) for every endomorphism f of A, $A/\operatorname{exker} f$ is finite ,

(21) A does not have any pair of non-principal complementary ideals .

Endo-rigid BAs are rigid in the usual sense, and possess even stronger rigidity properties, as we shall see. This notion is due to SHELAH [1979], where their existence, assuming CH, is proved. This assumption was removed in MONK [1980]. The strongest result is in SHELAH [1984]: for any $\lambda > \omega$ there is an endo-rigid ccc BA of power λ^{ω}. Our notation here differs from that in these articles.

Now we want to give the equivalent definition involving endomorphisms obtained from ultrafilters. An *endomorphism schema* for A is a sequence

$$\langle a_0, a_1, b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}, b_0^*, \ldots, b_{m-1}^*, c_0^*, \ldots, c_{n-1}^*,$$
$$I_0, \ldots, I_{m-1}, J_0, \ldots, J_{n-1}\rangle$$

such that the following conditions hold:

(22) $a_0, a_1, b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}$ are pairwise disjoint elements with sum 1; $b_i \neq 0 \neq c_j$ for all $i < m$, $j < n$,

(23) $b_0^*, \ldots, b_{m-1}^*, c_0^*, \ldots, c_{n-1}^*$ are pairwise disjoint non-zero elements with sum $a_0 + b_0 + \cdots + b_{m-1}$,

(24) for all $i < m$, I_i is a maximal ideal in $A \restriction b_i$,

(25) for all $i < n$, J_i is a maximal ideal in $A \restriction c_i$.

3.1. LEMMA. *Given an endomorphism schema as above, there is a unique endomorphism f of A with the following properties*:

(i) *For all* $x \leqq a_0$, $fx = 0$.
(ii) *For all* $i < m$ *and all* $x \in I_i$, $fx = 0$.
(iii) *For all* $x \leqq a_1$, $fx = x$.
(iv) *For all* $j < n$ *and all* $x \in J_j$, $fx = x$.
(v) *For all* $i < m$, $fb_i = b_i^*$.
(vi) *For all* $j < n$, $fc_j = c_j + c_j^*$.

PROOF. The following composition of homomorphisms clearly gives an endomorphism satisfying (i)–(vi):

$$\begin{aligned} A &\to (A \restriction a_0) \times (A \restriction a_1) \times \prod_{i<m} (A \restriction b_i) \times \prod_{j<n} (A \restriction c_j) \\ &\to (A \restriction a_1) \times \prod_{i<m} [(A \restriction b_i)/I_i] \times \prod_{j<n} (A \restriction c_j) \times \prod_{j<n} [(A \restriction c_j)/J_j] \\ &\to (A \restriction a_1) \times \prod_{i<m} (A \restriction b_i^*) \times \prod_{j<n} (A \restriction c_j) \times \prod_{j<n} (A \restriction c_j^*) \\ &\to A . \end{aligned}$$

Note here that $A \restriction c_j \to (A \restriction c_j) \times [(A \restriction c_j)/J_j]$ via the mapping sending x to $(x, x/J_j)$.

For uniqueness, note that if f is an endomorphism of A satisfying (i)–(vi), then f is uniquely determined on each factor $A \restriction a_0, A \restriction a_1, A \restriction b_0, \ldots, A \restriction b_{m-1}$, and $A \restriction c_0, \ldots, A \restriction c_{n-1}$, hence f itself is uniquely determined. In fact, $fx = 0$ for all $x \leqq a_0$, and $fx = x$ for all $x \leqq a_1$. Suppose $i < m$ and $x \leqq b_i$; if $x \in I_i$ then $fx = 0$, while if $x \not\in I$, then $b_i \cdot -x \in I_i$, hence $f(b_i \cdot -x) = 0$, and $fx = f(b_i \cdot -(b_i \cdot -x)) = fb_i = b_i^*$. Finally, suppose $j < n$ and $x \leqq c_j$. If $x \in J_j$, then $fx = x$, while if $x \not\in J_j$, then $c_j \cdot -x \in J_j$, hence $f(c_j \cdot -x) = c_j \cdot -x$, and

$$fx = f(c_j \cdot -(c_j \cdot -x)) = (c_j + c_j^*) \cdot -(c_j \cdot -x) = x + c_j^* . \quad \square$$

If f is as described in Lemma 3.1, we say that f is *determined* by the given schema.

3.2. THEOREM. *For any atomless BA A, A is endo-rigid* iff *every endomorphism of A is determined by some endomorphism schema.*

PROOF. $\Rightarrow$ Let f be any endomorphism of A. First suppose exker $f = A$. Then we can write $1 = a + b$ with $fa = 0$ and $fx = x$ for all $x \leqq b$. Thus, $a \cdot b = 0$, so $b = -a$. Hence, $b = fb = f(-a) = -fa = 1$, so $a = 0$. Thus, f is the identity on A, determined by the endomorphism schema $\langle 0, 1 \rangle$.

If exker $f \neq A$, the assumption that A is endo-rigid yields that $A/\text{exker } f$ is a finite non-trivial BA. Hence, there is a partition $\langle x_0, \ldots, x_{k-1} \rangle$ of A such that $x_0/\text{exker } f, \ldots, x_{k-1}/\text{exker } f$ are the atoms of $A/\text{exker } f$. Fix $l < k$. We show how x_l yields finitely many parts of our desired endomorphism schema. Let

$$I_0 = \{ y \leqq x_l \colon fy = 0 \} ,$$

$$I_1 = \{ y \leqq x_l \colon fz = z \text{ for all } z \leqq y \} ,$$

$$J = (\text{exker } f) \cap (A \restriction x_l) .$$

Then $I_0 \cap I_1 = \{0\}$, and $I_0 \cup I_1$ generates the maximal ideal J in $A \restriction x_l$. Thus, I_0 and I_1 are complementary ideals in $A \restriction x_l$. If I_0 and I_1 are non-principal, then I_0 and $\{a \in A: a \cdot x_l \in I_1\}$ are non-principal complementary ideals in A, contradicting A endo-rigid. Thus, I_0 is principal, or I_1 is. If they are both principal, then J is also, and so A has an atom, a contradiction. Thus, exactly one of I_0, I_1 is principal. We treat two cases separately.

Case 1. I_1 is principal, say generated by e. Since $x_l \not\in \text{exker } f$, we have $x_l \cdot -e \neq 0$. We let e be a part of a_1, $x_l \cdot -e$ be one of the b_i, $b_i^* = fb_i$, and $I_i = (\text{exker } f) \cap (A \restriction (x_l \cdot -e))$. Note that I_i is a maximal ideal in $A \restriction (x_l \cdot -e)$ and $fy = 0$ for all $y \in I_i$.

Case 2. I_0 is principal, say generated by e. We let e be a part of a_0. Clearly, now

(26) for all $y \leqq x_l \cdot -e$, if $y \in \text{exker } f$, then $y \in I_1$,

(27) $x_l \cdot -e \leqq f(x_l \cdot -e)$.

For, otherwise let $y = x_l \cdot -e \cdot -f(x_l \cdot -e)$; thus $y \neq 0$. Then $y \cdot fy = 0$, so $y \not\in I_1$. Hence, $y \not\in \text{exker } f$ by (26). Therefore $x_l \cdot -y \in \text{exker } f$, so $x_l \cdot -y \cdot -e \in I_1$ by (26). Since I_1 is non-principal, we can choose z with $x_l \cdot -y \cdot -e < z \in I_1$. Then $z \leqq x_l \cdot -e$ and $fz = z$, so

$$z \leqq x_l \cdot -e \cdot f(x_l \cdot -e) = x_l \cdot -e \cdot -y ,$$

a contradiction. So (27) holds.

(28) $x_l \cdot -e \neq 0$.

This is true since $x_l \not\in \text{exker } f$.

(29) $f(x_l \cdot -e) \cdot -(x_l \cdot -e) \neq 0$.

For, suppose (29) fails. Thus, by (27) we have $f(x_l \cdot -e) = x_l \cdot -e$. Now since $x_l \not\in \text{exker } f$, we have $x_l \cdot -e \not\in I_1$, and so there is a $y \leqq x_l \cdot -e$ with $fy \neq y$. Then $y \not\in I_1$, so $y \not\in \text{exker } f$ by (26). Hence, $x_l \cdot -e \cdot -y \in \text{exker } f$, hence $x_l \cdot -e \cdot -y \in I_1$ by (26). But then

$$\begin{aligned} fy &= f(x_l \cdot -e \cdot -(x_l \cdot -e \cdot -y)) \\ &= x_l \cdot -e \cdot -(x_l \cdot -e \cdot -y) \\ &= y , \end{aligned}$$

a contradiction.

Now we let $x_l \cdot -e$ be one of the c_j, with $f(x_l \cdot -e) \cdot -(x_l \cdot -e)$ the corresponding c_j^*. Furthermore, $J_j = (\text{exker } f) \cap (A \restriction (x_l \cdot -e))$.

It is clear that we obtain in this way an endomorphism schema which determines f.

$\Leftarrow$ We assume that f is determined by an endomorphism schema as in (22)–(25)

and Lemma 3.1. Then $A/\text{exker}\, f$ is a BA with atoms among $b_0/\text{exker}\, f, \ldots, b_{m-1}/\text{exker}\, f, c_0/\text{exker}\, f, \ldots, c_{n-1}/\text{exker}\, f$. Thus, it is finite, and (20) holds. To show (21), suppose on the contrary that I and J are non-principal complementary ideals. It is easily checked that A/J is isomorphic to the subalgebra $I \cup -I$ of A: if $fx = x/J$ for all $x \in I \cup -I$, then f is one-to-one since $x \in J$ combined with $x \in I$ gives $x = 0$, while combined with $x \in -I$ yields $1 \in \langle I \cup J \rangle^{\text{id}}$, a contradiction; and f maps onto A/J since it maps onto $\langle I \cup J \rangle^{\text{id}}/J$. Thus, we can consider the following endomorphism f of A:

$$A \to A/J \to I \cup -I \to A\,,$$

with all mappings natural. Assume that f is determined by an endomorphism schema as in Lemma 3.1. Note that $fx = 0$ for all $x \in J$ and $fx = x$ for all $x \in I$. Let $K = \langle I \cup J \rangle^{\text{id}}$; so K is a maximal ideal.

(30) For all $i < m$, $b_i \notin K$.

For otherwise, write $b_i = d + e$ with $d \in I$ and $e \in J$. Then $b_i^* = fb_i = d \leqq b_i$ and $d \neq 0$ since $b_i^* \neq 0$. For each $x \leqq d$ we have $x \in I$ and hence $fx = x$. But $A \restriction b_i$ atomless implies that I_i is non-principal and hence there is an $x \leqq d$ with $0 \neq x \in I_i$. Then $x = fx = 0$ by 3.1(ii), a contradiction.

(31) for all $j < n$, $c_j \notin K$.

For, otherwise, write $c_j = d + e$ with $d \in I$, $e \in J$. Then $fc_j = d \leqq c_j$, so $c_j^* = 0$, a contradiction.

Suppose $m > 0$. Then by (30) and (31), $m = 1$ and $n = 0$. Now $-b_0 \in K$, so we can write $-b_0 = d + e$ with $d \in I$ and $e \in J$. Hence, $e = a_0$, $d = a_1$, and $b_0^* = a_0 + b_0$. Since I is non-principal, choose d' with $d < d' \in I$. Then $0 \neq d' \cdot -d \leqq b_0$ so, since I_0 is a non-principal ideal of $A \restriction b_0$ ($A \restriction b_0$ being atomless), we can choose $u \in I_0$ such that $0 \neq u \leqq d' \cdot -d$. Then $0 = fu = u$, a contradiction. Thus, $m = 0$.

Similarly, $n = 0$. But clearly, $a_0 \in J$ and $a_1 \in I$, so $1 = a_0 + a_1 \in K$, a contradiction. This completes the proof of Theorem 3.2. □

Now we show that every endo-rigid BA is very rigid in the usual sense. Recall that a BA A is *onto-rigid* if every onto endomorphism is the identity, and *mono-rigid* if every one-to-one endomorphism is the identity.

3.3. THEOREM. *Every endo-rigid BA is onto-rigid and mono-rigid. Moreover, if A is endo-rigid, $a, b \in A$, f is an endomorphism of A, and $f \restriction (A \restriction a)$ embeds $A \restriction a$ into $A \restriction c$, then $x \leqq fx$ for all $x \leqq a$, and if $a = c$, then f is the identity on $A \restriction a$.*

PROOF. We prove the "moreover" part first: assume its hypothesis. If $x \leqq a$ and $x \nleqq fx$, let $d = x \cdot -fx$. Thus, $d \cdot fd = 0$, and $d \neq 0$. Clearly, if $u, v \leqq d$ and $u \neq v$, then $u \triangle v \notin \text{exker}\, f$, so $A/\text{exker}\, f$ is infinite, a contradiction. Suppose, additionally, that $a = c$. If $x \leqq a$ and $fx \not\leqq x$, let $d = fx \cdot -x$. Then the same argument gives a contradiction. Thus, $fx = x$ for all $x \leqq a$, as desired.

Taking $a = c = 1$, we get the mono-rigidity of A.

Now suppose f is an onto endomorphism of A, f not the identity. By the above, f is not one-to-one, so choose $a \neq 0$ such that $a \in \ker f$. For each $b \leqq a$ choose c_b so that $fc_b = b$. Then for distinct $b, d \leqq a$ we have $c_b \triangle c_d \not\in \operatorname{exker} f$. (This gives a contradiction as above.) For, suppose this fails for certain $b, d \leqq a$. Say $b \cdot -d \neq 0$. Since $c_b \cdot -c_d \in \operatorname{exker} f$, write $c_b \cdot -c_d = u + v$ with $u \in \ker f$ and $fx = x$ for all $x \leqq v$. Applying f, we get $v = b \cdot -da$, so $v \in \ker f$ and hence $v = 0$. Therefore $b \cdot -d = 0$, a contradiction. □

On the other hand, there is no relationship between endo-rigid and Bonnet-rigid BAs. Recall that A is Bonnet-rigid provided that if $f\colon A \rightarrowtail B$ and $g\colon A \twoheadrightarrow B$, then $f = g$. (Recall that $\rightarrowtail$ means one-to-one.) The endo-rigid BA we construct is not Bonnet-rigid (see below). SHELAH [1983] has shown assuming $\diamond$ that there is an endo-rigid, Bonnet-rigid BA of power ω_1. There exist Bonnet-rigid interval algebras; according to the following simple proposition they are not endo-rigid.

3.4. PROPOSITION. *If A is an interval algebra, then A is not endo-rigid.*

PROOF. Let A be the interval algebra on L. We may assume that L is infinite, and in fact that in L there is a strictly increasing sequence $a_0 < a_1 < a_2 < \cdots$. Then it is easily seen that there is an endomorphism f of A such that for any $x \in L$,

$$f[0, x) = \begin{cases} 0 & \text{if } x < a_0\,, \\ [0, a_i) & \text{if } a_i \leqq x < a_{i+1}\,, \\ 1 & \text{if } a_i < x \text{ for all } i\,. \end{cases}$$

Now it is easy to check that $\langle [a_i, a_{i+1})/\operatorname{exker} f\colon i < \omega \rangle$ is a system of distinct elements of $A/\operatorname{exker} f$, so A is not endo-rigid. (These are actually atoms of $A/\operatorname{exker} f$.) □

Before turning to the construction of an endo-rigid BA we give the following lemma concerning endomorphisms in general.

3.5. LEMMA. *If f is an endomorphism of a BA A, $a \in A$, and $fx \leqq x$ for all $x \leqq a$, then $a \in \operatorname{exker} f$.*

PROOF. The conclusion will be immediate from the following statements (32)–(34):

(32) if $x \leqq a$, then $x \cdot -fx \in \ker f$.

For, $f(x \cdot -fx) \leqq x \cdot -fx$ by assumption, and clearly $f(x \cdot -fx) \leqq fx$, so $f(x \cdot -fx) = 0$.

(33) If $x \leqq a$, then $fx = ffx$.

In fact, $fx \leqq ffx$ by (32), and $fx \leqq x \leqq a$, hence $ffx \leqq fx$ by assumption, so $fx = ffx$.

(34) If $x \leqq fa$, then $fx = x$.

For, assume that $x \leqq fa$. Now $fa = ffa$ by (33), $x \cdot -fx \leqq x \leqq fa$, and $fa \cdot -(x \cdot -fx) \leqq fa \leqq a$ by assumption. Hence,

$$\begin{aligned} fa = ffa &= f[fa \cdot -(x \cdot -fx) + x \cdot -fx] \\ &= f[fa \cdot -(x \cdot -fx)] + f(x \cdot -fx) \\ &= f[fa \cdot -(x \cdot -fx)] \text{ by (32)} \\ &\leqq fa \cdot -(x \cdot -fx) \text{ by assumption} \\ &\leqq -(x \cdot -fx) . \end{aligned}$$

But $x \cdot -fx \leqq x \leqq fa$, so $x \cdot -fx = 0$. Hence, $x \leqq fx \leqq x$ by assumption, so $fx = x$, as desired.

By (32) and (34), $a = a \cdot -fa + fa \in \operatorname{exker} f$, as desired in the lemma. □

Now to carry out the construction of an endo-rigid BA we need an auxiliary notion which is of independent interest. Let A be a BA and v a formal variable. Suppose $a \in {}^{\omega}A$ and $S \subseteq \omega$. Then by $[a_i, v]^{\text{if } i \in S}$ we mean the formula $a_i \leqq v$ if $i \in S$ and $a_i \cdot v = 0$ if $i \not\in S$. Of course we can define $[a_i, \tau]^{\text{if } i \in S}$ similarly for a more complicated term τ. A *standard type over* A is a set of the form:

$$\{[a_i, v]^{\text{if } i \in S} : i < \omega\} ,$$

where $\langle a_i : i \in \omega \rangle$ is a disjoint system of elements of A^+ and $S \subseteq \omega$. A *candidate over* A is a system $\langle (a_i, b_i) : i < \omega \rangle$ such that $\langle a_i : i < \omega \rangle$ and $\langle b_i : i < \omega \rangle$ are disjoint systems of elements of A^+ and for all $i < \omega$, $b_i \not\leqq a_i$. Finally, we call A *complicated* if, for every such candidate, there is an $S \subseteq \omega$ such that $\{[a_i, v]^{\text{if } i \in S} : i \in \omega\}$ is realized in A but $\{[b_i, v]^{\text{if } i \in S} : i \in \omega\}$ is omitted in A (i.e. there is an element $x \in A$ such that $a_i \leqq x$ for all $i \in S$ and $a_i \cdot x = 0$ for all $i \in \omega$ but there is no corresponding element for the b_i's).

3.6. LEMMA. *If A is complicated and f is an endomorphism of A, then $A/\operatorname{exker} f$ is finite.*

PROOF. Assume that $A/\operatorname{exker} f$ is infinite. Then there is a disjoint system $\langle a_n' : n \in \omega \rangle$ in A such that $a_n'/\operatorname{exker} f \neq 0$ for all $n \in \omega$. By Lemma 3.5, for every $n \in \omega$ there is an $a_n \leqq a_n'$ such that $fa_n \not\leqq a_n$. Thus, $\langle (a_n, fa_n) : n < \omega \rangle$ is a candidate over A, so we can choose $S \subseteq \omega$ so that $\{[a_n, v]^{\text{if } n \in S} : n \in \omega\}$ is realized in A, say by c, but $\{[fa_n, v]^{\text{if } n \in S} : n \in \omega\}$ is not. But clearly fc realizes this last type, a contradiction. □

Now we are ready for the main theorem concerning endo-rigid BAs.

3.7. THEOREM. *There is an endo-rigid BA of power 2^ω.*

PROOF. Let A be the free BA on 2^ω free generators and let $\bar{A}$ be its completion. Note that $|\bar{A}| = 2^\omega$. We may assume that $\bar{A} \subseteq 2^\omega$ as a set. Recall that $\bar{A}$ satisfies ccc. Let $\langle \langle (a_n^\alpha, b_n^\alpha) : n \in \omega \rangle : \alpha < 2^\omega \rangle$ list all members of ${}^\omega(2^\omega \times 2^\omega)$, each member repeated 2^ω times. Now we construct by induction two sequences $\langle B_\alpha : \alpha \leqq 2^\omega \rangle$ and $\langle Q_\alpha : \alpha \leqq 2^\omega \rangle$ such that, for all $\alpha, \beta < 2^\omega$.

(35) B_α is a subalgebra of $\bar{A}$, $|B_\alpha| \leqq |\alpha| + \omega$, and $\alpha < \beta$ implies $B_\alpha \subseteq B_\beta$,

(36) Q_α is a collection of standard types over B_α each omitted in B_α, $\alpha < \beta$ implies $Q_\alpha \subseteq Q_\beta$, and $|Q_\alpha| \leqq |\alpha| + \omega$.

We let B_0 be a denumerable atomless subalgebra of $\bar{A}$, and $Q_0 = 0$. For λ a limit ordinal $\leqq 2^\omega$ we let $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ and $Q_\alpha = \bigcup_{\alpha<\lambda} Q_\alpha$. The essential step is the successor step. So assume $\alpha < 2^\omega$, and B_α and Q_α have been defined satisfying (35) and (36). The constrution of $B_{\alpha+1}$ and $Q_{\alpha+1}$ takes two steps.

First we take care of a candidate, forming B_α' and Q_α'. If $\langle (a_n^\alpha, b_n^\alpha) : n < \omega \rangle$ is *not* a candidate over B_α, let $B_\alpha' = B_\alpha$, $Q_\alpha' = Q_\alpha$. So assume it is. For the rather lengthy considerations which follow we write a_n and b_n instead of a_n^α and b_n^α. First extend $\langle a_n : n < \omega \rangle$ to a partition $\langle a_\nu : \nu < \beta \rangle$, where β is a countable ordinal $\geqq \omega$. For each $S \subseteq \omega$ let $cS = \Sigma_{\alpha \in S} a_\nu$ (the sum in $\bar{A}$), and set $CS = \langle B_\alpha \cup \{cS\} \rangle$. We want to find S so tht

(37) CS omits $\{[b_n, v]^{\text{if } n \in S} : n \in \omega\}$,

(38) CS omits each member of Q_α.

Since CS obviously realizes $\{[a_n, v]^{\text{if } n \in S} : n < \omega\}$, this will take care of the current candidate. For each $n < \omega$ we have $b_n \not\leqq a_n$, and so there is a $\nu n < \beta$ such that $a_{\nu n} \cdot b_n \cdot -a_n \neq 0$ and hence $a_{\nu n} \cdot b_n \neq 0$ and $\nu n \neq n$. Now we claim:

(39) there is an $S^* \subseteq \omega$, S^* infinite, such that $\nu n \not\in S^*$ for all $n \in S^*$.

In fact, if $\omega \cap \operatorname{ran} \nu$ is finite, we can take $S^* = \omega \backslash \operatorname{ran} \nu$. Otherwise, by induction choose m_i for each $i < \omega$ such that both m_i and νm_i are members of $\omega \backslash (\{m_j : j < i\} \cup \{\nu m_j : j < i\})$. Clearly, $S^* = \{m_i : i < \omega\}$ is as desired in (39).

Now the following fact will enable us to take care of (37):

(40) if d, e and g are pairwise disjoint elements of B_α such that $d + e \cdot cS + g \cdot -cS$ realizes $\{[b_n, v]^{\text{if } n \in S} : n \in \omega\}$ in CS, and $S \subseteq S^*$, then $S = \{n \in S^* : b_n \cdot a_{\nu n} \leqq d + g\}$.

For, assume the hypothesis of (40). For any $n \in S^*$ we have $\nu n \not\in S^*$, hence $\nu n \not\in S$ and so $a_{\nu n} \cdot cS = 0$. Also, if $n \in S$, then $b_n \leqq d + e \cdot cS + g \cdot -cS$, so $b_n \cdot a_{\nu n} \leqq d + g$. If $n \in S^* \backslash S$, then $b_n \cdot (d + e \cdot cS + g \cdot -cS) = 0$, so

$$b_n \cdot a_{\nu n} \cdot (d + g) = b_n \cdot a_{\nu n} \cdot (d + g \cdot -cS) = 0 .$$

Thus (40) holds.

It is harder to take care of (38). To do so we shall use a combinatorial trick. Let K be a family of 2^ω infinite pairwise almost disjoint subsets of S^*. Now the following statement enables us to take care of (38):

(41) if d, e and g are pairwise disjoint elements of B_α, $q \in Q_\alpha$, $q = \{[h_n, v]^{\text{if } n\in T}: n \in \omega\}$, then there is at most one $S \in K$ such that $d + e \cdot cS + g \cdot -cS$ realizes q in CS.

To prove this, assume its hypotheses. Let

$$k = -d \cdot -e \cdot -g\,,$$

$$l = \sum_{n\in T} h_n \cdot (k + e) + \sum_{n\in\omega\setminus T} h_n \cdot (d + g)\,,$$

$$m = \sum_{n\in T} h_n \cdot (k + g) + \sum_{n\in\omega\setminus T} h_n \cdot (d + e)\,.$$

Then:

(42) if $B_\alpha \subseteq D \subseteq \bar{A}$ and $u \in D$, then $\{[h_n, d + e \cdot v + g \cdot -v]^{\text{if } n\in T}: n \in \omega\}$ is realized by u in D iff $l \leqq u$ and $m \cdot u = 0$.

In fact, for $\Rightarrow$ we have $h_n \leqq d + e \cdot u + g \cdot -u$ for $n \in T$, hence $h_n \cdot e = e \cdot u \leqq u$ and $h_n \cdot k = 0 \leqq u$. Furthermore, if $n \in \omega\setminus T$, then $h_n \cdot (d + e \cdot u + g \cdot -u) = 0$, hence $h_n \cdot d = 0 \leqq u$ and $h_n \cdot g \cdot -u = 0$ hence $h_n \cdot g \leqq u$. Thus, $l \leqq u$. Similarly, $m \cdot u = 0$. For $\Leftarrow$, note first that $l \cdot m = 0$. Suppose $n \in T$. Now $h_n \cdot k \leqq l \cdot m = 0$. Thus, $h_n \leqq d + e + g$. Furthermore, $h_n \cdot e \leqq l \leqq u$ and $h_n \cdot g \leqq m$, hence $h_n \cdot g \leqq -u$. Thus, $h_n \leqq d + e \cdot u + g \cdot -u$. Suppose $n \not\in T$. Then $h_n \cdot d \leqq l \cdot m$, so $h_n \cdot d = 0$. Also, $h_n \cdot e \leqq m$ so $h_n \cdot e \cdot u = 0$. Finally, $h_n \cdot g \leqq l \leqq u$, so $h_n \cdot g \cdot -u = 0$. Thus, $h_n \cdot (d + e \cdot u + g \cdot -u) = 0$, as desired. So, (42) holds. By (38) for B_α and (42) we obtain:

(43) there is no $u \in B_\alpha$ such that $l \leqq u$ and $m \cdot u = 0$.

Now let $L = \langle \{a_\nu : \nu < \beta\} \rangle^{\text{id}}$, an ideal in $\bar{A}$.

(44) if $l \in L$, then for any $S \in K$, $d + e \cdot cS + g \cdot -cS$ does not realize q in CS. (q is described in (41).)

For, otherwise, by (42) $l \leqq cS$ and $m \cdot cS = 0$. Since $l \in L$, there is a finite sum u of members of $\{a_\nu: \nu < \beta\}$ such that $l \leqq u$. Hence, $l \leqq u \cdot cS$ and $m \cdot u \cdot cS = 0$. But $u \cdot cS$ is a finite join of members of $\{a_n: n \in \omega\} \subseteq B_\alpha$, so $u \cdot cS \in B_\alpha$. This contradicts (43).

Now we show (41). By (44) we may assume that $l \not\in L$. Now suppose that S_1 and S_2 are distinct elements of K such that $d + e \cdot cS_i + g \cdot -cS_i$ realizes q in CS_i for $i = 1, 2$. Then by (42) $l \leqq cS_i$ for $i = 1, 2$, so $l \leqq cS_1 \cdot cS_2 = \Sigma\, \{a_n: n \in S_1 \cap S_2\}$; since $S_1 \cap S_2$ is finite, $l \in L$, a contradiction. Thus, (41) holds.

Since every element of CS has the form $d + e \cdot cS + g \cdot -cS$ for some pairwise disjoint $d, e, g \in B_\alpha$, we see by (40), (41), (35), and (36) that there are at most $|\alpha| + \omega$ subsets $S \in K$ for which (37) or (38) fails. We choose $S \in K$ for which (37) and (38) hold, and let $b'_\alpha = CS$, $Q'_\alpha = Q_\alpha \cup \{[b_n, v]^{\text{if } n \in S}: n < \omega\}$.

The second step in the construction is much shorter, and is intended to secure condition (21) for the final algebra. We need the following fact, which is a special case of a general result. Let A be freely generated by X.

(45) There is an $x \in X$ such that for every $y \in B'^{+}_\alpha$, $x \cdot y \neq 0 \neq -x \cdot y$.

In fact, for each $c \in B'_\alpha$, we can write $c = \Sigma D_c$ (sum in $\bar{A}$), where D_c is a countable subset of A. For each $y \in A$ there is a finite $E_y \subseteq X$ such that $y \in \langle E_y \rangle$. Let

$$F = \bigcup \{E_y : c \in B'_\alpha,\ y \in D_c\}.$$

Then $|F| < 2^\omega$, so there is an $x \in X \backslash F$. Clearly, x is as desired in (45).

We choose x as in (45) and let $B_{\alpha+1} = \langle B'_\alpha \cup \{x\} \rangle$. Let $Q_{\alpha+1}$ be Q'_α together with those of the two types:

$$\{[a^\alpha_n \cdot x, v]^{\text{if } n \text{ even}}: n < \omega\},$$

$$\{[a^\alpha_n \cdot -x, v]^{\text{if } n \text{ odd}}: n < \omega\},$$

which are standard types over $B_{\alpha+1}$ omitted in $B_{\alpha+1}$. We still must check that each member q of Q'_α is omitted in $B_{\alpha+1}$. But if $q = \{h_n, v]^{\text{if } n \in S}: n < \omega\}$ is realized by $c \cdot x + d \cdot -x$ with $c, d \in B'_\alpha$, it is easily checked that c realizes q too (so does d), a contradiction.

This completes the construction. Let $B' = B_\alpha$ with $\alpha = 2^\omega$; we claim that B' is endo-rigid. The second step in the above successor step stage assures us that B' is atomless. Clearly, $|B'| = 2^\omega$. Now any candidate for B' is a candidate for some B_α since $\omega < \mathrm{cf}(2^\omega)$; so we may choose α so that also the given candidate is $\langle (a^\alpha_n, b^\alpha_n): n \in \omega \rangle$ (since each such object was repeated 2^ω times). Then the first step in the $\alpha \to \alpha + 1$ construction "kills" this candidate, assuring us that B' is complicated and hence satisfies (20).

It remains only to check that B' satisfies (21). Suppose, on the contrary, that I_0 and I_1 are non-principal complementary ideals in B'. Note that if $x \in I_i$, then there is a $y \in I_i$ such that $x < y$, hence a $z \in I_i$ such that $x \cdot z = 0$ and $z \neq 0$ (take $z = y \cdot -x$), $i = 0, 1$. Therefore any maximal disjoint set of elements of I_i is infinite, $i = 0, 1$. Let $\langle a_n: n < \omega,\ n \text{ even} \rangle$ be a maximal disjoint set of non-zero elements of I_0, and $\langle a_n: n < \omega,\ n \text{ odd} \rangle$ one for I_1. Choose $\alpha < 2^\omega$ such that $\{a_n: n < \omega\} \subseteq B_\alpha$ and $\langle (a^\alpha_n, b^\alpha_n): n < \omega \rangle = \langle (a_n, a_n): n < \omega \rangle$. Recall that in the step $\alpha \to \alpha + 1$ of the construction we extended B_α to B'_α, then set $B_{\alpha+1} = \langle B'_\alpha \cup \{x\} \rangle$, where $x \cdot y \neq 0 \neq -x \cdot y$ for all $y \in B'^{+}_\alpha$. Now we claim:

(46) $\{[a_n, v]^{\text{if } n \text{ even}}: n < \omega\}$ is omitted in B'_α.

For, suppose $y \in B'_\alpha$ realizes this type. Without loss of generality $y \in \langle I_0 \cup I_1 \rangle^{\mathrm{id}}$ (since $\langle I_0 \cup I_1 \rangle^{\mathrm{id}}$ is a maximal ideal), so write $y = d + e$ with $d \in I_0$, $e \in I$. For n even we have $a_n \leqq y = d + e$, and $a_n \cdot e = 0$, so $a_n \leqq d$. Choose $s \in I_0$ with $d < s$. Then $s \cdot -d \neq 0$, and by the maximality of $\{a_n : n < \omega,\ v \text{ even}\}$ we get an even n with $s \cdot -d \cdot a_n \neq 0$, a contradiction. So (46) holds.

(47) $\{[a_n \cdot x, v]^{\text{if } n \text{ even}} : n < \omega\}$ and $\{|a_n \cdot -x, v]^{\text{ir } n \text{ even}} : n < \omega\}$ are omitted in $B_{\alpha+1}$.

To prove this, by symmetry we take the first type only, and suppose $b \cdot x + c \cdot -x$ realizes it, where b, $c \in B'_\alpha$. Then for n even we have $a_n \cdot x \leqq b \cdot x + c \cdot -x$, so $a_n \leqq b$. For n odd we have $a_n \cdot x \cdot (b \cdot x + c \cdot -x) = 0$, so $a_n \cdot b = 0$. Thus, $b \in B'_\alpha$ realizes $\{[a_n, v]^{\text{if } n \text{ even}} : n < \omega\}$ in B'_α, contradicting (46).

By (47), the types mentioned there are omitted in B' (see the construction). Say without loss of generality $x \in \langle I_0 \cup I_1 \rangle^{\mathrm{id}}$, and write $x = d + e$ with $d \in I_0$, $e \in I_1$. If n is even, then $a_n \cdot x = a_n \cdot d \leqq d$; if n is odd, then $a_n \cdot x = a_n \cdot e$, hence $a_n \cdot x \cdot d = 0$. Thus, d realizes $\{[a_n \cdot x, v]^{\text{if } n \text{ even}} : n < \omega\}$ in B', a contradiction. This finishes the proof of Theorem 3.7. □

As mentioned before, SHELAH [1984] has shown that there is an endo-rigid BA of each infinite cardinality λ such that $\lambda^\omega = \lambda$. On the other hand, MA implies that there is no endo-rigid BA of power $<2^\omega$. It is known that an endo-rigid BA always has at least 2^{ω_1} ultrafilters and hence at least 2^{ω_1} endomorphisms. For these two results see MONK [1980], where there are references to earlier papers. Note that the endo-rigid BA B' constructed above in the proof of Theorem 3.7 has a free subalgebra C of power 2^ω and hence 2^{2^ω} ultrafilters and endomorphisms. It appears to be open to construct an endo-rigid BA A with $|A| = |\mathrm{End}\, A|$.

Completing our earlier discussion (Proposition 3.4 and preceding remarks), we now show that the algebra B' constructed in Theorem 3.7 is not Bonnet-rigid. For, let C be a free subalgebra of B' of size 2^ω. Let h be a homomorphism from C onto B'. Then let $I = \langle \ker h \rangle^{\mathrm{id}}_{B'}$ and let $fhc = c/I$ for any $c \in C$. Clearly, f is a well-defined monomorphism from B' into B'/I. If g is the natural homomorphism from B' onto B'/I, then $f \neq g$, showing that B' is not Bonnet-rigid.

4. Hopfian Boolean algebras

A BA A is *hopfian* (*dual-hopfian*) if A is infinite and every onto (one-to-one) endomorphism is one-to-one (onto). Thus, every onto-rigid (mono-rigid) BA is hopfian (dual-hopfian). The most immediate question which arises is thus whether non-rigid hopfian or dual-hopfian BAs exist. Endo-rigid BAs give immediate answers, by the following two results of SHELAH [1984].

4.1. THEOREM. *If A is endo-rigid, then $A \times A$ is hopfian.*

PROOF. Suppose that f is an onto endomorphism of $A \times A$ which is not one-to-one. Then there is an $(a, b) \in (A \times A)^+$ with $f(a, b) = (0, 0)$. Say without loss of generality $a \neq 0$. Choose (d, e) such that $f(d, e) = (a, 0)$. Say without loss of generality $f(d, 0) \neq (0, 0)$; write $f(d, 0) = (m, 0)$; thus $m \leqq a$. Note that $f(d \cdot -a, 0) = (m, 0)$ also. Now clearly $d \cdot -a \neq 1$. Let I be a maximal ideal in A such that $d \cdot -a \in I$. For any $x \in A$ let $gx = (f(x, x/I))_0$ (first coordinate of $f(x, x/I)$). Thus, g is an endomorphism of A, and $g(d \cdot -a) = m$. For every $y \leqq m$ there is an $n_y \leqq d \cdot -a$ such that $f(n_y, 0) = (y, 0)$. Hence, $gn_y = y$ for any $y \leqq m$. Suppose $y, z \leqq m$ and $y \neq z$, while $n_y \triangle n_z \in \operatorname{exker} g$. Say $n_y \triangle n_z = u + v$ with $u \in \ker g$ and $gw = w$ for all $w \leqq v$. Applying g, $y \triangle z = v$. Now $v \leqq d \cdot -a$, and $v = gv \leqq m$, so $v \in I$. Hence, $v = gv = (f(v, 0))_0 = 0$, a contradiction. Therefore $A/\operatorname{exker} g$ is infinite, a contradiction. □

4.2. THEOREM. *If A is endo-rigid, then $A \times A$ is dual-hopfian.*

PROOF. Let f be a one-to-one endomorphism of $A \times A$. For all $x \in A$ let $gx = (f(x, x))_0$ (first coordinate of $f(x, x)$).

$(*)$ $\quad g$ is one-to-one.

Assume otherwise: say $ga = 0$, $a \neq 0$. Then $f(a, a)$ has the form $(0, b)$. Say $f(a, 0) = (0, c)$. Note that $a \neq 1$, and let I be a maximal ideal such that $a \in I$. For any $x \in A$ let $hx = (f(x, x/I))_1$. Then h is an endomorphism of A, and $h \restriction (A \restriction a)$ embeds $A \restriction a$ into $A \restriction c$. Hence, by Theorem 3.3 we get $a \leqq c$. Similarly, $f(0, a)$ has the form $(0, d)$ with $a \leqq d$. But $(a, 0) \cdot (0, a) = (0, 0)$, so $c \cdot d = 0$. Thus, $0 \neq a \leqq c \cdot d$ is a contradiction. So $(*)$ holds. Hence, g is the identity, by Theorem 3.3. By similar reasoning for the function $x \mapsto (f(x, x))_1$ we obtain:

$(**)$ $\quad f(x, x) = (x, x)$ for all $x \in A$.

Next we claim

$(***)$ $\quad f(1, 0)$ has the form $(b, -b)$ for some $b \in A$.

In fact, write $f(1, 0) = (b, c)$. Then $f(b \cdot c, b \cdot c) = (b \cdot c, b \cdot c) \leqq (b, c) = f(1, 0)$, so $b \cdot c = 0$. Now $f(0, 1) = (-b, -c)$, so $-b \cdot -c = 0$ by the same reasoning. Thus, $c = -b$, as desired in $(***)$.

(★) $\quad$ For all $x \leqq b$, $f(x, 0) = (x, 0)$.

In fact, $f(x, 0) = f((x, x) \cdot (1, 0)) = (x, x) \cdot (b, -b) = (x, 0)$. Similarly,

(★★) $\quad$ For all $y \leqq -b$, $f(0, y) = (y, 0)$.

It follows that every element $(z, 0)$ is in the range of f: $f(z \cdot b, z \cdot -b) = (z, 0)$. By symmetry every element $(0, z)$ is in the range of f, so f is onto, as desired. □

A further question now arises: Are there atomic hopfian or dual-hopfian BAs?

Call an atomic BA A *almost rigid* if every automorphism of A is induced by a finite permutation of its atoms. By modifying the construction of endo-rigid BAs we shall prove the existence of almost rigid hopfian and dual-hopfian BAs. On the other hand, there are no countable hopfian or dual-hopfian BAs. For these results, with different proofs, see LOATS [1979] and LOATS and ROITMAN [1981]

To prove the non-existence of countable hopfian or dual-hopfian BAs, we need the following lemma which is of independent interest.

4.3. LEMMA. *Let A be a denumerable BA with infinitely many atoms. Then there is an $X \in [\mathrm{At}\, A]^\omega$ such that for each $a \in A$, $\mathrm{At}(A \restriction a) \cap X$ is finite or $\mathrm{At}(A \restriction -a) \cap X$ is finite. (For any BA B, At B is the set of atoms of B.)*

PROOF. Let $A = \{a_0, a_1, \ldots\}$. It is easy to construct $\varepsilon \in {}^\omega\{-1, +1\}$ such that for all $n \in \omega$, $\mathrm{At}(A \restriction (\varepsilon_0 a_0 \cdot \cdots \cdot \varepsilon_n a_n))$ is infinite. Choose $x_n \in \mathrm{At}(A \restriction (\varepsilon_0 a_0 \cdot \cdots \cdot \varepsilon_n a_n)) \setminus \{x_i : i < n\}$. Clearly, $X = \{x_n : n \in \omega\}$ is as desired. □

4.4. THEOREM. *Let A be a denumerable BA with infinitely many atoms. Then there is a subalgebra B of A and an isomorphism of the semigroup* End B *into* End A *with the following properties*:

(i) *B is isomorphic to the finite–cofinite algebra on ω,*
(ii) *for each $f \in \mathrm{End}\, B$ we have $f \subseteq f^+$,*
(iii) *for each $f \in \mathrm{End}\, B$, f is one-to-one (onto) iff f^+ is one-to-one (onto).*

PROOF. Choose X as in Lemma 4.3, and let $B = \langle X \rangle$. Clearly, (i) holds. Let $I = \{a \in A : \mathrm{At}(A \restriction a) \cap X \text{ is finite}\}$. Clearly, I is a maximal ideal in A. For each $a \in I$ there is a unique finite $Sa \subseteq X$ and $ta \in A$ such that $a = \Sigma\, Sa + ta$ and $ta \cdot x = 0$ for all $x \in X$. For any $f \in \mathrm{End}\, B$ we then define

$$f^+ a = \sum_{b \in Sa} fb + ta ;$$

for $a \notin I$ we set $f^+ a = -f^+(-a)$. It is routine to check the desired conditions. □

4.5. COROLLARY. *There is no countable hopfian or dual-hopfian BA.*

PROOF. First note that the BA A of finite and cofinite subsets of ω is neither hopfian nor dual-hopfian. For, let $fx = \{n \in \omega : n + 1 \in x\}$ for all $x \in A$; f is an onto endomorphism of A but it is not one-to-one. On the other hand, let $gx = \{n \in \omega : n = 0 \text{ and } 0 \in x, \text{ or } n > 0 \text{ and } n - 1 \in x\}$; g is a one-to-one endomorphism of A but it is not onto.

Thus, by Theorem 4.4 it suffices to show that there is no denumerable hopfian or dual-hopfian BA having only finitely many atoms. Suppose that A is denumerable with only finitely many atoms; we construct $f, g \in \mathrm{End}\, A$ such that f (resp. g) is onto (resp. one-to-one) but not one-to-one (resp., onto). Let $a = \Sigma\, \mathrm{At}\, A$, $b = -a$. Then $A \restriction b$ is isomorphic to the free BA on ω generators; let $\langle x_n : n \in \omega \rangle$ be a system of free generators of $A \restriction b$. Let f' be the endomorphism of $A \restriction b$ such that $f' x_0 = x_0$ and $f' x_n = x_{n-1}$ for $n > 0$, and let g' be the one with $g' x_n = x_{2n}$ for all $n \in \omega$. Then f' is onto but not one-to-one, and g' is one-to-one but not onto. Since $A \cong (A \upharpoonright a) \times (A \upharpoonright b)$, f' and g' induce the desired endomorphisms of A. $\square$

The above results generalize to show that under MA there is no hopfian or dual-hopfian BA of size $<2^\omega$ with infinitely many atoms; see LOATS [1979].

To prove our existence theorem for almost rigid hopfian BAs we extend some of our terminology concerning endo-rigid BAs. We call a BA A *weakly endo-rigid* if (20) holds, i.e. if for every endomorphism f of A, $A/\text{exker } f$ is finite.

4.6. LEMMA. *Let κ be an infinite cardinal. If A is a weakly endo-rigid subalgebra of $\mathcal{P}\kappa$ containing all singletons, then A is almost rigid, hopfian, and dual-hopfian.*

PROOF. Suppose that f is an automorphism of A moving infinitely many atoms. If X is the set of all moved atoms, then clearly $x/\text{exker } f \neq y/\text{exker } f$ for all distinct $x, y \in X$, and hence A is not weakly endo-rigid.

Suppose f is an onto endomorphism which is not one-to-one. Then there is an $a \in A^+$ such that $fa = 0$, hence there is some $\alpha_0 \in \kappa$ such that $f\{\alpha_0\} = 0$. Since f maps onto A, there is a $d_0 \in A$ such that $\alpha_0 \in d_0$, $|d_0| > 1$, and $fd_0 = \{\alpha_0\}$. Choose $\alpha_1 \in d_0 \backslash \{\alpha_0\}$. Then there is a $d_1 \in A$ such that $d_0 \cap d_1 = 0$ and $fd_1 = \{\alpha_1\}$. We continue inductively: if distinct $\alpha_0, \ldots, \alpha_n$, $n \geqq 1$, and pairwise disjoint $d_0, \ldots, d_n$ have been defined such that $fd_i = \{\alpha_i\}$ and $\alpha_{j+1} \in d_j$ for all $i \leqq n$, $j < n$, choose $\alpha_{n+1} \in d_n$ – clearly $\alpha_{n+1} \neq \alpha_0, \ldots, \alpha_n$ – and choose d_{n+1} disjoint from $d_0, \ldots, d_n$ such that $fd_{n+1} = \{\alpha_{n+1}\}$.

Note that $\alpha_{n+1} \in fd_{n+1} \backslash d_{n+1}$ for all $n \in \omega$. Hence, $d_i/\text{exker } f \neq d_j/\text{exker } f$ if $1 \leqq i < j < \omega$, so again A is not weakly endo-rigid.

Finally, suppose that f is one-to-one but not onto. Let $\Gamma = \{\alpha : f\{\alpha\} \neq \{\alpha\}\}$. Then Γ is infinite: assume otherwise. If $\alpha \in \Gamma$ and $\beta \in \kappa \backslash \Gamma$, then $\{\beta\} \cap f\{\alpha\} = f\{\beta\} \cap f\{\alpha\} = 0$, so $\beta \not\in f\{\alpha\}$. Thus, $f\{\alpha\} \subseteq \Gamma$ for all $\alpha \in \Gamma$. Since Γ is finite, this implies that $|f\{\alpha\}| = 1$ for all $\alpha \in \Gamma$. Hence, f is an automorphism, a contradiction. Thus, Γ is infinite. Clearly, $\{\alpha\}/\text{exker } f \neq \{\beta\}/\text{exker } f$ for distinct $\alpha, \beta \in \Gamma$, so A is not weakly endo-rigid. $\square$

4.7. THEOREM. *There is a weakly endo-rigid subalgebra A of $\mathcal{P}\omega$ containing all singletons, with $|A| = 2^\omega$. Thus, A is almost rigid, hopfian, and dual-hopfian.*

PROOF. By Lemmas 3.6 and 4.6 it suffices to construct a complicated subalgebra A of $\mathcal{P}\omega$ containing all singletons, with $|A| = 2^\omega$. We can do this by modifying the proof of Theorem 3.7 just a little bit. Let $\langle\langle(a_n^\alpha, b_n^\alpha) : n \in \omega\rangle : \alpha < 2^\omega\rangle$ list all candidates over $\mathcal{P}\omega$, each one listed 2^ω times. Construct by induction two sequences $\langle B_\alpha : \alpha \leqq 2^\omega\rangle$ and $\langle Q_\alpha : \alpha \leqq 2^\omega\rangle$ such that, for all $\alpha, \beta < 2^\omega$,

(48) B_α is a subalgebra of $\mathcal{P}\omega$, $|B_\alpha| \leqq |\alpha| \cup \omega$, and $\alpha < \beta$ implies $B_\alpha \subseteq B_\beta$,

(49) Q_α is a collection of standard types over B_α each omitted in B_α, $\alpha < \beta$ implies $Q_\alpha \subseteq Q_\beta$, and $|Q_\alpha| \leqq |\alpha| + \omega$.

We let B_0 be the finite–cofinite algebra on ω, $Q_0 = 0$. Then we repeat the construction in the proof of 3.7, taking only the first of the two steps in passing from α to $\alpha+1$. The joins Σ mentioned there are now taken in $\mathscr{P}\omega$ and coincide with $\bigcup$. □

Now we consider the possibility of improving 4.7 by obtaining similar algebras in other cardinalities or with larger numbers of atoms. The proof of 3.7 can again be modified to give the following result.

4.8. THEOREM. *If $\kappa^{\omega}=2^{\omega}$, then there is a weakly endo-rigid subalgebra A of $\mathscr{P}\kappa$ containing all singletons, with $|A|=2^{\omega}$.*

PROOF. Let K be the BA of countable and cocountable subsets of κ; the construction takes place inside K. Let $\langle\langle(a_n^{\alpha}, b_n^{\alpha}): n\in\omega\rangle: \alpha<2^{\omega}\rangle$ list all functions from ω into $K\times K$, each repeated 2^{ω} times. We proceed as in the proof of 3.7, with the following changes. Let $B_0=\langle\{\{\alpha\}: \alpha\in\Gamma\}\rangle$, where $\Gamma\in[\kappa]^{\omega}$. In the step from α to $\alpha+1$, first part, the joins Σ are taken in K. Replace the second part by the adjunction of $\{\beta\}$, where β is the least ordinal $<\kappa$ such that $\{\beta\}\not\in B'_{\alpha}$. It is easily checked then that all members of Q'_{α} are still omitted in $B_{\alpha+1}$. □

The following two corollaries of this theorem were first proved in LOATS and ROITMAN [1981] in a somewhat weaker form, and directly.

4.9. COROLLARY. *Let μ be a cardinal with $\mathrm{cf}\,\mu>\omega$, and add μ many Cohen reals to a model of CH. Assume that $\omega\leqq\kappa\leqq 2^{\omega}$ in the extension. Then in the extension there is a weakly endo-rigid subalgebra A of $\mathscr{P}\kappa$ containing all singletons, with $|A|=2^{\omega}$.* □

4.10. COROLLARY (MA). *If $\omega\leqq\kappa\leqq 2^{\omega}$, then the conclusion of* 4.9 *holds.* □

Next, we want to give yet another application of the proof of 3.7, this time in a Cohen extension of a model of ZFC. Let A be a subalgebra of $\mathscr{P}\lambda$ (for some cardinal $\lambda\leqq\omega$) containing all singletons. A *weak candidate* over A is a candidate $\langle(a_i, b_i): i<\omega\rangle$ over A such that each a_i is a singleton. Then A is called *mildly complicated* if for every such weak candidate there is an $S\subseteq\omega$ such that $\{[a_i, v]^{\text{if } i\in S}: i\in\omega\}$ is realized in A but $\{[b_i, v]^{\text{if } i\in S}: i\in\omega\}$ is not.

4.11. LEMMA. *If A is a mildly complicated subalgebra of $\mathscr{P}\lambda$ containing all singletons, then A is almost rigid and dual hopfian.*

PROOF. Suppose f is an automorphism of A moving infinitely many atoms. Then clearly there is a weak candidate $\langle(a_i, b_i): i<\omega\rangle$ for which $fa_i=b_i$ for all $i<\omega$. This easily gives a contradition.

Next, suppose that f is a one-to-one endomorphism that is not onto. By the proof of Lemma 4.6, the set $\Gamma=\{\alpha<\lambda: f\{\alpha\}\neq\{\alpha\}\}$ is infinite. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of infinitely many members of Γ. Then $\{(\{\alpha_i\}, f\{\alpha_i\}): i<\omega\}$ is a weak candidate, which again gives a contradiction. □

We shall prove that there exist mildly complicated algebras in certain Cohen extensions – which gives the existence of "small" almost rigid, dual hopfian BAs by Lemma 4.11. ROITMAN [1986] has shown that there are even "small" almost rigid, hopfian, dual-hopfian BAs in certain forcing extensions.

If Γ is a set of ordinals, we denote by Fin Γ the set of all functions from a finite subset of Γ into 2. If G is generic over M with respect to Fin κ, κ a cardinal of M, we call $\{\alpha<\kappa\colon f\alpha=1 \text{ for some } f\in G\}$ a *Cohen subset of* κ *over* M.

For terminology concerning forcing we follow JECH [1978].

4.12. LEMMA. *Let C be a Cohen subset of κ over M. In M let f be a one-to-one function from some infinite subset Γ of κ into κ such that f moves infinitely many members of Γ. Then each of the following four sets is infinite*: $f\cap(C\times C)$, $f\cap(C\times(\kappa\backslash C))$, $f\cap((\kappa\backslash C)\times C)$, *and* $f\cap((\kappa\backslash C)\times(\kappa\backslash C)$.

PROOF. We take $f\cap(C\times(\kappa\backslash C))$ as an example. Inside M, for each $m\in\omega$ let

$$D_m=\{g\in\text{Fin }\kappa\colon |\{\alpha\in\Gamma\colon \alpha\in\text{dom } g,\ g(\alpha)=1,\ f(\alpha)\in\text{dom } g,\ g(f(\alpha))=0\}|\geqq m\}\,.$$

Clearly, each set D_m is dense in Fin κ. Hence, $f\cap(C\times(\kappa\backslash C))$ is infinite. □

4.13. COROLLARY. *Let C be a Cohen subset of κ over M, D an infinite subset of κ in M. Then $C\cap D$ is infinite.* □

4.14. LEMMA. *Let G be generic over M with respect to* $\text{Fin}(\lambda\times\omega_1)$, *$\lambda$ an infinite cardinal of M. For each $\beta<\omega_1$ let $G_\beta=\{f\restriction(\lambda\times\beta)\colon f\in G\}$, $G^\beta=\{f\restriction(\lambda\times(\omega_1\backslash\beta)\colon f\in G\}$. Suppose that $\beta0<\beta1\ldots$ and $\lambda\supseteq C_0\in M[G_{\beta0}]$, $\lambda\supseteq C_1\in M[G_{\beta1}],\ldots$. Then there is a $\gamma<\omega_1$ such that $\langle C_i\colon i\in\omega\rangle\in M[G_\gamma]$.*

PROOF. Let $\delta=\bigcup_{i\in\omega}\beta i$. Thus $\delta<\omega_1$ and $\langle C_i\colon i\in\omega\rangle\in{}^\omega M[G_\delta]$. Let $\boldsymbol{C}$ be a name for $\langle C_i\colon i\in\omega\rangle$ with $M[G_\delta]$ as base model. For each $n\in\omega$ choose $q_n\in G^\delta$ such that $q_m\Vdash \boldsymbol{C}n^\vee=(Cn)^\vee$, where $\Vdash$ is relative to $\text{RO}(\text{Fin}(\lambda\times(\omega\backslash\delta)))$ in $M[G_\delta]$. Finally, choose $\xi>\delta$ so that $q_n\in\text{Fin }\Gamma$ for each $n\in\omega$, with $\Gamma=(\lambda\times\xi)\backslash(\lambda\times\delta)$. We claim that $C\in M[G_\xi]$. To prove this, let $B=\text{RO}(\text{Fin }\Gamma)$; we define a name $\boldsymbol{D}\in(M[G_\delta])^B$:

(50) $$\text{dom }\boldsymbol{D}=\{(n,a)^\vee\colon n\in\omega,\ a\subseteq\lambda\},\ \boldsymbol{D}(n,a)^\vee=\sum^B\{e^Bp\colon p\Vdash \boldsymbol{C}n^\vee=a^\vee\}\,.$$

Let $H=\{f\restriction\Gamma\colon f\in G\}$. Then $M[G_\xi]=M[G_\delta][H]$. We claim that $i_h\boldsymbol{D}=C$, hence $C\in M[G_\xi]$, as desired. We have

(51) $$i_H\boldsymbol{D}=\{(n,a)\colon \text{there is an } r\in H \text{ such that } e^Br\leqq\boldsymbol{D}(n,a)^\vee\}\,.$$

Now the desired conclusion follows from the following statements (52) and (53):

(52) $$(n,C_n)\in i_H\boldsymbol{D} \text{ for any } n\in\omega\,.$$

Indeed, $q_n \Vdash Cn^\vee = (Cn)^\vee$ and $q_n \in \mathrm{Fin}\,\Gamma$, so $e^B q_n \leqq D(n, Cn)^\vee$. Furthermore, $q_n \in H$, so (52) holds.

(53) If $n \in \omega$, $a \subseteq \lambda$, and $a \neq C_n$, then $(n, a) \not\in i_H D$.

For, assume otherwise. Choose $r \in H$ such that $e^B r \leqq D(n, a)^\vee$. Since H is generic, q_n and r are compatible; say $q_n, r \subseteq s \in \mathrm{Fin}\,\Gamma$. Thus, $e^B s \leqq D(n, a)^\vee$, so there is a $p \in \mathrm{Fin}\,\Gamma$ with $p \Vdash Cn^\vee = a^\vee$ and p and s compatible, say $s, p \subseteq t \in \mathrm{Fin}\,\Gamma$. Thus, $t \Vdash Cn^\vee = a^\vee$ and, since $q_n \subseteq t$, $t \Vdash Cn^\vee = (Cn)^\vee$. Hence, $t \Vdash a^\vee = (Cn)^\vee$. Since $a \neq Cn$, this is a contradiction. □

4.15. LEMMA. *Let M be a model of ZFC, λ an infinite cardinal in M, A a subalgebra of $\mathcal{P}\lambda$ containing all singletons, and $\langle (a_i, b_i) : i \in \omega \rangle$ a weak candidate over A. Assume that $a_i \neq \{i\}$ for all $i \in \omega$. Let N be obtained from M by adding a Cohen subset C of λ. Let $S = \{i \in \omega : a_i \subseteq C\}$. Then $\{[b_i, v]^{\text{if } i \in S} : i \in \omega\}$ is not realized in $A(C)$. (Here $A(C) = \langle A \cup \{C\} \rangle$.)*

PROOF. Choose $\alpha : \beta \twoheadrightarrow \lambda$ such that $a_i = \{\alpha_i\}$ for all $i \in \omega$ (β some ordinal such that $|\beta| = \lambda$), so that $\alpha_i \neq i$ for all $i < \beta$. Then for each $i \in \omega$ there is a $\nu i < \beta$ such that $\alpha_{\nu i} \in b_i \backslash a_i$. Suppose that $\{[b_i, v]^{\text{if } i \in S} : i \in \omega\}$ is realized in $A(C)$, say by $(e \cap C) \cup (g \backslash C)$, where $e, g \in A$. Let $I = \{\alpha_i : i \in \omega, \alpha_i \in C\}$, $J = \{\alpha_i : i \in \omega, \alpha_{\nu i} \in e\}$, $K = \{\alpha_i : i \in \omega, \alpha_{\nu i} \in g\}$. By Lemma 4.12, I is infinite. If $\alpha_i \in I$, then $i \in S$, hence $b_i \subseteq (e \cap C) \cup (g \backslash C)$ and so $\alpha_{\nu i} \in (e \cap C) \cup (g \backslash C)$. Thus, $I \subseteq J \cup K$. Hence, J is infinite or K is infinite. If J is infinite, by Lemma 4.12 choose $\alpha_i \in J$ so that $\alpha_i \not\in C$ and $\alpha_{\nu i} \in C$. Then $i \not\in S$, so $b_i \cap e \cap C = 0$; but $\alpha_{\nu i} \in b_i \cap e \cap C$ since $\alpha_i \in J$, a contradiction. If K is infinite, by Lemma 4.12 choose $\alpha_i \in K$ so that $\alpha_i \not\in C$ and $\alpha_{\nu i} \not\in C$; again a contradiction is reached. □

4.16. LEMMA. *Let M, λ, A, N, C be as in Lemma 4.15. Suppose that $\{[h_n, v]^{\text{if } n \in T} : n \in \omega\}$ is a type over A omitted in A. Then it is also omitted in $A(C)$.*

PROOF. Suppose, on the contrary, that it is realized by $d \cup (e \cap C) \cup (g \backslash C)$, where d, e, g are pairwise disjoint elements of A. Introducing the notation following (41), we easily see, as in (42), that $l \subseteq C$ and $m \cap C = 0$. Since $l, m \in M$, we infer from Lemma 4.12 that l and m are finite. Let $t = d \cup ((l \cup m) \cap \bigcup_{n \in T} h_n)$. It is easily checked that $t \in A$ and t realizes $\{[h_n, v]^{\text{if } n \in T} : n \in \omega\}$, a contradiction. □

4.17. THEOREM. *Let M be a model of ZFC + CH, and let γ be a cardinal of M such that $\gamma^\omega = \gamma$. Add γ Cohen reals to M, forming the new model N (in which $2^\omega = \gamma$). Assume that κ, λ are cardinals in N such that $\omega \leqq \lambda \leqq \kappa \leqq 2^\omega$, $\kappa \geqq \omega_1$. Then in N there is a mildly complicated subalgebra A of $\mathcal{P}\lambda$ containing all singletons, with $|A| = \kappa$; thus A is almost rigid and dual hopfian.*

PROOF. We conceive the extension from M to N to take place in the following two steps: first add γ Cohen reals, forming M', then add $\lambda \cdot \omega_1$ further Cohen reals to get N. Write $N = M'[G]$, G generic with resepct to $\mathrm{Fin}(\lambda \times \omega_1)$, and for each $\beta < \omega_1$ let $G_\beta = \{f \restriction (\lambda \times \beta) : f \in G\}$, $G'_\beta = \{f \restriction (\lambda \times (\omega_1 \backslash \beta)) : f \in G\}$. Thus, $N = M'[G_\beta][G^\beta]$ for any $\beta < \omega_1$. Now we define two sequences $\langle \beta\alpha : \alpha < \omega_1 \rangle$, $\langle C_\alpha : \alpha < \omega_1 \rangle$ by induction. Let $\beta 0 = 0$, $\beta(\alpha + 1) = \beta\alpha + 1$, and for any $\alpha < \omega_1$ let $H = \{f \restriction (\lambda \times \{\beta\alpha\}) : f \in G\}$, generic over $M'[G_{\beta\alpha}]$, and set $C_\alpha = \{\alpha < \lambda : f(\gamma, \beta\alpha) = 1$ for some $f \in H\}$, so C_α is a generic subset of λ over $M'[G_{\beta\alpha}]$. For λ limit $< \omega_1$ we have $\langle C_\alpha : \alpha < \gamma \rangle \in M'[G_{\beta\gamma}]$ for some $\beta\gamma < \omega_1$ by Lemma 4.14. This finishes the construction of the two sequences.

For any $\alpha < \omega_1$ let B_α be the subalgebra of $\mathscr{P}\lambda$ generated by $\{\{\gamma\} : \gamma < \lambda\} \cup \{C_\gamma : \gamma < \alpha\}$, and let $A = \bigcup_{\alpha < \omega_1} B_\alpha$. We claim that A is the desired algebra. Let $\langle (a_i, b_i) : i < \omega \rangle$ be a weak candidate over A. Then there is an $\alpha < \omega_1$ such that it is a weak candidate over B_α. By Lemmas 4.15 and 4.16, it is omitted in A, as desired.

4.18. COROLLARY. *It is consistent with ZFC that if $\omega \leqq \lambda \leqq \kappa \leqq 2^\omega$, $\kappa \geqq \omega_1$, κ and λ cardinals, then there is a mildly complicated subalgebra A of $\mathscr{P}\lambda$ containing all singletons, with $|A| = \kappa$, and with 2^ω arbitrarily large.* □

Problems

We conclude this chapter with some problems concerning endomorphisms of Boolean algebras.

PROBLEM 1. *Associate with every BA a subsemigroup A' of* End A *such that every member of A' takes on infinitely many values, and such that $A' \cong B'$ implies that $A \cong B$.*

PROBLEM 2 (GCH). *If κ is singular or the successor of a singular cardinal, is there a BA A such that $|A| = |\mathrm{End}\ A| = \kappa$?*

PROBLEM 3. *Can one prove in ZFC that there are arbitrarily large cardinals κ for which there is a BA A with $|A| = |\mathrm{End}\ A| = \kappa$?*

PROBLEM 4. *Is $|\mathrm{End}\ A| \leqq |\mathrm{Sub}\ A|$ for infinite A?*

PROBLEM 5. *Can one prove in ZFC that there is a BA which is both endo-rigid and Bonnet-rigid?*

PROBLEM 6. *Is there an infinite endo-rigid BA A which has exactly $|A|$ endomorphisms?*

PROBLEM 7. *Is there an infinite endo-rigid, Bonnet-rigid BA A with exactly $|A|$ endomorphisms?*

References

JECH, T.

[1978] *Set Theory* (Academic Press) xii + 621 pp.

LOATS, J.

[1979] Hopfian Boolean algebras of power less than or equal to continuum, *Proc. Amer. Math. Soc.*, **77**, 186–190.

LOATS, J. and J. ROITMAN

[1981] Almost rigid Hopfian and dual Hopfian atomic Boolean algebras, *Pacific J. Math.*, **97**, 141–150.

MAGILL, K.

[1970] The semigroup of endomorphisms of a Boolean ring, *J. Austral. Math. Soc.*, **11**, 411–416.

MAXSON, C.

[1972] On semigroups of Boolean ring endomorphisms, *Semigroup Forum*, **4**, 78–82.

MONK, J.D.

[1980] A very rigid Boolean algebra, *Is. J. Math.*, **35**, 135–150.

ROITMAN, J.

[1986] A small Hopfian dual Hopfian almost rigid atomic BA, Preprint.

SCHEIN, B.

[1970] Ordered sets, semilattices, distributive lattices and Boolean algebras with homomorphic endomorphism semigroups, *Fund. Math.*, **68**, 31–50.

SHELAH, S.

[1979] Boolean algebras with few endomorphisms, *Proc. Amer. Math. Soc.*, **74**, 135–142.

[1983] Constructions of many complicated uncountable structures and Boolean algebras, *Israel J. Math.*, **45**, 100–146.

[1984] Existence of endo-rigid Boolean algebras, Preprint.

J. Donald Monk
University of Colorado

Keywords: Boolean algebra, endomorphism, reconstruction, endo-rigid, hopfian, dual-hopfian, almost rigid.

MOS subject classification: primary 06E05; secondary 03G05.