CHAPTER 12

The Number of Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
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0. Introduction

For almost all classes K of BAs which have been an object of intensive study, there are exactly 2^κ isomorphism types of members of K of each infinite power κ. In particular, this is true for the class of all BAs. This is evidence that the structure of members of such classes is complicated. In this chapter we prove several results of the above type. First we give five simple but somewhat special constructions, which apply to interval algebras, superatomic BAs, subalgebras of free BAs, subalgebras of $\mathscr{P}\kappa$, and complete BAs. In the second section of the chapter we present an instance of a general method of Shelah for producing many isomorphism types.

Here is a list of many of the theorems known about counting various kinds of BAs.

(1) For each $\kappa \geqq \omega$ there are 2^κ isomorphism types of interval BAs of power κ. See Section 1 below.

(2) For each $\kappa > \omega$ there are 2^κ isomorphism types of superatomic BAs of power κ; see Section 1 below. There are ω_1 isomorphism types of denumerable superatomic BAs; see Part I, Chapter 6, Theorem 17.11.

(3) For each $\kappa > \omega$ there are 2^κ isomorphism types of dense subalgebras of the free BA on κ generators; see Section 1 below. Any dense subalgebra of $\mathrm{Fr}\omega$ is atomless, and hence isomorphic to $\mathrm{Fr}\omega$ (recall that $\mathrm{Fr}\kappa$ is the free BA on κ free generators). But every countable BA can be isomorphically embedded in $\mathrm{Fr}\omega$, so $\mathrm{Fr}\omega$ has 2^ω pairwise non-isomorphic subalgebras.

(4) For each $\kappa \geqq \omega$ there are 2^{2^κ} pairwise nonisomorphic subalgebras of $\mathscr{P}\kappa$ each containing all singletons; see section 1.

(5) There are $2^{2^{2^\omega}}$ pairwise non-isomorphic countably complete subalgebras of $\mathscr{P}\boldsymbol{R}$ each containing all singletons, where $\boldsymbol{R}$ is the set of real numbers. See Freniche [1984], where further results along these lines are given (there are evidently some problems left, though).

(6) Let T be a complete theory of BAs, with infinite models and $T_1 \supseteq T$ in some language extending the language of BAs. Let K be the class of all BA-reducts of models of T_1. Then for each $\kappa > |T_1|$ there is a family of 2^κ pairwise non-elementarily-embeddable members of K of power κ. See Shelah [1978, pp. 9, 30–31, 364, 421]. In particular, if T is a complete theory of BAs with infinite models, and $\kappa > \omega$, there are 2^κ pairwise non-isomorphic models of T of power κ. For $\kappa = \omega$ the situation is simple: let T be a complete theory of BAs with infinite models. If all models of T have only finitely many atoms, then that number of atoms is constant and T has only one denumerable model, up to isomorphism. If T has models with infinitely many atoms, then T has 2^ω denumerable models. This can be seen by combining the first construction in Section 1 below with the construction given in the proof of Proposition 18.5 in Chapter 7, Part I.

(7) For each κ with $\kappa^\omega = \kappa$ there are 2^κ pairwise non-isomorphic rigid complete BAs of power κ. This is an unpublished result of Shelah which uses the methods of Shelah [1983]. See Section 1 below for a partial result along these lines.

(8) For each $\kappa > \omega$ there are 2^κ pairwise non-isomorphic rigid BAs of power κ; see MONK and RASSBACH [1979]. Recall from the article on automorphism groups that there is no denumerable rigid BA.

(9) More generally, for each $\kappa > \omega$ there are 2^κ pairwise non-isomorphic onto-rigid interval BA's of power κ; see LOATS and RUBIN [1978].

(10) TODORČEVIĆ [1979] showed that for each regular uncountable κ there are 2^κ pairwise non-isomorphic Bonnet-rigid interval BAs of power κ. (Bonnet, Loats, and Shelah independently worked along these lines.) There are evidently open problems here.

(11) If $\mu = \lambda^\omega$, then there is a family of power 2^μ of indecomposable endo-rigid BAs of power μ such that any homomorphism from one of them to another of them has finite range; see SHELAH [1984]. There still appear to be some small open problems in this connection.

The constructions given in the second section of this article may be considered to be an introduction to the methods used in (6), (7), (8), and (11) above, and to other constructions of this sort in SHELAH [1971], [1978], [1984].

1. Simple constructions

Our first construction is of a folklore nature. For each infinite cardinal κ we produce 2^κ pairwise non-isomorphic interval BAs of power k.

Let $\alpha_0 = \omega$ and $\alpha_1 = 1 + \eta + \omega$, where η is the order type of the rational numbers. For each $\varepsilon \in {}^\kappa 2$ we set

$$\beta_\varepsilon = \prod_{\xi<\kappa} \alpha_{\varepsilon\xi} ,$$

the *ordinal product*: β_ε consists of all functions $f \in \Pi_{\xi<\kappa} \alpha_{\varepsilon\xi}$ such that $\{\xi\colon f\xi \neq 0\}$ is finite, and $f < g$ iff $f\xi < g\xi$, where ξ is the greatest $\nu < \kappa$ such that $f\nu \neq g\nu$. Note that $|\beta_\varepsilon| = \kappa$. Let A_ε be the interval algebra over β_ε; so $|A_\varepsilon| = \kappa$. We shall show, eventually, that the algebras A_ε are pairwise non-isomorphic for $\varepsilon \in {}^\kappa 2$.

For any BA B let

$$JB = \langle \mathrm{At}\, B \cup \{x \in B\colon x \text{ is atomless}\}\rangle^{\mathrm{id}} ,$$

where At B is the set of all atoms of B. We repeat this construction transfinitely as follows:

$$I_0 B = \{0\} ;$$

$$I_\lambda B = \bigcup_{\xi<\lambda} I_\xi B \text{ for } \lambda \text{ a limit ordinal} ;$$

$$I_{\xi+1} B = \bigcup \; J(B/I_\xi B) .$$

If $a \in B$, we denote by $[a]$ the image of a under the natural homomorphism from B onto $B/I_\xi B$ (ξ is to be understood from context).

1.1. LEMMA. *Let $\varepsilon \in {}^{\kappa}2$. Then $\varepsilon 0 = 0$ iff A_ε is atomic.*

PROOF. $\Rightarrow$ Let $0 \neq x \in A_\varepsilon$; we want to find an atom $\leqq x$. We may assume that $x = [s, t)$ for some $s < t$ in β_ε. Say $s\nu = t\nu$ for all $\nu > \xi$, and $s\xi < t\xi$, where $\xi < \kappa$. Let $u0 = s0 + 1$ and $u\nu = s\nu$ for all $\nu > 0$. Clearly, $s < u \leqq t$. Since u is the successor of s in β_ε, it follows easily that $[s, u)$ is an atom $\leqq x$.

$\Leftarrow$ Suppose that $\varepsilon 0 = 1$; we show that A_ε is not atomic. Let $s, t \in \beta_\varepsilon$ be such that $s0 < t0$, both being in the η-part of $1 + \eta + \omega$, and $s\nu = t\nu$ for all $\nu > 1$. Then $[s, t)$ is atomless, as is easily checked. $\square$

1.2. LEMMA. *Let $\varepsilon \in {}^{\kappa}2$. For each $\xi < \kappa$ and each $s \in \Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ let s^+ be the member of $\Pi_{\nu < \kappa} \alpha_{\varepsilon\nu}$ such that $s^+\nu = 0$ for all $\nu < \xi$, and $s^+\nu = s\nu$ for $\xi \leqq \nu < \kappa$; and let $F_\xi s = ([0, s^+)]$. Then:*

(i) *F_ξ is an order-isomorphism into $A_\varepsilon / I_\xi A_\varepsilon$.*

(ii) *The range of F_ξ generates $A_\varepsilon / I_\xi A_\varepsilon$.*

(iii) *If $t \in \Pi_{\nu < \kappa} \alpha_{\varepsilon\nu}$ and $s \in \Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ is the restriction of t, then $[s^+, t) \in I_\xi A_\varepsilon$.*

PROOF. We proceed by induction on ξ. The case $\xi = 0$ is trivial. Assume the lemma for ξ, and let $s, t \in \Pi_{\xi+1 \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$. If $s \leqq t$, clearly $F_{\xi+1} s \leqq F_{\xi+1} t$. To show that $F_{\xi+1} \neq F_{\xi+1} t$ for $s < t$, it suffices to show that there are infinitely many atoms $\leqq [[s^+, t^+)]$ (here $[[s^+, t^+)] \in A_\varepsilon / I_\xi a_\varepsilon$, while $^+$ is relative to $\xi + 1$). For each $i \in \omega$, let u_i be like s^+ except that $u_i \xi = i + 1$ (in the ω-part of $\alpha_{\varepsilon\xi}$). By the induction hypothesis, $[[u_i^+, u_{i+1}^+)]$ is an atom of $A_\varepsilon / I_\xi A_\varepsilon$ and it is clearly $\leqq [[s^+, t^+)]$, as desired. To prove (iii), we assume that $t \in \Pi_{\nu < \kappa} \alpha_{\varepsilon\nu}$ and $s \in \Pi_{\xi+1 \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ is the restriction of t; also let u be the restriction of t to $\Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$, and let $v \in \Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ extend s so that $v\xi = 0$. Then $[u^+, t) \in I_\xi A_\varepsilon$ by the induction hypothesis. $[v, u)$ is a sum of an atomless element and finitely many atoms in $\mathrm{intalg}(\Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu})$, so by the induction hypothesis, $[v^+, u^+] \in I_{\xi+1} A_\varepsilon$. Hence $[v^+, t) \in I_{\xi+1} A_\varepsilon$. Since $v^+ = s^+$, this proves (iii). Clearly (ii) follows from (iii).

Now suppose that ξ is a limit ordinal $< \kappa$, and the lemma holds for all $\nu < \xi$. Clearly (i) holds. Assume the hypothesis of (iii). Choose $\nu < \xi$ so that $t\mu = 0$ for all $\mu \in [\nu, \xi)$, and let u be the restriction of t to $\Pi_{\nu \leqq \mu < \kappa} \alpha_{\varepsilon\mu}$. Then $[u^+, t) \in I_\nu A_\varepsilon \subseteq I_\xi A_\varepsilon$. Since $u^+ = s^+$, this proves (iii). Again, (ii) follows from (iii). $\square$

Lemmas 1.1 and 1.2 immediately give the desired result:

1.3. THEOREM. *For each $\kappa \geqq \omega$ there are 2^κ pairwise non-isomorphic interval algebras of power κ.* $\square$

Our second construction gives the number of superatomic BAs. By Theorem 17.11, Chapter 6 of Part I, there are exactly ω_1 denumerable superatomic BAs. Our construction gives 2^κ superatomic BAs for each uncountable cardinal κ. This result is due independently to BONNET [1977], CARPINTERO ORGANERO [1971], and WEESE [1976]; we follow the construction of Weese.

Recall the definition of the cardinal sequence of a superatomic BA A:

$$I_0A = \{0\},$$

$$I_{\beta+1}A = \bigcup \langle \mathrm{At}(A/I_\beta A)\rangle^{\mathrm{id}},$$

$$I_\lambda A = \bigcup_{\beta<\lambda} I_\beta A \text{ for } \lambda \text{ limit}.$$

We denote by $[a]_\alpha$ the image of $a \in A$ under the natural homomorphism of A onto $A/I_\alpha A$. A is superatomic iff $I_\beta A = A$ for some β. The least β such that $I_B A = A$ is a successor ordinal $\alpha A + 1$. Then $A/I_{\alpha A}$ is a finite non-trivial BA; we let nA be the number of atoms of $A/I_{\alpha A}$. The *cardinal sequence* of A is the sequence $\langle |\mathrm{At}(A/I_\xi A)|: \xi \leqq \alpha A\rangle$.

Our construction will use weak products; recall from 17.18, Chapter 6 of Part I, that a weak product of superatomic BAs is again superatomic, and the cardinal sequence of a weak product can be described in terms of the cardinal sequences of its factors. We also need the following lemma.

1.4. LEMMA. *Let $\langle A_i: i \in I\rangle$ be an infinite system of non-trivial BAs, and set $B = \Pi^{\mathrm{w}}_{i\in I} A_i$. Let σ be the least ordinal α such that $\{i \in I: \alpha \leqq \alpha A_i\}$ is finite. Then for each $\alpha < \sigma$ we have $B/I_\alpha B \cong \Pi^{\mathrm{w}}_{i\in I} A_i/I_\alpha A_i$.*

PROOF. We use the following elementary fact, easily established by induction on α:

(*) $I_\alpha C \cap (C \restriction c) = I_\alpha (C \restriction c)$ for any BA C, any $c \in C$, any ordinal α. Furthermore, $(C/I_\alpha C) \restriction [c]_\alpha$ is isomorphic to $(C \restriction c)/I_\alpha(C \restriction c)$ via $[x]_\alpha \mapsto [x]_\alpha$ for each $x \leq c$.

Now the desired isomorphism is given by $(f[x]_\alpha)_i = [x_i]_\alpha$ for each $x \in B$; the only non-trivial parts of the verfication of this fact are that f is well-defined and one-to-one. Well-definedness follows from (*). For one-to-one-ness, suppose that $f[x]_\alpha = 0$. If $\{i: x_i \neq 1\}$ is finite, then from $\{i: \alpha \leqq \rho_i\}$ infinite (which follows from the definition of σ) we find $i \in I$ such that $x_i = 1$ and $\alpha \leqq \rho_i$. But then $[x_i]_\alpha \neq 0$, a contradiction. So $\{i \in I: x_i \neq 0\}$ is finite. Since $x_i \in I_\alpha A_i$ for all $i \in I$, it then follows from (*) that $x \in I_\alpha B$, as desired. □

1.5. THEOREM. *For each $\kappa \geqq \omega_1$ there are 2^κ isomorphic types of superatomic BAs of power κ.*

PROOF. For any BA C we denote by $\Pi^{\mathrm{w}}_0 C$ the weak product of ω copies of C, and by $\Pi^{\mathrm{w}}_1 C$ that of ω_1 copies. Now for each $f \in {}^{\leqq\kappa}2$ we construct a superatomic BA A_f by induction on dom f:

$$A_0 = 2,$$

$$A_{f\varepsilon} = \prod_\varepsilon^{\mathrm{w}} A_f \ (\varepsilon = 0, 1),$$

$$A_f = \prod_{\alpha<\lambda}^{\mathrm{w}} A_{f\restriction\alpha} \text{ for dom } f = \lambda \text{ limit } \leq \kappa.$$

(Here $f\varepsilon$ is $f^\frown\langle\varepsilon\rangle$.) By induction, $|\mathrm{dom}\, f| \le |A_f| \le |\mathrm{dom}\, f| \cdot \omega_1$ for all $f \in {}^{\leq\kappa}2$, so $|A_f| = \kappa$ for all $f \in {}^{\kappa}2$. By induction using 17.18, Chapter 6 of Part I,

(1) If $\beta \leqq \kappa$ and $f \in {}^{\beta}2$, then $\alpha A_f = \beta$ and $nA_f = 1$.

Note that $A_{\langle 0\rangle}$ is a factor of each algebra A_f with $f \in {}^{\leq\kappa}2$, $0 \in \mathrm{dom}\, f$, $f0 = 0$. Hence

(2) If $f \in {}^{\leq\kappa}2$, $0 \in \mathrm{dom}\, f$, and $f0 = 0$, then there is an $a \in A_f$ such that $|A_f \restriction a| = \omega$.

On the other hand, we claim

(3) If $f \in {}^{\leq\kappa}2$, $0 \in \mathrm{dom}\, f$, and $f0 = 1$, then there is no $a \in A_f$ such that $|A_f \restriction a| = \omega$.

We prove (3) by induction on dom f. Since $A_{\langle 1\rangle}$ is isomorphic to the finite–cofinite algebra on ω_1, (3) holds for dom $f = 1$. If dom $f = \beta + 1$ and (3) is true for shorter functions, take any $a \in A_f$. If $\{i \in I: a_i \neq 1\}$ is finite, then there is a $b \leqq a$ with $A_f \restriction b \cong A_{f \restriction \beta}$, and the inductive hypothesis applies. If $\{i \in I: a_i \neq 0\}$ is finite, then

$$A_f \restriction a \cong (A_{f\restriction\beta} \restriction b_1) \times \cdots \times (A_{f\restriction\beta} \restriction b_m)$$

for certain $b_1, \ldots, b_m$, and again the inductive hypothesis applies. The final induction step – dom f a limit ordinal – is treated similarly. So (3) holds.

The major part of the proof is the following claim:

(4) If $\gamma + \delta = \beta \leqq \kappa$, $f \in {}^{\beta}2$, and $g\xi = f(\gamma + \xi)$ for all $\xi < \delta$, then $A_f/I_\gamma A_f \cong A_g$.

We prove (4) by induction on δ, with γ fixed. For $\delta = 0$ it follows from (1). Assume (4) for δ, let $f \in {}^{\gamma+\delta+1}2$, let $g\xi = f(\gamma + \xi)$ for all $\xi < \delta$, and let $h\xi = f(\gamma + \xi)$ for all $\xi < \delta + 1$. Then

$$\begin{aligned} A_f/I_\gamma A_f &= \Big(\prod\nolimits^{\mathrm{w}}_{f(\gamma+\delta)} A_{f\restriction(\gamma+\delta)}\Big)/I_\gamma A_f \\ &\cong \prod\nolimits^{\mathrm{w}}_{f(\gamma+\delta)} (A_{f\restriction(\gamma+\delta)}/I_\gamma A_{f\restriction(\gamma+\delta)}) \text{ (by (1) and Lemma 1.4)} \\ &\cong \prod\nolimits^{\mathrm{w}}_{f(\gamma+\delta)} A_g \text{ (induction hypothesis)} \\ &= A_h, \end{aligned}$$

as desired. Now assume that δ is a limit ordinal, (4) holds for all $\xi < \delta$, $f \in {}^{\gamma+\delta}2$, $g\xi = f(\gamma + \xi)$ for all $\xi < \delta$, and $(h\varepsilon)\xi = f(\gamma + \xi)$ for all $\xi < \varepsilon$, for each $\varepsilon < \delta$. Then

$$
\begin{aligned}
A_f/I_\gamma A_f &= \prod_{\xi<\beta}^{\mathrm{w}} A_{f\restriction\xi}/I_\gamma A_f \\
&\cong \prod_{\xi<\beta}^{\mathrm{w}} (A_{f\restriction\xi}/I_\gamma A_{f\restriction\xi}) \text{ (by (1) and Lemma 1.4)} \\
&\cong \prod_{\gamma\leq\xi<\delta}^{\mathrm{w}} (A_{f\restriction\xi}/I_\gamma A_{f\restriction\xi}) \text{ by (1)} \\
&\cong \prod_{\gamma\leq\xi<\delta} A_{h\xi} \text{ induction hypothesis} \\
&= A_g .
\end{aligned}
$$

We have established (4).

Now suppose that $f, g \in {}^\kappa 2$ and $f \neq g$. Let β be minimum such that $f\beta \neq g\beta$. By (4), $A_f/I_\beta A_f \cong A_h$ and $A_g/I_\beta A_g \cong A_k$, where $h\delta = f(\beta+\delta)$ and $k\delta = g(\beta+\delta)$ for all $\delta<\kappa$. So $h0 \neq k0$, and so by (2) and (3), $A_h \not\cong A_k$. This finishes the proof. □

The third construction of many non-isomorphic BAs gives the following remarkable theorem of EFIMOV and KUZNECOV [1970]: for each $\kappa>\omega$ there are 2^κ pairwise non-isomorphic dense subalgebras of the free BA on κ generators. The construction is based on the following general facts.

Let f be a homomorphism of a BA A onto a Ba B. We set

$$P_f = \{(x, y): x, y \in A \text{ and } fx = fy\} .$$

Thus, P_f is a subalgebra of $A \times A$. Set

$$I_f = \{(x, y): x, y \in A \text{ and } fx = fy = 0\} .$$

Then I_f is an ideal in P_f, and P_f/I_f is isomorphic to B.

1.6. LEMMA. *Suppose that A is a free BA, $0 \neq X \subseteq A$, and $I = \{a \in A: a\cdot x = 0$ for all $x \in X\}$. Then I is a countably generated ideal in A.*

PROOF. We may assume that A is uncountable. Say A is freely generated by $\langle x_\alpha: \alpha<\kappa\rangle$, κ an uncountable cardinal. For $\Gamma \in [\kappa]^{<\omega}$ and $f \in {}^\Gamma 2$ we set

$$x(f) = \prod_{\alpha\in\Gamma} f\alpha \cdot x_\alpha ,$$

where $1\cdot y = y$, $0\cdot y = -y$ for all y. Let

$$\mathcal{F} = \{f: f \in {}^\Gamma 2 \text{ for some } \Gamma \in [\kappa]^{<\omega}, x(f) \in I, \text{ and } x(f \restriction \Delta) \notin I \\ \text{if } \Delta \subset \Gamma\} .$$

Clearly, $\{x(f): f \in \mathcal{F}\}$ generates I, so it sffices to show that $\mathcal{F}$ is countable. Suppose not. Then by the Δ-system lemma plus the pigeon-hole principle there exist a finite $\Delta \subseteq \kappa$, an $h \in {}^\Delta 2$, and an uncountable $\mathcal{G} \subseteq \mathcal{F}$ such that for any two distinct $f, g \in \mathcal{G}$ we have $\operatorname{dom} f \cap \operatorname{dom} g = \Delta$ and $f \restriction \Delta = g \restriction \Delta = h$. Now we may assume that each member of X has the form $x(f)$ for some $f \in {}^{\Gamma}2$ with $\Gamma \in [\kappa]^{<\omega}$. Note that $x(h) \not\in I$ (since $\Delta \subset \Gamma$ for any Γ such that $f \in \mathscr{G}$ and $f \in {}^{\Gamma}2$ for some f). Hence, choose $\Gamma \in [\kappa]^{<\omega}$ and $f \in {}^{\Gamma}2$ so that $x(f) \in X$ and $x(f) \cdot x(h) \neq 0$. Say $\operatorname{dom} f = \{\alpha_0, \ldots, \alpha_{m-1}\}$. Choose $m+1$ distinct members $g_0, \ldots, g_m$ of $\mathscr{G}$. Since $x(f) \cdot x(g_i) = 0$ there is a $\beta_i \in \operatorname{dom} g_i \cap \operatorname{dom} f$ with $f\beta_i \neq g_i\beta_i$, for each $i \leqq m$. Now $|\operatorname{dom} f| < m+1$, so choose distinct $i, j \leqq m$ such that $\beta_i = \beta_j$. Then $\beta_i \in \operatorname{dom} g_i \cap \operatorname{dom} g_j = \Delta$, while $f\beta_i \neq g_i\beta_i = h\beta_i$, contradicting $x(f) \cdot x(h) \neq 0$. □

1.7. LEMMA. *If A is a free BA on $\kappa \geqq \omega$ generators, and if I is a maximal ideal in A, then I cannot be generated by $<\kappa$ elements.*

PROOF. Say A is freely generated by X, $|X| = \kappa$. Suppose that I is generated by Y, $|Y| < \kappa$. For each $y \in Y$ there is a finite $F_y \subseteq X$ such that $y \in \langle F_y \rangle$. Choose $x \in X \backslash \bigcup_{y \in Y} F_y$. Then $x \in I$ or $-x \in I$, and either possibility clearly gives a contradiction. □

1.8. LEMMA. *Suppose that A is an uncountable free BA, and f is a homomorphism of A onto a countable BA B. Then P_f is not isomorphic to A.*

PROOF. Suppose it is; we shall get a contradiction by finding a countably generated maximal ideal in P_f. (See Lemma 1.7.) Let J be a maximal ideal in B, and set $K = \{(a, b) \in P_f\colon fa \in J\}$. Clearly, K is a maximal ideal in P_f. To show that it is countably generated, first let $X = \{(x, 0) \in P_f\colon fx = 0\}$. Set

$$L = \{(u, v) \in P_f\colon (u, v) \cdot (x, 0) = (0, 0) \text{ for all } (x, 0) \in X\}\,.$$

Thus, by Lemma 1.6, L is a countably generated ideal in P_f. We claim

$$L = \{(0, v) \in P_f\colon fv = 0\} \tag{1}$$

For $\supseteq$, is obvious. For $\subseteq$, let $(u, v) \in L$. It suffices to show that $u = 0$. Suppose not. Now there is a non-zero $x \leqq u$ such that $fx = 0$, since otherwise $f \restriction (A \restriction u)$ would be one-to-one and B would be uncountable. Taking such an x, we have $(x, 0) \in X$ and $(x, 0) \cdot (u, v) \neq (0, 0)$, a contradiction. So (1) holds.

By symmetry, the set $L' \overset{\text{def}}{=} \{(u, 0) \in P_f\colon fu = 0\}$ is a countably generated ideal in P_f. Next, for each $z \in B$ choose $a_z \in A$ such that $fa_z = z$. To show that K is countably generated it now suffices to prove

$$K = \langle L \cup L' \cup \{(a_z, a_z)\colon z \in J\}\rangle^{\text{id}}\,. \tag{2}$$

Clearly, $\supseteq$ holds. For $\subseteq$, given $(x, y) \in K$ we have $(x, y) \cdot (-a_{fx}, -a_{fx}) \in \langle L \cup L' \rangle^{\text{id}}$ and $(x, y) \leqq (a_{fx}, a_{fx}) + (x, y) \cdot (-a_{fx}, -a_{fx})$, so (x, y) is in the right-hand side of (2), as desired. □

We are now ready for the theorem of Efimov and Kuznecov:

1.9. THEOREM. *For each $\kappa > \omega$ there are 2^κ pairwise non-isomorphic dense subalgebras of the free BA A on κ generators.*

PROOF. By Theorem 1.3 and its proof there is a family $\langle B_\alpha : \alpha < 2^\kappa \rangle$ of pairwise non-isomorphic BAs of power κ with the following property:

(1) For every $\alpha < 2^\kappa$ and every $x \in B_\alpha^+$ there is a non-zero $y \leqq x$ such that $B_\alpha \restriction y$ is countable.

Now for each $\alpha < 2^\kappa$ let f_α be a homomorphism from A onto B_α, and then set $P_\alpha = P_{f\alpha}$, $I_\alpha = I_{f\alpha}$. Recall from Theorem 9.14, Chapter 4 of Part I that $A \times A$ is isomorphic to A; so P_α can be considered to be a subalgebra of A. Now P_α is dense in $A \times A$, for suppose $(a, b) \in (A \times A)^+$; say $a \neq 0$. Then there is a non-zero $a' \leqq a$ with $f_\alpha a' = 0$ (otherwise $f_\alpha \restriction (A \restriction a)$ would be one-to-one and so B_α would be uncountable). So $(a', 0) \in P_\alpha$ and $0 \neq (a', 0) \leqq (a, b)$, as desired.

The proof will be completed by proving

(2) $P_\alpha \not\cong P_\beta$ for $\alpha \neq \beta$.

For, suppose that g is an isomorphism of P_α onto P_β. Since $P_\alpha / I_\alpha \cong B_\alpha$, it suffices to show that $g[I_\alpha] \subseteq I_\beta$. Suppose that $(a, b) \in I_\alpha$ but $g(a, b) \not\in I_\beta$. Now $(a, 0)$, $(0, b) \in P_\alpha$, and $g(a, b) = g(a, 0) + g(0, b)$, so say $g(a, 0) \not\in I_\beta$. Let $g(a, 0) = (c, d)$. Since $(c, d) \not\in I_\beta$, we have $f_\beta c \neq 0$. Choose $0 \neq e \leqq f_\beta c$ with $B_\beta \restriction e$ countable. Say $f_\beta c' = e$, and set $c'' = c' \cdot c \cdot d$; then also $f_\beta c'' = e$. Choose $a' \in A$ so that $g(a', 0) = (c'', c'')$. Hence,

(3) $A \cong P_\alpha \restriction (a', 0) \cong P_\beta \restriction (c'', c'')$.

Now $A \restriction c'' \cong A$; let h be such an isomorphism. Let $k = f_\beta \circ h^{-1}$. It is easily checked that $P_\beta \restriction (c'', c'') \cong P_k$. By Lemma 1.8, $P_k \not\cong A$, which contradicts (3). $\square$

VAN DOUWEN (unpublished) has improved Theorem 1.9 by showing that for each $\kappa > \omega$ there are 2^κ totally different rigid dense subalgebras of Frκ (totally different means no non-trivial isomorphic factors).

The fourth construction which we present also concerns subalgebras of specific algebras. Call a subalgebra of $\mathscr{P}\kappa$ *full* if it contains all singletons. We present the result of FRENICHE [1984] that for any infinite κ there are 2^{2^κ} pairwise non-isomorphic full subalgebras of $\mathscr{P}\kappa$. We give two proofs: one using the result just established, and a direct one.

Let $\kappa \geqq \omega$. Let A be a free subalgebra of $\mathscr{P}\kappa$ with $|A| = 2^\kappa$. Then by Theorem 1.9 let $\langle B_\alpha : \alpha < 2^{2^\kappa} \rangle$ be a system of pairwise non-isomorphic dense subalgebras of A. For each $\alpha < 2^{2^\kappa}$ let $C_\alpha = \langle B_\alpha \cup \{\{\xi\} : \xi < \kappa\} \rangle$. Suppose that $\alpha, \beta < 2^{2^\kappa}$, $\alpha \neq \beta$, and f is an isomorphism of C_α onto C_β. Then f induces an isomorphism from $C_\alpha / [\kappa]^{<\omega}$ onto $C_\beta / [\kappa]^{<\omega}$, and these are, respectively, isomorphic to B_α and B_β, a contradiction.

Now we turn to the direct construction.

1.10. LEMMA. *If $\mathscr{P}\kappa$ has λ full subalgebras in all, and $\lambda > 2^\kappa$, then $\mathscr{P}\kappa$ has λ pairwise non-isomorphic full subalgebras.*

PROOF. Any isomorphism between full subalgebras is induced by a permutation of κ, so every isomorphism class of full subalgebras has at most 2^κ members. The lemma follows. □

If F is a filter on a BA A, then, as is easily seen, $\langle F\rangle = F \cup \{a: -a \in F\}$.

1.11. LEMMA. *If F and G are distinct filters on A and neither is an ultrafilter, then $\langle F\rangle \neq \langle G\rangle$.*

PROOF. Say $a \in F\backslash G$. If $-a \not\in G$, then $a \in \langle F\rangle\backslash\langle G\rangle$. Assume that $-a \in G$. Choose $c \in A$ so that $c, -c \not\in F$ (since F is not an ultrafilter). Then $-c + -a \in G$ while $-c + -a$ and $c \cdot a$ are not in F; so $\langle G\rangle \neq \langle F\rangle$. □

1.12. THEOREM. *There are 2^{2^κ} full subalgebras of $\mathscr{P}\kappa$.*

PROOF. (For the first proof, see above.) By Lemma 1.10 it suffices to exhibit 2^{2^κ} full subalgebras without worrying about isomorphisms. The proof is now a consequence of Lemma 1.11 and the following two facts: (1) there are 2^{2^κ} non-principal ultrafilters on $\mathscr{P}\kappa$; and (2) if F_1, F_2, F_3, F_4 are distinct ultrafilters, then $F_1 \cap F_2 \neq F_3 \cap F_4$ (the desired family of full subalgebras is then $\{F \cap G: \{F, G\} \in \mathscr{P}\}$ for some partition $\mathscr{P}$ of the set of all non-principal ultrafilters into 2-element subsets). (1) is well known. For (2), say $a \in F_1\backslash F_3$, $b \in F_1\backslash F_4$, $c \in F_2\backslash F_3$, $d \in F_2\backslash F_4$. Then $a \cdot b + c \cdot d \in (F_1 \cap F_2)\backslash(F_3 \cap F_4)$. The proof is finished. □

The final construction in this section, taken from MONK and SOLOVAY [1972], is a construction of complete BAs derived from forcing conditions and using infinite combinatorics.

Let κ be an infinite cardinal. Let M be a family of independent subsets of κ with $|M| = 2^\kappa$, and let t be a one-to-one mapping from $\mathscr{P}\kappa$ onto M. For each $R \subseteq \mathscr{P}\kappa$ with $|\mathscr{P}\kappa\backslash R| = 2^\kappa$ we define a complete BA C_R as follows. Let $A_R = \{t_a: a \in \mathscr{P}\kappa\backslash R\}$. We also define a partial ordering on the set $\mathscr{P}_R = \{(k, K): k \in [\kappa]^{<\omega}, K \in [A_R]^{<\omega}\}$ by setting

$$(k_1, K_1) \leqq (k_2, K_2) \text{ iff } k_1 \subseteq k_2,\ K_1 \subseteq K_2, \text{ and } k_2 \cap \bigcup K_1 \subseteq k_1\,.$$

With this partial ordering we associate a complete BA in the usual way familiar to those used to forcing: for each $(k, K) \in \mathscr{P}_R$ we define

$$\mathcal{O}_{kK} = \{(k', K') \in \mathscr{P}_R: (k, K) \leqq (k', K')\}\,;$$

the sets $\mathcal{O}_{kK}$ form a base for a topology on $\mathscr{P}_R$, and C_R is the complete BA of regular open sets in this topology.

This construction of C_R is essentially found in MARTIN and SOLOVAY [1970]. We proceed to describe the basic properties of these algebras.

1.13. LEMMA. *C_R satisfies the κ^+-chain condition.*

PROOF. If $\mathcal{O}_{kK} \cap \mathcal{O}_{lL} = 0$, then $k \neq l$; since $|[\kappa]^{<\omega}| = \kappa$, the lemma follows. $\square$

Next, note the following properties of the topology above. For any $z \in \mathcal{P}_R$, let $b_R z$ be the interior of the closure of $\mathcal{O}_z$; thus $b_R z \in C_R$. Then, as is easily checked for any partial ordering,

$$b_R z = \{w \in \mathcal{P}_R : \forall w' \geqq w \exists z' \geqq z(z' \geqq w')\} ;$$

$$-b_R z = \{w \in \mathcal{P}_R : \forall z' \geqq z(z' \ngeqq w)\} .$$

We also need the following specific property of our partial ordering:

1.14. LEMMA. $b_R(k, K) = \left\{(l, L) \in \mathcal{P}_R : k \subseteq l \cup \left(\kappa \backslash \bigcup A_R\right), K \subseteq L, l \cap \bigcup K \subseteq k\right\}.$

PROOF. First suppose that $(l, L) \in b_R(k, K)$. Suppose that $\alpha \in k \cap \bigcup A_R$; we show that $\alpha \in l$, which thus establishes the first inclusion above. Say $\alpha \in x \in A_R$. Then $(l, L) \leqq (l, L \cup \{x\})$, so there is (m, M) such that $(l, L \cup \{x\}) \leqq (m, M)$ and $(k, K) \leqq (m, M)$. Now $\alpha \in m \cap \bigcup (L \cup \{x\})$, so $\alpha \in l$. Next, suppose that $K \backslash L \neq 0$; say $y \in K \backslash L$. By independence, choose $\alpha \in y \backslash (\bigcup L \cup l)$. Then $(l, L) \leqq (l \cup \{\alpha\}, L)$, so there is (m, M) with $(l \cup \{\alpha\}, L) \leqq (m, M)$ and $(k, K) \leqq (m, M)$. Then $\alpha \in m \cap \bigcup K$, so $\alpha \in k$. Also, $\alpha \in \bigcup A_R$, so $\alpha \in l$ by the above, a contradiction. Finally, suppose that $\alpha \in l \cap \bigcup K$. Choose (m, M) so that $(l, L) \leqq (m, M)$ and $(k, K) \leqq (m, M)$. Then $\alpha \in m \cap \bigcup K \subseteq \kappa$, as desired.

Conversely, let (l, L) satisfy the conditions in the braces. Suppose that $(l, L) \leqq (m, M)$. Then $(k, K) \subseteq (k \cup m, K \cup M)$, since

$$\begin{aligned}(k \cup m) \cap \bigcup K &\subseteq k \cup \left(m \cap \bigcup K\right) \subseteq k \cup \left(m \cap \bigcup L \cap \bigcup K\right) \\ &\subseteq k \cup \left(l \cap \bigcup K\right) \subseteq k\end{aligned}$$

and $(m, M) \subseteq (k \cup m, K \cup M)$, since

$$\begin{aligned}(k \cup m) \cap \bigcup M &\subseteq m \cup \left(k \cap \bigcup M\right) \\ &\subseteq m \cup \left[\left(l \cup \left(\kappa \backslash \bigcup A_R\right)\right) \cap \bigcup M\right] \\ &= m \cup \left(l \cap \bigcup M\right) \subseteq m .\end{aligned}$$

Thus $(l, L) \in b_R(k, K)$. This finishes the proof of Lemma 1.14. $\square$

For each $\alpha < \kappa$ let $a_R\alpha = b_R(\{\alpha\}, 0)$; these elements will be used in the proof of the following lemma.

1.15. LEMMA. *C_R is completely generated by a set with at most κ elements.*

PROOF. Lemma 1.14 yields the following:

(1) $\quad a_R\alpha = \{(l, L) \in \mathcal{P}_R : \alpha \in l\}$ if $\alpha \in \bigcup A_R$;

(2) $\quad a_R\alpha = \mathcal{P}_R$ if $\alpha \in \kappa \backslash \bigcup A_R$;

(3) $\quad -a_R\alpha = \left\{(l, L) : \alpha \in \bigcup L \backslash l\right\}$ if $\alpha \in \bigcup A_R$.

Hence, using Lemma 1.14 further, we get

(4) $$b_R(k, K) = \bigcap_{\alpha \in k} a_R\alpha \cap \bigcap_{\alpha \in \bigcup K \backslash k} -a_R\alpha = \prod_{\alpha \in k} a_R\alpha \cdot \prod_{\alpha \in \bigcup K \backslash k} -a_R\alpha .$$

Thus, C_R is generated by $\{a_R\alpha : \alpha < \kappa\}$, as desired. □

1.16. LEMMA. $|C_R| = 2^\kappa$.

PROOF. By Lemma 1.13, every join or meet is a join or meet over a subset of the index of power $\leqq \kappa$. Hence, by Lemma 1.15 it easily follows that $|C_R| \leqq 2^\kappa$. Now we exhibit 2^κ elements of C_R. For each $t \in A_R$ we have

$$b_R(0, \{t\}) = \{(l, L) \in \mathcal{P}_R : t \in L \text{ and } l \subseteq \kappa \backslash t\}$$

by Lemma 1.14. So $b_R(0, \{t\}) \neq b_R(0, \{t'\})$ if $t \neq t'$, as desired. □

Now let $R \subseteq \mathcal{P}\kappa$, $|\mathcal{P}\kappa \backslash R| = 2^\kappa$. We say that R is *represented* in a complete BA D by $x \in {}^\kappa D$ provided that

(*) $$R = \left\{a \subseteq \kappa : \sum \{x\alpha : \alpha \in t_a\} = 1\right\}.$$

(Remember that t maps $\mathcal{P}\kappa$ onto A_R.)

1.17. LEMMA. *If D is a complete* BA *of power 2^κ, then there are at most 2^κ sets $R \subseteq \mathcal{P}\kappa$ with $|\mathcal{P}\kappa \backslash R| = 2^\kappa$ which are representable in D by some $x \in {}^\kappa D$.*

PROOF. There are only 2^κ functions from κ into D. □

1.18. LEMMA. *For any $R \subseteq \mathcal{P}\kappa$ with $|\mathcal{P}\kappa \backslash R| = 2^\kappa$, the function a_R represents R in C_R.*

PROOF. Suppose that $a \in \mathcal{P}\kappa \backslash R$. Then by (4) in the proof of Lemma 1.15 we have

$$0 \neq b_R(0, \{t_a\}) = \prod \{-a_R\alpha : \alpha \in t_a\},$$

so that $\Sigma\{a_R\alpha: \alpha \in t_a\} \neq 1$. So $\supseteq$ in (*) holds. Now let $a \in R$; we need to show that $\bigcup\{a_R\alpha: \alpha \in t_a\}$ is dense in the topological space $\mathscr{P}_R$. Let $(k, K) \in \mathscr{P}_R$. Choose $\alpha \in t_a \setminus \bigcup K$ by independence. Then $(k \cup \{\alpha\}, K) \in \mathcal{O}_{kK} \cap a_R\alpha$, as desired. The proof is complete. $\square$

We are now prepared to prove the theorem.

1.19. THEOREM. *For each infinite κ there are exactly 2^{2^κ} pairwise non-isomorphic complete BAs of power 2^κ.*

PROOF. Define $R \equiv S$ iff $R, S \subseteq \mathscr{P}\kappa$, $|\mathscr{P}\kappa \setminus R| = |\mathscr{P}\kappa \setminus S| = 2^\kappa$, and $C_R \cong C_S$. Note that if f is an isomorphism from a complete BA D onto a complete BA E, and $x \in {}^\kappa D$ represents R in D, then $f \circ x$ represents R in E. So by Lemma 1.17, each $\equiv$-class has at most 2^κ elements. So the theorem follows from Lemma 1.18. $\square$

2. Construction of complicated Boolean algebras

As an illustration of the ideas in SHELAH [1983] we shall construct a large family of pairwise unembeddable rigid BAs (with strong notions of rigidity and unembeddability). The construction is in two parts: a purely combinatorial part, and a construction of BAs from certain combinatorial objects.

K_{tr} is the class of all relational structures I such that:

(1) the universe of I is a subset of ${}^{\omega\geq}\lambda$ for some λ, closed under initial segments;

(2) the relations of I are as follows:

$P_i = \{\eta \in I: \text{length}(\eta) = i\}$ for each $i \leq \omega$;

$\lessdot = \{(\eta, \nu) \in I \times I: \text{length}(\eta) < \text{length}(\nu)$ and $\eta = \nu \restriction \text{length}(\eta)\}$;

$< = \{(\eta^\frown\langle\alpha\rangle, \eta^\frown\langle\beta\rangle): \eta^\frown\langle\alpha\rangle, \eta^\frown\langle\beta\rangle \in I$ and $\alpha < \beta\}$;

$Eq_i = \{(\eta, \nu) \in I \times I: \text{length}(\eta), \text{length}(\nu) \geq i$ and $\eta \restriction i = \nu \restriction i\}$ for each $i < \omega$;

$z = \{\langle\cdot\rangle\}$, where $\langle\cdot\rangle$ is the empty sequence.

L is a language appropriate for K_{tr}. If $\langle I_t: t \in T\rangle$ is a system of L-structures, then $\Sigma_{t\in T} I_t$ is the *disjoint union* of them: its universe is $\bigcup_{t\in T} I_t \times \{t\}$ (I_t is identified with its universe), and if $\boldsymbol{R}$ is an n-ary relation symbol of L, then the corresponding relation of $\Sigma_{t\in T} I_t$ is

$$\{\langle(a_1, t), \ldots, (a_n, t)\rangle: t \in T, I_t \models \boldsymbol{R}[a_1, \ldots, a_n]\}.$$

For $t \in T$ we denote by I_t^- the structure $\Sigma_{s\neq t} I_s$.

Let L_{alg} be the language which has an m-ary operation symbol $\boldsymbol{F}_{mn}$ for all $m, n < \omega$. Let L' be a joint expansion of L and L_{alg} with an additional unary relation symbol $\boldsymbol{P}$. If I is an L-structure, then $M(I)$ is the following L'-structure: its L_{alg}-reduct is the absolutely free algebra generated by I, its L-reduct has all the relations of I, and $\boldsymbol{P}$ is interpreted as I. In the terms of L_{alg}, we always write the operation symbols to the left, and we use a standard sequence of variables $v_0, v_1, \ldots$. A term τ of L_{alg} is *initialized* if for some $m \in \omega$ the variables which occur in τ are $v_0, \ldots, v_{m-1}$, and they occur in that order in τ without repetitions; we call m the *type* of τ. If σ and τ are initialized terms of type m and n, respectively, $\bar{c} \in {}^mI$, $\bar{d} \in {}^nI$, and $\sigma\bar{c} = \tau\bar{d}$, then $\sigma = \tau$ and $\bar{c} = \bar{d}$ (proof by induction on σ). Every element of $M(I)$ can be written in the form $\tau\bar{c}$, where τ is an initialized term and $\bar{c} \in {}^mI$, m the type of τ, and this expression is unique. For each $a \in M(I)$ we denote this τ by τ_a, and $\bar{c}$ by $\bar{c}_a$.

Let $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1} \in M(I)$. We write

$$\langle a_0, \ldots, a_{m-1} \rangle \approx \langle b_0, \ldots, b_{m-1} \rangle \pmod{M(I)}$$

if $\tau_{ai} = \tau_{bi}$ for each $i < m$, and $\bar{c}_{a0}{}^\frown \cdots {}^\frown \bar{c}_{a(m-1)}$ satisfies the same quantifier-free formulas $\bar{c}_{b0}{}^\frown \cdots {}^\frown \bar{c}_{b(m-1)}$ in I.

Let I and J be L-structures. We say that I is $\psi(\bar{x}, \bar{y})$-*unembeddable* in J provided that: $\psi(\bar{x}, \bar{y})$ is an L-formula with $\bar{x}$ and $\bar{y}$ of the same length, and for every function $f: I \to M(J)$ there exist sequences $\bar{a}, \bar{b}$ in I both of the length of $\bar{x}$ such that $I \models \psi[\bar{a}, \bar{b}]$, fa_i has the same length as fb_i for each $i < \text{length}\,\bar{a}$, and $f(\bar{a}) \approx f(\bar{b}) \pmod{M(J)}$. ($f(\bar{a})$ is the concatenation of $f(a_0), \ldots, f(a_i), \ldots, i < \text{length}(\bar{a})$; similarly for $f(\bar{b})$.) Finally, we say that K_{tr} has the *full* (χ, λ)-ψ-*bigness property* if there are $I_i \in K_{\text{tr}}$ $(i < \chi)$ such that $|I_i| = \lambda$ and I_i is ψ-unembeddable in I for all $i < \chi$. We shall be interested only in the following formula $\psi(x_0, x_1, y_0, y_1)$:

$$\bigvee_{i<\omega} [P_{i+1}x_0 \wedge P_{i+1}y_0 \wedge P_\omega x_1 \wedge x_1 = y_1 \wedge x_0 < x_1 \\ \wedge Eq_i(x_0, y_0) \wedge y_0 < x_0] .$$

It expresses that x_0 and y_0 have the form $\eta^\frown\langle\alpha\rangle$ and $\eta^\frown\langle\beta\rangle$, respectively, with $\beta < \alpha$, and $x_1 = y_1$ has domain ω and extends x_0.

Now we prove a combinatorial theorem about these notions.

2.1. THEOREM. *If $\omega < \lambda \leqq \lambda^*$ with λ regular, then K_{tr} has the full (λ, λ^*)-ψ-bigness property.*

PROOF. Let $S = \{\delta < \lambda: \delta \text{ is a limit ordinal and } \operatorname{cf}\delta = \omega\}$. Thus, S is a stationary subset of λ. Write $S = \bigcup_{i<\lambda} S_i$ with the S_i stationary and pairwise disjoint. For each $\delta \in S$ choose $\eta_\delta \in {}^\omega\delta$ strictly increasing with sup δ. For all $i < \lambda$ let

$$I_i = \bigcup_{n<\omega} {}^n\lambda^* \cup \{\eta_\delta : \delta \in S_i\} .$$

Thus, $|I_i| = \lambda^*$, and I_i has a natural L-structure. The rest of the proof is devoted to showing that for an arbitrary $i < \lambda$, I_i is ψ-unembeddable in I_i^-.

To this end, let $f: I_i \to M(I_i^-)$ be given. For any $a \in I_i$ let

$$\operatorname{orco}(a) = \sup\{\gamma < \lambda: \text{there is a } k < \text{length}(\bar{c}_{fa}) \text{ such that } \bar{c}_{fa}(k) = (t, j) \\ \text{for some } t \text{ and } j, \text{ and } \gamma = j \text{ or } \gamma = \sup(\operatorname{ran}(t))\} .$$

Let

(3) $$C = \{\delta < \lambda\colon \text{for all } \eta \in {}^{\omega>}\delta,\ \mathrm{orco}(\eta) < \delta\}\,.$$

Clearly, C is club in λ. We shall use C later on.

Now with each $0 \neq \eta \in {}^{\omega>}\lambda$ we associate an equivalence relation E_η on λ. Let γ be the last value of η. For each $\alpha < \lambda$, let m_α be the length of $\bar{c}_{f(\eta^\frown\langle\alpha\rangle)}$, and for each $k < m_\alpha$ write $\bar{c}_{f(\eta^\frown\langle\alpha\rangle)}(k) = (t_{\alpha k}, j_{\alpha k})$. Now we set $\alpha E_\eta \beta$ iff $f(n^\frown\langle\alpha\rangle) \approx f(\eta^\frown\langle\beta\rangle) \pmod{M(I_i^-)}$, and for all $k < m_\alpha$, if $j_{\alpha k} < \gamma$ or $j_{\beta k} < \gamma$, then $j_{\alpha k} = j_{\beta k}$, and if $l < \mathrm{length}(t_{\alpha k})$ and $t_{\alpha k}(l) < \gamma$ or $t_{\beta k}(l) < \gamma$, then $t_{\alpha k}(l) = t_{\beta k}(l)$. Note that $f(\eta^\frown\langle\alpha\rangle) \approx f(\eta^\frown\langle\beta\rangle) \pmod{M(I_i^-)}$ implies that $m_\alpha = m_\beta$ and $\mathrm{length}(t_{\alpha k}) = \mathrm{length}(t_{\beta k})$ for all $k < m_\alpha$. Clearly, E_η is an equivalence relation on λ. Now we claim

(4) There are $<\lambda$ equivalence classes under E_η.

In fact, suppose that $\Gamma \in [\lambda]^\lambda$ consists of pairwise inequivalent elements under E_η. Since L' is countable while λ is regular and uncountable, we can assume that $f(\eta^\frown\langle\alpha\rangle) \approx f(\eta^\frown\langle\beta\rangle) \pmod{M(I_i^-)}$ for all $\alpha, \beta \in \Gamma$. Since $\gamma < \lambda$, we can assume that $j_{\alpha k} = j_{\beta k}$ if one of them is $<\gamma$, for all $\alpha, \beta \in \Gamma$ and all $k < m_\alpha$, and also that if $k < m_\alpha$ and $\mathrm{length}(t_{\alpha k})$ is finite, then $t_{\alpha k}(l) = t_{\beta k}(l)$ if one of them is $<\gamma$, for all $l < \mathrm{length}(t_{\alpha k})$ and all $\alpha, \beta \in \Gamma$. Now by construction, any infinite length $t_{\alpha k}$ has the form η_δ for some δ. Hence, we may assume that $t_{\alpha k} = t_{\beta k}$ if one of them has the form η_δ with $\delta \leqq \gamma$, for all $\alpha, \beta \in \Gamma$, and all $k < m_\alpha$. Now if $\alpha \in \Gamma$ and $k < m_\alpha$ with $t_{\alpha k}$ of infinite length, with some terms $>\gamma$, choose $l_{\alpha k}$ minimum such that $t_{\alpha k}(l_{\alpha k}) \geqq \gamma$. We may assume that for all $\alpha, \beta \in \Gamma$, $l_{\alpha k} = l_{\beta k}$ in these circumstances. Then we may assume that in these cases $t_{\alpha k}(l) = t_{\beta k}(l)$ for all $l < l_{\alpha k}$. But then $\alpha E_\eta \beta$ for any two members of Γ, a contradiction. Thus (4) holds.

Now we define a continuous function $\alpha\colon \lambda \to \lambda$: $\alpha 0 = 0$, and $\alpha\delta = \bigcup_{\kappa<\delta} \alpha\kappa$ for δ limit $<\lambda$. Now suppose that $\kappa < \lambda$ and $\alpha\kappa$ has been defined. We define $\beta\colon \omega \to \lambda$ by induction: $\beta 0 = \alpha\kappa$. Suppose that βj has been defined. For each $\eta \in {}^{\omega>}\beta j$ let $\Gamma_\eta \subseteq \lambda$ have exactly one element from each E_η-class. Thus, $|\Gamma_\eta| < \lambda$ by (4). Set

$$\beta(j+1) = \Big(\bigcup \{\Gamma_\eta\colon \eta \in {}^{\omega>}\beta j\} \cup \beta j\Big) + 1\,.$$

Finally, set $\alpha(\kappa+1) = \bigcup_{j\in\omega} \beta_j$. This defines α. For each $\kappa < \lambda$ we have:

(5) for all $0 \neq \eta \in {}^{\omega>}\alpha(\kappa+1)$ and all $\beta \in \lambda$ there is a $\gamma < \alpha(\kappa+1)$ such that $\beta E_\eta \gamma$.

Let $C_1 = \{\kappa < \lambda\colon \text{for all } j < \kappa,\ \alpha j < \kappa\}$. Thus, C_1 is club in λ. By stationarity of S_i, choose $\delta \in S_i \cap C \cap C_1$. Let $\bar{c}_{f\eta_\delta} = \langle (u_0, j_0), \ldots, (u_{k-1}, j_{k-1})\rangle$. Thus, $j_l \neq i$ for all $l < k$. Set

$$V = \Big(\bigcup_{l<k} \mathrm{ran}(u_l)\Big) \cup \{j_0, \ldots, j_{k-1}\}\,.$$

Now each u_l with infinite length is strictly increasing with $\sup \neq \delta$, by construction of the I_u's. Hence, $V \cap \delta$ is bounded in δ, say by $\eta_\delta n$. Since $\delta \in C_1$ we have

$\sup_{s<\delta} \alpha s = \delta$. Hence, we can choose $\kappa + 1 < \delta$ so that $\eta_\delta n < \alpha(\kappa + 1) < \delta$. Let m be maximum such that $\eta_\delta m < \alpha(k+1)$. Then by (5) choose $\gamma < \alpha(\kappa+1)$ so that $\eta_\delta(m+1)E_\nu\gamma$, with $\nu = \eta_\delta \restriction (m+1)$.

We claim that

$$(\eta_\delta \restriction (m+2), \eta_\delta, \nu^\frown\langle\gamma\rangle, \eta_\delta)$$

shows the ψ-unembeddability of I_i into I_i^- via f. Clearly, this quadruple satisfies ψ in I_i. Now we want to show that

$$\langle f(\eta_\delta \restriction (m+2)), f\eta_\delta\rangle \approx \langle f(\nu^\frown\langle\gamma\rangle), f\eta_\delta\rangle (\mathrm{mod}\, M(I_i^-)) .$$

Since $\eta_\delta(m+1)E_\nu\gamma$, we know that $f(\eta_\delta \restriction (m+2)) \approx f(\nu^\frown\langle\gamma\rangle)$ (mod $M(I_i^-)$). Let $a = f\eta_\delta$, $b = f(\eta_\delta \restriction (m+2))$, $d = f(\nu^\frown\langle\gamma\rangle)$. Thus, $\tau_b = \tau_d$, and $\bar{c}_b$ satisfies the same quantifier-free formulas as $\bar{c}_d$ in I. It remains to consider formulas relating a value of $\bar{c}_b$ with one of a and the corresponding formula for $\bar{c}_d$ and a. Say $\bar{c}_b = \langle(v_0, p_0), \ldots, (v_{l-1}, p_{l-1})\rangle$, $\bar{c}_d = \langle(w_0, q_0), \ldots, (w_{l-1}, q_{l-1})\rangle$. Note that length$(v_x)$ = length(w_x) since $\bar{c}_b$ and $\bar{c}_d$ realize the same quantifier-free type in I_i^-. Also, $\eta_\delta \restriction (m+2) \in {}^{\omega>}\delta$, so $\delta \in C$ implies that orco$(\eta_\delta(m+2)) < \delta$. Thus, ran$(v_x) \subseteq \delta$, and similarly ran$(w_x) \subseteq \delta$. These statements are true for any $x < l$. Now by symmetry it is enough to consider the following cases.

Case 1. $(v_x, p_x) = (u_y, j_y)$. Then ran$(v_x) \subseteq V \cap \delta$, so for any $s <$ length(v_x) we have $v_x(s) < \eta_\delta n \leqq \eta_\delta m$, so by $\eta_\delta(m+1)E_\nu\gamma$ we get $v_x(p) = w_x(p)$. Hence, $v_x = w_x$ and $w_x = u_y$. Now $p_x = j_y \in V \cap \delta$, hence $p_x < \eta_\delta n \leqq \eta_\delta m$, so $\eta_\delta(m+1)E_\nu\gamma$ yields $p_x = q_x$. Hence $(w_x, q_x) = (u_y, j_y)$.

Case 2. $(v_x, p_x) \lessdot (u_y, j_y)$. Just like Case 1.

Case 3. $(u_y, j_y) \lessdot (v_x, p_x)$. Similar to Case 1.

Case 4. $(v_x, p_x) < (u_y, j_y)$. Say length(v_x) = length$(u_y) = s+1$. Thus, $v_x \restriction s = u_y \restriction s$, and the argument of Case 1 gives $v_x \restriction s = w_x \restriction s$ and $p_x = q_x$. If $u_y s \leqq w_x s$, then $u_y s < \delta$ and we easily get $v_x s = w_x s$, a contradiction. Hence, $w_x s < u_y s$, so $(w_x, q_x) < (u_y, j_y)$.

Case 5. $(u_y, j_y) < (v_x, p_x)$. Similar to Case 4.

Case 6. $Eq_i((v_x, p_x), (u_y, j_y))$. Clearly, then, $Eq_i((w_x, q_x), (u_y, j_y))$. This completes the proof of Theorem 2.1. □

Now we turn to the construction of BAs from members of K_{tr}. For any $I \in K_{\mathrm{tr}}$ let $B_{\mathrm{tr}}(I)$ be the BA freely generated by $\langle x_\eta : \eta \in I\rangle$ except that $\eta \lessdot \nu$ implies that $x_\nu \leqq x_\eta$. That is, $B_{\mathrm{tr}}(I) = F/I$, where F is the free BA on $\langle x_\eta : \eta \in I\rangle$ and $I = \langle x_\nu \cdot -x_\eta : \eta \lessdot \nu\rangle^{\mathrm{id}}$, with x_η identified with its equivalence class under I.

2.2. LEMMA. *In $B_{\mathrm{tr}}(I)$, if $\eta_1, \ldots, \eta_m \in I$ are distinct and $\xi_1, \ldots, \xi_n \in I$ are distinct, then $x_{\eta 1} \cdot \cdots \cdot x_{\eta m} \cdot -x_{\xi 1} \cdot \cdots \cdot -x_{\xi m} = 0$ iff there exist i, j such that $\xi_j \lesseqqgtr \eta_i$.*

PROOF. Clearly $\Leftarrow$ holds. For $\Rightarrow$, suppose the implication fails. Then there exist $\nu_1, \rho_1, \ldots, \nu_p, \rho_p \in I$ with $\rho_1 \lessdot \nu_1, \ldots, \rho_p \lessdot \nu_p$ such that, in the free BA,

(6) $\qquad x_{\eta 1} \cdot \cdots \cdot x_{\eta m} \cdot -x_{\xi 1} \cdot \cdots \cdot -x_{\xi n} \leqq x_{\nu 1} \cdot -x_{\rho 1} + \cdots + x_{\nu p} \cdot -x_{\rho p}$.

Let f be the endomorphism of the free BA such that $fx_\sigma = 1$ if $\sigma \leqq \eta_i$ for some i, and $fx_\sigma = 0$ otherwise. Thus, $f(x_{\eta 1} \cdot \cdots \cdot x_{\eta m} \cdot -x_{\xi 1} \cdot \cdots \cdot -x_{\xi n}) = 1$. If $fx_{\nu j} = 1$ then $\nu_j \leqq \eta_i$ for some i; hence $\rho_j \leqq \eta_i$ and $fx_{\rho j} = 1$. So $f(x_{\nu 1} \cdot -x_{\rho 1} + \cdots + x_{\nu p} \cdot -x_{\rho p}) = 0$. This contradicts (6). □

Let $I \in K_{\mathrm{tr}}$ and let B be a BA. We say that B is *representable* in $M(I)$ if there is a function $f\colon B \to M(I)$ such that if $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1} \in B$ and $\langle fa_0, \ldots, fa_{m-1} \rangle \approx \langle fb_0, \ldots, fb_{m-1} \rangle$ (mod $M(I)$), then $\langle a_0, \ldots, a_{m-1} \rangle$ and $\langle b_0, \ldots, b_{m-1} \rangle$ satisfy the same quantifier-free formulas in B.

2.3. LEMMA. *Suppose I is ψ-unembeddable in J and B is a* BA *representable in $M(J)$. Then B_{tr} is not embeddable in a factor of B.*

PROOF. Let $g\colon B \to M(J)$ be a representation of B in $M(J)$, and suppose that h embeds $B_{\mathrm{tr}}(I)$ into a factor of B. Let $f\eta = ghx_\eta$ for all $\eta \in I$. Thus, $f\colon I \to M(J)$. Since I is ψ-unembeddable in J, there exist $\nu_1, \nu_2, \nu \in I$ and $n \in \omega$ such that length$(\eta) = \omega$, length$(\nu_1) =$ length$(\nu_2) = n+1$, $\nu_1 < \eta$, $\nu_1 \restriction n = \nu_2 \restriction n$, $\nu_2(n) < \nu_1(n)$, and $\langle f\nu_1, f\eta \rangle \approx \langle f\nu_2, f\eta \rangle$ (mod $M(J)$). Since g is a representation, $\langle hx_{\nu 1}, hx_\nu \rangle$ satisfies the same quantifier-free formulas as $\langle hx_{\nu 2}, hx_\eta \rangle$. In particular, $hx_\eta \leqq hx_{\nu 1}$ iff $hx_\eta \leqq hx_{\nu 2}$. Since h is an embedding, $x_\eta \leqq x_{\nu 1}$ iff $x_\eta \leqq x_{\nu 2}$. Now $x_\eta \leqq x_{\nu 1}$, so $x_\eta \leqq x_{\nu 2}$, contradicting Lemma 2.2. □

A BA B is called *embedding-rigid* if for all non-zero $a, b \in B$ with $a \nleqq b$, $B \restriction a$ cannot be embedded in $B \restriction b$.

2.4. LEMMA. *If B is embedding-rigid, then B is mono-rigid, hence rigid.*

PROOF. Recall that *mono-rigid* means that there do not exist non-trivial one-to-one endomorphisms. Suppose on the contrary that f is a non-trivial one-to-one endomorphism of B. Say $fx \neq x$. If $x \nleqq fx$, then $f \restriction (A \restriction x)$ embeds $A \restriction x$ into $A \restriction fx$, a contradiction. If $fx \nleqq x$, then $-x \nleqq f(-x)$, and again we get a contradiction. □

We shall use the following general construction for BAs. Let A and B be BAs and $b \in B$. The BA

$$(B \restriction -b) \times ((B \restriction b) * A)$$

is denoted by Att(A, b, B). It is called the *result of attaching A to B at b*, and is considered as a BA extending B.

Let λ be uncountable and regular. By Theorem 2.1, let $\langle I_i\colon i < \lambda \rangle$ attest to the full (λ, λ)-ψ_{tr}-bigness property. Let $\Gamma \in [\lambda]^\lambda$ also be given; say γ is a one-to-one mapping of λ onto Γ. We now construct a sequence $\langle B_i\colon i \leqq \lambda \rangle$ of BAs. Write $\lambda \backslash \{0, 1\} = \bigcup_{0<i<\lambda} \Delta_i$, the Δ_i's pairwise disjoint and of power λ, and let $\Delta_0 = \{0, 1\}$. Set $B_0 = \Delta_0$, a two-element BA. For i limit $\leqq \lambda$, let $B_i = \bigcup_{j<i} B_j$. Now

suppose that B_i has been defined with universe $\bigcup_{j<i} \Delta_j$. Let b_i be the first element of B_i different from $0,1$ and all b_j, $j<i$. We set $B_{i+1} = \mathrm{Att}(B_{\mathrm{tr}}(I_{\gamma i}), b_i, B_i)$, and we may take it to have universe $\bigcup_{j\leq i} \Delta_j$. This construction depends on Γ and γ, and if necessary we shall indicate this dependence by superscripts, e.g. $B_\lambda^{\Gamma\gamma}$.

2.5. LEMMA. *B_λ is representable in $M(\Sigma_{i<\lambda} I_{\gamma i})$.*

PROOF. With each Boolean term ρ with variables among $v_0, \ldots, v_{m-1}$, v_{m-1} actually occurring in ρ, associate an m-ary operation symbol $\boldsymbol{F}_\rho$ of L_{alg} in a one-to-one fashion.

We define fb for $b \in B_\lambda$ by induction on the first j such that $b \in B_j$. If $j=0$, then $b=0$ or $b=1$; we set $f0 = (\langle\ \rangle, \gamma 0)$, $f1 = (\langle 0\rangle, \gamma 0)$. Now suppose that $b \neq 0, 1$, with j as indicated. Thus, j is a successor ordinal. By the construction of B_j we can write

$$b = \left(a, \sum_{l<m} d_l \cdot e_l\right),$$

with $c \in B_{j-1} \restriction -b_{j-1}$, $d_l \in B_{j-1} \restriction b_{j-1}$, $e_l \in B_{\mathrm{tr}}(I_{\gamma(j-1)})$ for all $l<m$. For each $l<m$ let $e_l = \rho_l(t_{0l}, \ldots, t_{nl})$, ρ_l a Boolean term involving all of $v_0, \ldots, v_n$, and $t_{0l}, \ldots, t_{nl} \in I_{\gamma(j-1)}$. Let $\boldsymbol{G}$ be a $(2m+3)$-ary operation symbol of L_{alg}. Let the terms $\sigma_0, \ldots, \sigma_{2m+2}$ be obtained from $\tau_{fb(j-1)}, \tau_{f(-b(j-1))}, \tau_{fc}, \tau_{fd0}, \ldots, \tau_{fd(m-1)}$, $\boldsymbol{F}_{\rho 0}(v_0, \ldots, v_n), \ldots, \boldsymbol{F}_{\rho(m-1)}(v_0, \ldots, v_n)$ by simply increasing the indices of the variables so that in the sequence $\sigma_0, \ldots, \sigma_{2m+2}$ the variables form an initial segment of the standard sequence of variables, appearing in order from left to right, and let $\bar{d}$ be

$$\begin{gathered}\bar{c}_{fb(j-1)}{}^\frown \bar{c}_{f(-b(j-1))}{}^\frown \bar{c}_{fc}{}^\frown \bar{c}_{fd0}{}^\frown \cdots {}^\frown \bar{c}_{fd(m-1)}{}^\frown \\ \langle (t_{00}, \gamma(j-1)), \ldots, (t_{n0}, \gamma(j-1))\rangle^\frown \cdots {}^\frown \\ \langle (t_{0,m-1}, \gamma(j-1)), \ldots, (t_{n,m-1}, \gamma(j-1))\rangle .\end{gathered}$$

Finally, let $fb = (\boldsymbol{G}(\sigma_0, \ldots, \sigma_{2m+2}))(\bar{d})$. This finishes the definition of f.

Now suppose that $a_0, \ldots, a_{p-1}, c_0, \ldots, c_{p-1}$ are elements of B_λ and

$$\langle fa_0, \ldots, fa_{p-1}\rangle \approx \langle fc_0, \ldots, fc_{p-1}\rangle \left(\mathrm{mod}\, M\left(\sum_{i<\lambda} I_{\gamma i}\right)\right). \tag{7}$$

Assume that

$$a_0 \cdot \cdots \cdot a_h \cdot -a_{h+1} \cdot \cdots \cdot -a_{p-1} = 0 . \tag{8}$$

We want to show that

$$c_0 \cdot \cdots \cdot c_h \cdot -c_{h+1} \cdot \cdots \cdot -c_{p-1} = 0 . \tag{9}$$

If $a_i = 0$, then fa_i is $(\langle\ \rangle, \gamma 0)$, and $(\langle\ \rangle, \gamma 0) \in z$, in $M(\Sigma_{i<\lambda} I_{\gamma i})$. So (7) yields that $fc_i = fa_i$ and $c_i = 0$. Similarly, $a_i = 1$ implies $c_i = 1$. So we may assume that each a_i is $\neq 0, 1$.

Suppose that j is minimum such that for all $v < p$, $a_v \in B_j$; we proceed by induction on j. Say for $v < p$

$$(10) \qquad a_v = \left(a'_v, \sum_{l<mv} a''_{vl} \cdot a'''_{vl}\right),$$

with $a'_v \in B_{jv-1} \upharpoonright -b_{jv-1}$, $a''_{vl} \in B_{jv-1} \upharpoonright b_{jv-1}$, $a'''_{vl} \in B_{\mathrm{tr}}(I_{\gamma(jv-1)})$, $jv \leqq j$. Similarly, we assume that all $c_v \neq 0, 1$, and s is minimum such that for all $v < p$, $c_v \in B_s$. Say for $v < p$

$$(11) \qquad c_v = \left(c'_v, \sum_{l<nv} c''_{vl} \cdot c'''_{vl}\right),$$

with $c'_v \in B_{sv-1} \upharpoonright -b_{sv-1}$, $c''_{vl} \in B_{sv-1} \upharpoonright b_{sv-1}$, $c'''_{vl} \in B_{\mathrm{tr}}(I_{\gamma(sv-1)})$, $sv \leqq s$. Note by (7) that $m_v = n_v$ for all $v < p$. Applying (8) to the first coordinates in (10) we get

$$a'_0 \cdot \cdots \cdot a'_h \cdot -a'_{h+1} \cdot \cdots \cdot -a'_p \cdot -b_{j(h+1)-1} \cdot \cdots \cdot -b_{jp-1} = 0 .$$

Now (7) yields that

$$\begin{aligned} &\langle fa'_0, \ldots, fa'_{p-1}, f(-b_{j(h+1)-1}), \ldots, f(-b_{jp-1})\rangle \\ &\quad \approx \langle fc'_0, \ldots, fc'_{p-1}, f(-b_{s(h+1)-1}), \ldots, f(-b_{sp-1})\rangle \left(\mathrm{mod}\left(\sum_{i<\lambda} I_{\gamma i}\right)\right), \end{aligned}$$

so the induction hypothesis gives:

$$(12) \qquad c'_0 \cdot \cdots \cdot c'_h \cdot -c'_{h+1} \cdot \cdots \cdot -c'_p \cdot -b_{s(h+1)-1} \cdot \cdots \cdot -b_{sp-1} = 0 .$$

Now we proceed to the second coordinates. Suppose $g \in \prod_{v<h} m_v$ and $\Delta \in \prod_{h \leqq v < p} \mathscr{P} m_v$. Then by (8), (10),

$$\prod_{v<h} a''_{v,gv} \cdot a'''_{v,gv} \cdot \prod_{h \leqq v < n} \prod_{l \in \Delta v} - a''_{vl} \cdot b_{jv-1} \cdot \prod_{l \in mv/\Delta v} - a'''_{vl} = 0 ,$$

so by the free product property one of the following holds:

$$(13) \qquad \prod_{v<h} a''_{v,gv} \cdot \prod_{h \leqq v < n} \prod_{l \in \Delta v} - a'''_{vl} \cdot b_{jv-1} = 0 ,$$

$$(14) \qquad \prod_{v<h} a'''_{v,gv} \cdot \prod_{h \leqq v < n} \prod_{l \in mv \setminus \Delta v} - a'''_{vl} = 0 .$$

In the case where (13) holds, the argument for (12) gives

$$(15) \qquad \prod_{v<h} c''_{v,gv} \cdot \prod_{h \leqq v < n} \prod_{l \in \Delta v} - c''_{vl} \cdot b_{sv-1} = 0 .$$

If (14) holds, we use (7) to see that a'''_{vl} and c'''_{vl} are expressed by the same Boolean term applied to sequences satisfying the same quantifier-free Boolean formulas, using Lemma 2.2. So

$$(16) \qquad \prod_{v<h} c'''_{v,gv} \cdot \prod_{h \leq v<n} \prod_{l \in mv/\Delta v} -c'''_{vl} = 0$$

By (12) and all instances of (15) or (16), (9) follows. □

2.6. LEMMA. *For any $i<\lambda$, $B_\lambda \restriction -b_i$ is representable in $M(\Sigma_{i\neq j\in\lambda} I_{\gamma j})$.*

PROOF. The proof is very similar to that of Lemma 2.5, and we just sketch it. We define fb for $b \in (B_\lambda \restriction -b_i)$ by induction on the first j such that $b = b' \cdot -b_i$ for some $b' \in B_j$. The case $j=0$ is treated as before. In the main step, we are assured that $j-1 \neq i$, since otherwise $b' = c \in B_{j-1}$, contradicting the minimality of j. This assures that f maps into $M(\Sigma_{i\neq j\in\lambda} I_{\gamma j})$. The rest of the proof proceeds as before. □

Now we can give the main theorem.

2.7. THEOREM. *For each regular uncountable λ there is a family of 2^λ embedding-rigid BAs of power λ, none embeddable in a factor of another.*

PROOF. Let $\mathscr{A}$ be a family of 2^λ subsets of λ, each of power λ and none a subset of another. For each $\Gamma \in \mathscr{A}$ let B^Γ_λ be as above. Then $\langle B^\Gamma_\lambda : \Gamma \in \mathscr{A} \rangle$ is as desired. In fact, first suppose that $\Gamma, \Delta \in \mathscr{A}$ and $\Gamma \neq \Delta$. To show that B^Γ_λ cannot be embedded in a factor of B^Δ_λ, choose $i \in \Gamma \backslash \Delta$. Let γ be the one-to-one function mapping λ onto Δ used in the construction of B^Δ_λ. Since I_i is ψ-unembeddable in Γ_i, it is clearly ψ-unembeddable in $\Sigma_{j<\lambda} I_{\gamma j}$. By Lemma 2.5, B^Δ_λ is representable in $M(\Sigma_{j<\lambda} I_{\gamma j})$, so by Lemma 2.3, $B_{\mathrm{tr}}(I_i)$ is not embeddable in a factor of B^Δ_λ. Hence B^Γ_λ is not embeddable in a factor of B^Δ_λ.

Finally, fix $\Gamma \in \mathscr{A}$; we show that $B^\Gamma_\lambda = B_\lambda$ is embedding-rigid. Suppose $a \not\leq b$, but f embeds $B_\lambda \restriction a$ into $B_\lambda \restriction b$. Let $c = a \cdot -b$. Then $c \cdot fc = 0$, and f embeds $B_\lambda \restriction c$ into $B_\lambda \restriction fc$. Write $c = b_i$. Then $B_{\mathrm{tr}}(I_i)$ is embeddable in $B_\lambda \restriction c$, hence in $B \restriction fc$, hence in $B \restriction -c = B \restriction -b_i$. By Lemma 2.6, $B \restriction -b_i$ is representable in $M(\Sigma_{i\neq j<\lambda} I_j)$. This contradicts Lemma 2.3. □

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