VOLUME 2

HANDBOOK OF BOOLEAN ALGEBRAS

Edited by
J. Donald Monk
with *Robert Bonnet*

NORTH-HOLLAND

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Edited by

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Introduction to the Handbook

The genesis of the motion of a Boolean algebra (BA) is, of course, found in the works of George Boole; but his works are now only of historical interest – cf. HAILPERIN [1981] in the bibliography (elementary part). The notions of Boolean algebra were developed by many people in the early part of this century – Schröder, Löwenheim, etc. usually working with the concrete operations union, intersection, and complementation. But the abstract notion also appeared early, in the works of Huntington and others. Despite these early developments, the modern theory of BAs can only be considered to have started in the 1930s with works of M.H. Stone and A. Tarski. Since then there has been a steady development of the subject.

The present Handbook treats those parts of the theory of Boolean algebras of most interest to pure mathematicians: the set-theoretical abstract theory and applications and relationships to measure theory, topology, and logic. Aspects of the subject *not* treated here are discussion of axiom systems for BAs, finite Boolean algebras and switching circuits, Boolean functions, Boolean matrices, Boolean algebras with operators – including cylindric algebras and related algebraic forms of logic – and the role of BAs in ring theory and in functional analysis.

The Handbook is divided into two parts (published in three volumes). The first part (Volume 1) is a completely self-contained treatment of the fundamentals of the subject, which mathematicians in various fields may find interesting and useful. Here one will find the main results on disjointness (the Erdös–Tarski theorem), free algebras (the Gaifman–Hales, Shapirovskiĭ–Shelah, and Balcar–Franěk theorems), and the basic decidability and undecidability results for the elementary theory of BAs, as well as the systematic development of the abstract theory (ultrafilters, representation, subalgebras, ideals, topological duality, free algebras, free products, measure algebras, distributivity, interval algebras, superatomic algebras, tree algebras).

The second part of the Handbook (Volumes 2 and 3) is intended to indicate most of the frontiers of research in the subject; it consists of articles which are more or less independent of each other, although most of them assume a knowledge of at least the easier portions of Part I. The second part is arranged in four sections, with two appendices and a bibliography. Section A, Arithmetical properties of BAs, has two chapters: on distributive laws and their relationships to games on BAs, and on disjoint refinements, treating extensively this elementary notion discussed in Part I. Section B, Algebraic properties of BAs, treats subalgebras, particularly the lattice of all subalgebras and the existence of complements in this lattice; cardinal functions on Boolean spaces; the number of BAs of various sorts; endomorphisms of BAs, including the existence of endo-rigid BAs; automorphisms groups; reconstruction of BAs from their automorphism groups; embeddings and automorphisms, especially for complete rigid

BAs; rigid BAs; and homogeneous BAs. Section C is devoted to special classes of BAs: superatomic algebras, mainly thin-tall and related BAs; projective BAs; and two lengthy chapters on countable BAs, with Ketonen's theorem; and on measure algebras, giving an extensive survey of this topic which is perhaps the most important subfield of the theory of BAs for most mathematicians. Section D deals with logical questions: decidable and undecidable theories of BAs in various languages; recursive BAs; Lindenbaum–Tarski algebras; and Boolean-valued models of set theory. Two appendices, on set theory and on topology, explain some more advanced notions used in some places in the Handbook. There is a chart of topological duality. Finally, there is a comprehensive Bibliography on the aspects of the theory of Boolean algebras treated in the Handbook.

Many people contributed to the Handbook by checking manuscripts for mathematical and typographical errors; in addition to several of the authors of the Handbook, the editor is indebted to the following for help of this sort: Hajnal Andréka, Aleksander Błaszczyk, Tim Carlson, Ivo Düntsch, Francisco J. Freniche, Lutz Heindorf, Istvan Németi, Stevo Todorčević, and Petr Vojtaš. Thanks are also due to the North-Holland staff, especially Leland Pierce, for their editorial work.

Contents of Volume 2

Contents of the Handbook

VOLUME 2

VOLUME 3

Part II

TOPICS IN THE THEORY OF BOOLEAN ALGEBRAS

Section A

ARITHMETICAL PROPERTIES OF BOOLEAN ALGEBRAS

This Section contains two chapters on certain arithmetical properties of Boolean algebras: Chapter 8, Distributive laws, by Thomas Jech, extends the discussion of Chapter 5, Section 14, of Part I, connecting distributive laws with Boolean-valued models, κ-closed subsets, and Boolean games. Chapter 9, Disjoint refinements, by Bohuslav Balcar and Petr Simon, gives a comprehensive treatment of this topic, which also plays a prominent role in Part I.

CHAPTER 8

Distributive Laws

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HANDBOOK OF BOOLEAN ALGEBRA
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Let us review some definitions and basic facts from Section 14, Chapter 5 of Part I. A Boolean algebra B satisfies the (κ, λ)*-distributive law* (where κ and λ are cardinal numbers), if

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a(\alpha, \beta) = \sum_{f:\, \kappa\to\lambda} \prod_{\alpha<\kappa} a(\alpha, f(\alpha))\,, \tag{1}$$

for any collection $\{a(\alpha, \beta)\colon \alpha < \kappa, \beta < \lambda\}$ of elements of B. The sum on the right-hand side of (1) ranges over all functions from κ into λ, and the (possibly infinite) Boolean sums and product in (1) exist simultaneously on both sides of the equation. When $\kappa = \lambda = 2$, and $a(0,0) = a$, $a(0,1) = b$, $a(1,0) = a(1,1) = c$, then (1) is just

$$(a + b)\cdot c = a\cdot c + b\cdot c\,.$$

Every Boolean algebra satisfies (1) for finite values of κ; when $\kappa = 2$, (1) becomes

$$\sum_{\xi} a_\xi \cdot \sum_{\eta} b_\eta = \sum_{(\xi,\eta)} a_\xi \cdot b_\eta\,.$$

Thus, we restrict ourselves to the infinite values of κ. Also, to avoid problems with the existence of the sums and products in (1), we concern ourselves only with complete Boolean algebras.

Definition. A complete Boolean algebra B is (κ, λ)*-distributive* if it satisfies (1). B is κ*-distributive* if it is (κ, λ)-distributive for all λ.

Clearly, (κ, μ)-distributivity implies (κ, λ)-distributivity if $\lambda < \mu$. We recall also that if B is (κ, λ)-distributive for all $\lambda < \operatorname{sat}(B)$, then B is κ-distributive.

A *partition* of $a > 0$ is a maximal set W of mutually incompatible x such that $0 < x \leq a$. A partition W_1 is a *refinement* of a partition W_2 if every $x \in W_1$ is $\leq$ some $y \in W_2$.

Proposition 1. (a) *B is (κ, λ)-distributive if and only if*

(2) *for every $a > 0$ and for every set $\{W_\alpha\colon \alpha < \kappa\}$ of partitions of a, each of size $\leq\lambda$, there is a partition W of a that is a refinement of each W_α;*

(b) *B is κ-distributive if and only if*

(3) *for every $a > 0$, every set $\{W_\alpha\colon \alpha < \kappa\}$ of partitions of a has a common refinement.*

The properties (2) and (3) do not use infinite Boolean operations. In fact, they are formulated in terms of the partial order on B. Thus, these properties can be defined for any partial ordering. In fact, the property (3) is invariant under completion, that is if P is a separative partially ordered set and $B = \text{r.o.}(P)$ is the associated complete Boolean algebra, then P satisfies (3) if and only if B does; in particular, if B is the completion of a Boolean algebra A, then A has property (3) iff B does. (For obvious reason, property (2) is not invariant under completion.)

The next proposition assumes some knowledge of Boolean-valued models.

PROPOSITION 2. (a) *B is (κ, λ)-distributive if and only if, with Boolean value* 1, *every function from κ to λ in the generic extension $V[G]$ belongs to the ground model V.*

(b) *B is κ-distributive if and only if every function f: $\kappa \to V$ in $V[G]$ is in V.*

This equivalence is a consequence of the formulation of distributivity in terms of partitions. To see this (in one direction), let f be Boolean-valued name, and let

$$a = \| f \text{ is a function from } \kappa \text{ to } \lambda \| ,$$

and

$$a(\alpha, \beta) = a \cdot \| f(\alpha) = \beta \| .$$

For each $\alpha < \kappa$, $W_\alpha = \{a(\alpha, \beta): a(\alpha, \beta) \neq 0\}$ is a partition of a. If W is a common refinement of the W_α's and $u \in W$, then there is a function g: $\kappa \to \lambda$, defined as follows:

$$g(\alpha) = \text{the unique } \beta \text{ such that } u \leq a(\alpha, \beta) ,$$

and

$$u \Vdash f = g .$$

This shows that f is in the ground model. The other direction is similar.

As an application of this characterization of distributivity, let us prove the following. (This, as well as Proposition 4, can of course be proved without the use of forcing; see 14.10, Chapter 5, in Part I.)

PROPOSITION 3. *If B is $(\kappa, 2)$-distributive, then it is $(\kappa, 2^\kappa)$-distributive.*

PROOF. Since B is $(\kappa, 2)$-distributive, every function from κ into $\{0, 1\}$ in $V[G]$ is in V, and therefore every subset of κ in $V[G]$ is in V. Hence, every subset of $\kappa \times \kappa$ in $V[G]$ is in V.

Now let f be a function in $V[G]$ from κ into the power set of κ (whose size is 2^κ). Let

$$Y = \{(\alpha, \beta) \in \kappa \times \kappa : \beta \in f(\alpha)\} .$$

The set Y belongs to V by the assumption, and so does f, because

$$f(\alpha) = \{\beta < \kappa : (\alpha, \beta) \in Y\} . \quad \square$$

If $\lambda < \kappa$ and B is κ-distributive, then B is λ-distributive. The following example shows that as κ increases, κ-distributivity becomes stronger: for each regular cardinal κ there is a complete Boolean algebra B that is λ-distributive for all $\lambda < \kappa$ but not κ-distributive. For singular cardinals we have:

PROPOSITION 4. *If κ is a singular cardinal and B is λ-distributive for all $\lambda < \kappa$, then B is κ-distributive.*

PROOF. Again we use the Boolean-valued model characterization. Let $f: \kappa \to V$ be a function in $V[G]$. As $\kappa = \lim_{\alpha \to \mathrm{cf}(\kappa)} \kappa_\alpha$, we have $f = \bigcup_\alpha f \restriction \kappa_\alpha$. By the assumption, both the functions $f \restriction \kappa_\alpha$ and the sequence $\{f \restriction \kappa_\alpha : \alpha < \mathrm{cf}\, \kappa\}$ belong to V, and so does f. $\square$

EXAMPLE 1. Let κ be a regular cardinal. A complete Boolean algebra B, λ-distributive for all $\lambda < \kappa$, and not $(\kappa, 2)$-distributive.

Let P be the following partially ordered set: the elements of P are functions p whose domain is an ordinal less than κ, and whose values are 0 and 1:

$$(4) \qquad P = \{0, 1\}^{<\kappa} = \bigcup_{\alpha < \kappa} \{0, 1\}^\alpha .$$

This is the well-known notion of forcing that adjoins a new subset of κ.

Let $B = \mathrm{r.o.} P$ be the corresponding complete Boolean algebra; P is a dense subset of B.

The student of forcing is well aware that B is not $(\kappa, 2)$-distributive: a generic filter gives rise to a new subset of κ. The relevant forcing argument can be easily converted into a counterexample to property (2): Let, for all $\alpha < \kappa$,

$$a(\alpha, 0) = \sum \{p: p(\alpha) = 0\}, a(\alpha, 1) = \sum \{p: p(\alpha) = 1\} .$$

The partitions $\{a(\alpha, 0), a(\alpha, 1)\}$, $\alpha < \kappa$, do not have a common refinement because for any $p \in P$ there is $\alpha < \kappa$ such that neither $p \leq a(\alpha, 0)$ nor $p \leq a(\alpha, 1)$: let $\alpha \not\in \mathrm{dom}(p)$ and let p_0 and p_1 be extensions of p such that $p_0(\alpha) = 0$ and $p_1(\alpha) = 1$; we have $p_0 \leq a(\alpha, 0)$ and $p_1 \leq a(\alpha, 1)$.

That B is λ-distributive for all $\lambda < \kappa$ follows from a lemma that we shall now prove.

DEFINITION. Let λ be a cardinal number. A partially ordered set P is *λ-closed* if for every $\theta \leq \lambda$, every descending sequence

$$p_0 > p_1 > \cdots > p_\alpha > \cdots \quad (\alpha < \theta)$$

has a lower bound.

LEMMA 5. *Every λ-closed partially ordered set is λ-distributive.*

COROLLARY. *If B has a λ-closed dense subset, then it is λ-distributive.*

P is *κ-distributive* if it has property (3). We call a complete Boolean algebra *λ-closed* if it has a dense subset that is λ-closed. (Obviously, "λ-closed" is not invariant under completion, and in fact no B^+ is even ω-closed in the literal sense.)

PROOF. Let P be λ-closed and let W_α, $\alpha < \lambda$, be partitions of some $a \in P$. We find a common refinement W of the W_α. Let W be a maximal set of mutually incompatible p such that

(5) p is a lower bound of a descending sequence $\{p_\alpha : \alpha < \lambda\}$ such that for all α, $p_\alpha \leq$ some $w \in W_\alpha$.

It is enough to show that W is a partition of a since if it is, it is clearly a refinement of all the W_α. Thus, assume that some $q \leq a$ is incompatible with all $p \in W$. By induction we construct a descending sequence $\{p_\alpha : \alpha < \lambda\}$ such that for all α, $p_\alpha \leq$ some $w \in W_\alpha$. This can be done by using the fact that P is λ-closed at each limit step $\theta < \lambda$, and because the W_α are partitions of a. Finally, we let p be a lower bound of $\{p_\alpha : \alpha < \lambda\}$. Since p satisfies (5) we get a contradiction. □

Now we see why P, from the example above, is λ-distributive for all $\lambda < \kappa$: Any θ-sequence $(\theta < \kappa)$

$$p_0 \subset p_1 \subset \cdots \subset p_\alpha \subset \cdots \quad (\alpha < \theta)$$

of functions in P has a common extension $\bigcup_{\alpha<\theta} p_\alpha$ in P, and so P is λ-closed for all $\lambda < \kappa$.

We devote the rest of this chapter to $\aleph_0$-distributivity and its variants.

As $(\aleph_0, \lambda)$-distributivity become stronger with increasing λ, and by Proposition 3, $(\aleph_0, 2)$-distributivity implies $(\aleph_0, 2^{\aleph_0})$-distributivity, the natural question is whether $(\aleph_0, \aleph_2)$-distributivity is strictly stronger than $(\aleph_0, 2)$-distributivity. An affirmative answer is provided by the following

EXAMPLE 2 (Bukovský, Namba). Assuming that $2^{\aleph_0} = \aleph_1$, there exists a Boolean algebra B that is $(\aleph_0, 2)$-distributive but not $(\aleph_0, \aleph_2)$-distributive.

A description of the Boolean algebra B as well as a detailed proof of its distributivity properties can be found, for example, in JECH [1978a, pp. 289–291].

The question considered above can be generalized: given a cardinal κ, is there a Boolean algebra that is $(\aleph_0, \lambda)$-distributive for all $\lambda < \kappa$ but not κ-distributive?

For deep reasons related to Jensen's fine structure of the constructible universe, there is no example of such algebra unless we assume the existence of large cardinals.

The following example uses a measurable cardinal:

EXAMPLE 3 (Prikry). Let κ be a measurable cardinal. There is a Boolean algebra that is $(\aleph_0, \lambda)$-distributive for all $\lambda < \kappa$ but not $(\aleph_0, \kappa)$-distributive.

Again, we present a partially ordered set rather than a Boolean algebra. Let D be a normal κ-complete ultrafilter over κ.

Let P be the set of all pairs (s, A), where s is a finite increasing sequence of ordinals below κ, and $A \in D$. P is partially ordered as follows:

$$(t, B) \leq (s, A) \text{ if } t \supseteq s,\ B \subseteq A, \text{ and range } (t - s) \subseteq A\,.$$

As in the first two examples, it is rather easy to show that P is not $(\aleph_0, \kappa)$-distributive. For each $n < \omega$ and each $\alpha < \kappa$, let

$$a(n, \alpha) = \sum \{(s, A) \colon s(n) = \alpha\}$$

For any $n < \omega$, $W_n = \{a(n, \alpha) \colon \alpha < \kappa\}$ is a partition, and the partitions W_n, $n < \omega$, do not have a common refinement.

The proof that P is $(\aleph_0, \lambda)$-distributive uses combinatorial properties of measurable cardinals. The crucial fact is this: Let $B = \text{r.o.}P$.

LEMMA 6. *Let* $\{a_\alpha \colon \alpha < \lambda\}$, $\lambda < \kappa$, *be a partition of some* $a \in B$, *and let* $(s, A) \in P$. *Then there is* $A' \subset A$, $A' \in D$ *and some* $\alpha < \lambda$ *such that* $(s, A') \leq a_\alpha$.

Given this lemma it is easy to see that B is $(\aleph_0, \lambda)$-distributive for all $\lambda < \kappa$ (using the countable completeness of the ultrafilter D). The proof of the lemma can be found in JECH [1978a, pp. 470–471].

By Lemma 5, every complete Boolean algebra that has an $\aleph_0$-closed dense subset is $\aleph_0$-distributive. It is not immediately obvious that the converse fails. The following example has been discovered by several people; see BAUMGARTNER, HARRINGTON and KLEINBERG [1976].

EXAMPLE 4. Shooting a club through a stationary set. An $\aleph_0$-distributive complete Boolean algebra that does not have a dense $\aleph_0$-closed subset.

We assume that the reader is familiar with closed unbounded (*club*) and stationary subsets of ω_1.

Let E be a stationary subset of ω_1. We describe a partially ordered set P_E:

$$P_E = \{p \subset E \colon p \text{ is a closed set of ordinals}\}\,, \tag{7}$$

The partial ordering of P_E is by end-extension:

$$p \leq q \quad \text{if } q = p \cap \alpha \text{ for some } \alpha \ (p \textit{ extends } q)\,.$$

The partially ordered set P_E is $\aleph_0$-distributive. We shall not prove it at this point; we give the proof in Proposition 12.

If E is also co-stationary (i.e. its complement in ω_1 is stationary), then the complete Boolean algebra $B = \text{r.o.}(P_E)$ does not have an $\aleph_0$-closed dense subset. This will also be proved in Proposition 12.

Thus, we have an example showing that to have a dense $\aleph_0$-closed subset is strictly stronger than $\aleph_0$-distributive. The example is instructive in another respect. The stronger property is preserved by products of Boolean algebras, as the following lemma demonstrates:

LEMMA 7. *If partially ordered sets P and Q are $\aleph_0$-closed, then the product $P \times Q$ is $\aleph_0$-closed.*

PROOF. Obvious. We recall that the partial ordering of $P \times Q$ is

$$(p, q) \leq (p', q') \quad \text{iff } p \leq p' \text{ and } q \leq q' . \quad \square$$

Distributivity, however, is not closed under products, as Example 4 attests:

PROPOSITION 8. *If E and F are disjoint stationary sets, then $P_E \times P_F$ is not $\aleph_0$-distributive.*

PROOF. We use a forcing argument. (For a forcing-free proof, see Proposition 13.) The forcing notion P_E shoots a club $C_E \subset E$ through E, while P_F shoots a club C_F through F. Thus, in the generic extension by $P_E \times P_F$ there are two disjoint clubs C_E and C_F. Clearly, $(\omega_1)^V$ must be a countable ordinal in $V[G]$. Thus, $V[G]$ has a function from ω into $(\omega_1)^V$ that is not in the ground model. Ergo, $P_E \times P_F$ is not $\aleph_0$-distributive. $\square$

Let us consider the following infinite game $\mathscr{G}(B)$ played on a Boolean algebra B.

Two players, White and Black, take turns to choose successively the terms of a sequence

$$w_1 \geq b_1 \geq w_2 \geq b_2 \geq \cdots \geq w_n \geq b_n \geq \cdots \tag{8}$$

of nonzero elements of B. White wins the play (8) if and only if the sequence converges to zero, this is if it does not have a nonzero lower bound.

A *winning strategy* for a player (say for White) is a function σ from finite sequences in B into B such that every play (8) in which White follows σ (i.e. $w_n = \sigma(w_1, b_1, \ldots, w_{n-1}, b_{n-1})$) is a win for White. (A winning strategy for Black is defined similarly.)

There are three possibilities for each Boolean algebra: either White has a winning strategy in $\mathscr{G}(B)$, or Black has a winning strategy, or neither player has a winning strategy.

The game $\mathscr{G}$ is defined in terms of the partial order on B, and so can be defined for an arbitrary partially ordered set. It is easy to see that if P is a dense subset of a Boolean algebra B, then a winning strategy (for either player) exists in $\mathscr{G}(P)$ if and only if there exists one in $\mathscr{G}(B)$. (A caution is in order: This is not generally true when $B = \text{r.o.}(P)$, as the partial order on B may differ from the partial order on P if P is not separative.)

The game $\mathscr{G}$ is related to $\aleph_0$-distributivity:

THEOREM 9 (JECH [1978b]). *B is $\aleph_0$-distributive if and only if White does not have a winning strategy in the game $\mathscr{G}(B)$.*

PROOF. Without loss of generality we assume that B is complete.

(a) Let us assume that B is not $\aleph_0$-distributive. We are to show that White has a winning strategy in $\mathscr{G}(B)$. As B is not $\aleph_0$-distributive, there exists a collection $\{a(n, i)\colon n < \omega, i \in I\}$ such that

$$a = \prod_{n=1}^{\infty} \sum_{i \in I} a(n, i)$$

and

$$\prod_{n=1}^{\infty} a(n, f(n)) = 0 \tag{9}$$

for all $f\colon \omega \to I$. We describe a strategy σ for White.

First, we let $w_1 = \sigma(\emptyset) = a$. When

$$w_1 \geq b_1 \geq \cdots \geq w_n \geq b_n$$

has been played, we find $i_n \in I$ such that $b_n \cdot a(n, i_n) \neq 0$, and let $w_{n+1} = \sigma(w_1, b_1, \ldots, w_n, b_n) = b_n \cdot a(n, i_n)$.

The strategy we described is a winning strategy for White. If a play (8) is such that each w_n is played according to σ, we have

$$\prod_{n=1}^{\infty} w_n \leq \prod_{n=1}^{\infty} a(n, i_n) = 0$$

on account of (9).

(b) Now assume that White has a winning strategy σ in $\mathscr{G}(B)$; we prove that B is not $\aleph_0$-distributive.

Let $a = \sigma(\emptyset)$. For each n we find a partition W_n of a, such that $\{W_n\colon n = 1, 2, \ldots\}$ has no common refinement. For each n, let P_n be the set of all partial plays $p = \langle w_1, b_1, \ldots, w_n, b_n \rangle$ in which each w_k is played according to σ. By induction on n, we let Q_n be a maximal subset of P_n such that

(i) each $p \in Q_n$ extends some $q \in Q_{n-1}$, and

(ii) $W_n = \{\sigma(p)\colon p \in Q_n\}$ is a set of mutually disjoint elements of B.

It follows from the maximality of Q_n that W_n is a partition of a.

We shall show that if $z_n \in W_n$ for each n, then $\prod_{n=1}^{\infty} z_n = 0$; hence the W_n have no common refinement. Either some z_{n+1} is disjoint from z_n, or for all n, $z_{n+1} = \sigma(\ldots, z_n, b_n)$. In the latter case, each z_n is White's move according to σ in a play

$$a \geq b_1 \geq z_1 \geq b_2 \geq \cdots \geq z_{n-1} \geq b_n \geq \cdots.$$

Such a play is a win for White and so $\prod_{n=1}^{\infty} z_n = 0$. $\square$

In view of Theorem 9 we introduce the following property of Boolean algebras, stronger than $\aleph_0$-distributivity:

DEFINITION 10. A Boolean algebra B is *game-closed* if Black has a winning strategy in the game $\mathcal{G}(B)$.

Similarly, a partially ordered set P is *game-closed* if Black has a winning strategy in $\mathcal{G}(P)$. Being game-closed is a property intermediate between having a dense $\aleph_0$-closed subset and being $\aleph_0$-distributive:

PROPOSITION 11. *If B has a dense $\aleph_0$-closed subset, then it is game-closed.*

PROOF. Let P be a $\aleph_0$-closed dense subset of B. If Black's moves $b_1, b_2, b_3, \ldots$ are chosen so that each b_n belongs to P, then the sequence (8) has a lower bound and Black wins. □

We now return to Example 4.

PROPOSITION 12. *Let E be a stationary subset of ω_1, and let P_E be the partial ordering defined in* (7). *White does not have a winning strategy in $\mathcal{G}(P_E)$ and so P_E is $\aleph_0$-distributive. If E is co-stationary, then Black does not have a winning strategy in $\mathcal{G}(P_E)$ either, and so $B = \mathrm{r.o.}(P_E)$ does not have a dense $\aleph_0$-closed subset.*

PROOF. We prove the first statement; the other is similar. Let σ be any strategy for White; we show that σ is not a winning strategy by constructing a play (8) in which White follows σ and loses.

We define, for all finite increasing sequences $\alpha_1 < \cdots < \alpha_n$ of countable ordinals, two functions, $b(\alpha_1, \ldots, \alpha_n) \in P_E$ and $\beta(\alpha_1, \ldots, \alpha_n) < \omega_1$:

(10) Let $\alpha_1 < \omega_1$.
 (i) $b(\alpha_1) =$ some $b \in P_E$ such that b extends $w_1 = \sigma(\emptyset)$, and $\max(b) > \alpha_1$
 $\beta(\alpha_1) = \max(b(\alpha_1))$.
 (ii) Let $\alpha_1 < \cdots < \alpha_n$. Consider the partial play $w_1, b_1, w_2, b_2, \ldots, w_n$, where for each $k = 1, \ldots, n-1$, $b_k = b(\alpha_1, \ldots, \alpha_k)$ and $w_k = \sigma(w_1, b_1, \ldots, w_{k-1}, b_{k-1})$. Let $b(\alpha_1, \ldots, \alpha_n) =$ some $b \in P_E$ extending $w_n = \sigma(w_1, \ldots, b_{n-1})$ such that $\max(b) > \alpha_n$.

(11) $\beta(\alpha_1, \ldots, \alpha_n) = \max(b(\alpha_1, \ldots, \alpha_n))$.

Now consider the set

$$C = \{\lambda < \omega_1 \colon \lambda \text{ is a limit ordinal and } \beta(\alpha_1, \ldots, \alpha_n) < \lambda \text{ whenever } \alpha_1 < \cdots < \alpha_n < \lambda\}.$$

The set C is closed unbounded and so there exists $\lambda \in C$ such that $\lambda \in E$. Let $\lambda \in C \cap E$, and let $\alpha_1 < \cdots < \alpha_n < \cdots$ be a sequence of ordinals such that $\lim_n \alpha_n = \lambda$.

Now we consider the play (8) in which for each n, $b_n = b(\alpha_1, \ldots, \alpha_n)$, and $w_n = \sigma(w_1, \ldots, b_{n-1})$. This is a play in which White follows the strategy σ. Since $\lambda \in C$, we have

$$\lambda = \lim_n \max(b_n) .$$

Since $\lambda \in E$, the closed set of ordinals $p = \bigcup_{n=1}^{\infty} b_n \cup \{\lambda\}$ is a subset of E, and so p is a lower bound of (8). Hence White loses this play. □

The following is a forcing free proof of Proposition 8:

PROPOSITION 13. *If E and F are disjoint stationary sets, then White has a winning strategy in $\mathcal{G}(P_E \times P_F)$.*

PROOF. The following is a winning strategy for White. When Black plays $b_{n-1} = (p, q) \in P_E \times P_F$, let White play $w_n = (p_n, q_n) \in P_E \times P_F$ so that both $\max(p_n)$ and $\max(q_n)$ are greater than both $\max(p)$ and $\max(q)$. Then $\lim_n \max(p_n) = \lim_n \max(q_n) = \lambda$ and since λ cannot be both in E and in F, the sequence $\{(p_n, q_n): n < \omega\}$ does not have a lower bound in $P_E \times P_F$. □

While being game-closed is stronger than $\aleph_0$-distributivity, it is not known whether it is strictly weaker than having a dense $\aleph_0$-closed subset. The following theorem gives a partial solution of this problem:

THEOREM 14 (FOREMAN [1983]). *If B is a game-closed Boolean algebra of cardinality $\aleph_1$, then B has a dense $\aleph_0$-closed subset.*

(A generalization of this theorem is proved in VOJTÁŠ [1983].)

PROOF. Let B be a game-closed Boolean algebra, $|B| = \aleph_1$. By Theorem 9, B is $\aleph_0$-distributive, and so any countable set of partitions of unity has a common refinement. This enables us to do the following construction. Let $\{a_\alpha : \alpha < \omega_1\}$ enumerate $B - \{0\}$. Let W_α, $\alpha < \omega_1$, be partitions of unity such that

(i) if $\alpha < \beta$, then W_β is a refinement of W_α;

(ii) for each α there is $w \in W_\alpha$ such that $w \leq a_\alpha$.

Let $T = \bigcup_{\alpha < \omega_1} W_\alpha$. T is a dense subset of B, and:

(i) T is a tree: for all $t, s \in T$, either $t \leq s$ or $t \geq s$ or $t \cdot s = 0$;

(ii) for all $t \in T$, $|\{s \in T: s \geq t\}| \leq \aleph_0$;

(iii) every descending sequence $t_1 > t_2 > \cdots > t_n > \cdots$ that has a lower bound, has a maximal lower bound.

Let us fix a winning strategy σ for Black in $\mathcal{G}(T)$. We shall find a dense subset P of T that is $\aleph_0$-closed.

Let $t \in T$. A *partial play above* t is a finite sequence in T of even length, $p = \langle w_1, b_1, \ldots, w_n, b_n \rangle$ such that $w_1 \geq b_1 \geq \cdots \geq w_n \geq b_n > t$, where each b_k is obtained by the strategy σ.

CLAIM. For each $t \in T$ there exists $t^* < t$ such that for every partial play p above t^* and every $t' > t^*$ there is a partial play $q \supset p$ above t^* whose last move b is such that $t' > b > t^*$.

To prove the claim, we construct a sequence $t = t_0 > t_1 > \cdots > t_n > \cdots$ so that for every pair (p, t'), where p is a partial play above t_n and $t' > t_n$ (there are $\aleph_0$ of them), p is extended below t', above some t_m. The sequence $\{t_n\}$ has a lower bound, because there exists a play $\{w_n, b_n\}_n$ of $\mathscr{G}(T)$, cofinal in $\{t_n\}_n$, played according to σ. We let t^* be a maximal lower bound of $\{t_n\}_n$.

Now we let

$$P = \{s \in T\colon s \text{ is a maximal lower bound of } \{t_n^*\colon n < \omega\} \text{ for some sequence } \{t_n\colon n < \omega\}\} .$$

We show that P is $\aleph_0$-closed, and dense in T.

If $s_0 > s_1 > \cdots > s_n > \cdots$ is a descending sequence in P, then clearly there exists $t_n \in T$ such that $t_0^* > s_0 > t_1^* > s_1 > \cdots$. The sequence $\{t_n^*\}_n$ has a lower bound because there exists a play cofinal in $\{t_n^*\}_n$, won by Black. A maximal lower bound of $\{t_n^*\}_n$ is in P and is a lower bound for $\{s_n\}_n$.

If $t \in T$ is arbitrary, let $\{t_n\}_n$ be the sequence such that $t_0 = t$ and $t_{n+1} = t_n^*$ for all n. Again, there exists a play $\{w_n, b_n\}_n$ cofinal in $\{t_n^*\}_n$, won by Black, and so $\{t_n^*\}_n$ has a lower bound. Thus, there exists $s \in P$ below t and P is dense in T. $\square$

Theorem 14 shows that if game-closed does not imply a $\aleph_0$-closed dense subset then a counterexample has to be at least of size $\aleph_2$. Moreover, it provides the following equivalent formulation of "game-closed":

COROLLARY 15 (JECH [1984]). *A partially ordered set P is game-closed if and only if there exists a partially ordered set Q such that $P \times Q$ has a dense $\aleph_0$-closed subset.*

PROOF. (a) Let D be a $\aleph_0$-closed dense subset of $P \times Q$. Then Black has a winning strategy in $\mathscr{G}(P)$: when choosing the nth move $p_n \in P$ (below White's nth move), Black also chooses $q_n \in Q$ so that $q_n \leq q_{n-1}$ and that $(p_n, q_n) \in D$. The sequence $\{(p_n, q_n)\colon n < \omega\}$ has a lower bound and so does $\{p_n\colon n < \omega\}$.

(b) Let P be a game-closed partial ordering, and let σ be a winning strategy for Black in $\mathscr{G}(P)$. Let $\kappa = |P|$, and let Q be the standard notion of forcing for collapsing κ to $\aleph_1$ ($Q = \kappa^{<\omega_1}$, ordered by $\supset$). Q is $\aleph_0$-closed. In the generic extension $V[G]$ by Q, P has size $\aleph_1$, and σ is still a winning strategy for Black in $\mathscr{G}(P)$. By Theorem 14, P has (in $V[G]$) a $\aleph_0$-closed dense subset E.

Let

$$D = \{(p, q) \in P \times Q\colon q \Vdash p \in E\} .$$

D is a dense subset of $P \times Q$. We show that D is $\aleph_0$-closed. Let $\{(p_n, q_n)\colon n < \omega\}$ be descending sequence in D. The condition $q_\infty = \bigcup_{n=0}^{\infty} q_n$ forces that $\{p_n\colon n < \omega\}$ is a descending sequence in E and so there is $q \leq q_\infty$ and $p \in P$ such that $p \leq p_n$ for all n, and $q \Vdash p \in E$. Hence, $(p, q) \in D$ is a lower bound of $\{(p_n, q_n)\colon n < \omega\}$. $\square$

We shall now briefly discuss *Suslin algebras*. These are closely related to the well-known Suslin problem in point set topology.

DEFINITION 16. A *Suslin algebra* is a $\aleph_0$-distributive Boolean algebra that has the countable chain condition.

The existence of Suslin algebras is neither provable nor refutable in set theory. If T is a Suslin tree, then the partially ordered set $(T, \supset)$ has the countable chain condition (because every antichain in T is countable), and is also $\aleph_0$-distributive. If W_n, $n < \omega$, are partitions (i.e. maximal antichains in T), then because each W_n is bounded, all the W_n are below some level α of the tree T, and the αth level is a partition that is a refinement of each W_n. Hence, $B = \text{r.o.}(T)$ is a Suslin algebra.

On the other hand, if B is a Suslin algebra, we can find $T \subset B$ such that $(T, >)$ is (isomorphic to) a Suslin tree. We construct partitions of unity W_α, $\alpha < \omega_1$, such that if $\alpha < \beta$, then W_β is a refinement of W_α (this can be done because B is $\aleph_0$-distributive). The set $T = \bigcup_{\alpha<\omega_1} W_\alpha$ is a Suslin tree.

Thus, we have

PROPOSITION 17. *A Suslin algebra exists if and only if a Suslin tree exists.*

If T is a Suslin tree, then the complete Suslin algebra $B = \text{r.o.}(T)$ has cardinality $2^{\aleph_0}$. There can possibly exist larger Suslin algebras. The following theorems address this question. For proofs, the reader may consult JECH [1978a, p. 274].

THEOREM 18 (Solovay). *Every Suslin algebra has cardinality at most* $2^{\aleph_1}$.

THEOREM 19. *It is consistent that* $2^{\aleph_1}$ *is arbitrarily large and there exists a Suslin algebra of size* $2^{\aleph_1}$.

The rest of this chapter deals with a generalization of (κ, λ)-distributivity. Let κ, λ, ν be cardinal numbers. For and set A, let $[A]^\nu$ and $[A]^{<\nu}$ denote, respectively, the set of all subsets of size $\leq \nu$ and the set of all subsets of size $< \nu$.

DEFINITION 20. A Boolean algebra B is (κ, ν, λ) *distributive* if

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a(\alpha, \beta) = \sum_{f:\,\kappa\to[\lambda]^\nu} \prod_{\alpha<\kappa} \sum_{\beta\in f(\alpha)} a(\alpha, \beta)\,. \tag{12}$$

B is $(\kappa, <\nu, \lambda)$*-distributive* if

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a(\alpha, \beta) = \sum_{f:\,\kappa\to[\lambda]^{<\nu}} \prod_{\alpha<\kappa} \sum_{\beta\in f(\alpha)} a(\alpha, \beta)\,. \tag{13}$$

We shall only look at the case when $\kappa = \aleph_0$ and $\nu = \aleph_0$. When B is $(\aleph_0, \aleph_0, \lambda)$-distributive for all λ then B is called $(\aleph_0, \aleph_0, \infty)$*-distributive*. A Boolean algebra that is $(\aleph_0, <\aleph_0, \lambda)$-distributive for all λ is called *weakly distributive*.

These properties can be reformulated as follows:

PROPOSITION 21. *B is $(\aleph_0, \aleph_0, \infty)$-distributive (weakly distributive) if and only if for every $a > 0$ and a countable set $\{W_n\colon n < \omega\}$ of partitions of a there exist countable (finite) sets $E_n \subseteq W_n$ and $b > 0$ such that for all n, $b \leq \Sigma\, E_n$.*

Clearly, $\aleph_0$-distributive implies weakly distributive, which in turn implies $(\aleph_0, \aleph_0, \infty)$-distributive. The two properties considered are of some interest in the theory of forcing. For instance, B is $(\aleph_0, \aleph_0, \infty)$-distributive if and only if every countable set of ordinals in the generic extension is included in a countable set that is in the ground model.

The standard example of a weakly distributive, non-$\aleph_0$-distributive Boolean algebra is an atomless measure algebra:

EXAMPLE 5. An atomless measure algebra.

Let B be an atomless measure algebra, with measure μ. To see that B is not $(\aleph_0, 2)$-distributive, let $\{b_n\}$ be a countable set of independent elements of B, each of measure $\frac{1}{2}$ (hence $\mu(\pm b_1 \cdot \ldots \cdot \pm b_n) = 2^{-n}$). For each n, let $a(n, 0) = b_n$, $a(n, 1) = -b_n$. We have

$$\prod_{n=0}^{\infty} (a(n, 0) + a(n, 1)) = 1 .$$

But for each $f\colon \omega \to \{0, 1\}$,

$$\prod_{n=0}^{\infty} a(n, f(n)) = 0 .$$

To show that B is weakly distributive, let $a > 0$, and let W_n, $n < \omega$, be partitions of a. Let $r = \mu(a)$. For each n there is a finite set $E_n \subseteq W_n$ such that

$$\mu\Big(\sum E_n\Big) \geq r \cdot (1 - 2^{-(n+2)}) .$$

It follows that $\mu(\prod_{n=0}^{\infty} (\sum E_n)) \geq r/2$ and so $\prod_{n=0}^{\infty} (\sum E_n) > 0$. $\square$

The Boolean algebra from Example 1 (when $\kappa = \aleph_0$) is an example of an $(\aleph_0, \aleph_0, \infty)$-distributive algebra that is not weakly distributive. It is $(\aleph_0, \aleph_0, \infty)$-distributive because it has a dense subset that is countable. To show that it is not weakly distributive, we use the fact that $B = \text{r.o.}(P)$, where P is the following countable partial order (we recall that B is the unique atomless separable complete Boolean algebra):

(14) $$P = \omega^{<\omega} = \text{the set of all finite sequences of natural numbers},$$
$$p < q \quad \text{if } p \text{ extends } q .$$

For each $n < \omega$ and $m < \omega$, let

$$a(n, m) = \{p \in P\colon p(n) = m\} .$$

For each n, the set $W_n = \{a(n, m)\colon m < \omega\}$ is a partition. We show that

(15) $$\prod_{n=0}^{\infty} \Big(\sum E_n\Big) = 0$$

for any sequence $\{E_n\colon n < \omega\}$ of finite sets $E_n \subset W_n$.

Let $\{E_n: n<\omega\}$ be such a sequence, and let $p \in P$. It is enough to find $q \leq p$ and n such that

$$(16) \qquad q \cdot \left(\sum E_n\right) = 0 .$$

Let n be any $n \not\in \mathrm{dom}(p)$ and let m be such that $a(n, m) \not\in E_n$. Now if q extends p and $q(n) = m$, then (16) holds, proving (15). □

We conclude this chapter by mentioning one property of Boolean algebras stronger than $(\aleph_0, \aleph_0, \infty)$-distributive law (and neither stronger nor weaker than $\aleph_0$-distributivity). It is *properness*, introduced by Shelah. Properness can be formulated in several equivalent ways, including the one below.

DEFINITION 22. A Boolean algebra B is *proper* if, for every nonzero $u \in B$, the following game has a winning strategy for Black. Each move by White is a Boolean valued name a_n for an ordinal, and each move by Black is an ordinal α_n. Black wins if some nonzero $v \leq u$ forces $\forall n\, \exists k (a_n = \alpha_k)$.

We note that:

- every game-closed Boolean algebra is proper;
- every *ccc* Boolean algebra is proper;
- every proper Boolean algebra is $(\aleph_0, \aleph_0, \infty)$-distributive;
- the algebra from Example 4 is not proper.

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CHAPTER 9

Disjoint Refinement

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Contents

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HANDBOOK OF BOOLEAN ALGEBRA
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0. Introduction

Every mathematician knows how much easier it is to work with a disjoint family of sets instead of an arbitrary one. Let us illustrate this claim by comparing two proofs of one classical theorem, stating that there are 2^ω disjoint Bernstein subset of the reals. A Bernstein set is a subset $X \subseteq \mathbb{R}$ such that both X and $\mathbb{R} - X$ meet each closed uncountable set. Note that a Bernstein set is never Lebesgue measurable and never has the Baire property.

The first proof is straightforward and goes by a transfinite induction up to 2^ω as follows. Enumerate by $\{A_\alpha : \alpha \in 2^\omega\}$ all the closed uncountable subsets of the reals such that each one is listed cofinally many times. At the αth stage of the induction, choose distinct points $x_{\alpha,\beta}$ for $\beta \le \alpha$ from $A_\alpha - \{x_{\gamma,\delta} : \delta \le \gamma < \alpha\}$. There is plenty of room to do it, since $|A_\alpha| = 2^\omega$. Let $X_\beta = \{x_{\alpha,\beta} : \beta \le \alpha < 2^\omega\}$ for $\beta \in 2^\omega$. If $\beta < 2^\omega$ and $A \subseteq \mathbb{R}$ is an uncountable closed set, then for some $\alpha > \beta$, $A = A_\alpha$. Now $x_{\alpha,\beta} \in X_\beta \cap A$, $x_{\alpha,\alpha} \in A - X_\beta$ and the theorem follows.

There are many ways to reduce the general problem concerning an arbitrary family of sets to a problem dealing with a disjoint system, and several of them have become quite canonical. A good example is the Bernstein–Kuratowski–Sierpiński theorem: Given an infinite cardinal κ and a family $\mathcal{A} = \{A_\alpha : \alpha \in \kappa\}$ of sets with $|A_\alpha| = \kappa$ for each α, there is a family $\mathcal{D} = \{D_\alpha : \alpha \in \kappa\}$ such that $D_\alpha \cap D_\beta = \emptyset$ for distinct α, β and $|D_\alpha| = \kappa$, $D_\alpha \subseteq A_\alpha$ for each $\alpha \in \kappa$. We can say that $\mathcal{D}$ is a disjoint refinement of $\mathcal{A}$.

So, how does the second proof go? Apply the Bernstein–Kuratowski–Sierpiński theorem to the family $\{A_\alpha : \alpha \in 2^\omega\}$ of all uncountable closed subsets of the reals. The rest follows from the obvious fact that any $X \subseteq \mathbb{R}$ which selects one point from each D_α is Bernstein.

This chapter is devoted to the disjoint refinement property in a more general setting, namely in Boolean algebras, and to some related notions. The definitions which follow are quite simple.

0.1. Definition. Let B be a Boolean algebra. By a disjoint system we always mean a pairwise disjoint system of non-zero elements of B, i.e. a subset $D \subseteq B^+$ such that $d \cdot d' = \mathbb{0}$, whenever d, d' are distinct members of D.

If $A = \{a_\alpha : \alpha \in \tau\}$, $D = \{d_\alpha : \alpha \in \tau\} \subseteq B^+$, then the family D is a disjoint refinement of A provided that D is disjoint and for each $\alpha \in \tau$, $d_\alpha \le a_\alpha$. If there is a disjoint refinement $D = \{d_\alpha : \alpha \in \tau\}$ for $A = \{a_\alpha : \alpha \in \tau\}$, we say that A has a disjoint refinement.

If τ is a cardinal number, $A \subseteq B^+$, then the family A is called τ-decomposable (resp. strongly τ-decomposable) provided that there is a disjoint family $D \subseteq B^+$, $|D| = \tau$ such that for each $a \in A$, $|\{d \in D : a \cdot d \ne \mathbb{0}\}| = \tau$ (resp. for each $a \in A$ and for each $d \in D$, $a \cdot d \ne \mathbb{0}$).

The following easy theorem shows how closely these three notions are related.

0.2. Theorem. *Let τ be an infinite cardinal number, B a Boolean algebra, $A \in [B^+]^\tau$.*

(a) *If A is strongly τ-decomposable, then A is τ-decomposable.*

(b) *If there is a disjoint system $D \subseteq B^+$ such that for each $a \in A$, $|\{d \in D: a \cdot d \neq \mathbb{0}\}| \geq \tau$, then A is τ-decomposable.*

(c) *If A is τ-decomposable, then A has a disjoint refinement.*

(d) *If A has a disjoint refinement and* $\mathrm{hsat}(B) \geq \tau^+$, *then A is τ-decomposable.*

(e) *If A has a disjoint refinement and if B is τ^+-complete and* $\mathrm{hsat}(B) \geq \tau^+$, *then A is strongly τ-decomposable.*

PROOF. (a) is obvious.

(b) For $a \in A$, pick $D(a) \subseteq \{d \in D: a \cdot d \neq \mathbb{0}\}$ with $|D(a)| = \tau$, let $D' = \bigcup\{D(a): a \in A\}$. D' shows that A is τ-decomposable.

(c) Let P be a τ-decomposition of A. An easy induction enables us to choose $p(a) \in P$ for each $a \in A$ with $a \cdot p(a) \neq \mathbb{0}$, $p(a) \cdot p(a') = \mathbb{0}$ whenever $a \neq a'$, a, $a' \in A$. The family $D = \{a \cdot p(a): a \in A\}$ is the desired disjoint refinement of A.

(d) Let D be a disjoint refinement of A. For $d \in D$, let $P(d)$ be a family consisting of τ disjoint members of $B|d$. Then $P = \bigcup\{P(d): d \in D\}$ shows the τ-decomposability of A.

(e) Let D be a disjoint refinement of A. For $d \in D$, let $\{d_\alpha: \alpha \in \tau\}$ be a disjoint system of non-zero elements with $d_\alpha \leq d$ for each $\alpha \in \tau$. By $\mathrm{hsat}(B) \geq \tau^+$, such a system exists. Denote $p_\alpha = \Sigma\{d_\alpha: d \in D\}$. Then $a \cdot p_\alpha \neq \mathbb{0}$ for each $a \in A$ and $\alpha \in \tau$, therefore A is strongly τ-decomposable. Clearly, each $p_\alpha \in B$ by the τ^+-completeness of B. □

The theorem indicates the reason why the present chapter is devoted to the study of those three notions of decomposability. Since they do not coincide in general, we shall treat them separately, if necessary. All Boolean algebras are assumed to be non-degenerate, i.e. $\mathbb{0} \neq \mathbb{1}$.

The chapter is organized as follows.

Section 1. The disjoint refinement property in Boolean algebras. Given $A \subseteq B^+$, $|A| = \kappa$, $\mathrm{hsat}(B) = \kappa^+$. Must A have a disjoint refinement? Not if $\mathrm{h}\pi(B) \leq \kappa$. Yes, if $\mathrm{h}\pi(B) > \kappa$, with $\kappa = \omega$ or $\kappa > \omega$ and B is $(\omega, \cdot, \kappa)$-nowhere distributive. A characterization of $\mathrm{Col}(\lambda, \kappa)$ is given.

Section 2. The disjoint refinement property of centred systems in Boolean algebras. Given centred $A \subseteq B^+$, $|A| = \kappa$, $\mathrm{hsat}(B) = \kappa^+$. Must A have a disjoint refinement? If B is complete, then $\kappa = \omega$ or κ singular imply yes. The answer is independent on ZFC for regular uncountable κ. Examples: B is the completion of $\mathcal{P}(\omega)/\mathit{fin}$, B is $\mathrm{Col}(\omega, \omega_1)$.

Section 3. Non-distributivity of $\mathcal{P}(\omega)/\mathit{fin}$. The height of $\mathcal{P}(\omega)/\mathit{fin}$ is defined by $h = \min\{\tau: \mathcal{P}(\omega)/\mathit{fin}$ is not (τ, ∞)-distributive$\}$. There exists a base tree of height h. Basic inequalities for h and the other cardinal invariants concerning subsets of ω and functions on ω. Relations to Ellentuck topology.

Section 4. Refinements by countable sets. Theorem on the existence of almost disjoint refinement. Completely separable AD families on ω. Every uniform ultrafilter on ω has an almost disjoint refinement by a completely separable AD family. The property $\mathrm{RPC}(\omega)$ and its consequences. Komjáth's theorem.

Section 5. The algebra $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$; non-distributivity and decomposability. The basic properties of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$: $E(\kappa)$, $F(\kappa^+)$, saturatedness, $(2^\kappa, \kappa^+)$-

independent families, non-distributivity. Strong decomposability of ultrafilters. Decomposability of ultrafilters in $\mathscr{P}(\kappa)$.

1. The disjoint refinement property in Boolean algebras

1.0. DEFINITION. Let $\kappa \geq 2$ be a cardinal, B a Boolean algebra. We shall say that B satisfies the disjoint refinement property for systems of cardinality at most κ, in short, B has $\mathrm{Rp}(\kappa)$, provided that for each $A = \{a_\alpha : \alpha \in \kappa\} \subseteq B^+$, A has a disjoint refinement. The indexing is not supposed to be one-to-one.

When does B have $\mathrm{Rp}(\kappa)$? This question is studied in the present paragraph. Let us make first several self-evident remarks.

Any algebra B has $\mathrm{Rp}(\kappa)$ iff $\bar{B}$ has. For once there is some disjoint refinement, there is also another one which is contained in the given in advance dense part of the algebra.

If a Boolean algebra B satisfies $\mathrm{Rp}(2)$, then B is atomless.

There are two obvious obstructions for a Boolean algebra B to have $\mathrm{Rp}(\kappa)$. It may happen that there is no disjoint family of size κ in B; clearly, $\mathrm{Rp}(\kappa)$ fails then. And it may happen that B contains a dense subset A of size $\kappa \geq \omega$: if there were a disjoint refinement D for this particular A, pick a $d \in D$.

Since B is atomless, there is some $a \in A$ with $a < d$; now no $d' \in D$ can satisfy $d' \leq a$, $d' \cdot d = \mathbb{0}$.

Let us recall several notions.

1.1. DEFINITION. The saturatedness of a Boolean algebra B is a cardinal $\mathrm{sat}(B) = \min\{\tau\colon \text{if } A \in [B^+]^{\geq\tau}, \text{ then } A \text{ is not disjoint}\}$.

Hereditary saturatedness, $\mathrm{hsat}(B) = \min\{\mathrm{sat}(B|u)\colon u \in B^+\}$.

The density of a Boolean algebra B is a cardinal $\pi(B) = \min\{|D|\colon D \subseteq B^+ \text{ is dense in } B\}$ and its hereditary version, $\mathrm{h}\pi(B) = \min\{\pi(B|u)\colon u \in B^+\}$.

So in view of the previous remarks, necessary conditions for B to have $\mathrm{Rp}(\kappa)$ are: B is atomless, $\mathrm{hsat}(B) > \kappa$, $\mathrm{h}\pi(B) > \kappa$.

The basic result concerning $\mathrm{Rp}(\kappa)$ was proved in Part I, 3.14, namely:

1.2. THEOREM. *If* $\mathrm{hsat}(B) > \kappa^+$, *then* B *has* $\mathrm{Rp}(\kappa)$.

Using this theorem, one can prove the following strengthening of 0.2.

1.3. COROLLARY. *Let B be a complete BA*, $\mathrm{hsat}(B) \geq \kappa^+$, $A = \{a_\alpha\colon \alpha \in \kappa\} \subseteq B^+$. *Then A has a disjoint refinement if and only if A is* $\mathrm{cf}(\kappa)$*-decomposable.*

PROOF. Denote $\tau = \mathrm{cf}(\kappa)$. By 0.2, only the case $\tau < \kappa$ is to be considered.

Choose an increasing sequence $\{\kappa_\xi\colon \xi \in \tau\}$ of cardinals converging to κ.

If A is τ-decomposable, then there is a disjoint system $P = \{p_\xi\colon \xi \in \tau\}$ such that for each $\alpha \in \kappa$, $|\{\xi \in \tau\colon a_\alpha \cdot p_\xi \neq \mathbb{0}\}| = \tau$. By 1.2, each family $C_\xi = \{a_\alpha \cdot p_\xi\colon \alpha < \kappa_\xi \;\&\; a_\alpha \cdot p_\xi \neq \mathbb{0}\}$ has a disjoint refinement D_ξ. Clearly, $\bigcup \{D_\xi\colon \xi \in$

$\tau\}$ contains a disjoint refinement of A. The completeness of B was not needed in this part of the proof.

Let $\{d_\alpha: \alpha \in \kappa\}$ be a disjoint refinement of A. Since $\mathrm{hsat}(B) \geq \kappa^+$, there is a disjoint system $\{d_{\alpha,\xi}: \xi \in \tau\} \subseteq B|d_\alpha$. Let $P = \{\Sigma\{d_{\alpha,\xi}: \alpha \in \kappa\}: \xi \in \tau\}$. By the completeness of B, P is well defined and shows that A is τ-decomposable. □

1.4. PROPOSITION. *A Boolean algebra B has* $\mathrm{Rp}(\omega)$ *if and only if* $\mathrm{h}\pi(B) > \omega$.

PROOF. As already shown, the condition $\mathrm{h}\pi(B) > \omega$ is necessary.

We have to show that it is also sufficient. Let $A = \{a_n: n \in \omega\} \subseteq B^+$ and let $C_0 = \{a_n \cdot a_0: n \in \omega \ \&\ a_n \cdot a_0 \neq \mathbb{0}\}$. Since the set C_0 is not dense in $B|a_0$ as $\pi(B|a_0) > \omega$, there is some $d_0 < a_0$, $d_0 \neq \mathbb{0}$ with $a_n - d_0 \neq \mathbb{0}$ for each $n \in \omega$. If d_k have been defined for $k < i$, and if $a_n - (d_0 + d_1 + \cdots + d_{i-1}) \neq 0$ for each $n \in \omega$, there is some $d_i \neq \mathbb{0}$, $d_i < a_i - (d_0 + \cdots + d_{i-1})$ such that $a_n - (d_0 + d_1 + \cdots + d_i) \neq \mathbb{0}$, for $\pi(B|(a_i - (d_0 + \cdots + d_{i-1}))) > \omega$: the system $C_i = \{a_n \cdot a_i - (d_0 + \cdots + d_{i-1}): n \in \omega \ \&\ a_n \cdot a_i - (d_0 + \cdots + d_{i-1}) \neq \mathbb{0}\}$ cannot be dense in $B|(a_i - (d_0 + d_1 + \cdots + d_{i-1}))$. The set $D = \{d_n: n \in \omega\}$ is the desired disjoint refinement. □

1.5. EXAMPLE. Each measure algebra (i.e. a non-atomic complete Boolean algebra admitting a strictly positive σ-additive measure) has $\mathrm{Rp}(\omega)$.

From now on, we shall concentrate our attention on the case $\mathrm{hsat}(B) = \kappa^+$, $\mathrm{h}\pi(B) > \kappa$. By 1.4, only κ uncountable is of interest. It may be tempting to guess that B has $\mathrm{Rp}(\kappa)$ if $\mathrm{h}\pi(B) > \kappa$; unfortunately, this straightforward analogy of 1.4 does not hold. A counterexample will be given at 1.19. Yet the idea "almost" works: B really has $\mathrm{Rp}(\kappa)$ if $\mathrm{h}\pi(B) > \kappa$ and if, in addition, B has a κ-closed dense subset. Since a bit more is true, the theorem is formulated as follows.

1.6. THEOREM. *Let $\lambda \geq \omega$, $\kappa \geq 2$ be cardinal numbers, let B be a $(\lambda, \cdot, \kappa)$-nowhere distributive Boolean algebra having a λ-closed dense subset. Then the following conditions are equivalent*:

(a) *B has* $\mathrm{Rp}(\kappa^{<\lambda})$;

(b) $\mathrm{h}\pi(B) > \kappa^{<\lambda}$;

(c) *for each $x \in B^+$, $\overline{B|x}$ is not isomorphic to* $\mathrm{Col}(\lambda, \kappa)$;

(d) *for each $x \in B^+$, $\overline{B|x}$ is not isomorphic to* $\mathrm{Col}(\lambda, \kappa^{<\lambda})$.

The proof will be given later (1.17). For the reader's convenience, let us recall (Part I, Chapter 5, 14.13) the three-parameter distributivity and the other notions used in the theorem.

1.7. DEFINITION (Boolean matrix). Let B be a Boolean algebra, τ, κ cardinals. A collection $\mathcal{P} \subseteq \mathcal{P}(B^+)$ is called a matrix if each member of $\mathcal{P}$ is a maximal disjoint subset of B^+.

(Distributivity). The Boolean algebra B is (τ, λ, κ)-distributive if for every matrix $\mathcal{P} = \{P_\alpha: \alpha \in \tau\}$ with each $|P_\alpha| \leq \lambda$ there is some maximal disjoint system $Q \subseteq B^+$ such that for each $q \in Q$ and for each $\alpha \in \tau$, $|\{p \in P_\alpha: p \cdot q \neq \mathbb{0}\}| < \kappa$.

(Nowhere distributivity). The Boolean algebra B is called (τ, λ, κ)-nowhere distributive if for every $x \in B^+$, $B|x$ is not (τ, λ, κ)-distributive.

If we omit the condition $|P_\alpha| \leq \lambda$ in the above definitions, we obtain $(\tau, \cdot, \kappa)$-distributivity and $(\tau, \cdot, \kappa)$-nowhere distributivity.

1.8. DEFINITION (Closedness). Let B be a Boolean algebra, $C \subseteq B^+$, λ an infinite cardinal. The set C is λ-closed if for each $\gamma < \lambda$ and for each decreasing chain $c_0 \geq c_1 \geq \cdots \geq c_\alpha \geq \cdots (\alpha < \gamma)$ of elements of C there is some $b \in C$ with $b \leq c_\alpha$ for each $\alpha < \gamma$.

1.9. REMARK AND AN EXAMPLE. Notice that if $C \subseteq B^+$ is λ-closed and if λ is a singular cardinal, then C is λ^+-closed, too. Thus, when speaking about a λ-closed subset, we shall tacitly assume that λ is regular.

Let $\lambda \geq \omega$, $\kappa \geq 2$ be cardinal numbers. Consider the family $F(\lambda, \kappa)$ of all mappings f: $\xi \to \kappa$ for $\xi < \lambda$ ordered by the inverse inclusion. There is a unique (up to isomorphism) complete Boolean algebra, having a dense subset isomorphic to $F(\lambda, \kappa)$. This algebra will be denoted by $\mathrm{Col}(\lambda, \kappa)$. If λ is regular, then $\mathrm{Col}(\lambda, \kappa)$ has a λ-closed dense subset, namely $F(\lambda, \kappa)$; furthermore, $\mathrm{Col}(\lambda, \kappa)$ is $(\rho, \cdot, 2)$-distributive for each $\rho < \lambda$ and is $(\lambda, \cdot, \kappa)$-nowhere distributive.

1.10. LEMMA. *If a Boolean algebra has a λ-closed dense subset, then B is $(\rho, \cdot, 2)$-distributive for each $\rho < \lambda$.* □

1.11. LEMMA. *Let B be a $(\tau, \cdot, \kappa)$-nowhere distributive Boolean algebra. Then there is a matrix $\mathcal{P} = \{P_\alpha: \alpha \in \tau\}$ such that for each $x \in B^+$ there is some $\alpha \in \tau$ with $|\{p \in P_\alpha: p \cdot x \neq \mathbb{0}\}| \geq \kappa$.*

PROOF. Consider the set $C \subseteq B^+$ consisting of all $x \in B^+$ for which there is some matrix $\mathcal{P}(x) = \{P_\alpha(x): \alpha \in \tau\}$ in $B|x$ such that for each $y \leq x$ there is some $\alpha \in \tau$ with $|\{p \in P_\alpha(x): p \cdot y \neq \mathbb{0}\}| \geq \kappa$.

The set C is dense in B. If not, then for some $z \in B^+$ there is no member $c \in C$ with $c \leq z$. Thus, for each matrix $\{P_\alpha: \alpha \in \tau\}$ in $B|z$ the set $\{y \in B|z: (\forall \alpha \in \tau)|\{p \in P_\alpha: y \cdot p \neq \mathbb{0}\}| < \kappa\}$ is dense in $B|z$, hence there is a maximal disjoint set Q in $B|z$ with the property $((y \in Q, \alpha \in \tau) \to |\{p \in P_\alpha: y \cdot p \neq \mathbb{0}\}| < \kappa)$, which contradicts the $(\tau, \cdot, \kappa)$-nowhere distributivity of B.

Let S be a maximal disjoint subset of C. Since C is dense in B, S is maximal disjoint in B, too. For $x \in S$ let $\mathcal{P}(x)$ be as indicated above, and define $P_\alpha = \{u \in B$: for some $x \in S$, $u \in P_\alpha(x)\}$. The matrix $\mathcal{P} = \{P_\alpha: \alpha \in \tau\}$ has the required properties. □

Let us examine the non-distributivity property together with appropriate closedness in detail.

1.12. LEMMA. *Let τ, $\lambda \geq \omega$, $\kappa \geq 2$ be cardinal numbers, B a $(\tau, \cdot, \kappa)$-nowhere distributive Boolean algebra containing a λ-closed dense subset C. Suppose that B is $(\rho, \cdot, 2)$-distributive for each $\rho < \tau$. Then:*

(i) $\tau \geq \lambda$ *and* τ *is a regular cardinal*;

(ii) $\mathrm{hsat}(B) > \kappa^{<\lambda}$.

Let $\mathcal{P} = \{P_\alpha : \alpha \in \tau\}$ *be an arbitrary matrix on* B. *Then there is a matrix* $\mathcal{Q} = \{Q_\alpha : \alpha \in \tau\}$ *such that*

(iii) $\mathcal{Q}$ *witnesses to the* $(\tau, \cdot, \kappa)$*-nowhere distributivity of* B;

(iv) $\bigcup \mathcal{Q} \subseteq C$;

(v) *for each* $\alpha < \beta < \tau$, Q_β *refines* Q_α *and* Q_β *refines* P_β;

(vi) *for each* $\alpha \in \tau$ *and for each* $y \in Q_\alpha$, $|\{x \in Q_{\alpha+1} : x \leq y\}| \geq \kappa^{<\lambda}$;

(vii) *for each* $y \in B^+$, $|\{x \in \bigcup \mathcal{Q} : x \cdot y \neq \mathbb{0}\}| \geq \kappa^{<\lambda}$.

PROOF. (i) is an immediate consequence of the definitions and of 1.10.

(ii) follows from (vi) and from the fact that the lemma may be relativized to each $B|x$, $x \in B^+$.

So it suffices to find a matrix $\mathcal{Q}$ satisfying (iii)–(vii). This will be done by a transfinite induction.

Let $\mathcal{S} = \{S_\alpha : \alpha \in \tau\}$ be a matrix witnessing to the $(\tau, \cdot, \kappa)$-nowhere distributivity of B. Let $Q_0 \subseteq C$ be a maximal disjoint family refining both P_0, S_0.

Let $\alpha < \tau$ and suppose Q_β has been defined for all $\beta < \alpha$. Since B is $(|\alpha|, \cdot, 2)$-distributive, there is a maximal disjoint family U_α such that U_α refines P_α, S_α, as well as all Q_β for $\beta < \alpha$. For $x \in U_\alpha$, choose a maximal set $R_x \subseteq C$ such that R_x is disjoint, each member of R_x is smaller than x and cardinality of R_x is at least μ, where μ is some suitable cardinal.

As yet we know that $\mu \geq \kappa$ (for $\mathrm{hsat}(B) > \kappa$ according to the $(\tau, \cdot, \kappa)$-nowhere distributivity of B); we shall show later that $\mu \geq \kappa^{<\lambda}$ is possible. Let $Q_\alpha = \bigcup \{R_x : x \in U_\alpha\}$.

Having constructed the matrix $\mathcal{Q} = \{Q_\alpha : \alpha \in \tau\}$ it is easy to verify that (iii), (iv) and (v) hold for $\mathcal{Q}$; moreover, for each $\alpha \in \tau$ and for each $y \in Q_\alpha$,

$$|\{x : x \in Q_{\alpha+1} \;\&\; x \leq y\}| \geq \mu \geq \kappa\,. \tag{1}$$

Let us show now that $\mathrm{sat}(B|y) \geq \kappa^{<\lambda}$ for each $y \in \bigcup \mathcal{Q}$. Choose an arbitrary cardinal $\delta < \lambda$ and $y \in \bigcup \mathcal{Q}$. Then $y \in Q_\beta$ for some $\beta \in \tau$. By the validity of (1) for all $\alpha \in \tau$, by (iv) and by the λ-closedness of C, we obtain:

$$|\{x : x \in Q_{\beta+\delta} \;\&\; x \leq y\}| \geq \mu^\delta \geq \kappa^\delta\,. \tag{2}$$

Thus, $\mathrm{sat}(B|y) \geq \kappa^{<\lambda}$. Next we shall show that $\mathrm{sat}(B|y) > \kappa^{<\lambda}$.

If $\kappa^{<\lambda} = \kappa^\delta$ for some $\delta < \lambda$, then $\mathrm{sat}(B|y) \geq (\kappa^\delta)^+$ by (2). If for each $\delta < \lambda$, $\kappa^\delta < \kappa^{<\lambda}$, then $\kappa^{<\lambda}$ is a limit cardinal and $\lambda = \mathrm{cf}(\kappa^{<\lambda}) \leq \kappa^{<\lambda}$.

Two cases are possible. Either $\lambda < \kappa^{<\lambda}$, but then $\mathrm{sat}(B|y) \geq (\kappa^{<\lambda})^+$ since the saturatedness cannot be a singular cardinal by a well-known result of Erdös and Tarski (ERDÖS and TARSKI [1943], Part I, Theorem 3.10), or $\lambda = \kappa^{<\kappa}$. If $\lambda = \kappa^{<\lambda}$, let $y = c_\beta$, and choose by induction a chain $c_\beta \geq c_{\beta+1} \geq \cdots \geq c_{\beta+\alpha} \geq \cdots$ for each $\alpha < \lambda$, such that $c_{\beta+\alpha} \in Q_{\beta+\alpha}$. By (1), the inequalities are sharp, thence $D = \{c_{\beta+\alpha} - c_{\beta+\alpha+1} : \alpha < \lambda\} \subseteq B^+$ is a disjoint subset of $(B|y)^+$, which implies that $\mathrm{sat}(B|y) \geq \lambda^+$.

Thus, $\mu = \kappa^{<\lambda}$ could have been chosen in the construction of $\mathcal{Q}$, which proves (vi).

It remains to verify (vii). Let $y \in B^+$. The desired family $\{x \in \bigcup \mathcal{Q} : x \cdot y \neq \mathbb{0}\}$ can be found using a standard branching argument. For $\xi = 0$, choose $x_\emptyset \in Q_0$ arbitrary with $x_\emptyset \cdot y \neq \mathbb{0}$ and $c_\emptyset \in C$ with $c_\emptyset \leq x_\emptyset \cdot y$.

Let $\xi < \lambda$ and suppose that for each $f \in {}^\xi\kappa$ and for each $\eta < \xi$ an element $x_{f|\eta} \in \bigcup \mathcal{Q}$ and $c_{f|\eta} \in C$ have been found. We assume the following. If $f \in {}^\xi\kappa$, $\eta < \zeta < \xi$, then $x_{f|\eta} \geq x_{f|\zeta}$, $c_{f|\eta} \geq c_{f|\zeta}$ and $c_{f|\eta} \leq x_{f|\eta} \cdot y$, and if $f, g \in {}^\xi\kappa$, $\eta < \xi$ and $f|\eta \neq g|\eta$, then $x_{f|\eta} \cdot x_{g|\eta} = \mathbb{0}$.

Case ξ is a limit ordinal. Fix $f \in {}^\xi\kappa$. There is some $c \in C$ with $c \leq c_{f|\eta}$ for each $\eta < \xi$, since C is λ-closed. By (iii), there is some $\alpha \in \tau$ such that $|\{x \in Q_\alpha : x \cdot c \neq \mathbb{0}\}| \geq \kappa$. For this α, pick arbitrary $x_f \in Q_\alpha$ with $x_f \cdot c \neq \mathbb{0}$. For this x_f, clearly $x_f \leq x_{f|\eta}$ for all $\eta < \xi$. Consequently, $x_f \cdot y \neq \mathbb{0}$, for $c \leq c_\emptyset \leq y$ and $x_f \cdot c \neq \mathbb{0}$. Choose $c_f \in C$ with $c_f \leq x_f \cdot c$.

Case $\xi = \eta + 1$. If $f \in {}^\eta\kappa$, then by (iii), there is some $\alpha < \tau$ such that there is a disjoint set $\{b_i : i \in \kappa\} \subseteq Q_\alpha$ with $b_i \cdot c_f \neq \mathbb{0}$ for each $i \in \kappa$. For each $i \in \kappa$, let $x_{f \cup \{\langle \eta, i \rangle\}} = b_i$ and choose $c_{f \cup \{\langle \eta, i \rangle\}} \leq c_f \cdot b_i$, $c_{f \cup \{\langle \eta, i \rangle\}} \in C$.

Hence, for each $\xi < \lambda$ and for each $f \in {}^\xi\kappa$ we have found an element $x_f \in \bigcup \mathcal{Q}$ such that $x_f \cdot y \neq \mathbb{0}$, $x_f \neq x_g$ whenever $f \neq g$, $f, g \in \bigcup_{\xi<\lambda} {}^\xi\kappa$. Now (vii) follows and the proof is complete. □

In what follows we shall give several applications of Lemma 1.12. Another one will be found in Section 3.

1.13. THEOREM. *Let τ, $\lambda \geq \omega$, $\kappa \geq 2$ be cardinals, B a $(\tau, \cdot, \kappa)$-nowhere distributive Boolean algebra having a λ-closed dense subset C. Let B be $(\rho, \cdot, 2)$-distributive for each $\rho < \tau$. If $\pi(B) = \kappa^{<\lambda}$, then there is a dense subset $D \subseteq C$ of B such that $(D, \geq)$ is a tree of height τ and each $d \in D$ has $\kappa^{<\lambda}$ immediate successors.*

PROOF. Since $\pi(B) = \kappa^{<\lambda}$ and C is dense in B, we can assume $|C| = \kappa^{<\lambda}$. The assumptions of Lemma 1.12 are satisfied, so let $\mathcal{R} = \{R_\alpha : \alpha \in \tau\}$ be an arbitrary matrix satisfying 1.12(iii)–(vii). Using 1.12(vii), define a mapping $\varphi : C \to \bigcup \mathcal{R}$ with $c \cdot \varphi(c) \neq \mathbb{0}$ for each $c \in C$, $\varphi(c) \neq \varphi(c')$ for distinct $c, c' \in C$. For $\alpha \in \tau$, let P_α be a maximal disjoint family containing all elements $c \cdot \varphi(c)$, $\varphi(c) - c$ for $\varphi(c) \in R_\alpha$, and all $r \in R_\alpha - \{\varphi(c) : c \in C\}$. Apply 1.12 once more to the matrix $\mathcal{P} = \{P_\alpha : \alpha \in \tau\}$, let $\mathcal{Q} = \{Q_\alpha : \alpha \in \tau\}$ be the result. Since $\bigcup \mathcal{P}$ was dense in B, $\bigcup \mathcal{Q}$ is dense in B, too. Moreover, 1.12(v) implies that $(\bigcup \mathcal{Q}, \geq)$ is a tree. It remains to set $D = \bigcup \mathcal{Q}$. □

The forthcoming corollary was observed first by Alan Dow. The well-known notion from the model theory, when specialized to Boolean algebras, looks like this. For a regular cardinal κ, call a Boolean algebra B κ-homogeneous universal if, for each $A, C \subseteq B^+$, $|A| < \kappa$, $|C| < \kappa$ such that A is an increasing chain, C is a decreasing chain and $a < c$ for each $a \in A$, $c \in C$, there is a $b \in B^+$ with $a < b < c$ for all $a \in A$, $c \in C$. (The reader surely recognizes that this is a generalization of the DuBois–Reymond separating property.) It should be known that there always exists a κ-homogeneous universal Boolean algebra of size $\kappa^{<\kappa}$. Moreover, Sikorski's lemma (Part I, Chapter 2, 5.5) guarantees that each algebra of size $\leq\kappa$ embeds into a κ-homogeneous universal algebra.

1.14. COROLLARY (Dow). *For every regular cardinal κ, every κ-homogeneous universal Boolean algebra of size $\kappa^{<\kappa}$ has a dense subset D such that $(D, \geq)$ is a tree of height between κ and $\kappa^{<\kappa}$.*

PROOF. For $u \in B^+$ let $\mathrm{h}(B|u) = \min\{\mu\colon B|u$ is $(\mu, \cdot, \kappa)$-nowhere distributive$\}$. Clearly, if $u \leq v$, then $\mathrm{h}(B|u) \leq \mathrm{h}(B|v)$, so it suffices to split the unity $\mathbb{1}$ of B into a partition P such that for each $p \in P$, $\mathrm{h}(B|p) = \mathrm{h}(B|u)$ for all $u \leq p$, and then to show that each $B|p$ has a dense subset which is a tree under $\geq$.

Since $B|p$ is κ-homogeneous universal, no non-zero $v \leq p$ admits an infinite partition of size $<\kappa$. Therefore if $\rho < \mathrm{h}(B|p)$, then $B|p$ is $(\rho, \cdot, 2)$-distributive, since it is $(\rho, \cdot, \kappa)$-distributive. Furthermore, $B|p$ has a κ-closed dense subset. Thus, 1.13 applies with $\lambda = \kappa$, $\tau = \mathrm{h}(B|p)$. Clearly, $\kappa \leq \mathrm{h}(B|p) \leq |B| \leq \kappa^{<\kappa}$. □

A further corollary is a generalization of McAloon's theorem (Part I, Chapter 5, 14.17).

1.15. COROLLARY. *Let B be a complete $(\lambda, \cdot, \kappa)$-nowhere distributive Boolean algebra containing a λ-closed dense subset. If $\pi(B) = \kappa^{<\lambda}$, then B is isomorphic to* $\mathrm{Col}(\lambda, \kappa^{<\lambda})$.

PROOF. Indeed, by 1.10, the assumptions of 1.13 are satisfied in the special setting $\tau = \lambda$. If $\mathcal{Q}$ is the matrix given in 1.13, $\bigcup \mathcal{Q}$ is in this case isomorphic to the set $\{f\colon \lambda \to \kappa^{<\lambda}\colon \operatorname{dom} f \neq \emptyset$, $\operatorname{dom} f$ is a successor ordinal$\}$; but this set is dense in $\mathrm{Col}(\lambda, \kappa^{<\lambda})$. □

1.16. COROLLARY. *The algebras* $\mathrm{Col}(\lambda, \kappa)$ *and $Col(\lambda, \kappa^{<\lambda})$ are isomorphic whenever $\lambda \geq \omega$ is regular and $\kappa \geq 2$.* □

1.17. *Proof of 1.6.* Clearly, (c) and (d) are equivalent by 1.16. In the remarks following 1.0 and 1.1 it was proved that (a) implies (b) in every Boolean algebra. Similarly, (a) imples (d) since $\mathrm{h}\pi(\mathrm{Col}(\lambda, \kappa^{<\lambda})) = \pi(\mathrm{Col}(\lambda, \kappa^{<\lambda})) = \kappa^{<\lambda}$.

Assume (d) holds. If for some $x \in B^+$, $\pi(B|x) = \kappa^{<\lambda}$, then by 1.15, $\overline{B|x}$ is isomorphic to $\mathrm{Col}(\lambda, \kappa^{<\lambda})$. This contradicts (d) and shows that (d) implies (b).

So it remains to prove that (b) implies (a). Let $A \subseteq B^+$, $|A| = \kappa^{<\lambda}$. We have to find a disjoint refinement of A. Let $\mathcal{Q} = \{Q_\alpha\colon \alpha \in \lambda\}$ be a matrix having the properties 1.12(iii)–(vii). We may assume that $A \subseteq \bigcup \mathcal{Q}$ (see the proof of 1.13). Denote $P_\alpha = A \cap Q_\alpha$. If there is some $\mu < \lambda$ such that $P_\alpha = \emptyset$ whenever $\mu \leq \alpha < \lambda$, we are done – $Q_{\mu+1}$ can be used to show that A is $\kappa^{<\lambda}$-decomposable. Thus, suppose $P_\alpha \neq \emptyset$ for each $\alpha < \lambda$. Fix a λ-closed dense subset C of B.

We shall find a family of chains $\{c(x, \alpha)\colon \alpha < \lambda\}$, $x \in A$, with the following properties:

(a) $c(x, 0) \leq x$;
(b) for each $x \in A$ and $\alpha \in \lambda$, $c(x, \alpha) \in C$;
(c) if $x \in A$ and $\alpha < \beta < \lambda$, then $c(x, \alpha) \geq c(x, \beta) \neq \mathbb{0}$;
(d) if $x \in P_\alpha$, $y \in A$, $y \neq x$, then $c(x, \alpha + 1) \cdot c(y, \alpha + 1) = \mathbb{0}$.

Obviously, the set $\{c(x, \alpha+1): x \in P_\alpha, \alpha \in \lambda\}$ will be the desired disjoint refinement of A.

Transfinite induction. Let $\beta < \lambda$ and suppose $c(x, \alpha)$ to be defined for all $x \in A$ and all $\alpha < \beta$.

If $\beta < \lambda$, β limit, then choose $c(x, \beta) \in C$ such that $\mathbb{0} \neq c(x, \beta) \leq c(x, \alpha)$ for each $\alpha < \beta$ and each $x \in A$. The λ-closedness of C enables us to do this.

If $\beta = \gamma + 1$, proceed as follows. If $x \in P_\alpha$ with $\alpha < \gamma$, define $c(x, \gamma+1) = c(x, \gamma)$. Let $x \in P_\gamma$. The family $\{c(y, \gamma): y \in A\}$ is too small for being dense in $B|c(x, \gamma)$, since $\mathrm{h}\pi(B) > \kappa^{<\lambda}$, thus there is some $c(x, \gamma+1) \in C$ with $\mathbb{0} \neq c(x, \gamma+1) \leq c(x, \gamma)$ and $c(y, \gamma) - c(x, \gamma+1) \neq \mathbb{0}$ for each $y \in A$. Let $x \in P_\alpha$, $\alpha > \gamma$. By 1.12(v), there is at most one $y(x) \in P_\gamma$ such that $x \leq y(x)$. Let $c(x, \gamma+1)$ be an arbitrary element of C satisfying $\mathbb{0} \neq c(x, \gamma+1) \leq c(x, \gamma) - c(y(x), \gamma+1)$ (or $\mathbb{0} \neq c(x, \gamma+1) \leq c(x, \gamma)$, if no such $y(x) \in P_\gamma$ exist).

The proof of 1.6 is complete. □

Now, as a corollary, we can give a proof of a statement mentioned in 1.5.

1.18. EXAMPLE. Let B be a Boolean algebra having a λ-closed dense subset and let $\mathrm{h}\pi(B) > \lambda$. Then B has $\mathrm{Rp}(\lambda)$.

PROOF. Let $A \subseteq B^+$, $|A| = \lambda$. Using the same trick as in the proof of 1.14, we can consider two cases only:

(a) B is $(\lambda, \cdot, 2)$-distributive. Apply this fact to the matrix $\mathcal{Q} = \{\{a, \mathbb{1} - a\}: a \in A\}$. The existence of a disjoint refinement of A follows.

(b) B is $(\lambda, \cdot, 2)$-nowhere distributive. If $2^{<\lambda} > \lambda$, then by 1.12(ii), $\mathrm{hsat}(B) > \lambda^+$ and 1.2 applies. If $2^{<\lambda} = \lambda$, the result follows by 1.6 in the setting $\kappa = 2$. □

1.19. EXAMPLE. The assumption that B has a λ-closed dense subset is essential in 1.6. To show this, let κ be a cardinal such that $\kappa^\omega = \kappa$. Then there is a $(\omega_1, \cdot, \kappa)$-nowhere distributive Boolean algebra B with $\mathrm{h}\pi(B) > \kappa$ $(= \kappa^{<\omega_1})$; nevertheless, $\mathrm{Rp}(\kappa)$ does not hold for B.

Let $F = \{f \in {}^\alpha\kappa: \alpha < \omega_1\}$. F is a dense subset of $\mathrm{Col}(\omega_1, \kappa)$, $|F| = (\kappa \times \omega_1)^\omega = \kappa^\omega = \kappa$. Let B_1 be the free algebra generated by κ^+ generators. Then the free product $B = \mathrm{Col}(\omega_1, \kappa) \oplus B_1$ is the desired example. The algebra B is $(\omega_1, \cdot, \kappa)$-nowhere distributive since $\mathrm{Col}(\omega_1, \kappa)$ is; $\mathrm{h}\pi(B) > \kappa$ since $\mathrm{h}\pi(B_1) = \kappa^+$. Obviously, B has no ω_1-closed dense subset.

The system $\tilde{F} = \{\langle f, \mathbb{1}\rangle: f \in F\}$ is a system of cardinality κ having no disjoint refinement. Suppose the contrary, let D be a disjoint refinement of F. Without loss of generality we may assume that the members of D are of the form $\langle f, x\rangle$ with $f \in F$, $x \in B_1^+$. Choose arbitrary $\langle f_0, x_0\rangle \in D$, suppose $\langle f_\alpha, x_\alpha\rangle \in D$ have been found for $\alpha < \beta < \omega_1$ in such a way that $f_0 \subsetneqq f_1 \subsetneqq \cdots \subsetneqq f_\alpha \subsetneqq \cdots$. Let $g = \bigcup \{f_\alpha: \alpha < \beta\} \cup \{\langle \gamma, 0\rangle\}$, where $\gamma \notin \bigcup \{\mathrm{dom}(f_\alpha): \alpha < \beta\}$. Then $\langle g, \mathbb{1}\rangle \in \tilde{F}$ and $\langle g, \mathbb{1}\rangle$ contains no $\langle f_\alpha, x_\alpha\rangle$ with $\alpha < \beta$. Hence, there is some $\langle f_\beta, x_\beta\rangle \in D$ with $\langle g, \mathbb{1}\rangle \geq \langle f_\beta, x_\beta\rangle$. Obviously, $f_\beta \supsetneqq f_\alpha$ for all $\alpha < \beta$.

We have found a subset $\{\langle f_\alpha, x_\alpha\rangle: \alpha < \omega_1\}$ of D such that the f_α's form a nested sequence. However, D is disjoint, hence the set $\{x_\alpha: \alpha < \omega_1\} \subseteq B_1^+$ must be disjoint, too. But this contradicts the well-known fact that $\mathrm{hsat}(B_1) = \omega_1$.

1.20. GAME-THEORETICAL REMARK. Above we have frequently assumed the existence of a λ-closed dense subset of a Boolean algebra. This assumption cannot be dropped completely, by 1.19. However, it may be weakened.

For $A \subseteq B^+$ and a cardinal λ, consider the following game $G(\lambda, A)$ on B. Two players, named Empty and Non-empty, choose alternatively collections A_α of non-zero elements making a family of decreasing chains by their moves, i.e. at the αth stage the set $A_\alpha = \{a_\alpha : a \in A\} \subseteq B^+$ is chosen such that $a \geq a_\beta \geq a_\alpha$ for all $\beta < \alpha$ and all $a \in A$. The Empty player wins if the game finishes in less than λ steps; that means, if there is some $a \in A$ and $\alpha < \lambda$ with $\prod_{\beta<\alpha} a_\beta = \mathbb{0}$.

It is a routine matter to check that in all instances mentioned above, the assumption "B has a λ-closed dense subset" can be replaced by "The player Empty has no winning strategy in the game $G(\lambda, B^+)$".

2. The disjoint refinement property for centred systems in Boolean algebras

In this section we shall consider special families, namely those with the finite intersection property, from the disjoint refinement point of view. We shall show the close connection between the existence of a disjoint refinement and the lower bound for a character of an ultrafilter in Boolean algebra.

2.1. DEFINITION. Let B be a Boolean algebra, $A \subseteq B^+$. The set A is centred, equivalently, A has the finite intersection property (FIP), provided $\prod A_0 \neq \mathbb{0}$ whenever $A_0 \subseteq A$ is finite. A Boolean algebra has the refinement property for centred systems of power at most κ (B has Rfip(κ)) if each family $\{a_\alpha : \alpha \in \kappa\} \subseteq B^+$ with FIP has a disjoint refinement.

2.2. We shall investigate Rfip(κ) mainly in a special setting. The algebra B will be assumed to be complete and $\mathrm{hsat}(B) = \kappa^+$, $\pi(B) = \kappa$. The reason for this restriction stems from the previous results. By 1.2 the problem is solved if $\mathrm{hsat}(B) > \kappa^+$, and if $\pi(B) > \kappa$, then 1.6 gives the answer for enough nowhere distributive BAs. Since – contrary to the case of Rp(κ) – it may happen that B has Rfip(κ) and yet $\bar{B}$ does not, while if $\bar{B}$ has Rfip(κ), then so does B, the completeness of B seems to be a natural assumption. To illustrate these possibilities, we shall show that $\overline{\mathcal{P}(\omega)/fin}$ need not satisfy Rfip(2^ω) (see 2.15); nevertheless, Rfip(2^ω) is true in $\mathcal{P}(\omega)/fin$ as will be proved in Section 4.

When asking whether B has Rfip(κ), the easy case is $\kappa = \omega$ or κ singular. The complete answer is given in 2.3. For κ regular uncountable, the question is undecidable. We shall consider several combinatorial properties, which lead to a negative answer (2.5) as well as other ones, which imply an answer in the affirmative (2.11). Both cases are, for clarity, shown for the algebra $\mathrm{Col}(\omega, \omega_1)$.

2.3. THEOREM. *Let B be a complete Boolean algebra, let $\kappa = \omega$ or a singular cardinal. Then B has* Rfip(κ) *if and only if* $\mathrm{hsat}(B) \geq \kappa^+$.

PROOF. The condition $\mathrm{hsat}(B) \geq \kappa^+$ is obviously necessary.

Let F be a centred system in B, $|F| = \kappa$. If $\kappa = \omega$, then it is an easy exercise to show that F is ω-decomposable, so let $\kappa > \omega$ and $\tau = \mathrm{cf}(\kappa) < \kappa$. Let us assume that each finite meet of members of F belongs to F, too.

Claim. F is τ^+-decomposable.

For $\xi < \tau$ choose $F_\xi \subseteq F$ such that $|F_\xi| < \kappa$, $F_\xi \subseteq F_\eta$ for $\xi < \eta < \tau$ and $\bigcup \{F_\xi : \xi \in \tau\} = F$.

Transfinite recursion. Let $\gamma < \tau$ and suppose that for $\alpha < \beta < \gamma$ we have found ordinals $\xi(\alpha)$ and disjoint systems C_α, $C_{\alpha\beta}$ satisfying the following:

(i) $\xi(\alpha) < \xi(\beta) < \tau$ for $\alpha < \beta < \gamma$;

(ii) if $\alpha < \gamma$, $x \in F_{\xi(\alpha)}$, then $|C_\alpha| = |\{c \in C_\alpha : x \cdot c \neq \mathbb{0}\}| = \tau^+$;

(iii) if $\alpha < \beta < \gamma$, then $C_{\alpha,\beta} \subseteq C_\alpha$ and $|C_{\alpha,\beta}| \leq \tau$;

(iv) if $\alpha < \beta < \gamma$, then $(C_\alpha - C_{\alpha\beta}) \cup C_\beta$ is a disjoint system.

Define $W_\gamma = \bigcup \{C_\alpha - \bigcup \{C_{\alpha\beta} : \alpha < \beta < \gamma\} : \alpha < \gamma\}$. Then W_γ is disjoint by (iv) and $|W_\gamma| = \tau^+$ by (ii) and (iii). If for each $x \in F$, $|\{w \in W_\gamma : x \cdot w \neq \mathbb{0}\}| = \tau^+$, we are done: F is τ^+-decomposable.

If there is some $x \in F$ with $|\{w \in W_\gamma : x \cdot w \neq \mathbb{0}\}| \leq \tau$, then $x \not\in F_{\xi(\alpha)}$ for $\alpha < \gamma$ by (ii), hence there is some $\xi(\gamma) \geq \sup\{\xi(\alpha) : \alpha < \gamma\}$, $\xi(\gamma) < \tau$ such that $x \in F_{\xi(\gamma)}$. Define $C_{\alpha,\gamma} = \{c \in C_\alpha : c \cdot x \neq \mathbb{0}\}$. Clearly, $|C_{\alpha\gamma}| \leq \tau$ for $\alpha < \gamma$.

Since $|F_{\xi(\gamma)}|^+ < \kappa < \mathrm{hsat}(B)$, there is a disjoint refinement D of the system $\{x \cdot y : y \in F_{\xi(\gamma)}\}$ by 1.2. For $d \in D$ let $\{d_i : i \in \tau^+\}$ be an arbitrary disjoint family in $B|d$. Let $C_\gamma = \{\Sigma\{d_i : d \in D\} : i \in \tau^+\}$. Clearly, $|\{c \in C_\gamma : y \cdot c \neq \mathbb{0}\}| = \tau^+$ for each $y \in F_{\xi(\gamma)}$.

This completes the recursive definitions.

Suppose that the induction has not stopped before τ. We define $C = \bigcup \{C_\alpha - \bigcup \{C_{\alpha\beta} : \alpha < \beta < \tau\} : \alpha < \tau\}$. It is easy to see that $|C| = |\{c \in C : x \cdot c \neq \mathbb{0}\}| = \tau^+$ for each $x \in F$. The claim is proved.

Now we make use of the fact that F is centred. According to the claim, we know that $C = \{c_\eta : \eta \in \tau^+\}$ is a τ^+-decomposition of F. For $x \in F$, let $u_x = \{\eta \in \tau^+ : x \cdot c_\eta \neq \mathbb{0}\}$. The family $U = \{u_x : x \in F\}$ is a uniformly centred family of subsets of τ^+, since F is closed under finite meets. Moreover, τ is regular. Hence, U is τ-decomposable in $\mathcal{P}(\tau^+)$ according to 5.33. If the pairwise disjoint collection $\{a_i : i \in \tau\}$ of subsets of τ^+ witnesses to the τ-decomposability of U, then the system $\{\Sigma\{c_\eta : \eta \in a_i\} : i \in \tau\}$ shows the τ-decomposability of F. It remains only to apply 0.2. $\square$

2.4. REMARK. Recall that the character of an ultrafilter on a Boolean algebra is the minimal size of a family of its generators. Consider a complete infinite Boolean algebra B. It is well known that there is an ultrafilter on B whose character equals $|B|$, the greatest possible value.

It turns out that the lower bound for an ultrafilter character is closely related to the Rfip property. Trivially, no ultrafilter on a complete Boolean algebra is 2-decomposable. Consequently, if a complete B satisfies $\mathrm{Rfip}(\kappa)$, then no ultrafilter on B can have its character $\leq \kappa$. In particular, if $\mathrm{hsat}(B) > \kappa^+$, then κ^+ is a lower bound for the characters.

Since we are concerned with the case $\mathrm{hsat}(B) = \kappa^+$ and the $\mathrm{Rfip}(\kappa)$ property, this indicates why we shall deal with families generating an ultrafilter now.

2.5. THEOREM. *Let κ be an uncountable regular cardinal, B a κ-complete Boolean algebra with $\pi(B) \leq \kappa$. Suppose $\kappa^{<\kappa} = \kappa$.*

Then there is an ultrafilter on B generated by at most κ elements of B. Furthermore, B does not have Rfip(κ).

The theorem heavily depends on an important lemma which is due to K. Kunen, J. van Mill and C. Mills.

2.6. DEFINITION. Let B be a Boolean algebra, ξ an ordinal. A descending chain T of length ξ is a family $\{t_\alpha : \alpha < \xi\} \subseteq B^+$ sastisfying $t_\alpha > t_\beta$ for $\alpha < \beta < \xi$. A descending chain T is called a tower if for each $b \in B^+$ there is some $t_\alpha \in T$ with $b - t_\alpha \neq \mathbb{0}$.

2.7. LEMMA [KUNEN, VAN MILL and MILLS (1980)]. *Let κ be an uncountable regular cardinal, B a Boolean algebra, $|B| \leq \kappa$. Then there is an ultrafilter on B containing no tower of length κ.*

PROOF. If $|B| < \kappa$, the lemma holds trivially since there is no tower of length κ in B.

If $|B| = \kappa$, well-order $B = \{b_\alpha : \alpha \in \kappa\}$ and define B_α to be the smallest subalgebra of B generated by $\{b_\beta : \beta < \alpha\}$. We have $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots$, for $\mu < \kappa$, μ limit, $B_\mu = \bigcup \{B_\alpha : \alpha < \mu\}$.

Claim. Let T be a tower of length κ in B. Then there is some $\alpha < \kappa$ such that $T \cap B_\alpha$ is a tower in B_α.

Given arbitrary $\gamma < \kappa$, there is some $\gamma^* \geq \gamma$, $\gamma^* < \kappa$ such that for each $b \in B_\gamma^+$ there is a $t \in T \cap B_{\gamma^*}$ with $b - t \neq \mathbb{0}$. Indeed, for $b \in B_\gamma^+$ there is a $t(b) \in T$ with $b - t(b) \neq \mathbb{0}$ for T is a tower in B, and there is some $\beta(b) < \kappa$ such that $t(b) \in B_{\beta(b)}$. It suffices to set $\gamma^* = \sup\{\beta(b) : b \in B_\gamma^+\}$. The regularity of κ guarantees that $\gamma^* < \kappa$. So pick $\alpha_0 < \kappa$ arbitrary, and define $\alpha_{n+1} = \alpha_n^*$, $\alpha = \sup\{\alpha_n : n \in \omega\}$. If $b \in B_\alpha$, then $b \in B_{\alpha_n}$ for some $n \in \omega$ and there is a $t(b) \in T \cap B_{\alpha_{n+1}}$ with $b - t(b) \neq \mathbb{0}$. Thus, $T \cap B_\alpha$ is a tower.

For each tower T of length κ in B, let $\alpha(T) < \kappa$ be such that $T \cap B_{\alpha(T)}$ is a tower in $B_{\alpha(T)}$ and let $t(T) \in T$ be such that $t(T) \leq t$ for each $t \in T \cap B_{\alpha(T)}$.

Now, the family $F = \{\mathbb{1} - t(T) : T \text{ is a tower of length } \kappa \text{ in } B\}$ is centred: pick $n \in \omega$, choose $T_1, T_2, \ldots, T_n$ towers of length κ in B. Denote $B^i = B_{\alpha(T_i)}$. We can assume that $\alpha(T_1) \leq \alpha(T_2) \leq \cdots \leq \alpha(T_n)$, so $B^1 \subseteq B^2 \subseteq \cdots \subseteq B^n$. The system T_1 is a tower in B^1, thus there is some $t^1 \in T_1 \cap B^1$ with $\mathbb{1} - t^1 \neq \mathbb{0}$. But $\mathbb{1} - t^1 \in B^2$ and $T_2 \cap B^2$ is a tower in B^2, hence there is some $t^2 \in T_2 \cap B^2$ with $(\mathbb{1} - t^1) - t^2 \neq \mathbb{0}$, proceeding further, we shall find $t^3, t^4, \ldots, t^n$ such that $\mathbb{1} - (t^1 + t^2 + \cdots + t^n) \neq \mathbb{0}$.

As $t(T_i) \leq t^i$ for each $i = 1, 2, \ldots, n$, we have also $\mathbb{0} \neq \Pi\{\mathbb{1} - t^i : i = 1, 2, \ldots, n\} \leq \Pi\{\mathbb{1} - t(T_i) : i = 1, 2, \ldots, n\}$.

Obviously, if $U \in \mathrm{Ult}(B)$ and $U \supseteq F$, then U contains no tower of length κ in B. $\square$

2.8. *Proof of Theorem 2.5.* Let C be a dense subset of B, $|C| \leq \kappa$. Denote by D the smallest κ-complete subalgebra of B generated by C. Clearly, $|D| = |C|^{<\kappa} \leq \kappa^{<\kappa} = \kappa$. Thus, by 2.7, there is an ultrafilter $U \in \mathrm{Ult}(D)$ containing no tower of length κ in D. Clearly, $|U| \leq |D| \leq \kappa$ and $U \subseteq B$.

Claim. Let $\{p_\alpha: \alpha \in \kappa\} \subseteq C$ be a partition of unity in B. Then there is some $\beta \in \kappa$ with $\Sigma\{p_\alpha: \alpha < \beta\} \in U$.

Indeed, by κ-completeness of D, for each $\beta < \kappa$ both $\Sigma\{p_\alpha: \alpha < \beta\}$ and $\mathbb{1} - \Sigma\{p_\alpha: \alpha < \beta\}$ belong to D. Since $\{p_\alpha: \alpha < \beta\}$ is a partition of unity, the family $T = \{\mathbb{1} - \Sigma\{p_\alpha: \alpha < \beta\}: \beta < \kappa\}$ is a tower in D. Since $T \not\subseteq U$, the claim follows.

There is a unique ultrafilter in B which contains U. To see this, let $x, y \in B^+$ be such that $x \cdot y = \mathbb{0}$, $x + y = \mathbb{1}$. Let $Z = \{z_\alpha: \alpha < \kappa\}$ be a maximal disjoint family refining $\{x, y\}$, $Z \subseteq C$. According to the claim, there is some $\beta < \kappa$ with $\Sigma\{z_\alpha: \alpha < \beta\} \in U$. Since U is an ultrafilter in D, U must contain either $\Sigma\{z_\alpha: \alpha < \beta \;\&\; z_\alpha \leq x\}$ or $\Sigma\{z_\alpha: \alpha < \beta \;\&\; z_\alpha \leq y\}$. Thence U generates an ultrafilter in B.

Finally, assume on the contrary that B has $\mathrm{Rfip}(\kappa)$. Then $\mathrm{hsat}(B) = \mathrm{hsat}(D) = \kappa^+$, hence for the centred system U described above we have some disjoint refinement H with $H \subseteq C$. Extend H to some maximal disjoint family $Z = \{z_\alpha: \alpha \in \kappa\} \subseteq C$. By the claim, for some $\beta < \kappa$, $\Sigma\{z_\alpha: \alpha < \beta\} \in U$. For $\alpha < \beta$, select $x_\alpha, y_\alpha \in C$ such that $\mathbb{0} \neq x_\alpha, y_\alpha$, $x_\alpha \cdot y_\alpha = \mathbb{0}$, $x_\alpha + y_\alpha = z_\alpha$. Since U is an ultrafilter in D, either $x = \Sigma\{x_\alpha: \alpha < \beta\}$ or $y = \Sigma\{y_\alpha: \alpha < \beta\}$ belongs to U, but neither x nor y contains a member of H – a contradiction. □

2.9. Example. Assume CH. Then $\mathrm{Col}(\omega, \omega_1)$ does not have $\mathrm{Rfip}(\omega_1)$, for CH says that $\omega_1^{<\omega_1} = \omega_1$ and 2.5 may be applied.

2.10. Definition. Let λ be an infinite regular cardinal. The formula $f \leq^* g$ iff $|\{\alpha \in \lambda: f(\alpha) > g(\alpha)\}| < \lambda$ defines a preorder on ${}^\lambda\lambda$. A family $F \subseteq {}^\lambda\lambda$ is dominating if, for every $g \in {}^\lambda\lambda$, there is some $f \in F$ with $g \leq^* f$. If F is a dominating family in ${}^\lambda\lambda$, $|F| = \mu$ and $\leq^*$ on F is a well-ordering of type μ, then F is called a μ-scale.

2.11. Theorem. *Let $\lambda \leq \kappa$ be infinite cardinals, let B be a $(\lambda, \cdot, \kappa)$-nowhere distributive Boolean algebra which is $(\tau, \cdot, 2)$-distributive for each $\tau < \lambda$. Let $A = \{a_\xi: \xi < \kappa\} \subseteq B^+$ be λ-decomposable. Then A has a disjoint refinement provided there is no dominating family of power κ in ${}^\lambda\lambda$.*

Proof. Let $P = \{p(\alpha): \alpha \in \lambda\} \subseteq B^+$ be the disjoint set witnessing to the λ-decomposability of A. For each $\alpha \in \lambda$, there is a matrix $\mathcal{Q}(\alpha) = \{Q(\alpha, \beta): \beta \in \lambda\}$ in $B|p(\alpha)$ satisfying 1.12.

We shall construct a mapping $f_\xi \in {}^\lambda\lambda$ for each $\xi \in \kappa$ recursively as follows. Suppose $f_\xi(\gamma)$ has been defined for $\gamma < \alpha$. Let $\tilde{\alpha} \geq \alpha$ be the first ordinal number such that $a_\xi \cdot p(\tilde{\alpha}) \neq \mathbb{0}$. Let $\beta < \lambda$ be the smallest ordinal with $\beta \geq f_\xi(\gamma)$ for $\gamma < \alpha$ and $|\{q \in Q(\tilde{\alpha}, \beta): q \cdot a_\xi \neq \mathbb{0}\}| \geq \kappa$.

Define $f_\xi(\alpha) = \beta$.

The family $\{f_\xi: \xi \in \kappa\} \subseteq {}^\lambda\lambda$ is not dominating in ${}^\lambda\lambda$. Thus, there is some $g \in {}^\lambda\lambda$

satisfying $|\{\alpha \in \lambda: g(\alpha) > f_\xi(\alpha)\}| = \lambda$ for each $\xi \in \kappa$. According to the regularity of λ we may assume that g is strictly increasing. Let us define

$$Q = \bigcup \{Q(\alpha, g(\alpha)): \alpha \in \lambda\}\ .$$

Let us verify that for each $\xi \in \kappa$, $|\{q \in Q: a_\xi \cdot q \neq \mathbb{0}\}| \geq \kappa$.

Fix $\xi \in \kappa$, let $X_\xi = \{\alpha \in \lambda: g(\alpha) > f_\xi(\alpha)\}$, $Y_\xi = \{\alpha \in \lambda: a_\xi \cdot p(\alpha) \neq \mathbb{0}\}$. If $\alpha \in X_\xi \cap Y_\xi$, then $|\{q \in Q(\alpha, f_\xi(\alpha)): a_\xi \cdot q \neq \mathbb{0}\}| \geq \kappa$, thus $|\{q \in Q: a_\xi \cdot q \neq \mathbb{0}\}| \geq \kappa$, too. Thus, it suffices to show $X_\xi \cap Y_\xi \neq \emptyset$. Choose arbitrary $\alpha \in X_\xi$. By our construction of f_ξ, $f_\xi(\alpha) = f_\xi(\tilde{\alpha})$, where $\tilde{\alpha}$ was defined in the recursive procedure. The function g is increasing, thus we have $g(\tilde{\alpha}) \geq g(\alpha) > f_\xi(\alpha) = f_\xi(\tilde{\alpha})$, so $\tilde{\alpha} \in X_\xi$. Anyway, $\tilde{\alpha} \in Y_\xi$.

So applying 0.2, we obtain a disjoint refinement of A. □

2.12. COROLLARY. *Let B be an $(\omega, \cdot, \omega_1)$-nowhere distributive Boolean algebra, let $A \subseteq B^+$ have FIP and $|A| \leq \omega_1$. If there is no dominating family of power ω_1 in ${}^\omega\omega$, then A has a disjoint refinement.*

PROOF. Extend A arbitrarily to an ultrafilter U on B. Let W be a maximal disjoint system in B^+ such that for each $w \in W$, $\mathbb{1} - w \in U$. Clearly, $\{w \in W: a \cdot w \neq \mathbb{0}\}$ is infinite for each $a \in A$. If there is some $a_0 \in A$ such that $|\{w \in W: a_0 \cdot w \neq \mathbb{0}\}| = \omega$, let $P = \{w \in W: a_0 \cdot w \neq \mathbb{0}\}$. Since A has FIP, $|\{p \in P: a \cdot p \neq \mathbb{0}\}| = \omega$ for each $a \in A$, hence 2.11 applies.

If there is no such a_0, then for each $a \in A$, $|\{w \in W: a \cdot w \neq \mathbb{0}\}| = \omega_1$ and the existence of a disjoint refinement follows by 0.2. □

2.13. EXAMPLE. Assume that there is no ω_1-scale in ${}^\omega\omega$. Then $\mathrm{Col}(\omega, \omega_1)$ has $\mathrm{Rfip}(\omega_1)$.

To finish this part, we shall show what can happen in $\overline{\mathcal{P}(\omega)/fin}$. Notice that $\mathcal{P}(\omega)/fin$ always satisfies $\mathrm{Rfip}(2^\omega)$, which will be proved later. The following two propositions are immediate corollaries to 2.3 (resp. 2.5).

2.14. PROPOSITION. *Suppose $\mathbb{C}^{<\mathbb{C}} = \mathbb{C}$. Then there is an ultrafilter on $\overline{\mathcal{P}(\omega)/fin}$ generated by precisely $\mathbb{C}$ elements and therefore under $\mathbb{C}^{<\mathbb{C}} = \mathbb{C}$, the algebra $\overline{\mathcal{P}(\omega)/fin}$ does not have* $\mathrm{Rfip}(\mathbb{C})$. □

2.15. PROPOSITION. *If $\mathbb{C}$ is a singular cardinal, then $\overline{\mathcal{P}(\omega)/fin}$ has* $\mathrm{Rfip}(\mathbb{C})$. *Therefore no ultrafilter on $\overline{\mathcal{P}(\omega)/fin}$ is generated by $\leq\mathbb{C}$ elements.* □

One may ask whether $\mathrm{Rfip}(\mathbb{C})$ may hold for $\overline{\mathcal{P}(\omega)/fin}$, assuming $\mathbb{C}$ is a regular cardinal. We shall show conditions under which the answer is in the affirmative. Let us give first some combinatorial assumptions, which imply that $\overline{\mathcal{P}(\omega)/fin}$ has $\mathrm{Rfip}(\mathbb{C})$.

2.16. PROPOSITION. *Suppose $2^\omega = \omega_2$ and*

(i) *$\overline{\mathcal{P}(\omega)/fin}$ is isomorphic to* $\mathrm{Col}(\omega_1, \omega_2)$;

(ii) *each ultrafilter on $\mathcal{P}(\omega)/fin$ contains a tower in $\mathcal{P}(\omega)/fin$;*

(iii) *there is no ω_2-scale in ${}^{\omega_1}\omega_1$.*

Then $2^{\omega_1} > \omega_2$ *and* $\overline{\mathcal{P}(\omega)/fin}$ *has* Rfip(2^ω). *Therefore there is no ultrafilter on* $\overline{\mathcal{P}(\omega)/fin}$ *having* $\leq 2^\omega$ *generators.*

PROOF. The inequality $2^{\omega_1} > \omega_2$ follows from (iii). Denote $B = \overline{\mathcal{P}(\omega)/fin}$ and let $A = \{a_\alpha : \alpha \in \omega_2\}$ be a centred family of elements of B. Then there is an ultrafilter U on $\mathcal{P}(\omega)/fin$ which is compatible with A. By (ii), there is a tower $T \subseteq \mathcal{P}(\omega)/fin$, $T \subseteq U$. Clearly, $|T| > \omega$. If the length of T is ω_2, then A is ω_2-decomposable and hence A has a disjoint refinement by 0.1. If the length of T equals ω_1, then A is ω_1-decomposable. But then (i), (iii) imply the existence of a disjoint refinement for A via Theorem 2.11. □

2.17. A model of ZFC, where the assumption of 2.16 holds true.

Let $\mathcal{M}$ be a countable transitive model of ZFC + GCH. Let B_0, B_1, B_2 be the complete Boolean algebras in $\mathcal{M}$ such that B_0 is the Solovay–Tennenbaum algebra making $MA + 2^\omega = \omega_2$; B_1 is the well-known algebra for adding ω_1 Cohen reals, B_2 is the algebra with the basis Fn$(\omega_3, \omega_1, \omega_1)$.

Let $B = B_0 \oplus B_1 \oplus B_2$ be the free product, G a generic ultrafilter on B. Then in $\mathcal{M}[G]$, all the assumptions of Proposition 2.16 hold. We shall omit the details of the proof; the reader can find them in BALCAR, SIMON and VOJTÁŠ [1981].

3. Non-distributivity of $\mathcal{P}(\omega)/fin$

We shall be interested in disjoint refinement of systems in the Boolean algebra $\mathcal{P}(\omega)/fin$. The aim of this section is to build up the tools for it. We shall introduce the cardinal characteristics called the height of the algebra $\mathcal{P}(\omega)/fin$ and we shall show its position in the hierarchy of the other cardinal characteristics of systems of sets and functions on ω.

3.1. We know (cf. Part I) that the Boolean algebra $\mathcal{P}(\omega)/fin$ is atomless, homogeneous and its cardinality equals to 2^ω. The algebra $\mathcal{P}(\omega)/fin$ is not σ-complete. Furthermore, $\mathrm{sat}(\mathcal{P}(\omega)/fin) = \mathrm{hsat}(\mathcal{P}(\omega)/fin) = (2^\omega)^+$ and each countable family $F \subseteq \mathcal{P}(\omega)/fin$ with the finite intersection property has $\Pi F \neq \mathbb{0}$ by Cantor's property; hence, $(\mathcal{P}(\omega)/fin)^+$ is σ-closed and therefore $(\omega, \cdot, 2)$-distributive (see 1.10). Since $\mathcal{P}(\omega)/fin$ is atomless, there is some $\tau \leq 2^\omega$ such that $\mathcal{P}(\omega)/fin$ is not $(\tau, \cdot, 2)$-distributive. This leads to the following notion.

3.2. DEFINITION. The height of $\mathcal{P}(\omega)/fin$ is the cardinal number

$$h = \min\{\tau : \mathcal{P}(\omega)/fin \text{ is not } (\tau, \cdot, 2)\text{-distributive}\}.$$

Note that, according to the homogeneity of $\mathcal{P}(\omega)/fin$, we have also

$$h = \min\{\tau : \mathcal{P}(\omega)/fin \text{ is } (\tau, \cdot, 2)\text{-nowhere distributive}\}.$$

3.3. PROPOSITION. $\omega_1 \leq h \leq \mathrm{cf}(2^\omega)$ *and* h *is a regular cardinal.*

PROOF. As mentioned above, $\mathcal{P}(\omega)/fin$ is $(\omega, \cdot, 2)$-distributive, therefore $h \geq \omega_1$.

In order to show that $h \leq \mathrm{cf}(2^\omega)$, assume that $\kappa = \mathrm{cf}(2^\omega)$. Then $\mathcal{P}(\omega)/fin - \{\mathbb{0}\}$ can be expressed as $\bigcup \{A_\alpha : \alpha < \kappa\}$, where each A_α is of size less than 2^ω. Since $|A_\alpha| < 2^\omega < \mathrm{hsat}(\mathcal{P}(\omega)/fin\}$, each A_α has a disjoint refinement D_α. Let P_α be an arbitrary partition of unit containing D_α as a subset. Since $\mathcal{P}(\omega)/fin$ is atomless, there is no partition Q such that Q refines all P_α's simultaneously. Thus, $\mathcal{P}(\omega)/fin$ is not $(\kappa, \cdot, 2)$-distributive, so $h \leq \mathrm{cf}(2^\omega)$.

The regularity of h is an immediate consequence of its definition. □

3.4. THEOREM (Base tree). *There exists a family $T \subseteq \mathcal{P}(\omega)/fin$ such that*

(i) *T is a dense subset of $\mathcal{P}(\omega)/fin$;*

(ii) *$\langle T, \geq \rangle$ is a tree of height h, where $\leq$ is the canonical ordering of the algebra $\mathcal{P}(\omega)/fin$;*

(iii) *each level T_α is a partition of unity;*

(iv) *each $u \in T$ has 2^ω immediate successors.*

Any tree with the properties (i)–(iv) *will be called a base tree.*

PROOF. Apply Theorem 1.13 in the special setting $\mathcal{B} = \mathcal{P}(\omega)/fin$, $\lambda = \omega_1$, $\kappa = 2$, $\tau = h$. □

3.5. NOTATION. The element of $\mathcal{P}(\omega)/fin$ are determined by subsets of ω. The notation we use is the standard one. For $A \in \mathcal{P}(\omega)$ let $A^* \in \mathcal{P}(\omega)/fin$ be its equivalence class mod *fin*, i.e. $B \in A^*$ iff $(A - B) \cup (B - A)$ is finite. Obviously, $^*\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)/fin$ is the canonical projection. For A, $B \subseteq \omega$ denote $A \subseteq^* B$ if $A - B$ is finite. Clearly, $A \subseteq^* B$ if and only if $A^* \leq B^*$ in $\mathcal{P}(\omega)/fin$.

A family $\mathcal{A} \subseteq [\omega]^\omega$ is called an almost disjoint (AD) family if $\{A^* : A \in \mathcal{A}\}$ is a disjoint family in $\mathcal{P}(\omega)/fin$. Equivalently, $\mathcal{A} \subseteq [\omega]^\omega$ is an AD family if any two distinct members of $\mathcal{A}$ have finite intersection. A MAD family is a maximal AD family with respect to the inclusion.

Under the notation just introduced, we shall state the following consequence of the base tree theorem 3.4.

3.6. PROPOSITION. *There is a family $W \subseteq [\omega]^\omega$ such that*

(i) $(\forall A, B \in W)$ *(either $A \cap B$ is finite or $A \subseteq^* B$ or $B \subseteq^* A$);*

(ii) $(\forall A \in W)$ $|\{B \in W : A \subseteq^* B\}| < h$;

(iii) *for each $X \in [\omega]^\omega$ there is an $A \in W$ with $A \subseteq X$.*

PROOF. Let T be an arbitrary base tree in $\mathcal{P}(\omega)/fin$. For each $a \in T$ pick a set $A(a) \subseteq \omega$ such that $a = A(a)^*$. The family $W' = \{A(a) : a \in T\}$ satisfies (i) and (ii) since the height of T is h. By 3.4(i) and 3.4(iv) for each $X \in [\omega]^\omega$ there is an $\alpha < h$ such that the set $\{A(a) : a \in T_\alpha \ \&\ A(a) \subseteq^* X\}$ is of full cardinality 2^ω. Hence there is a one-to-one mapping f from $[\omega]^\omega$ into W' satisfying $f(X) \subseteq^* X$ for each $X \in [\omega]^\omega$. It suffices to set $W = \{X \cap f(X) : X \in [\omega]^\omega\} \cup (W' - \mathrm{Rng}(f))$. □

Let us show the relationship between the cardinal characteristics h and Baire number of special topological spaces. First, let us recall the general notions.

3.7. DEFINITION (Additivity and covering of a family of sets). Let X be a set, $\mathscr{A} \subseteq \mathscr{P}(X)$. The additivity of $\mathscr{A}$ is the cardinal number

$$\operatorname{add}(\mathscr{A}) = \min\{|\mathscr{S}|: \mathscr{S} \subseteq \mathscr{A} \ \& \ \bigcup \mathscr{S} \not\in \mathscr{A}\}\,.$$

The covering number of $\mathscr{A}$ is defined by

$$\operatorname{cov}(\mathscr{A}) = \min\{|\mathscr{S}|: \mathscr{S} \subseteq \mathscr{A} \ \& \ \bigcup \mathscr{S} = \bigcup \mathscr{A}\}\,.$$

If X is a topological space without isolated points, then the Baire number of X, denoted by $n(X)$, is the covering number of the family of all nowhere dense subsets of X.

The Baire number of a Boolean algebra is the Baire number of its Stone space.

3.8. PROPOSITION. *Let B be an atomless BA. Then $n(B) = \min\{|\mathscr{D}|$: each $D \in \mathscr{D}$ is a dense subset of B and for each ultrafilter F on B there is some $D \in \mathscr{D}$ with $F \cap D = \emptyset\}$.*

The proof of this proposition is clear as well as the fact that the system $\mathscr{D}$ in the statement can be replaced by a system of partitions of unity.

Let us return to the algebra $\mathscr{P}(\omega)/fin$.

3.9. PROPOSITION. *The additivity of a family of all nowhere dense subsets of $\beta\omega - \omega$ $(=\operatorname{Ult}(\mathscr{P}(\omega)/fin))$ equals the height of $\mathscr{P}(\omega)/fin$.*

We shall omit the easy proof.

3.10. THEOREM (Baire number of $\mathscr{P}(\omega)/fin$). *Denote $n = n(\mathscr{P}(\omega)/fin)$. Then:*

(i) *$n = h$ if and only if there is a base tree in $\mathscr{P}(\omega)/fin$ without cofinal branches;*

(ii) *if $h < 2^\omega$, then $h \le n \le h^+$;*

(iii) if $h = 2^\omega$, *then $2^\omega \le n \le 2^{2^\omega}$.*

PROOF. Since the additivity of nowhere dense subsets is h by 3.9, it follows that $h \le n$.

(i) Assume T to be a base tree without cofinal branches. For each $\alpha < h$, the level T_α is a partition of unity, therefore for any ultrafilter F in $\mathscr{P}(\omega)/fin$, the intersection $F \cap T_\alpha$, if non-empty , consists of precisely one element. If there is an ultrafilter F with $F \cap T_\alpha \neq \emptyset$ for each $\alpha < h$, then $\{F \cap T_\alpha : \alpha < h\}$ is a cofinal branch in T, a contradiction. Hence, $F \cap T_\alpha$ is empty for some $\alpha < h$, so $n = h$.

Next assume $n = h$. This means that there are partitions of unity, $P_\alpha (\alpha < h)$ such that each ultrafilter is disjoint with some P_α. We can construct a base tree T such that for each $\alpha < h$, the level T_α refines all P_β for $\beta \le \alpha$. Since each cofinal branch in T generates a filter which meets all P_α's, it follows that there is no cofinal branch in T.

(ii) Assume $h < 2^\omega$. We have to construct a family $\{D_\beta : \beta < h^+\}$ of dense subsets of $\mathscr{P}(\omega)/fin$ witnessing to the inequality $n \le h^+$. Let T be an arbitrary base tree. For each $t \in T$, pick a subset of h^+ immediate successors of t, denote it $\{d_\xi(t): \xi < h^+\}$. This is always possible, for t has 2^ω immediate successors in T and $2^\omega \geq h^+$. For $\beta < h^+$, define $D_\beta = \{d_\beta(t): t \in T\}$. Clearly, each D_β is dense in $\mathcal{P}(\omega)/fin$ for T is dense.

Aiming for a contradiction, suppose that there is an ultrafilter F in $\mathcal{P}(\omega)/fin$ with non-empty intersection with D_β for each $\beta < h^+$. Define then a mapping $f: h^+ \to h$ by the rule

$$f(\beta) = \min\{\alpha < h: \text{there is a } t \in T_\alpha \text{ such that } d_\beta(t) \in F\}.$$

Now it is easy to see that f is one-to-one, which is impossible for $h^+ > h$.

(iii) If $h = 2^\omega$, then there is no restriction on the size of n except the natural one, $n \leq |\mathrm{Ult}(\mathcal{P}(\omega)/fin)| = 2^{2^\omega}$. □

3.11. Completely Ramsey sets and Ellentuck space. Consider the set $[\omega]^\omega$ of all infinite subsets of ω. For a finite $s \subseteq \omega$ and infinite $X \subseteq \omega$ with $\max s < \min X$, denote $[s, X] = \{Y \in [\omega]^\omega$: s is an initial segment of Y and $Y - s \subseteq X\}$. If we topologize the set $[\omega]^\omega$ using the sets $[s, X]$ as an open base, we obtain the so-called Ellentuck space $\mathcal{E}$.

A set $S \subseteq [\omega]^\omega$ is called completely Ramsey if for each open base set $[s, X]$ there is a set $A \in [X]^\omega$ such that the family $[s, A]$ is either contained in $S \cap [s, X]$ or disjoint with $S \cap [s, X]$ (GALVIN and PRIKRY [1973]).

Ellentuck proved that completely Ramsey sets are just those with Baire property in the space $\mathcal{E}$ (ELLENTUCK [1974]).

We are interested in the question how complete the set algebra of completely Ramsey sets is. The full answer gives the following result due to Sz. Plewik.

3.12. THEOREM. *Let J denote the ideal of all nowhere dense subsets of the Ellentuck space $\mathcal{E}$. Then*

(i) *add* $(J) = h$;

(ii) $n(\mathcal{E}) = h$;

(iii) *the algebra B of all subsets of $\mathcal{E}$ having the Baire property is an h-complete set algebra which is not h^+-complete.*

PROOF. Let us start the proof with a claim, which shows an important property of open dense subsets in $\mathcal{E}$.

Claim. Let $S \subseteq \mathcal{E}$ be an open dense set, $X \in [\omega]^\omega$. Then there is a set $M \in [X]^\omega$ such that for each $Z \subseteq^* M$, $Z \in S$. (Equivalently, $[\emptyset, s \cup M] \subseteq S$ for each $s \in [\omega]^{<\omega}$.)

Denote $X_0 = X$, $n_0 = \min X_0$. Proceeding by induction, suppose $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{i-1}$ and $n_0 < n_1 < \cdots < n_{i-1}$ have been found, $n_k = \min X_k$. Enumerate $\{s_j: j < 2^{n_{i-1}}\} = \mathcal{P}(n_{i-1})$. The set $[s_0, X_{i-1}]$ is basic open in $\mathcal{E}$ and S is dense open. Since S is open, it is completely Ramsey, hence there is some set $Y_0 \in [X_{i-1}]^\omega$ with $[s_0, Y_0] \subseteq S \cap [s_0, X_{i-1}]$ (the other possibility, i.e. $[s_0, Y_0] \cap S \cap [s_0, X_{i-1}] = \emptyset$ is excluded by the density of S). Similarly, for $[s_1, Y_0]$ there is some $Y_1 \in [Y_0]^\omega$ with $[s_1, Y_1] \subseteq S \cap [s_1, Y_0]$. Proceeding further, we shall find $[s_j, Y_j] \subseteq S \cap [s_j, Y_{j-1}]$ and finish the induction step choosing $X_i \in [\bigcap \{Y_j: j < 2^{n_{i-1}}\}]^\omega$ with $n_i = \min X_i > n_{i-1}$.

Clearly, for each $s \subseteq n_{i-1}$, $[s, X_i] \subseteq S$. So it suffices to define $M = \{n_{2i}: i \in \omega\}$. If $Z \subseteq^* M$, then for some $i \in \omega$ and some $s \subseteq n_{2i-1}$, $Z \in [s, X_{2i}] \subseteq S$, which shows that M is as required.

Having proved the claim, we are allowed to introduce the following notation: If $\mathcal{P}$ is a MAD family on ω, let $\mathcal{P}^\wedge = \{A \subseteq \omega$: for some $Y \in \mathcal{P}$, $A \subseteq^* Y\}$. By the claim, if $S \subseteq \mathscr{E}$ is open and dense, then there is a MAD family $\mathcal{P}$ on ω with $\mathcal{P}^\wedge \subseteq S$. On the other hand, if $\mathcal{P}$ is an arbitrary MAD family on ω, then $\mathcal{P}^\wedge$ is an open dense subset of $\mathscr{E}$. To see this, pick $[s, X]$ from the open base of $\mathscr{E}$. For some $Y \in \mathcal{P}$, $X \cap Y$ is infinite, for $\mathcal{P}$ is maximal, and if $A \in [s, X \cap Y]$, then $A \in \mathcal{P}^\wedge$. Thus, $\mathcal{P}^\wedge$ is dense. It is open, too, for $\mathcal{P}^\wedge = \bigcup \{[\emptyset, s \cup Y]: Y \in \mathcal{P}$ & $s \in [\omega]^{<\omega}\}$.

Now we are ready to prove the theorem.

$\mathrm{add}(J) \geq h$: Let $\lambda < h$ and let $\{S_\alpha: \alpha < \lambda\}$ be a family of dense open subsets of $\mathscr{E}$. It suffices to show that $\bigcap \{S_\alpha: \alpha < \lambda\}$ contains a dense open set. To this end, let $\mathcal{P}_\alpha$ be a MAD family on ω with $\mathcal{P}_\alpha^\wedge \subseteq S_\alpha$. Then by $\lambda < h$, there is some MAD family $\mathcal{Q}$ refining all $\mathcal{P}_\alpha$'s. Clearly, $\mathcal{Q}^\wedge \subseteq \bigcap \{\mathcal{P}_\alpha^\wedge: \alpha < \lambda\} \subseteq \bigcap \{S_\alpha: \alpha < \lambda\}$.

$n(\mathscr{E}) \leq h$: Let T be a base tree as in 3.6. Each level T_α is a MAD family on ω; consider $\{T_\alpha^\wedge: \alpha < h\}$. This is a system of open dense sets in $\mathscr{E}$ with empty intersection, for if $X \in [\omega]^\omega$, then for some α, $|X - A| = \omega$ for each $A \in T_\alpha$, thus $X \notin T_\alpha^\wedge$.

Since $\mathrm{add}(J) \leq n(\mathscr{E})$, (i) and (ii) are proved, moreover, h-completeness of the ideal J implies the h-completeness of the algebra B of all subsets of $\mathscr{E}$ with the Baire property.

To finish the proof of (iii), pick an arbitrary subset of $\mathscr{E}$ which is not completely Ramsey (i.e. which does not have the Baire property). It is well known that such a set exists; any base tree is such an example. This set is a union of h-many nowhere dense subsets of $\mathscr{E}$, because the whole space $\mathscr{E}$ is such a union. Thus, B is not h^+-complete. □

The rest of this section will be devoted to the hierarchy of cardinal characteristics of subsets of ω and mappings from ω to ω. Our notation is the one used by VAN DOUWEN [1984].

3.13. DEFINITION.

$p = \min\{|F|: F \subseteq \mathcal{P}(\omega)/fin$, F has the finite intersection property and $\prod F = \mathbb{0}\}$,

$s = \min\{\tau: \mathcal{P}(\omega)/fin$ is not $(\tau, 2, 2)$-distributive$\}$

$(= \min\{\tau: \mathcal{P}(\omega)/fin$ is $(\tau, 2, 2)$-nowhere distributive$\}$, by the homogeneity of $\mathcal{P}(\omega)/fin)$

$(= \min\{|\mathcal{A}|: \mathcal{A} \subseteq [\omega]^\omega$ & $(\forall X \in [\omega]^\omega)(\exists A \in \mathcal{A})|X \cap A| = \omega = |X - A|\})$,

$a = \min\{|P|: P$ is an infinite partition of unity in $\mathcal{P}(\omega)/fin\}$

$(= \min\{|\mathcal{P}|: \mathcal{P}$ is an infinite MAD family on $\omega\})$,

$$b = \min\{|H|: H \subseteq {}^{\omega}\omega \text{ is unbounded with respect to } <^*\},$$

$$d = \min\{|D|: D \subseteq {}^{\omega}\omega \text{ is cofinal with respect to } <^*\},$$

$$a_s = \min\{|\mathcal{D}|: \mathcal{D} \subseteq \mathcal{P}(\omega \times \omega) \text{ is a maximal almost disjoint family consisting of partial functions from } \omega \text{ to } \omega\}.$$

The following diagram, where an arrow means "$\leq$ is provable in ZFC", shows the mutual relations between the cardinals just introduced.

$$\begin{tikzcd}
& & s \arrow[r] \arrow[drrr] & d \arrow[rr] & & 2^{\omega} \\
\omega_1 \arrow[r] & p \arrow[r] & h \arrow[r] \arrow[u] & b \arrow[r] \arrow[u] & a \arrow[r] & a_s \arrow[u]
\end{tikzcd}$$

We shall show the position of the cardinal h in this diagram and prove the inequality $s \leq a_s$. For the other inequalities, the reader is requested to consider VAN DOUWEN [1984].

3.14. PROPOSITION.
(i) $p \leq h \leq s$;
(ii) $h \leq b$.

PROOF. (i) is very easy. To show (ii), we shall use a trick due to Rothberger. For an infinite $X = \{x_0 < x_1 < \cdots < x_n < \cdots\} \in [\omega]^{\omega}$ the enumeration function e_X of X is defined by $e_X(n) = x_n$. The following two observations should be clear. If $X \in [\omega]^{\omega}$ and $g \in {}^{\omega}\omega$ are arbitrary, then there is a $Y \subseteq X$ with $e_Y > g$; if $Y \subseteq^* X$ and $X - Y$ is infinite, then $e_Y \, ^*\!> e_X$. Fix a family $\{f_\alpha: \alpha < b\} \subseteq {}^{\omega}\omega$ without an upper bound and let $\mathcal{T}_\alpha$ be an arbitrary MAD family on ω with $e_T > f_\alpha$ for each $T \in \mathcal{T}_\alpha$. The existence of such a MAD family follows from the first observation. Now, by the second one, there is no $Z \in [\omega]^{\omega}$ such that for each $\alpha < b$, $Z \subseteq T$ for some $T \in \mathcal{T}_\alpha$, otherwise any $e_{Z'}$, for $Z' \subseteq Z$, $|Z - Z'| = \omega$ would be a bound for $\{f_\alpha: \alpha < b\}$. Thus, $\{\mathcal{T}_\alpha: \alpha < b\}$ shows the non-distributivity of $\mathcal{P}(\omega)/fin$, thence $h \leq b$. □

Our aim now is to prove the inequality $s \leq a_s$. Before it, we need several prerequisites.

3.15. DEFINITION. Let $R = \{r_n: n \in \omega\}$ be a partition of ω into finite sets. Denote

$$\mathcal{J}^+(R) = \{X \subseteq \omega: (\forall k \in \omega)(\exists n \in \omega)|X \cap r_n| \geq k\}.$$

Therefore $\mathcal{J}^+(R)$ is non-empty iff $\limsup_{n\in\omega} |r_n| = \omega$. Let $\mu = \min\{|\mathcal{R}|$: each member of $\mathcal{R}$ is a partition of ω into finite sets and $[\omega]^{\omega} = \bigcup_{R\in\mathcal{R}} \mathcal{J}^+(R)\}$.

The forthcoming useful observation was implicitly used by many authors.

3.16. LEMMA. *Let f, $g \in {}^\omega\omega$ be strictly increasing functions, $f <^* g$. Define $h \in {}^\omega\omega$ by $h(0) = 0$, $h(n+1) = g(h(n)+1)$. Then for all but finitely many $n \in \omega$ there is some $k \in \omega$ with $f(k) \in [h(n), h(n+1))$.*

PROOF. Let k_0, $n_0 \in \omega$ be such that $f(k) < g(k)$ whenever $k \geq k_0$ and $h(n_0) > f(k_0)$. Let $n > n_0$ be arbitrary. Choose $k \in \omega$ satisfying $f(k-1) < h(n) \leq f(k)$. We need to show that $f(k) < h(n+1)$. Since $n > n_0$ and $f(k) \geq h(n)$, we have $k > k_0$. Now the definition of function h gives $h(n+1) = g(h(n)+1) > g(f(k-1)+1) \geq g(k-1+1) = g(k) > f(k)$. □

3.17. PROPOSITION. *$\mu = b$; moreover, there is a family $\{R_\alpha : \alpha < b\}$ of partitions of ω into finite sets such that for each $X \in [\omega]^\omega$ there is some $\alpha_0 < b$ such that $X \in \mathscr{I}^+(R_\alpha)$ whenever $\alpha \geq \alpha_0$.*

PROOF. $\mu \geq b$: Let $\mathscr{R} = \{R_\alpha : \alpha < \mu\}$ be a family of partitions with $\bigcup \{\mathscr{I}^+(R_\alpha) : \alpha < \mu\} = [\omega]^\omega$. For $\alpha < \mu$, define $f_\alpha \in {}^\omega\omega$ by induction as follows: $f_\alpha(0) = 0$, $f_\alpha(n) = \max s_n + 1$, where $s_n = r_n \cup \bigcup \{r \in R_\alpha : \min r \leq f_\alpha(n-1)\}$. Notice that for each $r \in R_\alpha$ there is some $n \in \omega$ with $r \subseteq [f_\alpha(n), f_\alpha(n+2))$.

The family $\{f_\alpha : \alpha < \mu\}$ is unbounded in ${}^\omega\omega$ under $<^*$. If not, there is a strictly increasing function $g \in {}^\omega\omega$ with $g(0) > 0$ such that $g \mathrel{{}^*{>}} f_\alpha$ for each $\alpha < \mu$. Define $h \in {}^\omega\omega$ by the rule $h(0) = g(0)$, $h(n+1) = g(h(n)+1)$ and let $X = \{h(n) : n \in \omega\}$. Let $\alpha < \mu$ be arbitrary. Since $g \mathrel{{}^*{>}} f_\alpha$, there is some $k \in \omega$ with $g(n) > f_\alpha(n)$ for each $n \geq k$. Now, for each $n > k$, $|X \cap [f_\alpha(n), f_\alpha(n+1))| \leq 1$: Indeed, let $n > k$, let $h(p) \in [f_\alpha(n), f_\alpha(n+1))$. Then $h(p+1) = g(h(p)+1) \geq g(f_\alpha(n)+1) \geq g(n+1) > f_\alpha(n+1)$. Consequently, $|X \cap r| \leq 2$ for almost all $r \in R_\alpha$, therefore $X \in [\omega]^\omega - \bigcup \{\mathscr{I}^+(R_\alpha) : \alpha < \mu\}$, a contradiction.

$b \geq \mu$: Choose $\{f_\alpha : \alpha < b\} \subseteq {}^\omega\omega$ without an upper bound. We may and shall assume that $f_\alpha(0) = 0$ for each $\alpha < b$, each f_α is strictly increasing and $\alpha < \beta$ implies $f_\alpha <^* f_\beta$.

Define $R_\alpha = \{[f_\alpha(n), f_\alpha(n+1)) : n \in \omega\}$ for $\alpha < b$. Let $X \in [\omega]^\omega$ be arbitrary, $X = \{x(0) < x(1) < \cdots < x(n) < \cdots\}$ and define $g(n) = x(n^2+1)$. Since g is not an upper bound for $\{f_\alpha : \alpha < b\}$, there is some $\alpha_0 < b$ with $f_{\alpha_0} \not<^* g$.

Now, if $\alpha \geq \alpha_0$, then $f_\alpha \not<^* g$, too, and we shall use this fact to show that $X \in \mathscr{I}^+(R_\alpha)$ proving thus $b \geq \mu$ as well as the "moreover" part of the proposition. Let $k \in \omega$ be arbitrary. Since $f_\alpha \not<^* g$, there is some $m \geq k$ such that $f_\alpha(m) \geq g(m) = x(m^2+1)$. Therefore $|X \cap [0, f_\alpha(m))| \geq m^2$, so for some $n < m$, $|X \cap [f_\alpha(n), f_\alpha(n+1))| \geq m \geq k$. Since k was arbitrary, $X \in \mathscr{I}^+(R_\alpha)$. □

3.18. PROPOSITION. *$s \leq a_s$.*

PROOF. Let $\mathscr{R} = \{R_\alpha : \alpha < \mu\}$ be a family of partitions of ω into finite sets with $[\omega]^\omega = \bigcup \{\mathscr{I}^+(R_\alpha) : \alpha < \mu\}$. Choose a family $\mathscr{D} \subseteq \omega \times \omega$ witnessing the definition of a_s. If $R_\alpha = \{r_n : n \in \omega\}$, let $\mathscr{A}_\alpha$ be a family of all infinite sets from $\{\{f(n) : n \in \omega, f(n) \in r_n\} : f \in \mathscr{D}\}$. Then $\mathscr{A}_\alpha$ is a MAD family on ω of size $\leq a_s$ and for each $r \in R_\alpha$ and $A \in \mathscr{A}_\alpha$, $|r \cap A| \leq 1$.

Now, the family $\bigcup\{\mathcal{A}_\alpha : \alpha < \mu\}$ is splitting: if $X \in [\omega]^\omega$, then $X \in \mathcal{J}^+(R_\alpha)$ for some $\alpha < \mu$. Since $\mathcal{A}_\alpha$ is maximal, $|X \cap A| = \omega$ for some $A \in \mathcal{A}_\alpha$, but $X - A \in \mathcal{J}^+(R_\alpha)$, too, hence $|X - A| = \omega$. So $s \leq |\bigcup\{\mathcal{A}_\alpha : \alpha < \mu\}|$. Using 3.17 and the inequality $b \leq a_s$ we obtain:

$$\left|\bigcup\{\mathcal{A}_\alpha : \alpha < \mu\}\right| \leq \mu \cdot a_s = b \cdot a_s = a_s \,. \quad \square$$

3.19. REMARKS ON CONSISTENCY RESULTS

(a) First, let us mention the basic properties of $V[G]$, where G is generic over the Boolean algebra $\mathcal{P}(\omega)/fin$.

If $\alpha < h$, the $(\alpha, \cdot, 2)$-distributivity of $\mathcal{P}(\omega)/fin$ implies that each function $f \in V[G]$ with $\mathrm{dom}(f) = \alpha$, $\mathrm{rng}(f) \subseteq V$ is, in fact, in V. In particular, $\mathcal{P}(\omega)^{V[G]} = \mathcal{P}(\omega)^V$ and all cardinals $\leq h$ are absolute.

On the other hand, $(2^\omega)^{V[G]} = h$.

The filter $\{A \subseteq \omega : A^* \in G\}$ is a selective ultrafilter with a tower of length h as its base.

The cardinal characteristic p is absolute for V and $V[G]$, and $h^{V[G]} = \mathrm{h}(\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin)^V$. The remaining ones introduced in 3.13 equal to 2^ω in $V[G]$.

(b) Second, let us discuss the diagram from 3.13. There is one open problem, namely CON$(a > d)$. Up to this exception, the diagram is complete in the sense that no arrow can be added. This will be clear by showing the consistency of:

(i) $a_s < 2^\omega$, $a_s < d$. Add ω_2 Cohen reals to CH.
(ii) $d < 2^\omega$. Add ω_2 random reals to CH.
(iii) $s < b$. Add ω_1 random reals to $MA + 2^\omega = \omega_2$.
(iv) $p < h$. This consistency can be found in DORDAL [1982].
(v) $a < s$. This is a deep result of SHELAH [1984].
(vi) The consistency of $d < a$ is still an open problem. The next best, CON$(b < a)$ was proved in SHELAH [1984].

Any other possible inequality between any pair of numbers in the diagram is covered by at least one of (i)–(vi).

(c) Finally, it may happen that there is no cofinal branch in the base tree for $\mathcal{P}(\omega)/fin$, i.e. $h = n$. Since obviously h must be greater than ω_1 in such a case, the model needs some care. Here is the rough description. Start with GCH. Denote B to be the free product of Solovay–Tennenbaum algebra for $MA + 2^\omega = \omega_2$ with the algebra for adding ω_3 subsets of ω_1 by countable conditions. In the generic extension $V[G]$ over B, collapse ω_3 to ω_2 by conditions of length ω_1. This works.

4. Refinements by countable sets

We shall now deal with families of sets which have an almost disjoint refinement by countable (infinite) sets. Throughout the whole section, by an almost disjoint or AD family on a set X we always mean a family consisting of countable subsets of X. Similarly, for a MAD family on X: its members are always countable.

4.1. DEFINITION. Let X be a set, $M \subseteq \mathscr{P}(X)$. The family M is said to have an almost disjoint refinement, briefly, M has ADR, provided there is an almost disjoint family D on X such that for each $C \in M$ there is some $A \in D$ with $A \subseteq C$.

There is no essential difference if the inclusion $\subseteq$ is replaced by $\subseteq^*$ in the definition of ADR.

In order to illustrate the spirit of the present section, let us start with a warming-up exercise.

4.2. EXAMPLE. Let X be a set of size 2^ω. Then any system $M \subseteq [X]^{>\omega}$ has an ADR.

PROOF. We can assume that $X = \mathbb{R}$, the set of reals. If $S \subseteq \mathbb{R}$ is uncountable and $C \subseteq S$ is countable and dense in S, then C has 2^ω many accumulation points. So it suffices to find an almost disjoint refinement for the family $\mathscr{C}$ consisting of all the countable subsets of $\mathbb{R}$ with uncountable closure. Since $|\mathscr{C}| = 2^\omega$, assign an accumulation point $f(C)$ to each $C \in \mathscr{C}$ in a one-to-one way, then for $C \in \mathscr{C}$, choose a convergent sequence $A(C) \subseteq C - \{f(C)\}$ with limit point $f(C)$. Obviously, the family $D = \{A(C)\colon C \in \mathscr{C}\}$ is as required. $\square$

4.3. If one looks for an almost disjoint refinement by countable sets, what is the best property one may hope for? Let us discuss this for a while.

Let κ be an infinite cardinal and suppose that a family $M \subseteq \mathscr{P}(\kappa)$ has an almost disjoint refinement D. Take some almost disjoint family $\mathscr{A}$ on κ such that for each $C \in D$, the set $\{A \in \mathscr{A}\colon A \subseteq C\}$ is infinite. Then every member of M meets infinitely many members of $\mathscr{A}$ in an infinite set. Denote by $J(\mathscr{A})$ the ideal $J(\mathscr{A}) = \{X \subseteq \kappa\colon |\{A \in \mathscr{A}\colon |A \cap X| = \omega\}| < \omega\}$; we have just observed that if a family $M \subseteq \mathscr{P}(\kappa)$ has an ADR, then for some infinite AD family $\mathscr{A}$ on κ, $M \subseteq J^+(\mathscr{A})$. (If J is an ideal on κ, then J^+ denotes $\mathscr{P}(\kappa) - J$.)

4.4. DEFINITION (RPC(κ)). The refinement property by countable subsets of a cardinal κ (RPC(κ)), is the statement: "For each infinite AD family $\mathscr{A}$ on κ, the family $J^+(\mathscr{A})$ has an ADR".

We have just seen that RPC(κ) is the strongest possible condition in the sense that every subsystem of $\mathscr{P}(\kappa)$ which can possibly have an ADR, really has one.

It is obvious that the following are equivalent;

(1) RPC(κ);

(2) if Q is an infinite MAD family on κ, then $J^+(Q)$ has an ADR;

(3) if J is a tall ideal on κ, then J^+ has an ADR. (Recall that an ideal J on κ is tall if each infinite $X \subseteq \kappa$ contains an infinite subset belonging to J.)

The basic open problem is whether RPC(ω) is provable in ZFC alone. The consistency of RPC(ω) relatively to ZFC is well known: in fact, the equality $a = 2^\omega$ (in particular, Martin's axiom) implies RPC(ω) (see 4.10 or ROITMAN [1975]). Moreover, $(\forall\kappa)$ RPC(κ) is consistent, as shown in 4.27.

Here we shall first investigate several tall ideals J on ω such that J^+ has an ADR. Second, we shall prove the existence of a completely separable AD family on ω (4.11) and deduce some consequences, e.g. every uniform ultrafilter on ω

has an almost disjoint refinement by a completely separable AD family. Finally, we show the implications:

$$\left.\begin{array}{l} s=\omega_1 \rightarrow \mathrm{RPC}(\omega) \\ b=d \rightarrow \mathrm{RPC}(\omega) \end{array}\right. \quad (4.18)\,,$$

and mention

$$\left.\begin{array}{l} \mathrm{RPC}(\omega) \rightarrow \mathrm{RPC}(2^\omega) \\ \mathrm{RPC}(\kappa) \rightarrow \mathrm{RPC}(\kappa^+) \end{array}\right. \quad (4.21)\,.$$

4.5. DEFINITION. Let X be an infinite set, $\mathcal{R}=\{R_n\colon n\in\omega\}$ a partition of X. Define

$$\mathcal{J}(\mathcal{R})=\{Y\subseteq X\colon (\exists k\in\omega)|\{n\in\omega\colon |R_n\cap Y|\geq k\}|<\omega\}\,.$$

Obviously, $\mathcal{J}(\mathcal{R})$ is a tall ideal on X whenever $\limsup_{n\in\omega}|R_n|\geq\omega$.

4.6. THEOREM. *Let $\mathcal{R}=\{R_n\colon n\in\omega\}$ be a partition of a set X. If $M\subseteq\mathcal{J}^+(\mathcal{R})$ and $|M|\leq 2^\omega$, then M has an almost disjoint refinement $\mathcal{A}$ such that each $A\in\mathcal{A}$ is a transversal of $\mathcal{R}$, i.e. $|A\cap R_n|\leq 1$ for each $n\in\omega$.*

PROOF. There is nothing to prove if $M=\emptyset$. In the opposite case, enumerate M as $\{M_\alpha\colon \alpha<2^\omega\}$. For each $\alpha<2^\omega$ choose a set $C(\alpha)\in[\omega]^\omega$ such that $\lim_{n\in C(\alpha)}|M_\alpha\cap R_n|\geq\omega$.

Fix some base tree T on ω.

The desired almost disjoint refinement will be constructed by a transfinite induction. We shall define $A_\alpha\subseteq X$ and $t(\alpha)\subseteq C(\alpha)$ such that

(i) $t(\alpha)\in T$;

(ii) $A_\alpha\in[M_\alpha]^\omega$, $\{n\in\omega\colon A_\alpha\cap R_n\neq\emptyset\}\subseteq t(\alpha)$ and for each $n\in\omega$, $|A_\alpha\cap R_n|\leq 1$;

(iii) if $\alpha\neq\beta$, then $|A_\alpha\cap A_\beta|<\omega$ and $t(\alpha)\neq t(\beta)$.

Choose $t(0)\in T$ with $t(0)\subseteq C(0)$. Then let A_0 be an arbitrary infinite transversal of $\{R_n\cap M_0\colon n\in t(0)\}$.

Let $\alpha<2^\omega$ and suppose that for all $\beta<\alpha$ the sets $t(\beta)$ and A_β have been found. Since $|\{t(\beta)\colon \beta<\alpha\}|<2^\omega$, there is some $t(\alpha)\in T$ such that $t(\alpha)\subseteq C(\alpha)$ and there is no $\beta<\alpha$ with $t(\beta)\subseteq^* t(\alpha)$.

Consider the set $M'_\alpha=M_\alpha\cap\bigcup\{R_n\colon n\in t(\alpha)\}$. By (ii) and by 3.6, if $|A_\beta\cap M'_\alpha|=\omega$, then $|t(\beta)\cap t(\alpha)|=\omega$, which in turn implies $t(\alpha)\subsetneqq t(\beta)$ by our choice of $t(\alpha)$. By (iii) and by 3.6(ii), there are fewer than h of such β's. Since all A_β's are transversals of $\mathcal{R}$, the set M'_α is not covered by a finite number of them. Using the inequality $h\leq a$, there is still room for an infinite selector $A_\alpha\subseteq M'_\alpha$ of $\{R_n\colon n\in t(\alpha)\}$ such that A_α is almost disjoint with all A_β having $t(\beta)\,{}^*\!\supset t(\alpha)$. Clearly, $t(\alpha)\cap t(\beta)=^*\emptyset$ implies $|A_\alpha\cap A_\beta|<\omega$, so (iii) will hold, too.

Thus, the family $\{A_\alpha\colon \alpha<2^\omega\}$ is an ADR for M, as stated in the theorem. □

4.7. APPLICATIONS. (i) The familly of all subsets of ω which have a positive upper Banach density has ADR.

A set $A \subseteq \omega$ has a positive upper Banach density if there is a sequence of intervals $\{I_i : i \in \omega\}$ of increasing lengths such that

$$\limsup_{i \in \omega} \frac{|I_i \cap A|}{|I_i|} > 0 .$$

Let $\mathscr{R} = \{R_n : n \in \omega\}$ be a partition of $\omega - \{0\}$ given by $R_n = [n^n, (n+1)^{n+1})$. Then for each set $M \in \mathscr{I}(\mathscr{R})$ there is some $k \in \omega$ such that no arithmetic progression contained in M has k or more members. By a deep result of SZEMERÉDI [1975] each set of positive upper Banach density contains arbitrarily long finite arithmetic progressions and therefore belongs to $\mathscr{I}^+(\mathscr{R})$. According to 4.6, $\mathscr{I}^+(\mathscr{R})$ has an ADR.

(ii) (Stronger version of 4.2.) The family S of all subsets of reals having infinitely many accumulation points has ADR.

Similarly as in 4.2, we can restrict ourselves to the family $\mathscr{C} \subseteq S$ consisting of countable sets only. Clearly, $|\mathscr{C}| = 2^\omega$. For a given $r \in \mathbb{R} \cup \{-\infty, +\infty\}$ consider a family $\mathscr{C}_r = \{X \in \mathscr{C} : r$ belongs to the second derived set of $X\}$.

Choose a decreasing sequence $\mathbb{R} = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_n \supseteq \cdots$ of neighborhoods of r with $\bigcap_{n \in \omega} W_n = \{r\}$. Define $R_n = W_n - W_{n+1}$, $\mathscr{R}_r = \{R_n : n \in \omega\}$. We have $\mathscr{C}_r \subseteq \mathscr{I}^+(\mathscr{R}_r)$, hence by the theorem there is an almost disjoint refinement $\mathscr{A}_r$ consisting of transversals of $\mathscr{R}_r$. Now $\mathscr{A} = \bigcup \{\mathscr{A}_r : r \in \mathbb{R} \cup \{-\infty, +\infty\}\}$ is the almost disjoint refinement we wanted.

(iii) The system $\{X \subseteq 2^\omega : \operatorname{otp}(X) \geq \omega^2\}$ has an ADR, where otp stands for order-type.

Choose for each $\alpha < 2^\omega$ with $\operatorname{cf}(\alpha) = \omega$ a strictly increasing sequence $\{\alpha_n : n \in \omega\}$ converging to α with $\alpha_0 = 0$. Let $\mathscr{R}_\alpha = \{[\alpha_n, \alpha_{n+1}] : n \in \omega\}$. Then for each set X with $\operatorname{otp}(X) = \omega^2$, $X \in \mathscr{I}^+(\mathscr{R}_\alpha)$, where $\alpha = \sup X$. Apply 4.6 to obtain $\mathscr{A}_\alpha$; then $\mathscr{A} = \bigcup \{\mathscr{A}_\alpha : \alpha < 2^\omega \ \&\ \operatorname{cf}(\alpha) = \omega\}$ is as required.

There is one special type of almost disjoint systems on ω, which seems to be relevant to RPC(ω). Let us introduce and study it now.

4.8. DEFINITION. An almost disjoint family $\mathscr{A} \subseteq [\kappa]^\omega$ is called completely separable provided $\mathscr{A}$ is infinite and for each $X \in J^+(\mathscr{A})$ there is an $A \in \mathscr{A}$ with $A \subseteq X$.

The notion of completely separable AD family on ω was introduced by HECHLER [1971]. He investigated many separability properties of AD families on ω and the complete separability turned out to be the strongest one.

The importance of completely separable families in the context of the present chapter stems from the straightforward observation: an AD family $\mathscr{A}$ is completely separable if and only if $\mathscr{A}$ is an almost disjoint refinement of $J^+(\mathscr{A})$. Further elementary properties of completely separable AD families can be summarized as follows.

4.9. PROPOSITION. *Let $\mathscr{A}$ be a completely separable almost disjoint family on ω. Then*

(i) $|\mathscr{A}| = 2^\omega$;

(ii) *for every $X \in J^+(\mathscr{A})$, the family $\{A \in \mathscr{A} : A \subseteq X\}$ is completely separable*;

(iii) *if* $\mathscr{A}' \subseteq \mathscr{A}$, $|\mathscr{A}'| < 2^\omega$, *then* $\mathscr{A} - \mathscr{A}'$ *is completely separable, too*;

(iv) *if for each* $A \in \mathscr{A}$ *one selects a set* $B(A) \in [A]^\omega$, *then the family* $\{B(A)\colon A \in \mathscr{A}\}$ *is completely separable*;

(v) *for any decreasing sequence* $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$ *of sets from* $J^+(\mathscr{A})$ *there is some* $Y \in J^+(\mathscr{A})$ *such that* $Y \subseteq^* X_n$ *for each* $n \in \omega$.

PROOF. (i) Choose distinct sets A_n, $n \in \omega$, from $\mathscr{A}$. Since $\mathscr{A}$ is infinite, this is possible. Pick up pairwise disjoint $C_n \subseteq A_n$ such that $A_n - C_n$ is infinite. Let $\mathscr{Q}$ be an AD family on ω of size 2^ω and let $C(Q) = \bigcup \{C_n\colon n \in Q\}$ for $Q \in \mathscr{Q}$. Clearly, each $C(Q)$ belongs to $J^+(\mathscr{A})$, hence by complete separability, there is some $A(Q) \in \mathscr{A}$ with $A(Q) \subseteq C(Q)$. Obviously, $A(Q) \neq A(Q')$ whenever $Q \neq Q'$, thus $|\mathscr{A}| = 2^\omega$.

(ii), (iii) and (iv) are obvious.

(v) For each $n \in \omega$ choose $A_n \in \mathscr{A}$ satisfying $A_n \subseteq X_n - \bigcup_{i<n} A_i$. Clearly, the set $A = \bigcup_{n\in\omega} A_n$ belongs to $J^+(\mathscr{A})$.

By (ii) we can find $\{B_i\colon i \in \omega\} \subseteq \mathscr{A}$ such that $B_i \subseteq A$ and $B_i \neq A_n$ for all i, $n \in \omega$. The set $Y = \bigcup_{n\in\omega} \bigcup_{i\leq n} B_i \cap A_n$ is as required. □

Notice that (i) and (ii) from the previous theorem imply that for each completely separable AD family $\mathscr{A}$ and for each $X \in J^+(\mathscr{A})$ we have $|\{A \in \mathscr{A}\colon |A \cap X| = \omega\}| = 2^\omega$. Though this condition is weaker than the complete separability, one can show that the loss is minimal.

4.10. PROPOSITION. *Let* $\mathscr{B}$ *be an infinite AD family on* ω *such that for each* $X \in J^+(\mathscr{B})$, $|\{B \in \mathscr{B}\colon |X \cap B| \doteq \omega\}| = 2^\omega$. *Then there is a completely separable AD family* $\mathscr{A}$ *on* ω *with* $J^+(\mathscr{A}) = J^+(\mathscr{B})$.

PROOF. Indeed, the property of $\mathscr{B}$ implies immediately the existence of a one-to-one mapping $\varphi\colon J^+(\mathscr{B}) \to \mathscr{B}$ such that for each $X \in J^+(\mathscr{B})$ the set $X \cap \varphi(X)$ is infinite. It suffices to define $\mathscr{A} = \{X \cap \varphi(X)\colon X \in J^+(\mathscr{B})\}$. □

ERDÖS and SHELAH [1972] raised the question of whether there is a completely separable MAD family on ω in ZFC. The problem is open till now. On the other hand, we shall show that completely separable AD families exist.

4.11. THEOREM. *Let* $\{A_n\colon n \in \omega\}$ *be an almost disjoint family on* ω *and let* J *be an arbitrary tall ideal on* ω. *Then there is a collection* $\mathscr{A} \subseteq J$ *such that* $\{A_n\colon n \in \omega\} \cup \mathscr{A}$ *is a completely separable AD family.*

The proof of the theorem will be given in 4.14, after proving Lemma 4.13. But let us first introduce several notions and mention a few facts concerning chains and towers (see 2.6) in Boolean algebra $\mathscr{P}(\omega)/fin$.

4.12. In accordance with 3.5, a chain (resp. tower) will be considered as a family of infinite subsets of ω well-ordered by $^*\supset$.

Let T be a chain of length γ. By the cofinality of T we understand the cardinal $\mathrm{cf}(\gamma)$.

A chain $T = \{B_0\}$ of length 1 will be identified with the set B_0.

If $T = \{B_\alpha: \alpha < \gamma\}$ is a chain and $X \in [\omega]^\omega$, then we shall say that

(a) X is below T if $X \subseteq^* B_\alpha$ for each $\alpha < \gamma$;

(b) X is compatible with T if the intersection $X \cap B_\alpha$ is infinite for each $\alpha < \gamma$;

(c) X meets the boundary of T if for each $\alpha < \gamma$ there is a $\beta < \gamma$, $\beta > \alpha$, with $|X \cap (B_\alpha - B_\beta)| = \omega$.

Notice that if some X meets the boundary of T, then the length of T is a limit ordinal.

Clearly, a chain T is a tower if and only if no infinite set is below T.

The chains T and T' will be called disjoint if there are members $B \in T$, $B' \in T'$ with finite intersection.

By Cantor's property (see 3.1), if a set X is compatibe with a chain T of countable cofinality, then there is some $Y \in [X]^\omega$ which is below T. This fact explains the important role which the chains of countable cofinality play in the sequel.

4.13. LEMMA. *Let T be a chain of cofinality ω, let $Z \in [\omega]^\omega$ be compatible with T. Then there are pairwise disjoint chains $\{V_\xi: \xi < b\}$ (b as in* 3.13!) *such that*:

(i) *all members of any V_ξ are below T*;

(ii) *if ξ is isolated, then V_ξ is a set (a chain of length* 1), *if ξ is limit, then* cf(ξ) *is the cofinality of V_ξ*;

(iii) *if $X \subseteq \omega$ is compatible with T, then the set $K_X = \{\xi < b$: X is compatible with $V_\xi\}$ is non-empty closed subset of b and if ξ is an accumulation point of K_X, then X meets the boundary of V_ξ*;

(iv) *if X meets the boundary of T, then the set K_X is moreover unbounded in b*;

(v) $V_0 \subseteq Z$.

PROOF. Let $T = \{B_\alpha: \alpha < \gamma\}$. Since $\mathrm{cf}(\gamma) = \omega$, choose a strictly increasing sequence $\{\alpha_n: n \in \omega\}$ cofinal to γ with $\alpha_0 > 0$ and define $R_0 = \omega - B_{\alpha_0}$, $R_n = \bigcap_{i<n} B_{\alpha_i} - B_{\alpha_n}$ for $n > 0$. The sets R_n are infinite and pairwise disjoint and (add a singleton to several R_n's, if necessary) we may and shall assume that $\bigcup_{n\in\omega} R_n = \omega$. Clearly, a set $Y \subseteq \omega$ is almost disjoint with all R_n's iff Y is below T. Fix some enumeration $R_n = \{y_{n,k}: k \in \omega\}$ for each $n \in \omega$, and let $f \in {}^\omega\omega$ be some mapping satisfying

$$|\{n \in \omega: y_{n,f(n)} \in Z\}| = \omega .$$

Choose a family $\{f_\xi: \xi < b\} \subseteq {}^\omega\omega$ such that $f <^* f_0$, each f_ξ is strictly increasing, for $\xi < \eta$ we have $f_\xi <^* f_\eta$ and there is no upper bound for $\{f_\xi: \xi < b\}$ in ${}^\omega\omega$. Define $S_\xi = \{y_{n,k}: n \in \omega, k \leq f_\xi(n)\}$ for $\xi < b$. Obviously, $\xi < \eta$ implies $S_\xi \subset^* S_\eta$ and every S_ξ is below T. Now let $V_0 = S_0$, $V_{\xi+1} = S_{\xi+1} - S_\xi$ and $V_\xi = \{S_\xi - S_\eta: \eta < \xi\}$ for ξ limit, $\xi < b$.

This definition easily implies (i) and (ii), as well as the pairwise disjointness of V_ξ's.

To show (iii), assume $X \subseteq \omega$ to be compatible with T. Define a function $h \in {}^\omega\omega$ by the rule $h(n) = \min\{k: y_{n,k} \in X \cap R_n\}$ if $X \cap R_n \neq \emptyset$, $h(n) = h(m)$ if $X \cap R_n = \emptyset$ and $m > n$ is the smallest one with $X \cap R_m \neq \emptyset$. The family $\{f_\xi: \xi < b\}$ is unbounded, hence there is some first ξ with $f_\xi \not<^* h$. Since $f_\xi(n) \geq h(n)$ for infinitely many n's and since f_ξ is strictly increasing, we have $|X \cap S_\xi| = \omega$. Consequently, K_X is non-empty. If ξ is an accumulation point of K_X and $\eta < \xi$ arbitrary, then there is some $\eta' \in K_X$, $\eta < \eta' < \xi$. Since $\eta' \in K_X$, $|X \cap (S_{\eta'} - S_\eta)| = \omega$. But this shows that X meets the boundary of V_ξ as well as that K_X is closed.

In order to prove (iv), let X meet the boundary of T. This means that the set $\{n \in \omega: |X \cap R_n| = \omega\}$ is infinite, thus for each $\xi < b$, the set $X - S_\xi$ is infinite and compatible with T. Now by (iii), K_X is unbounded.

According to our choice of the functions f and f_0 the set $Z \cap V_0$ is infinite. The simple redefining $V_0 \cap Z$ instead of V_0 and $V_1 \cup (V_0 - Z)$ instead of V_1 completes the proof. $\square$

4.14. *Proof of 4.11.* If T is a chain of countable cofinality, $Z \subseteq \omega$ a set compatible with T, then there is a family $\{V_\xi: \xi < b\}$ as indicated in Lemma 4.13. This can be viewed as follows. Applying Lemma 4.13 leads to an almost disjoint family $\mathscr{C}(T) = \{V_\xi: \xi < b,\ \xi \text{ isolated}\}$ and to a family of pairwise disjoint countably cofinal chains extending T, namely $\mathscr{T}(T) = \{T^\frown V_\xi: \xi < b, \operatorname{cf}(\xi) = \omega\}$. Using this notation, the proof goes by a transfinite induction to ω_1.

Our aim is to construct families $\mathscr{C}_\alpha$ and $\mathscr{T}_\alpha$ for $\alpha < \omega_1$ with the properties listed below:

(a) Each $\mathscr{C}_\alpha$ is an AD family on ω, each $\mathscr{T}_\alpha$ is a pairwise disjoint family of chains with countable cofinality, each $C \in \mathscr{C}_\alpha$ and $T \in \mathscr{T}_\alpha$ are disjoint;

(b) if $\alpha < \beta$, then $\mathscr{C}_\alpha \subseteq \mathscr{C}_\beta$;

(c) if $\alpha < \beta$, then each $T \in \mathscr{T}_\beta$ is an end-extension of some $T' \in \mathscr{T}_\alpha$ and each $C \in \mathscr{C}_\beta - \mathscr{C}_\alpha$ is below some $T' \in \mathscr{T}_\alpha$;

(d) if $Z \in [\omega]^\omega$ is compatible with 2^ω-many members of $\mathscr{T}_\alpha$, then there is some $C \in \mathscr{C}_{\alpha+1}$ with $C \subseteq Z$.

Case $\alpha = 0$. $\mathscr{C}_0 = \{A_n: n \in \omega\}$, $\mathscr{T}_0 = \{T\}$, where $T = \{\omega - \bigcup_{i<n} A_i: n \in \omega\}$.

Case $\alpha < \omega_1$, α limit. By the inductive assumption (c), the set $\bigcup_{\beta<\alpha} \mathscr{T}_\beta$ of chains ordered by end-extension is a tree of height $\alpha < \omega_1$. We define $\mathscr{T}_\alpha$ to be the set of all chains determined by branches of this tree and $\mathscr{C}_\alpha = \bigcup_{\beta<\alpha} \mathscr{C}_\beta$. It is obvious that (a)–(c) hold then; (d) need not be checked.

Case $\alpha = \beta + 1$. Define $D_\alpha = \{Z \in [\omega]^\omega: Z \text{ is compatible with } 2^\omega\text{-many chains of } \mathscr{T}_\beta\}$. Then there is a one-to-one mapping $\varphi: D_\alpha \to \mathscr{T}_\beta$ such that each $Z \in D_\alpha$ is compatible with $\varphi(Z)$. Using Lemma 4.13, for each $T \in \mathscr{T}_\beta$ there is a family $\mathscr{C}(T)$ consisting of sets below T and a family $\mathscr{T}(T)$ consisting of end-extensions of T into chains of countable cofinality; moreover, if $T = \varphi(Z)$ for some $Z \in D_\alpha$, then there is a $C \in \mathscr{C}(T)$ with $C \subseteq Z$. Define $\mathscr{C}_\alpha = \mathscr{C}_\beta \cup \bigcup \{\mathscr{C}(T): T \in \mathscr{T}_\beta\}$, $\mathscr{T}_\alpha = \bigcup \{\mathscr{T}(T): T \in \mathscr{T}_\beta\}$.

The inductive assumptions, the choice of a set D_α and Lemma 4.13 imply that (a)–(d) hold.

Finally, let $\mathscr{C} = \bigcup \{\mathscr{C}_\alpha: \alpha < \omega_1\}$.

We shall verify now that $\mathscr{C}$ is a completely separable almost disjoint family. Obviously, $\mathscr{C}$ is an infinite almost disjoint family, since $\{A_n: n \in \omega\} \subseteq \mathscr{C}$.

Let $X \in J^+(\mathscr{C})$.

We claim that then there is some $\alpha < \omega_1$ and $T \in \mathscr{T}_\alpha$ such that X meets the boundary of T.

Denote $\mathcal{M} = \{C \in \mathscr{C}: |X \cap C| = \omega\}$. Let I be the set of all $\beta < \omega_1$ such that there is some $T \in \mathscr{T}_\beta$ satisfying $|\{C \in \mathcal{M}: C$ is not below $T\}| < \omega$. The set I is bounded in ω_1 for $\mathcal{M}$ is infinite.

$I = \emptyset$. This means, for the tower $T \in \mathscr{T}_0$, X meets the boundary of T, so $\alpha = 0$.

$I \neq \emptyset$. Then either there is some $\beta = \max I$ and a tower $T' \in \mathscr{T}_\beta$ such that for each $T \in \mathscr{T}_{\beta+1}$ the set $\{C \in \mathcal{M}: C$ is not below $T\}$ is infinite. Since $\mathscr{T}(T')$ was constructed with the help of Lemma 4.13, $|K_X| \geq \omega$ then. By 4.13(iii), there is some $T \in \mathscr{T}(T')$ such that X meets the boundary of T. Set $\alpha = \beta + 1$.

Otherwise let $\alpha = \sup I$ and let $T = \bigcup_{\beta<\alpha} \{T' \in \mathscr{T}_\beta: |\{C \in \mathcal{M}: C$ is not below $T'\}| < \omega\}$. Clearly, X meets the boundary of T.

Having proved the claim, let $\alpha < \omega_1$ and $T \in \mathscr{T}_\alpha$ be such that X meets the boundary of T. Consider the situation at the stage $\alpha + 1$ of the inductive construction. There are b many extensions of the tower T, belonging to $\mathscr{T}_{\alpha+1}$, such that X meets the boundary of each of them. This follows by 4.13(iv), (iii). Moreover, we can continue. By the standard branching argument there are 2^ω many chains in $\mathscr{T}_{\alpha+\omega}$ compatible with X. Hence, $X \in D_{\alpha+\omega+1}$ and therefore by (d), there is some $C \in \mathscr{C}$, $C \subseteq X$.

Thus, we have succeeded in extending the family $\{A_n: n \in \omega\}$ into a completely separable AD family $\mathscr{C}$. It remains to make use of the fact that J is tall: For each $C \in \mathscr{C}$ pick up an infinite set $A(C) \subseteq C$, $A(C) \in J$. By 4.9(iv), the family $\mathscr{A} = \{A(C): C \in \mathscr{C} - \{A_n: n \in \omega\}\}$ has all the required properties. □

We shall mention a few applications of Theorem 4.11.

4.15. LEMMA. *Let T be a chain in $\mathscr{P}(\omega)/fin$ of the limit length. Then there is a completely separable AD family $\mathscr{A}$ refining the set $\{X \subseteq \omega$: X meets the boundary of $T\}$.*

PROOF. Let $T = \{B_\alpha: \alpha < \gamma\}$, where γ is a limit ordinal.

For $\alpha \leq \gamma$ with $\mathrm{cf}(\alpha) = \omega$ fix a strictly increasing sequence of ordinals $\{\alpha_n: n \in \omega\}$ converging to α, $\alpha_0 > 0$, and let $A_0^\alpha = \omega - B_{\alpha_0}$, $A_n^\alpha = B_{\alpha_{n-1}} - B_{\alpha_n}$ for $n > 0$. The family $\{A_n^\alpha: n \in \omega\}$ is almost disjoint, thus 4.11 applies; let $\mathscr{A}_\alpha$ be the result. Set $\mathscr{A}'_\alpha = \{A \in \mathscr{A}_\alpha: A \subseteq \omega - B_\alpha\}$ if $\alpha < \gamma$ with $\mathrm{cf}(\alpha) = \omega$, and $\mathscr{A}'_\gamma = \mathscr{A}_\gamma$ if $\mathrm{cf}(\gamma) = \omega$. The family $\mathscr{A} = \bigcup \{\mathscr{A}'_\alpha: \alpha \leq \gamma, \mathrm{cf}(\alpha) = \omega\}$ is as required.

Indeed, if X meets the boundary of T, then there is a sequence $\beta_0 < \beta_1 < \cdots$ such that $X \cap (B_{\beta_i} - B_{\beta_{i+1}})$ is infinite for each $i \in \omega$; this implies that $X - B_\alpha \in J^+(\{A_n^\alpha: n \in \omega\})$, where $\alpha = \sup\{\beta_i: i \in \omega\}$. Thus, $X - B_\alpha$ contains some $A \in \mathscr{A}_\alpha$ and this A belongs to $\mathscr{A}$.

The almost disjointness of $\mathscr{A}$ easily follows from T being a chain. Let $\alpha < \beta$. Then $B_\alpha \;{}^*\!\supset B_\beta$. By the construction, each member of $\mathscr{A}'_\alpha$ is disjoint with B_α and for each $A \in \mathscr{A}'_\beta$, $A \subseteq^* B_\alpha$.

It remains to show that $\mathscr{A}$ is completely separable. Suppose $X \in J^+(\mathscr{A})$. By the previous, it suffices to show that there is some α with $X \in J^+(\mathscr{A}'_\alpha)$. Let $\beta_0 = \min\{\beta \leq \gamma$: for some $A \in \mathscr{A}'_\beta$, $X \cap A$ is infinite$\}$. If $X \in J^+(\mathscr{A}'_{\beta_0})$, we are done; otherwise, $X \cap B_{\beta_0} \in J^+(\mathscr{A})$. Let $\beta_1 = \min\{\beta \leq \gamma$: for some $A \in \mathscr{A}'_\beta$, $X \cap B_{\beta_0} \cap A$ is infinite$\}$. Again, if $X \in J^+(\mathscr{A}'_{\beta_1})$, we are done, and in the opposite case, there is some $\beta_2 > \beta_1$ such that for some $A \in \mathscr{A}'_{\beta_2}$, $X \cap B_{\beta_1} \cap A$ is infinite.

Proceeding further as indicated, it may happen that for some $\beta_n < \gamma$, $X \in J^+(A'_{\beta_n})$ – and if this attempt fails, then $X \in J^+(\mathscr{A}'_\alpha)$ for $\alpha = \sup\{\beta_n : n \in \omega\}$. □

The first part of the next theorem shows that $\mathrm{Rfip}(2^\omega)$ holds in $\mathscr{P}(\omega)/fin$, as claimed in 2.2.

4.16. THEOREM. (i) *Every uniform ultrafilter on ω has an ADR by a completely separable AD family.*

(ii) *A union of less than 2^ω uniform ultrafilters on ω has an ADR.*

PROOF. If $\mathscr{U}$ is a uniform ultrafilter on ω, then there is a chain $T \subseteq \mathscr{U}$ such that each member of $\mathscr{U}$ meets the boundary of T – any maximal element in the set of all chains which are subsets of $\mathscr{U}$, ordered by an end-extension, is such.

Using 4.15, (i) follows.

We shall use (i) to show (ii). Let $\tau < 2^\omega$, let $\{\mathscr{U}_\alpha : \alpha < \tau\}$ be a family of uniform ultrafilters on ω. Let $\mathscr{A}_0$ be a completely separable AD family refining $\mathscr{U}_0$.

The proof goes by induction to τ. Suppose $\alpha < \tau$ and let $\{\mathscr{A}_\beta : \beta < \alpha\}$, $\{\mathscr{A}_{\beta,\gamma} : \beta < \gamma < \alpha\}$ have been found. We assume that each $\mathscr{A}_\beta$ is either a completely separable AD family or empty, $\mathscr{A}_{\beta,\gamma}$ is a finite subset of $\mathscr{A}_\beta$ and $\bigcup\{\mathscr{A}_\beta - \bigcup\{\mathscr{A}_{\beta,\gamma} : \beta < \gamma < \alpha\} : \beta < \alpha\}$ is almost disjoint. If for each $U \in \mathscr{U}_\alpha$ there is some $\beta < \alpha$ with $U \in J^+(\mathscr{A}_\beta)$, define $\mathscr{A}_\alpha = \emptyset$, $\mathscr{A}_{\beta,\alpha} = \emptyset$ for all $\beta < \alpha$.

Otherwise there is some $U_\alpha \in \mathscr{U}_\alpha$ such that for each $\beta < \alpha$, the set $\mathscr{A}_{\beta,\alpha} = \{A \in \mathscr{A}_\beta : |U_\alpha \cap A| = \omega\}$ is finite. By (i), there is a completely separable AD family $\mathscr{C}_\alpha$ refining $\mathscr{U}_\alpha$. Let $\mathscr{A}_\alpha = \{C \in \mathscr{C}_\alpha : C \subseteq U_\alpha\}$. By 4.9(ii), $\mathscr{A}_\alpha$ is completely separable, and by the finite intersection property of $\mathscr{U}_\alpha$, the family $\mathscr{A}_\alpha$ is an ADR for $\mathscr{U}_\alpha$, too.

Since $\tau < 2^\omega$, continuing this process, we may lose not more than τ elements from each non-empty family $\mathscr{A}_\alpha$. Thus, by 4.9(iii), the collection $\mathscr{A} = \bigcup\{\mathscr{A}_\alpha - \bigcup\{\mathscr{A}_{\alpha,\beta} : \alpha < \beta < \tau\} : \alpha < \tau\}$ is the desired ADR. □

4.17. COROLLARY. *For each uniform ultrafilter $\mathscr{U}$ on ω there is a completely separable AD family $\mathscr{A}$ such that for each $\mathscr{V} \in [\mathscr{U}]^\omega$ there is some $A \in \mathscr{A}$ with $A \subseteq^* U$ for each $U \in \mathscr{V}$.*

PROOF. Apply 4.16(i) and 4.9(v). □

Up to now we have investigated special families of sets which have ADR. Let us turn back to $\mathrm{RPC}(\omega)$. We shall show that $\mathrm{RPC}(\omega)$ follows from several equalities between cardinal characteristics introduced in 3.13.

4.18. THEOREM. *Each one of the following assumptions implies* $\mathrm{RPC}(\omega)$:

(i) $a = 2^\omega$;

(ii) $b = d$;

(iii) $s = \omega_1$.

PROOF. (i) follows from 4.10.

Before showing (ii) and (iii), let us mention two easy claims.

Claim 1. If $b = d$, then there is a family $\{\mathcal{R}_\alpha: \alpha < b\}$ of partitions of ω such that

(a) each $\mathcal{R}_\alpha$ is infinite and consists of infinite sets,

(b) for any $X \in [\omega]^\omega$ there is some $\alpha < b$ such that $|X \cap R| = \omega$ for every $R \in \mathcal{R}_\beta$ and every $\beta \geq \alpha$.

The equality $b = d$ implies that there is a dominating family $\{g_\alpha: \alpha < b\}$ consisting of strictly increasing functions and well-ordered by $<^*$.

For $\alpha < b$ define a mapping $h_\alpha \ {}^*\!\!\geq g_\alpha$ by the rule $h_\alpha(0) = 0$, $h_\alpha(n+1) = g_\alpha(h_\alpha(n) + 1)$. Let $r_n^\alpha = [h_\alpha(n), h_\alpha(n+1))$ for each $\alpha < b$ and $n \in \omega$. Since the family $\{g_\alpha: \alpha < b\}$ is dominating and well-ordered by $<^*$, if $X \in [\omega]^\omega$ is arbitrary, then by 3.16 there must be some $\alpha < b$ such that $X \cap r_n^\beta \neq \emptyset$ for all $\beta \geq \alpha$ and for all $n \geq n(\beta)$.

It suffices to choose some partition $\{P_n: n \in \omega\} \subseteq [\omega]^\omega$ of ω and to define $\mathcal{R}_\alpha = \{\bigcup \{r_i^\alpha: i \in P_n\}: n \in \omega\}$.

The claim is proved.

Claim 2. Let $\mathcal{A} \cup \mathcal{B}$ be an AD family on ω, $|\mathcal{B}| < b$, $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then for each $X \in J^+(\mathcal{A})$ there is a $Y \subseteq X$, $Y \in J^+(\mathcal{A})$ such that Y is almost disjoint with each member of $\mathcal{B}$.

Indeed, since $X \in J^+(\mathcal{A})$ there are $\{A_n: n \in \omega\} \subseteq \mathcal{A}$ such that $|X \cap A_n| = \omega$ for each n. For each $B \in \mathcal{B}$, define $f_B \in {}^\omega\omega$ by the rule $f_B(n) = \min\{k \in \omega: B \cap A_n \subseteq k\}$. The mapping f_B is well-defined because $B \cap A_n$ is always finite. By the assumption $|\mathcal{B}| < b$ there is some $g \in {}^\omega\omega$, $g \ {}^*\!\!> f_B$ for each $B \in \mathcal{B}$. The set $Y = \{k \in \omega: (\exists n \in \omega)(k \in A_n \cap X \ \& \ k > g(n))\}$ is as required.

Now we are ready to prove (ii). We have to show that $J^+(Q)$ has an ADR whenever Q is an infinite MAD family on ω.

Assuming $b = d$ choose a family $\{\mathcal{R}_\alpha: \alpha < b\}$ of partitions of ω satisfying (a), (b) from the Claim 1. Let $\mathcal{A}_\alpha$ be the result of an application of Theorem 4.11 to $\mathcal{R}_\alpha$ and $J(Q)$. So $\mathcal{R}_\alpha \cup \mathcal{A}_\alpha$ is a completely separable AD family on ω and $\mathcal{A}_\alpha \subseteq J(Q)$.

Define by induction $\mathcal{C}_0 = \mathcal{A}_0$, $\mathcal{C}_\alpha = \{A \in \mathcal{A}_\alpha: A$ is almost disjoint with each member of $\bigcup_{\beta<\alpha} \mathcal{C}_\beta\}$, $\mathcal{C} = \bigcup_{\alpha<b} \mathcal{C}_\alpha$.

The family $\mathcal{C}$ is obviously almost disjoint and we have to show that $\mathcal{C}$ refines $J^+(Q)$. To this end, let $X \in J^+(Q)$.

Since $X \in J^+(Q)$ we can choose $\{q_n: n \in \omega\} \subseteq Q$ such that $|q_n \cap X| = \omega$ for each n. Using Claim 1 we know that there is some $\mathcal{R}_\alpha = \{R_n: n \in \omega\}$ such that $|q_n \cap X \cap R_n| = \omega$ for each n. Denote $Y = \bigcup \{q_n \cap X \cap R_n - \bigcup_{i<n} q_i: n \in \omega\}$.

Now clearly $Y \in J^+(\mathcal{R}_\alpha) \cap J^+(Q)$, $Y \subseteq X$ and there are two possibilities:

(a) For each $\beta < \alpha$, $Y \in J(\mathcal{C}_\beta)$. Let $Q' = \{q \in Q$: there is some $\beta < \alpha$ and $C \in \mathcal{C}_\beta$ with $|Y \cap C \cap q| = \omega\} - \{q_n: n \in \omega\}$. Since $\mathcal{C} \subseteq J(Q)$ and $\alpha < b$, we have $|Q'| < b$, too. The family $\{q_n \cap R_n: n \in \omega\} \cup Q'$ is almost disjoint and $Y \in J^+(\{q_n \cap R_n: n \in \omega\})$, thus by Claim 2 there is some $Z \in J^+(\{q_n \cap R_n: n \in \omega\})$, $Z \subseteq Y$, Z almost disjoint with each member of Q'. If we choose arbitrary $A \in \mathcal{A}_\alpha$ with $A \subseteq Z$, then A is almost disjoint with Q' as well as with $\{q_n: n \in \omega\}$, so $A \in \mathcal{C}_\alpha$.

(b) There is some $\beta < \alpha$ with $Y \in J^+(\mathcal{C}_\beta)$. Pick the first β with this property. If $\beta = 0$, then from $\mathcal{A}_0 = \mathcal{C}_0$ and from the complete separability of $\mathcal{A}_0$ we have some $C \in \mathcal{C}_0$, $C \subseteq Y$.

If $\beta > 0$, then $|\{C \in \bigcup_{\gamma<\beta} \mathscr{C}_\gamma: |C \cap Y| = \omega\}| < b$. Another application of Claim 2 gives us a set $Z \subseteq Y$, $Z \in J^+(\mathscr{C}_\beta)$, Z almost disjoint with each member of $\bigcup_{\gamma<\beta} \mathscr{C}_\gamma$. Since $Z \in J^+(\mathscr{C}_\beta) \subseteq J^+(\mathscr{A}_\beta)$, there is some $A \in \mathscr{A}_\beta$, $A \subseteq Z$. Clearly, $A \in \mathscr{C}$, for A is almost disjoint with each C belonging to $\bigcup_{\gamma<\beta} \mathscr{C}_\gamma$.

The proof of (iii) will be more algebraic. We hope that Boolean counterparts of the notions defined in the language of infinite sets should be clear.

Fix a splitting family $\{s_\alpha: \alpha < \omega_1\} \subseteq \mathscr{P}(\omega)/\mathit{fin}$ witnessing to $s = \omega_1$. Let B be the subalgebra of $\mathscr{P}(\omega)/\mathit{fin}$ generated by $\{s_\alpha: \alpha < \omega_1\}$. Then B is atomless and $|B| = \omega_1$. Choose a continuous increasing chain $\{B_\alpha: \alpha < \omega_1\}$ of countable atomless subalgebras of B whose union is B.

Consider for a moment an arbitrary countable atomless subalgebra C of $\mathscr{P}(\omega)/\mathit{fin}$. If $\mathscr{U} \in \mathrm{Ult}(C)$, then there is a chain $T \subseteq \mathscr{U}$ of length ω which is a base for $\mathscr{U}$. For convenience, let us say that a non-zero element of $\mathscr{P}(\omega)/\mathit{fin}$ meets the boundary of $\mathscr{U}$ iff it meets the boundary of T. Using Theorem 4.15, for every $\mathscr{U} \in \mathrm{Ult}(C)$ there is a completely separable family $\mathscr{A}(\mathscr{U}) \subseteq \mathscr{P}(\omega)/\mathit{fin}$ such that for each $u \in \mathscr{P}(\omega)/\mathit{fin}$ which meets the boundary of $\mathscr{U}$ there is a $v \in \mathscr{A}(\mathscr{U})$ with $v \leq u$, and all members of $\mathscr{A}(\mathscr{U})$ are below $\mathscr{U}$.

We claim that $\mathscr{A}(C) = \bigcup\{\mathscr{A}(\mathscr{U}): \mathscr{U} \in \mathrm{Ult}(C)\}$ is a completely separable disjoint family in $\mathscr{P}(\omega)/\mathit{fin}$.

To see this, let $u \in \mathscr{P}(\omega)/\mathit{fin}$ and $\{v_n: n \in \omega\} \subseteq \mathscr{A}(C)$ be such that $u \cdot v_n \neq \mathbb{0}$ for each $n \in \omega$. It suffices to find some $\mathscr{U} \in \mathrm{Ult}(C)$ with $u \in J^+(\mathscr{A}(\mathscr{U}))$. Take arbitrary $\mathscr{U} \in \mathrm{Ult}(C)$ extending the family $\{w \in C: |\{n \in \omega: v_n \not\leq w\}| < \omega\}$. Then either $|\{n \in \omega: v_n \in \mathscr{A}(\mathscr{U})\}| = \omega$ or u meets the boundary of $\mathscr{U}$; in both cases, $u \in J^+(\mathscr{A}(\mathscr{U}))$.

Given an infinite partition Q of unity in $\mathscr{P}(\omega)/\mathit{fin}$, proceed as follows. For each $\alpha < \omega_1$ choose a completely separable disjoint family $\mathscr{A}_\alpha = \mathscr{A}(B_\alpha)$, $\mathscr{A}_\alpha \subseteq J(Q)$. Similarly as in the proof of (ii), define $\mathscr{C}_0 = \mathscr{A}_0$, $\mathscr{C}_\alpha = \{a \in \mathscr{A}_\alpha: c \cdot a = \mathbb{0}$ for all $c \in \bigcup_{\beta<\alpha} \mathscr{C}_\beta\}$, $\mathscr{C} = \bigcup_{\alpha<\omega_1} \mathscr{C}_\alpha$.

The proof will be finished by showing that the family $\mathscr{C}$ is the desired disjoint refinement of $J^+(Q)$.

Let $u_0 \in J^+(Q)$ be arbitrary. By the splitting property, there is some $\alpha_0 < \omega_1$ and $\mathscr{U}_0 \in \mathrm{Ult}(B_{\alpha_0})$ such that for some $w_0 \in \mathscr{U}_0$, $u_0 \cdot w_0 \neq \mathbb{0} \neq u_0 - w_0$ and for all $v \in \mathscr{U}_0$, $u_0 \cdot v \in J^+(Q)$.

There are two possibilities.

It may happen that there is some $\beta \leq \alpha_0$ such that $u_0 \in J^+(\mathscr{C}_\beta)$. Take the first β with this property. Now we may mimick part (b) from the proof of (ii) to show, using Claim 2, that some $c \in \mathscr{C}_\beta$ satisfies $c \leq u_0$. We will call this case good.

The bad case is to be treated as follows. Since $u_0 \in J(\mathscr{C}_\beta)$ for each $\beta \leq \alpha_0$, there is a countable subset $Q_0 \subseteq Q$ such that $c \cdot u_0 \cdot q = \mathbb{0}$ for all $c \in \bigcup_{\beta\leq\alpha_0} \mathscr{C}_\beta$ and all $q \in Q - Q_0$. Choose for each $v \in \mathscr{U}_0$ a member $q(v) \in Q - Q_0$ with $q(v) \cdot u_0 \cdot v \neq \mathbb{0}$ in a one-to-one manner. Then apply Claim 2 to $\mathscr{B} = Q_0$, $\mathscr{A} = \{q(v) \cdot v: v \in \mathscr{U}_0\}$, $X = u_0$. We obtain $u_0' \leq u_0$ such that $u_0' \cdot q = \mathbb{0}$ for all $q \in Q_0$, but still $u_0' \cdot v \in J^+(Q)$ whenever $v \in \mathscr{U}_0$. The element u_0' cannot meet the boundary of $\mathscr{U}_0$ – this would imply that $u_0 \in J^+(\mathscr{C}_{\alpha_0})$. But we are considering the bad case, therefore there must be some $v_0 \in \mathscr{U}_0$ such that $u_0' \cdot v_0 < v$ for all $v \in \mathscr{U}_0$. Pick some $q_0 \in Q$ with $u_0' \cdot v_0 \cdot q_0 \neq \mathbb{0}$.

Try once more with element $u_1 = u_0' \cdot v_0 - q_0$ in the place of u_0. There is some $\alpha_1 > \alpha_0$ and $\mathscr{U}_1 \in \mathrm{Ult}(B_{\alpha_1})$, $\mathscr{U}_1 \supseteq \mathscr{U}_0$ and some $w_1 \in \mathscr{U}_1$ such that $v \cdot u_1 \in J^+(Q)$ for all $v \in \mathscr{U}_1$, $w_1 \cdot u_1 \neq \mathbb{0} \neq u_1 - w_1$. If the good case happens, we are done; otherwise, we find Q_1, u_1', v_1 and q_1 similarly as before and we shall continue.

Suppose we never meet the good case. Then for $\alpha = \sup\{\alpha_n : n \in \omega\}$ we succeed in reaching $u_0 \in J^+(\mathscr{C}_\alpha)$. Indeed, consider $\mathscr{U} = \bigcup_{n\in\omega} \mathscr{U}_n \in \mathrm{Ult}(B_\alpha)$. We obtained two countable families, namely $I = \{u_n' \cdot v_n \cdot q_n : n \in \omega\}$ and $K = \{\mathbb{1} - q : q \in \bigcup_{i\in\omega} Q_i - \{q_n : n \in \omega\}\} \cup \{u_0\}$, and obviously $a < b$ whenever $a \in I$, $b \in K$. Thus, by Du Bois–Reymond separability property there is some member $u' \in \mathscr{P}(\omega)/\mathit{fin}$ such that $u' \geq u_n' \cdot v_n \cdot q_n$ for each $n \in \omega$, $u' \leq b$ for each $b \in K$. Clearly, u' meets the boundary of $\mathscr{U}$ and if $a \in \mathscr{A}(\mathscr{U})$, $a \leq u'$, then $a \cdot c = \mathbb{0}$ for all $c \in \bigcup_{\beta<\alpha} \mathscr{C}_\beta$. Thus, $a \in \mathscr{C}_\alpha$. □

The reader might notice that the AD families $\mathscr{C}$ constructed in the previous proof are completely separable. Although we did not prove it, the fact itself is not so surprising. The reason stems from the forthcoming theorem.

4.19. THEOREM. *The following are equivalent*:

(i) $\mathrm{RPC}(\omega)$;

(ii) *for each infinite MAD family Q on ω, $J^+(Q)$ has an almost disjoint refinement by a completely separable AD family*;

(iii) *there is a base tree $T \subseteq [\omega]^\omega$ such that any MAD family $Q \subseteq T$ is completely separable*;

(iv) *there is a dense subset $D \subseteq \mathscr{P}(\omega)/\mathit{fin}$ such that each partition of unity consisting of elements of D is completely separable.*

PROOF. The implications (iii)→(ii)→(i) are straightforward. A routine application of 1.13 shows (iv)→(iii). Let us prove (i)→(iv).

Choose an arbitrary base tree T of $\mathscr{P}(\omega)/\mathit{fin}$ of height h, denote its levels T_α. Proceeding by induction, construct a new base tree $\{Q_\alpha : \alpha < h\}$ such that $|Q_0| = 2^\omega$, Q_α refines T_α and $Q_{\alpha+1}$ is a disjoint refinement of $J^+(Q_\alpha)$ for each $\alpha < h$. For every $\alpha < h$ and every $u \in Q_\alpha$ choose a member $d(u) \in Q_{\alpha+1}$ satisfying $d(u) \leq u$. The set $D = \{d(u) : u \in \bigcup_{\alpha<h} Q_\alpha\}$ is dense because $\bigcup_{\alpha<h} Q_\alpha$ is.

It remains to show that each partition of unity $A \subseteq D$ is completely separable. Note that for any two distinct elements c, $d \in D$ either $c \cdot d = \mathbb{0}$ or $c < d$ or $d < c$.

Fix $x \in J^+(A)$ and denote $A_\alpha = \{a \in A \cap Q_\alpha : a \cdot x \neq \mathbb{0}\}$ for each $\alpha < h$. There is some first $\alpha < h$ such that $\bigcup_{\beta<\alpha} A_\beta$ is infinite.

If $\alpha = \beta + 1$, then A_β is infinite and $\bigcup_{\gamma<\beta} A_\gamma$ is finite. From the disjointness of A we immediately obtain that the element $y = x - \Sigma_{\gamma<\beta} \Sigma A_\gamma \in \mathscr{P}(\omega)/\mathit{fin}$ meets all members of A_β. Consequently, $y \in J^+(Q_{\beta-1})$ and therefore there is some $u \in Q_\beta$, $u \leq y$. The maximality of A implies the existence of some $a \in A$ with $a \cdot u \neq \mathbb{0}$; for this a, $a \leq u$ since a does not belong to $\bigcup_{\gamma<\beta} A_\gamma$.

If α is a limit ordinal, consider the family $\{q \in Q_\alpha : q \cdot a = \mathbb{0}$ for each $a \in \bigcup_{\beta<\alpha} A_\beta\} \cup \bigcup_{\beta<\alpha} A_\beta$. This family is clearly disjoint and maximal and $|\bigcup_{\beta<\alpha} A_\beta| = \omega < b$. Thus, by Claim 2 from the proof of 4.18 there is some $y \leq x$, $y \in J^+(Q_\alpha)$ and $y \cdot a = \mathbb{0}$ for all $a \in \bigcup_{\beta<\alpha} A_\beta$. By the construction, there is some

$u \in Q_{\alpha+1}$, $u \leq y$. From now on proceed as before: take $a \in A$ compatible with u, for this a we have $a \leq u \leq y \leq x$. $\square$

4.20. REMARKS AND COMMENTS. How strong or weak are the assumptions of 4.18? If one starts from an arbitrary ground model and adds at least ω_1 either Cohen or random reals, then in the extension $s = \omega_1$ and consequently RPC(ω) holds. It is fairly well known that $a = 2^\omega$ is a popular consequence of MA. In the model for $a = \omega_1$ given by HECHLER [1972] it is again true that $s = \omega_1$. Finally, one may frequently meet a scale in various models of ZFC.

The only model where no assumption of 4.18 holds – up to our knowledge – was given by SHELAH [1984]. Here $b = a = \omega_1$, $s = d = \omega_2 = 2^\omega$.

4.21. THEOREM. RPC(ω) *implies* RPC(2^ω). *Furthermore*, RPC(κ) *implies* RPC(κ^+) *for each cardinal* κ.

The proof can be found in BALCAR, DOČKÁLKOVÁ and SIMON [1984].

Up to now we have discussed several ideals J on ω such that J^+ has an almost disjoint refinement and considered the possibility that for each tall ideal J on ω, J^+ has an ADR. We shall close this section by the analogous question for higher cardinals. The very natural ideal on an uncountable κ is $[\kappa]^{<\kappa}$; we ask whether $[\kappa]^\kappa$ has an ADR by countable sets for $\kappa > \omega$. It is evident that Example 4.2 gives an affirmative answer for uncountable $\kappa \leq 2^\omega$. The following result was proved by KOMJÁTH [1984] under GCH; we present it under a bit weaker assumption using a different proof.

4.22. THEOREM. *If* $(\forall \lambda > 2^\omega)(\mathrm{cf}(\lambda) = \omega \rightarrow 2^\lambda = \lambda^+)$, *then for each* $\kappa > \omega$, $[\kappa]^\kappa$ *has an ADR by countable subsets of* κ.

PROOF. We shall give a brief sketch of a structure of the proof, postponing all non-trivial moments into the statements following this outline.

At first, the theorem is true for all $\omega_1 \leq \kappa \leq 2^\omega$ by 4.2. There are three steps in the induction: $\kappa = \lambda^+$, κ limit and $\mathrm{cf}(\kappa) = \omega$, κ limit and $\mathrm{cf}(\kappa) > \omega$.

We introduce an auxiliary notion of transversal trick on κ, $TT(\kappa)$, in the spirit of Theorem 4.6.

$TT(\kappa)$ is the statement: Given a countable partition $\mathcal{R} = \{R_n : n \in \omega\}$ of a set of size κ into infinite sets, then the family

$$\left\{X \subseteq \bigcup \mathcal{R} : \limsup_{n \in \omega} |X \cap R_n| = \kappa\right\}$$

has an almost disjoint refinement by transversals of $\mathcal{R}$.

We show:

$TT(\kappa)$ is true for $\kappa \leq 2^\omega$ by 4.6;

$TT(\kappa) \rightarrow TT(\kappa^+)$ for each κ (4.24);

$TT(\kappa) \rightarrow$ theorem holds for κ^+ (4.25);

for κ limit, $\mathrm{cf}(\kappa)=\omega$: $TT(\kappa)$ as well as the theorem follow from $2^\kappa=\kappa^+$ (4.26); for κ limit, $\mathrm{cf}(\kappa)>\omega$: $TT(\kappa)$ as well as the theorem for κ follow from

$$(\forall\lambda<\kappa,\ TT(\lambda))\quad (4.24, 4.25)\,. \quad\square$$

The forthcoming propositions are sometimes more general than needed for the purpose of 4.22, but we feel them to be of some interest.

4.23. PROPOSITION. *Let $n\in\omega$ and let $(2^\omega)^{+n}$ be the nth successor of continuum. The family $\{X\subseteq(2^\omega)^{+n}\colon \mathrm{otp}(X)\ge\omega^2\}$ has an ADR.*

PROOF. The proof goes by induction on n. We may and shall restrict our attention to the sets with order-type precisely ω^2.

For $n=0$ this is just 4.7(iii), and by 4.6 we know that the following version of the transversal trick holds for $\kappa=2^\omega$.

$TT_\omega(\kappa)$: Given a countable partition $\mathscr{R}=\{R_n\colon n\in\omega\}$ of a set of size κ with each R_n infinite, then the family $\{X\subseteq\bigcup\mathscr{R}\colon |\{n\in\omega\colon |X\cap R_n|\ge\omega\}|=\omega\}$ has an ADR by transversals of $\mathscr{R}$.

Assume that for $\kappa=(2^\omega)^{+n}$ both the proposition and $TT_\omega(\kappa)$ are true. We prove the same for κ^+.

Let $\alpha<\kappa^+$ be an ordinal with countable cofinality such that there is an $X\subseteq\alpha$ with $\mathrm{otp}(X)=\omega^2$ and $\sup X=\alpha$. Then there is an increasing sequence $\{\alpha_n\colon n\in\omega\}$ converging to α such that the partition $\mathscr{R}_\alpha=\{[\alpha_n,\alpha_{n+1})\colon n\in\omega\}$ consists of infinite sets. Let $\mathscr{A}_\alpha$ be an almost disjoint family of transversals of $\mathscr{R}_\alpha$ whose existence is guaranteed by $TT_\omega(\kappa)$. Obviously, $\mathscr{A}_\alpha$ refines the family $\{X\subseteq\alpha\colon \mathrm{otp}(X)=\omega^2$ and $\sup X=\alpha\}$; moreover, each $A\in\mathscr{A}_\alpha$ has order-type ω and $\sup A=\alpha$. Therefore $\mathscr{A}=\bigcup\{\mathscr{A}_\alpha\colon \alpha<\kappa^+$ and $\mathrm{cf}(\alpha)=\omega\}$ is the desired ADR, and the proposition holds for κ^+.

It remains to show $TT_\omega(\kappa^+)$. Let $\mathscr{R}=\{R_n\colon n\in\omega\}$ be a partition of κ^+, with infinite R_n's. We may assume that all R_n's have full size κ^+, the other possibilities being covered by $TT_\omega(\lambda)$ for $\lambda\le\kappa$. Represent R_n as $\{n\}\times\kappa^+$ and $\bigcup\mathscr{R}$ as $\omega\times\kappa^+$.

Denote $\mathscr{S}=\{X\subseteq\omega\times\kappa^+\colon |\{n\in\omega\colon |\{n\}\times\kappa^+\cap X|\ge\omega\}|=\omega\}$. For $X\in\mathscr{S}$ let $g(X)=\min\{\alpha<\kappa^+\colon (\forall n\in\omega)(\exists m>n)|\{m\}\times\alpha\cap X|\ge\omega\}$. Then $g(X)$ is a limit ordinal of countable cofinality. Moreover, one can choose an increasing sequence $\{\alpha_n\colon n\in\omega\}$ converging to $g(X)$ such that for $Y=X\cap\bigcup_{n\in\omega}\{n\}\times(g(X)-\alpha_n)$ we still have $Y\in\mathscr{S}$ and $g(Y)=g(X)$.

It means that every transversal A of $\mathscr{R}$ satisfying $A\subseteq Y$ is a sequence converging to $g(X)$.

By the induction hypothesis, for $\alpha<\kappa^+$ with $\mathrm{cf}(\alpha)=\omega$ there is an AD family $\mathscr{A}_\alpha$ which refines the family $\{X\in\mathscr{S}\colon g(X)=\alpha\}$, each member of $\mathscr{A}_\alpha$ is a transversal of $\mathscr{R}$ and as we have just noticed, we may assume that all members of $\mathscr{A}_\alpha$ are sequences converging to α.

Therefore $\bigcup\{\mathscr{A}_\alpha\colon \alpha<\kappa^+$ and $\mathrm{cf}(\alpha)=\omega\}$ witnesses to $TT_\omega(\kappa^+)$. $\square$

Both ideas exploited in the previous proof will be repeated in the next two lemmas.

4.24. LEMMA. (a) *For each* $\kappa \geq \omega$, $TT(\kappa)$ *implies* $TT(\kappa^+)$.

(b) *If* κ *is a limit cardinal and* $\mathrm{cf}(\kappa) > \omega$, *then* $((\forall \lambda < \kappa)(TT(\lambda)))$ *implies* $TT(\kappa)$.

PROOF. Imagine a partition $\mathcal{R}$ as $\{\{n\} \times \kappa^+ : n \in \omega\}$ (or $\{\{n\} \times \kappa : n \in \omega\}$ if the limit κ is considered). Let

$$\mathcal{S} = \left\{ X \subseteq \bigcup \mathcal{R} : \limsup_{n \in \omega} |X \cap \{n\} \times \kappa^+| = \kappa^+ \right\}.$$

For $X \in \mathcal{S}$, the set $\{\alpha < \kappa^+ : (\forall n \in \omega)(|X \cap \{n\} \times \kappa^+| = \kappa^+ \rightarrow (\forall \beta < \alpha)|X \cap \{n\} \times [\beta, \alpha)| = \kappa)\}$ is closed unbounded in κ^+, hence it contains some $g(X)$ of countable cofinality.

The respective case (b) is of course modified in an apparent way, considering the closed unbounded set of all $\alpha < \kappa$ such that if $|X \cap \{n\} \times \kappa| = \kappa$, then for each $\beta < \alpha$, $|X \cap \{n\} \times [\beta, \alpha)| = |\alpha|$.

So using $TT(\kappa)$ in case (a) and $(\forall \lambda < \kappa)(TT(\lambda))$ in case (b), we have families $\mathcal{A}_\alpha$ of transversals, where $\alpha < \kappa^+$ (resp. $\alpha < \kappa$) and $\mathrm{cf}(\alpha) = \omega$ and each $A \in \mathcal{A}_\alpha$ converges to α. Their union works to show $TT(\kappa^+)(TT(\kappa)$, if (b) is considered). □

4.25. LEMMA. (a) *If* $TT(\kappa)$ *holds, then* $[\kappa^+]^{\kappa^+}$ *has an almost disjoint refinement by countable subsets of* κ^+.

(b) *If* $\mathrm{cf}(\kappa) > \omega$, κ *is a limit cardinal and* $(\forall \lambda < \kappa)(TT(\lambda))$, *then* $[\kappa]^\kappa$ *has an ADR*.

PROOF. If $\alpha < \kappa^+$ (resp. $\alpha < \kappa$) has countable cofinality and if $\{\alpha_n : n \in \omega\}$ converges to α, consider the partition $\mathcal{R}_\alpha = \{[\alpha_n, \alpha_{n+1}) : n \in \omega\}$ in order to apply $TT(\kappa)$ (or resp. $TT(|\alpha|)$). It remains to notice that for each $X \in [\kappa^+]^{\kappa^+}$, $\{\alpha < \kappa^+ : (\forall \beta < \alpha)|X \cap [\beta, \alpha)| = \kappa\}$ is club in κ^+, and a similar statement for limit κ holds, too. □

4.26. LEMMA. *Let* $\kappa > \omega$, $\mathrm{cf}(\kappa) = \omega$ *and assume* $2^\kappa = \kappa^+$. *Then* $TT(\kappa)$ *holds and* $[\kappa]^\kappa$ *has ADR*.

PROOF. The classical diagonal argument proves both the statements. Choose an arbitrary partition $\{R_n : n \in \omega\}$ of κ with $|R_0| < |R_1| < \cdots < |R_n| < \cdots$. Enumerate $[\kappa]^\kappa = \{X_\alpha : \alpha < \kappa^+\}$. If for each $\beta < \alpha$ a set $A_\beta \subseteq X_\beta$, A_β a transversal of $\{R_n : n \in \omega\}$ have been found, then there is still room for one more in X_α which will be almost disjoint with all the previous.

4.27. CONSISTENCY RESULT. It is consistent with ZFC that for each infinite κ, $\mathrm{RPC}(\kappa)$ holds.

We are greatly indebted to members of Budapest Seminar on Set Theory and especially to A. Hajnál, who informed us about their result concerning the consistency of "On each infinite κ, there is a completely separable MAD family of countable subsets of κ". It turns out that the same model provides $\mathrm{CON}((\forall \kappa \geq \omega)\mathrm{RPC}(\kappa))$.

Let $Q = D * C$, where C is the standard poset for adding a Cohen real, D is the dominating real forcing, i.e. $(f, n) \in D$ iff $f \in {}^{\omega}\omega$, $n \in \omega$ and $(f, n) \geq (g, m)$ whenever $n \leq m$, $g \restriction n = f \restriction n$ and for all $i \in \omega$, $f(i) \leq g(i)$.

Let M be the ground model, B the ω_1-stage iteration of Q with finite support, G a generic ultrafilter on B over M. Let us fix some notation. $B_0 = Q$, $B_{\alpha+1} = B_\alpha * Q$, B_α is a direct limit of B_β's for limit α, $M_\alpha = M[G_\alpha] = M[G \cap B_\alpha]$. Clearly, B satisfies ccc, therefore $[\kappa]^\omega \cap M[G] = \bigcup_{\alpha<\omega_1} [\kappa]^\omega \cap M_\alpha$.

Denote by N_α the generic extension of M_α via D. We have $M_\alpha \subseteq N_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \omega_1$.

We shall make use from the following two basic facts.

(1) Suppose $X \in M_\alpha$, $\{A_n : n \in \omega\} \in M_\alpha$, all A_n's are countable and almost disjoint, $|X \cap A_n| = \omega$ for all $n \in \omega$. Then there is a $Y \in [X]^\omega$, $Y \in N_\alpha$ such that $|Y \cap A_n| < \omega$ for all $n \in \omega$ and if $|Z \cap A_n| < \omega$ for all $n \in \omega$ and $Z \in M_\alpha$, then $|Z \cap Y| < \omega$, too.

We can w.l.o.g. assume that all sets in question are subsets of $\omega = X$. The rule $f_Z(n) = \max((Z \cup \bigcup_{i<n} A_i) \cap A_n)$ defines in M_α a function from ω to ω. Then the set $Y = \{\min(A_n - [0, g(n)+1)) : n \in \omega\}$, where $g \in N_\alpha$ is the dominating function over M_α, is as required.

(2) The set $[\kappa]^\omega \cap N_\alpha$ has an almost disjoint refinement in $M_{\alpha+1}$. HECHLER [1978] proved that $[\omega]^\omega \cap M$ has an almost disjoint refinement in $M[c]$, where $M[c]$ is a generic extension of M by adding one Cohen real. Hechler's statement immediately implies Observation (2).

Using (1) and (2), one can find a collection $\{\mathscr{C}_\alpha : \alpha < \omega_1\}$ in $M[G]$ satisfying (a), (b) and (c) below.

(a) For each $\alpha < \omega_1$, $\mathscr{C}_\alpha \in M_{\alpha+1}$, $\mathscr{C}_\alpha \subseteq [\kappa]^\omega$ is an infinite AD family and $\mathscr{C}_\alpha$ refines $[\kappa]^\omega \cap N_\alpha$.

(b) For each $X \in [\kappa]^\omega$, $X \in M_{\alpha+1}$, no subfamily of $\bigcup_{\beta\leq\alpha} \mathscr{C}_\beta$ is an infinite MAD on X in $N_{\alpha+1}$.

(c) For $\alpha < \beta < \omega_1$, if $C \in \mathscr{C}_\alpha$ and $C' \in \mathscr{C}_\beta$, then either $C' \subseteq^* C$ or $|C \cap C'| < \omega$.

Since for each $X \in [\kappa]^\omega$ in $M[G]$ there is some $\alpha < \omega_1$ with $X \in M_\alpha$, the set $\bigcup_{\alpha<\omega_1} \mathscr{C}_\alpha$ represents a dense subset of an algebra $\mathscr{P}(\kappa)/\mathit{fin}$ by (a). If $\mathscr{A} \subseteq \bigcup_{\alpha<\omega_1} \mathscr{C}_\alpha$ is a MAD family on κ, then $\mathscr{A}$ is completely separable. Indeed, if $X \in J^+(\mathscr{A})$, choose $\{A_0, A_1, \ldots, A_n, \ldots\} \in [\mathscr{A}]^\omega$ such that $|X \cap A_i| = \omega$ for all $i \in \omega$. There is some α with X, $\{A_i : i \in \omega\} \in M_{\alpha+1}$. By (b), there is some $Y \subseteq X$, $Y \in N_{\alpha+1}$ such that $|Y \cap A| < \omega$ for all $A \in \mathscr{A} \cap \bigcup_{\beta\leq\alpha} \mathscr{C}_\beta$. Thus, by (a), there is some $C \in \mathscr{C}_{\alpha+1}$, $C \subseteq Y$ and by (c), if $A \in \mathscr{A}$ satisfies $|A \cap C| = \omega$, it must hold $A \subseteq^* C \subseteq^* X$.

Now the validity of RPC(κ) in $M[G]$ is clear.

5. The algebras $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$; non-distributivity and decomposability

We shall deal with the quotient algebras $\mathscr{P}_\kappa(\kappa) = \mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ of the power set algebra $\mathscr{P}(\kappa)$ modulo the ideal of all sets of size less than κ. We shall study the decomposability properties of ultrafilters in these algebras. Our aim is to investigate the questions analogous to those discussed in Sections 1 and 2 in a general setting and in Sections 3 and 4 in particular for $\mathcal{P}_\omega(\omega)$.

The properties of $\mathcal{P}_\kappa(\kappa)$ for uncountable κ dramatically differ from those of $\mathcal{P}_\omega(\omega)$ and we shall often require several additional set-theoretical assumptions.

At first, we shall mention some basic facts concerning disjoint families in $\mathcal{P}_\kappa(\kappa)$ and the saturatedness $\mathrm{sat}(\mathcal{P}_\kappa(\kappa))$. We show that for each κ there is a $(2^\kappa, \kappa^+)$-independent matrix in $\mathcal{P}_\kappa(\kappa)$. The cardinal characteristic h_κ, i.e. the non-distributivity of $\mathcal{P}_\kappa(\kappa)$ is used for the description of the completion of $\mathcal{P}_\kappa(\kappa)$. The characteristics b_κ – analogous to the number b defined in 3.13 – appears to be the same as the additivity of the ideal of all non-stationary sets in the algebra $\mathcal{P}_\kappa(\kappa)$ supposing κ to be uncountable and regular. Moreover, each ultrafilter in $\mathcal{P}_\kappa(\kappa)$ is strongly b_κ-decomposable.

At the end, we discuss the regularity and decomposability properties of uniform ultrafilters in the atomic algebra $\mathcal{P}(\kappa)$.

5.1. Let $\kappa \geq \omega$ be a cardinal number. The algebra $\mathcal{P}_\kappa(\kappa)$ is atomless, homogeneous and of size 2^κ. Furthermore, it has important separation properties; let us define them.

5.2. DEFINITION. Let τ be an infinite cardinal and B a Boolean algebra. We shall say that

(i) B satisfies $E(\tau)$ if $\mathrm{hsat}(B) > \tau$ and no $u \in B^+$ admits a partition of size τ;

(ii) B satisfies $F(\tau)$ if for any $C, D \subseteq [B]^{<\tau}$ such that each $c \in C$ and $d \in D$ are disjoint there is some $b \in B$ satisfying $c \leq b$ whenever $c \in C$ and $b \cdot d = \mathbb{0}$ whenever $d \in D$.

Clearly, both $E(\tau)$ and $F(\tau)$ are hereditary properties, i.e. if B satisfies them, then so does $B \upharpoonright u$ for all $u \in B^+$. Note that $F(\omega)$ is automatically true for an arbitrary B, $E(\omega)$ is the Cantor property, and $F(\omega_1)$ is Du Bois–Reymond property. It is known that every κ-homogeneous universal algebra (see the remarks preceding 1.14) satisfies $F(\kappa)$ and $E(\tau)$ for each $\tau < \kappa$.

5.3. PROPOSITION. *$\mathcal{P}_\kappa(\kappa)$ satisfies $E(\mathrm{cf}(\kappa))$ and $F(\mathrm{cf}(\kappa)^+)$. Furthermore, $\mathcal{P}_\kappa(\kappa)$ is $\mathrm{cf}(\kappa)$-complete.*

PROOF. Since $\mathcal{P}_\kappa(\kappa)$ is homogeneous, in order to prove $E(\mathrm{cf}(\kappa))$ it suffices to show that no disjoint family $\{a_\alpha: \alpha < \mathrm{cf}(\kappa)\}$ is maximal. Let $A_\alpha \in [\kappa]^\kappa$ represent a_α. For $\alpha < \mathrm{cf}(\kappa)$ define $A'_\alpha = A_\alpha - \bigcup_{\beta<\alpha} A_\beta$. Clearly, A'_α represents a_α, too, and $\{A'_\alpha: \alpha < \mathrm{cf}(\kappa)\}$ is a pairwise disjoint family of subsets of κ. For $\alpha < \mathrm{cf}(\kappa)$, choose $M_\alpha \subseteq A'_\alpha$ such that $|M_\alpha| < \kappa$ and the union $\bigcup \{M_\alpha: \alpha < \mathrm{cf}(\kappa)\} = M$ has the full cardinality κ. Obviously, the set M represents a member of $\mathcal{P}_\kappa(\kappa)$ disjoint with all a_α's.

The algebra $\mathcal{P}(\kappa)$ being complete, satisfies $F(\tau)$ for each τ, and the ideal $[\kappa]^{<\kappa}$ is $\mathrm{cf}(\kappa)$-complete. Hence, the quotient algebra $\mathcal{P}_\kappa(\kappa)$ satisfies $F(\mathrm{cf}(\kappa)^+)$. The same argument shows that $\mathcal{P}_\kappa(\kappa)$ is $\mathrm{cf}(\kappa)$-complete. □

Let us turn now to the saturatedness of $\mathcal{P}_\kappa(\kappa)$.

5.4. DEFINITION. A family $\mathscr{A} \subseteq [\kappa]^\kappa$ is called uniform almost disjoint (UAD) if for every two distinct members $X, Y \in \mathscr{A}$, $|X \cap Y| < \kappa$.

It is evident that UAD families of subsets of κ correspond to disjoint families in the algebra $\mathscr{P}_\kappa(\kappa)$.

When dealing with UAD families on κ, it is sometimes convenient, especially for κ singular, to imagine κ together with some partition and study the transversals of this partition. This motivates the forthcoming definition.

5.5. DEFINITION. Let $\{X_\alpha: \alpha \in \lambda\}$ be a family of non-empty sets. A set $S \subseteq \Pi\{X_\alpha: \alpha \in \lambda\}$ is called almost disjoint set of mappings if for distinct $f, g \in S$ there is some $\beta < \lambda$ such that $\{\alpha \in \lambda: f(\alpha) = g(\alpha)\} \subseteq \beta$.

5.6. EXAMPLE. Let κ be a singular cardinal with $\lambda = \mathrm{cf}(\kappa)$. Let $\{R_\alpha: \alpha < \lambda\}$ be a partition of κ such that $\{\kappa_\alpha = |R_\alpha|: \alpha < \lambda\}$ is an increasing sequence of infinite cardinals convergent to κ.

For $\alpha < \lambda$ take a partition $\{r(\alpha, \beta): \beta < \kappa_\alpha\}$ of R_α into κ_α sets of size κ_α. Now, if $S \subseteq \Pi_{\alpha<\lambda} \kappa_\alpha$ is an almost disjoint set of mappings, then the collection $\mathscr{A} = \{\bigcup \{r(\alpha, f(\alpha)): \alpha < \lambda\}: f \in S\}$ is a uniform almost disjoint family on κ and $|\mathscr{A}| = |S|$.

5.7. LEMMA. *Let κ be a singular cardinal, $\lambda = \mathrm{cf}(\kappa)$ and let $\{\kappa_\alpha: \alpha < \lambda\}$ be an increasing sequence of regular cardinals converging to κ, $\kappa_0 > \lambda$. Then there is an almost disjoint set of mappings $S \subseteq \Pi\{\kappa_\alpha: \alpha < \lambda\}$ with $|S| \geq \kappa^+$.*

PROOF. Let $\{f_\beta: \beta < \kappa\} \subseteq \Pi\{\kappa_\alpha: \alpha < \lambda\}$ be an almost disjoint set of mappings. For $\alpha < \lambda$, define $g(\alpha) = \min(\kappa_\alpha - \{f_\beta(\alpha): \beta < \bigcup_{\gamma<\alpha} \kappa_\gamma\})$. Clearly, $g \in \Pi_{\alpha<\lambda} \kappa_\alpha$ and g is almost disjoint with each f_β. Consequently, no almost disjoint set of mappings of size κ is maximal and the lemma follows. □

5.8. PROPOSITION. *For each infinite κ,* sat $\mathscr{P}_\kappa(\kappa) \geq \kappa^{++}$.

PROOF. Assume κ regular. Then sat $\mathscr{P}_\kappa(\kappa) \geq \kappa^{++}$ follows by $E(\kappa)$.

If κ is singular and $\lambda = \mathrm{cf}(\kappa)$ choose an increasing sequence $\{\kappa_\alpha: \alpha < \lambda\}$ of regular cardinals converging to κ. By 5.7, there is an almost disjoint set of mappings $S \subseteq \Pi_{\alpha<\lambda} \kappa_\alpha$ of size κ^+. It remains to use Example 5.6. □

More about sat $\mathscr{P}_\kappa(\kappa)$ can be proved under additional assumptions.

5.9. PROPOSITION. *Let κ be an infinite cardinal and assume that $\kappa^\nu = \kappa$ for each $\nu < \mathrm{cf}(\kappa)$. Then there is a disjoint family of size $\kappa^{\mathrm{cf}(\kappa)}$ in the algebra $\mathscr{P}_\kappa(\kappa)$.*

5.10. COROLLARY. (i) *If* $\mathrm{cf}(\kappa) = \omega$, *then* sat $\mathscr{P}_\kappa(\kappa) > \kappa^\omega$.

(ii) *If* $2^\nu \leq \kappa$ *for each* $\nu < \kappa$, *then* sat $\mathscr{P}_\kappa(\kappa) = (2^\kappa)^+$.

PROOF. Denote $\lambda = \mathrm{cf}(\kappa)$ and $R_\alpha = {}^\alpha\kappa$ for $\alpha < \lambda$. Under our assumptions, $|R_\alpha| = \kappa$ for each α. For each $\varphi \in {}^\lambda\kappa$ let $f_\varphi \in \Pi_{\alpha>\lambda} R_\alpha$ be defined by $f_\varphi(\alpha) = \varphi \restriction \alpha$. Then the family $\{f_\varphi : \varphi \in {}^\lambda\kappa\} \subseteq \Pi_{\alpha<\lambda} R_\alpha$ is an almost disjoint set of mappings and its size equals to κ^λ.

If κ is regular, then the graphs of f_φ's constitute a UAD family on the set $\bigcup_{\alpha<\lambda} R_\alpha$ and we finish.

If κ is singular, we need essentially the same trick as in 5.6. We know that there is a partition $\{R_\alpha : \alpha < \lambda\}$ of κ into the sets of size κ and that there is an almost disjoint set of mappings $S \subseteq \Pi_{\alpha<\lambda} R_\alpha$, $|S| = \kappa^\lambda$. Let $\{\kappa_\alpha : \alpha < \lambda\}$ be an increasing sequence of cardinals converging to κ and for each $\alpha < \lambda$, let $\{r(\alpha, \beta) : \beta < \kappa\}$ be a partition of R_α with $|r(\alpha, \beta)| = \kappa_\alpha$ for each $\beta < \kappa$. For $f \in S$, let $X_f = \bigcup \{r(\alpha, f(\alpha)) : \alpha < \lambda\}$. The family $\{X_f : f \in S\}$ is UAD on κ. □

The corollary is straightforward.

5.11. COMMENTS. (i) Even though $|\mathcal{P}_\kappa(\kappa)| = 2^\kappa$, it is not possible to prove in ZFC only that for uncountable κ there is a disjoint family in $\mathcal{P}_\kappa(\kappa)$ of size 2^κ, as an analogy to the case $\kappa = \omega$.

BAUMGARTNER [1976] showed CON(ZFC + 2^{ω_1} is arbitrarily large + there is no disjoint family in $\mathcal{P}_{\omega_1}(\omega_1)$ of size ω_3). In fact, if $V[G]$ is a generic extension over a Boolean algebra satisfying ccc, then in $V[G]$, sat $\mathcal{P}_{\omega_1}(\omega_1) \leq ((2^{\omega_1})^+)^V$ holds.

(ii) If κ is singular, then there may exist a maximal disjoint family in $\mathcal{P}_\kappa(\kappa)$ of size precisely κ. ERDÖS and HECHLER [1975] proved the existence of such a family from the assumption $(\forall \nu < \kappa)\nu^{\mathrm{cf}(\kappa)} < \kappa$.

Especially, assuming GCH, then for any κ and $0 < \nu \leq 2^\kappa$ there is a maximal disjoint family in $\mathcal{P}_\kappa(\kappa)$ of size ν iff $\nu \neq \mathrm{cf}(\kappa)$.

(iii) Let κ be a singular cardinal of uncountable cofinality λ. Let $\{\kappa_\alpha : \alpha < \lambda\}$ be an increasing and continuous sequence of cardinals converging to κ. If $\nu^\lambda < \kappa$ for each $\nu < \kappa$ (compare with the assumption in (ii) above), then every almost disjoint set of mappings in $\Pi\{\kappa_\alpha : \alpha < \lambda\}$ has size $\leq \kappa$. This explains why we needed to choose a sequence of regular cardinals in 5.7.

Furthermore, every maximal almost disjoint set of mappings in the product of successors, i.e. subset of $\Pi\{\kappa_\alpha^+ : \alpha < \lambda\}$, has the size exactly κ^+. This combinatorial fact is the key for the proof of famous Silver theorem stating that $2^\kappa = \kappa^+$ provided that $2^\tau = \tau^+$ for all $\tau < \kappa$.

Our next topics are independent matrices. They have proved to be a useful tool for the construction of ultrafilters with various extremal properties.

5.12. DEFINITION (Independent matrix). Let ν, $\tau \geq 2$ be cardinals. A family $\{a(\alpha, \beta) : \alpha < \nu, \beta < \tau\}$ of elements of a Boolean algebra B is called a (ν, τ)-independent matrix in B if

(i) each row $\{a(\alpha, \beta) : \beta < \tau\}$ is a disjoint family;

(ii) for each finite mapping $\varphi : X \to \tau$, where $X \in [\nu]^{<\omega}$, the meet $\Pi\{a(\alpha, \varphi(\alpha)) : \alpha \in X\}$ is non-zero.

A classical result of Hausdorff, and Engelking and Karłowicz, says that for each κ there is a $(2^\kappa, \kappa)$-independent matrix in the algebra $\mathcal{P}(\kappa)$. (See the Appendix on Set Theory in this Handbook for the proof.)

5.13. THEOREM. *Let κ be an infinite cardinal.*

(i) *There is a $(2^\kappa, \kappa^+)$-independent matrix in $\mathscr{P}_\kappa(\kappa)$.*

(ii) *Assuming κ to be strongly limit, i.e. $(\forall \nu < \kappa) 2^\nu < \kappa$, then there is a $(2^\kappa, 2^\kappa)$-independent matrix in $\mathscr{P}_\kappa(\kappa)$.*

In particular, there is a $(2^\omega, 2^\omega)$-independent matrix in $\mathscr{P}(\omega)/fin$.

PROOF. We prove (ii) first, then (i) for singular cardinals. The case (i) for regular cardinals will follow by the more general Theorem 5.14, due to A. Dow, and by 5.3.

Let κ be strongly limit. Consider the set $S = \{\langle \alpha, f\rangle : \alpha < \kappa$ & $f: \mathscr{P}(\alpha) \to \mathscr{P}(\alpha)\}$. Since $|\mathscr{P}(\alpha)| < \kappa$ for each $\alpha < \kappa$, we have $|S| = \kappa$ and we shall construct the desired matrix on S rather than on κ.

For X, $Y \subseteq \kappa$ define

$$A(X, Y) = \{\langle \alpha, f\rangle \in S: f(X \cap \alpha) = Y \cap \alpha\} .$$

We shall verify that $\{A(X, Y): X \subseteq \kappa, Y \subseteq \kappa\}$ determines a $(2^\kappa, 2^\kappa)$-independent matrix in $\mathscr{P}_\kappa(S)$.

Rows are almost disjoint families: pick X and $Y_1 \neq Y_2$ arbitrary. There is some $\alpha < \kappa$ such that $Y_1 \cap \alpha \neq Y_2 \cap \alpha$. If $\langle \beta, f\rangle \in A(X, Y_1) \cap A(X, Y_2)$, then $f(X \cap \beta) = Y_1 \cap \beta = Y_2 \cap \beta$, therefore $\beta < \alpha$ and $f \subseteq \mathscr{P}(\alpha) \times \mathscr{P}(\alpha)$. Thus, $|A(X, Y_1) \cap A(X, Y_2)| < \kappa$.

Let $k \in \omega$, let $X_0, \ldots, X_k$ be distinct, $Y_0, \ldots, Y_k$ be arbitrary. Choose $\alpha < \kappa$ such that $X_i \cap \alpha \neq X_j \cap \alpha$ for all $i < j \leq k$. For each $\beta < \kappa$, $\beta \geq \alpha$ and for each mapping $f: \mathscr{P}(\beta) \to \mathscr{P}(\beta)$ satisfying $f(X_i \cap \beta) = Y_i \cap \beta$ $(i \leq k)$, we have $\langle \beta, f\rangle \in \bigcap \{A(X_i, Y_i): i \leq k\}$. Therefore the cardinality of the intersection is κ and (ii) is proved.

Let κ be a singular cardinal, $\lambda = \mathrm{cf}(\kappa)$. Choose an increasing sequence $\{\kappa_\xi: \xi < \lambda\}$ of regular cardinals converging to κ and a partition $\{R_\xi: \xi < \lambda\}$ of κ such that $|R_\xi| = \kappa_\xi$. For each $\xi < \lambda$ fix a uniform $(2^{\kappa_\xi}, \kappa_\xi)$-independent matrix $\{a_\xi(\alpha, \beta): \alpha < 2^{\kappa_\xi}, \beta < \kappa_\xi\}$ in the power set algebra $\mathscr{P}(R_\xi)$.

Choose an almost disjoint set of mappings $S \subseteq \Pi\{\kappa_\xi: \xi < \lambda\}$ with $|S| = \kappa^+$ by 5.7.

For each $\xi < \lambda$ let φ_ξ be a bijection from $\mathscr{P}(\kappa_\xi)$ onto 2^{κ_ξ}. Now, for arbitrary $X \subseteq \kappa$ and $f \in S$ define

$$A(X, f) = \bigcup \{a_\xi(\varphi_\xi(X \cap \kappa_\xi), f(\xi)): \xi < \lambda\} .$$

It remains to show that the family $\{A(X, f): X \subseteq \kappa, f \in S\}$ represents the desired matrix.

If f, $g \in S$ are distinct, then there is some $\gamma < \lambda$ such that $f(\xi) \neq g(\xi)$ for all $\xi \geq \gamma$; consequently, the intersection $A(X, f) \cap A(X, g) \subseteq \bigcup \{a_\xi(\varphi_\xi(X \cap \kappa_\xi), f(\xi)): \xi < \gamma\} \subseteq \bigcup_{\xi < \gamma} R_\xi$ and $|\bigcup_{\xi<\gamma} R_\xi| < \kappa_\gamma < \kappa$. Thus, the rows are almost disjoint. The uniform independence property follows from the independence property of each factor. □

5.14. THEOREM (A. DOW). *Let κ be an infinite cardinal and suppose that a Boolean algebra B satisfies $E(\kappa)$ and $F(\kappa^+)$. Then there exists a $(2^\kappa, \kappa^+)$-independent matrix in B.*

PROOF. The proof will be broken into three claims.

Claim 1. There is a (κ, κ)-independent matrix in B.

Take an arbitrary disjoint family $\{d_\xi: \xi < \kappa\}$ in B, whose existence is guaranteed by $E(\kappa)$ and let $\{a(\alpha, \beta): \alpha < \kappa, \beta < \kappa\}$ be a (κ, κ)-independent matrix in $\mathcal{P}(\kappa)$. The idea is to blow up each $\xi \in \kappa$ to d_ξ thus carrying each $a(\alpha, \beta)$ into the desired $b(\alpha, \beta)$.

To be precise, let $b(\alpha, 0)$ be the element $c \in B$ such that $c \geq d_\xi$ for all $\xi \in a(\alpha, 0)$, $c \cdot d_\xi = \mathbb{0}$ for all $\xi \in \kappa - a(\alpha, 0)$.

Proceeding by induction, $b(\alpha, \beta)$ will be some $c \in B$ such that

$$\begin{aligned} &c \geq d_\xi && \text{for all } \xi \in a(\alpha, \beta)\,, \\ &c \cdot d_\xi = \mathbb{0} && \text{for all } \xi \in \kappa - a(\alpha, \beta) \end{aligned}$$

and

$$c \cdot b(\alpha, \gamma) = \mathbb{0} \quad \text{for all } \gamma < \beta\,.$$

It is evident that $F(\kappa^+)$ enables us to find all $b(\alpha, \beta)$'s as well as that the $\{b(\alpha, \beta): \alpha < \kappa, \beta < \kappa\}$ is a (κ, κ)-independent matrix in B.

Claim 2. There is a (κ, κ^+)-independent matrix in B.

Let $\{b(\alpha, \beta): \alpha < \kappa, \beta < \kappa\}$ be a (κ, κ)-independent matrix in B, $\alpha_0 < \kappa$ arbitrary. We shall show that there is some $b(\alpha_0, \kappa) \in B^+$ such that $b(\alpha_0, \beta)$. $b(\alpha_0, \kappa) = \mathbb{0}$ for all $\beta < \kappa$ and the matrix $\{b(\alpha, \beta): \alpha < \kappa, \alpha \neq \alpha_0, \beta < \kappa\} \cup \{b(\alpha_0, \beta): \beta < \kappa + 1\}$ is independent, too.

Consider the subalgebra $C \subseteq B$ generated by $\{b(\alpha, \beta): \alpha < \kappa, \beta < \kappa, \alpha \neq \alpha_0\}$. For every $c \in C^+$ and every $\beta < \kappa$ we have $c.\ b(\alpha_0, \beta) \neq \mathbb{0}$. By the $E(\kappa)$ property of B, no $c \in C^+$ is covered by $\{b(\alpha_0, \beta): \beta < \kappa\}$, therefore there is some non-zero $d(c) \leq c$, $d(c)$ disjoint with all $b(\alpha_0, \beta)$ for $\beta < \kappa$. By $F(\kappa^+)$ there is some $b(\alpha_0, \kappa)$ such that $b(\alpha_0, \kappa) \cdot b(\alpha_0, \beta) = \mathbb{0}$ for all $b < \kappa$ and $b(\alpha_0, \kappa) \geq d(c)$ for all $c \in C^+$.

We have just described one step, the straightforward induction proves the claim now.

Claim 3. There is a $(2^\kappa, \kappa^+)$-independent matrix in B.

Take an arbitrary disjoint family $\{d_\gamma: \gamma < \kappa\}$ in B. For each $\gamma < \kappa$, let $\{b_\gamma(\alpha, \beta): \alpha < \kappa, \beta < \kappa^+\}$ be a (κ, κ^+)-independent matrix in $B \restriction d_\gamma$.

Choose a finitely distinguishable family $F \subseteq {}^\kappa\kappa$ of size 2^κ (see Part I, Chapter 5, 13.9). For each $f \in F$ define by induction a disjoint family $\{c(f, \beta): \beta < \kappa^+\}$ in B such that:

$$c(f, \beta) \geq b_\gamma(f(\gamma), \beta) \qquad \text{for all } \gamma < \kappa$$

and

$$c(f, \beta) \cdot c(f, \delta) = \mathbb{0} \qquad \text{for all } \delta < \beta$$

and

$$c(f, \beta)\cdot(d_\gamma - b_\gamma(f(\gamma), \beta)) = \mathbb{0} \quad \text{for all } \gamma < \kappa .$$

Again, $F(\kappa^+)$ enables us to do this.

Then the matrix $\{c(f, \beta): f \in F, \beta < \kappa^+\}$ is $(2^\kappa, \kappa^+)$-independent. □

5.15. NON-DISTRIBUTIVITY OF $\mathscr{P}_\kappa(\kappa)$. The non-distributivity (or height) of $\mathscr{P}_\kappa(\kappa)$ is defined to be the cardinal number

$$h_\kappa = \min\{\tau: \mathscr{P}_\kappa(\kappa) \text{ is not } (\tau, \cdot, 2)\text{-distributive}\} .$$

This extends Definition 3.2 to higher cardinals; of course $h_\omega = h$. We have seen in Section 3 that the actual value of h depends on additional axioms of set theory. Now we shall show that for uncountable κ, the value of h_κ is either ω or ω_1, nothing else, and that the cofinality of κ is the decisive factor.

5.16. THEOREM. (i) *If* $\mathrm{cf}(\kappa) > \omega$, *then the algebra* $\mathscr{P}_\kappa(\kappa)$ *is* $(\omega, \cdot, (\mathrm{cf}(\kappa))^+)$-*nowhere distributive, hence* $h_\kappa = \omega$.

(ii) *If* $\kappa > \mathrm{cf}(\kappa) = \omega$, *then the algebra* $\mathscr{P}_\kappa(\kappa)$ *is* $(\omega_1, \cdot, 2^\omega)$-*nowhere distributive, hence* $h_\kappa = \omega_1$.

PROOF. Let us start with the following observation.

Claim. Let κ be an infinite cardinal, $X \in [\kappa]^\kappa$, $f: X \to \kappa$ a one-to-one mapping. Then there is a $Y \subseteq X$ and a one-to-one mapping $g: Y \to \kappa$ such that $|Y| = \kappa$ and $g(\xi) < f(\xi)$ for each $\xi \in Y$.

Indeed, let $\{\eta_i: i < \kappa\}$ be the strictly increasing enumeration of the set $f[X]$. It suffices to set

$$Y = \{\xi \in X: \text{for some } i < \kappa,\ f(\xi) = \eta_{i+1}\}$$

and define $g(\xi) = \eta_i$ whenever $f(\xi) = \eta_{i+1}$.

Having proved the claim, let us consider the case (i), $\mathrm{cf}(\kappa) > \omega$.

Using the claim and the maximality principle we shall construct, for each $n < \omega$, a maximal UAD family $\mathscr{A}_n$ on κ together with one-to-one mapping $f_A: A \to \kappa$ for each $A \in \mathscr{A}_n$, such that the following will be satisfied:

(a) $\mathscr{A}_0 = \{\kappa\}$, $f_\kappa = \mathrm{id}$;

(b) if $n < m < \omega$, then $\mathscr{A}_m$ is finer than $\mathscr{A}_n$, i.e. for each $B \in \mathscr{A}_m$ there is some $A \in \mathscr{A}_n$ with $B \subseteq A$;

(c) if $n < m < \omega$, $B \in \mathscr{A}_m$, $A \in \mathscr{A}_n$ and $B \subseteq A$, then for each $\xi \in B$, $f_B(\xi) < f_A(\xi)$.

Suppose we have constructed $\mathscr{A}_n$. Then by the claim, for each $A \in \mathscr{A}_n$ and $f_A: A \to \kappa$ there is some maximal UAD family $\mathscr{B}(A)$ on A together with one-to-one mappings $f_B: B \to \kappa$ for each $B \in \mathscr{B}(A)$ such that $f_B(\xi) < f_A(\xi)$ for each $\xi \in B$. The maximality of $\mathscr{B}(A)$ can be easily seen as follows. If $C \subseteq A$ is such that $|C| = \kappa$ and for each $B \in \mathscr{B}(A)$, $|C \cap B| < \kappa$, apply the claim to C and $f_A \restriction C$. Let $\mathscr{A}_{n+1} = \bigcup\{\mathscr{B}(A): A \in \mathscr{A}_n\}$.

We have to show that for each $X \in [\kappa]^\kappa$ there is some $n < \omega$ such that $|\{A \in \mathscr{A}_n\colon |X \cap A| = \kappa\}| \geq (\mathrm{cf}(\kappa))^+$.

Aiming for a contradiction, assume that $X \in [\kappa]^\kappa$ is a counterexample. According to 5.3, $E(\mathrm{cf}(\kappa))$ holds in the algebra $\mathscr{P}_\kappa(\kappa)$, therefore for each $n < \omega$ the size of the set $\mathscr{S}_n = \{A \in \mathscr{A}_n\colon |A \cap X| = \kappa\}$ is strictly smaller than $\mathrm{cf}(\kappa)$. Consequently, $|\bigcup \{A \cap B\colon A, B \in \mathscr{S}_n, A \neq B\}| < \kappa$. The maximality of $\mathscr{A}_n$ implies that $|X - \bigcup \mathscr{S}_n| < \kappa$, too. Making use of the fact that $\mathrm{cf}(\kappa) > \omega$ we obtain that the set $Z = (X \cap \bigcap_{n\in\omega} \bigcup \mathscr{S}_n) - \bigcup_{n\in\omega} \bigcup \{A \cap B\colon A, B \in \mathscr{S}_n, A \neq B\}$ is non-empty.

Pick $\xi \in Z$. Then for each $n \in \omega$ there is precisely one $A_n \in \mathscr{S}_n$ with $\xi \in A_n$ and these A_n's form a decreasing sequence. By (c), the corresponding mappings must satisfy $f_{A_0}(\xi) > f_{A_1}(\xi) > \cdots > f_{A_n}(\xi) > \cdots$. But this is an infinite decreasing sequence of ordinals, a contradiction.

Case (ii), $\kappa > \mathrm{cf}(\kappa) = \omega$.

Since $\mathscr{P}_\kappa(\kappa)$ is ω_1-closed, it is ω-distributive. Hence $h_\kappa \geq \omega_1$. Let us prove first $h_\kappa = \omega_1$.

By a transfinite induction to ω_1 we shall construct maximal UAD families $\mathscr{A}_\alpha$ together with one-to-one mappings $f_A\colon A \to \kappa$ for each $A \in \mathscr{A}_\alpha$, such that the following will hold.

(a) $\mathscr{A}_0 = \{\kappa\}$, $f_\kappa = \mathrm{id}$;

(b) if $\alpha < \beta < \omega_1$, then $\mathscr{A}_\beta$ is finer than $\mathscr{A}_\alpha$, i.e. for each $B \in \mathscr{A}_\beta$ there is some $A \in \mathscr{A}_\alpha$ with $B \subseteq^* A$ ($B \subseteq^* A$ means the obvious: $|B - A| < \kappa$);

(c) if $\alpha < \beta < \omega_1$, $B \in \mathscr{A}_\beta$, $A \in \mathscr{A}_\alpha$ and $B \subseteq^* A$, then for each $\xi \in A \cap B$, $f_B(\xi) < f_A(\xi)$.

Having constructed $\mathscr{A}_\alpha$, we proceed using the claim in the same manner as in case (i) in order to obtain $\mathscr{A}_{\alpha+1}$.

It remains to define $\mathscr{A}_\alpha$ for $\alpha < \omega_1$, α limit. Since $\mathscr{P}_\kappa(\kappa)$ is ω-distributive, there is some maximal UAD family $\mathscr{C}$ finer than all $\mathscr{A}_\beta$, $\beta < \alpha$; i.e. for each $C \in \mathscr{C}$ and each $\beta < \alpha$ there is some $A \in \mathscr{A}_\beta$ with $C \subseteq^* A$. For $C \in \mathscr{C}$, let us define $g_C\colon C \to \kappa$ by the rule $g_C(\xi) = \min\{f_A(\xi)\colon A \in \mathscr{A}_\beta$ is such that $C \subseteq^* A$ and $\xi \in A$, $\beta < \alpha\}$.

Then for each $i \in \kappa$, $g_C^{-1}(i) \subseteq \{f_A^{-1}(i)\colon C \subseteq^* A, A \in \mathscr{A}_\beta, \beta < \alpha\}$, therefore $|g_C^{-1}(i)| \leq \omega$, since α is a countable ordinal and each f_A is one-to-one. Thus, there is some $D \subseteq C$ such that $|D| = \kappa$ and $g_C \restriction D$ is one-to-one. So we may and shall assume that each mapping $g_C (C \in \mathscr{C})$ is one-to-one.

Now, passing from $\mathscr{C}$ to $\mathscr{A}_\alpha$ is quite analogous as from $\mathscr{A}_\beta$ to $\mathscr{A}_{\beta+1}$.

With the aid of the matrix $\{\mathscr{A}_\alpha\colon \alpha < \omega_1\}$ we shall show that $\mathscr{P}_\kappa(\kappa)$ is not $(\omega_1, \cdot, 2)$-distributive.

Suppose not. There is some $X \in [\kappa]^\kappa$ such that for each $\alpha < \omega_1$ there is some $A_\alpha \in \mathscr{A}_\alpha$ with $X \subseteq^* A_\alpha$. Let us denote $f_\alpha = f_{A_\alpha}$ and define for $\xi \in X$,

$$h(\xi) = \min\{f_\alpha(\xi)\colon \xi \in X \cap A_\alpha, \alpha < \omega_1\}\,.$$

For $\alpha < \omega_1$, let $X_\alpha = \{\xi \in X\colon h(\xi) = f_\alpha(\xi)\}$. Since $X = \bigcup_{\alpha<\omega_1} X_\alpha$, we have $|\bigcup_{\alpha<\omega_1} X_\alpha| = \kappa$. The cofinality of κ is countable, thus there must be some $\alpha < \omega_1$ with $|\bigcup_{\beta<\alpha} X_\beta| = \kappa$. Since we have $\bigcup_{\beta<\alpha} X_\beta \subseteq X \subseteq^* A_\alpha$, there is some $\beta < \alpha$ and $\xi \in X_\beta \cap A_\alpha$.

For this particular ξ, the definitions of a mapping h and of a set X_β imply:

$$f_\beta(\xi) = h(\xi) = \min\{f_\gamma(\xi): \xi \in X \cap A_\gamma, \gamma < \omega_1\} \leq f_\alpha(\xi).$$

But $\alpha > \beta$, hence by (c) we have also $f_\alpha(\xi) < f_\beta(\xi)$ – a contradiction.

We have proved that $\mathcal{P}_\kappa(\kappa)$ is not $(\omega_1, \cdot, 2)$-distributive. Since $\mathcal{P}_\kappa(\kappa)$ is homogeneous, it is $(\omega_1, \cdot, 2)$-nowhere distributive, too. Moreover, $\mathcal{P}_\kappa(\kappa)$ has an ω_1-closed dense subset. Therefore, by a standard branching argument, $\mathcal{P}_\kappa(\kappa)$ is $(\omega_1, \cdot, 2^\omega)$-nowhere distributive. $\square$

5.17. COROLLARY. (i) *If $\kappa > \omega$ is regular then $\mathcal{P}_\kappa(\kappa)$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive.*

(ii) *If $\kappa < 2^\omega$ is singular with countable cofinality, then $\mathcal{P}_\kappa(\kappa)$ is $(\omega_1, \cdot, \kappa^\omega)$-nowhere distributive.*

The non-distributivity of an algebra $\mathcal{P}_\kappa(\kappa)$ enables us to construct a base tree for κ analogously as it was done for ω (see 3.4, 3.6).

5.18. PROPOSITION. *Assume that $\kappa > \omega$ is a regular cardinal with $\kappa^+ = 2^\kappa$. Then there is a family $W \subseteq [\kappa]^\kappa$ such that*

(i) *if $A, B \in W$, then either $|A \cap B| < \kappa$ or $A \subseteq B$ or $B \subseteq A$;*

(ii) *for every $A \in W$, a set $\{B \in W: A \subseteq B\}$ is finite;*

(iii) *for each $X \in [\kappa]^\kappa$ there is an $A \in W$ with $A \subseteq X$.*

We call such a family W a base tree for κ.

PROOF. The assumption $\kappa^+ = 2^\kappa$ together with 5.17(i) enables us to apply 1.13. So $\mathcal{P}_\kappa(\kappa)$ has a dense subset D such that $(D, \geq)$ is a tree of height ω and each $d \in D$ has 2^κ immediate successors.

The desired family W is obtained from D by essentially the same trick as used in 3.6. $\square$

There is an open problem, whether it is provable in ZFC that $\mathcal{P}_\kappa(\kappa)$ is $(\omega, \cdot, \kappa^+)$- (resp. $(\omega_1, \cdot, \kappa^+)$-) nowhere distributive for each singular cardinal κ.

We shall show that this kind of non-distributivity holds under additional set-theoretical assumptions. The next theorem slightly generalizes a result from BALCAR and FRANĚK [1987].

5.19. THEOREM. (i) *Let κ be a singular cardinal with uncountable cofinality and assume that $2^{\mathrm{cf}(\kappa)} = (\mathrm{cf}(\kappa))^+$. Then $\mathcal{P}_\kappa(\kappa)$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive.*

(ii) *Let κ be a singular cardinal with countable cofinality and assume $h = \omega_1$. Then $\mathcal{P}_\kappa(\kappa)$ is $(\omega_1, \cdot, \kappa^+)$-nowhere distributive.*

PROOF. Denote $\lambda = \mathrm{cf}(\kappa)$. The cases $\lambda > \omega$ and $\lambda = \omega$ will be distinguished later.

Pick a strictly increasing sequence $\langle \kappa(i): i < \lambda \rangle$ of regular cardinals with $\kappa(0) = 0$, $\kappa(1) \geq \lambda$ converging to κ. Then $\{r(i): i < \lambda\}$, where $r(i) = \kappa(i) - \bigcup\{\kappa(\vartheta): \vartheta < i\}$ is a partition of κ.

If $X \in [\lambda]^\lambda$, let $X = \{X(\xi): \xi < \lambda\}$ be its one-to-one, strictly increasing enumeration.

Take a base tree W on λ; its existence follows by 5.18 for $\lambda > \omega$ and by 3.6 for $\lambda = \omega$. For $X \in W$, let

$$\mathcal{S}(X) = \{B \in [\kappa]^\kappa : B \subseteq \bigcup \{r(i) : i \in X\} \\ \text{and } (\forall \xi < \lambda)(|B \cap r(X(\xi))| < \kappa(\xi))\} .$$

For $\alpha < h_\lambda$, let W_α be the αth level of W. Choose a maximal UAD subfamily $\mathcal{A}_\alpha \subseteq \bigcup \{\mathcal{S}(X) : X \in W_\alpha\}$. Since all $\kappa(i)$'s are regular and since W_α is a maximal UAD family on λ, $\mathcal{A}_\alpha$ is a maximal UAD family on κ.

Claim. For each $C \in [\kappa]^\kappa$ there is an $X = \{X(\xi) : \xi < \lambda\} \in W$ such that

$$(\forall \xi < \lambda)(|C \cap r(X(\xi))| \geq \kappa(\xi)) .$$

Indeed, find by an induction on λ an increasing sequence $Y = \{Y(\xi) : \xi < \lambda\}$ of ordinals less than λ such that for each $\xi < \lambda$, $|C \cap r(Y(\xi))| \geq \kappa(\xi)$. Since $|C| = \kappa$, this is clearly possible. Then any $X \in W$ with $X \subseteq Y$ has the desired property.

Let $C \in [\kappa]^\kappa$ be arbitrary. Let $X \in W$ be the set from the claim. Then for each $\xi < \lambda$, $|C \cap r(X(\xi))| \geq \kappa(\xi)$, but for each $B \in \mathcal{S}(X)$, $|B \cap r(X(\xi))| < \kappa(\xi)$. Consequently, if $X \in W_\alpha$, then for this α, $|\{A \in \mathcal{A}_\alpha : |C \cap A| = \kappa\}| > \kappa$. Thus, we have proved that $\mathcal{P}_\kappa(\kappa)$ is $(h_\lambda, \cdot, \kappa^+)$-nowhere distributive. □

The previous considerations of non-distributivity of algebras $\mathcal{P}_\kappa(\kappa)$ give a simple characterization of a completion of $\mathcal{P}_\kappa(\kappa)$. The next theorem was proved for regular κ in BALCAR and VOPĚNKA [1972] (see also COMFORT and NEGREPONTIS [1974]). Recall that collapsing algebras were introduced in 1.9.

5.20. THEOREM. *Let κ be a cardinal number such that $\kappa^+ = 2^\kappa$, $(\mathrm{cf}(\kappa))^+ = 2^{\mathrm{cf}(\kappa)}$.*

(i) *If $\mathrm{cf}(\kappa) > \omega$, then the complete Boolean algebra $\overline{\mathcal{P}_\kappa(\kappa)}$ is isomorphic to* $\mathrm{Col}(\omega, 2^\kappa)$.

(ii) *If $\mathrm{cf}(\kappa) = \omega$, then $\overline{\mathcal{P}_\kappa(\kappa)}$ is isomorphic to* $\mathrm{Col}(\omega_1, 2^\kappa)$.

PROOF. Under the assumptions, the algebra $\overline{\mathcal{P}_\kappa(\kappa)}$ has a dense subset of size 2^κ and is $(\omega, \cdot, 2^\kappa)$- (resp. $(\omega_1, \cdot, 2^\kappa)$-) nowhere distributive. In the case (ii), the dense set is ω_1-closed. Thus, 1.15 applies. □

5.21. REMARK. The ideal $[\kappa]^{<\kappa}$ for a regular uncountable cardinal κ is κ-complete. If one considers an arbitrary κ-complete ideal $J \supseteq [\kappa]^{<\kappa}$, then it need not be true that $\mathcal{P}(\kappa)/[\kappa]^{<\kappa} \cong \mathcal{P}(\kappa)/J$. For instance, if $J = \mathcal{NS}$, the ideal of all non-stationary subsets of κ, then $\mathcal{P}(\kappa)/J$ does not satisfy $E(\kappa)$, whereas $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ does. Nevertheless, the completions can be isomorphic. We shall not give the proof of the next theorem; the reader may find it in BALCAR and FRANĚK [1987].

5.22. THEOREM. *Assume $V = L$, let κ be a regular uncountable cardinal, J a κ-complete ideal on κ, $J \supseteq [\kappa]^{<\kappa}$. Then the completion $\overline{\mathcal{P}(\kappa)/J}$ is isomorphic to* $\mathrm{Col}(\omega, 2^\kappa)$.

5.23. THE CARDINAL CHARACTERISTICS b_κ. For a regular cardinal κ, let us define

$$b_\kappa = \min\{|H|: H \subseteq {}^\kappa\kappa \text{ and } H \text{ has no upper bound under } \leq^*\} .$$

The preorder $\leq^*$ was defined in 2.10. Of course, $b_\omega = b$ as defined in 3.13. It is immediate to check that $b_\kappa > \kappa$ and b_κ is a regular cardinal.

If κ is uncountable and regular, then b_κ is connected with the properties of closed unbounded subsets in κ.

5.24. PROPOSITION. *Let $\kappa > \omega$ be a regular cardinal. Then there is a family $\{f_\alpha: \alpha < b_\kappa\} \subseteq {}^\kappa\kappa$ such that*

(i) *$\{f_\alpha: \alpha < b_\kappa\}$ has no upper bound*;

(ii) *each f_α is strictly increasing and continuous*;

(iii) *if $\alpha < \beta < b_\kappa$, then $f_\alpha \leq^* f_\beta$.*

Furthermore, each family of size less than b_κ consisting of continuous functions has an upper bound which is continuous, too.

PROOF. Let $\{h_\alpha: \alpha < b_\kappa\} \subseteq {}^\kappa\kappa$ be a family of functions without an upper bound. By a straightforward induction one can obtain functions g_α, $\alpha < b_\kappa$, such that each g_α is strictly increasing and each g_α is an upper bound for $\{g_\beta: \beta < \alpha\} \cup \{h_\beta: \beta \leq \alpha\}$. Clearly, the set $\{g_\alpha: \alpha < b_\kappa\}$ has no upper bound, too.

Now, for each $\alpha < b_\kappa$ modify g_α to reach a continuous f_α as follows: $f_\alpha(\xi) = g_\alpha(\xi)$ whenever $\xi < \kappa$ is not limit, $f_\alpha(\xi) = \sup\{f_\alpha(\eta): \eta < \xi\}$ otherwise.

Again we have $f_\alpha \leq^* f_\beta$ whenever $\alpha < \beta < b_\kappa$ and (ii), (iii) hold true. We shall verify that the family $\{f_\alpha: \alpha < b_\kappa\}$ has no upper bound.

Suppose not, let $f \in {}^\kappa\kappa$ be an upper bound for $\{f_\alpha: \alpha < b_\kappa\}$. We can assume that f is strictly increasing. For $\xi < \kappa$, define $g(\xi) = f(\xi + 1)$. Then g is an upper bound for $\{g_\alpha: \alpha < b_\kappa\}$ – a contradiction.

The rest of the proposition is clear. □

5.25. THEOREM. *Let $\kappa > \omega$ be a regular cardinal. Then b_κ is the largest cardinal λ satisfying the following.*

If $\mathscr{C} \subseteq \mathscr{P}(\kappa)$ is a family of closed unbounded subsets of κ with $|\mathscr{C}| < \lambda$, then there exists a closed unbounded set A such that for each $C \in \mathscr{C}$, $|A - C| < \kappa$.

PROOF. First, let $\tau < b_\kappa$ and let $\{C_\alpha: \alpha < \tau\}$ be a family of closed unbounded subsets of κ. We need to find a set A with the properties of theorem.

For $\alpha < \tau$ define $f_\alpha \in {}^\kappa\kappa$ by $f_\alpha(\xi) = \min\{\eta \in C_\alpha: \eta \geq \xi\}$. Since C_α is closed, f_α is continuous. Since $\tau < b_\kappa$, there is an upper bound $g \in {}^\kappa\kappa$ for $\{f_\alpha: \alpha < \tau\}$ and g may be chosen continuous by the previous 5.24. We shall show that $A = \{\xi \in \kappa: g(\xi) = \xi\}$ is as required.

Clearly, A is closed unbounded, for g is continuous and strictly increasing. Fix $\alpha < \tau$. Since $f_\alpha \leq^* g$, there is some $\eta < \kappa$ such that $f_\alpha(\xi) \leq g(\xi)$ for all $\xi \geq \eta$. Now, for each $\xi \geq \eta$ with $\xi \in A$ we have $\xi = g(\xi) \geq f_\alpha(\xi) \geq \xi$, thus $f_\alpha(\xi) = \xi$, therefore $\xi \in C_\alpha$. So $A - C_\alpha \subseteq \eta$.

Next, let us construct a family $\{C_\alpha: \alpha < b_\kappa\}$ of closed unbounded subsets of κ such that for no $X \in [\kappa]^\kappa$ we could have $|X - C_\alpha| < \kappa$ for all $\alpha < b_\kappa$.

Let $\{f_\alpha: \alpha < b_\kappa\}$ be a family from 5.24; set $C_\alpha = \{\xi < \kappa: f_\alpha(\xi) = \xi\}$. By the continuity, each C_α is closed unbounded. Choose $X \in [\kappa]^\kappa$ and suppose that for each $\alpha < b_\kappa$ there is some $\xi(\alpha) < \kappa$ with $X - C_\alpha \subseteq \xi(\alpha)$. Define $g(\xi) = \min\{\eta \in X: \eta \geq \xi\}$. Then $X = \{\xi \in \kappa: g(\xi) = \xi\}$ and $g(\xi) > \xi$ for each $\xi \in \kappa - X$. To obtain the desired contradiction, we shall show that g is an upper bound for all f_α's. Fix $\alpha < b_\kappa$. Then for $\xi > \xi(\alpha)$ we have $f_\alpha(\xi) = \xi \leq g(\xi)$ whenever $\xi \in C_\alpha$ and $\xi < f_\alpha(\xi) \leq f_\alpha(g(\xi)) = g(\xi)$ whenever $\xi \not\in C_\alpha$. Thus, $f_\alpha \leq^* g$. □

5.26. NOTATION. For $M \subseteq \kappa$, $\kappa \geq \omega$, denote by $[M]$ the element of $\mathcal{P}_\kappa(\kappa)$ determined by M, i.e. $[M] = \{Z \subseteq \kappa: |Z \triangle M| < \kappa\}$.

5.27. COROLLARY. *For a regular uncountable κ there is a family $\{C_\alpha: \alpha < b_\kappa\}$ of closed unbounded subsets of κ such that $\{[C_\alpha]: \alpha < b_\kappa\}$ is a tower in $\mathcal{P}_\kappa(\kappa)$.*

5.28. DECOMPOSABILITY IN THE COMPLETION OF $\mathcal{P}_\kappa(\kappa)$. The complete Boolean algebra $\overline{\mathcal{P}_\kappa(\kappa)}$ need not have the property Rfip(2^κ); there may even exist a centred system of size 2^κ which is not 2-decomposable. Suppose, for example, $\kappa = \omega_1$, $2^{\omega_1} = \omega_2$. Then $\overline{\mathcal{P}_{\omega_1}(\omega_1)}$ is isomorphic to $\overline{\text{Col}(\omega, \omega_2)}$ by 5.20(i), $\pi(\text{Col}(\omega, \omega_2)) = \omega_2$, so by 2.5 there is an ultrafilter in $\overline{\mathcal{P}_{\omega_1}(\omega_1)}$ generated by ω_2 elements.

The situation is different if one considers only those systems in $\overline{\mathcal{P}_\kappa(\kappa)}$ which are contained in $\mathcal{P}_\kappa(\kappa)$. The following example shows one such case.

5.29. PROPOSITION. *Let $\kappa > \omega$ be a regular cardinal. Then the family of all stationary subsets of κ is strongly b_κ-decomposable in $\overline{\mathcal{P}_\kappa(\kappa)}$.*

PROOF. By 5.27, there is a tower $\{c_\alpha: \alpha < b_\kappa\}$ in $\mathcal{P}_\kappa(\kappa)$ such that each $c_\alpha = [C_\alpha]$ with C_α closed unbounded in κ. The property $E(\kappa)$ holds in $\mathcal{P}_\kappa(\kappa)$ (see 5.3); therefore in $\overline{\mathcal{P}_\kappa(\kappa)}$, for each $\alpha < b_\kappa$ such that $\text{cf}(\alpha) = \kappa$, the element $b_\alpha = \Pi_{\xi<\alpha} c_\xi - c_\alpha$ is non-zero. Moreover, if $A \subseteq \kappa$ is stationary, then the set $\{\alpha < b_\kappa: \text{cf}(\alpha) = \kappa \ \&\ b_\alpha \cdot [A] \neq \mathbb{O}\}$ is κ-closed unbounded in b_κ. Fix a partition $\{I_i: i < b_\kappa\}$ of the set $\{\alpha < b_\kappa: \text{cf}(\alpha) = \kappa\}$ into b_κ stationary sets. The existence of such a partition follows by Solovay's theorem (see the Appendix on Set Theory, in this Handbook). The system $\{a_i: i < b_\kappa\}$, where $a_i = \Sigma\{b_\alpha: \alpha \in I_i\}$ shows the strong b_κ-decomposability of a family $\{[A]: A \subseteq \kappa \text{ is stationary}\}$. □

The next theorem, which generalizes the theorem of PRIKRY [1974], is proved in BALCAR and SIMON [1982].

5.30. THEOREM. *Let $\kappa > \omega$ be a regular cardinal. Then every ultrafilter in $\mathcal{P}_\kappa(\kappa)$ is strongly b_κ-decomposable in $\overline{\mathcal{P}_\kappa(\kappa)}$.*

Prikry's result immediately follows.

5.31. THEOREM. *If $\kappa > \omega$ is a regular cardinal and $2^\kappa = \kappa^+$, then every uniform ultrafilter on κ has a uniform almost disjoint refinement.*

5.32. DECOMPOSABILITY OF ULTRAFILTERS IN THE POWER SET ALGEBRA. We know that no ultrafilter on κ is strongly τ-decomposable for any $\tau \geq 2$. Trivially, each uniform ultrafilter on κ is κ-decomposable – the singletons provide the desired partition. Let us consider the situation $\omega \leq \tau < \kappa$, $\mathscr{U}$ a uniform ultrafilter on κ. What are the conditions guaranteeing the τ-decomposability of $\mathscr{U}$ in $\mathscr{P}(\kappa)$? Let us discuss it now.

Our first theorem on this subject deals with one special case.

5.33. THEOREM (K. Kunen, K. Prikry). *If κ is a regular cardinal, then each uniform ultrafilter on κ^+ is κ-decomposable in $\mathscr{P}(\kappa^+)$.*

PROOF. The proof makes use of Ulam matrices. A Ulam matrix on κ^+ is a family $\{A(\alpha, \beta)\colon \alpha < \kappa^+, \beta < \kappa\}$ of subsets of κ^+ such that

(i) for every $\alpha < \kappa^+$, $\{A(\alpha, \beta)\colon \beta < \kappa\}$ is a disjoint family;
(ii) for every $\alpha < \kappa^+$, $|\kappa^+ - \bigcup \{A(\alpha, \beta)\colon \beta < \kappa\}| \leq \kappa$;
(iii) for every $\beta < \kappa$, $\{A(\alpha, \beta)\colon \alpha < \kappa^+\}$ is a disjoint family.

Recall the classical proof of the existence of a Ulam matrix on κ^+. For each $\xi < \kappa^+$, $\xi \geq \kappa$, fix a one-to-one mapping f_ξ of κ onto ξ. It suffices to set $A(\alpha, \beta) = \{\xi\colon f_\xi(\beta) = \alpha\}$.

Take an arbitrary uniform ultrafilter $\mathscr{U}$ on κ^+. Two cases are possible.

It may happen that there is some $\alpha < \kappa^+$ such that for each $\beta < \kappa$, $\bigcup \{A(\alpha, \gamma)\colon \gamma > \beta\} \in \mathscr{U}$. Since κ is regular, the family $\{A(\alpha, \beta)\colon \beta < \kappa\}$ is a κ-decomposition of $\mathscr{U}$.

It remains to discuss the other case. For each $\alpha < \kappa^+$ there is some $g(\alpha) < \kappa$ such that $U_\alpha = \bigcup \{A(\alpha, \beta)\colon \beta < g(\alpha)\} \in \mathscr{U}$. Since g maps κ^+ into κ, there is a set $I \subseteq \kappa^+$ such that $|I| = \kappa^+$ and g is constant on I with value γ.

We claim that for each $X \subseteq I$ with $|X| \geq \kappa$ the intersection $\bigcap \{U_\alpha\colon \alpha \in X\}$ is empty. Suppose not, let $\xi \in \bigcap \{U_\alpha\colon \alpha \in X\}$. Since each row in a Ulam matrix is disjoint, for each $\alpha \in X$ there is a unique $\beta_\alpha < \gamma$ such that $\xi \in A(\alpha, \beta_\alpha)$. But each column of a Ulam matrix is disjoint, too, hence the map $\alpha \mapsto \beta_\alpha$ is a one-to-one mapping from X into γ. Since $|\gamma| < \kappa \leq |X|$, this is a contradiction.

Thus, we have showed that in this case, the ultrafilter $\mathscr{U}$ is (κ, κ^+)-regular and the rest of the proof as well as the definition of regularity can be found below (5.34, 5.36). □

5.34. DEFINITION. Let $\kappa \leq \lambda$ be infinite cardinals, S an infinite set. A family $\mathscr{A} \subseteq \mathscr{P}(S)$ is called (κ, λ)-regular, if there is a subfamily $\mathscr{B} \subseteq \mathscr{A}$ such that $|\mathscr{B}| = \lambda$ and for each $\mathscr{B}' \subseteq \mathscr{B}$, if $|\mathscr{B}'| \geq \kappa$, then $\bigcap \mathscr{B}' = \emptyset$.

5.35. REMARK. Notice that if $\kappa \leq \tau \leq \nu \leq \lambda$ and $\mathscr{A}$ is (κ, λ)-regular, then $\mathscr{A}$ is (τ, ν)-regular as well.

5.36. PROPOSITION. *If a uniform ultrafilter $\mathscr{U}$ on a set S is (κ, κ)-regular and κ is a regular cardinal, then $\mathscr{U}$ is κ-decomposable.*

PROOF. Let $\{U_\alpha: \alpha < \kappa\} \subseteq \mathcal{U}$ be the family witnessing to the (κ, κ)-regularity of $\mathcal{U}$. For $\xi < \kappa$ let $S_\xi = \{x \in S: \sup\{\alpha < \kappa: x \in U_\alpha\} = \xi\}$. We shall show that $\{S_\xi: \xi < \kappa\}$ κ-decomposes $\mathcal{U}$.

Immediately from the definition one concludes that $\{S_\xi: \xi < \kappa\}$ is disjoint. Pick a $U \in \mathcal{U}$. We must show that $|\{\xi < \kappa: U \cap S_\xi \neq \emptyset\}| = \kappa$. Since $\mathcal{U}$ is an ultrafilter, $U \cap U_{\alpha+1} \neq \emptyset$. Fix $x \in U \cap U_{\alpha+1}$. Then $x \not\in \bigcup \{S_\xi: \xi \leq \alpha\}$. The set $\{\beta < \kappa: x \in U_\beta\}$ is non-void, because it contains $\alpha + 1$, and its size is smaller than κ by (κ, κ)-regularity of $\mathcal{U}$. Since the cardinal κ is regular, $\sup\{\beta < \kappa: x \in U_\beta\} < \kappa$. So $x \in S_\xi$ for some $\xi > \alpha$. Hence, we see that the set $\{\xi < \kappa: U \cap S_\xi \neq \emptyset\}$ is cofinal in κ and again by the regularity of κ, $\mathcal{U}$ is κ-decomposable. □

The techniques used in the previous two proofs are strong enough to give a more general result, which is due to ČUDNOVSKIJ and ČUDNOVSKIJ [1971] and KUNEN and PRIKRY [1971].

5.37. THEOREM. *Let λ be an infinite cardinal, $\mathcal{U}$ an ultrafilter on a set X and suppose that $\mathcal{U}$ is λ^+-decomposable. Then*

(i) *If λ is regular, then $\mathcal{U}$ is λ-decomposable.*

(ii) *If λ is singular, then either*

(a) *$\mathcal{U}$ is $\mathrm{cf}(\lambda)$-decomposable, or*

(b) *there is some $\tau < \lambda$ such that $\mathcal{U}$ is (τ, λ^+)-regular. Therefore, $\mathcal{U}$ is μ-decomposable for each regular μ, $\tau \leq \mu < \lambda$.*

5.38. CONCLUDING REMARKS

(a) Except the trivial case of κ-decomposability in $\mathcal{P}(\kappa)$ of a uniform ultrafilter on κ one cannot prove anything more in general. Consider a κ-complete uniform ultrafilter $\mathcal{U}$ on a measurable cardinal κ. Obviously, $\mathcal{U}$ is not τ-decomposable for any $1 < \tau < \kappa$ in $\mathcal{P}(\kappa)$.

(b) On the other hand, if $\mathcal{U}$ is a uniform (ω, κ)-regular ultrafilter on a cardinal κ, then $\mathcal{U}$ is τ-decomposable in $\mathcal{P}(\kappa)$ for all τ, $\omega \leq \tau < \kappa$. It is well known that on each infinite κ there exists an (ω, κ)-regular ultrafilter.

(c) Notice that 5.33 is, in fact, a recurrent property. Therefore for each $n < \omega$, each uniform ultrafilter on ω_n is τ-decomposable for each τ, $\omega \leq \tau \leq \omega_n$. Thus, the first case where one may expect the failure of decomposability is a uniform ultrafilter on ω_ω. H. Woodin showed the consistency of "There exists a uniform ultrafilter $\mathcal{U}$ on ω_ω such that $\mathcal{U}$ is not ω_n-decomposable for all n, $1 \leq n < \omega$", relatively to large cardinals.

More detailed information concerning indecomposable ultrafilters can be found in KANAMORI [1986].

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Added in proof: Quite recently, the first author answered the problem preceding 5.19 in the affirmative.

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Section B

ALGEBRAIC PROPERTIES OF BOOLEAN ALGEBRAS

This Section consists of nine chapters which go into more detail than Part I concerning certain algebraic properties of Boolean algebras:

Chapter 10, Subalgebras, by Robert Bonnet, characterizes the lattice of subalgebras, and surveys results on complements and quasi-complements in this lattice.

Chapter 11, Cardinal functions on Boolean spaces, by Eric K. van Douwen, surveys important cardinal valued functions defined on Boolean spaces, giving proofs for many of the results mentioned.

Chapter 12, The number of Boolean algebras, by J. Donald Monk, gives several easy constructions of many non-isomorphic Boolean algebras, and as an illustration of the general methods of S. Shelah for these purposes, one more complicated construction.

Chapter 13, Endomorphisms of Boolean algebras, by J. Donald Monk, first shows that isomorphism of the endomorphism semigroups implies isomorphism of the BAs themselves (reconstruction). A brief section on the number of endomorphisms serves to formulate some problems. An endo-rigid algebra is then constructed. The method used for this purpose is then applied to describe some results about Hopfian Boolean algebras.

Chapter 14, Automorphism groups, by J. Donald Monk, gives some general information about this topic, which can serve as background for several later chapters (Bonnet's chapter on rigid algebras, Rubin's chapter on reconstruction, the chapter by Štěpánek and Rubin on homogeneous algebras, and Štěpánek's chapter on embeddings and automorphisms), although this chapter is not required in order to read those. The present chapter gives some general facts about automorphisms and direct product decompositions, relationships between automorphisms and extensions of Boolean algebras, and a discussion of the size of automorphism groups.

Chapter 15, Reconstruction of Boolean algebras from their automorphism groups, by Matatyahu Rubin, gives an exhaustive description of when such reconstruction is possible, and also some indications of generalizations of the methods outside the domain of Boolean algebras.

Chapter 16, Embeddings and automorphisms, by Petr Štěpánek, gives a construction of rigid complete Boolean algebras, and treats several questions concerning embedding algebras into rigid complete ones, and concerning the existence of Boolean algebras without rigid or homogeneous factors.

Chapter 17, Rigid Boolean algebras, by Mohamed Bekkali and Robert Bonnet, surveys the various constructions of rigid Boolean algebras, and gives several in full detail: the original Jónsson construction, Bonnet's construction, Todorčević's construction, and Jech's construction of a simple complete BA.

Chapter 18, Homogeneous Boolean algebras, by Petr Štěpánek and Matatyahu Rubin, gives several important facts about such algebras: that every complete weakly homogeneous algebra is a power of a homogeneous algebra, that infinite free algebras (which are homogeneous) have simple automorphism groups, and that, under CH, there is an infinite homogeneous algebra with non-simple automorphism group.

CHAPTER 10

Subalgebras

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Contents

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HANDBOOK OF BOOLEAN ALGEBRA
Edited by J.D. Monk, with R. Bonnet

0. Introduction

In this chapter we will develop the theory of the lattice $\mathrm{Sub}(\boldsymbol{B})$ of all subalgebras of a given algebra $\boldsymbol{B}$. The ordering $\leq$ on $\boldsymbol{L} = \mathrm{Sub}(\boldsymbol{B})$ is defined by $\boldsymbol{B}_1 \leq \boldsymbol{B}_2$ if and only if $\boldsymbol{B}_1 \subseteq \boldsymbol{B}_2$.

We should remark that $\boldsymbol{L}$ has a smallest element $\underline{\mathbf{2}} = \{\mathbf{0}, \mathbf{1}\}$ and a greatest element $\boldsymbol{B}$. Moreover, $\boldsymbol{L}$ is a complete lattice, with:

$$\bigwedge \{\boldsymbol{B}_i \colon i \in I\} = \bigcap \{\boldsymbol{B}_i \colon i \in I\} ,$$

and

$$\bigvee \{\boldsymbol{B}_i \colon i \in I\} = \left\langle \bigcup \{\boldsymbol{B}_i \colon i \in I\} \right\rangle$$

where $\bigvee \{\boldsymbol{B}_i \colon i \in I\}$ is the subalgebra of $\boldsymbol{B}$ generated by $\bigcup \{\boldsymbol{B}_i \colon i \in I\}$.

Recall that an element a of a lattice $\boldsymbol{L}$ is a compact element whenever for $\boldsymbol{X} \subseteq \boldsymbol{L}$, if $a \leq \bigvee \boldsymbol{X}$, then $a \leq \bigvee \boldsymbol{F}$, where $\boldsymbol{F}$ is some finite subset of $\boldsymbol{X}$. We denote by $\boldsymbol{K}(\boldsymbol{L})$ the set of compact elements of $\boldsymbol{L}$. Obviously, if $a, b \in \boldsymbol{K}(\boldsymbol{L})$, then $a \vee b \in \boldsymbol{K}(\boldsymbol{L})$. A lattice $\boldsymbol{L}$ is said to be *algebraic* if $x = \bigvee \{a \in \boldsymbol{K}(\boldsymbol{L}) \colon a \leq x\}$ for all $x \in \boldsymbol{L}$ (for developments of this notion, see BIRKHOFF [1948]).

Let $\boldsymbol{B}$ be a Boolean algebra and $\boldsymbol{L} = \mathrm{Sub}(\boldsymbol{B})$. Obviously, $\boldsymbol{B}' \in \boldsymbol{K}(\boldsymbol{L})$ if and only if $\boldsymbol{B}'$ is a finite subalgebra of $\boldsymbol{B}$. Consequently, $\mathrm{Sub}(\boldsymbol{B})$ is a complete algebraic lattice.

Let $\boldsymbol{L}$ be a lattice with a smallest element $\underline{\mathbf{0}}$ and a greatest element $\underline{\mathbf{1}}$. Let $a \in \boldsymbol{L}$. We say that $b \in \boldsymbol{L}$ is a *complement* of a in $\boldsymbol{L}$ if $a \wedge b = \underline{\mathbf{0}}$ and $a \vee b = \underline{\mathbf{1}}$ (see BIRKHOFF [1948]). Notice the symmetry of this notion. Call b a *quasi-complement* of a if $a \wedge b = \underline{\mathbf{0}}$ and there is no $c > b$ such that $a \wedge c = \underline{\mathbf{0}}$. Obviously, if $\boldsymbol{A}$ and $\boldsymbol{C}$ are subalgebras of $\boldsymbol{B}$, then $\boldsymbol{C}$ is a quasi-complement of $\boldsymbol{A}$ if and only if $\boldsymbol{A} \cap \boldsymbol{C} = \underline{\mathbf{2}}$, and $\boldsymbol{A} \cap \langle \boldsymbol{C} \cup \{x\} \rangle \neq \underline{\mathbf{2}}$, for every x in $\boldsymbol{B} \setminus \boldsymbol{C}$. So, in the complete lattice $\mathrm{Sub}(\boldsymbol{B})$ that the quasi-complement $\boldsymbol{C}$ of an element $\boldsymbol{A}$ exists, follows from Zorn's lemma. REMMEL [1980] has shown that $\boldsymbol{C}$ may be a quasi-complement of $\boldsymbol{A}$ without $\boldsymbol{A}$ being a quasi-complement of $\boldsymbol{C}$.

This chapter is divided into three sections.

Section 1 develops a characterization of the lattice $\mathrm{Sub}(\boldsymbol{B})$. The fact that $\mathrm{Sub}(\boldsymbol{B})$ is an algebraic lattice in which the compact elements are the finite algebras was proved by BIRKHOFF and FRINK [1948]; BIRKHOFF [1948] also observed that if $\boldsymbol{A}$ is the finite Boolean algebra with 2^n elements, then $\mathrm{Sub}(\boldsymbol{A})$ is dually isomorphic to the partition lattice $\mathcal{P}_n$ of a set with n elements. (This means there is a one-to-one function g from $\mathrm{Sub}(\boldsymbol{A})$ onto $\mathcal{P}_n$ satisfying $\boldsymbol{C}_1 \subseteq \boldsymbol{C}_2$ if and only if $g(\boldsymbol{C}_1) \geq g(\boldsymbol{C}_2)$.) In summary, $\boldsymbol{L} = \mathrm{Sub}(\boldsymbol{B})$ satisfies the following property: $(\boldsymbol{\sigma})$ $\boldsymbol{L}$ is an algebraic lattice and for each compact element $a \in \boldsymbol{L}$, $(-\infty, a] = \{x \in \boldsymbol{L} \colon x \leq a\}$ is dually isomorphic to a finite partition lattice. GRÄTZER, KOH and MAKKAI [1972] showed that if $\boldsymbol{L}$ is a lattice satisfying $(\boldsymbol{\sigma})$, then $\boldsymbol{L}$ is order-isomorphic to $\mathrm{Sub}(\boldsymbol{B})$ for some Boolean algebra which is unique up to isomorphism. The uniqueness of $\boldsymbol{B}$ had been independently proved earlier by SACHS [1962].

At the end of this section, we also develop some results due to Sachs concerning subalgebras generated by ideals.

Sections 2 and 3 are concerned with notions of complementation and of quasi-complementation in the lattice of subalgebras of a given Boolean algebra.

Section 2 relates the property that Sub($\boldsymbol{B}$) is complemented to the property that $\boldsymbol{B}$ is retractive. A lattice $\boldsymbol{L}$ is said to be *complemented* if each element has a complement. The Boolean algebra $\boldsymbol{B}$ is said to be *retractive* if, for every ideal $\boldsymbol{J}$ of $\boldsymbol{B}$, there is a subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$, such that $\boldsymbol{A}$ contains exactly one element from each equivalence class $x/\boldsymbol{J}$ for $x \in \boldsymbol{B}$. RAO and RAO [1979] have shown that if Sub($\boldsymbol{B}$) is complemented, then $\boldsymbol{B}$ is retractive (take $\boldsymbol{A}$ to be a complement of the subalgebra, $\langle \boldsymbol{J} \rangle$, of $\boldsymbol{B}$ generated by $\boldsymbol{J}$). REMMEL [1980] and, independently, JECH [1982] proved that if $\boldsymbol{B}$ is a countable algebra, then Sub($\boldsymbol{B}$) is complemented. MOSTOWSKI and TARSKI [1939] and, independently, ROTMAN [1972] showed that if $\boldsymbol{B}$ is an interval algebra, then $\boldsymbol{B}$ is retractive. The converse of this result is not true (we make this explicit below). More generally, Rubin in 1977, and published in 1983 (RUBIN [1983]), proved that every subalgebra $\boldsymbol{B}$ of an interval algebra is retractive. TODORČEVIĆ [1980] noticed that Rubin's proof could be modified to give the stronger result that, if $\boldsymbol{B}$ is a subalgebra of an interval algebra, then the lattice Sub($\boldsymbol{B}$) is complemented. These results suggest a strong connection between complementation and retractiveness, and raise the question whether for every retractive algebra $\boldsymbol{B}$, Sub($\boldsymbol{B}$) is complemented. RUBIN [1983], has proved, assuming Jensen's principle ($\Diamond$), that there is an algebra $\boldsymbol{B}$, of cardinality ω_1, such that:

(1) $\boldsymbol{B}$ is not embeddable in an interval algebra;

(2) $\boldsymbol{B}$ is retractive.

And TODORČEVIĆ [1980] has shown that $\boldsymbol{B}$ satisfies:

(3) Sub($\boldsymbol{B}$) is not complemented.

An interesting open question here is whether the fact that Sub($\boldsymbol{B}$) is complemented implies that $\boldsymbol{B}$ is embeddable in an interval algebra, and (weaker) whether that Sub($\boldsymbol{B}$) is complemented implies that Sub($\boldsymbol{A}$) is complemented for every subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$.

In the same part, we give some results of DÜNTSCH and KOPPELBERG [1985] about those subalgebras of $\boldsymbol{B} = \mathfrak{P}(\omega)$ which have a complement in Sub($\boldsymbol{B}$).

Section 3 concerns quasi-complements in Sub($\boldsymbol{B}$). The first study of quasi-complements was done by REMMEL [1980], and more recently by DÜNTSCH and KOPPELBERG [1985].

Section 4 is a survey on the congruence structure of the lattice Sub($\boldsymbol{B}$), developed by DÜNTSCH [1985a], [1985b].

We shall use the following notation: $f[X]$ and $f^{-1}[X]$ are, respectively, the image and the preimage of X under a function f. $a = a_1 \dotplus \cdots \dotplus a_n$ means $a = a_1 + \cdots + a_n$ and the a_i's are pairwise disjoint, and $a \triangle b$ denotes the symmetric difference of a and b. If $\boldsymbol{B}$ is an algebra, we denote by At($\boldsymbol{B}$) the set of atoms of $\boldsymbol{B}$. If $a \in \boldsymbol{B}$, then $\boldsymbol{B} \restriction a$ is the relative algebra of $\{x \in \boldsymbol{B}: x \leq a\}$. Such $\boldsymbol{B} \restriction a$ are called *factors* of $\boldsymbol{B}$. Recall that if X is a subset of $\boldsymbol{B}$, then $\langle X \rangle$ denotes the subalgebra generated by X. If $x \in \boldsymbol{B}$, then we write $\langle x \rangle$ instead of $\langle \{x\} \rangle$. Also, if $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$ and $u \not\in \boldsymbol{A}$, then we write $\boldsymbol{A}(u)$ for $\langle \boldsymbol{A} \cup \{u\} \rangle$. So $\underline{2}(u) = \langle u \rangle$. Finally, if X is a set, then $\mathfrak{P}(X)$ denotes the power set, viewed as a Boolean algebra, and $\boldsymbol{F}_c(X)$ is the algebra of finite and cofinite subsets of X.

1. Characterization of the lattice of subalgebras of a Boolean algebra

1.0. Let $\boldsymbol{B}$ be a finite algebra with 2^n elements. We identify $\boldsymbol{B}$ with the power set algebra $\mathfrak{P}(\underline{n})$, where $\underline{n} = \{\underline{0}, \underline{1}, \ldots, \underline{n-1}\}$. In this case the atoms of $\boldsymbol{B}$ are the singletons $\{\underline{i}\}$ for $i < n$.

Recall that a partition π defines an equivalence relation $\equiv_\pi$ on $\underline{n}$, where $i \equiv_\pi j$ if and only if $i, j \in c$, for some (unique) $c \in \pi$. We deliberately confuse the notation referring to π as a partition or the associated equivalence relation as required by the context. Moreover, if π' and π'' are partitions of $\underline{n}$, we set $\pi' \leq \pi''$ if $\pi' \subseteq \pi''$ as relations. In this way we obtain a natural dual isomorphism between $\mathrm{Sub}(\boldsymbol{B})$ and the set of partitions of $\underline{n}$ in associating with $\boldsymbol{C}$ the partition π_C determined by the atoms of $\boldsymbol{C}$.

Consequently (see BIRKHOFF [1948]): *If $\boldsymbol{B}$ is a finite algebra having 2^n elements, then* $\mathrm{Sub}(\boldsymbol{B})$ *is dually isomorphic to the partition lattice of the (finite) set* $\underline{n}$.

So if $\boldsymbol{B}$ is a finite algebra, then:

(1) $\mathrm{Sub}(\boldsymbol{B})$ is simple. Recall that a *congruence* on a lattice $\boldsymbol{L}$ is an equivalence relation on $\boldsymbol{L}$ compatible with $\wedge$ and $\vee$, and a lattice $\boldsymbol{L}$ is *simple* whenever $\boldsymbol{L}$ has only two (trivial) congruences;

(2) $\mathrm{Sub}(\boldsymbol{B})$ is a complemented lattice;

(3) $\mathrm{Sub}(\boldsymbol{B})$ is distributive if and only if $\boldsymbol{B}$ has at most 2^3 elements (RAO and RAO [1979]);

(4) If $\boldsymbol{B}$ has at least 2^4 elements, then $\mathrm{Sub}(\boldsymbol{B})$ is not semi-modular (DÜNTSCH [1985a]). Recall that a lattice $\boldsymbol{L}$ is *semi-modular* if $a \neq b$ and both a and b cover c in $\boldsymbol{L}$, then $a \vee b$ covers a and b (or equivalently if $a \neq b$ and, c covers both a and b in $\boldsymbol{L}$, then a and b cover $a \wedge b$).

We begin with some simple observations.

1.1. OBSERVATION. If either a or $-a$ is an atom of $\boldsymbol{B}$, then the subalgebra $\langle a \rangle = \{\boldsymbol{0}, a, -a, \boldsymbol{1}\}$ has exactly $n-1$ complements in $\mathrm{Sub}(\boldsymbol{B})$.

PROOF. If a is an atom, say $a = \{0\}$, then $\langle a \rangle$ has exactly $n-1$ complements in $\mathrm{sub}(\boldsymbol{B})$, namely the subalgebras $\boldsymbol{C}_i = \langle \{0, i\}, \{1\}, \{2\}, \ldots, \{i-1\}, \{i+1\}, \ldots, \{n-1\} \rangle$, for $0 < i < n$. □

The following result is not valid, if $|\mathrm{At}(\boldsymbol{B})| \leq 3$, i.e. $|\boldsymbol{B}| \leq 8$.

1.2. OBSERVATION. Let $b \in \boldsymbol{B}$, $b \neq \boldsymbol{0}, \boldsymbol{1}$. If neither b nor $-b$ is an atom of $\boldsymbol{B}$, then $\langle b \rangle$ has at least n complements in $\mathrm{Sub}(\boldsymbol{B})$.

PROOF. Indeed, using the above notation, there is no loss in assuming $b = \{0, \ldots, p-1\}$ and $-b = \{p, \ldots, n-1\}$. We have $p \geq 2$ and $n - p \geq 2$. For $0 \leq k < p$ and $p \leq l < n$ let $\boldsymbol{C}(k, l)$ be the subalgebra of $\boldsymbol{B}$, generated by $\{k, l\}$ and the set of $\{i\}$, for $0 \leq i < n$, $i \neq k$ and $i \neq l$. Obviously, each $\boldsymbol{C}(k, l)$ is a complement of $\langle b \rangle$ in $\mathrm{Sub}(\boldsymbol{B})$ and the number of these is $p(n-p)$. Now it is obvious that $p(n-p) \geq 2(n-2) \geq n$, for $p, n-p \geq 2$. □

From Observations 1.1 and 1.2, it follows immediately that:

1.3. OBSERVATION. Let $\boldsymbol{B}$ be a finite algebra having at least 4 elements. Let n be the smallest integer p such that there is a 4-element subalgebra of $\boldsymbol{B}$, with exactly $p-1$ complements in $\mathrm{Sub}(\boldsymbol{B})$. Then $\boldsymbol{B}$ has 2^n elements.

1.4. Let $\square(\boldsymbol{A}, \boldsymbol{B}, f)$ denote the following assertion: $\boldsymbol{A}$ and $\boldsymbol{B}$ are finite algebras, $|A| \geq 2^3$, and f is a one-to-one lattice homomorphism from $\mathrm{Sub}(A)$ onto a principal ideal of $\mathrm{Sub}(\boldsymbol{B})$.

We will prove that such an f determines a unique embedding ψ from $\boldsymbol{A}$ into $\boldsymbol{B}$ (Lemma 1.8).

Since $\mathrm{Sub}(A)$ and $\mathrm{Sub}(\boldsymbol{B})$ are finite lattices, and $\mathrm{Im}(f) = f[\mathrm{Sub}(A)]$ is a principal ideal, it follows that $\mathrm{Im}(f)$ is the ideal of $\mathrm{Sub}(\boldsymbol{B})$ generated by $f(A) \in \mathrm{Sub}(\boldsymbol{B})$, and so $\mathrm{Im}(f) = \mathrm{Sub}(f(A))$.

Notice that $\underline{2}$ has no proper subalgebra, and a 4-element algebra has only two subalgebras. Consequently:

1.5. OBSERVATION. If $\square(\boldsymbol{A}, \boldsymbol{B}, f)$ holds, then:
(P1) $f(\underline{2}) = \underline{2}$.
(P2) The image of a 4-element Boolean subalgebra of $\boldsymbol{A}$ is a 4-element subalgebra of $f(A)$.

1.6. LEMMA. *If $\square(\boldsymbol{A}, \boldsymbol{B}, f)$ holds, then for each atom p of $\boldsymbol{A}$ there is a unique atom $\psi(p)$ of the subalgebra $f(A)$ of $\boldsymbol{B}$ such that $f(\langle p \rangle) = \langle \psi(p) \rangle$. Moreover, ψ is a one-to-one function from the atoms of $\boldsymbol{A}$ onto the atoms of $f(A)$.*

PROOF. $\boldsymbol{B}$ has at least 8 elements, since f is one-to-one and $|\mathrm{Sub}(A)| \geq 8$. From $p \neq \mathbf{0}$, $\mathbf{1}$ and from observation 1.5, it follows that $f(\langle p \rangle)$ is a 4-element algebra. The existence of $\psi(p)$ follows from observations 1.1 through 1.3 and the fact that in $f(A)$ there is no element which is an atom and a coatom of $f(A)$. Now assume p and q are distinct atoms of $\boldsymbol{A}$ and $\psi(p) = \psi(q)$. The subalgebra $\langle p \rangle \vee \langle q \rangle$ of $\boldsymbol{A}$ has 3 atoms p, q and $1-(p+q)$ and thus has a non-trivial subalgebra. In $\mathrm{Sub}(\boldsymbol{B})$, we have:

$$f(\langle p \rangle \vee \langle q \rangle) = f(\langle p \rangle) \vee f(\langle q \rangle) = \langle \psi(p) \rangle$$

which has 4 elements and so has only trivial subalgebras, which contradicts $\square(\boldsymbol{A}, \boldsymbol{B}, f)$. Now let b be an atom of $f(A)$. There is $x \in \boldsymbol{A}$ such that $f(\langle x \rangle) = \langle b \rangle$. From observations 1.1 through 1.3, it follows that x is an atom of $\boldsymbol{A}$. $\square$

1.7. Suppose $\square(\boldsymbol{A}, \boldsymbol{B}, f)$ holds. For $c \in \boldsymbol{A}$, we denote by $\psi(c)$ the finite sum of $\psi(a)$ for $a \leq c$ and a an atom of $\boldsymbol{A}$. Notice that $\psi(c) \in f(A)$. Moreover, $\psi(\mathbf{0}) = \mathbf{0}$ by definition, and $\psi(\mathbf{1}) = \mathbf{1}$ follows from Lemma 1.6, and thus ψ is a Boolean homomorphism from $\boldsymbol{A}$ onto $f(A) \subseteq \boldsymbol{B}$.

1.8. LEMMA. *Assume $\square(\boldsymbol{A}, \boldsymbol{B}, f)$ holds. Then there is a one-to-one homomorphism ψ from $\boldsymbol{A}$ into $\boldsymbol{B}$ satisfying:*
(1) *for each $c \in \boldsymbol{A}$, we have $f(\langle c \rangle) = \langle \psi(c) \rangle$;*

(2) *if $\boldsymbol{D}$ is a subalgebra of $\boldsymbol{A}$, then $f(\boldsymbol{D})$, considered as a subalgebra of $\boldsymbol{B}$, is the image, $\psi[\boldsymbol{D}]$, of $\boldsymbol{D}$ under the one-to-one homomorphism ψ from $\boldsymbol{A}$ into $\boldsymbol{B}$.*

(3) *Moreover, ψ is uniquely determined by f.*

PROOF. We assume $\boldsymbol{A} = \mathfrak{P}(X)$ and $\boldsymbol{B} = \mathfrak{P}(Y)$. Notice that (1) holds for $c = \mathbf{0}$ or $\mathbf{1}$. Indeed, $f(\langle \mathbf{0}\rangle) = f(\{\mathbf{0}, \mathbf{1}\}) = \{\mathbf{0}, \mathbf{1}\} = \langle \mathbf{0}\rangle$, and thus we can set $\psi(\mathbf{0}) = \mathbf{0}$, and similarly $f(\langle \mathbf{1}\rangle) = \{\mathbf{0}, \mathbf{1}\} = \langle \mathbf{1}\rangle = \langle \psi(\mathbf{1})\rangle$. Hence, if $d \in \boldsymbol{A}$, then $d \subseteq \boldsymbol{X}$. Now if $c \in \boldsymbol{A}$, we define $\psi(c)$ as the supremum in $\boldsymbol{B}$ of $\psi(a)$ for all atoms a of $\boldsymbol{A}$ satisfying $a \leq c$. Assume for a contradiction that there is c in $\boldsymbol{A}$ such that $f(\langle c\rangle) \neq \langle \psi(c)\rangle$. There is no loss in assuming that the cardinality, $|c|$, of c is minimal. We have $|c| > 0$ since $c \neq \mathbf{0}, \mathbf{1}$, and so $|c| \geq 2$ since c is not an atom. We write $c = p \dotplus u = v \dotplus q = p \dotplus w \dotplus q$, where p, q are distinct atoms of $\boldsymbol{A}$ (and so $p \cdot u = q \cdot v = p \cdot w = q \cdot w = \mathbf{0}$).

Case 1. $w = 0$, i.e. $|c| = 2$. We have $c = p + q \neq \mathbf{1}$. We set $f(\langle c\rangle) = \{\mathbf{0}, b, -b, \mathbf{1}\}$, where $b \in f(A)$. We have $\langle b\rangle \neq \langle \psi(c)\rangle$. Then $\psi(p) + \psi(q) = \psi(c) \neq b$ and $\psi(c) \neq -b$. There is no loss in assuming that either: (i) $\psi(p) \leq b$ and $\psi(q) \leq -b$, (ii) $\psi(q) \leq b$ and $\psi(p) \leq -b$, or (iii) $\psi(p) + \psi(q) < b$. *Assume* (i) *holds*. First, we have $\psi(p) < b$ (and similarly $\psi(q) < -b$). Indeed, assume that $\psi(p) = b$ and b is an atom of $f(A)$. Consequently, $f(\langle c\rangle) = \langle b\rangle = \langle \psi(p)\rangle = f(\langle p\rangle)$ and thus $\langle c\rangle = \langle p\rangle$. So $c = p$ or $c = 1 - p$, which is a contradiction since $c = p + q$, which proves $\psi(p) < b$ and $\psi(q) < -b$. Now the subalgebra $\langle \psi(p), \psi(q), b\rangle$ of $f(A)$, has 4 atoms: $\psi(p)$, $\psi(q)$, $b - \psi(p)$, and $-b \cdot -\psi(q)$. But $\langle p, q, c\rangle$ has only 3 atoms p, q, and $-c$ and this gives a contradiction since $f(\langle p, q, c\rangle) = \langle \psi(p), \psi(q), b\rangle$. Case (ii) is handled similarly. Now *assume* (iii) *holds*. The algebra $\langle \psi(p), \psi(q), b\rangle$ has 4 atoms $\psi(p)$, $\psi(q)$, $b - (\psi(p) + \psi(q))$, and $-b$. But $\langle p, q, c\rangle$ has only the atoms p, q and $-c$, and this gives a contradiction since $f(\langle p, q, c\rangle) = \langle \psi(p), \psi(q), b\rangle$.

Case 2. $w \neq 0$, i.e. $|c| > 2$. We set $\boldsymbol{B}_0 = \langle \psi(p), \psi(q), \psi(w)\rangle$. The algebra $\boldsymbol{B}_0$ has 4 atoms, namely $\psi(p)$, $\psi(q)$, $\psi(w)$ and $d = \mathbf{1} - (\psi(p) + \psi(q) + \psi(w))$. Otherwise, $d = \mathbf{0}$, i.e. $\psi(p) + \psi(q) + \psi(w) = \mathbf{1}$. Consequently, $\psi(w) \in \langle \psi(p) + \psi(q)\rangle = \langle \psi(t)\rangle = f(\langle t\rangle)$, where $t = p + q$. So $\psi(w) = \psi(t)$ or $\psi(w) = -\psi(t)$. In the first case, $w = t$ follows from the definition of ψ, but this is impossible since $p \cdot w = \mathbf{0}$ and thus $t \cdot w = \mathbf{0}$. On the other hand, if $\psi(w) = -\psi(t)$, then $w = \mathbf{1} - t = \mathbf{1} - (p + q)$ and $c = w + (p + q) = \mathbf{1}$, contrary to our assumption. In $\boldsymbol{A}$, we have: $\langle c\rangle = (\langle u\rangle \vee \langle p\rangle) \cap (\langle v\rangle \vee \langle q\rangle)$. Consequently, from the minimality of $|c|$ and the properties of f, it follows that:

$$f(\langle c\rangle) = (f(\langle u\rangle) \vee f(\langle p\rangle)) \cap (f(\langle v\rangle) \vee f(\langle q\rangle)) = \boldsymbol{B}' \cap \boldsymbol{B}'',$$

where $\boldsymbol{B}' = \langle \psi(u)\rangle \vee \langle \psi(p)\rangle$ and $\boldsymbol{B}'' = \langle \psi(v)\rangle \vee \langle \psi(q)\rangle$ are subalgebras of $\boldsymbol{B}$, contained in $\boldsymbol{B}_0$. The algebra $\boldsymbol{B}'$ (resp. $\boldsymbol{B}''$) has 3 atoms: $\psi(p)$, $\psi(u)$ and d (resp. $\psi(q)$, $\psi(v)$ and d). Consequently,

$$f(\langle c\rangle) = \boldsymbol{B}' \cap \boldsymbol{B}'' = \langle d\rangle = \langle \psi(p) + \psi(q) + \psi(w)\rangle .$$

From the definition of ψ, it follows that $\psi(c) = \psi(p) + \psi(q) + \psi(w)$ and thus $f(\langle c\rangle) = \langle \psi(c)\rangle$, concluding the proof of (1).

Now we will prove (2). Let $\boldsymbol{D}$ be a subalgebra of $\boldsymbol{A}$. We will prove that $f(\boldsymbol{D}) = \psi[\boldsymbol{D}]$. First $\psi[\boldsymbol{D}] \subseteq f(\boldsymbol{D})$, since if $x \in \boldsymbol{D}$, then

$$\psi(x) \in \langle \psi(x) \rangle = f(\langle x \rangle) \subseteq f(\boldsymbol{D}) .$$

Conversely, assume $y \in f(\boldsymbol{D})$. Then $\langle y \rangle \subseteq f(\boldsymbol{D})$. From $\Box(\boldsymbol{A}, \boldsymbol{B}, f)$, $\mathrm{Im}(f)$ is an ideal of $\mathrm{Sub}(\boldsymbol{B})$, and so it follows that $\langle y \rangle \subseteq f(\boldsymbol{D}) \subseteq f(\boldsymbol{A})$. Consequently, $\langle y \rangle = f(\langle x \rangle)$ for some $x \in \boldsymbol{A}$. From $f(\langle x \rangle) = \langle y \rangle \subseteq f(\boldsymbol{D})$, and the fact that f is one-to-one and increasing, it follows that $\langle x \rangle \subseteq \boldsymbol{D}$ and thus $x \in \boldsymbol{D}$. From (1), it follows that

$$y \in \langle y \rangle = f(\langle x \rangle) = \langle \psi(x) \rangle \subseteq \psi[\boldsymbol{D}] ,$$

which proves (2).

Finally, we prove (3). Suppose that ϕ is an embedding from $\boldsymbol{A}$ into $\boldsymbol{B}$ such that $\phi(a)$ is an atom of $f(\boldsymbol{A})$ for each atom a of $\boldsymbol{A}$ and $f(\langle a \rangle) = \langle \phi(a) \rangle$. Then obviously $\psi(a) = \phi(a)$, because there is no atom which is also a coatom. Consequently, $\phi = \psi$, since ϕ is a Boolean homomorphism and $\boldsymbol{A}$ is an atomic finite algebra. $\Box$

1.9. Let $\boldsymbol{L}$ be a lattice. Recall that $\boldsymbol{K}(\boldsymbol{L})$ denotes the set of compact elements of $\boldsymbol{L}$, and recall that $\boldsymbol{L}$ satisfies the property $(\boldsymbol{\sigma})$ whenever:

(1) *$\boldsymbol{L}$ is an algebraic lattice*;

(2) *if $a \in \boldsymbol{K}(\boldsymbol{L})$, then $(-\infty, a]$ is dually-order-isomorphic to a finite partition lattice, i.e. $(-\infty, a]$ is order-isomorphic to* $\mathrm{Sub}(\boldsymbol{B}_a)$ *for some finite Boolean algebra $\boldsymbol{B}_a$.*

Notice that if $\boldsymbol{B}$ is a Boolean algebra, then $\mathrm{Sub}(\boldsymbol{B})$ satisfies $(\boldsymbol{\sigma})$.

Let $\boldsymbol{L}$ be a lattice satisfying $(\boldsymbol{\sigma})$. For $a \in \boldsymbol{K}(\boldsymbol{L})$, let α_a be an order-isomorphism from $(-\infty, a]$ onto $\mathrm{Sub}(\boldsymbol{B}_a)$, for some finite algebra $\boldsymbol{B}_a$. For $a \le b$ in $\boldsymbol{K}(\boldsymbol{L})$, the inclusion $i_{b,a}$ from $(-\infty, a]$ into $(-\infty, b]$ defines a unique one-to-one lattice homomorphism $f_{b,a}$ from $\mathrm{Sub}(\boldsymbol{B}_a)$ into $\mathrm{Sub}(\boldsymbol{B}_b)$ such that the following diagram is commutative:

$$\begin{tikzcd}
(-\infty, a] \arrow[r, "\alpha_a"] \arrow[d, "i_{b,a}"'] & \mathrm{Sub}(\boldsymbol{B}_a) \arrow[d, "f_{b,a}"] \\
(-\infty, b] \arrow[r, "\alpha_b"'] & \mathrm{Sub}(\boldsymbol{B}_b)
\end{tikzcd}$$

and thus $\Box(\boldsymbol{B}_a, \boldsymbol{B}_b, f_{b,a})$ holds if $|\boldsymbol{B}_a| \ge 8$. Also, $f_{c,a} = f_{c,b} \circ f_{b,a}$ for $a \le b \le c$ in $\boldsymbol{K}(\boldsymbol{L})$.

Now, with the preceding notations, we state the following theorem due to GRÄTZER, KOH and MAKKAI [1972]:

1.10. THEOREM. *Let $\boldsymbol{L}$ be a lattice satisfying $(\boldsymbol{\sigma})$. Then, there is a directed system $\langle \boldsymbol{B}_a, \psi_{b,a} \rangle$ for a, b in $\boldsymbol{K}(\boldsymbol{L})$ such that*:

(1) $\psi_{b,a}$ *induces* $f_{b,a}$, *i.e.* $\psi_{b,a}[C]=f_{b,a}(C)$ *for every* $C\in \mathrm{Sub}(B_a)$;

(2) *if* $a, b\in K(L)$, *with* $a\le b$, *and if* B_a *has at least* 8 *elements, then* $\psi_{b,a}$ *is uniquely determined by* $f_{b,a}$;

(3) L *is order-isomorphic to* $\mathrm{Sub}(B)$, *where* B *is the directed limit of* $\langle B_a, \psi_{b,a}\rangle$, *for* $a\le b$ *in* $K(L)$.

PROOF. In the following a, $\hat{a}$, b, $c, \ldots$ denote elements of $K(L)$. It follows from the commutativity of the diagram that $f_{a,a}$ is the identity on $\mathrm{Sub}(B_a)$ and, if $a\le b$, then $f_{b,a}$ is a one-to-one lattice homomorphism and the image of $\mathrm{Sub}(B_a)$ under $f_{b,a}$ is a principal ideal of $\mathrm{Sub}(B_b)$, since $(-\infty, a]$ is a principal ideal of $(-\infty, b]$. Notice that if $B_a=\{\mathbf{0},\mathbf{1}\}$, then $\psi(\mathbf{0})=\mathbf{0}$ and $\psi(\mathbf{1})=\mathbf{1}$. Now if $a\le b$ and $|B_a|\ne 4$, then $\Box(B_a, B_b, f_{b,a})$ holds. From Lemma 1.8 it follows that there is a unique one-to-one homomorphism $\psi_{b,a}$ from B_a into B_b satisfying $\psi_{b,a}[C]=f_{b,a}(C)$ for $C\in\mathrm{Sub}(B_a)$.

Now we will consider the case when a *is of height* 1, that means B_a is a 4-element algebra. In this case $\psi_{b,a}$ is not necessarily unique (since B_a has an element which is both an atom and a coatom). First notice that if there is no $b>a$ in $K(L)$, then L is isomorphic to $\mathrm{Sub}(B_a)$, and we are done. Now assume there is $b>a$ in $K(L)$. For each $c\in K(L)$ satisfying $c>a$, we must define $\psi_{c,a}$ such that $\psi_{d,a}=\psi_{d,c}\circ\psi_{c,a}$ whenever $a<c<d$ in $K(L)$. For this, we choose an immediate successor $\hat{a}$ of a in $K(L)$. So $|B_{\hat{a}}|=8$. Now we choose a one-to-one homomorphism $\psi_{\hat{a},a}$ from B_a into $B_{\hat{a}}$ which induces $f_{\hat{a},a}$: in fact, there are only two possibilities for $\psi_{\hat{a},a}$. Indeed, if p is an atom and a coatom of B_a, then $f_{\hat{a},a}(\langle p\rangle)=\{\mathbf{0}, x, -x, \mathbf{1}\}$ and thus we can define $\psi_{\hat{a},a}(p)=x$ or $\psi_{\hat{a},a}(p)=-x$. Now, via $\psi_{\hat{a},a}$, we will define $\psi_{b,a}$ for $b\ge a$ in $K(L)$. Let $c=b\vee\hat{a}$ and we define $\psi_{b,a}$ such that the following diagram is commutative:

$$\begin{tikzcd}
B_a \arrow[r, "\psi_{\hat{a},a}"] \arrow[d, "\psi_{b,a}"'] & B_{\hat{a}} \arrow[d, "\psi_{c,\hat{a}}"] \\
B_b \arrow[r, "\psi_{c,b}"'] & B_c
\end{tikzcd}$$

We must verify that $\psi_{b,a}$ is well defined. For this, we remark that:

$$D=\psi_{c,\hat{a}}\circ\psi_{\hat{a},a}[B_a]=f_{c,a}(B_a)=\alpha_c(a)\subseteq B_c$$

and

$$E=\psi_{c,b}[B_b]=f_{c,b}(B_b)=\alpha_c(b)\subseteq B_c\,,$$

hence $D\subseteq E$ and thus $\psi_{c,b}^{-1}$ is defined on E. Obviously, $\psi_{b,a}$ induces $f_{b,a}$. For $a\le b\le c$ we have $\psi_{c,a}=\psi_{c,b}\circ\psi_{b,a}$ follows from the definition of the $\psi_{u,v}$'s.

Now recall that $a\vee b\in K(L)$ for a, $b\in K(L)$. Let B be the directed limit of $\langle B_a, \psi_{b,a}\rangle$ for $a\le b$ in $K(L)$, and let ψ_a be the homomorphism from B_a into B. Each ψ_a is one-to-one since every $\psi_{b,a}$ is.

Now, we will show that $a \le b$ if and only if $\psi_a[\boldsymbol{B}_a] \subseteq \psi_b[\boldsymbol{B}_b]$. Let a, b be given and $c \ge a, b$. From the hypothesis we have:

$$\alpha_c(a) = f_{c,a}(\boldsymbol{B}_a) = \psi_{c,a}[\boldsymbol{B}_a]$$

and

$$\alpha_c(b) = \psi_{c,b}[\boldsymbol{B}_b] .$$

Now the following properties are equivalent: (α) $a \le b$, (β) $\alpha_c(a) \le \alpha_c(b)$, ($\gamma$) $\psi_{c,a}[\boldsymbol{B}_a] \subseteq \psi_{c,b}[\boldsymbol{B}_a]$, ($\delta$) $\psi_c \circ \psi_{c,a}[\boldsymbol{B}_a] \subseteq \psi_c \circ \psi_{c,b}[\boldsymbol{B}_b]$, and ($\varepsilon$) $\psi_a[\boldsymbol{B}_a] \subseteq \psi_b[\boldsymbol{B}_b]$.

Consequently, the function f from $\boldsymbol{K}(\boldsymbol{L})$ into $\mathrm{Sub}(\boldsymbol{B})$ defined by $f(a) = \psi_a[\boldsymbol{B}_a]$ is an isomorphism from $\boldsymbol{K}(\boldsymbol{L})$ into the ordered subset $\boldsymbol{K}(\mathrm{Sub}(\boldsymbol{B}))$ of compact elements of $\mathrm{Sub}(\boldsymbol{B})$. From the definition of $\boldsymbol{B}$, it follows that $f[\boldsymbol{K}(\boldsymbol{L})]$ is cofinal in $\boldsymbol{K}(\mathrm{Sub}(\boldsymbol{B}))$, i.e. for every $\boldsymbol{A}$ in $\boldsymbol{K}(\mathrm{Sub}(\boldsymbol{B}))$, there is $b \in \boldsymbol{K}(\boldsymbol{L})$ such that $\boldsymbol{A} \subseteq f(b) = \psi_b[\boldsymbol{B}_b]$. So there is $a \le b$ such that $f(a) = \psi_a[\boldsymbol{B}_a] = \boldsymbol{A}$.

Finally, f is a one-to-one isomorphism from $\boldsymbol{K}(\boldsymbol{L})$ onto $\boldsymbol{K}(\mathrm{Sub}(\boldsymbol{B}))$, and since $\boldsymbol{L}$ and $\mathrm{Sub}(\boldsymbol{B})$ are algebraic lattices, f can be extended to an isomorphism from $\boldsymbol{L}$ onto $\mathrm{Sub}(\boldsymbol{B})$, and the proof is complete. □

We now give two corollaries. The first follows directly from Theorem 1.10, the second is a result due to SACHS [1962].

1.11. COROLLARY 1. *The lattice* $\mathrm{Sub}(\boldsymbol{B})$ *of all subalgebras of a Boolean algebra* $\boldsymbol{B}$ *is characterized as a lattice satisfying* ($\boldsymbol{\sigma}$).

1.12. COROLLARY 2. *The lattice* $\boldsymbol{L} = \mathrm{Sub}(\boldsymbol{B})$ *determines the Boolean algebra* $\boldsymbol{B}$ *up to isomorphism.*

PROOF. Let us say that $\langle \boldsymbol{B}_a, \psi_{b,a} \rangle$ for $a \le b$ in $\boldsymbol{K}(\boldsymbol{L})$ is associated with $\boldsymbol{L}$, if this family satisfies the conclusion of the theorem for some isomorphism α_a from $(-\infty, a]$ onto $\mathrm{Sub}(\boldsymbol{B}_a)$, for $a \in \boldsymbol{K}(\boldsymbol{L})$. From the uniqueness part of the theorem, it follows that two directed families associated with $\boldsymbol{L}$, and thus their directed limits also, are isomorphic. Now we remark that if $\boldsymbol{L} = \mathrm{Sub}(\boldsymbol{B})$ for some algebra $\boldsymbol{B}$, then $\boldsymbol{L}$ satisfies ($\boldsymbol{\sigma}$). Consequently, if $\boldsymbol{L}$ is isomorphic to $\mathrm{Sub}(\boldsymbol{B})$, then some directed family associated with $\boldsymbol{L}$ has a directed limit isomorphic to $\boldsymbol{B}$. Consequently, if $\mathrm{Sub}(\boldsymbol{A})$ and $\mathrm{Sub}(\boldsymbol{B})$ are isomorphic lattices, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are isomorphic algebras. □

1.13. SACHS [1962] characterizes $\mathrm{Sub}(\boldsymbol{B})$ as being dually isomorphic to a certain subsystem of a partition lattice. Moreover, he gives some properties of $\mathrm{Sub}(\boldsymbol{B})$, that we will develop.

DEFINITION. $\boldsymbol{C}$ is a *dual atom* in $\mathrm{Sub}(\boldsymbol{B})$, or $\boldsymbol{C}$ is a *maximal subalgebra* of $\boldsymbol{B}$ whenever $\boldsymbol{C}$ is a proper subalgebra of $\boldsymbol{B}$, and there is no proper subalgebra between $\boldsymbol{C}$ and $\boldsymbol{B}$, and $\mathrm{Sub}(\boldsymbol{B})$ is *dually atomic* whenever every proper subalgebra is contained in a dual atom in $\mathrm{Sub}(\boldsymbol{B})$. $\boldsymbol{C}$ is a *dual subalgebra* whenever there is an ideal $\boldsymbol{I}$ of $\boldsymbol{B}$ such that $\boldsymbol{C}$ is the subalgebra of $\boldsymbol{B}$, generated by $\boldsymbol{I}$, i.e. $\boldsymbol{C}$ is the subalgebra of $\boldsymbol{B}$ consisting of $x \in \boldsymbol{B}$ such that $x \in \boldsymbol{I}$ or $-x \in \boldsymbol{I}$.

1.14. We will begin to construct a dual atom $\boldsymbol{C}$ in Sub($\boldsymbol{B}$).

1.14.1. Let $\boldsymbol{B}$ be an algebra, and $a \in \boldsymbol{B}$, with $a \neq \mathbf{0}$, $\mathbf{1}$. Let us begin to associate with $\langle a \rangle$ a dual atom $\boldsymbol{C}$ in Sub($\boldsymbol{B}$) such that $\langle a \rangle \cap \boldsymbol{C} = \underline{2}$.

Let $\boldsymbol{U}'$ and $\boldsymbol{U}''$ be maximal ideals of $\boldsymbol{B}$, satisfying $a \in \boldsymbol{U}'$ and $-a \in \boldsymbol{U}''$. Let $\boldsymbol{U} = \boldsymbol{U}' \cap \boldsymbol{U}''$ and $\boldsymbol{C}$ be the subalgebra of $\boldsymbol{B}$, generated by $\boldsymbol{U}$, i.e. $\boldsymbol{C}$ is the subalgebra of $\boldsymbol{B}$ consisting of $x \in \boldsymbol{B}$ such that $x \in \boldsymbol{U}$ or $-x \in \boldsymbol{U}$. Trivially, $\langle a \rangle \cap \boldsymbol{C} = \underline{2}$. We will prove that $\boldsymbol{C}(z) = \boldsymbol{B}$, for every $z \in \boldsymbol{B} \backslash \boldsymbol{C}$. First, we will show that $\boldsymbol{C}(a) = \boldsymbol{B}$. To see this, let $x \notin \boldsymbol{C}$. There is no loss in assuming that $x \in \boldsymbol{U}'$ and $x \notin \boldsymbol{U}''$. We have $x = (x \cdot a) + (x \cdot -a)$. First, $x \cdot -a \in \boldsymbol{C}$ follows from $x \cdot -a \in \boldsymbol{U}' \cap \boldsymbol{U}''$. Now, we will prove that $x \cdot a \in \boldsymbol{C}(a)$. To see this, we have $x \cdot a = (x \cdot a + -a) \cdot a$. It is sufficient to show that $u = x \cdot a + -a \in \boldsymbol{C}$. But this is trivial since $-u = a \cdot -x \in \boldsymbol{U}' \cap \boldsymbol{U}''$. Now, let $z \in \boldsymbol{B} \backslash \boldsymbol{C}$. We will prove that $\boldsymbol{C}(z) = \boldsymbol{B}$. It is sufficient to show that $a \in \boldsymbol{C}(z)$. We can assume $z \in \boldsymbol{U}'$ and $z \notin \boldsymbol{U}''$. We have $a = (z \cdot a) + (a \cdot -z)$. Trivially, $a \cdot -z \in \boldsymbol{U}' \cap \boldsymbol{U}''$. Now, we will prove that $z \cdot a \in \boldsymbol{C}(z)$. To see this, we have $z \cdot a = (z \cdot a + -z) \cdot z$. It is sufficient to show that $u = z \cdot a + -z \in \boldsymbol{C}$. But this is trivial since $-u = -a \cdot z \in \boldsymbol{U}' \cap \boldsymbol{U}''$.

1.14.2. Conversely, let $\boldsymbol{U}'$ and $\boldsymbol{U}''$ be distinct maximal ideals of $\boldsymbol{B}$, and $a \in \boldsymbol{B}$, with $a \in \boldsymbol{U}'$ and $-a \in \boldsymbol{U}''$. Then the subalgebra $\boldsymbol{C}$ of $\boldsymbol{B}$, generated by $\boldsymbol{U} = \boldsymbol{U}' \cap \boldsymbol{U}''$, is a dual atom $\boldsymbol{C}$ in Sub($\boldsymbol{B}$) such that $\langle a \rangle \cap \boldsymbol{C} = \underline{2}$.

Moreover in 1.18, we show that every dual atom is under this form.

1.15. LEMMA. *Let $\boldsymbol{A}$ be a subalgebra of $\boldsymbol{B}$, which is not maximal. Let $\boldsymbol{D}$ be a proper subalgebra of $\boldsymbol{B}$, properly containing $\boldsymbol{A}$. Then $\boldsymbol{A}$ is contained in a dual subalgebra $\boldsymbol{C}$ distinct from $\boldsymbol{A}$ which does not contain $\boldsymbol{D}$.*

PROOF. Let $z \in \boldsymbol{D} \backslash \boldsymbol{A}$. Let $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be the ideals of $\boldsymbol{A}$ of $x \in \boldsymbol{A}$ such that $x \cdot z = \mathbf{0}$ and $x \cdot -z = \mathbf{0}$, respectively. Obviously, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are proper ideals of $\boldsymbol{A}$, because $\mathbf{1} \in \boldsymbol{A}$. The ideal $\boldsymbol{I}$ of $\boldsymbol{A}$ generated by $\boldsymbol{I}_1 \cup \boldsymbol{I}_2$, is proper. Indeed, if $\mathbf{1} = x + y$, with $x \in \boldsymbol{I}_1$, and $y \in \boldsymbol{I}_2$, then $z = z \cdot y$, i.e. $z \leq y$; moreover, $y \cdot -z = \mathbf{0}$ implies $y \leq z$, and thus $y = z$, which contradicts $z \notin \boldsymbol{A}$. Let $\boldsymbol{J}$ be a maximal ideal of $\boldsymbol{A}$ containing $\boldsymbol{I}$, and $\boldsymbol{K}$ be the ideal of $\boldsymbol{B}$ generated by $\boldsymbol{J}$, i.e. $t \in \boldsymbol{K}$ if and only if $t \leq u$ for some $u \in \boldsymbol{J}$. We claim that $z \notin \boldsymbol{K}$ and $-z \notin \boldsymbol{K}$ (and so $\boldsymbol{K}$ is not a maximal ideal). By contradiction, let us suppose that $z \in \boldsymbol{K}$ (resp. $-z \in \boldsymbol{K}$). Then $z \leq u$ (resp. $-z \leq u$) for some $u \in \boldsymbol{J}$, and therefore $z \cdot -u = \mathbf{0}$ (resp. $-z \cdot -u = \mathbf{0}$). Since $-u \in \boldsymbol{A}$, the last equation implies that $-u \in \boldsymbol{I} \subseteq \boldsymbol{J}$, and this is impossible, since $u + -u = \mathbf{1} \notin \boldsymbol{J}$. Now the subalgebra $\boldsymbol{C}$ of $\boldsymbol{B}$, generated by $\boldsymbol{K}$, satisfies the conclusion of the lemma. $\square$

1.16. COMMENT. The above proof shows that *every dual atom in* Sub($\boldsymbol{B}$) *is a dual subalgebra of $\boldsymbol{B}$*. Indeed, let $\boldsymbol{A}$ be a dual atom in Sub($\boldsymbol{B}$). Using the same notations, $\boldsymbol{A}$ is the Boolean algebra generated by the ideal $\boldsymbol{K}$ of $\boldsymbol{B}$, since z, $-z \notin \boldsymbol{K}$.

1.17. LEMMA. *Every dual subalgebra of $\boldsymbol{B}$ is the intersection of maximal dual subalgebras of $\boldsymbol{B}$.*

The proof is a consequence of the fact that every ideal of $\boldsymbol{B}$ is the intersection of maximal ideals of $\boldsymbol{B}$.

1.18. COROLLARY. *Every dual atom of* Sub($\boldsymbol{B}$) *is the Boolean algebra generated by the intersection of two distinct maximal ideals of* $\boldsymbol{B}$.

PROOF. It is a direct consequence of 1.16, 1.17 and the following fact: let $\boldsymbol{I}$ and $\boldsymbol{J}$ be ideals of $\boldsymbol{B}$. Then $\boldsymbol{I} \subseteq \boldsymbol{J}$ if and only if the subalgebra of $\boldsymbol{B}$ generated by $\boldsymbol{I}$ is contained in the subalgebra of $\boldsymbol{B}$ generated by $\boldsymbol{J}$. □

1.19. THEOREM. *Every proper subalgebra of* $\boldsymbol{B}$ *is the intersection of maximal subalgebras of* $\boldsymbol{B}$. *In particular* Sub($\boldsymbol{B}$) *is dually atomic.*

PROOF. Let $\boldsymbol{C}$ be a proper subalgebra of $\boldsymbol{B}$. If $\boldsymbol{C}$ is a dual subalgebra, then the result follows from 1.17, 1.18. Now, let us suppose that $\boldsymbol{C}$ is not a dual subalgebra. In particular $\boldsymbol{C}$ is not a dual atom in Sub($\boldsymbol{B}$). Then $\boldsymbol{C}$ is contained in a maximal subalgebra of $\boldsymbol{B}$, follows from 1.15, 1.17 and 1.18. Now 1.15 shows that $\boldsymbol{C}$ is the intersection of all maximal subalgebras of $\boldsymbol{B}$ containing $\boldsymbol{C}$. □

1.20. Let $\boldsymbol{F}$ be a finite subalgebra of an algebra $\boldsymbol{B}$, having $n \geq 2$ atoms. In Observation 2.13, we construct a complement $\boldsymbol{C}$ in Sub($\boldsymbol{B}$), such that $\boldsymbol{C}$, generated by the intersection of n pairwise distinct maximal ideals of $\boldsymbol{B}$, is of height $n-1$ in (Sub($\boldsymbol{B}$), $\supseteq$), which generalizes 1.14.1 (the proof can be done by induction on n on the Boolean space).

2. Complementation and retractiveness in Sub($\boldsymbol{B}$)

Let $\boldsymbol{B}$ be a Boolean algebra, and $\boldsymbol{A} \in \mathrm{Sub}(\boldsymbol{B})$. Recall that $\boldsymbol{C} \in \mathrm{Sub}(\boldsymbol{B})$ is a complement of $\boldsymbol{A}$ if and only if $\boldsymbol{A} \cap \boldsymbol{C} = \{\mathbf{0}, \mathbf{1}\}$ and $\boldsymbol{A} \vee \boldsymbol{C} = \langle \boldsymbol{A} \cup \boldsymbol{C} \rangle = \boldsymbol{B}$. Let us recall:

2.1. DEFINITION. Let $\boldsymbol{B}$ be a Boolean algebra. We say that Sub($\boldsymbol{B}$) is *complemented* if and only if every member of Sub($\boldsymbol{B}$) has a complement in Sub($\boldsymbol{B}$).

The following result is found in TODORČEVIĆ [1980]. It should be noted that the proof is very closely related to Rubin's proof of Theorem 15.2 [Part I, Chapter 6, of this Handbook] which in turn is very closely related to MOSTOWSKI–TARSKI's [1939] proof of retractiveness of an interval algebra. First let us recall some notation. Let $\boldsymbol{I}$ be a linear ordering. We set $\boldsymbol{I}^0 = \boldsymbol{I} \cup \{-\infty\}$, $\boldsymbol{I}^+ = \boldsymbol{I} \cup \{-\infty, +\infty\}$ and $-\infty < t < +\infty$ for every $t \in \boldsymbol{I}$. We denote by $\boldsymbol{B}\langle \boldsymbol{I} \rangle$, the interval algebra generated by $\boldsymbol{I}$ (denoted by Intalg($\boldsymbol{I}^0$) in Section 15, Chapter 6 [Part 1]). So if a is a non-zero element of $\boldsymbol{B}\langle \boldsymbol{I} \rangle$, then there is a unique finite strictly increasing sequence $(a_k)_{k<2m}$ of elements of $\boldsymbol{I}^+$ such that $a = \bigcup \{[a_{2i}, a_{2i+1}): i < m\}$. This is called the *canonical decomposition* of a as the union of $[a_{2i}, a_{2i+1})$. We set $\sigma(a) = \{a_k: k < 2m\}$. We consider $\boldsymbol{B}\langle \boldsymbol{I} \rangle$ as a subalgebra of the power set algebra $\mathfrak{P}(\boldsymbol{I}^0)$. Now if $\boldsymbol{J} \subseteq \boldsymbol{I}$, we denote by $\boldsymbol{B}\langle \boldsymbol{J} \rangle$ the subalgebra of $a \in \boldsymbol{B}\langle \boldsymbol{I} \rangle$ satisfying $\sigma(a) \subseteq \boldsymbol{J}$.

2.2. THEOREM. *If $\boldsymbol{B}$ is a subalgebra of an interval algebra, then* Sub($\boldsymbol{B}$) *is a complemented lattice.*

PROOF. Let $\boldsymbol{I}$ be a chain, $\boldsymbol{B}$ be a subalgebra of $\boldsymbol{B}\langle \boldsymbol{I}\rangle$, and $\boldsymbol{A} \in \mathrm{Sub}(\boldsymbol{B})$. We will prove that $\boldsymbol{A}$ has a complement $\boldsymbol{C}$ in Sub($\boldsymbol{B}$). Let $\boldsymbol{J}$ be a maximal subchain of $\boldsymbol{I}$ with respect to $\boldsymbol{A} \cap \boldsymbol{B}\langle \boldsymbol{J}\rangle = \underline{2}$. We set $\boldsymbol{C} = \boldsymbol{B} \cap \boldsymbol{B}\langle \boldsymbol{J}\rangle$. Obviously, $\boldsymbol{A} \cap \boldsymbol{C} = \underline{2}$. We will prove that $\boldsymbol{A} \vee \boldsymbol{C} = \boldsymbol{B}$. First, if $a \in \boldsymbol{B}\langle \boldsymbol{I}\rangle$, we set $\underline{\sigma}(a) = \sigma(a)\backslash \boldsymbol{J}$. Notice that, $a \in \boldsymbol{B}\langle \boldsymbol{J}\rangle$ if and only if $\underline{\sigma}(a) = \emptyset$. We will prove that: if $a \in \boldsymbol{B}\backslash(\boldsymbol{A} \vee \boldsymbol{C})$, then there is $c \in \boldsymbol{B}\backslash(\boldsymbol{A} \vee \boldsymbol{C})$ such that $\underline{\sigma}(c) \subseteq \sigma(a)$ and $\underline{\sigma}(c) \neq \underline{\sigma}(a)$. We remark that a consequence of this, by descending induction, is that there is $d \in \boldsymbol{B}\backslash(\boldsymbol{A} \vee \boldsymbol{C})$ such that $\underline{\sigma}(d) = \emptyset$, which means $d \in \boldsymbol{B} \cap \boldsymbol{B}\langle \boldsymbol{J}\rangle = \boldsymbol{C}$, and this is a contradiction. Now let $a \in \boldsymbol{B}\backslash(\boldsymbol{A} \vee \boldsymbol{C})$. Then $\underline{\sigma}(a) \neq \emptyset$, say $a_k \in \underline{\sigma}(a)$. From the maximality of $\boldsymbol{J}$, it follows that $\boldsymbol{A} \cap \boldsymbol{B}\langle \boldsymbol{J} \cup \{a_k\}\rangle \neq \underline{2}$. Let $b \in \boldsymbol{A} \cap \boldsymbol{B}\langle \boldsymbol{J} \cup \{a_k\}\rangle$, with $b \neq \boldsymbol{0}, \boldsymbol{1}$. We must have $a_k \in \sigma(b)$, and $\sigma(b)\backslash\{a_k\} \subseteq \boldsymbol{J}$. Under its canonical decomposition we have $b = \bigcup \{[b_{2j}, b_{2j+1}): j \leq p\}$ and for a unique $l \leq 2p+1$, we have $b_l = a_k$.

Assume k and l have different parity, i.e. a_k is a left (resp. right) end-point and b_l is a right (resp. left) end-point of intervals in their canonical decompositions of a and b. We set $c_1 = a \cup b$ and $c_2 = a \cap b$. For $i = 1$, 2, we have $a_k \not\in \sigma(c_i)$ and $\sigma(c_i) \subseteq \sigma(a) \cup \sigma(b)$. Consequently, $\underline{\sigma}(c_i) \subseteq \underline{\sigma}(a)$, and $\underline{\sigma}(c_i) \neq \underline{\sigma}(a)$ for $i = 1$, 2. Now obviously $c_1, c_2 \in \boldsymbol{B}$. Observe that $a = ((a \cup b) - b) \cup (a \cap b)$. If $c_1, c_2 \in \boldsymbol{A} \vee \boldsymbol{C}$, then $a \in \boldsymbol{A} \vee \boldsymbol{C}$ since $b \in \boldsymbol{A}$, and this is a contradiction. Therefore, either c_1 or c_2 does not belong to $\boldsymbol{A} \vee \boldsymbol{C}$.

Suppose l and k have the same parity, i.e. a_k and b_l are both right (resp. left) end-points of intervals of the canonical decompositions of a and b, respectively. We set $c_1 = a - b$ and $c_2 = b - a$. As above, $a_k \not\in \sigma(c_i)$ and $\sigma(c_i) \subseteq \sigma(a) \cup \sigma(-b)$, and thus $\underline{\sigma}(c_i) \subseteq \underline{\sigma}(a)$ and $\underline{\sigma}(c_i) \neq \underline{\sigma}(a)$. We observe that $a = ((a-b) \cup b) - (b-a)$. From $b \in \boldsymbol{A}$ and $a \not\in \boldsymbol{A} \vee \boldsymbol{C}$, it follows that c_1 or c_2 does not belong to $\boldsymbol{A} \vee \boldsymbol{C}$. □

2.3. EXAMPLES

(1) If $\boldsymbol{B}$ is countable, then Sub($\boldsymbol{B}$) is complemented REMMEL [1980] and JECH [1982].

(2) Let $\boldsymbol{F}_c(\boldsymbol{X})$ be the algebra of finite or cofinite subsets of $\boldsymbol{X}$. Then Sub($\boldsymbol{F}_c(\boldsymbol{X})$) is complemented. Indeed, if $|\boldsymbol{X}| = \kappa$, then $F_c(\boldsymbol{X})$ can be considered as the subalgebra of the interval algebra, generated by the set of $\{\alpha\} = [\alpha, \alpha+1)$ for $\alpha < \kappa$ (this result was proved by Rotman).

(3) In 1.14, we have shown that if $a \in \boldsymbol{B}$, with $a \neq \boldsymbol{0}, \boldsymbol{1}$, then $\langle a\rangle$ has a complement in Sub($\boldsymbol{B}$).

2.4. OBSERVATION. Let $\boldsymbol{B}$ be an algebra, $\boldsymbol{J}$ an ideal of $\boldsymbol{B}$ and ψ the canonical homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}/\boldsymbol{J}$. Let $\boldsymbol{A}, \boldsymbol{C} \in \mathrm{Sub}(\boldsymbol{B})$ be such that $\boldsymbol{J} \subseteq \boldsymbol{A}$ and $\boldsymbol{C} \cap \boldsymbol{A} = \underline{2}$. Then the following are equivalent:

(i) $\boldsymbol{C}$ is a complement of $\boldsymbol{A}$ in Sub($\boldsymbol{B}$);

(ii) $\psi[\boldsymbol{C}]$ is a complement of $\psi[\boldsymbol{A}]$ in Sub($\boldsymbol{B}/\boldsymbol{J}$).

PROOF. (i) *implies* (ii). Assume (i). Obviously $\psi[\boldsymbol{A}] \vee \psi[\boldsymbol{C}] = \boldsymbol{B}/\boldsymbol{J}$. Now, assume there is $u \neq \boldsymbol{0}, \boldsymbol{1}$ with $u \in \psi[\boldsymbol{A}] \cap \psi[\boldsymbol{C}]$. We choose $a \in \boldsymbol{A}$ and $c \in \boldsymbol{C}$ such that $\psi(a) = \psi(c) = u$. So $c = a \triangle v$ for some $v \in \boldsymbol{J} \subseteq \boldsymbol{A}$, and thus $c \in \boldsymbol{A} \cap \boldsymbol{C}$ with $c \neq \boldsymbol{0}$, $\boldsymbol{1}$, a contradiction.

(ii) *implies* (i). Assume (ii). Let $x \in \boldsymbol{B}$. We have $\psi(x) \in \boldsymbol{B}/\boldsymbol{J} = \psi[\boldsymbol{A}] \vee \psi[\boldsymbol{C}] = \psi[\boldsymbol{A} \vee \boldsymbol{C}]$. Let $y \in \boldsymbol{A} \vee \boldsymbol{C}$ be such that $\psi(x) = \psi(y)$. So $x = y \triangle v$ for some $v \in \boldsymbol{J} \subseteq \boldsymbol{A}$, and thus $x \in \boldsymbol{A} \vee \boldsymbol{C}$. □

REMARK. The above proof shows that, if $\boldsymbol{J} \subseteq \boldsymbol{A}$ and $\boldsymbol{A} \cap \boldsymbol{C} = \underline{2}$, then $\psi[\boldsymbol{A}] \cap \psi[\boldsymbol{C}] = \underline{2}$.

As a consequence of the observation above, we obtain the following result of RAO and RAO [1979]:

2.5. PROPOSITION. *Let* $\boldsymbol{B}$ *be an algebra, and* $\boldsymbol{B}_1$ *be a quotient algebra of* $\boldsymbol{B}$. *If* Sub($\boldsymbol{B}$) *is a complemented lattice, then the same holds for* Sub($\boldsymbol{B}_1$).

PROOF. Let ψ be a homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}_1$, and $\boldsymbol{A}_1$ be a subalgebra of $\boldsymbol{B}_1$. We set $\boldsymbol{A} = \psi^{-1}[\boldsymbol{A}_1]$. We have $\boldsymbol{J} = \psi^{-1}(\boldsymbol{0}) \subseteq \boldsymbol{A}$. Let $\boldsymbol{C}$ be a complement of $\boldsymbol{A}$ in Sub($\boldsymbol{B}$). From Observation 2.4, it follows that $\boldsymbol{C}_1 = \psi[\boldsymbol{C}]$ is a complement of $\boldsymbol{A}_1$ in sub($\boldsymbol{B}_1$). □

Before giving the connection between $\boldsymbol{B}$ being retractive and Sub($\boldsymbol{B}$) being complemented, we make an observation.

2.6. OBSERVATION. Let $\boldsymbol{J}$ be an ideal of a Boolean algebra $\boldsymbol{B}$, and let ψ be the quotient homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}/\boldsymbol{J}$. The following properties are equivalent:

(i) there is a one-to-one homomorphism ϕ from $\boldsymbol{B}/\boldsymbol{J}$ into $\boldsymbol{B}$ such that $\psi \circ \phi(t) = t$ for every $t \in \boldsymbol{B}/\boldsymbol{J}$;

(ii) there is an endomorphism σ of $\boldsymbol{B}$ such that $\sigma(x) \equiv x \pmod{\boldsymbol{J}}$, and $\sigma(x) \equiv \sigma(y) \pmod{\boldsymbol{J}}$ implies $\sigma(x) = \sigma(y)$ for every $x, y \in \boldsymbol{B}$;

(iii) there is a subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$ such that $|\boldsymbol{A} \cap (x/\boldsymbol{J})| = 1$ for every $x \in \boldsymbol{B}$.

We recall that $x/\boldsymbol{J}$ denotes the equivalence class of $x \in \boldsymbol{A}$, modulo $\boldsymbol{J}$.

The proof is obvious, setting $\sigma = \phi \circ \psi$ and $\boldsymbol{A} = \phi[\boldsymbol{B}/\boldsymbol{J}]$. We say that ψ is *retractive* and, more simply, that $\boldsymbol{J}$ is *retractive*, and $\boldsymbol{A}$ is a *retract* of $\boldsymbol{B}/\boldsymbol{J}$. Let us recall:

2.7. DEFINITION. A Boolean algebra $\boldsymbol{B}$ is said to be *retractive* if for every ideal $\boldsymbol{J}$ of $\boldsymbol{B}$, there is a subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$ such that $|\boldsymbol{A} \cap (x/\boldsymbol{J})| = 1$ for every $x \in \boldsymbol{B}$.

We state the following theorem due to RAO and RAO [1979].

2.8. THEOREM. *Let* $\boldsymbol{B}$ *be an algebra. If* Sub($\boldsymbol{B}$) *is complemented, then* $\boldsymbol{B}$ *is retractive.*

PROOF. Let $\boldsymbol{J}$ be an ideal of $\boldsymbol{B}$ and ψ be the natural homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}/\boldsymbol{J}$. We have $\langle \boldsymbol{J} \rangle = \{t \in \boldsymbol{B}: t \in \boldsymbol{J} \text{ or } -t \in \boldsymbol{J}\}$. Let $\boldsymbol{A}$ be a complement of $\langle \boldsymbol{J} \rangle$ in Sub($\boldsymbol{B}$). Obviously, $\boldsymbol{J} \subseteq \langle \boldsymbol{J} \rangle$ and $\psi[\langle \boldsymbol{J} \rangle] = \{\boldsymbol{0}, \boldsymbol{1}\} \subseteq \boldsymbol{B}/\boldsymbol{J}$. From Observation 2.4, it follows that $\psi[\boldsymbol{A}] = \boldsymbol{B}/\boldsymbol{J}$, and thus for every $x \in \boldsymbol{B}$ there is $y \in \boldsymbol{A}$ satisfying $x \triangle y \in \boldsymbol{J}$. Moreover, $|x/\boldsymbol{J}| = 1$ for $x \in \boldsymbol{B}$, since $\boldsymbol{A} \cap \langle \boldsymbol{J} \rangle = \{\boldsymbol{0}, \boldsymbol{1}\}$. □

2.9. REMARK. Using the notations of Observation 2.6 and of the proof of Theorem 2.8, we have:

(1) If $x \in \boldsymbol{A} \vee \langle \boldsymbol{J} \rangle$, then there are unique a_x in $\boldsymbol{A}$ and u_x and v_x in $\boldsymbol{J}$ such that $x = a_x + u_x - v_x$.

(2) If $\boldsymbol{C} \cap \langle \boldsymbol{J} \rangle = \underline{2}$, then the following are equivalent: (i) $\boldsymbol{C}$ is isomorphic to $\boldsymbol{B}/\boldsymbol{J}$, (ii) $\boldsymbol{C}$ is a complement of $\langle \boldsymbol{J} \rangle$ in Sub($\boldsymbol{B}$) and (iii) $\boldsymbol{C}$ is a retract of $\boldsymbol{B}/\boldsymbol{J}$ (consider $\sigma(x) = a_x$).

2.10. COMMENT. We can define the dual notion of retractiveness. Let $\boldsymbol{B}$ be an algebra. $\boldsymbol{B}$ is said to be *co-retractive* whenever for every algebra $\boldsymbol{A}$ and every one-to-one homomorphism ϕ from $\boldsymbol{A}$ into $\boldsymbol{B}$, there is a homomorphism ψ from $\boldsymbol{B}$ into $\boldsymbol{A}$ satisfying $\psi \circ \phi(x) = x$, for every $x \in \boldsymbol{A}$. ROTMAN [1972] has proved that $\boldsymbol{B}$ is co-retractive if and only if $\boldsymbol{B}$ is isomorphic to some $\boldsymbol{F}_c(\boldsymbol{X})$.

2.11. As a direct consequence of Theorems 2.2 and 2.8, we obtain the following result of Rubin (see Theorem 15.18, Chapter 6, Part 1) obtained in 1977, and published later (RUBIN [1983]), that solved Conjecture (B) of ROTMAN [1972]:

THEOREM. *If $\boldsymbol{B}$ is a subalgebra of an interval algebra, then $\boldsymbol{B}$ is retractive.*

The following result is useful to prove Theorem 2.14.

2.12. OBSERVATION. Let $\boldsymbol{J}$ be an ideal of a Boolean algebra $\boldsymbol{B}$ such that $\boldsymbol{B}/\boldsymbol{J}$ is countable. Then there is a subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$ such that $|\boldsymbol{A} \cap (x/\boldsymbol{J})| = 1$ for every $x \in \boldsymbol{B}$.

PROOF. $\boldsymbol{B}/\boldsymbol{J}$ is a countable interval algebra. Let $\boldsymbol{D} = \{y_n \colon n < \omega\}$ be an enumeration of a countable chain generating $\boldsymbol{B}/\boldsymbol{J}$. By induction, construct a chain $\{x_n \colon n < \omega\}$ in $\boldsymbol{B}$ such that $\psi(x_n) = y_n$ for every $n < \omega$ (where ψ denotes the quotient homomorphism). Consider the algebra $\boldsymbol{A}$ generated by $\{x_n \colon n < \omega\}$. □

Before stating Theorem 2.14, let us remark that the observation above is connected with the following result, due to Rao and Rao:

2.13. OBSERVATION. If $\boldsymbol{F}$ is a finite subalgebra of an algebra $\boldsymbol{B}$, then $\boldsymbol{F}$ has a complement in Sub($\boldsymbol{B}$).

PROOF. Since $\boldsymbol{F}$ is finite, the atoms of $\boldsymbol{F}$ define a partition of $\mathbf{1} \in \boldsymbol{B}$. It follows that there is an ideal $\boldsymbol{J}$ of $\boldsymbol{B}$ such that $|\boldsymbol{F} \cap (x/\boldsymbol{J})| = 1$ for every $x \in \boldsymbol{B}$ (this can be shown directly by duality, on their corresponding Boolean spaces). Consequently, $\langle \boldsymbol{J} \rangle$ is a complement of $\boldsymbol{F}$ in Sub($\boldsymbol{B}$). □

Note that the proof of this observation is similar to the proof of 1.20, and can be proved, using Observation 2.12.

Now, we will show (for a statement of ($\diamond$), see the Appendix on Set Theory, this Handbook):

2.14. THEOREM. *Assume* ($\diamond$). *There is a Boolean algebra $\boldsymbol{B}$, of cardinality ω_1, such that*:

(1) *$\boldsymbol{B}$ is not embeddable in any interval algebra*;
(2) *every subalgebra of $\boldsymbol{B}$ is retractive*;
(3) *if $\boldsymbol{B}_1$ is a subalgebra of $\boldsymbol{B}$, of cardinality ω_1, then* Sub($\boldsymbol{B}_1$) *is not complemented.*

Statements (1) and (2) were proved by Rubin in 1977, and published in 1983 (RUBIN [1983]), and Todorčević has shown that Rubin's algebra above satisfies (3). Properties (1) and (2) solved Conjecture (A) of ROTMAN [1972].

PROOF. This will be given in two parts, assuming that ($\diamond$) implies the existence of a Boolean algebra $\boldsymbol{B}$, of cardinality ω_1, for which there is no uncountable somewhere dense subset in $\boldsymbol{B}$ (such subsets are defined and studied in Part B, below). Let us recall that a subset $\boldsymbol{X}$ of a Boolean algebra $\boldsymbol{B}$ is *dense* in $\boldsymbol{B}$ if for each $x \in \boldsymbol{B}$, $x \neq \boldsymbol{0}$, there is $y \in \boldsymbol{X}$, satisfying $\boldsymbol{0} \neq y \leq x$.

Part A. Let $\boldsymbol{B}$ be a Boolean algebra. We recall that an ideal $\boldsymbol{J}$ of $\boldsymbol{B}$ is said to be *countably generated* whenever there is a countable set $\boldsymbol{D}$ contained in $\boldsymbol{J}$ such that for each $x \in \boldsymbol{J}$, there is $y \in \boldsymbol{D}$ satisfying $x \leq y$ (i.e. $\boldsymbol{J}$ contains a countable subset cofinal in $\boldsymbol{J}$).

We will assume that there is a Boolean algebra $\boldsymbol{B}$ of cardinality ω_1 satisfying the following properties:

(H1) every chain and antichain of $\boldsymbol{B}$ is countable;
(H2) if $\boldsymbol{J}$ is a dense ideal of $\boldsymbol{B}$, then $\boldsymbol{B}/\boldsymbol{J}$ is countable;
(H3) if $\boldsymbol{B}'$ is an uncountable subalgebra of $\boldsymbol{B}$, then there is $a \neq \boldsymbol{0}$ in $\boldsymbol{B}$ such that $\boldsymbol{I}_a = \{x \in \boldsymbol{B}: x \leq a\} \subseteq \boldsymbol{B}'$. $\square$

We will show that such a $\boldsymbol{B}$ satisfies the conclusion of the theorem. First, that $\boldsymbol{B}$ is not embeddable in an interval algebra follows from (H1) and a theorem of RUBIN [1983] (see Theorem 15.22, Chapter 6, Part I).

Next we will prove that any subalgebra, $\boldsymbol{B}'$, of $\boldsymbol{B}$, is retractive. Let $\boldsymbol{J}'$ be an ideal of $\boldsymbol{B}'$. We choose an ideal $\boldsymbol{J}$ of $\boldsymbol{B}$ such that $\boldsymbol{J}' = \boldsymbol{J} \cap \boldsymbol{B}'$. Let $\underline{\boldsymbol{J}}$ be the set of $y \in \boldsymbol{B}$ satisfying $x \cdot y = \boldsymbol{0}$ for every $x \in \boldsymbol{J}$. Let $\boldsymbol{K}$ be the ideal generated by $\boldsymbol{J} \cup \underline{\boldsymbol{J}}$ in $\boldsymbol{B}$. Obviously, $\boldsymbol{K}$ is a dense ideal of $\boldsymbol{B}$. Let ψ be the natural homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}/\boldsymbol{K}$. Now (H2) implies $\boldsymbol{B}/\boldsymbol{K}$ is countable. By Observations 2.6 and 2.12, there is a one-to-one homomorphism ϕ from $\boldsymbol{B}/\boldsymbol{K}$ into $\boldsymbol{B}$ such that $\psi \circ \phi(t) = t$ for $t \in \boldsymbol{B}/\boldsymbol{K}$. Then $\boldsymbol{F} = \phi[\boldsymbol{B}/\boldsymbol{K}]$ is a subalgebra of $\boldsymbol{B}$. Let $\boldsymbol{A} = \langle \boldsymbol{F} \cup \underline{\boldsymbol{J}} \rangle \cap \boldsymbol{B}'$. We will prove that $\boldsymbol{A}$ satisfies $\boldsymbol{A} \cap \boldsymbol{J}' = \{\boldsymbol{0}\}$. For contradiction, assume there is $a \in \boldsymbol{A} \cap \boldsymbol{J}' = \boldsymbol{A} \cap \boldsymbol{J}$ satisfying $a \neq \boldsymbol{0}$. We have $a \in \boldsymbol{A} = \langle \boldsymbol{F} \cup \underline{\boldsymbol{J}} \rangle \cap \boldsymbol{B}'$, and thus $a = a_1 + a_2$ with $a_1 \in \underline{\boldsymbol{J}}$ and $a_2 = \Sigma\{f_k \cdot -\underline{i}_k: k < p\}$ with $f_k \in \boldsymbol{F}$ and $\underline{i}_k \in \underline{\boldsymbol{J}}$ for $k < p$. From $a \in \boldsymbol{J}$, $a_1 \in \underline{\boldsymbol{J}}$ and $a_1 \leq a$, it follows that $a_1 = a \cdot a_1 = \boldsymbol{0}$. Now $a \in \boldsymbol{J}$ implies $f_k \cdot -\underline{i}_k \in \boldsymbol{J}$ and thus $f_k = f_k \cdot \underline{i}_k + f_k \cdot -\underline{i}_k \in \boldsymbol{K}$, for every k (since $\boldsymbol{K}$ is the ideal generated by $\boldsymbol{J} \cup \underline{\boldsymbol{J}}$). Now $f_k = \boldsymbol{0}$ since $\boldsymbol{F} \cap \boldsymbol{K} = \{\boldsymbol{0}\}$. Consequently, $a_2 = \boldsymbol{0}$ and thus $a = \boldsymbol{0}$, which gives a contradiction.

Now, let $\boldsymbol{B}'$ be a subalgebra of $\boldsymbol{B}$, of cardinality ω_1. We will prove that Sub($\boldsymbol{B}'$) is not complemented. From the fact that every antichain and every chain of $\boldsymbol{B}$ is countable and from a Theorem of BAUMGARTNER and KOMJATH [1981] (see Theorem 4.25, Chapter 2, Part I), it follows that $\boldsymbol{B}'$ contains a dense countable subalgebra $\boldsymbol{A}$. For contradiction, assume that $\boldsymbol{A}$ has a complement $\boldsymbol{C}$ in Sub($\boldsymbol{B}'$). We have $|\boldsymbol{C}| = \omega_1$ since $|\boldsymbol{A}| = \omega$ and $|\boldsymbol{B}'| = \omega_1$. Now from (H3), let $b \in \boldsymbol{B}$, $b \neq \boldsymbol{0}$ be such that $\boldsymbol{I}_b \subseteq \boldsymbol{C}$. Since $\boldsymbol{A}$ is dense in $\boldsymbol{B}$, it follows that there is $x \neq \boldsymbol{0}$ in $\boldsymbol{A}$ such that $x \leq b$. So $\boldsymbol{0} \neq x \in \boldsymbol{A} \cap \boldsymbol{C}$, contrary to our assumption.

Part B. We will prove that the (narrow) algebra constructed by RUBIN [1983] has the properties (H1) through (H3). For this, let us recall some notation and definitions. As usual, in partial order theory, for $a < b$ in a Boolean algebra $\boldsymbol{B}$, we denote by (a, b) (resp. $[a, b]$) the interval of $x \in \boldsymbol{B}$ satisfying $a < x < b$ (resp. $a \leq x \leq b$); these are called respectively the open and closed intervals of $\boldsymbol{B}$ with end-points a and b. Now, if $n > 0$ and $a, b_1, \ldots, b_n, c_1, c_2 \in \boldsymbol{B}$, we denote by $\boldsymbol{R}(a, b_1, \ldots, b_n, c_1, c_2)$ the assertion:

$a, b_1, \ldots, b_n$ are pairwise disjoint, $c_1 \leq a + \Sigma\{b_i : 1 \leq i \leq n\}$, $a + c_1 \leq c_2$ and $(c_2 \cdot -c_1).\ b_i \neq \boldsymbol{0}$ for every $1 \leq i \leq n$.

A subset $\boldsymbol{P}$ is said to be *nowhere dense* if for every $n > 0$, and $a, b_1, \ldots, b_n \in \boldsymbol{B}$, such that $a, b_1, \ldots, b_n$ are pairwise disjoint, and $b_1, \ldots, b_n \neq \boldsymbol{0}$, there are $c_1, c_2 \in \boldsymbol{B}$ such that $\boldsymbol{R}(a, b_1, \ldots, b_n, c_1, c_2)$, and $(c_1, c_2) \cap \boldsymbol{P} = \emptyset$. If $\boldsymbol{P}$ is not nowhere dense, then $\boldsymbol{P}$ is called a *somewhere dense* subset of $\boldsymbol{B}$. Let $\boldsymbol{P}$ be a somewhere dense subset of $\boldsymbol{B}$. Then, there are $n > 0$, and $a, b_1, \ldots, b_n$ to witness this fact. So, for c, d^1 in $\boldsymbol{B}$, if $\boldsymbol{R}(a, b_1, \ldots, b_n, c, d^1)$, then there is $p \in \boldsymbol{P} \cap (c, d^1)$. Fig. 10.1 illustrates this.

Then Rubin's theorem is the following:

Assume ($\diamond$). *Then there is an atomless Boolean algebra* $\boldsymbol{B}$ *of cardinality* ω_1 *such that every nowhere dense subset is countable.*

We will prove that such an algebra satisfies (H1) through (H3) and

(H4) If $\boldsymbol{P}$ is an uncountable subset of $\boldsymbol{B}$, then there are distinct elements a, b and c in $\boldsymbol{P}$ such that $a \cdot b = c$.

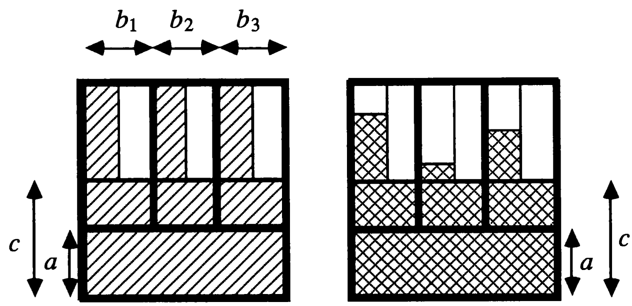

Fig. 10.1. Case $n = 3$. Legend: a and c are horizontal rectangles; b_1, b_2 and b_3 are vertical rectangles; and d^1 and p are the shaded parts as follows:

We remark that stronger properties than (H1) through (H4) can be found in RUBIN [1983]. We prefer to recall their direct proofs.

(H4) holds. Suppose $|\boldsymbol{P}| = \omega_1$ so that $\boldsymbol{P}$ is somewhere dense. Let $a, b_1, \ldots, b_n$ witness this fact. We set $c_1 = a$ and we choose $c_2 \leq c_1 + \Sigma \{b_i: 1 \leq i \leq n\}$ containing c_1 such that $(c_2 \cdot -c_1) \cdot b_i \neq \mathbf{0}$ and $b_i \cdot -c_2 \neq \mathbf{0}$ for $1 \leq i \leq n$ (i.e. c_2 is the union of $a = c_1$ and of c_2^i such that $\mathbf{0} \neq c_2^i < b_i$ for $1 \leq i \leq n$). Let $w \in \boldsymbol{P} \cap (c_1, c_2)$. Now, we set $w = c$. From the definition of c_2, it follows that $b_i \cdot -w \neq \mathbf{0}$ for $1 \leq i \leq n$. Let (d_i^1, d_i^2) be a non-trivial partition of $b_i \cdot -c$ (i.e. $d_i^1 \neq \mathbf{0} \neq d_i^2$). For $l = 1, 2$ we set $d^l = c + \Sigma \{d_i^l: 1 \leq i \leq n\}$. Obviously, $a + c = c < d^l$ and $d^l \cdot -c \cdot b_i \neq \mathbf{0}$ for $l = 1, 2$. Let $u_1 \in \boldsymbol{P} \cap (c, d^1)$ and $u_2 \in \boldsymbol{P} \cap (c, d^2)$. We have $c = u_1 \cdot u_2$ (since $d_i^1 \cdot d_i^2 = \mathbf{0}$ for $1 \leq i \leq n$) and c, u_1, u_2 are distinct elements of P. We conclude, setting $a = u_1$ and $b = u_2$.

(H1) holds. As a direct consequence of (H4).

(H2) holds. Let $\boldsymbol{J}$ be a dense ideal of $\boldsymbol{B}$. We assume $|\boldsymbol{B}/\boldsymbol{J}| = \omega_1$. Let $\boldsymbol{P}$ be a set of representative elements of $\boldsymbol{B}$ modulo $\boldsymbol{J}$. So $|\boldsymbol{P}| = \omega_1$ and if $x \neq y$, then $x \not\equiv y$ (mod $\boldsymbol{J}$). Let $a, b_1, \ldots, b_n$ witness the fact that $\boldsymbol{P}$ is somewhere dense. For $1 \leq i \leq n$, let $\mathbf{0} \neq b_i^1 < b_i^2 < b_i$, with $b_i^2 \in \boldsymbol{J}$. We define $d_l = a + \Sigma \{b_i^l: 1 \leq i \leq n\}$, and $l = 1, 2$. We set $c_1 = a$, $c_2 = d_1$. Let $u \in \boldsymbol{P} \cap (c_1, c_2)$ and $v \in \boldsymbol{P} \cap (d_1, d_2)$. We have $a < u < c_2 = d_1 < v < d_2$, and $a \equiv d_2$ (mod $\boldsymbol{J}$). Consequently, $u \equiv v$ (mod $\boldsymbol{J}$) and $u \neq b$ in $\boldsymbol{P}$, which contradicts the definition of $\boldsymbol{P}$.

(H3) holds. Let $\boldsymbol{B}'$ be a subalgebra of $\boldsymbol{B}$, of cardinality ω_1. By induction, we construct x_α, $\boldsymbol{P}_\alpha$ for $\alpha < \omega_1$ such that $x_\alpha \in \boldsymbol{B}'$, $\boldsymbol{P}_\alpha = \{x_\beta: \beta < \alpha\}$ and $x_\alpha \not\in \langle \boldsymbol{P}_\alpha \rangle$. Let $\boldsymbol{P} = \bigcup \{\boldsymbol{P}_\alpha: \alpha < \omega_1\}$. So $\boldsymbol{P}$ is somewhere dense. Let $a, b_1, \ldots, b_n$ witness this fact. We choose $b \in \boldsymbol{B}'$ such that $a \leq b \leq a + \Sigma \{b_i: 1 \leq i \leq n\}$ and $b_i \cdot -b \neq \mathbf{0} \neq b_i \cdot b$ for $1 \leq i \leq n$. Let $b \cdot -a = c \neq \mathbf{0}$. We will prove that $[a, b] \subseteq \boldsymbol{B}'$, that means $\boldsymbol{I}_c \subseteq \boldsymbol{B}'$. Let $d \in [a, b]$. We will prove that $d \in \boldsymbol{B}'$, using a similar argument to prove (H4). For $1 \leq i \leq n$, let (d_i^1, d_i^2) be a partition of $b_i \cdot -d$ such that $d_i^1 \neq \mathbf{0} \neq d_i^2$. We set $d^l = a + \Sigma \{d_i^l: 1 \leq i \leq n\}$ for $l = 1, 2$. Then $\boldsymbol{R}(a, b_1, \ldots, b_n, d, d^l)$ holds for $l = 1, 2$. Let $c^l \in \boldsymbol{P} \cap (d, d^l)$ for $l = 1, 2$. We have $c^1, c^2 \in \boldsymbol{P} \cap \boldsymbol{B}'$, $d^1 \cdot d^2 = d$ and thus $d = c^1 \cdot c^2 \in \boldsymbol{B}'$. $\square$

REMARK. As a consequence of the proof of (H3), we can show that, if $\boldsymbol{B}'$ is a subalgebra of $\boldsymbol{B}$, then there are an ideal $\boldsymbol{J}$ of $\boldsymbol{B}$ and a countable subset $\boldsymbol{D}$ of $\boldsymbol{B}$ such that $\boldsymbol{B}' = \langle \boldsymbol{J} \cup \boldsymbol{D} \rangle$ (that is proved in RUBIN [1983]). Indeed, this result is trivial if $|\boldsymbol{B}'| \leq \omega$ or $\boldsymbol{B}' = \boldsymbol{B}$. Now we suppose $\boldsymbol{B}' \neq \boldsymbol{B}$ and $|\boldsymbol{B}'| = \omega_1$. Let $\boldsymbol{J}$ be the set of $x \in \boldsymbol{B}$ such that $\boldsymbol{I}_x$ is contained in $\boldsymbol{B}'$. We remark that $\boldsymbol{J}$ is a non-trivial ideal of $\boldsymbol{B}'$. For contradiction, let us suppose that there is no countable set $\boldsymbol{D}$ of $\boldsymbol{B}$ such that $\boldsymbol{B}' = \langle \boldsymbol{J} \cup \boldsymbol{D} \rangle$. By induction we construct $x_\alpha, \boldsymbol{P}_\alpha$ for $\alpha < \omega_1$ such that $x_\alpha \in \boldsymbol{B}$, $\boldsymbol{P}_\alpha = \{x_\beta: \beta < \alpha\}$ and $x_\alpha \not\in \langle \boldsymbol{P}_\alpha \cup \boldsymbol{J} \rangle$. Let $\boldsymbol{P} = \bigcup \{\boldsymbol{P}_\alpha: \alpha < \omega_1\}$. So $\boldsymbol{P}$ is somewhere dense. Let $a, b_1, \ldots, b_n$ witness this fact. We choose $b \in \boldsymbol{B}$ such that $a \leq b \leq a + \Sigma \{b_i: 1 \leq i \leq n\}$ and $b_i \cdot -b \neq \mathbf{0} \neq b_i \cdot b$ for $1 \leq i \leq n$. As in the proof of (H3), we show that $[a, b] \subseteq \boldsymbol{B}'$ and there are c_1, c_2 in $\boldsymbol{P}$ satisfying $c_1 \neq c_2$ and $c_1, c_2 \in [a, b]$. Let $c = b \cdot -a$. We have $c \in \boldsymbol{J}$ and $c_1 \equiv a \equiv c_2$ (mod $\boldsymbol{J}$). This contradicts the definition of x_α's.

The following result can be found in JECH [1982]:

2.15. PROPOSITION. *There is a subalgebra* $\boldsymbol{B}$ *of* $\mathfrak{P}(\omega)$ *such that*:

(1) $\boldsymbol{B}$ *has cardinality* 2^{ω};
(2) $\boldsymbol{F}_c(\omega)$ *is a subalgebra of* $\boldsymbol{B}$; *and*
(3) *if* $\boldsymbol{C}$ *is a subalgebra of* $\boldsymbol{B}$ *satisfying* $\boldsymbol{C} \cap \boldsymbol{F}_c(\omega) = \underline{2}$, *then* $\boldsymbol{C}$ *verifies*:
 (i) $\boldsymbol{C}$ *is an atomic algebra*,
 (ii) $\mathrm{At}(\boldsymbol{C})$ *defines a partition of* ω (*so if* x *is an atom of* $\boldsymbol{C}$, *then* x *is an infinite subset of* ω), *and*
 (iii) $\boldsymbol{C} = \boldsymbol{F}_c(\mathrm{At}(\boldsymbol{C}))$, *i.e.* $x \in \boldsymbol{C}$ *if and only if* x *is a finite or cofinite union of members of* $\mathrm{At}(\boldsymbol{C})$.

In particular $\boldsymbol{C}$ *is countable*; *consequently*, $\boldsymbol{F}_c(\omega)$ *has no complement in* Sub($\boldsymbol{B}$).

PROOF. Let $\boldsymbol{A}$ be an uncountable almost-disjoint family of subsets of ω of cardinality 2^{ω}, i.e. if $x \in \boldsymbol{A}$, then $x \subseteq \omega$ and $|x| = \omega$ and $|x \cap y| < \omega$ for $x \neq y$ in $\boldsymbol{A}$. Let $\boldsymbol{B}$ be the subalgebra of $\mathfrak{P}(\omega)$, generated by $\boldsymbol{A} \subseteq \mathfrak{P}(\omega)$. We can assume that the algebra $\boldsymbol{F}_c(\omega)$ of finite and cofinite subsets of ω is contained in $\boldsymbol{B}$ (for instance, consider for $\boldsymbol{A}$ the set of all maximal branches of the binary tree on ω). We will prove that if $\boldsymbol{C} \in \mathrm{Sub}(\boldsymbol{B})$ and $\boldsymbol{C} \cap \boldsymbol{F}_c(\omega) = \underline{2}$, then $|\boldsymbol{C}| \leq \omega$. For this, let $\boldsymbol{J}$ be the ideal generated by $\mathrm{At}(\boldsymbol{B})$ in $\boldsymbol{B}$, then $\boldsymbol{F}_c(\omega) = \langle \mathrm{At}(\boldsymbol{B}) \rangle$. Let ψ be the natural homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}/\boldsymbol{J}$. We observe that: (1) the set $\mathrm{At}(\boldsymbol{B})$ is countable, (2) the set $\mathrm{At}(\boldsymbol{B}/\boldsymbol{J})$ is in one-to-one correspondence with $\boldsymbol{A}$, and $\boldsymbol{B}/\boldsymbol{J}$ is isomorphic to $\boldsymbol{F}_c(A)$ (note that, if $x \in \boldsymbol{A}$, then $x \in \boldsymbol{F}_c(A)$ and $x/\boldsymbol{J}$ is an atom of $\boldsymbol{B}/\boldsymbol{J}$). Consequently, $\boldsymbol{B}$ is superatomic.

Let $\boldsymbol{D} = \{a_k\colon k \in K\}$ be the set of atoms of $\boldsymbol{C}$. From $\boldsymbol{F}_c(\omega) \cap \boldsymbol{C} = \underline{2}$, it follows:
(1) $|a_k| = \omega$ for every $k \in K$,
(2) $a_k \cap a_l = \emptyset$ if $k \neq l$ in K,
(3) $\bigcup \{a_k\colon k \in K\} = \omega$.

Since $\boldsymbol{B}/\boldsymbol{J}$ is isomorphic to $F_c(\boldsymbol{A})$, it follows that $\boldsymbol{C}$ is isomorphic to $F_c(\boldsymbol{D})$. Consequently, $\boldsymbol{C}$ is countable (since $|\boldsymbol{D}| = |K| \leq \omega$ implies $|F_c(\boldsymbol{D})| \leq \omega$). □

The following result is due to Rao and Rao.

2.16. PROPOSITION. *Let* $\boldsymbol{I}$ *be the ideal of* $\boldsymbol{B} = \mathfrak{P}(\omega)$, *consisting of the set of finite subsets of* ω. *Then* $\boldsymbol{B}$ *is not a retract of* $\boldsymbol{B}/\boldsymbol{I}$.

PROOF. Let ψ be the natural homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{B}/\boldsymbol{I}$. We remark that $\boldsymbol{B}$ satisfies the countable chain condition (c.c.c.), and $\boldsymbol{B}/\boldsymbol{I}$ contains an uncountable set $\boldsymbol{Z}$ of pairwise disjoint non-zero elements. Now, if there is a retraction ϕ of ψ, then for every non-zero distinct elements u and v in $\boldsymbol{Z}$, we must have $\phi(u) \cap \phi(v) = \emptyset$ in ω, which is impossible. □

2.17. COROLLARY. *Let* $\boldsymbol{B}$ *be an infinite* σ*-complete Boolean algebra. Then there is an ideal* $\boldsymbol{J}$ *of* $\boldsymbol{B}$ *such that* $\langle \boldsymbol{J} \rangle$ *has no complement in* Sub($\boldsymbol{B}$). *In particular* Sub($\boldsymbol{B}$) *is not complemented.*

PROOF. Recall that $\boldsymbol{B}$ is σ-complete whenever every countable subset of $\boldsymbol{B}$ has a supremum in $\boldsymbol{B}$. We claim that $\mathfrak{P}(\omega)$ is a quotient algebra of $\boldsymbol{B}$. To prove the claim, let $(b_n)_{n<\omega}$ be a sequence of pairwise disjoint non-zero elements of $\boldsymbol{B}$, such that $\Sigma\{b_n: n<\omega\}=\mathbf{1}$, and U_k be an ultrafilter of $\boldsymbol{B}$ containing b_k, for $k<\omega$. Let h be the function from $\boldsymbol{B}$ into $\mathfrak{P}(\omega)$, defined by $h(x)=\{n<\omega: x\in U_n\}$. It is easy to verify that h is a homomorphism from $\boldsymbol{B}$ onto $\mathfrak{P}(\omega)$. We conclude, using Proposition 2.4 and 2.16. □

2.18. We will now give some results obtained by DÜNTSCH and KOPPELBERG [1985] on complements in $\mathrm{Sub}(\mathfrak{P}(\omega))$. Let $\boldsymbol{A}\in \mathrm{Sup}(\mathfrak{P}(\omega))$. If $a\in \boldsymbol{A}$, then we set $d(a)=a\setminus\bigcup \mathrm{At}(\boldsymbol{A})$. Recall that $a\in \mathrm{At}(\boldsymbol{A})$ is said to be *proper* if $|a|>1$, i.e. $a\notin \mathrm{At}(\mathfrak{P}(\omega))$. An element a of $\boldsymbol{A}$ is said to be $\boldsymbol{A}$*-bad* if a satisfies:

(1) $\boldsymbol{A}\restriction a$ is an atomic algebra,
(2) $d(a)$ is finite,
(3) each atom of $\boldsymbol{A}\restriction a$ is finite, and
(4) only finitely many atoms of $\boldsymbol{A}\restriction a$ are proper.

Let $\mathrm{Bad}(\boldsymbol{A})$ be the set of $\boldsymbol{A}$-bad elements of $\boldsymbol{A}$. Obviously $\mathrm{Bad}(\boldsymbol{A})$ is an *ideal* of $\boldsymbol{A}$. We have the following result from DÜNTSCH and KOPPELBERG [1985]:

2.19. THEOREM. *Let $\boldsymbol{A}$ be a countable subalgebra of $\mathfrak{P}(\omega)$. $\boldsymbol{A}$ does not have a complement in* $\mathrm{Sub}(\mathfrak{P}(\omega))$ *if, and only if,*

(1) *$\boldsymbol{A}$ is an atomic algebra;*
(2) *each atom of $\boldsymbol{A}$ is finite;*
(3) *$\boldsymbol{A}/\mathrm{Bad}(\boldsymbol{A})$ has at most two elements.*

In particular, $\boldsymbol{A}$ does not have a complement if $\boldsymbol{F}_c(\omega)$ is embeddable in $\boldsymbol{A}$ or $\boldsymbol{A}$ is embeddable in $\boldsymbol{F}_c(\omega)$.

From the theorem, it follows for instance that $\boldsymbol{F}_c(\omega)$ has no complement in $\mathrm{Sub}(\mathfrak{P}(\omega))$, which was proved directly in Proposition 2.16.

3. Quasi-complements

3.1. DEFINITION. Let $\boldsymbol{B}$ be a Boolean algebra, and $\boldsymbol{A}\in \mathrm{Sub}(\boldsymbol{B})$. A subalgebra $\boldsymbol{C}$ of $\boldsymbol{B}$ is said to be a *quasi-complement* of $\boldsymbol{A}$ in $\mathrm{Sub}(\boldsymbol{B})$ if $\boldsymbol{C}$ is maximal in $\mathrm{Sub}(\boldsymbol{B})$, with respect to $\boldsymbol{A}\cap\boldsymbol{C}=\underline{2}$.

So $\boldsymbol{C}$ is a quasi-complement of $\boldsymbol{A}$ in $\mathrm{Sub}(\boldsymbol{B})$ if and only if $\boldsymbol{A}\cap\boldsymbol{C}=\underline{2}$ and $\boldsymbol{A}\cap\boldsymbol{C}(x)\neq\underline{2}$ for every $x\notin\boldsymbol{C}$. The notion of "quasi-complement" is not symmetric, as noted by REMMEL [1980] (see Theorem 3.9, and see the example 3.10 due to DÜNTSCH and KOPPELBERG [1985]). Furthermore, there is very little connection between complement and quasi-complement:

3.2. EXAMPLES.

(1) Let $X=\{1,2,3\}$, $\boldsymbol{B}=\mathfrak{P}(X)$ and $\boldsymbol{A}=\langle\{1\}\rangle$. Then $\langle\{2\}\rangle$ and $\langle\{3\}\rangle$ are distinct quasi-complements and complements of $\boldsymbol{A}$.

(2) Let $X=\{1,2,3,4\}$, $\boldsymbol{B}=\mathfrak{P}(X)$, and $\boldsymbol{A}=\langle\{1,2\}\rangle$. Then $\boldsymbol{D}=\langle\{1,3\}\rangle$ is a complement of $\boldsymbol{A}$ in $\mathrm{Sub}(\boldsymbol{B})$, strictly included in $\boldsymbol{C}=\langle\{1\},\{3\}\rangle$.

(3) More generally, in 1.3 we have characterized the quasi-complements of $\langle a\rangle$ in $\mathrm{Sub}(\boldsymbol{B})$, for $a\in\boldsymbol{B}$, with $a\neq\mathbf{0},\mathbf{1}$.

(4) In Proposition 2.15, we gave an example of $A \in \mathrm{Sub}(B)$ for which $|A| = \omega$, $|B| = 2^\omega$ and every quasi-complement of A in $\mathrm{Sub}(B)$ is countable. So *a quasi-complement need not be a complement* in $\mathrm{Sub}(B)$.

3.3. REMARK. Let B be a Boolean algebra, and A, D be proper subalgebras of B. From Zorn's lemma it follows that there is a quasi-complement C of A in $\mathrm{Sub}(B)$, containing D, whenever $A \cap D = \underline{2}$.

The following result was proved by DÜNTSCH and KOPPELBERG [1985]:

3.4. THEOREM. *Let B be an algebra, and A, $C \in \mathrm{Sub}(B)$. If C is a quasi-complement of A in* $\mathrm{Sub}(B)$, *then $A \vee C$ is a dense subalgebra of B.*

PROOF. Assume that $A \vee C$ is not dense. This means there exists $b \in B$, with $b \neq 0$, such that there is no $x \in A \vee C$ satisfying $0 < x \leq b$. In particular, $b \not\in C$. We will prove that $A \cap C(b) = \underline{2}$, which contradicts the maximality of C. Let $D = A \cap C(b)$, and $a \in D$. We have $a = c_1 \cdot b \dotplus c_2 \cdot -b \dotplus c_3$, where $c_1, c_2, c_3 \in C$ are pairwise disjoint. So $1 = c_1 \dotplus c_2 \dotplus c_3 \dotplus c_4$. Now $a \cdot c_1 = c_1 \cdot b \leq b$. Since $a \cdot c_1 \in A \vee C$ and the hypothesis, we have $c_1 \cdot b = 0$. Now we have $-a = c_2 \cdot b \dotplus c_1 \cdot -b \dotplus c_4$. By the same proof, applied to $-a \in D$, we have $-a \cdot c_2 = c_2 \cdot b = 0$, and thus $c_2 \cdot -b = c_2$. Consequently, $a = c_2 + c_3$. So $a \in A \cap C = \underline{2}$. □

As a trivial consequence we have:

3.5. OBSERVATION. Let $B = F_c(X)$ be the algebra of finite or cofinite subsets of X. Let $A, C \in \mathrm{Sub}(B)$. If C is a quasi-complement of A in $\mathrm{Sub}(B)$, then C is a complement of A in $\mathrm{Sub}(B)$.

PROOF. $A \vee C$ is dense and $B = \langle \mathrm{At}(B) \rangle$ and so $A \vee C = B$. □

The following result, is analogous to Observation 2.4, and is implicitly proved in REMMEL [1980].

3.6. OBSERVATION. Let B be an algebra, J be an ideal of B and ψ be the canonical homomorphism from B onto B/J. Let A and C be subalgebras of B. We suppose $J \subseteq A$ and $A \cap C = \underline{2}$.

(1) If A is a quasi-complement of C in $\mathrm{Sub}(B)$, then $\psi[A]$ is a quasi-complement of $\psi[C]$ in $\mathrm{Sub}(B/J)$.

(2) If $\psi[C]$ is a quasi-complement of $\psi[A]$ in $\mathrm{Sub}(B/J)$, then C is a quasi-complement of A in $\mathrm{Sub}(B)$.

PROOF. (1) Assume $z \not\in \psi[A]$. So $z = \psi(x)$ for some $x \not\in A$. Let $c \in A(x) \cap C$, with $c \neq 0, 1$. Obviously, $c \not\in A$ and thus $c \not\in J$, i.e. $d = \psi(c) \neq 0$. We have $\psi(c) \neq 1$ because $\psi(c) = 1$ implies $-c \in J$, i.e. $-c \in A$. Moreover, $\psi[A(x)] = \psi[A](\psi(x))$ and thus $\psi(c) \in \psi[A(x)] \cap \psi[C]$.

(2) Assume $x \not\in C$. First we suppose $\psi(x) \in \psi[C]$, i.e. $\psi(x) = \psi(z)$ for some $z \in C$. We set $d = x \triangle z$. Consequently, $d \neq 0$ (since $x \neq z \in C$), and $\psi(d) = 0$ implies $d \in J \subseteq A$, and $d \neq 1$. Observe that $x \triangle z \in C(x) \cap A$. Now we assume

$\psi(x) \not\subseteq \psi[C](x)$. Let $t \neq \mathbf{0}, \mathbf{1}$ be such that $t \in \psi[C](\psi(x)) \cap \psi[A] = \psi[C(x)] \cap \psi[A]$. Let $u \in C(x)$, $v \in A$ be such that $\psi(u) = \psi(v) = t$. So $u \triangle v \in J \subseteq A$ and thus $u \in A$. Consequently, $u \in C(x) \cap A$ and $u \neq \mathbf{0}, \mathbf{1}$ since $\psi(u) = t \neq \mathbf{0}, \mathbf{1}$. □

3.7. We will establish one of the results of REMMEL [1980]. We thank Ivo Rosenberg and Robert Woodrow for helping to make the following argument more precise.

Let $B = F_c(X)$ be the Boolean algebra of finite and cofinite subsets of a set X. Let D be a subalgebra of B. Then D determines a natural partition π_D of X, whose cells are the atoms of D. The cells of π_D are the equivalence classes of the equivalence relation $\equiv_D$ defined by $x \equiv_D y$ whenever $x \in d$ if and only if $y \in d$ for each $d \in D$. Because at most one cofinite set can be an atom of C, we conclude that π_C has at most one infinite cell. Conversely, if π is a partition of X having at most one infinite cell, then there is a unique subalgebra D of B satisfying $\pi = \pi_D$: the elements of D are the finite and the cofinite unions of cells of π.

Now, let $\mathcal{P}$ be the set of partitions π of X having at most one infinite class. Obviously, $\mathcal{P}$ is ordered by refinement (π' is a refinement of π'', denoted by $\pi' \leq \pi''$, if every cell of π'' is a union of cells of π'). Noticing that C is a subalgebra of D if and only if π_D refines π_C, we establish a dual-isomorphism Ψ from the lattice Sub(B) onto the lattice $\mathcal{P}$.

Let us recall some of the properties of the lattice $\mathcal{P}$. Let π' and π'' be members of $\mathcal{P}$. Let $\equiv_{\pi'}$ and $\equiv_{\pi''}$ be the equivalence relations associated with them. We have $\pi' \leq \pi''$ whenever $\equiv_{\pi'}$ is a refinement of $\equiv_{\pi''}$, i.e. $\equiv_{\pi'} \leq \equiv_{\pi''}$ (recall that $\equiv_{\pi'} \leq \equiv_{\pi''}$ means $x \equiv y$ (mod $\equiv_{\pi'}$) implies $x \equiv y$ (mod $\equiv_{\pi''}$) for every x and y in X). Now, let $\pi(0)$ and $\pi(1)$ be members of $\mathcal{P}$. Then, in particular:

(i) $\pi(0) \wedge \pi(1) = \{U_0 \cap U_1\colon U_0 \in \pi(0),\ \ U_1 \in \pi(1)\ \text{ and }\ U_0 \cap U_1 \neq \emptyset\}$ this means that $\pi(0) \wedge \pi(1)$ is the partition associated with $\equiv_{\pi(0)} \cap \equiv_{\pi(1)}$ (i.e. $x \equiv y$ mod($\equiv_{\pi(0)} \cap \equiv_{\pi(1)}$) if and only if $x \equiv y$ (mod $\equiv_{\pi(0)}$) and $x \equiv y$ (mod $\equiv_{\pi(1)}$)).

(ii) In order to describe $\pi = \pi(0) \vee \pi(1)$, it is best to describe $\equiv_\pi$. Let $\equiv_i$ be $\equiv_{\pi(i)}$, for $i = 0, 1$. Then, we see that $\equiv_\pi$ is the transitive closure of $\equiv_0 \cup \equiv_1$. Equivalently, $x \equiv y$ (mod $\equiv_\pi$) just if there is a finite sequence $x = x_0, \ldots, x_n = y$ such that, for each $i < n$, we have $x_i \equiv x_{i+1}$ (mod $\equiv_\mu$) for $\mu = 0$ or 1. It is easy to see that a minimal length sequence is *alternating*, i.e. for $i + 2 \leq n$ we have, in addition, that $x_i \equiv x_{i+1}$ (mod $\equiv_\mu$) if and only if $x_{i+1} \equiv x_{i+2}$ (mod $\equiv_{1-\mu}$).

This dual-isomorphism, the notion of alternating sequence, and the related notion of *non-trivial alternating cycle* (that is, an alternating sequence of length $n \geq 2$ with $x_0 = x_n$, and $x_i \neq x_{i+1}$ for $i < n$) are the keys to a combinatorial proof of the theorem of REMMEL [1980], for which the original proof is very algebraic.

Before giving the proof, we must remark that, if $D = \underline{2} = \{\emptyset, X\}$, then $\pi_{\underline{2}} = \{X\}$ and $\equiv_{\underline{2}}$ is the full equivalence relation, i.e. $x \equiv y$ (mod $\equiv_{\underline{2}}$) for every $x, y \in X$, and if $D = B = F_c(X)$, then π_B is the set of all singletons $\{x\}$, for $x \in X$, and $\equiv_B$ is the trivial equivalence relation, i.e. $x \equiv y$ (mod $\equiv_B$) if and only if $x = y$ for all $x, y \in X$.

3.8.1. PROPOSITION. *Let A and C be subalgebras of $B = F_c(X)$. The following are equivalent*:

(i) *C is a complement of A in* Sub(B);

(ii) $\pi_A \wedge \pi_C = \pi_B$ *and* $\pi_A \vee \pi_C = \pi_{\underline{2}}$.

This is a direct consequence of the fact that Ψ is a dual-isomorphism from $\mathrm{Sub}(\boldsymbol{B})$ onto $\mathscr{P}$.

COMMENTS.

(1) $\pi_A \wedge \pi_C = \pi_B$ means that for every $x \in \boldsymbol{X}$, either $\{x\} \in \boldsymbol{C}$, or $\{x\} \in \boldsymbol{A}$, or there are unique atoms $\alpha \in \boldsymbol{A}$ and $\gamma \in \boldsymbol{C}$ such that: $|\alpha| \geq 2$, $|\gamma| \geq 2$ and $\alpha \cap \gamma = \{x\}$.

(2) $\pi_A \vee \pi_C = \pi_{\underline{2}}$ means that for $x \neq y$ in $\boldsymbol{X}$, there is an alternating path in $\pi_A \cup \pi_C$ joining x and y.

We now state and prove Remmel's theorem:

3.8.2. THEOREM. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{C}$ *be subalgebras of* $\boldsymbol{B} = \boldsymbol{F}_c(\boldsymbol{X})$. *The following are equivalent*:

(i) $\boldsymbol{C}$ *is a quasi-complement of* $\boldsymbol{A}$;

(ii) $\boldsymbol{A}$ *is a quasi-complement of* $\boldsymbol{C}$;

(iii) $\pi_A \wedge \pi_C = \pi_B$, $\pi_A \vee \pi_C = \pi_{\underline{2}}$ *and there is no non-trivial alternating cycle in* $\pi_A \vee \pi_C$, *i.e. in* $\equiv_A \cup \equiv_C$.

Notice that Proposition 3.8.1 and theorem 3.8.2 imply:

3.8.3. COROLLARY. $\mathrm{Sub}\boldsymbol{F}_c(\boldsymbol{X})$ *is a complemented lattice.*

Before proving the theorem, we must remark that (iii) means that we are in the situation illustrated by the Fig. 10.2, in which columns and squares represent atoms of $\boldsymbol{A}$, and horizontal lines and circles are atoms of $\boldsymbol{C}$ (squares and circles are also atoms of $\boldsymbol{F}_c(\boldsymbol{X})$).

COMMENTS (1) (iii) is equivalent to the following:

(iv) $\pi_A \vee \pi_C = \pi_{\underline{2}}$ (i.e. for $x \neq y$ in $\boldsymbol{X}$, there is an alternating path from x to y) and there is no non-trivial alternating cycle in $\pi_A \vee \pi_C$.

Trivially (iii) implies (iv). Conversely, let us suppose (iv) and $\pi_A \wedge \pi_C \neq \pi_B$. Let $x \neq y$ in $\boldsymbol{X}$ be such that $\{x, y\} \subseteq \alpha \cap \gamma$, where $\alpha \in \pi_A$ and $\gamma \in \pi_C$ (so α and γ are atoms of $\boldsymbol{A}$ and $\boldsymbol{C}$, respectively). Then x, y, x is a non-trivial alternating cycle in $\pi_A \vee \pi_C$.

(2) Assume $\boldsymbol{X}$ is finite. Then (iii) gives an algorithm to construct the quasi-complements $\boldsymbol{C}$ of a given $\boldsymbol{A} \in \mathrm{Sub}(\mathfrak{P}(\boldsymbol{X}))$.

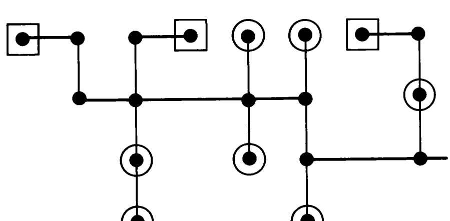

Fig. 10.2

Proof of Theorem 3.8.2. From (iii) and (iv) are symmetric conditions on $\boldsymbol{A}$ and $\boldsymbol{C}$, it follows that it is sufficient to show that (i) and (iv) are equivalent.

(i) implies (iv). Suppose (i) holds. Then trivially $\pi_A \vee \pi_C = \pi_{\underline{2}}$ holds. Suppose, for contradiction, that $x_0, \ldots, x_n$ is a non-trivial alternating cycle in $\boldsymbol{X}$ ($n \geq 2$, and $x_0 = x_n$). There is no loss in assuming that the length of this non-trivial alternating cycle is minimal, i.e. $i < j$ and $x_i = x_j$ imply $i = 0$ and $j = n$, and $\{x_0, x_1\} \subseteq \gamma$ for some $\gamma \in \pi_C$. Notice that $\{x_1, x_2\} \subseteq \alpha$ for some $\alpha \in \pi_A$. Now let $d = \{x_1\}$ and $\boldsymbol{C}' = \boldsymbol{C}(d)$. We have:

$$\pi_{C'} = (\pi_C \backslash \{\gamma\}) \cup \{\gamma \backslash d, d\} .$$

We have an alternating path $x_1, \ldots, x_n$ from x_1 to x_0. From this observation, it is easy to show that two distinct elements of $\boldsymbol{X}$ are joined by an alternating path in $\pi_A \vee \pi_{C'}$ since if such a path in $\pi_A \vee \pi_C$ uses the edge x_1x_0, we replace x_0x_1 by $x_1, \ldots, x_n$ (recall $x_0 = x_n$). Consequently, $\pi_A \vee \pi_{C'} = \pi_{\underline{2}}$, i.e. $\boldsymbol{A} \cap \boldsymbol{C}(d) = \underline{2}$ with $d \notin \boldsymbol{C}$, which contradicts the maximality of $\boldsymbol{C}$.

(iv) implies (i). Let us suppose $\boldsymbol{A} \cap \boldsymbol{C} = \underline{2}$, and $\boldsymbol{A} \cap \boldsymbol{C}(d) = \underline{2}$ for some $d \notin \boldsymbol{C}$. We can assume d is finite (otherwise, replace d by $X \backslash d$). Consequently, there is an atom δ of $\boldsymbol{C}$ such that $|\delta| \geq 2$, $\delta \cap d \neq \emptyset$ and $\delta \cap (X \backslash d) \neq \emptyset$. Let $x \in \delta \cap d$ and $y \in \delta \cap (X \backslash d)$. There is an alternating path $z_0, \ldots, z_n$ from x to y in $\pi_A \vee \pi_{C(d)}$ (so $z_0 = x$ and $z_n = y$). There is no loss in assuming that the alternating path is minimal, and $z_i \notin \delta$ for $0 < i < n$. From the choice of δ, we have $\{x, y\} \subseteq \delta$. Consequently, $x, z_1, \ldots, z_{n-1}, y, x$ induces a non-trivial alternating cycle in $\pi_A \vee \pi_C$ (draw a picture). □

3.9. THEOREM. *Let $\boldsymbol{B}$ be an algebra not isomorphic to some $\boldsymbol{F}_c(\boldsymbol{X})$. Then there are subalgebras $\boldsymbol{A}$ and $\boldsymbol{C}$ of $\boldsymbol{B}$ such that $\boldsymbol{A}$ is a quasi-complement of $\boldsymbol{C}$ and $\boldsymbol{C}$ is not a quasi-complement of $\boldsymbol{A}$ in* Sub($\boldsymbol{B}$).

The above result is due to REMMEL [1980], but we omit the proof. Now, we give an example, due to DÜNTSCH and KOPPELBERG [1985], which solves a problem of REMMEL [1980].

3.10. EXAMPLE. There are subalgebras $\boldsymbol{A}$ and $\boldsymbol{C}$ of $\mathfrak{P}(\omega)$ and a_0 in $\boldsymbol{A}$, such that:

(1) $\boldsymbol{A}$ is isomorphic to $\boldsymbol{F}_c(\omega)$;

(2) $\boldsymbol{C}$ is a quasi-complement of $\boldsymbol{A}$ in Sub($\mathfrak{P}(\omega)$), and $\boldsymbol{C}(a_0) = \mathfrak{P}(\omega)$;

(3) if $\boldsymbol{A}^c$ is the subalgebra of $\mathfrak{P}(\omega)$ completely generated by $\boldsymbol{A}$, then $\boldsymbol{C} \cap \boldsymbol{A}^c = \underline{2}$.

In particular $\boldsymbol{C}$ is a complement and a quasi-complement of $\boldsymbol{A}$ and, $\boldsymbol{A}$ is not a quasi-complement of $\boldsymbol{C}$ in Sub($\mathfrak{P}(\omega)$).

To show this, let $(a_n)_{n<\omega}$ be a partition of ω such that $|a_n| = \omega$ for every $n < \omega$. Let $\boldsymbol{A}$ (resp. $\boldsymbol{A}^c$) be the subalgebra of $\mathfrak{P}(\omega)$ generated (resp. completely generated) by the a_n's. Now let $(m_i)_{0<i<\omega}$ be a partition of a_0 such that $|m_i| = \omega$ for $0 < i < \omega$. We set $b_i = a_i \cup m_i$, for $0 < i < \omega$, and let $\boldsymbol{B}_0$ be the subalgebra of $\mathfrak{P}(\omega)$ completely generated by the b_i's.

Now let $e \subseteq \omega$ be such that $a_i \cap e$ has a unique element α_i and $m_i \cap e$ has a unique element μ_i for $0 < i < \omega$. Let $\boldsymbol{B}_1$ be the set of $b \subseteq \omega$ such that $b \cap e = b_0 \cap e$ for some $b_0 \in \boldsymbol{B}_0$. So $b \in \boldsymbol{B}_1$ if and only if b satisfies: (*) $\alpha_i \in b$ if and only if $\mu_i \in b$ for every $i > 0$. Consequently, if $\gamma \in \omega$, then $\{\gamma\} \in \boldsymbol{B}_1(a_0)$. Obviously,

$\boldsymbol{B}_1$ is complete and thus $\boldsymbol{B}_1(a_0)$ is complete too. Now $\boldsymbol{B}_1(a_0) = \mathfrak{P}(\omega)$ since for each $\gamma \in \omega$ we have $\{\gamma\} \in \boldsymbol{B}_1(a_0)$. That $\boldsymbol{A}^c \cap \boldsymbol{B}_1 = \underline{2}$ follows from the definition of $\boldsymbol{A}^c$, $\boldsymbol{B}_1$ and the property $(*)$. Now $\boldsymbol{A} \vee \boldsymbol{B}_1 = \mathfrak{P}(\omega)$ follows from $\boldsymbol{B}_1(a_0) = \mathfrak{P}(\omega)$. Since $\boldsymbol{B}_1$ is a complement of $\boldsymbol{A}^c$ too, let $\boldsymbol{C}$ be a quasi-complement of $\boldsymbol{A}^c$ containing $\boldsymbol{B}_1$ in $\mathrm{Sub}(\mathfrak{P}(\omega))$.

First, $\boldsymbol{C}$ is a complement of $\boldsymbol{A}$ in $\mathrm{Sub}(\mathfrak{P}(\omega))$, since $\boldsymbol{B}_1 \subseteq \boldsymbol{C}$ and $\boldsymbol{B}_1$ is a complement of $\boldsymbol{A}$.

Secondly, $\boldsymbol{A}$ is not a quasi-complement of $\boldsymbol{C}$ since $\boldsymbol{A} \neq \boldsymbol{A}^c$ and $\boldsymbol{A}^c \cap \boldsymbol{C} = \underline{2}$.

Now we will prove that $\boldsymbol{C}$ is a quasi-complement of $\boldsymbol{A}$. Let $\boldsymbol{C}' \in \mathrm{Sub}(\mathfrak{P}(\omega))$ be such that $\boldsymbol{C} \subseteq \boldsymbol{C}'$ and $\boldsymbol{C} \neq \boldsymbol{C}'$. Since $\boldsymbol{C}$ is a quasi-complement of $\boldsymbol{A}^c$, we can choose $a \in \boldsymbol{A}^c \cap \boldsymbol{C}'$ with $a \neq \mathbf{0}, \mathbf{1}$. We claim that

$$\boldsymbol{A} \cap \boldsymbol{B}_0(a) \neq \underline{2}\,.$$

For such an a, we have $a = \bigcup \{a_i \colon i \in I\}$ for some $I \subseteq \omega$, $\emptyset \neq I \neq \omega$. There is no loss in assuming $0 \notin I$ (otherwise consider $-a = \omega \setminus a$). Let $j \in I$ with $j \neq 0$. The element $c = b_j \cdot a = b_j \cdot a_j = a_j$ satisfies $a_j \in \boldsymbol{A}$, $b_j \cdot a \in \boldsymbol{B}_0(a)$, and thus $c \in \boldsymbol{A} \cap \boldsymbol{B}_0(a)$ with $c \neq \mathbf{0}, \mathbf{1}$, that completes the proof of the claim. We have $\boldsymbol{A} \cap \boldsymbol{B}_0(a) \subseteq \boldsymbol{A} \cap \boldsymbol{C}'$ since $a \in \boldsymbol{C}'$ and $\boldsymbol{B}_0 \subseteq \boldsymbol{B}_1 \subseteq \boldsymbol{C}'$. Consequently, $\boldsymbol{A} \cap \boldsymbol{C}' \neq \underline{2}$.

It was also noticed in DÜNTSCH and KOPPELBERG [1985] that every algebra $\boldsymbol{D}$ having at least four elements and embeddable into $\mathfrak{P}(\omega)$, can be embeddable into $\mathfrak{P}(\omega)$ in such a way it has a complement in $\mathfrak{P}(\omega)$. This is a consequence of $\boldsymbol{B}_1(a_0) = \mathfrak{P}(\omega)$ and $\boldsymbol{A}^c \cap \boldsymbol{B}_1 = \underline{2}$ (simply embed $\boldsymbol{D}$ into $\boldsymbol{A}^c$).

We conclude this section by stating, without proofs, some results of DÜNTSCH and KOPPELBERG [1985], concerning quasi-complements in $\mathrm{Sub}(\mathfrak{P}(\omega))$. The authors use Martin's axiom (M.A.), which is weaker than C.H.

3.11. PROPOSITION. *Assume (M.A.). Let* $\boldsymbol{A}, \boldsymbol{C} \in \mathrm{Sub}(\mathfrak{P}(\omega))$. *If* $|\boldsymbol{A}|, |\boldsymbol{C}| < 2^\omega$, *and* $\boldsymbol{A} \cap \boldsymbol{C} = \underline{2}$, *then* $\boldsymbol{C}$ *is not a quasi-complement of* $\boldsymbol{A}$ *in* $\mathrm{Sub}(\mathfrak{P}(\omega))$.

3.12. REMARK. $\mathfrak{P}(\omega)$ cannot be replaced by an arbitrary algebra $\boldsymbol{B}$ with cardinality 2^ω.

Assume (M.A.). There are an algebra $\boldsymbol{B}$ and $\boldsymbol{A}, \boldsymbol{C} \in \mathrm{Sub}(\boldsymbol{B})$ such that: $|\boldsymbol{B}| = 2^\omega$, $|\boldsymbol{A}| = \omega = |\boldsymbol{C}|$ and $\boldsymbol{A}$ and $\boldsymbol{C}$ are quasi-complement of each other in $\mathrm{Sub}(\boldsymbol{B})$. This can be related to Proposition 2.15.

The following result of DÜNTSCH and KOPPELBERG [1985] gives some information concerning quasi-complements of $\boldsymbol{F}_c(\omega)$ in $\mathrm{sub}(\mathfrak{P}(\omega))$.

3.13. PROPOSITION. *If* $\boldsymbol{C}$ *is a quasi-complement of* $\boldsymbol{F}_c(\omega)$ *in* $\mathrm{Sub}(\mathfrak{P}(\omega))$, *then* $\boldsymbol{C}$ *is a complete atomless algebra.*

PROOF. Assume b is an atom of $\boldsymbol{C}$. Then $b \subseteq \omega$ is infinite, and thus let (b_1, b_2) be a partition of b in two infinite subsets of ω. Obviously, $\boldsymbol{C}$ is a proper subalgebra of $\boldsymbol{C}(b_1)$ and each member of $\boldsymbol{C}(b_1)$ is an infinite subset of ω. Thus, $\boldsymbol{C}(b_1) \cap \boldsymbol{F}_c(\omega) = \underline{2}$, contradicting the fact that $\boldsymbol{C}$ is a quasi-complement of $\boldsymbol{F}_c(\omega)$; hence, $\boldsymbol{C}$ is atomless. Now assume that $\boldsymbol{C}$ is not complete, and let $\boldsymbol{C}^c$ be its completion. We have $C \neq C^c$. By Sikorski's extension theorem, there is a homomorphism f from C^c into $\mathfrak{P}(\omega)$, extending the inclusion map from C into $\mathfrak{P}(\omega)$. From the inclusion map is one-to-one, it follows that f is one-to-one. So we can assume $C \subseteq C^c \subseteq \mathfrak{P}(\omega)$. By density of C in C^c, it follows that if $c \in C^c$, then $c \subseteq \omega$ is infinite, which proves $F_c(\omega) \cap C^c = \underline{2}$. A contradiction. $\square$

REMARK. Let A be an algebra. Let us recall that $\pi(A)$ denotes the least cardinal λ such that A contains a dense subalgebra of cardinality λ. Düntsch and Koppelberg have shown, assuming Martin's Axiom (M.A.), that:

If A is a quasi-complement of $F_c(\omega)$ in Sub($\mathfrak{P}(\omega)$), *then* $\pi(A) > \kappa$, *for every* $\kappa < 2^\omega$.

3.14. EXAMPLE. Assume (M.A.). $F_c(\omega)$ has a quasi-complement C in Sub($\mathfrak{P}(\omega)$) such that C is the completion of the free Boolean algebra on 2^ω generators.

3.15. EXAMPLE. Assume (M.A.). $F_c(\omega)$ has a quasi-complement C in Sub($\mathfrak{P}(\omega)$) such that if A is a complete regular subalgebra of C, then $\pi(A) = 2^\omega$. In particular, the completion of the free Boolean algebra on 2^ω generators is not a regular subalgebra of C.

4. Congruences on the lattice of subalgebras

We will give a survey of congruences on Sub(B), which comes directly from DÜNTSCH [1985b]. Congruences on Sub(B) were intensively studied by DÜNTSCH [1985a]. The following results can be found in DÜNTSCH [1985a], [1985b]. Recall that a *congruence* on a lattice L is an equivalence relation on L compatible with $\wedge$ and $\vee$, and a lattice L is *simple* whenever L has only two (trivial) congruences. For classical notions, on congruences, see GRÄTZER [1978] and BIRKHOFF [1948].

4.1. PROPOSITION. *If θ is a non-trivial congruence on* Sub(B), *i.e. θ is not the identity, and if A is a finite subalgebra of B, then* $A \equiv \underline{2}$ (mod θ).

This result has several consequences for the structure of Sub(B); it follows that Sub(B) is subdirectly irreducible, weakly modular, and weakly complemented. Furthermore, if A is a subalgebra of B, and Sub(A) is simple, then $A \equiv \underline{2}$ (mod θ) for every non-trivial congruence θ on Sub(B).

There are a few positive cases which shall be treated first.

4.2. DEFINITION. A Boolean algebra B is said to be *λ-like* whenever $B \restriction a$ or $B \restriction -a$ is of cardinality less than λ, for every $a \in B$.

For example, if B is the interval algebra of an infinite cardinal λ, then B is λ-like. The only countable ω-like Boolean algebra is $F_c(\omega)$, and, more generally, it can be shown that an infinite Boolean algebra is ω-like if and only if B is a finite–cofinite algebra.

4.3. PROPOSITION. *Let B be an algebra of regular cardinality λ. If B is λ-like, then* Sub(B) *is not simple.*

We will make precise the case of an ω-like Boolean algebra:

4.4. PROPOSITION. *Let B be isomorphic to $F_c(\lambda)$, with $\lambda = \aleph_\gamma$. Then the congruences of* Sub(B) *form a chain of order-type $\gamma + 3$ if γ is finite, and of order-type $\gamma + 2$ otherwise.*

More precisely, each distributive ideal I of Sub(B) induces an equivalence relation $\theta(I)$ on Sub(B) defined by $A \equiv C \pmod{\theta(I)}$ whenever there is $D \in I$ satisfying $A \vee D = C \vee D$. For an infinite cardinal $\delta < \gamma$, let I_δ be an ideal of Sub(B), consisting of subalgebras of B, of cardinality less than $\aleph_\delta$. The fact that I_δ is a distributive element in the lattice of ideals of Sub(B) implies that $\theta(I_\delta)$ is a congruence on Sub(B). Now, Düntsch has proved that the proper congruences on Sub(B) are exactly the $\theta(I_\delta)$, for $0 \leq \delta < \gamma$.

The hypothesis of Theorem 4.3 is essential, since we have:

4.5. PROPOSITION. *Let B be a Boolean algebra of an ordinal, but not isomorphic to a Boolean algebra of a regular cardinal. Then* Sub(B) *is simple.*

4.6. PROPOSITION. Sub(B) *is a simple lattice under each of the following conditions*:

(i) *B is isomorphic to $A \times A$, for some A*;
(ii) *B is isomorphic to the free product of two algebras*;
(iii) *B contains a free subalgebra of its own cardinality.*

So, in particular, if B is homogeneous or complete, then Sub(B) is simple.

If B is countable, then either it contains the countable free algebra, hence Sub(B) is simple; or B is superatomic, and thus isomorphic to the interval algebra of a countable ordinal.

4.7. PROPOSITION. *If B is countable, then* Sub(B) *is simple if and only if B is isomorphic to $F_c(\omega)$.*

4.8. PROPOSITION. *If* Sub(B) *has a prime ideal, then* Sub(B) *is not simple.*

As a consequence of 4.3, 4.5, 4.7 and 4.8, we have:

4.9. COROLLARY. *If B is countable, then* Sub(B) *has no prime ideal.*

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CHAPTER 11

Cardinal Functions on Boolean Spaces

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Contents

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1. Introduction

The topic *cardinal functions* addresses the question how big or how small things can be. For example, if $\mathscr{B}$ is a BA we ask how big independent sets in $\mathscr{B}$ can be by investigating the cardinal function ind, *independence*, defined by

$$\operatorname{ind}(\mathscr{B}) = \operatorname{sub}\{|\mathscr{I}|: \mathscr{I} \text{ is an independent subset of } \mathscr{B}\},$$

and if X is a topological space we ask how long free sequences in X can be by investigating the cardinal function F, *freeness*, defined by

$$F(X) = \sup\{\kappa: X \text{ has a free } \kappa\text{-sequence}\}.$$

Formally, according to Stone's Duality Theorem, there is no real difference between BAs and *Boolean spaces*, i.e. compact zerodimensional spaces. But in fact there is a difference: if your interest is the algebras you care for independence, but probably not for freeness, and if your interest is the spaces you care for freeness but not for independence. (We hope to change this.)

There are several recent surveys. "Cardinal Functions in topology" has seen JUHÁSZ's [1971] book, completely revised (JUHÁSZ [1980]), and HODEL [1984] and JUHÁSZ [1984] each have a chapter about the topic in the *Handbook of Set-Theoretic Topology*. "Cardinal functions on BAs" has MONK's [1984] comprehensive survey. We wish not to do what has been done already. Fortunately, there is not yet a "Cardinal functions on Boolean spaces". We feel it is of interest for BAists to have a survey, with proofs, of cardinal functions on Boolean spaces that makes the topology as easy as possible by working almost exclusively within the class of zerodimensional spaces: of course there is no need at all to struggle to prove something for Hausdorff spaces if there is a much simpler proof for regular spaces. Also, proofs for zerodimensional spaces are cleaner than for regular spaces because one never has to work with pairs $\langle U, V\rangle$ of open sets such that $\bar{U} \subseteq V$: one simply works with clopen sets. Also, there are cases where working with Boolean spaces leads to a genuine simplification, and certain results become redundant because they follow from other results in the presence of compactness. Put differently, the title for this chapter could also be "Cardinal functions in topology for BAists".

Our interest in this chapter is mainly inequalities between cardinal functions. There are several purely BAic cardinal functions that we do not mention here, beause the known inequalities involving them are quite simple; we refer the reader to Monk's survey for a discussion of these functions. Topics we do not discuss are the sup = max problem and the behavior of cardinal functions under operations on Boolean spaces. Also, we do not give enough examples.

Finally, we should mention that this is not a history of the study of cardinal functions. Because of our emphasis of Boolean spaces the important work of Hajnal and Juhász is not mentioned.

2. Conventions

We follow, grudgingly, the convention of this Handbook to use CLOP(X), rather than CO(X), to denote the collection of all clopen subsets of X. But for simplicity the symbols X and $\mathscr{B}$ are always related by

$$\mathscr{B} \text{ is the BA of all clopen subsets of } X\,.$$

Because of our topological bias all our BAs will be algebras of clopen sets.

κ, λ, μ and τ always denote cardinals, and $\alpha, \gamma, \xi, \eta$ and ζ always (and ϕ occasionally) denote ordinals. For typographical reasons we occasionally write $\exp \kappa$ for 2^κ. Of course $\kappa^{<\lambda}$ denotes $\sup_{\mu<\lambda} \kappa^\mu$. For a set S we define, as usual, $[S]^\kappa$, $[S]^{<\kappa}$ and $[S]^{\leq\kappa}$ to be the collections of $K \subseteq S$ such that $|K| = \kappa$, $|K| < \kappa$ or $|K| \leq \kappa$.

For sets S and T we let ${}^S T$ denote the set of functions $S \to T$, and ${}^{<\kappa}2$ denotes $\bigcup_{\xi\in\kappa} {}^\xi 2$. For a function f and a set A we use $f^{\rightarrow}A$ and $f^{\leftarrow}A$ to denote the image and the inverse image of A under f. We call continuous functions *maps*.

For X clear from context we follow the convention that $\bigcap \emptyset = X$. Also, for sets F and G of ordinals, by convention $\max(F) < \min(G)$ holds if $F = \emptyset$ or $G = \emptyset$.

We should mention that we do not follow the convention that all cardinal functions are infinite. (Under this convention, if $\phi(X)$ would be finite, one artificially redefines it to be ω.) (By the remark following 3.7 for most cardinal functions ϕ we have $\phi(X) \geq \omega$ iff $|X| \geq \omega$. We feel that where $\phi(X) < \omega$ is possible, this is a useful warning sign that one does not want to throw away.)

3. A little bit of topology

We assume that the reader is familiar with basic topology, and therefore knows the following lemma. Since it plays such an important role we prove it anyway. Recall that if F is a subset of a space X, then a collection $\mathscr{U}$ of open sets in X is called a *neighborhood base for F in X* if $\mathscr{U}$ consists of neighborhoods of F and if for every neighborhood V of F there is $U \in \mathscr{U}$ with $U \subseteq V$.

3.1. LEMMA. *Let F be a closed subset of a Boolean space X.*

(a) *The collection of all clopen neighborhoods of F is a neighborhood base for F in X.*

(b) *If $\mathscr{U}$ is a collection of clopen neighborhoods of F, then $\mathscr{U}$ is a neighborhood base for F in X iff $\bigcap \mathscr{U} = F$ and $\mathscr{U}$ is a filterbase, i.e. $\forall U, V \in \mathscr{U}\ \exists W \in \mathscr{U}$ $[W \subseteq U \cap V]$.*

PROOF. (a) If V is an open set in X with $F \subseteq V$, then since F is compact and since $\mathscr{B}$ is a base for X there is a finite $\mathscr{U} \subseteq \mathscr{B}$ with $F \subseteq \bigcup \mathscr{U} \subseteq V$. Then $U = \bigcup \mathscr{U}$ is clopen and $F \subseteq U \subseteq V$.

(b) Necessity is clear. To prove sufficiency, let U be any open set in X with $F \subseteq U$. Since $\bigcap \mathscr{U} = F$, the collection $\{X - U\} \cup \mathscr{U}$, which consists of closed sets, has empty intersection, hence there is a finite $\mathcal{F} \subseteq \mathcal{U}$ with $\bigcap \mathcal{F} \subseteq U$. As $\mathcal{U}$ is a filter base there is $W \in \mathcal{U}$ with $W \subseteq \bigcap \mathcal{F}$. Then $F \subseteq W \subseteq U$. □

We also remind the reader of the following:

3.2. FACT. If $\mathcal{S}$ and $\mathcal{T}$ are topologies on a set X with $\mathcal{S}$ compact and $\mathcal{T}$ Hausdorff, then $\mathcal{S} = \mathcal{T}$ iff $\mathcal{S} \supseteq \mathcal{T}$. □

A surjection f from a space X onto a space Y is called *irreducible* if $f^{\rightarrow}A \neq Y$ whenever A is a proper closed subset of X. For a function $f: X \to Y$ one defines for $A \supseteq X$ the *small image* $f^{\#}A$ of A by

$$f^{\#}A = \{y \in Y: f^{\leftarrow}\{y\} \subseteq A\}, \text{ or, equivalently, } f^{\#}A = Y - F^{\rightarrow}(X - A)\,.$$

In the proofs of Theorems 9.3 and 12.6 we will use the following elementary:

3.3. FACT. Let f be a map from a compact space X onto a Hausdorff space Y.
(a) f is closed, i.e. $f^{\rightarrow}A$ is closed in Y for each closed A in X.
(b) f is irreducible iff $f^{\#}U$ is a nonempty open subset of Y for each nonempty open subset U of X.
(c) There is a closed $A \subseteq X$ such that $f \restriction A$ is an irreducible map from A onto Y.

PROOF. (a) If A is closed in X then A is compact, hence so is $f^{\rightarrow}A$. But compact subsets of Hausdorff spaces are compact.
(b) Apply (a) and the equivalent definition of $f^{\#}U$.
(c) Apply Zorn's Lemma and compactness to find a smallest member of

$$\{A \subseteq X: A \text{ is closed and } f^{\rightarrow}A = Y\}\,. \quad \square$$

The BAic significance of irreducibility is this: if X and Y are Boolean spaces and f is a map from X onto Y, then f is irreducible iff $\{f^{\leftarrow}B: B \in \mathrm{CLOP}(Y)\}$ is a dense subalgebra of $\mathrm{CLOP}(X)$.

We conclude this section with a remark we cannot find a better place for.

3.4. REMARK. Our cardinal functions are often defined as the supremum σ of some set K of cardinals. (As mentioned in the Introduction, we will not discuss the question of whether $\sigma \in K$.) Of course one proves $\lambda \geq \sigma$ by proving $\lambda \geq \kappa$ for all $\kappa \in K$. In certain cases the proof that $\lambda \geq \kappa$ requires κ to be regular. In those cases K will always satisfy $\forall \kappa \in K$, $\forall \mu < \kappa\ [\mu \in K]$, hence we prove $\lambda \geq \sigma$ if we prove that $\lambda \geq \kappa$ for all successor cardinals $\kappa \in K$.

4. New cardinal functions from old

Every cardinal function ϕ gives rise to several associated cardinal functions. In this section we define the ones we need.

For every cardinal function ϕ we define associated cardinal functions $h\phi$, *hereditary* ϕ, and $h_c\phi$, *closed hereditary* ϕ, by

$$h\phi = \sup\{\phi(Y)\colon Y \text{ is a subspace of } X\}; \text{ and}$$

$$h_c\phi = \sup\{\phi(Y)\colon Y \text{ is a closed subspace of } X\}\,.$$

One calls a cardinal function ϕ *hereditary* if $\phi = h\phi$, and *closed hereditary* if $\phi = h_c\phi$. If "$\phi = h\phi$" or "$\phi = h_c\phi$" is trivial, we will not mention this. Note that if $\phi_1 = h_c\phi_1$, then to prove $\phi_1 \geq h_c\phi_2$ or $\phi_1 \leq h_c\phi_2$ it suffices to prove $\phi_1 \geq \phi_2$, or $\phi_1 \leq \phi_2$. There is a similar remark for hereditary functions.

Clearly we have

4.1. Fact. If X is Boolean, then

$$h_c\phi(X) = \sup\{\phi(\mathcal{Q})\colon \mathcal{Q} \text{ is a quotient of } \mathcal{B}\}\,. \quad \square$$

There is no good translation of $h\phi$, but in many cases $h\phi$ is important, as we will see; in fact within the class of Boolean spaces we often have $h\phi = h_c\phi$.

To get the topological analogue for subalgebras of what we just did for subspaces, define for each cardinal function ϕ a new function $m\phi$, *mapping* ϕ, by

$$m\phi(X) = \sup\{\phi(Y)\colon \text{there is a map from } X \text{ onto } Y\}\,.$$

4.2. Fact. If X is Boolean, then

$$m\phi(X) = \sup\{\phi(\mathcal{A})\colon \mathcal{A} \text{ is a subalgebra of } \mathcal{B}\}\,. \quad \square$$

For a cardinal function ϕ the property $m\phi = \phi$ will remain unnamed.

Although it is not important to us we mention the following analogue of Fact 4.2, which is an easy corollary to Fact 3.4.

4.3. Fact. If X is a Boolean space then $m_i\phi(X) = ds\phi(\mathcal{B})$, where

$$m_i\phi(X) = \sup\{\phi(Y)\colon Y \text{ is a Boolean space and there is an irreducible map from } X \text{ onto } Y\}, \text{ and}$$

$$ds\phi(\mathcal{B}) = \sup\{\phi(\mathcal{A})\colon \mathcal{A} \text{ is a dense subalgebra of } \mathcal{B}\}\,. \quad \square$$

5. Topological cardinal functions: c, d, L, s, t, w, π, χ, χ_C, $\pi\chi$

We here introduce some basic topological cardinal functions, when possible give their Boolean algebraic translations, and prove some simple results.

The cardinal function *cardinality* needs no introduction. It does not have a good BAic translation. (We think good translations should not mention ultrafilters on $\mathcal{B}$.)

Our next cardinal function is *weight*, defined by

$$w(X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ is a base for } X\}.$$

Weight easily is seen to be hereditary. Its translation is:

5.1. THEOREM. *If X is an infinite Boolean space then $w(X) = |\mathcal{B}|$.*

PROOF. Obviously, $w(X) \leq |\mathcal{B}|$ since $\mathcal{B}$ is a base. Now let $\mathcal{U}$ be any base of X. Then every member of $\mathcal{B}$, being compact and open, is the union of some finite subcollection of $\mathcal{U}$. Hence $|\mathcal{B}| \leq |\mathcal{U}|$ since $\mathcal{U}$, being a base for an infinite Hausdorff space, is infinite. □

5.2. COROLLARY. *$mw = w$ (within the class of Boolean spaces).* □

Actually, the corollary holds within the class of Compact Hausdorff spaces, see Corrollary 6.3(c).

The natural thing to do after looking at bases is to look at neighborhood bases. For $F \subseteq X$ define the *character* of F in X to be

$$\chi(F, X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ is a neighborhood base for } F \text{ in } X\}.$$

We follow the habit of writing x, where $\{x\}$ is formally required, and write $\chi(x, X)$ instead of $\chi(\{x\}, X)$. We define the *character* of X to be

$$\chi(X) = \sup_{x \in X} \chi(x, X),$$

and define the *closed character* of X to be

$$\chi_C(X) = \sup\{\chi(F, X): F \text{ is a closed subset of } X\}.$$

Before we give the translations of these two functions, we define another cardinal function. The *Lindelöf degree* of X is defined by

$$L(X) = \min\{\kappa: \text{every open cover of } X \text{ has a subcover of cardinality at most } \kappa\}.$$

For compact spaces this function is of no interest. However, its offspring hL, *hereditary Lindelöf degree*, defined in Section 4, is important.

5.3. THEOREM. *Let X be an infinite Boolean space.*

(a) *For nonopen closed $F \subseteq X$ we have $\chi(F, X) = L(X - F) = \psi = \kappa$, where*

$$\psi = \min\left\{|\mathcal{U}|: \mathcal{U} \subseteq \mathcal{B} \text{ and } \bigcap \mathcal{U} = F\right\}, \text{ and}$$

$$\kappa = \min\{|\mathcal{K}|: \mathcal{K} \text{ is a cover of } X - F \text{ by compact sets}\}.$$

(b) *If f is a map from X onto a Hausdorff space Y, then* $\chi(y, Y) = \chi(f^{\leftarrow}\{y\}, X)$ *for each* $y \in Y$.
(c) $hL(X) = \chi_{\mathrm{C}}(X) = m\chi(X)$.

Proof. It is clear that $\chi(F, X) \geq \psi$.

Proof that $\psi \geq \kappa$. If $\mathcal{U} \subseteq \mathcal{B}$ satisfies $\bigcap \mathcal{U} = F$, then $\{X - U \colon U \in \mathcal{U}\}$ is a cover of $X - F$ by $|\mathcal{U}|$ compact sets.

Proof that $\kappa \geq \psi$. If $\mathcal{K}$ is a cover of $X - F$ by compact sets, then since F is closed, we can choose $\mathcal{C} \subseteq \mathcal{B}$ with $|\mathcal{C}| \leq |\mathcal{K}|$ which also covers $X - F$, and then $\mathcal{U} = \{X - C \colon C \in \mathcal{C}\}$ satisfies $|\mathcal{U}| \leq |\mathcal{K}|$ and $\mathcal{U} \subseteq \mathcal{B}$ and $\bigcap \mathcal{U} = F$.

Proof that $\psi \geq \chi(F, X)$. Because of 3.1 we can choose $\mathcal{U} \subseteq \mathcal{B}$ with $|\mathcal{U}| = \psi$ such that $\bigcap \mathcal{U} = F$. Then $\mathcal{U}$ is infinite since F is not open, hence if

$$\mathcal{V} = \left\{\bigcap \mathcal{F} \colon \mathcal{F} \in [\mathcal{U}]^{<\omega}\right\},$$

then $|\mathcal{V}| = |\mathcal{U}|$, and $\mathcal{V}$ is a neighborhood base for F because of Lemma 3.1.

Proof of (b). Since X is compact and Y is Hausdorff, f is closed; recall that this is equivalent to the statement that for every $y \in Y$ and for every neighborhood U of $f^{\leftarrow}\{y\}$ in X there is a neighborhood V of y in Y such that $f^{\leftarrow}V \subseteq U$. Of course, (b) is an immediate consequence of this.

Proof that $hL(X) = \chi_{\mathrm{C}}(X)$. Because of the part "$\chi(F, X) = L(X - F)$" of (a) it suffices to show for any space Y that $hL(Y) = h_0L(Y)$, where $h_0L(Y)$ is defined by

$$h_0L(Y) = \sup\{L(S) \colon S \text{ is open in } Y\}.$$

$h_0L(Y) \leq hL(Y)$ is obvious. To prove $hL(Y) \leq h_0L(Y)$, consider any subspace S of Y and any open cover $\mathcal{U}$ of S. Find an open family $\mathcal{U}^{\sim} = \{U^{\sim} \colon U \in \mathcal{U}\}$ in Y with $U = U^{\sim} \cap S$ for $U \in \mathcal{U}$. As $\bigcup \mathcal{U}^{\sim}$ is open there is $\mathcal{V} \subseteq \mathcal{U}$ with $|\mathcal{V}| \leq h_0L(Y)$ such that $\bigcup_{V \in \mathcal{V}} V^{\sim} = \bigcup \mathcal{U}^{\sim}$; clearly $\bigcup \mathcal{V} = S$.

Proof that $\chi_{\mathrm{C}}(X) \leq m\chi(X)$. If F is closed in X, the quotient X/F obtained from X by collapsing F to a point is a Boolean space since $\{B \in \mathcal{B} \colon F \subseteq B\}$ is a neighborhood base for F, by Lemma 3.1. Using (b) with the obvious map $X \to X/F$ we see that $\chi(F, X) \leq \chi(X/F)$.

Finally, the fact that $m\chi(X) \leq \chi_{\mathrm{C}}(X)$ is an immediate consequence of (b). □

The BAic translation of (part of) this theorem is that for every filter $\mathcal{F}$ on $\mathcal{B}$ the minimal number of a generating set for $\mathcal{F}$ is $\chi(\bigcap \mathcal{F}, X)$; in particular, $\chi(\mathcal{B})$ is the smallest cardinal such that every ultrafilter on $\mathcal{B}$ is $\leq\kappa$-generated.

We next weaken the concept of a base: a π*-base* for a space X is a collection of nonempty open sets $\mathcal{A}$ of X such that for every nonempty open U in X there is $A \in \mathcal{A}$ such that $A \subseteq U$. (The difference with a base is that for some $x \in U$ there is $A \in \mathcal{A}$ with $x \in A \subseteq U$, but we do not require this to hold for every $x \in U$.) One defines $\pi(X)$, the π-weight of X by

$$\pi(X) = \min\{|\mathcal{A}| \colon \mathcal{A} \text{ is a } \pi\text{-base for } X\}.$$

(Long ago π-weight was called pseudo-weight; this offers an explanation of sorts for the notation.) BAists are in the habit of calling a subset $\mathscr{A}$ of a BA $\mathscr{B}$ *dense* if

$$\forall B \in \mathscr{B}\ \exists A \in \mathscr{A}[B \neq \emptyset \Rightarrow \emptyset \neq A \subseteq B] .$$

The BAic translation of π-weight is given by:

5.4. PROPOSITION. *(a) If $\mathscr{B}$ is an infinite BA, then*

$$\pi(\mathscr{B}) = \min\{|\mathscr{A}|\colon \mathscr{A} \textit{ is a dense subalgebra of } \mathscr{B}\} .$$

(b) *If $\mathscr{A}$ is a dense subalgebra of $\mathscr{B}$, then $\pi(\mathscr{A}) = \pi(\mathscr{B})$.*

PROOF. Since for each infinite $\mathscr{A} \subseteq \mathscr{B}$ the subalgebra of $\mathscr{B}$ generated by $\mathscr{A}$ has cardinality $|\mathscr{A}|$, this is a simple consequence of 5.9 below, which says that if $\mathscr{C}$ is a dense subset of $\mathscr{B}$ then

$$\pi(\mathscr{B}) = \min\{|\mathscr{A}|\colon \mathscr{A} \subseteq \mathscr{C} \text{ and } \mathscr{A} \text{ is a } \pi\text{-base for } \mathscr{B}\} . \quad \square$$

We have no information about $m\pi$; $h\pi$ will be discussed in Theorem 8.7.

The π-analogue of character is straightforward. First, for $x \in X$ call a family $\mathscr{A}$ a *neighborhood π-base* if $\mathscr{A}$ consists of nonempty open sets and if for every neighborhood U of x there is $A \in \mathscr{A}$ with $A \subseteq U$ (but not necessarily with $x \in A$), then define the *π-character* of x in X to be

$$\pi\chi(x, X) = \min\{|\mathscr{A}|\colon \mathscr{A} \text{ is a neighborhood } \pi\text{-base at } x\} .$$

Finally, define the *π-character* of X to be

$$\pi\chi(X) = \sup_{x \in X} \pi\chi(x, X) .$$

These do not have good BAic translations.

The next cardinal function is what topologists call density. Recall that topologists call a subset D of a space X *dense* in X if $\bar{D} = X$. The *density* of a space X is defined by

$$d(X) = \min\{|D|\colon D \text{ is dense in } X\} .$$

A not very good BAic translation of density is that for a BA $\mathscr{B}$:

$$d(\mathscr{B}) = \min\{|\mathscr{F}|\colon \mathscr{F} \text{ is a collection of ultrafilters with } \bigcup \mathscr{F} = \mathscr{B}\} .$$

A better translation is given by:

5.5. THEOREM. *If X is a Boolean space, then*

$$d(X) = \min\{\kappa\colon \mathscr{B} \textit{ embeds into } \mathscr{P}(\kappa)\} .$$

Proof. Since $md(Y) \leq d(Y)$ for every Y, it suffices to take dense $D \subseteq X$ and observe that $B \mapsto B \cap D$ ($B \in \mathscr{B}$) is an embedding of $\mathscr{B}$ into $\mathscr{P}(D)$. (The topological version of this proof is this: let Δ be D with the discrete topology. Then id_D extends to a map $\beta\Delta \to X$.) □

Information about hd will be given in Theorem 8.2.

An important cardinal function, which can be thought of as the local form of hereditary density, is tightness: For $x \in X$ define the *tightness of X at x* by

$$t(x, X) = \min\{\kappa: \forall A \subseteq X[x \in \bar{A} \Rightarrow \exists B \subseteq A[x \in \bar{B} \ \& \ |B| \leq \kappa]]\},$$

and define the tightness of X by

$$t(X) = \sup_{x \in X} t(x, X).$$

This function does not have a good BAic translation at all, but in Theorem 8.7 we will see that $t(X) = F(\mathscr{B})$, the freeness of $\mathscr{B}$, is a function that can be defined BAically, see Theorem 7.2. A simple result we use in Section 10 is:

5.6. Lemma. *If $\mathscr{F}$ is a chain of closed sets such that $\{\mathscr{F}, \subset\}$ has order type $t(X)^+$, then $\bigcup \mathscr{F}$ is closed.* □

A cardinal function that measures how wide a space or BA is, is cellularity. Call a *cellular family* for a space X, any pairwise disjoint collection of nonempty open sets. The *cellularity* of X is defined by

$$c(X) = \sup\{|\mathscr{C}|: \mathscr{C} \text{ is a cellular family for } X\}.$$

If one defines a cellular subset of a BA the obvious way, then the BAic translation of cellularity is trivial. We have the following simple result about cellular families:

5.7. Proposition. *Let Y be an infinite subset of the zerodimensional space X. There is an infinite cellular $\mathscr{U}$ such that $\forall U \in \mathscr{U}\ [U \cap Y \neq \emptyset]$.*

Proof. Let I denote the set of isolated points of the subspace Y.

Case 1. I is infinite.

Let $x: \omega \to I$ be an injection. With recursion on $n \in \omega$ pick $U_n \in \mathscr{B}$ such that $X_n \in U_n$ but $U_n \cap ((I - \{x_n\}) \cup \bigcup_{k \in n} U_k) = \emptyset$.

Case 2. I is finite.

Then with recursion on $n \in \omega$ pick $U_n \in \mathscr{B}$ such that $U_n \cap Y \neq \emptyset$ and $(Y - I) - \bigcup_{k \in n} U_k \neq \emptyset$.

In both cases $\mathscr{U} = \mathrm{ran}(U)$ is as required. □

At this point we mention that if X is a Boolean space, then for most cardinal functions ϕ one has $\phi(X)$ infinite iff X is infinite; one quickly proves this using the fact that each infinite subset of X has a limit point, or using Proposition 5.7.

The cardinal function hc, hereditary cellularity, is usually called *spread*, and is defined by

$$s(X) = \sup\{|S|: S \text{ is a discrete subspace of } X\}.$$

5.8. THEOREM. $hc(X) = h_c c(X) = s(X)$.

PROOF. *Proof that* $hc(X) \le s(X)$. If $\mathcal{U}$ is a cellular family for $Y \subseteq X$, and if $S \subseteq \bigcup \mathcal{U}$ is such that $\forall U \in \mathcal{U}[|S \cap U| = 1]$, then S is a discrete subspace of Y, hence of X.

Proof that $h_c c(X) \subseteq hc(X)$. Obvious.

Proof that $s(X) \le h_c(X)$. If S is a discrete subspace of X, then S is open in $\bar{S}$, hence $\{\{x\}: x \in S\}$ is a cellular family in $\bar{S}$. □

We use cellularity in Theorems 10.6 and 10.8, and use spread in Theorems 10.5 and 10.9.

We conclude this section with pointing out that bases and similar collections can often be chosen small and nice:

5.9. LEMMA. (a) *For every base $\mathcal{A}$ for X there is $\mathcal{C} \subseteq \mathcal{A}$ with $|\mathcal{C}| = w(X)$ which is a base for X.*

(b) *For every π-base $\mathcal{A}$ for X there is $\mathcal{C} \subseteq \mathcal{A}$ with $|\mathcal{C}| = \pi(X)$ which is a π-base for X.*

(c) *Let $F \subseteq X$. For every neighborhood base $\mathcal{A}$ for F in X there is $\mathcal{C} \subseteq \mathcal{A}$ with $|\mathcal{C}| = \chi(F, X)$ which is a neighborhood base for F in X.*

(d) *Let $x \in X$. For every π-base $\mathcal{A}$ for X there is $\mathcal{U} \subseteq \mathcal{A}$ with $|\mathcal{U}| = \pi\chi(x, X)$ which is a neighborhood π-base for x.*

PROOF. (b), (c) and (d) are simple consequences of the Axiom of Choice.

Proof of (a). If $w(X)$ is finite, then X has a smallest base, so we may assume without loss of generality that $w(X) \ge \omega$. Let $\mathcal{Q}$ be a base for X. As $|\mathcal{Q}| \ge \omega$ the collection

$$\mathbb{P} = \{\langle P, Q\rangle \in \mathcal{Q} \times \mathcal{Q}: \exists A \in \mathcal{A}[P \subseteq A \subseteq Q]\}$$

has $|\mathbb{P}| \le |\mathcal{Q}|$. Let $c: \mathbb{P} \to \mathcal{A}$ be the obvious choice function. Then $\mathcal{C} = \operatorname{ran}(c)$ is a base for X with $|\mathcal{C}| \le |\mathbb{P}|$. □

In 5.9(d) one cannot weaken "π-base" to "neighborhood π-base". Let X be a space containing points x and y with $\pi\chi(x, X) < \pi\chi(y, X)$. (For example, let X be $\omega_1 + 1$ in the order topology, then $\pi\chi(\omega, X) = \omega$ and $\pi\chi(\omega_1, X) = \omega_1$.) Let $P = \{x, y\}$, and as in the proof of 5.3 let X/P denote the quotient obtained from X by collapsing P to a point. Then $\pi\chi(P, X/P) = \pi\chi(x, X)$, and if $\mathcal{U}$ is a neighborhood π-base for y in X, then $\mathcal{U}$ is a neighborhood π-base for P in X/P (since we may assume $P \cap \bigcup \mathcal{U} = \emptyset$) no subfamily of which with cardinality less than $\pi\chi(y, X)$ is a neighborhod π-base for P in X/P.

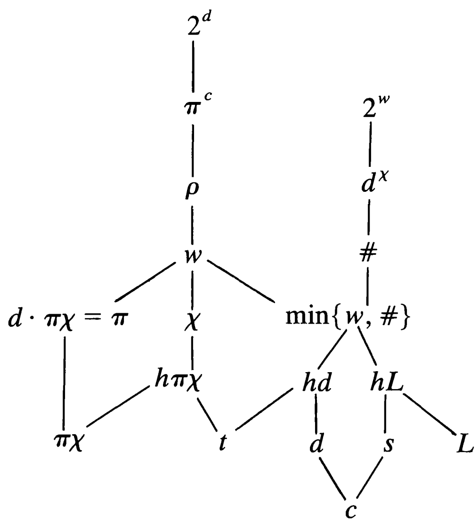
6. Basic results

Let us define one more cardinal function. Recall that a subset U of a space X is called *regularly open* if $U = \mathrm{int}_X \mathrm{cl}_X U$. Let $\mathrm{RO}(X)$ denote the family of regularly open subsets of X. The unnamed function ρ is defined by

$$\rho(X) = |\mathrm{RO}(X)|\,.$$

This is one of the few cardinal functions that is not a minimum or a supremum. It is of interest because $\rho(X)$ is the cardinality of the completion of $\mathscr{B}$.

6.1. Theorem. *For the class of infinite zerodimensional spaces the following diagram holds, with* $\begin{smallmatrix}\psi\\ |\\ \phi\end{smallmatrix}$ *signifying that* $\psi \geq \phi$ *and with* # *denoting cardinality*:



Proof. The inequalities we prove are not completely trivial consequences of the definitions.

Proof that $\rho(X) \leq \pi(X)^{c(X)}$. For $A \subseteq X$, A^0 denotes the interior of A. We need to know that $A^0 \in \mathrm{RO}(X)$ whenever A is closed: using the facts that $A^- \subseteq B^-$ and $A^0 \subseteq B^0$ whenever $A \subseteq B$ and $A^0 \subseteq A \subseteq A^-$ for all $A \subseteq X$ we see that if A is closed then $A^0 \subseteq A$ hence $A^{0-} \subseteq A^- = A$ hence $A^{0-0} \subseteq A^0$, and $A^0 \subseteq A^{0-}$ hence $A^0 = A^{00} \subseteq A^{0-0}$. We prove our inequality by showing that if $\mathscr{V}$ is a π-base for X and if

$$\mathscr{R} = \left\{ \left(\bigcup \mathscr{U} \right)^{-0} : \mathscr{U} \text{ is a cellular subfamily of } \mathscr{V} \right\},$$

then $\mathscr{R} = \mathrm{RO}(X)$. By what we just observed, $\mathscr{R} \subseteq \mathrm{RO}(X)$. Also, for each $S \in \mathrm{RO}(X)$, if $\mathscr{U}$ is a maximal cellular subfamily of $\{V \in \mathscr{V} : V \subseteq S\}$, and $A = \bigcup \mathscr{U}$, then $S \subseteq A^-$ by maximality, and of course $A \subseteq S$, hence $S^- = A^-$, and therefore $S = S^{-0} = A^{-0} \in \mathscr{R}$.

Proof that $w(X) \leq \rho(X)$. $\mathscr{B} \subseteq \mathrm{RO}(X)$.

Proof that $\pi(X)^{c(X)} \leq 2^{d(X)}$. First note that $w(X) \leq 2^{d(X)}$: if $\mathscr{A}$ is a base for X with $\mathscr{A} \subseteq \mathscr{B}$ and $|\mathscr{A}| = w(X)$ (which exists by 5.9(a)), and if D is dense in X, then clearly $A \mapsto A \cap D$ $(A \in \mathscr{A})$ is an injection $\mathscr{A} \to \mathscr{P}(D)$. As $\pi(X) \leq w(X)$ and $c(X) \leq d(X)$ it follows that $\rho(X) \leq \pi(X)^{c(X)} \leq (2^{d(X)})^{d(X)} = 2^{d(X)}$.

Proof that $|X| \leq d(X)^{\chi(X)}$. Let D be dense in X and for $x \in X$ choose a neighborhood base $\mathscr{A}_x$ with $|\mathscr{A}_x| \leq \chi(X)$. For $x \in X$ and $A \in \mathscr{A}_x$ we can choose $c(x, A) \subseteq A \cap D$ with cardinality at most $\chi(X)$ (indeed, with cardinality at most $t(X)$) such that $x \in \overline{c(x, A)}$. For $x \in X$ define

$$\phi(x) = \{c(x, A) \colon A \in \mathscr{A}_x\}\ .$$

Then ϕ is a function $X \to [[D]^{\leq \chi(X)}]^{\leq \chi(X)}$, so we prove our inequality if we prove ϕ is an injection. Since X is Hausdorff we see from our choice of c that

$$\{x\} \subseteq \bigcap_{C \in \phi(x)} \bar{C} \subseteq \bigcap_{A \in \mathscr{A}_x} \bar{A} = \{x\}\ .$$

Proof that $t(X) \leq h\pi\chi(X)$. Consider any $A \subseteq X$ and any $x \in \bar{A}$. If κ abbreviates $h\pi\chi(X)$ we must find $C \in [A]^{\leq\kappa}$ with $x \in \bar{C}$. So without loss of generality assume $x \notin A$. Let $\mathscr{G}$ be a neighborhood π-base for x in the subspace $\{x\} \cup A$ with $|\mathscr{G}| \leq \kappa$. Since x is not isolated in the subspace $\{x\} \cup A$, every member of $\mathscr{G}$ meets A. Hence we can choose $C \subseteq A$ with $|C| \leq |\mathscr{G}|$ such that $\forall G \in \mathscr{G}\ [C \cap G \neq \emptyset]$. Then clearly x is in the closure of C in the subspace $\{x\} \cup A$ of X, hence $x \in \bar{C}$. □

Our next objective is to give a useful method, due to ARHANGEL'SKIĬ [1959], to find upper bounds on weight. Call a *network* for a space X, any collection $\mathscr{A}$ of subsets of X such that for every $x \in X$ and for every neighborhood U of x there is $A \in \mathscr{A}$ such that $x \in A \subseteq U$; the difference with a base is that the members of $\mathscr{A}$ are not required to be open. Define the *netweight*, $nw(X)$, of X by

$$nw(X) = \min\{|\mathscr{A}| \colon \mathscr{A} \text{ is a network for } X\}\ .$$

6.2. LEMMA. *If X is compact Hausdorff, then $w(X) = nw(X)$.*

PROOF. Trivially $nw(Y) \leq w(Y)$ for every space Y. (nw fits below $\min(w, \#)$ and above hd and hL in Theorem 6.1.) Also, $w(X) = |X| = nw(X)$ if $nw(X)$ is finite. Now let $\mathscr{A}$ be an infinite network for X, and let τ denote the topology of X. The family

$$\begin{aligned} \mathbb{P} = \{\langle A, B\rangle \in \mathscr{A} \times \mathscr{A} \colon &\exists \text{ open } U, V \text{ in } X \\ &[U \cap V = \emptyset\ \&\ A \subseteq U\ \&\ B \subseteq V]\} \end{aligned}$$

has cardinality at most $|\mathscr{A}|$. There is a collection $\mathscr{U}$ consisting of at most $|\mathscr{A}| = (|\mathscr{A}|^2)^{<\omega}$ open sets such that

$$\forall \langle A, B\rangle \in \mathbb{P}\ \exists U, V \in \mathscr{U}\ [U \cap V = \emptyset\ \&\ A \subseteq U \text{ and } B \subseteq V]$$

which is closed under finite intersection. So $\mathscr{U}$ is a base for a topology $\mathscr{S}$ on X. This topology is easily seen to be Hausdorff since X is. Also, $\mathscr{S} \subseteq \mathscr{T}$ since $\mathscr{U} \subseteq \mathscr{T}$. As $\langle X, \mathscr{T} \rangle$ is compact it follows from Lemma 3.2 that $\mathscr{S} = \mathscr{T}$. □

6.3. COROLLARIES. *If X is a compact Hausdorff space, then*

(a) $w(X) < |X|$.

(b) (The Addition Theorem). *If $\mathscr{F}$ is any collection of subspaces of X with* $\bigcup \mathscr{F} = X$, then $w(X) \leq \Sigma_{F \in \mathscr{F}}\, w(X)$.

((c) *If Y is a continuous Hausdorff image of X, then* $w(Y) \leq w(\bar{X})$.)

PROOF. (a) Note that $nw(Y) \leq |Y|$ for every space Y.

(b) Simply note that $nw(X) \leq \Sigma_{F \in \mathscr{F}}\, nw(F)$.

((c) If f is a map from any space S onto any space T then $nw(T) \subseteq nw(S)$ since if $\mathscr{A}$ is a network for S then $\{f^{\rightarrow}A\colon A \in \mathscr{A}\}$ is a network for T.

All of this is parenthetical since Corollary 5.2 already gives us the BAic case of (c).) □

Note how simple this becomes once one has the right concept, namely that of nw.

In Theorem 10.9 we will prove that if X is a Boolean space, then $w(X) \leq 2^{s(X)}$. For the proof we need the following simple but powerful lemma twice:

6.4. LEMMA (ŠAPIROVSKIĬ [1972]; HAJNAL and JUHÁSZ [1973]). *For every open cover $\mathscr{U}$ of an arbitrary space X there are $S \subseteq X$ and $\mathscr{V} \subseteq \mathscr{U}$ with $|S|, |\mathscr{V}| \leqslant s(X)$ such that $\bar{S} \cup \bigcup \mathscr{V} = X$.*

PROOF. Choose $U\colon X \rightarrow \mathscr{U}$ with $\forall x \in X\ [x \in U_x]$. Let P be a maximal member of the poset

$$\{P\colon P \text{ is a partial function } X \rightarrow \mathscr{B} \text{ such that } \forall x \in X\ [x \in P_x \subseteq U_x] \text{ and } \forall x \neq y \in \mathrm{dom}(P)\ [x \not\in P_y]\}\,,$$

ordered by inclusion. Clearly, $S = \mathrm{ran}(P)$ is a discrete subspace of X, hence $|S| \leq s(X)$, and therefore $\mathscr{V} = U^{\rightarrow}S$ also satisfies $|\mathscr{V}| \leq s(X)$. Suppose there is $x \in X$ with $x \not\in \bigcup \mathscr{V} \cup \bar{S}$, hence with $\forall y \in S\ [X \not\in P_y]$. There is $B \in \mathscr{A}$ with $x \in B \subseteq U_x$ and $B \cap S = \emptyset$. Then $P \cup \{\langle x, B\rangle\}$ shows P is not maximal. □

We also need the following weak version of the inequality $w \leq 2^s$.

6.5. LEMMA (ŠAPIROVSKIĬ [1972]). *If X is a Boolean space, then $\chi(X) \leq 2^{s(X)}$.*

PROOF. Let $x \in X$. Since X is zerodimensional, $\mathscr{U} = \{U \in \mathscr{B}\colon x \not\in B\}$ covers $X - \{x\}$. By 6.4 there are $S \subseteq X$ and $\mathscr{V} \subseteq \mathscr{U}$ with $|S|, |\mathscr{V}| \leq s(X)$ such that $X - \{x\} \subseteq \bar{S} \cup \bigcup \mathscr{V}$; while we could take $S \subseteq X - \{x\}$, we take an $S \subseteq X$ with $x \in S$. From Theorem 6.1 we see that

$$\chi(x, \bar{S}) \leq w(\bar{S}) \leq \exp(d(\bar{S})) \leq \exp(|S|)\,.$$

As X is zerodimensional it follows that there is $\mathscr{G} \subseteq \mathscr{B}$ with $|\mathscr{G}| \leq \exp(|S|)$ such that $\bar{S} \cap \bigcap \mathscr{G} = \{x\}$. The collection $\mathscr{H} = \{X - V: V \in \mathscr{V}\}$ satisfies $|\mathscr{H}| = |\mathscr{V}| \leq s(X)$ and $x \in \bigcap \mathscr{H}$ and $\bigcap \mathscr{H} \subseteq \bar{S}$. Hence, $\mathscr{K} = \mathscr{G} \cup \mathscr{H}$ is a subcollection of $\mathscr{B}$ with $|\mathscr{K}| \leq \exp(s(X))$ and $\bigcap \mathscr{K} = \{x\}$. It follows from 5.3(a) that $\chi(x, X) \leq \exp(s(X))$. $\square$

We conclude this section with an elementary result about dense suspaces.

6.6. LEMMA. *Let Y be a dense subspace of the zerodimensional space X and let $y \in Y$. Then*
(a) $c(Y) = c(X)$;
(b) $\pi(Y) = \pi(X)$;
(c) $\chi(y, Y) = \chi(y, X)$;
(d) $\pi\chi(y, Y) = \pi\chi(y, X)$.

PROOF. For any collection $\mathscr{F}$ of subsets the restriction of $\mathscr{F}$ to Y is defined to be $\mathscr{F} \restriction Y = \{F \cap Y: F \in \mathscr{F}\}$. Then $\mathscr{B} \restriction Y$ need not be $\mathrm{CLOP}(Y)$, but of course $\mathscr{B} \restriction Y$ is a base for Y, and

$$(*) B \mapsto B \cap Y,\ (B \in \mathscr{B}),\ \text{is a monomorphism } \mathscr{B} \to \mathrm{CLOP}(Y)$$

(it is one-to-one since Y is dense).

Proof that $c(X) \leq c(Y)$. $(*)$.

Proof that $c(Y) \leq c(X)$. For every cellular $\mathscr{C} \subseteq \mathrm{CLOP}(Y)$ there is a cellular $\mathscr{A} \subseteq \mathscr{B}$ such that $\mathscr{A} \restriction Y$ is cellular in Y with $|\mathscr{A} \restriction Y| = |\mathscr{C}|$ since $\mathscr{B} \restriction Y$ is a base. But $\mathscr{A}$ is cellular in X because of $(*)$.

Proof that $\pi(Y) \leq \pi(X)$. If $\mathscr{A}$ is a π-base for X, then $\mathscr{A} \restriction Y$ is easily seen to have the property that for every nonempty open set U of Y there is $A \in \mathscr{A} \restriction Y$ with $A \subseteq U$, and $\emptyset \not\in A \restriction Y$ since Y is dense.

Proof that $\pi(X) \leq \pi(Y)$. Similar to the proof below that $\pi\chi(y, X) \leq \pi\chi(y, X)$.

Proof that $\pi\chi(y, Y) \leq \pi\chi(y, X)$. Similar to the proof that $\pi(Y) \leq \pi(X)$.

Proof that $\pi\chi(y, X) \leq \pi\chi(y, Y)$. Since $\mathscr{B} \restriction Y$ is a (π-)base for Y, there is $\mathscr{A} \subseteq \mathscr{B}$, with $|\mathscr{A}| = \pi\chi(y, Y)$ such that $\mathscr{A} \restriction Y$ is a neighborhood π-base for y in Y, because of 5.9(d). To show $\mathscr{A}$ is a neighborhood π-base for y in X consider any $B \in \mathscr{B}$ with $y \in B$. There is $A \in \mathscr{A}$ with $A \cap Y \subseteq B \cap Y$, and then $A \subseteq B$ by $(*)$.

Proof that $\chi(y, X) = \chi(Y, Y)$. Similar to the proof that $\pi\chi(y, X) = \pi\chi(y, Y)$. $\square$

One can also prove that $\rho(X) = \rho(Y)$ if Y is dense in X, but we manage to avoid using this.

6.7. COROLLARY. *If $\phi \in \{c, \pi, \pi\chi\}$, then $h\phi = h_c\phi$.* $\square$

The corollary is trivially true for χ since χ is hereditary: if $x \in Y \subseteq X$, then $\chi(x, Y) \leq \chi(x, X)$.

7. Variations of independence

A subset $\mathcal{I}$ of a BA $\mathcal{B}$ is called *independent* if

$$\forall F \in \mathcal{S}\, \forall \mathcal{G} \in [\mathcal{S}]^{<\omega}\Big[F \not\subseteq \mathcal{G} \Rightarrow F - \bigcup \mathcal{G} \neq \emptyset\Big],$$

(Recall that, by convention, $\bigcap \emptyset = X$).) The *independence number* of a BA $\mathcal{B}$ is defined by

$$\operatorname{ind}(\mathcal{B}) = \sup\{|\mathcal{I}|\colon \mathcal{I} \text{ is an independent subset of } \mathcal{B}\}.$$

We have information about ind in Theorems 7.7 and 9.3, Corollary 9.5, and Theorems 10.8, 11.6 and 14.2.

In this section we play with $|\mathcal{F}|$ and $|\mathcal{G}|$, and also with wellordering, and get some other cardinal functions:

$\vert\mathcal{F}\vert$	$\vert\mathcal{G}\vert$	(Not well-ordered)	Well-ordered variation
$<\omega$	$<\omega$	independence	freeness
$<\omega$	1	spread	hereditary density
1	$<\omega$	spread	hereditary Lindelöf degree
1	1	incomparability	hereditary cofinality

While freeness is known in its topological form, its BAic form seems not to have been pointed out before. The other functions are known already (with *h-cof* in the form of the right-hand side of Theorem 7.5), and those that have a direct topological translation, s, hd and hL, are known as such; see Monk [1984].

A first variation is to have $|\mathcal{F}| = 1$ and $|\mathcal{G}| < \omega$: Call $\mathcal{S} \subseteq \mathcal{B}$ *discrete* if

$$\forall F \in \mathcal{S}\, \forall \mathcal{G} \in [\mathcal{S}]^{<\omega}\Big[F \not\subseteq \mathcal{G} \Rightarrow F - \bigcup \mathcal{G} \neq \emptyset\Big],$$

and define the *spread* of $\mathcal{B}$ to be

$$s(\mathcal{B}) = \sup\{|\mathcal{S}|\colon \mathcal{S} \text{ is a discrete subset of } \mathcal{B}\}.$$

Recall from Section 5 that the *spread* of a space X is defined by

$$s(X) = \sup\{|S|\colon S \text{ is a discrete subspace of } X\}.$$

The following result shows that the definition of $s(\mathcal{B})$ is not capricious.

7.1. Theorem. $s(\mathcal{B}) = s(X)$.

Proof. A subset S of X is discrete iff there is $\Phi\colon S \to \mathcal{B}$ such that

(1) $\forall x, y \in S[x \in \Phi(y) \text{ iff } x = y]$,

and a subset $\mathcal{S}$ of $\mathcal{B}$ is discrete iff there is $\phi\colon \mathcal{S} \to X$ such that

(2) $\forall A, B \in \mathcal{S}[\phi(A) \in B \text{ iff } A = B]$.

Clearly, if $\Phi\colon S\to\mathscr{B}$ satisfies (1), then Φ is an injection and $\phi=\Phi^{-1}$ is a function that shows $\mathscr{S}=\operatorname{ran}(\Phi)$ satisfies (2). Also, if $\phi\colon\mathscr{S}\to X$ satisfies (2), then ϕ is an injection, then $\Phi=\phi^{-1}$ is a function that shows $S=\operatorname{ran}(\phi)$ satisfies (1). □

The second variation has $|\mathscr{F}|<\omega$ and $|G|=1$. As our table suggests, this variation also yields spread. Indeed, one quickly verifies that

$$s(\mathscr{B})=\sup\Big\{|\mathscr{S}|\colon \mathscr{S}\subseteq\mathscr{B}\text{ satisfies }\forall\mathscr{F}\in[\mathscr{S}]^{<\omega}\,\forall G\in\mathscr{S}\\ \Big[G\notin\mathscr{F}\Rightarrow\bigcap\mathscr{F}-G\neq\emptyset\Big]\Big\}.$$

Since the dual of $F-\bigcup\mathscr{G}$ is $\bigcap\mathscr{G}-F$, we interpret this equality as saying that spread is self-dual.

In our third variation of independence we have both $|\mathscr{F}|=1$ and $|\mathscr{G}|=1$: a subset $\mathscr{P}$ of $\mathscr{B}$ is called a *pie*, i.e. set of pairwise incomparable elements if

$$\forall F,G\in\mathscr{P}[F\neq G\Rightarrow F-G\neq\emptyset\ (\text{i.e. } F\nsubseteq G)].$$

There are no further bad puns: the *incomparability number* of $\mathscr{B}$ is defined by

$$\operatorname{inc}(\mathscr{B})=\sup\{|\mathscr{P}|\colon \mathscr{P}\subseteq\mathscr{B}\text{ is a pie}\}.$$

We have results about inc in Theorems 7.7, 8.5 and 8.6.

Let us now consider the variations of the above variations that result if we wellorder our special sets. We begin with the analogue of independence. So for a cardinal κ call *free κ-sequence* in $\mathscr{B}$ any function $\Phi\colon\kappa\to\mathscr{B}$ such that

$$\forall\text{ finite }F,G\subseteq\kappa\Big[\max(F)<\min(G)\Rightarrow\bigcap_{\phi\in F}\Phi(\phi)-\bigcup_{\gamma\in G}\Phi(\gamma)\neq\emptyset\Big],$$

and define the *freeness* of $\mathscr{B}$ to be

$$F(\mathscr{B})=\sup\{\kappa\colon\text{there is a free }\kappa\text{-sequence in }\mathscr{B}\}.$$

Let us define the topological analogues. For a cardinal κ call *free κ-sequence* in X any function $\Psi\colon\kappa\to X$ such that

$$\forall\alpha\in\kappa[\operatorname{cl}\Psi^{\to}[0,\alpha)\cap\operatorname{cl}\Psi^{\to}[\alpha,\kappa)=\emptyset]$$

and define the *freeness* of X to be

$$F(X)=\sup\{\kappa\colon\text{there is a free }\kappa\text{-sequence in }X\}.$$

This is an important cardinal function due to ARHANGEL'SKIĬ [1969].

7.2. THEOREM. $F(X)=F(\mathscr{B})$.

PROOF. The proof is like the proof of $s(X)=s(\mathscr{B})$:

Call $\mathscr{C}\subseteq\mathscr{C}$ *centered* if $\forall\mathscr{F}\subseteq\mathscr{C}\,[1\leq|\mathscr{F}|<\omega\Rightarrow\bigcap\mathscr{F}\neq\emptyset]$. Note that $\Phi\colon\kappa\to\mathscr{B}$ is free iff

(1) $\forall\alpha\in\kappa[\{\Phi(\phi)-\Phi(\gamma)\colon\phi\leq\alpha<\gamma\}$ is centered]

iff we can choose $\Psi\colon\kappa\to X$ such that

(1*) $\forall\alpha\in\kappa[\Psi(\alpha)\in\bigcap_{\phi\leq\alpha}\Phi(\phi)-\bigcup_{\gamma>\alpha}\Phi(\gamma)]$.

Similarly, from Lemma 3.1(a) we see that $\Psi\colon\kappa\to X$ is free iff

(2) $\forall\xi\in\kappa\,\exists B\in\mathscr{B}[\Psi^{\to}[\xi,\kappa)\subseteq B\;\&\;\Psi[0,\xi)\subseteq X-B]$,

i.e. iff we can choose $\Phi\colon\kappa\to\mathscr{B}$ such that

(2*) $\forall\alpha\in\kappa[\Psi^{\to}[0,\alpha)\subseteq X-\Phi(\alpha)$ and $\Psi^{\to}[\alpha,\kappa)\subseteq\Phi(\alpha)]$.

Now notice that (1*) iff

$$\forall\alpha,\,\xi\in\kappa[\Psi(\xi)\in\Phi(\alpha)\text{ iff }\alpha\leq\xi]$$

iff (2*), i.e. Φ witnesses that Ψ is free iff Ψ witnesses that Φ is free. $\square$

Note that freeness, like spread, is self-dual, i.e.

$$F(\mathscr{B})=\sup\Big\{\kappa\colon\text{there is a function }\phi\colon\kappa\to\mathscr{B}\text{ such that }\forall\text{ finite}$$
$$F,G\subseteq\kappa\Big[\min(F)>\max(G)\Rightarrow\bigcap_{\phi\in F}\Phi(\phi)-\bigcup_{\gamma\in G}\Phi(\gamma)\neq\emptyset\Big]\Big\}.$$

Next we consider the analogues of spread: For a cardinal κ call *left-separated κ-sequence* in $\mathscr{B}$ any function $\Phi\colon\kappa\to\mathscr{B}$ such that

$$\forall\phi\in\kappa\,\forall G\in[\kappa]^{<\omega}\Big[\phi<\min(G)\Rightarrow\Phi(\phi)-\bigcup_{\gamma\in G}\Phi(\gamma)\neq\emptyset\Big]$$

(no collection $\Phi(\phi,\kappa)$ covers its left end-point $\Phi(\phi)$) and call *right-separated κ-sequence* in $\mathscr{B}$ any function $\Phi\colon\kappa\to\mathscr{B}$ such that

$$\forall\phi\in\kappa\,\forall G\in[\kappa]^{<\omega}\Big[\phi>\max(G)\Rightarrow\Phi(\phi)-\bigcup_{\gamma\in G}\Phi(\gamma)\neq\emptyset\Big]$$

(no collection $\Phi[0,\phi)$ covers its right end-point $\Phi(\phi)$). Define cardinal functions with the unlikely names *hereditary density* and *hereditary Lindelöf number* by

$$hd(\mathscr{B})=\sup\{\kappa\colon\text{there is a left-separated }\kappa\text{-sequence in }\mathscr{B}\},$$

$$hL(\mathscr{B})=\sup\{\kappa\colon\text{there is a right-separated }\kappa\text{-sequence in }\mathscr{B}\}.$$

(The direct well-ordered variation of the $|\mathscr{F}|<\omega\;\&\;|\mathscr{G}|=1$ form of spread would be what could be called dual hereditary density, i.e.

$$\text{dual }hd(\mathscr{B})=\sup\Big\{\kappa\colon\text{there is a function }\Phi\colon\kappa\to\mathscr{B}\text{ such that}$$
$$\forall F\in[\kappa]^{<\omega}\,\forall\gamma\in\kappa\Big[\max(F)<\gamma\Rightarrow\bigcap_{\phi\in F}\Phi(\phi)-\Phi(\gamma)\neq\emptyset\Big]\Big\}.$$

It is easy to see that dual hd equals hL, and similarly the obvious dual hL equals hd.)

Before justifying these names, let us do the topological analogues. So for a cardinal κ call *left-separated κ-sequence* in X any function $\Psi\colon \kappa \to X$ such that

$$\forall \alpha \in \kappa[\Psi^{\to}[\alpha, \kappa) \text{ is open in } \operatorname{ran}(\Psi)]$$

(intervals with α at the left are open), and call *right-separated κ-sequence* in X any function $\Psi\colon \kappa \to X$ such that

$$\forall \alpha \in \kappa[\Psi^{\to}[0, \alpha] \text{ is open in } \operatorname{ran}(\Psi)]$$

(intervals with α at the right are open). Define cardinal functions $Hd(X)$ and $HL(X)$ by

$$Hd(X) = \sup\{\kappa\colon \text{there is a left-separated } \kappa\text{-sequence in } X\}\ ;$$

$$HL(X) = \sup\{\kappa\colon \text{there is a right-separated } \kappa\text{-sequence in } X\}\ .$$

Let us also define the BAic cardinal functions *ideal depth*, *ideal height* and *ideal generating number* of $\mathscr{B}$ by:

$$id(\mathscr{B}) = \sup\{\kappa\colon \text{there is a strictly decreasing } \kappa\text{-chain of ideals in } \mathscr{B}\}\ ;$$

$$ih(\mathscr{B}) = \sup\{\kappa\colon \text{there is a strictly increasing } \kappa\text{-chain of ideals in } \mathscr{B}\}\ ;$$

$$ig(\mathscr{B}) = \min\{\kappa\colon \text{every nonprincipal ideal of } \mathscr{B} \text{ is } \leq\kappa\text{-generated}\}\ .$$

7.3. THEOREM. (*d*) $hd(X) = Hd(X) = hd(\mathscr{B}) = id(\mathscr{B})$. (*L*) $hL(X) = HL(X) = hL(\mathscr{B}) = ih(\mathscr{B}) = ig(\mathscr{B})$.

PROOF. We first point the topological versions of id, ih and ig: these are given by

$$id(X) = \sup\{\kappa\colon \text{there is a strictly decreasing } \kappa\text{-chain of open sets in } X\}\ ;$$

$$ih(X) = \sup\{\kappa\colon \text{there is a strictly increasing } \kappa\text{-chain of open sets in } X\}\ ;$$

$$ig(X) = \min\{\kappa\colon \text{every open set in } X \text{ is the union of a collection of at most } \kappa \text{ clopen sets}\}\ .$$

Proof that $hd(\mathscr{B}) = Hd(X)$. This is similar to the proof of $F(\mathscr{B}) = F(X)$: $\Phi\colon \kappa \to \mathscr{B}$ is left-separated iff

(1) $\forall \alpha \in \kappa[\{\Phi(\alpha) - \Phi(\gamma)\colon \gamma > \alpha\} \text{ is centered}]$

iff we can choose $\Psi\colon \kappa \to X$ such that

(1*) $\forall \alpha \in \kappa[\Psi(\alpha) \in \Phi(\alpha) - \bigcup_{\gamma > \alpha} \Phi(\gamma)]$

and $\Psi\colon \kappa \to X$ is left-separated iff

(2) $\forall \alpha \in \kappa\ \exists B \in \mathscr{B}[\Psi(\alpha) \in B\ \&\ B \cap \operatorname{ran}(\Psi) \subseteq \Psi^{\to}[\alpha, \kappa)]$

iff we can choose $\Phi\colon \kappa \to \mathscr{B}$ such that

(2*) $\forall \alpha \in \kappa[\min\{\xi \in \kappa\colon \Psi(\xi) \in \Phi(\alpha)\} = \alpha]$.

Since clearly $(1^*) \Leftrightarrow (2^*)$ it follows that Φ witnesses Ψ is left-separated iff Ψ witnesses Φ is left-separated.

Proof that $hd(\mathcal{B}) \le id(\mathcal{B})$. If Φ: $\kappa \to \mathcal{B}$ is left-separated, then for $\xi \in \kappa$ let $\mathcal{I}_\xi$ be the ideal generated by $\Phi^{\rightarrow}[\xi, \kappa)$. Then for $\xi < \eta \in \kappa$ we have $\mathcal{I}_\xi \supseteq \mathcal{I}_\eta$ and also $\Phi(\xi) \in \mathcal{I}_\xi$ but $\Phi(\xi) \not\in \mathcal{I}_\eta$, hence $\mathcal{I}_\xi \supset \mathcal{I}_\eta$.

Proof that $id(X) \le hd(X)$. Consider a successor $\kappa \le id(X)$. Let $\{U_\xi: \xi \in \kappa\rangle$ be a κ-sequence of open sets with $\forall \xi < \eta \in \kappa\ [U_\xi \supset U_\eta]$. Choose Ψ: $\kappa \to X$ such that $\forall \xi \in \kappa\ [\Psi(\xi) \in U_\xi - U_{\xi+1}]$. Then $d(\mathrm{ran}(\Psi)) \ge \kappa$: consider any $D \in [\kappa]^{<\kappa}$, and let $\delta = \sup(D)$. Then $\delta < \kappa$, so the open subset $U_\delta \cap \mathrm{ran}(\Psi)$ of $\mathrm{ran}(\Psi)$, which clearly is disjoint from $\{\Psi(\xi)$: $\xi \in D\}$, is nonempty. (We here used Remark 3.5.)

Proof that $hd(X) \le Hd(X)$. If Y is a subspace of X with $d(Y) \ge \kappa$ choose Ψ: $\kappa \to Y$ such that $\forall \alpha \in \kappa\ [\Psi(\alpha) \not\in \overline{\Psi^{\rightarrow}[0, \alpha)}]$. Then Ψ is a left-separated κ-sequence.

The proofs that $hL(\mathcal{B}) = HL(X)$ and $hL(\mathcal{B}) \le ih(\mathcal{B})$ are similar to the proofs of $hd(\mathcal{B}) = Hd(X)$ and $hd(\mathcal{B}) \le id(\mathcal{B})$.

Proof that $ih(X) \le hL(X)$. Consider a successor cardinal $\kappa \le ih(X)$. Let $\langle U_\xi: \xi \in \kappa\rangle$ be a κ-sequence of open sets with $\forall \xi < \eta \in \kappa\ [U_\xi \subset U_\eta]$, and let $Y = \bigcup_{\xi \in \kappa} U_\xi$. Then $L(Y) \ge \kappa$: $\langle U_\xi: \xi \in \kappa\}$ covers Y, and for $D \in [\kappa]^{<\kappa}$ we have that if $\delta = \sup(D)$ then $\delta \in \kappa$, hence $\bigcup_{\xi \in D} U_\xi \subseteq U_\delta$, but $U_\delta \ne X$ since U_δ is a proper subset of $U_{\delta+1}$. (We here used Remark 3.5.)

Proof that $hL(X) \le HL(X)$. If $\mathcal{U}$ is an open cover of a subspace Y of X such that $\mathcal{U}$ has no subcover of size less than κ find Ψ: $\kappa \to Y$ and Φ: $\kappa \to \mathcal{U}$ as follows: at stage η first choose $\Psi(\eta) \in Y - \bigcup_{\phi < \eta} \Phi(\phi)$ and then choose $\Phi(\eta) \in \mathcal{U}$ with $\Psi(\eta) \in \Phi(\eta)$. Then Φ witnesses that Ψ is right-separated.

Proof that $hL(X) = ig(X)$. By the proof of 5.3(a), is U is open and not closed in X, then $L(U) = \min\{|\mathcal{V}|: \mathcal{V} \subseteq \mathcal{B}\ \&\ \bigcup \mathcal{V} = U\}$. □

We point out the following variation of the inequality $hd(X) \le Hd(X)$:

7.4. LEMMA. *Any space Y has a left-separated $d(Y)$-sequence with dense range.*

PROOF. Let $\kappa = d(Y)$, and let Ψ: $\kappa \to Y$ be such that $\mathrm{ran}(\Psi)$ is dense in Y. Since

$$\forall D \in [\kappa]^{<\kappa}[\Psi^{\rightarrow}D \text{ is not dense}]$$

we can define ϕ: $\kappa \to \kappa$ by

$$\phi(\eta) = \min\{\xi \in \kappa: \Psi(\xi) \not\in \overline{\{\Psi(\phi(\zeta)): \zeta \in \eta\}}\}\,.$$

Then $\Psi \circ \phi$ is left-separated, and $\mathrm{ran}(\Phi \circ \phi)$ is dense since

$$\forall \eta \in \kappa[\Psi(\eta) \in \overline{\{\Psi(\phi(\xi)): \phi(\xi) \le \eta\}}]\,. \quad \square$$

Our last variation is the analogue of the incomparability number. For κ a cardinal call *pie κ-sequence* in $\mathcal{B}$ any function Φ: $\kappa \to \mathcal{B}$ such that

$$\forall \phi, \gamma \in \kappa[\phi < \gamma \to \Phi(\phi) \not\subseteq \Phi(\gamma)]\,,$$

and define the *hereditary cofinality* of $\mathscr{B}$ to be

$$h-\mathrm{cof}(\mathscr{B}) = \sup\{\kappa\colon \text{there is a pie } \kappa\text{-sequence in } \mathscr{B}\}\,.$$

This name is justified by:

7.5. THEOREM. $h-\mathrm{cof}(\mathscr{B}) = \min\{\kappa\colon \forall \mathscr{A} \subseteq \mathscr{B}\ \exists \mathscr{C} \in [\mathscr{A}]^{\leq\kappa}$ [*$\mathscr{C}$ is cofinal in $\mathscr{A}$*]$\}$.

PROOF. For convenience we work with "coinitial" rather than "cofinal".

If $\mathscr{A} \subseteq \mathscr{B}$ has $\kappa = \min\{|\mathscr{C}|\colon \mathscr{C} \subseteq \mathscr{A}$ is a coinitial in $\mathscr{A}\}$, then by minimality of κ we can find a pie κ-sequence: $\kappa \to \mathscr{A}$.

For successor $\kappa \leq h-\mathrm{cof}(\mathscr{B})$, if Φ is a pie κ-sequence, then for all $F \in [\kappa]^{<\kappa}$ we have that $\Phi^{\to}F$ is not cofinal in $\mathrm{ran}(\Phi)$, for if $\gamma = \sup(C)$ then $\gamma \in \kappa$ and $\forall \phi \in F\ [\Phi(\phi) \not\subseteq \Phi(\gamma+1)]$. (We here used the remark at the end of Section 3.) $\square$

The following corollary will be used in the proof of Theorem 8.5, which gives the stronger inequality $\pi(\mathscr{B}) \leq \mathrm{inc}(\mathscr{B})$.

7.6. COROLLARY. $\pi(\mathscr{B}) \leq h-\mathrm{cof}(\mathscr{B})$. (*Hence* $h_c\pi(\mathscr{B}) \leq h-\mathrm{cof}(\mathscr{B})$.)

PROOF. (The parenthetical remark follows from the observation that $h-\mathrm{cof}(\mathscr{Q}) \leq h-\mathrm{cof}(\mathscr{B})$ whenever $\mathscr{Q}$ is a quotient of $\mathscr{B}$.)

$$\pi(\mathscr{B}) = \min\{|\mathscr{A}|\colon \mathscr{A} \text{ is coinitial in } \mathscr{B}-\{\emptyset\}\}\,. \quad \square$$

We hope that people who call $\pi(\mathscr{B})$ "algebraic density" will not also call h-cof$(\mathscr{B})$ "algebraic hereditary density".

We conclude with describing the inequalities that hold between the functions introduced in this section, where we use the conventions of theorem 6.1.

7.7. THEOREM. (a)

$$\begin{tikzcd}
& h\text{-cof} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
hd \arrow[dr, dash] & \mathrm{inc} \arrow[d, dash] & hL \arrow[dl, dash] \\
& s \arrow[d, dash] & \\
& F \arrow[d, dash] & \\
& \mathrm{ind} &
\end{tikzcd}$$

(b) *If ϕ is any of the functions mentioned under (a), then $m\phi = \phi$.*

(c) $h_c\phi = \phi$ *for* $\phi \in \{h-\mathrm{cof},\ \mathrm{inc},\ F,\ \mathrm{ind}\}$ *and* $h\phi = \phi$ *for* $\phi \in \{hd,\ hL,\ s\}$.

PROOF. *Proof of* (a). Let h denote hd or hL. Then obviously inc, s, and ind satisfy inc $\geq s \geq$ ind, and their well-ordered analogues $h -$ cof, h and F satisfy $h -$ cof $\geq h \geq F$. Moreover, each of inc, s and ind is $\leq$ its well-ordered analogue, hence $h -$ cof $\geq$ inc, $h \geq s$ and $F \geq$ ind. Hence all inequalities are clear except $s \geq F$.

(*Topological*) *proof that* $s(X) \geq F(X)$. If $\Psi: \kappa \to X$ is free, then ran(Ψ) is a discrete subspace of X since

$$\forall \eta \in \kappa[\text{both } \Psi(\eta) \not\in \overline{\{\Psi(\phi): \phi < \eta\}} \text{ and } \Psi(\eta) \not\in \overline{\{\Psi(\gamma): \gamma > \eta\}}].$$

(*BAic*) *proof that* $s(\mathscr{B}) \geq F(\mathscr{B})$. If $\Phi: \kappa \to \mathscr{B}$ is free, define

$$\mathscr{S} = \{\Phi(\eta) - \Phi(\eta + 1): \eta \in \kappa\}.$$

A straightforward calculation shows $\mathscr{S}$ is discrete.

Proof of (b) *and* (c). Clearly, for $\phi \in \{h -\text{cof}, hd, \text{inc}, hL, s, F, \text{ind}\}$ one has $\phi(\mathscr{A}) \leq \phi(\mathscr{B})$ for each subalgbra or quotient $\mathscr{A}$ of $\mathscr{B}$, hence $m\phi = \phi$ and $h_c\phi = \phi$, by 4.1 and 4.2. It is trivial that $h\phi = \phi$ for $\phi \in \{hd, hL, s\}$. □

Theorem 8.4 below will give the additional inequality *inc* $\geq$ *hd*.

8. π-weight and π-character

We begin our study of π-weight with showing that every BA has a left-separated π-base, i.e.

8.1. LEMMA. *β has a left-separated $\pi(\mathscr{B})$-sequence the range of which is a π-base.*

PROOF. Let π abbreviate $\pi(\mathscr{B})$. If $\pi < \omega$ the lemma is trivial, so without loss of generality we assume $\pi \geq \omega$.

Claim. There is a function B: $\pi \to \mathscr{B} - \{\emptyset\}$ with ran(B) a π-base such that

$$(1) \qquad \forall \eta \in \pi[|\{\xi \in \eta: B_\xi \cap B_\eta \neq \emptyset\}| < \pi(B_\eta)].$$

Indeed, call $B \in \mathscr{B} - \{\emptyset\}$ π-homogeneous if $\forall A \in \mathscr{B}$ $[\emptyset \neq A \subseteq B \Rightarrow \pi(A) = \pi(B)]$. Since every set of cardinals is well-ordered, the collection of π-homogeneous sets is a π-base. Hence, if $\mathscr{A}$ is a maximal pairwise disjoint collection of π-homogeneous sets, then $\bigcup \mathscr{A}$ is dense. Let $\alpha = |\mathscr{A}|$, and let A be a surjection $\alpha \to \mathscr{A}$, and let $\Delta = \bigcup_{\xi \in \alpha} \{\xi\} \times \pi(A_\xi)$. There clearly is a function Φ: $\Delta \to \mathscr{B} - \{\emptyset\}$ such that for each $\xi \in \alpha$ we have that $\Phi^{\to}(\{\xi\} \times \pi(A_\xi))$ is a π-base for A_ξ; note that

$$(2) \qquad \forall \zeta \in \alpha\, \forall \langle \xi, \eta \rangle \in \Delta[\Phi(\xi, \eta) \subseteq A_\zeta \text{ iff } \xi = \zeta \text{ iff } \Phi(\xi, \eta) \cap A_\zeta \neq \emptyset].$$

Since $\bigcup \mathscr{A}$ is dense, ran(Φ) is a π-base, hence $|\Delta| \geq \pi$. Also, clearly $\alpha \leq c(\mathscr{B})$, hence $\alpha \leq \pi$ since $c(\mathscr{B}) \leq \pi(\mathscr{B})$ by Theorem 6.1, and $\sup_{\xi \in \alpha} \pi(A_\xi) \leq \pi$, hence $\Delta \subseteq \pi \times \pi$. It follows that there is a bijection $D\colon \Delta \to \pi$ such that

$$\forall \xi, \eta, \zeta \in \pi\, [\eta < \zeta \Rightarrow D(\xi, \eta) < D(\xi, \zeta)]\,.$$

Then $B = \Phi \circ D^{-1}$ is as required, because of (2).

With B as in the claim we will construct a function $A\colon \pi \to \mathscr{B} - \{\emptyset\}$ such that

$$(3_\alpha) \qquad A_\alpha \subseteq B_\alpha$$

for $\alpha < \pi$, and such that

$$(4_\alpha) \qquad \forall \xi \in \alpha\, \forall F \in [(\xi, \alpha)]^{<\omega}\Big[A_\xi \not\subseteq \bigcup_{\eta \in F} A_\eta\Big]$$

for $\alpha \leq \pi$. Let $\beta < \pi$, and assume A_α known for $\alpha \in \beta$, and that (3_α) holds for $\alpha \in \beta$, and that (4_α) holds whenever $\alpha + 1 < \beta$. Then (4_β) also holds. From (1) and the (3_α)'s we see that

$$I = \{\alpha \in \beta\colon A_\alpha \cap B_\beta \neq \emptyset\}$$

has cardinality less than $\pi(B_\beta)$, hence $|I \times [I]^{<\omega}| < \pi(\mathscr{B})$ too, since $\pi(B_\beta) = 1$ or $\pi(B_\beta) \geq \omega$. It follows that there is an $A_\beta \in \mathscr{B}$ with $\emptyset \neq A_\beta \subseteq B_\beta$ such that

$$\forall \phi \in I\, \forall G \in [I]^{<\omega}\Big[A_\phi - \bigcup_{\gamma \in G} A_\gamma \not\subseteq A_\beta \Big(\text{i.e. } A_\phi \not\subseteq \bigcup_{\gamma \in G \cup \{\beta\}} A_\gamma\Big)\Big]\,.$$

Therefore we have (3_β) and $(4_{\beta+1})$. This completes the construction.

From the (3_α)'s we see that ran(A) is a π-base, and since the (4_α)'s imply that (4_π) holds, A is left-separated. $\square$

From this we get the following corollaries (to proof):

8.2. THEOREM (ŠAPIROVSKIĬ [1975], [1976]). *Let X be a Boolean space. Then*
(1) *there is a left-separated $\pi(B)$-sequence in X with dense range* (*cf.* 7.4);
(2) $hd(X) = h_c d(X) = h\pi(X) = h_c \pi(X)$.

PROOF. *Proof of* (1). If A is as in Lemma 8.1, we can choose $D\colon \pi(X) \to X$ such that $D(\xi) \in A_\xi - \bigcup_{\eta \in (\xi, \pi(X))} A_\eta$ for all $\xi \in \pi(X)$. Then ran(D) is dense since ran(A) is a π-base, and D is left-separated by the proof of $hd(\mathscr{B}) = Hd(X)$ of Theorem 7.3.

Proof that $hd(X) \geq h_c d(X)$. This is trivial.

Proof that $h_c d(X) \geq h_c \pi(X)$. If Y is closed in X, then Y has a left-separated $\pi(Y)$-sequence by (1), hence $hd(Y) \geq \pi(Y)$ by 7.3.

Proof that $h_c \pi(X) = h\pi(X)$. Corollary 6.7.

Proof that $h\pi(X) \geq hd(X)$. $\pi \geq d$, by 6.1. $\square$

Let us have a short topological intermezzo. One defines the *strong density* of a space X by

$$sd(X) = \sup\{d(Y)\colon Y \text{ is a dense subspace of } X\}\,.$$

By Lemma 6.6(b), if Y is dense in X, then $\pi(Y) = \pi(X)$. Also, $d \leq \pi$ by Theorem 6.1. Therefore $sd \leq \pi$. There are easy examples of noncompact X with $sd(X) < \pi(X)$; indeed, there is a countable zerodimensional space X with $\pi(X) = \mathfrak{c}$. However, it is a well-known open question whether $sd(X) = \pi(X)$ for compact X. (It is easy to show that this is equivalent to asking whether $sd(X) = \pi(X)$ for Boolean X.) The following corollary to Theorem 8.2(1) is the only information available.

8.3. Proposition. *If X is a Boolean space then* $\operatorname{cf}(\pi(X)) \leqq sd(X)$.

Proof. By Theorem 8.2(1) there is a dense D in X with a well-order $\prec$ such that $\langle D, \prec\}$ has order type $\pi(X)$ and such that

$(*)$ final segments of $\langle D, \prec\rangle$ are open in D.

There is $S \subseteq D$ with $|S| \leq sd(X)$ with S dense. It follows from $(*)$ that I is cofinal in $\langle D, \prec\rangle$. $\square$

8.4. Corollary. *If GCH then $\pi(X) = sd(X)$ for all Boolean X.*

Proof. For any zerodimensional space X we have $d(X) \leq sd(X) \leq \pi(X) \leq w(X) \leq \exp(d(X))$, by 6.1, hence $sd(X) \leq \pi(X) \leq \exp(sd(X))$. So if also $\operatorname{cf}(\pi(X)) \leq sd(X)$ and if $\exp(sd(X))$ is less than the first limit cardinal bigger than $sd(X)$, which certainly is true under GCH, then $\pi(X) = sd(X)$. $\square$

This completes the intermezzo.

Since $\operatorname{inc} \leq h - \operatorname{cof}$, by 7.7, the following result improves Corollary 7.6, which says that $\pi \leq h - \operatorname{cof}$ (and in fact $h_c\pi \leq h - \operatorname{cof}$).

8.5. Theorem (Baumgartner and Komjáth [1981]. *See also Part I, Theorem* 4.25.) *For every BA $\mathscr{B}$* $\operatorname{inc}(\mathscr{B}) \geq h_c\pi(\mathscr{B})$. (*Hence* $\operatorname{inc}(\mathscr{B}) \geq hd(\mathscr{B})$.)

Proof. (The parenthetical remark is an immediate consequence of 8.2(2).) Since $\operatorname{inc}(\mathscr{B}) = h_c\operatorname{inc}(\mathscr{B})$, by 7.7(c), it suffices to prove that $\operatorname{inc}(\mathscr{B}) \geq \pi(\mathscr{B})$. Let π abbreviate $\pi(\mathscr{B})$. As in the proof of Lemma 8.1 call $B \in \mathscr{B} - \{\emptyset\}$ π-homogeneous if

$$\forall A \in \mathscr{B}\ [\emptyset \neq A \subseteq B \Rightarrow \pi(A) = \pi(B)]\,,$$

and let $\mathscr{A}$ be a maximal pairwise disjoint collection of π-homogeneous sets. If $|A| = \pi(X)$, then $\mathscr{A}$ is a pie of cardinality $\pi(X)$. Otherwise, let κ be a regular infinite cardinal $\leq \pi$. Since $\pi = \Sigma_{A \in \mathscr{A}}\, \pi(A)$, there is $A \in \mathscr{A}$ with $\pi(A) \leq \kappa$. There are two disjoint nonempty clopen subsets B_0 and B_1 of A. For each $i \in 2$ we have that $\pi(B_i) = \pi(A) \geq \kappa$ since $A \in \mathscr{A}$, hence $h-\operatorname{cof}(\mathrm{CLOP}(B_i)) \geq \kappa$ by 7.6. For $i \in 2$ choose a pie κ-sequence Φ_i for B_i. Then $\{\Phi_0(\xi) \cup (B_1 - \Phi_1(\xi)): \xi \in \kappa\}$ is a pie for $\mathscr{B}$ of size κ. Since κ was arbitrary, this shows $\operatorname{inc}(\mathscr{B}) \geq \pi$. $\square$

•

We now have another brief intermezzo, and use the inequality just proved to prove the result about a modification of incomparability. Recall that a poset $T = \{T, \leq\rangle$ is called a *tree* if $\{s \in T: s < t\}$ is well-ordered for every $t \in T$. This suggests that one define a cardinal function *treeness* of $\mathscr{B}$ by

$$T(\mathscr{B}) = \sup\{|\mathscr{T}|: \mathscr{T} \subseteq \mathscr{B} \text{ is such that } \langle \mathscr{T}, \subseteq\rangle \text{ is a tree}\}.$$

Fortunately, there is no need for a cardinal function with this atrocious name:

8.6. THEOREM (MONK [1984]). $\operatorname{inc}(\mathscr{B}) = T(\mathscr{B})$.

PROOF. *Proof that* $\operatorname{inc}(\mathscr{B}) \leq T(\mathscr{B})$. Every pie is a tree since each element has zero predecessors.

Proof that $T(\mathscr{B}) \leq \operatorname{inc}(\mathscr{B})$. Let ι abbreviate $\operatorname{inc}(\mathscr{B})$, and assume there is a tree $\mathscr{T} \subseteq \mathscr{B}$ of cardinality ι^+. For $\eta \in \iota^+$ define the ηth level $\mathscr{T}_\eta$ of $\mathscr{T}$ to be

$$\mathscr{T}_\eta = \{B \in \mathscr{T}: \{A \in \mathscr{T}: A \subset B\} \text{ has order type } \eta\}.$$

Clearly, each $\mathscr{T}_\eta$ is a pie, hence has cardinality at most ι. Therefore $\mathscr{T}_\eta \neq \emptyset$ for $\eta \in \iota^+$. For $\eta \in \iota^+$ choose $B_\eta \in \mathscr{T}_{\eta+1}$ and then $A_\eta \in \mathscr{T}_\eta$ with $A_\eta \subset B_\eta$. Then $B_\eta - A_\eta \neq \emptyset$ for each $\eta \in \iota^+$. Since $\mathscr{B}$ has a π-base of cardinality at most ι, by Theorem 8.5, it follows that there is $I \subseteq \iota^+$ of cardinality ι^+ such that

$$\forall \xi \neq \eta \in I\,[(B_\eta - A_\eta) \cap (B_\xi - A_\xi) \neq \emptyset], \text{ hence such that}$$
$$\forall \xi \neq \eta \in I\,[B_\xi \not\subseteq A_\eta].$$

To see $\{B_\eta: \eta \in I\}$ is a pie consider any ξ, $\eta \in I$ with $\xi < \eta$. Then $B_\eta \not\subseteq B_\xi$ since $B_\xi \in \mathscr{T}_{\xi+1}$ and $B_\eta \in T_{\eta+1}$, and $B_\xi \not\subseteq B_\eta$ since otherwise $B_\xi \subseteq A_\eta$ because $\mathscr{T}$ is a tree and

$$B_\xi \in \mathscr{T}_{\xi+1} \text{ and } A_\eta \in \mathscr{T}_\eta \text{ and } B_\eta \in T_{\eta+1}\ \xi + 1 \leq \eta \leq \eta + 1 \text{ and}$$
$$A_\eta \subseteq B_\eta. \quad \square$$

The final result of this section tells us that the cardinal functions t, $h\chi\pi$ and $h_c\pi\chi$, which do not have a good BAic translation, are within the class of Boolean spaces equal to the cardinal function F, which by Theorem 7.2 has a good BAic translation.

8.7. THEOREM. *If X is a Boolean space then* $F(X) = t(X) = h\pi\chi(X) = h_c\pi\chi(X)$. *($F = t$ is due to* ARHANGEL'SKIĬ [1971], *and $t = h\pi\chi$ is due to* ŠAPIROVSKIĬ [1975].)

PROOF. *Proof that* $F(X)\leqq t(X)$. It suffices to prove that $t(X)\geqq\kappa$ for every successor $\kappa\leq F(X)$. So consider any free κ-sequence x in X with κ a successor, hence regular. By compactness there is $p\in\bigcap_{\xi\in\kappa}\overline{\{x_\eta:\eta\in(\xi,\kappa)\}}$. Then $p\in\overline{\operatorname{ran}(x)}$ but for $A\in[\operatorname{ran}(x)]^{<\kappa}$, if $\alpha=\sup\{\xi\in\kappa: x_\xi\in A\}$, then $p\not\in\overline{\{x_\xi:\xi\leq\alpha\}}$ (hence $x\not\in\bar{A}$) since $p\in\overline{\{x_\xi:\xi\in(\alpha,\kappa)\}}$ and since x is free. (We here used Remark 3.5.)

Proof that $t(X)\leq h\pi\chi(X)$. This is Theorem 6.1.

Proof that $h\pi\chi(X)\leq h_c\pi\chi(X)$. Corollary 6.7.

Proof that $h_c\pi\chi(X)\leq F(X)$. Since $h_cF(X)\leq F(X)$ by 7.7(c), it suffices to prove that $\pi\chi(X)\leq F(X)$. So consider any point x of X and let κ abbreviate $\pi\chi(x,X)$. We will prove the BAic statement that $F(\mathscr{B})\geq\kappa$, i.e. we will construct a function $B:\kappa\to\mathscr{B}$ such that if

$$G*F \text{ abbreviates } \bigcap_{\gamma\in G}B_\gamma-\bigcup_{\phi\in F}B_\phi \text{ for } F,G\subseteq\kappa\,,$$

then for $\eta=\kappa$

$$(1_\eta)\qquad \forall F,G\subseteq[\eta]^{<\omega}[\max(F)<\min(G)\Rightarrow G*F\neq\emptyset]\,.$$

(Recall our convention that $\max(F)<\min(G)$ if $F=\emptyset$ or $G=\emptyset$, and that $\bigcap\emptyset=X$.)

We will construct B_ξ with recursion on $\xi\in\kappa$. In order to prevent our construction from terminating prematurely we will also require

$$(2_\eta)\qquad \forall\xi\in\eta\,[x\not\in B_\xi]\,.$$

Now let $\eta\in\kappa$, and suppose B_ξ to be constructed for $\xi\in\eta$. Then (1_η) tells us that if

$$\mathscr{A}=\{G*F: F,G\in[\eta]^{<\omega} \text{ and } \max(F)<\min(G)\}\,,$$

then $\emptyset\not\in\mathscr{A}$. Since $\kappa=1$ (if x is isolated) or $\kappa\geq\omega$ (if x is not isolated), $|\mathscr{A}|<\kappa$, so $\mathscr{A}$ is not a neighborhood π-base at x. Choose $U\in\mathscr{B}$ with $x\in U$ such that $\forall A\in\mathscr{A}\ [A\not\subseteq U]$, and let $B_\eta=X-U$. Then clearly $(2_{\eta+1})$. To verify $(1_{\eta+1})$ consider any finite $F,G\subseteq\eta+1$ with $\max(F)<\min(G)$.

Case 1. $\eta\not\in F\cup G$: then $G*F\neq\emptyset$ by (1_η).

Case 2. $\eta\in G$: if A denotes $(G-\{\eta\})*F$, then $G*F=A\cap B_\eta$, and $A\in\mathscr{A}$, hence $A\not\subseteq U$, and therefore $A\cap B_\eta=A\cap(X-U)\neq\emptyset$.

Case 3. $\eta\in F$: then $G=\emptyset$ since $\max(F)<\min(G)$, and now (2_η) tells us that

$$G*F=\bigcap\emptyset-\bigcup_{\xi\in F}B_\xi=\bigcap_{\xi\in F}(X-B_\xi)\supseteq\{x\}\neq\emptyset\,.\quad\square$$

8.8. COROLLARY. *If X is a Boolean space, then* $t(X)\leq s(X)$.

PROOF. $F(X)\leq s(X)$ by 7.7, since $s(X)=s(\mathscr{B})$ and $F(X)=F(\mathscr{B})$ by 7.1 and 7.2. $\square$

9. Character and cardinality, independence and π-character

We begin by giving a lower bound on the cardinality in terms of the *minimum character*, defined by

$$\min \chi(X) = \min_{x \in X} \chi(x, X) ;$$

in Theorem 10.3 we give an upper bound on the cardinality in terms of character. (This is a rare case where both the minimum and the supremum of a certain set of cardinals leads to a useful cardinal function.)

9.1. THEOREM (ČECH-POSPÍŠIL [1938]). *If X is an infinite Boolean space, then* $|X| \geq 2^{\min \chi(X)}$.

PROOF. Let $\kappa = \min \chi(X)$. We use the BAic equivalent hypothesis that no ultrafilter on $\mathcal{B}$ is $<\kappa$-generated. To prove that $|X| \geq 2^\kappa$ it suffices to show that there is $F\colon {}^\kappa 2 \to \mathcal{P}(\mathcal{B})$ such that

(1) $\forall s \in {}^\kappa 2$ [$F(s)$ is centered (i.e. $\forall \mathcal{F} \subseteq F(s)[0 < |\mathcal{F}| < \omega \Rightarrow \bigcap \mathcal{F} \neq \emptyset]$)], and

(2) $\forall s \neq t \in {}^\kappa 2\, \exists B \in \mathcal{B}$ [$B \in F(s)$ and $X - B \in F(t)$],

for then $\{\bigcap_{\xi \in \kappa} F(s \restriction \xi)\colon s \in {}^\kappa 2\}$ is a pairwise disjoint collection consisting of 2^κ nonempty sets. To this end we will construct a function $f\colon {}^{<\kappa}2 \to \mathcal{B}$ such that

(1′) $\forall s \in {}^{<\kappa}2$ [$\{f(s \restriction \alpha)\colon \alpha \in \operatorname{dom}(s)\}$ is centered];

(2′) $\forall s \in {}^{<\kappa}2$ [$f(s^\frown 0) \cap f(s^\frown 1) = \emptyset$].

(For $s \in {}^{<\kappa}2$ and $i \in 2$ the concatenation $s^\frown i$ of s and i is the function $[0, \operatorname{dom}(s)] \to 2$ that extends s and takes the value i at $\operatorname{dom}(s)$, i.e. $s^\frown i = s \cup \{\langle \operatorname{dom}(s), i\rangle\}$.)

To see f can be constructed, let $s \in {}^{<\kappa}2$, and assume $f(s \restriction \sigma)$ known for $\alpha \in \operatorname{dom}(s)$. From our tacit induction hypothesis we see that $\mathcal{F} = \{f(s \restriction \alpha)\colon \alpha \in \operatorname{dom}(s)\}$ is centered. As $|\mathcal{F}| = |\alpha| < \kappa$, $\mathcal{F}$ does not generate an ultrafilter. Hence there is $B \in \mathcal{B}$ such that both $\mathcal{B} \cup \{B\}$ and $\mathcal{B} \cup \{X - B\}$ are centered. Let $f(s^\frown 0) = B$ and $f(s^\frown 1) = X - B$.

Now define $F\colon {}^\kappa 2 \to \mathcal{P}(\mathcal{B})$ by

$$F(s) = \{f(s \restriction \alpha)\colon \alpha \in \kappa\} \quad \text{for } s \in {}^\kappa 2 .$$

It is clear from (1′) and (2′) that (1) and (2) hold. □

9.2. COROLLARY TO PROOF. *If* $\min \chi(X) \geqq \omega$, *i.e. if $\mathcal{B}$ is nonatomic, then $\mathcal{B}$ has an infinite independent subset. (In fact, the conclusion holds iff $\mathcal{B}$ is not superatomic.)*

PROOF. In the proof of 9.1 we found $f\colon {}^{<\omega}2 \to \mathcal{B}$ such that

$$\forall s \in {}^{<\omega}2\, [f(s^\frown 0) \cap f(s^\frown 1) = \emptyset] ,$$

and such that if we define $\phi\colon {}^{<\omega}2 \to \mathcal{B}$ by

$$\phi(s) = \bigcap_{n \leq \operatorname{dom}(s)} f(s \restriction n) ,$$

then $\forall s \in {}^{<\omega}2\ [\phi(s) \neq \emptyset]$ [because of (2′)]. Define $F\colon \omega \to \mathcal{B}$ by

$$F(n) = \bigcup \{f(s)\colon \mathrm{dom}(s) = n+1 \text{ and } s(n) = 0\}\,.$$

Then $\mathrm{ran}(F)$ is an infinite independent set since for each $s \in {}^{<\omega}2$ we have

$$\bigcap \{F(n)\colon n \in s^{-1}\{0\}\} \cap \bigcap \{X - F(n)\colon n \in s^{-1}\{1\}\} \supseteq \phi(s) \neq \emptyset\,.$$

(Since "$\mathcal{B}$ superatomic" means that $\mathcal{B}$ has no nonatomic quotient, the "if" part of the parenthetical remark is clear. The "only if" part will be proved when we do the easy part of the proof of Theorem 9.3.) □

The main result of this section is the following generalization of 9.2, where the minimum π-character $\min \pi\chi$ is defined in the obvious way by

$$\min \pi\chi(X) = \min_{x \in X} \pi\chi(x, X)\,.$$

9.3. Theorem (Šapirovskiĭ [1980]. See also Part I, Theorem 10.16). *If $\mathcal{B}$ is not superatomic then* $\mathrm{ind}(\mathcal{B}) = h_c \min \pi\chi(\mathcal{B})$.

Proof. *Proof of $h_c \min \pi\chi(X) \geq \mathrm{ind}(\mathcal{B})$.* This is the easy part of the proof. Let $\mathcal{I}$ be an independent subset of $\mathcal{B}$, and let $\mathcal{A}$ be the subalgebra of $\mathcal{B}$ generated by $\mathcal{I}$. If A denotes the Stone space of $\mathcal{A}$, then there is a map f from X onto A, and $\min \pi\chi(A) = |\mathcal{I}|$ by 14.1. By 3.4(c) there is a closed subspace F of X such that the closed map $\phi = f \restriction F$ is an irreducible map from F onto A. By 3.4(b) $\phi^{\#}U$ is a nonempty open subset of A for every nonempty open U in Y.

The following is easy to see: For any spaces S and T, if there is a map $\phi\colon S \to T$ such that for each nonempty open U in S the small image $\phi^{\#}U$ is a nonempty open subset of T, then for each $x \in S$ we have $\pi\chi(x, S) \geq \pi\chi(\phi(x), T)$. It follows that $\min \pi\chi(Y) \geq \min \pi\chi(A) = \kappa$.

Proof of $\mathrm{ind}(\mathcal{B}) \geq h_c \min \pi\chi(X)$. Let κ be a successor cardinal with $\kappa \leq h_c \min \pi\chi(X)$. We will prove $\mathrm{ind}(\mathcal{B}) \geq \kappa$. As $h_c \mathrm{ind}(\mathcal{B}) = \mathrm{ind}(\mathcal{B})$, by 7.7, we may assume without loss of generality that

$$(A)\ \min \pi X(X) \geq \kappa\,.$$

Because of Corollary 9.2 we may also assume without loss of generality that $\kappa > \omega$. The proof that $\mathrm{ind}(\mathcal{B}) \geq \kappa$ will be purely BAic, except for the following claim, which extracts a useful BAic property out of the topological (A).

Claim. $\forall \mathcal{F} \in [\mathcal{B} - \{\emptyset\}]^{<\kappa}\ \exists K, L \in \mathcal{B}\ [\forall F \in \mathcal{F}[F \cap K \neq \emptyset \neq F \cap L]$ but $\exists F \in \mathcal{F}[K \cap F \subseteq X - L]]$.

To prove this, consider any $\mathcal{F} \in [\mathcal{B} - \{\emptyset\}]^{<\kappa}$. Because of (A) $\mathcal{F}$ is not a neighborhood π-base at any point of X. Hence, each point has a clopen neighborhood U such that $\forall F \in \mathcal{F}\ [F \not\subseteq U]$. It follows that

$$\mathcal{G} = \{G \in \mathcal{B} : \forall F \in \mathcal{F}\ [F \cap G \neq \emptyset]\}$$

satisfies $\bigcap \mathcal{G} = \emptyset$. Find $\mathcal{S} \subseteq \mathcal{G}$ and $H \in \mathcal{F}$, with $\mathcal{S}$ of minimal cardinality, such that $H \cap \bigcap \mathcal{S} = \emptyset$; of course, $2 \leq |\mathcal{S}| < \omega$ since F is compact. Pick any $K \in \mathcal{S}$, put $L = \bigcap (\mathcal{S} - \{K\})$. Then $\forall F \in \mathcal{F}\ [F \cap K \neq \emptyset \neq F \cap L]$ by the minimality of $|\mathcal{S}|$, but $H \cap K \cap L = H \cap \bigcap \mathcal{S} = \emptyset$, hence $K \cap H \subseteq X - L$. This completes the proof of the claim.

For a set S let $\mathrm{Fn}(S)$ denote the set of finite partial functions $S \to 2$. We will first construct a function Q: $\kappa \times 2 \to \mathcal{B}$ and a function q with $\mathrm{dom}(q) = \kappa$ such that

$$\forall \xi \in \kappa\, [q_\xi \in \mathrm{Fn}(\xi)]$$

and such that if

$$Q^{\#}(p) \text{ abbreviates } \bigcap_{\xi \in \mathrm{dom}(p)} Q(\xi, p(\xi)), \text{ for } p \in \mathrm{Fn}(\kappa)\,,$$

then

(1) $\forall p \in \mathrm{Fn}(\kappa)\ [Q^{\#}(p) \neq \emptyset]$, and

(2) $\forall \xi \in \kappa\, [Q(\xi, 1) \cap Q^{\#}(q_\xi) \subseteq X - Q(\xi, 0)]$.

(Note that (1) does not imply $\{Q(\xi, 0)\colon \xi \in \kappa\}$ is independent since (2) does not tell us that $Q(\xi, 1) \subseteq X - Q(\xi, 0)$ for $\xi \in \kappa$.)

The construction is straightforward. Let $\eta \in \kappa$, assume $Q(\xi, i)$ is known for $\xi \in \eta$ and $i \in 2$, and assume that

$$\mathcal{F} = \{Q^{\#}(p)\colon p \in \mathrm{Fn}(\eta)\}$$

does not contain $\emptyset$; then by the claim there are $Q(\eta, 0)$, $Q(\eta, 1) \in \mathcal{B}$ and $P \in \mathcal{F}$ such that

$$\forall F \in \mathcal{F}\ \forall i \in 2 [F \cap Q(\eta, i) \neq \emptyset] \text{ but } Q(\eta, 1) \cap P \subseteq X - Q(\eta, 0)\,;$$

let $q_\eta \in \mathrm{Fn}(\eta)$ be such that $P = Q^{\#}(q_\eta)$.

To manufacture an independent family of size κ from Q and q we use the Pressing Down Lemma (see the Appendix on Set Theory in this Handbook): since κ is regular and uncountable, and since $\max(\mathrm{dom}(q_\xi)) < \xi$ for $\xi \in \kappa$, there is $\sigma < \kappa$ such that $S = \{\xi \in \lambda\colon \mathrm{dom}(q_\xi) \subseteq \sigma\}$ has cardinality κ. Obviously, $|\mathrm{Fn}(\sigma)| < \kappa$. Hence, there is $f \in \mathrm{Fn}(\sigma)$ such that $I = \{\xi \in \lambda\colon q_\xi = f\}$ has cardinality κ. Note that $I \cap \mathrm{dom}(f) = \emptyset$.

Define $\mathcal{Q}: I \to \mathcal{B}$ by

$$\mathcal{Q}(\xi) = Q^{\#}(f) \cap Q(\xi, 0) \quad (\xi \in I)\,.$$

Clearly $\mathrm{ran}(\mathcal{Q})$ is a collection of cardinality κ. To show $\mathrm{ran}(\mathcal{Q})$ is independent consider any $p \in \mathrm{Fn}(I)$. We must show that

$$\mathcal{Q}^{\#}(p) = \bigcap \{\mathcal{Q}(\xi)\colon \xi \in p^{-1}\{0\}\} \cap \bigcap \{X - \mathcal{Q}(\xi)\colon \xi \in p^{-1}\{1\}\}$$

is nonempty. From (2) we see that for each $\xi \in I$ we have

$$\begin{aligned} X - \mathcal{Q}(\xi) = X - (Q^{\#}(f) \cap Q(\xi, 0)) &\supseteq Q^{\#}(f) - Q(\xi, 0) \\ &\supseteq Q^{\#}(f) \cap Q(\xi, 1)\,, \end{aligned}$$

hence

$$\begin{aligned}\mathcal{Q}^{\#}(p) &\supseteq \bigcap \{Q^{\#}(f) \cap Q(\xi, 0)\colon \xi \in p^{-1}\{0\} \\ &\quad \cap \bigcap \{Q^{\#}(f) \cap Q(\xi, 1)\colon \xi \in p^{-1}\{1\}\} \\ &= Q^{\#}(f) \cap Q^{\#}(p) .\end{aligned}$$

As $I \cap \mathrm{dom}(f) = \emptyset$, we have $f \cup p \in \mathrm{Fn}(\kappa)$. It follows that

$$\mathcal{Q}^{\#}(p) \supseteq Q^{\#}(f) \cap Q^{\#}(p) = Q^{\#}(f \cap p) \neq \emptyset . \quad \square$$

9.4. REMARK. In fact, one can prove for every nonsuperatomic BA $\mathcal{B}$ and for every cardinal κ that $\mathcal{B}$ has an independent family of cardinality κ iff X has a closed subspace with min $\pi\chi = \kappa$. The proof that h_c min $\pi\chi(X) \geq \mathrm{ind}(X)$ establishes the "if" part, and the proof that $\mathrm{ind}(X) \geq h_c$ min $\pi\chi(X)$ establishes the "only if" part if κ is regular. We outline the proof for singular κ: it is easy to modify the proof that $\mathrm{ind}(\mathcal{B}) \geq h_c$ min $\pi\chi(X)$ and prove

(*) for every regular uncountable $\lambda \leq \kappa$ and every independent $\mathcal{H} \in [\mathcal{B}]^{<\lambda}$ there is an independent $\mathcal{I} \in [\mathcal{B}]^{\lambda}$ such that $\mathcal{H} - \mathcal{I}$ is finite .

Next, let $\gamma = \mathrm{cf}(\kappa)$, and let $\{\lambda_\xi\colon \xi \in \gamma\rangle$ be a strictly increasing γ-sequence of regular uncountable cardinals with $\sup_{\xi \in \gamma} \lambda_\xi = \kappa$. One next constructs a $(\gamma + 1)$-sequence $\langle \mathcal{I}_\xi\colon \xi \in \gamma + 1\}$ of independent subcollections of $\mathcal{B}$ and a γ-sequence $\langle \mathcal{F}_\xi\colon \xi \in \gamma\rangle$ of finite subsets of $\mathcal{B}$ satisfying

$$\forall \xi \in \gamma[|\mathcal{I}_{\xi+1}| = \lambda_{\xi+1} \text{ and } \mathcal{I}_\xi - \mathcal{F}_\xi \subseteq \mathcal{I}_{\xi+1}] ;$$

$$\forall \eta \in \gamma + 1\left[\eta \text{ a limit} \Rightarrow \mathcal{I}_\eta = \bigcup_{\xi \in \eta} \left(\mathcal{I}_\xi - \bigcup_{\zeta \in \eta} \mathcal{F}_\zeta\right)\right] .$$

Then $\mathcal{I}_\gamma$ is an independent family of cardinality κ.

This (including the argument in 9.4) is the zerodimensional form of the Gerlits–Nagy proof of 9.3 as presented by JUHÁSZ [1980], with a simplification in the proof of $\mathrm{ind}(X) \geq h_c$ min $\pi\chi(X)$: We construct Q so that (1) holds, and then have to "thin out" Q because we do not have (*) $Q(\xi, 1) = X - Q(\xi, 0)$ for $\xi \in \lambda$; while they construct Q so that (*) holds but have to "thin out" Q because (1) does not hold, and that is harder to do. (We have not seen Šapirovskiĭ's proof.)

For later use we point out the following corollary:

9.5. COROLLARY. *If X is an infinite Boolean space, then $\{x \in X\colon \pi\chi(x, X) \leq \mathrm{ind}(\mathcal{B})\}$ is dense in X.*

PROOF. If $D = \{x \in X\colon \pi\chi(x, X) \leq \mathrm{ind}(\mathcal{B})\}$ is not dense, consider any nonempty $Y \in \mathcal{B}$ with $Y \cap D = \emptyset$. Since Y is open we have $\forall x \in Y\, [\pi\chi(x, Y) = \pi\chi(x, X) \geq \mathrm{ind}(\mathcal{B})^+]$. As Y is closed, it now follows from 9.3 that $\mathrm{ind}(\mathcal{B}) \geq \mathrm{ind}(\mathcal{B})^+$, which is absurd. $\square$

One can interpret this result as saying that $\min_d \pi\chi(X) \leq \mathrm{ind}(\mathscr{B})$, with the variation $\min_d \pi\chi(X)$ of $\min \pi\chi(X)$ defined by

$$\min_d \pi\chi(X) = \min\{\kappa\colon \{x \in X\colon \pi\chi(x, X) \leq \kappa\} \text{ is dense in } X\}\,.$$

We should point out that it is not generally true that $\min_d \pi\chi(X) = \mathrm{ind}(\mathscr{B})$: by Theorem 14.2 $\mathrm{ind}(\mathscr{P}(\kappa)) = 2^\kappa$, but of course $\min_d \pi\chi(\mathscr{P}(\kappa)) = 1$.

10. Getting small dense subsets by killing witnesses

The technique of killing winesses is this: we want to construct a dense subset D of a space Y which has "small" cardinality, κ say. We construct D as $\bigcup_{\xi\in\lambda} Q(\xi)$, where λ is a cardinal $\leq\kappa$ and where $\langle Q(\xi)\colon \xi \in \lambda\rangle$ is an increasing λ-sequence of subsets of Y each of cardinality at most κ. Then trivially $|D| \leq \kappa$. How do we ensure D is dense? By killing all witnesses that D is not dense: if D is not dense then there will be a set W of a special form which witnesses that $\bar{D} \neq Y$ in the sense that $D \subseteq \bar{W}$ and $\bar{W} \neq Y$. However, because of the special form of W and the way we construct D from $\langle Q(\xi)\colon \xi \in \lambda\rangle$ there will be $\eta \in \lambda$ such that W is a witness that $\overline{Q(\eta)} \neq Y$. But we construct $Q(\eta + 1)$ so as to kill W, i.e. $Q(\eta + 1) \not\subseteq \bar{W}$, and so W does not witness that D is not dense after all. This technique was first used by ŠAPIROVSKIĬ [1972], and became widely known after it was used by POL [1974], to give new proofs of known inequalities.

In one application of the technique we must do this with Y a subspace of another space X; this explains the X and Y in the following two lemmata.

10.1. LEMMA. *Let λ, μ and τ be cardinals with $\tau \geq \omega$ and $\lambda \leq \tau^+$, and let $\kappa = (\tau\cdot\mu)^{<\lambda}$. Let $Y \subseteq X$, and assume there is $\phi\colon [Y]^{\leq\kappa} \to \mathscr{P}(\mathscr{B})$ such that*

(1) $\forall A \in [Y]^{\leq\kappa}\ [|\phi(A)| \leq \mu]$;

(2) $\forall$ *nondecreasing* $Q\colon \tau^+ \to [Y]^{\leq\kappa}$ [*if* $Y \not\subseteq \overline{\bigcup_{\xi\in\tau^+} Q(\xi)}$, *then* $\exists \mathscr{U} \subseteq \bigcup_{\xi\in\tau^+} \phi(Q(\xi))$ $[|\mathscr{U}| < \lambda$ *and* $\bigcup_{\xi\in\tau^+} Q(\xi) \subseteq \overline{\bigcup \mathscr{U}}$ *and* $Y \not\subseteq \overline{\bigcup \mathscr{U}}$]].

Then $d(Y) \leq \kappa\cdot\tau^+$.

PROOF. Construct $Q\colon \tau^+ \to [Y]^{\leq\kappa}$ as follows: $Q(0)$ is a singleton subset of Y (unless $Y = \emptyset$, in which case the lemma is trivial). For limit $\eta \in [1, \tau^+)$ put $Q(\eta) = \bigcup_{\xi\in\eta} Q(\xi)$; note that $|Q(\eta)| \leq \kappa$ since $\kappa \geq \tau \geq |\eta|$. For $\eta \in \tau^+$, if $Q(\eta)$ is known, then by (1) the collection

$$\mathbb{T}_\eta = \left\{\mathscr{U} \subseteq \bigcup_{\xi\in\eta} \phi(Q(\xi))\colon |\mathscr{U}| < \lambda \text{ and } Q(\eta) \subseteq \overline{\bigcup \mathscr{U}} \text{ and } Y \not\subseteq \overline{\bigcup \mathscr{U}}\right\}$$

of witnesses that $Q(\eta)$ is not dense has cardinality at most $(\tau\cdot\mu)^{<\lambda} = \kappa$. Hence, we can choose $Q(\eta+1) \subseteq Y$ with $Q(\eta+1) \supseteq Q(\eta)$ and $|Q(\eta+1)| \leq \kappa$ which kills all witnesses that $Q(\eta)$ is not dense, i.e. which satisfies

$$(*)\qquad \forall \mathscr{U} \in \mathbb{T}_\eta \left[Q(\eta+1) \not\subseteq \overline{\bigcup \mathscr{U}}\right].$$

This completes the construction.

Clearly, $D = \bigcup_{\xi\in\tau^+} Q(\xi)$ has cardinality at most $\kappa\cdot\tau^+$. Suppose $Y \not\subseteq \bar{D}$. Then by (2) there is $\mathcal{U} \subseteq \bigcup_{\xi\in\tau^+} \phi(Q(\xi))$ with $|\mathcal{U}| < \lambda$ such that $D \subseteq \overline{\bigcup \mathcal{U}}$ and $Y \not\subseteq \overline{\bigcup \mathcal{U}}$. As $\lambda \leq \tau^+$, it follows that there is $\eta\in\tau^+$ with $\mathcal{U} \subseteq \bigcup_{\xi\in\eta} \phi(Q(\xi))$. But now $Q(\eta) \subseteq D \subseteq \overline{\bigcup \mathcal{U}}$ and $Y \not\subseteq \overline{\bigcup \mathcal{U}}$, hence $\mathcal{U} \in \mathbb{T}_\eta$, but also $Q(\eta+1) \subseteq D \subseteq \overline{\bigcup \mathcal{U}}$, which contradicts (*). □

10.2. LEMMA. *Let X be infinite, let μ be a cardinal, and let $\kappa = \mu\cdot t(X)^+$.*

(a) *If $\chi(\bar{A}, X) \leq \mu$ for every $A \in [X]^{\leq\kappa}$ then $d(X) \leq \kappa$.*

More generally:

(b) *Let $Y \subseteq X$, and assume there is $\phi\colon [Y]^{\leq\kappa} \to \mathcal{P}(\mathcal{B})$ such that*

(1) $\forall A \in [Y]^{\leq\kappa}\, [|\phi(A)| \leq \mu]$;

(2) $\forall A \in [Y]^{\leq\kappa}\, [\bar{A} \subseteq \bigcap \phi(A)$ *and* $Y \cap \bar{A} = Y \cap \bigcap \phi(A)]$.

Then $d(Y) \leq \kappa$.

PROOF. We verify that Lemma 10.1 holds, with $\lambda = \omega$ and $\tau = t(X)$. Condition (1) of 10.1 is trivially satisfied. To verify (2) of 10.1 consider any nondecreasing $Q\colon \tau^+ \to [Y]^{\leq\kappa}$ such that if S abbreviates $\overline{\bigcup_{\xi\in\tau^+} Q(\xi)}$, then $Y \not\subseteq \bar{S}$. Then S is closed in X, by 5.6, since it is the union of an nondecreasing $t(X)^+$-sequence of closed subsets of X, and hence S is compact. Pick any $y \in Y - \bar{S}$. By (2) of 10.2(b) we can choose for each $\xi\in\tau^+$ a $U_\xi \in \phi(Q(\xi))$ such that $y \not\in U_\xi$ and $\overline{Q(\xi)} \subseteq U_\xi$. Hence, $S \subseteq \bigcup_{\xi\in\tau^+} U_\xi$, and therefore there is a finite $\mathcal{U} \subseteq \{U_\xi\colon \xi\in\tau^+\}$ which covers the compact set S. Then $\mathcal{U}$ is a finite subcollection of $\bigcup_{\xi\in\tau^+} \phi(Q(\xi))$ with $\overline{\bigcup_{\xi\in\tau^+} Q(\xi)} \subseteq \bigcup\mathcal{U}$ and $Y \not\subseteq \overline{\bigcup \mathcal{U}}$ since $Y \not\subseteq \bigcup\mathcal{U}$ and since $\mathcal{U}$ is a finite collection of clopen sets. □

The hard work has been done now. Our first application is a celebrated result of ARHANGEL'SKIĬ [1969].

10.3. THEOREM. *If X is a Boolean space, then $|X| < 2^{\chi(X)}$.*

PROOF. Let $\kappa = 2^{\chi(X)}$, and for $x\in X$ choose a neighborhood base $\mathcal{U}_x$ with $|\mathcal{U}_x| \leq \chi(X)$. From the inequality $\# \leq d^\chi$ in Theorem 6.1 we know that

$$(*)\qquad \forall A \in [X]^{\leq\kappa}\, [|\bar{A}| \leq \kappa]\,.$$

It follows that we prove the theorem if we prove $d(X) \leq \kappa$. Obviously, since X is compact for each closed $F \subseteq X$ the collection $\mathcal{C}(F) = \{\bigcup \mathcal{V}\colon \mathcal{V} \in [\bigcup_{x\in F} \mathcal{U}_x]^{<\omega}\}$ is a neighborhood base at F of cardinality at most $|F|\cdot\chi(X)$. It follows from (*) that $\forall A \in [X]^{\leq\kappa}\, [\chi(\bar{A}, X) \leq \kappa]$. Hence $d(X) \leq \kappa$ by 10.2(a). □

An almost immediate consequence of Lemma 10.2(a) is the following:

10.4. PROPOSITION (ŠAPIROVSKIĬ [1974]). *If X is a Boolean space, then $hd(X) \leq t(X)^+\cdot hL(X)$.*

PROOF. Since t and hL are monotone, and since $hd(X) = h_c d(X)$ by Theorem 8.2, it suffices to prove $d(X) \leq t(X)^+\cdot hL(X)$. As $hL(X) = \chi_C(X)$ by 5.3(c), this inequality is an immediate consequence of 10.2(a). □

Since $hL(X) \geq s(X)$, as pointed out in Theorem 7.7, the following result is stronger; to prove it we use the full generality of Lemma 10.2.

10.5. THEOREM (ŠAPIROVSKIĬ [1974]). *If X is a Boolean space, then $hd(X) \leq s(X) \cdot t(X)^+$. (Hence then $hd(X) \leq s(X)^+$.)*

PROOF. (For the parenthetical remark recall from 7.7 and 8.7 that $F(X) \leq s(X)$ and that $F(X) = t(X)$.)

Let κ abbreviate $s(X) \cdot t(X)^+$. Since s and t are monotone, and since $hd(X) = h_C d(X)$, by 8.2(2), it suffices to prove that $d(X) \leq \kappa$.

By 7.4, and also by 8.2, there is a dense $Y \subseteq X$ which has a well-order $<_1$ such that

(1) every final $<_1$-segment is open in Y.

Since we want to prove $d(X) \leq \kappa$ we may without loss of generality assume that $|Y| > \kappa$.

We claim that $hL(Y) \leq \kappa$. Indeed, if not there are by 7.3 an $S \in [Y]^{\kappa^+}$ and a well-order $<_2$ on S such that

(2) every initial $<_2$-segment is open in S.

As is well known, the arrow relation $\kappa^+ \to (\kappa^+, \omega)^2$ implies that there is $D \in [S]^{\kappa^+}$ such that $\forall x \neq y \in S\ [x <_1 y \Leftrightarrow x <_2 y]$. But then D is a discrete subspace of Y, because of (1) and (2), which contradicts $s(Y) \leq s(X) \leq \kappa$.

Since $hL(Y) \leq \kappa$ we can choose for each $A \subseteq Y$ a collection $\phi(A)$ of neighborhoods of $\bar{A}$ in X such that $Y \cap \bar{A} = Y \cap \bigcap \phi(A)$: find $\mathcal{U} \subseteq \{B \in \mathcal{B} : B \cap A = \emptyset\}$ with $|\mathcal{U}| \leq L(Y - \bar{A}) \leq \kappa$ such that $Y - \bar{A} \subseteq \bigcup \mathcal{U}$, and let $\phi(A) = \{X - U : U \in \mathcal{U}\}$. Hence $d(X) \leq \kappa$ by Lemma 10.2(b). □

Our next result is valid for all zerodimensional spaces, not just for Boolean spaces, and this extra generality will be used in the proof of Theorem 10.8.

10.6. THEOREM (ŠAPIROVSKIĬ [1974]). *If X is a nondiscrete zerodimensional space, then $d(X) \leq \pi\chi(X)^{c(X)}$. (Hence then $\rho(X) \leq \pi\chi(X)^{c(X)}$.)*

PROOF. (To prove the parenthetical statement recall from 6.1 that $\pi(X) = d(X) \cdot \pi\chi(X)$ and $\rho(X) \leq \pi(X)^{c(X)}$, hence

$$\rho(X) \leq (d(X) \cdot \pi\chi(X))^{c(X)} \leq (\pi\chi(X)^{c(X)} \cdot \pi\chi(X))^{c(X)} = \pi\chi(X)^{c(X)}\ .)$$

The point of the requirement that X be nondiscrete is that $\pi\chi(X) > 1$.

We will apply Lemma 10.1 with $X = Y$ and with λ, μ and τ defined by

$$\tau = c(X)\,, \quad \lambda = \tau^+\,, \quad \mu = \pi\chi(X)^{c(X)}\,, \quad \text{so if } \kappa = (\tau \cdot \mu)^{<\lambda} \text{ then}$$

$$\kappa = \pi\chi(X)^{c(X)}\ .$$

For each $x \in X$ choose a neighborhood π-base $\mathcal{B}_x$ with $|\mathcal{B}_x| \leq \pi\chi(X)$, and for all $A \subseteq X$ define $\phi(A) = \bigcup_{x \in A} \mathcal{B}_x$. Obviously, $\phi(A) \leq \kappa \cdot \pi\chi(X) = \kappa$ for all $A \in [X]^{\leq\kappa}$, hence (1) of 10.(1) holds. We verify (2) of 10.(1).

Consider any nondecreasing $Q: \tau^+ \to [X]^{\leq\kappa}$, such that if D abbreviates $\bigcup_{\xi\in\tau^+} Q(\xi)$, then D is not dense. As X is zerodimensional there is $B \in \mathcal{B}$ with $\overline{D} \subseteq B \neq X$. Choose a maximal cellular $\mathcal{V} \subseteq \{U \in \phi(D): U \subseteq B\}$; of course $\overline{\bigcup \mathcal{V}} \neq X$ since $\bigcup \mathcal{V} \subseteq B = \bar{B}$ and $B \neq X$. A moment's reflection reveals that $D \subseteq \overline{\bigcup \mathcal{V}}$ by maximality. Clearly, $|\mathcal{V}| \leq c(X) = \tau$. So $|\mathcal{V}| < \lambda$. Also, since clearly $\phi(D) = \bigcup_{\xi\in\tau^+} \phi(Q(\xi))$, there is $\eta \in \tau^+$ such that $\mathcal{V} \subseteq \bigcup_{\xi\in\eta} \phi(Q(\xi))$. $\square$

The above result improves the inequality $\rho \leq \pi^c$ of Theorem 6.1, for there are easy examples of Boolean spaces X with $\pi X(X) < \pi(X)$. For example, if X is the one-point compactification of a discrete space of cardinality κ (so $\mathcal{B}$ is the finite–cofinite algebra on κ) then $\pi\chi(X) = \omega$ but $\pi(X) = \kappa$. However, in this example $\pi\chi(X)^{c(X)} = \pi(x)^{c(X)} = 2^\kappa$. This is no coincidence, as our next observation shows.

10.7. PROPOSITION. *If X is a nondiscrete zerodimensional space, then*

$$\rho(X)^{c(X)} = w(X)^{c(X)} = \pi(X)^{c(X)} = \pi\chi(X)^{c(X)} .$$

PROOF. Since $\pi \leq w \leq \rho$, by 6.1, we have $\pi^c \leq w^c \leq \rho^c \leq (\pi\chi^c)^c = \pi\chi^c$. $\square$

In Example 14.3(b) we will see that $\rho(X)$ can be strictly less than $\pi(X)^{c(X)}$.

In spite of Proposition 10.7, Theorem 10.6 is useful, namely in situations where one has information about $\pi\chi$ but not about π. The proof of the following theorem illustrates this.

10.8. THEOREM (Probably ŠAPIROVSKIĬ [1980]). *If $\mathcal{B}$ is a BA, then $\rho(\mathcal{B}) \leq \operatorname{ind}(\mathcal{B})^{c(\mathcal{B})}$.*

PROOF. Let $\kappa = \operatorname{ind}(\mathcal{B})$. By Corollary 9.5 the set

$$Y = \{y \in X: \pi\chi(y, X) \leq \kappa\}$$

is dense in X. We use this to prove $\rho(X) \leq \kappa^{c(X)}$:

By 6.6, $\pi\chi(Y) \leq \kappa$ and $c(Y) = c(X)$, hence $d(Y) \leq \kappa^{c(X)}$ by 10.6, and therefore $\pi(Y) \leq \kappa \cdot \kappa^{c(X)} = \kappa^{c(X)}$ since $\pi(Y) = d(Y) \cdot \pi\chi(Y)$ by 6.1. But $\pi(X) = \pi(Y)$ by 6.6. So $\rho(X) \leq \kappa^{c(X)}$ by 10.7. (Alternatively, $\rho(Y) \leq \kappa^{c(X)}$, and, as mentioned after 6.6, $\rho(X) = \rho(S)$ whenever S is dense in X.) $\square$

By Lemma 6.5, if X is a Boolean space, then $\chi(X) \leq 2^{s(X)}$. The final result of this section is the promised improvement that $|\mathcal{B}| \leq 2^{s(\mathcal{B})}$, which, unlike its proof, is purely BAic. (By this we mean that the cardinal function s has a satisfactory BAic translation, see Theorem 5.8 or Theorem 7.1.)

10.9. THEOREM (ŠAPIROVSKIĬ [1972]). *If X is a Boolean space, then $w(X) \leq 2^{s(X)}$. (Hence $\rho(X) \leq 2^{s(X)}$.)*

PROOF. (The parenthetical remark follows from the inequalities $\rho \le \pi^c$, $\pi \le w$ and $c \le s$ from 6.1.)

We need some notation. For $S \subseteq X$ and for a cardinal τ define

$$\bar{S}^\tau = \bigcup \{\bar{A} : A \in [S]^{\le\tau}\} .$$

(So $t(X) = \min\{\tau : \forall S \subseteq X[\bar{S} = \bar{S}^\tau]\}$.)

Let $\sigma = s(X)$. Because of the Addition Theorem for weight, 6.3, it suffices to find a collection $\mathscr{C}$ of subspaces of X such that $\bigcup \mathscr{C} = X$ and $\Sigma_{C\in\mathscr{C}} w(C) \le 2^\sigma$. As $w(Y) \le 2^{d(Y)}$ for every zerodimensional space Y, by 6.1, and as $|[2^\lambda]^{\le\lambda}| = 2^\lambda$ for every infinite cardinal λ, it follows that the following claim implies the theorem.

Claim. There is $S \subseteq X$ with $|S| \le 2^\sigma$ such that $X = \bar{S}^\sigma$

The proof is a variation of the proof of Lemma 10.1.

Since $\chi(X) \le 2^{s(X)} = 2^\sigma$, by 6.5, we can choose for each $x \in X$ a neighborhood base $\mathscr{B}_x \subseteq \mathscr{B}$ with $|\mathscr{B}_x| \le 2^\sigma$. For $A \subseteq X$ define $\mathscr{B}_A = \bigcup_{x\in A} \mathscr{B}_x$; clearly, $|\mathscr{B}_A| \le |A| \cdot 2^\sigma$.

Construct a nondecreasing $Q : \sigma^+ \to [X]^{\le \exp \sigma}$ as follows: $Q(0)$ is a singleton. For limit $\eta \in \sigma^+$, if $Q(\eta)$ is known for $\xi \in \eta$, then $Q(\eta) = \bigcup_{\xi\in\eta} Q(\xi)$; then $|Q(\eta)| \le 2^\sigma$ since $|\eta| \le \sigma$. For $\eta \in \sigma^+$, if $Q(\eta)$ is known, then the set

$$\mathbb{T}_\eta = \Big\{\langle K, \mathscr{V}\rangle \in [Q(\eta)]^{\le\Sigma} \times [\mathscr{B}_{Q(\eta)}]^{\le\sigma} : Q(\eta) \subseteq \bar{K} \cup \bigcup \mathscr{V} \text{ and } \bar{K} \cup \bigcup \mathscr{V} \ne X\Big\}$$

of witnesses that $\overline{Q(\eta)}^\sigma \ne X$ has cardinality at most $(2^\sigma)^\sigma \cdot (2^\sigma)^\sigma = 2^\sigma$. Hence we can choose $Q(\eta+1) \supseteq Q(\eta)$ with $|Q(\eta+1)| \le 2^\sigma$ such that

$$(*) \qquad \forall \langle K, \mathscr{V}\rangle \in \mathbb{T}_\eta \Big[Q(\eta+1) \not\subseteq \bar{K} \cup \bigcup \mathscr{V} \Big] .$$

This completes the construction.

Let $S = \bigcup_{\xi\in\sigma^+} Q(\xi)$. We claim $\bar{S}^\sigma = X$. Suppose not, pick $p \in X - \bar{S}^\sigma$, and let $\mathscr{U} = \{B \in \mathscr{B}_S : p \not\in B\}$. Then $\mathscr{U}$ covers S, hence by 6.4 there is $\langle K, \mathscr{V}\rangle \in [S]^{\le\sigma} \times [\mathscr{U}]^{\le\sigma}$ such that $S \subseteq \bar{K} \cup \bigcup \mathscr{V}$. Clearly, $p \not\in \bigcup \mathscr{V}$, and also $p \not\in \bar{K}$ since $p \not\in \bar{S}^\sigma$. As evidently $\mathscr{B}_S = \bigcup_{\xi\in\sigma^+} \mathscr{B}_{Q(\xi)}$, there is $\eta \in \sigma^+$ such that $\langle K, \mathscr{V}\rangle \in [Q(\eta)]^{\le\sigma} \times [\mathscr{B}_{Q(\eta)}]^{\le\sigma}$. So now $Q(\eta) \subseteq S \subseteq \bar{K} \cup \bigcup \mathscr{V}$ and $\bar{K} \cup \bigcup \mathscr{V} \ne X$, so $\langle K, \mathscr{V}\rangle \in \mathbb{T}_\eta$, but also $Q(\eta+1) \subseteq S \subseteq \bar{K} \cup \bigcup \mathscr{V}$, which contradicts $(*)$. □

11. Weakly countably complete algebras

In this section we show that for certain infinite Boolean spaces X there are restrictions on the values $\phi(X)$ can take if ϕ is weight, cardinality or independence. See also Theorem 12.7.

We call a BA $\mathscr{B}$ WCC (= weakly countably complete) if for all countable families $\mathscr{F}, \mathscr{G} \subseteq \mathscr{B}$, If $\forall F \in \mathcal{F}\ \forall G \in \mathcal{G}\ [F \cap G = \emptyset]$, then $\exists S \in \mathcal{B}\ \forall F \in \mathcal{F}\ \forall G \in \mathcal{G}$ $[F \subseteq S$ and $G \subseteq X - S]$.

(WCC algebras are also said to have the countable separation property. See also Part I, Sections 5 and 12.)

Clearly (countably) complete BAs are WCC. The topological significance of WCC BAs is that they are precisely the clopen algebras of compact zerodimensional F-spaces. For the purpose of this chapter we call a space X and *F-space* if for every two F_σ-subsets F and G of X, if F and G are *separated*, i.e. if $\bar{F} \cap G = \emptyset = F \cap \bar{G}$, then $\bar{F} \cap \bar{G} = \emptyset$.

11.1. Proposition (see also Part I, Proposition 12.1). *For a Boolean space X the following are equivalent*:

(1) *$\mathcal{B}$ is WCC*;

(2) *X is an F-space*; *and*

(3) *every two disjoint open F_σ-subsets of X have disjoint closures.*

Proof. That (2)$\Rightarrow$(3) and that (1)$\Leftrightarrow$(3) should be clear.

Proof of (3)$\Rightarrow$(2): Let K and L be two separated F_σ-subsets of X. From the proof that regular Lindelöf spaces are normal we see that there are countable $\mathcal{F}, \mathcal{G} \subseteq \mathcal{B}$ with $K \subseteq \bigcup \mathcal{F}$ and $L \subseteq \bigcup \mathcal{G}$ such that $\forall F \in \mathcal{F}\ \forall G \in \mathcal{G}\ [F \cap G = \emptyset]$. Hence, there is $S \in \mathcal{B}$ with $K \subseteq S$ and $L \subseteq X - S$. □

11.2. Corollary. *Every closed subspace of a compact zerodimensional F-space is an F-space.* □

One calls a cardinal κ an *ω-power* if $\kappa^\omega = \kappa$. We will show for infinite Boolean F-spaces X that $\phi(X)$ is an ω-power for ϕ weight, cardinality and independence.

We will need the following easy set theoretic fact:

11.3. Fact. If $\kappa \geq \omega$ then κ is an ω-power iff $\lambda^\omega \leq \kappa$ for every $\lambda \leq \kappa$ with $\mathrm{cf}(\lambda) = \omega$. □

Our strategy to prove $\phi(X)$ is an ω-power if X is an infinite Boolean F-space is this: for $x \in X$ define the *local ϕ at x* by

$$l\phi(x) = \min\{\phi(B) \colon x \in B \in \mathcal{B}\},$$

and consider any $\lambda \leq \phi(X)$ with $\mathrm{cf}(\lambda) = \omega$. If there is an injection $x \colon \omega \to X$ with $\sup_{n\in\omega} l\phi(x_n) \geq \lambda$ we can easily show that $\phi(X) \geq \lambda^\omega$. If there is no such injection, then obviously there are a finite $G \subseteq X$ and a $\mu < \lambda$ such that $\forall x \in X - F\ [l\phi(X) \leq \mu]$. For each of the three functions cardinality, weight and independence we do something else to take care of the hard case.

We also need the following information about WCC BAs and their Stone spaces.

11.4. LEMMA. *Let X be an infinite zerodimensional compact F-space.*

(a) *For every countable cellular $\mathscr{A} \subseteq \mathscr{B}$ and every $U \in \Pi_{A\in\mathscr{A}}$ CLOP(A) there is $B \in \mathscr{B}$ such that $\forall A \in \mathscr{A}\ [B \cap A = U(A)]$.*

(b) *For every countable discrete subspace D of X we have $\bar{D} = \beta D$, i.e. $\bar{K} \cap \bar{L} = \emptyset$ for every two disjoint $K, L \subseteq D$.*

(c) *$\beta\omega$ embeds into X, or dually, $\mathscr{P}(\omega)$ is a quotient of $\mathscr{B}$.*

(d) *X has infinitely many nonisolated points.*

Proof of (*a*). For $U \in \Pi_{A\in\mathscr{A}}$ CLOP(A) use (1) or (3) of 11.1 to find $B \in \mathscr{B}$ with $\bigcup_{A\in\mathscr{A}} U(A) \subseteq B$ and $\bigcup_{A\in\mathscr{A}} (A - U(A)) \subseteq X - B$.

Proof of (*b*). Clear from 11.1.

Proof of (*c*). X has a countably infinite discrete subspace, by 5.7.

Proof of (*d*). Clear from (c) once we show that $|\beta\omega - \omega| \geq \omega$. (In fact $|\beta\omega - \omega| = 2^{\mathfrak{c}}$, but we do not need that much.) There is an infinite pairwise disjoint $\mathscr{A} \subseteq [\omega]^{\omega}$, each member of A has a limit point, and $\bar{A} \cap \bar{B} = \emptyset$ for $A \neq B \in \mathscr{A}$. $\square$

We now are ready for the first result about ω-powers and WCC algebras.

11.5. THEOREM (KOPPELBERG [1975]. See also Part 1, Theorem 12.2). *If $\mathscr{B}$ is an infinite WCC BA, then $|\mathscr{B}|$ is an ω-power.*

PROOF. That $|\mathscr{B}|$ is an ω-power if $\mathscr{B}$ is an infinite countably complete BA is an earlier result of COMFORT and HAGER [1972], and the following proof is based on their proof. See Corollary 11.6 for an even earlier result.

We prove the theorem with induction. Hence we may assume that

(1) $\qquad \forall B \in \mathscr{B}\ [\omega \leq w(B) < w(X) \Rightarrow w(B)$ is an ω-power$]$.

For a Boolean space Y we let $\gamma(Y)$ abbreviate $|\text{CLOP}(Y)|$. (Recall from 5.1 that $\gamma(Y) = w(Y)$ if Y is infinite; the reason for using γ is that $\mathscr{B}$ may contain finite members.)

We use the following fact:

(2) $\qquad \forall$ countable cellular family $\mathscr{A}$ in $\mathscr{B}$ $\left[\gamma\left(\overline{\bigcup \mathscr{A}}\right) = \prod_{A\in\mathscr{A}} \gamma(A)\right]$.

Indeed, let D abbreviate $\bigcup \mathscr{A}$ and let Y abbreviate $\bar{D}$. Since D is dense in Y the function $U \mapsto D \cap U$, ($U \in \text{CLOP}(Y)$), is an injection CLOP$(Y) \to$ CLOP(D), hence $\gamma(Y) \leq \gamma(D)$; since obviously $\gamma(D) = \Pi_{A\in\mathscr{A}} \gamma(A)$ it follows that $\gamma(Y) \leq \Pi_{A\in\mathscr{A}} \gamma(A)$. The inequality $\Pi_{A\in\mathscr{A}} \gamma(A) \leq \gamma(Y)$ follows from 11.4(a).

From 11.4(c) we get $|\mathscr{B}| \geq |\mathscr{P}(\omega)|$, hence

(3) $\qquad w(X) \geq \mathfrak{c}$.

We prove $|\mathscr{B}|$ is an ω-power by using Fact 11.3. So consider any $\lambda \leq |\mathscr{B}|$ with cf$(\lambda) = \omega$; we will show $\lambda^{\omega} \leq |\mathscr{B}|$. Let $\langle \lambda_n : n \in \omega \rangle$ be a nondecreasing sequence of cardinals with $\sup_{n\in\omega} \lambda_n = \lambda$; note that $\Pi_{n\in\omega} \lambda_n = \lambda^\omega$. For $x \in X$ define the local number of clopen sets of X in x by

$$l\gamma(x) = \min\{\gamma(B)\colon x \in B \in \mathcal{B}\}\ .$$

Case 1. There is an injection $s\colon \omega \to X$ such that $\forall n \in \omega\ [l\gamma(s_n) \geq \lambda_n]$.

Because of 5.7 we may assume there is $A\colon \omega \to \mathcal{B}$ with $\forall n \in \omega\ [s_n \in A_n]$ and $\forall k \neq n \in \omega\ [A_k \cap A_n = \emptyset]$. Then also $\forall n \in \omega\ [\gamma(A_n) \geq \lambda_n]$. It follows from (2) that $|\mathcal{B}| \geq \Pi_{n\in\omega} \lambda_n = \lambda^\omega$.

Case 2. Not Case 1.

Then there are a finite $F \subseteq X$ and a $\mu < \lambda$ such that if

$$\mathcal{A} = \{B \in \mathcal{B}\colon \gamma(B) \leq \mu\}\ ,$$

then $\bigcup \mathcal{A} = X - F$. For later use we point out that $\mu \geq \omega$. Since X has infinitely many nonisolated points, by 11.4(d), and since F is finite, the points of $X - F$ cannot be all isolated.

The crux of the matter is that there is a countable $\mathcal{Q} \subseteq \mathcal{A}$ such that if

$$Y = \overline{\bigcup \mathcal{Q}} \text{ and } \mathcal{L} = \{A \in \mathcal{A}\colon A \cap Y = \emptyset\}$$

then

$$\forall \mathcal{C} \in [\mathcal{L}]^{\leq\omega}\ \exists L \in \mathcal{L}\left[\bigcup \mathcal{C} \subseteq L\right]. \tag{4}$$

Indeed, since F is finite, there is a countable $\mathcal{Q} \subseteq \mathcal{A}$ such that

$$\forall x \in F\left[\text{if } \exists \mathcal{C} \in [\mathcal{A}]^{\leq\omega}\left[x \in \overline{\bigcup \mathcal{C}}\right], \text{then } x \in \overline{\bigcup \mathcal{Q}}\right].$$

Now if $\mathcal{C} \subseteq \mathcal{L}$ is countable, then $\bigcup \mathcal{C}$ and $\bigcup \mathcal{Q}$ are disjoint open F_σ's, so $\overline{\bigcup \mathcal{C}} \cap Y = \emptyset$, while $\overline{\bigcup \mathcal{C}} \cap F = \emptyset$ by our choice of $\mathcal{Q}$. Hence, $\overline{\bigcup \mathcal{C}} \subseteq \bigcup \mathcal{L}$. As $\mathcal{L}$ is closed under finite union, because $\mu \geq \omega$, and as X is compact, it follows that there is $L \in \mathcal{L}$ with $\bigcup \mathcal{C} \subseteq L$.

Subcase (a). $w(Y) = w(X)$.

We take care of this subcase by showing that $w(Y)$ is an ω-power. Indeed, since $\mathcal{Q}$ is a countable subcollection of $\mathcal{A}$ there is a countable cellular $\mathcal{K} \subseteq \mathcal{A}$ with $\bigcup \mathcal{K} = \bigcup \mathcal{Q}$, and it now follows from (2) that

$$\gamma(Y) = \prod_{K\in\mathcal{K}} \gamma(K)\ .$$

If $w(Y) = \mathfrak{c}$, then trivially it is an ω-power, and otherwise $w(Y) > \mathfrak{c}$, because of (3), and now

$$w(Y) = \prod\{w(K)\colon K \in \mathcal{K} \text{ and } w(K) \in \omega\}\ ,$$

hence $w(Y)$ is an ω-power because of (1).

Subcase (b). $w(Y) < w(X)$.

Of course $X = F \cup Y \cup \bigcup \mathscr{L}$, hence $w(X) \leq |F| + w(Y) + \Sigma_{L \in \mathscr{L}} w(L)$ by the Addition Theorem for weight, 6.3. As $|F| < \omega \leq w(X)$ and $w(Y) < w(X)$ and $\forall L \in \mathscr{L}\ [w(L) \leq \mu < w(X)]$, it follows that $|\mathscr{L}| = w(X)$. Because of (4) we have

$$^{\omega}\mathscr{L} = \bigcup_{L \in \mathscr{L}} {}^{\omega}\{B \in \mathscr{B} : B \subseteq L\} .$$

Since $\mu < w(X)$ it follows from (1) that

$$|\mathscr{L}|^{\omega} \leq \sum_{L \in \mathscr{L}} \gamma(L)^{\omega} \leq |\mathscr{L}| \cdot \sup_{L \in \mathscr{L}} \mathfrak{c} \cdot \gamma(L) \leq |\mathscr{L}| \cdot \mathfrak{c} \cdot \mu ,$$

hence $w(X)$ is an ω-power since $\mathfrak{c} \leq w(X)$ and since $\mu < |\mathscr{L}| = w(X)$. □

11.6. COROLLARY (PIERCE [1958]). *If $\mathscr{A}$ is an infinite complete BA, then $|\mathscr{A}|$ is an ω-power. In particular, if X is an infinite Boolean space, then $\rho(X)$ is an ω-power.* □

Recall that a family $\mathscr{A}$ of countable sets is called *almost disjoint* if $\forall A \neq B \in \mathscr{A}$ $[A \cap B$ is finite$]$. We need the following result of Tarski.

11.7. TARSKI'S LEMMA. *Let $\langle \lambda_n : n \in \omega \rangle$ be a sequence of nonzero cardinals such that*

$$\forall n \in \omega \left[\lambda_{n+1} \geq \prod_{k \leq n} \lambda_k \right] ,$$

and let $\lambda = \sup_{n \in \omega} \lambda_n$. For every pairwise disjoint sequence $\langle L_n : n \in \omega \rangle$ of sets with $\forall n \in \omega\ [|L_n| \geq \lambda_n]$ there is an almost disjoint family $\mathscr{A}$ of subsets of $\bigcup_{n \in \omega} L_n$ with $|\mathscr{A}| = \lambda^{\omega}$ such that $\forall A \in \mathscr{A}$, $\forall n \in \omega\ [|A \cap L_n| = 1]$.

PROOF. To prove this note that without loss of generality $L_n = \Pi_{k \leq n} [0, \lambda_k)$, hence if $A_f = \{f \restriction [0, n] : n \in \omega\}$ for $f \in \Pi_{n \in \omega} [0, \lambda_n)$, then $\mathscr{A} = \{A_f : f \in \Pi_{n \in \omega} [0, \lambda_n)\}$ is as required since $|\Pi_{n \in \omega} \lambda_n| = \lambda^{\omega}$ because $\langle \lambda_n : n \in \omega \rangle$ is nondecreasing and $\lambda = \sup_{n \in \omega} \lambda_n$. □

11.8. THEOREM (VAN DOUWEN [1981]). *If X is an infinite compact Boolean F-space, then $|F|$ is an ω-power.*

PROOF. For $x \in X$ define the local cardinality of X at x by

$$l(x) = \min\{|B| : x \in B \in \mathscr{B}\} .$$

Let λ be a cardinal with $\lambda \leq |X|$ and $\mathrm{cf}(\lambda) = \omega$. We will prove $|X| \geq \lambda^{\omega}$. Let $\langle \lambda_n : n \in \omega \rangle$ be a nondecreasing sequence of infinite cardinals with $\sup_{n \in \omega} \lambda_n = \lambda$.

Claim. There is an almost disjoint family $\mathscr{A}$ of countably infinite subsets of X with $|\mathscr{A}| \geq \lambda^{\omega}$ such that

(*) $\quad \forall A \neq B \in \mathscr{A}\ [A \cup B$ is a discrete subspace of $X]$.

The theorem follows easily from the claim. For all $A \neq B \in \mathscr{A}$ we first see from (*) that $A \cup B$ is a discrete subspace of X, hence that $A - B$ and $B - A$ are separated, and then from almost disjointness that A and B do not have any limit points in common. Therefore $|X| \geq |\mathscr{A}| \geq \lambda^{\omega}$.

Now we are ready to prove the claim.

Case 1. There is an injection $s: \omega \to X$ such that $\forall n \in \omega\ [l(s_n) \geqq \lambda_n]$.

As in the proof of 11.5, because of 5.7 we may assume that there is $U: \omega \to \mathscr{B}$ such that

$$\forall n \in \omega\ [s_n \in U_n] \text{ and } \forall k \neq n \in \omega\ [U_k \cap U_n = \emptyset].$$

Apply Tarski's Lemma with $L_n = U_n$ for $n \in \omega$, and note that (*) follows from the fact that the U_n's are open and pairwise disjoint.

Case 2. Not Case 1.

Then there are a finite $F \subseteq X$ and a $\mu < \lambda$ and a $B: X - F: \to \mathscr{B}$ such that

$$\forall x \in X - F\ [x \in B_x \text{ and } |B_x| \leq \mu].$$

As $\mu < \lambda \leq |X|$, it now follows from Hajnal's Free Set Lemma (HAJNAL [1961]) (see the Appendix on Set Theory in this Handbook), that there is $D \subseteq X - F$ with $|D| = |X|$ such that $\forall x \neq y \in D\ [x \not\in B_y]$. Clearly, D is a discrete subspace of X. Now apply Tarski's Lemma with $\langle L_n: n \in \omega \rangle$ any pairwise disjoint sequence of subsets of D of big enough cardinality. □

Of course, if X is an infinite Boolean F-space, then also $|X| \geq 2^{\mathfrak{c}}$ since $\beta\omega$ embeds into X by 11.4(c), and $|\beta\omega| = 2^{\mathfrak{c}}$ by 14.2.

Our next proof is quite different from the original proof.

11.9. THEOREM (MONK [1983]). *If $\mathscr{B}$ is an infinite WCC BA then* ind($\mathscr{B}$) *is an ω-power.*

PROOF. We emphasize that for a BA $\mathscr{A}$ one defines ind($\mathscr{A}$) to be the supremum of the cardinalities of independent sets; it need not be true that $\mathscr{A}$ has an independent family of cardinality ind($\mathscr{A}$); see MONK [1983].

We take it for granted that

(1) $\forall$ infinite BA $\mathscr{A}$ $[\mathrm{ind}(\mathscr{A}) \geq \omega]$.

For each collection $\mathscr{K}$ and each set A we use $\mathscr{K} \upharpoonright A$ to denote $\{K \cap A: K \in \mathscr{K}\}$. We repeatedly use the following observation

(2) if $\mathscr{I}$ is independent and $\mathscr{F}, \mathscr{G} \in [\mathscr{I}]^{<\omega}$ are disjoint, and B denotes $\bigcap \mathscr{F} - \bigcup \mathscr{G}$, then $\mathscr{K} = (\mathscr{I} - (\mathscr{F} \cup \mathscr{G})) \upharpoonright B$ is an independent subcollection of CLOP(B) with $|\mathscr{K} = |\mathscr{I} - (\mathscr{F} \cup \mathscr{G})|$.

We prove the theorem with induction on ind($\mathscr{B}$). So we may assume:

$$\forall \text{ infinite } B \in \mathscr{B}\ [\mathrm{ind}(B) < \mathrm{ind}(\mathscr{B}) \Rightarrow \mathrm{ind}(B) \text{ is an } \omega\text{-power}]\,.$$

(Of course ind(B) means ind(CLOP(B)).)

For $x \in X$ define the local independence of $\mathscr{B}$ at x to be

$$\mathrm{lind}(x) = \min\{\mathrm{ind}(B)\colon x \in B \in \mathscr{B}\}\,.$$

Consider any $\lambda \leq \mathrm{ind}(\mathscr{B})$ with $\mathrm{cof}(\lambda) = \omega$, and pick a sequence $\langle \lambda_n\colon n \in \omega\rangle$ of cardinals such that

$$\lambda = \sup_{n\in\omega} \lambda_n \text{ and } \forall n \in \omega\Big[\lambda_{n+1} \geq \prod_{k\leq n} \lambda_k\Big]\,.$$

Claim 1. $\forall A, B \in \mathscr{B}$ [if $A \cup B$ is infinite, then $\mathrm{ind}(A \cup B) = \max \times \{\mathrm{ind}(A), \mathrm{ind}(B)\}$].

Let $\mu = \max\{\mathrm{ind}(A), \mathrm{ind}(B)\}$. Let $\mathscr{I}$ be an infinite independent subfamily of CLOP($A \cup B$). Since we want to prove $|\mathscr{I}| \leq \mu$ we may assume without loss of generality that $|\mathscr{I}|$ is regular. Therefore one of the following two cases occurs.

Case 1. There is $\mathscr{J} \subseteq \mathscr{I}$ with $|\mathscr{J}| = |\mathscr{I}|$ such that $\forall I, J \in \mathscr{J}\ [I \cap A = J \cap A]$. Then $\mathscr{J} \restriction B$ is an independent subcollection of CLOP(B) of cardinality $|\mathscr{I}|$.

Case 2. There is $\mathscr{J} \subseteq \mathscr{I}$ with $|\mathscr{J}| = |\mathscr{I}|$ such that $\forall I, J \in \mathscr{J}\ [I \cap A \neq J \cap A]$. We may assume without loss of generality that $\mathscr{J} \restriction A$, which has cardinality $|\mathscr{I}|$, is not independent. So there are disjoint finite $\mathscr{F}, \mathscr{G} \subseteq \mathscr{J}$ such that if $K = \bigcap \mathscr{F} - \bigcup \mathscr{G}$, then $K \cup A = \emptyset$, so $K \subseteq B$, and therefore $\mathscr{K} = (\mathscr{J} - (\mathscr{F} \cup \mathscr{G})) \restriction B$ is independent with $|\mathscr{K}| = |\mathscr{I}|$, because of (2).

Claim 2. For every $F \in [X]^{<\omega}$, if σ abbreviates $\sup_{x\in X-F} \mathrm{lind}(x)$, then $\sigma \geq \lambda$.

First, from 11.4(d) we see that $X - F$ has a nonisolated point, hence $\sigma \geq \omega$ by (1). Consider any $\mu < \lambda$ with $\mu \geq \omega$. There is an independent family $\mathscr{I} \subseteq \mathscr{B}$ with $|\mathscr{I}| \geq \mu + |F|$. There are disjoint $\mathscr{F}, \mathscr{G} \subseteq \mathscr{I}$ with $|\mathscr{F}| + |\mathscr{G}| \leq |F|$ such that if $B = \bigcap \mathscr{F} - \bigcup \mathscr{G}$, then $B \cap F = \emptyset$. Then $\mathrm{ind}(B) \geq \mu$ by (2). Since B is compact it follows from Claim 1 that there is $x \in B$ with $\mathrm{lind}(x) \geq \mu$.

Because of Claim 2 there is an injection $x\colon \omega \to X$ such that $\forall n \in \omega\ [\mathrm{lind}(x_n) > \lambda_n]$. Because of 5.7 we may assume that there is an injection $U\colon \omega \to \mathscr{B}$ with cellular range such that $\forall n \in \omega\ [x_n \in U_n]$, hence such that $\forall n \in \omega\ [\mathrm{ind}(U_n) > \lambda_n]$. Since we were careful to use strict inequality we can choose for each $n \in \omega$ an independent $\mathscr{L}_n \subseteq \mathrm{CLOP}(U_n)$ with $|\mathscr{L}_n| = \lambda_n$; note that $\forall k \neq n \in \omega\ [\mathscr{L}_k \cap \mathscr{L}_n = \emptyset]$. By Tarski's Lemma there is an almost disjoint subcollection Σ of $\bigcup_{n\in\omega} \mathscr{L}_n$ with $|\Sigma| = \lambda^\omega$ such that

(3) $\qquad \forall \mathscr{S} \in \Sigma, \forall n \in \omega\ [|\mathscr{S} \cap \mathscr{L}_n| = 1]\,.$

By 11.4(a) we can choose $\phi\colon \Sigma \to \mathscr{B}$ such that

(4) $\qquad \forall \mathscr{S} \in \Sigma,\ \forall n \in \omega\ [\phi(\mathscr{S}) \cap U_n \in \mathscr{S} \cap \mathscr{L}_n]\,.$

We complete the proof by showing that ϕ is an injection with independent range: consider any two disjoint finite $\Phi, \Gamma \subseteq \Sigma$. Since Σ is almost disjoint, we see from

(3) that there is $n \in \omega$ such that $\forall \mathcal{S} \neq \mathcal{T} \in \Phi \cap \Gamma\ [\mathcal{S} \cap \mathcal{L}_n \neq \mathcal{T} \cap \mathcal{L}_n]$. It now follows from (4) and the fact that $\mathcal{L}_n$ is independent that

$$\bigcap_{\mathcal{S} \in \Phi} \phi(\mathcal{S}) - \bigcup_{\mathcal{T} \in \Gamma} \phi(\mathcal{T}) \subseteq \bigcap_{\mathcal{S} \in \Phi} (\phi(\mathcal{S}) \cap U_n) - \bigcup_{\mathcal{T} \in \Gamma} (\phi(\mathcal{T}) \cap U_n) \neq \emptyset . \quad \square$$

11.10. Corollary (to proof). (a) *If $\mathcal{A}$ is an infinite WCC BA, then for every $\kappa < ind(\mathcal{A})$ there is an independent subcollection of $\mathcal{A}$ that has cardinality κ^ω.*

(b) *If $\mathcal{A}$ is an infinite WCC BA, in particular if $\mathcal{A} = \mathcal{P}(\omega)$, there $\mathcal{A}$ has an independent subcollection of cardinality* $\mathfrak{c}$. $\square$

In 14.2 we will generalize part of this: if $\kappa \geq \omega$, then $\mathcal{P}(\kappa)$ has an independent subcollection of cardinality 2^κ.

Information about other cardinal functions is given in van Douwen [1981]. For example, it is shown that if X is an infinite Boolean F-space, then $s(X)$ is not a strong limit of countable cofinality, hence under the Singular Cardinal Hypothesis $s(X)$ is an ω-power. In this context we should mention that it is not known whether or not $\chi(X) = hL(X) = \cdots = s(X) = w(X)$ if X is an infinite Boolean F-space.

12. Cofinality of Boolean algebras and some other small cardinal functions

If $\mathcal{B}$ is infinite, one defines $\mathrm{cf}(\mathcal{B})$, the *cofinality* of $\mathcal{B}$ (which is completely unrelated to the function h-cof of Section 7), by

$$\mathrm{cf}(\mathcal{B}) = \min\Big\{\kappa\colon \kappa \geq \omega \text{ and there is a strictly increasing } \kappa\text{-sequence } \langle \mathcal{B}_\xi\colon \xi \in \kappa\rangle \text{ of subalgebras such that } \bigcup_{\xi \in \kappa} \mathcal{B}_\xi = \mathcal{B}\Big\} .$$

(It is easy to see that $\mathrm{cf}(\mathcal{B})$ is well-defined, and that $\mathrm{cf}(\mathcal{B}) \leq |\mathcal{B}|$. Below we give another proof that it is well-defined.) This cardinal function was introduced by Koppelberg [1977]. We here use topology to give new proofs of her results.

The obvious translation of $\mathrm{cf}(\mathcal{B})$ into topology involves inverse limits of Boolean spaces; because this is so obvious it is useless as a tool to study $\mathrm{cf}(\mathcal{B})$. For a zerodimensional space X call two functions $x, y\colon \kappa \to X$ *parallel* if

$$\forall B \in \mathcal{B},\ \exists \xi \in \kappa,\ \forall \eta \in (\xi, \kappa)\ [x_\eta \in B \Leftrightarrow y_\eta \in B] ,$$

and define $\mathrm{cf}(X)$ by

$$\mathrm{cf}(X) = \min\{\kappa\colon \kappa \geq \omega \text{ and there are two parallel functions } x, y\colon \kappa \to X \text{ with } \mathrm{ran}(x) \cap \mathrm{ran}(y) = \emptyset \text{ each of which is an injection or is constant}\} .$$

(We show in Theorem 12.3(b) that $\mathrm{cf}(X)$ is well-defined.) Note that if x and y are as in this definition, they cannot both be constant. Also note that x and y are parallel iff

$$\forall B \in \mathscr{B}, \exists \xi \in \kappa, \forall \eta \in (\xi, \kappa)\ [x_\eta \in B \Rightarrow y_\eta \in B]$$

12.1. THEOREM. $\mathrm{cf}(\mathscr{B}) = \mathrm{cf}(X)$.

PROOF. Our proof will show that if one of $\mathrm{cf}(\mathscr{B})$ and $\mathrm{cf}(X)$ is defined, then so is the other. Of course, each of $\mathrm{cf}(\mathscr{B})$ and $\mathrm{cf}(X)$ is regular if it is defined.

Proof that $\mathrm{cf}(\mathscr{B}) \geq \mathrm{cf}(X)$. Consider any regular infinite cardinal κ and any strictly increasing κ-sequence $\langle \mathscr{B}_\xi : \xi \in \kappa \rangle$ of subalgebras such that $\bigcup_{\xi\in\kappa} \mathscr{B}_\xi = \mathscr{B}$. For each $\xi \in \kappa$, since $\mathscr{B}_\xi$ is a proper subalgebra of $\mathscr{B}$ and X is compact, the topology on X that has $\mathscr{B}_\xi$ as a base is not Hausdorff, because of 3.2. It follows that we can find a strictly increasing $s: \kappa \to \kappa$ and functions $x, y: \kappa \to X$ such that

(a) $\forall B \in \mathscr{B}_{s(\xi)}\ [x_\xi \in B$ iff $y_\xi \in B]$; and

(b) $\exists B \in \mathscr{B}_{s(\xi+1)}\ [x_\xi \in B$ and $y_\xi \notin B]$.

Since $\langle \mathscr{B}_\xi : \xi \in \kappa \rangle$ is strictly increasing, $\forall \xi \neq \eta \in \kappa\ [x_\xi \neq x_\eta$ or $y_\xi \neq y_\eta]$. Hence in two steps we find $K \in [\kappa]^\kappa$ such that each of $x \restriction K$ and $y \restriction K$ is constant or is one-to-one. In one more step we find $L \in [K]^\kappa$ with $x^{\to}L \cap y^{\to}L = \emptyset$. We may assume without loss of generality that $L = \kappa$. Of course x and y are parallel since $\bigcup_{\xi\in\kappa} \mathscr{B}_{s(\xi)} = \mathscr{B}$.

Proof that $\mathrm{cf}(\mathscr{B}) \leq \mathrm{cf}(X)$. Let κ be a regular cardinal, and let x and y be two parallel functions $\kappa \to X$ with $\mathrm{ran}(x) \cap \mathrm{ran}(y) = \emptyset$ such that each of x and y is an injection or is constant. For each $\xi \in \kappa$ define

$$\mathscr{B}_\xi = \{B \in \mathscr{B} : \forall \eta \in [\xi, \kappa)\ [x_\eta \in B \text{ iff } y_\eta \in B]\}.$$

Then $\langle \mathscr{B}_\xi : \xi \in \kappa \rangle$ is a nondecreasing κ-sequence of proper subalgebras with $\bigcup_{\xi\in\kappa} \mathscr{B}_\xi = \mathscr{B}$. (To see they are proper subalgebras consider any $\xi \in \kappa$. As $x_\xi \neq y_\xi$ there is $B \in \mathscr{B}$ with $x_\xi \in B$ but $y_\xi \notin B$, hence with $B \notin \mathscr{B}_\xi$.) Hence, it can be thinned out to a strictly increasing κ-sequence of subalgebras with union $\mathscr{B}$. □

Let us define three more small cardinal functions. For $x \in {}^\kappa X$ and for $p \in X$ we say

$$x \textit{ has limit } p \text{ if } \bigcap_{\xi\in\kappa} \overline{\{x_\eta : \eta \in (\xi, \kappa)\}} = \{p\}.$$

(If X is compact then for $\kappa = \omega$ this coincides with the usual definition of limit, by the proof of 12.3 below.)

The *altitude* of X is defined by

$$a(X) = \min\{\kappa : \kappa \geq \omega \text{ and there is an injection } x: \kappa \to X \text{ which has a limit}\},$$

or, equivalently, by

$$a(X) = \min\Big\{\kappa\colon \kappa \geq \omega \text{ and there is a strictly decreasing } \kappa\text{-sequence } \langle F_\xi\colon \xi \in \kappa\rangle \text{ of closed sets with } \Big|\bigcap_{\xi\in\kappa} F_\xi\Big| = 1\Big\},$$

the *pseudoaltitude* of X is defined by

$$pa(X) = \min\{\chi(y, Y)\colon Y \text{ is closed in } X \text{ and } y \text{ is a nonisolated point of } Y\},$$

and the *homomorphism type* of $\mathscr{B}$ is defined by

$$h(\mathscr{B}) = \min\{|\mathscr{Q}|\colon \mathscr{Q} \text{ is an infinite quotient of } \mathscr{B}\},$$

or, equivalently, by

$$h(X) = \min\{w(Y)\colon Y \text{ is an infinite closed subspace of } X\}.$$

12.3. THEOREM. *Let X be an infinite Boolean space.*
(a) $\mathrm{cf}(X) \leq a(X) \leq pa(X) \leq h(X) \leq \mathfrak{c}$; *hence*
(b) $\mathrm{cf}(X)$ *and* $a(X)$ *are well-defined.*
(c) $a(X) = \omega$ *iff* $pa(X) = \omega$ *iff* $h(X) = \omega$.

PROOF.
Proof that $\mathrm{cf}(X) \leq a(X)$. Consider an injection $x\colon \kappa \to X$ which has a limit, p say, and for $\xi \in \kappa$ let F_ξ abbreviate $\overline{\{x_\eta\colon \eta \in (\xi, \kappa)\}}$. Ley $y_\xi = p$ for $\xi \in \kappa$. To see x and y are parallel consider any $B \in \mathscr{B}$ with $p \in B$. Then $(X - B) \cap \bigcap_{\xi\in\kappa} F_\xi = \emptyset$, hence since X is compact and $\langle F_\xi\colon \xi \in \kappa\rangle$ is decreasing there is $\xi \in \kappa$ such that $F_\xi \subseteq B$. Then $\forall \eta \in [\xi, \kappa)\ [y_\eta \in B]$, and of course $\forall \xi \in \kappa\ [x_\xi = p \in B]$.
Proof that $a(X) \leq pa(X)$. Let p be a nonisolated point of a closed subspace Y of X. Let κ abbreviate $\chi(p, Y)$, let $B\colon \kappa \to \mathrm{CLOP}(Y)$ be such that $\mathrm{ran}(B)$ is a neighborhood base at p in Y, and define $F\colon \kappa \to \mathscr{P}(X)$ by $F(\eta) = \bigcap_{\xi\leq\eta} B_\xi$. By 5.3 $F_\eta \neq \{p\}$ for all $\eta \in \kappa$, hence the nonincreasing κ-sequence $\langle F_\xi\colon \xi \in \kappa\rangle$ is not eventually constant. Therefore it can be thinned out to a strictly decreasing $\mathrm{cf}(\kappa)$-sequence of closed sets with one-point intersection.
It is clear that $pa(X) \leq h(X)$.
Proof that $h(X) \leq \mathfrak{c}$. Since X is infinite it has an infinite separable closed subspace Y, and $w(Y) \leq \mathfrak{c}$ by 6.1.
Proof of (c). If $a(X) = \omega$ then there is an injection $x\colon \omega \to X$ which converges to a point, p say, of X. Then $Y = \mathrm{ran}(x) \cup \{p\}$ is a countable closed subspace of X, and $w(Y) = \omega$, e.g. because Y is homeomorphic to $[0, \omega]$. □

12.4. COROLLARY (to proof). *If X is a Boolean space then*

$$\mathrm{cf}(X) = \min\{\kappa\colon \kappa \geq \omega \textit{ and there is an injection } \kappa \to X \textit{ that has a limit or there are two parallel injections } \kappa \to X \textit{ with disjoint ranges}\}.\ \square$$

One can give a condition equivalent to "cf$(X) \leq \kappa$" using only one function. This seems useful only in the case $\kappa = \omega$, for then we can get a function with the additional property that its range is a discrete subspace:

12.5. THEOREM. *If X is an infinite Boolean space, then* cf$(X) = \omega$ *iff there is an injection $z: \omega \to X$ such that* ran(z) *is a discrete subspace of X and such that*

$$(*) \qquad \forall B \in \mathscr{B},\ \exists m \in \omega,\ \forall n \geq m\ [z_{2n} \in B \textit{ iff } z_{2n+1} \in B]\,.$$

PROOF. Sufficiency is clear. We prove necessity:

Case 1. There is an injection x: $\omega \to X$ which has a limit.

Let $z = x$. This works since x converges to some point of X in the usual meaning of "converges to".

Case 2. Not Case 1.

Let x and y be parallel injections $\omega \to X$ with $\mathrm{ran}(x) \cap \mathrm{ran}(y) = \emptyset$. Apply 5.7 twice to get an infinite $I \subseteq \omega$ and U, V: $\omega \to \mathscr{B}$ such that

$$(1) \qquad \forall n \in I\ [x_n \in U_n\ \&\ y_n \in V_n] \text{ and } \forall k \neq n \in I \\ [U_k \cap U_n = \emptyset\ \&\ V_k \cap V_n = \emptyset]\,.$$

Then use the fact that x and y are parallel twice to find an infinite $J \subseteq I$ such that

$$(2) \qquad \forall k, n \in J\ [k < n \Rightarrow ((x_n \in U_k \Leftrightarrow y_n \in U_k)\ \&\ (x_n \in V_k \Leftrightarrow y_n \in V_k)]\,.$$

Note that since (1) implies $\forall k \neq n \in \omega\ [x_k \not\in U_n$ and $y_k \not\in V_n]$, (2) implies

$$(3) \qquad \forall k, n \in J\ [(x_n \in V_k \text{ or } y_n \in U_k) \Rightarrow n \leq k]\,.$$

Let j be any strictly increasing function $\omega \to J$, and define z: $\omega \to X$ by

$$z_{2n} = x_{j(n)} \text{ and } z_{2n+1} = y_{j(n)} \quad \text{for } n \in \omega\,.$$

Then $(*)$ is clear. To prove ran(z) is a discrete subspace it suffices, by symmetry, to show that each $x_{j(n)}$ is an isolated point of ran(z). So consider any $n \in \omega$. Then $U_{j(n)}$ is a neighborhod of $x_{j(n)}$ such that

$$U_{j(n)} \cap \{x_{j(k)}: k \in \omega\} = \{x_{j(n)}\}\,,$$

because of (1), and such that

$$U_{j(n)} \cap \{y_{j(k)}: k \in \omega\} \subseteq \{y_{j(k)}: k \leq n\}\,,$$

because of (3) and the fact that j is a strictly increasing. Therefore $U_{j(n)} \cap \mathrm{ran}(z)$ is finite. □

Most results involving two cardinal functions, ϕ_1 and ϕ_2, tell us that if ϕ_1 gets big, then so does ϕ_2. The following result, due to Balcar and Simon [∞], to Kunen (unpublished), and to Shelah (unpublished), independently, is different.

12.6. Theorem. *If X is a Boolean space such that* $\operatorname{ind}(\mathscr{B}) \geq \omega_1$, *then* $pa(X) \leq \omega_1$.

Proof. The theorem is an easy consequence of the following:
Claim. There are an injection $x\colon \omega_1 \to X$ and a $p \in X$ such that

(1) $\quad \forall \xi \in \omega_1,\ \exists B \in \mathscr{B}\ [p \in B\ \&\ B \cap \operatorname{ran}(x) = x^{\to}[\xi, \omega_1)]$; and

(2) $\quad \forall$ neighborhood U of p $\exists \xi \in \omega_1,\ \forall \eta \in \omega_1\ [\xi \leq \eta \Rightarrow x_\eta \in U]$.

To prove the theorem from the claim, let Y be the subspace $\{p\} \cup \operatorname{ran}(x)$ of X. Clearly, $\{\{p\} \cup x^{\to}[\xi, \omega_1)\colon \xi \in \omega_1\}$ is a neighborhood base at p in Y, and $\forall A \in [Y - \{p\}]^{\leq\omega}\ [p \not\in \bar{A}]$, hence $\chi(p, Y) = \omega_1$. It follows from 6.6 that $\chi(p, Y) = \omega_1$.

As in the proof of 9.3 there is a closed subspace Y of X which admits an irreducible map f onto ${}^{\omega_1}2$, the product of ω_1 factors 2. Since clearly $pa(X) \leq pa(Y)$ we assume without loss of generality that $Y = X$. Then $\mathscr{B}$ has a dense subalgebra $\mathscr{S}$ which is generated by an independent set of cardinality ω_1. Let U be an injection $\omega_1 \to \mathscr{S}$ with $\operatorname{ran}(U)$ an independent collection which generates $\mathscr{S}$.

We would like to have for each $\eta \in \omega_1$ be automorphism h_η of $\mathscr{B}$ satisfying

(3) $\quad \forall \xi, \eta \in \omega_1$ [if $\xi \leq \eta$, then $h_\eta^{\to} U_\xi = U_\xi$, & if $\xi > \eta$, then
$$h_\eta^{\to} U_\xi = X - U_\xi].$$

However, such h_η's need not exist. Now let $\mathscr{Q}$ denote the completion of $\mathscr{B}$. Then $\mathscr{Q}$ also is the completion of $\mathscr{S}$. As $\operatorname{ran}(U)$ is an independent collection that generates $\mathscr{S}$ it follows that for each $\eta \in \omega_1$ there is an automorphism h_η of $\mathscr{Q}$ such that (3) holds.

Pick $p \in \bigcap_{\xi \in \omega_1} U_\xi$, and extend the ultrafilter $\{B \in \mathscr{B}\colon p \in B\}$ on $\mathscr{B}$ to an ultrafilter $\mathscr{F}$ on $\mathscr{Q}$. From $\mathscr{F}$ define $x\colon \kappa \to X$ as follows. For each $\xi \in \kappa$ the collection $\mathscr{F}_\eta = \{h_\eta(F)\colon F \in \mathscr{F}\}$ is an ultrafilter on $\mathscr{Q}$, hence $\mathscr{F}_\eta \cap \mathscr{B}$ is an ultrafilter on $\mathscr{B}$; let x_η be the unique point in $\bigcap (\mathscr{F}_\eta \cap \mathscr{B})$. Then

(4) $\quad \forall \xi, \eta \in \omega_1\ [x_\eta \in U_\xi$ iff $\xi \leq \eta]$.

Indeed, if $\xi \leq \eta$ then $h_\eta(U_\gamma) = U_\gamma$, so $x_\eta \in U_\xi$, while if $\xi > \eta$ then $h_\eta(U_\xi) = X - U_\xi$, so $x_\eta \in X - U_\xi$.

Since $p \in \bigcap_{\xi \in \omega_1} U_\xi$, (1) is clear from (4). To verify (2), let V be a clopen neighborhood of q, and let $\mathscr{A}$ be a maximal cellular subcollection of $\{S \in \mathscr{S}\colon S \subseteq V\}$. As $\mathscr{S}$ is a dense subalgebra of $\mathscr{B}$, $\overline{\bigcup \mathscr{A}} = V$. As $c(\mathscr{S}) = \omega$, by Theorem 14.1(e), $|\mathscr{A}| \leq \omega$, hence there is $\xi \in \omega_1$ such that $\mathscr{A}$ is a subcollection of the subalgebra $\mathscr{K}$ of $\mathscr{Q}$ generated by $\mathscr{U}^{\to}[0, \xi]$. From (3) we see that $\forall A \in \mathscr{K}\ \forall \eta \in [\xi, \omega_1)\ [h_\eta^{\to} A = A]$. As $\mathscr{A} \subseteq \mathscr{K}$ and $\overline{\bigcup \mathscr{A}} = V$ it follows that $\forall \eta \in [\xi, \omega_1)\ [h_\eta V = V]$. Therefore $\forall \eta \in [\xi, \omega_1)\ [x_\xi = h_\xi(p) \in V]$. $\square$

The part $\mathrm{cf}(X)=\omega_1$ of the following Theorem was proved by KOPPELBERG [1977].

12.7. THEOREM. *If X is an infinite Boolean F-space, then* $pa(X)=a(X)=\mathrm{cf}(X)=\omega_1$.

PROOF.

Proof that $\mathrm{cf}(X)\geq\omega_1$. If z is any injection $\omega\to X$ such that $\mathrm{ran}(z)$ is a discrete subspace of X, then $E=\{z_n: n \text{ Even}\}$ and $O=\{z_n: n \text{ Odd}\}$ are separated, i.e. $\bar{E}\cap O=\emptyset=E\cap\bar{O}$, hence $\bar{E}\cap\bar{O}=\emptyset$ since X is an F-space. It follows from 12.5 that $\mathrm{cf}(X)\geq\omega_1$.

Proof that $pa(X)\leq\omega_1$. $\mathrm{ind}(\mathscr{B})=\mathfrak{c}$, by 11.10, hence this follows from 12.6.

The theorem now follows from 12.3(a). □

12.8. EXAMPLE (KOPPELBERG [1977]). There is a Boolean space X with $\mathrm{cf}(X)=\omega$ and $a(X)=pa(X)=\omega_1$ and $h(X)=\mathfrak{c}$.

PROOF. Let X be the space obtained from $\beta\omega$ by replacing each $n\in\omega$ by two points z_{2n} and z_{2n+1}: z is an injection with domain ω such that $\beta\omega-\omega$ and $\mathrm{ran}(z)$ are disjoint, the underlying set of X is $(\beta\omega-\omega)\cap\mathrm{ran}(z)$, the points of $\mathrm{ran}(z)$ are isolated, and basic neighborhoods of $p\in\beta\omega-\omega$ in X have the form $(U-\omega)\cup\{z_{2n+i}: i\in 2\ \&\ n\in U\}$, where U is a neighborhood of p in $\beta\omega$. Then X is a Boolean space, and z witnesses that $\mathrm{cf}(X)=\omega$, because of 12.5. (Indeed, this example and the proof of $\mathrm{cf}(\mathscr{B})\geq\omega_1$ in 12.7 have motivated both our definition of $\mathrm{cf}(X)$ and 12.5.) The other claims about X should be clear. □

We do not know whether $a=pa$, nor whether always $\mathrm{cf}(\mathscr{B})\leq\omega_1$. By an unpublished result of Just it is consistent with $\neg$CH that $\forall\mathscr{B}\ [a(\mathscr{B})\leq\omega_1]$ (recall from 12.3 that $\mathrm{cf}(\mathscr{B})\leq a(\mathscr{B})$). It also is consistent that there is a BA $\mathscr{B}$ with $\omega_1<h(\mathscr{B})<\mathfrak{c}$.

13. Survey of results

With exception of cardinality our notation is such that $\phi(X)=\phi(\mathscr{B})$ for all cardinal functions ϕ, for all infinite Boolean spaces X. The following list indicates where the proofs of equalities of this type and related equalities may be found:

(12.1) $\mathrm{cf}(\mathscr{B})=\mathrm{cf}(X)$;
(5.5) $d(X)=\min\{\kappa: \mathscr{B}$ embeds into $\mathscr{P}(\kappa)\}$;
(4.1) $h_c\phi(X)=\sup\{\phi(\mathscr{Q})$: $\mathscr{Q}$ is a quotient algebra of $\mathscr{B}\}$;
(7.2) $F(X)=F(\mathscr{B})$;
(7.3) $hd(X)=hd(\mathscr{B})=id(\mathscr{B})$;
(7.3) $hL(X)=hL(\mathscr{B})=ih(\mathscr{B})=ig(\mathscr{B})$;
(4.2) $m\phi(X)=\sup\{\phi(\mathscr{A})$: $\mathscr{A}$ is a subalgebra of $\mathscr{B}\}$;
(7.1) $s(X)=s(\mathscr{B})$;
(5.1) $w(X)=|\mathscr{B}|$;
(5.4) $\pi(X)=\pi(\mathscr{B})$.

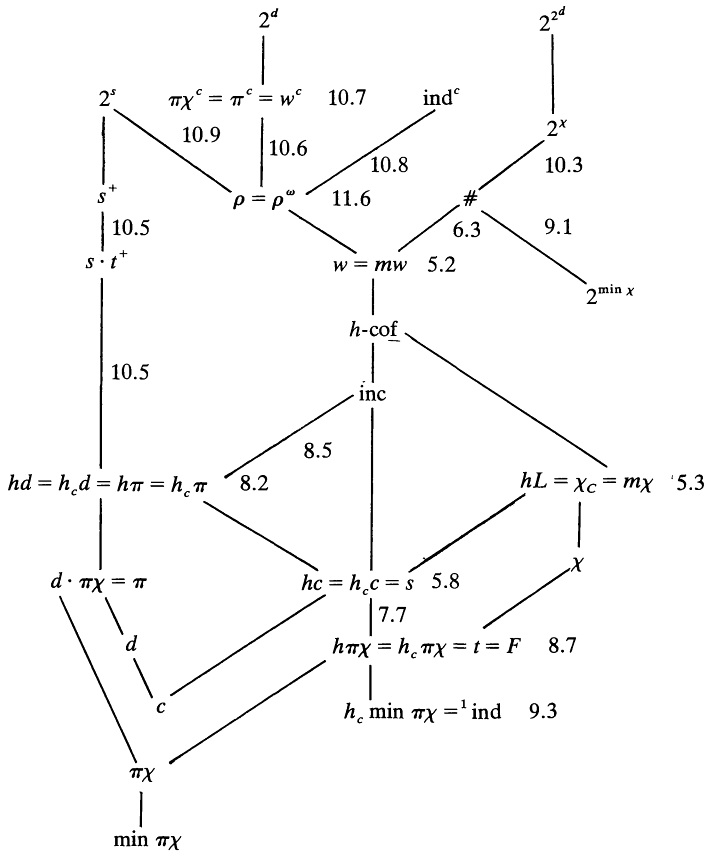


Fig. 11.1. *Note*: [1]For nonsuperatomic BAs. Clearly $\mathscr{B}$ is superatomic iff $h_c \min \pi\chi(\mathscr{B}) = 1$. Also, $\mathscr{B}$ is superatomic iff $\mathscr{B}$ has no infinite independent subsets, by 9.2, but $\mathrm{ind}(\mathscr{B}) \geq \omega$ iff $|\mathscr{B}| \geq \omega$.

Figure 11.1 contains all other equalities and inequalities, except those from Section 12. As before, $\#$ abbreviates $|X|$. The numbers to the right of equalities or inequalities are the numbers of the theorems; in order not to clutter the diagram too much, references to 6.1 and, with one exception, to 7.7 have been omitted. Some unexplained inequalities are trivial or follow from other inequalities. For example, from $w \leq 2^d$ and $\chi \leq w$ one gets $2^\chi \leq 2^{2^d}$, an inequality we put in the diagram to get the well-known inequality $\# \leq 2^{2^d}$.

14. The free BA on κ generators

In the final section we calculate our cardinal functions for the free BA on κ generators, or, equivalently, for ${}^\kappa 2$, the product of κ factors $2 = \{0, 1\}$ (which of course carries the discrete topology). This is quite easy with all the information from Theorem 13.1:

14.1. THEOREM. *Let* $\kappa \geq \omega$. *Then*

(a) $|{}^\kappa 2| = 2^\kappa$;

(b) $\rho({}^\kappa 2) = \kappa^\omega$;

(c) $\phi({}^{\kappa}2) = \kappa$ *for* $\phi \in \{F, h\text{-cof}, hd, hL, h\pi, h\pi\chi, \text{inc}, \text{ind}, F, \min \pi\chi, s, t, w, \pi, \pi\chi\}$;

(d) (HAUSDORFF [1936]). $d({}^{\kappa}2) = \log(\kappa) = \min\{\lambda: 2^{\lambda} \geq \kappa\}$; *and*

(e) $c({}^{\kappa}2) = \omega$.

PROOF. Let $\mathbb{P}$ be the set of all finite partial functions $\kappa \to 2$, and define

$$U(p) = \{x \in {}^{\kappa}2: x \supseteq p\} \quad \text{for } p \in \mathbb{P}\,,$$

$$\mathcal{U}(A) = \{U(p): p \in A\} \quad \text{for } A \subseteq \mathbb{P}\,.$$

Then $\mathcal{U}(\mathbb{P})$ is the canonical base for ${}^{\kappa}2$.

Proof of (*a*). The definition of 2^{κ}.

Proof of (*b*). Use $w \leq \rho \leq w^{c}$ and $\rho = \rho^{\omega}$, together with $w({}^{\kappa}2) = \kappa$ from (c) and $c({}^{\kappa}2) = \omega$ from (e).

Proof of (*c*). Since CLOP$({}^{\kappa}2)$ has a free set of κ generators and $\kappa \leq \omega$ it is clear that $w({}^{\kappa}2) \leq \kappa$ and $\text{ind}({}^{\kappa}2) \geq \kappa$. It follows from Fig. 11.1 that it suffices to prove $\min \pi\chi({}^{\kappa}2) \geq \kappa$. Consider any $x \in {}^{\kappa}2$. Since $\mathcal{U}(\mathbb{P})$ is a (π-)base for ${}^{\kappa}2$ it suffices to prove $|A| \geq \kappa$ for any $A \subseteq \mathbb{P}$ such that $\mathcal{U}(A)$ is a neighborhood π-base at x. Well, if $|A| < \kappa$, then since κ is infinite there is $\xi \in \kappa$ such that $\xi \not\in \bigcup_{p \in A} \text{dom}(p)$, and then $\forall p \in A\ [U(p) \not\subseteq U(x \restriction \{\xi\})$.

Proof of (*d*). Let $\lambda = \log(\kappa)$. Since $w({}^{\kappa}2) = \kappa$, it is an immediate consequence of the inequality $w \leq 2^{d}$ that $d({}^{\kappa}2) \geq \lambda$.

To prove $d({}^{\kappa}2) \leq \lambda$ note that $2^{\lambda} \geq \kappa$ so that $x \mapsto x \restriction \kappa$, $(x \in {}^{2^{\lambda}}2)$, is a function from ${}^{2^{\lambda}}2$ onto ${}^{\kappa}2$, which obviously is continuous. Hence, it suffices to prove $d({}^{2^{\lambda}}2) \leq \lambda$. We use Pondiczery's elegant proof: topologize $\mathcal{P}({}^{\lambda}2)$ by declaring the collection of all sets of the form

$$W(F, G) = \{A \in \mathcal{P}({}^{\lambda}2): F \subseteq A \ \&\ A \cap G = \emptyset\}$$
$$\text{with } F, G \in [{}^{\lambda}2]^{<\omega} \text{ disjoint}\,,$$

to be a base; then $\mathcal{P}({}^{\lambda}2)$ is homeomorphic to ${}^{(2^{\lambda})}2$ via the correspondence between subsets of ${}^{\lambda}2$ and their characteristic functions. Since $|\text{CLOP}({}^{\lambda}2)| = w({}^{\lambda}2) = \lambda$, by 5.1 and (c), it therefore suffices to show that CLOP$({}^{\lambda}2)$ is dense in $\mathcal{P}({}^{\lambda}2)$. For disjoint $F, G \in [{}^{\lambda}2]^{<\omega}$ there is a clopen B in ${}^{\lambda}2$ such that $F \subseteq B$ and $B \subseteq {}^{\lambda}2 - G$, i.e. such that $B \in W(F, G)$.

Proof of (*e*). Since $\mathcal{U}(\mathbb{P})$ is a base for ${}^{\kappa}2$ it suffices to consider any $A \subseteq \mathbb{P}$ with $|A| = \omega_1$ and prove there are $p \neq q \in A$ with $U(p) \cap U(q) \neq \emptyset$. If $K = \bigcup_{p \in A} \text{dom}(p)$, then $|K| = \omega_1$, hence we assume without loss of generality that $K = \omega_1$. As $\forall p, q \in \mathbb{P}$ we have that $U(p) \cap U(q) = \emptyset$ iff $\exists \xi \in \text{dom}(p) \cap \text{dom}(q)$ $[p_{\xi} \neq q_{\xi}]$, we now may further assume without loss of generality that $\kappa = \omega_1$. But $d(2^{\omega_1}) = \omega$ by (d), so $c(2^{\omega_1}) = \omega$. $\square$

An obvious corollary to 14.1 and 5.5, which improves 11.10, is:

14.2. THEOREM. (a) $\mathcal{P}(\kappa)$ *has an independent subcollection of cardinality* 2^{κ}.

(b) $|\beta\kappa| = \exp(2^{\kappa})$. $\square$

Two other corollaries, promised after 8.2 and after 10.7, are:

14.3. Examples. (a) There is a countable space X with $\pi(X) = \mathfrak{c}$.
(b) There is a BA $\mathscr{B}$ with $\rho(\mathscr{B}) \neq \rho(\mathscr{B})^{c(\mathscr{B})}$.

Proof. *Proof of* (*a*). ${}^{\mathfrak{c}}2$ has a dense countable subspace X. By 6.6(b) and 14.1(c), $\pi(X) = \pi({}^{\mathfrak{c}}2) = \mathfrak{c}$.

Proof of (b). Let κ be a strong limit (i.e. $\forall\lambda < \kappa\ [2^{\lambda} < \kappa]$) with $\mathrm{cf}(\kappa) = \omega_1$, and let $\mathscr{F}$ be the free BA on κ generators. Then $c(\mathscr{F}) = \omega$ and $\rho(\mathscr{F}) = \kappa^{\omega} = \kappa$.

Let $\mathscr{A}$ be the finite–cofinite algebra on ω_1, so $c(\mathscr{A}) = \omega_1$, and $\rho(\mathscr{A}) = 2^{\omega_1}$.

Let $\mathscr{B} = \mathscr{A} \times \mathscr{F}$. Then $\rho(\mathscr{B}) = \rho(\mathscr{A}) \cdot \rho(\mathscr{F}) = \kappa$, and $c(\mathscr{B}) = c(\mathscr{A}) \cdot c(\mathscr{F}) = \omega_1$, therefore $\rho(\mathscr{B})^{c(\mathscr{B})} > \kappa$ since $\mathrm{cf}(\kappa) = \omega_1$.

(Topologists who are puzzled by this need to recall that BAic products correspond to topological sums.) (This trick is due to Comfort and Hager [1970].) □

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⟦This indicates location of references or names of mathematicians; "x.0" means the introduction to Section x, and "$x.y^{-}$" means "just before $x.y$" and "$x.y^{+}$" means "immediately following $x.y$".⟧

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CHAPTER 12

The Number of Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

For almost all classes $\boldsymbol{K}$ of BAs which have been an object of intensive study, there are exactly 2^κ isomorphism types of members of $\boldsymbol{K}$ of each infinite power κ. In particular, this is true for the class of all BAs. This is evidence that the structure of members of such classes is complicated. In this chapter we prove several results of the above type. First we give five simple but somewhat special constructions, which apply to interval algebras, superatomic BAs, subalgebras of free BAs, subalgebras of $\mathscr{P}\kappa$, and complete BAs. In the second section of the chapter we present an instance of a general method of Shelah for producing many isomorphism types.

Here is a list of many of the theorems known about counting various kinds of BAs.

(1) For each $\kappa \geqq \omega$ there are 2^κ isomorphism types of interval BAs of power κ. See Section 1 below.

(2) For each $\kappa > \omega$ there are 2^κ isomorphism types of superatomic BAs of power κ; see Section 1 below. There are ω_1 isomorphism types of denumerable superatomic BAs; see Part I, Chapter 6, Theorem 17.11.

(3) For each $\kappa > \omega$ there are 2^κ isomorphism types of dense subalgebras of the free BA on κ generators; see Section 1 below. Any dense subalgebra of $\mathrm{Fr}\omega$ is atomless, and hence isomorphic to $\mathrm{Fr}\omega$ (recall that $\mathrm{Fr}\kappa$ is the free BA on κ free generators). But every countable BA can be isomorphically embedded in $\mathrm{Fr}\omega$, so $\mathrm{Fr}\omega$ has 2^ω pairwise non-isomorphic subalgebras.

(4) For each $\kappa \geqq \omega$ there are 2^{2^κ} pairwise nonisomorphic subalgebras of $\mathscr{P}\kappa$ each containing all singletons; see section 1.

(5) There are $2^{2^{2^\omega}}$ pairwise non-isomorphic countably complete subalgebras of $\mathscr{P}\boldsymbol{R}$ each containing all singletons, where $\boldsymbol{R}$ is the set of real numbers. See Freniche [1984], where further results along these lines are given (there are evidently some problems left, though).

(6) Let T be a complete theory of BAs, with infinite models and $T_1 \supseteq T$ in some language extending the language of BAs. Let $\boldsymbol{K}$ be the class of all BA-reducts of models of T_1. Then for each $\kappa > |T_1|$ there is a family of 2^κ pairwise non-elementarily-embeddable members of $\boldsymbol{K}$ of power κ. See Shelah [1978, pp. 9, 30–31, 364, 421]. In particular, if T is a complete theory of BAs with infinite models, and $\kappa > \omega$, there are 2^κ pairwise non-isomorphic models of T of power κ. For $\kappa = \omega$ the situation is simple: let T be a complete theory of BAs with infinite models. If all models of T have only finitely many atoms, then that number of atoms is constant and T has only one denumerable model, up to isomorphism. If T has models with infinitely many atoms, then T has 2^ω denumerable models. This can be seen by combining the first construction in Section 1 below with the construction given in the proof of Proposition 18.5 in Chapter 7, Part I.

(7) For each κ with $\kappa^\omega = \kappa$ there are 2^κ pairwise non-isomorphic rigid complete BAs of power κ. This is an unpublished result of Shelah which uses the methods of Shelah [1983]. See Section 1 below for a partial result along these lines.

(8) For each $\kappa > \omega$ there are 2^κ pairwise non-isomorphic rigid BAs of power κ; see MONK and RASSBACH [1979]. Recall from the article on automorphism groups that there is no denumerable rigid BA.

(9) More generally, for each $\kappa > \omega$ there are 2^κ pairwise non-isomorphic onto-rigid interval BA's of power κ; see LOATS and RUBIN [1978].

(10) TODORČEVIĆ [1979] showed that for each regular uncountable κ there are 2^κ pairwise non-isomorphic Bonnet-rigid interval BAs of power κ. (Bonnet, Loats, and Shelah independently worked along these lines.) There are evidently open problems here.

(11) If $\mu = \lambda^\omega$, then there is a family of power 2^μ of indecomposable endo-rigid BAs of power μ such that any homomorphism from one of them to another of them has finite range; see SHELAH [1984]. There still appear to be some small open problems in this connection.

The constructions given in the second section of this article may be considered to be an introduction to the methods used in (6), (7), (8), and (11) above, and to other constructions of this sort in SHELAH [1971], [1978], [1984].

1. Simple constructions

Our first construction is of a folklore nature. For each infinite cardinal κ we produce 2^κ pairwise non-isomorphic interval BAs of power k.

Let $\alpha_0 = \omega$ and $\alpha_1 = 1 + \eta + \omega$, where η is the order type of the rational numbers. For each $\varepsilon \in {}^\kappa 2$ we set

$$\beta_\varepsilon = \prod_{\xi<\kappa} \alpha_{\varepsilon\xi} ,$$

the *ordinal product*: β_ε consists of all functions $f \in \Pi_{\xi<\kappa} \alpha_{\varepsilon\xi}$ such that $\{\xi\colon f\xi \neq 0\}$ is finite, and $f < g$ iff $f\xi < g\xi$, where ξ is the greatest $\nu < \kappa$ such that $f\nu \neq g\nu$. Note that $|\beta_\varepsilon| = \kappa$. Let A_ε be the interval algebra over β_ε; so $|A_\varepsilon| = \kappa$. We shall show, eventually, that the algebras A_ε are pairwise non-isomorphic for $\varepsilon \in {}^\kappa 2$.

For any BA B let

$$JB = \langle \mathrm{At}\, B \cup \{x \in B\colon x \text{ is atomless}\}\rangle^{\mathrm{id}} ,$$

where At B is the set of all atoms of B. We repeat this construction transfinitely as follows:

$$I_0 B = \{0\} ;$$

$$I_\lambda B = \bigcup_{\xi<\lambda} I_\xi B \text{ for } \lambda \text{ a limit ordinal} ;$$

$$I_{\xi+1} B = \bigcup\ J(B/I_\xi B) .$$

If $a \in B$, we denote by $[a]$ the image of a under the natural homomorphism from B onto $B/I_\xi B$ (ξ is to be understood from context).

1.1. LEMMA. *Let $\varepsilon \in {}^{\kappa}2$. Then $\varepsilon 0 = 0$ iff A_ε is atomic.*

PROOF. $\Rightarrow$ Let $0 \neq x \in A_\varepsilon$; we want to find an atom $\leqq x$. We may assume that $x = [s, t)$ for some $s < t$ in β_ε. Say $s\nu = t\nu$ for all $\nu > \xi$, and $s\xi < t\xi$, where $\xi < \kappa$. Let $u0 = s0 + 1$ and $u\nu = s\nu$ for all $\nu > 0$. Clearly, $s < u \leqq t$. Since u is the successor of s in β_ε, it follows easily that $[s, u)$ is an atom $\leqq x$.

$\Leftarrow$ Suppose that $\varepsilon 0 = 1$; we show that A_ε is not atomic. Let $s, t \in \beta_\varepsilon$ be such that $s0 < t0$, both being in the η-part of $1 + \eta + \omega$, and $s\nu = t\nu$ for all $\nu > 1$. Then $[s, t)$ is atomless, as is easily checked. $\square$

1.2. LEMMA. *Let $\varepsilon \in {}^{\kappa}2$. For each $\xi < \kappa$ and each $s \in \Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ let s^+ be the member of $\Pi_{\nu < \kappa} \alpha_{\varepsilon\nu}$ such that $s^+\nu = 0$ for all $\nu < \xi$, and $s^+\nu = s\nu$ for $\xi \leqq \nu < \kappa$; and let $F_\xi s = ([0, s^+))$. Then*:

(i) *F_ξ is an order-isomorphism into $A_\varepsilon / I_\xi A_\varepsilon$.*

(ii) *The range of F_ξ generates $A_\varepsilon / I_\xi A_\varepsilon$.*

(iii) *If $t \in \Pi_{\nu < \kappa} \alpha_{\varepsilon\nu}$ and $s \in \Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ is the restriction of t, then $[s^+, t) \in I_\xi A_\varepsilon$.*

PROOF. We proceed by induction on ξ. The case $\xi = 0$ is trivial. Assume the lemma for ξ, and let $s, t \in \Pi_{\xi+1 \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$. If $s \leqq t$, clearly $F_{\xi+1}s \leqq F_{\xi+1}t$. To show that $F_{\xi+1} \neq F_{\xi+1}t$ for $s < t$, it suffices to show that there are infinitely many atoms $\leqq [[s^+, t^+)]$ (here $[[s^+, t^+)] \in A_\varepsilon / I_\xi a_\varepsilon$, while $^+$ is relative to $\xi + 1$). For each $i \in \omega$, let u_i be like s^+ except that $u_i \xi = i + 1$ (in the ω-part of $\alpha_{\varepsilon\xi}$). By the induction hypothesis, $[[u_i^+, u_{i+1}^+)]$ is an atom of $A_\varepsilon / I_\xi A_\varepsilon$ and it is clearly $\leqq [[s^+, t^+)]$, as desired. To prove (iii), we assume that $t \in \Pi_{\nu < \kappa} \alpha_{\varepsilon\nu}$ and $s \in \Pi_{\xi+1 \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ is the restriction of t; also let u be the restriction of t to $\Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$, and let $v \in \Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu}$ extend s so that $v\xi = 0$. Then $[u^+, t) \in I_\xi A_\varepsilon$ by the induction hypothesis. $[v, u)$ is a sum of an atomless element and finitely many atoms in intalg$(\Pi_{\xi \leqq \nu < \kappa} \alpha_{\varepsilon\nu})$, so by the induction hypothesis, $[v^+, u^+] \in I_{\xi+1} A_\varepsilon$. Hence $[v^+, t) \in I_{\xi+1} A_\varepsilon$. Since $v^+ = s^+$, this proves (iii). Clearly (ii) follows from (iii).

Now suppose that ξ is a limit ordinal $< \kappa$, and the lemma holds for all $\nu < \xi$. Clearly (i) holds. Assume the hypothesis of (iii). Choose $\nu < \xi$ so that $t\mu = 0$ for all $\mu \in [\nu, \xi)$, and let u be the restriction of t to $\Pi_{\nu \leqq \mu < \kappa} \alpha_{\varepsilon\mu}$. Then $[u^+, t) \in I_\nu A_\varepsilon \subseteq I_\xi A_\varepsilon$. Since $u^+ = s^+$, this proves (iii). Again, (ii) follows from (iii). $\square$

Lemmas 1.1 and 1.2 immediately give the desired result:

1.3. THEOREM. *For each $\kappa \geqq \omega$ there are 2^κ pairwise non-isomorphic interval algebras of power κ.* $\square$

Our second construction gives the number of superatomic BAs. By Theorem 17.11, Chapter 6 of Part I, there are exactly ω_1 denumerable superatomic BAs. Our construction gives 2^κ superatomic BAs for each uncountable cardinal κ. This result is due independently to BONNET [1977], CARPINTERO ORGANERO [1971], and WEESE [1976]; we follow the construction of Weese.

Recall the definition of the cardinal sequence of a superatomic BA A:

$$I_0A = \{0\},$$

$$I_{\beta+1}A = \bigcup \langle \mathrm{At}(A/I_\beta A)\rangle^{\mathrm{id}},$$

$$I_\lambda A = \bigcup_{\beta<\lambda} I_\beta A \text{ for } \lambda \text{ limit}.$$

We denote by $[a]_\alpha$ the image of $a \in A$ under the natural homomorphism of A onto $A/I_\alpha A$. A is superatomic iff $I_\beta A = A$ for some β. The least β such that $I_B A = A$ is a successor ordinal $\alpha A + 1$. Then $A/I_{\alpha A}$ is a finite non-trivial BA; we let nA be the number of atoms of $A/I_{\alpha A}$. The *cardinal sequence* of A is the sequence $\langle |\mathrm{At}(A/I_\xi A)|: \xi \leqq \alpha A\rangle$.

Our construction will use weak products; recall from 17.18, Chapter 6 of Part I, that a weak product of superatomic BAs is again superatomic, and the cardinal sequence of a weak product can be described in terms of the cardinal sequences of its factors. We also need the following lemma.

1.4. LEMMA. *Let $\langle A_i: i \in I\rangle$ be an infinite system of non-trivial BAs, and set $B = \Pi^{\mathrm{w}}_{i\in I} A_i$. Let σ be the least ordinal α such that $\{i \in I: \alpha \leqq \alpha A_i\}$ is finite. Then for each $\alpha < \sigma$ we have $B/I_\alpha B \cong \Pi^{\mathrm{w}}_{i\in I} A_i/I_\alpha A_i$.*

PROOF. We use the following elementary fact, easily established by induction on α:

(*) $I_\alpha C \cap (C \restriction c) = I_\alpha(C \restriction c)$ for any BA C, any $c \in C$, any ordinal α. Furthermore, $(C/I_\alpha C) \restriction [c]_\alpha$ is isomorphic to $(C \restriction c)/I_\alpha(C \restriction c)$ via $[x]_\alpha \mapsto [x]_\alpha$ for each $x \leq c$.

Now the desired isomorphism is given by $(f[x]_\alpha)_i = [x_i]_\alpha$ for each $x \in B$; the only non-trivial parts of the verfication of this fact are that f is well-defined and one-to-one. Well-definedness follows from (*). For one-to-one-ness, suppose that $f[x]_\alpha = 0$. If $\{i: x_i \neq 1\}$ is finite, then from $\{i: \alpha \leqq \rho_i\}$ infinite (which follows from the definition of σ) we find $i \in I$ such that $x_i = 1$ and $\alpha \leqq \rho_i$. But then $[x_i]_\alpha \neq 0$, a contradiction. So $\{i \in I: x_i \neq 0\}$ is finite. Since $x_i \in I_\alpha A_i$ for all $i \in I$, it then follows from (*) that $x \in I_\alpha B$, as desired. □

1.5. THEOREM. *For each $\kappa \geqq \omega_1$ there are 2^κ isomorphic types of superatomic BAs of power κ.*

PROOF. For any BA C we denote by $\Pi^{\mathrm{w}}_0 C$ the weak product of ω copies of C, and by $\Pi^{\mathrm{w}}_1 C$ that of ω_1 copies. Now for each $f \in {}^{\leqq\kappa}2$ we construct a superatomic BA A_f by induction on dom f:

$$A_0 = 2,$$

$$A_{f\varepsilon} = \prod_\varepsilon^{\mathrm{w}} A_f \ (\varepsilon = 0, 1),$$

$$A_f = \prod_{\alpha<\lambda}^{\mathrm{w}} A_{f\restriction\alpha} \text{ for dom } f = \lambda \text{ limit } \leq\kappa.$$

(Here $f\varepsilon$ is $f^\frown\langle\varepsilon\rangle$.) By induction, $|\mathrm{dom}\ f| \leq |A_f| \leq |\mathrm{dom}\ f| \cdot \omega_1$ for all $f \in {}^{\leqq\kappa}2$, so $|A_f| = \kappa$ for all $f \in {}^{\kappa}2$. By induction using 17.18, Chapter 6 of Part I,

(1) If $\beta \leqq \kappa$ and $f \in {}^{\beta}2$, then $\alpha A_f = \beta$ and $nA_f = 1$.

Note that $A_{\langle 0\rangle}$ is a factor of each algebra A_f with $f \in {}^{\leqq\kappa}2$, $0 \in \mathrm{dom}\ f$, $f0 = 0$. Hence

(2) If $f \in {}^{\leqq\kappa}2$, $0 \in \mathrm{dom}\ f$, and $f0 = 0$, then there is an $a \in A_f$ such that $|A_f \upharpoonright a| = \omega$.

On the other hand, we claim

(3) If $f \in {}^{\leqq\kappa}2$, $0 \in \mathrm{dom}\ f$, and $f0 = 1$, then there is no $a \in A_f$ such that $|A_f \upharpoonright a| = \omega$.

We prove (3) by induction on dom f. Since $A_{\langle 1\rangle}$ is isomorphic to the finite–cofinite algebra on ω_1, (3) holds for dom $f = 1$. If dom $f = \beta + 1$ and (3) is true for shorter functions, take any $a \in A_f$. If $\{i \in I: a_i \neq 1\}$ is finite, then there is a $b \leqq a$ with $A_f \upharpoonright b \cong A_{f \upharpoonright \beta}$, and the inductive hypothesis applies. If $\{i \in I: a_i \neq 0\}$ is finite, then

$$A_f \upharpoonright a \cong (A_{f\upharpoonright\beta} \upharpoonright b_1) \times \cdots \times (A_{f\upharpoonright\beta} \upharpoonright b_m)$$

for certain $b_1, \ldots, b_m$, and again the inductive hypothesis applies. The final induction step – dom f a limit ordinal – is treated similarly. So (3) holds.

The major part of the proof is the following claim:

(4) If $\gamma + \delta = \beta \leqq \kappa$, $f \in {}^{\beta}2$, and $g\xi = f(\gamma + \xi)$ for all $\xi < \delta$, then $A_f/I_\gamma A_f \cong A_g$.

We prove (4) by induction on δ, with γ fixed. For $\delta = 0$ it follows from (1). Assume (4) for δ, let $f \in {}^{\gamma+\delta+1}2$, let $g\xi = f(\gamma + \xi)$ for all $\xi < \delta$, and let $h\xi = f(\gamma + \xi)$ for all $\xi < \delta + 1$. Then

$$\begin{aligned} A_f/I_\gamma A_f &= \Big(\prod^{\mathrm{w}}_{f(\gamma+\delta)} A_{f\upharpoonright(\gamma+\delta)}\Big)/I_\gamma A_f \\ &\cong \prod^{\mathrm{w}}_{f(\gamma+\delta)} (A_{f\upharpoonright(\gamma+\delta)}/I_\gamma A_{f\upharpoonright(\gamma+\delta)}) \text{ (by (1) and Lemma 1.4)} \\ &\cong \prod^{\mathrm{w}}_{f(\gamma+\delta)} A_g \text{ (induction hypothesis)} \\ &= A_h , \end{aligned}$$

as desired. Now assume that δ is a limit ordinal, (4) holds for all $\xi < \delta$, $f \in {}^{\gamma+\delta}2$, $g\xi = f(\gamma + \xi)$ for all $\xi < \delta$, and $(h\varepsilon)\xi = f(\gamma + \xi)$ for all $\xi < \varepsilon$, for each $\varepsilon < \delta$. Then

$$\begin{aligned}
A_f/I_\gamma A_f &= \prod_{\xi<\beta}^{\mathrm{w}} A_{f\upharpoonright\xi}/I_\gamma A_f \\
&\cong \prod_{\xi<\beta}^{\mathrm{w}} (A_{f\upharpoonright\xi}/I_\gamma A_{f\upharpoonright\xi}) \text{ (by (1) and Lemma 1.4)} \\
&\cong \prod_{\gamma\leqq\xi<\delta}^{\mathrm{w}} (A_{f\upharpoonright\xi}/I_\gamma A_{f\upharpoonright\xi}) \text{ by (1)} \\
&\cong \prod_{\gamma\leqq\xi<\delta} A_{h\xi} \text{ induction hypothesis} \\
&= A_g\,.
\end{aligned}$$

We have established (4).

Now suppose that $f, g \in {}^\kappa 2$ and $f \neq g$. Let β be minimum such that $f\beta \neq g\beta$. By (4), $A_f/I_\beta A_f \cong A_h$ and $A_g/I_\beta A_g \cong A_k$, where $h\delta = f(\beta+\delta)$ and $k\delta = g(\beta+\delta)$ for all $\delta < \kappa$. So $h0 \neq k0$, and so by (2) and (3), $A_h \not\cong A_k$. This finishes the proof. □

The third construction of many non-isomorphic BAs gives the following remarkable theorem of EFIMOV and KUZNECOV [1970]: for each $\kappa > \omega$ there are 2^κ pairwise non-isomorphic dense subalgebras of the free BA on κ generators. The construction is based on the following general facts.

Let f be a homomorphism of a BA A onto a Ba B. We set

$$P_f = \{(x, y) \colon x, y \in A \text{ and } fx = fy\}\,.$$

Thus, P_f is a subalgebra of $A \times A$. Set

$$I_f = \{(x, y) \colon x, y \in A \text{ and } fx = fy = 0\}\,.$$

Then I_f is an ideal in P_f, and P_f/I_f is isomorphic to B.

1.6. LEMMA. *Suppose that A is a free BA, $0 \neq X \subseteq A$, and $I = \{a \in A \colon a \cdot x = 0$ for all $x \in X\}$. Then I is a countably generated ideal in A.*

PROOF. We may assume that A is uncountable. Say A is freely generated by $\langle x_\alpha \colon \alpha < \kappa \rangle$, κ an uncountable cardinal. For $\Gamma \in [\kappa]^{<\omega}$ and $f \in {}^\Gamma 2$ we set

$$x(f) = \prod_{\alpha\in\Gamma} f\alpha \cdot x_\alpha\,,$$

where $1 \cdot y = y$, $0 \cdot y = -y$ for all y. Let

$$\mathscr{F} = \{f \colon f \in {}^\Gamma 2 \text{ for some } \Gamma \in [\kappa]^{<\omega},\ x(f) \in I, \text{ and } x(f \upharpoonright \Delta) \not\in I \text{ if } \Delta \subset \Gamma\}\,.$$

Clearly, $\{x(f) \colon f \in \mathscr{F}\}$ generates I, so it sffices to show that $\mathscr{F}$ is countable. Suppose not. Then by the Δ-system lemma plus the pigeon-hole principle there exist a finite $\Delta \subseteq \kappa$, an $h \in {}^\Delta 2$, and an uncountable $\mathscr{G} \subseteq \mathscr{F}$ such that for any two distinct $f, g \in \mathscr{G}$ we have $\operatorname{dom} f \cap \operatorname{dom} g = \Delta$ and $f \upharpoonright \Delta = g \upharpoonright \Delta = h$. Now we may assume that each member of X has the form $x(f)$ for some $f \in {}^{\Gamma}2$ with $\Gamma \in [\kappa]^{<\omega}$. Note that $x(h) \notin I$ (since $\Delta \subset \Gamma$ for any Γ such that $f \in \mathcal{G}$ and $f \in {}^{\Gamma}2$ for some f). Hence, choose $\Gamma \in [\kappa]^{<\omega}$ and $f \in {}^{\Gamma}2$ so that $x(f) \in X$ and $x(f) \cdot x(h) \neq 0$. Say $\operatorname{dom} f = \{\alpha_0, \ldots, \alpha_{m-1}\}$. Choose $m+1$ distinct members $g_0, \ldots, g_m$ of $\mathcal{G}$. Since $x(f) \cdot x(g_i) = 0$ there is a $\beta_i \in \operatorname{dom} g_i \cap \operatorname{dom} f$ with $f\beta_i \neq g_i\beta_i$, for each $i \leqq m$. Now $|\operatorname{dom} f| < m+1$, so choose distinct $i, j \leqq m$ such that $\beta_i = \beta_j$. Then $\beta_i \in \operatorname{dom} g_i \cap \operatorname{dom} g_j = \Delta$, while $f\beta_i \neq g_i\beta_i = h\beta_i$, contradicting $x(f) \cdot x(h) \neq 0$. □

1.7. LEMMA. *If A is a free BA on $\kappa \geqq \omega$ generators, and if I is a maximal ideal in A, then I cannot be generated by $<\kappa$ elements.*

PROOF. Say A is freely generated by X, $|X| = \kappa$. Suppose that I is generated by Y, $|Y| < \kappa$. For each $y \in Y$ there is a finite $F_y \subseteq X$ such that $y \in \langle F_y \rangle$. Choose $x \in X \backslash \bigcup_{y \in Y} F_y$. Then $x \in I$ or $-x \in I$, and either possibility clearly gives a contradiction. □

1.8. LEMMA. *Suppose that A is an uncountable free BA, and f is a homomorphism of A onto a countable BA B. Then P_f is not isomorphic to A.*

PROOF. Suppose it is; we shall get a contradiction by finding a countably generated maximal ideal in P_f. (See Lemma 1.7.) Let J be a maximal ideal in B, and set $K = \{(a, b) \in P_f \colon fa \in J\}$. Clearly, K is a maximal ideal in P_f. To show that it is countably generated, first let $X = \{(x, 0) \in P_f \colon fx = 0\}$. Set

$$L = \{(u, v) \in P_f \colon (u, v) \cdot (x, 0) = (0, 0) \text{ for all } (x, 0) \in X\}\,.$$

Thus, by Lemma 1.6, L is a countably generated ideal in P_f. We claim

(1) $\qquad L = \{(0, v) \in P_f \colon fv = 0\}$

For $\supseteq$, is obvious. For $\subseteq$, let $(u, v) \in L$. It suffices to show that $u = 0$. Suppose not. Now there is a non-zero $x \leqq u$ such that $fx = 0$, since otherwise $f \restriction (A \restriction u)$ would be one-to-one and B would be uncountable. Taking such an x, we have $(x, 0) \in X$ and $(x, 0) \cdot (u, v) \neq (0, 0)$, a contradiction. So (1) holds.

By symmetry, the set $L' \overset{\text{def}}{=} \{(u, 0) \in P_f \colon fu = 0\}$ is a countably generated ideal in P_f. Next, for each $z \in B$ choose $a_z \in A$ such that $fa_z = z$. To show that K is countably generated it now suffices to prove

(2) $\qquad K = \langle L \cup L' \cup \{(a_z, a_z) \colon z \in J\} \rangle^{\text{id}}\,.$

Clearly, $\supseteq$ holds. For $\subseteq$, given $(x, y) \in K$ we have $(x, y) \cdot (-a_{fx}, -a_{fx}) \in \langle L \cup L' \rangle^{\text{id}}$ and $(x, y) \leqq (a_{fx}, a_{fx}) + (x, y) \cdot (-a_{fx}, -a_{fx})$, so (x, y) is in the right-hand side of (2), as desired. □

We are now ready for the theorem of Efimov and Kuznecov:

1.9. THEOREM. *For each $\kappa > \omega$ there are 2^κ pairwise non-isomorphic dense subalgebras of the free BA A on κ generators.*

PROOF. By Theorem 1.3 and its proof there is a family $\langle B_\alpha : \alpha < 2^\kappa \rangle$ of pairwise non-isomorphic BAs of power κ with the following property:

(1) For every $\alpha < 2^\kappa$ and every $x \in B_\alpha^+$ there is a non-zero $y \leqq x$ such that $B_\alpha \restriction y$ is countable.

Now for each $\alpha < 2^\kappa$ let f_α be a homomorphism from A onto B_α, and then set $P_\alpha = P_{f\alpha}$, $I_\alpha = I_{f\alpha}$. Recall from Theorem 9.14, Chapter 4 of Part I that $A \times A$ is isomorphic to A; so P_α can be considered to be a subalgebra of A. Now P_α is dense in $A \times A$, for suppose $(a, b) \in (A \times A)^+$; say $a \neq 0$. Then there is a non-zero $a' \leqq a$ with $f_\alpha a' = 0$ (otherwise $f_\alpha \restriction (A \restriction a)$ would be one-to-one and so B_α would be uncountable). So $(a', 0) \in P_\alpha$ and $0 \neq (a', 0) \leqq (a, b)$, as desired.

The proof will be completed by proving

(2) $P_\alpha \ncong P_\beta$ for $\alpha \neq \beta$.

For, suppose that g is an isomorphism of P_α onto P_β. Since $P_\alpha / I_\alpha \cong B_\alpha$, it suffices to show that $g[I_\alpha] \subseteq I_\beta$. Suppose that $(a, b) \in I_\alpha$ but $g(a, b) \not\in I_\beta$. Now $(a, 0)$, $(0, b) \in P_\alpha$, and $g(a, b) = g(a, 0) + g(0, b)$, so say $g(a, 0) \not\in I_\beta$. Let $g(a, 0) = (c, d)$. Since $(c, d) \not\in I_\beta$, we have $f_\beta c \neq 0$. Choose $0 \neq e \leqq f_\beta c$ with $B_\beta \restriction e$ countable. Say $f_\beta c' = e$, and set $c'' = c' \cdot c \cdot d$; then also $f_\beta c'' = e$. Choose $a' \in A$ so that $g(a', 0) = (c'', c'')$. Hence,

(3) $A \cong P_\alpha \restriction (a', 0) \cong P_\beta \restriction (c'', c'')$.

Now $A \restriction c'' \cong A$; let h be such an isomorphism. Let $k = f_\beta \circ h^{-1}$. It is easily checked that $P_\beta \restriction (c'', c'') \cong P_k$. By Lemma 1.8, $P_k \ncong A$, which contradicts (3). $\square$

VAN DOUWEN (unpublished) has improved Theorem 1.9 by showing that for each $\kappa > \omega$ there are 2^κ totally different rigid dense subalgebras of Frκ (totally different means no non-trivial isomorphic factors).

The fourth construction which we present also concerns subalgebras of specific algebras. Call a subalgebra of $\mathscr{P}\kappa$ *full* if it contains all singletons. We present the result of FRENICHE [1984] that for any infinite κ there are 2^{2^κ} pairwise non-isomorphic full subalgebras of $\mathscr{P}\kappa$. We give two proofs: one using the result just established, and a direct one.

Let $\kappa \geqq \omega$. Let A be a free subalgebra of $\mathscr{P}\kappa$ with $|A| = 2^\kappa$. Then by Theorem 1.9 let $\langle B_\alpha : \alpha < 2^{2^\kappa} \rangle$ be a system of pairwise non-isomorphic dense subalgebras of A. For each $\alpha < 2^{2^\kappa}$ let $C_\alpha = \langle B_\alpha \cup \{\{\xi\} : \xi < \kappa\} \rangle$. Suppose that $\alpha, \beta < 2^{2^\kappa}$, $\alpha \neq \beta$, and f is an isomorphism of C_α onto C_β. Then f induces an isomorphism from $C_\alpha / [\kappa]^{<\omega}$ onto $C_\beta / [\kappa]^{<\omega}$, and these are, respectively, isomorphic to B_α and B_β, a contradiction.

Now we turn to the direct construction.

1.10. LEMMA. *If $\mathscr{P}\kappa$ has λ full subalgebras in all, and $\lambda > 2^\kappa$, then $\mathscr{P}\kappa$ has λ pairwise non-isomorphic full subalgebras.*

PROOF. Any isomorphism between full subalgebras is induced by a permutation of κ, so every isomorphism class of full subalgebras has at most 2^κ members. The lemma follows. □

If F is a filter on a BA A, then, as is easily seen, $\langle F \rangle = F \cup \{a: -a \in F\}$.

1.11. LEMMA. *If F and G are distinct filters on A and neither is an ultrafilter, then $\langle F \rangle \neq \langle G \rangle$.*

PROOF. Say $a \in F \backslash G$. If $-a \not\in G$, then $a \in \langle F \rangle \backslash \langle G \rangle$. Assume that $-a \in G$. Choose $c \in A$ so that $c, -c \not\in F$ (since F is not an ultrafilter). Then $-c + -a \in G$ while $-c + -a$ and $c \cdot a$ are not in F; so $\langle G \rangle \neq \langle F \rangle$. □

1.12. THEOREM. *There are 2^{2^κ} full subalgebras of $\mathscr{P}\kappa$.*

PROOF. (For the first proof, see above.) By Lemma 1.10 it suffices to exhibit 2^{2^κ} full subalgebras without worrying about isomorphisms. The proof is now a consequence of Lemma 1.11 and the following two facts: (1) there are 2^{2^κ} non-principal ultrafilters on $\mathscr{P}\kappa$; and (2) if F_1, F_2, F_3, F_4 are distinct ultrafilters, then $F_1 \cap F_2 \neq F_3 \cap F_4$ (the desired family of full subalgebras is then $\{F \cap G: \{F, G\} \in \mathscr{P}\}$ for some partition $\mathscr{P}$ of the set of all non-principal ultrafilters into 2-element subsets). (1) is well known. For (2), say $a \in F_1 \backslash F_3$, $b \in F_1 \backslash F_4$, $c \in F_2 \backslash F_3$, $d \in F_2 \backslash F_4$. Then $a \cdot b + c \cdot d \in (F_1 \cap F_2) \backslash (F_3 \cap F_4)$. The proof is finished. □

The final construction in this section, taken from MONK and SOLOVAY [1972], is a construction of complete BAs derived from forcing conditions and using infinite combinatorics.

Let κ be an infinite cardinal. Let M be a family of independent subsets of κ with $|M| = 2^\kappa$, and let t be a one-to-one mapping from $\mathscr{P}\kappa$ onto M. For each $R \subseteq \mathscr{P}\kappa$ with $|\mathscr{P}\kappa \backslash R| = 2^\kappa$ we define a complete BA C_R as follows. Let $A_R = \{t_a: a \in \mathscr{P}\kappa \backslash R\}$. We also define a partial ordering on the set $\mathscr{P}_R = \{(k, K): k \in [\kappa]^{<\omega}, K \in [A_R]^{<\omega}\}$ by setting

$$(k_1, K_1) \leqq (k_2, K_2) \text{ iff } k_1 \subseteq k_2,\ K_1 \subseteq K_2, \text{and } k_2 \cap \bigcup K_1 \subseteq k_1 .$$

With this partial ordering we associate a complete BA in the usual way familiar to those used to forcing: for each $(k, K) \in \mathscr{P}_R$ we define

$$\mathcal{O}_{kK} = \{(k', K') \in \mathscr{P}_R: (k, K) \leqq (k', K')\} ;$$

the sets $\mathcal{O}_{kK}$ form a base for a topology on $\mathscr{P}_R$, and C_R is the complete BA of regular open sets in this topology.

This construction of C_R is essentially found in MARTIN and SOLOVAY [1970]. We proceed to describe the basic properties of these algebras.

1.13. LEMMA. *C_R satisfies the κ^+-chain condition.*

PROOF. If $\mathcal{O}_{kK} \cap \mathcal{O}_{lL} = 0$, then $k \neq l$; since $|[\kappa]^{<\omega}| = \kappa$, the lemma follows. $\square$

Next, note the following properties of the topology above. For any $z \in \mathcal{P}_R$, let $b_R z$ be the interior of the closure of $\mathcal{O}_z$; thus $b_R z \in C_R$. Then, as is easily checked for any partial ordering,

$$b_R z = \{w \in \mathcal{P}_R : \forall w' \geqq w \exists z' \geqq z(z' \geqq w')] ;$$

$$-b_R z = \{w \in \mathcal{P}_R : \forall z' \geqq z(z' \not\geqq w)\} .$$

We also need the following specific property of our partial ordering:

1.14. LEMMA. $b_R(k, K) = \left\{(l, L) \in \mathcal{P}_R : k \subseteq l \cup \left(\kappa \setminus \bigcup A_R\right), K \subseteq L, l \cap \bigcup K \subseteq k\right\}.$

PROOF. First suppose that $(l, L) \in b_R(k, K)$. Suppose that $\alpha \in k \cap \bigcup A_R$; we show that $\alpha \in l$, which thus establishes the first inclusion above. Say $\alpha \in x \in A_R$. Then $(l, L) \leqq (l, L \cup \{x\})$, so there is (m, M) such that $(l, L \cup \{x\}) \leqq (m, M)$ and $(k, K) \leqq (m, M)$. Now $\alpha \in m \cap \bigcup (L \cup \{x\})$, so $\alpha \in l$. Next, suppose that $K \setminus L \neq 0$; say $y \in K \setminus L$. By independence, choose $\alpha \in y \setminus (\bigcup L \cup l)$. Then $(l, L) \leqq (l \cup \{\alpha\}, L)$, so there is (m, M) with $(l \cup \{\alpha\}, L) \leqq (m, M)$ and $(k, K) \leqq (m, M)$. Then $\alpha \in m \cap \bigcup K$, so $\alpha \in k$. Also, $\alpha \in \bigcup A_R$, so $\alpha \in l$ by the above, a contradiction. Finally, suppose that $\alpha \in l \cap \bigcup K$. Choose (m, M) so that $(l, L) \leqq (m, M)$ and $(k, K) \leqq (m, M)$. Then $\alpha \in m \cap \bigcup K \subseteq \kappa$, as desired.

Conversely, let (l, L) satisfy the conditions in the braces. Suppose that $(l, L) \leqq (m, M)$. Then $(k, K) \subseteq (k \cup m, K \cup M)$, since

$$\begin{aligned}(k \cup m) \cap \bigcup K &\subseteq k \cup \left(m \cap \bigcup K\right) \subseteq k \cup \left(m \cap \bigcup L \cap \bigcup K\right) \\ &\subseteq k \cup \left(l \cap \bigcup K\right) \subseteq k\end{aligned}$$

and $(m, M) \subseteq (k \cup m, K \cup M)$, since

$$\begin{aligned}(k \cup m) \cap \bigcup M &\subseteq m \cup \left(k \cap \bigcup M\right) \\ &\subseteq m \cup \left[\left(l \cup \left(\kappa \setminus \bigcup A_R\right)\right) \cap \bigcup M\right] \\ &= m \cup \left(l \cap \bigcup M\right) \subseteq m .\end{aligned}$$

Thus $(l, L) \in b_R(k, K)$. This finishes the proof of Lemma 1.14. $\square$

For each $\alpha<\kappa$ let $a_R\alpha = b_R(\{\alpha\}, 0)$; these elements will be used in the proof of the following lemma.

1.15. LEMMA. *C_R is completely generated by a set with at most κ elements.*

PROOF. Lemma 1.14 yields the following:

$$(1)\qquad a_R\alpha = \{(l, L)\in \mathcal{P}_R\colon \alpha\in l\} \text{ if } \alpha\in \bigcup A_R\,;$$

$$(2)\qquad a_R\alpha = \mathcal{P}_R \text{ if } \alpha\in \kappa\backslash\bigcup A_R\,;$$

$$(3)\qquad -a_R\alpha = \left\{(l, L)\colon \alpha\in \bigcup L\backslash l\right\} \text{ if } \alpha\in \bigcup A_R\,.$$

Hence, using Lemma 1.14 further, we get

$$(4)\qquad b_R(k, K) = \bigcap_{\alpha\in k} a_R\alpha \cap \bigcap_{\alpha\in\bigcup K\backslash k} -a_R\alpha = \prod_{\alpha\in k} a_R\alpha\cdot \prod_{\alpha\in\bigcup K\backslash k} -a_R\alpha\,.$$

Thus, C_R is generated by $\{a_R\alpha\colon \alpha<\kappa\}$, as desired. □

1.16. LEMMA. $|C_R| = 2^\kappa$.

PROOF. By Lemma 1.13, every join or meet is a join or meet over a subset of the index of power $\leqq\kappa$. Hence, by Lemma 1.15 it easily follows that $|C_R|\leqq 2^\kappa$. Now we exhibit 2^κ elements of C_R. For each $t\in A_R$ we have

$$b_R(0, \{t\}) = \{(l, L)\in\mathcal{P}_R\colon t\in L \text{ and } l\subseteq\kappa\backslash t\}$$

by Lemma 1.14. So $b_R(0, \{t\})\neq b_R(0, \{t'\})$ if $t\neq t'$, as desired. □

Now let $R\subseteq\mathcal{P}\kappa$, $|\mathcal{P}\kappa\backslash R| = 2^\kappa$. We say that R is *represented* in a complete BA D by $x\in{}^\kappa D$ provided that

$$(*)\qquad R = \left\{a\subseteq\kappa\colon \Sigma\,\{x\alpha\colon \alpha\in t_a\} = 1\right\}.$$

(Remember that t maps $\mathcal{P}\kappa$ onto A_R.)

1.17. LEMMA. *If D is a complete* BA *of power 2^κ, then there are at most 2^κ sets $R\subseteq\mathcal{P}\kappa$ with $|\mathcal{P}\kappa\backslash R| = 2^\kappa$ which are representable in D by some $x\in{}^\kappa D$.*

PROOF. There are only 2^κ functions from κ into D. □

1.18. LEMMA. *For any $R\subseteq\mathcal{P}\kappa$ with $|\mathcal{P}\kappa\backslash R| = 2^\kappa$, the function a_R represents R in C_R.*

PROOF. Suppose that $a\in\mathcal{P}\kappa\backslash R$. Then by (4) in the proof of Lemma 1.15 we have

$$0\neq b_R(0, \{t_a\}) = \prod\{-a_R\alpha\colon \alpha\in t_a\}\,,$$

so that $\Sigma\{a_R\alpha: \alpha \in t_a\} \neq 1$. So $\supseteq$ in $(*)$ holds. Now let $a \in R$; we need to show that $\bigcup\{a_R\alpha: \alpha \in t_a\}$ is dense in the topological space $\mathcal{P}_R$. Let $(k, K) \in \mathcal{P}_R$. Choose $\alpha \in t_a \setminus \bigcup K$ by independence. Then $(k \cup \{\alpha\}, K) \in \mathcal{O}_{kK} \cap a_R\alpha$, as desired. The proof is complete. □

We are now prepared to prove the theorem.

1.19. THEOREM. *For each infinite κ there are exactly 2^{2^κ} pairwise non-isomorphic complete BAs of power 2^κ.*

PROOF. Define $R \equiv S$ iff $R, S \subseteq \mathcal{P}\kappa$, $|\mathcal{P}\kappa \setminus R| = |\mathcal{P}\kappa \setminus S| = 2^\kappa$, and $C_R \cong C_S$. Note that if f is an isomorphism from a complete BA D onto a complete BA E, and $x \in {}^\kappa D$ represents R in D, then $f \circ x$ represents R in E. So by Lemma 1.17, each $\equiv$-class has at most 2^κ elements. So the theorem follows from Lemma 1.18. □

2. Construction of complicated Boolean algebras

As an illustration of the ideas in SHELAH [1983] we shall construct a large family of pairwise unembeddable rigid BAs (with strong notions of rigidity and unembeddability). The construction is in two parts: a purely combinatorial part, and a construction of BAs from certain combinatorial objects.

K_{tr} is the class of all relational structures I such that:

(1) the universe of I is a subset of ${}^{\omega \geq}\lambda$ for some λ, closed under initial segments;

(2) the relations of I are as follows:

$P_i = \{\eta \in I: \text{length}(\eta) = i\}$ for each $i \leqq \omega$;

$\lessdot = \{(\eta, \nu) \in I \times I: \text{length}(\eta) < \text{length}(\nu) \text{ and } \eta = \nu \restriction \text{length}(\eta)\}$;

$< = \{(\eta^\frown\langle\alpha\rangle, \eta^\frown\langle\beta\rangle): \eta^\frown\langle\alpha\rangle, \eta^\frown\langle\beta\rangle \in I \text{ and } \alpha < \beta\}$;

$Eq_i = \{(\eta, \nu) \in I \times I: \text{length}(\eta), \text{length}(\nu) \geqq i \text{ and } \eta \restriction i = \nu \restriction i\}$ for each $i < \omega$;

$z = \{\langle\cdot\rangle\}$, where $\langle\cdot\rangle$ is the empty sequence.

L is a language appropriate for K_{tr}. If $\langle I_t: t \in T\rangle$ is a system of L-structures, then $\Sigma_{t\in T} I_t$ is the *disjoint union* of them: its universe is $\bigcup_{t \in T} I_t \times \{t\}$ (I_t is identified with its universe), and if $\boldsymbol{R}$ is an n-ary relation symbol of L, then the corresponding relation of $\Sigma_{t\in T} I_t$ is

$$\{\langle(a_1, t), \ldots, (a_n, t)\rangle: t \in T, I_t \models \boldsymbol{R}[a_1, \ldots, a_n]\}.$$

For $t \in T$ we denote by I_t^- the structure $\Sigma_{s \neq t} I_s$.

Let L_{alg} be the language which has an m-ary operation symbol $\boldsymbol{F}_{mn}$ for all m, $n < \omega$. Let L' be a joint expansion of L and L_{alg} with an additional unary relation symbol $\boldsymbol{P}$. If I is an L-structure, then $M(I)$ is the following L'-structure: its L_{alg}-reduct is the absolutely free algebra generated by I, its L-reduct has all the relations of I, and $\boldsymbol{P}$ is interpreted as I. In the terms of L_{alg}, we always write the operation symbols to the left, and we use a standard sequence of variables $v_0, v_1, \ldots$. A term τ of L_{alg} is *initialized* if for some $m \in \omega$ the variables which occur in τ are $v_0, \ldots, v_{m-1}$, and they occur in that order in τ without repetitions; we call m the *type* of τ. If σ and τ are initialized terms of type m and n, respectively, $\bar{c} \in {}^mI$, $\bar{d} \in {}^nI$, and $\sigma\bar{c} = \tau\bar{d}$, then $\sigma = \tau$ and $\bar{c} = \bar{d}$ (proof by induction on σ). Every element of $M(I)$ can be written in the form $\tau\bar{c}$, where τ is an initialized term and $\bar{c} \in {}^mI$, m the type of τ, and this expression is unique. For each $a \in M(I)$ we denote this τ by τ_a, and $\bar{c}$ by $\bar{c}_a$.

Let $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1} \in M(I)$. We write

$$\langle a_0, \ldots, a_{m-1} \rangle \approx \langle b_0, \ldots, b_{m-1} \rangle \ (\mathrm{mod}\ M(I))$$

if $\tau_{ai} = \tau_{bi}$ for each $i < m$, and $\bar{c}_{a0}{}^\frown \cdots {}^\frown \bar{c}_{a(m-1)}$ satisfies the same quantifier-free formulas $\bar{c}_{b0}{}^\frown \cdots {}^\frown \bar{c}_{b(m-1)}$ in I.

Let I and J be L-structures. We say that I is $\psi(\bar{x}, \bar{y})$-*unembeddable* in J provided that: $\psi(\bar{x}, \bar{y})$ is an L-formula with $\bar{x}$ and $\bar{y}$ of the same length, and for every function $f: I \to M(J)$ there exist sequences $\bar{a}$, $\bar{b}$ in I both of the length of $\bar{x}$ such that $I \models \psi[\bar{a}, \bar{b}]$, fa_i has the same length as fb_i for each $i <$ length $\bar{a}$, and $f(\bar{a}) \approx f(\bar{b})$ (mod $M(J)$). ($f(\bar{a})$ is the concatenation of $f(a_0), \ldots, f(a_i), \ldots, i <$ length$(\bar{a})$; similarly for $f(\bar{b})$.) Finally, we say that K_{tr} has the *full* (χ, λ)-ψ-*bigness property* if there are $I_i \in K_{\mathrm{tr}}$ $(i < \chi)$ such that $|I_i| = \lambda$ and I_i is ψ-unembeddable in I for all $i < \chi$. We shall be interested only in the following formula $\psi(x_0, x_1, y_0, y_1)$:

$$\bigvee_{i<\omega} [P_{i+1}x_0 \wedge P_{i+1}y_0 \wedge P_\omega x_1 \wedge x_1 = y_1 \wedge x_0 \lessdot x_1$$
$$\wedge\ Eq_i(x_0, y_0) \wedge y_0 < x_0] \,.$$

It expresses that x_0 and y_0 have the form $\eta^\frown\langle \alpha \rangle$ and $\eta^\frown\langle \beta \rangle$, respectively, with $\beta < \alpha$, and $x_1 = y_1$ has domain ω and extends x_0.

Now we prove a combinatorial theorem about these notions.

2.1. THEOREM. *If* $\omega < \lambda \leqq \lambda^*$ *with* λ *regular, then* K_{tr} *has the full* (λ, λ^*)-ψ-*bigness property*.

PROOF. Let $S = \{\delta < \lambda : \delta$ is a limit ordinal and $\mathrm{cf}\,\delta = \omega\}$. Thus, S is a stationary subset of λ. Write $S = \bigcup_{i<\lambda} S_i$ with the S_i stationary and pairwise disjoint. For each $\delta \in S$ choose $\eta_\delta \in {}^\omega\delta$ strictly increasing with sup δ. For all $i < \lambda$ let

$$I_i = \bigcup_{n<\omega} {}^n\lambda^* \cup \{\eta_\delta : \delta \in S_i\} \,.$$

Thus, $|I_i| = \lambda^*$, and I_i has a natural L-structure. The rest of the proof is devoted to showing that for an arbitrary $i < \lambda$, I_i is ψ-unembeddable in I_i^-.

To this end, let $f: I_i \to M(I_i^-)$ be given. For any $a \in I_i$ let

$$\mathrm{orco}(a) = \sup\{\gamma < \lambda : \text{there is a } k < \mathrm{length}(\bar{c}_{fa}) \text{ such that } \bar{c}_{fa}(k) = (t, j) \text{ for some } t \text{ and } j, \text{ and } \gamma = j \text{ or } \gamma = \sup(\mathrm{ran}(t))\} \,.$$

Let

(3) $\quad C=\{\delta<\lambda\colon \text{for all } \eta\in{}^{\omega>}\delta,\ \operatorname{orco}(\eta)<\delta\}$.

Clearly, C is club in λ. We shall use C later on.

Now with each $0\neq\eta\in{}^{\omega>}\lambda$ we associate an equivalence relation E_η on λ. Let γ be the last value of η. For each $\alpha<\lambda$, let m_α be the length of $\bar{c}_{f(\eta^\frown\langle\alpha\rangle)}$, and for each $k<m_\alpha$ write $\bar{c}_{f(\eta^\frown\langle\alpha\rangle)}(k)=(t_{\alpha k}, j_{\alpha k})$. Now we set $\alpha E_\eta\beta$ iff $f(n^\frown\langle\alpha\rangle)\approx f(\eta^\frown\langle\beta\rangle) \pmod{M(I_i^-)}$, and for all $k<m_\alpha$, if $j_{\alpha k}<\gamma$ or $j_{\beta k}<\gamma$, then $j_{\alpha k}=j_{\beta k}$, and if $l<\operatorname{length}(t_{\alpha k})$ and $t_{\alpha k}(l)<\gamma$ or $t_{\beta k}(l)<\gamma$, then $t_{\alpha k}(l)=t_{\beta k}(l)$. Note that $f(\eta^\frown\langle\alpha\rangle)\approx f(\eta^\frown\langle\beta\rangle) \pmod{M(I_i^-)}$ implies that $m_\alpha=m_\beta$ and $\operatorname{length}(t_{\alpha k})=\operatorname{length}(t_{\beta k})$ for all $k<m_\alpha$. Clearly, E_η is an equivalence relation on λ. Now we claim

(4) $\quad$ There are $<\lambda$ equivalence classes under E_η.

In fact, suppose that $\Gamma\in[\lambda]^\lambda$ consists of pairwise inequivalent elements under E_η. Since L' is countable while λ is regular and uncountable, we can assume that $f(\eta^\frown\langle\alpha\rangle)\approx f(\eta^\frown\langle\beta\rangle) \pmod{M(I_i^-)}$ for all $\alpha,\beta\in\Gamma$. Since $\gamma<\lambda$, we can assume that $j_{\alpha k}=j_{\beta k}$ if one of them is $<\gamma$, for all $\alpha,\beta\in\Gamma$ and all $k<m_\alpha$, and also that if $k<m_\alpha$ and $\operatorname{length}(t_{\alpha k})$ is finite, then $t_{\alpha k}(l)=t_{\beta k}(l)$ if one of them is $<\gamma$, for all $l<\operatorname{length}(t_{\alpha k})$ and all $\alpha,\beta\in\Gamma$. Now by construction, any infinite length $t_{\alpha k}$ has the form η_δ for some δ. Hence, we may assume that $t_{\alpha k}=t_{\beta k}$ if one of them has the form η_δ with $\delta\leqq\gamma$, for all $\alpha,\beta\in\Gamma$, and all $k<m_\alpha$. Now if $\alpha\in\Gamma$ and $k<m_\alpha$ with $t_{\alpha k}$ of infinite length, with some terms $>\gamma$, choose $l_{\alpha k}$ minimum such that $t_{\alpha k}(l_{\alpha k})\geqq\gamma$. We may assume that for all $\alpha,\beta\in\Gamma$, $l_{\alpha k}=l_{\beta k}$ in these circumstances. Then we may assume that in these cases $t_{\alpha k}(l)=t_{\beta k}(l)$ for all $l<l_{\alpha k}$. But then $\alpha E_\eta\beta$ for any two members of Γ, a contradiction. Thus (4) holds.

Now we define a continuous function $\alpha\colon\lambda\to\lambda$: $\alpha0=0$, and $\alpha\delta=\bigcup_{\kappa<\delta}\alpha\kappa$ for δ limit $<\lambda$. Now suppose that $\kappa<\lambda$ and $\alpha\kappa$ has been defined. We define $\beta\colon\omega\to\lambda$ by induction: $\beta0=\alpha\kappa$. Suppose that βj has been defined. For each $\eta\in{}^{\omega>}\beta j$ let $\Gamma_\eta\subseteq\lambda$ have exactly one element from each E_η-class. Thus, $|\Gamma_\eta|<\lambda$ by (4). Set

$$\beta(j+1)=\Big(\bigcup\{\Gamma_\eta\colon\eta\in{}^{\omega>}\beta j\}\cup\beta j\Big)+1\,.$$

Finally, set $\alpha(\kappa+1)=\bigcup_{j\in\omega}\beta_j$. This defines α. For each $\kappa<\lambda$ we have:

(5) $\quad$ for all $0\neq\eta\in{}^{\omega>}\alpha(\kappa+1)$ and all $\beta\in\lambda$ there is a $\gamma<\alpha(\kappa+1)$ such that $\beta E_\eta\gamma$.

Let $C_1=\{\kappa<\lambda\colon \text{for all } j<\kappa,\ \alpha j<\kappa\}$. Thus, C_1 is club in λ. By stationarity of S_i, choose $\delta\in S_i\cap C\cap C_1$. Let $\bar{c}_{f\eta\delta}=\langle(u_0,j_0),\ldots,(u_{k-1},j_{k-1})\rangle$. Thus, $j_l\neq i$ for all $l<k$. Set

$$V=\Big(\bigcup_{l<k}\operatorname{ran}(u_l)\Big)\cup\{j_0,\ldots,j_{k-1}\}\,.$$

Now each u_l with infinite length is strictly increasing with $\sup\neq\delta$, by construction of the I_u's. Hence, $V\cap\delta$ is bounded in δ, say by $\eta_\delta n$. Since $\delta\in C_1$ we have

$\sup_{s<\delta} \alpha s = \delta$. Hence, we can choose $\kappa + 1 < \delta$ so that $\eta_\delta n < \alpha(\kappa + 1) < \delta$. Let m be maximum such that $\eta_\delta m < \alpha(k+1)$. Then by (5) choose $\gamma < \alpha(\kappa + 1)$ so that $\eta_\delta(m+1) E_\nu \gamma$, with $\nu = \eta_\delta \restriction (m+1)$.

We claim that

$$(\eta_\delta \restriction (m+2), \eta_\delta, \nu^\frown\langle\gamma\rangle, \eta_\delta)$$

shows the ψ-unembeddability of I_i into I_i^- via f. Clearly, this quadruple satisfies ψ in I_i. Now we want to show that

$$\langle f(\eta_\delta \restriction (m+2)), f\eta_\delta \rangle \approx \langle f(\nu^\frown\langle\gamma\rangle), f\eta_\delta \rangle (\mathrm{mod}\ M(I_i^-)) .$$

Since $\eta_\delta(m + 1)E_\nu\gamma$, we know that $f(\eta_\delta \restriction (m + 2)) \approx f(\nu^\frown\langle\gamma\rangle)$ (mod $M(I_i^-)$). Let $a = f\eta_\delta$, $b = f(\eta_\delta \restriction (m + 2))$, $d = f(\nu^\frown\langle\gamma\rangle)$. Thus, $\tau_b = \tau_d$, and $\bar{c}_b$ satisfies the same quantifier-free formulas as $\bar{c}_d$ in I. It remains to consider formulas relating a value of $\bar{c}_b$ with one of a and the corresponding formula for $\bar{c}_d$ and a. Say $\bar{c}_b = \langle(v_0, p_0), \ldots, (v_{l-1}, p_{l-1})\rangle$, $\bar{c}_d = \langle(w_0, q_0), \ldots, (w_{l-1}, q_{l-1})\rangle$. Note that $\mathrm{length}(v_x) = \mathrm{length}(w_x)$ since $\bar{c}_b$ and $\bar{c}_d$ realize the same quantifier-free type in I_i^-. Also, $\eta_\delta \restriction (m + 2) \in {}^{\omega>}\delta$, so $\delta \in C$ implies that $\mathrm{orco}(\eta_\delta(m + 2)) < \delta$. Thus, $\mathrm{ran}(v_x) \subseteq \delta$, and similarly $\mathrm{ran}(w_x) \subseteq \delta$. These statements are true for any $x < l$. Now by symmetry it is enough to consider the following cases.

Case 1. $(v_x, p_x) = (u_y, j_y)$. Then $\mathrm{ran}(v_x) \subseteq V \cap \delta$, so for any $s < \mathrm{length}(v_x)$ we have $v_x(s) < \eta_\delta n \leqq \eta_\delta m$, so by $\eta_\delta(m+1)E_\nu\gamma$ we get $v_x(p) = w_x(p)$. Hence, $v_x = w_x$ and $w_x = u_y$. Now $p_x = j_y \in V \cap \delta$, hence $p_x < \eta_\delta n \leqq \eta_\delta m$, so $\eta_\delta(m+1)E_\nu\gamma$ yields $p_x = q_x$. Hence $(w_x, q_x) = (u_y, j_y)$.

Case 2. $(v_x, p_x) \lessdot (u_y, j_y)$. Just like Case 1.

Case 3. $(u_y, j_y) \lessdot (v_x, p_x)$. Similar to Case 1.

Case 4. $(v_x, p_x) < (u_y, j_y)$. Say $\mathrm{length}(v_x) = \mathrm{length}(u_y) = s + 1$. Thus, $v_x \restriction s = u_y \restriction s$, and the argument of Case 1 gives $v_x \restriction s = w_x \restriction s$ and $p_x = q_x$. If $u_y s \leqq w_x s$, then $u_y s < \delta$ and we easily get $v_x s = w_x s$, a contradiction. Hence, $w_x s < u_y s$, so $(w_x, q_x) < (u_y, j_y)$.

Case 5. $(u_y, j_y) < (v_x, p_x)$. Similar to Case 4.

Case 6. $Eq_i((v_x, p_x), (u_y, j_y))$. Clearly, then, $Eq_i((w_x, q_x), (u_y, j_y))$. This completes the proof of Theorem 2.1. □

Now we turn to the construction of BAs from members of K_{tr}. For any $I \in K_{\mathrm{tr}}$ let $B_{\mathrm{tr}}(I)$ be the BA freely generated by $\langle x_\eta : \eta \in I\rangle$ except that $\eta \lessdot \nu$ implies that $x_\nu \leqq x_\eta$. That is, $B_{\mathrm{tr}}(I) = F/I$, where F is the free BA on $\langle x_n : \eta \in I\rangle$ and $I = \langle x_\nu \cdot -x_\eta : \eta \lessdot \nu\rangle^{\mathrm{id}}$, with x_η identified with its equivalence class under I.

2.2. LEMMA. *In* $B_{\mathrm{tr}}(I)$, *if* $\eta_1, \ldots, \eta_m \in I$ *are distinct and* $\xi_1, \ldots, \xi_n \in I$ *are distinct, then* $x_{\eta 1} \cdot \cdots \cdot x_{\eta m} \cdot -x_{\xi 1} \cdot \cdots \cdot -x_{\xi m} = 0$ iff *there exist* i, j *such that* $\xi_j \trianglelefteq \eta_i$.

PROOF. Clearly $\Leftarrow$ holds. For $\Rightarrow$, suppose the implication fails. Then there exist $\nu_1, \rho_1, \ldots, \nu_p, \rho_p \in I$ with $\rho_1 \lessdot \nu_1, \ldots, \rho_p \lessdot \nu_p$ such that, in the free BA,

(6) $$x_{\eta 1}\cdot \cdots \cdot x_{\eta m}\cdot -x_{\xi 1}\cdot \cdots \cdot -x_{\xi n} \leqq x_{\nu 1}\cdot -x_{\rho 1} + \cdots + x_{\nu p}\cdot -x_{\rho p}\,.$$

Let f be the endomorphism of the free BA such that $fx_\sigma = 1$ if $\sigma \leqq \eta_i$ for some i, and $fx_\sigma = 0$ otherwise. Thus, $f(x_{\eta 1}\cdot \cdots \cdot x_{\eta m}\cdot -x_{\xi 1}\cdot \cdots \cdot -x_{\xi n}) = 1$. If $fx_{\nu j} = 1$ then $\nu_j \leqq \eta_i$ for some i; hence $\rho_j \leqq \eta_i$ and $fx_{\rho j} = 1$. So $f(x_{\nu 1}\cdot -x_{\rho 1} + \cdots + x_{\nu p}\cdot -x_{\rho p}) = 0$. This contradicts (6). $\square$

Let $I \in K_{\mathrm{tr}}$ and let B be a BA. We say that B is *representable* in $M(I)$ if there is a function $f\colon B \to M(I)$ such that if $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1} \in B$ and $\langle fa_0, \ldots, fa_{m-1}\rangle \approx \langle fb_0, \ldots, fb_{m-1}\rangle \pmod{M(I)}$, then $\langle a_0, \ldots, a_{m-1}\rangle$ and $\langle b_0, \ldots, b_{m-1}\rangle$ satisfy the same quantifier-free formulas in B.

2.3. LEMMA. *Suppose I is ψ-unembeddable in J and B is a* BA representable in *$M(J)$. Then B_{tr} is not embeddable in a factor of B.*

PROOF. Let $g\colon B \to M(J)$ be a representation of B in $M(J)$, and suppose that h embeds $B_{\mathrm{tr}}(I)$ into a factor of B. Let $f\eta = ghx_\eta$ for all $\eta \in I$. Thus, $f\colon I \to M(J)$. Since I is ψ-unembeddable in J, there exist $\nu_1, \nu_2, \nu \in I$ and $n \in \omega$ such that length$(\eta) = \omega$, length$(\nu_1) =$ length$(\nu_2) = n+1$, $\nu_1 < \eta$, $\nu_1 \restriction n = \nu_2 \restriction n$, $\nu_2(n) < \nu_1(n)$, and $\langle f\nu_1, f\eta\rangle \approx \langle f\nu_2, f\eta\rangle \pmod{M(J)}$. Since g is a representation, $\langle hx_{\nu 1}, hx_\nu\rangle$ satisfies the same quantifier-free formulas as $\langle hx_{\nu 2}, hx_\eta\rangle$. In particular, $hx_\eta \leqq hx_{\nu 1}$ iff $hx_\eta \leqq hx_{\nu 2}$. Since h is an embedding, $x_\eta \leqq x_{\nu 1}$ iff $x_\eta \leqq x_{\nu 2}$. Now $x_\eta \leqq x_{\nu 1}$, so $x_\eta \leqq x_{\nu 2}$, contradicting Lemma 2.2. $\square$

A BA B is called *embedding-rigid* if for all non-zero $a, b \in B$ with $a \nleqq b$, $B \restriction a$ cannot be embedded in $B \restriction b$.

2.4. LEMMA. *If B is embedding-rigid, then B is mono-rigid, hence rigid.*

PROOF. Recall that *mono-rigid* means that there do not exist non-trivial one-to-one endomorphisms. Suppose on the contrary that f is a non-trivial one-to-one endomorphism of B. Say $fx \neq x$. If $x \nleqq fx$, then $f \restriction (A \restriction x)$ embeds $A \restriction x$ into $A \restriction fx$, a contradiction. If $fx \nleqq x$, then $-x \nleqq f(-x)$, and again we get a contradiction. $\square$

We shall use the following general construction for BAs. Let A and B be BAs and $b \in B$. The BA

$$(B \restriction -b) \times ((B \restriction b) * A)$$

is denoted by Att(A, b, B). It is called the *result of attaching A to B at b*, and is considered as a BA extending B.

Let λ be uncountable and regular. By Theorem 2.1, let $\langle I_i\colon i < \lambda\rangle$ attest to the full (λ, λ)-ψ_{tr}-bigness property. Let $\Gamma \in [\lambda]^\lambda$ also be given; say γ is a one-to-one mapping of λ onto Γ. We now construct a sequence $\langle B_i\colon i \leqq \lambda\rangle$ of BAs. Write $\lambda\backslash\{0, 1\} = \bigcup_{0<i<\lambda} \Delta_i$, the Δ_i's pairwise disjoint and of power λ, and let $\Delta_0 = \{0, 1\}$. Set $B_0 = \Delta_0$, a two-element BA. For i limit $\leqq \lambda$, let $B_i = \bigcup_{j<i} B_j$. Now

suppose that B_i has been defined with universe $\bigcup_{j<i} \Delta_j$. Let b_i be the first element of B_i different from $0, 1$ and all b_j, $j<i$. We set $B_{i+1} = \mathrm{Att}(B_{\mathrm{tr}}(I_{\gamma i}), b_i, B_i)$, and we may take it to have universe $\bigcup_{j \leq i} \Delta_j$. This construction depends on Γ and γ, and if necessary we shall indicate this dependence by superscripts, e.g. $B_\lambda^{\Gamma\gamma}$.

2.5. LEMMA. *B_λ is representable in $M(\Sigma_{i<\lambda} I_{\gamma i})$.*

PROOF. With each Boolean term ρ with variables among $v_0, \ldots, v_{m-1}$, v_{m-1} actually occurring in ρ, associate an m-ary operation symbol $\boldsymbol{F}_\rho$ of L_{alg} in a one-to-one fashion.

We define fb for $b \in B_\lambda$ by induction on the first j such that $b \in B_j$. If $j=0$, then $b=0$ or $b=1$; we set $f0 = (\langle\ \rangle, \gamma 0)$, $f1 = (\langle 0 \rangle, \gamma 0)$. Now suppose that $b \neq 0, 1$, with j as indicated. Thus, j is a successor ordinal. By the construction of B_j we can write

$$b = \left(a, \sum_{l<m} d_l \cdot e_l\right),$$

with $c \in B_{j-1} \restriction -b_{j-1}$, $d_l \in B_{j-1} \restriction b_{j-1}$, $e_l \in B_{\mathrm{tr}}(I_{\gamma(j-1)})$ for all $l<m$. For each $l<m$ let $e_l = \rho_l(t_{0l}, \ldots, t_{nl})$, ρ_l a Boolean term involving all of $v_0, \ldots, v_n$, and $t_{0l}, \ldots, t_{nl} \in I_{\gamma(j-1)}$. Let $\boldsymbol{G}$ be a $(2m+3)$-ary operation symbol of L_{alg}. Let the terms $\sigma_0, \ldots, \sigma_{2m+2}$ be obtained from $\tau_{fb(j-1)}, \tau_{f(-b(j-1))}, \tau_{fc}, \tau_{fd0}, \ldots, \tau_{fd(m-1)}$, $\boldsymbol{F}_{\rho 0}(v_0, \ldots, v_n), \ldots, \boldsymbol{F}_{\rho(m-1)}(v_0, \ldots, v_n)$ by simply increasing the indices of the variables so that in the sequence $\sigma_0, \ldots, \sigma_{2m+2}$ the variables form an initial segment of the standard sequence of variables, appearing in order from left to right, and let $\bar{d}$ be

$$\begin{aligned}
&\bar{c}_{fb(j-1)}{}^\frown \bar{c}_{f(-b(j-1))}{}^\frown \bar{c}_{fc}{}^\frown \bar{c}_{fd0}{}^\frown \cdots {}^\frown \bar{c}_{fd(m-1)}{}^\frown \\
&\langle (t_{00}, \gamma(j-1)), \ldots, (t_{n0}, \gamma(j-1)) \rangle^\frown \cdots {}^\frown \\
&\langle (t_{0,m-1}, \gamma(j-1)), \ldots, (t_{n,m-1}, \gamma(j-1)) \rangle .
\end{aligned}$$

Finally, let $fb = (\boldsymbol{G}(\sigma_0, \ldots, \sigma_{2m+2}))(\bar{d})$. This finishes the definition of f.

Now suppose that $a_0, \ldots, a_{p-1}, c_0, \ldots, c_{p-1}$ are elements of B_λ and

$$(7) \qquad \langle fa_0, \ldots, fa_{p-1} \rangle \approx \langle fc_0, \ldots, fc_{p-1} \rangle \left(\operatorname{mod} M\left(\sum_{i<\lambda} I_{\gamma i}\right)\right).$$

Assume that

$$(8) \qquad a_0 \cdot \cdots \cdot a_h \cdot -a_{h+1} \cdot \cdots \cdot -a_{p-1} = 0 .$$

We want to show that

$$(9) \qquad c_0 \cdot \cdots \cdot c_h \cdot -c_{h+1} \cdot \cdots \cdot -c_{p-1} = 0 .$$

If $a_i = 0$, then fa_i is $(\langle\ \rangle, \gamma 0)$, and $(\langle\ \rangle, \gamma 0) \in z$, in $M(\Sigma_{i<\lambda} I_{\gamma i})$. So (7) yields that $fc_i = fa_i$ and $c_i = 0$. Similarly, $a_i = 1$ implies $c_i = 1$. So we may assume that each a_i is $\neq 0, 1$.

Suppose that j is minimum such that for all $v < p$, $a_v \in B_j$; we proceed by induction on j. Say for $v < p$

$$(10) \qquad a_v = \left(a'_v, \sum_{l<mv} a''_{vl} \cdot a'''_{vl} \right),$$

with $a'_v \in B_{jv-1} \restriction -b_{jv-1}$, $a''_{vl} \in B_{jv-1} \restriction b_{jv-1}$, $a'''_{vl} \in B_{\mathrm{tr}}(I_{\gamma(jv-1)})$, $jv \leqq j$. Similarly, we assume that all $c_v \neq 0$, 1, and s is minimum such that for all $v < p$, $c_v \in B_s$. Say for $v < p$

$$(11) \qquad c_v = \left(c'_v, \sum_{l<nv} c''_{vl} \cdot c'''_{vl} \right),$$

with $c'_v \in B_{sv-1} \restriction -b_{sv-1}$, $c''_{vl} \in B_{sv-1} \restriction b_{sv-1}$, $c'''_{vl} \in B_{\mathrm{tr}}(I_{\gamma(sv-1)})$, $sv \leqq s$. Note by (7) that $m_v = n_v$ for all $v < p$. Applying (8) to the first coordinates in (10) we get

$$a'_0 \cdot \cdots \cdot a'_h \cdot -a'_{h+1} \cdot \cdots \cdot -a'_p \cdot -b_{j(h+1)-1} \cdot \cdots \cdot -b_{jp-1} = 0 .$$

Now (7) yields that

$$\begin{aligned} &\langle fa'_0, \ldots, fa'_{p-1}, f(-b_{j(h+1)-1}), \ldots, f(-b_{jp-1}) \rangle \\ &\quad \approx \langle fc'_0, \ldots, fc'_{p-1}, f(-b_{s(h+1)-1}), \ldots, f(-b_{sp-1}) \rangle \left(\mathrm{mod} \left(\sum_{i<\lambda} I_{\gamma i} \right) \right), \end{aligned}$$

so the induction hypothesis gives:

$$(12) \qquad c'_0 \cdot \cdots \cdot c'_h \cdot -c'_{h+1} \cdot \cdots \cdot -c'_p \cdot -b_{s(h+1)-1} \cdot \cdots \cdot -b_{sp-1} = 0 .$$

Now we proceed to the second coordinates. Suppose $g \in \Pi_{v<h} m_v$ and $\Delta \in \Pi_{h \leqq v<p} \mathscr{P} m_v$. Then by (8), (10),

$$\prod_{v<h} a''_{v,gv} \cdot a'''_{v,gv} \cdot \prod_{h \leqq v<n} \prod_{l \in \Delta v} - a''_{vl} \cdot b_{jv-1} \cdot \prod_{l \in mv/\Delta v} - a'''_{vl} = 0 ,$$

so by the free product property one of the following holds:

$$(13) \qquad \prod_{v<h} a''_{v,gv} \cdot \prod_{h \leqq v<n} \prod_{l \in \Delta v} - a'''_{vl} \cdot b_{jv-1} = 0 ,$$

$$(14) \qquad \prod_{v<h} a'''_{v,gv} \cdot \prod_{h \leqq v<n} \prod_{l \in mv \setminus \Delta v} - a'''_{vl} = 0 .$$

In the case where (13) holds, the argument for (12) gives

$$(15) \qquad \prod_{v<h} c''_{v,gv} \cdot \prod_{h \leqq v<n} \prod_{l \in \Delta v} - c''_{vl} \cdot b_{sv-1} = 0 .$$

If (14) holds, we use (7) to see that a'''_{vl} and c'''_{vl} are expressed by the same Boolean term applied to sequences satisfying the same quantifier-free Boolean formulas, using Lemma 2.2. So

$$(16)\qquad \prod_{v<h} c'''_{v,gv} \cdot \prod_{h\leqq v<n} \prod_{l\in mv/\Delta v} -c'''_{vl} = 0$$

By (12) and all instances of (15) or (16), (9) follows. □

2.6. LEMMA. *For any $i<\lambda$, $B_\lambda \restriction -b_i$ is representable in $M(\Sigma_{i\neq j\in\lambda} I_{\gamma j})$.*

PROOF. The proof is very similar to that of Lemma 2.5, and we just sketch it. We define fb for $b\in(B_\lambda \restriction -b_i)$ by induction on the first j such that $b=b'\cdot -b_i$ for some $b'\in B_j$. The case $j=0$ is treated as before. In the main step, we are assured that $j-1\neq i$, since otherwise $b'=c\in B_{j-1}$, contradicting the minimality of j. This assures that f maps into $M(\Sigma_{i\neq j\in\lambda} I_{\gamma j})$. The rest of the proof proceeds as before. □

Now we can give the main theorem.

2.7. THEOREM. *For each regular uncountable λ there is a family of 2^λ embedding-rigid BAs of power λ, none embeddable in a factor of another.*

PROOF. Let $\mathscr{A}$ be a family of 2^λ subsets of λ, each of power λ and none a subset of another. For each $\Gamma\in\mathscr{A}$ let B^Γ_λ be as above. Then $\langle B^\Gamma_\lambda\colon \Gamma\in\mathscr{A}\rangle$ is as desired. In fact, first suppose that $\Gamma,\Delta\in\mathscr{A}$ and $\Gamma\neq\Delta$. To show that B^Γ_λ cannot be embedded in a factor of B^Δ_λ, choose $i\in\Gamma\backslash\Delta$. Let γ be the one-to-one function mapping λ onto Δ used in the construction of B^Δ_λ. Since I_i is ψ-unembeddable in Γ_i, it is clearly ψ-unembeddable in $\Sigma_{j<\lambda} I_{\gamma j}$. By Lemma 2.5, B^Δ_λ is representable in $M(\Sigma_{j<\lambda} I_{\gamma j})$, so by Lemma 2.3, $B_{\mathrm{tr}}(I_i)$ is not embeddable in a factor of B^Δ_λ. Hence B^Γ_λ is not embeddable in a factor of B^Δ_λ.

Finally, fix $\Gamma\in\mathscr{A}$; we show that $B^\Gamma_\lambda=B_\lambda$ is embedding-rigid. Suppose $a\nleqq b$, but f embeds $B_\lambda \restriction a$ into $B_\lambda \restriction b$. Let $c=a\cdot -b$. Then $c\cdot fc=0$, and f embeds $B_\lambda \restriction c$ into $B_\lambda \restriction fc$. Write $c=b_i$. Then $B_{\mathrm{tr}}(I_i)$ is embeddable in $B_\lambda \restriction c$, hence in $B \restriction fc$, hence in $B \restriction -c = B \restriction -b_i$. By Lemma 2.6, $B \restriction -b_i$ is representable in $M(\Sigma_{i\neq j<\lambda} I_j)$. This contradicts Lemma 2.3. □

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CHAPTER 13

Endomorphisms of Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

We describe in this chapter the main results concerning endomorphisms of Boolean algebras, giving proofs for some of them. We divide the discussion into four parts.

(1) Reconstruction: the very satisfying result here is that if A and B have isomorphic endomorphism semigroups, then A and B are isomorphic.

(2) Number of endomorphisms: it is clear that $|\text{Ult } A| \leq |\text{End } A|$, where End A is the semigroup of endomorphisms of A. The main question here seems to be to construct BAs A with $|A| = |\text{End } A|$ of various sizes or with additional properties. We give some easy results, and some questions, along these lines.

(3) Endo-rigid BAs: A BA A is endo-rigid if, roughly speaking, it has no endomorphisms except the obvious ones. We show that there *is* one of power continuum.

(4) Hopfian BAs: a BA A is *hopfian* if every onto endomorphism of A is one-to-one. We show that there is an atomic BA of power continuum, and give some results concerning hopfian BAs of other powers.

1. Reconstruction

The main theorem here, mentioned in the introduction to this article, was proved by Schein [1970]. Magill [1970] proved an analogous theorem for Boolean rings, where the endomorphisms preserve Δ and $\cdot$ but not necessarily 1. His proof is topological. Maxson [1972] proved Magill's theorem in a direct algebraic way. We follow Schein's proof. His result applies to rather general ordered sets, and for BAs gives a result, Theorem 1.2, which is stronger than mere reconstruction. In fact, reconstruction is possible just from two- and four-valued endomorphisms.

A subsemigroup S of End A is *sufficient* if the following conditions hold:

(1) For any two distinct $a, b \in A$ there is a two-valued $f \in \text{End } A$ such that $fa \neq fb$.

(2) If $f_1, f_2, f_3, f_4 \in S$ and $f_1 \neq f_2$, and all are two-valued, then there is a $g \in S$ such that $f_1 \circ g = f_3$ and $f_2 \circ g = f_4$.

(3) For each $a \in A$ there is a four-valued $f \in S$ such that f takes on the value a.

Now we first note that End A itself is sufficient:

1.1. Lemma. *If* $|A| > 2$, *then* End A *is sufficient.*

PROOF. Condition (1) is obvious. Recall that if I and J are distinct maximal ideals on A then $|A/(I \cap J)| = 4$. Hence (3) is clear. Now assume the hypothesis of (2). If $f_3 = f_4$, then we can take $g = f_3$, since $f_1 \circ f_3 = f_3 = f_2 \circ f_3$. So, assume that $f_3 \neq f_4$. Let the kernels of f_1, f_2, f_3, and f_4 be I_1, I_2, I_3, and I_4, respectively. Choose $a \in I_1 \backslash I_2$ and $b \in I_3 \backslash I_4$. Since $|A/(I_3 \cap I_4)| = 4$, there is an endomorphism g of A with range $\{0, a, -a, 1\}$ and kernel $I_3 \cap I_4$ such that $gb = a$. Note that $x \in I_3 \cap I_4 \Rightarrow gx = 0$, $x \in I_3 \backslash I_4 \Rightarrow gx = a$, $x \in I_4 \backslash I_3 \Rightarrow gx = -a$, and $x \in A \backslash (I_3 \cup I_4) \Rightarrow gx = 1$. Hence, it is clear that $f_1 \circ g = f_3$ and $f_2 \circ g = f_4$, as desired. □

1.2. THEOREM. *Let A and B be BAs, and S and T sufficient semigroups of endomorphisms of A and B, respectively. Suppose that p is a homomorphism of S onto T. Then either $|B| \leqq 2$, or $|B| > 2$ and p is an isomorphism induced by an isomorphism of A onto B.*

PROOF. Let S_2 and S_4 be the set of all two-valued and four-valued members of S, respectively, and similarly define T_2 and T_4. Assume that $|B| > 2$. Recall that an element u of a semigroup U is a *right zero* if $v \cdot u = u$ for all $v \in U$. Now we claim

(4) $\quad S_2$ is the set of all right zeros of S .

For, clearly every member of S_2 is a right zero. Now let f be a right zero of S. Let $g \in S$ be two-valued; there is such a g by (1). Then $g \circ f = f$ and $g \circ f$ is two-valued, so $f \in S_2$. Thus, (4) holds. Similarly

(5) $\quad T_2$ is the set of all right zeros of T .

Next we claim

(6) $\quad p[S_2] = T_2$.

In fact, since p maps onto T it is clear that pf is a right zero of T for each $f \in S_2$, using (4). Thus $\subseteq$ holds. Now let $g \in T_2$; say $pf = g$. Choose $h \in S_2$, using (1). Then $p(h \circ f) = ph \circ pf = ph \circ g$; $ph \circ g = g$ since g is two-valued, and $h \circ f$ is two-valued, so $g \in p[S_2]$, as desired.

Next, by (1) and $|B| > 2$ we have

(7) $\quad |T_2| \geqq 2$.

(8) $\quad p \restriction S_2$ is one-to-one .

For, suppose f_1, f_2 are distinct members of S_2, and $pf_1 = pf_2$. Let $f_3, f_4 \in S_2$; we show that $pf_3 = pf_4$; by (6) and (7), this is not possible. By (2), choose $g \in S$ so that $f_1 \circ g = f_3$ and $f_2 \circ g = f_4$. Applying p, we do then obtain $pf_3 = pf_4$, as claimed.

(9) $\quad h \in S_4$ iff $|S_2 \circ h| = 2$, for any $h \in S$.

For, let $h \in S_4$; say $\operatorname{ran}(h) = \{0, a, -a, 1\}$. By (1) there exist $f_1, f_2 \in S_2$ with

$f_1 a \neq 0$, $f_2 a \neq 1$, hence $f_1 a \neq f_2 a$. Thus, $f_1 \circ h \neq f_2 \circ h$. Clearly, $|S_2 \circ h| \leq 2$, so $|S_2 \circ h| = 2$. Conversely, suppose $h \in S$ and $|S_2 \circ h| = 2$. Since $S_2 \circ k = \{k\}$ for any two-valued k, it follows that h is not two-valued. Suppose $|\text{ran}(h)| > 4$. Then there exist $0 < a_1 < a_2 < 1$ in $\text{ran}(h)$. By (1), choose two-valued f_1, f_2, and f_3 such that $f_1 a_1 \neq f_1 0$, $f_2 a_1 \neq f_2 a_2$, and $f_3 a_2 \neq f_3 1$. Thus $f_1 a_1 = 1$, $f_2 a_1 = 0$, $f_2 a_2 = 1$, and $f_3 a_2 = 0$. Hence, $f_1 \circ h$, $f_2 \circ h$, and $f_3 \circ h$ are three distinct members of $S_2 \circ h$, a contradiction. So $h \in S_4$, as desired.

Similarly,

(10) $\quad h \in T_3$ iff $|T_2 \circ h| = 2$, for any $h \in T$.

From (6)–(10), using also $S_2 \circ h \subseteq S_2$ and $T_2 \circ h \subseteq T_2$, we obtain:

(11) $\quad p[S_4] = T_4$,

(12) $\quad p \restriction S_4$ is one-to-one.

For, suppose $f_1, f_2 \in S_4$ and $pf_1 = pf_2$. Then for any $g \in S_2$ we have $p(g \circ f_1) = pg \circ pf_1 = pg \circ pf_2 = p(g \circ f_2)$, hence $g \circ f_1 = g \circ f_2$ by (8). Then from (1) it follows that $f_1 = f_2$.

(13) $\quad$ If $a_1, a_2 \in A$, $0 < a_1 < 1$, $a_2 \not\in \{0, a_1, -a_1, 1\}$, then there exist $g_1, g_2 \in S_2$ such that $g_1 a_1 = g_2 a_1$ and $g_1 a_2 \neq g_2 a_2$.

To prove (13) we consider three cases.

Case 1. $a_1 < a_2$. Then $a_1 + -a_2 \neq 1$. Choose $g_1, g_2 \in S_2$ so that $g_1 a_2 = 0$, $g_2(a_1 + -a_2) = 0$.

Case 2. $a_1 < -a_2$. Similarly.

Case 3. $a_1 \cdot -a_2 \neq 0 \neq a_1 \cdot a_2$. Choose $g_1, g_2 \in S_2$ so that $g_1(a_1 \cdot -a_2) = 1 = g_2(a_1 \cdot a_2)$.

Similarly we have

(14) $\quad$ If $b_1, b_2 \in B$, $0 < b_1 < 1$, $b_2 \not\in \{0, b_1, -b_1, 1\}$, then there exist $g_1, g_2 \in T_2$ such that $g_1 b_1 = g_2 b_1$ and $g_1 b_2 \neq g_2 b_2$.

(15) $\quad$ If $g \in S_4$, $\text{ran}(g) = \{0, a, -a, 1\}$, $\text{ran}(pg) = \{0, b, -b, 1\}$, and $f_1, f_2 \in S_2$, then $f_1 a = f_2 a$ iff $(pf_1)b = (pf_2)b$.

For, using (8) we have $f_1 a = f_2 a$ iff $f_1 \circ g = f_2 \circ g$ iff $p(f_2 \circ g) = p(f_2 \circ g)$ iff $pf_1 \circ pg = pf_2 \circ pg$ iff $(pf_1)b = (pf_2)b$.

(16) $\quad$ If $g_1, g_2 \in S_4$, then $\text{ran}(g_1) = \text{ran}(g_2)$ iff $\text{ran}(pg_1) = \text{ran}(pg_2)$.

For $\Rightarrow$, suppose $\text{ran}(g_1) = \text{ran}(g_2) = \{0, a, -a, 1\}$, $\text{ran}(pg_1) = \{0, b, -b, 1\}$, and $c \in \text{ran}(pg_2)$ with $c \not\in \{0, b, -b, 1\}$. By (14), choose $f'_1, f'_2 \in T_2$ such that $f'_1 b = f'_2 b$ and $f'_1 c \neq f'_2 c$. Say $pf_1 = f'_1$, $pf_2 = f'_2$, with $f_1, f_2 \in S_2$. Then $f_1 a = f_2 a$ by (15), and then $f'_1 c = f'_2 c$ by (15), a contradiction. The direction $\Leftarrow$ is similar.

Now we are ready to define the desired isomorphism l from A onto B. Of course, we let $l0 = 0$ and $l1 = 1$. Now suppose $0 < a < 1$ in A. By (3) there is a four-valued $h \in S_4$ such that $a \in \mathrm{ran}(h)$. By (11) say $\mathrm{ran}(ph) = \{0, b, -b, 1\}$. Let $f \in S_2$ be such that $fa \neq 0$. By (6) we have $(pf)b = 0$ or $(pf)b = 1$, and we set $la = -b$ or $la = b$ in these respective cases. By (15) this definition does not depend on the choice of f, and by (16) it does not depend on the choice of h.

(17) $\quad l(-a) = -la$ for any $a \in A$.

In fact, (17) is obvious for $a \in \{0, 1\}$. Now suppose that $0 < a < 1$. Let $h \in S_4$ with $a \in \mathrm{ran}(h)$, and say $\mathrm{ran}(ph) = \{0, b, -b, 1\}$. Let $f_1, f_2 \in S_2$ with $f_1 a \neq 0, f_2(-a) \neq 0$. Then by (15) we have $(pf_1)b \neq (pf_2)b$. It follows that $l(-a) = -la$.

Now l maps onto B: suppose $0 < b < 1$ in B. Choose $f' \in T_4$ such that $b \in \mathrm{ran}(f')$, and choose $f \in S_4$ such that $pf = f'$. Say $\mathrm{ran}\, f = \{0, a, -a, 1\}$. Then using (17), we have $la = b$ or $l(-a) = b$.

By (16) and (17) it is clear that l is one-to-one.

Now we show that $a_1 \leqq a_2$ iff $la_1 \leqq la_2$, for any $a_1, a_2 \in A$. First suppose that $a_1 \cdot -a_2 \neq 0$. Choose $f \in S_2$ such that $f(a_1 \cdot -a_2) = 1$. Then by the definition of l, $(pf)(la_1) = 1$ and $(pf)(l(-a_2)) = 1$, i.e. by (17) $(pf)(la_2) = 0$. Hence, $la_1 \cdot -la_2 \neq 0$. Conversely, suppose $la_1 \cdot -la_2 \neq 0$. Say $f' \in T_2$ and $f'(la_1 \cdot -la_2) = 1$. Say $f' = pf$ with $f \in S_2$. If $fa_1 = 0$, then $f(-a_1) = 1$ and $(pf)(la_1) = f'la_1 = 1$, so $l(-a_1) = la_1$, contradicting (17). Thus, $fa_1 = 1$. Similarly, $fa_2 = 0$, so $a_1 \cdot -a_2 \neq 0$.

Thus, l is an isomorphism from A onto B. It remains only to show that l induces p, i.e.

(18) $\quad$ for any $f \in S$, and any $a \in A$ we have $(pf)(la) = lfa$.

For, if $a \in \{0, 1\}$, then the conclusion of (18) is clear. Assume that $0 < a < 1$. Let $h \in S_4$ with $a \in \mathrm{ran}(h)$. Suppose $fa = 1$, but $(pf)(la) = 0$. Let $g' \in T_2$; say $g' = pg$ with $g \in S_2$. Then $(p(g \circ f))la = 0$, and $(g \circ f)a = 1$, so by the definition of l, $la = -la$, a contradiction. Thus, $fa = 1$ implies $(pf)(la) = 1$. If $fa = 0$, then $f(-a) = 1$, hence $(pf)(l(-a)) = 1$ and $(pf)(la) = 0$. Now suppose $0 < fa < 1$. Suppose $(pf)(la) \neq lfa$. Then there is a $g \in S_2$ with $(pg)(pf)(la) \neq (pg)(lfa)$.

Case 1. $gfa = 1$. Then by the definition of la, $(p(g \circ f))(la) = 1$, i.e. $(pg)((pf)(la)) = 1$. By the definition of lfa, $(pg)(lfa) = 1$, a contradiction.

Case 2. $gfa = 0$. Then, as in case 1, $(pg)((pf)(l(-a))) = (pg)(lf(-a))$, a contradiction. □

1.3. COROLLARY. *If A and B are non-trivial and* End $A \cong$ End B, *then* $A \cong B$.

PROOF. For any BA C, $|C| \leqq 2$ iff $|\mathrm{End}\ C| = 1$, so the corollary follows from 1.1 and 1.2. □

Reconstruction does not work for automorphism groups, since, for example, there are rigid BAs of all uncountable cardinalities. Perhaps there is some natural kind of endomorphisms other than the entire semigroup End A or the two- and four-valued endomorphisms which still yields reconstruction. Note that the class of all one-to-one endomorphisms is ruled out by the existence of many mono-rigid BAs; similarly, the class of all onto endomorphisms is ruled out by the existence of many onto-rigid BAs.

2. Number of endomorphisms

Since $|A| \leqq |\text{Ult } A| \leqq |\text{End } A|$, one of the most natural questions about the number of endomorphisms is: In which cardinalities κ does there exist a BA A with $A| = |\text{End } A| = \kappa$? A complete answer to this question is not known, but we present some simple results concerning it.

2.1. THEOREM. *Suppose L is a complete dense linear ordering of power $\lambda \geqq \omega$, and D is a dense subset of L of power κ, where $\lambda^\kappa = \lambda$. Let A be the interval algebra on L. Then $|A| = |\text{End } A| = \lambda$.*

PROOF. Ult A is a linearly-ordered space with a dense subspace of power κ, while $|\text{Ult } A| = \lambda$. Hence, there are at most $\lambda^\kappa = \lambda$ continuous functions from Ult A into Ult A, as desired. □

2.2. COROLLARY. *If A is the interval algebra on $\boldsymbol{R}$, then $|A| = |\text{End } A| = 2^\omega$.* □

Now recall that if μ is any infinite cardinal and ν is minimum such that $\mu^\nu > \mu$, then there is a complete dense linear ordering L of power μ^ν with a dense subset D of power μ. Namely, we can take

$$L = {}^\nu(\mu+1) \backslash \{f \in {}^\nu(\mu+1) \colon \exists \beta < \nu(f_\beta < \mu \text{ and } \\ \forall \gamma > \beta(f_\gamma = \mu))\}$$

and

$$D = \{f \in {}^\nu(\mu+1) \colon \exists \beta < \nu(f_\beta = \mu \text{ and } \forall \gamma > \beta(f_\gamma = 0))\}$$

under the lexicographic ordering. Hence, we obtain:

2.3. COROLLARY. *If μ is an infinite cardinal and $\forall \nu < \mu(\mu^\nu = \mu)$, then there is a BA A such that $|A| = |\text{End } A| = 2^\mu$.* □

2.4. COROLLARY (GCH). *If κ is infinite and regular, then there is a BA A such that $|A| = |\text{End } A| = \kappa^+$.* □

We do not know, even under GCH, the situation for κ singular or the successor of a singular cardinal.

The following simple result is relevant when CH is not assumed.

2.5. THEOREM. *If $|A| = \omega$, then $|\text{End } A| \geqq 2^\omega$.*

PROOF. If A has an atomless subalgebra, clearly $|\text{End } A| \geqq |\text{Ult } A| \geqq 2^\omega$. Suppose A is superatomic. Then there is a homomorphism f from A onto B, the finite–cofinite algebra on ω: if a is an atom of $A/\langle \text{At } A\rangle^{\text{id}}$, then f can be taken to be the composition of the natural mappings

$$A \twoheadrightarrow A \restriction a \rightarrowtail\!\!\!\!\twoheadrightarrow C \twoheadrightarrow B\ ,$$

where C is the finite–cofinite algebra on ω or ω_1. ("$\twoheadrightarrow$" means "onto", and "$\rightarrowtail\!\!\!\!\twoheadrightarrow$" means "one-to-one and onto".) There is an isomorphism g of B into A. If X is any subset of ω with $\omega\backslash X$ infinite, then $B/\langle\{i\}: i \in X\rangle^{\text{id}}$ is isomorphic to B, and so there is an endomorphism k_X of B with kernel $\langle\{i\}: i \in X\rangle^{\text{id}}$. Clearly, the endomorphisms $g \circ k_X \circ f$ of A are distinct for distinct X's. □

2.6. COROLLARY ($\omega_1 < 2^\omega$). *There is no BA A with $|A| = |\text{End } A| = \omega_1$.* □

Another unresolved question concerning $|\text{End } A|$ is whether $|\text{End } A| \leqq |\text{Sub } A|$ for all infinite A, where Sub A is the set of all subalgebras of A.

3. Endo-rigid algebras

Recall that a BA is *rigid* if it has no automorphisms except the identity. In general algebra a stronger rigidity has been extensively studied: an algebra is *strongly rigid* if it has no endomorphism except the identity. Now a BA with more than two elements is not strongly rigid in this sense. For, if $|A| > 2$ and F is an ultrafilter on A, then $|A/F| = 2$, and there is a natural embedding of A/F into A. Then $A \to A/F \to A$ gives a non-trivial endomorphism of A. Using this simple fact, one can build more complicated endomorphisms. For example, if $\langle a, b, c\rangle$ is a partition of unity in A, F is an ultrafilter on $A \restriction b$, and G is an ultrafilter on $A \restriction c$, then the composition of natural maps,

$$\begin{aligned} A &\to (A \restriction a) \times (A \restriction b) \times (A \restriction c) \\ &\to (A \restriction a) \times [(A \restriction b)/F] \times [(A \restriction c)/G] \\ &\to (A \restriction a) \times [(A \restriction c)/F] \times [(A \restriction b)/G] \\ &\to (A \restriction a) \times (A \restriction b) \times (A \restriction c) \\ &\to A\ , \end{aligned}$$

gives an endomorphism f of A defined by

$$fx = \begin{cases} x \cdot a + b + c \text{ if } x \cdot b \in F \text{ and } x \cdot c \in G\ , \\ x \cdot a + b \text{ if } x \cdot b \not\in F \text{ and } x \cdot c \in G\ , \\ x \cdot a + c \text{ if } x \cdot b \in F \text{ and } x \cdot c \not\in G\ , \\ x \cdot a \text{ if } x \cdot b \not\in F \text{ and } x \cdot c \not\in G\ . \end{cases}$$

It is possible to describe the most general kind of endomorphism constructible from ultrafilters in the above way; we do this below. A BA A is called *endo-rigid* if it has only endomorphisms of this sort. One can give a definition of endo-rigid which is not so complicated, and we now do this. Only this simpler definition is actually used later.

If f is an endomorphism of A, then the *extended kernel of* f is

$$\operatorname{exker} f = \{a + b\colon fa = 0 \text{ and } fx = x \text{ for all } x \leqq b\}\,.$$

Clearly $\operatorname{exker} f$ is an ideal of A.

Two ideals I and J of A are *complementary* if $I \cap J = \{0\}$, while $\langle I \cap J\rangle^{\mathrm{id}}$ is a maximal ideal of A.

We call A *endo-rigid if* it satisfies the following three conditions:

(19) A is atomless ,

(20) for every endomorphism f of A, $A/\operatorname{exker} f$ is finite ,

(21) A does not have any pair of non-principal complementary ideals .

Endo-rigid BAs are rigid in the usual sense, and possess even stronger rigidity properties, as we shall see. This notion is due to SHELAH [1979], where their existence, assuming CH, is proved. This assumption was removed in MONK [1980]. The strongest result is in SHELAH [1984]: for any $\lambda > \omega$ there is an endo-rigid ccc BA of power λ^{ω}. Our notation here differs from that in these articles.

Now we want to give the equivalent definition involving endomorphisms obtained from ultrafilters. An *endomorphism schema* for A is a sequence

$$\langle a_0, a_1, b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}, b_0^*, \ldots, b_{m-1}^*, c_0^*, \ldots, c_{n-1}^*,$$
$$I_0, \ldots, I_{m-1}, J_0, \ldots, J_{n-1}\rangle$$

such that the following conditions hold:

(22) $a_0, a_1, b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}$ are pairwise disjoint elements with sum 1; $b_i \neq 0 \neq c_j$ for all $i < m$, $j < n$,

(23) $b_0^*, \ldots, b_{m-1}^*, c_0^*, \ldots, c_{n-1}^*$ are pairwise disjoint non-zero elements with sum $a_0 + b_0 + \cdots + b_{m-1}$,

(24) for all $i < m$, I_i is a maximal ideal in $A \upharpoonright b_i$,

(25) for all $i < n$, J_i is a maximal ideal in $A \upharpoonright c_i$.

3.1. LEMMA. *Given an endomorphism schema as above, there is a unique endomorphism f of A with the following properties*:

(i) *For all* $x \leqq a_0$, $fx = 0$.
(ii) *For all* $i < m$ *and all* $x \in I_i$, $fx = 0$.
(iii) *For all* $x \leqq a_1$, $fx = x$.
(iv) *For all* $j < n$ *and all* $x \in J_j$, $fx = x$.
(v) *For all* $i < m$, $fb_i = b_i^*$.
(vi) *For all* $j < n$, $fc_j = c_j + c_j^*$.

PROOF. The following composition of homomorphisms clearly gives an endomorphism satisfying (i)–(vi):

$$\begin{aligned} A &\to (A \restriction a_0) \times (A \restriction a_1) \times \prod_{i<m} (A \restriction b_i) \times \prod_{j<n} (A \restriction c_j) \\ &\to (A \restriction a_1) \times \prod_{i<m} [(A \restriction b_i)/I_i] \times \prod_{j<n} (A \restriction c_j) \times \prod_{j<n} [(A \restriction c_j)/J_j] \\ &\to (A \restriction a_1) \times \prod_{i<m} (A \restriction b_i^*) \times \prod_{j<n} (A \restriction c_j) \times \prod_{j<n} (A \restriction c_j^*) \\ &\to A\,. \end{aligned}$$

Note here that $A \restriction c_j \to (A \restriction c_j) \times [(A \restriction c_j)/J_j]$ via the mapping sending x to $(x, x/J_j)$.

For uniqueness, note that if f is an endomorphism of A satisfying (i)–(vi), then f is uniquely determined on each factor $A \restriction a_0, A \restriction a_1, A \restriction b_0, \ldots, A \restriction b_{m-1}$, and $A \restriction c_0, \ldots, A \restriction c_{n-1}$, hence f itself is uniquely determined. In fact, $fx = 0$ for all $x \leqq a_0$, and $fx = x$ for all $x \leqq a_1$. Suppose $i < m$ and $x \leqq b_i$; if $x \in I_i$ then $fx = 0$, while if $x \not\in I_i$ then $b_i \cdot -x \in I_i$, hence $f(b_i \cdot -x) = 0$, and $fx = f(b_i \cdot -(b_i \cdot -x)) = fb_i = b_i^*$. Finally, suppose $j < n$ and $x \leqq c_j$. If $x \in J_j$, then $fx = x$, while if $x \not\in J_j$, then $c_j \cdot -x \in J_j$, hence $f(c_j \cdot -x) = c_j \cdot -x$, and

$$fx = f(c_j \cdot -(c_j \cdot -x)) = (c_j + c_j^*) \cdot -(c_j \cdot -x) = x + c_j^*\,. \quad \square$$

If f is as described in Lemma 3.1, we say that f is *determined* by the given schema.

3.2. THEOREM. *For any atomless BA A, A is endo-rigid* iff *every endomorphism of A is determined by some endomorphism schema.*

PROOF. $\Rightarrow$ Let f be any endomorphism of A. First suppose exker $f = A$. Then we can write $1 = a + b$ with $fa = 0$ and $fx = x$ for all $x \leqq b$. Thus, $a \cdot b = 0$, so $b = -a$. Hence, $b = fb = f(-a) = -fa = 1$, so $a = 0$. Thus, f is the identity on A, determined by the endomorphism schema $\langle 0, 1\rangle$.

If exker $f \neq A$, the assumption that A is endo-rigid yields that $A/\text{exker } f$ is a finite non-trivial BA. Hence, there is a partition $\langle x_0, \ldots, x_{k-1}\rangle$ of A such that $x_0/\text{exker } f, \ldots, x_{k-1}/\text{exker } f$ are the atoms of $A/\text{exker } f$. Fix $l < k$. We show how x_l yields finitely many parts of our desired endomorphism schema. Let

$$I_0 = \{y \leqq x_l \colon fy = 0\}\,,$$

$$I_1 = \{y \leqq x_l \colon fz = z \text{ for all } z \leqq y\}\,,$$

$$J = (\text{exker } f) \cap (A \restriction x_l)\,.$$

Then $I_0 \cap I_1 = \{0\}$, and $I_0 \cup I_1$ generates the maximal ideal J in $A \restriction x_l$. Thus, I_0 and I_1 are complementary ideals in $A \restriction x_l$. If I_0 and I_1 are non-principal, then I_0 and $\{a \in A\colon a \cdot x_l \in I_1\}$ are non-principal complementary ideals in A, contradicting A endo-rigid. Thus, I_0 is principal, or I_1 is. If they are both principal, then J is also, and so A has an atom, a contradiction. Thus, exactly one of I_0, I_1 is principal. We treat two cases separately.

Case 1. I_1 is principal, say generated by e. Since $x_l \not\in$ exker f, we have $x_l \cdot -e \neq 0$. We let e be a part of a_1, $x_l \cdot -e$ be one of the b_i, $b_i^* = fb_i$, and $I_i = (\text{exker } f) \cap (A \restriction (x_l \cdot -e))$. Note that I_i is a maximal ideal in $A \restriction (x_l \cdot -e)$ and $fy = 0$ for all $y \in I_i$.

Case 2. I_0 is principal, say generated by e. We let e be a part of a_0. Clearly, now

(26) $\quad$ for all $y \leqq x_l \cdot -e$, if $y \in$ exker f, then $y \in I_1$,

(27) $\quad$ $x_l \cdot -e \leqq f(x_l \cdot -e)$.

For, otherwise let $y = x_l \cdot -e \cdot -f(x_l \cdot -e)$; thus $y \neq 0$. Then $y \cdot fy = 0$, so $y \not\in I_1$. Hence, $y \not\in$ exker f by (26). Therefore $x_l \cdot -y \in$ exker f, so $x_l \cdot -y \cdot -e \in I_1$ by (26). Since I_1 is non-principal, we can choose z with $x_l \cdot -y \cdot -e < z \in I_1$. Then $z \leqq x_l \cdot -e$ and $fz = z$, so

$$z \leqq x_l \cdot -e \cdot f(x_l \cdot -e) = x_l \cdot -e \cdot -y ,$$

a contradiction. So (27) holds.

(28) $\quad$ $x_l \cdot -e \neq 0$.

This is true since $x_l \not\in$ exker f.

(29) $\quad$ $f(x_l \cdot -e) \cdot -(x_l \cdot -e) \neq 0$.

For, suppose (29) fails. Thus, by (27) we have $f(x_l \cdot -e) = x_l \cdot -e$. Now since $x_l \not\in$ exker f, we have $x_l \cdot -e \not\in I_1$, and so there is a $y \leqq x_l \cdot -e$ with $fy \neq y$. Then $y \not\in I_1$, so $y \not\in$ exker f by (26). Hence, $x_l \cdot -e \cdot -y \in$ exker f, hence $x_l \cdot -e \cdot -y \in I_1$ by (26). But then

$$\begin{aligned} fy &= f(x_l \cdot -e \cdot -(x_l \cdot -e \cdot -y)) \\ &= x_l \cdot -e \cdot -(x_l \cdot -e \cdot -y) \\ &= y , \end{aligned}$$

a contradiction.

Now we let $x_l \cdot -e$ be one of the c_j, with $f(x_l \cdot -e) \cdot -(x_l \cdot -e)$ the corresponding c_j^*. Furthermore, $J_j = (\text{exker } f) \cap (A \restriction (x_l \cdot -e))$.

It is clear that we obtain in this way an endomorphism schema which determines f.

$\Leftarrow$ We assume that f is determined by an endomorphism schema as in (22)–(25)

and Lemma 3.1. Then $A/\text{exker}\, f$ is a BA with atoms among $b_0/\text{exker}\, f, \ldots, b_{m-1}/\text{exker}\, f, c_0/\text{exker}\, f, \ldots, c_{n-1}/\text{exker}\, f$. Thus, it is finite, and (20) holds. To show (21), suppose on the contrary that I and J are non-principal complementary ideals. It is easily checked that A/J is isomorphic to the subalgebra $I \cup -I$ of A: if $fx = x/J$ for all $x \in I \cup -I$, then f is one-to-one since $x \in J$ combined with $x \in I$ gives $x = 0$, while combined with $x \in -I$ yields $1 \in \langle I \cup J \rangle^{\text{id}}$, a contradiction; and f maps onto A/J since it maps onto $\langle I \cup J \rangle^{\text{id}}/J$. Thus, we can consider the following endomorphism f of A:

$$A \to A/J \to I \cup -I \to A\,,$$

with all mappings natural. Assume that f is determined by an endomorphism schema as in Lemma 3.1. Note that $fx = 0$ for all $x \in J$ and $fx = x$ for all $x \in I$. Let $K = \langle I \cup J \rangle^{\text{id}}$; so K is a maximal ideal.

(30) For all $i < m$, $b_i \not\in K$.

For otherwise, write $b_i = d + e$ with $d \in I$ and $e \in J$. Then $b_i^* = fb_i = d \leqq b_i$ and $d \neq 0$ since $b_i^* \neq 0$. For each $x \leqq d$ we have $x \in I$ and hence $fx = x$. But $A \upharpoonright b_i$ atomless implies that I_i is non-principal and hence there is an $x \leqq d$ with $0 \neq x \in I_i$. Then $x = fx = 0$ by 3.1(ii), a contradiction.

(31) for all $j < n$, $c_j \not\in K$.

For, otherwise, write $c_j = d + e$ with $d \in I$, $e \in J$. Then $fc_j = d \leqq c_j$, so $c_j^* = 0$, a contradiction.

Suppose $m > 0$. Then by (30) and (31), $m = 1$ and $n = 0$. Now $-b_0 \in K$, so we can write $-b_0 = d + e$ with $d \in I$ and $e \in J$. Hence, $e = a_0$, $d = a_1$, and $b_0^* = a_0 + b_0$. Since I is non-principal, choose d' with $d < d' \in I$. Then $0 \neq d' \cdot -d \leqq b_0$ so, since I_0 is a non-principal ideal of $A \upharpoonright b_0$ ($A \upharpoonright b_0$ being atomless), we can choose $u \in I_0$ such that $0 \neq u \leqq d' \cdot -d$. Then $0 = fu = u$, a contradiction. Thus, $m = 0$.

Similarly, $n = 0$. But clearly, $a_0 \in J$ and $a_1 \in I$, so $1 = a_0 + a_1 \in K$, a contradiction. This completes the proof of Theorem 3.2. □

Now we show that every endo-rigid BA is very rigid in the usual sense. Recall that a BA A is *onto-rigid* if every onto endomorphism is the identity, and *mono-rigid* if every one-to-one endomorphism is the identity.

3.3. THEOREM. *Every endo-rigid BA is onto-rigid and mono-rigid. Moreover, if A is endo-rigid, $a, b \in A$, f is an endomorphism of A, and $f \upharpoonright (A \upharpoonright a)$ embeds $A \upharpoonright a$ into $A \upharpoonright c$, then $x \leqq fx$ for all $x \leqq a$, and if $a = c$, then f is the identity on $A \upharpoonright a$.*

PROOF. We prove the "moreover" part first: assume its hypothesis. If $x \leqq a$ and $x \not\leqq fx$, let $d = x \cdot -fx$. Thus, $d \cdot fd = 0$, and $d \neq 0$. Clearly, if $u, v \leqq d$ and $u \neq v$, then $u \triangle v \not\in \text{exker}\, f$, so $A/\text{exker}\, f$ is infinite, a contradiction. Suppose, additionally, that $a = c$. If $x \leqq a$ and $fx \not\leqq x$, let $d = fx \cdot -x$. Then the same argument gives a contradiction. Thus, $fx = x$ for all $x \leqq a$, as desired.

Taking $a = c = 1$, we get the mono-rigidity of A.

Now suppose f is an onto endomorphism of A, f not the identity. By the above, f is not one-to-one, so choose $a \neq 0$ such that $a \in \ker f$. For each $b \leqq a$ choose c_b so that $fc_b = b$. Then for distinct $b, d \leqq a$ we have $c_b \triangle c_d \not\in \operatorname{exker} f$. (This gives a contradiction as above.) For, suppose this fails for certain $b, d \leqq a$. Say $b \cdot -d \neq 0$. Since $c_b \cdot -c_d \in \operatorname{exker} f$, write $c_b \cdot -c_d = u + v$ with $u \in \ker f$ and $fx = x$ for all $x \leqq v$. Applying f, we get $v = b \cdot -da$, so $v \in \ker f$ and hence $v = 0$. Therefore $b \cdot -d = 0$, a contradiction. □

On the other hand, there is no relationship between endo-rigid and Bonnet-rigid BAs. Recall that A is Bonnet-rigid provided that if $f\colon A \rightarrowtail B$ and $g\colon A \twoheadrightarrow B$, then $f = g$. (Recall that $\rightarrowtail$ means one-to-one.) The endo-rigid BA we construct is not Bonnet-rigid (see below). SHELAH [1983] has shown assuming $\diamond$ that there is an endo-rigid, Bonnet-rigid BA of power ω_1. There exist Bonnet-rigid interval algebras; according to the following simple proposition they are not endo-rigid.

3.4. PROPOSITION. *If A is an interval algebra, then A is not endo-rigid.*

PROOF. Let A be the interval algebra on L. We may assume that L is infinite, and in fact that in L there is a strictly increasing sequence $a_0 < a_1 < a_2 < \cdots$. Then it is easily seen that there is an endomorphism f of A such that for any $x \in L$,

$$f[0, x) = \begin{cases} 0 & \text{if } x < a_0\,, \\ [0, a_i) & \text{if } a_i \leqq x < a_{i+1}\,, \\ 1 & \text{if } a_i < x \text{ for all } i\,. \end{cases}$$

Now it is easy to check that $\langle [a_i, a_{i+1})/\operatorname{exker} f\colon i < \omega \rangle$ is a system of distinct elements of $A/\operatorname{exker} f$, so A is not endo-rigid. (These are actually atoms of $A/\operatorname{exker} f$.) □

Before turning to the construction of an endo-rigid BA we give the following lemma concerning endomorphisms in general.

3.5. LEMMA. *If f is an endomorphism of a BA A, $a \in A$, and $fx \leqq x$ for all $x \leqq a$, then $a \in \operatorname{exker} f$.*

PROOF. The conclusion will be immediate from the following statements (32)–(34):

(32) if $x \leqq a$, then $x \cdot -fx \in \ker f$.

For, $f(x \cdot -fx) \leqq x \cdot -fx$ by assumption, and clearly $f(x \cdot -fx) \leqq fx$, so $f(x \cdot -fx) = 0$.

(33) If $x \leqq a$, then $fx = ffx$.

In fact, $fx \leqq ffx$ by (32), and $fx \leqq x \leqq a$, hence $ffx \leqq fx$ by assumption, so $fx = ffx$.

(34) If $x \leqq fa$, then $fx = x$.

For, assume that $x \leqq fa$. Now $fa = ffa$ by (33), $x \cdot -fx \leqq x \leqq fa$, and $fa \cdot -(x \cdot -fx) \leqq fa \leqq a$ by assumption. Hence,

$$\begin{aligned} fa = ffa &= f[fa \cdot -(x \cdot -fx) + x \cdot -fx] \\ &= f[fa \cdot -(x \cdot -fx)] + f(x \cdot -fx) \\ &= f[fa \cdot -(x \cdot -fx)] \text{ by (32)} \\ &\leqq fa \cdot -(x \cdot -fx) \text{ by assumption} \\ &\leqq -(x \cdot -fx) . \end{aligned}$$

But $x \cdot -fx \leqq x \leqq fa$, so $x \cdot -fx = 0$. Hence, $x \leqq fx \leqq x$ by assumption, so $fx = x$, as desired.

By (32) and (34), $a = a \cdot -fa + fa \in \text{exker } f$, as desired in the lemma. □

Now to carry out the construction of an endo-rigid BA we need an auxiliary notion which is of independent interest. Let A be a BA and v a formal variable. Suppose $a \in {}^{\omega}A$ and $S \subseteq \omega$. Then by $[a_i, v]^{\text{if } i \in S}$ we mean the formula $a_i \leqq v$ if $i \in S$ and $a_i \cdot v = 0$ if $i \notin S$. Of course we can define $[a_i, \tau]^{\text{if } i \in S}$ similarly for a more complicated term τ. A *standard type over A* is a set of the form:

$$\{[a_i, v]^{\text{if } i \in S} : i < \omega\} ,$$

where $\langle a_i : i \in \omega \rangle$ is a disjoint system of elements of A^+ and $S \subseteq \omega$. A *candidate over A* is a system $\langle (a_i, b_i) : i < \omega \rangle$ such that $\langle a_i : i < \omega \rangle$ and $\langle b_i : i < \omega \rangle$ are disjoint systems of elements of A^+ and for all $i < \omega$, $b_i \not\leqq a_i$. Finally, we call A *complicated* if, for every such candidate, there is an $S \subseteq \omega$ such that $\{[a_i, v]^{\text{if } i \in S} : i \in \omega\}$ is realized in A but $\{[b_i, v]^{\text{if } i \in S} : i \in \omega\}$ is omitted in A (i.e. there is an element $x \in A$ such that $a_i \leqq x$ for all $i \in S$ and $a_i \cdot x = 0$ for all $i \in \omega$ but there is no corresponding element for the b_i's).

3.6. LEMMA. *If A is complicated and f is an endomorphism of A, then $A/\text{exker } f$ is finite.*

PROOF. Assume that $A/\text{exker } f$ is infinite. Then there is a disjoint system $\langle a'_n : n \in \omega \rangle$ in A such that $a'_n/\text{exker } f \neq 0$ for all $n \in \omega$. By Lemma 3.5, for every $n \in \omega$ there is an $a_n \leqq a'_n$ such that $fa_n \not\leqq a_n$. Thus, $\langle (a_n, fa_n) : n < \omega \rangle$ is a candidate over A, so we can choose $S \subseteq \omega$ so that $\{[a_n, v]^{\text{if } n \in S} : n \in \omega\}$ is realized in A, say by c, but $\{[fa_n, v]^{\text{if } n \in S} : n \in \omega\}$ is not. But clearly fc realizes this last type, a contradiction. □

Now we are ready for the main theorem concerning endo-rigid BAs.

3.7. THEOREM. *There is an endo-rigid BA of power* 2^ω.

PROOF. Let A be the free BA on 2^ω free generators and let $\bar{A}$ be its completion. Note that $|\bar{A}| = 2^\omega$. We may assume that $\bar{A} \subseteq 2^\omega$ as a set. Recall that $\bar{A}$ satisfies ccc. Let $\langle\langle(a_n^\alpha, b_n^\alpha): n \in \omega\rangle: \alpha < 2^\omega\rangle$ list all members of ${}^\omega(2^\omega \times 2^\omega)$, each member repeated 2^ω times. Now we construct by induction two sequences $\langle B_\alpha: \alpha \leqq 2^\omega\rangle$ and $\langle Q_\alpha: \alpha \leqq 2^\omega\rangle$ such that, for all α, $\beta < 2^\omega$.

(35) B_α is a subalgebra of $\bar{A}$, $|B_\alpha| \leqq |\alpha| + \omega$, and $\alpha < \beta$ implies $B_\alpha \subseteq B_\beta$,

(36) Q_α is a collection of standard types over B_α each omitted in B_α, $\alpha < \beta$ implies $Q_\alpha \subseteq Q_\beta$, and $|Q_\alpha| \leqq |\alpha| + \omega$.

We let B_0 be a denumerable atomless subalgebra of $\bar{A}$, and $Q_0 = 0$. For λ a limit ordinal $\leqq 2^\omega$ we let $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ and $Q_\alpha = \bigcup_{\alpha<\lambda} Q_\alpha$. The essential step is the successor step. So assume $\alpha < 2^\omega$, and B_α and Q_α have been defined satisfying (35) and (36). The constrution of $B_{\alpha+1}$ and $Q_{\alpha+1}$ takes two steps.

First we take care of a candidate, forming B'_α and Q'_α. If $\langle(a_n^\alpha, b_n^\alpha): n < \omega\rangle$ is *not* a candidate over B_α, let $B'_\alpha = B_\alpha$, $Q'_\alpha = Q_\alpha$. So assume it is. For the rather lengthy considerations which follow we write a_n and b_n instead of a_n^α and b_n^α. First extend $\langle a_n: n < \omega\rangle$ to a partition $\langle a_\nu: \nu < \beta\rangle$, where β is a countable ordinal $\geqq \omega$. For each $S \subseteq \omega$ let $cS = \Sigma_{\alpha \in S} a_\nu$ (the sum in $\bar{A}$), and set $CS = \langle B_\alpha \cup \{cS\}\rangle$. We want to find S so tht

(37) CS omits $\{[b_n, v]^{\text{if } n\in S}: n \in \omega\}$,

(38) CS omits each member of Q_α .

Since CS obviously realizes $\{[a_n, v]^{\text{if } n\in S}: n < \omega\}$, this will take care of the current candidate. For each $n < \omega$ we have $b_n \nleqq a_n$, and so there is a $\nu n < \beta$ such that $a_{\nu n} \cdot b_n \cdot -a_n \neq 0$ and hence $a_{\nu n} \cdot b_n \neq 0$ and $\nu n \neq n$. Now we claim:

(39) there is an $S^* \subseteq \omega$, S^* infinite, such that $\nu n \not\in S^*$ for all $n \in S^*$.

In fact, if $\omega \cap \operatorname{ran} \nu$ is finite, we can take $S^* = \omega \backslash \operatorname{ran} \nu$. Otherwise, by induction choose m_i for each $i < \omega$ such that both m_i and νm_i are members of $\omega \backslash(\{m_j: j < i\} \cup \{\nu m_j: j < i\})$. Clearly, $S^* = \{m_i: i < \omega\}$ is as desired in (39).

Now the following fact will enable us to take care of (37):

(40) if d, e and g are pairwise disjoint elements of B_α such that $d + e \cdot cS + g \cdot -cS$ realizes $\{[b_n, v]^{\text{if } n\in S}: n \in \omega\}$ in CS, and $S \subseteq S^*$, then $S = \{n \in S^*: b_n \cdot a_{\nu n} \leqq d + g\}$.

For, assume the hypothesis of (40). For any $n \in S^*$ we have $\nu n \not\in S^*$, hence $\nu n \not\in S$ and so $a_{\nu n} \cdot cS = 0$. Also, if $n \in S$, then $b_n \leqq d + e \cdot cS + g \cdot -cS$, so $b_n \cdot a_{\nu n} \leqq d + g$. If $n \in S^* \backslash S$, then $b_n \cdot (d + e \cdot cS + g \cdot -cS) = 0$, so

$$b_n \cdot a_{\nu n} \cdot (d + g) = b_n \cdot a_{\nu n} \cdot (d + g \cdot -cS) = 0 .$$

Thus (40) holds.

It is harder to take care of (38). To do so we shall use a combinatorial trick. Let K be a family of 2^{ω} infinite pairwise almost disjoint subsets of S^*. Now the following statement enables us to take care of (38):

(41) if d, e and g are pairwise disjoint elements of B_α, $q \in Q_\alpha$, $q = \{[h_n, v]^{\text{if } n \in T}: n \in \omega\}$, then there is at most one $S \in K$ such that $d + e \cdot cS + g \cdot -cS$ realizes q in CS.

To prove this, assume its hypotheses. Let

$$k = -d \cdot -e \cdot -g ,$$

$$l = \sum_{n \in T} h_n \cdot (k + e) + \sum_{n \in \omega \setminus T} h_n \cdot (d + g) ,$$

$$m = \sum_{n \in T} h_n \cdot (k + g) + \sum_{n \in \omega \setminus T} h_n \cdot (d + e) .$$

Then:

(42) if $B_\alpha \subseteq D \subseteq \bar{A}$ and $u \in D$, then $\{[h_n, d + e \cdot v + g \cdot -v]^{\text{if } n \in T}: n \in \omega\}$ is realized by u in D iff $l \leqq u$ and $m \cdot u = 0$.

In fact, for $\Rightarrow$ we have $h_n \leqq d + e \cdot u + g \cdot -u$ for $n \in T$, hence $h_n \cdot e = e \cdot u \leqq u$ and $h_n \cdot k = 0 \leqq u$. Furthermore, if $n \in \omega \setminus T$, then $h_n \cdot (d + e \cdot u + g \cdot -u) = 0$, hence $h_n \cdot d = 0 \leqq u$ and $h_n \cdot g \cdot -u = 0$ hence $h_n \cdot g \leqq u$. Thus, $l \leqq u$. Similarly, $m \cdot u = 0$. For $\Leftarrow$, note first that $l \cdot m = 0$. Suppose $n \in T$. Now $h_n \cdot k \leqq l \cdot m = 0$. Thus, $h_n \leqq d + e + g$. Furthermore, $h_n \cdot e \leqq l \leqq u$ and $h_n \cdot g \leqq m$, hence $h_n \cdot g \leqq -u$. Thus, $h_n \leqq d + e \cdot u + g \cdot -u$. Suppose $n \not\in T$. Then $h_n \cdot d \leqq l \cdot m$, so $h_n \cdot d = 0$. Also, $h_n \cdot e \leqq m$ so $h_n \cdot e \cdot u = 0$. Finally, $h_n \cdot g \leqq l \leqq u$, so $h_n \cdot g \cdot -u = 0$. Thus, $h_n \cdot (d + e \cdot u + g \cdot -u) = 0$, as desired. So, (42) holds. By (38) for B_α and (42) we obtain:

(43) there is no $u \in B_\alpha$ such that $l \leqq u$ and $m \cdot u = 0$.

Now let $L = \langle \{a_\nu: \nu < \beta\} \rangle^{\text{id}}$, an ideal in $\bar{A}$.

(44) if $l \in L$, then for any $S \in K$, $d + e \cdot cS + g \cdot -cS$ does not realize q in CS. (q is described in (41).)

For, otherwise, by (42) $l \leqq cS$ and $m \cdot cS = 0$. Since $l \in L$, there is a finite sum u of members of $\{a_\nu: \nu < \beta\}$ such that $l \leqq u$. Hence, $l \leqq u \cdot cS$ and $m \cdot u \cdot cS = 0$. But $u \cdot cS$ is a finite join of members of $\{a_n: n \in \omega\} \subseteq B_\alpha$, so $u \cdot cS \in B_\alpha$. This contradicts (43).

Now we show (41). By (44) we may assume that $l \not\in L$. Now suppose that S_1 and S_2 are distinct elements of K such that $d + e \cdot cS_i + g \cdot -cS_i$ realizes q in CS_i for $i = 1, 2$. Then by (42) $l \leqq cS_i$ for $i = 1, 2$, so $l \leqq cS_1 \cdot cS_2 = \Sigma\, \{a_n: n \in S_1 \cap S_2\}$; since $S_1 \cap S_2$ is finite, $l \in L$, a contradiction. Thus, (41) holds.

Since every element of CS has the form $d + e \cdot cS + g \cdot -cS$ for some pairwise disjoint d, e, $g \in B_\alpha$, we see by (40), (41), (35), and (36) that there are at most $|\alpha| + \omega$ subsets $S \in K$ for which (37) or (38) fails. We choose $S \in K$ for which (37) and (38) hold, and let $b'_\alpha = CS$, $Q'_\alpha = Q_\alpha \cup \{[b_n, v]^{\text{if } n \in S}: n < \omega\}$.

The second step in the construction is much shorter, and is intended to secure condition (21) for the final algebra. We need the following fact, which is a special case of a general result. Let A be freely generated by X.

(45) There is an $x \in X$ such that for every $y \in B'^{+}_\alpha$, $x \cdot y \neq 0 \neq -x \cdot y$.

In fact, for each $c \in B'_\alpha$, we can write $c = \Sigma\, D_c$ (sum in $\bar{A}$), where D_c is a countable subset of A. For each $y \in A$ there is a finite $E_y \subseteq X$ such that $y \in \langle E_y \rangle$. Let

$$F = \bigcup \{E_y : c \in B'_\alpha,\ y \in D_c\}.$$

Then $|F| < 2^\omega$, so there is an $x \in X \backslash F$. Clearly, x is as desired in (45).

We choose x as in (45) and let $B_{\alpha+1} = \langle B'_\alpha \cup \{x\} \rangle$. Let $Q_{\alpha+1}$ be Q'_α together with those of the two types:

$$\{[a^\alpha_n \cdot x, v]^{\text{if } n \text{ even}}: n < \omega\},$$

$$\{[a^\alpha_n \cdot -x, v]^{\text{if } n \text{ odd}}: n < \omega\},$$

which are standard types over $B_{\alpha+1}$ omitted in $B_{\alpha+1}$. We still must check that each member q of Q'_α is omitted in $B_{\alpha+1}$. But if $q = \{h_n, v]^{\text{if } n \in S}: n < \omega\}$ is realized by $c \cdot x + d \cdot -x$ with c, $d \in B'_\alpha$, it is easily checked that c realizes q too (so does d), a contradiction.

This completes the construction. Let $B' = B_\alpha$ with $\alpha = 2^\omega$; we claim that B' is endo-rigid. The second step in the above successor step stage assures us that B' is atomless. Clearly, $|B'| = 2^\omega$. Now any candidate for B' is a candidate for some B_α since $\omega < \operatorname{cf}(2^\omega)$; so we may choose α so that also the given candidate is $\langle (a^\alpha_n, b^\alpha_n): n \in \omega \rangle$ (since each such object was repeated 2^ω times). Then the first step in the $\alpha \to \alpha + 1$ construction "kills" this candidate, assuring us that B' is complicated and hence satisfies (20).

It remains only to check that B' satisfies (21). Suppose, on the contrary, that I_0 and I_1 are non-principal complementary ideals in B'. Note that if $x \in I_i$, then there is a $y \in I_i$ such that $x < y$, hence a $z \in I_i$ such that $x \cdot z = 0$ and $z \neq 0$ (take $z = y \cdot -x$), $i = 0, 1$. Therefore any maximal disjoint set of elements of I_i is infinite, $i = 0, 1$. Let $\langle a_n : n < \omega,\ n \text{ even} \rangle$ be a maximal disjoint set of non-zero elements of I_0, and $\langle a_n : n < \omega,\ n \text{ odd} \rangle$ one for I_1. Choose $\alpha < 2^\omega$ such that $\{a_n : n < \omega\} \subseteq B_\alpha$ and $\langle (a^\alpha_n, b^\alpha_n): n < \omega \rangle = \langle (a_n, a_n): n < \omega \rangle$. Recall that in the step $\alpha \to \alpha + 1$ of the construction we extended B_α to B'_α, then set $B_{\alpha+1} = \langle B'_\alpha \cup \{x\} \rangle$, where $x \cdot y \neq 0 \neq -x \cdot y$ for all $y \in B'^{+}_\alpha$. Now we claim:

(46) $\{[a_n, v]^{\text{if } n \text{ even}}: n < \omega\}$ is omitted in B'_α.

For, suppose $y \in B'_\alpha$ realizes this type. Without loss of generality $y \in \langle I_0 \cup I_1 \rangle^{\text{id}}$ (since $\langle I_0 \cup I_1 \rangle^{\text{id}}$ is a maximal ideal), so write $y = d + e$ with $d \in I_0$, $e \in I$. For n even we have $a_n \leqq y = d + e$, and $a_n \cdot e = 0$, so $a_n \leqq d$. Choose $s \in I_0$ with $d < s$. Then $s \cdot -d \neq 0$, and by the maximality of $\{a_n: n < \omega, \nu \text{ even}\}$ we get an even n with $s \cdot -d \cdot a_n \neq 0$, a contradiction. So (46) holds.

(47) $\{[a_n \cdot x, v]^{\text{if } n \text{ even}}: n < \omega\}$ and $\{|a_n \cdot -x, v]^{\text{ir } n \text{ even}}: n < \omega\}$ are omitted in $B_{\alpha+1}$.

To prove this, by symmetry we take the first type only, and suppose $b \cdot x + c \cdot -x$ realizes it, where $b, c \in B'_\alpha$. Then for n even we have $a_n \cdot x \leqq b \cdot x + c \cdot -x$, so $a_n \leqq b$. For n odd we have $a_n \cdot x \cdot (b \cdot x + c \cdot -x) = 0$, so $a_n \cdot b = 0$. Thus, $b \in B'_\alpha$ realizes $\{[a_n, v]^{\text{if } n \text{ even}}: n < \omega\}$ in B'_α, contradicting (46).

By (47), the types mentioned there are omitted in B' (see the construction). Say without loss of generality $x \in \langle I_0 \cup I_1 \rangle^{\text{id}}$, and write $x = d + e$ with $d \in I_0$, $e \in I_1$. If n is even, then $a_n \cdot x = a_n \cdot d \leqq d$; if n is odd, then $a_n \cdot x = a_n \cdot e$, hence $a_n \cdot x \cdot d = 0$. Thus, d realizes $\{[a_n \cdot x, v]^{\text{if } n \text{ even}}: n < \omega\}$ in B', a contradiction. This finishes the proof of Theorem 3.7. □

As mentioned before, SHELAH [1984] has shown that there is an endo-rigid BA of each infinite cardinality λ such that $\lambda^\omega = \lambda$. On the other hand, MA implies that there is no endo-rigid BA of power $<2^\omega$. It is known that an endo-rigid BA always has at least 2^{ω_1} ultrafilters and hence at least 2^{ω_1} endomorphisms. For these two results see MONK [1980], where there are references to earlier papers. Note that the endo-rigid BA B' constructed above in the proof of Theorem 3.7 has a free subalgebra C of power 2^ω and hence 2^{2^ω} ultrafilters and endomorphisms. It appears to be open to construct an endo-rigid BA A with $|A| = |\text{End } A|$.

Completing our earlier discussion (Proposition 3.4 and preceding remarks), we now show that the algebra B' constructed in Theorem 3.7 is not Bonnet-rigid. For, let C be a free subalgebra of B' of size 2^ω. Let h be a homomorphism from C onto B'. Then let $I = \langle \ker h \rangle^{\text{id}}_{B'}$ and let $fhc = c/I$ for any $c \in C$. Clearly, f is a well-defined monomorphism from B' into B'/I. If g is the natural homomorphism from B' onto B'/I, then $f \neq g$, showing that B' is not Bonnet-rigid.

4. Hopfian Boolean algebras

A BA A is *hopfian* (*dual-hopfian*) if A is infinite and every onto (one-to-one) endomorphism is one-to-one (onto). Thus, every onto-rigid (mono-rigid) BA is hopfian (dual-hopfian). The most immediate question which arises is thus whether non-rigid hopfian or dual-hopfian BAs exist. Endo-rigid BAs give immediate answers, by the following two results of SHELAH [1984].

4.1. THEOREM. *If A is endo-rigid, then $A \times A$ is hopfian.*

PROOF. Suppose that f is an onto endomorphism of $A \times A$ which is not one-to-one. Then there is an $(a, b) \in (A \times A)^+$ with $f(a, b) = (0, 0)$. Say without loss of generality $a \neq 0$. Choose (d, e) such that $f(d, e) = (a, 0)$. Say without loss of generality $f(d, 0) \neq (0, 0)$; write $f(d, 0) = (m, 0)$; thus $m \leqq a$. Note that $f(d \cdot -a, 0) = (m, 0)$ also. Now clearly $d \cdot -a \neq 1$. Let I be a maximal ideal in A such that $d \cdot -a \in I$. For any $x \in A$ let $gx = (f(x, x/I))_0$ (first coordinate of $f(x, x/I)$). Thus, g is an endomorphism of A, and $g(d \cdot -a) = m$. For every $y \leqq m$ there is an $n_y \leqq d \cdot -a$ such that $f(n_y, 0) = (y, 0)$. Hence, $gn_y = y$ for any $y \leqq m$. Suppose $y, z \leqq m$ and $y \neq z$, while $n_y \triangle n_z \in \operatorname{exker} g$. Say $n_y \triangle n_z = u + v$ with $u \in \ker g$ and $gw = w$ for all $w \leqq v$. Applying g, $y \triangle z = v$. Now $v \leqq d \cdot -a$, and $v = gv \leqq m$, so $v \in I$. Hence, $v = gv = (f(v, 0))_0 = 0$, a contradiction. Therefore $A/\operatorname{exker} g$ is infinite, a contradiction. □

4.2. THEOREM. *If A is endo-rigid, then $A \times A$ is dual-hopfian.*

PROOF. Let f be a one-to-one endomorphism of $A \times A$. For all $x \in A$ let $gx = (f(x, x))_0$ (first coordinate of $f(x, x)$).

$(*)$ $\quad g$ is one-to-one.

Assume otherwise: say $ga = 0$, $a \neq 0$. Then $f(a, a)$ has the form $(0, b)$. Say $f(a, 0) = (0, c)$. Note that $a \neq 1$, and let I be a maximal ideal such that $a \in I$. For any $x \in A$ let $hx = (f(x, x/I))_1$. Then h is an endomorphism of A, and $h \restriction (A \restriction a)$ embeds $A \restriction a$ into $A \restriction c$. Hence, by Theorem 3.3 we get $a \leqq c$. Similarly, $f(0, a)$ has the form $(0, d)$ with $a \leqq d$. But $(a, 0) \cdot (0, a) = (0, 0)$, so $c \cdot d = 0$. Thus, $0 \neq a \leqq c \cdot d$ is a contradiction. So $(*)$ holds. Hence, g is the identity, by Theorem 3.3. By similar reasoning for the function $x \mapsto (f(x, x))_1$ we obtain:

$(**)$ $\quad f(x, x) = (x, x)$ for all $x \in A$.

Next we claim

$(***)$ $\quad f(1, 0)$ has the form $(b, -b)$ for some $b \in A$.

In fact, write $f(1, 0) = (b, c)$. Then $f(b \cdot c, b \cdot c) = (b \cdot c, b \cdot c) \leqq (b, c) = f(1, 0)$, so $b \cdot c = 0$. Now $f(0, 1) = (-b, -c)$, so $-b \cdot -c = 0$ by the same reasoning. Thus, $c = -b$, as desired in $(***)$.

$(\bigstar)$ $\quad$ For all $x \leqq b$, $f(x, 0) = (x, 0)$.

In fact, $f(x, 0) = f((x, x) \cdot (1, 0)) = (x, x) \cdot (b, -b) = (x, 0)$. Similarly,

$(\bigstar\bigstar)$ $\quad$ For all $y \leqq -b$, $f(0, y) = (y, 0)$.

It follows that every element $(z, 0)$ is in the range of f: $f(z \cdot b, z \cdot -b) = (z, 0)$. By symmetry every element $(0, z)$ is in the range of f, so f is onto, as desired. □

A further question now arises: Are there atomic hopfian or dual-hopfian BAs?

Call an atomic BA A *almost rigid* if every automorphism of A is induced by a finite permutation of its atoms. By modifying the construction of endo-rigid BAs we shall prove the existence of almost rigid hopfian and dual-hopfian BAs. On the other hand, there are no countable hopfian or dual-hopfian BAs. For these results, with different proofs, see LOATS [1979] and LOATS and ROITMAN [1981]

To prove the non-existence of countable hopfian or dual-hopfian BAs, we need the following lemma which is of independent interest.

4.3. LEMMA. *Let A be a denumerable BA with infinitely many atoms. Then there is an $X \in [\mathrm{At}\, A]^{\omega}$ such that for each $a \in A$, $\mathrm{At}(A \restriction a) \cap X$ is finite or $\mathrm{At}(A \restriction -a) \cap X$ is finite. (For any BA B, $\mathrm{At}\, B$ is the set of atoms of B.)*

PROOF. Let $A = \{a_0, a_1, \ldots\}$. It is easy to construct $\varepsilon \in {}^{\omega}\{-1, +1\}$ such that for all $n \in \omega$, $\mathrm{At}(A \restriction (\varepsilon_0 a_0 \cdot \cdots \cdot \varepsilon_n a_n))$ is infinite. Choose $x_n \in \mathrm{At}(A \restriction (\varepsilon_0 a_0 \cdot \cdots \cdot \varepsilon_n a_n)) \backslash \{x_i : i < n\}$. Clearly, $X = \{x_n : n \in \omega\}$ is as desired. □

4.4. THEOREM. *Let A be a denumerable BA with infinitely many atoms. Then there is a subalgebra B of A and an isomorphism of the semigroup* End B *into* End A *with the following properties*:

(i) *B is isomorphic to the finite–cofinite algebra on ω,*
(ii) *for each $f \in \mathrm{End}\, B$ we have $f \subseteq f^+$,*
(iii) *for each $f \in \mathrm{End}\, B$, f is one-to-one (onto) iff f^+ is one-to-one (onto).*

PROOF. Choose X as in Lemma 4.3, and let $B = \langle X \rangle$. Clearly, (i) holds. Let $I = \{a \in A : \mathrm{At}(A \restriction a) \cap X$ is finite$\}$. Clearly, I is a maximal ideal in A. For each $a \in I$ there is a unique finite $Sa \subseteq X$ and $ta \in A$ such that $a = \Sigma\, Sa + ta$ and $ta \cdot x = 0$ for all $x \in X$. For any $f \in \mathrm{End}\, B$ we then define

$$f^+ a = \sum_{b \in Sa} fb + ta\,;$$

for $a \notin I$ we set $f^+ a = -f^+(-a)$. It is routine to check the desired conditions. □

4.5. COROLLARY. *There is no countable hopfian or dual-hopfian BA.*

PROOF. First note that the BA A of finite and cofinite subsets of ω is neither hopfian nor dual-hopfian. For, let $fx = \{n \in \omega : n + 1 \in x\}$ for all $x \in A$; f is an onto endomorphism of A but it is not one-to-one. On the other hand, let $gx = \{n \in \omega : n = 0$ and $0 \in x$, or $n > 0$ and $n - 1 \in x\}$; g is a one-to-one endomorphism of A but it is not onto.

Thus, by Theorem 4.4 it suffices to show that there is no denumerable hopfian or dual-hopfian BA having only finitely many atoms. Suppose that A is denumerable with only finitely many atoms; we construct $f, g \in \mathrm{End}\, A$ such that f (resp. g) is onto (resp. one-to-one) but not one-to-one (resp., onto). Let $a = \Sigma\, \mathrm{At}\, A$, $b = -a$. Then $A \restriction b$ is isomorphic to the free BA on ω generators; let $\langle x_n : n \in \omega \rangle$ be a system of free generators of $A \restriction b$. Let f' be the endomorphism of $A \restriction b$ such that $f' x_0 = x_0$ and $f' x_n = x_{n-1}$ for $n > 0$, and let g' be the one with $g' x_n = x_{2n}$ for all $n \in \omega$. Then f' is onto but not one-to-one, and g' is one-to-one but not onto. Since $A \cong (A \upharpoonright a) \times (A \upharpoonright b)$, f' and g' induce the desired endomorphisms of A. □

The above results generalize to show that under MA there is no hopfian or dual-hopfian BA of size $<2^\omega$ with infinitely many atoms; see LOATS [1979].

To prove our existence theorem for almost rigid hopfian BAs we extend some of our terminology concerning endo-rigid BAs. We call a BA A *weakly endo-rigid* if (20) holds, i.e. if for every endomorphism f of A, $A/\text{exker } f$ is finite.

4.6. LEMMA. *Let κ be an infinite cardinal. If A is a weakly endo-rigid subalgebra of $\mathscr{P}\kappa$ containing all singletons, then A is almost rigid, hopfian, and dual-hopfian.*

PROOF. Suppose that f is an automorphism of A moving infinitely many atoms. If X is the set of all moved atoms, then clearly $x/\text{exker } f \neq y/\text{exker } f$ for all distinct $x, y \in X$, and hence A is not weakly endo-rigid.

Suppose f is an onto endomorphism which is not one-to-one. Then there is an $a \in A^+$ such that $fa = 0$, hence there is some $\alpha_0 \in \kappa$ such that $f\{\alpha_0\} = 0$. Since f maps onto A, there is a $d_0 \in A$ such that $\alpha_0 \in d_0$, $|d_0| > 1$, and $fd_0 = \{\alpha_0\}$. Choose $\alpha_1 \in d_0 \backslash \{\alpha_0\}$. Then there is a $d_1 \in A$ such that $d_0 \cap d_1 = 0$ and $fd_1 = \{\alpha_1\}$. We continue inductively: if distinct $\alpha_0, \ldots, \alpha_n$, $n \geqq 1$, and pairwise disjoint $d_0, \ldots, d_n$ have been defined such that $fd_i = \{\alpha_i\}$ and $\alpha_{j+1} \in d_j$ for all $i \leqq n$, $j < n$, choose $\alpha_{n+1} \in d_n$ – clearly $\alpha_{n+1} \neq \alpha_0, \ldots, \alpha_n$ – and choose d_{n+1} disjoint from $d_0, \ldots, d_n$ such that $fd_{n+1} = \{\alpha_{n+1}\}$.

Note that $\alpha_{n+1} \in fd_{n+1} \backslash d_{n+1}$ for all $n \in \omega$. Hence, $d_i/\text{exker } f \neq d_j/\text{exker } f$ if $1 \leqq i < j < \omega$, so again A is not weakly endo-rigid.

Finally, suppose that f is one-to-one but not onto. Let $\Gamma = \{\alpha: f\{\alpha\} \neq \{\alpha\}\}$. Then Γ is infinite: assume otherwise. If $\alpha \in \Gamma$ and $\beta \in \kappa \backslash \Gamma$, then $\{\beta\} \cap f\{\alpha\} = f\{\beta\} \cap f\{\alpha\} = 0$, so $\beta \not\in f\{\alpha\}$. Thus, $f\{\alpha\} \subseteq \Gamma$ for all $\alpha \in \Gamma$. Since Γ is finite, this implies that $|f\{\alpha\}| = 1$ for all $\alpha \in \Gamma$. Hence, f is an automorphism, a contradiction. Thus, Γ is infinite. Clearly, $\{\alpha\}/\text{exker } f \neq \{\beta\}/\text{exker } f$ for distinct $\alpha, \beta \in \Gamma$, so A is not weakly endo-rigid. □

4.7. THEOREM. *There is a weakly endo-rigid subalgebra A of $\mathscr{P}\omega$ containing all singletons, with $|A| = 2^\omega$. Thus, A is almost rigid, hopfian, and dual-hopfian.*

PROOF. By Lemmas 3.6 and 4.6 it suffices to construct a complicated subalgebra A of $\mathscr{P}\omega$ containing all singletons, with $|A| = 2^\omega$. We can do this by modifying the proof of Theorem 3.7 just a little bit. Let $\langle\langle(a_n^\alpha, b_n^\alpha): n \in \omega\rangle: \alpha < 2^\omega\rangle$ list all candidates over $\mathscr{P}\omega$, each one listed 2^ω times. Construct by induction two sequences $\langle B_\alpha: \alpha \leqq 2^\omega\rangle$ and $\langle Q_\alpha: \alpha \leqq 2^\omega\rangle$ such that, for all $\alpha, \beta < 2^\omega$,

(48) B_α is a subalgebra of $\mathscr{P}\omega$, $|B_\alpha| \leqq |\alpha| \cup \omega$, and $\alpha < \beta$ implies $B_\alpha \subseteq B_\beta$,

(49) Q_α is a collection of standard types over B_α each omitted in B_α, $\alpha < \beta$ implies $Q_\alpha \subseteq Q_\beta$, and $|Q_\alpha| \leqq |\alpha| + \omega$.

We let B_0 be the finite–cofinite algebra on ω, $Q_0 = 0$. Then we repeat the construction in the proof of 3.7, taking only the first of the two steps in passing from α to $\alpha+1$. The joins Σ mentioned there are now taken in $\mathscr{P}\omega$ and coincide with $\bigcup$. $\square$

Now we consider the possibility of improving 4.7 by obtaining similar algebras in other cardinalities or with larger numbers of atoms. The proof of 3.7 can again be modified to give the following result.

4.8. THEOREM. *If $\kappa^\omega = 2^\omega$, then there is a weakly endo-rigid subalgebra A of $\mathscr{P}\kappa$ containing all singletons, with $|A| = 2^\omega$.*

PROOF. Let K be the BA of countable and cocountable subsets of κ; the construction takes place inside K. Let $\langle\langle(a_n^\alpha, b_n^\alpha)\colon n\in\omega\rangle\colon \alpha<2^\omega\rangle$ list all functions from ω into $K\times K$, each repeated 2^ω times. We proceed as in the proof of 3.7, with the following changes. Let $B_0 = \langle\{\{\alpha\}\colon \alpha\in\Gamma\}\rangle$, where $\Gamma\in[\kappa]^\omega$. In the step from α to $\alpha+1$, first part, the joins Σ are taken in K. Replace the second part by the adjunction of $\{\beta\}$, where β is the least ordinal $<\kappa$ such that $\{\beta\}\not\in B'_\alpha$. It is easily checked then that all members of Q'_α are still omitted in $B_{\alpha+1}$. $\square$

The following two corollaries of this theorem were first proved in LOATS and ROITMAN [1981] in a somewhat weaker form, and directly.

4.9. COROLLARY. *Let μ be a cardinal with* cf $\mu>\omega$, *and add μ many Cohen reals to a model of CH. Assume that $\omega\leqq\kappa\leqq 2^\omega$ in the extension. Then in the extension there is a weakly endo-rigid subalgebra A of $\mathscr{P}\kappa$ containing all singletons, with $|A|=2^\omega$.* $\square$

4.10. COROLLARY (MA). *If $\omega\leqq\kappa\leqq 2^\omega$, then the conclusion of* 4.9 *holds.* $\square$

Next, we want to give yet another application of the proof of 3.7, this time in a Cohen extension of a model of ZFC. Let A be a subalgebra of $\mathscr{P}\lambda$ (for some cardinal $\lambda\leqq\omega$) containing all singletons. A *weak candidate* over A is a candidate $\langle(a_i, b_i)\colon i<\omega\rangle$ over A such that each a_i is a singleton. Then A is called *mildly complicated* if for every such weak candidate there is an $S\subseteq\omega$ such that $\{[a_i, v]^{\text{if } i\in S}\colon i\in\omega\}$ is realized in A but $\{[b_i, v]^{\text{if } i\in S}\colon i\in\omega\}$ is not.

4.11. LEMMA. *If A is a mildly complicated subalgebra of $\mathscr{P}\lambda$ containing all singletons, then A is almost rigid and dual hopfian.*

PROOF. Suppose f is an automorphism of A moving infinitely many atoms. Then clearly there is a weak candidate $\langle(a_i, b_i)\colon i<\omega\rangle$ for which $fa_i=b_i$ for all $i<\omega$. This easily gives a contradition.

Next, suppose that f is a one-to-one endomorphism that is not onto. By the proof of Lemma 4.6, the set $\Gamma=\{\alpha<\lambda\colon f\{\alpha\}\neq\{\alpha\}\}$ is infinite. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of infinitely many members of Γ. Then $\{(\{\alpha_i\}, f\{\alpha_i\})\colon i<\omega\}$ is a weak candidate, which again gives a contradiction. $\square$

We shall prove that there exist mildly complicated algebras in certain Cohen extensions – which gives the existence of "small" almost rigid, dual hopfian BAs by Lemma 4.11. ROITMAN [1986] has shown that there are even "small" almost rigid, hopfian, dual-hopfian BAs in certain forcing extensions.

If Γ is a set of ordinals, we denote by Fin Γ the set of all functions from a finite subset of Γ into 2. If G is generic over M with respect to Fin κ, κ a cardinal of M, we call $\{\alpha < \kappa\colon f\alpha = 1 \text{ for some } f \in G\}$ a *Cohen subset of* κ *over* M.

For terminology concerning forcing we follow JECH [1978].

4.12. LEMMA. *Let C be a Cohen subset of κ over M. In M let f be a one-to-one function from some infinite subset Γ of κ into κ such that f moves infinitely many members of Γ. Then each of the following four sets is infinite: $f \cap (C \times C)$, $f \cap (C \times (\kappa \backslash C))$, $f \cap ((\kappa \backslash C) \times C)$, and $f \cap ((\kappa \backslash C) \times (\kappa \backslash C)$.*

PROOF. We take $f \cap (C \times (\kappa \backslash C))$ as an example. Inside M, for each $m \in \omega$ let

$$D_m = \{g \in \text{Fin}\,\kappa\colon |\{\alpha \in \Gamma\colon \alpha \in \text{dom}\, g,\ g(\alpha) = 1,\ f(\alpha) \in \text{dom}\, g,$$
$$g(f(\alpha)) = 0\}| \geqq m\}\,.$$

Clearly, each set D_m is dense in Fin κ. Hence, $f \cap (C \times (\kappa \backslash C))$ is infinite. □

4.13. COROLLARY. *Let C be a Cohen subset of κ over M, D an infinite subset of κ in M. Then $C \cap D$ is infinite.* □

4.14. LEMMA. *Let G be generic over M with respect to* $\text{Fin}(\lambda \times \omega_1)$, *$\lambda$ an infinite cardinal of M. For each $\beta < \omega_1$ let $G_\beta = \{f \restriction (\lambda \times \beta)\colon f \in G\}$, $G^\beta = \{f \restriction (\lambda \times (\omega_1 \backslash \beta)\colon f \in G\}$. Suppose that $\beta 0 < \beta 1 \ldots$ and $\lambda \supseteq C_0 \in M[G_{\beta 0}]$, $\lambda \supseteq C_1 \in M[G_{\beta 1}], \ldots$. Then there is a $\gamma < \omega_1$ such that $\langle C_i\colon i \in \omega\rangle \in M[G_\gamma]$.*

PROOF. Let $\delta = \bigcup_{i \in \omega} \beta i$. Thus $\delta < \omega_1$ and $\langle C_i\colon i \in \omega\rangle \in {}^\omega M[G_\delta]$. Let $\boldsymbol{C}$ be a name for $\langle C_i\colon i \in \omega\rangle$ with $M[G_\delta]$ as base model. For each $n \in \omega$ choose $q_n \in G^\delta$ such that $q_m \Vdash \boldsymbol{C}n^\vee = (Cn)^\vee$, where $\Vdash$ is relative to $\text{RO}(\text{Fin}(\lambda \times (\omega \backslash \delta)))$ in $M[G_\delta]$. Finally, choose $\xi > \delta$ so that $q_n \in \text{Fin}\,\Gamma$ for each $n \in \omega$, with $\Gamma = (\lambda \times \xi) \backslash (\lambda \times \delta)$. We claim that $C \in M[G_\xi]$. To prove this, let $B = \text{RO}(\text{Fin}\,\Gamma)$; we define a name $\boldsymbol{D} \in (M[G_\delta])^B$:

$$\text{(50)} \qquad \text{dom}\,\boldsymbol{D} = \{(n, a)^\vee\colon n \in \omega,\ a \subseteq \lambda\},\ \boldsymbol{D}(n, a)^\vee = \sum^B \{e^B p\colon p \Vdash \boldsymbol{C}n^\vee = a^\vee\}\,.$$

Let $H = \{f \restriction \Gamma\colon f \in G\}$. Then $M[G_\xi] = M[G_\delta][H]$. We claim that $i_h\boldsymbol{D} = C$, hence $C \in M[G_\xi]$, as desired. We have

$$\text{(51)} \qquad i_H\boldsymbol{D} = \{(n, a)\colon \text{there is an } r \in H \text{ such that } e^B r \leqq \boldsymbol{D}(n, a)^\vee\}\,.$$

Now the desired conclusion follows from the following statements (52) and (53):

$$\text{(52)} \qquad (n, C_n) \in i_H\boldsymbol{D} \text{ for any } n \in \omega\,.$$

Indeed, $q_n \Vdash Cn^{\vee} = (Cn)^{\vee}$ and $q_n \in \mathrm{Fin}\,\Gamma$, so $e^B q_n \leqq \boldsymbol{D}(n, Cn)^{\vee}$. Furthermore, $q_n \in H$, so (52) holds.

(53) If $n \in \omega$, $a \subseteq \lambda$, and $a \neq C_n$, then $(n, a) \not\in i_H \boldsymbol{D}$.

For, assume otherwise. Choose $r \in H$ such that $e^B r \leqq \boldsymbol{D}(n, a)^{\vee}$. Since H is generic, q_n and r are compatible; say $q_n, r \subseteq s \in \mathrm{Fin}\,\Gamma$. Thus, $e^B s \leqq \boldsymbol{D}(n, a)^{\vee}$, so there is a $p \in \mathrm{Fin}\,\Gamma$ with $p \Vdash Cn^{\vee} = a^{\vee}$ and p and s compatible, say $s, p \subseteq t \in \mathrm{Fin}\,\Gamma$. Thus, $t \Vdash Cn^{\vee} = a^{\vee}$ and, since $q_n \subseteq t$, $t \Vdash Cn^{\vee} = (Cn)^{\vee}$. Hence, $t \Vdash a^{\vee} = (Cn)^{\vee}$. Since $a \neq Cn$, this is a contradiction. □

4.15. LEMMA. *Let M be a model of ZFC, λ an infinite cardinal in M, A a subalgebra of $\mathcal{P}\lambda$ containing all singletons, and $\langle (a_i, b_i) : i \in \omega \rangle$ a weak candidate over A. Assume that $a_i \neq \{i\}$ for all $i \in \omega$. Let N be obtained from M by adding a Cohen subset C of λ. Let $S = \{i \in \omega : a_i \subseteq C\}$. Then $\{[b_i, v]^{\text{if } i \in S} : i \in \omega\}$ is not realized in $A(C)$. (Here $A(C) = \langle A \cup \{C\} \rangle$.)*

PROOF. Choose $\alpha : \beta \rightarrowtail\!\!\!\!\rightarrow \lambda$ such that $a_i = \{\alpha_i\}$ for all $i \in \omega$ (β some ordinal such that $|\beta| = \lambda$), so that $\alpha_i \neq i$ for all $i < \beta$. Then for each $i \in \omega$ there is a $\nu i < \beta$ such that $\alpha_{\nu i} \in b_i \backslash a_i$. Suppose that $\{[b_i, v]^{\text{if } i \in S} : i \in \omega\}$ is realized in $A(C)$, say by $(e \cap C) \cup (g \backslash C)$, where $e, g \in A$. Let $I = \{\alpha_i : i \in \omega, \alpha_i \in C\}$, $J = \{\alpha_i : i \in \omega, \alpha_{\nu i} \in e\}$, $K = \{\alpha_i : i \in \omega, \alpha_{\nu i} \in g\}$. By Lemma 4.12, I is infinite. If $\alpha_i \in I$, then $i \in S$, hence $b_i \subseteq (e \cap C) \cup (g \backslash C)$ and so $\alpha_{\nu i} \in (e \cap C) \cup (g \backslash C)$. Thus, $I \subseteq J \cup K$. Hence, J is infinite or K is infinite. If J is infinite, by Lemma 4.12 choose $\alpha_i \in J$ so that $\alpha_i \not\in C$ and $\alpha_{\nu i} \in C$. Then $i \not\in S$, so $b_i \cap e \cap C = 0$; but $\alpha_{\nu i} \in b_i \cap e \cap C$ since $\alpha_i \in J$, a contradiction. If K is infinite, by Lemma 4.12 choose $\alpha_i \in K$ so that $\alpha_i \not\in C$ and $\alpha_{\nu i} \not\in C$; again a contradiction is reached. □

4.16. LEMMA. *Let M, λ, A, N, C be as in Lemma* 4.15. *Suppose that $\{[h_n, v]^{\text{if } n \in T} : n \in \omega\}$ is a type over A omitted in A. Then it is also omitted in $A(C)$.*

PROOF. Suppose, on the contrary, that it is realized by $d \cup (e \cap C) \cup (g \backslash C)$, where d, e, g are pairwise disjoint elements of A. Introducing the notation following (41), we easily see, as in (42), that $l \subseteq C$ and $m \cap C = 0$. Since $l, m \in M$, we infer from Lemma 4.12 that l and m are finite. Let $t = d \cup ((l \cup m) \cap \bigcup_{n \in T} h_n)$. It is easily checked that $t \in A$ and t realizes $\{[h_n, v]^{\text{if } n \in T} : n \in \omega\}$, a contradiction. □

4.17. THEOREM. *Let M be a model of ZFC + CH, and let γ be a cardinal of M such that $\gamma^{\omega} = \gamma$. Add γ Cohen reals to M, forming the new model N (in which $2^{\omega} = \gamma$). Assume that κ, λ are cardinals in N such that $\omega \leqq \lambda \leqq \kappa \leqq 2^{\omega}$, $\kappa \geqq \omega_1$. Then in N there is a mildly complicated subalgebra A of $\mathcal{P}\lambda$ containing all singletons, with $|A| = \kappa$; thus A is almost rigid and dual hopfian.*

PROOF. We conceive the extension from M to N to take place in the following two steps: first add γ Cohen reals, forming M', then add $\lambda \cdot \omega_1$ further Cohen reals to get N. Write $N = M'[G]$, G generic with resepct to $\mathrm{Fin}(\lambda \times \omega_1)$, and for each $\beta < \omega_1$ let $G_\beta = \{f \restriction (\lambda \times \beta): f \in G\}$, $G'_\beta = \{f \restriction (\lambda \times (\omega_1 \backslash \beta)): f \in G\}$. Thus, $N = M'[G_\beta][G^\beta]$ for any $\beta < \omega_1$. Now we define two sequences $\langle \beta\alpha: \alpha < \omega_1 \rangle$, $\langle C_\alpha: \alpha < \omega_1 \rangle$ by induction. Let $\beta 0 = 0$, $\beta(\alpha + 1) = \beta\alpha + 1$, and for any $\alpha < \omega_1$ let $H = \{f \restriction (\lambda \times \{\beta\alpha\}): f \in G\}$, generic over $M'[G_{\beta\alpha}]$, and set $C_\alpha = \{\alpha < \lambda$: $f(\gamma, \beta\alpha) = 1$ for some $f \in H\}$, so C_α is a generic subset of λ over $M'[G_{\beta\alpha}]$. For λ limit $<\omega_1$ we have $\langle C_\alpha: \alpha < \gamma \rangle \in M'[G_{\beta\gamma}]$ for some $\beta\gamma < \omega_1$ by Lemma 4.14. This finishes the construction of the two sequences.

For any $\alpha < \omega_1$ let B_α be the subalgebra of $\mathscr{P}\lambda$ generated by $\{\{\gamma\}: \gamma < \lambda\} \cup \{C_\gamma: \gamma < \alpha\}$, and let $A = \bigcup_{\alpha < \omega_1} B_\alpha$. We claim that A is the desired algebra. Let $\langle (a_i, b_i): i < \omega \rangle$ be a weak candidate over A. Then there is an $\alpha < \omega_1$ such that it is a weak candidate over B_α. By Lemmas 4.15 and 4.16, it is omitted in A, as desired.

4.18. COROLLARY. *It is consistent with ZFC that if $\omega \leqq \lambda \leqq \kappa \leqq 2^\omega$, $\kappa \geqq \omega_1$, κ and λ cardinals, then there is a mildly complicated subalgebra A of $\mathscr{P}\lambda$ containing all singletons, with $|A| = \kappa$, and with 2^ω arbitrarily large.* □

Problems

We conclude this chapter with some problems concerning endomorphisms of Boolean algebras.

PROBLEM 1. *Associate with every BA a subsemigroup A' of* End A *such that every member of A' takes on infinitely many values, and such that $A' \cong B'$ implies that $A \cong B$.*

PROBLEM 2 (GCH). *If κ is singular or the successor of a singular cardinal, is there a BA A such that $|A| = |\mathrm{End}\ A| = \kappa$?*

PROBLEM 3. *Can one prove in ZFC that there are arbitrarily large cardinals κ for which there is a BA A with $|A| = |\mathrm{End}\ A| = \kappa$?*

PROBLEM 4. *Is $|\mathrm{End}\ A| \leqq |\mathrm{Sub}\ A|$ for infinite A?*

PROBLEM 5. *Can one prove in ZFC that there is a BA which is both endo-rigid and Bonnet-rigid?*

PROBLEM 6. *Is there an infinite endo-rigid BA A which has exactly $|A|$ endomorphisms?*

PROBLEM 7. *Is there an infinite endo-rigid, Bonnet-rigid BA A with exactly $|A|$ endomorphisms?*

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CHAPTER 14

Automorphism Groups

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

In this chapter we describe what is known about automorphism groups of BAs, exclusive of results concerning rigid BAs which are treated in other chapters in this Handbook. No characterization is known of those groups isomorphic to Aut A for some BA A. In Section 1 we show how the general study of automorphism groups can be reduced to several cases: automorphism groups of products of rigid BAs, of products of homogeneous BAs, and of BAs with no rigid or homogeneous elements. Section 2 is devoted to the study of the relative automorphism group $\mathrm{Aut}_A B = \{f \in \mathrm{Aut}\, B\colon f \upharpoonright A$ is the identity$\}$, when B is obtained from A by adjoining a single element (and hence all Boolean combinations of it with elements of A). These groups turn out to be very simple to describe and work with. Section 3 does the same when B is obtained from A by adjoining finitely many elements; then the situation is more complicated. Finally, in Section 4 we discuss the size of automorphism groups. In the general case a fairly complete description of the relationship between $|A|$ and $|\mathrm{Aut}\, A|$ is known, but there are still open problems when we restrict attention to classes of BAs such as interval algebras or superatomic BAs. Section 4 can be read directly after Section 1.

1. General properties

Direct product decompositions of BAs enable us to break the analysis of automorphism groups into several cases. Most of the results of this section are taken from McKenzie and Monk [1975]. We begin with these considerations.

1.1. Lemma. *If $\langle A_j\colon i \in I\rangle$ is a system of similar algebras, then $\Pi_{i\in I}\,\mathrm{Aut}\, A_i$ can be isomorphically embedded in $\mathrm{Aut}\,\Pi_{i\in I} A_i$.*

Proof. For each $f \in \Pi_{i\in I}\,\mathrm{Aut}\, A_i$, each $i \in I$, and each $x \in \Pi_{i\in I} A_i$ let $(Ff)_x i = f_i x_i$. It is easily verified that F is the desired isomorphic embedding. □

Now we shall call BAs A, B *totally different* if, for all $a \in A^+$ and $b \in B^+$, we have $A \upharpoonright a \not\cong B \upharpoonright b$.

In several of the proofs below we shall use the following construction. Given a system $\langle A_i\colon i \in I\rangle$ of BAs, an index $j \in I$, and an element $a \in A_j$, we denote by $\xi_j a$ the element of $\Pi_{i\in I} A_i$ such that $(\xi_j a)_i = 0$ if $i \neq j$, while $(\xi_j a)_j = a$.

1.2. Theorem. *If $\langle A_i\colon i \in I\rangle$ is a system of pairwise totally different BAs, then*

$$\prod_{i\in I} \mathrm{Aut}\, A_i \cong \mathrm{Aut} \prod_{i\in I} A_i\,.$$

Proof. We show that the function F defined in the proof of Lemma 1.1 is onto. Let $g \in \mathrm{Aut}\,\Pi_{i\in I} A_i$; we want to find $f \in \Pi_{i\in I}\,\mathrm{Aut}\, A_i$ such that $Ff = g$. To do this, we need three auxiliary statements.

(1) If $i \in I$, $x \in \prod_{i\in I} A_i$, and $(gx)_i \neq 0$, then $x_i \neq 0$.

For, assume otherwise. Let $y = \xi_i(gx)_i$. Thus, $y \leq gx$, so $g^{-1}y \leq x$. Choose j so that $(g^{-1}y)_j \neq 0$; thus $j \neq i$. Now $\xi_j(g^{-1}y)_j \leq g^{-1}y \leq x$, so $\langle (g\xi_j a)_i : a \leq (g^{-1}y)_j \rangle$ is an isomorphism from $A_j \restriction (g^{-1}y)_j$ onto $A_i \restriction (g\xi_j(g^{-1}y)_j)_i$, a contradiction. So (1) holds.

(2) If $i, j \in I$, $i \neq j$, and $a \in A_i$, then $(g\xi_i a)_j = 0$.

This is immediate from (1).

(3) If $i \in I$, then $g\xi_i 1 = \xi_i 1$.

For, by (2) write $g\xi_i 1 = \xi_i a$ and $g^{-1}\xi_i(-a) = \xi_i b$. Then $\xi_i b = \xi_i b \cdot \xi_i 1 = g^{-1}\xi_i(-a) \cdot g^{-1}\xi_i a = 0$, so $b = 0$ and $a = 1$. So (3) holds.

Now define $f_i a = (g\xi_i a)_i$ for any $i \in I$ and $a \in A_i$. By (2) and (3) it is clear that $f \in \Pi_{i\in I} \operatorname{Aut} A_i$. To show that $Ff = g$, let $x \in \Pi_{i\in I} A_i$ and $i \in I$; we show that $(Ff)_x i = (gx)_i$. Now $(Ff)_x i = f_i x_i = (g\xi_i x_i)_i$. Since $\xi_i x_i \leq x$, we have $(g\xi_i x_i)_i \leq (gx)_i$. Also, $(x \cdot -\xi_i x_i)_i = 0$, so by (1) $(gx \cdot -g\xi_i x_i)_i = 0$, i.e. $(gx)_i \leq (g\xi_j x_j)_i$, as desired. □

Theorem 1.2 has several useful corollaries. Thus, if $\langle A_i : i \in I \rangle$ is a system of pairwise totally different rigid BAs, then $\Pi_{i\in I} A_i$ is rigid. If A and B are totally different and B is rigid, then $\operatorname{Aut}(A \times B) \cong \operatorname{Aut} A$. If A is infinite and homogeneous and B is rigid, then $\operatorname{Aut}(A \times B) \cong \operatorname{Aut} A$.

At least for complete BAs, the study of Aut A breaks into three cases by the next theorem.

1.3. THEOREM. *Let A be a complete BA. Then there exist B, C, D such that $A \cong B \times C \times D$, B is a product of homogeneous BAs, C is a product of rigid BAs, and D has no rigid or homogeneous direct factors. Furthermore,* $\operatorname{Aut} A \cong \operatorname{Aut} B \times \operatorname{Aut} C \times \operatorname{Aut} D$.

PROOF. Let $a = \Sigma\{x : \forall y \in (A \restriction x)^+ \, \exists z \in (A \restriction y)^+ \, (A \restriction z$ is homogeneous$)$, and let b be defined similarly with "homogeneous" replaced by "rigid". Then $A \restriction a$, $A \restriction b$, $A \restriction (-a \cdot -b)$ may be taken for B, C, D. □

The decomposition in Theorem 1.3 is clearly unique. It is natural now to consider the three cases in 1.3 in turn, even for non-complete BAs. First, however, we give a general fact about isomorphism of direct powers which will be used below.

1.4. THEOREM. *Let $|I| \leq \kappa \geq |J|$, let A be a κ^+-complete BA, and suppose that $\langle a_{ij} : i \in I, j \in J \rangle$ is a system of elements of A such that $\forall i \in I \, \langle A_{ij} : j \in J \rangle$ is a partition of unity and $\forall j \in J \, \langle a_{ij} : i \in I \rangle$ is a partition of unity (we allow zeros in a partition of unity). For any $x \in {}^I A$ and any $j \in J$ let $(fx)_j = \Sigma_{i\in I} x_i \cdot a_{ij}$. Then f is an isomorphism from ${}^I A$ onto ${}^J A$.*

PROOF. We define the inverse g of f. For any $y \in {}^J A$ and $i \in I$ let $(gy)_i = \Sigma_{i \in J} y_j \cdot a_{ij}$. Then for any $y \in {}^J A$ and $j \in J$ we have

$$(fgy)_j = \sum_{i \in I} (gy)_i \cdot a_{ij} = \sum_{i \in I} y_j \cdot a_{ij} = y_j .$$

Thus, $fgy = y$, so $f \circ g$ is the identity. Similarly, $g \circ f$ is the identity. Clearly, $x \leq z$ implies that $fx \leq fz$ for $x, z \in {}^I A$, and analogously for g, so f is the desired isomorphism. □

1.1. *Products of rigid BAs*

The following simple lemma will be fundamental for what follows.

1.5. LEMMA. *Let $a, b \in A^+$, $a \neq b$, and let f be an isomorphism from $A \restriction a$ onto $A \restriction b$. Then there exist disjoint non-zero $c \leq a$, $d \leq b$ such that $fc = d$.*

PROOF. If $a \not\leq b$ we let $c = a \cdot -b$, $d = f(a \cdot -b)$, while if $b \not\leq a$ we let $c = f^{-1}(b \cdot -a)$, $d = b \cdot -a$. □

This lemma has two immediate corollaries worth mentioning. If $f \in \text{Aut } A$ is non-trivial (i.e. not the identity), then there is an $a \in A^+$ with $a \cdot fa = 0$; hence, $A \cong B \times B \times C$, where $B = A \restriction a$ and $C = A \restriction (-a \cdot -fa)$. If A is rigid, then $A \restriction a \not\cong A \restriction b$ for any two distinct elements $a, b \in A$.

It can also be shown that for an BA A, Aut A has a non-trivial center iff $A \cong B \times B \times C$ for some non-trivial rigid B and some C such that B and C are totally different; see MCKENZIE and MONK [1975, Theorem 1.16].

From Lemma 1.5 it follows in particular that if Aut A is non-trivial, then it has an element of order 2 – hence not every group is the automorphism group of a BA. This result can be generalized as follows.

1.6. THEOREM. *Let A be an infinite BA, and G the direct sum of $|A|$ copies of the two-element group. Then* Aut$(A \times A)$ *has a subgroup isomorphic to G. In the case where A is rigid,* Aut$(A \times A)$ *is actually isomorphic to G.*

PROOF. For each $a \in A$, let f_a be the automorphism of $A \times A$ pictured as follows:

$$\begin{aligned}(x, y) &\mapsto (x \cdot a, x \cdot -a, y \cdot a, y \cdot -a)\\ &\mapsto (y \cdot a, x \cdot -a, x \cdot a, y \cdot -a)\\ &\mapsto (y \cdot a + x \cdot -a, x \cdot a + y \cdot -a) .\end{aligned}$$

If $a, b \in A$ and $a \neq b$, say $a \cdot -b \neq 0$, then $f_b(a, 0) = (a \cdot -b, a \cdot b) \neq (0, a) = f_a(a, 0)$, so $f_a \neq f_b$. Clearly, f_a has order 2, and $f_a \circ f_b = f_b \circ f_a = f_{a \triangle b}$ for any $a, b \in A$. Hence, $\{f_a \colon a \in A\}$ is isomorphic to G.

Now let A be rigid, and let g be any automorphism of $A \times A$. Say $g(1, 0) = (a, b)$, and then choose c so that $g(c, 0) = (a, 0)$. The mapping

$x \mapsto (x, 0) \mapsto_g (y, 0) \mapsto y$ is an isomorphism of $A \restriction c$ onto $A \restriction a$, so by the above remarks, $a = c$. Thus, $g(a, 0) = (a, 0)$ and $g(1, 0) = g(a, 0) + g(-a, 0)$, so $g(-a, 0) = (0, b)$, and the above remarks give $b = -a$. Also by the same arguments, $g(0, 1) = (-a, a)$, $g(0, a) = (0, a)$, $g(0, -a) = (-a, 0)$. Hence,

$$\begin{aligned} g(x, y) &= g(x \cdot a + x \cdot -a, y \cdot a + y \cdot -a) \\ &= g(x \cdot a, 0) + g(x \cdot -a, 0) + g(0, y \cdot a) + g(0, y \cdot -a) \\ &= (x \cdot a, 0) + (0, x \cdot -a) + (0, y \cdot a) + (y \cdot -a, 0) \\ &= f_{-a}(x, y), \end{aligned}$$

and $g = f_{-a}$, finishing the proof. $\square$

The subgroup of $\mathrm{Aut}(A \times A)$ constructed in the proof of 1.6 is not in general normal; for example, $\mathrm{Fr}\,\kappa \times \mathrm{Fr}\,\kappa \cong \mathrm{Fr}\,\kappa$, and $\mathrm{Aut\,Fr}\,\kappa$ is simple, for $\kappa \geq \omega$.

1.7. COROLLARY. *If A is atomless and* Aut A *is non-trivial, then* $|\mathrm{Aut}\, A| \geq \aleph_1$.

1.8. COROLLARY. *If A is infinite and* Aut A *is finite, then A is isomorphic to some product $B \times C$, where B is finite, and C is infinite, atomless, and rigid. Furthermore,* Aut A *is then isomorphism to some finite symmetric group – namely to the group of all permutations of the atoms of B.*

After these preliminaries, we now discuss automorphisms of products of rigid BAs. The following result describes automorphisms of a power of a rigid (complete) BA A in terms of elements of A.

1.9. THEOREM. *Let $|I| \cup |J| = \kappa$, let A be a rigid κ^+-complete BA, and let $f: {}^IA \to {}^JA$. Then the following conditions are equivalent:*
(i) *f is an isomorphism from ${}^IA$ onto ${}^JA$.*
(ii) *There is an a satisfying the conditions of* 1.4 *with respect to f.*

PROOF. (ii)$\Rightarrow$(i) is given by Theorem 1.4. Now assume (i). For any $i \in I$ and $j \in J$ let $a_{ij} = (f\xi_i 1)_j$. Now $\langle \xi_1 1: i \in I\rangle$ is a partition of unity, so

(4) $\langle a_{ij}: i \in I\rangle$ is a partition of unity for each $j \in J$.

(5) For any $x \in A$, $i \in I$, and $j \in J$, $f\xi_i(x \cdot a_{ij}) = \xi_j(x \cdot a_{ij})$.

For, we have $\xi_j(x \cdot a_{ij}) = \xi_j(x \cdot (f\xi_i 1)_j) \leq f\xi_i 1$, so choose u so that $f\xi_i u = \xi_j(x \cdot a_{ij})$. Now $\langle (f\xi_i y)_y: y \leq u\rangle$ is an isomorphism of $A \restriction u$ onto $A \restriction (x \cdot a_{ij})$, so $u = x \cdot a_{ij}$. Hence (5) follows.

(6) $\langle a_{ij}: j \in J\rangle$ is a partition of unity for each $i \in I$.

In fact, if $j, k \in J$ and $j \neq k$, then

$$0=\xi_j a_{ij}\cdot\xi_k a_{ik}=f\xi_i a_{ij}\cdot f\xi_i a_{ik}\quad\text{by (5)},$$

so $\xi_i a_{ij}\cdot\xi_i a_{ik}=0$ and hence $a_{ij}\cdot a_{ik}=0$. Furthermore,

$$\begin{aligned}f\xi_i 1&=\sum_{j\in J}\xi_j(f\xi_i 1)_j=\sum_{j\in J}\xi_j a_{ij}\\&=\sum_{j\in J}f\xi_i a_{ij}\quad\text{by (5)}\\&=f\xi_i\sum_{j\in J}a_{ij},\end{aligned}$$

and hence $1=\Sigma_{j\in J}a_{ij}$. So (6) holds.

(7) For any $x\in A$, $i\in I$, and $j\in J$, $(f\xi_i x)_j=x\cdot a_{ij}$.

For,

$$\begin{aligned}(f\xi_i x)_j&=\Big(f\sum_{k\in J}\xi_i(x\cdot a_{ij})\Big)_j\\&=\sum_{k\in J}(f\xi_i(x\cdot a_{ik}))_j\\&=\sum_{k\in J}(\xi_k(x\cdot a_{ik}))_j\quad\text{by (5)}\\&=x\cdot a_{ij},\end{aligned}$$

as desired. Finally, if $x\in {}^I\!A$ and $j\in J$, then

$$\begin{aligned}(fx)_j&=\Big(f\sum_{i\in I}\xi_i x_i\Big)_j=\sum_{i\in I}(f\xi_i x_i)_j\\&=\sum_{i\in I}x_i\cdot a_{ij}\quad\text{by (7)},\end{aligned}$$

and the proof is complete. □

The case $I=J$ in Theorem 1.9 gives a characterization of $\mathrm{Aut}({}^I\!A)$. In particular, we obtain:

1.10. COROLLARY. *If A is an infinite rigid BA and $2\le m<\omega$, then* $|\mathrm{Aut}({}^m\!A)|=|A|$.

A characterization of $\mathrm{Aut}({}^m\!A)$ different from that in 1.9 can be given, still for $2\le m<\omega$: for A rigid, $\mathrm{Aut}({}^m\!A)$ is isomorphic to the subgroup of ${}^{\mathrm{Ult}\,A}\mathrm{Sym}(m)$ consisting of all continuous functions f: Ult $A\to\mathrm{Sym}(m)$, where $\mathrm{Sym}(m)$ has the discrete topology: see MCKENZIE and MONK [1975, Theorem 1.12].

Having given a kind of characterization of $\mathrm{Aut}({}^I\!A)$ for A rigid, the next natural problem is to describe when ${}^I\!A$ and ${}^I\!B$ are isomorphic for A and B rigid. To do this, we make a slight digression.

An important notion for any BA A is its *invariant subalgebra*: Inv $A=\{a\in$

A: $fa = a$ for every $f \in \text{Aut } A\}$. Note that $0, 1 \in \text{Inv } A$ and if $0 < a < 1$, then $a \in \text{Inv } A$ iff $A \upharpoonright a$ and $A \upharpoonright (-a)$ are totally different.

1.11. THEOREM. *Let A be rigid and let I be a non-empty set. Then* $\text{Inv}({}^I A)$ *is the diagonal subalgebra of ${}^I A$, consisting of all constant functions in ${}^I A$; in particular,* $\text{Inv}({}^I A) \cong A$.

PROOF. For each $a \in A$, let c_a be the member of ${}^I A$ such that $c_a i = a$ for all $i \in I$. Then

$$(8) \qquad fc_a = c_a \quad \text{for all } a \in A \text{ and all } f \in \text{Aut } A\,.$$

(We cannot obtain this from 1.9, since we have no completeness assumptions.) In fact, suppose that $fc_a \neq c_a$. Choose $i \in I$ so that $(fc_a)i \neq a$. Then $(fc_a)i \nleq a$ or $a \nleq (fc_a)i$. Assume that $(fc_a)i \nleq a$, and let $b = (fc_a)i \cdot -a$. Thus, $\xi_i b \leq fc_a$, so $f^{-1}(\xi_i b) \leq c_a$. Since $b \neq 0$, there is a $j \in I$ such that $(f^{-1}(\xi_i b))j \neq 0$. Now $(f^{-1}(\xi_i b))j \leq a$, $\xi_j(f^{-1}(\xi_i b))j \leq f^{-1}(\xi_i b)$ and hence $f\xi_j(f^{-1}(\xi_i b))j \leq \xi_i b \leq \xi_i(-a)$. Hence, a and $-a$ are not totally different, contradicting the rigidity of A. $a \nleq (fc_a)i$ is treated similarly. So (8) holds.

Now suppose that $x \in {}^I A$ and x is not a constant mapping. Choose $i, j \in I$ so that $x_i \nleq x_j$. Let $a = x_i \cdot -x_j$, so that $a \neq 0$. Then $\xi_i a \leq x$ and $\xi_j a \leq -x$, so x and $-x$ are not totally different. Hence $x \notin \text{Inv}({}^I A)$. $\square$

1.12. THEOREM. *If A, B are rigid, $I \neq 0 \neq J$, and ${}^I A \cong {}^J B$, then $A \cong B$. Moreover, if I is finite and $|A| > 1$, then $|I| = |J|$.*

PROOF. By 1.11, $A \cong B$. Now assume that I is finite. We show that $|J| \leq |I|$ so that, by symmetry, $|I| = |J|$. Suppose that $|J| > |I|$. Then ${}^I A$ has a system $\langle x^k \colon k \leq |I| \rangle$ of non-zero, pairwise disjoint, pairwise isomorphic elements. For any fixed $m \leq |I|$ it is easy to convert such a system into a similar one $\langle \bar{x}^k \colon k \leq |I| \rangle$ with the same properties, where additionally $\bar{x}^m \leq \xi_i 1$ for some i. Thus, in $|I| + 1$ steps we obtain such a system with $\forall m \leq |I|\ \exists im \in I\ (\bar{x}^m \leq \xi_{im} 1)$. Since A is rigid, it follows that for each $j \in I$, $\xi_j 1$ does not contain two disjoint non-zero isomorphic elements. Hence, i is a one-to-one function, a contradiction. $\square$

Note that for any non-trivial finite BA A, if ${}^I A \cong {}^J A$ then $|I| = |J|$. For any infinite BA A, if $|I| \leq |J| \leq \text{sat } A$ and ${}^I A \cong {}^J A$, then $|I| = |J|$. In fact, $\text{sat}({}^I A) = |I|^+ \cup \text{sat } A$ and $\text{sat}({}^J A) = |J|^+$, so this is clear. (Recall that $\text{sat } A = \min\{\kappa \colon |X| < \kappa$ for every disjoint system $X \subseteq A\}$.) In the remaining cases we can have ${}^I A \cong {}^J A$ with $|I| \neq |J|$:

1.13. THEOREM. *Suppose $\omega \leq \mu \leq \lambda$. Then there is a rigid complete BA A such that ${}^\mu A \cong {}^\lambda A$.*

PROOF. Let B be freely generated by $\langle x_{\alpha\beta}: \alpha<\mu, \beta<\lambda\rangle$, and set

$$I=\langle\{x_{\alpha\beta}\cdot x_{\alpha\gamma}: \alpha<\mu, \beta, \gamma<\lambda, \beta\neq\gamma\}\cup$$
$$\{x_{\alpha\beta}\cdot x_{\gamma\beta}: \beta<\lambda, \alpha, \gamma<\mu, \alpha\neq\gamma\}\rangle^{\mathrm{id}}.$$

Set $C=B/I$ and $a_{\alpha\beta}=[x_{\alpha\beta}]$ for all $\alpha<\mu$, $\beta<\lambda$, where $[x_{\alpha\beta}]$ is the image of $x_{\alpha\beta}$ under the natural map $B\to C$. We claim now:

$$\text{(9)}\qquad \forall\alpha<\mu\Big(\sum_{\beta<\lambda}{}^{\bullet} a_{\alpha\beta}=1\Big) \quad\text{and}\quad \forall\beta<\lambda\Big(\sum_{\alpha<\mu} a_{\alpha\beta}=1\Big).$$

By symmetry it suffices to prove the first part of (9). Assume that $\alpha<\mu$, $c\in C$, and $c\cdot a_{\alpha\beta}=0$ for all $\beta<\lambda$. Say $c=[y]$. There is a finite $\Gamma\subseteq\mu\times\lambda$ such that $y\in\{x_{\gamma\delta}: (\gamma,\delta)\in\Gamma\}$. Pick $\beta<\lambda$ such that $(\gamma,\beta)\not\in\Gamma$ for all γ. Now $y\cdot x_{\alpha\beta}\in I$, so we can write

$$\text{(10)}\qquad y\cdot x_{\alpha\beta}\leq x_{\gamma 1,\delta 1}\cdot x_{\gamma 1,\varepsilon 1}+\cdots+x_{\gamma m,\delta m}\cdot x_{\gamma m,\varepsilon m}$$
$$+x_{\xi 1,\eta 1}\cdot x_{\gamma 1,\eta 1}+\cdots+x_{\xi n,\eta n}\cdot x_{\gamma n,\eta n},$$

with obvious assumptions. There is an endomorphism f of B such that $fx_{\theta\psi}=x_{\theta\psi}$ for all $(\theta,\psi)\in\Gamma$, $fx_{\alpha\beta}=1$, and $fx_{\theta\psi}=0$ if $(\theta,\psi)\not\in\Gamma$ and $(\theta,\psi)\neq(\alpha,\beta)$. Note that if $(\xi i,\eta i)=(\alpha,\beta)$, then $fx_{\gamma i,\eta i}=0$, and if $(\gamma i,\eta i)=(\alpha,\beta)$, then $fx_{\xi i,\eta i}=0$ $(i=1,\ldots,n)$. It follows that if we apply f to (10) and then apply the natural homomorphism of B onto C we get an expression of the form:

$$[y]\leq[x_{\alpha,\rho 1}]+\cdots+[x_{\alpha,\rho p}].$$

Since $[y]\cdot[x_{\alpha,\rho i}]=0$ for each $i=1,\ldots,p$, it follows that $[y]=0$, as desired.

Now it is known that C can be completely embedded in a rigid complete BA A. By (9) and Theorem 1.4 our theorem follows. □

The above results give a fairly complete picture of possibilities for ${}^{I}A$, A rigid. Now we discuss how the powers ${}^{I}A$, ${}^{J}B$ can be combined.

1.14. LEMMA. *Let A be a complete BA with at least one non-trivial rigid element. (An element a is **rigid** provided that $A\restriction a$ is rigid.) Then there is a non-empty collection C of rigid, pairwise disjoint and isomorphic, non-zero elements of A such that $\Sigma\, C$ and $-\Sigma\, C$ are totally different.*

PROOF. Let y be a non-zero rigid element of A. By Zorn's lemma let D be a maximal family of pairwise disjoint elements of A each isomorphic to y, and with $y\in D$. For each $d\in D$ let f_d be an isomorphism of $A\restriction y$ onto $A\restriction d$. Let E be a maximal collection of pairwise disjoint elements $(d,e)\in(A\restriction y)\times(A\restriction -\Sigma\, D)$ such that $A\restriction d\cong A\restriction e$. For each $(d,e)\in E$ let g_{de} be an isomorphism of $A\restriction d$ onto $A\restriction e$. Let $z=\Sigma_{(d,e)\in E}\, d$, and set $x=y\cdot -z$. Then $x\neq 0$ since D is maximal, and x is rigid since $x \leq y$. Let $C = \{f_u x: u \in D\}$. Thus, C is a collection of pairwise disjoint elements of A each isomorphic to x; also, $x \in C$ since $x = f_y x$ and $y \in D$. To complete the proof it suffices to derive a contradiction from the assumptions $0 \neq v \leq \Sigma\, C$, $w \leq -\Sigma\, C$, h an isomorphism from $A \restriction v$ onto $A \restriction w$. Choose $u \in D$ such that $v \cdot f_u x \neq 0$. Now there are three cases.

Case 1. $\exists t \in D[h(v \cdot f_u x) \cdot t \neq 0]$. Thus, $s \overset{\mathrm{def}}{=} h(v \cdot f_u x) \cdot f_t z \neq 0$, since $f_t y = t$ and $f_t(y \cdot -z) = f_t x \leq \Sigma\, C$ while $h(v \cdot f_u x) \leq -\Sigma\, C$. Then $f_t^{-1} s$ and $f_u^{-1} h^{-1} s$ are isomorphic disjoint non-zero subelements of y, a contradiction; $f_t^{-1} s \leq z$ and $f_u^{-1} h^{-1} s \leq x$, so $f_t^{-1} s \cdot f_u^{-1} h^{-1} s = 0$.

Case 2. $\forall t \in D[h(v \cdot f_u x) \cdot t = 0]$ but $\exists (d, e) \in E[h(v \cdot f_u x) \cdot e \neq 0]$. This time $g_{de}^{-1}(h(v \cdot f_u x) \cdot e)$ and $f_u^{-1} h^{-1}(h(v \cdot f_u x) \cdot e)$ are isomorphic disjoint non-zero elements of y, a contradiction.

Case 3. $h(v \cdot f_u x) \leq -\Sigma\, D \cdot -\Sigma_{(d,e)\in E}\, e$. Then $(f_u^{-1}(v \cdot f_u x), h(v \cdot f_u x)) \in (A \restriction y) \times (A \restriction -\Sigma\, D)$, $A \restriction f_u^{-1}(v \cdot f_u x) \cong A \restriction h(v \cdot f_u x)$, and $(f_u^{-1}(v \cdot f_u x), h(v \cdot f_u x)) \cdot (d, e) = 0$ for all $(d, e) \in E$, contradicting the maximality of E. □

1.15. THEOREM. *Let A be a complete BA in which the rigid elements are dense. Then there exists a system $\langle B_\alpha: \alpha < B \rangle$ of non-trivial pairwise totally different rigid BAs and a strictly increasing sequence $\langle \kappa\alpha: \alpha < \beta \rangle$ of non-zero cardinals such that $A \cong \Pi_{\alpha<\beta}\, {}^{\kappa\alpha}B_\alpha$.*

PROOF. By an easy transfinite construction using Lemma 1.14 we can write $A \cong \Pi_{\alpha<\beta}\, {}^{\kappa\alpha}B_\alpha$, as in the theorem, except that $\langle \kappa\alpha: \alpha < \beta \rangle$ is just a sequence of non-zero cardinals. But we can assume that $\alpha < \gamma < \beta$ implies $\kappa\alpha \leq \kappa\gamma$. Now for any $\alpha < \beta$ we have

$$\Pi \{{}^{\kappa\gamma}B_\gamma: \kappa\alpha = \kappa\gamma\} = {}^{\kappa\alpha}\Pi \{B_\gamma: \kappa\alpha = \kappa\gamma\}\,,$$

and by Theorem 1.2, $\Pi \{B_\gamma: \kappa\alpha = \kappa\gamma\}$ is rigid. The theorem follows. □

The representation in Theorem 1.15 is not unique, by Theorem 1.13. It is possible to refine this representation so as to obtain uniqueness; see MCKENZIE and MONK [1975, Theorem 1.21].

1.2. Products of homogeneous BAs

Products of homogeneous algebras, and their automorphisms, can be analyzed much as for products of rigid algebras.

1.16. THEOREM. *If A is a complete homogeneous BA and A has a disjoint family of size $|I|$, then ${}^I A \cong A$.*

1.17. THEOREM. *If A and B are non-trivial homogeneous BAs, A has no disjoint family of size $|I|$, and ${}^I A \cong {}^J B$, then $|I| = |J|$ and $A \cong B$.*

PROOF. Let f be the isomorphism from ${}^{I}A$ onto ${}^{J}B$. A and B have isomorphic non-zero elements, so $A \cong B$. Suppose $|I| \neq |J|$; wlog say $|I| > |J|$. For each $j \in J$ let $K_j = \{i \in I\colon f(\xi_i 1)_j \neq 0\}$. Since A has no disjoint family of size $|I|$, each set K_j has power $<|I|$. Thus, for every $j \in J$, A has a disjoint family of size $|K_j| < |I|$, and $|I| = \Sigma_{j \in J} |K_j|$. Thus, $|I|$ is singular, so by the Erdös–Tarski theorem A has a disjoint family of size $|I|$, a contradiction. □

1.18. THEOREM. *Let A be a complete BA in which the homogeneous elements are dense. Then there is a non-decreasing sequence $\langle \kappa_\alpha\colon \alpha < \beta \rangle$ of non-zero cardinals and a system $\langle B_\alpha\colon \alpha < \beta \rangle$ of pairwise totally different, non-trivial, homogeneous BAs such that for every $\alpha < \beta$ with $\kappa_\alpha > 1$, B_α has no disjoint family of size κ_α, and $A \cong \Pi_{\alpha<\beta} {}^{\kappa_\alpha}B_\alpha$. The representation is unique: if $A \cong \Pi_{\alpha<\gamma} {}^{\lambda_\alpha}C_\alpha$ with similar conditions, then $\beta = \gamma$, $\kappa_\alpha = \lambda_\alpha$ for each $\alpha < \beta$, and for each $\alpha < \beta$ there is a permutation π of $\{\delta\colon \kappa_\delta = \kappa_\alpha\}$ such that $B_\delta \cong C_{\pi\delta}$ for each such δ.*

PROOF. Given any homogeneous element a of A, there is a maximal disjoint family C such that $a \in C$ and all elements of C are isomorphic. Thus, $\Sigma\, C$ and $-\Sigma\, C$ are totally different. Repeating this construction transfinitely, we easily arrive at the indicated representation.

Now suppose that another representation is given, as indicated; say f is an isomorphism from $\Pi_{\alpha<\beta} {}^{\kappa_\alpha}B_\alpha$ onto $\Pi_{\alpha<\beta} {}^{\lambda_\alpha}C_\alpha$. By Theorem 1.17 it suffices now to take any $\alpha < \beta$ and find $\delta < \gamma$ such that ${}^{\kappa_\alpha}B_\alpha \cong {}^{\lambda_\delta}C_\delta$. Since the C_δ's are pairwise totally different, we know that there is a unique $\delta < \gamma$ such that $(f\xi_\alpha 1)_\delta \neq 0$. Thus, ${}^{\kappa_\alpha}B_\alpha$ is isomorphic to an element of ${}^{\lambda_\delta}C_\delta$. Hence, $B_\alpha \cong C_\delta$ and $\kappa_\alpha \leq \lambda_\delta$. By symmetry, there is an $\varepsilon < \beta$ such that $C_\delta \cong B_\varepsilon$ and $\lambda_\delta \leq k_\varepsilon$. So $\alpha = \varepsilon$ and $\kappa_\alpha = \lambda_\delta$, as desired. □

This representation theorem has the following consequence for automorphisms: Aut A is isomorphic to $\Pi_{\alpha<\beta}\,\mathrm{Aut}({}^{\kappa_\alpha}B_\alpha)$. Thus, for a complete BA in which the homogeneous elements are dense, the automorphism problem reduces to considering automorphisms of complete BAs ${}^{I}A$, A homogeneous. Not much is known about these automorphisms. If A is homogeneous and complete, then Aut A is simple. See ŠTĚPÁNEK [Ch. 16 in this Handbook] and RUBIN [Ch. 15 in this Handbook] for more on automorphism of homogeneous BAs.

1.3. Products of BAs with no rigid or homogeneous factors

Not much is known about BAs of the kind mentioned. Complete BAs with no rigid or homogeneous factors were shown to exist in ŠTĚPÁNEK and BALCAR [1977]. See also KOPPELBERG [1978] and ŠTĚPÁNEK [1982]. An easy example of an incomplete BA with no rigid or homogeneous factor was given in BRENNER [1983]. See also ŠTĚPÁNEK [Ch. 16 in this Handbook].

To close this section we mention some open problems.

PROBLEM 1. If the isomorphism in the proof of Lemma 1.1 is onto, are the BAs pairwise totally different?

PROBLEM 2. Does the decomposition in Theorem 1.3 extend to incomplete BAs in some form?

PROBLEM 3. What can one say about $|\mathrm{Aut}({}^{I}A)|$ for A homogeneous?

2. Galois theory of simple extensions

Given BAs $A \subseteq B$, we let $\mathrm{Aut}_A B = \{f \in \mathrm{Aut}\, B: f \restriction A \text{ is the identity}\}$. Connections between $\mathrm{Aut}_A B$ and Boolean algebraic properties of the extension relation between A and B may loosely be called *Galois theory* for Boolean algebras. In this section and the next one we deal with this theory. As will be seen, the results are easy and may be considered as part of the folklore of this subject (especially by ring theorists). In this section we take the case in which B is a simple extension of A: $B = A(u) \overset{\text{def}}{=} \langle A \cup \{u\}\rangle$. In both the sections our treatment is quite elementary. Some of the results and formulations may seem more natural when expressed in terms of the sheaf theory described in Part I of this Handbook.

If $A \subseteq B$ and $u \in B$, we define two ideals I_0^u and I_1^u of A:

$$I_0^u = \{a \in A: a \cdot u = 0\}\ ,$$

$$I_1^u = \{a \in A: a \cdot -u = 0\}\ .$$

Ideals I and J in a BA A are *disjoint* if $I \cap J = \{0\}$.

2.1. THEOREM. (i) *Let $A(u)$ be a simple extension of A. Then I_0^u and I_1^u are disjoint ideals of A. Furthermore, $u \in A$ iff $I_0^u + I_1^u = A$.*

(ii) *Conversely, let J_0 and J_1 be disjoint ideals of A. Then there is a simple extension $A(u)$ of A such that $I_0^u = J_0$ and $I_1^u = J_1$. If $A(v)$ is any other simple extension of A with $I_0^v = J_0$, $I_1^v = J_1$, then there is an isomorphism of $A(u)$ onto $A(v)$ which is the identity on A.*

PROOF. The first part of (i) is clear. If $u \in A$, let $x \in A$ be arbitrary. Then $x = x \cdot -u + x \cdot u \in I_0^u + I_1^u$. So $I_0^u + I_1^u = A$. Conversely, suppose that $I_0^u + I_1^u = A$. Say $1 = x + y$ with $x \in I_0^u$, $y \in I_1^u$. Then $x \cdot y = 0$ since I_0^u and I_1^u are disjoint. Thus, $y = -x$. Now $x \cdot u = 0$ and $-x \cdot -u = 0$, so $x = -u$ and $u \in A$.

For (ii), define $F: A \to (A/J_0) \times (A/J_1)$ by $fa = (a/J_0, a/J_1)$ for all $a \in A$. Clearly, f is an isomorphism into. Since $(A/J_0) \times (A/J_1)$ is generated by $\mathrm{Rng}\, f \cup \{(1/J_0, 0/J_1)\}$, the existence of the desired $A(u)$ is clear. The uniqueness part follows from the Sikorski extension criterion. □

The automorphism groups $\mathrm{Aut}_A A(u)$ are characterized in the following theorem. In this theorem and several others below, Σ and Π refer to operations in the completion of A.

2.2. THEOREM. *Let $A(u)$ be a simple extension of A. Let $F = \{a \in A: a \cdot \Sigma\,(I_0^u + I_1^u) = 0\}$. Then $\langle F, \Delta\rangle$ is an abelian 2-group, and it is isomorphic to* $\mathrm{Aut}_A A(u)$

PROOF. For each $a \in F$ define f_a: $A(u) \to A(u)$ by setting $f_a x = x$ for all $x \in A$ and $f_a u = a \triangle u$; f_a extends to an endomorphism of $A(u)$ by Sikorski's extension criterion. In fact, if $x \in A$ and $x \cdot u = 0$, then $x \in I_0^u$, so $x \cdot a = 0$, hence $x \cdot (a \triangle u) = 0$; if $x \in A$ and $x \cdot -u = 0$, then $x \in I_1^u$, so $x \cdot a = 0$, hence $x \cdot -(a \triangle u) = x \cdot (a \cdot u + -a \cdot -u) = 0$. Now $f_a \circ f_a$ = identity, so f_a is an automorphism. Thus, f: $F \to \mathrm{Aut}_A\, A(u)$. It is easily checked that F is closed under Δ, hence $\langle F, \Delta \rangle$ is an abelian 2-group, and f is an isomorphism from F into $\mathrm{Aut}_A\, A(u)$. Now let $g \in \mathrm{Aut}_A\, A(u)$ be arbitrary. Write $gu = b \cdot u + c \cdot -u$ with b, $c \in A$. Now $b \cdot c \le gu$, so $b \cdot c = g^{-1}(b \cdot c) \le u$. Similarly, $-b \cdot -c \cdot gu = 0$ implies that $-b \cdot -c \cdot u = 0$. So $gu = (b + -c) \cdot u + (c \cdot -b) \cdot -u$. Let $a = c \cdot -b$. We claim that $a \in F$. To show this first let $x \in I_0^u$. Thus, $x \cdot u = 0$, so $x \cdot gu = 0$, hence $x \cdot c \cdot -b \cdot -u = 0$; but $x \cdot u = 0$ then yields $0 = x \cdot c \cdot -b = x \cdot a$. Second, let $x \in I_1^u$. Now $g(-u) = -gu = c \cdot -b \cdot u + (b + -c) \cdot -u$, so the same proof yields $x \cdot a = 0$ again. Thus, $x \in F$. Clearly, $f_a = g$. □

By Theorems 2.1 and 2.2 the relative automorphism groups $\mathrm{Aut}_A\, A(u)$ can be of any size κ for which there a BA of size κ, namely 2^m for any $m \in \omega$, and any infinite κ.

Given a simple extension $A(u)$ of A, we denote by $F = F^{Au}$ the ideal F defined in Theorem 2.2, and by $f = f^{Au}$ the isomorphism defined there.

If $G \subseteq \mathrm{Aut}\, B$, we set $\mathrm{Fix}\, G = \{b \in B\colon gb = b \text{ for all } g \in G\}$.

2.3. THEOREM. *Let G be a subset of* $\mathrm{Aut}_A\, A(u)$ *and set* $F' = \{a \in F^{Au}\colon f_a^{Au} \in G\}$. *Then* $\mathrm{Fix}\, G = \{c \cdot u + d \cdot -u\colon c, d \in A \text{ and } (c \triangle d) \cdot \Sigma F' = 0\}$.

PROOF. To prove $\subseteq$, let $a \in F'$, c, $d \in A$, and $c \cdot u + d \cdot -u \in \mathrm{Fix}\, G$; we show that $(c \triangle d) \cdot a = 0$. We have

$$c \cdot u + d \cdot -u = f_a(c \cdot u + d \cdot -u)$$
$$= c \cdot -a \cdot u + c \cdot a \cdot -u + d \cdot a \cdot u + d \cdot -a \cdot -u\,.$$

Hence, $c \cdot u = c \cdot -a \cdot u + d \cdot a \cdot u$, so $[(c \cdot a) \triangle (d \cdot a)] \cdot u = 0$. Similarly, $[(c \cdot a) \triangle (d \cdot a)] \cdot -u = 0$, so $(c \triangle d) \cdot a = 0$.

The inclusion $\supseteq$ is treated similarly. □

Now we can characterize "closed" groups:

2.4. THEOREM. *Let $A(u)$ be a simple extension of A, let G be a subset of* $\mathrm{Aut}_A\, A(u)$, *and set* $F' = \{a \in F^{Au}\colon f_a \in G\}$. *Then the following conditions are equivalent*:

(i) $G = \mathrm{Aut}_{\mathrm{Fix}\, G}\, A(u)$.

(ii) $F' = \{a \in F^{Au}\colon a \le \Sigma F'\}$.

PROOF. (i)$\Rightarrow$(ii). Assume (i) and take any $a \in F^{Au}$ with $a \le \Sigma F'$; we show that $a \in F'$. To this end it suffices to take any $x \in \mathrm{Fix}\, G$ and show that $f_a x = x$. By Theorem 2.3, say $x = c \cdot u + d \cdot -u$ with $(c \triangle d) \cdot \Sigma F' = 0$. So $(c \triangle d) \cdot a = 0$, hence $x \in \mathrm{Fix}\{f_a\}$ by 2.3.

(ii)$\Rightarrow$(i). Assume (ii), and let $g \in \mathrm{Aut}_{\mathrm{Fix}\, G} A(u)$; we want to show that $g \in G$. Say $g = f_a$ with $a \in F^{Au}$; we need to show that $a \in F'$. Suppose $a \notin F'$; then by (ii) $a \cdot -\Sigma F' \neq 0$; say $b \in A^+$ and $b \leq a \cdot -\Sigma F'$. By 2.3 we have $b \cdot u \in \mathrm{Fix}\, G$, so

$$b \cdot u = f_a(b \cdot u) = b \cdot -a \cdot u + b \cdot a \cdot -u .$$

So $b \cdot -u = 0$. Similarly, $b \cdot -u \in \mathrm{Fix}\, G$, hence $b \cdot u = 0$, so $b = 0$, a contradiction. □

Using 2.4 it is easy to construct examples of closed groups, and examples of non-closed groups. Also we can show:

2.5. THEOREM. *Let $A(u)$ be a simple extension of A. Then the following conditions are equivalent*:

(i) *For every subgroup G of $\mathrm{Aut}_A A(u)$ we have $G = \mathrm{Aut}_{\mathrm{Fix}\, G} A(u)$.*

(ii) $|\mathrm{Aut}_A A(u)| \leq 2$.

PROOF. Trivially (ii)$\Rightarrow$(i). Now suppose that $|\mathrm{Aut}_A A(u)| > 2$. By Theorem 2.2 choose $0 < b < a$ in F^{Au}. Let $G = \{Id, f_a\}$. By Theorem 2.3 it is clear that $f_b \in \mathrm{Aut}_{\mathrm{Fix}\, G} A(u)$. Hence $G \neq \mathrm{Aut}_{\mathrm{Fix}\, G} A(u)$. □

Now we consider the other aspect of Galois theory, namely closed algebras.

2.6. THEOREM. *For any simple extension $A(u)$ of A the following conditions are equivalent*:

(i) $A = \mathrm{Fix}\,\mathrm{Aut}_A A(u)$.

(ii) $I_0^u + I_1^u = \{x \in A: x \leq \Sigma\, (I_0^u + I_1^u)\}$.

PROOF. (i)$\Rightarrow$(ii). Assume (i), and let $x \in A$ with $x \leq \Sigma\, (I_0^u + I_1^u)$; we are supposed to show that $x \in I_0^u + I_1^u$. Now $x \cdot \Sigma\, F^{Au} = 0$, so by Theorem 2.3, $x \cdot -u \in \mathrm{Fix}\,\mathrm{Aut}_A A(u) = A$. Hence, also $x \cdot u \in A$, and $x = x \cdot -u + x \cdot u \in I_0^u + I_1^u$.

(ii)$\Rightarrow$(i). Assume (ii), and let $x \in \mathrm{Fix}\,\mathrm{Aut}_A A(u)$; we show that $x \in A$. By Theorem 2.3 write $x = c \cdot u + d \cdot -u$ with $c, d \in A$ and $(c \triangle d) \cdot \Sigma\, F = 0$. Now $\Sigma\, F = -\Sigma\, (I_0^u + I_1^u)$, so by (ii) choose $a \in I_0^u$ and $b \in I_1^u$ with $c \triangle d = a + b$. Then

$$\begin{aligned} x &= c \cdot -d \cdot u + d \cdot -c \cdot -u + c \cdot d \\ &= c \cdot -d \cdot b \cdot u + d \cdot -c \cdot a \cdot -u + c \cdot d \\ &= c \cdot -d \cdot b + d \cdot -c \cdot a + c \cdot d \in A . \quad \square \end{aligned}$$

Some special cases of simple extensions are of interest. There are two extreme cases: in the first, $|\mathrm{Aut}_A A(u)| = 1$, or equivalently, by Theorem 2.2, $I_0^u + I_1^u$ is dense in A. We then call $A(u)$ a *rigid* simple extension. The second extreme case occurs when u is independent over A, i.e. $a \cdot u \neq 0 \neq a \cdot -u$ for all $a \in A^+$. This is equivalent to saying that $I_0^u = I_1^u = \{0\}$. In this case we have $F^{Au} = A$ and $\mathrm{Fix}\,\mathrm{Aut}_A A(u) = A$.

It is of interest to see the connection of the observations in this section with the known Galois theory of commutative rings. We give some indications along these lines; see CHASE, HARRISON and ROSENBERG [1965], DEMEYER and INGRAHAM [1971], MAGID [1974], and VILLAMAYOR and ZELINSKY [1969] for the general theory and further references.

Suppose that $A \subseteq B$. We form the amalgamated free product $B \oplus_A B$. There is a homomorphism μ from $B \oplus_A B$ into B such that $\mu(b \oplus c) = b \cdot c$ for all $b, c \in B$. (For clarity we write $b \oplus c$ for $b \cdot c$, where b comes from the first factor B, and c from the second.) We say that B is a *separable extension* of A provided that there is a $u \in B \oplus_A B$ such that $\mu u = 1$ and $(\ker u) \cdot u = \{0\}$.

2.7. THEOREM. *If $A \subseteq B$, then B is a separable extension of A iff B is a finite extension of A.*

PROOF. $\Rightarrow$. Let $u \in B \oplus_A B$ be such that $\mu u = 1$ and $(\ker \mu) \cdot u = \{0\}$. Write $u = \Sigma_{i=1}^m b_i \oplus c_i$ with $c_i \cdot c_j = 0$ for $i \neq j$, each $c_i \neq 0$. Since $1 = \mu u = \Sigma_{i=1}^m b_i \cdot c_i$, it follows that $c_i \leq b_i$ for all i, and $\Sigma_{i=1}^m c_i = 1$. We may assume that $c_i = b_i$ for all i. Now we claim that $B = \langle A \cup \{c_1, \ldots, c_m\}\rangle$. Let $d \in B$. Then, since $(\ker \mu) \cdot u = \{0\}$, for any i, $(d \oplus (-d)) \cdot (c_i \oplus c_i) = 0$, so there is an $s_i \in A$ such that $d \cdot c_i \leq s_i$ and $-d \cdot s_i \cdot c_i = 0$. Hence $d \cdot c_i = s_i \cdot c_i$. Therefore

$$d = d \cdot \sum_{i=1}^{m} c_i = \sum_{i=1}^{m} s_i \cdot c_i ,$$

hence $d \in \langle A \cup \{c_1, \ldots, c_m\}\rangle$.

$\Leftarrow$. Let $B = \langle A \cup \{u_1, \ldots, u_m\}\rangle$. We may assume that $u_i \cdot u_j = 0$ for $i \neq j$, and $u_1 + \cdots + u_m = 1$. Set $v = \Sigma_{i=1}^m u_i \oplus u_i$. Thus, $\mu v = 1$. Now it is easily verified that $\ker \mu$ is generated by $\{d \oplus (-d)\colon d \in B\}$. So to check that $\ker \mu \cdot v = \{0\}$ it suffices to take any $d \in B$ and $1 \leq i \leq m$ and show that $(d \oplus (-d)) \cdot (u_i \oplus u_i) = 0$. Write $d = \Sigma_{i=1}^m s_i \cdot u_i$ with each $s_i \in A$. Then $d \cdot u_i = s_i \cdot u_i \leq s_i$ and $-d \cdot u_i = -s_i \cdot u_i \leq -s_i$, hence $[d \oplus -d] \cdot (u_i \oplus u_i) = 0$. □

Let B be any BA. Automorphisms f and g of B are *strongly distinct* if for every non-zero $b \in B$ there is an $s \in B$ such that $fs \cdot b \neq gs \cdot b$.

2.8. LEMMA. *If I_0^u or I_1^u is non-trivial, then no members of* Aut_A *$A(u)$ are strongly distinct.*

PROOF. Say $0 \neq a \in I_0^u$. Let g and h be distinct members of $\mathrm{Aut}_A A(u)$. By 2.2, write $g = f_d$, $h = f_e$, with $d, e \in F^{Au}$. Then for any $s \in A(u)$ write $s = s_0 \cdot u + s_1 \cdot -u$. Then

$$\begin{aligned} f_d s \cdot a &= [s_0 \cdot (d \triangle u) + s_1 \cdot -(d \triangle u)] \cdot a \\ &= s_1 \cdot a , \end{aligned}$$

since $a \cdot u = 0$ and $a \cdot d = 0$. Similarly, $f_e s = s_1 \cdot a$. This shows that f_d and f_e are not strongly distinct. □

2.9. LEMMA. *Assume that u is independent over A [hence $I_0^u = I_1^u = \{0\}$ and $F^{Au} = A$]. Let $d, e \in A$. Then the following conditions are equivalent*:

(i) *f_d and f_e are strongly distinct.*

(ii) $d = -e$.

PROOF. (i)$\Rightarrow$(ii). Suppose that $d \neq -e$. Then there are two possibilities.

Case 1. $d + e \neq 1$. Say $0 \neq a \in A$ and $a \cdot d = 0 = a \cdot e$. Thus, $a \cdot u \neq 0$. Then for any $s \in A(u)$, say with $s = s_0 \cdot u + s_1 \cdot -u$, with $s_0, s_1 \in A$,

$$\begin{aligned} f_d s \cdot a \cdot u &= [s_0 \cdot (d \triangle u) + s_1 \cdot -(d \triangle u)] \cdot a \cdot u \\ &= s_0 \cdot a \cdot u , \end{aligned}$$

and similarly $f_e s \cdot a \cdot u = s_0 \cdot a \cdot u$, so f_d and f_e are not strongly distinct.

Case 2. $d \cdot e \neq 0$. Then $d \cdot e \cdot u \neq 0$ and for any $s \in A(u)$ as above, $f_d s \cdot d \cdot e \cdot u = s_1 \cdot d \cdot e \cdot u = f_e s \cdot d \cdot e \cdot u$, again showing that f_d and f_e are not strongly distinct.

(ii)$\Rightarrow$(i). Given $0 \neq b \in A(u)$, say wlog b has the form $c \cdot u$ with $c \in A$. Then $f_d(-u) \cdot c \cdot u = -(d \triangle u) \cdot c \cdot u = c \cdot d \cdot u$ and $f_{-d}(-u) \cdot c \cdot u = c \cdot -d \cdot u \neq c \cdot d \cdot u$. Hence, f_d and f_{-d} are strongly distinct. □

Let $A \leq B$. We say that B is *Galois over A* if B is a separable extension of A and there is a finite subgroup G of strongly distinct members of $\operatorname{Aut}_A B$ such that $\operatorname{Fix} G = A$.

2.10. THEOREM. *For $A(u)$ a simple extension of A the following conditions are equivalent*:

(i) *$A(u)$ is Galois over A*;

(ii) *$u \in A$ or u is independent over A.*

PROOF. (i)$\Rightarrow$(ii): by Lemma 2.8. (ii)$\Rightarrow$(i): assume that u is independent over A. By Lemma 2.9, f_0 and f_1 are strongly distinct, so it suffices to show that $\operatorname{Fix}\{f_0, f_1\} = A$. Suppose that $b \in \operatorname{Fix}\{f_0, f_1\}$; say $b = b_0 \cdot u + b_1 \cdot -u$, with b_0, $b_1 \in A$. Then $b = f_1 b = b_0 \cdot (1 \triangle u) + b_1 \cdot -(1 \triangle u) = b_0 \cdot -u + b_1 \cdot u$, so $b_0 \cdot u = b_1 \cdot u$, hence $b_0 \triangle b_1 \in I_0^u = \{0\}$, hence $b_0 = b_1$. Thus $b \in A$. □

Given $A \leq B$, we call B *weakly Galois over A* if there is a finite partition of unity $\langle a_i : i < m \rangle$ in A such that for each $i < m$, $B \restriction a_i$ is Galois over $A \restriction a_i$.

2.11. THEOREM. *$A(u)$ is weakly Galois over A iff I_0^u and I_1^u are principal.*

PROOF. $\Rightarrow$. Let $\langle a_i : i < n \rangle$ be a partition of unity in A such that $A(u) \restriction a_i$ is Galois over $A \restriction a_i$ for all $i < n$. Note that $A(u) \restriction a_i = (A \restriction a_i)(u \cdot a_i)$ for each $i < n$. Say by Theorem 2.10 that $u \cdot a_i \in A \restriction a_i$ for all $i < m$, while $u \cdot a_i$ is independent over $A \restriction a_i$ for $m \leq i < n$. Let $x = \Sigma_{i<m} -u \cdot a_i$, $y = \Sigma_{i<m} u \cdot a_i$. Thus, $x \in I_0^u$ and $y \in I_1^u$. Suppose that z is any member of I_0^u. Thus, $z \leq -u$. Suppose $z \cdot a_i \neq 0$ with $m \leq i < n$. Then $z \cdot a_i \cdot u \neq 0$, a contradiction. Hence, $z \leq x$. This shows that $I_0^u = \langle \{x\} \rangle^{\mathrm{id}}$. Similarly, $I_1^u = \langle \{y\} \rangle^{\mathrm{id}}$.

$\Leftarrow$. Say $I_0^u = \langle\{x\}\rangle^{\mathrm{id}}$ and $I_1^u = \langle\{y\}\rangle^{\mathrm{id}}$. Let $z = -x \cdot -y$. Then $\langle x+y, z\rangle$ is a partition of unity in A. Now $A(u) \restriction (x+y) = A \restriction (x+y)$. For, suppose that $b \in A(u) \restriction (x+y)$. Thus, $b \le x+y$ and, say, $b = c \cdot u + d \cdot -u$ with $c, d \in A$. Then $b \cdot x = d \cdot x \in A$ and similarly $b \cdot y \in A$, so $b \in A$. Also, $A(u) \restriction z$ is Galois over $A \restriction z$, in fact $u \cdot z$ is independent over $A \restriction z$. For, suppose that $0 \ne b \in A \restriction z$. If $b \cdot u \cdot z = 0$, then $b \cdot u = 0$, hence $b \in I_0^u$ and $b \le x$ so $b = 0$, a contradiction. Similarly, $b \cdot -a \cdot z = 0$ is impossible. $\square$

3. Galois theory of finite extensions

It is more complicated to analyze the relative automorphism groups for arbitrary finite extensions. This was carried out by KOPPELBERG [1982] using sheaf theory. We present these results here in a non-sheaf setting. We begin with a generalization of Theorem 2.1. If $B = \langle A \cup F\rangle$ for some finite set F, where $A \le B$, then we write $B = A(F)$. We call F *reduced* if it is a partition of unity, and for all distinct $u, v \in F$ we have $u \notin \langle A \cup (F \setminus \{u, v\})\rangle$. Any finite extension $A(F)$ can be written in the form $A(G)$, G reduced: just let G be a partition of unity such that $A(F) = A(G)$ of smallest cardinality – clearly possible. Note that if F is reduced, then $0 \notin F$. We call $\langle b_i: i < m\rangle$ reduced if $\{b_i: i < m\}$ is and the b_i's are distinct. If $\langle b_i: i < m\rangle$ is a finite system of elements, we set $A(b_0, \ldots, b_{m-1}) = A(\{b_0, \ldots, b_{m-1}\})$. A finite system $\langle I_i: i < m\rangle$ of ideals is an *extender* if the following conditions hold:

(1) $\quad I_0 \cap \cdots \cap I_{m-1} = \{0\}$.

(2) $\quad$ For all $i, j < m$ and all $a \in A$, if $a \in I_i$, then $-a \notin I_j$.

Given a finite partition of unity $u = \langle u_i: i < m\rangle$ in B, and $A \le B$, we define an associated sequence of ideals $\langle J_i^u: i < m\rangle$ by

$$J_i^u = \{a \in A: a \cdot u_i = 0\} .$$

The following extension of Theorem 2.1 holds.

3.1. THEOREM. (i) *Let $\langle u_i: i < m\rangle$ be reduced in $A(u_0, \ldots, u_{m-1})$. Then $\langle J_i^u: i < m\rangle$ is an extender.*

(ii) *Conversely, let $\langle K_i: i < m\rangle$ be an extender. Then there is an extension B of A and a reduced system $\langle u_i: i < m\rangle$ in B such that $B = A(u_0, \ldots, u_{m-1})$ and $J_i^u = K_i$ for all $i < m$. If $C = A(v_0, \ldots, v_{m-1})$ with $\langle v_i: i < m\rangle$ a partition of unity and $J_i^v = K_i$ for all $i < m$, then there is an isomorphism g of B onto C such that $ga = a$ for all $A \in A$ and $gu_i = v_i$ for all $i < m$.*

PROOF. For (i), clearly $J_0^u \cap \cdots \cap J_{m-1}^u = \{0\}$. Now suppose that $i, j < m$, $a \in J_i^u$, and $-a \in J_j^u$. Then $a \cdot u_i = 0 = -a \cdot u_j$, so $i \ne j$ since $u_i \ne 0$. Then $u_i = (-\Sigma_{k \ne i,j} u_k) \cdot -a$, so $u_i \in \langle A \cup \{u_k: k \ne i, j\}\rangle$, a contradiction.

For (ii), let $B = \Pi_{i<m} A/K_i$, and define $g\colon A \to B$ by setting $(ga)_i = a/K_i$ for each $i<m$. Clearly, g is an isomorphism into. For each $i<m$ let $u_i \in B$ be defined by $u_i i = 1$, $u_i j = 0$ if $j \neq i$. Then $\langle u_i\colon i<m\rangle$ is a partition of unity, and $B = (g[A])(u_0, \ldots, u_{m-1})$. Also, clearly $J_i^u = g[K_i]$ for all $i<m$. To show that $\langle u_i\colon i<m\rangle$ is reduced, suppose that $i, j<m$, $i \neq j$, and $u_i \in \langle g[A] \cup \{u_k\colon k \neq i, j\}\rangle$. It is easily seen that

$$(*) \qquad \langle g[A] \cup \{u_k\colon k \neq i, j\}\rangle = \{b \in B\colon \text{for some } a \in A,\ b_i = a/K_i \text{ and } b_j = a/K_j\}\,.$$

Hence there is an $a \in A$ with $u_i i = a/K_i$ and $u_i j = a/K_j$. Thus, $a \in K_j$ and $-a \in K_i$, a contradiction. The final assertion of (ii) is clear by the Sikorski extension criterion. □

It is worth noting that not every finite extension is a simple extension. For, suppose that u is independent over A and $v \notin A(u)$. Then we claim that $A(u, v)$ is not a simple extension of A. For, suppose that $A(u, v) = A(w)$. Write $u = a \cdot w + b \cdot -w$, where $a, b \in A$. Then $a \cdot b \cdot -u = 0$, so $a \cdot b = 0$. Similarly, $-a \cdot -b = 0$, so $b = -a$. Thus, $u = a \cdot w + -a \cdot -w$. Hence, $-u = -a \cdot w + a \cdot -w$, so $w = a \cdot u + -a \cdot -u \in A(u)$. Therefore $v \in A(u)$, a contradiction.

Now we shall give a (rather complicated) description of the members of $\mathrm{Aut}_A B$, where $B = A(u_0, \ldots, u_{m-1})$, with $\langle u_i\colon i<m\rangle$ reduced. For each $p \in \mathrm{Ult}\, A$, let $B_p = B/\langle p\rangle^{\mathrm{fi}}$, and let pr_p be the natural homomorphism from B onto B_p. Thus B_p is a finite BA, and its atoms are the non-zero elements in $\{pr_p u_0, \ldots, pr_p u_{m-1}\}$. Now if $g \in \mathrm{Aut}_A B$, then g induces an automorphism of B_p, hence a permutation of the atoms of B_p, hence a permutation of $\{0, \ldots, m-1\}$. Our description hinges on a characterization of these permutations. We denote by S_m the symmetric group of all permutations of $\{0, \ldots, m-1\}$.

Temporarily fix $p \in \mathrm{Ult}\, A$. If $i, j<m$, we write $i \sim j$ *at* p provided that there is a $c \in p$ such that for all $q \in \mathrm{Ult}\, A$ with $c \in q$ we have $-u_i \in \langle q\rangle^{\mathrm{fi}}$ iff $-u_j \in \langle q\rangle^{\mathrm{fi}}$. Clearly, this is an equivalence relation on m. It is easily checked that $i \sim j$ at p iff there is a $c \in p$ such that for all $a \in A \restriction c$, $a \in J_i^u$ iff $a \in J_j^u$. Next, if $\rho \in S_m$ we say that ρ is *compatible with* p if $i \sim \rho i$ at p for all $i<m$. Finally, if $g \in \mathrm{Aut}_A B$ and $\rho \in S_m$, we say that g *is induced by* ρ *at* p if for all $i<m$ we have $pr_p g u_i = pr_p u_{\rho i}$. (Note that g is uniquely determined "on B_p" by this last condition.) If one of these three relations holds – $i \sim j$ at p, ρ is compatible with p, g is induced by ρ at p – then there is a $c \in p$ such that the given relation holds for any q with $c \in q$ in place of p.

3.2. LEMMA. *Let $p \in \mathrm{Ult}\, A$ and $\rho \in S_m$. Then ρ is compatible with p iff there is a $g \in \mathrm{Aut}_A B$ which is induced by ρ at p.*

PROOF. First suppose that ρ is compatible with p. By a remark before this lemma, choose $c \in p$ such that for all $a \in A \restriction c$ and all $i<m$, $a \in J_i^u$ iff $a \in J_{\rho i}^u$. Sikorski's extension criterion yields an isomorphism g of B into B such that $g \restriction A$ is the identity and $g u_i = -c \cdot u_i + c \cdot u_{\rho i}$ for all $i<m$. Clearly, g maps onto B, and so

$g \in \mathrm{Aut}_A B$. To show that g is induced by ρ at p, take any $i < m$. Clearly, $-(gu_i \triangle u_{\rho i}) \in \langle p \rangle^{\mathrm{fi}}$, so $pr_p gu_i = pr_p u_{\rho i}$, as desired.

Now suppose that $g \in \mathrm{Aut}_A B$ is induced by ρ at p. Let $i < m$. Choose $c \in p$ such that $c \leq (-gu_i + u_{\rho i}) \cdot (gu_i + -u_{\rho i})$. Then for any $a \in A \restriction c$, $a \in J_i^u$ iff $a \cdot u_i = 0$ iff $a \in J_{\rho i}^u$. Hence, ρ is compatible with p. □

Now we are ready for the theorem characterizing members of $\mathrm{Aut}_A B$. If T is a finite partition of unity in A, then a function $h\colon T \to S_m$ is *compatible with* T if for all $t \in T$ and all $p \in \mathrm{Ult}\, A$ with $t \in p$ we have that h_t is compatible with p.

3.3. THEOREM. (i) *Let T be a finite partition of unity in A, and suppose that $h\colon T \to S_m$ is compatible with T. Then there is a $g = g_h \in \mathrm{Aut}_A B$ such that, for all $i < m$,*

$$gu_i = \sum_{t \in T} t \cdot u_{h_t i} .$$

Moreover, g is induced by h_t at p whenever $t \in p$.

(ii) *For T a finite partition of unity in A, let $G_T = \{g_h\colon h\colon T \to S_m$ is compatible with $T\}$. Then G_T is a finite subgroup of $\mathrm{Aut}_A B$.*

(iii) *If H is a finite subset of $\mathrm{Aut}_A B$, then there is a finite partition T of unity in A such that $H \subseteq G_T$.*

PROOF. (i) For the existence of g it suffices to show that if $i < m$ and $t \in T$, then there is an isomorphism k of $B \restriction (t \cdot u_i)$ onto $B \restriction (t \cdot u_{h_t i})$ such that $k(a \cdot t \cdot u_i) = a \cdot t \cdot u_{h_t i}$ for all $a \in A$. To do this, it is enough to check that $a \cdot t \cdot u_i = 0$ iff $a \cdot t \cdot u_{h_t i} = 0$. Suppose that $a \cdot t \cdot u_i \neq 0$. Say $a \cdot t \cdot u_i \in q \in \mathrm{Ult}\, B$. Let $p = A \cap q$; so $p \in \mathrm{Ult}\, A$. Now $t \in p$, so h_t is compatible with p. Choose $c \in p$ so that for all $x \in A \restriction c$, $x \in J_i^u$ iff $x \in J_{h_t i}^u$. Now $a \cdot t \cdot u_i \cdot c \in q$, so $a \cdot t \cdot u_i \cdot c \neq 0$, hence $a \cdot t \cdot c \not\in J_i^u$. Therefore $a \cdot t \cdot c \not\in J_{h_t i}^u$, so $a \cdot t \cdot u_{h_t i} \neq 0$. The converse is similar. For the final statement of (i) assume that $t \in p$. Then for any $i < m$, $t \cdot gu_i = t \cdot u_{h_t i} \leq u_{h_t i}$ and $t \cdot u_{h_t i} \leq gu_i$, so $t \leq (-gu_i + u_{h_t i}) \cdot (-u_{h_t i} + gu_i)$, consequently $pr_p gu_i = pr_p u_{h_t i}$.

(ii) Obviously G_T is finite. The identity element of $\mathrm{Aut}_A B$ is g_h, where h_t is the identity for each $t \in T$. If $h, k\colon T \to S_m$ are compatible with T, then so is l, where $l_t = h_t \circ k_t$ for each $t \in T$, and $g_l = g_h \circ g_k$. If h is compatible with T, then so is k, where $k_t = h_t^{-1}$ for each $t \in T$, and $g_h^{-1} = g_k$. Thus, (ii) holds.

(iii) Temporarily fix $k \in H$; we construct a finite partition of unity T_k in A. For each $\rho \in S_m$ let

$$v_\rho^k = \{p \in \mathrm{Ult}\, A\colon \rho \text{ induces } k \text{ at } p\} .$$

Then v_ρ^k is an open subset of $\mathrm{Ult}\, A$ by the comment before Lemma 3.2. Furthermore,

$$(\dagger) \qquad \mathrm{Ult}\, A = \bigcup_{\rho \in S_m} v_\rho^k .$$

In fact, let $p \in \mathrm{Ult}\, A$. Define $\rho \in S_m$ as follows; let $i < m$. If $-u_i \in \langle p \rangle^{\mathrm{fi}}$, set $\rho i = i$; otherwise $ku_i / \langle p \rangle^{\mathrm{fi}}$ is an atom $u_j / \langle p \rangle^{\mathrm{fi}}$ of B_p, and we set $\rho i = j$. Clearly, ρ induces k at p, and so $p \in v^k_\rho$. That is, (†) holds.

By (†) and the compactness of Ult A, there is a partition of unity $\langle c^k_\rho \colon \rho \in S_m \rangle$ in A, $c^k_\rho = 0$ allowed, such that $sc^k_\rho \subseteq v^k_\rho$ for all $\rho \in S_m$. Clearly, if $c^k_\rho \in p \in \mathrm{Ult}\, A$, then ρ is compatible with p – this is the content of Lemma 3.2. We let T_k be the set of non-zero elements of $\{c^k_\rho \colon \rho \in S_m\}$. Then let T be the common refinement of all the partitions T_k for $k \in H$. To prove that $H \subseteq G_T$, let $k \in H$ be arbitrary. We define $h\colon T \to S_m$. Given $t \in T$, choose $h_t \in S_m$ so that $t \leq c^k_{h_t}$. Now h is compatible with T, since if $t \in T$ and $p \in \mathrm{Ult}\, A$ with $t \in p$, then $c^k_{h_t} \in p$ and so h_t is compatible with p. We claim that $g_h = k$. To show this, take any $i < m$ and $t \in T$; we show that $g_h(t \cdot u_i) = k(t \cdot u_i)$. Now $g_h(t \cdot u_i) = t \cdot u_{h_t i}$ and $k(t \cdot u_i) = t \cdot ku_i$. Suppose that these are not the same. Say $t \cdot (u_{h_t i} \Delta ku_i) \in q \in \mathrm{Ult}\, B$ and set $p = A \cap q$. Thus $t \in p$, so $c^k_{h_t} \in p$. By construction, then, h_t induces k at p. Hence, $-(ku_i \Delta u_{h_t i}) \in \langle p \rangle^{\mathrm{fi}} \subseteq q$, a contradiction. This finishes the proof of Theorem 3.3. □

Recall that a group G is *locally finite* if every finitely generated subgroup of G is finite.

3.4. COROLLARY. *$\mathrm{Aut}_A\, B$ is locally finite.*

Now we discuss in the general finite extension case the general notions of ring theory mentioned in the previous section – Galois and weakly Galois BAs. First we have a simple lemma generalizing Lemma 2.8.

3.5. LEMMA. *If $a \leq u_i$ for some $i < m$ and some $a \in A^+$, then no members of $\mathrm{Aut}_A\, B$ are strongly distinct.*

PROOF. By Theorem 3.3, we may assume that our two arbitrary members of $\mathrm{Aut}_A\, B$ have the form g_h, g_k, where $h, k\colon T \to S_m$ are compatible with T, T some finite partition of unity in A. Choose $t \in T$ so that $a \cdot t \neq 0$. We show that $g_h s \cdot a \cdot t = g_k s \cdot a \cdot t$ for any $s \in B$, so that g_h and g_k are not strongly distinct. Say $s = e_0 \cdot u_0 + \cdots + e_{m-1} \cdot u_{m-1}$, with $e_0, \ldots, e_{m-1} \in A$. Then, since $a \leq u_i$,

$$g_h s \cdot a \cdot t = g_h(s \cdot a \cdot t) = g_h(s_i \cdot a \cdot t \cdot u_i) = g_h(s_i \cdot a \cdot t) = s_i \cdot a \cdot t\,.$$

Similarly, $g_k s \cdot a \cdot t = s_i \cdot a \cdot t$. □

For the next theorem, we say that C is a *relatively complete* subalgebra of D if $C \leq D$ and for every $d \in D$ there is a smallest $c \in C$ such that $d \leq c$; see, for example, HALMOS [1955].

3.6. THEOREM. *Let A and B be as above. Then the following conditions are equivalent*:

(i) *For every $i < m$, J^u_i is principal.*
(ii) *A is relatively complete in B.*

(iii) *There is a* $g \in \mathrm{Aut}_A B$ *such that* $gb \neq b$ *for all* $b \in B \backslash A$.
(iv) *There is a finite subgroup* G *of* $\mathrm{Aut}_A B$ *such that* $\mathrm{Fix}\, G = A$.

PROOF. (i)$\Rightarrow$(ii). Say $J_i^u = \langle a_i \rangle^{\mathrm{id}}$ for each $i < m$. Let $b \in B$; say $b = c_0 \cdot u_0 + \cdots + c_{m-1} \cdot u_{m-1}$ with $c_0, \ldots, c_{m-1} \in A$. Set $d = c_0 \cdot -a_0 + \cdots + c_{m-1} \cdot -a_{m-1}$. Thus, $b \leq d \in A$. Suppose $b \leq e \in A$. Then for any $i < m$, $c_i \cdot u_i = b \cdot u_i \leq e \cdot u_i$, so $c_i \cdot -e \cdot u_i = 0$, hence $c_i \cdot -e \leq a_i$, or $c_i \cdot -a_i \leq e$. Thus, $d \leq e$, as desired.

(ii)$\Rightarrow$(iii). Define $p \equiv q$ iff $p, q \in \mathrm{Ult}\, A$ and for all $i < m$, $-u_i \in \langle p \rangle^{\mathrm{fi}}$ iff $-u_i \in \langle q \rangle^{\mathrm{fi}}$. This is an equivalence relation on Ult A. Each equivalence class is open: let $p \in \mathrm{Ult}\, A$. For each $i < m$ let a_i be the smallest element of $A \geq u_i$ and let c_i be a_i or $-a_i$ depending on which is in p. Then $s(c_0 \cdot \cdots \cdot c_{m-1})$ is contained in the equivalence class of p. In fact, let $c_0 \cdot \cdots \cdot c_{m-1} \in q \in \mathrm{Ult}\, A$ and let $i < m$. If $-u_i \in \langle p \rangle^{\mathrm{fi}}$, then $b \leq -u_i$ for some $b \in p$. Thus, $u_i \leq -b$, so $a_i \leq -b$ and so $c_i = -a_i$. Hence, $-a_i \in q$ and so $-u_i \in \langle q \rangle^{\mathrm{fi}}$. The converse is similar. Thus, indeed, each equivalence class is open. Hence, there is a finite partition of unity T such that for each $t \in T$, st is contained in an equivalence class.

Let $t \in T$. Let $p \in \mathrm{Ult}\, A$ with $t \in p$. There is a permutation h_t of $\{0, \ldots, m-1\}$ such that $h_t i = i$ whenever $pr_p u_i = 0$, while $pr_p u_i \mapsto pr_p u_{h_t i}$ is a cyclic permutation of the atoms of B_p for $pr_p u_i \neq 0$. The equivalence property assures us that this definition does not need to depend on p. Clearly, h is compatible with T.

Now let $b \in B \backslash A$. Let $p \in \mathrm{Ult}\, A$ such that $b, -b \not\in \langle p \rangle^{\mathrm{fi}}$. Say $t \in T$ with $t \in p$. By Theorem 3.3(i), g_h is induced by h_t at p. Since h_t is cyclic and $pr_p b$ is the sum of some, but not all, atoms of B_p, g_h moves b.

(iii)$\Rightarrow$(iv). This is clear by Theorem 3.3(ii), (iii).

(iv)$\Rightarrow$(i). Let $i < m$, and suppose that J_i^u is not principal. Then

$$(*) \qquad \{a \in A: -a \cdot u_i = 0\} \cup \{a \in A: \forall x \in (A \restriction -a)^+ (x \cdot u_i \neq 0)\}$$

has the finite intersection property. In fact, otherwise we obtain:

$$a_1 \cdot \cdots \cdot a_n \cdot c_1 \cdot \cdots \cdot c_p = 0\,,$$

with $a_1, \ldots, a_n, c_1, \ldots, c_p \in A$, $-a_k \cdot u_i = 0$ for $k = 1, \ldots, n$, and $\forall x \in (A \restriction -c_k)^+$ $(x \cdot u_i \neq 0)$ for $k = 1, \ldots, p$. So $c_1 \cdot \cdots \cdot c_p \leq -a_1 + \cdots + -a_n$, and hence $c_1 \cdot \cdots \cdot c_p \cdot u_i = 0$. Choose $d \in J_i^u$ with $c_1 \cdot \cdots \cdot c_p < d$. Then $d \cdot -(c_1 \cdot \cdots \cdot c_p) \neq 0$; say $d \cdot -c_k \neq 0$. But $d \cdot u_i = 0$, contradicting the choice of the c_k's. So $(*)$ holds.

Let $p \in \mathrm{Ult}\, A$ contain the set $(*)$. Now by Theorem 3.3(iii), we may assume that the subgroup G described in (iv) has the form G_T for some finite partition of unity T in A. Say $t \in T \cap p$. Let α be the equivalence class of i under the relation $\sim$ at p. By the remark before Lemma 3.2, choose $a \in p$ such that for every $q \in \mathrm{Ult}\, A$, if $a \in q$, then for any $j, k \in \alpha$ we have $j \sim k$ at q. Now we claim

$$(\dagger) \qquad \text{there is a } j \not\in \alpha \text{ such that } -u_j \not\in \langle p \rangle^{\mathrm{fi}}\,.$$

For, otherwise, for each $j \not\in \alpha$ there is a $c_j \in p$ such that $c_j \leq -u_j$. By the definition of p (see $(*)$), there is then an element $x \in (A \restriction \Pi_{j \not\in \alpha}\, c_j \cdot a)^+$ such that

$x \cdot u_i = 0$. If $x \in q \in \mathrm{Ult}\, A$, then $pr_q u_k = 0$ for all $k < m$, a contradiction. So (†) holds.

Let $b = a \cdot t \cdot \Sigma_{k \in \alpha} u_k$. Then $0 < pr_p b < 1$, using (†), so $b \notin A$. We claim that $gb = b$ for each $g \in G_T$ (contradicting (iv)). Say $g = g_h$, where $h: T \to S_m$ is compatible with T. Thus, h_t is compatible with p, and so h_t maps α into α. By Theorem 3.3(i) we have $g(a \cdot t \cdot u_k) = a \cdot t \cdot u_{h_t k}$ for each $k \in \alpha$, so $gb = b$. This finishes the proof of Theorem 3.6. □

We call $\langle u_i : i < m \rangle$ *independent over* A if for all $i < m$ and all $a \in A$ we have $a \cdot u_i \neq 0 \neq a \cdot -u_i$.

3.7. LEMMA. *If $\langle u_i : i < m \rangle$ is independent over A and A is a relatively complete subalgebra of B, then B is Galois over A.*

PROOF. Let σ be a cyclic permutation of $\{0, \ldots, m-1\}$. Then using the proof of Theorem 3.6(ii)⇒(iii), we choose $h: T \to S_m$ compatible with T such that $h_t = \sigma$ for all $t \in T$, and $\mathrm{Fix}\{g_h\} = A$. It suffices to show that the powers of g_h are strongly distinct. Suppose that k and l are such that $\sigma^k \neq \sigma^l$. Assume that v and w are associated with σ^k and σ^l, respectively, and let $0 \neq b \in B$. Say $b = e_0 \cdot u_0 + \cdots + e_{m-1} \cdot u_{m-1}$ with $e_0, \ldots, e_{m-1} \in A$. Say $e_i \cdot u_i \neq 0$, and choose $t \in T$ with $t \cdot e_i \cdot u_i \neq 0$. Let $\sigma^k j = i$. Then $v(t \cdot u_j) \cdot b = t \cdot e_i \cdot u_i$, while $w(t \cdot u_j) \cdot b = t \cdot e_s \cdot u_s$ for some $s \neq i$, so $v(t \cdot u_j) \cdot b \neq w(t \cdot u_j) \cdot b$, so v and w are strongly distinct. □

The converse of Lemma 3.7 does not hold. To see this, let A be a complete atomless BA, and suppose that $0 < a < 1$ in A. Using Theorem 3.1 it is easy to find an extension $B = A(u_0, u_1, u_2, u_3)$ of A such that $I_0^u = I_1^u = \langle a \rangle^{\mathrm{id}}$ and $I_2^u = I_3^u = \{0\}$, with $\langle u_i : i < 4 \rangle$ reduced. Then A is a relatively complete subalgebra of B, $\langle u_i : i < 4 \rangle$ is not independent over A, but B is Galois over A. The last statement is seen by the argument of the proof of Lemma 3.7, letting σ be the permutation $(0, 1)(2, 3)$.

We do not have a characterization of the Galois extensions, but there is one for the weakly Galois extensions:

3.8. THEOREM. *B is weakly Galois over A iff A is relatively complete in B.*

PROOF. ⇒. Say $\langle a_i : i < n \rangle$ is a finite partition of unity in A such that $B \upharpoonright a_i$ is Galois over $A \upharpoonright a_i$ for each $i < n$. By Theorem 3.6, $A \upharpoonright a_i$ is relatively complete in $B \upharpoonright a_i$ for all $i < n$. Hence, A is relatively complete in B.

⇐. For each $i < m$ choose $a_i \in A$ maximum such that $a_i \leq -u_i$. Then for each $E \subseteq m$ set

$$c_E = \prod_{i \in E} a_i \cdot \prod_{i \in m \setminus E} -a_i \,.$$

Let $X = \{c_E : E \subseteq m\} \setminus \{0\}$. Thus, X is a finite partition of unity. For $E \subseteq m$ and $c_E \neq 0$, $B \upharpoonright c_E$ is generated by $A \upharpoonright c_E \cup \{u_i \cdot c_E : i \in m \setminus E\}$. If $x \in A \upharpoonright c_E$, $i \in m \setminus E$, and $x \cdot u_i \cdot c_E = 0$, then $x \leq -u_i$, so $x \leq a_i$ and hence $x = 0$. Thus, for each such E there is a subset F of $\{u_i \cdot c_E : i \in m \setminus E\}$ which is reduced, and independent over A – so $B \upharpoonright c_E$ is Galois over $A \upharpoonright c_E$. This finishes the proof. □

4. The size of automorphism groups

In this section we present the known results (mainly from McKENZIE and MONK [1975]) concerning the relationships between $|A|$ and $|\mathrm{Aut}\, A|$, more precisely, those provable in ZFC. As we shall see, under GCH all the a priori possible relations between these two cardinals are true.

The results depend on the following basic theorem. (It is also found in MONK [Ch. 13 in this Handbook] in a stronger form, with a more complicated proof.)

4.1. THEOREM. *There is an atomic BA A of power 2^ω with $|\mathrm{Aut}\, A| = \omega$.*

PROOF. Let $\langle f_\xi : \xi < 2^\omega \rangle$ enumerate all of the permutations of ω which move infinitely many integers. Now we shall define two sequences: a sequence $\langle A_\xi : \xi \leq 2^\omega \rangle$ of subalgebras of $\mathscr{P}\omega$, and a sequence $\langle B_\xi : \xi \leq 2^\omega \rangle$ of subsets of $\mathscr{P}\omega$, so that for all $\xi \leq 2^\omega$ we have $|A_\xi|, |B_\xi| \leq |\xi| + \omega$ and $A_\xi \cap B_\xi = 0$. To start with, we let A_0 be the BA of finite and cofinite subsets of ω, and $B_0 = 0$. For λ a limit ordinal $\leq 2^\omega$ we set $A_\lambda = \bigcup_{\xi<\lambda} A_\xi$, $B_\lambda = \bigcup_{\xi<\lambda} B_\xi$. The step from $\xi < 2^\omega$ to $\xi + 1$ is the essential thing.

Choose I infinite, $I \subseteq \omega$, so that $I \cap f_\xi[I] = 0$. The following claim is the heart of the proof:

(1) there is an infinite $J \in \mathscr{P}I \backslash A_\xi$ such that $\langle A_\xi \cup \{J\} \rangle \cap (B_\xi \cup \{f_\xi[J]\}) = 0$.

Suppose that (1) fails. Thus,

(2) for every infinite $J \in \mathscr{P}I \backslash A_\xi$ there exist $C, D \in A_\xi$ such that $(C \cap J) \cup (D \backslash J) \in B_\xi \cup \{f_\xi[J]\}$.

Let K_0 be a family of pairwise almost disjoint infinite subsets of I with $|K_0| = 2^\omega$. Then by (2) and the condition $|A_\xi| \leq |\xi| + \omega$ there exist $C, D \in A_\xi$ such that the set

$$K_1 = \{J \in K_0 \backslash A_\xi : (C \cap J) \cup (D \backslash J) \in B_\xi \cup \{f_\xi[J]\}\}$$

has power $>|A_\xi| \cup |B_\xi|$. Now

(3) there exist at most two $J \in K_1$ such that $(C \cap J) \cup (D \backslash J) = f_\xi[J]$.

For suppose: there are at least three such, J_0, J_1, J_2. Now $f_\xi[J_i] \cap J_i = 0$, so $D \backslash J_i = f_\xi[J_i]$ for $i < 3$. Hence,

$$\begin{aligned} f_\xi[J_0 \cap J_1] \cup f_\xi[J_0 \cap J_2] &= (f_\xi[J_0] \cap f_\xi[J_1]) \cup (f_\xi[J_0] \cap f_\xi[J_2]) \\ &= [D \backslash (J_0 \cup J_1)] \cup [D \backslash (J_0 \cup J_2)] \\ &= (D \backslash J_0) \cap [\omega \backslash (J_1 \cap J_2)] ; \end{aligned}$$

but the first set is finite and the last cofinite, a contradiction. So (3) holds.

Let K_2 be K_1 without the J's of (3). Since $|K_2| > |B_\xi| + \omega$, there is an $E \in B_\xi$ such that the set

$$K_3 = \{J \in K_2 \colon (C \cap J) \cup (D \setminus J) = E\}$$

has at least two elements J, H. Then

$$E = [(C \cap J \cap H) \cup D] \setminus [(D \cap H \cap J) \setminus E] \in A_\xi ,$$

a contradiction. Hence (1) holds.

Choose J as in (1). Let $A_{\xi+1} = \langle A_\xi \cup \{J\} \rangle$, $B_{\xi+1} = B_\xi \cup \{f_\xi[J]\}$. This finishes the construction. Let $A = A_\alpha$, where $\alpha = 2^\omega$. Clearly, $|A| = 2^\omega$. A is a subalgebra of $\mathcal{P}\omega$ containing all singletons. As such, each automorphism of A is induced by a permutation of ω. Each finite permutation of ω induces an automorphism of A. These are the only automorphisms of A. For suppose that g is an automorphism of A induced by the non-finite permutation f of ω. Say $f = f_\xi$ with $\xi < \omega$. Then with J as in the construction we have $J \in A$ but $gJ = f_\xi[J] \not\in A$, a contradiction. □

From Theorem 4.1 we can obtain at once the main facts about the size of automorphism groups:

4.2. THEOREM. (i) *If $m \in \omega$, $m \neq 0$, and $\kappa > \omega$, then there is a BA A with $|A| = \kappa$ and $|\mathrm{Aut}\ A| = m!$. For any infinite BA B with $|\mathrm{Aut}\ B| < \omega$ we have $|\mathrm{Aut}\ B| = m!$ for some positive integer m.*

(ii) *If $2^\omega \leq \kappa$, then there is a BA A such that $|\mathrm{Aut}\ A| = \omega$ and $|A| = \kappa$.*

(iii) *If $\omega < \kappa \leq \lambda$, then there is a BA A with $|A| = \lambda$ and $|\mathrm{Aut}\ A| = \kappa$.*

(iv) *If $\omega \leq \kappa$, then there is a BA A with $|A| = \kappa$ and $|\mathrm{Aut}\ A| = 2^\kappa$.*

PROOF. We already proved (i) in Corollary 1.8. For (ii), let A be the BA given in Theorem 4.1: $|A| = 2^\omega$, $|\mathrm{Aut}\ A| = \omega$, A atomic, and let B be an atomless rigid BA of power κ. Then $A \times B$ is as desired, by Theorem 1.2. For (iii), let A be a rigid BA of power κ and B a rigid BA of power λ such that A and B are totally different. Then $A \times A \times B$ is the desired algebra, by Theorems 1.2 and 1.6. Finally, for (iv) take A to be the free BA on κ generators. □

Theorem 4.2 does not say anything about denumerable BAs. We now show that in this case the automorphism groups are always of size 2^ω. Actually, the proof also gives one of the consistency results concerning possible improvements of Theorem 4.2.

4.3. THEOREM. *If the κ-Martin's axiom holds, and A is a BA with infinitely many atoms with $|A| = \kappa$, then the symmetric group on ω can be isomorphically embedded in* Aut A.

PROOF. We shall apply Theorem 2.2 of MARTIN and SOLOVAY [1970]. To this end let a be a one-to-one mapping of ω into the set of atoms of A. Clearly, there is an ultrafilter F on A such that $x \in F$ whenever $\{i \in \omega: a_i \leq -x\}$ is finite. Set $B = \{J \subseteq \omega: \text{for some } x \in F, J = \{i \in \omega: a_i \leq x\}\}$ and $C = \{J \subseteq \omega: \text{for some } x \in F, J = \{i \in \omega: a_i \cdot x = 0\}\}$. If $J \in B$ and $\mathcal{K}$ is a finite subset of C, say $J = \{i \in \omega: a_i \leq x_J\}$ with $x_J \in F$, and $K = \{i \in \omega: a_i \cdot x_K = 0\}$ with $x_K \in F$, for each $K \in \mathcal{K}$. Then $x_J \cdot \Pi_{k \in \mathcal{K}} x_K \in F$, and $a_i \leq x_J \cdot \Pi_{k \in \mathcal{K}} x_K$ implies that $i \in J \backslash \bigcup \mathcal{K}$. It follows that $J \backslash \bigcup \mathcal{K}$ is infinite, since otherwise $-(x_J \cdot \Pi_{K \in \mathcal{K}} x_K) \in F$. So Theorem 2.2 of MARTIN and SOLOVAY [1970] applies, and we infer:

(1) there is a $D \subseteq \omega$ such that $J \cap D$ is finite for each $J \in C$ and infinite if $J \in B$.

Let $E = \{a_i: i \in D\}$. Then by (1),

(2) for all $x \in A \backslash F$ the set $\{e \in E: e \leq x\}$ is finite; and E is infinite.

By (2), we can write each $x \in A \backslash F$ in the form $t_x + \Sigma M_x$, where no member of E is $\leq t_x$, and M_x is a finite subset of E. For any permutation f of E and any $x \in A \backslash F$ we set

$$f^+ x = t_x + \sum_{e \in M_x} fe ;$$

if $x \in F$ we set $f^+ x = -f^+(-x)$. It is routine to check that f^+ is an automorphism of A and f is an isomorphism of the symmetric group on E into Aut A. □

4.4. COROLLARY. *If $|A| = \omega$, then $|\text{Aut } A| = 2^\omega$.*

4.5. COROLLARY. *If Martin's axiom holds and $|\text{Aut } A| = \omega$, then $|A| \geq 2^\omega$.*

PROOF. By Corollary 1.7, A has infinitely many atoms. Hence, the corollary follows from 4.3. □

These are all the results provable in ZFC that we know concerning the size of automorphism groups of arbitrary BAs. Under GCH, the results are complete. Thus, let κ and λ be cardinals, with λ infinite. Then the following conditions are equivalent under GCH:

(A) There is a BA A such that $|\text{Aut } A| = \kappa$ and $|A| = \lambda$.

(B) One of the following holds:

(1) $\kappa = m!$ for some positive integer m, and $\lambda > \omega$;

(2) $\lambda = \omega$ and $\kappa = \omega_1$;

(3) $\lambda > \omega$ and $\omega \leq \kappa \leq \lambda^+$.

There are consistency results with not(GCH) also. Thus, from Theorem 4.17 in MONK [Ch. 13 in this Handbook] it follows that it is consistent that if $\omega < \kappa \leq 2^\omega$, then there is a BA A with $|\text{Aut } A| = \omega$ and $|A| = \kappa$, where 2^ω can be large. For further results along these lines see VAN DOUWEN [1980] and ROITMAN [1981].

Now we consider the size of automorphism groups of special kinds of BAs, namely complete BAs, atomic BAs, interval algebras, and superatomic BAs.

For complete BAs, the main new fact is as follows (KOPPELBERG [1981]):

4.6. THEOREM. *If A is a complete BA and $|\text{Aut } A|$ is infinite, then* $|\text{Aut } A|^{\omega} = |\text{Aut } A|$.

PROOF. By Theorem 1.3 we can write $A \cong B_0 \times B_1 \times C \times D$, where B_0 is atomic, B_1 is an atomless product of homogeneous algebras, C is a product of rigid BAs, and D has no rigid or homogeneous direct factors. Note that if E is homogeneous, then

$$\begin{aligned} |{}^{\omega}\text{Aut } E| &\leq |\text{Aut}^{\omega} E| && (1.1)\\ &= |\text{Aut } E| && (1.16)\\ &\leq |{}^{\omega}\text{Aut } E| . \end{aligned}$$

Hence, by 1.2 and 1.18, $|\text{Aut } B_1|^{\omega} = |\text{Aut } B_1|$.

Hence, it suffices to show that $|\text{Aut}(C \times D)|^{\omega} = |\text{Aut}(C \times D)|$. Now $C \times D$ has no homogeneous direct factors. Hence, from ŠTĚPÁNEK [Ch. 16, 3.13, in this Handbook] we know that $\text{inv}(C \times D)$ is atomless (see the definition before 1.11). Let X be a partition of unity in $\text{inv}(C \times D)$ satisfying the following conditions:

(1) for all $x \in X$ and all $a, b \in ((C \times D) \restriction x)^{+}$, $|\text{Aut}((C \times D) \restriction a)| = |\text{Aut}((C \times D) \restriction b)|$.

(2) if $x \in X$, then there are infinitely many $y \in X$ such that $|\text{Aut}((C \times D) \restriction x)| = |\text{Aut}((C \times D) \restriction y)|$.

(We need $\text{inv}(C \times D)$ atomless to get (2).) Since x and y are totally different for distinct $x, y \in X$, we get

$$|\text{Aut}(C \times D)| = \prod_{x \in X} |\text{Aut}((C \times D) \restriction x)| = \kappa^{\omega}$$

for some κ, as desired. □

Theorem 4.6 shows that the following theorem cannot be improved in ZFC.

4.7. THEOREM. (i) *If $m \in \omega$, $m \neq 0$, $\kappa \geq \omega$, and $\kappa^{\omega} = \kappa$, then there is a complete BA A with $|A| = \kappa$ and $|\text{Aut } A| = m!$.*

(ii) *If $\omega < \kappa \leq \lambda$, $\kappa^{\omega} = \kappa$, $\lambda^{\omega} = \lambda$, then there is a complete BA A with $|A| = \lambda$ and $|\text{Aut } A| = k$.*

(iii) *If $\omega \leq \kappa = \kappa^{\omega}$, then there is a complete BA A with $|A| = \kappa$ and $|\text{Aut } A| = 2^{\kappa}$.*

Turning to atomic BAs, note first that if A is atomic, then any finite permutation of the atoms of A extends to an automorphism of A. Hence, in this case we have the restrictions $|\text{At } A| \leq |\text{Aut } A| \leq 2^{|A|}$. Our main result for atomic BAs is the following generalization of Theorem 4.1 and its proof. We use this notation – if κ and λ are infinite cardinals, then

$$\mathrm{Sym}\,\kappa = \{f\colon f \text{ is a permutation of } \kappa\}\,,$$

$$\mathrm{Supp}\, f = \{\alpha < \kappa\colon f\alpha \neq \alpha\} \text{ for } f \in \mathrm{Sym}\,\kappa\colon$$

$$\mathrm{Sym}_{<\lambda}\,\kappa = \{f \in \mathrm{Sym}\,\kappa\colon |\mathrm{Supp}\, f| < \lambda\}\,.$$

4.8. THEOREM. *Let $\omega \leq \lambda \leq \kappa$. Then there is a BA $A \subseteq \mathscr{P}\kappa$ with $|A| = 2^\kappa$ such that $[\kappa]^{<\lambda} \subseteq A$ and* Aut A *is naturally isomorphic to* $\mathrm{Sym}_{<\lambda}\,\kappa$.

PROOF. Let $\langle f_\xi\colon \xi < 2^\kappa\rangle$ enumerate $\mathrm{Sym}\,\kappa \backslash \mathrm{Sym}_{<\lambda}\,\kappa$. We call two subsets X, $Y \subseteq \kappa$ *equivalent modulo* $[\kappa]^{<\lambda}$ if $X \triangle Y \in [\kappa]^{<\lambda}$. Let $\langle X_\alpha\colon \alpha < 2^\kappa\rangle$ be a system of subsets of κ which are independent modulo $[\kappa]^{<\kappa}$, that is, so that $\langle X_\alpha/[\kappa]^{<\kappa}\colon \alpha < 2^\kappa\rangle$ is a system of independent elements of $\mathscr{P}\kappa/[\kappa]^{<\kappa}$.

Now we construct by transfinite recursion two sequences $\langle Y_\alpha\colon \alpha < 2^\kappa\rangle$ and $\langle B_\alpha\colon \alpha < 2^\kappa\rangle$; each Y_α will be a subset of κ, and each B_α a subset of 2^κ such that $|B_\alpha| \leq \omega + |\alpha|$. The only essential part of the construction is to do this so that the following condition holds:

(1) for each $\beta < 2^\kappa$, $\langle Y_\alpha\colon \alpha \leq \beta\rangle \cup \langle X_\alpha\colon \beta < \alpha < 2^\kappa, \alpha \notin B_\beta\rangle$ is independent modulo $[\kappa]^{<\kappa}$, and for all $\xi \leq \beta$, $f_\xi[Y_\xi] \notin \langle\{Y_\alpha\colon \alpha \leq \beta\} \cup [\kappa]^{<\lambda}\rangle$.

Let γ be $<2^\kappa$ so that Y_β and B_β have been defined for all $\beta < \gamma$ so that (1) holds. The rest of the proof is to construct Y_γ and B_γ. To begin with, let $B^\gamma = \bigcup_{\alpha<\gamma} B_\alpha \cup (\gamma + 1)$. Then clearly

(2) $\langle Y_\alpha\colon \alpha < \gamma\rangle \cup \langle X_\alpha\colon \alpha \in 2^\kappa \backslash B^\gamma\rangle$ is independent modulo $[\kappa]^{<\kappa}$; $|B^\gamma| \leq \omega + |\gamma|$; and for all $\xi < \gamma$, $f_\xi[Y_\xi] \notin \langle\{Y_\alpha\colon \alpha < \gamma\} \cup [\kappa]^{<\lambda}\rangle$.

Now let δ_0 and δ_1 be the two least elements of $2^\kappa \backslash B^\gamma$. We claim

(3) there exist disjoint non-empty Z_0, $Z_1 \subseteq \kappa$, $Z \subseteq \kappa$, and $\bar{X} \in \langle\{X_{\delta_0}, X_{\delta_1}\}\rangle \backslash \{\kappa\}$ such that $Z = Z_0 \cup Z_1 \subseteq \bar{X}$, $|Z_0| = \lambda$, and $f_\gamma[Z_0] = Z_1$.

In fact, since $f_\gamma \notin \mathrm{Sym}_{<\lambda}\,\kappa$, there are disjoint C, $D \subseteq \kappa$ with $|C| = \lambda$ and $f_\gamma[C] = D$. There is an atom T_0 of $\langle\{X_{\delta_0}, X_{\delta_1}\}\rangle$ such that $|T_0 \cap C| = \lambda$, and there is an atom T_1 of $\langle\{X_{\delta_0}, X_{\delta_1}\}\rangle$ such that $|T_1 \cap f_\gamma[T_0 \cap C]| = \lambda$. Let $Z_0 = f_\gamma^{-1}[T_1 \cap f_\gamma[T_0 \cap C]]$, $Z_1 = T_1 \cap f_\gamma[T_0 \cap C]$, $\bar{X} = T_0 \cup T_1$, and note that $\langle\{X_{\delta_0}, X_{\delta_1}\}\rangle$ has four atoms; (3) follows.

Now for each $\delta \in 2^\kappa \backslash (B^\gamma \cup \{\delta_0, \delta_1\})$ and each $S \subseteq Z_0$ we set

$$Y^{\delta S} = (X_\delta \backslash Z) \cup S\,;$$

$$B^{\delta S} = B^\gamma \cup \{\delta, \delta_0, \delta_1\}\,.$$

Note that $Y^{\delta S} \cap (\kappa \backslash \bar{X}) = X_\delta \cap (\kappa \backslash \bar{X})$. Let $C = (\mathscr{P}\kappa/[\kappa]^{<\kappa}) \restriction ((\kappa \backslash \bar{X})/[\kappa]^{<\kappa})$. For each $W \subseteq \mathscr{P}\kappa$ let $hW = (W \backslash [\kappa]^{<\kappa}) \cdot ((\kappa \backslash \bar{X})/[\kappa]^{<\kappa})$. Then h is a homomorphism from $\mathscr{P}\kappa$ onto C, and it takes the elements Y_α, $\alpha<\gamma$, $Y^{\delta S}$, X_β, $\beta\in 2^\kappa\backslash B^{\delta S}$ to independent elements. Hence,

(4) for all $\delta\in 2^\kappa\backslash(B^\gamma\cup\{\delta_0,\delta_1\})$ and all $S\subseteq Z_0$, $\langle Y_\alpha:\alpha<\gamma\rangle$ $\cup\{(\gamma, Y^{\delta S})\}\cup\langle X_\alpha:\alpha\in 2^\kappa\backslash B^{\delta S}\rangle$ is independent.

[Eventually we will let $Y_\delta = Y^{\delta S}$, $B_\gamma = B^{\delta S}$ for some such δ and S.] Now we need

(5) let $\xi<\gamma$ and $A, B\in\langle Y_\alpha:\alpha<\gamma\rangle$. Then there is at most one $\delta\in 2^\kappa\backslash(B^\gamma\cup\{\delta_0,\delta_1\})$ such that for some $S\subseteq Z_0$ we have $f_\xi[Y_\xi]$ equivalent to $(A\cap Y^{\delta S})\cup(B\backslash Y^{\delta S})$ modulo $[\kappa]^{<\lambda}$.

For, suppose not; say $\delta^i\in 2^\kappa\backslash(B^\gamma\cup\{\delta_0,\delta_1\})$, $S^i\subseteq Z_0$, $\delta^0\neq\delta^1$, and

(6) $f_\xi[Y_\xi]\equiv(A\cap Y^{\delta^i S^i})\cup(B\backslash Y^{\delta^i S^i})$, $i=1,2$.

Then

$$(\kappa\backslash\bar{X})\cap(A\triangle B)\cap(X_{\delta^1}\triangle X_{\delta^2})\subseteq(\kappa\backslash Z)\cap(A\triangle B)\cap(X_{\delta^1}\triangle X_{\delta^2})$$
$$\subseteq[(A\cap Y^{\delta^1 S^1})\cup(B\backslash Y^{\delta^1 S^1})]\triangle[(A\cap Y^{\delta^2 S^2})\cup(B\backslash Y^{\delta^2 S^2})],$$

from which it follows by (2) that $A\triangle B=0$ and so $A=B$, so (6) yields $f_\xi[Y_\xi]$ equivalent to A modulo $[\kappa]^{<\lambda}$, contradicting (2). Thus (5) holds.

There are at most $\omega+|\gamma|$ triples (ξ, A, B) as in (5); so let δ be the least member of $2^\kappa\backslash(B^\gamma\cup\{\delta_0,\delta_1\})$ such that there are not ξ, A, B, S as described in (5). Thus

(7) for all $\xi<\gamma$ and all $S\subseteq Z_0$ we have $f_\xi[Y_\xi]\not\in\langle\{Y_\alpha:\alpha<\gamma\}$ $\cup\{Y^{\delta S}\}\cup[\kappa]^{<\lambda}\rangle$.

Now we claim that for $S=0$ or $S=Z_0$ we have $f_\gamma[Y^{\delta S}]\not\in\langle\{Y_\alpha:\alpha<\gamma\}\cup\{Y^{\delta S}\}\cup[\kappa]^{<\lambda}\rangle$. (This will finish the construction.) Suppose that this is not true. Then there are $A_i, B_i\in\langle\{Y_\alpha:\alpha<\gamma\}\rangle$, $i=1,2$, such that

(8) $$f_\gamma[Y^{\delta 0}]\equiv(A_1\cap Y^{\delta 0}\cup(B_1\backslash Y^{\delta 0})\bmod[\kappa]^{<\lambda},$$
$$f_\gamma[Y^{\delta Z_0}]\equiv(A_2\cap Y^{\delta Z_0})\cup(B_2\backslash Y^{\delta Z_0})\bmod[\kappa]^{<\lambda}.$$

Note that $f_\gamma[y^{\delta 0}]\backslash\bar{X}=f_\gamma[Y^{\delta Z_0}]\backslash\bar{X}$, $Y^{\delta 0}\backslash\bar{X}=Y^{\delta Z_0}\backslash\bar{X}=X_\delta\backslash\bar{X}$, and (consequently) $(\kappa\backslash Y^{\delta 0})\backslash\bar{X}=(\kappa\backslash Y^{\delta Z_0})\backslash\bar{X}=(\kappa\backslash X_\delta)\backslash\bar{X}$. Hence, intersecting both sides of the congruences (8) with $\kappa/\bar{X}$ we get

$$(A_1\cap X_\delta\backslash\bar{X})\cup(B_1\cap(\kappa\backslash X_\delta)\backslash\bar{X})\equiv(A_2\cap X_\delta\backslash\bar{X})$$
$$\cup(B_2\cap(\kappa\backslash X_\delta)\backslash\bar{X})\bmod[\kappa]^{<\lambda}.$$

Since $\langle Y_\alpha \colon \alpha < \gamma \rangle \cup \langle X_\delta, X_{\delta_0}, X_{\delta_1} \rangle$ are independent modulo $[\kappa]^{<\kappa}$, this equivalence is actually an equality, and implies that $A_1 = A_2$ and $B_1 = B_2$.

Intersecting the congruences (8) with Z_1 we get

$$0 \equiv B_1 \cap Z_1 \bmod [\kappa]^{<\lambda} ,$$

$$Z_1 \equiv B_1 \cap Z_1 \bmod [\kappa]^{<\lambda} ,$$

so $|Z_1| < \lambda$, a contradiction.

So we choose $S = 0$ or $S = Z_0$ so that $f_\gamma[Y^{\delta S}] \not\in \langle \{Y_\alpha \colon \alpha < \gamma\rangle \cup \{Y^{\delta s}\} \cup [\kappa]^{<\lambda}\rangle$. Let $Y_\gamma = Y^{\delta S}$, $B_\gamma = B^{\delta S}$. Then (1) holds for γ.

This finishes the construction. Let $A = \langle \{Y_\alpha \colon \alpha < Z_\kappa\} \cup [\kappa]^{<\lambda}\rangle$. The desired conclusions are clear from (1). $\square$

Concerning the size of automorphism groups of atomic BAs, we also recall a corollary of Theorem 4.8 of MONK [Ch. 13 in this Handbook] there is an atomic BA A with $|A| = |\mathrm{Aut}\, A| = |\mathrm{At}\, A| = 2^\omega$.

Assume GCH for the following remarks. For any atomic BA we have $|\mathrm{At}\, A| \leqq |\mathrm{Aut}\, A| \leqq |A|^+$ and $|A| \leqq |\mathrm{At}\, A|^+$. By Theorem 4.8, for any infinite κ there is a BA A such that $|\mathrm{At}\, A| = |\mathrm{Aut}\, A| = \kappa$ and $|A| = \kappa^+$. If B is the finite–cofinite algebra on κ, then $|\mathrm{At}\, B| = |B| = \kappa$ and $|\mathrm{Aut}\, B| = \kappa^+$. And $|\mathscr{P}\kappa| = \kappa^+ = |\mathrm{Aut}\, \mathscr{P}\kappa|$, while $|\mathrm{At}\, \mathscr{P}\kappa| = \kappa$. The essential missing possibility here is an atomic algebra A such that $|\mathrm{At}\, A| = |\mathrm{Aut}\, A| = |A| = \kappa$. By the remark of the preceding paragraph, there is such an algebra for $\kappa = \omega_1$. Of course, there is none for $\kappa = \omega$. We do not know whether there are such for $\kappa > \omega_1$.

Turning to automorphism groups of interval algebras, first recall that for each $\kappa > \omega_1$ there is a rigid, cardinality-homogeneous interval algebra of power κ. Hence, parts (i) and (iii) of Theorem 4.2 hold for interval algebras. Part (iv) also holds, by taking the interval algebra on an ordered set S of power κ which has 2^κ order-automorphisms (for the existence of such an ordered set see, for example, MONK [1976, p. 451]). We do not know whether part (ii) holds; in particular, we do not know whether there is an interval algebra of power 2^ω with automorphism group of power ω.

We know even less about the size of automorphism groups of superatomic BAs. Perhaps it is even true that always $|\mathrm{Aut}\, A| = 2^{|\mathrm{At}\, A|}$ for A superatomic.

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CHAPTER 15

On the Reconstruction of Boolean Algebras from their Automorphism Groups

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
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1. Introduction

This chapter is concerned mainly with the following problem. Given two Boolean algebras B_1 and B_2 whose automorphism groups are isomorphic, does it follow that B_1 and B_2 are isomorphic?

In its full generality the above statement is clearly false, because, for example, all rigid BAs have the same automorphism group, namely the trivial one. So in order to conclude that $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2) \Rightarrow B_1 \cong B_2$, one should impose some restrictions on B_1 and B_2. A typical theorem of this type is that: if B_1 and B_2 are homogeneous BAs, then the fact that $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$ implies that $B_1 \cong B_2$. (Recall that B is homogeneous if for every $a \in B - \{0\}$ $B \cong B \upharpoonright a$.)

Another way to bypass the above counterexample, is to prove that the fact $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$ implies that B_1 and B_2, though not necessarily isomorphic, have at least some similarities. A representative to this type of theorem is the following: if B_1 and B_2 are weakly homogeneous and $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$, then the completion of B_1 is isomorphic to the completion of B_2. (Recall that B is weakly homogeneous if for every non-zero $a, b \in B$ there are non-zero $a' \leq a$, $b' \leq b$ such that $B \upharpoonright a' \cong B \upharpoonright b'$.)

The question whether the isomorphism type of the automorphism group of a structure determines its own isomorphism type is of course not unique to Boolean algebras. Even though the framework within which such theorems are proved, in many cases, happens to be the same, namely the method of interpretation, the proofs themselves depend very much on the type of the structures that are being considered. So, solving such a problem for one type of structures does not help in general in solving the same problem for another type of structures.

Nevertheless the reconstruction of Boolean algebras from their automorphism groups happens to be applicable in several analogous questions for other types of structures. The following state of affairs demonstrates the above. Let X be a topological space, and suppose we wish to reconstruct X from its group of autohomeomorphisms $H(X)$. The set $\mathrm{Ro}(X)$ of regular open subsets of X is a complete Boolean algebra, each homeomorphism of X induces an automorphism of $\mathrm{Ro}(X)$, hence assuming that X is regular, $H(X)$ can be regarded as a subgroup of $\mathrm{Aut}(\mathrm{Ro}(X))$. If $\mathrm{Ro}(X)$ could be reconstructed from $H(X)$, then we would have a first step towards reconstructing X from $H(X)$.

A similar situation occurs when dealing with automorphism groups of linear orderings, trees or measure algebras. (See RUBIN [1988].)

Note that in the application to topological spaces, it was required to reconstruct $\mathrm{Ro}(X)$ from a subgroup of $\mathrm{Aut}(\mathrm{Ro}(X))$, and this differs from the original setting where it was attempted to reconstruct a BA from its full automorphism group. This calls for a more general setting.

1.1. DEFINITION. Let K be a class of pairs $\langle B, G\rangle$, where B is a BA and $G \subseteq \mathrm{Aut}(B)$. K is *faithful* if for every $\langle B_1, G_1\rangle, \langle B_2, G_2\rangle \in K$: if $G_1 \cong G_2$, then $B_1 \cong B_2$. K is *strongly faithful* if for every $\langle B_1, G_1\rangle, \langle B_2, G_2\rangle \in K$ and an isomorphism φ between G_1 and G_2, there is an isomorphism τ between B_1 and B_2 such that for every $f \in G_1$, $\varphi(f) = \tau \circ f \circ \tau^{-1}$.

If K is a class of BAs, we say that K is *faithful* (*strongly faithful*) when $\{\langle B, \mathrm{Aut}(B)\rangle \mid B \in K\}$ is faithful (strongly faithful).

We thus wish to find large and natural faithful classes.

In some cases we consider classes K which are not faithful. However, for some members $\langle B_0, G_0\rangle$ of K it is still true that within K, G_0 determines B_0. We next introduce a notion to describe this state of affairs.

1.2. DEFINITION. Let K be a class of pairs of the form $\langle B, G\rangle$, where B is a BA and $G \subseteq \mathrm{Aut}(B)$, and let $\langle B_0, G_0\rangle \in K$. $\langle B_0, G_0\rangle$ is *group-categorical* in K if for every $\langle B, G\rangle \in K$ and any isomorphism φ between G_0 and G there is an isomorphism $\tilde{\varphi}$ between B_0 and B which induces φ, that is, for every $f \in G_0$, $\varphi(f) = \tilde{\varphi} \circ f \circ \tilde{\varphi}^{-1}$.

Questions concerning faithfulness of classes of BAs were first raised by MONK [1975] and by MCKENZIE and MONK [1973]. MCKENZIE [1977] showed that the class of non-atomless countable BAs with a maximal atomic element is faithful and that the class of non-atomless countable BAs is not faithful. In RUBIN [1979] it was shown that the class of homogeneous BAs is faithful. The class of complete BAs was considered in RUBIN [1980].

The elementary equivalence of automorphism groups of Boolean algebras was considered in RUBIN [1979], [1980a]; RUBIN and SHELAH [1980] and SHELAH [1978], [1983].

In Section 2 we introduce the formal notion of interpretation, and make some easy observations related to this notion. The definition is rather technical, whereas its intuitive meaning may be known to the reader. The understanding of the technical definition may thus be postponed to a later stage.

Section 3 deals with faithfulness results concerning complete BAs. The main theorem in this section is Theorem 3.4 which says that the class K_C of complete flexible $\langle B, G\rangle$'s for which $a^{[\leq 2]}(B, G) = 0$ is faithful. ($\langle B, G\rangle$ is complete if B is a complete BA, movability is defined in 3.2(d), flexibility in 3.2(h) and $a^{[\leq 2]}(B, G)$ in 3.1(e).) Theorem 3.4 is later applied in Sections 4, 5 and 6. It is also the basic theorem in RUBIN [1988].

The final result of Section 3 is Theorem 3.23 where a necessary and sufficient condition is given for when the automorphism groups of two complete BAs are isomorphic. The following is the central open problem of Section 3.

QUESTION 1. Is the class $\{\langle B, G\rangle \mid B$ is complete, $a^{[\leq 2]}(B, G) = 0$ and $\langle B, G\rangle$ is movable$\}$ faithful?

In Section 4 we deal with the faithfulness problem of BAs which are not necessarily complete. The class of homogeneous BAs is faithful (RUBIN [1979]). This class is, however, too restrictive. In Theorem 4.5 we show that the class $\{\langle B, \mathrm{Aut}(B)\rangle \mid 1_B$ is a finite sum of elements a for which there are $f, g \in \mathrm{Aut}(B)$ such that $a, f(a), g(a)$ are pairwise disjoint$\}$ is faithful. This theorem is later applied in Section 6.

In Section 5 we prove McKenzie's theorem (MCKENZIE [1977]), which states that the class of non-atomless countable BAs that have a maximal atomic element is faithful.

Section 6 deals with measure algebras. We deal with objects of the form $\langle B, \mu\rangle$; B is a σ-algebra (not necessarily a field of sets) and μ is a σ-additive σ-finite function to $[0, \infty]$. Under appropriate assumptions on $\langle B, \mu\rangle$, we reconstruct $\langle B, \mu\rangle$ from its group of measure preserving automorphisms $\mathrm{MP}(B, \mu)$. μ is of course reconstructed up to a multiplicative constant. This is done in Theorems 6.1, 6.5 and 6.7. Let K be a locally compact σ-compact group, let $\mathrm{Bl}(K)$ be the Borel field of K and μ be a left Haar measure on K. Let G^K be the group of piecewise left translations of K. (An automorphism f of $\mathrm{Bl}(K)$ is a piecewise left translation if there is a partition of $\{a_i \mid i \in \omega\}$ $1_{\mathrm{Bl}(K)}$ and elements $\{k_i \mid i \in \omega\}$ of K such that for every $i \in \omega$ and $x \in a_i$ $f(x) = k_i x$.) We prove (Corollary 6.5) that $\langle \mathrm{Bl}(K), \mu\rangle$ is the unique σ-finite measure algebra $\langle B_0, \mu_0\rangle$ which has a B-closed group $G \subseteq \mathrm{MP}(B_0, \mu_0)$ such that $I^{[\geq 2]}(B_0, G) = B_0$ and $G \cong G^K$. $I^{[\geq 2]}(B, G)$ is defined in 3.1(e) and B-closedness in 4.3. If $a \in \mathrm{Bl}(K)$ has a non-empty interior, then the above is true for the group $G^K(a)$ of all piecewise translations which are the identity outside a, and the measure algebra $\langle \mathrm{Bl}(K) \restriction a, \mu \restriction a\rangle$.

Our faithful classes include the classical measure algebras: (1) the fields of Borel and Lebesgue measurable subsets of $\mathbb{R}$ with their Lebesgue measure; (2) strictly positive measure algebras $\langle B, \mu\rangle$ in which B is homogeneous; (3) measures obtained by Carathéodory extension theorem from homogeneous BAs; and (4) Haar measures of locally compact σ-compact groups.

Let $I_Z(B, \mu) = \{a \in B \mid \mu(a) = 0\}$ and $\mathrm{MZP}(B, \mu) = \{f \in \mathrm{Aut}(B) \mid f(I_Z(B, \mu)) = I_Z(B, \mu)\}$. In Theorems 6.8–6.10 we prove that under appropriate assumptions on $\langle B, \mu\rangle$, the structure $\langle B; I_Z(B, \mu), \leq\rangle$ can be reconstructed from $\mathrm{MZP}(B, \mu)$. The faithful classes presented in 6.8–6.10 include the classical measure spaces.

The function $\rho_\mu(a, b) = \mu(a \triangle b)$ is a semi-metric on B. The members of $\mathrm{MZP}(B, \mu)$ are the continuous automorphisms of B with respect to this semi-metric, and they are always uniformly continuous. A set Γ of moduli of continuity defines an intermediate group $\mathrm{MP}_\Gamma(B, \mu)$ which contains $\mathrm{MP}(B, \mu)$ and is contained in $\mathrm{MZP}(B, \mu)$. In 6.13 we prove that for countable Γ's $\mathrm{MP}_\Gamma(B, \mu)$ determines Γ. This theorem is due to RUBIN and YOMDIN [198?].

Faithfulness problems for measure algebras have not been thoroughly investigated, and there are many open questions which arise in the context of Section 6. However, we cannot comment on the difficulty of these questions.

QUESTION 2. Let $a \subseteq \mathbb{R}$ be a Borel set of positive measure and G_a be the group of Borel-piecewise translations of $\mathbb{R}$ which are the identity outside a. Is $\langle \mathrm{Bl}(\mathbb{R}) \restriction a, \mu_{\text{Lebesgue}} \restriction a\rangle$ the only σ-finite measure algebra that has a movable σ-closed group of measure preserving automorphisms isomorphic to G_a. Recall that this is true when a has a non-empty interior.

QUESTION 3. Let $S = \{z \in \mathbb{C} \mid |z| = 1\}$ and $Z = \{0, 1\}^\omega$. S and Z are compact groups, so let G^S, G^Z be the groups of piecewise translations of S and Z, respectively, with respect to their Borel fields. Is G^S isomorphic to Z^S? More generally, when is it true for locally compact groups K and H that $G^K \cong G^H$?

QUESTION 4. Develop the results of Section 6 for measure rings.

QUESTION 5. Does Theorem 6.13 remain true for Γ's which are not necessarily countable?

2. The method

A pair $\langle B, G\rangle$, where B is a BA and G is a subgroup of $\mathrm{Aut}(B)$, is called a *BG-pair*. For a BG-pair $\langle B, G\rangle$, let Op – the operation function from $G \times B$ to B be the following function: $\mathrm{Op}(g, b) = g(b)$. Let $M(B, G) = \langle B, G; \mathrm{Op}\rangle$ be the following structure. The universe of $M(B, G)$ is $B \cup G$; $M(B, G)$ has unary predicates to represent B and G; $M(B, G)$ has function symbols, individual constants and predicates to represent all the function's relations and constants of B and G, and finally $M(B, G)$ has a function symbol to represent Op.

If K is a class of BG-pairs, let $K^G = \{G \mid \exists B(\langle B, G\rangle \in K)\}$ and $K^{BG} = \{M(B, G) \mid \langle B, G\rangle \in K\}$.

In order to show that a class K is faithful we will find a way to reconstruct from any member G of K^G its corresponding $M(B, G)$. The method of reconstruction will resemble the way in which fields of fractions are reconstructed from integral domains, or the way that Stone spaces are reconstructed from Boolean algebas.

We now formalize the notion of reconstruction.

2.1. DEFINITION. (a) Let K and K^* be classes of structures in the languages L and L^*, respectively, and $R \subseteq K \times K^*$ be a relation. Let $\varphi_U(x_1, \ldots, x_n)$, $\varphi_{\mathrm{Eq}}(x_1, \ldots, x_n, y_1, \ldots, y_n)$, $\{\varphi_P(x_1^1, \ldots, x_n^1, \ldots, x_1^k, \ldots, x_n^k) \mid k \in \omega,\ P \in L^*$ is a k-place predicate$\}$, $\{\varphi_F(x_1^1, \ldots, x_n^1, \ldots, x_1^k, \ldots, x_n^k, y_1, \ldots, y_n) \mid k \in \omega$, $F \in L^*$ is a k-place function symbol$\}$ be a family of first-order formulas in L with the following property: for every $\langle M, M^*\rangle \in R$ there is a function $h\colon \{\langle a_1, \ldots, a_n\rangle \mid M \models \varphi_U[a_1, \ldots, a_n]\} \xrightarrow{\text{onto}} |M^*|$ such that:

(1) for every $\langle a_1, \ldots, a_n\rangle$, $\langle b_1, \ldots, b_n\rangle \in \mathrm{Dom}(h)$ $h(\langle a_1, \ldots, a_n\rangle) = h(\langle b_1, \ldots, b_n\rangle)$ iff $M \models \varphi_{\mathrm{Eq}}[a_1, \ldots, a_n, b_1, \ldots, b_n]$;

(2) for every m-place relation symbol $P \in L^*$ and $\langle a_1^i, \ldots, a_n^i\rangle \in \mathrm{Dom}(h)$, $i = 1, \ldots, m$: $M \models \varphi_p[a_1^1, \ldots, a_n^1, \ldots, a_1^m, \ldots, a_n^m]$ iff $\langle h(\langle a_1^1, \ldots, a_n^1\rangle), \ldots, h(\langle a_1^m, \ldots, a_n^m\rangle)\rangle \in P^{M^*}$; and

(3) for every m-place function symbol $F \in L^*$, $\langle a_1^i, \ldots, a_n^i\rangle \in \mathrm{Dom}(h)$, $i = 1, \ldots, m$, and $\langle b_1, \ldots, b_n\rangle \in \mathrm{Dom}(h)$: $M \models \varphi_F[a_1^1, \ldots, a_n^1, \ldots, a_1^m, \ldots, a_n^m, b_1, \ldots, b_n]$ iff $F^M(h(\langle a_1^1, \ldots, a_n^1\rangle), \ldots, h(\langle a_1^m, \ldots, a_n^m\rangle)) = h(\langle b_1, \ldots, b_n\rangle)$.

Then the above family of formulas is called a *first-order interpretation* of K^* in K relative to R, and K^* is said to be *first-order interpretable in K relative to R*.

(b) Let K, K^* and R be as above and suppose, in addition, that there is a unary predicate S of L^* such that for every $\langle M, M^*\rangle \in R$ $M = S^{M^*} \restriction L$, that is, M is the reduct to L of the submodel of M^* whose universe is $\{a \in |M^*| \mid M^* \models S[a]\}$. Let $\varphi_{\mathrm{U}}, \varphi_{\mathrm{Eq}}, \ldots$ be an interpretation of K^* in K, and suppose that there is a formula $\varphi_h(x_1, \ldots, x_n, y)$ in L^* such that for every $\langle M, M^* \rangle \in R$: $h \stackrel{\text{def}}{=} \{\langle\langle a_1, \ldots, a_n\rangle, b\rangle \mid M^* \models \varphi_h[a_1, \ldots, a_n, b]\}$ is a function from $\{\langle a_1, \ldots, a_n\rangle \mid M \models \varphi_{\mathrm{U}}[a_1, \ldots, a_n]\}$ onto M^* satisfying the requirements in part (a) of the definition. Then K^* is *strongly first-order interpretable in K relative to R*.

If K^* is first-order interpretable in K relative to R and $\langle M, M^* \rangle \in R$, then we represent the elements of $|M^*|$ by elements of $|M|^n$; there are cases in which M^* can be reconstructed from M but its elements are represented by objects more complicated than n-tuples from M. A typical such case is Stone spaces and Boolean algebras; the elements of the Stone space of a Boolean algebra B are represented by ultrafilters which are elements of the power set of $|M|$ rather than elements of $|M|^n$. Our next goal is thus to define a notion of interpretation more general than first-order interpretation. Let $P(A)$ denote the power set of A. If M is a model, we define the second-order model based on M, and it will be denoted by $M^{[2]}$.

$$M^{[2]} = \langle M, P(|M|), P(|M|^2), \ldots, ; \varepsilon_1, \varepsilon_2, \ldots \rangle ,$$

where $\varepsilon_i \subseteq |M|^i \times P(|M|^i)$ with $\langle a_1, \ldots, a_i, r\rangle \in \varepsilon_i$ iff $\langle a_1, \ldots, a_i\rangle \in r$.

REMARK. It turns out that all relations on a set A can be encoded by binary relations on A, so as far as we are concerned $M^{[2]}$ can be replaced by $\langle M, P(|M|^2); \varepsilon_2\rangle$.

2.2. DEFINITION. Let K, K^*, L, L^*, and R be as in Definition 1.1. We say that K^* is (*strongly*) *interpretable in K relative to R*, if K^* is (strongly) first-order interpretable in $\{M^{[2]} \mid M \in K\}$ relative to R.

2.3. PROPOSITION. (a) *If K^* is interpretable in K relative to R, then for every $\langle M_1, M_1^*\rangle$, $\langle M_2, M_2^*\rangle \in R$ if $M_1 \underset{\varphi}{\cong} M_2$, then $M_1^* \cong M_2^*$. Moreover, if K^* is strongly interpretable in K and $M_1 \cong M_2$, then there is a unique $\varphi^* \supseteq \varphi$ which is an isomorphism between M_1^* and M_2^*.*

(b) *Let K be a class of BG-pairs then if K^{BG} is* (*strongly*) *interpretable in K^G then K is* (*strongly*) *faithful.*

The highly technical definition of interpretation and the proposition following it summarize the framework in which faithfulness results are proved in this chapter. The proof of Proposition 2.3 is simple; its intuitive meaning is that if in two isomorphic structures one uses the same set of formulas to obtain definable structures, then the two definable structures are isomorphic.

At this point it is worthwhile to notice that strong faithfulness is meaningful also for classes containing only one element. The fact that $\{\langle B, G\rangle\}$ is strongly faithful is equivalent to saying that every automorphism of G is a conjugation by a member of $\mathrm{Aut}(B)$. If, in addition, we know that G is equal to its normalizer in $\mathrm{Aut}(B)$, this will imply that every automorphism of G is inner.

As an instance of such a state of affairs consider the group $\mathrm{MP}(B, \mu)$ of measure preserving automorphisms of the field of Borel subsets of $[0, 1]$, where μ is the Lebesgue measure on B. We shall prove that $\{\langle B, \mathrm{MP}(B, \mu)\rangle\}$ is strongly faithful, that is, if $\varphi \in \mathrm{Aut}(\mathrm{MP}(B, \mu))$, then there is $\tau \in \mathrm{Aut}(B)$ such that for every $f \in \mathrm{MP}(B, \mu)$ $\varphi(f) = \tau \circ f \circ \tau^{-1}$. It will be easy to see that every $\tau \in \mathrm{Aut}(B)$ which satisfies $\tau \circ \mathrm{MP}(B, \mu) \circ \tau^{-1} = \mathrm{MP}(B, \mu)$ must belong to $\mathrm{MP}(B, \mu)$, so it follows that every automorphism of $\mathrm{MP}(B, \mu)$ is inner.

3. Faithfulness in the class of complete Boolean algebras

In this section we show how complete BAs can be reconstructed from subgroups of their automorphism groups. It so happens that the main theorem in this chapter, Theorem 3.4, deals with the reconstruction of complete BAs. All faithfulness results appearing in this chapter use 3.4 as a first step in the interpretation.

We start with some definitions needed in the formulation of 3.4.

Let *Id* denote the identity function. A Boolean algebra B is rigid if $\mathrm{Aut}(B) = \{Id\}$. The completion of a Ba B is denoted by $\bar{B}$.

3.1. DEFINITION. (a) Let $f \in \mathrm{Aut}(B)$, then $\mathrm{var}(f) \stackrel{\mathrm{def}}{=} \Sigma\, \{a \in B \mid f(a) \cdot a = 0\}$ and $\mathrm{fix}(f) \stackrel{\mathrm{def}}{=} \Sigma\, \{a \in B \mid (\forall b \le a)(f(b) = b)\}$.

Note that $\mathrm{var}(f)$ and $\mathrm{fix}(f)$ belong to $\bar{B}$, and that $\mathrm{fix}(f) = -\mathrm{var}(f)$.

(b) Let $\langle B, G\rangle$ be a BG-pair, a, $b \in B$ and $g \in G$. We denote $a \stackrel{g}{\cong} b$, if $g(a) = b$; we say that $a \cong b$ in $\langle B, G\rangle$ if there is $g \in G$ such that $a \stackrel{g}{\cong} b$.

Note that $a \cong b$ in $\langle B, G\rangle$ implies that $B \restriction a \cong B \restriction b$.

(c) Let $\langle B, G\rangle$ be a BG-pair and $a \in B$. a is *rigid in* $\langle B, G\rangle$ if there are no pairwise disjoint non-zero a_1, $a_2 \in B$ such that a_1, $a_2 \le a$ and $a_1 \cong a_2$.

(d) Let $\langle B, G\rangle$ be a BG-pair, $a \in B$ and $n \in \omega$. We say that the *multiplicity of a is greater than or equal to n*, if there are pairwise disjoint $a_1, \ldots, a_n \in B$ such that $a = a_1 \cong a_2 \cong \cdots \cong a_n$ in $\langle B, G\rangle$. We denote this fact by $m(a) \ge n$ in $\langle B, G\rangle$.

(e) Let $I^{[\ge n]}(B, G)$ be the ideal generated by $\{b \in B \mid m(b) \ge n\}$, and $a^{[\ge n]}(B, G) \stackrel{\mathrm{def}}{=} \Sigma\, I^{[\ge n]}(B, G)$.

When $\langle B, G\rangle$ is understood from the context we omit its mention, when $G = \mathrm{Aut}(B)$ we omit the mention of G. Let $a^{[\le n]} = -a^{[\ge n+1]}$; $a^{[n]} = a^{[\ge n]} - a^{[\ge n+1]}$ and $G^{[\le n]} = \{g \in G \mid \mathrm{var}(g) \le a^{[\le n]}\}$. $G^{[\ge n]}$ and $G^{[n]}$ are defined similarly.

We now wish to motivate and state Theorem 3.4.

We say that $\langle B, G\rangle$ is a *complete BG-pair* if B is complete.

Theorem 3.4 states that for a certain class K_{C^-} of complete BG-pairs, $K_{C^-}^{BG}$ is strongly interpretable in $K_{C^-}^{G}$. Naturally, we wish to include in K_{C^-} as many BAs as possible. In order to motivate the conditions appearing in the definition of K_{C^-}, we first check which BAs have to be excluded from K_{C^-}.

We have already noted that rigid BAs cannot be reconstructed from their automorphism groups; similarly, G does not carry any information about $a^{[1]}(B, G)$, since for every $g \in G$, $\mathrm{var}(g) \cdot a^{[1]}(B, G) = 0$.

After understanding our notations, the reader will find it easy to verify that the center of G, $Z(G)$, always contains $G^{[2]}$, and that every element of $G^{[2]}$ has order 2. (See 3.3(h).) Thus, $G^{[2]}$ is isomorphic to a vector space over the field $\{0, 1\}$, and its isomorphism type is determined solely by its cardinality. All of this means that G carries very little information about $a^{[2]}$, and that in order to reconstruct B from G we have to assume that $a^{[1]} = a^{[2]} = 0$ or equivalently that $I^{[\geq 3]}$ is dense in B. For complete pairs of the form $\langle B, \operatorname{Aut}(B)\rangle$ the above requirement suffices, that is, $\{\langle B, \operatorname{Aut}(B)\rangle \mid B$ is complete and $a^{[1]}(B) = a^{[2]}(B) = 0\}$ is faithful.

In the following definition we introduce the main notions and notations of Section 3.

3.2. DEFINITION. (a) $a \in B$ is *locally movable in* $\langle B, G\rangle$ if for every non-zero $b \leq a$ there is $g \in G - \{Id\}$ such that $\operatorname{var}(g) \leq b$.

(b) Let $a^R(B, G) = \Sigma\{a \in B \mid a$ is rigid in $\langle B, G\rangle\}$ and $a^{\mathrm{LM}}(B, G) = \Sigma\{a \in B \mid a$ is locally movable in $\langle B, G\rangle\}$.

(c) $f \in G$ is a *transposition* if $f \neq Id$ and there are no non-zero pairwise disjoint $a_1 \cong a_2 \cong a_3$ in B such that $a_1, a_2, a_3 \leq \operatorname{var}(f)$. An element $a \in \bar{B}$ is a *transpose* of f if $a \cdot f(a) = 0$ and $a + f(a) = \operatorname{var}(f)$.

(d) $\langle B, G\rangle$ is *movable* if for every non-rigid $a \in B$ there is $g \in G - \{Id\}$ such that $\operatorname{var}(g) \leq a$.

(e) For $a \in \bar{B}$ and $g \in G$ let $B \upharpoonright a = \{b \cdot a \mid b \in B\}$, $g \upharpoonright a = g \upharpoonright (B \upharpoonright a)$ and $G(a) = \{g \in G \mid \operatorname{var}(g) \leq a\}$. $\langle B \upharpoonright a, \leq\rangle$ is a BA, and $G(a)$ is sometimes regarded as a subgroup of $\operatorname{Aut}(B \upharpoonright a)$.

(f) $a_1 \cong a_2 (\operatorname{mod} b)$ in $\langle B, G\rangle$ if there is $g \in G(-b)$ such that $g(a_1) = a_2$.

(g) a *is flexible with respect to* b in $\langle B, G\rangle$ ($\mathrm{Fl}(a; b)$) if for every non-zero $a_1, a_2 \leq a$: if $a_1 \cong a_2$ then there are non-zero $b_1 \leq a_1$, $b_2 \leq a_2$ such that $b_1 \cong b_2 (\operatorname{mod} b)$.

(h) $\langle B, G\rangle$ is *flexible* if for every non-zero $a \leq a^{\mathrm{LM}}(B, G)$ there are non-zero $a_1, a_2 \leq a$ such that $a_1 \cong a_2(\operatorname{mod}(-a))$ and $\mathrm{Fl}(a_1; a_2)$.

The following proposition summarizes some trivial or almost trivial observations.

3.3. PROPOSITION. (a) (1) *Let* $\langle B, G\rangle$ *be a BG-pair and* $f \in G$. *Then* f *induces an automorphism* $\tilde{f}$ *of* $M(B, G)$ *in the following way: for every* $a \in B$, $\tilde{f}(a) = f(a)$ *and for every* $g \in G$, $\tilde{f}(g) = g^f$. (g^f *denotes* fgf^{-1}.)

(2) *All the notions defined in* 3.1 *and* 3.2 *are definable in* $M(B, G)$, *hence they are preserved by every* $f \in G$.

For example, $\operatorname{var}(g^f) = f(\operatorname{var}(g))$; $f(a^{[n]}) = a^{[n]}$; *a is locally movable iff* $f(a)$ *is locally movable*; *and* $G(a)^f = G(f(a))$. *The same observation applies to other notions definable in* $M(B, G)$ *that will be presented later.*

(b) $\operatorname{var}(fg) \leq \operatorname{var}(f) + \operatorname{var}(g)$, *hence* $\operatorname{var}(f^n) \leq \operatorname{var}(f)$.

(c) *If* $f(a) \neq a$, *then there is a non-zero* $b \leq a$ *such that* $f(b) \cdot b = 0$. *If* $a \in B$ *and* $f_1, \ldots, f_k \in G$ *are such that for every* $i < j \leq k$ *and a nonzero* $b \leq a$ *there is a non-zero* $c \leq b$ *such that* $f_i(c) \neq f_j(c)$, *then there is a non-zero* $b \leq a$ *such that* $f_1(b), \ldots, f_k(b)$ *are pairwise disjoint.*

(d) *If* $\mathrm{Fl}(a_1; a_2)$ *holds, then* $a_1 \cdot a_2 = 0$.

(e) *If f is a transposition, then* (1) $\mathrm{var}(f) \le a^{\mathrm{R}}(B, G)$; (2) $f^2 = Id$; (3) *f has a transpose*; *and* (4) *if a is a transpose of f and $B \ni b \le a$ then b is rigid in $\langle B, G \rangle$. If f and g are transpositions and* $\mathrm{var}(f) = \mathrm{var}(g)$, *then* $f = g$.

(f) *$\langle B, G \rangle$ is movable iff $a^{\mathrm{R}}(B, G) + a^{\mathrm{LM}}(B, G) = 1$, and for every non-zero disjoint rigid $a_1, a_2 \in B$ such that $a_1 \cong a_2$ there is a transposition $g \in G$ such that* $\mathrm{var}(g) \le a_1 + a_2$.

(g) *For every $n \ge 1$ $G^{[\le n]}$, $G^{[n]}$ and $G^{[\ge n]}$ are normal subgroups of G.*

(h) (1) *Every member of $G^{[2]}$ has order* 2. (2) *$G^{[2]}$ is a subgroup of $Z(G)$.* (3) *If $\langle B, G \rangle$ is movable, then $G^{[2]} = Z(G)$.*

PROOF. The only less trivial part of 3.3 is (h)(3). We prove only (h). (1) Let $f \in G^{[2]}$ and suppose by contradiction that $f^2 \ne Id$. By part (c) there is $a \in B - \{0\}$ such that a, $f(a)$ and $f^2(a)$ are pairwise disjoint. This means that $a \le \mathrm{var}(f) \cdot a^{[\ge 3]}(B, G)$ contradicting the fact that $f \in G^{[2]}$.

(2) Let $f \in G^{[2]}$ and $g \in G$ and suppose by contradiction that $fg \ne gf$. By part (c) there is $a \in B - \{0\}$ such that $fg(a) \cdot gf(a) = 0$. Let $A = \{Id, f, g, fg, gf\}$. By a repeated application of part (c) we can find $0 < a' \le a$ such that for every h_1, $h_2 \in A$ either $h_1(a') \cdot h_2(a') = 0$ or $h_1 \restriction a' = h_2 \restriction a'$. $a' \le a^{[2]}$, for otherwise $a' - a^{[2]}$, $g(a' - a^{[2]}) \le \mathrm{fix}(f)$ which means that $fg(a') \cdot gf(a') \ne 0$. Repeating the argument of (h)(1) we conclude that $g^2 \restriction a' = Id$. Either $f(a') \cdot a' = 0$ or $g(a') \cdot a' = 0$ for otherwise $f \restriction a' = g \restriction a' = Id$ whence $fg(a') = gf(a')$. W.l.o.g. $f(a') \cdot a' = 0$. If also $g(a') \cdot a' = 0$, then since $a' \le a^{[2]}$ $f(a') = g(a')$, hence $fg(a') = ff(a') = a' = gg(a') = gf(a')$, a contradiction. If, on the other hand, $g(a') = a'$, then $g(f(a')) \cdot a' = 0$, and since $a' \le a^{[2]}$ $gf(a') = f(a')$, hence $gf(a') = f(a') = fg(a')$, a contradiction.

(3) Let $\langle B, G \rangle$ be movable and $f \not\in G^{[2]}$.
We wish to show that $f \not\in Z(G)$. For this it is convenient to use Lemma 3.5 which will be proved later. If $\mathrm{var}(f) \cdot a^{\mathrm{LM}} \ne 0$, then by 3.5(c) $f \not\in Z(G)$. Hence, $\mathrm{var}(f) \cdot (a^{\mathrm{R}} - a^{[\le 2]}) \ne 0$. By 3.5(d) there is a good sequence $\langle a_1, a_2, a_3 \rangle$ such that $f(a_1) = a_2$. Let $g \in G$ be such that $\mathrm{var}(g) = a_2 + a_3$ and $g(a_2) = a_3$. $gfg^{-1}(a_1) = a_3 \ne f(a_1)$, hence g and f do not commute. □

There are two key facts in the proof that $K^{BG}_{C^-}$ is interpretable in $K^{G}_{C^-}$. The first one is that for $f, g \in G(a^{\mathrm{LM}})$ it is expressible in group theoretic terms that $\mathrm{var}(f) \le \mathrm{var}(g)$. The second fact is that being a transposition is expressible in group theoretic terms in G.

We shall thus represent elements of $B \restriction a^{\mathrm{LM}}$ as the var of members of $G(a^{\mathrm{LM}})$, that is, the function h appearing in the definition of interpretation will send each member of $G(a^{\mathrm{LM}})$ to its var.

Elements of $B \restriction a^{\mathrm{R}}$ will be represented by certain pairs of transpositions. Let $\langle f, g \rangle$ be a pair of transpositions. $\langle f, g \rangle$ is called a *representative* if $\mathrm{var}(f) \cdot \mathrm{var}(g)$ is a transpose of both f and g. A representative $\langle f, g \rangle$ will represent $\mathrm{var}(f) \cdot \mathrm{var}(g)$.

Let $D(B, G) = \{\mathrm{var}(f) \mid f \in G(a^{\mathrm{LM}})\} \cup \{\mathrm{var}(f) \cdot \mathrm{var}(g) \mid \langle f, g \rangle$ is a representative$\}$. It so happens that $D(B, G)$ is dense in B iff $\langle B, G \rangle$ is movable and $a^{[\le 2]}(B, G) = 0$. So trying to interpret B with the help of $D(B, G)$ forces us to assume that $\langle B, G\rangle$ is movable and that $a^{[\leq 2]}(B, G) = 0$. Since a complete BA is reconstructible from any of its dense subsets, it will indeed suffice to interpret $D(B, G)$ in G.

It is unlikely that $B \restriction a^{\mathrm{LM}}$ will be interpretable in $G(a^{\mathrm{LM}})$, unless some further transitivity assumptions on the action of G on a^{LM} are made. The assumption that we make is the flexibility of $\langle B, G\rangle$ as defined in 3.2(h).

Flexibility is an important notion in this section. It seems to be a rather technical notion; hereafter we try to explain how this notion arose.

If $g \in \mathrm{Aut}(B)$, $a \in B$ and $g(a)\cdot a = 0$, then $g \restriction a \cup g^{-1} \restriction (g(a)) \cup Id \restriction (1 - a - g(a))$ can be uniquely extended to an automorphism of B. This is a strong closure property which the full automorphism group of B always possesses; it implies that for every $a \in B$, $\mathrm{Fl}(a; -a)$ holds, and thus $\langle B, \mathrm{Aut}(B)\rangle$ is always flexible. If the applications of Theorem 3.4 were confined only to the full automorphism groups of general Boolean algebras then we could have replaced the flexibility requirement by the above closure property. However, in the application of 3.4 to homeomorphism groups of topological spaces the above closure property does not hold, and one has to seek a weaker homogeneity requirement that usually holds in that context, and which is still strong enough to enable reconstruction.

Let us take $H(\mathbb{R}^n)$ as an example. $H(\mathbb{R}^n) \subseteq \mathrm{Aut}(\mathrm{Ro}(\mathbb{R}^n))$. As a subgroup of $\mathrm{Aut}(\mathrm{Ro}(\mathbb{R}^n))$, $H(\mathbb{R}^n)$ does not have the closure property stated above. However, every ball D in $\mathbb{R}^n$ is flexible with respect to $-D \stackrel{\mathrm{def}}{=} \mathrm{int}(\mathbb{R}^n - D)$. This implies that $\langle \mathrm{RO}(\mathbb{R}^n), H(\mathbb{R}^n)\rangle$ is flexible, and thus Theorem 3.4 can be used as a first step in the reconstruction of $\mathbb{R}^n$ from $H(\mathbb{R}^n)$. (See RUBIN [1988].)

There is still another complication which happens when $a^{[6]}(B, G) \neq 0$. The symmetric group S_6 has automorphisms which are not inner. This phenomenon makes it difficult to reconstruct the action of $\mathrm{Aut}(B)$ on $a^{[6]}(B)$, and makes it necessary to impose special conditions on $G \restriction a^{[6]}(B, G)$. We shall deal with the case that $a^{[6]}(B, G) \neq 0$ in Theorems 3.4, 3.23 and 4.1. We take the liberty to ignore this case in Theorem 4.5. In Sections 5 and 6 the problem of $a^{[6]}(B, G) \neq 0$ does not arise at all.

Let

$$K_C = \{\langle B, G\rangle \mid \text{(1) } B \text{ is complete, (2) } a^{[\leq 2]}(B, G) = 0\,,$$
$$\text{(3) } \langle B, G\rangle \text{ is movable, and (4) } \langle B, G\rangle \text{ is flexible}\}\,,$$

and let $K_{C^-} = \{\langle B, G\rangle \in K_C \mid a^{[6]}(B, G) = 0\}$. We have already noted that K_C^{BG} is not strongly interpretable in K_C^G. We now describe a model $M^-(B, G)$ that encompasses the information strongly capturable from G when $\langle B, G\rangle \in K_C$. Let $Op^- = \{\langle g, a, b\rangle \mid g \in B,\ a \in B \restriction (-a^{[6]}(B, G))$ and $b = g(a)\}$. Let $M^-(B, G) = \langle B, G; \leq, Op^-\rangle$, and let $K^{BG^-} = \{M^-(B, G) \mid \langle B, G\rangle \in K\}$.

Let $K_D = \{\langle B, G\rangle \mid \langle B, G\rangle$ is complete and movable and $a^{[1]}(B, G) = 0\}$.

3.4. THEOREM. (a) *$K_C^{BG^-}$ is interpretable in K_C^G.*

(b) *$K_{C^-}^{BG}$ is strongly interpretable in $K_{C^-}^G$.*

(c) *Every member of K_{C^-} is group-categorical in K_D.* (*Group categoricity was defined in* 1.2.)

Note that for every complete B, $\langle B, \mathrm{Aut}(B)\rangle$ satisfies (3) and (4) in the definition of K_C. Hence,

$$K_C \cap \{\langle B, \mathrm{Aut}(B)\rangle \mid B \text{ is complete}\} = \{\langle B, \mathrm{Aut}(B)\rangle \mid B \text{ is complete and } a^{[\leq 2]}(B) = 0\}\,.$$

A more general conclusion is obtained when we consider arbitrary Boolean algebras. If B is an arbitrary BA then every automorphism of B can be uniquely extended to an automorphism of $\bar{B}$, hence $\mathrm{Aut}(B)$ may be regarded as a subgroup of $\mathrm{Aut}(\bar{B})$. If B is an arbitrary BA, then $\langle \bar{B}, \mathrm{Aut}(B)\rangle$ automatically satisfies conditions (3) and (4) above. Hence

$$K_C \cap \{\langle \bar{B}, \mathrm{Aut}(B)\rangle \mid B \text{ is a BA}\} = \{\langle \bar{B}, \mathrm{Aut}(B)\rangle \mid a^{[\leq 2]}(B) = 0\}\,.$$

We need some group-theoretic notation. $\bar{f}, \bar{g}, \bar{h}$ denote subsets of G. $f^h \overset{\text{def}}{=} hfh^{-1}$, $[f, h] \overset{\text{def}}{=} fhf^{-1}h^{-1}$, $[f, h, g] = [[f, h], g]$. $f \cong g$ means that $\exists h(f^h = g)$, $Z(f) \overset{\text{def}}{=} \{g \in G \mid [g, f] = Id\}$, $\bar{f}^{\bar{h}} \overset{\text{def}}{=} \{f^h \mid f \in \bar{f}, h \in \bar{h}\}$. $[\bar{f}, \bar{h}]$, $\bar{f}^h$, $[\bar{f}, h]$ and $Z(\bar{f})$ are defined similarly. S_n denotes the group of all permutations of $\{1, \ldots, n\}$.

3.5. LEMMA. (a) *If* $0 < a \leq a^{\mathrm{LM}}$ *and* $n \in \omega$, *then there is* $h \in G(a)$ *such that* $h^n \neq Id$.

(b) *Let* $0 < a \leq \mathrm{var}(f) \cdot \mathrm{var}(g) \cdot a^{\mathrm{LM}}$, *then there is* $h \in G(a)$ *such that* $[f^h, g] \neq Id$.

(c) *Let* $0 < a < \mathrm{var}(f) \cdot a^{\mathrm{LM}}$, *then there is* $h \in G(a)$ *such that* $[h, f] \neq Id$.

(d) *Suppose* $\langle B, G\rangle$ *is complete and movable, and let* $a_1, \ldots, a_n$ *be pairwise disjoint non-zero rigid elements such that* $a_1 \cong a_2 \cong \cdots \cong a_n$. *Then there are* $0 < b_i \leq a_i$, $i = 1, \ldots, n$, *such that for every* $\pi \in S_n$ *there is* $g \in G(\Sigma_{i=1}^n b_i)$ *such that for every* $i \in \{1, \ldots, n\}$ $g(b_i) = b_{\pi(i)}$.

A sequence $\langle b_1, \ldots, b_n\rangle$ *with the above property is called a* ***good sequence.***

PROOF. (a) We prove by induction on $n \geq 1$ that for every $0 < b \leq a^{\mathrm{LM}}$ there is $h \in G(b)$ and $0 < a < b$ such that $a, h(a), \ldots, h^n(a)$ are pairwise disjoint. For $n = 1$ the claim follows from the fact that $b \leq a^{\mathrm{LM}}$. Suppose the claim is true for n. Let $0 < b \leq a^{\mathrm{LM}}$ and let $h \in G(b)$ and a be as assured by the induction hypothesis. If $a \cdot \mathrm{var}(h^{n+1}) \neq 0$, then there is $0 < a' \leq a$ such that $a' \cdot h^{n+1}(a') = 0$, hence a', $h(a'), \ldots, h^{n+1}(a')$ are pairwise disjoint and the claim is proved. Otherwise, $h^{n+1} \restriction a = Id$. Let $h' \in G(a)$ and $0 < a' < a$ be such that $h'(a') \cdot a' = 0$. $(hh')^{n+1}(a') = h'(a')$, so it follows that $a', hh'(a'), \ldots, (hh')^{n+1}(a')$ are pairwise disjoint, hence h', a' satisfy the induction claim for $n+1$, thus (a) is proved.

(b) If $[f, g] \neq Id$, then $h = Id$ satisfies the claim of part (b). Suppose $[f, g] = Id$. Since $a \leq \mathrm{var}(f), \mathrm{var}(g)$, there is non-zero $b_1 \leq a$ such that $b_1 \cdot (f(b_1) + g(b_1)) = 0$. If (1) $b_1 \cdot \mathrm{var}(fg) \neq 0$, let $b \leq b_1$ be a non-zero element such that $b \cdot fg(b) = 0$; (2) otherwise let $b = b_1$. By part (a) there is $h \in G(b)$ such that $h^2 \neq Id$. Let c be a non-zero element such that c, $h(c)$, $h^2(c)$ are pairwise disjoint. $gf^h(h(c)) = ghfh^{-1}h(c) = ghf(c) = gf(c) = fg(c)$. The equality $h(f(c)) = f(c)$ follows from the fact that $f(c) \leq \mathrm{fix}(h)$. $f^hg(h(c)) = hfh^{-1}gh(c) = hfgh(c)$. If (1) happens then $fg(h(c)) \leq \text{fix}(h)$, and hence $hfgh(c) = fgh(c)$. So $gf^h(h(c)) = fg(c) \neq fg(h(c)) = f^h g(h(c))$.

If (2) happens, then $fg(c) = c$ and so $gf^h(h(c)) = fg(c) = c \neq h^2(c) = hfgh(c) = f^h g(h(c))$. Hence, g and f^h do not commute, so (b) is proved.

(c) Let $g \in G(a) - \{Id\}$. By (b) there is $h \in G(a)$ such that $[g^h, f] \neq Id$. Hence g^h is as required.

(d) Let $a_1, \ldots, a_n$ be pairwise disjoint non-zero rigid elements such that $a_1 \cong a_2 \cong \cdots \cong a_n$. We prove by induction on $k \leq n$ that there are $0 < b_i \leq a_i$, $i = 1, \ldots, k$, and $g \in G(\Sigma_{i=1}^{k} b_i)$ such that for every $i < k$, $g(b_i) = b_{i+1}$, and $g(b_k) = b_1$. The claim for $k = 2$ follows from the movability of $\langle B, G \rangle$. Assume the claim is true for k, and let $b_1, \ldots, b_k$ and g be as assured. There is $b_{k+1} \leq a_{k+1}$ such that $b_{k+1} \cong b_1$. By the movability of $\langle B, G \rangle$, there is a transposition $h \in G(b_1 + b_{k+1})$. Let $c_1 = \text{var}(h) \cdot b_1$, c_i be the unique $c \leq b_i$ such that $c \cong c_1$ and $f = (hg)^{-k}$. It is easy to see that $c_1, \ldots, c_{k+1}$ and f fulfill the induction claim for $k + 1$.

Let $b_1, \ldots, b_n$ and g as assured by the claim for $k = n$. Let $f \in G(b_1 + b_2)$ be a transposition, $c_1 = \text{var}(f) \cdot b_1$ and $c_i \leq b_i$ be such that $c_i \cong c_1$. Let H be a subgroup of G generated by $\{f^{(g^i)} \mid i = 1, \ldots, n\}$. It is easy to see that $H \subseteq G(\Sigma_{i=1}^{n} c_i)$ and that for every $\pi \in S_n$ there is $h \in H$ such that for every $i \in \{1, \ldots, n\}$ $h(c_i) = c_{\pi(i)}$. So (d) is proved. □

3.6. DEFINITION. (a) Let $a, b \in \bar{B}$; a and b are *totally different* in $\langle B, G \rangle$ if for every $f \in G$ $f(a) \cdot b = 0$. We denote the above fact by $D(a, b)$.

(b) If $a \in \bar{B}$, then $\text{conv}(a) \stackrel{\text{def}}{=} \Sigma\{f(b) \mid b \leq a \text{ and } f \in G\}$.

(c) $a \preccurlyeq b$ if $\text{conv}(a) \leq \text{conv}(b)$, and $a \approx b$ if $\text{conv}(a) = \text{conv}(b)$.

Note that $a \preccurlyeq b$ iff for every non-zero $c \leq a$ there is $g \in G$ such that $c \cdot g(b) \neq 0$.

(d) Let $D(f, g)$ denote that $D(\text{var}(f), \text{var}(g))$, $\text{conv}(f) = \text{conv}(\text{var}(f))$, $f \preccurlyeq g$ denote that $\text{var}(f) \preccurlyeq \text{var}(g)$ and $f \approx g$ denote that $\text{var}(f) \approx \text{var}(g)$.

(e) h is a *component* of f if $D(h, fh^{-1})$. We denote this relation by $C(h, f)$. Note that $C(h, f)$ holds iff $f \restriction \text{conv}(h) = h \restriction \text{conv}(h)$.

(f) Recall that S_n denotes the symmetric group of $\{1, \ldots, n\}$. If $\pi \in S_n$, then $[\pi]$ denotes the conjugacy class of π in S_n, that is $[\pi] = \{\pi^\sigma \mid \sigma \in S_n\}$.

(g) Let $f \in G$ and $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$ be a sequence of non-zero pairwise disjoint rigid elements such that $f(\{a_1, \ldots, a_n\}) = \{a_1, \ldots, a_n\}$, then $\pi_{f,\boldsymbol{a}}$ denotes the permutation π of $\{1, \ldots, n\}$ such that for every $i \leq n$ $f(a_i) = a_{\pi(i)}$.

(h) f is *permutation-like* if for some $\boldsymbol{a}$ as above $\text{var}(f) \leq \Sigma_{i=1}^{n} a_i$, $\pi_{f,\boldsymbol{a}}$ is defined, $a_1 \cong \cdots \cong a_n$ and $\text{conv}(a_1) = \Sigma_{i=1}^{n} a_i$. We denote $[\pi_{f,\boldsymbol{a}}]$ by $[\pi]_f$. Note that if f is permutation-like, then $[\pi_{f,\boldsymbol{a}}]$ is uniquely determined by f although $\pi_{f,\boldsymbol{a}}$ itself, is not.

Let $K_{C'} = \{\langle B, G \rangle \mid \langle B, G \rangle$ is complete and movable and $a^{[\leq 2]}(B, G) = 0\}$.

3.7. LEMMA. *There are formulas $\phi_D(f, g)$, $\phi_{\preccurlyeq}(f, g)$, $\phi_{\approx}(f, g)$, and $\phi_C(f, g)$ such that for every $\langle B, G \rangle \in K_{C'}$ and $f, g \in G$ $G \models \phi_D[f, g]$ iff $D(f, g)$, $G \models \phi_{\preccurlyeq}[f, g]$ iff $f \preccurlyeq g$, $G \models \phi_{\approx}[f, g]$ iff $f \approx g$ and $G \models \phi_C[f, g]$ iff $C(f, g)$.*

PROOF. Once we prove the existence of $\phi_D(f, g)$, the other parts of the lemma follow easily. We may define $\phi_{\preccurlyeq}$, $\phi_{\approx}$ and ϕ_C as follows: $\phi_{\preccurlyeq}(f, g) \equiv$

$\forall h(\phi_D(g,h) \to \phi_D(f,h))$, $\phi_{\approx}(f,g) \equiv \phi_{\leqslant}(f,g) \wedge \phi_{\leqslant}(g,f)$ and $\phi_C(f,g) \equiv \phi_D(f, gf^{-1})$.

Let $\varphi'_D(f,g) \equiv (\forall g_1 \cong g)([f, g_1] = Id)$. It is trivial that if $\text{conv}(f) \cdot \text{conv}(g) = 0$, then $G \models \varphi'_D[f,g]$. The converse is almost always true.

Let $b(f,g) = \text{conv}(f) \cdot \text{conv}(g) \cdot (a^{[3]} + a^{[4]})$ and $t(f,g) = \text{conv}(f) \cdot \text{conv}(g) - (a^{[3]} + a^{[4]})$.

We shall prove the following claim.

Claim 1. If $t(f,g) \neq 0$, then $G \not\models \varphi'_D[f,g]$.

So a "correction" to φ'_D is needed for the case when $\text{var}(f) \cdot \text{var}(g) \leq a^{[3]} + a^{[4]}$.

Let $\varphi''_D(f,g) \equiv (\forall g_1 \cong g)\ \forall h([h, fg_1] = Id \to [h, f] = Id)$. It is again quite easy to see that if $\text{conv}(f) \cdot \text{conv}(g) = 0$, then $G \models \varphi''_D[f,g]$. We shall prove the following claim.

Claim 2. If $b(f,g) \neq 0$, then $G \not\models \varphi'_D[f,g]$ or $G \not\models \varphi''_D[f,g]$.

So let $\varphi_D(f,g) \equiv \varphi'_D(f,g) \wedge \varphi''(f,g)$. Hence, if $\text{conv}(f) \cdot \text{conv}(g) = 0$, then $G \models \varphi_D[f,g]$. If $\text{conv}(f) \cdot \text{conv}(g) \neq 0$, then either $t(f,g) \neq 0$, in which case by Claim 1 $G \not\models \varphi'_D[f,g]$ and hence $G \not\models \varphi_D[f,g]$, or $b(f,g) \neq 0$, in which case by Claim 2 $G \not\models \varphi_D[f,g]$.

The proofs of Claims 1 and 2 are easy but lengthy. The reader may check the proof by himself or see the detailed proof in the sequel.

So it remains to prove that Claims 1 and 2 are true.

Proof of Claim 1.

Case 1. $t(f,g) \cdot a^{\text{LM}} \neq 0$. Hence, for some $h \in G$ $\text{var}(f) \cdot \text{var}(g^h) \cdot a^{\text{LM}} \neq 0$. By 3.5(b) there is $h_1 \in G$ such that $[f, (g^h)^{h_1}] \neq Id$ and hence $G \not\models \varphi'_D[f,g]$.

Case 2. $t(f,g) \cdot a^{\text{LM}} = 0$. Recall that if $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$ is a sequence of pairwise disjoint non-zero rigid elements and $h \in G$ satisfies $h(\{a_1, \ldots, a_n\}) = \{a_1, \ldots, a_n\}$, then $\pi_{h,\boldsymbol{a}}$ denotes the permutation π of $\{1, \ldots, n\}$ such that for every i, $h(a_i) = a_{\pi(i)}$. For $h \in G$ let $NC_G(h)$ denote the normal closure of h in G, that is, $NC_G(h)$ is the subgroup of G generated by all conjugates of h. Clearly, $G \models \varphi'_D[f,g]$ iff $[NC_G(f), NC_G(g)] = \{Id\}$.

Let A_n denote the group of even permutations of $\{1, \ldots, n\}$. We shall show: $(*)$ If $f \in G$, $n \geq 5$ and $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$ is a sequence of pairwise disjoint non-zero rigid elements such that $a_1 \cong \cdots \cong a_n$ and $a_1 \leq \text{conv}(f)$, then there is $\boldsymbol{a}^1 \overset{\text{def}}{=} \langle a_1^1, \ldots, a_n^1 \rangle$ such that for every $i \leq n$ $0 \neq a_i^1 \leq a_i$ and

$$N(f, \boldsymbol{a}^1) \overset{\text{def}}{=} \{\pi_{h,\boldsymbol{a}^1} \mid h \in NC_G(f) \text{ and } h(\{a_1^1, \ldots, a_n^1\}) = \{a_1^1, \ldots, a_n^1\}\} \supseteq A_n\,.$$

Let us first see why $(*)$ implies that $G \not\models \varphi'_D[f,g]$, and then prove $(*)$. Let $0 \neq a \leq \text{conv}(f) \cdot \text{conv}(g) - b(f,g)$ be rigid. Hence, there is $\boldsymbol{a} = \langle a_1, \ldots, a_5 \rangle$ such that $0 \neq a_1 \leq a$, $a_1, \ldots, a_5$ are pairwise disjoint and $a_1 \cong \cdots \cong a_5$. By applying $(*)$ first for f and $\boldsymbol{a}$ and then for g and the sequence $\boldsymbol{a}^1$ resulting from the first application of $(*)$, we obtain a sequence $\boldsymbol{a}^2$ such that $N(f, \boldsymbol{a}^2)$, $N(g, \boldsymbol{a}^2) \supseteq A_5$. Since $[A_5, A_5] \neq \{Id\}$ also $[NC_G(f), NC_G(g)] \neq \{Id\}$, and hence $G \not\models \varphi'_D[f,g]$.

We prove $(*)$. Recall that a sequence $\boldsymbol{c} = \langle c_1, \ldots, c_n \rangle$ of pairwise disjoint non-zero rigid elements is called a good sequence if for every $\pi \in S_n$ there is $h \in G$ such that $h(\{c_1, \ldots, c_n\}) = \{c_1, \ldots, c_n\}$, $\text{var}(h) \leq \Sigma_{i=1}^n c_i$ and $\pi = \pi_{h,\boldsymbol{c}}$. (See 3.5(d).)

Let $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$ and f be as in $(*)$. Let f_1 be a conjugate of f such that $a_1 \cdot \mathrm{var}(f_1) \neq 0$. Let $a_i^1 \leq a_i$ be such that $a_1^1 \leq \mathrm{var}(f_1)$ and $\langle a_1^1, \ldots, a_n^1 \rangle$ is good. We can construct inductively $0 \neq a_i^2 \leq a_i^1$, $i = 1, \ldots, n$, such that $a_1^2 \cong \cdots \cong a_n^2$ and for every $i \in \{1, \ldots, n\}$ and $l \in \{1, 2\}$ $(f_1)^l(a_i^2) \in \{a_1^2, \ldots, a_n^2\}$ or $(f_1)^l(a_i^2) \cdot \Sigma_{j=1}^n a_j^2 = 0$. Let $\{a_{n+1}^2, \ldots, a_m^2\} = \{(f_1)^l(a_i^2) \mid i = 1, \ldots, n,\ l = 1, 2\} - \{a_1^2, \ldots, a_n^2\}$. It is easy to see that $a_1^2, \ldots, a_m^2$ are pairwise disjoint. Let $a_i^3 \leq a_i^2$ be such that $\boldsymbol{a}^3 \stackrel{\mathrm{def}}{=} \langle a_1^3, \ldots, a_m^3 \rangle$ is good.

We again distinguish between cases.

Case 2.1. For every $i = 1, \ldots, m$, $f_1(a_i^3) \in \{a_1^3, \ldots, a_m^3\}$. Hence, $\pi_1 \stackrel{\mathrm{def}}{=} \pi_{f_1, \boldsymbol{a}^3}$ is defined and $\pi_1 \neq Id$. Since $m \geq n \geq 5$ $NC_{S_m}(\pi_1) \supseteq A_m$. Hence

$$N(f, \boldsymbol{a}^3) = N(f_1, \boldsymbol{a}^3) \supseteq NC_{S_m}(\pi_1) \supseteq A_m .$$

Case 2.2. Case 2.1 does not happen. So there is $i \leq n$ such that a_i^3, $f_1(a_i^3)$ and $f_1^2(a_i^3)$ are pairwise disjoint, for otherwise we are back in Case 2.1. Hence, there are distinct $i, j, k \leq m$ such that $f_1(a_i^3) = a_j^3$ and $f_1(a_j^3) = a_k^3$. Let $g \in G$ be a transposition with transposes a_i^3 and a_j^3. Hence, g^{f_1} is a transposition with transposes a_j^3 and a_k^3. Hence $f_2 \stackrel{\mathrm{def}}{=} g^{f_1} g^{-1}$ has the following properties: (1) $\mathrm{var}(f_2) = a_i^3 + a_j^3 + a_k^3$; (2) $f_2(a_i^3) = a_k^3$, $f_2(a_k^3) = a_j^3$ and $f_2(a_j^3) = a_i^3$, and (3) $f_2 = f_1(f_1^{-1})^g \in NC_G(f_1) = NC_G(f)$. By Case 2.1, $A_m \subseteq N(f_2, \boldsymbol{a}^3) \subseteq N(f, \boldsymbol{a}^3)$.

This concludes the proof of $(*)$ and thus Claim 1 is proved.

Proof of Claim 2. Let f and g be as in Claim 2 and suppose that $G \models \varphi'_D[f, g]$. By Claim 1, $t(f, g) = 0$, and hence $\mathrm{conv}(f) \cdot \mathrm{conv}(g) \leq a^{[3]} + a^{[4]}$. Let $0 \neq a \leq \mathrm{conv}(f) \cdot \mathrm{conv}(g)$ be rigid. It is easy to see that there is $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$ such that $\boldsymbol{a}$ is good, $\mathrm{conv}(a_1) = \Sigma_{i=1}^n a_i$, $a_1 \leq a$ and both $\pi_{f,\boldsymbol{a}}$ and $\pi_{g,\boldsymbol{a}}$ are defined. Clearly, $n \in \{3, 4\}$ and $\pi_{f,\boldsymbol{a}}, \pi_{g,\boldsymbol{a}} \neq Id$. Since $G \models \varphi'_D[f, g]$ $[NC_G(f), NC_G(g)] = \{Id\}$ and hence also $[N(f, \boldsymbol{a}), N(g, \boldsymbol{a})] = \{Id\}$. Since $\boldsymbol{a}$ is good $N(f, \boldsymbol{a})$ and $N(g, \boldsymbol{a})$ are normal subgroups of S_n. The only pair of non-trivial normal commuting subgroups of S_3 is A_3 and A_3, and for S_4 it is K and K, where K is the group of all permutations conjugate to the permutation $(1,2)(3,4)$. In either case $\pi_{f,\boldsymbol{a}}$ and $\pi_{g,\boldsymbol{a}}$ are conjugate, and so also $(\pi_{f,\boldsymbol{a}})^{-1}$ and $\pi_{g,\boldsymbol{a}}$ are conjugate. Let $\sigma \in S_n$ be such that $(\pi_{g,\boldsymbol{a}})^\sigma = (\pi_{f,\boldsymbol{a}})^{-1}$. Since $\boldsymbol{a}$ is good, there is $k \in G$ such that $\pi_{k,\boldsymbol{a}}$ is defined and is equal to σ, and so $\pi_{fg^k,\boldsymbol{a}} = Id$. This implies that $fg^k \restriction \Sigma_{i=1}^n a_i = Id$. Let $\tau \in S_n$ be a permutation not commuting with $\pi_{f,\boldsymbol{a}}$, and $h \in G$ be such that $\mathrm{var}(h) \leq \Sigma_{i=1}^n a_i$ and $\pi_{h,\boldsymbol{a}} = \tau$. Then $[h, fg^k] = Id$ but $[h, f] \neq Id$. Hence, $G \models \varphi''_D[f, g]$.

We have proved Claim 2, hence the lemma is proved. □

Recall that we intend to represent rigid elements by pairs of transpositions. The set of transposition is, however, not definable in G unless $a^{[6]} = 0$. The reason is that the symmetric group on $\{1, \ldots, 6\}$ has an automorphism that takes the transposition $(1,2)$ to the permutation $(1,2)(3,4)(5,6)$, which is not a transposition. See SCOTT [1964, p. 311].

3.8. DEFINITION. Let $g \in G$. g is called a *pseudo-transposition* if g is permutation-like and $[\pi]_g = [(1,2)(3,4)(5,6)]$. Hence, if g is permutation-like then $\mathrm{var}(g) \leq a^{[6]}(B)$.

If $G = \mathrm{Aut}(B)$ and $a^{[6]}(B) \neq 0$, then there are automorphisms of G which take transpositions to pseudo-transpositions.

3.9. Lemma. *There is a formula $\varphi_{\mathrm{Tr}}(f)$ such that for every $\langle B, G\rangle \in K_{C'}$ and $f \in G$, $G \models \varphi_{\mathrm{Tr}}[f]$ iff $(*)$ $f \neq Id$ and for every $a \preccurlyeq \mathrm{var}(f)$ there is a component h of f such that $\mathrm{var}(h) \preccurlyeq a$ and h is a transposition or a pseudo-transposition.*

Proof. Let $\varphi'_{\mathrm{Tr}}(f) \equiv (f \neq Id) \wedge (f^2 = Id) \wedge (\forall f_1 \approx f)((f_1 f)^6 = Id))$. We shall prove the following claim.

Claim 1. If f satisfies $(*)$ then $G \models \varphi'_{\mathrm{Tr}}[f]$.

The converse of Claim 1 is almost always true. We shall prove the following restricted converse of Claim 1.

Claim 2. If $G \models \varphi'_{\mathrm{Tr}}[f]$ and f does not have a permutation-like component h such that $[\pi]_h = [(1,2)(3,4)]$, then f satisfies $(*)$.

Hence, we need a "correction" to φ'_{Tr} to exclude f's with a permutation-like component as above. Let $\varphi''_{\mathrm{Tr}}(f) \equiv \forall h((C(h, f) \wedge h \neq Id) \to (\forall h_1 \cong h)(h_1 h \ncong h))$. We shall prove the following claims.

Claim 3. If f satisfies $(*)$, then f satisfies φ''_{Tr}.

Claim 4. If $G \models \varphi''_{\mathrm{Tr}}[f]$, then f does not have a permutation-like component h such that $[\pi]_h = [(1,2)(3,4)]$.

Now we define $\varphi_{\mathrm{Tr}}(f) \equiv \varphi'_{\mathrm{Tr}}(f) \wedge \varphi''_{\mathrm{Tr}}(f)$, and it follows from Claims 1–4 that $G \models \varphi_{\mathrm{Tr}}[f]$ iff f satisfies $(*)$.

The proofs of the above claims are easy, though long. For the sake of completeness we bring here the proofs of Claims 1 and 2. The proofs of the other claims are left to the reader.

We first call the attention of the refer to the following observation.

Observation 5. Let $a_1, \ldots, a_n$ be pairwise disjoint non-zero rigid elements such that $a_1 \cong \cdots \cong a_n$. Then

(a) If $f \in G$, $f(\{a_1, \ldots, a_n\}) = \{a_1, \ldots, a_n\}$, and for $i = 1, \ldots, n$, $a_i^0 \leq a_i$ are such that $a_1^0 \cong \cdots \cong a_n^0$, then $f(\{a_1^0, \ldots, a_n^0\}) = \{a_1^0, \ldots, a_n^0\}$.

(b) If $f_1, \ldots, f_k \in G$, then there are non-zero $a_i^0 \leq a_i$, $i = 1, \ldots, n$, such that $a_1^0 \cong \cdots \cong a_n^0$, and for every $i \leq n$ and $j \leq k$ either $f_j(a_i^0) \in \{a_1^0, \ldots, a_n^0\}$ or $f_j(a_i^0) \cdot \Sigma_{l=1}^n a_l^0 = 0$. If, in addition, $\mathrm{conv}(a_1) = \Sigma_{l=1}^n a_l$, then the a_i^0's chosen above also satisfy that for every $i \leq n$ and $j \leq k$, $f_j(a_i^0) \in \{a_1^0, \ldots, a_n^0\}$.

The proof of part (b) of the above observation is by a simple inductive process.

Proof of Claim 1. Let f satisfy $(*)$. Clearly, $f \neq Id$ and $f^2 = Id$. Let $f_1 \cong f$. In order to show that $(f_1 f)^6 = Id$ it suffices to prove that $\{a \mid (f_1 f)^6(a) = a\}$ is dense in B. Let $a \in B - \{0\}$. If $a - (\mathrm{var}(f) + \mathrm{var}(f_1)) \neq 0$, then clearly there is a non-zero $b \leq a$ such that $(f_1 f)^6(b) = b$. So suppose $a \leq \mathrm{var}(f) + \mathrm{var}(f_1)$. W.l.o.g. $a \cdot \mathrm{var}(f) \neq 0$, and let $a_1 \leq a \cdot \mathrm{var}(f)$ be a non-zero rigid element. Let h be a component of f such that $\mathrm{conv}(h) \preccurlyeq a_1$ and h is either a transposition or a pseudo-transposition. By the existence of the above h there are $n \in \{2, 6\}$ and non-zero rigid elements $a_1^0, \ldots, a_n^0$ such that $a_1^0 \leq a_1$, $a_1^0 \cong \cdots \cong a_n^0$, for every $i = 1, \ldots, n/2$, $f(a_{2i-1}^0) = a_{2i}^0$ and $f(a_{2i}^0) = a_{2i-1}^0$ and $\mathrm{var}(f) \cdot \mathrm{conv}(a_1^0) = \Sigma_{i=1}^n a_i^0$. Let $g \in G$ be such that $f_1 = f^g$.

Case 1. $n = 6$. In this case $\mathrm{conv}(a_1^0) = \Sigma_{i=1}^6 a_i^0$. Hence, by Observation 5(b) there are non-zero $a_i^1 \leq a_i^0$, $i = 1, \ldots, 6$, such that $a_1^1 \cong \cdots \cong a_6^1$ and $f(\{a_1^1, \ldots, a_6^1\}) = g(\{a_1^1, \ldots, a_6^1\}) = \{a_1^1, \ldots, a_6^1\}$. Let $\boldsymbol{a}^1 = \langle a_1^1, \ldots, a_6^1\rangle$, then clearly $\pi \stackrel{\mathrm{def}}{=} \pi_{f,\boldsymbol{a}^1} = (1,2)(3,4)(5,6)$. $\pi_1 \stackrel{\mathrm{def}}{=} \pi_{f_1}$ is clearly defined, and is equal to

$\pi^{\pi_{g,a^1}}$, and so π_1 is conjugate to $(1,2)(3,4)(5,6)$. By direct computation $(\pi_1\pi)^6 = Id$, so $\pi_{(f_1 f)^6, a^1} = Id$, and hence $(f_1 f)^6(a_1^1) = a_1^1$. So in the case that $n=6$ we found a non-zero $a_1^1 \leq a$ such that $(f_1 f)^6(a) = a$.

Case 2. $n=2$. By Observation 5 there are non-zero $a_i^1 \leq a_i^0$, $i=1, 2$, such that $a_1^1 \cong a_2^1$, $f(a_1^1) = a_2^1$, $f(a_2^1) = a_1^1$ and for every $i \leq 2$ either $g(a_i^1) \in \{a_1^1, a_2^1\}$ or $g(a_i^1)\cdot(a_1^1 + a_2^1) = 0$. Let us denote the set $\{g(a_i^1) \mid i=1,2\} - \{a_1^1, a_2^1\}$ by $\{a_3^1, \ldots, a_m^1\}$, (clearly, $2 \leq m \leq 4$). Since f is a transposition, for every $i>2$ $f(a_i^1) = a_i^1$. So if we denote $\boldsymbol{a}^1 = \langle a_1^1, \ldots, a_m^1\rangle$, then $\pi \stackrel{\text{def}}{=} \pi_{f,\boldsymbol{a}^1} = (1,2)$ and it is easy to see that $\pi_1 \stackrel{\text{def}}{=} \pi_{f_1,\boldsymbol{a}}$ is also defined and it is a transposition. By direct computation we conclude again that $(\pi_1\pi)^6 = Id$, and hence $(f_1 f)^6(a_1^1) = a_1^1$. This concludes the proof of Claim 1.

Proof of Claim 2. Let $f \in G$ satisfy φ'_{Tr}, and f does not have a permutation-like component h with $[\pi]_h = [(1,2)(3,4)]$. We show that f satisfies $(*)$.

We first prove that $\mathrm{var}(f)\cdot a^{\mathrm{LM}} = 0$. If not, let $0 \neq b < a^{\mathrm{LM}}$ be such that $f(b)\cdot b = 0$. We shall see that there is $h \in G$ and pairwise disjoint non-zero $b_1, \ldots, b_7 < b$ such that $\mathrm{var}(h) \leq b$ and for every $i<7$, $h(b_i) = b_{i+1}$. Assuming the existence of such h, let $f_1 = f^h$. A direct computation shows that for every $i<7$, $f_1(b_{i+1}) = f(b_i)$. So $f_1^{-1} f(b_i) = b_{i+1}$, and since $f_1 = f_1^{-1}$, $f_1 f(b_i) = b_{i+1}$. It thus follows that $(f_1 f)^6(b_1) = b_7$ and hence $(f_1 f)^6 \neq Id$.

Now we prove the existence of the above h. By 3.5(a) there is $h \in G(b)$ such that $h^{6!} \neq Id$. There is a non-zero $c \leq b$ such that for every non-zero $d \leq c$ and $0 \leq i < j \leq 6$ $h^i(d) \neq h^j(d)$, for otherwise $\{d \mid h^{6!}(d) = d\}$ is dense in B which implies that $h^{6!} = Id$. By a repeated application of Proposition 3.3 we find $0 \neq b_1 \leq c$ such that $b_1, h(b_1), \ldots, h^6(b_1)$ are pairwise disjoint.

We have proved that $\mathrm{var}(f) \leq a^{\mathrm{R}}$. Now assume by contradiction that f does not satisfy $(*)$. Let $0 \neq \mathrm{a} \preccurlyeq \mathrm{var}(f)$ be such that if h is a component of f and $\mathrm{conv}(h) \preccurlyeq a$, then h is not a transposition and not a pseudo-transposition. Also by our assumption on f, f does not have a permutation-like component h such that $[\pi]_h = (1,2)(3,4)$. It follows that there is a good sequence $\boldsymbol{a} = \langle a_1, \ldots, a_n\rangle$ (see 3.5(d)), such that $a_1 \preccurlyeq a$, $\pi \stackrel{\text{def}}{=} \pi_{f,\boldsymbol{a}}$ is defined and (1) $\pi^2 = Id \neq \pi$; (2) π is not a transposition; (3) either $\pi \not\in S_6$, or $\pi \in S_6$ but it is not conjugate to $(1,2)(3,4)(5,6)$; and (4) either $\pi \not\in S_4$, or $\pi \in S_4$ but it is not conjugate to $(1,2)(3,4)$.

We have reduced the proof of Claim 2 to a problem in finite symmetric groups. A direct computation shows that if π satisfies (1)–(4), then there is $\sigma \in S_n$ such that $(\pi^\sigma \pi)^6 \neq Id$. If $h \in G$ is such that $\pi_{h,\boldsymbol{a}} = \sigma$, then $(f^h f)^6 \neq Id$. However, this contradicts the fact that f satisfies φ'_{Tr}. Since the existence of h is assured by the goodness of $\boldsymbol{a}$ we obtain a contradiction. We have thus proved Claim 2.

Claims 3 and 4 are proved by similar methods, so their proofs are left to the reader.

There is no strong interpretation of $B \restriction a^{[6]}$ in G. We thus have to treat $B \restriction a^{[6]}$ separately. Let us explain how. We first find a formula $\varphi_{[6]}(f)$ which says that $\mathrm{var}(f) \leq a^{[6]}$. Let $a_1, \ldots, a_6$ be a partition of $a^{[6]}$ such that $a_1 \cong a_2 \cong \cdots \cong a_6$. Then $B \restriction a_1$ is isomorphic to the subalgebra of $B \restriction a^{[6]}$ whose universe is $\{\mathrm{conv}(a) \mid a \leq a^{[6]}\}$, and thus $B \restriction a^{[6]} = \{\mathrm{conv}(a) \mid a \leq a^{[6]}\}^6$. So in order to interpret $B \restriction a^{[6]}$ in G it suffices that we interpret $\{\mathrm{conv}(a) \mid a \leq a^{[6]}\}$ in G. $\square$

3.10. LEMMA. *There is a formula* $\varphi_{[6]}(f)$ *such that for every* $\langle B, G\rangle \in K_{C'}$ *and* $f \in G$, $G \models \varphi_{[6]}[f]$ *iff* $\mathrm{var}(f) \le a^{[6]}$.

PROOF. Let

$$E(f_1, f_2) \equiv (f_1 \approx f_2) \wedge (\forall h \preccurlyeq f_1)\Big(h \neq Id \to \exists g_1 g_2$$
$$\Big(\bigwedge_{i=1}^{2} (C(g_i, f_i) \wedge g_i \preccurlyeq h \wedge g_i \neq Id) \wedge g_1 \cong g_2\Big)\Big).$$

To understand the meaning of $E(f_1, f_2)$ note that if $\langle B, G\rangle \in K_{C'}$ and f_1 and f_2 are transpositions, then $G \models E[f_1, f_2]$ iff $\mathrm{conv}(f_1) = \mathrm{conv}(f_2)$.

Let

$$\varphi_{[\neq 6]}(f) \equiv \forall g_1 g_2\Big(\Big(\bigwedge_{i=1}^{2} ((g_i \preccurlyeq f) \wedge \varphi_{\mathrm{Tr}}(g_i)) \wedge g_1 \approx g_2\Big) \to E(g_1, g_2)\Big).$$

It is easy to see that if $\langle B, G\rangle \in K_{C'}$ and $f \in G$, then $G \models \varphi_{[\neq 6]}[f]$ iff $\mathrm{var}(f) \cdot a^{[6]} = 0$.

Clearly, $\varphi_{[6]}(f) \overset{\text{def}}{=} \forall g(\varphi_{[\neq 6]}(g) \to \varphi_D(f, g))$ is as required. □

3.11. COROLLARY. $\{B \restriction a^{[6]}(B, G) \mid \langle B, G\rangle \in K_{C'}\}$ *is interpretable in* $K^G_{C'}$.

PROOF. We represent elements of $B \restriction a^{[6]}$ be sixtuples $\langle f_1, \dots, f_6\rangle$ such that for every $i = 1, \dots, 6$, $\mathrm{var}(f_i) \le a^{[6]}$.

$\langle f_1, \dots, f_6\rangle$ and $\langle g_1, \dots, g_6\rangle$ represent the same element of $B \restriction a^{[6]}$ if for $i = 1, \dots, 6$, $f_i \approx g_i$.

$\langle f_1, \dots, f_6\rangle \le \langle g_1, \dots, g_6\rangle$ if for $i = 1, \dots, 6$, $f_i \preccurlyeq g_i$. Note that we have already proved the existence of φ_{U}, φ_{Eq} and $\varphi_{\le}$.

Our next goal is to interpret a dense set of rigid elements. Recall that a pair of transpositions $\langle f, g\rangle$ is called a representative if $\mathrm{var}(f) \cdot \mathrm{var}(g)$ is a transpose of both f and g. Let

$$\varphi_{\mathrm{Rp}}(f_1, f_2) \equiv \varphi_{\mathrm{Tr}}(f_1) \wedge \varphi_{\mathrm{Tr}}(f_2) \wedge f_1 \approx f_2 \approx f_1 f_2 \wedge (f_1 f_2)^3 = Id.$$

For two representatives $\langle f_1, f_2\rangle$ and $\langle f_3, f_4\rangle$ we wish to express the fact that $\mathrm{var}(f_1) \cdot \mathrm{var}(f_2) = \mathrm{var}(f_3) \cdot \mathrm{var}(f_4)$. Let $\mathrm{var}(f_1) \cdot \mathrm{var}(f_2) = a = \mathrm{var}(f_3) \cdot \mathrm{var}(f_4)$ and let $a_1, \dots, a_4$ be transposes of $f_1, \dots, f_4$, respectively, such that $a_i \cdot a = 0$, $i = 1, \dots, 4$. The relationships between $a_1, \dots, a_4$ is a mixture of the following basic behaviors: (1) $a_1 = a_3$ and $a_2 = a_4$; (2) $a_1 = a_4$ and $a_2 = a_3$; (3) $a_1 = a_3$ and $a_2 \cdot a_4 = 0$; (4) $a_2 = a_4$ and $a_1 \cdot a_3 = 0$; (5) $a_1 = a_4$ and $a_2 \cdot a_3 = 0$; (6) $a_2 = a_3$ and $a_1 \cdot a_4 = 0$; and lastly (7) $(a_1 + a_2) \cdot (a_3 + a_4) = 0$.

The above basic behaviors can be expressed in G by formulas $\chi_1, \dots, \chi_7$ as follows. Let

$$\chi_1(f_1, f_2, f_3, f_4) \equiv f_1 = f_3 \wedge f_2 = f_4,$$

$$\chi_2(f_1, f_2, f_3, f_4) \equiv \chi_1(f_2, f_1, f_3, f_4),$$

$$\chi_3(f_1, f_2, f_3, f_4) \equiv f_1 = f_3 \wedge \varphi_{\mathrm{Rp}}(f_2, f_4) \wedge f_2 f_3 f_4 \neq Id$$

χ_4, χ_5, χ_6 are the appropriate permutations of χ_3,

and

$$\chi_7 \equiv \bigwedge_{i=1}^{2} \bigwedge_{j=3}^{4} \varphi_{\mathrm{Rp}}(f_i, f_j).$$

We are ready to define the formula $\varphi^{\mathrm{Rp}}_{\mathrm{Eq}}(f_1, f_2, f_3, f_4)$ which says that $\langle f_1, f_2 \rangle$ and $\langle f_3, f_4 \rangle$ represent the same element. Let

$$\varphi^{\mathrm{Rp}}_{\mathrm{Eq}}(f_1, f_2, f_3, f_4) \equiv f_1 \approx f_3 \wedge (\forall g \leqslant f_1)\Big(g \neq Id \rightarrow \exists h_1 h_2 h_3 h_4 \Big(Id \neq h_1 \approx h_2 \approx h_3 \approx h_4 \leqslant g \wedge \bigwedge_{i=1}^{4} C(h_i, f_i) \wedge \bigvee_{j=1}^{7} \chi_j(h_1, h_2, h_3, h_4)\Big)\Big).$$

$\varphi^{\mathrm{Rp}}_{\mathrm{Eq}}(f_1, f_2, f_3, f_4)$ says that if $0 < a \leqslant \mathrm{var}(f_1)$, then there are components $h_1, \ldots, h_4$ of $f_1, \ldots, f_4$, respectively, such that $0 < \mathrm{var}(h_i) \leqslant a$ and $h_1, \ldots, h_4$ behave according to one of the basic behaviors (1)–(7). □

3.12. LEMMA. *Let* $\langle B, G \rangle \in K_{C'}$.

(a) *Let* $f_1, f_2 \in G(-a^{[6]})$, *then* $\langle f_1, f_2 \rangle$ *is a representative iff* $G \models \varphi_{\mathrm{Rp}}[f_1, f_2]$.

(b) *Let* $\langle f_1, f_2 \rangle$, $\langle f_3, f_4 \rangle$ *be representatives and* $f_i \in G(-a^{[6]})$, $i = 1, \ldots, 4$. *Then* $\mathrm{var}(f_1) \cdot \mathrm{var}(f_2) = \mathrm{var}(f_3) \cdot \mathrm{var}(f_4)$ *iff* $G \models \varphi^{\mathrm{Rp}}_{\mathrm{Eq}}[f_1, \ldots, f_4]$.

The proof of the lemma is left to the reader.

We wish to express the facts that $f \in G(a^{\mathrm{LM}})$ and that $f \in G(a^{\mathrm{R}})$. Let $\varphi_{\mathrm{LM}}(f) \equiv \forall g(\varphi_{\mathrm{Tr}}(g) \rightarrow \varphi_{\mathrm{D}}(f, g))$, and $\varphi_{\mathrm{R}}(f) \equiv \forall g(\varphi_{\mathrm{LM}}(g) \rightarrow \varphi_{D}(f, g))$. Clearly, if $\langle B, G \rangle \in K_{C'}$ and $f \in G$, then $f \in G(a^{\mathrm{LM}})$ iff $G \models \varphi_{\mathrm{LM}}[f]$ and $f \in G(a^{\mathrm{R}})$ iff $G \models \varphi_{\mathrm{R}}[f]$.

We shall next represent all elements of $B \restriction (a^{\mathrm{R}} - a^{[6]})$ of the form $\mathrm{var}(f) \cdot \mathrm{var}(g)$, where $f, g \in G(a^{\mathrm{R}} - a^{[6]})$. We need formulas $\varphi^{\mathrm{R}}_{\mathrm{Eq}}(f_1, f_2, g_1, g_2)$ and $\varphi^{\mathrm{R}}_{\leq}(f_1, f_2, g_1, g_2)$ to express the facts that $\mathrm{var}(f_1) \cdot \mathrm{var}(f_2) = \mathrm{var}(g_1) \cdot \mathrm{var}(g_2)$ and that $\mathrm{var}(f_1) \cdot \mathrm{var}(f_2) \leq \mathrm{var}(g_1) \cdot \mathrm{var}(g_2)$. Let

$$\psi^{\mathrm{R}}_{\leq}(f, g_1, g_2) \equiv \forall h_1 h_2 \Big(\Big(\bigwedge_{i=1}^{2} C(h_i, g_i) \wedge \varphi_{\mathrm{Rp}}(h_1, h_2)\Big) \rightarrow \neg \varphi^{\mathrm{Rp}}_{\mathrm{Eq}}(h_1, h_2, h_1^f, h_2^f)\Big).$$

It is easy to see that if $f, g_1, g_2 \in G(a^{\mathrm{R}} - a^{[6]})$ and $\langle g_1, g_2 \rangle$ is a representative, then $G \models \psi^{\mathrm{R}}_{\leq}[f, g_1, g_2]$ iff $\mathrm{var}(g_1) \cdot \mathrm{var}(g_2) \leq \mathrm{var}(f)$. Now let

$$\varphi^{\mathrm{R}}_{\leq}(f_1, f_2, g_1, g_2) \equiv \forall h_1 h_2\Big(\Big(\varphi_{\mathrm{Rp}}(h_1, h_2) \wedge \bigwedge_{i=1}^{2} \psi^{\mathrm{R}}_{\leq}(f_i, h_1, h_2)\Big) \to \bigwedge_{i=1}^{2} \psi^{\mathrm{R}}_{\leq}(g_i, h_1, h_2)\Big)$$

and

$$\varphi^{\mathrm{R}}_{\mathrm{Eq}}(f_1, f_2, g_1, g_2) \equiv \varphi^{\mathrm{R}}_{\leq}(f_1, f_2, g_1, g_2) \wedge \varphi^{\mathrm{R}}_{\leq}(g_1, g_2, f_1, f_2)\,.$$

The easy proof of the next lemma is left to the reader.

3.13. LEMMA. *Let* $\langle B, G\rangle \in K_{C'}$ *and* $f_1, f_2 \in G(a^{\mathrm{R}} - a^{[6]})$, *then* $G \models \varphi^{\mathrm{R}}_{\leq}[f_1, f_2, g_1, g_2]$ *iff* $\mathrm{var}(f_1)\cdot\mathrm{var}(f_2) \leq \mathrm{var}(g_1)\cdot\mathrm{var}(g_2)$, *and* $G \models \varphi^{\mathrm{R}}_{\mathrm{Eq}}[f_1, f_2, g_1, g_2]$ *iff* $\mathrm{var}(f_1)\cdot\mathrm{var}(f_2) = \mathrm{var}(g_1)\cdot\mathrm{var}(g_2)$.

What we have done so far implies that $B \restriction a^{\mathrm{R}}$ can be reconstructed from G. We turn now to the more difficult task of reconstructing $B \restriction a^{\mathrm{LM}}$ from G.

Recall that $M(B, G) = \langle B, G; \mathrm{Op}\rangle$. Let $h \in G$, it is a trivial but important observation that h induces an automorphism $\tilde{h}$ of $M(B, G)$: for $b \in B$ $\tilde{h}(b) = h(b)$ and for $g \in G$ $\tilde{h}(g) = g^h$. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in M(B, G)$ and $h \in G$, we denote $\langle x_1, \ldots, x_n\rangle \overset{h}{\cong} \langle y_1, \ldots, y_n\rangle$ if for $i = 1, \ldots, n$, $\tilde{h}(x_i) = y_i$. $\langle x_1, \ldots, x_n\rangle \cong \langle y_1, \ldots, y_n\rangle$ means that there is $h \in G$ such that $\langle x_1, \ldots, x_n\rangle \overset{h}{\cong} \langle y_1, \ldots, y_n\rangle$.

For $\bar{f} \subseteq G$ let $\mathrm{var}(\bar{f}) = \Sigma\{\mathrm{var}(f) \mid f \in \bar{f}\}$ and $\mathrm{fix}(\bar{f}) = -\mathrm{var}(\bar{f})$. Let $\mathrm{Cm}(\bar{f}, \bar{g})$ mean that $[\bar{f}, \bar{g}] = \{Id\}$ and $\bar{f} \preccurlyeq \bar{g}$ mean that $\mathrm{var}(\bar{f}) \preccurlyeq \mathrm{var}(\bar{g})$.

3.14. DEFINITION. (a) Let $\bar{f} \subseteq G$ and $b \in B$. We say that $\bar{f}$ is *mixed on* b if for every $0 < c \leq b$ there are $f \in \bar{f}$ and $0 < d \leq c\cdot\mathrm{var}(f)$ such that $f^{G(d)} \subseteq \bar{f}$.

(b) f is *mixed* if it is mixed on $\mathrm{var}(\bar{f})$.

3.15. LEMMA. *Let* $\langle B, G\rangle \in K_{C'}$ *and* $\bar{f} \subseteq G(a^{\mathrm{LM}})$.

(a) If $\bar{f}$ *is mixed on* b *and* $g \in Z(\bar{f})$, *then* $\mathrm{var}(g)\cdot b = 0$.

(b) *If* $\bar{f}$ *is mixed, then* $Z(\bar{f}) = G(\mathrm{fix}(\bar{f}))$.

PROOF. (a) follows from 3.5(b), and (b) follows easily from (a). □

Let us now explain the general method for reconstructing $B \restriction a^{\mathrm{LM}}$ from G. We would like to represent in G a dense subset D of B together with its partial ordering. Every $d \in D$ will be represented by $G(d)$. For $f, g \in G(a^{\mathrm{LM}})$ let $V(f, g) = Z(g^{Z(f)}) \cap G(a^{\mathrm{LM}})$. It follows from 3.5(b), or more explicitly from 3.15(b), that if $\mathrm{var}(f)\cdot\mathrm{var}(g) = 0$, then $V(f, g)$ is of the form $G(b)$, namely $V(f, g) = G(a^{\mathrm{LM}}) - \mathrm{var}(g^{Z(f)})$. It is thus desirable to find a formula ψ in the language of groups such that $G \models \psi[f, g]$ iff $\mathrm{var}(f)\cdot\mathrm{var}(g) = 0$. At this stage of the proof we cannot find such a ψ, instead we find a formula $\psi_3(f, g)$ which never holds for f and g satisfying $\mathrm{var}(f)\cdot\mathrm{var}(g) \neq 0$, but holds in some cases in which $\mathrm{var}(f)\cdot\mathrm{var}(g) = 0$ (Lemma 3.20).

We define D' to be $\{\mathrm{var}(V(f, g)) \mid G \models \psi_3[f, g]\}$, and D as $\{d_1 - d_2 \mid d_1, d_2 \in$

$D'\}$. The assumption that $\langle B, G\rangle$ is flexible implies that D is dense in B. We have just to notice that the set $\{G(d) \mid d \in D\}$ is group-theoretically definable in G. It follows from the definition of $G(b)$ that $G(b \cdot c) = G(b) \cap G(c)$, and it follows from 3.15(b) that $G(-b) = Z(G(b))$. Hence, $G(b - c) = G(b) \cap Z(G(c))$. The definability of $\{G(d) \mid d \in D'\}$ in G thus implies that $\{G(d) \mid d \in D\}$ is also definable in G.

The requirement that $\langle B, G\rangle$ be flexible appearing in the definition of K_C is needed only to assure that D be dense. For the main lemmas, 3.18, 3.19 and 3.20, we have only to assume that $\langle B, G\rangle \in K_{C'}$, namely that B is complete, movable and $a^{[\leq 2]}(B, G) = 0$.

3.16. DEFINITION. Let $a_1, a_2, b \in B$ and $\bar{f}_1, \bar{f}_2, \bar{g} \subseteq G$.

(a) $D(a_1, a_2; b) \stackrel{\text{def}}{\equiv} (\forall f \in G(-b))(f(a_1) \cdot a_2 = 0)$ and $D(a_1, a_2; \bar{g}) \stackrel{\text{def}}{\equiv} (\forall f \in Z(\bar{g}))(f(a_1) \cdot a_2 = 0)$.

(b) $\operatorname{conv}(a; b) \stackrel{\text{def}}{=} \Sigma\,\{f(a) \mid f \in G(-b)\}$ and $\operatorname{conv}(a; \bar{g}) \stackrel{\text{def}}{=} \Sigma\,\{f(a) \mid f \in Z(\bar{g})\}$.

(c) $\mathrm{Fl}(a; b) \stackrel{\text{def}}{=} (\forall a_1 \leq a)(\forall a_2 \leq a)(D(a_1, a_2; b) \rightarrow D(a_1, a_2))$ and $\mathrm{Fl}(a; \bar{g}) \equiv (\forall a_1 \leq a)(\forall a_2 \leq a)(D(a_1, a_2; \bar{g}) \rightarrow D(a_1, a_2))$.

(d) $D(\bar{f}_1, \bar{f}_2; b)$, $D(\bar{f}_1, \bar{f}_2; \bar{g})$, $\operatorname{conv}(\bar{f}; b)$, $\operatorname{conv}(\bar{f})$, $\operatorname{conv}(\bar{f}; \bar{g})$, $\mathrm{Fl}(\bar{f}; b)$ and $\mathrm{Fl}(\bar{f}; \bar{g})$ denote, respectively, $D(\operatorname{var}(\bar{f}_1), \operatorname{var}(\bar{f}_2); b)$. $D(\operatorname{var}(\bar{f}_1), \operatorname{var}(\bar{f}_2); \bar{g})$, etc.

Note that we have redefined the notion "a is flexible with respect to b", but the two definitions are equivalent. Note also that $D(a_1, a_2; \bar{g})$ implies $D(a_1, a_2; \operatorname{var}(\bar{g}))$ and that $\mathrm{Fl}(a; b) \wedge \operatorname{var}(\bar{g}) \leq b$ implies $\mathrm{Fl}(a; \bar{g})$.

3.17. DEFINITION. (a) Let

$$V(\bar{f}, \bar{g}, \bar{h}) = Z(\bar{g}^{Z(\bar{f})}) \cap G(a^{\mathrm{LM}}) \cap \{f' \mid f' \lessdot \bar{h}\}\,.$$

(b) Let

$$\psi_1(\bar{f}, \bar{f}') \equiv \forall g(\neg \mathrm{Cm}(g, \bar{f}') \rightarrow (\exists g_1 \lessdot g)$$
$$(g_1 \neq Id \wedge \mathrm{Cm}(V(\bar{f}, g, g_1), \bar{f}')))\,.$$

In 3.18, 3.19 and 3.20 we assume that $\langle B, G\rangle \in K_{C'}$.

3.18. LEMMA. (a) $V(\bar{f}, \bar{g}, \bar{h}) \subseteq G(\operatorname{conv}(\bar{h}) \cdot a^{\mathrm{LM}} - \operatorname{conv}(\operatorname{var}(\bar{g}) \cdot \operatorname{fix}(\bar{f}); \bar{f}))$.

(b) $V(\bar{f}, \bar{g}, \bar{h}) \cap G(\operatorname{fix}(\bar{f})) = G(a^{\mathrm{LM}} \cdot \operatorname{conv}(\bar{h}) \cdot \operatorname{fix}(\bar{f}) - \operatorname{conv}(\bar{g}; \bar{f}))$.

(c) *If* $\operatorname{var}(\bar{g}) \cdot \operatorname{var}(\bar{f}) \cdot \operatorname{conv}(\bar{h}) \cdot a^{\mathrm{LM}} = 0$, *then* $V(\bar{f}, \bar{g}, \bar{h}) = G(\operatorname{conv}(\bar{h}) \cdot a^{\mathrm{LM}} - \operatorname{conv}(\bar{g}; \bar{f}))$

(d) *Let* $\bar{f}, \bar{f}' \subseteq G(a^{\mathrm{LM}})$ *and* $G \models \psi_1[\bar{f}, \bar{f}']$, *then* $\mathrm{Fl}(\operatorname{var}(\bar{f}'); \bar{f})$ *holds, in particular* $D(\operatorname{var}(\bar{f}') \cdot \operatorname{var}(\bar{f}), \operatorname{var}(\bar{f}') \cdot \operatorname{fix}(\bar{f}))$ *holds.*

(e) *Let* $\bar{f}, \bar{f}' \subseteq G(a^{\mathrm{LM}})$. *If* $\operatorname{var}(\bar{f}') \cdot \operatorname{var}(\bar{f}) = 0$ *and* $\mathrm{Fl}(\operatorname{var}(\bar{f}'); \bar{f})$ *holds, then* $G \models \psi_1[\bar{f}, \bar{f}']$.

PROOF. (a), (b), (c) are trivial consequences of 3.5(b) and 3.15.

(d) Suppose $\neg \mathrm{Fl}(\operatorname{var}(\bar{f}'); \bar{f})$ holds and we show that $G \not\models \psi_1[\bar{f}, \bar{f}']$. Let a_1,

$a_2 \leq \mathrm{var}(\bar{f}')$ be such that $a_1 \cong a_2$ but $D(a_1, a_2; \bar{f})$. Since $\langle B, G\rangle$ is movable, by 3.5(c) there is $g \in G(a_1)$ such that $\neg\mathrm{Cm}(g, \bar{f}')$. Let $Id \neq g_1 \preccurlyeq g$. We show that $\neg\mathrm{Cm}(V(\bar{f}, g, g_1), \bar{f}'))$ holds.

Let $a_3 = a_2 \cdot \mathrm{conv}(g_1)$. Clearly, $a_3 \neq 0$. $\mathrm{var}(g^{Z(\bar{f})}) \cdot a_3 = 0$, hence $Z(g^{Z(\bar{f})})$ is mixed on a_3, i.e. $V(\bar{f}, g, g_1)$ is mixed on a_3. Hence, since $\mathrm{var}(\bar{f}') \cdot a_3 \neq 0$, by 3.15(a) $\neg\mathrm{Cm}(V(\bar{f}, g, g_1), \bar{f}')$ holds. So (d) is proved.

(e) Suppose $\bar{f}, \bar{f}' \subseteq G(a^{\mathrm{LM}})$, $\mathrm{var}(\bar{f}') \cdot \mathrm{var}(\bar{f}) = 0$ and $\mathrm{Fl}(\mathrm{var}(\bar{f}'); \bar{f})$. Let $g \in G$ satisfy $\neg\mathrm{Cm}(g, \bar{f}')$. Hence, $\mathrm{var}(g) \cdot \mathrm{var}(\bar{f}') \neq 0$. Let $g_1 \neq Id$ be such that $\mathrm{var}(g_1) \leq \mathrm{var}(g) \cdot \mathrm{var}(\bar{f}')$. By (a) and (c): $V(\bar{f}, g, g_1) \subseteq G(\mathrm{conv}(g_1) - \mathrm{conv}(\mathrm{var}(g) \cdot \mathrm{fix}(\bar{f}); \bar{f})) \subseteq G(\mathrm{conv}(g_1) - \mathrm{conv}(\mathrm{var}(g_1); \bar{f}))$.

Since $\mathrm{var}(g_1) \leq \mathrm{var}(\bar{f}')$ and $\mathrm{Fl}(\mathrm{var}(\bar{f}'); \bar{f})$, $\mathrm{var}(\bar{f}') \cdot \mathrm{conv}(g_1) = \mathrm{var}(\bar{f}') \cdot \mathrm{conv}(g_1; \bar{f})$, and hence $\mathrm{var}(\bar{f}') \cdot (\mathrm{conv}(g_1) - \mathrm{conv}(g_1; \bar{f})) = 0$. Hence, $\mathrm{var}(\bar{f}') \cdot \mathrm{var}(V(\bar{f}, g, g_1)) = 0$, and so $\mathrm{Cm}(V(\bar{f}, g, g_1), \bar{f}')$. Thus, $G \models \psi_1[\bar{f}, \bar{f}']$. □

3.19. LEMMA. *Let $k_0, \ldots, k_n \in \omega$, $f \in G$ and $a \in B$, and suppose that $f^{k_0}(a), \ldots, f^{k_n}(a)$ are pairwise disjoint, then for every $h_1, \ldots, h_n \in Z(f)$ and $0 < b \leq a$, $\Sigma_{i=0}^{n} f^{k_i}(b) \not\leq \Sigma_{i=1}^{n} h_i(a)$.*

PROOF. By induction on n.

$n = 1$: Suppose by contradiction that for some $h_1 \in Z(f)$ and $0 < b \leq a$, $f^{k_0}(b) + f^{k_1}(b) \leq h_1(a)$. Hence, $0 \neq b \leq f^{-k_0}(h_1(a)) \cdot f^{-k_1}(h_1(a))$. Since $\langle a, f\rangle \cong \langle h_1(a), f\rangle$, $f^{-k_0}(a) \cdot f^{-k_1}(a) \neq 0$. So $f^{k_1}(a) \cdot f^{k_0}(a) = f^{k_0+k_1}(f^{-k_0}(a) \cdot f^{-k_1}(a)) \neq 0$; this contradicts our assumption, so the case $n = 1$ is proved.

$n + 1$: Suppose the claim has been proved for $n \geq 1$, and we prove it for $n + 1$. Let f, $k_0, \ldots, k_{n+1}$, $h_1, \ldots, h_{n+1}$ and b be as in the claim. If $h_{n+1}(a) \cdot \Sigma_{i=0}^{n+1} f^{k_i}(b) = 0$, then by the induction hypothesis $\Sigma_{i=1}^{n+1} h_i(a) \not\geq \Sigma_{i=0}^{n+1} f^{k_i}(b)$. Otherwise, let $c \overset{\mathrm{def}}{=} h_{n+1}(a) \cdot f^{k_i}(b) \neq 0$. Let $b' = f^{-k_i}(c)$. For each $j \neq i$ we apply the case $n = 1$ to f, k_i, k_j, h_{n+1} and b', and conclude that $h_{n+1}(a) \cdot f^{k_j}(b') = 0$. By the induction hypothesis, $\Sigma_{i=1}^{n} h_i(a) \not\geq \Sigma_{j\neq i} f^{k_j}(b')$, hence $\Sigma_{i=1}^{n+1} h_i(a) \not\geq \Sigma_{j\neq i} f^{k_j}(b') \leq \Sigma_{i=0}^{n+1} f^{k_i}(b)$. This proves the lemma. □

Let

$$\psi_2(f, f') \equiv \forall h([h, f] \neq Id \rightarrow (\exists f_1 f_2 \in Z(f'))([h, f_1, f_2] \neq Id \wedge \psi_1([h, f_1, f_2], f'))) .$$

Let

$$\psi_3(f, f') \equiv (\forall g \preccurlyeq f')((g \neq Id) \rightarrow (\exists f'' \preccurlyeq g)(((f'')^{12} \neq Id) \wedge \psi_1(f, \{f', f''\}) \wedge \psi_2(f, f''))) .$$

3.20. LEMMA. (a) *Let $f, f' \in G(a^{\mathrm{LM}})$ and $\mathrm{var}(f) \cdot \mathrm{var}(f') \neq 0$, then $G \not\models \psi_3[f, f']$.*
(b) *Let $f, f' \in G(a^{\mathrm{LM}})$ and $\mathrm{Fl}(\mathrm{var}(f'); \mathrm{var}(f))$, then $G \models \psi_3[f, f']$.*

PROOF. (a) We first prove the following claim.

Claim 1. If $f \in G(a^{\mathrm{LM}})$, $\mathrm{var}(f'') \le \mathrm{var}(f)$ and $(f'')^{12} \ne Id$, then $G \not\models \psi_2[f, f'']$.

Proof of Claim 1. Let f'' and f be as in the claim. Since $(f'')^{12} \ne Id$, there is $a > 0$ such that for every $0 < b \le a$, b, $f''(b)$, $(f'')^2(b)$, $(f'')^3(b)$ and $(f'')^4(b)$ are pairwise distinct. It follows that there is $0 < b \le a$ such that b, $f''(b)$, $(f'')^2(b)$, $(f'')^3(b)$ and $(f'')^4(b)$ are pairwise disjoint. $b \le \mathrm{var}(f'') \le \mathrm{var}(f)$, so by 3.5(c) there is $h \in G(b)$ such that $[h, f] \ne Id$; let $c = \mathrm{var}(h)$. Let $f_1, f_2 \in Z(f'')$. We denote $[h, f_1, f_2] = g$. Suppose $g \ne Id$, and we shall show that $\psi_1(g, f'')$ cannot hold:

$$\begin{aligned}(1)\qquad g &= [h, f_1, f_2] = [h \cdot (h^{-1})^{f_1}, f_2] = h(h^{-1})^{f_1} \cdot ((h \cdot (h^{-1})^{f_1})^{-1})^{f_2} \\ &= h \cdot (h^{-1})^{f_1} \cdot h^{f_2 f_1} \cdot (h^{-1})^{f_2} .\end{aligned}$$

From (1) it follows that

$$(2)\qquad \mathrm{var}(g) \le c + f_1(c) + f_2 f_1(c) + f_2(c) .$$

Since $c \le \mathrm{var}(f'')$ and $f_1, f_2 \in Z(f'')$, and by (2)

$$(3)\qquad \mathrm{var}(g) \le \mathrm{var}(f'') .$$

By 3.19 using (2) it follows that for every $0 \ne c_1 \le c$, $\Sigma_{i=0}^{4} (f'')^i(c_1) - (c_1 + f_1(c_1) + f_2 f_1(c_1) + f_2(c_1)) \ne 0$. Hence, $\mathrm{conv}(\mathrm{var}(g)) \le \mathrm{conv}(c) \le \mathrm{conv}(\Sigma_{i=0}^{4} (f'')^i(c) - (c + f_1(c) + f_2 f_1(c) + f_2(c)) \le \mathrm{conv}(\mathrm{var}(f'') - \mathrm{var}(g))$, namely

$$(4)\qquad \mathrm{var}(g) \preccurlyeq \mathrm{var}(f'') - \mathrm{var}(g) .$$

By (3), (4) and the fact $g \ne Id$, $\neg D(\mathrm{var}(f'') \cdot \mathrm{var}(g), \mathrm{var}(f'') \cdot \mathrm{fix}(g))$, hence by 3.18(d) $G \not\models \psi_1[g, f'']$. We have thus proved Claim 1.

We are now ready to prove (a). Suppose $\mathrm{var}(f) \cdot \mathrm{var}(f') \ne 0$, and let $g \in G(\mathrm{var}(f) \cdot \mathrm{var}(f')) - \{Id\}$. Hence, $Id \ne g \preccurlyeq f'$; we shall show that there is no f'' as required in ψ_3.

Let $f'' \preccurlyeq g$, $(f'')^{12} \ne Id$ and $G \models \psi_1[f', \{f', f''\}]$; we shall show that $\mathrm{var}(f'') \le \mathrm{var}(f)$, and hence it will follow from Claim 1 that $G \not\models \psi_2[f, f'']$.

Suppose by contradiction that $\mathrm{var}(f'') \cdot \mathrm{fix}(f) \ne 0$; then $0 \ne \mathrm{var}(f'') \cdot \mathrm{fix}(f) \preccurlyeq \mathrm{var}(f'') \preccurlyeq \mathrm{var}(g) \preccurlyeq \mathrm{var}(f') \cdot \mathrm{var}(f)$. Hence, $\neg D(\mathrm{var}(\{f'', f'\}) \cdot \mathrm{var}(f), \mathrm{var}(\{f'', f'\}) \cdot \mathrm{fix}(f))$. By 3.18(d), $G \not\models \psi_1[f, \{f'', f'\}]$, a contradiction. Hence, $\mathrm{var}(f'') \le \mathrm{var}(f)$, so by Claim 1, $G \not\models \psi_2[f, f'']$. This proves (a).

(b) We first prove the following claim.

Claim 2. Suppose that $f, f' \in G(a^{\mathrm{LM}})$ and $\mathrm{Fl}(\mathrm{var}(f'); \mathrm{var}(f))$, then $G \models \psi_2[f, f']$.

Proof of Claim 2. Let h be such that $[h, f] \ne Id$. Hence, there is $0 < a \le \mathrm{var}(f)$ such that $h(a) \cdot a = 0$. Let $f_1 \in G(a) - \{Id\}$. We check that $\mathrm{var}([h, f_1]) \le a + h(a)$, $[h, f_1](a) = a$ and $\mathrm{var}([h, f_1]) \cdot a \ne 0$. $[h, f_1] = f_1^h f_1^{-1}$, hence $\mathrm{var}([h, f_1]) \le h(\mathrm{var}(f_1)) + \mathrm{var}(f_1^{-1}) \le h(a) + a$. $[h, f_1] \restriction a = f_1^{-1} \restriction a$, this implies that $\mathrm{var}([h, f_1]) \cdot a \ne 0$ and that $[h, f_1](a) = a$. Let $0 < a_1 \le a$ be such that

$[h, f_1](a_1) \cdot a_1 = 0$, and let $f_2 \in G(a_1) - \{Id\}$. Arguing as above we conclude that $[h, f_1, f_2] \neq Id$, and that $\mathrm{var}([h, f_1, f_2] \leq \mathrm{var}(f_2) + [h, f_1](\mathrm{var}(f_2)) \leq a$. Hence, $\mathrm{var}([h, f_1, f_2]) \leq \mathrm{var}(f)$. Since by assumption $\mathrm{Fl}(\mathrm{var}(f'); \mathrm{var}(f))$, it follows that $G \models \psi_1([h, f_1, f_2], f')$. Also, since $\mathrm{var}(f_1)$ and $\mathrm{var}(f_2)$ are disjoint from $\mathrm{var}(f')$, $f_1, f_2 \in Z(f')$, hence $G \models \psi_2[f, f']$ and Claim 2 is proved.

We are now ready to prove (b). Suppose $\mathrm{Fl}(\mathrm{var}(f'); \mathrm{var}(f))$ holds, and let $Id \neq g \preccurlyeq f'$. By 3.5(a) there is $f'' \preccurlyeq g$ such that $\mathrm{var}(f'') \leq \mathrm{var}(f')$ and $(f'')^{12} \neq Id$. By Claim 2, $G \models \psi_2[f, f'']$; and since $\mathrm{Fl}(\mathrm{var}(\{f', f''\}); \mathrm{var}(f))$ holds $G \models \psi_1[f, \{f', f''\}]$. It follows that $G \models \psi_3[f, f']$, so (b) is proved. □

We are now ready to present the interpretation of $B \restriction a^{\mathrm{LM}}$ in G. Up to this point we did not use the flexibility of $\langle B, G \rangle$. We shall need this requirement now in order to assure the existence of sufficiently many pairs $\langle f, g \rangle$ for which $G \models \psi_3[f, g]$.

Let $V(f, g) = Z(g^{Z(f)}) \cap G(a^{\mathrm{LM}})$, $U(f, g, h) = V(f, g) \cap Z(V(f^h, g^h))$, $v(f, g) = \mathrm{var}(V(f, g))$ and $u(f, g, h) = \mathrm{var}(U(f, g, h))$. We have already found a formula $\varphi_{\mathrm{LM}}(f)$ expressing the fact that $f \in G(a^{\mathrm{LM}})$, hence $V(f, g)$ and $U(f, g, h)$ are first-order definable in G from f, g and f, g, h, respectively. We intend to use $V(f, g)$ and $U(f, g, h)$ only in cases when $f, g \in G(a^{\mathrm{LM}})$ and $\mathrm{var}(f) \cdot \mathrm{var}(g) = 0$. In such cases $V(f, g) = G(v(f, g))$, $u(f, g, h) = v(f, g) - v(f^h, g^h)$ and $U(f, g, h) = G(u(f, g, h))$. Let

$$DT^{\mathrm{LM}}(B, G) = \{\langle f, g, h \rangle \mid h \in G,\ f, g \in G(a^{\mathrm{LM}}) \text{ and } G \models \psi_3[f, g]\}\,.$$

Clearly the formula $\varphi_{DT}^{\mathrm{LM}}(f, g, h) \equiv \varphi_{\mathrm{LM}}(f) \wedge \varphi_{\mathrm{LM}}(g) \wedge \psi_3(f, g)$ defines $DT^{\mathrm{LM}}(B, G)$. We shall see that for flexible $\langle B, G \rangle$, $\{u(f, g, h) \mid \langle f, g, h \rangle \in DT^{\mathrm{LM}}(B, G)\}$ is dense in $B \restriction a^{\mathrm{LM}}$.

We now wish to find formulas $\varphi_{\mathrm{Eq}}^{\mathrm{LM}}(f, g)$ and $\varphi_{\leq}^{\mathrm{LM}}(f, g)$ which for flexible $\langle B, G \rangle$ express the facts that $\mathrm{var}(f) = \mathrm{var}(g)$ and that $\mathrm{var}(f) \leq \mathrm{var}(g)$, respectively. Recall that $\mathrm{Cm}(\bar{f}, \bar{g})$ means that every member of $\bar{f}$ commutes with every member of $\bar{g}$. Let

$$\begin{aligned} \varphi_{\leq}^{\mathrm{LM}}(f, g) \equiv (\forall f_1 g_1 h_1)((\varphi_{DT}^{\mathrm{LM}}(f_1, g_1, h_1) \wedge \mathrm{Cm}(g, U(f_1, g_1, h_1)) \\ \to \mathrm{Cm}(f, U(f_1, g_1, h_1)))\,, \end{aligned}$$

and let

$$\varphi_{\mathrm{Eq}}^{\mathrm{LM}}(f, g) \equiv \varphi_{\leq}^{\mathrm{LM}}(f, g) \wedge \varphi_{\leq}^{\mathrm{LM}}(g, f)\,.$$

Recall that $K_C = \{\langle B, G \rangle \mid B$ is complete, $a^{[\leq 2]}(B, G) = 0$ and $\langle B, G \rangle$ is movable and flexible$\}$. Also $\langle B, G \rangle$ is flexible if for every non-zero $a \leq a^{\mathrm{LM}}(B, G)$ there are non-zero $a_1, a_2 \leq a$ such that $a_1 \cong a_2 \bmod (-a)$ and $\mathrm{Fl}(a_1; a_2)$.

3.21. LEMMA. (a) *If* $\langle B, G \rangle \in K_{C'}$, *then for every* $\langle f, g, h \rangle \in DT^{\mathrm{LM}}(B, G)$: $V(f, g) = G(v(f, g))$, $u(f, g, h) = v(f, g) - v(f^h, g^h)$ *and* $U(f, g, h) = G(u(f, g, h))$.

(b) *If* $\langle B, G\rangle \in K_C$, *then* $\{u(f, g, h) \mid \langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)\}$ *is dense in* $B \restriction a^{\mathrm{LM}}$.

(c) *If* $\langle B, G\rangle \in K_C$, *then for every* $f, g \in G(a^{\mathrm{LM}})$ $G \models \varphi_{\le}^{\mathrm{LM}}[f, g]$ *iff* $\mathrm{var}(f) \le \mathrm{var}(g)$, *and* $G \models {}_{\mathrm{Eq}}^{\mathrm{LM}}[f, g]$ *iff* $\mathrm{var}(f) = \mathrm{var}(g)$.

PROOF. (a) Let $\langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)$, hence $f, g \in G(a^{\mathrm{LM}})$ and $G \models \psi_3[f, g]$ by 3.20 $\mathrm{var}(f) \cdot \mathrm{var}(g) = 0$. Hence, $g^{Z(f)}$ is mixed, and so by 3.15(b) $Z(g^{Z(f)}) = G(-\mathrm{var}(g^{Z(f)}))$, and so

$$\begin{aligned} V(f, g) &= G(a^{\mathrm{LM}}) \cap G(-\mathrm{var}(g^{Z(f)})) \\ &= G(a^{\mathrm{LM}} \cap \mathrm{var}(Z(g^{Z(f)}))) = G(v(f, g)) . \end{aligned}$$

It follows easily from 3.15(b) that for every $a, b \le a^{\mathrm{LM}}$, $G(a-b) = G(a) \cap Z(G(b))$, hence

$$\begin{aligned} U(f, g, h) &= V(f, g) \cap Z(V(f^h, g^h)) = G(v(f, g)) \cap Z(G(v(f^h, g^h))) \\ &= G(v(f, g) - v(f^h, g^h)) . \end{aligned}$$

Since $U(f, g, h)$ has the form $G(b)$, $U(f, g, h) = G(\mathrm{var}(U(f, g, h))) = G(u(f, g, h))$. This proves (a).

(b) Let $\langle B, G\rangle \in K_C$ and $0 < a \le a^{\mathrm{LM}}$. By the flexibility of $\langle B, G\rangle$ there are non-zero a_1, $a_2 \le a$ and $h \in G(a)$ such that $h(a_2) = a_1$ and $\mathrm{Fl}(a_2; a_1)$. Let $f \in G(a_1) - \{Id\}$ and $g = f^{h^{-1}}$. We show that $0 < v(f, g) - v(f^h, g^h) \le a$. $\mathrm{var}(g) = h^{-1}(\mathrm{var}(f))$. Since $g^{Z(f)}$ is mixed and $\mathrm{var}(g) \le \mathrm{var}(g^{Z(f)})$, $v(f, g) = \mathrm{var}(Z(g^{Z(f)}))$ is disjoint from $\mathrm{var}(g)$. $\mathrm{var}(g^{Z(f)}) \cdot \mathrm{var}(f) = 0$, hence $\mathrm{var}(f) \le v(f, g)$. $v(f^h, g^h) = h(v(f, g))$, and since $v(f, g) \cdot \mathrm{var}(g) = 0$, $v(f^h, g^h) \cdot \mathrm{var}(g^h) = 0$. $g^h = f$, so $v(f^h, g^h) \cdot \mathrm{var}(f) = 0$, hence $v(f, g) - v(f^h, g^h) \ge \mathrm{var}(f) \ne 0$ and so $u(f, g, h) \ne 0$. $h \restriction (-a) = Id$, and so $v(f, g) - v(f^h, g^h) = v(f, g) - h(v(f, g)) \le a$, that is, $u(f, g, h) \le a$. $\mathrm{var}(f) \le a_1$, $\mathrm{var}(g) \le a_2$ and $\mathrm{Fl}(a_2; a_1)$ holds, so $\mathrm{Fl}(\mathrm{var}(g); \mathrm{var}(f))$ holds. By 3.20, $G \models \psi_3[f, g]$, hence $\langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)$. We have thus proved that $\{u(f, g, h) \mid \langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)\}$ is dense in $B \restriction a^{\mathrm{LM}}$.

(c) Let $\mathrm{var}(f) \le \mathrm{var}(g) \le a^{\mathrm{LM}}$. Suppose that $\langle f_1, g_1, h_1\rangle \in DT^{\mathrm{LM}}(B, G)$ and g commutes with $U(f_1, g_1, h_1)$. Since $U(f_1, g_1, h_1) = G(u(f_1, g_1, h_1))$, it follows from 3.15(b) that $\mathrm{var}(g) \cdot u(f_1, g_1, h_1) = 0$, hence $\mathrm{var}(f) \cdot u(f_1, g_1, h_1) = 0$, and so f commutes with $U(f_1, g_1, h_1)$.

Suppose that $f, g \in G(a^{\mathrm{LM}})$ and $\mathrm{var}(f) \not\le \mathrm{var}(g)$. By (b) there is $\langle f_1, g_1, h_1\rangle \in DT^{\mathrm{LM}}(B, G)$ such that $0 \ne u(f_1, g_1, h_1) \le \mathrm{var}(f) - \mathrm{var}(g)$. Since $U(f_1, g_1, h_1) = G(u(f_1, g_1, h_1))$, g commutes with $U(f_1, g_1, h_1)$ but f does not. So $G \not\models \varphi_{\le}^{\mathrm{LM}}[f, g]$. This proves (c). $\square$

We now combine 3.11, 3.13 and 3.21 to obtain a first-order interpretation of a dense subset of B in G. Let

$$D(B,G)=(\{\mathrm{conv}(\mathrm{var}(f))\mid f\in a^{[6]}(B,G)\})^6$$
$$\times\{\mathrm{var}(f)\cdot\mathrm{var}(g)+\mathrm{var}(h)\mid f,g\in G(a^{\mathrm{R}}(B,G)-a^{[6]}(B,G)),$$
$$h\in G(a^{\mathrm{LM}}(B,G))\}\,.$$

Let $M^D(B,G)=\langle D(B,G),G;\leq,Op^-\rangle$. Clearly, $\langle D(B,G),\leq\rangle$ is isomorphic to a dense subset of B. Let $K_C^D=\{M^D(B,G)\mid\langle B,G\rangle\in K_C\}$ and $K_{C^-}^D=\{M^D(B,G)\mid\langle B,G\rangle\in K_{C^-}\}$.

Let

$$\varphi_{\mathrm{U}}^D(f_1,\ldots,f_6,f_7,f_8,f_9)\equiv\bigwedge_{i=1}^{6}\varphi_{[6]}(f_i)\wedge\bigwedge_{i=7}^{8}(\varphi_{\mathrm{R}}(f_i)$$
$$\wedge\,\varphi_{[\neq6]}(f_i))\wedge\varphi_{\mathrm{LM}}(f_9)\,,$$

$$\varphi_{\mathrm{Eq}}^D(f_1,\ldots,f_9,g_1,\ldots,g_9)\equiv\bigwedge_{i=1}^{6}(f_i\approx g_i)\wedge\varphi_{\mathrm{Eq}}^{\mathrm{R}}(f_7,f_8,g_7,g_8)$$
$$\wedge\,\varphi_{\mathrm{Eq}}^{\mathrm{LM}}(f_9,g_9)\,,$$

$$\varphi_{\leq}^D(f_1,\ldots,f_9,g_1,\ldots,g_9)\equiv\bigwedge_{i=1}^{6}(f_i\preccurlyeq g_i)\wedge\varphi_{\leq}^{\mathrm{R}}(f_7,f_8,g_7,g_8)$$
$$\wedge\,\varphi_{\leq}^{\mathrm{LM}}(f_9,g_9)$$

and

$$\varphi_{Op^-}^D(h,f_1,\ldots,f_9,g_1,\ldots,g_9)$$
$$\equiv\bigwedge_{i=1}^{6}(f_i=g_i=Id)\wedge\varphi_{\mathrm{Eq}}^D(f_1,\ldots,f_6,f_7^h,f_8^h,f_9^h,g_1,\ldots,g_9)\,.$$

3.22. COROLLARY. *$\langle\varphi_{\mathrm{U}}^D,\varphi_{\mathrm{Eq}}^D,\varphi_{\leq}^D,\varphi_{Op}^D\rangle$ is a first-order interpretation of K_C^D in K_C^G and a strong first order interpretation of $K_{C^-}^D$ in $K_{C^-}^G$.*

Let D be a dense subset of a complete Boolean algebra B. A subset $b\subseteq D$ is a *complete ideal* in D if: (1) for every $d_1\in b$ and $d_2\leq d_1$, $d_2\in b$, and (2) if $x\in D$ and for every $0\neq y\leq x$ there is $0\neq z\leq y$ such that $z\in b$, then $x\in b$. Let $I(D)=\{b\subseteq D\mid b$ is a complete ideal in $D\}$. It is easy to see that $\langle I(D),\subseteq\rangle\cong\langle B,\leq\rangle$. Theorem 3.4 now follows easily.

3.4. THEOREM. *$K_C^{BG^-}$ is interpretable in K_C^G and $K_{C^-}^{BG}$ is strongly interpretable in $K_{C^-}^G$. Every member of K_{C^-} is group-categorical in K_D.*

PROOF. By 3.22, K_C^D is first-order interpretable in K_C^G. To show that $K_C^{BG^-}$ is interpretable in K_C^D, we use the discussion preceding this lemma. Let $\langle B,G\rangle\in K_C$, then $M^-(B,G)\cong\langle I(D(B,G)),G;\subseteq,Op^-\rangle$. To show that $\langle I(D(B,G)),G;\subseteq,Op^-\rangle$ is interpretable in $M^D(B,G)$ it remains to find a formula $\varphi_{Op^-}^I$ representing Op^-. It is easy to see that $\varphi_{Op^-}^I(f,b,c)\equiv c=\{h^f\mid h\in b\}$ is as required.

The interpretation of K_C^D in K_C^G which is described in 3.22 is a strong interpretation when restricted to $K^D_{C^-}$. The interpretation of K^{BG}_C in K^D_C is again a strong interpretation, hence $K^{BG}_{C^-}$ is strongly interpretable in $K^G_{C^-}$.

To prove the third part of Theorem 3.4, we shall show that there is a first-order sentence φ_C in the language of groups such that for every $\langle B, G\rangle \in K_D$, $G \models \varphi_C$ iff $\langle B, G\rangle \in K_C$. If follows easily from Lemma 3.10 that there is a sentence $\psi_{[6]}$ such that for every $\langle B, G\rangle \in K_C$ $G \models \psi_{[6]}$ iff $\langle B, G\rangle \in K_{C^-}$.

Now assume we have already established the existence of φ_C. Let $\langle B, G\rangle \in K_C$ and $\langle B_1, G_1\rangle \in K_D$ and $G \cong G_1$. Since $G \models \varphi_C$, so does G_1. Hence, $\langle B_1, G_1\rangle \in K_C$. Similarly, $G_1 \models \psi_{[6]}$, and hence $\langle B_1, G_1\rangle \in K_{C^-}$. By the strong interpretability of $K^{BG}_{C^-}$ in $K^G_{C^-}$, every isomorphism between G and G_1 is induced by an isomorphism between B and B_1.

Let φ^1 be the sentence saying that the group is centerless. Hence, for every $\langle B, G\rangle \in K_D$, $G \models \varphi^1$ iff $\langle B, G\rangle \in K_{C'}$. In 3.21 we presented a first-order definable set $DT^{\mathrm{LM}}(B, G) \subseteq G \times G \times G$. For every $\langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)$ we defined the element $u(f, g, h) \in B$; we also defined a formula $\chi(f, g, h, k)$ such that for every $\langle B, G\rangle \in K_{C'}$ and $\langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)$, $G(u(f, g, h)) = \{k \in G \mid G \models \chi[f, g, h, k]\}$. The interpretation presented in part (a) of this theorem remains valid for the bigger class $\{\langle B, G\rangle \in K_{C'} \mid \{u(f, g, h) \mid \langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)\}$ is dense in $B \restriction a^{\mathrm{LM}}\}$. Hence, it suffices to find a formula saying that the above set is indeed dense in $B \restriction a^{\mathrm{LM}}$. Let P be a set of subgroups of G and each member of P has the form $G(a)$ for some $a \in B \restriction a^{\mathrm{LM}}$. Let $\alpha \equiv (\forall f \in G(a^{\mathrm{LM}}))(\exists A \in P)(A \neq \{Id\} \wedge \mathrm{Cm}(A, A^f))$. We leave it to the reader to check that if $\langle B, G\rangle \in K_{C'}$, then $\langle G, P\rangle \models \alpha$ iff $\{a \in B \mid G(a) \in P\}$ is dense in $B \restriction a^{\mathrm{LM}}$. Replacing in α the variable set P by the set $\{G(u(f, g, h)) \mid \langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)\}$ we obtain a first-order sentence φ^2 which says that $\{u(f, g, h) \mid \langle f, g, h\rangle \in DT^{\mathrm{LM}}(B, G)\}$ is dense in $B \restriction a^{\mathrm{LM}}$. This concludes the proof of 3.4. $\square$

Theorem 3.4 gives full information on the reconstruction of complete Boolean algebras from their automorphism groups. We summarize this information in the following theorem.

3.23. THEOREM. (a) *Let B_1, B_2 be complete BAs, then* $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$ *iff* $B_1 \restriction a^{[\geq 3]}(B_1) \cong B_2 \restriction a^{[\geq 3]}(B_2)$ *and* $|B_1 \restriction a^{[2]}(B_1)| = |B_2 \restriction a^{[2]}(B_2)|$.

(b) $\{M(B, \mathrm{Aut}(B)) \mid B$ *is complete and* $a^{[\leq 2]}(B) = 0\}$ *is first-order interpretable in* $\{\mathrm{Aut}(B) \mid B$ is complete and $a^{[\leq 2]}(B) = 0\}$.

(c) $\{M(B, \mathrm{Aut}(B)) \mid B$ *is complete and* $a^{[\leq 2]}(B) = a^{[6]}(B) = 0\}$ *is strongly first-order interpretable in* $\{\mathrm{Aut}(B) \mid B$ *is complete and* $a^{[\leq 2]}(B) = a^{[6]}(B) = 0\}$.

PROOF. (a) Let B be a complete BA, then $Z(\mathrm{Aut}(B)) = \mathrm{Aut}(B)(a^{[2]}(B))$, $|Z(\mathrm{Aut}(B))| = |\{a \leq a^{[2]}(B) \mid a = \mathrm{conv}(a)\}|$ and every element of $Z(\mathrm{Aut}(B))$ has order two. Hence, if $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$, then $|Z(\mathrm{Aut}(B_1))| = |Z(\mathrm{Aut}(B_2))|$ and hence $|B_1 \restriction a^{[2]}(B_1)| = |B_2 \restriction a^{[2]}(B_2)|$. $\mathrm{Aut}(B)/Z(\mathrm{Aut}(B)) \cong \mathrm{Aut}(B \restriction a^{[\geq 3]}(B))$. Hence, if $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$, then $\mathrm{Aut}(B_1 \restriction a^{[\geq 3]}(B_1)) \cong \mathrm{Aut}(B_2 \restriction a^{[\geq 3]}(B_2))$. It thus follows from 3.4 that $B_1 \restriction a^{[\geq 3]}(B_1) \cong B_2 \restriction a^{[\geq 3]}(B_2)$.

The other direction in the proof of (a) is easy.

(b) and (c) follow from the fact that if B is complete and $a^{[\leq 2]}(B)=0$, then $B\cong\langle D(B, \mathrm{Aut}(B)), \leq\rangle$. □

4. Faithfulness of incomplete Boolean algebras

There does not seem to be an exact condition like in 3.23 which is equivalent to the fact that $\mathrm{Aut}(B_1)\cong\mathrm{Aut}(B_2)$, when B_1 and B_2 are not necessarily complete. We have thus to settle for weaker results. In Theorem 4.1 we assume very little about B_1 and B_2, but then the fact that $\mathrm{Aut}(B_1)\cong\mathrm{Aut}(B_2)$ implies only that $\bar{B}_1\cong\bar{B}_2$. If we wish to conclude that $\mathrm{Aut}(B_1)\cong\mathrm{Aut}(B_2)\Rightarrow B_1\cong B_2$, then we have to make more assumptions on B_1 and B_2. This is done in Theorem 4.5, and the assumptions on B_i are that $a^{[6]}(B_i)=0$, and that $I^{[\geq 3]}(B_i)=B_i$, that is, 1_{B_i} is a finite sum of elements a for which there are $f, g\in\mathrm{Aut}(B_i)$ such that $a, f(a), g(a)$ are pairwise disjoint. These assumptions which are much weaker than the homogeneity of B_i are later used in Section 6 in order to show that certain groups of measure preserving automorphisms are faithful.

Recall that $\bar{B}$ denotes the completion of B, and if $a\in\bar{B}$, then $B\restriction a$ denotes $\{b\cdot a\mid b\in B\}$, hence $B\restriction a$ is a Boolean algebra and $B\restriction a\subseteq\bar{B}$.

4.1. THEOREM. *Let B_1 and B_2 be Boolean algebras such that* $\mathrm{Aut}(B_1)\cong\mathrm{Aut}(B_2)$, *then* (a) $\bar{B}_1\restriction a^{[\geq 3]}(B_1)\cong\bar{B}_2\restriction a^{[\geq 3]}(B_2)$, *and* (b) *there are partitions* $\{b_i^l\mid i\in I\}\subseteq B_l$ *of* $a^{[\geq 3]}(B_l)$, $l=1, 2$, *such that for every* $i\in I$, $B_1\restriction b_i^1\cong B_2\restriction b_i^2$.

PROOF. Since (b) implies (a) it is sufficient to prove (b). Let B_1 and B_2 be as in the theorem. Let $B_l'=B_l\restriction a^{[\geq 3]}(B_l)$. The group $\mathrm{Aut}(B_l)/Z(\mathrm{Aut}(B_l))$ is naturally isomorphic to a subgroup of $\mathrm{Aut}(B_l')$. We denote this subgroup by G_l. Hence, $\mathrm{Aut}(B_l)(a^{[\geq 3]}(B_l))\subseteq G_l\subseteq\mathrm{Aut}(B_l')$. It is easy to see that if $a\in B_l$ and $a\leq -a^{[\leq 2]}(B_l)$ and $f\in G_l$, then $f(a)\in B_l$. It is also easy to see that $\langle\bar{B}_l', G_l\rangle\in K_C$. $\mathrm{Aut}(B_1)\cong\mathrm{Aut}(B_2)$, hence $G_1\cong\mathrm{Aut}(B_1)/Z(\mathrm{Aut}(B_1))\cong\mathrm{Aut}(B_2)/Z(\mathrm{Aut}(B_2))\cong G_2$. Let $\varphi\colon G_1\to G_2$ be an isomorphism. The strong interpretability of $K_C^{BG^-}$ in K_C^G which is established in 3.4 implies that there is an isomorphism $\tilde{\varphi}\colon\bar{B}_1\to\bar{B}_2$ such that for every $f\in G_1(-a^{[6]}(\bar{B}_1', G_1))$, $\varphi(f)=\tilde{\varphi}\circ f\circ\tilde{\varphi}^{-1}$. Note also that $a^{[6]}(\bar{B}_l', G_l)=a^{[6]}(B_l)$.

We shall show: (∗) for every $0<a\in\bar{B}_1'$ there is $0<b\leq a$ such that $b\in B_1$, $\tilde{\varphi}(b)\in B_2$ and $B_1\restriction b\cong B_2\restriction\tilde{\varphi}(b)$. (b) follows trivially from this claim.

Proof of (∗).

Case 1. $a\not\leq a^{[6]}(B_1)$. Let $c_1\in B_1$ and $f_1\in G_1$ such that $0<c_1\leq a-a^{[6]}(B_1)$ and $f_1(c_1)\cdot c_1=0$. Since $c_1\leq a\in B_1'$, $c_1\leq -a^{[\leq 2]}(B_1)$. Let $c_2\in B_2$ such that $\tilde{\varphi}^{-1}(c_2)\leq c_1$. We prove that $\tilde{\varphi}^{-1}(c_2)\in B_1$. W.l.o.g. $\mathrm{var}(f_1)\cdot a^{[6]}(B_1)=0$, hence $f_1^{\tilde{\varphi}}\in G_2$, and hence $f_1^{\tilde{\varphi}}(c_2)\in B_2$. Let $g_2\in\mathrm{Aut}(\bar{B}_2')$ be such that $g_2\restriction c_2=f_1^{\tilde{\varphi}}\restriction c_2$, $g_2\restriction f_1^{\tilde{\varphi}}(c_2)=(f_1^{\tilde{\varphi}})^{-1}\restriction f_1(c_2)$ and $g_2\restriction -(c_2+f_1^{\tilde{\varphi}}(c_2))=Id\restriction -(c_2+f_1^{\tilde{\varphi}}(c_2))$. It is easy to check that $g_2\in G_2$, and hence $g_1\overset{\mathrm{def}}{=}g_2^{\tilde{\varphi}^{-1}}\in G_1$. A computation shows that $g_1(c_1)=c_1-\tilde{\varphi}^{-1}(c_2)+f_1(\tilde{\varphi}^{-1}(c_2))$. Since $c_1\in B_1$ and $g_1\in G_1$, $g_1(c_1)\in B_1$. Hence, $c_1-g_1(c_1)\in B_1$, and since $\tilde{\varphi}^{-1}(c_2)=c_1-g_1(c_1)$, $\tilde{\varphi}^{-1}(c_2)\in B_1$.

If in the above argument we let B_2 take the role of B_1, c_2 take the role of c_1, and $f_1^{\tilde{\varphi}}$ take the role of f_1, we may conclude that for every $d \in B_1$: if $d \leq \tilde{\varphi}^{-1}(c_2)$ then $\tilde{\varphi}(d) \in B_2$. Hence,

$$\tilde{\varphi}^{-1}(c_2) \in B_1,\ c_2 \in B_2 \text{ and } B_1 \restriction \tilde{\varphi}^{-1}(c_2) \overset{\tilde{\varphi}}{\cong} B_2 \restriction c_2 .$$

Case 2. $a \leq a^{[6]}(B_1)$. There are pairwise disjoint $a_1^1 \cong a_2^1 \cong \cdots \cong a_6^1 \in B_1$ such that $0 < a^1 \leq a$. Hence, $a^1 \overset{\text{def}}{=} \Sigma_{i=1}^6 a_i^1 = \text{conv}(a_1^1)$. Let ρ_1: $B_1 \restriction a_1^1 \to B_1' \restriction a^1$ be defined as follows: $\rho_1(b) = \text{conv}(b)$. Then ρ is an isomorphism between $\langle B_1 \restriction a_1^1, \leq \rangle$ and $\langle \{x \in B_1 \mid x = \text{conv}(x) \leq a^1\}, \leq \rangle \overset{\text{def}}{=} C_1$. Let $f_1 \in \text{Aut}(B_1)$ be such that $\text{conv}(f_1) = a^1$, then it is easy to check that ($\approx$ is defined in 3.6)

$$D_1 \overset{\text{def}}{=} \langle \{f/_{\approx} \mid f \preccurlyeq f_1\}, \leq \rangle \overset{\tau_1}{\cong} C_1, \text{ where } \tau_1(f/_{\approx}) = \text{conv}(f) .$$

Let $a_1^2 = a_2^2 \cong \cdots \cong a_6^2 \in B_2$ be such that $0 < a_1^2 \leq \tilde{\varphi}(a_1^1)$. We define a^2, C_2, D_2 and f_2 in analogy to the definition of a_1, C_1, D_1 and f_1, so $\langle B_2 \restriction a^2, \leq \rangle \cong C_2 \cong D_2$. It follows from the proof of 3.4 that for every $g \in G_1$, $\tilde{\varphi}(\text{conv}(g)) = \text{conv}(\varphi(g))$, and since the same holds for G_2 and $\tilde{\varphi}^{-1}$ we may conclude that $\tilde{\varphi}^{-1}(a^2) = \tilde{\varphi}^{-1}(\text{conv}(f_2)) = \text{conv}(\varphi^{-1}(f_2))$. $\tilde{\varphi}^{-1}(a^2) \leq a^1$ and it has the form $\text{conv}(g)$ for some $g \in G_1$, hence $\tilde{\varphi}^{-1}(a^2) = \tau_1(\varphi^{-1}(f_2)) \in B_1$. Let $\hat{a}^1 = \tilde{\varphi}^{-1}(a^2)$, $\hat{a}_i^1 = \hat{a}^1 \cdot a_i^1$ and $\hat{f}_1 = \varphi^{-1}(f_2)$. $\tilde{\varphi}(\hat{a}_1^1) = a_1^2$, and if we define $\hat{C}_1$ and $\hat{D}_1$ as above, then

$$B_1 \restriction \hat{a}_1^1 = \hat{C}_1 = \hat{D}_1 \overset{\varphi^*}{\cong} \hat{D}_2 \cong C_2 \cong B_2 \restriction a_1^2 ,$$

where for every $g/_{\approx} \in \hat{D}_1$, $\varphi^*(g/_{\approx}) = \varphi(g)/_{\approx}$. The reason that φ^* is indeed an isomorphism is that $\leq$, $\approx$ are definable in G_1 and G_2, $\varphi(\hat{f}_1) = f_2$ and φ is an isomorphism between G_1 and G_2. Recall that $a_1^2 = \tilde{\varphi}(\hat{a}_1^1)$, thus we have found $0 < \hat{a}_1^1 < a$ such that $\hat{a}_1^1 \in B_1$, $\tilde{\varphi}(\hat{a}_1^1) \in B_2$ and $B_1 \restriction a_1^1 \cong B_2 \restriction \tilde{\varphi}(\hat{a}_1^1)$. This concludes the proof of $(*)$, hence the theorem is proved. □

Theorem 4.1 tells all what is known about the relationship between general Boolean algebra which have isomorphic groups of automorphisms. We next present two examples of non-isomorphic weakly homogeneous Boolean algebras that have isomorphic groups of automorphisms.

4.2. EXAMPLES. The algebra B_1 of all subsets of κ, and its subalgebra B_2 generated by all subsets of κ of cardinality $< \mu$, are non-isomorphic if $\aleph_0 \leq \mu \leq \kappa$, but both have the symmetric group on κ as their automorphism group.

We can easily modify the above example to obtain atomless B_1 and B_2. Let B_0 be the countable atomless BA and $B_1 = B_0^\kappa$, $\kappa > \aleph_0$. Let B_2 be the subalgebra of B_1 generated by $\{\langle a_i \mid i < \kappa \rangle \mid |\{i \mid a_i \neq 0\}| < \mu\}$. It is easy to see that $B_1 \not\cong B_2$ and that $\text{Aut}(B_1) \cong \text{Aut}(B_2)$.

We now wish to obtain a faithfulness result for a class K containing Boolean algebras which are not necessarily complete. The statement and proof of the main theorem will be better understood in their topological form, so we have to switch to topological language. X will denote a Boolean space, that is, a zero-dimensional compact Hausdorff space, $\mathrm{Clop}(X)$ denotes the Boolean algebra of clopen subsets of X, and $\mathrm{Ro}(X)$ denotes the complete Boolean algebra of regular open subsets of X. Hence $\mathrm{Ro}(X)$ can be considered to be the completion of $\mathrm{Clop}(X)$. Let $H(X)$ be the homeomorphism group of X and G be a subgroup of $H(X)$. $H(X)$ can be identified with $\mathrm{Aut}(\mathrm{Clop}(X))$ and hence can be regarded as a subgroup of $\mathrm{Aut}(\mathrm{Ro}(X))$. Consequently, G is also regarded as a subgroup of $\mathrm{Aut}(\mathrm{Ro}(X))$.

To reconstruct $M(\mathrm{Clop}(X), G)$ from G, we shall first reconstruct $M(\mathrm{Ro}(X), G)$ from G using 3.4(b), and then find a formula $\phi(x)$ in the language of $M(\mathrm{Ro}(X), G)$ such that for every $b \in \mathrm{Ro}(X)$ $M(\mathrm{Ro}(X), G) \models \phi[b]$ iff $b \in \mathrm{Clop}(X)$. Hence, we shall show that $\mathrm{Clop}(X)$ is definable in $M(\mathrm{Ro}(X), G)$. The facts that $M(\mathrm{Ro}(X), G)$ is reconstructable from G and that $\mathrm{Clop}(X)$ is definable in $M(\mathrm{Ro}(X), G)$ imply that $M(\mathrm{Clop}(X), G)$ is reconstructable from G.

We shall define two classes, K_{B^1} and K_{B^2}, and show (Theorem 4.5) that $K_{B^1} \cup K_{B^2}$ is faithful. Prior to this, however, we wish to introduce the topological duals of some of the Boolean algebraic notions and facts that we have used.

For $x \in X$ let $G(x) = \{g(x) \mid g \in G\}$. Let $U^{[\geq n]}(X, G) = \{x \in X \mid |G(x)| \geq n\}$, $U^{[\leq n]}(X, G) = \{x \in X \mid |G(x)| \leq n\}$ and $U^{[n]}(X, G) = \{x \in X \mid |G(x)| = n\}$. Recall that $I^{[\leq n]}(B, G)$ and $a^{[n]}(B, G)$ were defined in 3.1(e). The easy checking of the following facts is left to the reader. $U^{[\geq n]}(X, G) = \bigcup \{a \mid a \in I^{[\geq n]}(\mathrm{Ro}(X), G)\}$ and $\mathrm{int}(U^{[n]}(X, G)) = a^{[n]}(\mathrm{Ro}(X), G)$. $U^{[\geq n]}(X, G)$ is dense in X iff $I^{[\geq n]}(\mathrm{Ro}(X), G)$ is a dense ideal in $\mathrm{Ro}(X)$. $U^{[\geq n]}(X, G) = X$ iff $I^{[\geq n]}(\mathrm{Clop}(X), G) = \mathrm{Clop}(X)$.

Let I be an ideal in a BA B. I is called an indecomposable ideal if, for every two ideals I_1 and I_2: if $I_1 \cap I_2 = \{0\}$ and the ideal generated by $I_1 \cup I_2$ contains I, then there is a principal ideal J such that $J \supseteq I_1$ and $J \cap I_2 = \{0\}$.

The following are equivalent: (1) $U^{[\geq n]}(X, G)$ intersects the boundary of every regular open non-clopen set. (2) $I^{[\geq n]}(\mathrm{Clop}(X), G)$ is indecomposable.

Let us next explain the definition of K_{B^i}. Recall that $K_{C^-} = \{\langle B, G\rangle \mid$ (1) B is complete, (2) $a^{[\leq 2]}(B, G) = a^{[6]}(B, G) = 0$, and (3) $\langle B, G\rangle$ is movable and flexible$\}$. 3.4(b) said that $K_{C^-}^{BG}$ is strongly interpretable in $K_{C^-}^{G}$. In order to apply 3.4(b) we have thus to include in the definition of K_{B^i} the requirements that (1) $a^{[\leq 2]}(B, G) = 0$ and (2) $a^{[6]}(B, G) = 0$. These requirements have the following topological translation: (1) $U^{[\geq 3]}(X, G)$ is dense in X and (2) $\mathrm{int}(U^{[6]}(X, G)) = \emptyset$.

4.3. DEFINITION. $\langle X, G\rangle$ is *B-closed* if for every $g \in G$ and $b \in \mathrm{Clop}(X)$: (1) if $g(b) = b$, and $g \restriction b \cup Id \restriction -b \in G$; and (2) if $g(b) \cdot b = 0$, then $g \restriction b \cup g^{-1} \restriction g(b) \cup Id \restriction -(b + g(b)) \in G$.

We say that $\langle B, G\rangle$ is *B-closed* if $\langle \mathrm{Ult}(B), G\rangle$ is B-closed.

4.4. PROPOSITION. *If $\langle X, G\rangle$ is B-closed, then $\langle \mathrm{Ro}(X), G\rangle$ is movable and flexible.*

We leave the trivial verification of 4.4 to the reader. In the definition of K_{B^i} we shall require that $\langle \mathrm{Ult}(B), G\rangle$ be B-closed. Together with the previous requirements this will imply that $K_{B^i} \subseteq K_{C^-}$.

Note that $\langle X, H(X)\rangle$ is always B-closed. Also, the groups of measure preserving or non-singular automorphisms of a measure algebra are B-closed.

Let $\phi'_B(v)$ be the following formula in the language of $M(\mathrm{Ro}(X), G)$: $\phi'_B(v) \equiv (\forall g \in G)((g^2 = Id) \rightarrow (\exists h \in G)(h \supseteq g \restriction (v \cdot g(v)) \cup Id \restriction -(v \cdot g(v))),$ and $\phi_B(v) = \phi'_B(v) \wedge \phi'_B(-v)$.

We shall use ϕ_B to define $\mathrm{Clop}(X)$ in $M(\mathrm{Ro}(X), G)$. Clearly, if $\langle X, G\rangle$ is B-closed, then for every $b \in \mathrm{Clop}(X)$ $M(\mathrm{Ro}(X), G) \models \phi_B[b]$.

We have so far listed three requirements to be included in the definition of K_{B^i}. However, the examples in 4.2 satisfy the above requirements but refute faithfulness. Both K_{B^1} and K_{B^2} will be defined by adding an additional requirement.

Let $K_{T^-} = \{\langle X, G\rangle \mid \langle X, G\rangle$ is B-closed, $U^{[\geq 3]}(X, G)$ is dense in X and $\mathrm{int}(U^{[6]}(X, G) = \emptyset\}$, $K_{T^1} = \{\langle X, G\rangle \in K_{T^-} | U^{[\geq 3]}(X, G)$ intersects the boundary of every regular open non-clopen set$\}$, $K_{T^2} = \{\langle X, G\rangle \in K_{T^-} | U^{[\geq 2]}(X, G) = X\}$ and $K_T = K_{T^1} \cup K_{T^2}$. Let $\mathrm{Ult}(B)$ denote the Stone space of B and $K_B = \{\langle B, G\rangle \mid \langle \mathrm{Ult}(B), G\rangle \in K_T\}$. K_{B^1} and K_{B^2} are defined similarly.

4.5. THEOREM. (a) *K_B is faithful.*

(b) *$K^{BG}_{B^1}$ is strongly interpretable in $K^G_{B^1}$, thus K_{B^1} is strongly faithful. Let $K_{T^3} = \{\langle X, G\rangle \mid \langle X, G\rangle$ is B-closed, $U^{[\geq 2]}(X, G)$ is dense in X, and $U^{[\geq 2]}(X, G)$ intersects the boundary of every regular open non-clopen set$\}$ and $K_{B^3} = \{\langle B, G\rangle \mid \langle \mathrm{Ult}(B), G\rangle \in K_{T^3}\}$. Then every member of K_{B^1} is group-categorical in K_{B^3}. (Group-categoricity was defined in* 1.2.)

(c) *Let $K_{T^0} = \{\langle X, G\rangle \in K_{T^-} \mid U^{[\geq 3]}(X, G) = X\}$ and $K_{B^0} = \{\langle B, G\rangle \mid \langle \mathrm{Ult}(B), G\rangle \in K_{T^0}\}$. Then $K^{BG}_{B^0}$ is strongly first-order interpretable in $K^G_{B^0}$.*

REMARK. Note that $K_{B^0} \subseteq K_{B^1} \cap K_{B^2}$ and that $K_{B^1} \cup K_{B^2} \subseteq K_{B^3}$.

A summary of both positive and negative faithfulness results formulated in Boolean algebraic terms appears at the end of this section.

One may try to define a class $K_{T'}$ using a common weakening of the conditions appearing in the definitions of K_{T^1} and K_{T^2}. Let $K_{T'} = \{\langle X, G\rangle \in K_{T^-} | U^{[\geq 2]}(X, G)$ intersects the boundary of every regular open non-clopen set$\}$. Let $K_{B'} = \{\langle B, G\rangle \mid \langle \mathrm{Ult}(B), G\rangle \in K_{T'}\}$. Unfortunately, $K_{B'}$ is not faithful; this will be shown in Example 4.12.

The definition of K_B is quite complicated, this is however compensated by the fact that K_B is a large class, and it contains some interesting subclasses. The following classes can be easily checked to be contained in K_B. (When G is not mentioned, it is assumed to be $\mathrm{Aut}(B)$.) $\{B \times B \mid a^{[1]}(B) = a^{[3]}(B) = 0\} \subseteq K_{B^2}$. $\{B \times B \times B \mid a^{[2]}(B) = 0\} \subseteq K_{B^1}$. More generally, $\{\Pi_{i \in I} B_i^{\lambda_i} \mid \lambda_i \geq 2$; if $\lambda_i = 2$, then $a^{[1]}(B_i) = a^{[3]}(B_i) = 0$; if $\lambda_i = 3$, then $a^{[2]}(B_i) = 0$; and if $\lambda_i = 6$, then $a^{[1]}(B_i) = 0\} \subseteq K_{B^2}$. If $\langle B, \mu\rangle$ is a σ-finite measure algebra (see Section 6), and for every a, $b \in B$ such that $0 < \mu(a) = \mu(b) < \mu(1_B)$, there is $f \in \mathrm{MP}(B, \mu)$ such that $f(a) =$

b, then $\langle B, \mathrm{MP}(B, \mu)\rangle \in K_{B^1}$. $\{B \mid B$ is complete and $a^{[\leq 2]}(B) = a^{[6]}(B) = 0\} \subseteq K_B$.

Theorem 4.5 contains all the information known on faithfulness of classes of incomplete BAs. The only exception to this is the exclusion of BAs in which $a^{[6]} \neq 0$ for which some additional information may be obtained. The most central result in 4.5 is the fact that K_B is faithful (4.5(a)). The proof of 4.5(b) is very simple. By Lemma 4.7, if $\langle B, G\rangle \in K_{B^1}$ and $M(\bar{B}, G) \models \phi_B[a]$, then $a \in B$. So $M(B, G)$ is strongly interpretable in $M(\bar{B}, G)$.

For a set $b \subseteq X$ let $\mathrm{cl}(b)$ denote the closure of b and $\mathrm{bd}(b)$ the boundary of b.

4.6. DEFINITION. (a) Let $\mathrm{Bad}(X, G) = \bigcup \{\mathrm{bd}(b) \mid M(\mathrm{Ro}(X), G) \models \phi_B[b]\}$.

(b) Let $a \in \mathrm{Ro}(X)$ and $x \in X$. x belongs to the *hidden boundary* of a ($\mathrm{hbd}(a)$) if $x \in \mathrm{bd}(a)$ and for every $g \in G$: if $g(x) \neq x$, then there is a clopen neighborhood b of x such that $g(b \cdot a) = g(b) - a$ and $g(b - a) = g(b) \cdot a$.

We shall first prove some lemmas.

4.7. LEMMA. *Let $\langle X, G\rangle$ be B-closed, then*:

(a) *For every $b \in \mathrm{Clop}(X)$ $M(\mathrm{Ro}(X), G) \models \phi_B[b]$.*

(b) *If $a \in \mathrm{Ro}(X)$ and $M(\mathrm{Ro}(X), G) \models \phi_B[a]$, then $\mathrm{bd}(a) = \mathrm{hbd}(a)$, and hence if $x \in \mathrm{bd}(a)$, then $G(x) \subseteq \mathrm{bd}(a)$.*

(c) *If $a \in \mathrm{Ro}(X)$ and $\mathrm{bd}(a) = \mathrm{hbd}(a)$, then for every $x \in \mathrm{bd}(a) |G(x)| \leq 2$.*

(d) *$\mathrm{Bad}(X, G) \subseteq U^{[\leq 2]}(X, G)$, and if $\langle X, G\rangle \in K_{T^1}$, then $\mathrm{Bad}(X, G) = \emptyset$.*

PROOF. The proof of (a) is trivial.

(b) Let $M(\mathrm{Ro}(X), G) \models \phi_B[a]$, let $x \in \mathrm{bd}(a)$ and $g \in G$ be such that $g(x) \neq x$. Let c be a clopen neighborhood of x such that $g(c) \cdot c = 0$. By the B-closedness of $\langle X, G\rangle$ it can be assumed that $g^2 = Id$ and $\mathrm{var}(g) = c + g(c)$. Since $M(\mathrm{Ro}(X), G) \models \phi'_B[a]$ there is $h_1 \in G$ such that $h_1 \supseteq g \restriction (a \cdot g(a)) \cup Id \restriction -(a \cdot g(a))$. Let $b^1 = c \cdot h_1(c)$. Since $h_1 \in G \subseteq H(X)$ $b^1 \in \mathrm{Clop}(X)$. Since $h_1 \restriction -a = Id \restriction -a$, $b^1 - a = c - a$. Hence $x \in \mathrm{bd}(b^1 - a)$ and since b^1 is clopen, $x \in b^1$. We wish to show that $g(b^1 \cdot a) \leq -a$.

$$a \cdot g(b^1 \cdot a) = a \cdot g(c \cdot h_1(c) \cdot a) = a \cdot g(a) \cdot g(c) \cdot gh_1(c)\,.$$

$gh_1 \restriction (a \cdot g(a)) = Id \restriction (a \cdot g(a))$, hence

$$\begin{aligned} a \cdot g(a) \cdot g(c) \cdot gh_1(c) &= a \cdot g(a) \cdot g(c) \cdot (gh_1(c - a) + gh_1(c \cdot a)) \\ &= a \cdot g(a) \cdot g(c) \cdot (g(c - a) + c \cdot a) = 0\,. \end{aligned}$$

Hence, $g(b^1 \cdot a) \leq -a$.

By letting $-a$ take the role of a and b^1 take the role of c and repeating the above argument, we obtain $b \in \mathrm{Clop}(X)$ such that $b \leq b^1$, $b \ni x$ and $g(b - a) \leq a$. It thus follows that $b \in \mathrm{Clop}(X)$, $x \in b$, $g(b \cdot a) = g(b) - a$ and $g(b - a) = g(b) \cdot a$. Hence, b shows that $x \in \mathrm{hbd}(a)$.

(c) Let $a \in \mathrm{Ro}(X)$, $x \in \mathrm{hbd}(a)$ and suppose by contradiction that $|G(x)| \geq 3$. Let $g_1, g_2 \in G$ be such that $x \neq g_1(x) \neq g_2(x) \neq x$. By the definition of $\mathrm{hbd}(a)$ there are neighborhoods of x, $b_1, b_2 \in \mathrm{Clop}(X)$, such that $g_i(b_i \cdot a) = g_i(b_i) - a$ and $g_i(b_i - a) = g_i(b_i) \cdot a$, $i = 1, 2$. $g_1(x) \in \mathrm{bd}(a) = \mathrm{hbd}(a)$. $g_1(b_1 \cdot b_2) \stackrel{\text{def}}{=} b_3$ is a clopen neighborhood of $g_1(x)$, $g_2 g_1^{-1}(g_1(x)) \neq g_1(x)$, $g_2 g_1^{-1}(b_3 \cdot a) = g_2 g_1^{-1}(b_3) \cdot a$ and $g_2 g_1^{-1}(b_3 - a) = g_2 g_1^{-1}(b_3) - a$. Hence, there is no clopen neighborhood b of $g_1(x)$ such that $g_2 g_1^{-1}(b \cdot a) = g_2 g_1^{-1}(b) - a$. This contradicts the fact that $g_1(x) \in \mathrm{hbd}(a)$, hence (c) is proved.

(d) follows trivially from (b) and (c). □

REMARK. The converse of 4.7(b) is true if $\langle X, G \rangle$ is assumed to be closed in the following stronger sense: (1) it is B-closed; and (2) for every $g \in G$ and $b \in \mathrm{Ro}(X)$ if $g(b) = b$ and $g \restriction \mathrm{bd}(b) = Id \restriction \mathrm{bd}(b)$, then $g \restriction b \cup Id(X - b) \in G$.

It follows from (d) and the definition of K_{T^2} that for every $\langle X, G \rangle \in K_{T^1} \cup K_{T^2}$ $\mathrm{cl}(\mathrm{Bad}(X, G)) \subseteq U^{[2]}(X, G)$. Hence, if we define $K_{T^*} = \{\langle X, G \rangle \in K_{T^-} \mid \mathrm{cl}(\mathrm{Bad}(X, G)) \subseteq U^{[2]}(X, G)\}$ and $K_{B^*} = \{\langle B, G \rangle \mid \langle \mathrm{Ult}(B), G \rangle \in K_{T^*}\}$, then $K_B \subseteq K_{B^*}$.

We shall prove that K_{B^*} is faithful, this will imply that K_B is faithful.

Let $g \in G$ and $g^2 = Id$, and let $a \in \mathrm{Ro}(X)$. a is called a transpose of g if $a \cdot g(a) = 0$ and $a + g(a) = \mathrm{var}(g)$.

4.8. LEMMA. *Let $\langle X, G \rangle$ be B-closed and* $\mathrm{cl}(\mathrm{Bad}(X, G)) \subseteq U^{[2]}(X, G)$. *Then there are $g \in G$ and $a \in \mathrm{Clop}(X)$ such that* (1) $g^2 = Id$; (2) *a is a transpose of g; and* (3) $\mathrm{Bad}(X, G) \subseteq \mathrm{var}(g)$.

PROOF. Since $\mathrm{cl}(\mathrm{Bad}(X, G)) \subseteq U^{[2]}(X, G)$ and since $\langle X, G \rangle$ is B-closed for every $x \in \mathrm{cl}(\mathrm{Bad}(X, G))$, there are $g_x \in G$ and $a_x \in \mathrm{Clop}(X)$ such that $x \in a_x$, $g_x^2 = Id$ and a_x is a transpose of g_x. Since $\mathrm{cl}(\mathrm{Bad}(X, G))$ is compact, there are $x_1, \ldots, x_n$ such that $\bigcup_{i=1}^{n} a_{x_i} \supseteq \mathrm{Bad}(X, G)$. Denote $g_i = g_{x_i}$. We define by induction on $i \leq n$, $h_i \in G$. $h_0 = Id$. Suppose h_i has been defined and let

$$h_{i+1} = h_i \restriction \mathrm{var}(h_i) \cup g_{i+1} \restriction \{x \in X \mid x, g_{i+1}(x) \not\in \mathrm{var}(h_i)\}\,.$$

It is easy to see that for every i, $h_i \in G$, $h_i^2 = Id$ and h_i has a transpose belonging to $\mathrm{Clop}(X)$. We show that if $x \in \mathrm{var}(g_i) \cap U^{[2]}(X, G)$, then $x \in \mathrm{var}(h_i)$. Suppose $x \not\in \mathrm{var}(h_{i-1})$, then $g_i(x) \not\in \mathrm{var}(h_{i-1})$, for if $g_i(x) \in \mathrm{var}(h_{i-1})$, then $h_{i-1}(g_i(x)) \neq g_i(x)$, hence $x \in U^{[\geq 3]}(X, G)$, a contradiction. Hence, $g_i(x) \not\in \mathrm{var}(h_{i-1})$, so $x \in \mathrm{var}(h_i)$. Let $g = h_n$ and $a = a_n$, then g and a are as required. □

Let $\phi_B^+(v) \equiv \forall u(\phi_B(u) \to \phi_B(v \cdot u))$ and $\phi_B^*(v) \equiv \phi_B^+(v) \wedge (\forall u \leq v)(\phi_B(u) \to \phi_B^+(u))$. □

4.9. LEMMA. *Let $\langle X, G \rangle$ be B-closed, and for every $b \in \mathrm{Ro}(X) - \mathrm{Clop}(X)$* $\mathrm{bd}(b) \cap U^{[\geq 2]}(X, G) \neq \emptyset$. *Then*

(a) *For every $a \in \mathrm{Ro}(X)$, $M(\mathrm{Ro}(X), G) \models \phi_B^+[a]$ iff $a \in \mathrm{Clop}(X)$ and for every $x \in a \cap \mathrm{Bad}(X, G)$, $G(x) \subseteq a$.*

(b) *For every* $a \in \mathrm{Ro}(X)$, $M(\mathrm{Ro}(X), G) \models \phi^*_B[a]$ *iff* $a \in \mathrm{Clop}(X)$ *and* a *is disjoint from* $\mathrm{Bad}(X, G)$.

PROOF. (a) Let $M(\mathrm{Ro}(X), G) \models \phi^+_B[a]$. Suppose by contradiction that a is not clopen. Let $x \in \mathrm{bd}(a) \cap U^{[\geq 2]}(X, G)$ and let $y \in G(x) - \{x\}$. Let b be a clopen neighborhood of x not containing y. Then by 4.7, $M(\mathrm{Ro}(X), G) \models \phi_B[b]$ but $M(\mathrm{Ro}(X), G) \not\models \phi_B(a \cdot b)$. A contradiction; hence, a is clopen.

Suppose by contradiction that $x \in a \cap \mathrm{Bad}(X, G)$ and $y \in G(x) - a$. Let $b \in \mathrm{Ro}(X)$, $M(\mathrm{Ro}(X), G) \models \phi_B[b]$ and $x \in \mathrm{bd}(b)$. Since $y \not\in \mathrm{bd}(a \cdot b)$ by Lemma 4.7(b), $M(\mathrm{Ro}(X), G) \not\models \phi_B[a \cdot b]$.

Next we assume that $a \in \mathrm{Clop}(X)$ and for every $x \in a \cap \mathrm{Bad}(X, G)$, $G(x) \subseteq a$, and prove that $M(\mathrm{Ro}(X), G) \models \phi^+_B[a]$. Let $b \in \mathrm{Ro}(X)$, $g \in \mathrm{Aut}(\mathrm{Ro}(X))$, and $g^2 = Id$. Regarding g as a homeomorphism of $\mathrm{Ult}(\mathrm{Ro}(X))$, the following identity holds. Let

$$g_1 = g \restriction b \cdot g(b) \cup Id \restriction -(b \cdot g(b)),$$

then

$$\begin{aligned} h &\overset{\text{def}}{=} g \restriction ((a \cdot b) \cdot g(a \cdot b)) \cup Id \restriction -(a \cdot b \cdot g(a \cdot b)) \\ &= g_1 \restriction (a \cdot g_1(a)) \cup Id \restriction -(a \cdot g_1(a)). \end{aligned}$$

Suppose now that $M(\mathrm{Ro}(X), G) \models \phi'_B[b]$ and $g \in G$. Then $g_1 \in G$, and thus by the B-closedness of $\langle X, G \rangle$, $h \in G$. Hence, $M(\mathrm{Ro}(X), G) \models \phi'_B[a \cdot b]$.

Suppose now that $M(\mathrm{Ro}(X), G) \models \phi_B[b]$ and we wish to show that $M(\mathrm{Ro}(X), G) \models \phi'_B[-(a \cdot b)]$. Let $g \in G$ and $g^2 = Id$. Let $c = (-(a \cdot b)) \cdot g(-(a \cdot b))$; we wish to show that $f \overset{\text{def}}{=} g \restriction c \cup Id \restriction (-c) \in G$. Let $a_1 = -a$ and $b_1 = -b$. Hence,

$$\begin{aligned} c &= (a_1 + b_1) \cdot g(a_1 + b_1) \\ &= a_1 \cdot g(a_1) + a_1 \cdot g(b_1) + b_1 \cdot g(a_1) + b_1 \cdot g(b_1). \end{aligned}$$

Since a_1 is clopen, $\mathrm{bd}(c) \subseteq \mathrm{bd}(b_1) \cup \mathrm{bd}(g(b_1))$, and since b_1 satisfies ϕ_B, $\mathrm{bd}(b_1) \cup \mathrm{bd}(g(b_1)) \subseteq \mathrm{Bad}(X, G)$. Hence, $\mathrm{bd}(c) \subseteq \mathrm{Bad}(X, G)$. Thus, for every $x \in \mathrm{bd}(c)$ $x \in a_1 \Leftrightarrow x \in g(a_1)$. Since $\mathrm{bd}(c) \cap (a_1 \cap g(a_1)) = \emptyset$, it follows that $\mathrm{bd}(c) \cap (a_1 \cup g(a_1)) = \emptyset$. Let $d = b_1 \cdot g(b_1)$ and $e = -a_1 \cdot -g(a_1)$. Hence, e is clopen, $g(e) = e$, $c \cdot e = d \cdot e$ and $c \cdot -e$ is clopen.

$$\begin{aligned} f &= g \restriction (c \cdot e) \cup g \restriction (c - e) \cup Id \restriction (-c \cdot e) \cup Id \restriction (-c \cdot -e) \\ &= (g \restriction (d \cdot e) \cup Id \restriction -(d \cdot e)) \circ (g \restriction (c - e) \cup Id \restriction -(c - e)). \end{aligned}$$

Since b_1 satisfies ϕ'_B, $h \overset{\text{def}}{=} g \restriction d \cup Id \restriction -d \in G$. e is clopen and $h(e) = e$; hence, by the B-closedness of G, $h_1 \overset{\text{def}}{=} h \restriction e \cup Id \restriction (-e) \in G$. But $h_1 = g \restriction (d \cdot e) \cup Id \restriction -(d \cdot e)$, so $g \restriction (d \cdot e) \cup Id \restriction -(d \cdot e) \in G$.

$c - e$ is clopen and $g(c - e) = c - e$. Hence, by the B-closedness of G, $g \restriction (c - e) \cup Id \restriction -(c - e) \in G$, thus $f \in G$. This concludes the proof of (a).

(b) It is trivial that a clopen set disjoint from $\mathrm{Bad}(X, G)$ satisfies ϕ_B^*. Suppose that $M(\mathrm{Ro}(X), G) \models \phi_B^*[a]$. Clearly, a is clopen. Suppose by contradiction that $x \in a \cap \mathrm{Bad}(X, G)$. Since $I^{[\geq 2]}(X, G)$ intersects the boundary of every non-clopen regular open set, it can be assumed that $|G(x)| \geq 2$. Let $b \subseteq a$ be a clopen set which contains x but $G(x) \not\subseteq b$. Hence, b satisfies ϕ_B but not ϕ_B^+. A contradiction, hence (b) is proved. □

Let $\chi(g, v)$ be the formula in the language of $M(\mathrm{Ro}(X), G)$ which is the conjunction of the following statements: (1) $g^2 = Id$; (2) v is a transpose of g; (3) $\phi_B(v)$; and (4) $\phi_B^*(-\mathrm{var}(g))$.

4.10. DEFINITION. $\langle X, G\rangle$ is *strongly closed* if it is B-closed and for every $a \in \mathrm{Ro}(X)$ and $g \in G$ if $g(a) = a$ and $g \restriction \mathrm{bd}(a) = Id \restriction \mathrm{bd}(a)$, then $g \restriction a \cup Id \restriction (X - a) \in G$.

4.11. LEMMA. *Let $\langle B, G\rangle \in K_{B^*}$ and $X = \mathrm{Ult}(B)$. Then*:

(a) *There are $g \in G$ and $a \in \mathrm{Ro}(X)$ such that $M(\mathrm{Ro}(X), G) \models \chi[g, a]$.*

(b) *If $g \in G$ and there is a such that $\chi(g, a)$ holds, then there is $a \in \mathrm{Clop}(X)$ such that $\chi(g, a)$ holds.*

(c) *If $M(\mathrm{Ro}(X), G) \models \chi[g, a]$ and $c = \mathrm{var}(g)$, then $B_{g,a} \overset{\mathrm{def}}{=} \{b \in \mathrm{Ro}(X) \mid M(\mathrm{Ro}(X), G) \models \phi_B[b \cdot a] \wedge \phi_B[b \cdot (c - a)] \wedge \phi_B[b - c]\}$ is a Boolean algebra isomorphic to B, and $G \subseteq \mathrm{Aut}(B_{g,a})$. If, in addition, $\langle B, G\rangle$ is strongly closed, then $M(B_{g,a}, G) \cong M(B, G)$.*

PROOF. (a) follows from 4.8, 4.7 and 4.9.

(b) Suppose that $\chi(g, a)$ holds. If $a \in \mathrm{Clop}(X)$, there is nothing to prove. Suppose otherwise. We show that for every $x \in \mathrm{bd}(a)$, $g(x) \neq x$. Since a satisfies ϕ_B, $\mathrm{bd}(a) \subseteq \mathrm{Bad}(X, G) \subseteq U^{[2]}(X, G)$. Suppose by contradiction that $x \in \mathrm{bd}(a)$ and $g(x) = x$. Let $h \in G$ be such that $h(x) \neq x$. Since a satisfies ϕ_B, there is $c \in \mathrm{Clop}(X)$ such that $x \in c$, $h(c \cdot a) = h(c) - a$ and $h(c - a) = h(c) \cdot a$. Since $g(x) = x$, it can be assumed that $g(c) = c$. Since $g(a) \cdot a = 0$, $g(c \cdot a) = g(c) - a$ and $g(c - a) = g(c) \cdot a$. But then $hg(c \cdot a) = h(g(c) - a) = h(c - a) = h(c) \cdot a$. This contradicts the fact that a satisfies ϕ_B, and hence for every $x \in \mathrm{bd}(a)$, $g(x) \neq x$. It follows that for every $x \in \mathrm{var}(g)$, $g(x) \neq x$. Since $-\mathrm{var}(g)$ satisfies ϕ_B^*, $\mathrm{var}(g)$ is clopen. Using the compactness of $\mathrm{var}(g)$ and the fact that for every $x \in \mathrm{var}(g)$, $g(x) \neq x$, it is easy to find a clopen transpose d of g. It is easy to see that $M(\mathrm{Ro}(X), G) \models \chi[g, d]$.

(c) Let $M(\mathrm{Ro}(X), G) \models \chi[g, a]$.

Let b be a clopen transpose of g, and let $c = \mathrm{var}(g)$, $g(a - b) = b - a$ and hence $g(a \triangle b) = a \triangle b$. Let $h = g \restriction (a \triangle b) \cup Id \restriction -(a \triangle b)$. $h \in \mathrm{Aut}(\mathrm{Ro}(X))$, we shall show that $h(\mathrm{Clop}(X)) = B_{g,a}$, and this will imply that $B_{g,a} \cong B$. Note that $h(b) = a$. Let $d \in \mathrm{Clop}(X)$.

$$\begin{aligned} h(d) \cdot a = h(d) \cdot h(b) &= h(d \cdot b) = h(d \cdot b \cdot a) + h(d \cdot b - a) \\ &= d \cdot b \cdot a + g(d \cdot b - a) = d \cdot b \cdot a + (g(d \cdot b) \cdot g(-a)) \\ &= d \cdot b \cdot a + (g(d \cdot b) \cdot (-c + a)) = d \cdot b \cdot a + g(d \cdot b) \cdot a \\ &= (d \cdot b + g(d \cdot b)) \cdot a\,. \end{aligned}$$

$e \overset{\text{def}}{=} d \cdot b + g(d \cdot b)$ is clopen and it is easy to check that for every $x \in e \cap \operatorname{Bad}(X, G)$, $G(x) \subseteq e$. Hence, e satisfies ϕ_B^+, and since a satisfies ϕ_B, it follows that $h(d) \cdot a = e \cdot a$ satisfies ϕ_B. The same argument can be applied to $c - a$ and $c - b$, thus concluding that $h(d) \cdot (c - a)$ satisfies ϕ_B. $h(d) - c = d - c$ is clopen and thus satisfies ϕ_B. Hence, $h(d) \in B_{g,a}$. We have thus shown that $h(\operatorname{Clop}(X)) \subseteq B_{g,a}$.

Let $d \in \operatorname{Ro}(X) - \operatorname{Clop}(X)$ and we show that $h(d) \not\in B_{g,a}$. Either $d - c$ or $d \cdot b$ or $d \cdot (c - b)$ is not clopen. If $d - c$ is not clopen, then $h(d) - c = d - c$ is not clopen. Since $\operatorname{Bad}(X, G) \cap (h(d) - c) = \emptyset$, it follows that $h(d) - c$ does not satisfy ϕ_B. Suppose that $d \cdot b$ is not clopen and we show that $e \overset{\text{def}}{=} -(h(d) \cdot a)$ does not satisfy ϕ'_B. It is easy to see that $e \cdot g(e) = -(d \cdot b + g(d \cdot b))$. But then $g \restriction (e \cdot g(e)) \cup Id \restriction -(e \cdot g(e)) \not\in G$, since it moves the clopen set b to the non-clopen set $d \cdot b + g(b - d \cdot b)$. This implies that $-(h(d) \cdot a)$ does not satisfy ϕ'_B. The same argument applied to $c - a$ and $c - b$ shows that if $d - b$ is not clopen, then $-(h(d) - a)$ does not satisfy ϕ'_B. Hence, $h(\operatorname{Ro}(X) - \operatorname{Clop}(X)) \subseteq \operatorname{Ro}(X) - B_{g,a}$. This implies that h is an isomorphism between $\operatorname{Clop}(X)$ and $B_{g,a}$.

We leave it to the reader to verify that $G \subseteq \operatorname{Aut}(B_{g,a})$, and that if $\langle X, G \rangle$ is strongly closed, then the automorphism h of $\operatorname{Ro}(X)$ defined above normalizes G, that is $G^h = G$. This means that h is an isomorphism between $M(B, G)$ and $M(B_{g,a}, G)$. Q.E.D. □

Proof of Theorem 4.5. (a) Since $K_B \subseteq K_{B^*}$, it suffices to show that K_{B^*} is faithful. Let $\langle B_1, G_1 \rangle$, $\langle B_2, G_2 \rangle \in K_{B^*}$ and $G_1 \cong G_2$. Let $X_i = \operatorname{Ult}(B_i)$. By Lemma 3.4(b), $M(\operatorname{Ro}(X_1), G_1) \simeq M(\operatorname{Ro}(X_2), G_2)$. Let τ denote the above isomorphism. Let $M(\operatorname{Ro}(X_1), G_1) \models \chi[g_1, a_1]$, $g_2 = \tau(g_1)$ and $a_2 = \tau(a_1)$. Hence, $\tau(G_{a_1, g_1}) = G_{a_2, g_2}$, and thus $B_1 \cong B_{a_1, g_1} \cong B_{a_2, g_2} \cong B_2$.

(b) Let $\langle B, G \rangle \in K_{B^1}$. By 3.4(b), $M(\bar{B}, G)$ is strongly interpretable in G. By 4.7 for every $a \in \bar{B}$, $M(\bar{B}, G) \models \phi_B[a]$ iff $a \in B$; hence B is definable in $M(\bar{B}, G)$. This implies that $M(B, G)$ is strongly interpretable in $M(\bar{B}, G)$. So $K_{B^1}^{BG}$ is strongly interpretable in $K_{B^1}^{G}$.

We next prove that every member of K_{B^1} is group-categorical in K_{B^3}. As a first step we use Theorem 3.4(c). $\{\langle \bar{B}, G \rangle \mid \langle B, G \rangle \in K_{B^1}\} \subseteq K_{C^-}$ and $\{\langle \bar{B}, G \rangle \mid \langle B, G \rangle \in K_{B^3}\} \subseteq K_D$, hence by 3.4(c) if

$$\langle B, G \rangle \in K_{B^1},\ \langle B_1, G_1 \rangle \in K_{B^3} \text{ and } G \overset{\phi}{\simeq} G_1\,,$$

then there is an isomorphism $\tilde{\phi}$ between $M(\bar{B}, G)$ and $M(\bar{B}_1, G_1)$ extending ϕ. Let us show that $\tilde{\phi} \restriction B$ is an isomorphism between B and B_1. Since $\langle B, G \rangle \in K_{B^1}$, by 4.7 $B = \{a \in \bar{B} \mid M(\bar{B}, G) \models \phi_B[a]\}$, hence $\tilde{\phi}(B) = \{a \in \bar{B}_1 \mid M(\bar{B}_1, G_1) \models \phi_B[a]\} \supseteq B_1$. Moreover, the fact that $M(\bar{B}_1, G_1) \cong M(\bar{B}, G)$ implies that $\{a \in \bar{B}_1 \mid M(\bar{B}_1, G_1) \models \phi_B[a]\}$ is a subalgebra of $\bar{B}_1$.

Let $X_1 = \operatorname{Ult}(B_1)$. Hence, $M(\operatorname{Ro}(X_1), G_1) \cong M(\bar{B}_1, G_1)$. Let $a \in \operatorname{Ro}(X_1)$ be such that $M(\operatorname{Ro}(X_1), G_1) \models \phi_B[a]$, and suppose by contradiction that $a \in \operatorname{Clop}(X_1)$. Since $\langle B_1, G_1 \rangle \in K_{T^3}$, $\operatorname{bd}(a) \cap U^{[\geq 2]}(X_1, G_1) \neq \emptyset$. Hence, let $x \in \operatorname{bd}(a)$ and $y \in G_1(x) - \{x\}$. Let b be a clopen neighborhood of x not containing y. Then $M(\operatorname{Ro}(X_1), G_1) \models \phi_B[b]$, but by 4.7(b), $M(\operatorname{Ro}(X_1), G_1) \not\models \phi_B[a \cdot b]$. This contradicts the fact that $\{a \in \bar{B}_1 \mid M(\bar{B}_1, G_1) \models \phi_B[a]\}$ is a subalgebra of $\bar{B}_1$. Hence a is clopen. We have thus proved that $\{a \in \bar{B}_1 \mid M(\bar{B}_1, G_1) \models \phi_B[a]\} = B_1$, so $\tilde{\phi} \restriction B$ is an isomorphism between B and B_1.

REMARK. It is not true that every member of K_{B^1} is group-categorical in the class $\{\langle B, G\rangle \mid \langle \mathrm{Ult}(B), G\rangle$ is B-closed and $U^{[\geq 2]}(\mathrm{Ult}(B), G)$ is dense in $\mathrm{Ult}(B)\}$. So a strengthening of 4.5(c) in this direction is impossible.

(c) Let $\langle B, G\rangle \in K_{B^0}$. Then it is easy to see that

$$B \subseteq \{\mathrm{var}(f_1)\cdot \mathrm{var}(f_2) + \mathrm{var}(g) \mid f_1, f_2, g \in G \text{ and } \langle f_1, f_2\rangle \text{ is a representative}\} \stackrel{\mathrm{def}}{=} D_1(B, G)\,.$$

Let $M_1(B, G) = \langle D_1(B, G), G; Op\rangle$. It follows from the proof of 3.4 that $M_1(B, G)$ is strongly first-order interpretable in G. ϕ_B is a first-order formula and it is easy to see that for every $a \in D_1(B, G)$, $M_1(B, G) \models \phi_B[a]$ iff $a \in B$. This implies that $M(B, G)$ is strongly first-order interpretable in G. Thus (c) is proved. □

REMARK. It is not true that $K^{BG}_{B^*}$ is interpretable in $K^{G}_{B^*}$. However, if K denotes the class consisting of all members of K_{B^*} which are either strongly closed or belong to K_{B^1}, then K^{BG} is interpretable in K^G.

In the discussion following Theorem 4.5 we considered the following class. $K_{B'} \stackrel{\mathrm{def}}{=} \{\langle B, G\rangle \mid \langle \mathrm{Ult}(B), G\rangle$ is B-closed, $U^{[\geq 3]}(\mathrm{Ult}(B), G)$ is dense in $\mathrm{Ult}(B)$ and $I^{[\geq 2]}(B, G)$ is indecomposable$\}$. We conclude this section with an example showing that $K_{B'}$, is not faithful.

4.12. EXAMPLE. Let $X = \{0, 1\}^{\aleph_1}$, let $\{x_i\}_{i\in\omega}$ be a convergent sequence in X, and denote $x_\omega = \lim_{i\in\omega} x_i$. Let $g \in H(X)$ be such that x_ω is the only fixed point of g and $g^2 = \mathrm{Id}$. W.l.o.g. $g(x_{2i}) = x_{2i+1}$. For $l \in \{0, 1\}$ let $X^l = \{l\} \times X$, and for every $x \in X$ let $x^l = \langle l, x\rangle$. Let $X^* = X^0 \cup X^1$. Now we define spaces X_0 and X_1 by identifying points of X^*. Let $X_0 = \{\{x\} \mid x \in X^*$ and for every $i < \omega$ and $l \in \{0, 1\}$ $x \neq x^l_i\} \cup \{\{x^l_{2i}, x^l_{2i+1}\} \mid i < \omega, l \in \{0, 1\}\}$. $V \subseteq X_0$ is open in X_0 if $\bigcup V$ is open in X^*.

Let $X_1 = \{\{x\} \mid x \in X^*$ and for every $i \leq \omega$ and $l \in \{0, 1\}$ $x \neq x^l_i\} \cup \{\{x^0_i, x^1_i\} \mid i \leq \omega\}$. $V \subseteq X_1$ is open in X_1 if $\bigcup V$ is open in X^*.

We first show that X_0 and X_1 are non-homeomorphic. Let $V = \{\{x\} \mid x \in X^0 - \{x^0_i \mid i \leq \omega\}\}$. Clearly, $V \in \mathrm{Ro}(X_1)$ and $\mathrm{bd}(V) = \{\{x^0_i, x^1_i\} \mid i \leq \omega\}$. Hence, in X_1 there is a regular open set whose boundary is countable. It is not difficult to see that the boundary of every regular open subset of X_0 is either finite or has cardinality $2^{\aleph_1}$. Hence, X_0 and X_1 are non-homeomorphic. If $h \in H(X^*)$, $l \in \{0, 1\}$ and for every $x \in X_l$ $h(x) \in X_l$, then h induces a homeomorphism of X_l which we denote by $h^{[l]}$. Let $k\colon X^* \to X^*$ be defined as follows: $k(\langle l, x\rangle) = \langle 1 - l, g(x)\rangle$.

Let $V \subseteq X^*$, V is said to be good if it is clopen, $k(V) = V$ and for every $x \in X$, $x^0 \in V$ iff $x^1 \in V$. Let $G = \{f \in H(X^*) \mid$ there are V, W such that V and W are good, $V \cup W \supseteq \{x^l_i \mid l \in \{0, 1\}, i \leq \omega\}, f \restriction V = k \restriction V$ and $f \restriction W = Id \restriction W\}$. It is easy to check that for every $f \in G$, $l \in \{0, 1\}$ and $x \in X_l$, $f(x) \in X_l$. Let $G_l = \{f^{[l]} \mid f \in G\}$. We regard G_l as a subgroup of $\mathrm{Aut}(\mathrm{Clop}(X_l))$, and show that $\langle \mathrm{Clop}(X_l), G_l\rangle \in K_{B'}$.

It is easy to see that for $l = 1, 2$,

$$U^{[\geq 3]}(X_l, G_l) = \{\{x^t\} \mid x \in X - \{x_i \mid i \leq \omega\}, t = 0, 1\}\,.$$

Hence, $U^{[\geq 3]}(X_l, G_l)$ is dense in X_l. For every $x \in X_0$ $|G_0(x)| \geq 2$, hence $I^{[\geq 2]}(\mathrm{Clop}(X_0), G_0)$ is indecomposable. For every $x \in X_1 - \{\{x_\omega^0, x_\omega^1\}\}$, $|G_1(x)| \geq 2$. Since $\{\{x_\omega^0, x_\omega^1\}\}$ is not the boundary of any regular open set in X_1, $I^{[\geq 2]}(\mathrm{Clop}(X_1), G_1)$ is indecomposable.

It remains to show that $\langle X_l, G_l \rangle$ is B-closed. For $U \subseteq X_l$ let $U^* = \bigcup U$, hence $U^* \subseteq X^*$. Let $Y = \{x_i^l \mid l \in \{0,1\}, i \leq \omega\}$. It is easy to see that if $U \in \mathrm{Clop}(X_l)$ and $k(U^*) = U^*$, then there is a good $V \subseteq U^*$ such that $V \cap Y = U^* \cap Y$. Let $f^{[l]} \in G_l$, $u \in \mathrm{Clop}(X_l)$ and $f^{[l]}(U) = U$. We wish to show that $h \stackrel{\mathrm{def}}{=} f^{[l]} \restriction U \cup Id \restriction (X_l - U) \in G_l$. Let $h_0 = f \restriction U^* \cup Id \restriction (X^* - U^*)$, then $h = h_0^{[l]}$. Thus, it suffices to show that $h_0 \in G$.

Let V, W be good subsets of X^* demonstrating the fact that $f \in G$. For some $V_1 \in \mathrm{Clop}(X_l)$, $V = V_1^*$. Hence, $V \cap U^* = (V_1 \cap U)^*$ and $V - U^* = (V_1 - U)^*$. $f \restriction (V \cap U^*) = k \restriction (V \cap U^*)$, hence $k(V \cap U^*) = V \cap U^*$. Similarly, $k(V - U^*) = V - U^*$. Hence, there are good $U_1 \subseteq V \cap U^*$ and $U_2 \subseteq V - U^*$ such that $Y \cap U_1 = Y \cap (V \cap U^*)$ and $Y \cap U_2 = Y \cap (V - U^*)$. Hence, $Y \subseteq U_1 \cup U_2 \cup W$, $h_0 \restriction U_1 = k \restriction U_1$ and $h_0 \restriction (U_2 \cup W) = Id \restriction (U_2 \cup W)$. Thus, $h_0 \in G$.

Let $f^{[l]} \in G_l$, $U \in \mathrm{Clop}(X_l)$ and $f^{[l]}(U) \cap U = \emptyset$. We wish to show that

$$h \stackrel{\mathrm{def}}{=} f^{[l]} \restriction U \cup (f^{[l]})^{-1} \restriction f^{[l]}(U) \cup Id \restriction (X_l - U - f^{[l]}(U)) \in G_l$$

$$f(U^*) \cap U^* = \emptyset .$$

Let

$$h_0 = f \restriction U^* \cup f^{-1} \restriction f(U^*) \cup Id \restriction (X^* - U^* - f(U^*)) .$$

$h = h_0^{[l]}$, hence it suffices to show that $h_0 \in G$.

Let V, W be good sets demonstrating the fact that $f \in G$. $V_1 \stackrel{\mathrm{def}}{=} (V \cap U^*) \cup k(V \cap U^*)$ and $V_2 \stackrel{\mathrm{def}}{=} (V - U^*) \cup k(V - U^*)$ are closed under k, and for some $T_i \in \mathrm{Clop}(X_l)$, $V_i = T_i^*$. Hence, there are good $U_i \subseteq V_i$ such that $U_i \cap Y = V_i \cap Y$. $h_0 \restriction U_1 = k \restriction U_1$, $h_0 \restriction (U_2 \cup W) = Id \restriction (U_2 \cup W)$ and $Y \subseteq U_1 \cup U_2 \cup W$. Hence, $h_0 \in G$.

In the above example let $B_i = \mathrm{Clop}(X_i)$, $i = 1, 2$. It is easy to see that $\langle B_0, G_0 \rangle \in K_{B^2}$ and that $\langle B_1, G_1 \rangle \in K_{B'}$. This shows that it is not true that every member of K_{B^2} is group-categorical in $K_{B'}$. Since $K_{B'} \subseteq K_{B^3}$ it follows that 4.5(b) cannot be strengthened to say that every member of K_B is group-categorical in K_{B^3}.

4.13. EXAMPLE. Let B_i, $i \leq 2$, be three complete rigid atomless BAs such that

$$B_0 \overset{h_1}{\cong} B_1 \overset{h_2}{\cong} B_2 ,$$

let $X_i = \mathrm{Ult}(B_i)$, and let $x_0 \in B_0$, $x_1 = h_1(x_0)$ and $x_2 = h_2(x_1)$. Let $X^0 = \bigcup_{i \leq 2} X_i$ and X^1 be obtained from X^0 by identifying x_0, x_1 and x_2. $X^0 \not\cong X^1$ but $H(X^0) \cong H(X^1)$. $\langle \mathrm{Clop}(X^0), H(X^0) \rangle \in K_{B^0}$ and $\langle \mathrm{Clop}(X^1), H(X^1) \rangle \in \{\langle B, G \rangle \mid \langle \mathrm{Ult}(B), G \rangle$ is B-closed and $U^{[\geq 2]}(\mathrm{Ult}(B), G)$ is dense in $\mathrm{Ult}(B)\}$. This example verifies the remark following the proof of 4.5(b).

Summary

We have considered the following classes:

$$K_{B^3} = \{\langle B, G\rangle \mid \langle B, G\rangle \text{ is } B\text{-closed and } I^{[\geq 2]}(B, G) \text{ is dense and indecomposable}\}\,.$$

$$K_{B^1} = \{\langle B, G\rangle \mid \langle B, G\rangle \text{ is } B\text{-closed, } a^{[6]}(B, G) = 0 \text{ and } I^{[\geq 3]}(B, G) \text{ is dense and indecomposable}\}\,.$$

$$K_{B^2} = \{\langle B, G\rangle \mid \langle B, G\rangle \text{ is } B\text{-closed, } a^{[6]}(B, G) = 0,\ I^{[\geq 3]}(B, G) \text{ is dense and } I^{[\geq 2]}(B, G) = B\}\,.$$

$$K_{B^0} = \{\langle B, G\rangle \mid \langle B, G\rangle \text{ is } B\text{-closed, } a^{[6]}(B, G) = 0 \text{ and } I^{[\geq 3]}(B, G) = B\}\,.$$

In the examples and remarks we have considered the following additional classes: in 4.12:

$$K_{B'} = \{\langle B, G\rangle \mid \langle B, G\rangle \text{ is } B\text{-closed, } a^{[6]}(B, G) = 0,\ I^{[\geq 3]}(B, G) \text{ is dense and } I^{[\geq 2]}(B, G) \text{ is indecomposable}\}\,;$$

in 4.13:

$$K_{B^4} \stackrel{\text{def}}{=} \{\langle B, G\rangle \mid \langle B, G\rangle \text{ is } B\text{-closed and } I^{[\geq 2]}(B, G) \text{ is dense}\}\,;$$

and in the remark following the proof of 4.5(c) we considered the class

$$K_{B^{2+}} \stackrel{\text{def}}{=} \{\langle B, G\} \mid \langle B, G\rangle \text{ is strongly closed, } a^{[6]}(B, G) = 0,\ I^{[\geq 3]}(B, G) \text{ is dense and } I^{[\geq 2]}(B, G) = B\}\,.$$

The containment relations among the above classes are as follows: $K_{B^0} \subseteq K_{B^1} \cap K_{B^2}$, $K_{B^1} \cup K_{B^2} \subseteq K_{B'} \subseteq K_{B^3} \subseteq K_{B^4}$ and $\{\langle B, \mathrm{Aut}(B)\rangle \mid B \text{ is a } BA\} \cap K_{B^2} \subseteq K_{B^{2+}} \subseteq K_{B^2}$.

We have proved the following facts.

(1) $K_{B^1} \cup K_{B^2}$ is faithful.

(2) $K_{B^1}^{BG} \cup K_{B^2}^{BG}$ is interpretable in $K_{B^1}^{G} \cup K_{B^{2+}}^{G}$.

(3) $K_{B^1}^{BG}$ is strongly interpretable in $K_{B^1}^{G}$.

(4) $K_{B^0}^{BG}$ is strongly first-order interpretable in $K_{B^0}^{G}$.

(5) Every member of K_{B^1} is group-categorical in K_{B^3}.

(6) There are $\langle B_2, G_2\rangle \in K_{B^2}$ and $\langle B', G'\rangle \in K_{B'}$ such that $G_2 \cong G'$ but $B_2 \not\cong B'$. Hence, it is not true that every member of K_{B^2} is group-categorical in K_{B^3}.

(7) There are $\langle B_0, G_0\rangle \in K_{B^0}$ and $\langle B_4, G_4\rangle \in K_{B^4}$ such that $G_0 \cong G_4$ but $B_0 \not\cong B_4$. Indeed, G_i can be taken to be $\mathrm{Aut}(B_i)$, $i = 0, 4$. Hence, it is not true that every member of K_{B^0} is group-categorical in K_{B^4}.

5. Countable Boolean algebras

Faithfulness problems in the class of countable BAs were first considered by Monk [1975]. The countable atomless BAs, B_L and $B_L \times \{0,1\}$, have up to isomorphism the same automorphism group, hence in order to obtain a faithful class one of the above BAs has to be excluded. But this is not the only example of two non-isomorphic countable BAs that have the same automorphism group. McKenzie [1977], and independently Shelah, found the following example.

5.1. Example. Let B be a countable atomic BA in which there are two second-order definable ultrafilters, F_1 and F_2. Let F be an ultrafilter in B_L, and for $i = 1, 2$ let B_i be the subalgebra of $B \times B_L$ generated by $\{\langle a, 0\rangle \mid a \in B - F_i\} \cup \{\langle 0, a\rangle \mid a \in B_L - F\}$. It is easy to see that $B_1 \not\cong B_2$ but $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$.

It only remains to show that indeed there is a countable atomic BA which has two second-order definable ultrafilters.

Let B_{FC} denote the BA of all finite and cofinite subsets of ω. Let $\mathrm{At}(B)$ denote the set of atoms of B. Let I be a maximal ideal in B_L and $B_{[1]}$ be the subalgebra of $B_{\mathrm{FC}} \times B_L$ generated by $\{\langle a, 0\rangle \mid a \in \mathrm{At}(B_{\mathrm{FC}})\} \cup \{\langle 0, a\rangle \mid a \in I\}$. Let $I_{\mathrm{At}}(B)$ denote the ideal of B generated by $\mathrm{At}(B)$. Let B^1 be a countable atomic BA such that $B^1/I_{\mathrm{At}}(B^1) \cong B_{[1]}$, let B^2 be a countable BA such that $B^2/I_{\mathrm{At}}(B^2) \cong B_{\mathrm{FC}}$, and let $B = B^1 \times B^2$. Let

$$F_1 = \{a \in B \mid (B \restriction a)/I_{\mathrm{At}}(B \restriction a) \cong B_{[1]}\}$$

and

$$F_2 = \{a \in B \mid (B \restriction a)/I_{\mathrm{At}}(B \restriction a) \cong B_{FC}\}\,.$$

B, F_1 and F_2 are as required in the example.

The main result of this section is Theorem 5.2(b) stating that the class of countable BAs B that have a maximal atomic element and for which $|\mathrm{At}(B)| \neq 1$ is faithful. This theorem is due to McKenzie [1977].

An element $a \in B$ is *atomic* if $B \restriction a$ is an atomic BA, $a \in B$ is *atomless* if $B \restriction a$ is atomless. Let $\mathrm{As}(B)$ and $\mathrm{Al}(B)$ denote, respectively, the ideals of atomic and atomless elements of B. Let $I_{\mathrm{SL}}(B)$ denote the ideal generated by $\mathrm{As}(B) \cup \mathrm{Al}(B)$. Let $a^{\mathrm{AT}}(B) = \Sigma\, \mathrm{At}(B)$ and $a^{\mathrm{AL}}(B) = \Sigma\, \mathrm{Al}(B)$. Clearly, $a^{\mathrm{AT}}(B), a^{\mathrm{AL}}(B) \in \bar{B}$. Let B^{TL} be the subalgebra of $\bar{B}$ generated by $B \cup \{a^{\mathrm{AT}}(B)\}$, let $B^{\mathrm{AT}} = B^{\mathrm{TL}} \restriction a^{\mathrm{AT}}(B)$ and $B^{\mathrm{AL}} = B^{\mathrm{TL}} \restriction a^{\mathrm{AL}}(B)$.

It is easy to see that $B/I_{\mathrm{SL}}(B) \cong B^{\mathrm{AT}}/\mathrm{As}(B) \cong B^{\mathrm{AL}}/\mathrm{Al}(B)$, and that if B_1, B_2 are of power $\leq \aleph_0$ and $\langle B_1^{\mathrm{AT}}, \mathrm{As}(B_1)\rangle \cong \langle B_2^{\mathrm{AT}}, \mathrm{As}(B_2)\rangle$, then $B_1 \cong B_2$.

We can now state the main theorem of this section. Let $M^{\mathrm{TL}}(B) = \langle B^{\mathrm{TL}}, \mathrm{Aut}(B); \mathrm{Al}(B), \leq, Op\rangle$.

Let $K_{\mathrm{CL}} = \{B \mid |B| \leq \aleph_0$ and $|\mathrm{At}(B)| \neq 1\}$, let $K_{\mathrm{CL}^-} = \{B \in K_{\mathrm{CL}} \mid 2 \neq |\mathrm{At}(B)| \neq 6\}$. Recall that for a class K of BAs $K^G = \{\mathrm{Aut}(B) \mid B \in K\}$. Let $K^{\mathrm{TL}} = \{M^{\mathrm{TL}}(B) \mid B \in K\}$.

5.2. THEOREM. (a) *$K^{\mathrm{TL}}_{\mathrm{CL}}$ is first-order interpretable in K^{G}_{CL}; the above interpretation is strong when restricted to K_{CL^-}.*

(b) (MCKENZIE [1977]). *Let $K_{\mathrm{CLM}} = \{B \mid |B| \le \aleph_0$ and $a^{\mathrm{AT}}(B) \in B\}$, then K_{CLM} is faithful.*

PROOF. If $B \in K_{\mathrm{CLM}}$, then $B^{\mathrm{TL}} = B$, so B is definable in $M^{\mathrm{TL}}(B)$. Since $K_{\mathrm{CLM}} \subseteq K_{\mathrm{CL}}$, (b) follows trivially from (a). We concentrate on the proof of (a).

As a first step in the proof of (a) we wish to use Theorem 3.4. The class K_C was defined just before Theorem 3.4; and it is easy to see that $\{\langle \bar{B}, \mathrm{Aut}(B)\rangle \mid B \in K_{\mathrm{CL}}\} \subseteq K_C$. Hence, by Theorem 3.4 $K^{\bar{B}G}_{\mathrm{CL}} \stackrel{\mathrm{def}}{=} \{\langle \bar{B}, \mathrm{Aut}(B); \le, Op\rangle \mid B \in K_{\mathrm{CL}}\}$ is interpretable in $K^{G}_{\mathrm{CL}} \stackrel{\mathrm{def}}{=} \{\mathrm{Aut}(B) \mid B \in K_{\mathrm{CL}}\}$.

Since we wish to obtain a first-order interpretation, Theorem 3.4 is not exactly sufficient. Corollary 3.22 states that $\{\langle D(B, G); G; \le, Op\rangle \mid \langle B, G\rangle \in K_C\}$ is first-order interpretable in K^{G}_{C}, where $D(B, G)$ was defined before Corollary 3.22 as

$$(\{\mathrm{conv}(\mathrm{var}(f)) \mid f \in a^{[6]}(B, G)\})^6 \times \{\mathrm{var}(f)\cdot \mathrm{var}(g) + \mathrm{var}(h) \mid f, g \in G(a^{\mathrm{R}}(B, G) - a^{[6]}(B, G)), h \in G(a^{\mathrm{LM}}(B, G))\}\,.$$

For $B \in K_{\mathrm{CL}}$, $a^{\mathrm{R}}(\bar{B}, \mathrm{Aut}(B)) = a^{\mathrm{AT}}(B)$ and $a^{\mathrm{LM}}(\bar{B}, \mathrm{Aut}(B)) = a^{\mathrm{AL}}(B)$. We shall show that for $B \in K_{\mathrm{CL}}$, $D(\bar{B}, \mathrm{Aut}(B)) \cong \bar{B}$; it will thus follow from 3.22 that $K^{\bar{B}G}_{\mathrm{CL}}$ is first-order interpretable in K^{G}_{CL}, and that $K^{\bar{B}G}_{\mathrm{CL}^-}$ is strongly first-order interpretable in $K^{G}_{\mathrm{CL}^-}$.

If B is a BA such that: (1) $|\mathrm{At}(B)| \ne 1$ and (2) for every $a \in \bar{B}$ if $a \cdot a^{\mathrm{AT}}(B) \not\subseteq \mathrm{At}(B)$, then there is $f \in \mathrm{Aut}(B)$ such that $a = \mathrm{var}(f)$; *then* $D(\bar{B}, \mathrm{Aut}(B)) \cong \bar{B}$. We thus prove the following lemma.

5.3. LEMMA. *If $B \in K_{\mathrm{CL}}$, $a \in \bar{B}$, and $a \cdot a^{\mathrm{AT}}(B) \not\subseteq \mathrm{At}(B)$, then there is $f \in \mathrm{Aut}(B)$ such that* $\mathrm{var}(f) = a$.

PROOF. Let $I_0(B)$ denote the ideal generated by $\mathrm{At}(B) \cup \mathrm{Al}(B)$. We shall show that for every B and a as above there is $f \in \mathrm{Aut}(B)$ such that $a = \mathrm{var}(f)$ and for every $b \in B$, $b \,\triangle\, f(b) \in I_0(B)$.

Case 1. $a \le a^{\mathrm{AT}}(B)$ and $a \notin I_{\mathrm{AT}}(B)$. Let $\{e_n \mid n \in \omega\}$ be an enumeration of B. We define by induction functions f_n with the following properties: (1) $\mathrm{Dom}(f_n) = \mathrm{Rng}(f_n)$ is a finite subalgebra of B denoted by B_n and $f_n \in \mathrm{Aut}(B_n)$; (2) $f_n^2 = Id$; (3) $\mathrm{At}(B_n) \cap I_0(B) \subseteq \mathrm{At}(B) \cup \mathrm{Al}(B)$; (4) for every $b \in \mathrm{At}(B_n)$: $f_n(b) \ne b$ iff $b \le a$ and $b \in \mathrm{At}(B)$; (5) $\{e_i \mid i < n\} \subseteq B_n$; and (6) if $b \in \mathrm{At}(B_n) - \mathrm{At}(B)$, then $\{d \in \mathrm{At}(B) \mid d \le b \cdot a\}$ is either infinite or empty.

If $a^{\mathrm{AT}}(B) \in B$ let

$$f_0 = \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle a^{\mathrm{AT}}(B), a^{\mathrm{AT}}(B)\rangle, \langle a^{\mathrm{AL}}(B), a^{\mathrm{AL}}(B)\rangle\}\,,$$

otherwise let $f_0 = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$. Suppose f_n has been defined. Let $\{c_0, \ldots, c_{k-1}\}$ be an enumeration of $\{e_n \cdot b \mid b \in \mathrm{At}(B_n)$ and $0 \ne e_n \cdot b \ne b\}$. We define by induction on $i \le k$ functions $f_{n,i}$ satisfying (1)–(5). Let $B_{n,i}$ denote

$\mathrm{Dom}(f_{n,i})$. Let $f_{n,0} = f_n$. Suppose $f_{n,i}$ has been defined. Let $b \in \mathrm{At}(B_n)$ and $c_i \leq b$. If the subalgebra of $\bar{B} \restriction b$ generated by $\{c_i, a^{\mathrm{AT}}(B) \cdot b, a \cdot b\}$ does not intersect $I_{\mathrm{At}}(B)$, then we define $B_{n,i+1}$ to be the intersection of B with the subalgebra of $\bar{B}$ generated by $B_{n,i} \cup \{c_i, a^{\mathrm{AT}}(B)\}$, and we define $f_{n,i+1}$ to be the automorphism of $B_{n,i+1}$ extending $f_{n,i}$ and such that for every $d \in B_{n,i+1} \restriction b$, $f_{n,i+1}(d) = d$. Otherwise, there is a finite set $T \subseteq \mathrm{At}(B) \cap B \restriction b$ and a set $L \subseteq \mathrm{Al}(B) \cap B \restriction b$ with at most one element, such that the subalgebra B' of B generated by $B_{n,i} \cup T \cup L$ contains c_i and satisfies requirement (6) in the induction hypothesis. W.l.o.g. $|T \cap \{d \in \mathrm{At}(B) \mid d \leq a\}|$ is even, so let $T = \{d_0, \ldots, d_{2l-1}, \ldots, d_m\}$, where $d_0, \ldots, d_{2l-1} \leq a$ and $d_{2l}, \ldots, d_m \leq -a$. Let $B_{n,i+1} = B'$ and $f_{n,i+1}$ be the automorphism of $B_{n,i+1}$ extending $f_{n,i} \cup \{\langle d_{2j}, d_{2j+1}\rangle \mid j < l\} \cup Id \restriction (L \cup \{d_{2l}, \ldots, d_m\})$. It is easy to see that $f_{n,i+1}$ satisfies requirements (1)–(6).

Let $f_{n+1} = f_{n,k}$. It is easy to see that f_{n+1} satisfies the induction hypotheses and that $f \stackrel{\mathrm{def}}{=} \bigcup_{n \in \omega} f_n$ is as required.

Case 2. $a \in I_{\mathrm{At}}(B)$. Let $a = \Sigma_{i<k} d_i$, where $d_0, \ldots, d_{k-1} \in \mathrm{At}(B)$. Let $f \in \mathrm{Aut}(B)$ be such that $f \restriction -a = Id$, for every $i < k-1$, $f(d_i) = d_{i+1}$ and $f(d_{k-1}) = d_0$. Then f is as required.

Case 3. $a \leq a^{\mathrm{AL}}(B)$. The proof is similar to the proof of Case 1. We define by induction functions f_n satisfying (1), (2), (3) and (5) and (4)$'$ for every $b \in \mathrm{At}(B_n)$, $f_n(b) \neq b$ iff $b \leq a$ and $b \in \mathrm{Al}(B)$.

The construction of the f_n's is simpler than in Case 1 and $f \stackrel{\mathrm{def}}{=} \bigcup_{n\in\omega} f_n$ is as required.

Case 4. The general case. Let $a_1 = a \cdot a^{\mathrm{AT}}(B)$ and $a_2 = a \cdot a^{\mathrm{AL}}(B)$. By Cases 1 or 2 there is $f \in \mathrm{Aut}(B)$ such that $a_1 = \mathrm{var}(f)$, by Case 3 there is $g \in \mathrm{Aut}(B)$ such that $a_2 = \mathrm{var}(g)$. $\mathrm{var}(g \circ f) = a$. □

It follows from Lemmas 3.22 and 5.3 that $K_{\mathrm{CL}}^{\bar{B}G}$ is first-order interpretable in K_{CL}^{G}.

Our next goal is to show that if $B \in K_{\mathrm{CL}}$, then $\mathrm{Al}(B)$ is a definable subset of $M^{\bar{B}G}(B) \stackrel{\mathrm{def}}{=} \langle \bar{B}, \mathrm{Aut}(B); \leq, Op\rangle$. Recall that $a \cong b$ in $\langle B, G\rangle$ means that $(\exists g \in G)(g(a) = b)$.

5.4. PROPOSITION. (a) *Let $\varphi'_{\mathrm{Al}}(v)$ be the following formula in the language of $K^{\bar{B}G}$:*

$$\varphi'_{\mathrm{Al}}(v) \equiv (v \textit{ is an atomless element}) \wedge v \neq a^{\mathrm{AL}}(\bar{B})$$
$$\wedge\, \forall u(u \cong v \to ((u - v = 0) \vee (u - v \cong v)))\,.$$

Then for every $B \in K_{\mathrm{CL}}$ and $b \in \bar{B}$, $M^{\bar{B}G}(B) \models \varphi'_{\mathrm{Al}}[b]$ iff $b \in \mathrm{Al}(B)$ and $b \neq a^{\mathrm{AL}}(B)$.

(b) *Let $\varphi_{\mathrm{Al}}(v) \equiv \exists v_1 v_2\ (\varphi'_{\mathrm{Al}}(v_1) \wedge \varphi'_{\mathrm{Al}}(v_2) \wedge (v = v_1 + v_2))$, then for every $B \in K_{\mathrm{CL}}$ and $b \in \bar{B}$, $M^{\bar{B}G}(B) \models \varphi_{\mathrm{Al}}[b]$ iff $b \in \mathrm{Al}(B)$.*

PROOF. (b) follows trivially from (a).

(a) It is easy to see that if $b \in \mathrm{Al}(B)$ and $b \neq a^{\mathrm{AL}}(B)$, then $M^{\bar{B}G}(B) \models \varphi'_{\mathrm{Al}}[b]$. Suppose by contradiction that $M^{\bar{B}G}(B) \models \varphi'_{\mathrm{Al}}[b]$ but $b \notin \mathrm{Al}(B)$. Clearly, $b \in \mathrm{Al}(\bar{B})$ and $b \neq a^{\mathrm{AL}}(B)$. Hence, there are $b_1, b_2 \in \mathrm{Al}(B) - \{0\}$ such that $b_1 \leq b$

and $b_2 \leq -b$. Let $f \in \mathrm{Aut}(B)$ be such that $f(b_1) = b_2$, $f(b_2) = b_1$ and $f \restriction -(b_1 + b_2) = Id$. Hence, $f(b) - b = b_2$, but $0 \neq b_2 \not\leq b$, a contradiction.

A set $D \subseteq B$ is called a disjoint family if for every distinct $d_1, d_2 \in D$, $d_1 \cdot d_2 = 0$. If $D \subseteq B$ is a disjoint family and $b \in \bar{B}$ let $D \uparrow b \overset{\mathrm{def}}{=} \{d \in D \mid d \cdot b \neq 0\}$. Let D_1 and D_2 be disjoint families, then D_1 is a refinement of D_2 if there is a one-to-one function r: $D_1 \to D_2$ such that for every $d \in D_1$, $d \leq r(d)$.

The following definition is due to MONK [1975].

5.5. DEFINITION. Let $E \subseteq B$ be a disjoint family. E is called an *excellent subset* of B if: (1) $E \subseteq \mathrm{At}(B)$ or $E \subseteq \mathrm{Al}(B) - \{0\}$; (2) $\Sigma\, E \not\in B^{\mathrm{TL}}$; and (3) for every $b \in B^{\mathrm{TL}}$, either $E \uparrow b$ is finite or $E \uparrow -b$ is finite.

Note that by 5.5(2) every excellent set is infinite. The following lemma deals with the existence of excellent sets. Its easy proof is left to the reader.

5.6. LEMMA. *Let $B \in K_{\mathrm{CL}}$, then*

(a) *If $D \subseteq B$ is an infinite disjoint family, then D has a refinement E which is excellent.*

(b) *If $b \in \bar{B} - B^{\mathrm{TL}}$, then there is an excellent set E such that for every $e \in E$, $e \leq b$ or $e \leq -b$ and both $E \uparrow b$ and $E \uparrow -b$ are infinite.*

(c) *If E is excellent, then there is $E_1 \supseteq E$ such that E_1 is excellent and $E_1 - E$ is infinite.*

(d) *If E is excellent and E_1 is an infinite refinement of E and $0 \not\in E_1$, then E_1 is excellent.*

We wish to show that B^{TL} is a first-order definable subset of $M^{\bar{B}G}(B)$. By the definition of an excellent set and by 5.6(b) of $B \in K_{\mathrm{CL}}$, then for every $b \in \bar{B}$: $b \in B^{\mathrm{TL}}$ iff there is no excellent E such that $E \uparrow b$ and $E \uparrow -b$ are both infinite.

In order to express the above facty by a first-order formula, we shall have to represent excellent sets by members of $M^{\bar{B}G}(B)$, and then show that finiteness is expressible by a first-order formula.

If $D_1, D_2 \subseteq B$ let $D_1 \cong D_2$ denote the fact that there is $f \in \mathrm{Aut}(B)$ such that $f(D_1) = D_2$. For $f \in \mathrm{Aut}(B)$ let $\mathrm{Inv}(f) \overset{\mathrm{def}}{=} \{b \in B \mid f(b) = b\}$. An ideal I of B can be regarded as a lattice, hence $\mathrm{Aut}(I)$ denotes the group of automorphisms of the lattice I. If X is a subset of B, let $\langle X \rangle$ denote the subalgebra generated by X and $\langle X \rangle^{\mathrm{id}}$ denote the ideal generated by X.

5.7. LEMMA. *Let $B \in K_{\mathrm{CL}}$.*

(a) *If a disjoint family $D \subseteq B$ satisfies* (1) *and* (3) *of* 5.5 *and $f \in \mathrm{Aut}(\langle D \rangle^{\mathrm{id}})$, then there is a unique $g \in \mathrm{Aut}(B)$ extending $f \cup Id \restriction \{b \in B \mid b \leq -\Sigma\, D\}$.*

(b) *Let D be a disjoint family, $D \subseteq \mathrm{Al}(B) - \{0\}$, and D satisfies* 5.5(3), *then there is $f \in \mathrm{Aut}(B)$ such that $D = \mathrm{At}(\mathrm{Inv}(f)) \cap \mathrm{Al}(B)$.*

(c) *If E is excellent, E_1 is an infinite refinement of E and $0 \not\in E_1$, then $E_1 \cong E$.*

PROOF. (a) Let I be an ideal in any BA B_0, and let $J(I) = \{a \in B_0 \mid \Sigma\, \{b \in I \mid b \leq a\} \in I\}$. It is easy to see that for every $f \in \mathrm{Aut}(I)$ there is a unique $g_0 \in \mathrm{Aut}(J(I))$

extending $f \cup Id \restriction \{a \in J(I) \mid a \cdot \Sigma I = 0\}$. If we take I to be $\langle D\rangle^{\mathrm{id}}$, where D is as in (a), then $J(I)$ is either a maximal ideal in B or is equal to B, hence g_0 can be uniquely extended to an automorphism g of B.

(b) Let D be as in (b). For every $d \in D$ let $f_d \in \mathrm{Aut}(B \restriction d)$ be such that $\mathrm{Inv}(f_d) = \{0, d\}$. $\bigcup \{f_d \mid d \in D\}$ can be extended to some $f \in \mathrm{Aut}(\langle D\rangle^{\mathrm{id}})$. Let $g \in \mathrm{Aut}(B)$ extend $f \cup Id \restriction \{b \in B \mid b \leq -\Sigma D\}$. It is easy to check that $\mathrm{At}(\mathrm{Inv}(G)) \cap \mathrm{Al}(B) = D$.

(c) Let E and E_1 be as in (c). By 5.6(c) there is an excellent set $E_2 \supseteq E$ such that $E_2 - E$ is infinite. Let π: $E \to E_2 - E$ and π_1: $E_1 \to E_2 - E$ be one-to-one and onto. For every $e \in E$ let f_e be an isomorphism beftween $B \restriction e$ and $B \restriction \pi(e)$; for every $e \in E_1$ let f_e^1 be an isomorphism between $B \restriction e$ and $B \restriction \pi_1(e)$. By part (a) there is $f \in \mathrm{Aut}(B)$ extending $\bigcup \{f_e \cup f_e^{-1} \mid e \in E\} \cup Id \restriction \{b \in B \mid b \leq -\Sigma E_2\}$. Similarly, there is $f^1 \in \mathrm{Aut}(B)$ which extends $\bigcup \{f_e^1 \cup (f_e^1)^{-1} \mid e \in E_1\} \cup Id \restriction \{b \in B \mid b \leq -(\Sigma E_1 + \Sigma (E_2 - E))\}$. $f \circ f^1(E_1) = E$, hence $E_1 \cong E$. $\square$

For $b \in \bar{B}$ and $f \in \mathrm{Aut}(B)$ let

$$\mathrm{Dt}(b, f) = \{a \in \mathrm{At}(B) \mid a \leq b\} \cup (\mathrm{At}(\mathrm{Inv}(f)) \cap \mathrm{Al}(B)) ;$$

let

$$\mathrm{Dt}(B) = \{\mathrm{Dt}(b, f) \mid b \in \bar{B},\ f \in \mathrm{Aut}(B)\} .$$

Let $M^{\bar{B}GD}(B) = \langle \bar{B}, \mathrm{Aut}(B), \mathrm{Dt}(B); \leq, Op, \in\rangle$ and $K_{\mathrm{CL}}^{\bar{B}GD} = \{M^{\bar{B}GD}(B) \mid B \in K_{\mathrm{CL}}\}$.

5.8. COROLLARY. (a) *$K_{\mathrm{CL}}^{\bar{B}GD}$ is strongly first-order interpretable in $K_{\mathrm{CL}}^{\bar{B}G}$.*

(b) *If $B \in K_{\mathrm{CL}}$, then every excellent subset of B and every finite disjoint family contained in* $\mathrm{At}(B) \cup \mathrm{Al}(B) - \{0\}$ *belongs to* $\mathrm{Dt}(B)$.

PROOF. Left to the reader. $\square$

We are now ready to show that B^{TL} is definable in $M^{\bar{B}GD}(B)$.

5.9. LEMMA. *Let $\varphi_{\mathrm{Fin}}(D)$, $\varphi_{\mathrm{Exc}}(D)$ and $\varphi_{\mathrm{TL}}(b)$ be the following formulas in the language of $K_{\mathrm{CL}}^{\bar{B}GD}$:*

$$\varphi_{\mathrm{Fin}}(D) \equiv (\forall D_1, D_2)(((D_1 \text{ is a refinement of } D) \wedge (D_2 \subsetneq D_1)) \to (D_1 \not\cong D_2)) ,$$

$$\varphi_{\mathrm{Exc}}(D) \equiv (\forall D_1)(((D_1 \text{ is a refinement of } D) \wedge \neg \varphi_{\mathrm{Fin}}(D_1)) \to (D_1 \cong D)) ,$$

$$\varphi_{\mathrm{TL}}(b) \equiv \forall D(\varphi_{\mathrm{Exc}}(D) \to (\varphi_{\mathrm{Fin}}(D \uparrow b) \vee \varphi_{\mathrm{Fin}}(D \uparrow -b))) .$$

Then for every $B \in K_{\mathrm{CL}}$, $D \in \mathrm{Dt}(B)$ and $b \in \bar{B}$: $M^{\bar{B}GD}(B) \models \varphi_{\mathrm{Fin}}[D]$ iff D is finite, $M^{\bar{B}GD}(B) \models \varphi_{\mathrm{Exc}}[D]$ iff D is excellent, and $M^{\bar{B}GD}(B) \models \varphi_{\mathrm{TL}}[b]$ iff $b \in B^{\mathrm{TL}}$.

PROOF. The proof follows easily from 5.6 and 5.7. □

This concludes the proof of Theorem 5.2.

6. Faithfulness of measure algebras

Given a measure algebra $\langle B, \mu \rangle$ we may consider the group $\mathrm{MP}(B, \mu)$ of measure preserving automorphisms of $\langle B, \mu \rangle$, or the group $\mathrm{MZP}(B, \mu)$ of measure-zero preserving automorphisms of $\langle B, \mu \rangle$. (An automorphism f is measure-zero preserving if for every $a \in B$: $\mu(a) = 0$ iff $\mu(f(a)) = 0$. Such automorphisms are also called non-singular.)

Our usual goal is to prove that the group $\mathrm{MP}(B, \mu)$ determines the measure algebra $\langle B, \mu \rangle$, and that the group $\mathrm{MZP}(B, \mu)$ determines $\langle B, I_Z(B, \mu) \rangle$, where $I_Z(B, \mu) = \{a \in B \mid \mu(a) = 0\}$. In addition we shall also conclude that every automorphism of $\mathrm{MP}(B, \mu)$ and of $\mathrm{MZP}(B, \mu)$ is inner.

The faithful classes we shall define contain the following well-known measure algebras: the fields of Borel and Lebesgue measurable sets of $\mathbb{R}$ and $[0, 1]$ with their Lebesgue measure; the fields of Baire and Borel sets of locally compact σ-compact groups with their Haar measure; the strictly positive homogeneous measure algebras; and measure algebras obtained by Carathéodory extension theorem from σ-additive functions on homogeneous BAs.

Every member of $\mathrm{MZP}(B, \mu)$ is a uniformly continuous function with respect to the following semimetric ρ_μ on B: $\rho_\mu(a, b) = \mu(a \triangle b)$. Certain sets Γ of moduli of continuity define groups $\mathrm{MP}_\Gamma(B, \mu)$ such that $\mathrm{MP}(B, \mu) \subseteq \mathrm{MP}_\Gamma(B, \mu) \subseteq \mathrm{MZP}(B, \mu)$. In the last theorem of this section we show that for every countable Γ, $\mathrm{MP}_\Gamma(B, \mu)$ determines Γ.

The problem of faithfulness of measure algebras has not been thoroughly investigated. In the sequel we shall mention some of the many unsolved questions.

Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid 0 \leq x\}$ and $\mathbb{R}^* = \mathbb{R}^+ \cup \{\infty\}$. A function μ from a BA B to R^* is σ-additive if for every countable disjoint family $A \subseteq B$: if $\Sigma A \in B$, then $\mu(\Sigma A) = \Sigma \{\mu(a) \mid a \in A\}$; μ is a measure on B if μ is σ-additive, $\mu(0) = 0$ and B is a σ-algebra, that is, every countable subset of B has a supremum. μ is finite if $\mu(1) < \infty$; μ is σ-finite if there is a countable subset $A \subseteq B$ such that $\Sigma A = 1$ and for every $a \in A$, $\mu(a) < \infty$. μ is positive if for some $a \in B$, $\mu(a) > 0$, μ is strictly positive if for every $a \in B - \{0\}$, $\mu(a) > 0$. μ is atomless if for every $a \in B$ if $\mu(a) > 0$, then there is $b < a$ such that $\mu(b)$, $\mu(a - b) > 0$.

Let us first find the appropriate translation of faithfulness problems to the context of groups of measure preserving automorphisms. If $0 < c \in \mathbb{R}^+$, then $\mathrm{MP}(B, \mu) = \mathrm{MP}(B, c \cdot \mu)$, hence μ may be determined by a subgroup $G \subseteq \mathrm{MP}(B, \mu)$ only up to a multiplicative constant. Let us devise a structure $\mathrm{MAM}(B, \mu, G)$ which will represent just that information which is capturable from $\mathrm{MP}(B, \mu)$. For a measure algebra $\langle B, \mu \rangle$ and $G \subseteq \mathrm{Aut}(B)$ let the "measurable action model" $\mathrm{MAM}(B, \mu, G)$ be the following model:

$$\mathrm{MAM}(B, \mu, G) = \langle B, G; \leq, Op, P_0, P_\infty, R_\alpha \rangle_{\alpha \in \mathbb{R}^+},$$

where

$$P_0 = \{a \in B \mid \mu(a) = 0\}, P_\infty = \{a \in B \mid \mu(a) = \infty\}$$

and

$$R_\alpha = \{\langle a, b\rangle \mid a, b \in B \ \mu(a) < \infty \text{ and } \mu(b) = \alpha \cdot \mu(a)\} .$$

Clearly, $\mathrm{MAM}(B_1, \mu_1, G_1) \overset{\tau}{\cong} M(B_2, \mu_2, G_2)$ iff τ is an isomorphism between B_1 and B_2, $G_1^\tau = G_2$, and for some $0 < c < \infty$, $\mu_2(\tau(a)) = c \cdot \mu_1(a)$ for every $a \in B_1$.

Let K be any class of triples of the form $\langle B, \mu, G\rangle$, where $\langle B, \mu\rangle$ is a measure algebra and $G \subseteq \mathrm{Aut}(B)$. We denote $K^{\mathrm{MAM}} = \{\mathrm{MAM}(B, \mu, G) \mid \langle B, \mu, G\rangle \in K\}$ and $K^G = \{G \mid (\exists B, \mu)(\langle B, \mu, G\rangle \in K\}$. A member $\langle B, \mu, G\rangle$ of K is group-categorical in K if for every $\langle B_1, \mu_1, G_1\rangle \in K$ and an isomorphism φ between G and G_1 there is an isomorphism $\tilde{\varphi}$ between $\mathrm{MAM}(B, \mu, G)$ and $\mathrm{MAM}(B_1, \mu_1, G_1)$ extending φ. Note that if $\langle B, \mu, G\rangle$ is group-categorical in K and $\mu(1) < \infty$, then every automorphism of G is induced by a member of $\mathrm{MP}(B, \mu)$.

Our results have the following form. We define a class K_{M} of triples $\langle B, \mu, G\rangle$ of the above form, and prove that for various subclasses $K \subseteq K_{\mathrm{M}}$ every member of K is group-categorical in K_{M}.

Let

$$K_{\mathrm{M}} = \{\langle B, \mu, G\rangle \mid \mu \text{ is a positive } \sigma\text{-finite measure on } B,\ G \subseteq \mathrm{MP}(B, \mu), \text{ and } \langle B, G\rangle \in K_{B^3}\} .$$

(Recall that K_{B^3} was defined in 4.5(b) as $\{\langle B, G\rangle \mid \langle B, G\rangle$ is B-closed and $I^{[\geq 2]}(B, G)$ is dense and indecomposable$\}$. $I^{[\geq 2]}(B, G)$ was defined in 3.1 and B-closedness in 4.3.)

Let $G \subseteq \mathrm{Aut}(B)$ and μ be a measure on B. We say that μ is G-invariant if $G \subseteq \mathrm{MP}(B, \mu)$. We say that $\langle B, G\rangle$ is μ-homogeneous if for every $a, b \in B$ such that $0 < \mu(a) \leq \mu(b) < \mu(1)$ there is $g \in G$ such that $g(a) \leq b$. We say that $\langle B, G\rangle$ is μ-weakly homogeneous if for every $a, b \in B$: if $0 < \mu(a)$, $\mu(b)$ then there is $g \in G$ such that $\mu(g(a) \cdot b) > 0$.

Let B be a σ-algebra, $G \subseteq \mathrm{Aut}(B)$ and $f \in \mathrm{Aut}(B)$, we say that f is *piecewise in* G if there are $\{g_i \mid i \in \omega\} \subseteq G$ and $\{a_i \mid i \in \omega\} \subseteq B$ such that $\Sigma\, a_i = 1_B$ and for every $i \in \omega$, $f \restriction a_i = g_i \restriction a_i$. We denote $G^B = \{f \in \mathrm{Aut}(B) \mid f$ is piecewise in G with respect to $B\}$, and we say that G is σ-closed with respect to B if $G = G^B$.

We start with the following class. Let $K_{\mathrm{H}} = \{\langle B, \mu, G\rangle \mid \mu$ is a positive σ-finite atomless measure on B, and $\langle B, G\rangle$ is B-closed and μ-homogeneous$\}$.

6.1. THEOREM. *$K_{\mathrm{H}} \subseteq K_{\mathrm{M}}$ and every member of K_{H} is group categorical in K_{M}.*

Even though the requirement that $\langle B, G\rangle$ be μ-homogeneous seems to be too restrictive, K_{H} still includes some important measure algebras. If the Lebesgue measure on $\mathbb{R}$, $\mathrm{Bl}(\mathbb{R})$ and $\mathrm{Lg}(\mathbb{R})$ are, respectively, the algebras of Borel measurable and Lebesgue measurable subsets of $\mathbb{R}$, then $\langle \mathrm{Bl}(\mathbb{R}), \mu, \mathrm{MP}(\mathrm{Bl}(\mathbb{R}), \mu)\rangle$,

$\langle \mathrm{Lg}(\mathbb{R}), \mu, \mathrm{MP}(\mathrm{Lg}(\mathbb{R}), \mu)\rangle$ and the restrictions of the above to $[0, 1]$ all belong to K_{H}. Also if B is a homogeneous BA and μ is an atomless strictly positive σ-finite measure on B, then by MAHARAM [1942] $\langle B, \mu, \mathrm{MP}(B, \mu)\rangle \in K_{\mathrm{H}}$.

Proof of 6.1. We first see that if $\langle B, \mu, G\rangle \in K_{\mathrm{H}}$, then $\langle B, G\rangle \in K_{B^1}$. We leave it to the reader to check that $a \in B$ is rigid iff $a \in \mathrm{At}(B)$ and that $\mathrm{At}(B)$ is either empty or infinite. This implies that $I^{[\geq 3]}(B, G)$ is dense in B, and that $a^{[6]}(B, G) = 0$.

We next show that $I^{[\geq 3]}(B, G)$ is indecomposable. It is easy to see that there is $\{a_i \mid i \in \omega\} \subseteq I^{[\geq 3]}(B, G)$ such that $\Sigma_{i\in\omega}\, a_i = 1_B$. Let I_1, I_2 be ideals such that $I_1 \cap I_2 = \{0\}$ and for no principal ideal J, $J \supseteq I_1$ and $J \cap I_2 = \{0\}$. Let I_3 be the ideal generated by $I_1 \cup I_2$ and we show that for some $i \in \omega$, $a_i \not\in I_3$. If, in contrary, $\{a_i \mid i \in \omega\} \subseteq I_3$, then for every $i \in \omega$ there are $a_{i,l} \in I_l$, $l = 1, 2$, such that $a_{i,1} + a_{i,2} = a_i$. Let J be the principal ideal generated by $\Sigma_{i\in\omega}\, a_{i,1}$, then $I_1 \subseteq J$ and $I_2 \cap J = \{0\}$, a contradiction. So $I^{[\geq 3]}(B, G)$ is indecomposable and hence $\langle B, G\rangle \in K_{B^1}$.

We next show that $K_{\mathrm{H}}^{\mathrm{MAM}}$ is strongly interpretable in K_{H}^{G}. By 4.5(b) K_{H}^{BG} is strongly interpretable in K_{H}^{G}. It remains to see that if $\langle \mathrm{B}, \mu, \mathrm{G}\rangle \in \mathrm{K}_{\mathrm{H}}$, then the relations P_0, P_∞ and $\{R_\alpha \mid \alpha \in \mathbb{R}^+\}$ are definable in $M(B, G)$. Clearly, $M(B, G^B)$ is strongly interpretable in $M(B, G)$. Let $a \in B$, then $\mu(a) = 0$ iff $M(B, G^B)$ satisfies: "there is no $\{a_i \mid i \in \omega\} \subseteq B$ such that $\Sigma_{i\in\omega}\, a_i = 1$ and for every $i \in \omega$, $a_i \cong a$". $\mu(a) = \infty$ iff $\mu(a) \neq 0$ and $M(B, G^B)$ satisfies: "there are a_0, a_1, a_2 such that $\Sigma_{i\leq 2}\, a_i = a$ and $a_0 \cong a_1 \cong a_2 \cong a_0 + a_1$". $\mu(a) \leq (m/n)\mu(b)$ iff $\mu(a) = 0$ or there are pairwise disjoint $a_1 \cong \cdots \cong a_m$, pairwise disjoint $b_1 \cong \cdots \cong b_n$ and $g \in G^B$ such that $a \leq \Sigma_{i=1}^m a_i$, $\Sigma_{i=1}^n b_i \leq b$ and $g(\Sigma_{i=1}^m a_i) \leq \Sigma_{i=1}^n b_i$.

It follows easily that for every $\alpha \in \mathbb{R}^+$, R_α is definable in $M(B, G^B)$. All of this implies that $K_{\mathrm{H}}^{\mathrm{MAM}}$ is strongly interpretable in K_{H}^{G}.

Let $\langle B, \mu, G\rangle \in K_{\mathrm{H}}$, $\langle B_1, \mu_1, G_1\rangle \in K_M$ and φ: $G \cong G_1$. By 4.5(b) there is an isomorphism $\tilde{\varphi}$ between $M(B, G)$ and $M(B_1, G_1)$ extending φ. Using the fact that $G_1 \subseteq \mathrm{MP}(B_1, \mu_1)$ it is easy to see that $\tilde{\varphi}$ is indeed an isomorphism between $\mathrm{MAM}(B, \mu, G)$ and $\mathrm{MAM}(B_1, \mu_1, G)$. Q.E.D.

We next turn to an important well-known class of measure spaces – the Haar measures of locally compact groups. Since our faithfulness results deal with Boolean algebras and not with rings, we shall deal here only with σ-compact groups. The reader is referred to HALMOS [1950, Chs. 10–12] for the definition and theorems on Haar measures. Some of the arguments we give here are adapted from HALMOS [1950].

For a topological space X let $\mathrm{Bl}(X)$ denote the subalgebra of $P(X)$ σ-generated by the closed subsets of X, and $\mathrm{Br}(X)$ denote the subalgebra of $P(X)$ σ-generated by all compact G_δ subsets of X. Note that if X is σ-compact, then $\mathrm{Bl}(X)$ is σ-generated by the compact subsets of X. Let $\hat{B}(X)$ denote either $\mathrm{Bl}(X)$ or $\mathrm{Br}(X)$.

If K is a topological group, then K may be regarded as a group of permutations of K, that is, $k \in K$ is identified with the left translation t_k which maps each $x \in K$ to kx. Hence, we regard K as a subgroup of $\mathrm{Aut}(\hat{B}(K))$. Let μ be a left Haar measure on $\hat{B}(K)$. We may define now two additional σ-algebras denoted by

$\mathrm{Bc}(K)$ and $\mathrm{Bp}(K)$. $\mathrm{Bc}(K)$ is the σ-algebra generated by $\mathrm{Br}(K) \cup \{b \subseteq K \mid$ there is $a \in \mathrm{Br}(K)$ such that $\mu(a)=0$ and $b \subseteq a\}$. If K is locally compact and σ-compact, then $\mathrm{Bl}(K) \subseteq \mathrm{Bc}(K)$. (See Halmos [1950, p. 287, Theorem H].) Let $I_Z(B, \mu)$ be the ideal of measure-zero elements of B, and let $\mathrm{Bp}(K) = \mathrm{Br}(K)/I_Z(\mathrm{Br}(K), \mu)$. It is easy to see that K may be naturally regarded as a subgroup of $\mathrm{Aut}(\mathrm{Bc}(K))$ or $\mathrm{Aut}(\mathrm{Bp}(K))$. The measure induced by μ on $\mathrm{Bc}(K)$ or $\mathrm{Bp}(K)$ is called a Haar measure on $\mathrm{Bc}(K)$ or $\mathrm{Bp}(K)$. We use $\hat{B}(K)$ to denote any of the four algebras defined above.

6.2. THEOREM. *Let K be a locally compact σ-compact group and μ be a left Haar meaasure on $\hat{B}(K)$. Let G be a B-closed subgroup of* $\mathrm{MP}(B(K), \mu)$ *and $G \supseteq K$. Let $a \in \hat{B}(K)$ be such that*: (1) *a is not the sum of one, two or six atoms, and* (2) *if $\hat{B}(K) \neq \mathrm{Bp}(K)$, then $\mathrm{int}_K(a) \neq \emptyset$, and if $\hat{B}(K) = \mathrm{Bp}(K)$, then for some $a' \in \mathrm{Br}(K)$ $\mathrm{int}_K(a') \neq \emptyset$ and $a = a'/I_Z(\mathrm{Br}(K), \mu)$. Then $\langle \hat{B}(K) \restriction a, \mu \restriction a, G(a)\rangle$ is group-categorical in K_{M}.* ($\mathrm{int}_K(a)$ *denotes the interior of a with respect to the topology of K. Recall that for $G \subseteq \mathrm{Aut}(B)$ and $a \in B$, $G(a)$ was defined to be $\{g \in G \mid \mathrm{var}(g) \leq a\}$, hence we may regard $G(a)$ as a subgroup of* $\mathrm{Aut}(B \restriction a)$.)

REMARK. For $a = K$ we may use the proof of the uniqueness of the Haar measure as a step in the proof of 6.2. However, since Theorem 6.2 also deals with cases in which $a \neq K$, we shall have to adapt arguments appearing in the uniqueness proof to our context, rather than taking that proof as a whole.

6.3. QUESTION. Does Theorem 6.2 remain true if the requirement that $\mathrm{int}_K(a) \neq \emptyset$ is replaced by the weaker requirement that $\mu(a) > 0$? We even do not know the answer to the following special case. Let $a \subseteq \mathbb{R}$ be a Borel set of positive Lebesgue measure, and let G be the group of piecewise translation automorphisms of $\mathrm{Bl}(\mathbb{R}) \restriction a$. Is it true that the only positive σ-finite G-invariant measure on $\mathrm{Bl}(\mathbb{R}) \restriction a$ is the restriction of the Lebesgue measure to $\mathrm{Bl}(\mathbb{R}) \restriction a$? (A function $f: a \to \mathbb{R}$ is a piecewise translation, if there are Borel sets $\{a_i \mid i \in \omega\}$ and real numbers $\{\alpha_i \mid i \in \omega\}$ such that $a = \bigcup_{i \in \omega} a_i$, and for every $i \in \omega$ and $x \in a_i$, $f(x) = \alpha_i + x$.)

The proof of Theorem 6.2 is divided into three parts. We first describe this division. Let $K_{\mathrm{U}} = \{\langle B, \mu, G\rangle \in K_{\mathrm{M}} \mid \langle B, G\rangle \in K_{B^1}$ and for every G-invariant positive σ-finite measure ν on B: $\langle B, G\rangle$ is ν-weakly homogeneous$\}$.

We shall show the following facts. (1) If K, $\hat{B}(K)$, μ, G and a satisfy the conditions of 6.2 and $\hat{B}(K) = \mathrm{Br}(K)$, then $\langle \hat{B}(K) \restriction a, \mu \restriction a, G(a)\rangle \in K_U$. (2) Every member of K_U is group-categorical in K_{M}. (3) The restriction of Theorem 6.2 to the case when $\hat{B}(K) = \mathrm{Br}(K)$ already implies the full statement of 6.2.

6.4. LEMMA. *If K,* $\mathrm{Br}(K)$, *μ, G and a satisfy the conditions of* 6.2, *then* $\langle \mathrm{Br}(K) \restriction a, \mu \restriction a, G(a)\rangle \in K_{\mathrm{U}}$.

PROOF. Let K, μ, G and $a \in \mathrm{Br}(K)$ be as in 6.2. Let $H \overset{\mathrm{def}}{=} G(a)$, and $\mathrm{cl}_K(a)$ denote the closure of a with respect to the topology of K. We first see that

$\langle \mathrm{Br}(K) \restriction a, H\rangle \in K_{B^1}$. For every $x \in \mathrm{cl}_K(a)$ there are a neighborhood V_x of x and $y_x^l \in K$, $l<2$, such that $V_x \in \mathrm{Br}(K)$, $y_x^l V_x \subseteq \mathrm{int}_K(a)$, $l=0,1$, and V_x, $y_x^0 V_x$, $y_x^1 V_x$ are pairwise disjoint. Since K is σ-compact, so is $\mathrm{cl}_K(a)$, hence there are x_i, $i \in \omega$, such that $\bigcup_{i\in\omega} V_{x_i} = \mathrm{cl}_K(a)$. For every $i \in \omega$ let $a_i = V_{x_i} \cdot a$. Since $G \supseteq K$ and G is B-closed, $a_i \in I^{[\geq 3]}(\mathrm{Br}(K) \restriction a, H)$, and clearly in $\mathrm{Br}(K) \restriction a$, $\Sigma_{i\in\omega} a_i = 1$. Since $\mathrm{Br}(K)$ is a σ-algebra this implies that $I^{[\geq 3]}(\mathrm{Br}(K), H)$ is dense and indecomposable. (This argument has already appeared in the proof of Theorem 6.1.) It is trivial that $\langle \mathrm{Br}(K) \restriction a, H\rangle$ satisfies the other conditions in the definition of K_{B^1}.

We next prove that if ν is a positive σ-finite H-invariant measure on $\mathrm{Br}(K) \restriction a$, then $\langle \mathrm{Br}(K) \restriction a, H\rangle$ is ν-weakly homogeneous. Let e_K denote the unit of K. W.l.o.g. int $(a) \ni e_k$, hence there is an open $V \subseteq a$ such $e_k \in V \in \mathrm{Br}(K)$, $V = V^{-1}$ and $V \cdot V \cdot V \subseteq a$. We now apply the method used in HALMOS [1950, §59 pp. 257–261]. Let A, $B \subseteq V$ and $0 < \nu(A)$, $\nu(B) < \infty$. Then by the fact that $A \cdot B \subseteq a$ and by Fubini's theorem:

$$0 < \nu(A)\cdot \nu(B) = (\nu \times \nu)(A \times B) = \int_A \nu(B)\,\mathrm{d}\nu(x) = \int_A \nu(xB)\,\mathrm{d}\nu(x)$$

$$= \int \chi_{\{\langle x, xy\rangle \mid x\in A,\, y\in B\}}(x, y)\,\mathrm{d}(\nu \times \nu)(x, y)$$

$$= \int_{A\cdot B} \nu(A \cap yB^{-1})\,\mathrm{d}\nu(y)\,.$$

Hence, for some $y \in AB$, $\nu(A \cap yB^{-1}) > 0$. It follows that $\nu(B^{-1}) > 0$; that is, if $B \subseteq V$ and $\nu(B) > 0$, then $\nu(B^{-1}) > 0$.

Now, assuming that $\nu(A)$, $\nu(B) > 0$ and A, $B \subseteq V$ we may use the above identity for A and B^{-1}. We obtain that

$$0 < \nu(A)\cdot \nu(B^{-1}) = \int_{AB^{-1}} \nu(A \cap yB)\,\mathrm{d}\nu(y)\,,$$

and hence for some $y \in AB^{-1}$, $\nu(A \cap yB) > 0$. We conclude that if A, $B \subseteq V$ and $\nu(A)$, $\nu(B) > 0$, then there is $h \in H$ such that $\nu(A \cap h(B)) > 0$, for if $\nu(A \cap B) > 0$, take h to be the identity function; otherwise, let $y \in AB^{-1}$ be such that $\nu(A \cap yB) > 0$, let

$$h = t_y \restriction (B - A) \cup t_y^{-1} \restriction t_y(B - A) \cup Id \restriction (a - (B - A) - t_y(B - A))\,.$$

It is clear that h is as required.

Let A, $B \subseteq a$ and $\nu(A)$, $\nu(B) > 0$. It is easy to see that there are $A' \subseteq A$, $B' \subseteq B$ and x, $y \in K$ such that $\nu(A')$, $\nu(B') > 0$ and xA', $yB' \subseteq V$. We already know that for some $h \in H$, $\nu(xA' \cap h(yB')) > 0$; it follows easily that there is $h \in H$ such that $\nu(A \cap h(B)) > 0$, and so $\langle \mathrm{Br}(K) \restriction a, H\rangle$ is ν-weakly homogeneous.

This concludes the proof that $\langle \mathrm{Br}(K) \restriction a, \mu \restriction a, H\rangle \in K_U$.

6.5. LEMMA. *Every member of K_U is group-categorical in K_M.*

PROOF. Let $\langle B_1, \mu_1, G_1\rangle \in K_U$ and $\langle B_2, \mu_2, G_2\rangle \in K_M$, and let $\varphi: G_1 \to G_2$ be an isomorphism. Since $\langle B_1, G_1\rangle \in K_{B^1}$ and $\langle B_2, G_2\rangle \in K_{B^3}$, by Theorem 4.5(b) φ can be extended to an isomorphism $\tilde{\varphi}$ between $M(B_1, G_1)$ and $M(B_2, G_2)$. It follows trivially from the definition of K_U that $\langle B_2, \mu_2, G_2\rangle \in K_U$. Recall that $M(B, G) = \langle B, G; \le, Op\rangle$ and that $\mathrm{MAM}(B, \mu, G)$ is an expansion of $M(B, G)$. It thus remains to show that $\tilde{\varphi}$ is an isomorphism between $\mathrm{MAM}(B_1, G_1, \mu_1)$ and $\mathrm{MAM}(B_2, G_2, \mu_2)$. W.l.o.g. $B_1 = B_2$, $G_1 = G_2$ and $\tilde{\varphi} = Id$.

Let ν_1 and ν_2 be positive σ-finite G_1-invariant measures on B_1 and ν_1 is absolutely continuous with respect to ν_2. Let $a_1, a_2 \in B_1$ be such that $\nu_2(a_1) = \nu_2(a_2) < \infty$. We show that $\nu_1(a_1) = \nu_1(a_2)$. It follows from the ν_2-weak homogeneity of $\langle B_1, G_1\rangle$ that there are $a_1' \le a_1$ and $g \in G_1$ such that $\nu_2(a_1 - a_1') = 0$ and $g(a_1') \le a_2$. So $\nu_2(a_2 - g(a_1')) = 0$. By the absolute continuity of ν_1 with respect to ν_2, $\nu_1(a_1 - a_1') = \nu_1(a_2 - g(a_1')) = 0$, and since $\nu_1(a_1') = \nu_1(g(a_1'))$, $\nu_1(a_1) = \nu_1(a_2)$.

We next prove that if $\nu_2(a) > 0$, then $\nu_1(a) > 0$. Suppose the contrary. W.l.o.g. $\alpha = \nu_2(a) < \infty$. By the ν_2-weak homogeneity of $\langle B_1, G_1\rangle$ and the σ-finiteness of ν_2 there is $\{a_i \mid i \in \omega\} \subseteq B_1$ such that $\Sigma_{i=1}^n a_i = 1$ and for every $i \in \omega$, $\nu_2(a_i) \le \alpha$. By the ν_2-weak homogeneity of $\langle B_1, G_1\rangle$, $\nu_1(a_i) = 0$, and hence $\nu_1(1) \le \Sigma_{i\in\omega} \nu_1(a_i) = 0$. This contradicts the positivity of ν_1. It follows that $I_Z(B_1, \nu_1) = I_Z(B_1, \nu_2)$.

We next show that for every $a \in B_1$ if $\nu_2(a) = \infty$, then $\nu_1(a) = \infty$. Let $a \in B_1$ and $\nu_2(a) = \infty$. Let $b \in B_1$ be such that $0 < \nu_2(b) < \infty$. By the ν_2-weak homogeneity of $\langle B_1, G_1\rangle$ there are pairwise disjoint $\{b_i \mid i \in \omega\}$ such that for every $i \in \omega$, $\nu_2(b_i) = \nu_2(b)$ and $b_i \le a$, hence $\nu_1(b_i) = \nu_1(b) > 0$, and so $\nu_1(a) = \infty$.

We have already proved that $I_Z(B_1, \mu_1) = I_Z(B_1, \mu_2)$, so ν_2 is absolutely continuous with respect to ν_1; by switching the roles of ν_1 and ν_2 in the above argument we conclude that if $a \in B_1$ and $\nu_1(a) = \infty$, then $\nu_2(a) = \infty$.

Let $\langle B, \nu\rangle$ be a measure algebra and $a \in B$; a is called a ν-atom if $\nu(a) > 0$ and for every $b \le a$ either $\nu(b) = 0$ or $\nu(a - b) = 0$.

If B_1 contains a ν_2-atom, then it is easy to see that there are ν_2-atoms $\{a_i \mid i \in \omega\}$ such that $\Sigma_{i\in\omega} a_i = 1$ and for every $i, j \in \omega$, $\nu_2(a_i) = \nu_2(a_j) < \infty$. It easily follows that for every $i, j \in \omega$, a_i is a ν_1-atom and $\nu_1(a_i) = \nu_1(a_j) < \infty$. This implies that for some $c > 0$, $\nu_1 = c \cdot \nu_2$.

Now assume that B_1 does not contain ν-atoms. It is easy to see that for every $a \in B_1$, if $\alpha < \nu_2(a)$ then there is $b < a$ such that $\nu_2(b) = \alpha$. It follows that for every positive rational number m/n and $a, b \in B_1$ such that $\nu_2(a), \nu_2(b) < \infty$: $m/n \cdot \nu_2(a) \le \nu_2(b)$ iff there are pairwise disjoint $a_1, \dots, a_n \le a$ and pairwise disjoint $b_1, \dots, b_m \le b$ such that $\nu_2(a - \Sigma_{i=1}^n a_i) = 0$, and for every $i \le n$ and $j \le m$, $\nu_2(a_i) = \nu_2(b_j)$ iff $(m/n)\nu_1(a) \le \nu_1(b)$. We may now conclude that for some $c > 0$, $\nu_1 = c \cdot \nu_2$.

Let us now return to μ_1 and μ_2. $\mu_1 + \mu_2$ is a positive σ-finite G_1-invariant measure on B_1 and μ_1 and μ_2 are absolutely continuous with respect to $\mu_1 + \mu_2$. Hence, there are $0 < c_1, c_2 < \infty$ such that $\mu_i = c_i(\mu_1 + \mu_2)$, $i = 1, 2$. Thus, $\mu_1 = (c_2/c_1)\mu_2$. This implies that $\mathrm{MAM}(B_1, \mu_1, G_1) = \mathrm{MAM}(B_1, \mu_2, G_1)$, so the lemma is proved. □

Proof of Theorem 6.2. Let K, μ, G and $a \in \hat{B}(K)$ be as in 6.2. If $B(K) = \mathrm{Br}(K)$, then by 6.4 and 6.5 $\langle \mathrm{Br}(K) \restriction a, \mu \restriction a, G(a)\rangle$ is group-categorical in K_{M}. So let $\hat{B}(K) \neq \mathrm{Br}(K)$. Repeating the argument of 6.4 we can conclude that $\langle \hat{B}(K) \restriction a, G(a)\rangle \in K_{B^1}$. Let $\langle B_1, \mu_1, G_1\rangle \in K_{\mathrm{M}}$ and φ: $G(a) \cong G_1$. Hence, by 4.5(b) φ can be extended to an isomorphism $\tilde{\varphi}$: $M(\hat{B}(K) \restriction a, G(a)) \cong M(B_1, G_1)$. W.l.o.g. $\tilde{\varphi} = Id$. It suffices to show that there is $c \in \mathbb{R}$ such that for every $b \in \hat{B}(K) \restriction a$, $\mu_1(b) = c \cdot \mu(b)$.

Let us first deal with the case when $\hat{B}(K) = \mathrm{Bl}(K)$ or $\hat{B}(K) = \mathrm{Bc}(K)$. By HALMOS [1950, p. 287, Theorem H], $\mathrm{Bl}(K) \subseteq \mathrm{Bc}(K)$. By the first part of the proof there is $c \in \mathbb{R}$ such that $\mu_1 \restriction (\mathrm{Br}(K) \restriction a) = c \cdot \mu \restriction (\mathrm{Br}(K) \restriction a)$, but since $\hat{B}(K) \subseteq \mathrm{Bc}(K)$, μ_1 is the unique extension of $\mu_1 \restriction (\mathrm{Br}(K) \restriction a)$ to $B(K) \restriction a$. Hence, $\mu_1 = c \cdot \mu \restriction a$.

The case when $B(K) = \mathrm{Bp}(K)$ is even simpler and is left to the reader. This concludes the proof of 6.2. □

Let T denote the circle. It follows from Theorem 6.2 that $T^{\mathrm{Bl}(T)} \ncong \mathbb{R}^{\mathrm{Bl}(\mathbb{R})}$. However, in general we do not know whether for two groups K_1 and K_2 $K_1^{\hat{B}(K_1)} \cong K_2^{\hat{B}(K_2)}$.

QUESTION: (a) For which pairs of locally compact σ-compact groups K_1 and K_2 is it true that $K_1^{\hat{B}(K_1)} \cong K_2^{\hat{B}(K_2)}$?

(b) Let K be a locally compact σ-compact group is it true that every automorphism of $K^{\hat{B}(K)}$ is a composition of members of $\mathrm{Aut}(K)$ and inner automorphisms of $K^{\hat{B}(K)}$? In particular is it true that every automorphism of $\mathbb{R}^{\mathrm{Bl}(\mathbb{R})}$ is generated by the functions of the form $y = cx$ and the piecewise translations?

(c) Investigate group-categoricity problems for groups which are not necessarily σ-compact.

There is another classical construction giving rise to many measure algebras, namely the construction in the Carathéodory extension theorem. This theorem, adapted to our framework, may be fomulated as follows. Let B be a σ-algebra, B_0 be a subalgebra of B and μ_0: $B_0 \to \mathbb{R}^*$. We say that μ_0 is B-σ-additive if for every countable disjoint family $A \subseteq B_0$: if $\Sigma A \in B_0$, then $\mu(\Sigma A) = \Sigma \{\mu(a) \mid a \in A\}$.

6.6. THEOREM. (Carathéodory). *Let B be a σ-algebra, B_0 be a subalgebra of B, σ-generating B. Let μ_0: $B_0 \to \mathbb{R}^*$ be B-σ-additive, σ finite and $\mu_0(0) = 0$. Then there is a unique measure μ on B extending μ_0; and μ is σ-finite.*

PROOF. The proof can be found in HALMOS [1950, §10–12]. It is given there for σ-fields of sets, but the proof transfers without changes to our setting. □

Let $K_{\mathrm{FM}} = \{\langle B, \mu, G\rangle \in K_{\mathrm{M}} \mid \mu(1) < \infty\}$. It turns out that if $\langle B, \mu\rangle$ is obtained from $\langle B_0, \mu_0\rangle$ according to Theorem 6.6, and $\langle B_0, \mu_0\rangle$ is sufficiently homogeneous, then $\langle B, \mu, \mathrm{MP}(B, \mu)\rangle$ is group-categorical in K_{FM}. We give two instances of this state of affairs.

Let B be a Boolean algebra and $\mu: B \to \mathbb{R}^*$, we say that μ is a finitely additive measure on B if $\mu(0)=0$, and for every pairwise disjoint $a_1, \ldots, a_n \in B$, $\mu(\Sigma_{i=1}^n a_i) = \Sigma_{i=1}^n \mu(a_i)$. We may regard B as a subalgebra of the power set of $\mathrm{Ult}(B)$, and consider the subalgebra B_1 of $P(\mathrm{Ult}(B))$ σ-generated by B. It is trivial to check that μ is B_1-σ-additive. Suppose that μ is positive finite and atomless, and that $\langle B, G\rangle$ is μ-homogeneous. Let μ_1 denote the extension of μ to B_1, then $\langle B_1, \mu_1, G^{B_1}\rangle$ is group-categorical in K_{FM}.

Let μ be a positive, finite, atomless, finitely additive measure on an $\aleph_1$-saturated BA B. Let $G \subseteq \mathrm{MP}(B, \mu)$ and $\langle B, G\rangle$ be μ-homogeneous. Let B_1 be the σ-completion of B; clearly, μ is B_1-σ-additive, and hence there is a measure μ_1 on B_1 extending μ. We shall prove that $\langle B_1, \mu_1, G^{B_1}\rangle$ is group-categorical in K_{FM}.

Let $K_{\mathrm{T}} = \{\langle B, \mu, G\rangle \mid \mu$ is a positive finite atomless measure on B, $G \subseteq \mathrm{MP}(B, \mu)$, $\langle B, G\rangle$ is B-closed, and there is a subalgebra B_0 of B which σ-generates B and such that for every $a, b \in B_0$ iff $\mu(a) < \mu(b)$, then there is $g \in G$ such that $B_0 \ni g(a) < b\}$.

6.7. THEOREM. *Every member of K_{T} is group-categorical in K_{FM}.*

QUESTION. We do not know if Theorem 6.7 remains true if K_{FM} is replaced by K_{M}, and in the definition of K_{T} the finiteness of μ is replaced by σ-finiteness.

Proof of Theorem 6.7. Let $\langle B, G, \mu\rangle \in K_{\mathrm{T}}$ and $B_0 \subseteq B$ be as in the definition of K_{T}.

We first show that if $a \in B_0$ and $\alpha < \beta < \mu(a)$, then there is $b \in B_0$, $b < a$ and $\alpha < \mu(b) < \beta$. Since μ is atomless there is $c < a$ such that $\mu(c) = (\beta + \alpha)/2$. By HALMOS [1950, §10–§12] there is a disjoint family $\{a_i \mid i \in \omega\} \subseteq B_0$ such that $c \leq \Sigma_{i\in\omega} a_i \leq a$ and $\mu(\Sigma_{i\in\omega} a_i) < \beta$. For some n, $\mu(\Sigma_{i<n} a_i) > \alpha$. Hence, $b = \Sigma_{i<n} a_i$ is as required.

We now show that $I^{[\geq 7]}(B, G) = B$. Let $\{a_i \mid i < 8\} \subseteq B_0$ be such that $\Sigma_{i<8} a_i = 1$ and for every $i < 8$, $\mu(a_i) < \frac{1}{7}\mu(1)$. It is easy to see that each a_i belongs to $I^{[\geq 7]}(B, G)$, hence $1 \in I^{[\geq 7]}(B, G)$. This shows that $\langle B, G\rangle \in K_{B^0}$, as defined before Theorem 4.5. Hence, by Theorem 4.5(b), $\langle B, G\rangle$ is group-categorical in K_{B^3}. By the definition of K_{FM}, if $\langle B_1, \mu_1, G_1\rangle \in K_{\mathrm{FM}}$, then $\langle B_1, G_1\rangle \in K_{B^0}$, and hence if τ is an isomorphism between G and G_1, then it can be extended to an isomorphism $\tilde{\tau}$ between $M(B, G)$ and $M(B_1, G_1)$. We shall show that $\tilde{\tau}$ is indeed an isomorphism between $\mathrm{MAM}(B, \mu, G)$ and $\mathrm{MAM}(B_1, \mu_1, G_1)$. To prove this we shall define for every $\alpha \in \mathbb{R}^+$ a formula $\varphi_\alpha(x, y)$ in the language of K_{B^3} such that: (1) if $\langle B_1, \mu_1, G_1\rangle \in K_{\mathrm{FM}}$, a, $b \in B_1$, and $M(B_1, G_1) \models \varphi_\alpha[a, b]$, then $\mu(a) = \alpha \cdot \mu(b)$; and (2) if $\langle B, \mu, G\rangle \in K_{\mathrm{T}}$, $B_0 \subseteq B$ is as in the definition of K_{T}, a, $b \in B_0$, and $\mu(a) = \alpha \cdot \mu(b)$, then $M(B, G) \models \varphi_\alpha[a, b]$. Let τ be the above isomorphism between $M(B, G)$ and $M(B_1, G_1)$. It will follow that for every a, $b \in B_0$ and $\alpha \in \mathbb{R}^+$: $\mu_1(\tilde{\tau}(a)) = \alpha \cdot \mu_1(\tau(b))$ iff $\mu(a) = \alpha \cdot \mu(b)$. Since $\tilde{\tau}(B_0)$ σ-generates B_1 it will follow that for every a, $b \in B$ and $\alpha \in \mathbb{R}^+$: $\mu_1(\tilde{\tau}(a)) = \alpha \cdot \mu_1(\tilde{\tau}(b))$ iff $\mu(a) = \alpha \cdot \mu(b)$; this implies that $\tilde{\tau}$ is an isomorphism between $\mathrm{MAM}(B, \mu, G)$ and $\mathrm{MAM}(B_1, \mu_1, G_1)$.

Let

$$\chi_{\leq 1/n}(x) \equiv (\exists g_1, \ldots, g_n \in G)\Big(\bigwedge_{i<j<n} g_i(x) \cdot g_j(x) = 0\Big).$$

Let m/n be a positive rational number and let

$$\psi_{\leq m/n}(x, y) \equiv \bigwedge_{0<k<\omega} (\exists z)(\exists g_1, \ldots, g_n, h_1, \ldots, h_m \in G)$$

$(\{g_1(z), \ldots, g_n(z)\} \subseteq B \restriction x$ and is a disjoint family, $\chi_{\leq 1/k}(x - \Sigma_{i=1}^n g_i(z))$, and $\{h_1(z), \ldots, h_m(z)\} \subseteq B \restriction y$ and is a disjoint family).

For every $\alpha \in \mathbb{R}^+$ let

$$\varphi_\alpha(x, y) \equiv \bigwedge \{\psi_{\leq m/n}(y, x) \mid 0 < m/n < \alpha\}$$
$$\wedge \bigwedge \{\psi_{\leq n/m}(x, y) \mid m/n > \alpha\}.$$

We leave it to the reader to check the proof of the following claims.

(1) If $\langle B_1, \mu_1, G_1\rangle \in K_{\mathrm{FM}}$ and $a, b \in B$, then: $M(B_1, G_1) \models \chi_{\leq 1/n}[a]$ implies that $\mu_1(a) \leq (1/n)\mu_1(1)$; $M(B_1, G_1) \models \psi_{\leq m/n}[a, b]$ implies that $m/n \cdot \mu_1(a) \leq \mu_1(b)$; and $M(B_1, G_1) \models \varphi_\alpha[a, b]$ implies that $\mu_1(a) = \alpha\mu_1(b)$.

(2) If $\langle B, \mu, G\rangle \in K_{\mathrm{T}}$, $B_0 \subseteq B$ is as in the definition of K_{T} and $a, b \in B_0$, then $M(B, G) \models \chi_{\leq 1/n}[a]$ iff $\mu(a) \leq (1/n)\mu(1)$, $M(B, G) \models \psi_{\leq m/n}[a, b]$ iff $m/n \cdot \mu(a) \leq \mu(b)$, and $M(B, G) \models \varphi_\alpha[a, b]$ iff $\mu(a) = \dot{\alpha} \cdot \mu(b)$.

This concludes the proof of the theorem. □

We now turn to groups of measure-zero preserving automorphisms. Recall that MZP(B, μ) denotes the group of measure-zero preserving automorphism of B. If μ is absolutely continuous with respect to ν ($\mu \ll \nu$) and $\nu \ll \mu$, then MZP$(B, \mu) =$ MZP(B, ν); thus we expect to reconstruct from MZP(B, μ) only the structure $\langle B, G; I_Z(B, \mu), \leq, Op\rangle$. We denote this structure by MZM(B, μ, G). Let K be a class of $\langle B, \mu, G\rangle$'s and $\langle B_0, \mu_0, G_0\rangle \in K$ we say that $\langle B_0, \mu_0, G_0\rangle$ is group-categorical in K if for every $\langle B, \mu, G\rangle \in K$ and an isomorphism φ between G_0 and G, φ can be extended to an isomorphism $\tilde{\varphi}$ between MZM(B_0, μ_0, G_0) and MZM(B, μ, G). We shall prove the counterparts of 6.1 and 6.2 in the present context. We do not know an analogue of 6.7 for the context of measure-zero preserving automorphisms.

Let

$$K_{\mathrm{ZM}} = \{\langle B, \mu, G\rangle \mid \mu \text{ is a positive } \sigma\text{-finite measure on } B,$$
$$\langle B, G\rangle \in K_{B^3} \text{ and } G \subseteq \mathrm{MZP}(B, \mu)\}.$$

Let

$$K_{\mathrm{ZH}} = \{\langle B, \mu, G\rangle \in K_{\mathrm{ZM}} \mid \mu \text{ is atomless and for every } a, b \in B \text{ if}$$
$$0 < \mu(a), \mu(b) < \mu(1), \text{ then there is } g \in G \text{ such that } g(a) = b\}.$$

If ν is a measure on B and $G \subseteq \mathrm{Aut}(B)$, we say that ν is G-non-singular if $G \subseteq \mathrm{MZP}(B, \nu)$.

6.8. THEOREM. *Every member of K_{ZH} is group-categorical in K_{ZM}.*

PROOF. Let $\langle B, \mu, G\rangle \in K_{\mathrm{ZH}}$ and $\langle B_1, \mu_1, G_1\rangle \in K_{\mathrm{ZM}}$, and let φ: $G \to G_1$ be an isomorphism. By Theorem 4.5(b), φ can be extended to an isomorphism $\tilde{\varphi}$ between $M(B, G)$ and $M(B_1, G_1)$. It suffices to show that $\tilde{\varphi}(I_Z(B, \mu)) = I_Z(B_1, \mu_1)$. Let

$$\chi_p(x) \equiv (\exists\{g_i \mid i \in \omega\} \subseteq G)\Big(\sum_{i\in\omega} g_i(x) = 1\Big).$$

It is clear that for every $\langle B_0, \mu_0, G_0\rangle \in K_{\mathrm{ZM}}$ and $a \in B_0$ if $M(B_0, G_0) \models \chi_p[a]$, and $\mu_0(a) > 0$. Also, if $a \in B$ and $\mu(a) > 0$, then $M(B, G) \models \chi_p[a]$. Thus, $\tilde{\varphi}(B - I_Z(B, \mu)) \subseteq B_1 - I_Z(B_1, \mu_1)$ and so $\tilde{\varphi}(I_Z(B, \mu)) \supseteq I_Z(B_1, \mu_1)$. W.l.o.g. $B_1 = B$ and $\tilde{\varphi} = Id$. Suppose by contradiction that $I_Z(B, \mu) - I_Z(B, \mu_1) \neq \emptyset$. Let $a \in I_Z(B, \mu) - I_Z(B, \mu_1)$ be such that for every $b \in I_Z(B, \mu)$ if $b \cdot a = 0$, then $\mu_1(b) = 0$. Let c, $d \in B$ be such that $a \leq c$, $c \cdot d = 0$ and $0 < \mu(c)$, $\mu(d) < \mu(1)$. Since $\langle B, \mu, G\rangle \in K_{\mathrm{ZH}}$ there is $g \in G$ such that $g(c) = d$, and since μ_1 is G-non-singular and $\mu_1(a) > 0$, $\mu_1(g(a)) > 0$. But $g(a) \cdot a = 0$ and $\mu(g(a)) = 0$. This contradicts the choice of a. Hence, $I_Z(B, \mu) = I_Z(B, \mu_1)$ and the proof is complete. □

We now return to Haar measures. The next theorem is an analogue of 6.2.

6.9. THEOREM. *Let K be a locally compact σ-compact group and μ be a left Haar measure on $\hat{B}(K)$. Let G be a B-closed subgroup of* $\mathrm{MZP}(\hat{B}(K), \mu)$ *and $G \supseteq K$. Let $a \in \hat{B}(K)$ be such that*: (1) *a is not the sum of one, two or six atoms, and* (2) *if $\hat{B}(K) \neq \mathrm{Bp}(K)$, then $\mathrm{int}_K(a) \neq \emptyset$, and if $\hat{B}(K) = \mathrm{Bp}(K)$, then for some $a' \in \mathrm{Br}(K)$, $\mathrm{int}_K(a') \neq \emptyset$ and $a = a'/I_Z(\mathrm{Br}(K), \mu)$. Then $\langle \hat{B}(K) \restriction a, \mu \restriction a, G(a)\rangle$ is group-categorical in K_{ZM}.*

The proof of 6.9 resembles the proof of 6.2. Let $K_{\mathrm{ZU}} = \{\langle B, \mu, G\rangle \mid \langle B, G\rangle \in K_{B^1}$, μ is atomless and for every positive σ-finite G-non-singular measure ν on B $\langle B, G\rangle$ is ν-weakly homogeneous$\}$.

6.10. LEMMA. (a) *If K,* $\mathrm{Br}(K)$, *μ, G and a satisfy the conditions of* 6.9, *then* $\langle \mathrm{Br}(K) \restriction a, \mu \restriction a, G(a)\rangle \in K_{\mathrm{ZU}}$.

(b) *Every member of K_{ZU} is group-categorical in K_{ZM}.*

PROOF. (a) The proof is similar to the proof of 6.4, and is left to the reader.

(b) The proof resembles that of 6.5. The only point where the argument differs is the following. Let $\langle B, \mu, G\rangle \in K_{\mathrm{ZU}}$ and ν_1, ν_2 be positive σ-finite G-non-singular measures on B such that $\nu_1 \ll \nu_2$, then $I_Z(B, \nu_1) = I_Z(B, \nu_2)$. Suppose $\nu_2(a) > 0$ and we prove that $\nu_1(a) > 0$. By the σ-finiteness of ν_2 there is a subset $\{g_i \mid i \in \omega\} \subseteq G$ such that for every $g \in G$, $\nu_2(g(a) - \Sigma_{i\in\omega}\, g_i(a)) = 0$. By the

ν_2-weak homogeneity of $\langle B, G\rangle$, $\nu_2(1-\Sigma_{i\in\omega} g_i(a))=0$, and since $\nu_1 \ll \nu_2$ $\nu_1(1-\Sigma_{i\in\omega} g_i(a))=0$. Hence, $\nu_1(\Sigma_{i\in\omega} g_i(a))=\nu_1(1)>0$, so $\nu_1(a)>0$. This concludes the proof of 6.10. □

In the same way that in metric spaces the uniform continuity of a function can be measured by its modulus of continuity, we can measure the non-singularity of a measurable function by its so-called non-singularity modulus (NS-modulus).

In what follows measure algebras are assumed to be positive and atomless; for simplicity we also assume that the measures are finite rather than being σ-finite. Let $\langle B_i, \mu_i\rangle$ be measure algebras and g be an isomorphism between B_1 and B_2 such that $g(I_Z(B_1, \mu_1))=I_Z(B_2, \mu_2)$; the NS-modulus of g, α_g: $\mathbb{R}^+\to\mathbb{R}^+$, is defined as follows: $\alpha_g(x)=\operatorname{Sup}(\{\mu_2(g(a)) \mid \mu_1(a)\le x\})$.

6.11. LEMMA. (a) *Let $\langle B_i, \mu_i\rangle$, $i=1, 2$, and g be as above, then* (1) $\alpha_g(0)=0$; (2) *α_g is continuous and concave, that is, for every $x, y\in\mathbb{R}^+$ and $0\le\lambda\le 1$, $\alpha_g(\lambda x+(1-\lambda)y)\ge\lambda\alpha_g(x)+(1-\lambda)\alpha_g(y)$*; (3) *for every $0\le x\le\mu_1(1)$, $\alpha_g(x)\ge[\mu_2(1)/\mu_1(1)]\cdot x$; and* (4) *$\alpha_g\restriction[0, \mu_1(1)]$ is strictly increasing.*

(b) *Let α be a non-zero continuous concave increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\alpha(0)=0$ and α is constant on some ray of the form $[a, \infty)$, then there are $\langle B_i, \mu_i\rangle$, $i=1, 2$, and g as above such that $\alpha_g=\alpha$.*

PROOF. (a) (1) follows from the fact that $g(I_Z(B_1, \mu_1))=I_Z(B_2, \mu_2)$. (2) Let ν: $B_1\to\mathbb{R}^2$ be defined by $\nu(a)=\langle\mu_1(a), \mu_2(g(a))\rangle$, then ν is a countably additive atomless vector-valued measure to a finite dimensional vector space. The Liapounoff convexity theorem [DIESTEL and UHL [1977, p. 264, Corollary 5] states that the range of ν is a compact convex set. Let $C=\nu(B_1)$ be the range of ν, then for $x\in[0, \mu_1(1)]$, $\alpha_g(x)=\max(\{y \mid \langle x, y\rangle\in C\})$. This implies that α_g is continuous and concave, and that the value of $\alpha_g(x)$ is attained. (3) follows from the concavity of α_g and the fact that $\alpha_g(0)=0$ and $\alpha_g(\mu_1(1))=\mu_2(1)$. (4) Let $0\le x<y<\mu_1(1)$. Let $a\in B_1$ be such that $\mu_1(a)=x$ and $\mu_2(g(a))=\alpha_g(x)$, let $a<b\in B_1$ be such that $\mu_1(b)=y$, then $\alpha_g(x)=\mu_2(g(a))<\mu_2(g(b))\le\alpha_g(y)$.

(b) Let $m_1=\min(\{m \mid \alpha\mid[m, \infty)$ is constant$\})$, and let $m_2=\alpha(m_1)$. If $\langle B_i, \mu_i\rangle$ is the Borel field of the interval $[0, m_i]$ with its Lebesgue measure, $i=1, 2$, and g is the isomorphism between B_1 and B_2 induced by α, then it is easy to see that $\alpha_g=\alpha$. □

A continuous concave increasing function α: $\mathbb{R}^+\to\mathbb{R}^+$ for which $\alpha(0)=0$ is called a *non-singularity modulus* (NS-modulus). Using sets of NS-moduli Γ, we can define a fine hierarchy of intermediate groups $\mathrm{MP}_\Gamma(B, \mu)$ lying between $\mathrm{MP}(B, \mu)$ and $\mathrm{MZP}(B, \mu)$. The goal of the last theorem in this chapter is to show that the group $\mathrm{MP}_\Gamma(B, \mu)$ characterizes Γ and that every automorphism of $\mathrm{MP}_\Gamma(B, \mu)$ is inner.

Let $\langle B_i, \mu_i\rangle$, $i=1, 2$, be measure algebras, g: $B_1\to B_2$ be an isomorphism and α be an NS-modulus, g is α-non-singular (α-NS) if $\alpha_g\le\alpha$, that is, for every $x\in\mathbb{R}^+$, $\alpha_g(x)\le\alpha(x)$; let Γ be a set of NS-moduli, g is Γ-NS if g is α-NS for some $\alpha\in\Gamma$, g is a Γ-isomorphism if both g and g^{-1} are Γ-NS. We denote the set of Γ-automorphisms of $\langle B, \mu\rangle$ by $\mathrm{MP}_\Gamma(B, \mu)$. If $Id\in\Gamma$, then $\mathrm{MP}(B, \mu)\subseteq$

$\mathrm{MP}_\Gamma(B, \mu)$ and if, in addition, Γ is closed under composition, then $\mathrm{MP}_\Gamma(B, \mu)$ is a subgroup of $\mathrm{MZP}(B, \mu)$.

We next describe the relationship between non-singular isomorphisms and uniformly continuous functions. The algebra $B/I_Z(B, \mu)$ carries the following metric ρ_μ: $\rho_\mu(a/I_Z(B, \mu), b/I_Z(B, \mu)) = \mu(a \triangle b)$. It is easy to see that ρ_μ is a complete metric. If f is a non-singular isomorphism between B_1 and B_2, that is, $f(I_Z(B_1, \mu_1)) = I_Z(B_2, \mu_2)$, then f induces an isomorphism $\tilde{f}$ between $B_1/I_Z(B_1, \mu_1)$ and $B_2/I_Z(B_2, \mu_2)$ and f is α-NS iff $\tilde{f}$ is α-continuous, that is, for every $x, y \in B_1/I_Z(B_1, \mu_1)$, $\rho_{\mu_2}(\tilde{f}(x), \tilde{f}(y)) \le \alpha(\rho_{\mu_1}(x, y))$.

From now on Γ denotes a set of NS-moduli containing the identity and closed under composition. There are trivial reasons why two different Γ's may define the same $\mathrm{MP}_\Gamma(B, \mu)$.

6.12. DEFINITION. Let Γ_1, Γ_2 be as above and $x \in (0, \infty)$, $\Gamma_1 \le_x \Gamma_2$, if for every $\alpha_1 \in \Gamma_1$ there is $\alpha_2 \in \Gamma_2$ such that $\alpha_1 \restriction [0, x] \le \alpha_2 \restriction [0, x]$. $\Gamma_1 \equiv_x \Gamma_2$ if $\Gamma_1 \le_x \Gamma_2$ and $\Gamma_2 \le_x \Gamma_1$, $\Gamma_1 \le \Gamma_2$ if for every x $\Gamma_1 \le_x \Gamma_2$ and $\Gamma_1 \equiv \Gamma_2$ if for every x, $\Gamma_1 \equiv_x \Gamma_2$.

We leave it to the reader to check that if for some $x \in (0, \infty)$, $\Gamma_1 \le_x \Gamma_2$, then $\Gamma_1 \le \Gamma_2$ and that if $\Gamma_1 \le_x \Gamma_2$ and $\langle B, \mu \rangle$ is a measure algebra such that $\mu(1) = x$, then $\mathrm{MP}_{\Gamma_1}(B, \mu) \subseteq \mathrm{MP}_{\Gamma_2}(B, \mu)$. By 6.11(b) the converse is true, at least if $\langle B, \mu \rangle$ is the Borel field of an interval in $\mathbb{R}$ with its Lebesgue measure.

Let $\Gamma_E = \{Id\}$, Γ_U denote the set of all NS-moduli and $\Gamma_L = \{nx \mid n \in \omega - \{0\}\}$. We leave it to the reader to check that if $\Gamma \not\equiv \Gamma_E$, then $\Gamma_L \le \Gamma$.

Let $K_{\mathrm{NS}} = \{\langle B, \mu \rangle \mid \mu$ is positive finite and atomless, for every $a, b \in B$ such that $\mu(a) < \mu(b)$ there is $f \in \mathrm{MP}(B, \mu)$ such that $f(a) < b$, and every $g \in \mathrm{MP}(B/I_Z(B, \mu), \mu)$ is induced by some member of $\mathrm{MP}(B, \mu)\}$.

Note that if a is a Borel subset of $\mathbb{R}$ of positive finite Lebesgue measure, then $\langle \mathrm{Bl}(R) \restriction a, \mu \rangle \in K_{\mathrm{NS}}$; the same is true for Lebesgue measurable sets. If μ is a strictly positive finite measure on B, and B is homogeneous, then $\langle B, \mu \rangle \in K_{\mathrm{NS}}$. See Maharam [1942].

Note that if $\langle B, \mu \rangle \in K_{\mathrm{NS}}$ for every Γ_1 and Γ_2: $\Gamma_1 \equiv \Gamma_2$ iff $MP_{\Gamma_1}(B, \mu) = MP_{\Gamma_2}(B, \mu)$. See Maharam [1942].

We are now ready to formulate our result on groups of Γ-automorphisms.

6.13. THEOREM. (RUBIN and YOMDIN [198?]). (a) *For $i = 1, 2$ let $\langle B_i, \mu_i \rangle \in K_{\mathrm{NS}}$ and Γ_i be either a countable set of NS-moduli containing the identity and closed under composition or $\Gamma_i = \Gamma_U$. Let τ be an isomorphism between* $\mathrm{MP}_{\Gamma_1}(B_1, \mu_1)$ *and* $\mathrm{MP}_{\Gamma_2}(B_2, \mu_2)$, *then $\Gamma_1 \equiv \Gamma_2$ and τ is induced by an isomorphism $\tilde{\tau}$ between B_1 and B_2. Either $\Gamma_1 \equiv \Gamma_E$ and for every $a \in B_1$, $\mu_2(\tilde{\tau}(a)) = [\mu_2(1)/\mu_1(1)]$. $\mu_1(a)$ or $\tilde{\tau}$ is a Γ_1-isomorphism.*

(b) *If $\langle B, \mu \rangle \in K_{\mathrm{NS}}$ and Γ is as in* (a), *then every automorphism of* $\mathrm{MP}_\Gamma(B, \mu)$ *is inner.*

The central argument in the proof of 6.13 is formulated in the following lemma.

6.14. LEMMA. *Let $\langle B_1, \mu_1 \rangle, \langle B_2, \mu_2 \rangle \in K_{\mathrm{NS}}$ and μ_1, μ_2 be strictly positive; let Γ be a countable set of NS-moduli containing all functions of the form $\alpha(x) = nx$,*

$n \in \omega - \{0\}$, and closed under composition; and let τ: $B_1 \to B_2$ be an isomorphism such that $\mathrm{MP}(B_1, \mu_1)^\tau \subseteq \mathrm{MP}_\Gamma(B_2, \mu_2)$, *then τ is a Γ-isomorphism.*

PROOF. We may regard $\mathrm{MP}(B_1, \mu_1)$ as a metric space using the following metric d: $d(f, g) = \mathrm{Sup}(\{\rho_{\mu_1}(f(a), g(a)) \mid a \in B\})$. It is easy to check that d is a complete metric on B_1. Hence, we shall be able to apply Baire category theorem. For every $\gamma_1, \gamma_2 \in \Gamma$, let $A_{\gamma_1\gamma_2} = \{f \in \mathrm{MP}(B_1, \mu_1) \mid f^\tau$ is γ_1-NS and $(f^{-1})^\tau$ is γ_2-NS$\}$. $\bigcup \{A_{\gamma_1\gamma_2} \mid \gamma_1, \gamma_2 \in \Gamma\} = \mathrm{MP}(B_1, \mu_1)$, and since Γ is countable there are $\gamma_1, \gamma_2 \in \Gamma$ such that $A_{\gamma_1\gamma_2}$ is somewhere dense in $\mathrm{MP}(B_1, \mu_1)$. Since τ is a continuous function from $\langle B_1, \rho_{\mu_1}\rangle$ to $\langle B_2, \rho_{\mu_2}\rangle$ it follows that $A_{\gamma_1\gamma_2}$ is closed, and hence it contains a non-empty open set V. VV^{-1} is a symmetric neighborhood of the identity contained in $A_{\gamma_1\gamma_2}$. Let $U_r^{\mathrm{MP}}(Id)$ be a ball with radius r and center at Id contained in V, and let $\gamma = \gamma_1 \circ \gamma_2$. Let $W = U_{r/2}^B(0)$ be a ball of $\langle B_1, \rho_{\mu_1}\rangle$ with radius $r/2$ and center at 0. We show that $\tau \restriction W$ is $n \cdot \gamma$-NS for some $n \in \omega$. Suppose by contradiction this is not true. Then for every $n \in \omega - \{0\}$ there is $a_n \in W$ such that $\mu_2(\tau(a_n)) > n \cdot \gamma(\mu_1(a_n))$. There is an interval of the form $[0, x_0]$ on which γ is strictly increasing. Since some member of $\mathrm{MZP}(B_2, \mu_2)$ is γ-NS, $\gamma \restriction [0, \mu_2(1)] \geq Id \restriction [0, \mu_2(1)]$, hence $\mu_2(1) \geq \mu_2(\tau(a_n)) > n \cdot \gamma(\mu_1(a_n)) \geq n \cdot \mu(a_n)$, hence $\mu_1(a_n) \to 0$. Hence, it can be assumed that $n \cdot \mu_1(a_n) \leq x_0$. Let $a \in B_1$ and $\mu_1(a) = m > r/2$. We show that $\mu_2(\tau(a)) = \infty$, this will contradict the finiteness of μ_2. Let $m_n = \mu_1(a_n)$. Let us fix n. There is $\{g_i \mid i = 1, \ldots, [m/m_n]\} \subseteq \mathrm{MP}(B_1, \mu_1)$ such that $\{g_i(a_n) \mid i = 1, \ldots, [m/m_n]\}$ is a disjoint family, and $g_i(a_n) \leq a$ and $g_i \restriction (1 - a_n - g_i(a_n)) = Id$ for every $i = 1, \ldots, [m/m_n]$. Let $c = \tau(a)$, $c_n = \tau(a_n)$ and $b_i = \tau(g_i(a_n))$. Hence $\{b_i \mid i = 1, \ldots, [m/m_n]\}$ is a disjoint family of subelements of c and $(g_i^{-1})^\tau(b_i) = c_n$. $g_i \in U_r^{\mathrm{MP}}(Id)$, and hence $(g_i^{-1})^\tau$ is γ-NS. Hence, $\mu_2(c_n) \leq \gamma(\mu_2(b_i))$. By the choice of a_n, $n \cdot \gamma(\mu_1(a_n)) < \mu_2(c_n)$, and by the concavity of γ, $\gamma(n \cdot \mu_1(a_n)) \leq n \cdot \gamma(\mu_1(a_n))$. Combining the above inequalities we obtain that $\gamma(n \cdot \mu_1(a_n)) \leq \gamma(\mu_2(b_i))$; and since γ is strictly increasing in $[0, n \cdot \mu_1(a_n)]$, $n \cdot \mu_1(a_n) \leq \mu_2(b_i)$. Now recall that $\mu_1(a_n) = m_n$ so $n \cdot m_n \leq \mu_2(b_i)$. $b_i, \ldots, b_{[m/m_n]}$ are pairwise disjoint subelements of c, and hence

$$\mu_2(c) \geq \sum_{i=1}^{[m/m_n]} \mu_2(b_i) \geq [m/m_n] \cdot m_n n > (m - m_n) \cdot n\,.$$

When n lends to infinity, m_n tends to zero and hence $(m - m_n) \cdot n$ tends to infinity. Thus, $\mu_2(c) = \infty$, and this contradiction implies that for some $n_0 \in \omega$, $\tau \restriction W$ is $n_0 \cdot \gamma$-NS.

Let $d_1, \ldots, d_l \in B_1$ be such that $\mu_i(d_i) = t < r/2$ and $\Sigma_{i=1}^l d_i = 1$. We show that τ is $l \cdot n_0 \cdot \gamma$-NS. Let $b \in B_1$ if $\mu_1(b) < r/2$, then $b \in W$ and hence $\mu_2(\tau(b)) \leq n_0 \cdot \gamma(\mu_1(b)) \leq l \cdot n_0 \cdot \gamma(\mu_1(b))$. If $\mu_1(b) \geq r/2$, then $\mu_2(\tau(b)) \leq \mu_2(1) = \Sigma_{i=1}^l \mu_2(d_i) \leq l \cdot n_0 \cdot \gamma(t) < l \cdot n_0 \cdot \gamma\, \mu_1((b))$. $l \cdot n_0 \cdot \gamma \in \Gamma$ since Γ contains the functions γ and $\alpha(x) = l \cdot n_0 \cdot x$ and is closed under composition. We have thus proved that τ is Γ-NS.

Using the same choice of γ, $U_r^{\mathrm{MP}}(Id)$ and W, we next show that $\tau^{-1} \restriction \tau(W)$ is $n \cdot \gamma$-NS for some $n \in \omega$. Suppose by contradiction the above is not true. Then for every $n \in \omega - \{0\}$ there is $a_n \in W$ such that $\mu_1(a_n) > n \cdot \gamma(\mu_2(\tau(a_n)))$. Let $a \in B_1$

and $\mu_1(a) = m > 0$. We show that $\mu_2(\tau(a)) = 0$, and this will contradict the fact that μ_2 is strictly positive. Let $m_n = \mu_1(a_n)$. We now fix n. Let $k = [m/m_m] + 1$, there is $\{g_i \mid i = 1, \ldots, k\} \subseteq \mathrm{MP}(B_1, \mu_1)$ such that $a \leq \Sigma_{i=1}^k g_i(a_n)$ and $g_i \restriction (1 - a_n - g_i(a_n)) = Id$ for every $i = 1, \ldots, k$. Clearly, each g_i belongs to $U_r^{\mathrm{MP}}(Id)$. Let $c_n = \tau(a_n)$, $c = \tau(a)$ and $b_i = \tau(g_i(a_n))$. Hence, $c \leq \Sigma_{i=1}^k b_i$. g_i^τ are γ-NS and $b_i = g_i^\tau(c_n)$, so $\mu_2(b_i) \leq \gamma(\mu_2(c_n))$. a_n was chosen so that $\gamma(\mu_2(c_n)) < (1/n)\mu_1(a_n)$. Combining the above inequalities we obtain that $\mu_2(b_i) < (1/n)\mu_1(a_n)$. Hence

$$\begin{aligned}\mu_2(c) &\leq \sum_{i=1}^k \mu_2(b_i) < \frac{k}{n}\,\mu_1(a_n) = \frac{1}{n}\left(\left[\frac{m}{m_n}\right] + 1\right) \cdot m_n \\ &\leq \frac{1}{n} \cdot (m + m_n) \leq \frac{1}{n} \cdot (m + \mu_1(1))\,.\end{aligned}$$

Hence, for every $n \in \omega - \{0\}$, $\mu_2(c) \leq (1/n) \cdot (m + \mu_1(1))$, that is, $\mu_2(c) = 0$. However, since $a \neq 0$ and $c = \tau(a)$, $c \neq 0$. This contradicts the fact that μ_2 is strictly positive. We have thus shown that $\tau^{-1} \restriction \tau(W)$ is $n_0 \cdot \gamma$-NS for some $n_0 \in \omega - \{0\}$. Let $\varepsilon > 0$ be such that $\alpha_{\tau^{-1}}(\varepsilon) < r/2$. Hence, for every $b \in B_2$, if $\mu_2(b) \leq \varepsilon$, then $b \in \tau(W)$. Let $d_1, \ldots, d_l \in B_2$ be such that $\Sigma_{i=1}^l d_i = 1$ and $\mu_2(d_i) = \varepsilon$, $i = 1, \ldots, l$. Let $d \in B_2$ if $\mu_2(d) \leq \varepsilon$, then $\tau^{-1}(d) \in W$ so $\mu_1(\tau^{-1}(d)) \leq n_0 \cdot \gamma(\mu_2(d)) \leq l \cdot n_0 \cdot \gamma(\mu_2(d))$. If $\mu_2(d) > \varepsilon$, then $\mu_1(\tau^{-1}(d)) \leq \Sigma_{i=1}^l \mu_1(\tau^{-1}(d_i)) \leq l \cdot n_0 \cdot \gamma(\varepsilon) \leq l \cdot n_0 \cdot \gamma(d)$. We have shown that for every $d \in B_2$, $\mu_1(\tau^{-1}(d)) \leq l \cdot n_0 \cdot \gamma(d)$, namely τ^{-1} is $l \cdot n_0 \cdot \gamma$-NS. Since $l \cdot n_0 \cdot \gamma \in \Gamma$, τ^{-1} is Γ-NS. We have completed the proof of the lemma.

REMARK. The method of 6.14 can be used to prove more difficult claims of the same type. For example. Let K_1 and K_2 be locally compact groups, and τ: $\mathrm{Br}(K_1) \cong \mathrm{Br}(K_2)$ be such that for every $k \in K_1$, $(t_k)^\tau$ is Γ-NS, then τ and τ^{-1} are Γ-NS.

Proof of Theorem 6.13. Let $\langle B_i, \mu_i \rangle$, Γ_i, $i = 1, 2$, and τ be as in Theorem 6.13. By Theorem 4.5(c), τ is induced by an isomorphism $\tilde{\tau}$: $B_1 \to B_2$. Suppose first that $\Gamma_1, \Gamma_2 \neq \Gamma_E$. By MAHARAM [1942], $\mathrm{MP}_{\Gamma_i}(B_i, \mu_i) \in K_{\mathrm{ZH}}$, $i = 1, 2$, hence by Theorem 6.8, $\tilde{\tau}(I_Z(B_1, \mu_1)) = I_Z(B_2, \mu_2)$. Let $B_i^{\mathrm{SP}} = B_i / I_Z(B_i, \mu_i)$; $\tilde{\tau}$ induces an isomorphism $\tilde{\tau}^{\mathrm{SP}}$ between B_1^{SP} and B_2^{SP} and $\alpha_{\tilde{\tau}^{\mathrm{SP}}} = \alpha_{\tilde{\tau}}$, hence if $\tilde{\tau}^{\mathrm{SP}}$ is a Γ-isomorphism for some Γ so is $\tilde{\tau}$. Since $\Gamma_2 \not\equiv \Gamma_E$, $\Gamma_L \leq \Gamma_2$, hence w.l.o.g. $\Gamma_L \subseteq \Gamma_2$. $(\mathrm{MP}(B_1^{\mathrm{SP}}, \mu_1))^{\tilde{\tau}^{\mathrm{SP}}} \subseteq \mathrm{MP}_{\Gamma_2}(B_2^{\mathrm{SP}}, \mu_2)$. If $\Gamma_2 = \Gamma_U$, then by 6.11(a), $\tilde{\tau}^{\mathrm{SP}}$ is a Γ_2-isomorphism; otherwise Γ_2 is countable and then by 6.14, $\tilde{\tau}^{\mathrm{SP}}$ is a Γ_2-isomorphism. Since Γ_2 is closed under composition, $\mathrm{MP}_{\Gamma_2}(B_1, \mu_1) = (\mathrm{MP}_{\Gamma_2}(B_2, \mu_2))^{\tilde{\tau}^{-1}} = \mathrm{MP}_{\Gamma_1}(B_1, \mu_1)$. Since $\langle B_1, \mu_1 \rangle \in K_{\mathrm{NS}}$, the above implies that $\Gamma_1 \equiv \Gamma_2$. We have proved the claim of the theorem for the case that $\Gamma_1, \Gamma_2 \not\equiv \Gamma_E$.

Suppose that $\Gamma_1 \equiv \Gamma_E$. Recall that $M(B, G) = \langle B, G; \leq, Op \rangle$. Let ψ be the following sentence in the language of $M(B, G)$:

$$\begin{aligned}&(\exists a_1, a_2 \in B)(\exists g_1, g_2 \in G)\Big(\bigwedge_{i=1}^{2} (a_i < 1) \wedge \bigwedge_{i=1}^{2} (a_i + g_i(a_i) = 1) \\ &\quad \wedge (\forall f \in G)(f(a_1) \neq a_2)\Big)\,.\end{aligned}$$

Let $G_i = \mathrm{MP}_\Gamma(B_i, \mu_i)$, $i=1,2$. Since $\Gamma_1 \equiv \Gamma_E$, $G_1 = \mathrm{MP}(B_1, \mu_1)$, hence $M(B_1, G_2) \models \psi$. Since $\tilde{\tau}$ is an isomorphism between $M(B_1, G_1)$ and $M(B_2, G_2)$, $M(B_2, G_2) \models \psi$. This can happen only if $G_2 = \mathrm{MP}(B_2, \mu_2)$, and it follows that $\Gamma_2 \equiv \Gamma_E$. Hence, $\langle B_1, \mu_1, G_1\rangle, \langle B_2, \mu_2, G_2\rangle \in K_{\mathrm{H}}$ and by Theorem 6.1 for every $a \in B_1$:

$$\mu_2(\tilde{\tau}(a)) = \frac{\mu_2(1)}{\mu_1(1)} \cdot \mu_1(a) .$$

(b) is a special case of (a).

6.15. QUESTION. Is Theorem 6.13 true when Γ is not restricted to be countable?

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CHAPTER 16

Embeddings and Automorphisms

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

General theorems concerning embeddings for certain classes of Boolean algebras are discussed, in particular embeddings into rigid algebras, homogeneous algebras and into algebras with no rigid or homogeneous factors. Extensibility of automorphisms of the embedded algebra to the larger algebra is one of the aspects our discussion is centered around. Other properties of algebras, namely distributivity and saturatedness, are also taken into account. It is obvious that no non-trivial automorphism of a subalgebra of a rigid algebra can be extended. More generally, if we define the center of an algebra as the subalgebra consisting of all elements that are left fixed by every automorphism, then no non-trivial automorphism of the center extends to an automorphism of the whole algebra.

Homogeneous and weakly homogeneous algebras have a trivial center consisting of the zero and unit elements and complete weakly homogeneous algebras are characterized by this condition. On the other hand, a non-trivial center always indicates some restrictions on the automorphisms of the algebra in question. Most results presented here deal with embeddings into algebras with a non-trivial center. In all cases, these embeddings preserve saturatedness. This contrasts with embeddings into complete (weakly) homogeneous algebras which have a trivial center, where no general saturatedness-preserving embedding theorem is provable in ZFC.

As we have already noted, embeddings into rigid Boolean algebras result in non-extensibility of automorphisms. On the other hand, Kripke's Theorem (13.3, 14.18 of Part I) shows that there are embeddings into homogeneous algebras such that every automorphism of the embedded algebra extends to the larger algebra. We shall show that there are two types of embeddings into algebras with no rigid or homogeneous factors: one with extensibility of automorphisms and the other one with non-extensibility of automorphisms. It is still open whether the same is true for homogeneous algebras, namely whether every algebra can be embedded in a homogeneous algebra in such a way that no non-trivial automorphism of the embedded algebra extends. Homogeneous algebras have trivial centers; however, results of Koppelberg and Monk [1983] indicate that embedding into the center is not necessary to prevent extensibility of automorphisms.

Most of the techniques presented here can be traced back to the first model-theoretical construction of complete rigid Boolean algebras due to Shelah [1975]. The existence of such algebras was implied by the results of McAloon [1970] about ordinal definable sets in Boolean-valued models. It was difficult, however, to extract a direct construction from his forcing argument. Shelah attacked the problem from a different angle. His construction is based on ideas from stability theory.

The organization of this chapter is as follows. Section 1 describes Shelah's construction of rigid complete Boolean algebras and Section 2 deals with complete embeddings of Boolean algebras into rigid complete Boolean algebras. Section 3 is concerned with embeddings into the center of a Boolean algebra. It is shown that every complete Boolean algebra is the center of another complete

Boolean algebra. In certain cases, the larger algebra has no rigid and no homogeneous factors. This gives the positive answer to a problem of McKENZIE and MONK [1975] who asked whether there are Boolean algebras without rigid or homogeneous factors. These algebras are studied in detail in Section 4, where a simpler construction is described. It is shown that for every uncountable cardinal κ, there are 2^κ isomorphism types of Boolean algebras of power κ with no rigid or homogeneous factors. Every Boolean algebra can be completely embedded in a complete algebra with no rigid or homogeneous factors in such a way that every automorphism of the smaller algebra extends to an automorphism of the larger one. It turns out that the automorphism group of the smaller algebra is a subgroup of the automorphism group of the larger algebra. On the other hand, the results of Section 3 give embeddings into the center of a Boolean algebra with no rigid or homogeneous factors, hence embeddings with non-extensibility of automorphisms. Section 5 lists some open problems.

Several results presented here are the joint work of B. Balcar and the author. I would like to thank R. Bonnet, J.D. Monk and S. Shelah for many helpful discussions concerning the material and open problems.

1. Rigid complete Boolean algebras

The first examples of rigid Boolean algebras were given by JÓNSSON [1951], KATĚTOV [1951] and RIEGER [1951]. The problem whether there are rigid algebras in the class of complete Boolean algebras was raised by Jónsson. It was answered positively by MCALOON [1970] by means of Boolean-valued models of set theory. In all known examples, the power of rigid algebras was at least continuum. McKENZIE and MONK [1975] asked whether there are rigid Boolean algebras of cardinality $\aleph_1$ independently of the size of the continuum. Such an algebra cannot be complete if $2^{\aleph_0} > \aleph_1$, since the infinite powers of complete algebras satisfy the condition $\kappa^\omega = \kappa$. SHELAH [1975] showed that for every uncountable cardinal κ, there is a rigid Boolean algebra of power κ with a rigid completion. His construction was extended in several ways. BALCAR and ŠTĚPÁNEK [1977] proved that every Boolean algebra can be completely embedded in a rigid complete Boolean algebra. Similar embeddings preserving certain types of distributivity were constructed by ŠTĚPÁNEK [1978]. MONK and RASSBACH [1979] showed that there are 2^κ isomorphism types of rigid Boolean algebras in any uncountable power κ. In particular, for every regular κ, $\kappa^\omega = \kappa$, there are 2^κ isomorphism types of complete rigid Boolean algebras of power κ. Recently, Shelah extended this result to all cardinals κ, $\kappa^\omega = \kappa$.
We shall start with the fundamental construction.

1.1. THEOREM (SHELAH). *For every uncountable cardinal κ, there is a rigid Boolean algebra of power κ with rigid completion. If κ is regular, the algebra satisfies the countable chain condition.*

1.2. The first step is to prove the theorem for regular cardinals. We shall construct an algebra of power κ for each regular uncountable κ and prove that it is rigid in 1.8. We prove in 1.9 that it satisfies CCC and that the algebra has rigid completion in 1.13. Rigid algebras of singular powers are easily obtained as subalgebra of products of rigid algebras of regular powers.

Given a regular uncountable cardinal κ, let A be a free Boolean algebra with κ free generators $a(\alpha)$, $\alpha < \kappa$. For every subset s of κ, let $A(s)$ denote the subalgebra of A generated by $a(\alpha)$, $\alpha \in s$. Hence $A = A(\kappa)$ and every element a of A belongs to $A(s)$ for some finite s. The least finite s such that $a \in A(s)$ is uniquely determined by a. We shall denote it by $s(a)$ and we shall call it the support of a. Every element of A is a finite join of elements $\varepsilon_1 a(\alpha_1) \cdot \cdots \cdot \varepsilon_n a(\alpha_n)$, where α_i, $i \le n$, are distinct ordinals, $\varepsilon_i \in \{-1, 1\}$ and $\varepsilon_i a(\alpha_i)$ is $a(\alpha_i)$ whenever $\varepsilon_i = 1$, or $-a(\alpha_i)$ if $\varepsilon_i = -1$. We can identify every such element with the finite mapping d, such that $\mathrm{dom}(d) = \{\alpha_1, \ldots, \alpha_n\}$ and $d(\alpha_i) = \varepsilon_i$ for every $i \le n$. Let D be the set of all mappings from finite subsets of κ to the two-element set $\{-1, 1\}$. Then D corresponds to a dense subset of A and the partial ordering induced on D by the canonical ordering of A is the reversed inclusion, i.e. $d \le e$ iff $e \subseteq d$. If d, $e \in D$ are compatible mappings, then $e \cup d$ is their common extension which corresponds to $e \cdot d$. *If* d and e are incompatible mappings, then $e \cdot d = 0$. Clearly, $d \in A(\mathrm{dom}(d))$ and $s(d) = \mathrm{dom}(d)$ holds for every $d \in D$.

1.3. We are constructing a rigid algebra R as a quotient of A modulo a suitable ideal I. To define I, let W be the stationary subset of κ defined as follows

$$W = \{\alpha < \kappa \colon \mathrm{cf}(\alpha) = \omega \text{ and } \alpha \text{ is divisible by } |\alpha|\}. \tag{0}$$

Thus, $\alpha \in W$ iff $\alpha < \kappa$ is a cardinal of cofinality ω or $\alpha = |\alpha| \cdot \delta$ for some ordinal δ, $\mathrm{cf}(\delta) = \omega$. Let $\{W_\alpha \colon \alpha < \kappa\}$ be a partition of W into disjoint stationary subsets. If we put

$$S_\alpha = W_\alpha - (\alpha + 1) \quad \text{for every non-zero } \alpha < \kappa,$$

and

$$S_0 = W - \bigcup \{S_\alpha \colon 0 < \alpha < \kappa\},$$

we get a new partition of W into disjoint stationary subsets S_α such that for every $\alpha < \kappa$, $\delta \in S_\alpha$ implies $\delta > \alpha$.

For every $\alpha \in W$, the set

$$D(\alpha) = \{d \in D \colon \mathrm{dom}(d) \subseteq \alpha\}$$

has power $|\alpha|$. As every $\alpha \in W$ is divisible by $|\alpha|$, there is a mapping φ from κ onto D such that φ maps every $\alpha \in W$ onto $D(\alpha)$. Thus, for every $\alpha \in W$, φ provides an indexing of elements of $D(\alpha)$ by ordinals $\beta < \alpha$. Now A is a free algebra, hence the filter

$$F(a, A(\alpha)) = \{b \in A(\alpha) \colon b \ge a\}$$

is principal for every non-zero $a \in A$ and every $\alpha < \kappa$.

The ideal I has two types of generators. The first ones make non-principal every filter corresponding to $F(a(\alpha), A(\alpha))$ for $\alpha \in W$ in the quotient $R = A/I$, while the second ones put every element of R above stationary many $a(\alpha)$'s. Let I be the ideal in A generated by

$$a(\delta) - a(\xi(\delta, n)) \quad \text{for } \delta \in W,\ n \leqslant \omega ,$$

and

$$a(\delta) - \varphi(\alpha) \qquad \text{for } \alpha < \kappa \text{ and } \delta \in S_\alpha ,$$

where $\langle \xi(\delta, n)\colon n < \omega \rangle$ is an increasing sequence of isolated ordinals with limit δ and disjoint from $\operatorname{dom}(\varphi(\alpha))$, where α is the unique ordinal such that $\delta \in S_\alpha$.

If we denote by $[a]$ the equivalence class of $a \in A$ in the quotient $R = A/I$, we have

(1) $\quad [a(\delta)] \leq [a(\xi(\delta, n))]$ for every $\delta \in W,\ n < \omega$,

(2) $\quad [a(\delta)] \leq [\varphi(\alpha)]$ for every $\alpha < \kappa$ and $\delta \in S_\alpha$.

Let $R(\alpha)$ denote the subalgebra of R generated by $[\alpha(\beta)]$, $\beta < \alpha$. We will prove in Lemma 1.7 that the filter

(3) $\quad F([a(\delta)], R(\delta)) = \{b \in R(\delta)\colon b \geq [a(\delta)]\}$

is non-principal for every $\delta \in W$, and that the filter

$$F(a, R(\delta)) = \{b \in R(\delta)\colon b \geq a\}$$

is principal whenever $a \in R$ is disjoint from $[a(\delta)]$.

Now, if G is an automorphism of R, then there is a closed unbounded $C \subseteq \kappa$ such that

(4) $\quad G(R(\delta)) = R(\delta)$ for every $\delta \in C$.

Suppose that G is non-trivial. Then there exists a non-zero a such that a and $G(a)$ are disjoint. We may assume that $a = [\varphi(\alpha)]$ for some $\alpha < \kappa$ since D is dense in A. Consequently, for every $\delta \in S_\alpha$, we have $[a(\delta)] \leq [\varphi(\alpha)]$ and the filter $F([a(\delta)], R(\delta))$ is non-principal.

If $\delta \in S_\alpha \cap C$, then $G \restriction R(\delta)$ is an automorphism of $R(\delta)$ and

$$G(F([a(\delta)], R(\delta))) = F(G([a(\delta)]), R(\delta)) .$$

We get a contradiction, the filter of the left-hand side is non-principal while the filter on the right-hand side is principal, since $G([a(\delta)])$ is disjoint from $[a(\delta)]$. Hence, there is no non-trivial automorphism of R and R is rigid.

We shall examine the properties of R in detail and we break the argument in a sequence of lemmas. We shall start with the ideal I. To every finite set of support $s \subseteq \kappa$ we associate an element $I(s)$ of I bigger than every $a \in A(s) \cap I$. Given s, if there is an ordinal $\delta_0 \in s \cap W$ that belongs to some S_α, then $a(\delta_0) - \varphi(\alpha)$ belongs to I and $s(\varphi(\alpha))$ consists of ordinals below δ_0. Similarly, if there is an ordinal $\delta_1 \in s(\varphi(\alpha)) \cap W$ that belongs to some S_β, then $\delta_1 < \delta_0$ and $s(\varphi(\beta))$ consists of ordinals below δ_1. In this way, every $\delta \in s \cap W$ determines a tree, the branches of which are decreasing sequences of ordinals. Hence, there is a finite t such that $s \subseteq t$ and $s(\varphi(\alpha)) \subseteq t$ for every α, $S_\alpha \cap t \neq 0$. Let t_s be the least set with this property.

If we put

$$I(s) = \sum \{a(\delta) - \varphi(\alpha) \colon \delta \in S_\alpha \cap t_s\} + \sum \{a(\delta) - a(\xi(\delta, n)) \colon \delta \in W \cap t_s \text{ and } \xi(\delta, n) \in t_s\}$$

then $I(s) \in I$. Moreover, we have

1.4. LEMMA. *For every $a \in A$, $a \in I$ iff $a \leq I(s(a))$.*

PROOF. Clearly, if $a \leq I(s(a))$, then $a \in I$. Conversely, suppose that $a \not\leq I(s(a))$. Then there is a non-zero $d \in A(t_{s(a)})$, $d \leq a$ disjoint from $I(s(a))$ since both a and $I(s(a))$ belong to $A(t_{s(a)})$. It suffices to show that d does not belong to I. We can assume that $d \in D$ since D is dense in A. Suppose that $d \in I$; then there are ordinals $\delta_1, \ldots, \delta_k, \eta_1, \ldots, \eta_m \in W$ and natural numbers $n_1, \ldots, n_m$ such that

$$\begin{aligned} (5) \qquad d \leq {} & \sum \{a(\delta_i) - \varphi(\alpha_i) \colon \delta_i \in S_{\alpha_i} \text{ and } i \leq k\} \\ & + \sum \{a(\eta_j) - a(\xi(\eta_j, n_j)) \colon j \leq m\}\,, \end{aligned}$$

where no δ_i belongs to $t_{s(a)}$ and $\xi(\eta_j, n_j) \not\in t_{s(a)}$ whenever $\eta_j \in t_{s(a)}$. We extend the mapping d giving the value -1 to every δ_i, $i \leq k$, and to every $\eta_j \not\in t_{s(a)}$. We give the value -1 to every $\xi(\eta_j, n_j) \not\in t_{s(a)}$. Consequently, we get a non-zero $d_1 \leq d$ disjoint from the right-hand side of (5). This contradicts our assumption on d, hence $a \not\leq I(s(a))$ implies $a \not\in I$. □

1.5. We shall write $I(a)$ instead of $I(s(a))$. With this notation we have $s(I(a)) = t_{s(a)}$. If a and b are compatible elements of A, then $s(I(a \cdot b)) = t_{s(a \cdot b)} = t_{s(a)} \cup t_{s(b)}$. Moreover, if $a \not\in I$, then there is a non-zero $d \in D$, $d \leq a$ and $s(d) = s(I(a)) = s(I(d))$ such that d is disjoint from $I(a)$.

Now, let $R = A/I$ and for every $a \in A$ let $[a] \in R$ be the equivalence class modulo I corresponding to a. For every $\alpha < \kappa$, let $R(\alpha)$ be the subalgebra of R generated by $[a(\xi)]$, $\xi < \alpha$. Similarly, let $D(\alpha) = D \cap A(\alpha)$.

1.6. LEMMA. *For every $\delta \in W$ and $d \in D$ such that $[d] \cdot [a(\delta)] = 0$, there is $d_1 \in D(\delta)$, $[d] \leq [d_1]$ such that*

(6) *for every $e \in D(\delta)$, if $[e]$ is disjoint from $[d]$ then it is disjoint from $[d_1]$.*

PROOF. The case $[d]=0$ is trivial, let $[d]$ be non-zero and disjoint from $[a(\delta)]$. Then either $d(\delta)=-1$ or δ does not belong to the domain of d. Let $T=\{\alpha\in s(d)\colon \alpha\geq\delta\}$. If T is empty, then $d\in D(\delta)$ and there is nothing to prove. Suppose that T is non-empty and that $\tau=\max(T)$. We shall construct $d'\in D(\tau)$, $[d]\leq[d']$ which satisfies (6). By iterating the same construction finitely many times, we get an element d_1.

First, we consider the case where $\tau\in W$ and $d(\tau)=-1$ and the case $\tau\not\in W$. In both cases, we put $d'=d\restriction\tau$. Then $[d]\leq[d']$ and (6) immediately follows. There remains the case $\tau\in W$ and $d(\tau)=1$. Then $\tau\in S_\alpha$ for some α and by applying (1) and (2), we have $[\varphi(\alpha)]\geq[d]$ and $[a(\xi(\tau,n))]\geq[d]$ for every $n\leq\omega$. There are only finitely many natural numbers $n_1,\ldots,n_k$, $k\geq 0$, such that $\xi(\tau,n_i)<\delta<\tau$. Moreover, $s(\varphi(\alpha))=\mathrm{dom}(\varphi(\alpha))\subseteq\tau$ since $\alpha<\tau$, and the mappings d, $\varphi(\alpha)$ are compatible since $[\varphi(\alpha)]\geq[d]\neq 0$ in R. Hence $d''=d\cup\varphi(\alpha)$ belongs to D. If we put

$$d'=a(\xi(\tau,n_1))\cdot\cdots\cdot a(\xi(\tau,n_k))\cdot(d''\restriction\tau)\,,$$

then $d'\in D(\tau)$. We shall show that d' satisfies (6). Let $e\in D(\delta)$ be an arbitrary element such that $[e]$ and $[d]$ are disjoint in R. If e and d are incompatible in D, then the same holds for $d''\restriction\tau$ and e, and so for d' and e. Then there is nothing to prove. Suppose that e and d are compatible in D, then $e\cdot d=e\cup d$ and $e\cdot d\leq I(e\cdot d)$, where

$$(7)\qquad I(e\cdot d)=\textstyle\sum\{a(\beta)-\varphi(\gamma)\colon \beta\in S_\gamma\cap t\}$$
$$+\textstyle\sum\{a(\varepsilon)-a(\xi(\varepsilon,n))\colon \varepsilon\in W\cap t \text{ and } \xi(\varepsilon,n)\in t\}$$

and

$$t=t_{s(e\cdot d)}\,.$$

Now we consider e and d'. If they are incompatible in D, then there is nothing to prove. If e and d' are compatible, then $e\cdot d'=e\cup d'\in D$ and

$$s(I(e\cdot d'))\subseteq(t-\{\tau\})\cup\{\xi(\tau,n_1),\ldots,\xi(\tau,n_k)\}\,,$$

since $e\in D(\delta)$ implies $s(I(e))\subset\delta$ and $\delta\leq\tau$. Now we will show that $e\cdot d\leq I(e\cdot d)$ implies that $[e]$ is disjoint from $[d']$ in R. If $e\cdot d\leq a(\tau)-a(\xi(\tau,n))$ for some n, then $\xi(\tau,n)$ belongs to $\mathrm{dom}(e)$ since $[d]$ is non-zero. In this case $\xi(\tau,n)<\delta$ and e, d' are incompatible mappings. Similarly, if $e\cdot d\leq a(\tau)-\varphi(\alpha)$, then e and d' are incompatible. In both cases $[e]$ is disjoint from $[d']$. The rest follows from the fact that all other elements from the right-hand side of (7) are used in the definition of $I(e\cdot d')$. □

1.7. LEMMA. *For every $\delta\in W$,*

(i) *$F([a(\delta)],R(\delta))=\{v\in R(\delta)\colon v\geq[a(\delta)]\}$ is a non-principal filter,*

(ii) *If u is an element of R disjoint from $[a(\delta)]$, then*

$$F(u, R(\delta)) = \{v \in R(\delta) : v \geq u\}$$

is a principal filter.

PROOF. (i) For every $d \in D(\delta)$, there is $n < \omega$ such that $[d]$ has a non-empty intersection with $-[a(\xi(\delta, n))]$, hence there is no $[a] \in R(\delta)$ such that $[a] \leq [a(\xi(\delta, n))]$ holds for all $n < \omega$. It is not difficult to show that

(8) $$[a(\delta)] = \prod \{[a(\xi(\delta, n))] : n < \omega\}$$

holds in R and so in its completion as well.

To prove (ii), let $\delta \in W$ and let u be any element of R disjoint from $[a(\delta)]$. There are $d_1, \ldots, d_n \in D$ such that

$$u = [d_1] + \cdots + [d_n] .$$

It follows from Lemma 1.6 that there are $d_1^\delta, \ldots, d_n^\delta \in D(\delta)$, such that for every $i \leq n$, we have $[d_i^\delta] \geq [d_i]$ and for every $e \in D(\delta)$, $[e]$ is disjoint from $[d_i]$ iff it is disjoint from $[d_i^\delta]$. If we put

$$u^\delta = [d_1^\delta] + \cdots + [d_n^\delta] ,$$

then $u^\delta \geq u$ and for every $v \in R(\delta)$, we have

(9) $$u \leq v \quad \text{iff } u^\delta \leq v .$$

Clearly, $-v = [e_1] + [e_2] + \cdots + [e_m]$ for some $e_1, e_2, \ldots, e_m \in D(\delta)$, and $u \leq v$ iff u is disjoint from $-v$. Hence, (9) follows from Lemma 1.6. □

1.8. If G is an automorphism of R, then there is a closed unbounded set $C \subseteq \kappa$ such that each $R(\delta)$, $\delta \in C$, is closed with respect to G and G^{-1}. Hence, $G(R(\delta)) = R(\delta)$ for every $\delta \in C$. We have proved all claims stated in 1.3 and thus we have completed the proof that R is a rigid Boolean algebra.

We want to extend this argument to the completion of R, which requires more information about the powers of antichains in R.

1.9. LEMMA. *R satisfies the countable chain condition.*

PROOF. Suppose that there is an antichain $\langle \alpha_\alpha : \alpha < \omega_1 \rangle$ in R. According to 1.5 we may suppose that each $\alpha_\alpha = [d_\alpha]$ for some $d_\alpha \in D$, where $s(d_\alpha) = s(I(d_\alpha))$ and d_α is disjoint from $I(d_\alpha)$. If $\beta < \gamma < \omega_1$, then one of the following conditions hold:

(10) d_β, d_γ are incompatible in D, i.e. $d_\beta \cup g_\gamma \not\in D$;

(11) there exists $\delta \in W$, $n < \omega$ such that $d_\beta(\delta) = 1$ and $d_\gamma(\xi(\delta, n)) = -1$ or vice versa.

Indeed, assume that d_β and d_γ are compatible in D. If we put $d = d_\beta \cdot d_\gamma$, then $s(d) = s(d_\beta) \cup s(d_\gamma) = s(I(d))$. Now, $d_\beta \cdot I(d_\beta) = 0$ and $d_\gamma \cdot I(d_\gamma) = 0$ implies

$$d \cdot \sum \{a(\delta) - \varphi(\alpha) \colon \delta \in S_\alpha \cap s(d)\} = 0 .$$

Consequently, $d = d_\beta \cdot d_\gamma \leq I(d)$ implies $d(\delta) = 1$ and $d(\xi(\delta, n)) = -1$ for some $\delta \in W$ and $n < \omega$, which gives (11).

A standard Δ-lemma argument shows that the case (10) is inessential, namely that there is a subset $J \subseteq \omega_1$, $|J| = \omega_1$, such that any pair d_α, d_β, $\alpha, \beta \in J$, $\alpha \neq \beta$, satisfies (11). The domain $s(d_\alpha)$ of each d_α, $\alpha < \omega_1$, is a finite set, hence there is a finite r and $J \subseteq \omega_1$, $|J| = \omega_1$, such that $s(d_\alpha) \cap s(d_\beta) = r$ whenever α, β are different elements of J. Moreover, we may assume that $d_\alpha \restriction r = d_\beta \restriction r$ since there are only finitely many choices for $d_\alpha \restriction r = d_\beta \restriction r$ since there are only finitely many choices for $d_\alpha \restriction r$. Then $\{d_\alpha \colon \alpha \in J\}$ is a family of pairwise compatible mappings and both the ordinals δ, $\xi(\delta, n)$ required by (11) must be outside of r, for otherwise one of the elements d_α, d_β would belong to I. Thus, we may leave out the whole r from the domains of mappings d_α, $\alpha \in J$, and assume that any two of them are compatible.. For every $\alpha \in J$, we put

$$A_\alpha = \{\xi(\delta, n) \in s(d_\alpha) \colon d_\alpha(\xi(\delta, n)) = -1\}$$

and

$$B_\alpha = \{\xi(\delta, m) \colon \delta \in W, m < \omega \text{ and } d_\alpha(\delta) = 1\} ,$$

then $\{A_\alpha, \alpha \in J\}$ is a family of pairwise disjoint finite sets of ordinals and the order-type of each B_α, $\alpha \in J$, is less than $\omega \cdot \omega$. It follows from (11) that for every $\alpha, \beta \in J$, $\alpha \neq \beta$, one of the intersections $A_\alpha \cap B_\beta$, $A_\beta \cap B_\alpha$ is non-empty. The rest of the proof follows from the next lemma.

1.10. LEMMA. *Let μ be an infinite ordinal. For every j, $j \in J$, let $A(j)$, $B(j)$ be sets of ordinals satisfying the following properties.*

(i) *$\{A(j) \colon j \in J\}$ is a pairwise disjoint family of finite sets,*

(ii) *The order-type of each $B(j)$, $j \in J$, is at most μ, and*

(iii) *for every $j, k \in J$, $j \neq k$, we have $A(j) \cap B(k) \neq 0$ or $A(k) \cap B(j) \neq 0$.*

Then the cardinality of J is at most that of μ.

PROOF. Let $\Lambda = |\mu|$. Suppose that the power of J is at least Λ^+. It follows from the regularity of Λ^+ that there is a natural number k and a subset K of J, $|K| = \Lambda^+$, such that each $A(j)$, $j \in K$, consists exactly of k elements $\alpha_0(j) < \alpha_1(j) < \cdots < \alpha_{k-1}(j)$. Since the $A(j)$'s are pairwise disjoint, the order-type of the set $\{\alpha_i(j) \colon j \in K\}$ is at least Λ^+, we can shrink down the set K in k steps to $K' \subseteq K$, $|K'| = \Lambda^+$, with a "coherent" ordering of columns, i.e. such that

$$\alpha_i(j) < \alpha_i(j') \quad \text{iff } \alpha_0(j) < \alpha_0(j')$$

holds for every $j, j' \in K'$ and every $i < k$.

We may identify the elements of K' with ordinals less than Λ^+ and construct an increasing sequence $\langle \beta(\gamma), \gamma < \Lambda^+ \rangle$ such that

$$(12) \qquad A(\beta(\gamma)) \cap B(\beta(\delta)) \neq 0 \text{ whenever } \gamma < \delta < \Lambda^+ .$$

We proceed by induction on γ and put

$$\beta(0) = 0 ,$$

$$\beta(\gamma + 1) = \sup\{\delta + 1 \colon \delta = \beta(\gamma) \text{ or } B(\beta(\gamma)) \cap A(\delta) \neq 0\} ,$$

$$\beta(\gamma) = \sup\{\beta(\delta) + 1 \colon \delta < \gamma\} \text{ if } \gamma \text{ is a limit ordinal} .$$

Note that each $\beta(\gamma + 1)$ is defined correctly since $|B(\beta(\gamma))| \leq \Lambda$ and the $A(\delta)$'s are pairwise disjoint, hence $\beta(\xi)$ are defined for every $\xi < \Lambda^+$.

Now, we can derive a contradiction. Let $\delta = \omega^{\mu+1}$ (ordinal power). If we put

$$\Gamma_i = \{\gamma < \omega^{\mu+1} \colon a_i(\beta(\gamma)) \in B(\beta(\delta))\} \quad \text{for every } i < k ,$$

then

$$\omega^{\mu+1} = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_{k-1}$$

and some Γ_i has order type $\omega^{\mu+1} \geq \mu + 1$, contradicting (ii). Hence, $|J| \leq \Lambda$ and the proof is complete. □

1.11. Now we can define a sequence $Q(\alpha)$, $\alpha < \kappa$, of subsets of the completion Q of R which will play a similar role for Q as did the sequence of subalgebras $R(\alpha)$ for R.

For every $\alpha < \kappa$, let $Q(\alpha) \subseteq Q$ be the set of all joins of subsets of $R(\alpha)$. Hence, $a \in Q(\alpha)$ iff $a = \Sigma P$ for some $P \subseteq R(\alpha)$. It is clear that $R(\alpha) \subseteq Q(\alpha)$ and that every element of $Q(\alpha)$ is a join of a subset of $\{[d] \colon d \in D(\alpha)\}$. Each set $Q(\alpha)$ is closed with respect to arbitrary joins and it follows from (8) that it need not be closed under arbitrary meets and complements if $\alpha \in W$. According to Lemma 1.9, R satisfies CCC and the same holds for its completion Q. Since κ is an uncountable regular cardinal, we have

$$(13) \qquad Q = Q(\kappa) ,$$

and Lemma 1.7 generalizes to Q immediately.

1.12. LEMMA. *For every $\delta \in W$,*

(i) *$F([a(\delta)], Q(\delta)) = \{v \in Q(\delta) \colon v \geq [a(\delta)]\}$ is a non-principal filter;*

(ii) *if $q \in Q$ is disjoint from $[a(\delta)]$, then the filter $F(q, Q(\delta)) = \{v \in Q(\delta) \colon v \geq q\}$ is principal.*

1.13. LEMMA. *The completion Q of R is a rigid complete Boolean algebra.*

PROOF. Let G be an automorphism of Q. It follows from (13) that there is a closed unbounded set $C \subseteq \kappa$ such that $G(Q(\delta)) = Q(\delta)$ for every $\delta \in C$. The rest of the proof follows by the same argument as in 1.3 using Lemma 1.12 instead of 1.7. □

1.14. We have shown that for every regular uncountable cardinal κ, there is a rigid Boolean algebra of power κ which satisfies CCC and has a rigid completion. In fact, the algebra and its completion have no non-trivial one-to-one countably complete endomorphisms. It remains to show that there is a rigid algebra of power κ with a rigid completion for every singular κ.

We shall first observe that the above construction gives totally different rigid algebras for different regular cardinals. Every non-trivial factor $R(\kappa) \restriction a$ of a rigid algebra of regular power κ described above has power κ. Hence, rigid algebras $R(\kappa)$, $R(\Lambda)$ of different regular powers have no isomorphic nontrivial factors, i.e. are totally different. We shall show that the same holds for their completions.

1.15. LEMMA. *Let κ, Λ be regular cardinals, $\kappa < \Lambda$ and let $Q(\kappa)$, $Q(\Lambda)$ be the completions of rigid Boolean algbras $R(\kappa)$, $R(\Lambda)$ constructed in the proof of* Theorem 1.1 *for cardinals κ and Λ. If a, b are non-zero elements of $Q(\kappa)$, $Q(\Lambda)$, respectively, then $Q(\kappa) \restriction a$ is not isomorphic to $Q(\Lambda) \restriction b$.*

PROOF. Suppose that F is an isomorphism that maps $Q(\kappa) \restriction a$ onto $Q(\Lambda) \restriction b$. Let us recall that $Q(\Lambda) = \bigcup \{Q(\beta)\colon \beta < \Lambda\}$ and that $Q(\kappa) = \bigcup \{Q(\alpha)\colon \alpha < \kappa\}$, where $Q(\alpha)$ consists of all joins of subsets of $\{[d]\colon d \in D(\alpha)\}$. The isomorphism F determines a non-decreasing mapping $f: \kappa \to \Lambda$ defined as follows: for every $\alpha < \kappa$, $f(\alpha)$ is the least $\beta < \Lambda$ such that $F([d]) \in Q(\beta)$ for each $d \in D(\alpha)$, $[d] \leq a$. Now f is bounded below Λ, since Λ is regular and $\kappa < \Lambda$. Consequently, $F(Q(\kappa) \restriction a) \subseteq Q(\gamma)$ for $\gamma < \Lambda$. This contradicts our assumption on F, since $Q(\Lambda) \restriction b$ contains elements of $Q(\delta)$ for arbitrary large $\delta < \Lambda$. □

1.16. LEMMA. *For every singular cardinal κ, there is a rigid Boolean algebra of power κ with a rigid completion.*

PROOF. Given a singular cardinal κ, let $\nu = \mathrm{cf}(\kappa)$ and let κ_α, $\alpha < \nu$, be an increasing sequence of regular cardinals with limit κ. Let R be the product of algebras $R(\kappa_\alpha)$, $\alpha < \nu$, where each $R(\kappa_\alpha)$ is a rigid algebra of power κ_α constructed as above. Clearly, R is rigid since it is a product of totally different rigid algebras. It is not difficult to observe that the completion Q of R is isomorphic to the product of completions $Q(\kappa_\alpha)$ of algebras $R(\kappa_\alpha)$. It follows from the previous lemma that Q is rigid too. It remains to cut R down to size κ. Every element r of R is a sequence $\langle r_\alpha\colon \alpha < \nu\rangle$, where $r_\alpha \in R(\kappa_\alpha)$ for every $\alpha < \nu$. Let J be the ideal on R of all sequences with finitely many non-zero members. Clearly, J is non-principal and $J \cup \{-r\colon r \in J\}$ is a subalgebra of R of power κ. The rest of the proof follows from the following simple lemma. □

1.17. LEMMA. *Let R be a rigid algebra and J a non-principal ideal on R. Then $R_J = J \cup \{-r\colon r \in J\}$ is a rigid Boolean algebra.*

PROOF. Let F be a non-trivial automorphism of R_J, then there is a non-zero r disjoint from $F(r)$. Note that $R \restriction r = R_J \restriction r$ for every $r \in J$. Hence, both r and $F(r)$ cannot belong to J. Suppose that $r \in J$ and $F(r)$ is a complement of some element of J. Then $-F(r) \in J$ and there is an $s \in J$, $-F(r) < s$ since J is not principal. Hence, $s \cdot F(r)$ is a non-zero element of J and $F^{-1}(s.F(r))$ belongs to J, a contradiction. The other two cases are similar. □

1.18. It is easy to see that there are no countable rigid algebras, since any rigid algebra cannot have more than one atom and all countable atomless algebras are homogeneous. Theorem 1.1 shows that there are rigid algebras in all uncountable powers. It is natural to ask what is the number of isomorphism types of rigid algebras in each power. By an easy modification of the basic construction, MONK and RASSBACH [1979] showed that the number of isomorphism types of rigid algebras is always maximal possible.

1.19. THEOREM (Monk, Rassbach). *For every uncountable cardinal κ, there are 2^κ isomorphism types of rigid Boolean algebras of power κ with non-isomorphic completions.*

Moreover, if κ is regular and $\kappa^{\aleph_0} = \kappa$ then there are 2^κ isomorphism types of complete rigid Boolean algebras of power κ.

PROOF. We shall show that for every uncountable cardinal κ, there are 2^κ Boolean algebras with non-isomorphic rigid completions. Given an uncountable cardinal κ, we shall construct a rigid algebra $R(X)$ of power κ for every $X \subseteq \kappa$, $|X| = \kappa$, in such a way that their corresponding completions are non-isomorphic. It suffices to modify properly the definition of the ideal I used in the basic construction. Let W be the stationary subset of κ defined in (0) and let $\langle S_\alpha : \alpha < \kappa \rangle$ be a partition of W to disjoint stationary subsets such that for each α, $\delta \in S_\alpha$ implies $\alpha < \delta$. For every $X \subseteq \kappa$, $|X| = \kappa$ let $\langle \nu(X, \alpha) : \alpha < \kappa \rangle$ be an increasing sequence of ordinals enumerating all elements of X. Clearly, $\alpha \leq \nu(X, \alpha)$ for every $\alpha \in X$. Let

$$(14) \qquad W(X) = \bigcup \{S_\alpha : \alpha \in X\} = \bigcup \{S_{\nu(X,\alpha)} : \alpha < \kappa\} .$$

As in 1.2, let A be a free Boolean algebra with κ free generators $a(\alpha)$, $\alpha < \kappa$, and let D be the dense subset of A which corresponds to the set of all mappings from finite subsets of κ to $\{-1, 1\}$. For every $\alpha < \kappa$, let $A(\alpha)$ be the subalgebra of A generated by all $a(\xi)$, $\xi < \alpha$, and let $D(\alpha) = A(\alpha) \cap D$. Let φ be a mapping from κ onto D which maps every $\delta \in W$ onto $D(\delta)$. For every $a \in A$, let $s(a)$ be the support of a (so $a \in A(s(a))$). For every $\alpha < \kappa$ and every $\delta \in S_{\nu(X,\alpha)}$, let $\xi(X, \delta, n)$, $n < \omega$, be an increasing sequence of non-limit ordinals disjoint from the support of $\varphi(\alpha)$ and with limit δ.

Now we can define the ideal $I(X)$ on A specifying its generators as follows: $a(\delta) - a(\xi(X, \delta, n))$ for every $\delta \in W(X)$, $n < \omega$, and $a(\delta) - \varphi(\alpha)$ for $\alpha < \kappa$ and $\delta \in S_{\nu(X,\alpha)}$. Let $R(X)$ be the quotient $A/I(X)$. Now it is easy to prove that $R(X)$ is a rigid Boolean algebra of power κ satisfying the countable chain condition and that its completion $Q(X)$ is rigid as well.

We shall show that $Q(X)$, $Q(Y)$ are non-isomorphic whenever $X \neq Y$. Let X, Y be subsets of κ of power κ such that there is a γ, $\gamma \in X - Y$. Suppose that there is an isomorphism G which maps $Q(X)$ onto $Q(Y)$. For every $\alpha < \kappa$, let

$$Q(X, \alpha) = \left\{ q \in Q(X),\ q = \sum P \text{ for some } P \subseteq A(\alpha)/I(X) \right\}.$$

Then

$$K_G = \{\alpha < \kappa,\ G(Q(X, \alpha)) = Q(Y, \alpha)\}$$

is closed unbounded in κ. It follows from (14) that S_γ is a stationary subset of $W(X)$ disjoint from $W(Y)$. If $\delta \in K_G \cap S_\gamma$, then the filter $F([a(\delta)], Q(X, \delta)) = \{q \in Q(X, \delta)\colon q \geq [a(\delta)]\}$ is non-principal in $Q(X, \delta)$, and its image is a subset of $Q(Y, \delta)$. As $\delta \not\in W(Y)$, $Q(Y, \delta)$ is closed under arbitrary meets and $G(F([a(\delta)], Q(X, \delta)))$ is a principal filter, a contradiction. Hence, $Q(X)$ and $Q(Y)$ are not isomorphic. This completes the proof for regular κ. Note that $|Q(X)| = \kappa^{\aleph_0}$ since $R(X)$ satisfies CCC. If $\kappa^{\aleph_0} = \kappa$, we have 2^κ isomorphism types of rigid complete Boolean algebras of size κ.

It remains to prove the theorem for singular cardinals. If κ is a singular cardinal of cofinality μ, let κ_α, $\alpha < \mu$, be an increasing sequence of regular cardinals with limit κ. We have shown that for every $\alpha < \mu$, there are 2^{κ_α} algebras of power κ_α with rigid non-isomorphic completions. Using the method of Lemma 1.16, we can construct $\Pi\{2^{\kappa_\alpha}\colon \alpha < \mu\} = 2^\kappa$ algebras of power κ with non-isomorphic completions. □

1.20. It is well known that the powers of complete Boolean algebras satisfy the condition $\kappa^{\aleph_0} = \kappa$. Theorem 1.1 leaves open the question of whether there is a rigid complete Boolean algebra of power κ for every κ, $\kappa^{\aleph_0} = \kappa$. It has been shown by Shelah that for every such cardinal κ, there are 2^κ isomorphism types of rigid complete Boolean algebras.

2. Embeddings into complete rigid algebras

We shall extend the construction from Section 1 to a general method of embeddings into rigid complete Boolean algebras. Let us recall that $\mathrm{sat}(B)$ is the least cardinal κ such that no subset of the Boolean algebra B of power κ consists of pairwise disjoint elements.

2.1. THEOREM (Balcar, Štěpánek). *Every Boolean algebra can be completely embedded in a complete rigid Boolean algebra Q. If B is infinite, then* $\mathrm{sat}(Q) = \mathrm{sat}(B)$.

Note that $\mathrm{sat}(B)$ is finite whenever B is a finite algebra. In this case, B can be completely embedded in any rigid complete Boolean algebra. We shall assume that B is infinite. Then $\mathrm{sat}(B) \geq \omega_1$ is a regular cardinal.

2.2. Given an algebra B, let κ be an infinite cardinal such that $|B| \leq \kappa$. The rigid complete algebra is constructed as the completion of a quotient $R = (B \oplus A)/I$ of the free product of B and a free algebra A with κ^+ free generators $a(\alpha)$, $\alpha < \kappa^+$. As in 1.2, for every $s \subseteq \kappa^+$, let $A(s)$ be the subalgebra generated by $a(\alpha)$, $\alpha \in s$, and for every $a \in A$, let $s(a)$ be the uniquely determined least finite s, $a \in A(s)$. Let D be the dense subset of A consisting of elements $d = \varepsilon(\alpha_1)a(\alpha_1) \cdot \varepsilon(\alpha_2)a(\alpha_2) \cdot \cdots \cdot \varepsilon(\alpha_n)a(\alpha_n)$, where $\{\alpha_1, \ldots, \alpha_n\}$ is a set of distinct ordinals less than κ^+ and $\varepsilon(\alpha_i) = \pm 1$. We shall identify d with the mapping $\varepsilon : \{\alpha, \ldots, \alpha_n\} \to \{-1, 1\}$. Let C be the set of all ordered pairs $\langle b, d \rangle$, where b is a non-zero element of B and $d \in D$. Clearly, C corresponds to a dense subset of the free product $B \oplus A$. We can identify every $b \in B$ with the pair $\langle b, 0 \rangle$, where 0 is the empty mapping, and every $d \in D$ with the pair $\langle 1_B, d \rangle$. Every element u of $B \oplus A$ is a finite join of elements of C. Let $s(u)$ be the least finite subset of κ^+ such that u belongs to $B \oplus A(s(u))$. In particular, $s(\langle b, d \rangle) = \mathrm{dom}(d)$ for every $\mathrm{d} \in D$. For every $\alpha < \kappa^+$, let $C(\alpha)$ be the subset of C consisting of all pairs $\langle b, d \rangle$, $d \in A(\alpha)$.

Let W be the stationary subset of κ^+ consisting of all ordinals α, $\mathrm{cf}(\alpha) = \omega$, which are divisible by $|\kappa|$. For every $\alpha \in W$, $|C(\alpha)| = \kappa$ and there is a mapping $\varphi : \kappa^+ \to C$ which maps every $\alpha \in W$ onto $C(\alpha)$. Let $\langle S_\alpha : \alpha < \kappa^+ \rangle$ be a partition of W to disjoint stationary subsets such that for every $\alpha < \kappa^+$, $\delta \in S_\alpha$ implies $\alpha < \delta$. For every α and every $\delta \in S_\alpha$, let $\langle \xi(\delta, n) : n < \omega \rangle$ be an increasing sequence of odd ordinals, disjoint from $s(\varphi(\alpha))$ and with limit δ.

Let I be the ideal on $B \oplus A$ generated by

$$a(\delta) - a(\xi(\delta, n)) \quad \text{for } \delta \in W,\ n < \omega ,$$

and

$$a(\delta) - \varphi(\alpha) \qquad \text{for } \alpha < \kappa^+ \text{ and } \delta \in S_\alpha .$$

Let $R = (B \oplus A)/I$ and let $[u]$ be the equivalence class of $u \in B \oplus A$ modulo I.

We have the following analogue of Lemma 1.4.

2.3. LEMMA. *For every $d \in D$, there exists $I(d) \in I$ such that $\langle b, d \rangle \in I$ iff $\langle b, d \rangle \le I(d)$ holds for every $b \in B$.*

PROOF. Given $d \in D$, let t be the least finite subset of κ^+ with the following properties: $s(d) \subseteq t$ and $s(\varphi(\alpha)) \subseteq t$ whenever $S_\alpha \cap t \ne 0$. If we put

$$\begin{aligned} I(d) = \textstyle\sum &\{a(\varphi) - \varphi(\alpha) : \delta \in S_\alpha \cap t\} \\ &+ \textstyle\sum \{a(\delta) - a(\xi(\delta, n)) : \delta \in W \cap t \text{ and } \xi(\delta, n) \in t\} , \end{aligned}$$

we can check the conclusion of the lemma by essentially the same method as in the proof of 1.4. Again, $I(d)$ depends on $s(d)$ rather than on d and if $\langle b, d \rangle \not\le I(d)$, then there are $b' \in B$, $0_B < b' \le b$, and $d' \in D$, $d' \le d$, such that $s(\langle b', d' \rangle) = S(I(d))$ and $\langle b', d' \rangle$ is disjoint from $I(d)$. $\square$

2.4. LEMMA. *B is completely embedded in $R = (B \oplus A)/I$.*

PROOF. For every $b \in B$, we put $e(b) = [b]$. It is not difficult to see that e is an embedding preserving finite operations. We shall show that it preserves all infinite meets. In fact, it suffices to show that for any family $\{b_j: j \in J\}$ of elements of B, $\Pi_B\, b_j = 0_B$ implies $\Pi_R\, e(b_j) = 0_R$. Suppose that there is $\langle b, d\rangle \in C$ such that $[\langle b, d\rangle]$ is less than every $e(b_j)$, $j \in J$. It follows from 2.3 that $\langle b, d\rangle - b_j = \langle b - b_j, d\rangle \leq I(d)$ for every $j \in J$. If $\Pi_B\, b_j = 0_B$, then $\Sigma_B\, (b - b_j) = b$ and we have $\langle b, d\rangle \in I(d)$. This completes the proof. □

2.5. LEMMA. *If B is infinite, then* $\mathrm{sat}(R) = \mathrm{sat}(B)$.

PROOF. Let $\mu = \mathrm{sat}(B)$. Suppose that there is an antichain of power μ in R. According to 2.3, we may assume that the antichain consists of elements $\langle b_\alpha, d_\alpha\rangle$, $\alpha < \mu$ such that $s(d_\alpha) = s(I(d_\alpha))$ and $\langle b_\alpha, d_\alpha\rangle$ is disjoint from $I(d_\alpha)$. If $\alpha < \beta < \mu$, then one of the following conditions holds

(15) $\quad b_\alpha \cdot b_\beta = 0_B\,,$

(16) $\quad d_\alpha, d_\beta$ are incompatible mappings, i.e. $d_\alpha \cup d_\beta \notin D$,

(17) $\quad$ there exist $\delta \in W$, $n < \omega$, such that $d_\alpha(\delta) = 1$ and $d_\beta(\xi(\delta, n)) = -1$ or vice versa.

A standard Δ-lemma argument shows that there is $J \subseteq \mu$, $|J| = \mu$, and a finite $r \subseteq \kappa^+$ such that all mappings d_α are compatible and their domains make a Δ-system with the root r. As in the proof of 1.9, we shall make the domains of mappings d_α, $\alpha \in J$, pairwise disjoint, leaving out the root r from each domain. Hence, we have (15) or (17) for any distinct $\alpha, \beta \in J$. If we put

$$A_\alpha = \{\xi(\delta, n) \in s(d_\alpha): d_\alpha(\xi(\delta, n)) = -1\}$$

and

$$B_\alpha = \{\xi(\delta, m): \delta \in W, m < \omega \text{ and } d_\alpha(\delta) = 1\}\,,$$

then $\{A_\alpha, \alpha \in J\}$ is a family of pairwise disjoint finite set of ordinals and the order-type of each B_α, $\alpha \in J$, is less than $\omega \cdot \omega$. We may assume that there is a natural number k, such that each $A(j)$, $j \in J$, consists exactly of k elements $\alpha_0(j) < \alpha_1(j) < \cdots < \alpha_{k-1}(j)$ since μ is uncountable and regular. As the $A(j)$'s are pairwise disjoint, the order-type of the set $\{\alpha_i(j): j \in J\}$ is at least μ for every $i \leq k$. Using regularity of μ again, we can make all columns $\{\alpha_i(j), j \in J\}$ "coherent", i.e. in k steps, we can construct a subset $K \subseteq J$, $|K| = \mu$, such that for every $j, j' \in K$, and every $i < k$, we have

$$\alpha_i(j) < \alpha_i(j') \quad \text{iff } \alpha_0(j) < \alpha_0(j')\,.$$

It follows from (15) and (17) that for every $\alpha, \beta \in K$, $\alpha < \beta$,

(18) b_α, b_β are disjoint in B or $A_\alpha \cap B_\beta \neq 0$,

or $A_\beta \cap B_\alpha \neq 0$.

To complete the proof, we shall construct an increasing sequence $\langle \eta_\alpha : \alpha < \mu \rangle$ or ordinals from K such that

(19) for every $\beta, \gamma \in K$, $\beta < \gamma$, there is an ordinal $\delta \in K$, $\eta_\beta \leq \delta < \eta_{\beta+1}$, with $A_\delta \cap B_{\eta_\gamma} \neq 0$.

We put $\eta_0 = \min K$. If we have defined η_α, the set $\{\beta \in K, \eta_\alpha < \beta < \mu\}$ splits into two parts according to (18). Let U_α consists of all $\beta \in K$, $\beta > \eta_\alpha$, and η_α, β do not satisfy (17). Choose a maximal subset $V_\alpha \subseteq U_\alpha$ such that every couple of elements of V_α does not satisfy (17). Then the family $\{b_\xi : \xi \in V_\alpha\}$ is an antichain in B and hence $|V_\alpha| < \mu$. Let $\nu_\alpha \in K$ be an upper bound for V_α, then for every $\gamma > \nu_\alpha$ there is a $\delta \in K$, $\eta_\alpha \leq \delta < \nu_\alpha$, such that γ, δ satisfy (17). Now, all the A_γ's are non-empty and pairwise disjoint and the union of all B_δ, $\eta_\alpha \leq \delta < \nu_\alpha$, has power less than μ. Hence, there is an $\varepsilon \in K$, $\varepsilon < \mu$, such that $A_\gamma \cap B_\delta = 0$ for every $\gamma \geq \varepsilon$, $\gamma \in K$, and every $\delta \in K$, $\eta_\alpha \leq \delta < \nu_\alpha$. Let $\eta_{\alpha+1}$ be the least such ε. Then for every $\gamma, \eta_{\alpha+1} \leq \gamma$, there is some $\delta, \eta_\alpha \leq \delta < \eta_{\alpha+1}$, such that $A_\delta \cap B_\gamma \neq 0$. If α is a limit ordinal and all η_ξ, $\xi < \alpha$, have been defined, let η_α be the supremum of all η_ξ, $\xi < \alpha$. It is clear that the sequence $\langle \eta_\alpha : \alpha < \mu \rangle$ satisfies (19). Now, choose γ, $\gamma > \omega^{\omega^2+1}$ (ordinal power). It follows from (19) that there is an increasing sequence of ordinals $\delta_\beta \in K$, $\beta < \gamma$, such that B_{η_γ} has a common element with every A_{δ_β}, $\beta < \gamma$. But the order-type of B_{η_γ} is less than $\omega \cdot \omega$, hence it is too small to satisfy the last requirement. Hence, R satisfies the μ-chain condition. $\square$

2.6. It is clear that $\mathrm{sat}(R) \leq \kappa^+$. We can define a hierarchy $Q(\alpha)$, $\alpha < \kappa^+$, of subsets of the completion Q of R in such a way that $Q(\alpha)$ is the set of all joins (in Q) of subsets of $\{[c]: c \in C(\alpha)\}$. Then $Q = \bigcup \{Q(\alpha): \alpha < \kappa^+\}$, and if G is an automorphism, there is a closed unbounded subset $K_G \subseteq \kappa^+$ such that

(20) $G(Q(\alpha)) = Q(\alpha)$ for every $\alpha \in K_G$.

Similarly as in Section 1, we can prove the following analogue of Lemma 1.6.

2.7. LEMMA. *Let $\delta \in W$ and let $\langle b, d \rangle \in C$ be such that $[b, d]$ is disjoint from $[a(\delta)]$. Then there is $\langle b', d' \rangle \in C(\delta)$, $[b, d] \leq [b', d']$ such that for every $\langle c, e \rangle \in C(\delta)$, if $[c, e]$ is disjoint from $[b, d]$, then it is disjoint from $[b', d']$.*

2.8. From Lemma 2.7, we get both statements of Lemma 1.12 and we can prove that Q is a rigid complete Boolean algebra using (20). This completes the proof of Theorem 2.1.

In fact, it is possible to prove a little stronger statement about Q in both Theorems 1.1 and 2.1. Using the argument of BALCAR and ŠTĚPÁNEK [1977], we can prove that there is no non-trivial σ-complete one-to-one endomorphism of Q.

2.9. A modified construction gives embeddings that preserve a particular type of distributivity, namely the existence of κ-closed dense subsets for any infinite cardinal κ. Let us recall that any algebra with a κ-closed dense subset is (μ, ∞)-distributive for every $\mu < \kappa$. As a corollary, we get the existence of distributive rigid complete Boolean algebras.

2.10. THEOREM. *Let κ be an uncountable cardinal, B a complete Boolean algebra with a κ-closed dense subset D. Then B can be completely embedded in a rigid complete Boolean algebra R which has a κ-closed dense subset.*

Moreover, if κ is regular and $\Lambda = \max(|D|, \kappa)$, then R satisfies the $(\Lambda^{<\kappa})^+$-chain condition.

2.11. COROLLARY. *For every infinite cardinal μ, there exist (μ, ∞)-distributive rigid complete Boolean algebras.*

2.12. The proof of Theorem 2.10 can be found in Štěpánek [1978]. The upper bound on the saturatedness of R does not imply that sat(B) and sat(R) are the same in all cases. The next example shows that there are cases when it does.

If κ is a regular cardinal and $\mu > \kappa$, $\mathrm{cf}(\mu) \geq \kappa$, then the standard set $\mathbb{P}$ of forcing conditions to collapse μ to κ while preserving all cardinals below κ consists of all mappings p from ordinals $\alpha < \kappa$ to μ. $\mathbb{P}$ is ordered by reverse inclusion, it is κ-closed and $|\mathbb{P}| = \Lambda^{<\kappa}$. The corresponding complete Boolean algebra B satisfies the $(\Lambda^{<\kappa})^+$-chain condition and it is exactly the same bound for antichains in R, that we get from Theorem 2.10.

3. Embeddings into the center of a Boolean algebra

3.1. Let us recall that the center of a Boolean algebra B is the set of all elements that are left fixed by every automorphism of B. We shall denote it by center(B). Note that the center of B is a subalgebra of B and that it is a regular complete subalgebra of B whenever B is a complete algebra. In particular, if B is rigid, we have center$(B) = B$. Hence, the results of Section 2 deal with embeddings of a Boolean algebra into the center of a rigid Boolean algebra. We shall show that every Boolean algebra B can be completely embedded as a center of a Boolean algebra E. If B is atomless, then E has no rigid or homogeneous factors. This was an affirmative answer to a problem of MCKENZIE and MONK [1975] who asked whether there are such algebras. These algebras will be studied in more detail in Section 4. We shall start with the following theorem.

3.2. THEOREM. *Any Boolean algebra B can be completely embedded as the center of a Boolean algebra E. Moreover, E is complete whenever B is a complete algebra and sat$(E) = \max(\aleph_1, \mathrm{sat}(B))$.*

Given an algebra, B, let κ be an infinite cardinal, $\kappa \geq |B|$. We shall construct a quotient E of the free product of B and of a free algebra A with κ^+ free generators. To fix every $b \in B$ with respect to automorphisms of E, we put stationary many generators of A below b. A similar argument as in Section 2 will show that B is a subset of the center of E. To show that every other element of E is moved by an automorphism, we shall repeat the situation ω times.

Let A be the free Boolean algebra with $\omega \times \kappa^+$ free generators $a(m, \alpha)$, $\alpha < \kappa^+$, $m < \omega$. For every subset $s \subseteq \omega \times \kappa^+$ let $A(s)$ be the subalgebra of A generated by $a(m, \alpha)$, $\langle m, \alpha \rangle \in s$. For every $a \in A$, let $s(a)$ be the least uniquely determined finite s, such that $a \in A(s)$. Let D be the dense subset of A consisting of elements

$$d = \varepsilon(m_1, \alpha_1)a(m_1, \alpha_1) \cdot \varepsilon(m_2, \alpha_2)a(m_2, \alpha_2) \cdot \cdots \cdot \varepsilon(m_k, \alpha_\kappa)a(m_k, \alpha_k)\,.$$

As above, we shall identify the elements of D with mappings from finite subsets of $\omega \times \kappa^+$ to $\{-1, 1\}$. Then $C = (B - \{0_B\}) \times D$ corresponds to a dense subset of the free product $B \oplus A$. As usual, we shall identify every $b \in B$ with the pair $\langle b, 0 \rangle$, where 0 is the empty mapping, and every $d \in D$ with the pair $\langle 1_B, d \rangle$. Every element u of $B \oplus A$ is a finite join of elements of C. Let $s(u)$ be the least finite subset of $\omega \times \kappa^+$ such that u belongs to $B \oplus A(s(u))$. In particular, $s(\langle b, d \rangle) = \mathrm{dom}(d)$ for every $d \in D$. For every $\alpha < \kappa^+$, let $C(\alpha)$ be the subset of C consisting of all pairs $\langle b, d \rangle$, $d \in A(\alpha)$. We call $s(u)$ the support of u in $B \oplus A$.

Let W be the stationary subset of κ^+ consisting of all ordinals α, $\mathrm{cf}(\alpha) = \omega$, which are divisible by κ. Let $\langle S_b, b \in B \rangle$ be a partition of W into pairwise disjoint stationary sets. For every $b \in B$ and every $\delta \in S_b$, choose an increasing sequence $\langle \xi(\delta, n)\colon n < \omega \rangle$ of odd ordinals with limit δ, such that $\{\delta \in S_b\colon \xi(\delta, 0) > \alpha \rangle$ is stationary for every $\alpha < \kappa^+$. To satisfy the last condition, we can split each S_b into κ^+ disjoint stationary subsets $S(b, \alpha)$, such that $\delta \in S(b, \alpha)$ implies $\delta > \alpha$ and chose $\xi(\delta, 0)$ appropriately.

Let I be the ideal on $B \oplus A$ generated by elements

$$a(m, \delta) - a(m, \xi(\delta, n)) \quad \text{for } \delta \in W,\ m, n < \omega\,,$$

and

$$a(m, \delta) - u \qquad \text{for } u \in B,\ \delta \in S_u \text{ and } m < \omega\,.$$

Let $E = (B \oplus A)/I$ and let $[u]$ be the equivalence class of $u \in B \oplus A$ modulo I.

3.3. LEMMA. *For every $d \in D$, there exists $I(d) \in I$ such that for every $b \in B$, we have*

$$\langle b, d \rangle \in I \quad \textit{iff } \langle b, d \rangle \le I(d)\,.$$

PROOF. Given $d \in D$, we put

$$I(d) = \sum \{a(m, \delta) - u\colon \langle m, \delta \rangle \in \mathrm{dom}(d) \text{ and } \delta \in S_u\}$$
$$+ \sum \{a(m, \delta) - a(m, \xi(\delta, n))\colon \delta \in W \text{ and } \langle m, \delta \rangle,$$
$$\langle m, \xi(\delta, n) \rangle \in \mathrm{dom}(d))\}\,.$$

Note that both joins are finite and that $\langle b, d\rangle \in I$ iff one of the following conditions hold:

(21) there is $\delta \in W$ and $m, n < \omega$ such that $d(m, \delta) = 1$ and $d(m, \xi(\delta, n)) = -1$,

or

(22) there are no δ, m, n satisfying (21) and

$$b \leq \sum_B \{-u\colon d(m, \delta) = 1 \text{ for some } \delta \in S_u \text{ and } m < \omega\},$$

or equivalently, b is disjoint from

$$\prod_B \{u \in B\colon d(m, \delta) = 1 \text{ for some } \delta \in S_u \text{ and } m < \omega\}.$$

The statement of the lemma follows immediately. Note that computing $I(d)$, we need not take the closure of $\mathrm{dom}(d)$ with respect to domains of other elements of C related to d through the generators of I. This fact greatly simplifies the proofs of subsequent lemmas which can be derived by similar steps as the corresponding lemmas of Section 2. □

3.4. LEMMA. *If we put $e(b) = [b]$ for every element of B, then e is a complete embedding of B to $E = (B \oplus A)/I$.*

3.5. LEMMA. $\mathrm{sat}(E) = \max(\aleph_1, \mathrm{sat}(B))$.

It is useful to consider the lexicographic order of the set of $\omega \times \kappa^+$ generators. If we follow the steps of the proof of Lemma 2.5, we get the sets A_α, B_α defined as follows:

$$A_\alpha = \{\langle m, \xi(\delta, n)\rangle \in s(d_\alpha)\colon d_\alpha(m, \xi(\delta, n)) = -1\},$$

$$B_\alpha = \{\langle m, \xi(\delta, n)\rangle\colon \delta \in W, n < \omega \text{ and } d_\alpha(m, \delta) = 1\}.$$

Then each A_α is finite and all A_α's are pairwise disjoint and the order-type of each B_α is less than $\omega \cdot \omega \cdot \omega$. The proof can be completed as that of Lemma 2.5.

3.6. Let $\bar{E}$ be the completion of E and let for every $\alpha < \kappa^+$, $E(\alpha)$ and $\bar{E}(\alpha)$, respectively, be the sets of finite and arbitrary joins of subsets of $\{[b, d]\colon \langle b, d\rangle \in C(\alpha)\}$. Then for every automorphism H of E and every automorphism G of $\bar{E}$ there are closed unbounded sets K_G, $K_H \subseteq \kappa^+$ such that

$$\begin{aligned} &H(E(\alpha)) = E(\alpha) \quad \text{for every } \alpha \in K_H, \\ (23)\qquad &G(\bar{E}(\alpha)) = \bar{E}(\alpha) \quad \text{for every } \alpha \in K_G. \end{aligned}$$

We get the following analogue of Lemma 2.7.

3.7. LEMMA. *Let $\delta \in W$ and let $\langle b, d \rangle \in C$ be such that $[b, d]$ is disjoint from $[a(m, \delta)]$ for every $m < \omega$. Then there is $\langle b', d' \rangle \in C(\delta)$, $[b, d] \le [b', d']$ such that for every $\langle c, e \rangle \in C(\delta)$, if $[c, e]$ is disjoint from $[b, d]$ then it is disjoint from $[b', d']$.*

3.8. If $v \in B$ is such that $\delta \in S_v$, it follows from (22) that a non-zero $[b, d]$ is disjoint from all $[a(m, \delta)]$, $m < \omega$, iff v is disjoint from

$$b \cdot \prod_B \{u \in B\colon \text{for some } \varepsilon \in S_u \text{ and } m < \omega,\ d(m, \varepsilon) = 1\}\ .$$

Let H be an automorphism of E which moves an element $[u]$, $u \in B$. Then $0 < u < 1$ and there is a non-zero $[b, d] \le [u]$ such that $H([b, d])$ is disjoint from $[u]$. We may assume that $b \le u$. Let δ be an element of the stationary set $K_H \cap S_b$ such that $\xi(\delta, 0) > \varepsilon$ for every ε, $\langle n, \varepsilon \rangle \in \mathrm{dom}(\dot{d})$. Then $a = [a(0, \delta)] \cdot [b, d]$ is non-zero and less than $[b, d]$. It follows from Lemma 3.7 that the filter $F(a, E(\delta))$ consisting of all elements $[c]$, $c \in C(\delta)$, $[c] \ge a$, is non-principal and that $H(a)$ is disjoint from $[u]$ and hence from $[b]$. Now $H(F(a, E(\delta))) = F(H(a), E(\delta))$ and it follows from Lemma 3.7 that the filter on the right-hand side is principal, a contradiction. We have shown that B is embedded as a subset of the center of E. If B is a complete Boolean algebra, a similar argument using G and the $\bar{E}(\alpha)$'s instead of H and the $E(\alpha)$'s shows that B is embedded as a subset of the center of $\bar{E}$. It remains to show that any other element of E or $\bar{E}$ is moved.

3.9. LEMMA. *If $e \in E$ is different from every $[b]$, $b \in B$, then there is an automorphism H of E such that $H(e) \ne e$.*

PROOF. Let e be an arbitrary element of E satisfying the assumption of lemma. Then e is a join of finitely many elements $[c_j]$, $j \in J$, where $c_j \in C$. Suppose that for each $j \in J$ there were an element b_j of B with $[c_j] \le [b_j] \le e$. Then $e = \Sigma\, [b_j]$ would correspond to an element of B. Hence, there exists $j \in J$ such that there is no $b \in B$ with $[c_j] \le [b] \le e$. Clearly, such $[c_j]$ must be non-zero and we may assume that $[c_j] = [b, d]$ and $\langle b, d \rangle$ is disjoint from $I(d)$. It follows from (22) that then

$$b \le \prod_B \{u \in B\colon \text{for some } \delta \in S_u \text{ and } m < \omega,\ d(m, \delta) = 1\}\ ,$$

and for every non-zero $b' \in B$, $b' \le b$, we have $\langle b', d \rangle \notin I$. Now, $[b] \ge [b, d]$ and $[b] \not\le e$. Hence, there is a non-zero $b' \le b$ and $d' \in D$ such that $[b', d']$ is a non-zero element less than $[b]$ and disjoint from e. We shall show that there is an automorphism G of E such that G moves $[b, d]$ to an element which has a non-empty intersection with $-e$.

Let s be the set of all $n < \omega$ such that $\langle n, \alpha \rangle \in \mathrm{dom}(d) \cup \mathrm{dom}(d')$ for some α. Clearly, s is finite and there is a permutation φ of ω such that $\varphi(s)$ is disjoint from s. If we put $\pi(a(n, \alpha)) = a(\varphi(n), \alpha)$ for every $n < \omega$, $\alpha < \kappa^+$, then π generates an automorphism of A and, consequently, an automorphism G_1 of $B \oplus A$ which maps I onto itself. If G is the automorphism of E corresponding to G_1, then

$$G([b, d]) = [b, d \circ \pi] \leq G(e),$$

and the domains of mappings $d \circ \pi$ and d' are disjoint. Hence $G([b, d]) \cdot [b', d'] = [b', (d \circ \pi) \cup d']$ is a non-zero element less than $G(e)$ and disjoint from e. This completes the proof of Lemma 3.9 and of Theorem 3.2, since Lemma 3.9 can be extended to the completion of E by a similar argument. □

3.10. We have shown that every Boolean algebra can be the center of another Boolean algebra. If the center is non-trivial, the algebra is not homogeneous. We shall show that if the center is atomless, the algebra has no homogeneous factors. It follows from the construction that the algebra has no rigid factors.

3.11. DEFINITION. We say that a Boolean algebra B has no rigid or homogeneous factors if the relative algebra $B \restriction b$ is neither rigid nor homogeneous for every non-zero $b \in B$.

3.12. THEOREM. *If B is atomless, then there is a Boolean algebra E with no rigid or homogeneous factors such that $B = \text{center}(E)$.*

Moreover, $\text{sat}(E) = \text{sat}(B)$ *and E is complete whenever B is a complete algebra.*

PROOF. Let E, $\text{center}(E) = B$, be as in the proof of Theorem 3.2. We shall show that it has no homogeneous and no rigid factors. Let u be a non-zero element of E. Let us recall that every automorphism of $E \restriction u$ can be extended by identity to an automorphism of E. There is a non-zero $b \in B$ and $d \in D$ such that $[b, d] \leq u$, since the set of such elements of E is dense in E. Now, b is not an atom of B and there are two disjoint non-zero elements $b_1, b_2 \in B$ below b. We may assume that the corresponding elements $[b_1, d], [b_2, d]$ are not isomorphic, since every such isomorphism could be extended to an automorphism G of E with $G(b_1) \neq b_1$. Hence, $E \restriction u$ is not homogeneous. It is not difficult to see that $E \restriction u$ is not rigid. We shall use the same notation as in 3.3. Let $\eta, \eta < \kappa^+$, be an even non-limit ordinal higher than every ordinal from $s(u) \subseteq \kappa^+$. Then η does not belong to W and it is not a member of any sequence $\xi(\delta, n)$, $n < \omega$. Let us recall that E is a quotient of the free product $B \oplus A$ modulo an ideal I, where A is a free algebra with $\omega \times \kappa^+$ free generators $a(m, \alpha)$, $m < \omega$, $\alpha < \kappa^+$. If π is the automorphism of A, which interchanges $a(0, \eta)$ and $-a(0, \eta)$ leaving all other generators of A fixed, then π generates an automorphism G_1 of $B \oplus A$ which maps the ideal I onto itself. It suffices to put $G_1([b, d]) = [b, d \circ \pi]$ for every $b \in B$ and $d \in D$. If G is the automorphism of E corresponding to G_1, then $G(u) = u$ and $G \restriction (E \restriction u)$ is a non-trivial automorphism of $E \restriction u$. Hence, E has no rigid and no homogeneous factors. A similar argument shows that the same holds for the completion of E. This completes the proof. □

3.13. We shall show that the center of every complete algebra with no homogeneous factors is atomless. Hence, the only assumption of Theorem 3.12 cannot be removed at least in the case of complete Boolean algebras. Let c be an atom of the center of a complete Boolean algebra C. Then every non-zero element $d \in C$, $d \leq c$, is moved by an automorphism G of C and $G(d) \leq c$ since $G(c) = c$. Thus,

$G \restriction (C \restriction c)$ is an automorphism of $C \restriction c$ and the center of $C \restriction c$ is trivial. It follows from Lemma 2.5 of Chapter 18 that $C \restriction c$ is a complete weakly homogeneous algebra which has a homogeneous factor acording to a theorem of Solovay and Koppelberg (see Theorem 4.1 of Chapter 18). Hence, an arbitrary Boolean algebra can be embedded as a subalgebra of the center of a Boolean algebra with no rigid or homogeneous factors.

3.14. THEOREM. *Any Boolean algebra B can be completely embedded as a subalgebra of the center of an algebra C which has no rigid or homogeneous factors. If B is infinite*, $\text{sat}(B) = \text{sat}(C)$. [*Note that no non-trivial automorphism of B extends to C.*]

PROOF. Given an algebra B, we can apply Theorem 3.12 if B has no atoms. Otherwise, we have to embed B to an atomless algebra B' with the same saturatedness. To this end, it suffices to take for B' the free product of B with a countable free algebra. Then we have $\text{sat}(B) = \text{sat}(B')$ by a standard argument. If we apply Theorem 3.12 to the completion of B' we get a complete algebra C with no rigid or homogeneous factors such that B is isomorphic to a regular subalgebra of the center of C. $\square$

4. Boolean algebras with no rigid or homogeneous factors

Let us recall that an algebra B has no rigid or homogeneous factors if the relative algebra $B \restriction b$ is neither rigid nor homogeneous for every non-zero element b of B. MCKENZIE and MONK [1975] asked whether such algebras do exist. A positive answer was given by BALCAR and ŠTĚPÁNEK [1977] by a construction described in Theorem 3.12 of the previous section. We have shown that the center of every complete algebra with no homogeneous factors is atomless and that every Boolean algebra can be completely embedded in the center of a complete algebra with no rigid or homogeneous factors. We shall describe a simpler construction of algebras without rigid or homogeneous factors and show that there are 2^κ isomorphism types of such algebras in every uncountable power κ. We shall prove another embedding theorem which guarantees extensions of all automorphisms. Moreover, such embeddings can preserve distributivity. The basic construction is much simpler than those that have been used so far.

4.1. We shall show that for every uncountable regular cardinal κ there is an algebra of power κ with no rigid or homogeneous factors. Note that every countable algebra contains a homogeneous factor since countable atomless algebras are homogeneous and every atom constitutes a homogeneous factor.

Given an uncountable regular cardinal κ, let W be the stationary subset of κ, which consists of all ordinals $\alpha < \kappa$, $\text{cf}(\alpha) = \omega$, which are divisible by $|\alpha|$. To every $\alpha \in W$ choose an increasing sequence $\langle \xi(\alpha, n) : n < \omega \rangle$ of odd ordinals with limit α such that for every β, $\beta < \kappa$, the set $\{\alpha \in W : \xi(\alpha, 0) > \beta\}$ is stationary in κ. Let A be a free algebra with κ free generators $a(\kappa)$, $\alpha < \kappa$. For every $s \subseteq \kappa$, let

$A(s)$ denote the subalgebra of A generated by $a(\alpha)$, $\alpha \in s$, and for every $a \in A$, let $s(a)$ be the least finite subset of κ with $a \in A(s(a))$. Every element of A is a finite join of elements $d = \varepsilon(\alpha_1) \cdot a(\alpha_1) \cdot \dots \cdot \varepsilon(\alpha_n) a(\alpha_n)$, where $\alpha_1, \dots, \alpha_n$ are distinct and $\varepsilon(\alpha_i) = \pm 1$. As above, we shall identify d with the mapping from $\{\alpha_1, \dots, \alpha_n\}$ to $\{-1, 1\}$ defined by $d(\alpha_i) = \varepsilon(\alpha_i)$ for $i \leq n$. Let D be the dense subset of A which consists of all such elements d. Clearly, $s(d) = \mathrm{dom}(d)$ for every $d \in D$.

Let I be the ideal on A generated by the set $a(\delta) - a(\xi(\delta, n))$: $\delta \in W$, $n < \omega$, and let B be the quotient A/I. As above, we shall denote by $[u]$ the equivalence class of $u \in A$ modulo I. For each subset $s \subseteq \kappa$, let $B(s)$ denote the set of all $[u]$, $u \in A(s)$. Similarly, let $D(\alpha) = A(\alpha) \cap D$ for every $\alpha < \kappa$. First, we shall show that B, and hence the completion of B, has no rigid factors. □

4.2. LEMMA. *For every non-zero b, $B \restriction b$ is not rigid.*

PROOF. Given b, let s be a finite subset of κ such that b belongs to $B(s)$. Choose an even non-limit ordinal $\eta < \kappa$ bigger than every element of s. Then η belongs neither to W nor to any sequence $\xi(\alpha, n)$. Let π be the automorphism of A which interchanges $a(\eta)$ and $-a(\eta)$ while all other generators of A are left fixed. Then π maps the ideal I onto itself determines a non-trivial automorphism of $B \restriction b$.

We shall show that B and its completion have no homogeneous factors. □

4.3. LEMMA. *Let $\delta \in W$ and $d \in D$ be such that $[d]$ is disjoint from $[a(\delta)]$. Then there is $d' \in D(\delta)$, $[d] \leq [d']$, such that for every $e \in D(\delta)$, if $[e]$ is disjoint from $[d]$, then it is disjoint from $[d']$.*

PROOF. If $[d] = 0_B$, there is nothing to prove. Suppose that $[d]$ is non-zero, then δ does not belong to $\mathrm{dom}(d)$ or $d(\delta) = -1$, since $[d]$ is disjoint from $[a(\delta)]$. For every $\varepsilon \in W$, $\varepsilon > \delta$, and $d(\varepsilon) = 1$, there are finitely many natural numbers $n_1, \dots, n_k$ such that $\xi(\varepsilon, n_i) < \delta$ for $i \leq n$. Let d' be the meet of $d \restriction \delta$ with all $a(\xi(\varepsilon, n))$, where $\varepsilon \in W$, $\varepsilon > \delta$ and $\xi(\varepsilon, n) < \delta$. Then $d' \in D(\delta)$, $[d] \leq [d']$ and for every $e \in D(\delta)$ such that $[e]$ is disjoint with $[d]$, $[e]$ is disjoint with $[d']$ as well.

As every element of $B(\delta)$ is a finite join of some $[d]$, $d \in D(\delta)$, we have immediately the following analogue of Lemma 1.12. □

4.4. LEMMA. *Let $\delta \in W$ and $b \in B(\delta)$ be disjoint from $[a(\delta)]$, then*
(i) *the filter $F([a(\delta)], B(\delta)) = \{u \in B(\delta)\colon u \geq [a(\delta)]\}$ is non-principal,*
(ii) *$F(b, B(\delta)) = \{u \in B(\delta)\colon u \geq b\}$ is a principal filter.*

4.5. LEMMA. *For every non-zero $b \in B$, $B \restriction b$ is not homogeneous.*

PROOF. Suppose that $B \restriction b$ is homogeneous. Now, $b \in B(s)$ for some finite $s \subseteq \kappa$ and it follows from our assumption on the sequences $\xi(\delta, n)$ that the set,

$$W' = \{\delta \in W\colon \xi(\delta, 0) \text{ is above every element of } s\},$$

is stationary in κ. As $\xi(\delta, 0) < \delta$ for every $\delta \in W'$, it follows from the well-known theorem of Fodor that there is a stationary subset $W'' \subseteq W'$ and an ordinal $\eta < \kappa$

such that

(24) $\xi(\delta, 0) = \eta$ for every $\alpha \in W''$.

If we put $b_1 = b \cdot [a(\eta)]$ and $b_2 = b - [a(\eta)]$, then b_1, b_2 are disjoint non-zero elements of $B \upharpoonright b$. Suppose that G is an automorphism of $B \upharpoonright b$ which maps b_1 onto b_2. We may extend G by identity outside of b to an automorphism of B. Then

$$X_G = \{\delta < \kappa \colon G \text{ maps } B(\delta) \text{ onto itself}\}$$

is closed unbounded in κ and hence

$$S = X_G \cap W'' \text{ is stationary in } \kappa .$$

Let $\delta \in S$, then $G(b_1) = b_2$ is a non-zero element disjoint from $[a(\delta)]$. If we put $c = b_1 \cdot [a(\delta)]$, then the filter $F(c, B(\delta)) = \{u \in B(\delta) \colon u \geq c\}$ is non-principal and $G(F(c, B(\delta))) = F(G(c), B(\delta))$ is a principal filter, according to Lemma 4.4(ii). Hence, there is no automorphism which maps b_1 onto b_2 and $B \upharpoonright b$ is not homogeneous. □

4.6. LEMMA. *B satisfies the countable chain condition.*

The proof of the lemma is a simplified version of the proof of Lemma 1.9. It can be reduced to Lemma 1.10.

4.7. Now we can extend the statement of Lemma 4.5 to the completion C of B. For every $\alpha < \kappa$, we denote by $C(\alpha)$ the set of all joins (in C) of subsets of $D(\alpha)/I = \{[d] \colon d \in D(\alpha)\}$. Then $C = \bigcup \{C(\alpha) \colon \alpha < \kappa\}$ follows from Lemma 4.6 and we get both statements of Lemma 4.4 for $C(\alpha)$'s instead of $B(\alpha)$'s. We can prove that no factor $C \upharpoonright c$ is homogeneous by a similar argument as in the proof of Lemma 4.5. Hence, the complete algebra C has no rigid or homogeneous factors.

4.8. THEOREM. *For every uncountable cardinal κ, there are Boolean algebras B_α, $\alpha < 2^\kappa$ of power κ, such that for $\alpha, \beta < 2^\kappa$, we have*

(i) *B_α and its completion have no rigid or homogeneous factors;*

(ii) *For distinct α, β, B_α and B_β are not isomorphic and the same holds for the corresponding complete algebras;*

(iii) *if κ is regular, B_α satisfies the countable chain condition. Moreover, if $\kappa^{\aleph_0} = \kappa$, then there are 2^κ isomorphism types of complete algebras with no rigid or homogeneous factors.*

PROOF. We shall first assume that κ is uncountable and regular. We shall construct a Boolean algebra $B(X)$ of power κ for every non-empty X, $X \subseteq \kappa$. It suffices to modify the definition of the ideal I used in the basic construction. Let W be the stationary subset of κ defined in 4.1 and let W_α, $\alpha < \kappa$, be pairwise disjoint stationary subsets of W. For every $\delta \in W_\alpha$ choose an increasing sequence of odd ordinals $\xi(\delta, n)$, $n < \omega$, with limit δ such that the set $\{\delta \in W_\alpha : \xi(\delta, 0) > \beta\}$ is stationary in κ for every α, $\beta < \kappa$. For every non-empty $X \subseteq \kappa$, let

$$W(X) = \bigcup \{W_\alpha : \alpha \in X\} .$$

As above, let A be a free Boolean algebra with κ free generators $a(A)$, $\alpha < \kappa$, and let $I(X)$ be the ideal on A generated by elements $(a(\delta) - a(\xi(\delta, n))$ for $\delta \in W(X)$, $n < \omega$. Let $B(X)$ denote the quotient $A/I(X)$. Now it is easy to prove that $B(X)$ and its completion have no rigid or homogeneous factors using the arguments of 4.2–4.7. It remains to prove that for different X, Y the algebras $B(X)$, $B(Y)$ and their completions $C(X)$, $C(Y)$ are not isomorphic. Let X, Y be distinct non-empty subsets of κ. We may assume that there is a γ, $\gamma \in X - Y$. Suppose that there is an isomorphism G which maps $C(X)$ onto $C(Y)$. For every $\alpha < \kappa$, let $C(X, \alpha)$ consists of all joins (in $C(X)$) of subsets of $\{[u] : u \in A(\alpha)\}$. $C(Y, \alpha)$ is defined similarly. Then

$$X_G = \{\alpha < \kappa : G \text{ maps } C(X, \alpha) \text{ onto } C(Y, \alpha)\}$$

is closed unbounded in κ. It follows from our assumption on X and Y that W_γ is a subset of $W(X)$ disjoint from $W(Y)$. Hence, $S = X_G \cap W_\gamma$ is a stationary subset of $W(X)$ disjoint from $W(Y)$. If $\delta \in X_G \cap S_\gamma$, then the filter $F([a(\delta)], C(X, \delta)) = \{c \in C(X, \delta) : c \geq [a(\delta)]\}$ is non-principal and G maps it onto a subset of $C(Y, \delta)$. As $\delta \not\in W(Y)$, $C(Y, \delta)$ is closed under arbitrary meets and $G(F([a(\delta)], C(Y, \delta)))$ is a principal filter, a contradiction. Hence, $C(X)$ and $C(Y)$ are not isomorphic and the same holds for $B(X)$ and $B(Y)$. Since every $B(X)$ satisfies the countable chain condition, $C(X)$ has power $\kappa^{\aleph_0}$. A modified argument shows that $C(X)$ and $C(Y)$ are totally different for distinct X, Y, i.e. they have no isomorphic factors. The same applies to $B(X)$ and $B(Y)$. If $\kappa^{\aleph_0} = \kappa$, we have 2^κ isomorphism types of complete algebras of power κ with no rigid or homogeneous factors. This completes the proof for regular κ.

Let κ be a singular cardinal and $\langle \kappa_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be an increasing sequence of regular uncountable cardinals with limit κ. For every $\alpha < \mathrm{cf}(\kappa)$ there are 2^{κ_α} totally different algebras $B(\alpha, \beta)$, $\beta < 2^{\kappa_\alpha}$, of power κ_α with no rigid or homogeneous factors. Using a method similar to that of Lemma 1.16, we can construct 2^κ algebras of power κ with no rigid or homogeneous factors with non-isomorphic completions. □

4.9. PROBLEM. It remains open whether there is a complete Boolean algebras of power κ with no rigid or homogeneous factors for every singular κ satisfying $\kappa^{\aleph_0} = \kappa$.

4.10. We shall describe another type of embeddings to algebras with no rigid or homogeneous factors. These embeddings make it possible to extend every automorphism of the smaller algebra in a uniform way to an automorphism of the larger algebra while preserving the saturatedness.

4.11. THEOREM. *Every Boolean algebra B can be completely embedded in a complete Boolean algebra C with no rigid or homogeneous factors and with the following properties*:

(i) *Every automorphism of B can be extended in a uniform way to an automorphism of C and, consequently, the automorphism group of B is a subgroup of the automorphism group of C.*

(ii) $\mathrm{sat}(B) = \mathrm{sat}(C)$ *whenever B is infinite.*

PROOF. The construction of C is particularly simple. Given B, let κ be the power of B if B is infinite, otherwise let $\kappa = \aleph_0$. If F is an algebra of power κ^+ with no rigid or homogeneous factors constructed in 4.1, it suffices to define C as the free product $B \oplus F$. Then (i) follows immediately. It remains to prove that C has no rigid or homogeneous factors. The key step in this direction is to prove (ii), or more precisely, $\mathrm{sat}(C) = \max(\aleph_1, \mathrm{sat}(B))$. This can be done as in the proof of Lemma 2.5. The ideal used here is much simpler, however. Then we can prove analogues of Lemmas 4.2–4.7 in a straightforward manner using the set of pairs $[b, d]$, $b \in B - \{0_B\}$, $d \in D$, which corresponds to a dense subset of C, instead of D. Then the proof is complete. $\square$

4.12. It is possible to prove that for every uncountable cardinal κ, there are complete algebras with no rigid or homogeneous factors and with a κ-closed dense subset. Hence, there are (κ, ∞)-distributive algebras without rigid or homogeneous factors for every infinite cardinal κ.

5. Embeddings into algebras with a trivial center

So far we have dealt with embeddings to Boolean algebras with a non-trivial center. Let us recall that in the class of complete algebras, the trivial center characterizes the subclass of all weakly homogeneous algebras. Moreover, we have noted in 3.13 that the center of any complete algebra with no homogeneous factors is atomless.

Now we can compare Theorems 3.2, 3.14 and 4.11 which give embeddings into rigid complete algebras and two types of embeddings to complete algebras with no rigid or homogeneous factors, one with the automorphism extension property and the other one without. All these embeddings preserve saturatedness, except the trivial case when the smaller algebra is finite. This contrasts with the properties of Kripke's embeddings to complete homogeneous algebras (Theorems 13.3, 14.8 of Part I), where the saturatedness need not be preserved. It was shown in Theorem 3.5 of Chapter 18 that no general saturatedness-preserving embedding theorem to complete algebras with trivial center is provable in ZFC.

Theorem 4.11, however, bears a certain resemblance to Kripke's embedding theorem, since it gives a uniform way of extension of automorphisms of the embedded algebra to the larger algebra with no rigid or homogeneous factors. Hence, in both embeddings, the automorphism group of the smaller algebra is a subgroup of the automorphism group of the larger algebra. It seems natural to ask whether an analogue of Theorem 3.14 is provable for embeddings to complete homogeneous algebras.

5.1. PROBLEM. Is it possible to embed any Boolean algebra B into a homogeneous algebra H in such a way that no non-trivial automorphism of B extends to an automorphism of H?

A positive answer to the following simpler question seems to be the first step in this direction.

5.2. PROBLEM. Is there a homogeneous algebra H with a subalgebra B which is not rigid and no non-trivial automorphism of B extends to H?

An affirmative answer was obtained by KOPPELBERG and MONK [1983]. Assuming the combinatorial principle $\diamond$, Koppelberg has obtained the following result.

5.3. THEOREM (Koppelberg). *Assuming $\diamond$, there is an $\aleph_1$-Suslin tree T such that the corresponding complete Boolean algebra B_T is homogeneous and has a regular complete subalgebra onto which no non-trivial automorphism of B restricts.*

Hence, the positive solution of Problem 5.2 in the class of complete algebras is consistent with ZFC. Monk has proved the following with no additional set-theoretic assumptions.

5.4. THEOREM (Monk). *For every Boolean algebra A, there are homogeneous algebras B and C such that $A \subseteq B \subseteq C$ and every endomorphism or automorphism of A extends to an endomorphism or automorphism of B, and no non-trivial one-to-one endomorphism of B extends to an endomorphism of C.*

Theorem 5.4 gives an affirmative answer to Problem 5.2 in ZFC. However, B and C are not complete and it is still open whether no non-trivial automorphism of A extends to C.

Hence, Problem 5.2 is still open for complete algebras and no answer to Problem 5.1 is known even for non-complete algebras.

Some open problems are related to embeddings preserving distributivity. Theorem 2.10 describes embeddings into rigid complete Boolean algebras which preserve the existence of a κ-closed dense subset for every uncountable cardinal κ. A similar theorem on embeddings to complete homogeneous algebras was proved by KOPPELBERG [1980]. It is still open whether these results can be generalized to (κ, ∞)-distributivity.

5.5. PROBLEM. Are there embeddings into rigid and into homogeneous complete Boolean algebras preserving (κ, ∞)-distributivity for every infinite cardinal κ?

Results about ordinal definable sets in Boolean-valued models of McAloon and others indicate that a positive answer is possible for embeddings into complete rigid algebras.

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CHAPTER 17

Rigid Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

KURATOWSKI [1926] constructed a space X such that, using a result of Čech [1937], $\beta(X)$ is a rigid Boolean space. But, people working on lattices and Boolean algebras did not know the Kuratowski result, and BIRKHOFF [1948] asked the question of the existence of a rigid algebra. KATETOV [1951] constructed a rigid algebra of cardinality 2^{ω}, and independently RIEGER [1951] and JÓNSSON [1951] found the same example of a rigid algebra of some singular cardinals. LOZIER [1969] solved a problem of DE GROOT and MCDOWELL [1963], constructing a rigid algebra of cardinality 2^{κ} for each $\kappa > \omega$. DE GROOT [1959] constructed a rigid algebra of cardinal $2^{2^{\omega}}$, and MCKENZIE and MONK [1975] constructed a rigid algebra for every strong limit cardinal, and $2^{2^{\kappa}}$ isomorphic types of rigid algebra of cardinality 2^{κ} for every $\kappa > \omega$. MONK and RASSBACH [1979] proved that for $\kappa > \omega$, there are 2^{κ} isomorphic types of algebras of cardinality κ. We say that an algebra is mono-rigid (resp. epi-rigid) if $\boldsymbol{B}$ has no non-trivial one-to-one endomorphism (resp. onto endomorphism). We should remark that a mono-rigid interval algebra $\boldsymbol{B}$ is epi-rigid (since $\boldsymbol{B}$ is retractive). BONNET [1978], [1980] has shown that there is a mono-rigid interval algebra. LOATS and RUBIN [1978] have proved that for each uncountable cardinal κ, there are 2^{κ} isomorphic types of onto-rigid interval algebras. TODORČVIĆ [1980], as a continuation of his work (TODORČEVIĆ [1979]), has shown that for each uncountable regular cardinal there are 2^{κ} isomorphic types of mono-rigid algebras of cardinal κ.

MCALOON [1970] proved the existence of a rigid complete algebra, and the existence of a rigid complete algebra of cardinality κ for each regular cardinal κ satisfying $\kappa^{\omega} = \kappa$, was proved by SHELAH [1975]. MONK and RASSBACH [1979] have proved that for each regular cardinal, there are 2^{κ} rigid algebras of cardinality κ, with rigid completion of cardinality κ^{ω}. Now the existence of simple complete algebras (which are also rigid) was shown by JENSEN (see MCALOON [1970]), and completed by a theorem of JECH [1974].

We have made the following choices to develop this chapter. The main tools to obtain rigid algebras are trees, chains and stationary sets (and also connected notions such as the Jensen's principle ($\diamondsuit$)), and we restrict to these tools. In the "Odds and ends" paragraph, other results are mentioned and the Kuratowski construction is given.

We construct linear orderings on trees. More precisely, let $\boldsymbol{T}$ be a rooted ω-tree, i.e. a partial order having a smallest element 0_T and of height ω. For such a tree $\boldsymbol{T}$, we have: for each x in $\boldsymbol{T}$, the set $[0_T, x)$ of $y \in \boldsymbol{T}$ satisfying $y < x$ is a finite chain, and for each $x \in \boldsymbol{T}$ the set of $y \in \boldsymbol{T}$ satisfying $y > x$ is non-empty. Let $\mathrm{Br}(\boldsymbol{T})$ be the set of maximal branches of $\boldsymbol{T}$. We can obtain rigid interval algebras, using linear orderings on $\mathrm{Br}(\boldsymbol{T})$. One of the possibilities is the following. We choose a "specific" linear ordering on the set of immediate successors of each element of level n. In this way, we can lexicographically order $\mathrm{Br}(\boldsymbol{T})$, and we obtain Jónsson's construction of rigid algebra, which is related to the Benner construction of a rigid tree algebra given in Part I.

Another way to obtain a linear ordering on $\mathrm{Br}(\boldsymbol{T})$, developed by Todorčević, is to define it directly in a purely set theoretical way. For each ordinal α of cofinal type ω, we chose a strictly increasing function f_α from ω into α, such that the sequence $(f_\alpha(n))_{n<\omega}$, converges to α. If we lexicographically order the f_α's, then this chain gives a mono-rigid interval algebra.

Another possibility is the construction of a "rigid" subchain P of the real line, developed by Bonnet, in such a way that the interval algebra generated by P is rigid.

Using ($\diamondsuit$), we develop Jech's construction of a simple complete algebra (i.e. an atomless complete algebra without proper atomless complete subalgebras).

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1. Basic concepts concerning orderings and trees

1.1. *Ordered sets*

Let $\langle P, \leq \rangle$ be a partial ordering (i.e. $\leq$ is a reflexive, transitive and anti-symmetric binary relation on P). We denote by $\langle P, \leq^* \rangle$ its converse ordering, i.e. $x \leq^* y$ if and only if $y \leq x$. If there is no confusion we write P instead $\langle P, \leq \rangle$ and P^* instead $\langle P, \leq^* \rangle$. If $x, y \in P$, then x and y are said to be *comparable* whenever $x \leq y$ or $y \leq x$. If $x, y \in P$ are not comparable, then we say that they are *incomparable*. Finally, x and y are *disjoint* whenever there is no $z \in P$ such that $z \leq x$ and $z \leq y$ (and thus x, y are incomparable).

A *chain* (resp. *antichain*) is a partial ordering for which two distinct elements are comparable (resp. incomparable). If A is a subset of P, then A is a partial ordering under the induced order relation on A. So we can define *subchains*, and *subantichains*, ... of a partial ordering.

Let P be a partial ordering, and $A \subseteq P$. We denote by $\mathrm{Sup}(A)$ and $\mathrm{Inf}(A)$, if they exist, the least upper bound and the greatest lower bound of A in P.

Let P be a partial ordering. If $p \in P$, we denote by $\mathrm{Succ}(p)$ the set of *immediate successors* of p in P, i.e. $q \in \mathrm{Succ}(p)$ whenever $q > p$, and there is no $r \in P$ such that $p < r < q$. By duality, we can define the notion of set of *immediate predecessors* of a member of P.

An element a of A, is *minimal* whenever there is no $x \in A$ such that $x < a$. By duality, we define the notion of a *maximal* element. So, for instance, if $\mathrm{Inf}(A) = a \in A$, then a is the unique minimal element of A. A subset A of P is a *final interval* or a *final segment* of P if $x \in A$ and $x \leq y$ in P, implies $y \in A$. For instance, the set $[x, +\infty) = \{y \in P\colon y \geq x\}$ (resp. $(x, +\infty) = \{y \in P\colon y > x\}$) is called the *principal closed* (resp. *principal open*) *final interval* of P, generated by x. A subset A of P is an *interval* or a *segment* of P if $x, y \in A$ and $x \leq z \leq y$ in P, implies $z \in A$. A subset A of P is *cofinal* in P whenever for every $x \in P$, there is $y \in A$ satisfying $y \geq x$. By duality, we define the notion of *initial intervals* and *coinitial* subsets of P.

A partial ordering P is *well-founded* whenever every non-empty subset has a minimal element, which means (assuming the axiom of choice) there is no strictly decreasing sequence. A well-founded chain is called a *well-ordering*, and thus is order-isomorphic to an ordinal.

A chain P is *scattered* whenever P does not contain a subchain order-isomorphic to the rational chain. A *totally disconnected* chain $\boldsymbol{P}$ is a chain for which between two distinct elements of $\boldsymbol{P}$ there are two consecutive elements. So $\boldsymbol{P}$ is scattered if and only if every subchain is totally disconnected.

Let κ be a regular cardinal. A chain $\boldsymbol{P}$ is said to be *κ-dense* whenever $\boldsymbol{P}$ has no first nor last element and for every $a < b$ in $\boldsymbol{P}$, the open interval (a, b) of $\boldsymbol{P}$ is of cardinality κ. A chain $\boldsymbol{P}$ is said to be *dense* whenever $\boldsymbol{P}$ has no first nor last element and for every $a < b$ in $\boldsymbol{P}$, there is c in $\boldsymbol{P}$ such that $a < c < b$.

1.2. Basic facts about interval Boolean algebras

1.2.1.

Let $\boldsymbol{C}$ be a chain (i.e. a total linear ordering). We put $\boldsymbol{C}^0 = \boldsymbol{C}$ and $\boldsymbol{C}^+ = \boldsymbol{C} \cup \{+\infty\}$ if $\boldsymbol{C}$ has a first element, and $\boldsymbol{C}^0 = \boldsymbol{C} \cup \{-\infty\}$, $\boldsymbol{C}^+ = \boldsymbol{C} \cup \{-\infty, +\infty\}$ otherwise, where $-\infty < x < +\infty$ for all $x \in \boldsymbol{C}$. We denote by $\boldsymbol{B}\langle \boldsymbol{C}\rangle$ the algebra of all the subsets of $\boldsymbol{C}^0$ which are finite unions of intervals $[u, v)$ for $u < v$ in $\boldsymbol{C}^+$ (by definition $[u, v)$ is the set of $t \in \boldsymbol{C}^+$ such that $u \leqq t < v$). Note that $+\infty \notin [u, v)$. Such an algebra $\boldsymbol{B}\langle \boldsymbol{C}\rangle$ is called an *interval algebra* (generated by the chain $\boldsymbol{C}$). A non-zero element a of $\boldsymbol{B}\langle \boldsymbol{C}\rangle$ has a unique decomposition (called the *canonical decomposition*), in the form:

$$a = \bigcup \{[a_{2i}, a_{2i+1}) \colon i < n\} ,$$

where $0 \leq n < \omega$, $-\infty \leq a_0 < a_1 < a_2 < \cdots < a_{2n-1} \leq +\infty$ and $a_k \in \boldsymbol{C}^+$ ($k = 0, 1, \ldots, 2n-1$). We put $\sigma(a) = \{a_0, a_1, a_2, \ldots, a_{2n-1}\} \subseteq \boldsymbol{C}^+$. a_{2i} (resp. a_{2i+1}, a_k) is called a *left* (resp. *right*, *end*) point of a.

If $\boldsymbol{C}^*$ denotes the chain obtained from $\boldsymbol{C}$ by reversing the order, then $\boldsymbol{B}\langle \boldsymbol{C}\rangle$ and $\boldsymbol{B}\langle \boldsymbol{C}^*\rangle$ are isomorphic.

Let $\boldsymbol{B}$ be a Boolean algebra. $\boldsymbol{B}$ is isomorphic to an interval algebra if and only if there is a chain $\boldsymbol{C} \subseteq \boldsymbol{B}$ which is a set of generators of $\boldsymbol{B}$ (under the Boolean operations).

1.2.2. Remarks

(1) *Let $\boldsymbol{P}$ and $\boldsymbol{P}_1$ be chains. We assume that $\boldsymbol{P}_1 \subseteq \boldsymbol{P}$.* We set $\boldsymbol{B} = \boldsymbol{B}\langle \boldsymbol{P}\rangle$ and $\boldsymbol{B}_1 = \boldsymbol{B}\langle \boldsymbol{P}_1\rangle$. We will develop the relationship between members of $\boldsymbol{B}_1$ and $\boldsymbol{B}$. *We can view $\boldsymbol{B}_1$ as a subalgebra of $\boldsymbol{B}$* in the following manner. If $a = \bigcup \{[x_{2i}, x_{2i+1}) \colon i \leq m\}$ is an element of $\boldsymbol{B}_1$ (and thus $x_k \in \boldsymbol{P}_1 \subseteq \boldsymbol{P}$), then a is an element of $\boldsymbol{B}$, interpreting the $[x_{2i}, x_{2i+1})$ in $\boldsymbol{P}$, and an element of $\boldsymbol{B}$ with end points in $\boldsymbol{P}_1$, by restricting the interval to $\boldsymbol{P}_1$. With this identification, we observe that *$\boldsymbol{B}_1$ is a subalgebra of $\boldsymbol{B}$*.

(2) We recall that every interval algebra is retractive.

These remarks are useful in Sections 3 and 4.

1.3. Topology and linear ordering

Let $\boldsymbol{C}$ be a chain. The *interval topology* on $\boldsymbol{C}$ is defined by considering open sets as unions of open intervals of $\boldsymbol{C}$, and $\boldsymbol{C}$ endowed with the interval topology is said to be an *interval space*. For instance, the classical topology on the real line is the interval topology.

Let us recall the duality in interval Boolean algebras. Let C be a chain. Recall that $I \subseteq C^+$ is a prime ideal whenever: I is a proper non-empty initial interval of C^+. The prime ideals of C^+ are exactly the initial intervals of C, and therefore the set Spec(C) of prime ideals of C^+ *is the set of initial intervals of* C, *endowed with the induced topology of* 2^C, *which is also the interval topology on* Spec(C) *ordered by inclusion*. Moreover, *the spaces* Spec(C) *and the Boolean space of* $B\langle C\rangle$ *are homeomorphic*.

1.3.1. FACT. Let C be a chain. The following properties are equivalent:

(i) C is a complete chain (resp. complete and totally disconnected).

(ii) The interval space C is compact (resp. compact and totally disconnected).

Interval spaces are used in Sections 2, 3 and 5.

Let C be a chain, and $x \in C$. We denote by cf(x) the *cofinal type* of $(-\infty, x)$. So cf(x) = 0 if x has an immediate predecessor, or x is the smallest element of C, while otherwise cf(x) is the smallest cardinal ξ ($\xi \geq \omega$) such that there is a strictly increasing sequence of length ξ cofinal in $(-\infty, x)$. By duality, we define the *coinitial type* ci(x) of $(x, +\infty)$. Now $\tau_C(x) = (\mathrm{cf}(x), \mathrm{ci}(x))$, or more simply $\tau(x)$, is called the *character* of x in C.

1.3.2. FACT. Let C and D be chains considered as interval spaces, f be a one-to-one continuous function from C into D, and $a \in C$. We set $\tau(a) = (\lambda, \mu)$. Then:

(1) if $0 \neq \lambda \neq \mu \neq 0$, then $\tau(f(a)) = (\lambda, \mu)$ or (μ, λ);

(2) if $0 = \lambda \neq \mu$, then $\tau(f(a)) = (\nu, \mu)$ or (μ, ν) for some ν;

(3) if $\lambda \neq 0 = \mu$, then $\tau(f(a)) = (\nu, \lambda)$ or (λ, ν) for some ν;

(4) if $\lambda = \mu > \omega$ then $\tau(f(a)) = (\nu, \lambda)$ or (λ, ν) for some ν;

(5) if $\lambda = \mu = \omega$, then $\tau(f(a)) = (\omega, \nu)$ or (ν, ω) for some ν;

(6) if $\lambda = \mu = 0$, i.e. a is isolated in C, then $\tau(f(a)) = (\xi, \zeta)$ for some ξ, ζ.

Moreover, if f is an homeomorphism from C onto D, then:

(2′) if $0 = \lambda \neq \mu > \omega$, then $\tau(f(a)) = (0, \mu)$, $(\mu, 0)$, or (μ, μ);

(3′) if $\omega < \lambda \neq 0 = \mu$, then $\tau(f(a)) = (0, \lambda)$, $(\lambda, 0)$, or (λ, λ);

(5′) if $\tau(a) = (\omega, \omega)$, $(\omega, 0)$ or $(0, \omega)$, then $\tau(f(a)) = (\omega, \omega)$, $(\omega, 0)$ or $(0, \omega)$;

(6′) if $\lambda = \mu = 0$, i.e. a is isolated in C, then $\tau(f(a)) = (0, 0)$.

The proofs are trivial since μ and ν are 0 or regular, and if $\lambda = \mu > \omega$, then the intersection of two clubs of λ is a club too (see the Appendix on Set Theory in this Handbook).

1.4. Trees

A *rooted tree* T is a partial ordering such that:

(1) T has a smallest element, denoted by 0_T, and

(2) for each $t \in T$, the set $[0_T, t)$ is well-ordered.

Throughout this section, *trees are assumed to be rooted*.

Let T be a tree.

(i) For $t \in T$, the *height of an element t of T*, is the order-type of $[0_T, t)$.

(ii) For each ordinal α, the αth *level* of T, denoted by $\mathrm{Lev}_\alpha(T)$, is the set of $t \in T$ of height α in T. So if $t \in \mathrm{Lev}_\alpha(T)$, then $\mathrm{Succ}(t) \subseteq \mathrm{Lev}_{\alpha+1}(T)$.

(iii) The *height of a tree T* is the smallest ordinal α such that $\mathrm{Lev}_\alpha(T) = \emptyset$.

For instance, let X be a set, and δ be an ordinal. The set $X^{<\delta}$ of all γ-sequences of members of X, for $\gamma < \delta$, is a tree, under the order relation $s \leq t$ whenever $s \subseteq t$, i.e. $\mathrm{dom}(s) = \alpha \leq \mathrm{dom}(t) = \beta$, and $s(\nu) = t(\nu)$, for $\nu < \alpha$.

A *branch* b of T is an initial interval of T, and so b is a chain (for instance, for $t \in T$, the sets $[0_T, t)$ and $[0_T, t]$ are branches of T). A *maximal branch* b is a branch of T, which is maximal under the inclusion relation. *We denote by* $\mathrm{Br}(T)$ *the set of all maximal branches b of T.*

A *subtree T' of T* is an initial interval of T. If $t \in T$, then T_t denotes the set of $t_1 \in T$ comparable with t. So T_t is a subtree of T.

An *ω-tree* is a tree T of level ω such that $\mathrm{Succ}(t) \neq \emptyset$ for $t \in T$.

1.5. REMARK. In Sections 2 and 4, branches are supposed to be maximal (under the inclusion relation).

1.6. FACT. Let T be an ω-tree such that $\mathrm{Br}(T)$ is of cardinality $\lambda > \omega$, where λ is regular. Assume that $\mathrm{Br}(T_u)$ or $\mathrm{Br}(T) \backslash \mathrm{Br}(T_u)$ is of cardinality $<\lambda$, for $u \in T$. Then there is t in T such that $\mathrm{Succ}(t)$ is of cardinality λ.

Indeed, let $u \in \mathrm{Lev}_m(T)$ and $v \in \mathrm{Lev}_n(T)$ satisfying $|\mathrm{Br}(T_u)| = |\mathrm{Br}(T_v)| = \lambda$. If $m = n$, then $u = v$, and thus, if $m \leq n$, then $u \leq v$. Now, by contradiction, we construct a strictly increasing sequence $u(k)$, for $k < \omega$, satisfying $|\mathrm{Succ}(u(k))| < \lambda$ and $|\mathrm{Br}(T_{u(k)})| = \lambda$. Note that for such a $u(k)$, we have $|\mathrm{Br}(T) \backslash \mathrm{Br}(T_{u(k)})| < \lambda$. Now let b be the branch of T consisting of the $u(k)$'s. We have:

$$\bigcap \{\mathrm{Br}(T_{u(k)}) \colon k < \omega\} = \{b\}\,,$$

and so

$$\mathrm{Br}(T) \backslash \{b\} = \bigcup \{\mathrm{Br}(T) \backslash \mathrm{Br}(T_{u(k)}) \colon k < \omega\}$$

is of cardinality $<\lambda$, which contradicts $|\mathrm{Br}(T)| = \lambda$.

2. The Jónsson construction of a rigid algebra

We will prove, using a linear ordering on the set of branches of an ω-tree, that there is a rigid Boolean algebra of cardinality ω_α, where ω_α is the smallest cardinal satisfying $\omega_\alpha = \alpha$ (JÓNSSON [1951] or RIEGER [1951]).

The BRENNER [1983] construction (see [Part I]) to obtain the existence of a rigid tree algebra and JÓNSSON's [1951] construction are quite similar.

Briefly, the basic idea is to construct a tree (rooted) T, such that: first, every branch of T is of length ω; secondly, if $t \in T$, then $\mathrm{Succ}(t)$ is a set of regular cardinality $\lambda(t) > \omega$; and thirdly, if $t' \neq t''$ in T, then $\lambda(t') \neq \lambda(t'')$. Now, we consider a linear ordering on $\mathrm{Succ}(t)$ of order-type $\lambda(t) + 1$. The set $\mathrm{Br}(T)$, lexicophically ordered, is a Boolean interval space, satisfying: the set of $b \in \mathrm{Br}(\boldsymbol{T})$ of character $(\lambda(t), 0)$ is topologically dense in $\mathrm{Br}(\boldsymbol{T})$. That shows the rigidity of the interval space $\mathrm{Br}(\boldsymbol{T})$.

Let $\langle \boldsymbol{T}, \leq \rangle$ be an ω-tree. Let us suppose that for each $t \in \boldsymbol{T}$, we have a linear ordering $\leq_t$ on $\mathrm{Succ}(t)$. By induction on $m < \omega$, we define a linear ordering $\leq_m$ on $\mathrm{Lev}_m(\boldsymbol{T})$, considering $(\mathrm{Lev}_m(\boldsymbol{T}), \leq_m)$ as the lexicographic sum of $(\mathrm{Succ}(t), \leq_t)$ under $(\mathrm{Lev}_{m-1}(\boldsymbol{T}), \leq_{m-1})$. Let $\mathrm{Br}(\boldsymbol{T})$ be the set of maximal branches of $\langle \boldsymbol{T}, \leq \rangle$. We will identify each branch b of $\boldsymbol{T}$, with the canonical strictly increasing enumeration $\langle b_n \rangle_{n<\omega}$ of its elements. Now if b' and b'' are distinct members of $\mathrm{Br}(\boldsymbol{T})$, then we set $b' < b''$ if and only if for the smallest integer m satisfying $b'_m \neq b''_m$ (so $b'_{m-1} = b''_{m-1} = t$) we have $b'_m <_t b'_m$ in $\mathrm{Succ}(t)$, or equivalently $b'_m <_m b''_m$ in $\mathrm{Lev}_m(\boldsymbol{T})$. Obviously, the relation $\leq$ on $\mathrm{Br}(\boldsymbol{T})$ defined by $b' \leq b''$ whenever $b' = b''$ or $b' < b''$ is a linear ordering. Moreover, if for each $t \in \boldsymbol{T}$, the chain $\langle \mathrm{Succ}(t), \leq_t \rangle$ is complete, then $\langle \mathrm{Br}(\boldsymbol{T}), \leq \rangle$ is complete too (see, for instance, TODORČEVIĆ [1984]).

We will construct a rooted tree $\boldsymbol{T}$ and $\boldsymbol{F} = \langle F_m \rangle_{m<\omega}$, with $F_m \subseteq \mathrm{Lev}_m(\boldsymbol{T})$, satisfying:

(J1) $\mathrm{Lev}_0(\boldsymbol{T}) = F_0 = \{0_T\}$,

(J2) if $t \in F_m \subseteq \mathrm{Lev}_m(\boldsymbol{T})$, then $\mathrm{Succ}(t)$ has a unique element $t^+(\infty)$ and $t^+(\infty) \in F_{m+1}$,

(J3) if $t \in \mathrm{Lev}_m(\boldsymbol{T}) \setminus F_m$, then $\mathrm{Succ}(t)$ is a chain of order-type $\lambda(t) + 1$, where $\lambda(t) > \omega$ is a regular cardinal, and thus $\mathrm{Succ}(t)$ has a greatest element $t^+(\infty)$, and $F_{m+1} \cap \mathrm{Succ}(t) = \{t^+(\infty)\}$, and

(J4) if $t' \neq t''$ in $\boldsymbol{T} \setminus \bigcup \{F_m : m < \omega\}$, then $\lambda(t') \neq \lambda(t'')$.

We will give the main steps of the construction of such a tree.

(i) $\mathrm{Succ}(0_T)$ is a chain of order-type $\omega_1 + 1$ and $F_1 = \{t^+(0_T)\}$, where $t^+(0_T)$ is the greatest element of $\mathrm{Succ}(0_T)$.

(ii) Assume that $\boldsymbol{T}_n = \bigcup \{\mathrm{Lev}_k(\boldsymbol{T}) : k < n\}$ and $F_0, \ldots, F_{n-1}$ are defined satisfying (J1) through (J4). Let L_n be the set of $\lambda(t)$ for $t \in T_{n-1}$, and $(x_\nu)_{\nu<\theta}$ be an enumeration of elements of $\mathrm{Lev}_{n-1}(\boldsymbol{T}) \setminus F_{n-1}$.

By induction, we will define $\lambda(x_\nu)$ for $\nu < \theta$, satisfying $\lambda(t) \neq \lambda(x_\nu) \neq \lambda(x_\mu)$ for $t \in \boldsymbol{T}_{n-1} \setminus \bigcup \{F_k : k < n-1\}$, $\nu, \mu < \theta$ and $\nu \neq \mu$. This is sufficient to construct the chains $\mathrm{Succ}(t)$ for $t \in \mathrm{Lev}_{n-1}(\boldsymbol{T})$ and thus the chain $\mathrm{Lev}_n(\boldsymbol{T})$ and to define F_n using properties (J2) and (J3). Let $\mu < \theta$ be given, and assume $L_n^\mu = L_n \cup \{\lambda(x_\nu) : \nu < \mu\}$ is defined, and satisfies (J4). Let $\lambda(x_\mu)$ be a regular cardinal satisfying $\lambda(x_\mu) \not\in L_n^\mu$. For example, we can assume that L_n^μ is an initial segment of the class of regular cardinals (for instance $\mathrm{Lev}_1(\boldsymbol{T}) \setminus F_1$ is of order-type ω_1, and $L_1^{\omega_1}$ is the set of the first ω_1 regular cardinals $>\omega_1$). In this case, the tree $\boldsymbol{T}$ is of cardinal κ, where κ is the first cardinal ω_α satisfying $\omega_\alpha = \alpha$.

Notice that such a tree $\boldsymbol{T}$ satisfies:

(J5) $\mathrm{Lev}_m(\boldsymbol{T})$ is order-isomorphic to a successor ordinal for every $m < \omega$.

Consequently, $\mathrm{Br}(\boldsymbol{T})$ is complete totally disconnected chain and thus $\mathrm{Br}(\boldsymbol{T})$ is a Boolean interval space.

We will prove that the interval space $\mathrm{Br}(\boldsymbol{T})$ has no non-trivial homeomorphism onto itself.

This is a consequence of the following observations:

OBSERVATION (a). *If* $t \in \mathrm{Lev}_m(\boldsymbol{T}) \setminus F_m$, *then* $t^+(\infty) \in F_{m+1}$ *and the character of* $t^+(\infty)$ *in* $\mathrm{Lev}_{m+1}(\boldsymbol{T})$ *is* $(\lambda(t), 0)$.

Consequently, $t^+(\infty)$ defines a unique branch of $\boldsymbol{T}$, denoted by b_t. Obviously, the character of b_t is the chain $\mathrm{Br}(\boldsymbol{T})$ is $(\lambda(t), 0)$. The converse property of (a) is studied in (c).

OBSERVATION (b). *Let* $b \in \mathrm{Br}(\boldsymbol{T})$ *and thus* $b \subseteq \boldsymbol{T}$. *If* b *is disjoint from* $\bigcup \{F_m: m < \omega\}$, *then the coinitial type of* b *in* $\mathrm{Br}(\boldsymbol{T})$ *is* ω.

Indeed, let $m < \omega$ be fixed. Let $t \in b \cap \mathrm{Lev}_m(\boldsymbol{T})$ and $u \in b \cap \mathrm{Lev}_{m+1}(\boldsymbol{T})$. We have $u \in \mathrm{Succ}(t)$, $u < t^+(\infty)$ (because $b \cap F_{m+1} = \emptyset$). Let v be the successor of u in $\mathrm{Succ}(t)$, which is order-isomorphic to $\lambda(t) + 1$. Let $b[m]$ be a branch containing $v \in \mathrm{Lev}_{m+1}(\boldsymbol{T})$. Notice that the first m elements of b and $b[m]$ are identical, and thus: first $b < b[n] < b[m]$ for $m < n < \omega$ and, secondly the infimum of $b[m]$ for $m < \omega$ is b. Consequently, the coinitial type of b is ω.

OBSERVATION (c). *Let* L *be the set of* $\lambda(t)$ *for* $t \in \boldsymbol{T} \setminus \bigcup \{F_m: m < \omega\}$. *If* $b \in \mathrm{Br}(\boldsymbol{T})$ *is of character* $(\lambda, 0)$ *in* $\mathrm{Br}(\boldsymbol{T})$ *for some* $\lambda \in L$, *then* $b = b_t$ *for a unique* $t \in T$.

This follows from (J4) and observations (a) and (b). To see the following result, draw a picture.

OBSERVATION (d). *The set* $\mathrm{Br}_F(\boldsymbol{T})$ *of* b_t *for* $t \in \boldsymbol{T} \setminus \bigcup \{F_m: m < \omega\}$, *is topologically dense in the space* $\mathrm{Br}(\boldsymbol{T})$.

Now we conclude the argument by contradiction. Assume f is a non-trivial homeomorphism from $\mathrm{Br}(\boldsymbol{T})$ onto itself. There is $t \in \boldsymbol{T} \setminus \bigcup \{F_m: m < \omega\}$ such that $b = f(b_t) \neq b_t$; this follows from (d). The character of b is $(\lambda(t), 0)$, $(0, \lambda(t))$, or $(\lambda(t), \lambda(t))$; this follows from (a) and Fact 1.3.2.

The characters $(0, \lambda(t))$ and $(\lambda(t), \lambda(t))$ cannot occur, by (J3) and (b). We have $b = b_{t'}$ for some $t' \in \boldsymbol{T}$ using (c), and thus $t = t'$, applying (J4). Consequently, $f(b_t) = b_t$.

REMARK. Let κ be the first cardinal ω_α satisfying $\omega_\alpha = \alpha$, and L_0 be the set of regular cardinals less than κ. For each $L \subseteq L_0$ satisfying $|L| = \kappa$, there is a tree $\boldsymbol{T}_L$ of cardinality κ, and a family $\boldsymbol{F} = \langle F_m \rangle_{m<\omega}$ satisfying (J1) through (J4), and $\lambda \in L$ if and only if there is a (unique) $t \in \boldsymbol{T}_L$ such that $\lambda(t) = \lambda$. The interval space $\mathrm{Br}(\boldsymbol{T}_L)$ has no non-trivial homeomorphism onto itself, and thus the Boolean algebra associated with $\mathrm{Br}(\boldsymbol{T}_L)$ which is of cardinality κ is a rigid interval algebra.

Consequently, there are 2^κ pairwise non-isomorphic rigid interval algebras of power κ, where κ is the first cardinal ω_α satisfying $\omega_\alpha = \alpha$.

3. Bonnet's construction of mono-rigid interval algebras

3.0. In this section we develop the construction of mono-rigid interval algebras due to BONNET [1978], [1980]. Before stating the theorem, let us recall and introduce some definitions and comments.

DEFINITION. A Boolean algebra $\boldsymbol{B}$ is said to be *embedding-rigid* whenever for every $a, b \in \boldsymbol{B}$ such that $a \not\leq b$, there is no one-to-one homomorphism from $\boldsymbol{B} \upharpoonright a$ into $\boldsymbol{B} \upharpoonright b$. More generally, $\boldsymbol{B}$ is said to be an *order-embedding-rigid* algebra if for every $a, b \in \boldsymbol{B}$ such that $a \not\leq b$, there is no on-to-one order-isomorphism from $\boldsymbol{B} \upharpoonright a$ into $\boldsymbol{B} \upharpoonright b$ (here $\boldsymbol{B} \upharpoonright c$ is considered as a partial ordering, for $c \in \boldsymbol{B}$).

DEFINITION. Let $\boldsymbol{B}$ be a Boolean algebra. $\boldsymbol{B}$ is said to be *mono-rigid* (resp. *onto-rigid*) whenever every one-to-one endomorphism (resp. onto endomorphism) is the identity.

DEFINITION. Let $\boldsymbol{B}$ be a Boolean algebra. $\boldsymbol{B}$ is *order-mono-rigid* whenever the ordered set $\boldsymbol{B}$ has no non-trivial one-to-one order-preserving function from $\boldsymbol{B}$ into itself.

3.1. COMMENTS

(1) If $\boldsymbol{B}$ is order-mono-rigid, order-embedding-rigid, then trivially, $\boldsymbol{B}$ is mono-rigid, embedding-rigid, respectively.

(2) If $\boldsymbol{B}$ is mono-rigid, then $\boldsymbol{B}$ is rigid.

(3) If $\boldsymbol{B}$ is an embedding-rigid algebra, then $\boldsymbol{B}$ is mono-rigid.

To see this, assume for a contradiction, that f is a non-trivial one-to-one endomorphism and that $a \in \boldsymbol{B}$ satisfies $f(a) \neq a$. If $a \not\leq f(a)$, then the restriction g of f on $\boldsymbol{B} \upharpoonright a$ is a one-to-one embedding from $\boldsymbol{B} \upharpoonright a$ into $\boldsymbol{B} \upharpoonright f(a)$, contradicting the hypothesis. If $f(a) \not\leq a$, then $-a \not\leq f(-a)$, and then we conclude as above.

(4) Let $\boldsymbol{B}$ be an order-embedding-rigid algebra. Then $\boldsymbol{B}$ is order-mono-rigid, and thus mono-rigid and rigid.

To justify this, let f be a one-to-one, increasing, and $a \in \boldsymbol{B}$ satisfy $f(a) \neq a$. First, if $a \not\leq f(a)$, then the restriction g of f on $\boldsymbol{B} \upharpoonright a$ is a one-to-one increasing function from $\boldsymbol{B} \upharpoonright a$ into $\boldsymbol{B} \upharpoonright f(a)$, contradicting the hypothesis. Now assume $f(a) \not\leq a$. Let $c = f(a) \cdot -a \neq \mathbf{0}$. Let g be the function from $\boldsymbol{B} \upharpoonright c$ into $\boldsymbol{B} \upharpoonright -c$ defined by $g(x) = f(a \dotplus x) \cdot -c$ (note that a and x are disjoint). Trivially the range of g is contained in $\boldsymbol{B} \upharpoonright -c$. Now g is increasing and one-to-one because f is increasing and one-to-one, a and x are disjoint, and $c \leq f(a) \leq f(a \dotplus x)$.

Being motivated by some results on Banach spaces, developed at the end of this section, we will investigate the notion of Bonnet-rigid algebra, starting by:

3.2. DEFINITION. A Bolean algebra $\boldsymbol{B}$ is said to be *Bonnet-rigid* whenever, for every algebra $\boldsymbol{B}_1$ and every one-to-one homomorphism φ from $\boldsymbol{B}$ into $\boldsymbol{B}_1$ and every homomorphism ψ from $\boldsymbol{B}$ onto $\boldsymbol{B}_1$, it follows that $\varphi = \psi$.

We can introduce the dual definition. A Boolean algebra $\boldsymbol{B}$ is said to be *dual-Bonnet-rigid* if for every algebra $\boldsymbol{B}_1$, and every one-to-one homomorphism φ from $\boldsymbol{B}_1$ into $\boldsymbol{B}$ and every homomorphism ψ from $\boldsymbol{B}_1$ onto $\boldsymbol{B}$, it follows that $\varphi = \psi$.

If $\boldsymbol{B}$ is a mono-rigid interval algebra, then $\boldsymbol{B}$ is Bonnet-rigid (since $\boldsymbol{B}$ is retractive). Note that every Bonnet-rigid Boolean algebra is both mono- and onto-rigid.

We will prove that the notions of Bonnet-rigid and dual-Bonnet-rigid are identical, and that there is a Bonnet-rigid interval algebra $\boldsymbol{B}$ of cardinality $\kappa = \mathrm{cf}(2^\omega)$, satisfying: for every subalgebra $\boldsymbol{B}'$ of $\boldsymbol{B}$, if $\boldsymbol{B}' \restriction a$ is of cardinality κ for every $a \neq \boldsymbol{0}$ in $\boldsymbol{B}'$, then $\boldsymbol{B}'$ is order-embedding-rigid.

3.3. PROPOSITION. *Let $\boldsymbol{B}$ be a Boolean algebra. $\boldsymbol{B}$ is Bonnet-rigid if and only if $\boldsymbol{B}$ is dual-Bonnet-rigid.*

PROOF. We denote by $\boldsymbol{K}$ the Boolean space associated with $\boldsymbol{B}$. Now $\boldsymbol{F}, \boldsymbol{L}, \ldots$ are Boolean spaces, $\varphi, \psi, \ldots$ are continuous functions. By duality, $\boldsymbol{K}$ is a Bonnet-rigid space if and only if $\boldsymbol{K}$ satisfies:

(*) for every space $\boldsymbol{L}$, and all continuous functions φ and ψ from $\boldsymbol{L}$ into $\boldsymbol{K}$ such that φ is one-to-one and ψ is onto, it follows that $\varphi = \psi$.

First assume $\boldsymbol{K}$ satisfies (*), $\boldsymbol{L}$ is a Boolean space φ and ψ are continuous mappings from $\boldsymbol{K}$ into $\boldsymbol{L}$ such that φ is one-to-one and ψ is onto. We set $\boldsymbol{K}' = \varphi[\boldsymbol{K}]$ and $\boldsymbol{L}' = \psi^{-1}[\boldsymbol{K}']$. Let θ be the unique continuous function from $\boldsymbol{K}'$ onto $\boldsymbol{K}$ such that $\theta \circ \varphi$ is the identity on $\boldsymbol{K}$. So $\boldsymbol{L}'$ is a closed subspace of $\boldsymbol{K}$ and the restriction ψ' of $\theta \circ \psi$ on $\boldsymbol{L}'$ is a continuous function from $\boldsymbol{L}'$ onto $\boldsymbol{K}$. From (*), it follows that $\boldsymbol{L}' = \boldsymbol{K}$ and ψ' is the identity on $\boldsymbol{K}$. Consequently, $\theta \circ \psi$ is the identity on $\boldsymbol{K}$, and hence is equal to $\theta \circ \varphi$. So $\varphi = \psi$.

Conversely, let $\boldsymbol{B}_1$ be a Boolean algebra, and φ and ψ be homomorphisms from $\boldsymbol{B}$ into $\boldsymbol{B}_1$ such that φ is one-to-one and ψ is onto. We set $\boldsymbol{B}_2 = \varphi[\boldsymbol{B}]$ and $\boldsymbol{B}_3 = \psi^{-1}[\boldsymbol{B}_2]$. Trivially, $\boldsymbol{B}$ and $\boldsymbol{B}_2$ are isomorphic algebras. Let φ' and ψ' be the restrictions of φ and ψ on $\boldsymbol{B}_3 \subseteq \boldsymbol{B}$, respectively. From the hypothesis, it follows that $\varphi' = \psi'$. Now, $\boldsymbol{B}_3 = \boldsymbol{B}$, and hence $\varphi = \psi$ follows from $\varphi'[\boldsymbol{B}_3] = \psi'[\boldsymbol{B}_3] = \boldsymbol{B}_2$ and φ is one-to-one. □

3.4. RESIDUALS SUBSET $\boldsymbol{R}(\mathscr{A})$ OF A SET $\mathscr{A}$

3.4.1. Let $\boldsymbol{P}$ be a subchain of the real line $\boldsymbol{R}$. We suppose that $\boldsymbol{P}$ is of cardinality ν, where ν is an uncountable regular cardinal. So $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ is of cardinality ν too.

Now let $\mathscr{A}$ be a subset of $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ of uncountable regular cardinality μ. For $\boldsymbol{U} \in \mathscr{A}$, we have $\boldsymbol{U} = \bigcup \{[a^{\boldsymbol{U}}_{2k}, a^{\boldsymbol{U}}_{2k+1}) \colon k < m(i)\}$, with $a^{\boldsymbol{U}}_0 < a^{\boldsymbol{U}}_1 < \cdots < a^{\boldsymbol{U}}_{2m(i)-1}$ in $\boldsymbol{P} \cup \{-\infty, +\infty\}$. Now there are $m < \omega$ and a subset $\mathscr{A}'$ of $\mathscr{A}$, of cardinality μ such that $m(\boldsymbol{U}) = m$ for $\boldsymbol{U}$ in $\mathscr{A}'$. For each $\boldsymbol{U}$ in $\mathscr{A}'$, we choose rational numbers $q^{\boldsymbol{U}}_k$, $r^{\boldsymbol{U}}_k$ for $k < 2m - 1$ such that:

$$a_0^U < q_0^U < r_0^U < a_1^U < q_1^U < r_1^U < a_2^U < \cdots < a_{2m-2}^U < q_{2m-2}^U < r_{2m-2}^U < a_{2m-1}^U \,.$$

From $\boldsymbol{Q}^{4m-2}$ is countable, it follows that there are rational numbers q_k, r_k and a subset $\mathscr{A}''$ of $\mathscr{A}'$, of cardinality μ such that $q_k^U = q_k$ and $r_k^U = r_k$ for every U in $\mathscr{A}''$.

Now there is a subset $\mathscr{A}_0$ of $\mathscr{A}''$ of cardinality μ such that: either $a_0^U = a_0^V$ for distinct members U and V of $\mathscr{A}_0$, or $a_0^U \neq a_0^V$ for distinct members U and V of $\mathscr{A}_0$. There is a subset $\mathscr{A}_1$ of $\mathscr{A}_0$ of cardinality μ such that: either $a_1^U = a_1^V$ for distinct members U and V of $\mathscr{A}_1$, or $a_1^U \neq a_1^V$ for distinct members U and V of $\mathscr{A}_1$. Repeating this procedure $2m$ times, we obtain $\mathscr{A}_{2m} \subseteq \mathscr{A}_{2m-1}$ of cardinality μ. We set $\boldsymbol{R}(\mathscr{A}) = \mathscr{A}_{2m}$.

3.4.2. The set $\boldsymbol{R}(\mathscr{A})$ is called a *residual subset* of $\mathscr{A}$. Recall that $\mathscr{A}$ and $\boldsymbol{R}(\mathscr{A})$ have the same cardinality. We denote by $m(\boldsymbol{R}(\mathscr{A}))$ the integer m, by $\rho(\boldsymbol{R}(\mathscr{A}))$ the set of indexes $k < 2m$ such that $a_k^U \neq a_k^V$ for distinct members U and V of $\boldsymbol{R}(\mathscr{A})$.

3.4.3. REMARKS.

(1) From the construction of $\boldsymbol{R}(\mathscr{A}) = \mathscr{A}_{2m}$, it follows that: $a_p^U = a_p^V$ if $p < 2m$ with $p \not\in \rho(\boldsymbol{R}(\mathscr{A}))$, and $a_q^U \neq a_q^V$ if $q \in \rho(\boldsymbol{R}(\mathscr{A}))$ for distinct members U and V of $\boldsymbol{R}(\mathscr{A})$.

(2) We have $U \subseteq V$ in $\boldsymbol{R}(\mathscr{A})$ if and only if $a_{2k}^U \geq a_{2k}^V$ and $a_{2k+1}^U \leq a_{2k+1}^V$ for $k < m$. In particular if $\rho(\boldsymbol{R}(\mathscr{A}))$ has a unique element , then $\boldsymbol{R}(\mathscr{A})$ is a chain.

(3) Let $-\mathscr{A}$ be the set of $-V$ for $V \in \mathscr{A}$. Then the set $-\boldsymbol{R}(\mathscr{A})$ of $-U$ for $U \in \boldsymbol{R}(\mathscr{A})$ is a residual subset of $-\mathscr{A}$. Moreover, $|\rho(\boldsymbol{R}(\mathscr{A}))| = |\rho(-\boldsymbol{R}(\mathscr{A}))|$.

3.5. In what follows we set $\kappa = \mathrm{cf}(2^\omega)$, and we recall that $\kappa > \omega$.

DEFINITION. Let $\boldsymbol{P}$ be a subset of the chain of real numbers. We say that $\boldsymbol{P}$ is *strongly rigid* whenever:

(1) $\boldsymbol{P}$ is κ-dense, and

(2) If $\boldsymbol{P}_1$ is a subset of $\boldsymbol{P}$ such that there is a one-to-one increasing or decreasing function from $\boldsymbol{P}_1$ into $\boldsymbol{P}$ satisfying $f(x) \neq x$ for every $x \in \boldsymbol{P}_1$, then $\boldsymbol{P}_1$ is of cardinality $< \kappa$.

Notice that if such a $\boldsymbol{P}$ exists, then $\boldsymbol{P}$ satisfies:

(3) $\boldsymbol{R} \backslash \boldsymbol{P}$ is topologically dense in $\boldsymbol{R}$ (since $\boldsymbol{P}$ does not contain a non-trivial segment of the real line).

3.6. LEMMA. *Let $\boldsymbol{P}$ be a subchain of the real line. We suppose that $\boldsymbol{P}$ is strongly rigid. Then there is a strongly rigid subchain $\boldsymbol{P}_1$ of $\boldsymbol{R}$ order-isomorphic to $\boldsymbol{P}$ satisfying*:

(1′) *if $a < b$ in $\boldsymbol{R}$, then $\boldsymbol{P}_1 \cap (a, b)$ is of cardinality κ.*

PROOF. Let $\boldsymbol{S}$ be the bounded completion by cuts of $\boldsymbol{P}$, i.e. $\boldsymbol{S}$ has no first or last element, $\boldsymbol{P}$ is dense in $\boldsymbol{S}$, and any bounded subset of $\boldsymbol{S}$ has a sup and inf in $\boldsymbol{S}$. Then $\boldsymbol{S}$ is isomorphic to $\boldsymbol{R}$, and, with $\boldsymbol{P}_1$ the image of $\boldsymbol{P}$ under this isomorphism, it is clear that (1′) and (2) hold. $\square$

Now, we have:

3.7. PROPOSITION. *There is a strongly rigid subchain of the real line.*

A proof can be found in BONNET [1980] or in the Appendix on Set Theory in this Handbook. The key of the proof is:

(a) if A is a subset of the real line and f is a one-to-one monotonic function from A into R, then f is extendable to a monotonic function $\bar{f}$ from the topological closure of A into $\{-\infty\} \cup R \cup \{+\infty\}$ such that $\bar{f}^{-1}(t)$ is finite for every t;

(b) the set of monotonic functions from a closed subset of R into $\{-\infty\} \cup R \cup \{+\infty\}$ is of cardinality 2^ω.

So the set F of f such that f is a one-to-one monotonic function from A into R satisfying $f(t) \neq t$ for $t \in A$ and $|A| = 2^\omega$ is of cardinality 2^ω. We construct P using diagonal argument.

Notice that the set of points of discontinuity of $\bar{f}$ is countable (since f is monotonic) and thus we can extend f in a one-to-one continuous and monotonic function from a G_δ-subset of R into R, and thus the set of monotonic continuous functions from a G_δ-subset of R into $\{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$ is of cardinality 2^ω. In this way, this is connected to Kuratowski's construction of a rigid Boolean space developed in Section 6.

Now we will prove two lemmas about residual subsets of chains or sublattices of $B\langle P\rangle$, where P is strongly rigid; these lemmas will be useful to prove results concerning Bonnet's construction.

3.8. LEMMA. *Let P be a strongly rigid subchain of the real line. Let $\mathscr{A}$* be a chain *of $B\langle P\rangle$, of cardinality κ. Then, each residual subset $R(\mathscr{A})$ of $\mathscr{A}$ is of cardinality κ, and $|\rho(R(\mathscr{A}))| = 1$.*

PROOF. By contradiction, let us suppose that $\rho(R(\mathscr{A}))$ has at least two elements k and l. Let P_k and P_l be the set of a_k^U and of a_l^U, for U in $R(\mathscr{A})$. Let g be the function from P_k into P_l defined by $g(a_k^U) = a_l^U$. Obviously, g is one-to-one. Now if k and l have the same parity, then g is increasing and if k and l do not have the same parity, then g is decreasing. In both cases, we obtain a contradiction since P_k and P_l are disjoint and $|P_k| = \kappa$. □

Note that this result cannot be extended from a chain to a sublattice of $B\langle P\rangle$. Indeed, let $a < b < c < d$ in P. So (a, b) and (c, d) are of cardinality κ, and let f be a one-to-one function from (a, b) onto (c, d). For each $t \in (a, b)$, we set $A_t = [t, f(t))$. Let $\mathscr{R}$ be the set of A_t for $t \in (a, b)$, and $\mathscr{L}$ be the lattice generated by $\mathscr{R}$ in $B\langle P\rangle$. Then $\mathscr{R}$ is a residual subset of $\mathscr{L}$ with $|\rho(\mathscr{R})| = 2$. However, we have:

3.9. LEMMA. *Let P be a strongly rigid subchain of the real line, and $\mathscr{L}$ be a sublattice of $B\langle P\rangle$ of cardinality κ, i.e. if U and V are member of $\mathscr{L}$, then $U \cap V$ and $U \cup V \in \mathscr{L}$. Then there is a residual subset $R(\mathscr{L})$ of $\mathscr{L}$ of cardinality κ with $|\rho(R(\mathscr{L}))| = 1$.*

PROOF. Before proving the Lemma, let us establish the following fact:

FACT 1. Let λ be a regular cardinal and $\boldsymbol{T}$ be a partial ordered set of cardinality λ. We suppose that for each $x \in \boldsymbol{T}$, the set $\boldsymbol{T}(x)$ of $y \in \boldsymbol{T}$ such that y is incomparable to x (i.e. $x \not\leq y$ and $y \not\leq x$) is a set of cardinality $<\lambda$. Then $\boldsymbol{T}$ contains a chain of cardinality λ.

PROOF. Indeed, we construct by induction a family $(x_\alpha)_{\alpha<\lambda}$ of pairwise comparable distinct elements of $\boldsymbol{T}$. Let $x_0 \in \boldsymbol{T}$. Assume $(x_\alpha)_{\alpha<\beta}$, with $\beta < \lambda$, be defined. Then we choose $x_\beta \in \boldsymbol{T}$ such that $x_\beta \not\in \bigcup \{\boldsymbol{T}(x_\alpha): \alpha < \beta\}$. $\square$

Now, let $\mathcal{R}_0$ be a residual subset of $\mathcal{L}$ such that $|\rho(\mathcal{R}_0)|$ is minimal. We will show that $\mathcal{R}_0$ contains a chain $\mathcal{C}$ of cardinality κ. Then by lemma 3.8, the set $\rho(\boldsymbol{R}(\mathcal{C}))$ has a unique element.

For $\boldsymbol{U} \in \mathcal{R}_0$, we have $\boldsymbol{U} = \bigcup \{[a^U_{2k}, a^U_{2k+1}): k < m\}$, and $\rho(\mathcal{R}_0) \subseteq \{0, 1, \ldots, 2m-1\}$. For $\boldsymbol{V} \in \mathcal{R}_0$, we denote by Inc($\boldsymbol{V}$) the set of $\boldsymbol{W} \in \mathcal{R}_0$ satisfying $\boldsymbol{W}$ is incomparable to $\boldsymbol{V}$ (under the inclusion relation).

Case 1. Inc($\boldsymbol{V}$) is of cardinality $<\kappa$, for every $\boldsymbol{V} \in \mathcal{R}_0$. Then $\mathcal{R}_0$ contains a chain of cardinality κ, which follows from Fact 1, and thus the conclusion is a consequence of Lemma 3.8.

Case 2. Assume that for some $\boldsymbol{V} \in \mathcal{R}_0$, the set Inc($\boldsymbol{V}$) is of cardinality κ. Let k and l be the first and the second element of $\rho(\mathcal{R}_0)$, respectively. This means that for distinct members $\boldsymbol{U}$ and $\boldsymbol{V}$ of $\mathcal{R}_0$ we have $a^U_j = a^V_j$ for $j < k$ or $k < j < l$, $a^U_k \neq a^V_k$ and $a^U_l \neq a^V_l$. There is no loss in assuming k is even (otherwise, replace $\boldsymbol{U}$ by its complement). Denote by $\boldsymbol{P}(k)$ and $\boldsymbol{P}(l)$ the set of a^U_k and a^U_l for $\boldsymbol{U} \in \mathcal{R}_0$, respectively. So $\boldsymbol{P}(k)$ and $\boldsymbol{P}(l)$ are subsets of cardinality κ.

First, assume l is even. Notice that if $\boldsymbol{U} \supset \boldsymbol{V}$ in $\mathcal{R}_0$, then $a^U_k \leq a^V_k$ and $a^U_l \leq a^V_l$. In this case, we define a partial ordering on $\boldsymbol{P}(k)$ by: $a^U_k \leq_1 a^V_k$ whenever $a^U_k \leq a^V_k$ and $a^U_l \leq a^V_l$, where $\leq$ is the natural ordering of the real line.

Secondly, assume l is odd. Note that if $\boldsymbol{U} \supset \boldsymbol{V}$ in $\mathcal{R}_0$, then $a^U_k \leq a^V_k$ and $a^U_l \geq a^V_l$. Now, we define a partial ordering $\leq_1$ on $\boldsymbol{P}(k)$ by: $a^U_k \leq_1 a^V_k$ whenever $a^U_k \leq a^V_k$ and $a^U_l \geq a^V_l$.

FACT 2. There is a $t \in \boldsymbol{P}(k)$ such that the set of $S \in \boldsymbol{P}(k)$ which are incomparable to t under $\leq_1$ is of cardinality κ.

PROOF. Otherwise, let $\boldsymbol{D}$ be a chain of cardinality κ, contained in $\langle \boldsymbol{P}(k), \leq_1 \rangle$ and given by Fact 1. Let h be the function from $\boldsymbol{D}$ into $\boldsymbol{P}$ defined by $h(a^U_k) = a^U_l$. Obviously, h is one-to-one (since k, $l \in \rho(\mathcal{R}_0)$), monotonic (more precisely increasing if l is even and decreasing if l is odd), and satisfies $h(a^U_k) = a^U_l \neq a^U_k$, which contradicts the strong rigidity of $\boldsymbol{P}$. $\square$

Proof of Lemma 3.9. We recall that $\boldsymbol{P}(k)$ is a subset of $\boldsymbol{P}$ of cardinality $\kappa > \omega$. So, if l is even (resp. odd), then we can choose $a^U_k \in \boldsymbol{P}(k)$ such that: either the set $\boldsymbol{L}_1$ of $a^V_k \in \boldsymbol{P}(k)$ satisfying $a^V_k \leq a^U_k$ and $a^V_l \geq a^U_l$ (resp. $a^V_k \leq a^U_k$ and $a^V_l \leq a^U_l$) is of cardinality κ, or the set $\boldsymbol{L}_2$ of $a^V_k \in \boldsymbol{P}(k)$ satisfying $a^V_k \geq a^U_k$ and $a^V_l \leq a^U_l$ (resp. $a^V_k \geq a^U_k$ and $a^V_l \geq a^U_l$) is of cardinality κ.

For the index $i = 1$ or 2 such that $\boldsymbol{L}_i$ is of cardinality κ, we consider the set $\mathcal{S}$ *of* $\boldsymbol{W} = \boldsymbol{U} \cup \boldsymbol{V}$ for $a^V_k \in \boldsymbol{L}_i$. From the definition of $\boldsymbol{L}_i$, it follows that $\mathcal{S}$ is of cardinality κ. Let $\boldsymbol{W} = \boldsymbol{U} \cup \boldsymbol{V}$ be in $\mathcal{S}$, and

$$W = \bigcup \{[b^W_{2n}, b^W_{2n+1}) \colon n < m\}$$

be its canonical decomposition. For distinct members $W' = U \cup V'$ and $W'' = U \cup V''$, we have:

$$b^{W'}_p = a^U_p = a^{V'}_p = a^{V''}_p = b^{W''}_p \quad \text{if } p < 2m, \text{ with } p \not\in \rho(\mathcal{R}_0) .$$

Moreover, if the set L_1 is of cardinality κ, then:

$$b^{W'}_k = a^{V'}_k \neq a^{V''}_k = b^{W''}_k \quad \text{and} \quad b^{W'}_l = a^U_l = b^{W''}_l ;$$

if the set L_2 is of cardinality κ, then:

$$b^{W'}_k = a^U_k = b^{W''}_k \quad \text{and} \quad b^{W'}_l = a^{V'}_l \neq a^{V''}_l = b^{W''}_l .$$

In both cases, the residual subset $\mathcal{S}_0$ of $\mathcal{S}$ which is of cardinality κ satisfies: $\rho(\mathcal{S}_0) \subseteq \rho(\mathcal{S}) \subseteq \rho(\mathcal{R}_0)$, $\mathcal{S}_0$ is a residual subset of $\mathcal{L}$, and $|\rho(\mathcal{S}_0)| \leq |\rho(\mathcal{S})| < |\rho(\mathcal{R}_0)|$, contradicting the minimality of $|\rho(\mathcal{R}_0)|$. □

3.10. NOTATION. Let P be a strongly rigid subchain of the real line. Let $\mathcal{A}$ be a subset of $B\langle P\rangle$ of cardinality κ, and $R(\mathcal{A})$ be its residual subset. Assume $|\rho(R(\mathcal{A}))| = 1$. Then there are $m < \omega$ and $k \leq 2m - 1$, such that: first, $U = \bigcup \{[a^U_{2n}, a^U_{2n+1}) \colon n < m\}$, with $a^U_i \in P \cup \{-\infty, +\infty\}$, for $U \in R(\mathcal{A})$, and secondly, for distinct members U and V in $R(\mathcal{A})$, we have:
(i) $a^U_j = a^V_j$, for $j \leq 2m - 1$ and $j \neq k$, and
(ii) $a^U_k \neq a^V_k$.
In what follows we assume that $\rho(R(\mathcal{A}))$ has a unique element k, and so we can consider the set P' of a^U_k for $U \in R(\mathcal{A})$. Note that $U \subseteq V$ in $R(\mathcal{A})$ if and only if $a^U_k \leq a^V_k$ whenever k is odd (resp. $a^U_k \geq a^V_k$ whenever k is even).

3.11. PROPOSITION. *Let P be a strongly rigid subchain of the real line. Let P_1 be a subchain of P. We assume that P_1 is κ-dense and topologically dense in P. Then, there is no non-trivial one-to-one monotonic function from $B\langle P_1\rangle$ into $B\langle P\rangle$.*

PROOF. By contradiction. Let f be a non-trivial one-to-one monotonic function from $B_1 = B\langle P_1\rangle$ into $B = B\langle P\rangle$. We can assume f is one-to-one and increasing. Let $a \in B_1$ be such that $f(a) \neq a$. Note that B_1 is a dense subalgebra of B.

Case 1. $a \not\leq f(a)$. We set $c = a \setminus f(a) \neq \emptyset$. So there is $c_1 \leq c$ in B_1 such that $c_1 \neq \emptyset$ in B_1. So the restriction g of f on $B_1 \upharpoonright c_1$ is a one-to-one increasing function from $B_1 \upharpoonright c_1$ into $B \upharpoonright f(a)$. Since $c_1 \subseteq P_1$, the set $P' = c_1$ is a subchain of P_1. Trivially, $B\langle P'\rangle$ and $B_1 \upharpoonright c_1$ are isomorphic algebra, and so P' can be considered as a subchain of $B \upharpoonright c_1$, identifying $t \in P'$ with $[-\infty, t) \cap c_1$. Thus, $g[P']$ is a subchain of $B\langle P\rangle$ of cardinality κ. From Lemma 3.8 it follows that $R(g[P'])$ is of cardinality κ and $|\rho(R(g[P']))| = 1$. Using notation 3.10, let k be the unique element of $\rho(R(g[P']))$, and P'' be the set of a^U_k for $U \in R(g[P'])$, which is of cardinality κ. Let S be the set of $t \in P'$ satisfying $g(t) \in R(g[P'])$. Note that S is a subchain of P_1, contained in c_1, of cardinality κ. Let h be the function from S into P'' defined by $h(t) = a^{f(t)}_k$. Obviously h is one-to-one. Moreover, h is increasing if k is odd, and h is decreasing otherwise. Now, S and P'' are pairwise disjoint subsets of P and of cardinality κ, contradicting the strong-rigidity of P.

Case 2. $f(a) \not\leq a$. Let $c = f(a) \setminus a \neq \emptyset$. There is $c_1 \leq c$ in $\boldsymbol{B}_1$ such that $c_1 \neq \emptyset$ in $\boldsymbol{B}_1$. Let g be the function from $\boldsymbol{B}_1 \restriction c_1$ into $\boldsymbol{B} \restriction -c_1$ defined by $g(x) = f(a \cup x) \setminus c_1$ for $x \leq c_1$. The function g is increasing and one-to-one, since:

(1) f is one-to-one and increasing,

(2) a and x are disjoint elements, and

(3) $c_1 \subseteq c \subseteq f(a) \subseteq f(a \cup x)$.

Now we conclude as in Case 1, since $c_1 \not\leq g(c_1)$. □

3.11.1. REMARK. In the above result the condition $\boldsymbol{P}_1$ is topologically dense in $\boldsymbol{P}$ is necessary. Indeed, let $\boldsymbol{P}$ satisfy (1′) and (2), $\boldsymbol{P}_1 = (0, 1) \cap \boldsymbol{P}$ and $a < 0$ in $\boldsymbol{P}$. Let $\boldsymbol{B}_1$ be the subalgebra of $\boldsymbol{B} = \boldsymbol{B}\langle \boldsymbol{P} \rangle$ of $\boldsymbol{U}$ such that:

$$\boldsymbol{U} = \bigcup \{[a^{\boldsymbol{U}}_{2k}, a^{\boldsymbol{U}}_{2k+1}) \colon k < m\} ,$$

with $a^{\boldsymbol{U}}_n$ in $\{-\infty\} \cup \boldsymbol{P}_1 \cup \{+\infty\}$. Then the function f from $\boldsymbol{B}_1$ into $\boldsymbol{B}$, defined by $f(\boldsymbol{U}) = \boldsymbol{U} \cap [a, +\infty)$, is one-to-one and increasing.

3.12. LEMMA. *Let $\boldsymbol{P}'$ and $\boldsymbol{P}''$ be strongly rigid subchains of a strongly rigid subchain $\boldsymbol{P}$ of $\boldsymbol{R}$. If there is a one-to-one monotonic function from $\boldsymbol{B}\langle \boldsymbol{P}' \rangle$ into $\boldsymbol{B}\langle \boldsymbol{P}'' \rangle$, then $\boldsymbol{P}' \setminus \boldsymbol{P}''$ is of cardinality less than κ.*

PROOF. For instance, let H be a strictly increasing function from $\boldsymbol{B}\langle \boldsymbol{P}' \rangle$ into $\boldsymbol{B}\langle \boldsymbol{P}'' \rangle$. By contradiction, we suppose that $\boldsymbol{L} = \boldsymbol{P}' \setminus \boldsymbol{P}''$ is of cardinality κ. For $t \in \boldsymbol{L}$, we set $b_t = [-\infty, t) \in \boldsymbol{B}\langle \boldsymbol{P}' \rangle$. Let $\mathscr{C}$ be the set of $H(b_t)$ for $t \in \boldsymbol{L}$, and $\mathfrak{R} = \boldsymbol{R}(\mathscr{C})$. The set $\rho(\mathfrak{R})$ has a unique element k, defined by notation 3.10, since $\mathscr{C}$, and so $\mathfrak{R}$, are chains.

Let $\boldsymbol{L}_1$ be the set of $t \in \boldsymbol{L}$ satisfying $H(b_t) \in \mathfrak{R}$. The function h from $\boldsymbol{L}_1$ into $\boldsymbol{P}''$, defined by $h(t) = a^{H(b_t)}_k$, is one-to-one and monotonic (in fact h is increasing if k is odd, and h is decreasing otherwise), and satisfies $h(t) \neq t$, since $t \not\in \boldsymbol{P}''$: this contradicts the strong rigidity of $\boldsymbol{P}$. □

3.13. PROPOSITION. *Let $\boldsymbol{P}$ be a strongly rigid subchain of the real line, and $\boldsymbol{B}$ be a subalgebra of $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ of cardinality κ. Then there is a strongly rigid subchain $\boldsymbol{P}'$ of $\boldsymbol{P}$ such that $\boldsymbol{B}\langle \boldsymbol{P}' \rangle$ is isomorphic to a subalgebra of $\boldsymbol{B}$.*

PROOF. From Lemma 3.9 it follows that we can choose a residual subset $\mathfrak{R}$ of $\boldsymbol{B}$ of cardinality κ such that $\rho(\mathfrak{R})$ has a unique element k defined by notation 3.10. Let $\boldsymbol{P}_1$ be the set of $a^{\boldsymbol{U}}_k \in \boldsymbol{P}$ for $\boldsymbol{U}$ in $\mathfrak{R}$. Let $\boldsymbol{P}'$ be a strongly rigid subchain of $\boldsymbol{R}$, contained in $\boldsymbol{P}_1$ (the existence of $\boldsymbol{P}'$ follows from Lemma 2.3, in the Appendix on Set Theory in this Handbook). Note that the algebra $\boldsymbol{B}\langle \boldsymbol{P}' \rangle$ and the subalgebra $\boldsymbol{B}'$ of $\boldsymbol{B}$ generated by $\{\boldsymbol{U} \in \mathfrak{R} \colon a^{\boldsymbol{U}}_k \in \boldsymbol{P}'\}$ are isomorphic. □

3.14. THEOREM. *For each subset $\boldsymbol{I}$ of κ, we can associate a strongly rigid subchain $\boldsymbol{P}_{\boldsymbol{I}}$ of $\boldsymbol{R}$ such that the interval Boolean algebra $\boldsymbol{B}_{[\boldsymbol{I}]} = \boldsymbol{B}\langle \boldsymbol{P}_{\boldsymbol{I}} \rangle$ satisfies:*

(1) For each $a, b \neq \boldsymbol{0}$ in $\boldsymbol{B}_{[\boldsymbol{I}]}$, with $a \not\leq b$, there is no one-to-one increasing function from $\boldsymbol{B}_{[\boldsymbol{I}]} \restriction a$ into $\boldsymbol{B}_{[\boldsymbol{I}]} \restriction b$.

(2) Let $\boldsymbol{I}$ and $\boldsymbol{J}$ be subsets of κ. The following properties are equivalent:

(i) $I \subseteq J$,
(ii) *there is a one-to-one increasing function from* $B_{[I]}$ *into* $B_{[J]}$, *and*
(iii) *there is a one-to-one homomorphism from* $B_{[I]}$ *into* $B_{[J]}$,
(3) *If* $I \cap J = \emptyset$, *and* B *is an algebra embeddable in* $B_{[I]}$ *and* $B_{[J]}$, *then* $|B| < \kappa$.

PROOF. Let P be a strongly rigid subchain of R such that $|P \cap (a, b)| = \kappa$ whenever $a < b$ in R. Let $\langle Q_\beta : \beta < \kappa \rangle$ enumerate all intervals (a, b) with rational endpoints, each interval repeated κ times. Let $\{(\delta(\gamma), \varepsilon(\gamma)) : \gamma < \kappa\}$ enumerate $\kappa \times \kappa$. By induction choose $y_\gamma \in (P \cap Q_{\varepsilon(\gamma)}) \setminus \{y_\delta : \delta < \gamma\}$. For $\alpha < \kappa$ set $P_\alpha = \{y_\gamma : \delta(\gamma) = \alpha\}$. Then $(P_\alpha)_{\alpha<\kappa}$ is a family of pairwise disjoint subsets of P and
(i) P_α is a strongly rigid chain,
(ii) $(a, b) \cap P_\alpha$ is of cardinality κ for every $a < b$ in R.

For non-empty $I \subseteq \kappa$, we set $P_I = \bigcup \{P_\alpha : \alpha \in I\}$ and $B_{[I]} = B\langle P_I \rangle$. The chain P_I is strongly rigid, since each P_α is topologically dense in P_I. By Remark 1.2.2, $B_{[I]}$ can be viewed as a subalgebra of $B\langle P \rangle$.

We will prove (1) by contradiction. Fix $\emptyset \neq I \subseteq \kappa$, and assume there are a and b in $B_{[I]}$ such that $a \not\leq b$ in $B_{[I]}$ and a one-to-one increasing function H from $B_{[I]} \upharpoonright a$ into $B_{[I]} \upharpoonright b$. Let $c = a \cdot -b$. Let P' (resp. P'') be the set of $t \in P$ such that $[-\infty, t) \cap c \in B_{[I]} | c$ (resp. of $t \in P$ such that $[-\infty, t) \cap b \in B_{[I]} | b$). The sets P' and P'' are disjoint strongly rigid subchains of P. This follows because P is strongly rigid. Obviously, the algebras $B\langle P' \rangle$ and $B_{[I]} \upharpoonright c$ (resp. $B\langle P'' \rangle$ and $B_{[I]} \upharpoonright b$) are isomorphic algebras. The restriction of H on $B_{[I]} \upharpoonright c$ induces a one-to-one increasing function from $B\langle P' \rangle$ into $B\langle P'' \rangle$, giving a contradiction to Lemma 3.12.

We next prove (2). Clearly, (iii) implies (ii) while (i) implies (ii) and (iii) for if $\emptyset \neq I \subseteq J \subseteq \kappa$, then P_I is a subchain of P_J.

Now we prove that (ii) implies (i), by contradiction. Let $\alpha \in I \setminus J$ and assume that H be a one-to-one increasing function from $B_{[I]}$ into $B_{[J]}$. Since $P_\alpha \subseteq P_I$, H induces a one-to-one increasing function from $B\langle P_\alpha \rangle$ into $B\langle P_J \rangle$, which contradicts Lemma 3.12 since $P_\alpha \cap P_J = \emptyset$.

To prove (3) let us suppose, for contradiction, that I and J are pairwise disjoint subsets of κ, and B is an algebra of power κ embeddable in both $B\langle P_I \rangle$ and $B\langle P_J \rangle$. From Proposition 3.13, it follows that there is a strongly rigid subchain S of P_I such that $B\langle S \rangle$ is isomorphic to a subalgebra of B. By Lemma 3.12, the sets $S \setminus P_I$ and $S \setminus P_J$ are of cardinality less than κ, and thus $S \setminus (P_I \cap P_J)$ is of cardinality less than κ. Consequently, $P_I \cap P_J$ is of cardinality κ, which contradicts $P_I \cap P_J = \emptyset$. $\square$

REMARK. Using the above notation, we can improve the statement (1) of the Theorem, as follows: if a in $B_{[I]}$ and b in $B_{[J]}$ are such that $a \not\leq b$ in $B\langle P \rangle$, then there is no one-to-one monotonic function from $B_{[I]} \upharpoonright a$ into $B_{[J]} \upharpoonright b$, for all non-empty subsets I and J of κ.

3.15. PROPOSITION. *We assume* $\sup\{2^\lambda : \lambda < \kappa\} = \kappa$. *There is a family* $(B_i)_{i \in I}$, *with* $|I| = 2^\kappa$, *of rigid interval algebras of cardinality* κ *such that*:
(1) *if* B *is an algebra embeddable in* B_i *and* B_j, *with* $i \neq j$, *then* $|B| < \kappa$,
(2) *if* B *is a quotient algebra of both* B_i *and* B_j, *with* $i \neq j$, *then* $|B| < \kappa$.

PROOF. Let $\boldsymbol{P}$ be a strongly rigid subchain of the real line. Let $(\boldsymbol{P}_i)_{i\in I}$ be a family of subsets of $\boldsymbol{P}$ satisfying:

(i) I is of cardinality 2^κ,

(ii) $|\boldsymbol{P}_i| = \kappa$ for $i \in I$, and

(iii) $|\boldsymbol{P}_i \cap \boldsymbol{P}_j| < \kappa$ for $i \neq j$ in I.

We recall that such a family is called an almost disjoint family of sets, and the existence of such a family is proved in the Appendix on Set Theory in this Handbook. Moreover, from the fact that $\boldsymbol{R}$ has a countable basis of open sets, it follows that we can suppose:

(iv) $\boldsymbol{P}_i$ is a strongly rigid subchain of the real line, for every $i \in I$, and

(v) $|(a, b) \cap \boldsymbol{P}_i| = \kappa$ for $a < b$ in $\boldsymbol{R}$, and $i \in I$.

Now, we set $\boldsymbol{B}_i = \boldsymbol{B}\langle \boldsymbol{P}_i \rangle$. Let $i \neq j$ in I. We conclude, using the arguments developed in the proof of (3) of Theorem 3.14.

Statement (2) of the theorem is a consequence of retractiveness property in interval algebras. □

3.16. COMMENT. Let μ be a cardinal satisfying $\sup\{2^\theta : \theta < \mu\} = \mu$. Let $\lambda = \mathrm{cf}(\mu)$. Then we can extend the above results to cardinality λ, replacing the rational chain by the Hausdorff chain η_α, where $\mu = \aleph_\alpha$ and the chain of real numbers by the relative Dedekind completion of η_α.

3.17. COMMENTS. We can improve the above results in the two following ways:

(1) There is a subchain $\boldsymbol{P}$ of the real line, of cardinality $\kappa = \mathrm{cf}(2^\omega)$ such that the interval algebra $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ satisfies: $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ is of cardinality κ and every subset of $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ of pairwise distinct elements is of cardinality less than κ. Such a Boolean algebra is said to be narrow (for development, see BONNET and SHELAH [1985]). Moreover each factor of $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ is of cardinality κ too, and thus $\boldsymbol{B}\langle \boldsymbol{P} \rangle$ is mono-rigid.

(2) Let $\boldsymbol{B}$ be an algebra and f be an endomorphism of $\boldsymbol{B}$. We denote by $\boldsymbol{I}(f)$ the set of $a \in \boldsymbol{B}$ such that $f(a) = \boldsymbol{0}$ or $f(x) = x$ for every $x \leq a$ in $\boldsymbol{B}$. If $\boldsymbol{B}$ is an interval algebra then $\boldsymbol{B}/\boldsymbol{I}(f)$ is infinite for some endomorphism f of $\boldsymbol{B}$. We assume C.H. There is a ω_1-dense subchain $\boldsymbol{P}$ of the real line, such that the interval algebra $\boldsymbol{B} = \boldsymbol{B}\langle \boldsymbol{P} \rangle$ satisfies: $\boldsymbol{B}/\boldsymbol{I}(f)$ is countable for every endomorphism of $\boldsymbol{B}$, in particular $\boldsymbol{B}$ is mono-rigid and thus Bonnet-rigid.

To show this, we begin to construct a subset $\boldsymbol{P}_1$ of $\boldsymbol{R}$ such that every nowhere dense subset of $\boldsymbol{P}_1$ is countable. To construct $\boldsymbol{P}_1$ consider an enumeration $\boldsymbol{F}_\alpha$ for $\alpha < \omega_1$ of all uncountable closed nowhere dense subsets of $\boldsymbol{R}$ and by induction pick x_α in $\boldsymbol{R}$ which does not in the meager set $\{x_\mu : \mu < \alpha\} \cup \bigcup \{\boldsymbol{F}_\mu : \mu < \alpha\}$, and set $\boldsymbol{P}_1 = \{x_\alpha : \alpha < \omega_1\}$. Now apply the method developed in 3.7 or in §2 of the Appendix on Set Theory of this Handbook, replacing the real line by $\boldsymbol{P}_1$, to obtain a strongly-rigid subchain $\boldsymbol{P}$ of $\boldsymbol{R}$, contained in $\boldsymbol{P}_1$ (see BELHASSAN [1984]).

3.18. REMARK. The introduction of the notion of Bonnet-rigid algebra was motivated by a Theorem of GEBA and SEMADENI [1960]. A compact space $\boldsymbol{K}$ is said to be very strongly rigid whenever for every compact space $\boldsymbol{L}$, for every one-to-one continuous function φ from $\boldsymbol{L}$ into $\boldsymbol{K}$ and every continuous function ψ from $\boldsymbol{L}$ onto $\boldsymbol{K}$, then $\varphi = \psi$. Let $\boldsymbol{K}$ be a compact Hausdorff space. We denote by $C(\boldsymbol{K})$ the space of real-valued functions on $\boldsymbol{K}$, with the uniform convergence norm. Let T be a linear isometry from $C(\boldsymbol{K})$ into itself. T is said to be positive – or isotonical – whenever $f \geq 0$ if and only if $T(f) \geq 0$ for $f \in C(\boldsymbol{K})$. The following result is a consequence of a theorem of Geba and Semadeni.

Let $\boldsymbol{K}$ be a compact space. The following properties are equivalent:

(i) $C(\boldsymbol{K})$ has only one positive and linear isometry, namely the identity;

(ii) if $\boldsymbol{L}$ is a compact subspace of $\boldsymbol{K}$ and if θ is a continuous function from $\boldsymbol{L}$ onto $\boldsymbol{K}$, then $\boldsymbol{L} = \boldsymbol{K}$ and θ is the identity; and

(iii) $\boldsymbol{K}$ is a very strongly rigid compact space.

THEOREM. *Let $\boldsymbol{B}$ be a Bonnet-rigid algebra, and $\boldsymbol{K}$ be its Stone space. Assume that $\boldsymbol{K}$ is of cardinality 2^{ω}. Then $C(\boldsymbol{K})$ has only one positive and linear isometry, namely the identity. Moreover, there are 2^{ω} compact spaces, pairwise non-homeomorphic, of cardinality 2^{ω}, having this property.*

REMARK. (1) Let $\boldsymbol{P}$ be a subset of $\boldsymbol{R}$. We denote by $C_{\boldsymbol{P}}(\boldsymbol{R})$ the set of functions f from the extended real line into $\boldsymbol{R}$, such that:

(i) f is a regulated function, i.e. f is a uniform limit of step functions, which means f is continuous except for jump discontinuities, and for every $x \in \boldsymbol{R}$, the function f has a right limit $f(x^{+})$ and a left limit $f(x^{-})$ in x;

(ii) $f(-\infty)$ and $f(+\infty)$ exist;

(iii) $f(x) = (f(x^{+}) + f(x^{-}))/2$ for every $x \in \boldsymbol{R}$; and

(iv) the set of jumps Σ_f of f is a countable subset of $\boldsymbol{P}$. Recall that x is a jump of f whenever $f(x^{+}) \neq f(x^{-})$.

Now, let $\boldsymbol{P}$ be a strongly rigid subset of $\boldsymbol{R}$, and $\boldsymbol{K}$ be the Stone space associated with $\boldsymbol{B}\langle \boldsymbol{P} \rangle$. Recall that $\boldsymbol{K}$ is the interval space consisting of all initial intervals of $\boldsymbol{P}$. The ordered Banach space $C(\boldsymbol{K})$ of continuous functions from $\boldsymbol{K}$ into $\boldsymbol{R}$, with the uniform norm, is identified under H with $C_{\boldsymbol{P}}(\boldsymbol{R})$, in the following way. Let $f \in C(\boldsymbol{K})$. We set: $H(f)(x) = f(\boldsymbol{I}_x)$ if $\boldsymbol{I}_x$ is an initial interval of $\boldsymbol{P}$ determined by a point x which does not belong to $\boldsymbol{P}$, and $H(f)(x) = (f((-\infty, x)) + f((-\infty, x]))/2$ for $x \in \boldsymbol{P}$. So $H(f) \in C_{\boldsymbol{P}}(\boldsymbol{R})$, and H is an isometry from $C(\boldsymbol{K})$ onto $C_{\boldsymbol{P}}(\boldsymbol{R})$.

(2) We only can consider positive and linear isometries on $C(\boldsymbol{K})$, since $\boldsymbol{K}$ is a Boolean space.

4. Todorčević's construction of many mono-rigid interval algebras

The Todorčević idea to construct rigid algebra is completely different. It is a purely set-theoretical construction different from the ideas in Sections 2 and 3. He uses some results of BAUMGARTNER [1976], and exploits a trick concerning stationary subsets of $\boldsymbol{S}_{\omega} \subseteq \kappa$. We thank Mati Rubin for information concerning results of Baumgartner.

We recall a consequence of Fodor's theorem (see the Appendix on Set Theory in this Handbook):

4.1.1. FACT. Let κ be a regular cardinal, $\boldsymbol{S}$ be a stationary subset of κ and f be a function from $\boldsymbol{S}$ into κ. We denote by $\boldsymbol{S}^{-}$ (resp. $\boldsymbol{S}^{0}$, $\boldsymbol{S}^{+}$) the set of $\alpha \in \boldsymbol{S}$ satisfying $f(\alpha) < \alpha$ (resp. $f(\alpha) = \alpha$, $f(\alpha) > \alpha$). Then at least one of these sets is stationary, and:

(1) if $\boldsymbol{S}^{-}$ is stationary, then f is constant on a stationary subset of $\boldsymbol{S}^{-}$;

(2) if $\boldsymbol{S}^{+}$ is stationary, then f is strictly increasing on a stationary subset of $\boldsymbol{S}^{+}$.

4.1.2. SELECTION PROPERTY. *Let κ be a regular cardinal, S be a stationary subset of κ, and $(S_i)_{i\in I}$ be a partition of S in non-empty and non-stationary sets. Then there is a subset $\sigma(S)$ of S such that:*

(a) *$\sigma(S) \cap S_i$ has exactly one element, and*

(b) *$S\setminus\sigma(S)$ is non-stationary in κ, and $\sigma(S)$ is stationary.*

Because of (a), $\sigma(S)$ is called a *selector* of the partition $(S_i)_{i\in I}$. The natural way to obtain $\sigma(S)$ is to consider the set of first elements of each S_i.

4.1.3. Let us make a digression to explain the selection property in terms of measure theory, in the case ω_1. Consider ω_1 as interval space and let C_0 be the Boolean algebra generated by the set of clubs of ω_1 (under the finite operations). Let μ_0 be the finitely additive 2-valued measure on C_0 generated by $\mu_0(A) = 1$ whenever A is a club. Now, let $\mathrm{Bor}(\omega_1)$ be the σ-algebra of Borel subsets of ω_1 (generated by C_0). For $A \subseteq \omega_1$, denote by $\mu^*(A)$ and $\mu_*(A)$ the outer and the inner measure of A. Recall that $\mu^*(A)$ is the infimum of $\Sigma_{n<\omega}\mu_0(A_n)$, $A_n \in C_0$, such that $A \subseteq \bigcup\{A_n: n<\omega\}$, and $\mu_*(A)$ is the supremum of $\mu_0(B)$, $B \in C_0$, such that $A \supseteq B$. For every $A \in \mathrm{Bor}(\omega_1)$, we have $\mu^*(A) = \mu_*(A) = 0$ or 1, because every countable intersection of clubs is a club too. So $\mu = \mu^* = \mu_*$ is a 2-valued σ-measure defined on $\mathrm{Bor}(\omega_1)$, and called Dieudonné's measure. Notice that for every $A \in \mathrm{Bor}(\omega_1)$, we have:

(i) $\mu(A) = 1$ if and only if A contains a club, and

(ii) $\mu(A) = 0$ if and only if A is disjoint from a club.

Moreover for a subset S of ω_1 we have:

(iii) $\mu^*(S) = 1$ if and only if S meets every club of ω_1, i.e. S is stationary, which is equivalent to:

(iv) $\mu^*(S) = 0$ if and only if S is disjoint from a club of ω_1, i.e. S is non-stationary.

We can generalize this notion for every uncountable regular cardinal, using κ-complete measures. So the selection property can be viewed as Fubini's theorem.

4.2. Let $\kappa > \omega$ be a regular cardinal, and S_ω be the stationary set of α of cofinal type ω. For each $\alpha \in S_\omega$, we fix a continuous strictly increasing function f_α from $\omega+1$ into α such that $f_\alpha(\omega) = \alpha$, i.e. $f_\alpha(m) < f_\alpha(n) < f_\alpha(\omega)$ for $m<n<\omega$ and $\alpha = f_\alpha(\omega) = \sup\{f_\alpha(m): m<\omega\}$. Since f_α is continuous, we identify f_α with the sequence $\langle f_\alpha(m)\rangle_{m<\omega}$. We denote by $f_\alpha \restriction m$ the restriction of f_α to $m = \{0,1,\ldots,m-1\}$. So $f_\alpha \restriction m$ is identified with $\langle f_\alpha(k)\rangle_{k<m}$.

Let $S \subseteq S_\omega$. We denote by $\underline{T}(S)$ the set of $f_\alpha \restriction m$ for $m<\omega$ and $\alpha \in S$. Let $\le$ be the partial ordering on $\underline{T}(S)$ defined by $f_\alpha \restriction m \le f_\beta \restriction n$ if and only if $m \le n$ and $f_\alpha(k) = f_\beta(k)$ for $k<m$. Obviously, $\langle \underline{T}(S), \le\rangle$ is a tree and $\mathrm{Lev}_m(\underline{T}(S))$ is the set of $f_\alpha \restriction m$ for $\alpha \in S$. So $\mathrm{Lev}_0(\underline{T}(S))$ has a unique element $\emptyset$. Consequently, $\mathrm{Br}(\underline{T}(S))$ can be identified with the set of f_α for $\alpha \in S$. Let $m<n$ and $u \in \mathrm{Lev}_m(\underline{T}(S))$. Let $\le$ be the linear ordering on $\mathrm{Succ}(u)$ defined by $f_\alpha \restriction (m+1) \le f_\beta \restriction (m+1)$ if and only if $f_\alpha \restriction m = f_\beta \restriction m = u$ and $f_\alpha(m) \le f_\beta(m)$. So $\mathrm{Succ}(u)$ is a well-ordered chain, and thus induces a well-order on $\mathrm{Lev}_m(\underline{T}(S))$ in the natural way. Also, $\mathrm{Br}(\underline{T}(S))$ is linearly ordered, in a natural way, denoted by $L(S)$. So

$f_\alpha < f_\beta$ in $L(S)$ if and only if for the smallest $m < \omega$ satisfying $f_\alpha(m) \neq f_\beta(m)$, then $f_\alpha(m) < f_\beta(m)$. For α and β in S such that $f_\alpha < f_\beta$, we denote by $S[\alpha, \beta]$ the set of all $\gamma \in S$ such that $f_\alpha \leqq f_\gamma \leqq f_\beta$.

4.3. LEMMA. *$L(S)$ has no strictly decreasing ω_1-sequence.*

This is an easy consequence of the fact that each $\mathrm{Lev}_m(\underline{T}(S))$ is a well-ordered chain.

4.4.1. NOTATIONS. Let $S \subseteq S_\omega$, $\alpha \in S$, and $m < \omega$ be given. We denote by $S_{[\alpha/m]}$ (resp. $\underline{S}_{[\alpha/m]}$) the set of $\beta \in S$ satisfying $f_\beta \restriction m = f_\alpha \restriction m$ (resp. $f_\beta \restriction m = f_\alpha \restriction m$ and $f_\beta > f_\alpha$).

4.4.2. REMARKS.

(1) $\underline{S}_{[\alpha/m]} \subseteq S_{[\alpha/m]}$ for $\alpha \in S$, and $m < \omega$.

(2) $S_{[\alpha/n]} \subseteq S_{[\alpha/m]}$ and $\underline{S}_{[\alpha/n]} \subseteq \underline{S}_{[\alpha/m]}$ for $\alpha \in S$, and $m < n < \omega$.

(3) If $m < \omega$, $\alpha', \alpha'' \in S$ satisfy $f_{\alpha'} \restriction m \neq f_{\alpha''} \restriction m$, then $S_{[\alpha'/m]} \cap S_{[\alpha''/m]} = \emptyset$, and thus $\underline{S}_{[\alpha'/m]} \cap \underline{S}_{[\alpha''/m]} = \emptyset$.

Now we prove Lemma 2.1 of TODORČEVIĆ [1979], [1980]. The first study of this kind of chains, was done by BAUMGARTNER [1976]. For pedagogical reasons, we give two proofs of Lemmas 4.6, 4.7; one, the original proof of S. Todorčević (using basically Fodor's theorem), the other observed by M. Rubin (using essentially the selection property; see Comment 4.10).

4.5. LEMMA. *Let κ be a regular cardinal, S be a stationary subset of S_ω. Denote by $r(S)$ (resp. $\underline{r}(S)$) the set of $\alpha \in S$ such that $S_{[\alpha/m]}$ (resp. $\underline{S}_{[\alpha/m]}$) is stationary in κ, for every $m < \omega$. Then:*

(1) $\underline{r}(S) \subseteq r(S)$, and

(2) $S \setminus \underline{r}(S)$ is a non-stationary subset of κ. In particular $\underline{r}(S)$ and thus $r(S)$ are stationary in κ.

PROOF. (1) is a trivial consequence of Remark 4.4.2. Now, we will prove (2) by contradiction. Assume that $H = S \setminus \underline{r}(S)$ is stationary in κ. For each $\alpha \in H$ there is $n(\alpha) < \omega$ such that $\underline{S}_{[\alpha/n(\alpha)]}$ is non-stationary in κ. From the fact that every countable union of non-stationary subsets of κ is still non-stationary, it follows, setting H_m equal to the set of $\alpha \in H$ such that $n(\alpha) = m$, that H_m is stationary for some $m < \omega$. Now the function g_0 from H_m into κ, defined by $g_0(\alpha) = f_\alpha(0)$, is regressive, and thus constant on a stationary subset H_m^0 of H_m. We set $t_0 = g_0(\alpha) = g_0(\beta)$ for $\alpha, \beta \in H_m^0$. Now the function g_1 from H_m^0 into κ, defined by $g_1(\alpha) = f_\alpha(1)$, is regressive, and thus constant on a stationary subset H_m^1 of H_m^0. We set $t_1 = g_1(\alpha) = g_1(\beta)$ for $\alpha, \beta \in H_m^1$. Repeating this procedure m times, we obtain a stationary set $T = H_m^{m-1}$ and a finite sequence $\underline{t} = (t_k)_{k<m}$ such that $\underline{t} = f_\alpha \restriction m$ for every $\alpha \in T$. Notice that for $\alpha \in T$, the sets $\underline{S}_{[\alpha/m]}$ and thus $\underline{T}_{[\alpha/m]}$ are non-stationary. From the fact that every countable union of non-stationary subsets of κ is still non-stationary, it follows that:

(i) either $L(T)$ has a smallest element f_δ and then $T \setminus \{\delta\} = \underline{T}_{[\delta/m]} \subseteq \underline{S}_{[\delta/m]}$; consequently, $\underline{S}_{[\delta/m]}$ is stationary, which contradicts our assumption, or

(ii) $L(T)$ is of coinitial type ω, and then $T = \bigcup \{T_k: k < \omega\}$, wnere $(f_{\alpha(k)})_{k<\omega}$ is coinitial in $L(T)$, and $T_k = \underline{T}_{[\alpha(k)/m]}$. Consequently, some $T_k = \underline{T}_{[\alpha(k)/m]}$ is stationary, which contradicts our assumption. □

REMARK. Let S be a stationary subset of S_ω. Then $r(S)_{[\alpha/m]}$ and $\underline{r(S)}_{[\alpha/m]}$ are stationary, for all $\alpha \in \underline{r}(S)$ and $m < \omega$.

4.6. LEMMA. *Assume that $L(S)$ satisfies: for every α and β in S such that $f_\alpha < f_\beta$ the set $S[\alpha, \beta]$ is non-stationary in κ. Then S is non-stationary in κ.*

PROOF. By contradiction. By Lemma 4.5, let α, β, and γ be distinct members of $r(S)$. We can suppose $f_\alpha < f_\beta < f_\gamma$. Let $n_{\alpha,\beta}$, $n_{\alpha,\gamma}$, $n_{\beta,\gamma}$ be the smallest integer m satisfying $f_\alpha(m) \neq f_\beta(m)$, $f_\alpha(m) \neq f_\gamma(m)$, $f_\beta(m) \neq f_\gamma(m)$, respectively, let p be the maximum of $n_{\alpha,\beta}$, $n_{\alpha,\gamma}$, $n_{\beta,\gamma}$. Then obviously $S_{[\beta/p+1]} \subseteq S[\alpha, \gamma]$, $S_{[\beta/p+1]}$ is stationary (since $\beta \in r(S)$), contradicting $S[\alpha, \gamma]$ is non-stationary. □

4.7. LEMMA. *Let $S \subseteq S_\omega$ be such that $L(S)$ is a well-ordered chain. Then S is a non-stationary subset of κ.*

PROOF. Otherwise, apply Lemma 4.6 to S, where $L(S)$ is the least order-type ρ realized in this way. □

4.8. DEFINITION. Let $S \subseteq \kappa$. The chain $L(S)$ is said to have the *stationary interval property* whenever for every $f_\alpha < f_\beta$ with α, β in S, the set $S[\alpha, \beta]$ is stationary in κ.

4.9. LEMMA. *Let S be stationary in κ. There is a subset S' of S such that $S \backslash S'$ is non-stationary, and $L(S')$ has the stationary interval property (note that S' is therefore stationary).*

PROOF. Let $S' = \underline{r}(S)$; so $S \backslash S'$ is non-stationary by Lemma 4.5. Let α and β be in $\underline{r}(S)$ with $f_\alpha < f_\beta$. Let p be the smallest integer such that $f_\alpha(p) \neq f_\beta(p)$; so $f_\alpha(p) < f_\beta(p)$. If $\delta \in r(S)_{[\alpha/p+1]}$, then $f_\delta < f_\beta$, and thus $\underline{r(S)}_{[\alpha/p+1]}$ is contained in $r(S)[\alpha, \beta]$. The desired conclusions follow since $\underline{r(S)}_{[\alpha/p+1]}$ is stationary (see the Remark above). □

4.10. COMMENTS. We will develop Rubin's approach to the above results.

(1) Let $S \subseteq S_\omega$ be such that $L(S)$ is a well-ordered chain of order-type κ. Then S is a non-stationary subset of κ.

PROOF. For a contradiction, we suppose that S is stationary in κ. For each $u \in \underline{T}(S)$ let Σ_u be the set of $\alpha \in S$ satisfying $u \subseteq f_\alpha$. Clearly,

$$L(\Sigma_u) \stackrel{\text{def}}{=} \{f_\alpha: \alpha \in \Sigma_u\}$$

is an interval of $L(S)$. The hypotheses of Fact 1.6 hold, so we can choose

$u \in \underline{T}(S)$ so that Succ(u) is of cardinality κ. Thus, $L(\Sigma_u)$ is an interval of cardinality κ, and so is a final segment of $L(S)$. Hence, $|S \setminus \Sigma_u| < \kappa$, so Σ_u is stationary in κ. Now Σ_u is the disjoint union of the Σ_v for $v \in$ Succ(u), and each $L(\Sigma_v)$ is an interval of $L(S)$, so $|\Sigma_v| < \kappa$ for each such v. Thus, each Σ_v is non-stationary in κ. From the selection property it follows that there is a stationary subset Σ of Σ_u such that $\Sigma \cap \Sigma_v$ has exactly one point for $v \in$ Succ(u). Let $m < \omega$ be such that $u \in \mathrm{Lev}_m(\underline{T}(S))$. Thus, $f_\alpha(m) \neq f_\beta(m)$ for $\alpha \neq \beta$ in Σ. Now the function g from Σ into κ defined by $g(\alpha) = f_\alpha(m)$ is regressive and one-to-one. This finally contradicts Fodor's theorem. □

(2) Let $S \subseteq S_\omega$ be such that $L(S)$ has a first element $f_{\alpha(0)}$, a cofinal sequence of order type κ, and is such that $S[\alpha, \beta]$ is non-stationary in κ whenever $\alpha, \beta \in S$ and $f_\alpha < f_\beta$. Then S is a non-stationary subset of κ.

PROOF. To prove this, let $(f_{\alpha(\nu)})_{\nu<\kappa}$ be a cofinal sequence in $L(S)$. For $\nu < \kappa$ let

$$I_\nu = S[\alpha(0), \alpha(\nu)] \setminus \bigcup \{S[\alpha(0), \alpha(\mu)] \colon \mu < \nu\}\ .$$

For each $\nu < \kappa$, $S[\alpha(0), \alpha(\nu)]$ is non-stationary. Thus, $(I_\nu)_{\nu<\kappa}$ is a partition of S into non-stationary sets. By the selection property 4.1.2, choose $\underline{S} \subseteq S$ such that $S \setminus \underline{S}$ is non-stationary (and thus $\underline{S}$ is stationary), and so that $\underline{S} \cap I_\nu$ has a unique point. So $L(\underline{S})$ is order-isomorphic to κ, contradicting comment (1). □

(3) Assume that $L(S)$ satisfies: for every α and β in S such that $f_\alpha < f_\beta$, the set $S[\alpha, \beta]$ is non-stationary in κ. Then S is non-stationary in κ (Lemma 4.6).

PROOF. Let $\alpha = \alpha(0) \in S$ be fixed. Let S^- be the set of $\beta \in S$ satisfying $f_\beta \leq f_{\alpha(0)}$. From $L(S)$ and thus $L(S^-)$ has a smallest element, or is of coinitial type ω, it follows that S^- is non-stationary in κ. Now, let $S^+ = S \setminus S^-$, and $(f_{\alpha(\nu)})_{\nu<\lambda}$ be a cofinal sequence in $L(S^+)$ and thus in $L(S)$, where $\lambda \leq \kappa$ is regular (the case where $L(S)$ has a greatest element is trivial). If $\lambda < \kappa$ then we conclude trivially, and if $\lambda = \kappa$, then we conclude by comment (2). □

(4) If $L(S)$ is a well-ordered chain, then S is non-stationary in κ (Lemma 4.7).

Otherwise, apply comment (3) to S, where $L(S)$ is the least order-type ρ realised in this way.

4.11. LEMMA. *Let S and T be subsets of S_ω. If $L(S)$ is order-isomorphic to a subchain of $L(T)$, then $S \setminus T$ is non-stationary in κ.*

PROOF. By contradiction. We assume $\Sigma_0 = S \setminus T$ is stationary in κ. Let h be an isomorphism from $L(S)$ into $L(T)$. Let φ be the function from Σ_0 into κ defined $h(f_\alpha) = f_{\varphi(\alpha)}$. If $\alpha \neq \beta$ in Σ_0, then $\varphi(\alpha) \neq \varphi(\beta)$. From Fact 4.1.1 it follows that there is a stationary subset Σ_1 of Σ_0 such that $\varphi \restriction \Sigma_1$ is strictly increasing, and $\varphi(\alpha) \geq \alpha$ for $\alpha \in \Sigma_1$). We have $\varphi(\alpha) > \alpha$ for $\alpha \in \Sigma_1$ since $\Sigma_0 \cap T \neq \emptyset$. Consequently, $\Gamma = \varphi(\Sigma_1)$ is non-stationary, since $\varphi^{-1} \restriction \Gamma$ is one-to-one and regressive (see Theorem 5.6 of the Appendix on Set Theory in this Handbook). Let C

be a club disjoint from $\varphi(\Sigma_1)=\Gamma$, and let $(c_\beta)_{\beta<\kappa}$ be the canonical enumeration of elements of C (we may assume $c_0=0$). For $\alpha\in\Sigma_1$ let $\beta(\alpha)$ such that $c_{\beta(\alpha)}<\varphi(\alpha)<c_{\beta(\alpha)+1}$. For each $\delta<\kappa$, the set N_δ of $\eta\in\Sigma_1$ satisfying $c_\delta<\varphi(\eta)<c_{\delta+1}$ is of cardinality $<\kappa$ and this N_δ is non-stationary in κ. Let Σ_2 be a selector stationary subset of Σ_1 associated with the partition of the non-empty N_δ's. So for $\alpha'\neq\alpha''$ in Σ_2, we have $\varphi(\alpha')\neq\varphi(\alpha'')$. Now, for $\alpha\in\Sigma_2$ let $n(\alpha)$ be the smallest integer $k<\omega$ satisfying

$$c_{\beta(\alpha)}<f_{\varphi(\alpha)}(k)<\varphi(\alpha)<c_{\beta(\alpha)+1}\,.$$

Let $m<\omega$ and let Σ_3 be a stationary subset of Σ_2 satisfying $n(\alpha)=m$ for $\alpha\in\Sigma_3$ (see Corollary 5.3 of the Appendix on Set Theory). Now if $\alpha'\neq\alpha''$ in Σ_3, then $c_{\beta(\alpha')}\neq c_{\beta(\alpha'')}$ and thus $f_{\varphi(\alpha')}(m)\neq f_{\varphi(\alpha'')}(m)$. Consequently, $f_{\varphi(\alpha')}\leq f_{\varphi(\alpha'')}$ if and only if $f_{\varphi(\alpha')}\restriction(m+1)\leq f_{\varphi(\alpha'')}\restriction(m+1)$ in the lexicographic ordering κ^{m+1}. Since $h(f_\alpha)=f_{\varphi(\alpha)}$, we see that h induces an isomorphism from $L(\Sigma_3)$ into a well-ordered subchain of κ^{m+1}. That Σ_3 is stationary contradicts Lemma 4.7. $\square$

4.12. NOTATION. Let $\boldsymbol{S}\subseteq\boldsymbol{S}_\omega$. We denote by $\boldsymbol{B}_{\boldsymbol{S}}$ the interval algebra $\boldsymbol{B}\langle L(\boldsymbol{S})\rangle$.

4.13. PROPOSITION. *Let $\boldsymbol{S}$ and $\boldsymbol{T}$ be subsets of $\boldsymbol{S}_\omega$. If there is a strictly monotonic function from $\boldsymbol{B}_{\boldsymbol{S}}$ into $\boldsymbol{B}_{\boldsymbol{T}}$, then $\boldsymbol{S}\setminus\boldsymbol{T}$ is non-stationary in κ.*

PROOF. For instance, let H be a strictly increasing function from $\boldsymbol{B}_{\boldsymbol{S}}$ into $\boldsymbol{B}_{\boldsymbol{T}}$. By contradiction, we suppose that $\Sigma_0=\boldsymbol{S}\setminus\boldsymbol{T}$ is stationary in κ. For $\alpha\in\Sigma_0$, we set $b_\alpha=[-\infty,f_\alpha)\in\boldsymbol{B}_{\boldsymbol{S}}$. If $f_\alpha<f_\beta$ with $\alpha,\beta\in\Sigma_0$, then $H(b_\alpha)<H(b_\beta)$. For $\alpha\in\Sigma_0$, we set:

$$H(b_\alpha)=\bigcup\{[x_\alpha^{2i},x_\alpha^{2i+1}):i<n(\alpha)\}\,,$$

where $x_\alpha^j<x_\alpha^k$ in $L(\boldsymbol{T})\cup\{-\infty,+\infty\}$, for $j<k<2n(\alpha)$. Let $m<\omega$ and Σ be a stationary subset of Σ_0 satisfying $n(\alpha)=m$ for $\alpha\in\Sigma$. Let $\alpha,\beta\in\Sigma_1$, and assume $f_\alpha<f_\beta$. Then:

(i) $x_\alpha^0\geq x_\beta^0$,

(ii) $x_\alpha^{2i}=x_\beta^{2i}$ implies $x_\alpha^{2i+1}\leq x_\beta^{2i+1}$, and

(iii) $x_\alpha^{2i-1}=x_\beta^{2i-1}$ implies $x_\alpha^{2i}\geq x_\beta^{2i}$.

CLAIM. There is a stationary subset $\Sigma^0\subseteq\Sigma$ satisfying $x_\alpha^0=x_\beta^0$ for $\alpha,\beta\in\Sigma^0$.

PROOF. By contradiction. Otherwise, from (i) we obtain a stationary subset $\boldsymbol{S}'$ of Σ_1 such that $f_\alpha<f_\beta$ in $L(\boldsymbol{S}')$ implies $x_\alpha^0>x_\beta^0$ in $L(\boldsymbol{T})\cup\{-\infty,+\infty\}$ (for $\alpha,\beta\in\boldsymbol{S}'$). We will prove:

(∗) *$L(\boldsymbol{S}')$ contains a strictly increasing ω_1-sequence.*

This will give the desired contradiction, as $L(\boldsymbol{T})$ would then contain a strictly decreasing ω_1-sequence (Lemma 4.3).

If $(*)$ is not true, then it follows by induction, that each $\mathrm{Lev}_m(\underline{T}(S'))$ is a countable set of ordinals (see 4.2), and hence

$$D = \bigcup \{\mathrm{ran}(f) \colon f \in \mathrm{Lev}_m(\underline{T}(S')) \text{ for some } m < \omega\}$$

is bounded by some $\delta < \kappa$. So, if $\alpha \in S'$ and $\alpha > \delta$, then f_α is not in $\mathrm{Br}(\underline{T}(S')) = L(S')$. This is the desired contradiction, giving the Claim. □

Next, we show there is a stationary subset Σ^1 of Σ^0 satisfying $x^1_\alpha = x^1_\beta$ for $\alpha, \beta \in \Sigma^1$. Again by contradiction. Assume there is a stationary subset S' of Σ^0 satisfying $x^1_\alpha \neq x^1_\beta$ for $\alpha \neq \beta$ in S'. Consequently, if $f_\alpha < f_\beta$, with $\alpha, \beta \in S'$, then $x^1_\alpha < x^1_\beta$. But then the function G from $L(S')$ into $L(T)$ defined by $G(f_\alpha) = x^1_\alpha$ is one-to-one and increasing, contradicting Lemma 4.11, since $S' \setminus T$ is stationary. This contradiction gives Σ^1.

Finally, repeating this procedure $2m$ times we obtain a stationary set $\Sigma^{2m} \subseteq \Sigma^{2m-1} \subseteq \cdots \subseteq \Sigma^0$ satisfying $x^k_\alpha = x^k_\beta$ for $\alpha, \beta \in \Sigma^{2m}$ and $k < 2m$. Consequently, $H(b_\alpha) = H(b_\beta)$ for $\alpha, \beta \in \Sigma^{2m}$, which finally contradicts H is one-to-one. □

Now we state the Theorem of TODORČEVIĆ [1979], [1980]. Namely, the following result is really proved in TODORČEVIĆ [1979], [1980] is merely an explanation.

4.14. THEOREM. *Let $\kappa > \omega$ be a regular cardinal. For each subset I of κ, we can associate an interval Boolean algebra $B_{[I]}$ satisfying*:

(1) *For each $a, b \neq 0$ in $B_{[I]}$, with $a \not\leq b$, there is no one-to-one increasing function from $B_{[I]} \restriction a$ into $B_{[I]} \restriction b$.*

(2) *Let I and J be subsets of κ. The following properties are equivalent*:

(i) $I \subseteq J$,

(ii) *there is a one-to-one increasing function from $B_{[I]}$ into $B_{[J]}$, and*

(iii) *there is a one-to-one homomorphism from $B_{[I]}$ into $B_{[J]}$.*

(3) *If $I \cap J = \emptyset$, then there is no stationary subset S of κ, contained in S_ω, such that B_S is embeddable in both $B_{[I]}$ and $B_{[J]}$.*

PROOF. Let $(S_\alpha)_{\alpha<\kappa}$ be a family of pairwise disjoint stationary subsets of S_ω. By Lemma 4.5 and its Remark, we can assume that $\underline{r}(S_\alpha) = S_\alpha$ for every α, and thus for every $\xi \in S_\alpha$ and $m < \omega$, the set $\underline{S}_{\alpha[\xi/m]}$ is stationary in κ. For non-empty $I \subseteq \kappa$, we set $S_I = \bigcup \{S_\alpha \colon \alpha \in I\}$, and we denote by $B_{[I]}$ the interval algebra $B\langle L(S_I)\rangle$.

$L(S_I)$ has the stationary interval property. Indeed, let $f_\xi < f_\zeta$ in S_I. Let α be such that $f_\xi \in S_\alpha$, and n be the greatest integer such that $f_\xi \restriction n = f_\zeta \restriction n$. So $f_\xi(n) < f_\zeta(n)$. Then $\underline{S}_{\alpha[\xi/n+1]}$ is stationary, $\underline{S}_{\alpha[\xi/n+1]} \subseteq \underline{S}_{I[\xi/n+1]}$ and so $\underline{S}_{I[\xi/n+1]}$ is stationary.

If the reader prefers another approach to this, developed in Comment 4.10, then he can deduce that $L(S_I)$ has the stationary interval property, from the Theorem's comment (that is a standard argument to prove that S_ω can be split in κ stationary sets).

We will prove (1) by contradiction. Fix $\emptyset \neq I \subseteq \kappa$, and assume there are $a \not\leq b$ in

$\boldsymbol{B}_{[I]}$ and a one-to-one increasing function H from $\boldsymbol{B}_{[I]} \upharpoonright a$ into $\boldsymbol{B}_{[I]} \upharpoonright b$. Let $c = a \cdot -b$. Let $\boldsymbol{S}'$ (resp. $\boldsymbol{S}''$) be the set of $\alpha \in \boldsymbol{S}_I$ with $[-\infty, f_\alpha) \cap c \in \boldsymbol{B}_{[I]}|c$ (resp. of $t \in \boldsymbol{P}$ such that $[-\infty, f_\alpha) \cap b \in \boldsymbol{B}_{[I]}|b$). The sets $\boldsymbol{S}'$ and $\boldsymbol{S}''$ are disjoint stationary sets of $\boldsymbol{S}_I$. This follows because $L(\boldsymbol{S}_I)$ has the stationary interval property. Obviously, the algebras $\boldsymbol{B}_{\boldsymbol{S}'}$ and $\boldsymbol{B}_{[I]} \upharpoonright c$ (resp. $\boldsymbol{B}_{\boldsymbol{S}''}$ and $\boldsymbol{B}_{[I]} \upharpoonright b$) are isomorphic algebras. The restriction of H on $\boldsymbol{B}_{[I]} \upharpoonright c$ induces a one-to-one increasing function from $\boldsymbol{B}_{\boldsymbol{S}'}$ into $\boldsymbol{B}_{\boldsymbol{S}''}$, giving a contradiction to Proposition 4.13.

We next prove (2). Clearly, (iii) implies (ii) while (i) implies (ii) and (iii), for $\emptyset \neq \boldsymbol{I} \subseteq \boldsymbol{J} \subseteq \kappa$, then $L(\boldsymbol{S}_I)$ is a subchain of $L(\boldsymbol{S}_J)$ and $L(\boldsymbol{S}_I)$ is a dense chain, and every interval algebra is retractive. Consequently, by observation 1.2.2, $\boldsymbol{B}_{[I]}$ can be viewed as a subalgebra of $\boldsymbol{B}_{[J]}$.

Now, we prove that (ii) implies (i), by contradiction. Let $\alpha \in \boldsymbol{I} \backslash \boldsymbol{J}$ and assume that H is a one-to-one increasing function from $\boldsymbol{B}_{[I]}$ into $\boldsymbol{B}_{[J]}$. Via the identification recalled above, there is no loss in assuming that H is a one-to-one increasing function from the subalgebra $\boldsymbol{B}_{[\{\alpha\}]}$ into $\boldsymbol{B}_{[J]}$. Since $\boldsymbol{S}_{\{\alpha\}} \backslash \boldsymbol{S}_J = \boldsymbol{S}_\alpha$ is stationary, this contradicts Proposition 4.13.

We next prove (3). For a contradiction, let us suppose that $\boldsymbol{I}$ and $\boldsymbol{J}$ are pairwise disjoint subsets of κ, and $\boldsymbol{S}$ is a stationary subset of κ such that there are one-to-one increasing functions from $\boldsymbol{B}\langle L(\boldsymbol{S})\rangle$ into $\boldsymbol{B}\langle L(\boldsymbol{S}_I)\rangle$ and $\boldsymbol{B}\langle L(\boldsymbol{S}_J)\rangle$, respectively. From Proposition 4.13, it follows that $\boldsymbol{S} \backslash \boldsymbol{S}_I$ and $\boldsymbol{S} \backslash \boldsymbol{S}_J$ are non-stationary in κ, which contradicts the fact that $\boldsymbol{S}_I$ and $\boldsymbol{S}_J$ are pairwise disjoint stationary sets of κ. $\square$

COMMENT. Let us give another proof of the fact that we can assume that $L(\boldsymbol{S}_I)$ has the stationary interval property for every non-empty subset $\boldsymbol{I}$ of κ (using a standard argument to show that $\boldsymbol{S}_\omega$ can be split in κ stationary sets).

Recall that $\boldsymbol{S}_\omega$ is the set of $\alpha < \kappa$ of cofinality ω.

FACT. There is $m < \omega$ such that for every $\eta < \kappa$, the set $\boldsymbol{S}(m, \eta)$ of $\alpha \in \boldsymbol{S}_\omega$ satisfying $f_\alpha(m) \geq \eta$ is stationary in κ.

We prove this fact by contradiction. Assume thatfor every $q < \omega$, there are $\eta_q < \kappa$ and a club $\boldsymbol{C}_q$ such that $\boldsymbol{S}(q, \eta_q) \cap \boldsymbol{C}_q = \emptyset$. So $f_\alpha(q) < \eta_q$ for $\alpha \in \boldsymbol{S}_\omega \cap \boldsymbol{C}_q$. Let $\boldsymbol{C}$ be the club, intersection of $\boldsymbol{C}_q$ for $q < \omega$, and $\eta = \sup\{\eta_q : q < \omega\}$. For $\alpha \in \boldsymbol{C} \cap \boldsymbol{S}_\omega$, we have $f_\alpha(q) < \eta$ for every $q < \omega$, which means $\alpha \leq \eta$ for $\alpha \in \boldsymbol{S}_\omega \cap \boldsymbol{C}$, which is impossible.

Now let $m < \omega$ and $\eta < \kappa$ satisfy the conclusion of the Fact. Let g be the function from $\boldsymbol{S}_\omega$ into κ defined by $g(\alpha) = f_\alpha(m) \geq \eta$. From Fodor's theorem and the fact that $\boldsymbol{S}(m, \eta)$ is stationary, it follows that there are $\gamma_\eta \geq \eta$ and a stationary subset $\boldsymbol{U}_\eta$ of $\boldsymbol{S}(m, \eta)$ such that $g(\beta) = \gamma_\eta$ for every $\beta \in \boldsymbol{U}_\eta$. In this way we construct a family $(\boldsymbol{S}_\alpha)_{\alpha<\kappa}$ of pairwise disjoint stationary subsets of κ and a family $(\gamma_\alpha)_{\alpha<\kappa}$ of pairwise distinct ordinals such that:

(i) $f_\beta(m) = \gamma_\alpha$ for every $\beta \in \boldsymbol{S}_\alpha$, and

(ii) $\gamma_\delta < \gamma_\varepsilon$ for every $\delta < \varepsilon < \kappa$.

Now replacing each f_β by $\underline{f}_\beta$ defined by $\underline{f}_\beta(k) = f_\beta(m + k)$, we have:

(iii) if $\delta < \varepsilon < \kappa$, $\xi \in \boldsymbol{S}_\delta$, $\zeta \in \boldsymbol{S}_\varepsilon$, then $\underline{f}_\xi(0) = \gamma_\delta < \underline{f}_\zeta(0) = \gamma_\varepsilon$, which implies:

(iv) if $\delta < \varepsilon < \kappa$, $f_\xi \in L(\boldsymbol{S}_\delta)$ and $f_\zeta \in L(\boldsymbol{S}_\varepsilon)$, then $f_\xi < f_\zeta$.

By shrinking each S_α (by non-stationary many points), we can assume that for each $\alpha < \kappa$, the set $L(S_\alpha)$ has the stationary interval property. For $I \subseteq \kappa$, we set $S_I = \bigcup \{S_\alpha : \alpha \in I\}$. From the facts that each $L(S_\alpha)$ has the stationary interval property and $L(S_I)$ is the lexicographic sum of $L(S_\alpha)$, under the subchain I of κ, it follows that $L(S_I)$ has the stationary interval property.

4.15. COROLLARY. *Let $\kappa > \omega$ be a regular cardinal. There are 2^κ pairwise non-isomorphic order-embedding-rigid interval algebras of cardinality κ.* □

4.16. COMMENT. (1) Let us give some information, proved by Todorčević, concerning Theorem 4.14(3), whenever $\kappa = \omega_1$. Let I and J be non-empty disjoint subsets of ω_1 and P be a chain. Let us suppose that P is embeddable in both $L(S_I)$ and $L(S_J)$. Then P is the union of countably many well-orderings.

PROOF. Assume P is not the union of countably many well-orderings. Thus there exist isomorphic chains $L_0 \subseteq L(S_I)$ and $L_1 \subseteq L(S_J)$ which are not the union of countably many well-orderings. Thus, $L_0 = L(S_0)$ for some $S_0 \subseteq S_I$, so by Lemma 4.13, S_0 is not stationary in ω_1. So it suffices to prove that if S' is a non-stationary subset of ω_1, then $L(S')$ is the union of countably many well-orderings. For each $\alpha < \omega_1$, $S' \cap \alpha$ is countable, and thus is the union of countably many well-orderings. For each $\alpha < \omega_1$, let F_α be a function from $L(S' \cap \alpha)$ into ω, such that $F_\alpha^{-1}(k)$ is a well-ordered subchain of $L(S')$, with $L(S' \cap \alpha) = \bigcup \{F_\alpha^{-1}(k) : k \in \omega\}$. Since S' is non-stationary, let C be a club of ω_1 disjoint from S'. Let $c(\beta)$, for $\beta < \omega_1$, be the canonical enumeration of elements of C. We can suppose $c(0) = 0$. For $\alpha \in S'$ there is a unique $\beta(\alpha) < \omega_1$ such that $c(\beta(\alpha)) < \alpha < c(\beta(\alpha) + 1)$. For $\alpha \in S'$, let $n(\alpha)$ be the smallest $k < \omega$ such that $f_\alpha(k) > c(\beta(\alpha))$. For $\alpha \in S'$, we set $m(\alpha) = F_{c(\beta(\alpha)+1)}(f_\alpha)$, and $H(f_\alpha) = (m(\alpha), n(\alpha))$. So H is a function from $L(S')$ into $\omega \times \omega$. It is sufficient to prove that for each $(m, n) \in \omega \times \omega$, the subset $H^{-1}((m, n))$ is a well-ordered subchain of $L(S')$. To see this, assume $(f_{\alpha(k)})_{k<\omega}$ is a decreasing sequence in $H^{-1}((m, n))$. There is no loss in assuming $f_{\alpha(p)} \restriction n + 1 = f_{\alpha(q)} \restriction n + 1$, for $p, q < \omega$. So there is $\gamma < \omega_1$ satisfying:

$$c(\gamma) < f_{\alpha(p)}(n) = f_{\alpha(q)}(n) < c(\gamma + 1)\,.$$

Note that $c(\gamma) < f_{\alpha(p)}(n) < \alpha(p) < c(\gamma + 1)$ (and thus $\gamma = \beta(\alpha(p))$). Consequently, the sequence $(f_{\alpha(k)})_{k<\omega}$ is contained in $F_{c(\gamma+1)}^{-1}(m)$, which is a well-ordered chain.

(2) Using a similar argument, TODORČEVIĆ [1980] has proved the following generalization of the above remark. Assume $\kappa > \omega$ is a regular cardinal, and that $S', S'' \subseteq S_\omega \cap \kappa$ have the property that $S' \cap \lambda$ and $S'' \cap \lambda$ are non-stationary in κ for any ordinal $\lambda \leq \kappa$ and $\lambda > \omega$. Let P be a chain such that P is embeddable in both $L(S')$ and $L(S'')$. Then P is the union of countably well-orderings.

4.17. PROPOSITION. *Let κ be a singular cardinal. There are 2^κ pairwise non-isomorphic onto-rigid interval algebras.*

PROOF. Let $\rho = \mathrm{cf}(\kappa) < \kappa$, and κ_α, $\alpha < \rho$, be a strictly increasing sequence of uncountable successor cardinals with supremum κ. Recall that κ_α^+ denotes the successor cardinal of κ. For each $\alpha < \rho$, let $S(\kappa_\alpha^+)$ be the set of $\xi < \kappa_\alpha^+$ of cofinal type κ_α. For each $\alpha < \rho$, let $S_{\zeta(\alpha)}(\kappa_\alpha^+)$ for $\zeta(\alpha) < \kappa_\alpha^+$ be a family of pairwise disjoint stationary subsets of κ_α^+, included in $S(\kappa_\alpha^+)$ and such that $L(S_{\zeta(\alpha)}(\kappa_\alpha^+))$ has the stationary interval property. For a family $\zeta = \langle \zeta(\alpha) \colon \alpha < \rho \rangle$, let $L_\zeta(S) = \Sigma\{L(S_{\zeta(\alpha)}(\kappa_\alpha^+)) \colon \alpha < \rho\}$ be the lexicographic sum of $L(S_{\zeta(\alpha)}(\kappa_\alpha^+))$ over ρ. Let B_ζ be the interval algebra generated by $L_\zeta(S)$. The rigidity of B_ζ follows from the following fact. First, two Boolean algebras B' and B'' are said to be totally onto-different whenever, for every $a' \in B'$ and $a'' \in B''$, with $a' \neq 0$ and $a'' \neq 0$, there is no homomorphism from $B' \restriction a'$ onto $B'' \restriction a''$, and there is no homomorphism from $B'' \restriction a''$ onto $B' \restriction a'$. Now, reasoning on the Boolean spaces, the reader can easily check the following fact.

Let $B_\alpha = B\langle C_\alpha \rangle$, for $\alpha < \kappa$, be a family of interval algebras, where κ is a regular cardinal. Let C be the lexicographic sum of C_α, for $\alpha < \kappa$, and $B = B\langle C \rangle$.

(1) If $(B_\alpha)_{\alpha<\kappa}$ are pairwise totally different rigid algebras, then B is a rigid algebra.

(2) If $(B_\alpha)_{\alpha<\kappa}$ are pairwise totally onto-different onto-rigid algebras, then B is an onto-rigid algebra.

The sequence of onto-rigid interval algebras generated by $L(S_{\zeta(\alpha)}(\kappa_\alpha^+))$ for $\alpha < \rho$ are pairwise totally onto-different and thus B_ζ is onto-rigid. Now, let $\zeta' \neq \zeta''$ be given. By duality, it is easy to show that there is no homomorphism from $B_{\zeta'}$ onto $B_{\zeta''}$. The existence of 2^κ such algebras follows from the existence of 2^κ pairwise distinct ζ. $\square$

In addition, TODORČEVIĆ [1979] has studied for singular cardinals the numbers of pairwise non-isomorphic mono-rigid interval algebras, under some set-theoretical assumptions.

5. Jech's construction of simple complete algebras

5.0. The problem of the existence of simple complete algebras was discussed in MCALOON [1970] and the construction of such algebra, under some set-theoretical assumptions, was developed by JECH [1974], who showed in JECH [1972] that if an atomless complete algebra is not simple, then it contains a non-rigid atomless complete subalgebra.

We will give Jech's construction of a simple complete algebra as the completion of a Suslin tree, using Jensen's principle ($\diamondsuit$). This example is of cardinality ω_1. The generalization to cardinals κ, for $\kappa > \omega_1$, regular and not weakly compact, is obtained using similar methods. Since our purpose is to construct rigid algebras, we only mention the following results (JECH [1974]): If κ is a weakly compact cardinal, then there is no simple complete algebra in cardinality κ, and if we assume G.C.H. plus κ is singular, then there is no simple complete algebra of cardinality κ either.

5.1.1. DEFINITION. An atomless complete algebra is said to be *simple* whenever it has no non-trivial proper atomless complete subalgebra.

Recall that if $\boldsymbol{B}$ is a complete algebra, and if $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{B}$, then $\boldsymbol{C}$ is said to be a complete subalgebra of $\boldsymbol{B}$ whenever for every subset $\boldsymbol{X}$ of $\boldsymbol{C}$, the supremum (and so the infimum) of $\boldsymbol{X}$ in $\boldsymbol{B}$, belongs to $\boldsymbol{C}$.

5.1.2. PROPOSITION. *If $\boldsymbol{B}$ is a simple complete algebra, then $\boldsymbol{B}$ is rigid and, more precisely, $\boldsymbol{B}$ has no non-trivial one-to-one complete endomorphism.*

PROOF. By contradiction. Assume that f is a non-trivial one-to-one complete endomorphism of $\boldsymbol{B}$. Let $a \in \boldsymbol{B}$ satisfy $a \neq \mathbf{0}$ and $f(a) \neq a$. We can suppose $a \cdot f(a) = 0$. Let $\boldsymbol{D}$ be the set of all $x \in \boldsymbol{B}$ of the form $x = y + f(y) + z$, where $y \leq a$ and $z \cdot (a + f(a)) = \mathbf{0}$. Notice that for such an x, this decomposition is unique. Obviously, $\boldsymbol{D}$ is an atomless complete subalgebra of $\boldsymbol{B}$. A contradiction follows, since $a \not\in \boldsymbol{D}$. $\square$

5.2. Using Diamond ($\diamond$), a simple algebra can be constructed as follows: first we construct a Suslin tree $\langle \boldsymbol{T}, \leq \rangle$, and we define the notion of a good equivalence relation on $\boldsymbol{T}$. The notion entails that each member p of $\boldsymbol{T}$ is majorized by an element p^+ of $\boldsymbol{T}$, with trivial equivalence class $\{p^+\}$. We take the completion algebra $\boldsymbol{B}$ of the regular open sets of $\langle \boldsymbol{T}, \geq \rangle$. Then we show that each atomless complete subalgebra $\boldsymbol{C}$ of $\boldsymbol{B}$ induces a good equivalence relation. Consequently, every non-zero element p of $\boldsymbol{T} \subseteq \boldsymbol{B}$ is less than the non-zero element p^+ of $\boldsymbol{C}$. This is sufficient to prove that $\boldsymbol{C}$ is dense in $\boldsymbol{B}$ and so $\boldsymbol{C} = \boldsymbol{B}$.

NOTATION. $p, q, r, \ldots$ are members of a tree, $u, v, w, \ldots, \boldsymbol{U}, \boldsymbol{V}, \ldots$ are subsets of a tree, and $x, y, z, \ldots, \boldsymbol{X}, \boldsymbol{Y}, \ldots$ are elements of a Boolean algebra.

5.3. *Suslin trees*

DEFINITION. A (*normal*) *Suslin tree* $\langle \boldsymbol{T}, \leq \rangle$ is a (rooted) tree satisfying the following properties:

(S1) $\boldsymbol{T}$ is of cardinality ω_1 ;

(S2) each element of $\boldsymbol{T}$ has at least two and at most countably many immediate successors ;

(S3) if p and q are both at a limit level α and if $\{r \in \boldsymbol{T}: r < p\} = \{r \in \boldsymbol{T}: r < q\}$, then $p = q$;

(S4) for every $\alpha < \omega_1$ and every $p \in \boldsymbol{T}$ of level α, there is $q > p$ of level β in $\boldsymbol{T}$ for every $\beta > \alpha$, $\beta < \omega_1$; and

(S5) every antichain of $\boldsymbol{T}$ is countable .

We recall that:

(i) $\mathrm{Lev}_\alpha(\boldsymbol{T})$ is the set of $p \in \boldsymbol{T}$ of height α,

(ii) $\boldsymbol{T}_\alpha$ is the set of $p \in \boldsymbol{T}$ of height less than α,

(iii) a branch of $\boldsymbol{T}$ is a linear ordered subset of $\boldsymbol{T}$ containing all predecessors of all its elements (and thus a branch is not necessarily a maximal chain of $\boldsymbol{T}$),

(iv) an antichain of $\boldsymbol{T}$ is a set of pairwise incomparable elements of $\boldsymbol{T}$.

Notice that (S2), (S4) and (S5) imply:

(S6) every subchain of $\boldsymbol{T}$ is countable.

We will prove (S6) by contradiction. Assume there is a strictly increasing sequence x_α for $\alpha < \omega_1$ in $\boldsymbol{T}$. We can suppose $x_\alpha \in \mathrm{Lev}_\alpha(\boldsymbol{T})$. For each α, choose $y_{\alpha+1} > x_\alpha$ of level $\alpha + 1$, incomparable to $x_{\alpha+1}$. The set of y_α for $1 \le \alpha < \omega_1$ is an uncountable antichain of $\boldsymbol{T}$, contradicting (S5).

A construction, with ($\Diamond$) of a Suslin tree, can be found in the Appendix of Set Theory in this Handbook.

5.4. *Embedding of a Suslin tree into its completion algebra*

5.4.1. Let $\langle \boldsymbol{P}, \le \rangle$ be a partially ordered set. We recall that $p, q \in \boldsymbol{P}$ are *incompatible* whenever there is no $r \in \boldsymbol{P}$ satisfying $r \le p$, and $r \le q$. We say that $\boldsymbol{P}$ is *separative* whenever $p \not\le q$ implies that there is $r \in \boldsymbol{P}$ satisfying $r \le p$, with r and q incompatible.

A tree $\langle \boldsymbol{T}, \le \rangle$ is said to be *branched* whenever each member of $\boldsymbol{T}$ has at least two immediate successors, i.e. $|\mathrm{Succ}(p)| \ge 2$ for $p \in \boldsymbol{T}$.

Let $\langle \boldsymbol{T}, \le \rangle$ be a tree. Recall that $\langle \boldsymbol{T}, \le^* \rangle$ denotes the converse ordering of $\langle \boldsymbol{T}, \le \rangle$, i.e. $p \le^* q$ if and only if $p \ge q$. Now $\langle \boldsymbol{T}, \le^* \rangle$ satisfies the following properties:

(a) for $p, q \in \boldsymbol{T}$, p and q are incomparable if and only if p and q are incompatible;

(b) if $\langle \boldsymbol{T}, \le \rangle$ is branched, then $\langle \boldsymbol{T}, \le^* \rangle$ is a separative ordered set.

As an illustration of Lemma 4.19 (Part I of this Handbook), we have:

5.4.2. Let $\langle \boldsymbol{T}, \le \rangle$ be a branched tree. Then there is a complete algebra $\boldsymbol{B}$ and a mapping $\boldsymbol{i}$ from $\langle \boldsymbol{T}, \le^* \rangle$ into $\boldsymbol{B}$ with the following properties:

(i) $\boldsymbol{i}[\boldsymbol{T}]$ is dense in $\boldsymbol{B}$,

(ii) $p \le^* q$ in $\langle \boldsymbol{T}, \le^* \rangle$ if and only if $\mathbf{0} \ne \boldsymbol{i}(p) \le \boldsymbol{i}(q)$,

(iii) p and q are incompatible in $\langle \boldsymbol{T}, \le^* \rangle$ (i.e. p and q are incomparable) if and only if $\boldsymbol{i}(p) \cdot \boldsymbol{i}(q) = \mathbf{0}$.

We give in detail some facts concerning $\boldsymbol{i}$ and $\boldsymbol{B}$. Recall that if $\boldsymbol{X}$ is a topological space, we may define the complete Boolean algebra $\mathrm{RO}(\boldsymbol{X})$ of $\boldsymbol{X}$. The elements of $\mathrm{RO}(\boldsymbol{X})$ are the regular open subsets $\boldsymbol{Y} \subseteq \boldsymbol{X}$ (recall that $\boldsymbol{Y}$ is regular if and only if $\boldsymbol{Y} = \mathrm{Int}(\mathrm{Cl}(\boldsymbol{Y}))$. So $\boldsymbol{Y} \le \boldsymbol{Z}$ if and only if $\boldsymbol{Y} \subseteq \boldsymbol{Z}$. The algebraic operations are $\boldsymbol{Y} \cdot \boldsymbol{Z} = \boldsymbol{Y} \cap \boldsymbol{Z}$, $\boldsymbol{Y} + \boldsymbol{Z} = \mathrm{Int}(\mathrm{Cl}(\boldsymbol{Y} \cup \boldsymbol{Z}))$, $-\boldsymbol{Y} = \mathrm{Int}(\boldsymbol{X} \setminus \boldsymbol{Y})$, $\Sigma\{\boldsymbol{Y}_i : i \in I\} = \mathrm{Int}(\mathrm{Cl}(\bigcup \{\boldsymbol{Y}_i : i \in I\}))$ and $\Pi\{\boldsymbol{Y}_i : i \in I\} = \mathrm{Int}(\bigcap \{\boldsymbol{Y}_i : i \in I\})$.

Now let $\langle \boldsymbol{T}, \le \rangle$ be a branched tree. For $p \in \boldsymbol{T}$, we consider $\boldsymbol{U}_p = \{q \in \boldsymbol{T} : q \le^* p$. The set of $\boldsymbol{U}_p$ for $p \in \boldsymbol{T}$ is a basis of a topology on $\boldsymbol{T}$. We set $\boldsymbol{B} = \mathrm{RO}(\boldsymbol{T})$, and $\boldsymbol{i}(p) = \mathrm{Int}(\mathrm{Cl}(\boldsymbol{U}_p))$ for $p \in \boldsymbol{T}$.

5.4.3. OBSERVATION. *The open sets and closed sets of T are exactly the initial segments and final segments of $\langle T, \le^* \rangle$, respectively.*

5.4.4. OBSERVATION. *Let U be a non-empty open subset of T and $m(U)$ be the set of maximal elements of U in $\langle T, \leq^* \rangle$.* Then:

(2.1) *U is the union of U_p for $p \in m(U)$,*

(2.2) $\mathrm{Cl}(U) = U \cup \bigcup \{F_p \colon p \in m(U)\}$, *where F_q is the closed principal final segment of $\langle T, \leq^* \rangle$ generated by q, for $q \in T$. Notice that F_q is a closed initial segment in $\langle T, \leq \rangle$, and thus F_q is a well-ordered chain of $\langle T, \leq \rangle$.*

5.4.5. OBSERVATION. *$i(p) = U_p$ for $p \in T$, since T is branched and* $\mathrm{Cl}(Z)$ *is the final segment of $\langle T, \leq^* \rangle$ generated by Z, for $Z \subseteq T$.*

5.4.6. OBSERVATION. *Let U be a regular subset of T, and $p \in T$. We have $p \in U$ if and only if* $\mathrm{Succ}(p) \subseteq U$.

We recall that $\mathrm{Succ}(p)$ denotes the set of all immediate successors of p in $\langle T, \leq \rangle$.

5.4.7. OBSERVATION. *If $\langle T, \leq \rangle$ is a branched tree, then* $\mathrm{RO}(\langle T, \leq^* \rangle)$ *is atomless.*

5.4.8. OBSERVATION. *If we assume that there is no uncountable set of pairwise incomparable elements in $\langle T, \leq \rangle$, then* $\mathrm{RO}(\langle T, \leq^* \rangle)$ *satisfies the countable chain condition (c.c.c.).*

5.5. Suslin Boolean algebras

5.5.1. Let us recall that a complete Boolean algebra is *σ-distributive* whenever

$$\Pi_{n<\omega} \Sigma_{\alpha \in I(n)} u_{n,\alpha} = \Sigma_{f \in \Pi I(n)} \Pi_{n<\omega} u_{n,f(n)}$$

where the $u_{n,\alpha}$'s are members of B. Now let us recall that a subset D of B is said to be *open dense* if D is dense (i.e. for $x \neq 0$ in B, there is $y \in D$ such that $0 \neq y \leq x$) and if $0 \neq u \leq v$ and $v \in D$, then $u \in D$.

Propositions 5.5.2 and 5.5.3 are proved as a part of the proof of Theorem 14.20 in Part 1 of this Handbook:

5.5.2. PROPOSITION. *For complete algebra B, the following properties are equivalent*:

(i) *B is σ-distributive,*

(ii) *the intersection of countably many open dense subsets of B is an open dense subset of B.*

5.5.3. PROPOSITION. *If $\langle T, \leq \rangle$ is a normal Suslin tree, then $B =$* $\mathrm{RO}(\langle T, \leq^* \rangle)$ *is a σ-distributive, atomless and complete Boolean algebra satisfying the countable chain condition.*

5.5.4. DEFINITION. A Boolean algebra B is said to be a *Suslin* algebra whenever B is a σ-distributive, atomless and complete algebra, and B satisfies the countable chain condition.

Now, trivially if $\langle T, \leq \rangle$ is a normal Suslin tree, then $\mathrm{RO}(\langle T, \leq^* \rangle)$ is a Suslin algebra. Conversely, we have (see Exercise 22.3, Jech [1978] and 14.20 of Part I):

5.5.5. PROPOSITION. *Let $\boldsymbol{B}$ be a Suslin algebra. Then there is a Suslin tree $\langle T, \leq \rangle$ such that*:

(i) $T \subseteq \boldsymbol{B} \setminus \{\mathbf{0}\}$, *and* $\leq^*$ (*the converse order relation of* $\leq$) *is the induced order relation on T by the order relation on* $\langle \boldsymbol{B}, \leq \rangle$,

(ii) *$\boldsymbol{B}$ is isomorphic to the algebra* $\boldsymbol{B} = \mathrm{RO}(\langle T, \leq^* \rangle)$.

PROOF. Let $\boldsymbol{B}$ be a Suslin algebra. We consider $\boldsymbol{B}$ as a partial ordering $\langle \boldsymbol{B}, \leq \rangle$. We will construct $\langle T, \leq^* \rangle$, with $T \subseteq \boldsymbol{B} \setminus \{\mathbf{0}\}$, by induction on levels of T. To set successors of $p \in T$, pick a countable partition of $p \in \boldsymbol{B}$, i.e. a family $(p_n)_{n<\omega}$ of members of $\boldsymbol{B}$ such that: (i) $\mathbf{0} \neq p_n < p$, (ii) $p_m \cdot p_n = \mathbf{0}$ for $m < n < \omega$, and (iii) $\Sigma\{p_n : n < \omega\} = p$. At limit levels, take all possible non-zero Boolean infimum along branches of the tree constructed so far. By σ-distributivity, each level of T is non-empty and is a partition of $p \in \boldsymbol{B}$. Indeed, assume $p \in \mathrm{Lev}_\alpha(T)$, and $T_\lambda = \bigcup \{\mathrm{Lev}_\beta(T) : \beta < \lambda\}$ is defined, with $\alpha < \lambda$, and λ is limit. Let $(\alpha(n))_{n<\omega}$ be a strictly increasing sequence, cofinal in λ, with $\alpha(0) > \alpha$. Assume that for each β, $\alpha < \beta < \gamma$, the set

$$T_\beta(p) = \{q \in \mathrm{Lev}_\beta(T) : q < p \text{ in } \boldsymbol{B}\}$$

is a partition of $p \in \boldsymbol{B}$. We write

$$T_{\alpha(n)}(p) = \{u_{\alpha(n),k} : k \in I(n)\} .$$

We have

$$p = \Pi_{n<\omega} \Sigma_{k \in I(n)} u_{\alpha(n),k} = \Sigma_{f \in \Pi I(n)} \Sigma_{n<\omega} u_{\alpha(n),f(n)} .$$

Now, we remark that if $f \in \Pi I_n$ satisfies $\Pi_{n<\omega} u_{n,f(n)} \neq \mathbf{0}$, then f defines a maximal branch of T_λ containing t, and the set of these f is countable. $\square$

5.6. The good equivalence relation associated with a Suslin algebra

We will develop the connection between subalgebras of $\boldsymbol{B}$ and good relations on the tree $\langle T, \leq \rangle$.

Let $\boldsymbol{B}$ be an algebra, and $x \in \boldsymbol{B}$, $x \neq 0$. We say that $y \in \boldsymbol{B}$ *slices* x whenever $x \cdot y \neq \mathbf{0}$ and $x \cdot -y \neq \mathbf{0}$. For subsets $\boldsymbol{X}$ and $\boldsymbol{Y}$ of $\boldsymbol{B}$, we say that $\boldsymbol{Y}$ *slices* $\boldsymbol{X}$ whenever for each $x \in \boldsymbol{X}$, $x \neq \mathbf{0}$ there is $y \in \boldsymbol{Y}$ such that y slices x.

5.6.1. LEMMA. *Let $\boldsymbol{B}$ be a complete algebra, and $\boldsymbol{C}$ be a complete subalgebra of $\boldsymbol{B}$. Then $\boldsymbol{C}$ is atomless if and only if $\boldsymbol{C}$ slices $\boldsymbol{B}$.*

PROOF. Obviously, if $\boldsymbol{C}$ slices $\boldsymbol{B}$, then $\boldsymbol{C}$ is atomless. Now to prove the converse. Assume $x \neq \mathbf{0}$ in $\boldsymbol{B}$ is not sliced by $\boldsymbol{C}$. Let $z = \Pi\{y \in \boldsymbol{C} : y \geq x\}$. First $z \in \boldsymbol{C}$, and $\mathbf{0} \neq z \geq x$. We will prove that z is an atom of $\boldsymbol{C}$. Let $y \in \boldsymbol{C}$ be such that $\mathbf{0} \leq y \leq z$.

So if $y \geq x$, then $y \geq z$ and $y = z$. Now, assume $y \cdot x = \mathbf{0}$. So $x \leq -y$, and thus $y \leq z \leq -y$ follows from $-y \in \boldsymbol{C}$. Consequently, $y = \mathbf{0}$. □

Now, we assume that $\boldsymbol{B} = \mathrm{RO}(\boldsymbol{T})$, *where* $\boldsymbol{T}$ *is a Suslin tree*. For $u \subseteq \boldsymbol{T}$, we set

$$\Sigma u = \Sigma\{\boldsymbol{i}(p) \colon p \in u\} .$$

From the fact that $\boldsymbol{i}[\boldsymbol{T}]$ is dense in $\boldsymbol{B}$, it follows that:

5.6.2. LEMMA. (a) *Let* $\alpha < \beta < \omega_1$ *and* $p \in \mathrm{Lev}_\alpha(\boldsymbol{T})$. *Then* $\boldsymbol{i}(p) = \Sigma\{\boldsymbol{i}(q) \colon q \in \mathrm{Lev}_\beta(\boldsymbol{T}),\ q \leq^* p\}$.

(b) *If* $u \subseteq \mathrm{Lev}_\alpha(\boldsymbol{T})$ *and* v *is the set of* $q \in \mathrm{Lev}_\beta(\boldsymbol{T})$ *comparable to some member of* u, *then* $\Sigma u = \Sigma v$.

Next, let $\boldsymbol{B}'_\alpha$ be the set of Σu for $u \subseteq \mathrm{Lev}_\alpha(\boldsymbol{T})$. For $u, v \subseteq \mathrm{Lev}_\alpha(\boldsymbol{T})$, we have $u \subseteq v$ if and only if $\Sigma u \leq \Sigma v$ in $\boldsymbol{B}$ and $\Sigma\ \mathrm{Lev}_\alpha(\boldsymbol{T}) = \mathbf{1}$ in $\boldsymbol{B}$ since $i[\boldsymbol{T}]$ is dense in $\boldsymbol{B}$. Consequently, $\boldsymbol{B}'_\alpha$ is isomorphic to the power set algebra $\mathfrak{P}(\mathrm{Lev}_\alpha(\boldsymbol{T}))$, and so $\boldsymbol{B}'_\alpha$ is an atomic complete algebra.

For $\alpha < \beta$, we have $\boldsymbol{B}'_\alpha \subseteq \boldsymbol{B}'_\beta$; this follows from Lemma 5.6.2. For $0 \neq \alpha < \omega_1$, we set $\boldsymbol{B}_\alpha = \bigcup\{\boldsymbol{B}'_\beta \colon \beta < \alpha\}$. Obviously, $\boldsymbol{B}'_\alpha = \boldsymbol{B}_{\alpha+1}$ and $\boldsymbol{B}_\alpha$ is a subalgebra of $\boldsymbol{B}$ (but not necessarily complete for α limit).

5.6.3. LEMMA. *Assume* ($\diamond$). $\boldsymbol{B}$ *is of cardinality* ω_1 *and* $\boldsymbol{B} = \bigcup\{\boldsymbol{B}_\beta \colon \beta < \omega_1\}$.

PROOF. Let $x \in \boldsymbol{B}$. Using Zorn's lemma, we can find a maximal subset $\boldsymbol{X}$ of $\boldsymbol{i}[\boldsymbol{T}]$ formed of pairwise disjoint elements $\boldsymbol{i}(p)$ satisfying $\boldsymbol{i}(p) \leq x$. By maximality of $\boldsymbol{X}$, we have $x = \Sigma\boldsymbol{X}$. Now, $\boldsymbol{X}$ is countable, since $\boldsymbol{T}$ is a Suslin tree. Now, recall that ($\diamond$) implies the existence of both a Suslin tree $\boldsymbol{T}$ and C.H. So $\boldsymbol{B}$ is of cardinality ω_1. Moreover, there is $\alpha < \omega_1$ such that $\boldsymbol{X} \subseteq i[\boldsymbol{T}_\alpha]$, since $\boldsymbol{X}$ is countable. Consequently, $x \in \boldsymbol{B}'_\alpha = \boldsymbol{B}_{\alpha+1}$, and thus $\boldsymbol{B}$ is the union of the $\boldsymbol{B}_\alpha$'s. □

In what follows $\boldsymbol{C}$ *is a complete atomless subalgebra of* $\boldsymbol{B}$.

For $\alpha < \omega_1$, we set $\boldsymbol{C}'_\alpha = \boldsymbol{C} \cap \boldsymbol{B}'_\alpha$ and $\boldsymbol{C}_\alpha = \boldsymbol{C} \cap \boldsymbol{B}_\alpha$. Notice that $\boldsymbol{C} = \bigcup\{\boldsymbol{C}_\alpha \colon \alpha < \omega_1\}$, and $\boldsymbol{C}'_\alpha$ is a complete subalgebra of $\boldsymbol{B}$.

5.6.4. NOTATION. We denote by $\equiv_C$ the binary relation on $\boldsymbol{T}$ defined by $p \equiv_C q$ if and only if there is $\alpha < \omega_1$ such that $p, q \in \mathrm{Lev}_\alpha(\boldsymbol{T})$ and for each $u \subseteq \mathrm{Lev}_\alpha(\boldsymbol{T})$ satisfying $\Sigma u \in \boldsymbol{C}$, we have $p \in u$ if and only if $q \in u$.

5.6.5. OBSERVATION. $\equiv_C$ *is an equivalence relation on* $\boldsymbol{T}$.

5.6.6. NOTATION. We denote by $\mathrm{cl}_C(p) = \{q \in \boldsymbol{T} \colon q \equiv_C p\}$ the equivalence class of $p \in \boldsymbol{T}$, modulo $\equiv_C$.

5.6.7. OBSERVATION. *For* $p \in \mathrm{Lev}_\alpha(T)$ we have

$$\mathrm{cl}_C(p) = \bigcap\{u \colon u \subseteq \mathrm{Lev}_\alpha(\boldsymbol{T}),\ p \in u \textit{ and } \Sigma u \in \boldsymbol{C}\} .$$

Since $\boldsymbol{C}'_\alpha$ is a complete subalgebra of $\boldsymbol{B}'_\alpha$ and $\boldsymbol{B}'_\alpha$ is canonically isomorphic to the power set algebra $\mathfrak{P}(\mathrm{Lev}_\alpha(\boldsymbol{T}))$, it follows that:

5.6.8. OBSERVATION. *We have* $\Sigma\, \mathrm{cl}_C(p) \in \boldsymbol{C}$ *for every* $p \in \mathrm{Lev}_\alpha(\boldsymbol{T})$.

Note that $a_p = \Sigma\, \mathrm{cl}_C(p)$ is an atom of $\boldsymbol{C}'_\alpha = \boldsymbol{C} \cap \boldsymbol{B}'_\alpha$ for every $p \in \mathrm{Lev}_\alpha(\boldsymbol{T})$.
Obviously, a_p belongs to $\boldsymbol{C}'_\alpha$ and $a_p \neq \mathbf{0}$. If a_p is not an atom of $\boldsymbol{C}'_\alpha$, then there are $u, v \subseteq \mathrm{Lev}_\alpha(\boldsymbol{T})$ such that $x = \Sigma u$ and $y = \Sigma v$ belong to $\boldsymbol{C}$, $x \cdot y = \mathbf{0}$ and $a_p = x \dotplus y$. By isomorphism (u, v) is a non-trivial partition of $\mathrm{cl}_C(p)$. Now let $q \in u$ and $r \in v$. We have $q \equiv_C r$, which contradicts $r \not\in u$ and $x = \Sigma u \in \boldsymbol{C}$.

5.6.9. DEFINITION. Let $\equiv$ be an equivalence relation on a tree $\langle \boldsymbol{T}, \leq \rangle$. We say that $\equiv$ *slices* $\boldsymbol{T}$ whenever, for $p \in \boldsymbol{T}$, there are q and r in $\boldsymbol{T}$, at the same level, such that $q, r > p$ and $q \not\equiv r$.

5.6.10. OBSERVATION. *The relation* $\equiv_C$ *slices* $\boldsymbol{T}$. *More precisely, let* A *be the set of all limit ordinals* $\alpha < \omega_1$ *such that the restriction of* $\equiv_C$ *to* $\boldsymbol{T}_\alpha$ *slices* $\langle \boldsymbol{T}_\alpha, \leq \rangle$. *Then* A *is a club.*

Before starting the proof of Observation 5.6.10, we introduce a technical definition. Let $\alpha \leq \beta < \omega_1$. We say that $\boldsymbol{T}_\alpha$ *is* $\equiv_C$*-sliced in* $\boldsymbol{T}_\beta$ if and only if for every $p \in \boldsymbol{T}_\alpha$ there are $\gamma < \beta$ and $q, r \in \mathrm{Lev}_\gamma(\boldsymbol{T}_\beta) = \mathrm{Lev}_\gamma(\boldsymbol{T})$ such that $q, r \geq p$ in $\langle \boldsymbol{T}, \leq \rangle$ and $q \not\equiv_C r$. For instance, $\equiv_C$ slices $\boldsymbol{T}$ if and only if $\boldsymbol{T}$ is $\equiv_C$-sliced in $\boldsymbol{T}$.

Proof of Observation 5.6.10. First notice that $\boldsymbol{C}$ slices $\boldsymbol{B}$ since $\boldsymbol{C}$ is atomless (Lemma 5.6.1). Now $\boldsymbol{C} = \bigcup \{\boldsymbol{C}_\alpha : \alpha < \omega_1\}$, and we also have $\boldsymbol{C}'_\gamma = \boldsymbol{C}_{\gamma+1}$ and $\boldsymbol{C}'_\gamma \subseteq \boldsymbol{C}'_\delta$ for $\gamma < \delta < \omega_1$. We next use this observation to prove that $\equiv_C$ slices $\boldsymbol{T}$.
To this end, let $\alpha < \omega_1$, and $p \in \mathrm{Lev}_\alpha(\boldsymbol{T})$. Let $y \in \boldsymbol{C}$ slice $\boldsymbol{i}(p)$, i.e. $\boldsymbol{i}(p) \cdot y \neq \mathbf{0} \neq \boldsymbol{i}(p) \cdot -y$. There is $\beta > \alpha$ satisfying $y \in \boldsymbol{C}'_\beta$. We have $y = \Sigma u$ for a unique $u \subseteq \mathrm{Lev}_\beta(\boldsymbol{T})$ since $y \in \boldsymbol{C}'_\beta = \boldsymbol{C} \cap \boldsymbol{B}'_\beta$. The fact that y slices $\boldsymbol{i}(p)$ means $u \cap \boldsymbol{i}(p)$ and $(\mathrm{Lev}_\beta(\boldsymbol{T}) \setminus u) \cap \boldsymbol{i}(p)$ are non-empty sets. Let q and r be members of the above sets, respectively. We have $q, r \geq p$ in $\langle \boldsymbol{T}, \leq \rangle$ and $q \not\equiv_C r$ by definition. Notice that p is $\equiv_C$-sliced in $\boldsymbol{T}_{\beta+1} \subseteq \boldsymbol{T}_{\beta+\omega}$, since $q, r \in \mathrm{Lev}_\beta(\boldsymbol{T})$.
Now we will prove that A is a club. Obviously, A is closed. We will prove that A is unbounded. Let $\delta = \alpha(0) < \omega_1$. Since $\boldsymbol{T}_{\alpha(0)}$ is countable, it is obvious from the proof above that there is $\alpha(1)$ such that $\alpha(0) < \alpha(1) < \omega_1$, and $\boldsymbol{T}_{\alpha(0)}$ is $\equiv_C$-sliced in $\boldsymbol{T}_{\alpha(1)}$. Repeating this procedure ω-times, we construct $\alpha(0) < \alpha(1) < \alpha(2) < \cdots$ such that $\boldsymbol{T}_{\alpha(n)}$ is $\equiv_C$-sliced in $\boldsymbol{T}_{\alpha(n+1)}$. Let $\alpha = \sup(\alpha(n))$. Obviously $\boldsymbol{T}_\alpha$ is $\equiv_C$-sliced in $\boldsymbol{T}_\alpha$, i.e. $\alpha \in A$. □

5.6.11. DEFINITION. Let $\equiv$ be an equivalence relation on a tree $\langle \boldsymbol{T}, \leq \rangle$. We say that $\equiv$ is *lower-compatible* whenever for every $\alpha < \beta < \omega_1$, $p', q' \in \mathrm{Lev}_\beta(\boldsymbol{T})$, $p'', q'' \in \mathrm{Lev}_\alpha(\boldsymbol{T})$, such that $p'' < p'$, $q'' < q'$, if $p' \equiv q'$, then $p'' \equiv q''$.

Finally,

5.6.12. DEFINITION. Let $\equiv$ be an equivalence relation on a tree $\langle \boldsymbol{T}, \leq \rangle$. We say that $\equiv$ is a *good equivalence relation* whenever $\equiv$ is lower-compatible, slices $\boldsymbol{T}$, and respects levels, i.e. if $p \equiv q$ and $p \in \mathrm{Lev}_\alpha(\boldsymbol{T})$, then $q \in \mathrm{Lev}_\alpha(\boldsymbol{T})$.

5.6.13. LEMMA. *Let $\boldsymbol{C}$ be a complete atomless subalgebra of $\boldsymbol{B}$. Then $\equiv_C$ is a good equivalence relation on $\boldsymbol{T}$.*

PROOF. It is a consequence of Observation 5.6.10 and the definition. Indeed, the remaining point is to verify that $\equiv_C$ is lower-compatible. Let $\alpha < \beta < \omega_1$, p', $q' \in \mathrm{Lev}_\beta(\boldsymbol{T})$, p'', $q'' \in \mathrm{Lev}_\alpha(\boldsymbol{T})$. We assume $p'' < p'$ and $q'' < q'$. Now suppose $p'' \not\equiv_C q''$. Then we can choose $u \subseteq \mathrm{Lev}_\alpha(\boldsymbol{T})$ such that $\Sigma u \in \boldsymbol{C}$, $p'' \in u$ and $q'' \not\in u$. By Lemma 5.6.2, there is $v \subseteq \mathrm{Lev}_\beta(\boldsymbol{T})$ such that $p' \in v$ and $q' \not\in v$, and $\Sigma u = \Sigma v$. So $p' \not\equiv_C q'$. □

5.7. *Construction of the Suslin tree* $\langle \boldsymbol{T}, \leq \rangle$

Let us recall that an equivalence $\equiv$ on a (branched) tree $\langle \boldsymbol{T}, \leq \rangle$ is said to be good whenever $\equiv$ is lower-compatible, slices $\boldsymbol{T}$, and $\equiv$ respects levels, i.e. if $p \equiv q$ and $p \in \mathrm{Lev}_\alpha(\boldsymbol{T})$, then $q \in \mathrm{Lev}_\alpha(\boldsymbol{T})$.

Let $\langle \boldsymbol{T}, \leq \rangle$ be a tree and $\equiv$ be a good equivalence relation on $\boldsymbol{T}$. Let α be a limit ordinal and $\boldsymbol{T}_\alpha = \bigcup \{\mathrm{Lev}_\beta(\boldsymbol{T}): \beta < \alpha\}$. We denote by $\equiv_\alpha$ the induced relation of $\equiv$ on $\boldsymbol{T}_\alpha$. We note that if $\equiv_\alpha$ slices the subtree $\langle \boldsymbol{T}_\alpha, \leq \rangle$, then $\equiv_\alpha$ is a good equivalence relation on $\boldsymbol{T}_\alpha$.

5.7.1. Let us recall Jensen's principle:
($\diamond$): there is a family (S_α) for $\alpha < \omega_1$ of subsets of ω_1 satisfying:
  (i) $S_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$,
  (ii) if A is a subset of ω_1, then the set of α satisfying $A \cap \alpha = S_\alpha$ is a stationary subset of ω_1.

S_α for $\alpha < \omega_1$, is called a ($\diamond$)-*sequence of* ω_1. So, a ($\diamond$)-sequence can be thought of "as capturing" all subsets of ω_1 as well, or as an "oracle" (see the Appendix on Set Theory in this Handbook or KUNEN [1981]).

5.7.2. Now, we will construct $\boldsymbol{T}$. Let (W_0, W_1) be a partition of ω_1 in two subsets of cardinality ω_1. We construct $\langle \boldsymbol{T}, \leq \rangle$, with $\boldsymbol{T} \subseteq W_0$ and save W_1 for coding good partitions of $\boldsymbol{T}$ by subsets of W_1. Moreover, let f be a one-to-one function from $\omega_1 \times (\omega_1 \times \omega_1)$ onto W_1. Let S_α for $\alpha < \omega_1$, be a ($\diamond$)-sequence of ω_1. We will construct $\boldsymbol{T}$ by induction, assuming (S_4) inductively.

Let $p \in W_0$. We set $\mathrm{Lev}_0(\boldsymbol{T}) = \{p\}$. Assume that $\mathrm{Lev}_\beta(\boldsymbol{T})$ for $\beta < \alpha$ are defined, with $\alpha < \omega_1$. We set $\boldsymbol{T}_\alpha = \bigcup \{\mathrm{Lev}_\beta(\boldsymbol{T}): \beta < \alpha\}$. We assume $\boldsymbol{T}_\alpha$ is countable (since we must obtain a Suslin tree).

First, if $\alpha = \beta + 1$, then $\mathrm{Lev}_\alpha(\boldsymbol{T})$ is formed by appointing exactly two immediate successors for each element of $\mathrm{Lev}_\beta(\boldsymbol{T})$.

Now we assume that α is limit. In this case, we use a sophistication of the classical construction of a Suslin tree (see JECH [1978, Lemma 22.8 and Lemma 22.5], and the Appendix on Set Theory in this Handbook). Note that a maximal branch in a tree can be disjoint from a maximal antichain. For each $p \in \boldsymbol{T}_\alpha$, we choose a maximal branch b_p of $\boldsymbol{T}_\alpha$ containing p and of order type α, using $\mathrm{cf}\,\alpha = \omega$ and (S_4).

We distinguish three cases:

Case 1. S_α is a maximal antichain of T_α. We want each member of $\mathrm{Lev}_\alpha(T)$ to be greater than some member of S_α. In this case S_α remains a maximal antichain of $T_{\alpha+1} = T_\alpha \cup \mathrm{Lev}_\alpha(T)$. To obtain this, let Z be the final segment of T_α generated by S_α: i.e. $r \in Z$ if and only if $r \geq q$ for some $q \in S_\alpha$. For each $p \in Z$, we add a greatest element p^+ to the branch b_p and we set $\mathrm{Lev}_\alpha(T)$ the set of p^+ for $p \in Z$.

Case 2. Assume $S_\alpha \subseteq W_1$, $f^{-1}[S_\alpha] = \{q\} \times E$, where $q \in T_\alpha$ and $E \subseteq \omega_1 \times \omega_1$ is a good equivalence relation on T_α. In particular $f^{-1}[S_\alpha] \subseteq T_\alpha \times (T_\alpha \times T_\alpha) \subseteq \omega_1 \times (\omega_1 \times \omega_1)$. In this case let $V_q \subseteq T_\alpha$ be the set of the equivalent elements, modulo E, to some element of the branch b_q. Notice that for $r \in b_q$ the equivalence class $\mathrm{cl}_E(r)$ of r, modulo E, is contained in $\mathrm{Lev}_\beta(T)$, where β is the level of r in T_α, and V_q can be interpreted as a "neighborhood" of b_q in T_α. For each $p \in T_\alpha \setminus V_q$ or $p = q$, we add a greatest element p^+ to the branch b_p, and set $\mathrm{Lev}_\alpha(T)$ equal to the set of p^+ for $p \in (T_\alpha \setminus V_q) \cup \{q\}$.

The introduction of V_q and q^+ is the key to construct the desired dense subalgebra (see Comment 5.7.3 below).

Case 3. Otherwise. We construct $\mathrm{Lev}_\alpha(T)$ by adding an immediate successor p^+ to each branch b_p for $p \in T_\alpha$.

This completes the construction of T, considering $T = \bigcup \{T_\alpha : \alpha < \omega_1\}$. We begin to verify that T is a normal Suslin tree. The properties (S1) through (S3) follow from the construction. For (S4), we prove by induction on α that if $p \in \mathrm{Lev}_\gamma(T)$ and $\gamma < \alpha$, then there is $q \in \mathrm{Lev}_\alpha(T)$ with $p < q$. This is trivial unless α is a limit stage, and Case 2 applies. There, the only problem is to show that if $r \in V_q$, there is s with r dominated by $s^+ \in \mathrm{Lev}_\alpha(T)$. Since E is a good equivalence relation on T_α, there are $\beta < \alpha$ and $s, t \in \mathrm{Lev}_\beta(T)$, with $s, t > r$ and $s \not{E} t$. We may assume $s \not\in V_q$, and so $r < s < s^+$ with $s^+ \in \mathrm{Lev}_\alpha(T)$. Finally, we will verify (S5). Let S be a maximal antichain of T. We use the classical techniques (($\diamond$) and Case 1) to prove that S is countable. We set

$$A = \{\alpha < \omega_1 : S \cap \alpha \text{ is a maximal antichain of } T_\alpha\} .$$

First, we will prove that A is a club. A is closed. Indeed, let $(\alpha(n))_{n<\omega}$ be an increasing sequence of members of A, converging to $\alpha < \omega_1$. Obviously, $S \cap \alpha$ is an antichain of T_α. To prove that $\alpha \in A$, it is sufficient to show that each $p \in T_\alpha$ is comparable to some element of $S \cap \alpha$. We have $p \in T_{\alpha(n)}$ for some $\alpha(n) < \alpha$. From $\alpha(n) \in A$ and thus $S \cap \alpha(n)$ is a maximal antichain of $T_{\alpha(n)}$, it follows that p is comparable to some element of $S \cap \alpha(n) \subseteq S \cap \alpha$.

A is unbounded. For $p \in T$, let $g(p)$ be an element of S, comparable to p. Let $\beta < \omega_1$, we set $\alpha(0) = \beta$, and for each $k < \omega$, we choose:

(i) $\alpha(3k+1) > \alpha(3k)$ such that $S \cap \alpha(3k) \subseteq T_{\alpha(3k+1)}$;

(ii) $\alpha(3k+2) > \alpha(3k+1)$ such that $S \cap T_{\alpha(3k+1)} \subseteq \alpha(3k+2)$;

(iii) $\alpha(3k+3) > \alpha(3k+2)$ such that $g[T_{\alpha(3k+2)}] \subseteq \alpha(3k+3)$.

All these conditions are feasible by simple cardinality arguments. Let $\alpha = \sup\{\alpha(n) : n < \omega\}$. By (i) and (ii), we have $S \cap \alpha = S \cap T_\alpha \subseteq T_\alpha$, and $S \cap \alpha$ is an antichain of T_α just because S is. Now, it is sufficient to show that every $p \in T_\alpha$ is comparable to a member of $\boldsymbol{S} \cap \alpha$. Let $p \in \boldsymbol{T}_\alpha$. For some k, we have $p \in \boldsymbol{T}_{\alpha(3k+2)}$, and thus p is comparable to $g(p)$ which is an element of

$$\boldsymbol{S} \cap g[\boldsymbol{T}_{\alpha(3k+2)}] \subseteq \boldsymbol{S} \cap \alpha(3k+3) \subseteq \boldsymbol{S} \cap \alpha .$$

Thus, A is a club, as desired.

Now we will finish the proof of (S5) using ($\diamond$). Let $\gamma \in A$, γ limit, satisfying $\boldsymbol{S} \cap \gamma = S_\gamma$ and $\boldsymbol{S} \cap \gamma$ is a maximal antichain of $\boldsymbol{T}_\gamma$. Now if $p \in \mathrm{Lev}_\delta(\boldsymbol{T})$ with $\delta > \gamma$, then $p > q$ for some $q \in \mathrm{Lev}_\gamma(\boldsymbol{T})$. But $q > r$ for some $r \in S_\gamma$ by the construction in Case 1. Consequently, each member of $\mathrm{Lev}_\delta(\boldsymbol{T})$ is greater than some element of $\boldsymbol{S} \cap \gamma$ and thus $\boldsymbol{S} \subseteq \boldsymbol{S} \cap \gamma$ is countable.

5.7.3. COMMENTS. (1) Let us return to Case 2 in order to investigate its effect on good equivalences relations. Let H be a good equivalence relation on $\boldsymbol{T}$. Let E be its restriction to $\boldsymbol{T}_\alpha$. Assume that α is limit and $f^{-1}[S_\alpha] = \{q\} \times E$; so Case 2 applies. Using the notations of Case 2, we claim that the equivalence class $\mathrm{cl}_E(q^+)$ of q^+, modulo E, in $\boldsymbol{T}$ is $\{q^+\}$. Now $q^+ \in \mathrm{Lev}_\alpha(\boldsymbol{T})$. Let $p^+ \in \mathrm{Lev}_\alpha(\boldsymbol{T})$ with $p^+ \neq q^+$. Since $p \in \boldsymbol{T} \setminus V_q$, it follows that there are $\beta < \alpha$, $p_1 < p^+$, $q_1 < q^+$, with $p_1, q_1 \in \mathrm{Lev}_\beta(\boldsymbol{T})$, $p_1 \not{E} q_1$, i.e. $p_1 \not{H} q_1$, and since H is lower compatible, we have $p^+ \not{H} q^+$. As H respects levels, the equivalence class of q^+ in $\boldsymbol{T}$ is $\{q^+\}$.

(2) Now the reader can adapt the above reasoning to show that for each $p \in \boldsymbol{T}$, there is $q \in \boldsymbol{T}$ with $p \leq q$ and $\mathrm{cl}_H(q) = \{q\}$. It suffices to consider $\boldsymbol{S} = \{f(p, r, s)\colon rHs\}$, and the club A of $\alpha < \omega_1$ such that:

(i) α is limit,

(ii) $p \in \boldsymbol{T}_\alpha$,

(iii) the restriction E_α of H on $\boldsymbol{T}_\alpha$ is a good equivalence relation, and

(iv) $f(p, r, s) < \alpha$ for $rE_\alpha s$.

Obviously, A is closed. To show that A is unbounded, let $\beta = \alpha(0) < \omega_1$, and for every $k < \omega$, we choose:

(i) $\alpha(3k+1) > \alpha(3k)$ such that $p \in \boldsymbol{T}_{\alpha(3k+1)}$ and $f[\{p\} \times E_{\alpha(3k)}] \subseteq \boldsymbol{S} \cap \alpha(3k+1)$;

(ii) $\alpha(3k+2) > \alpha(3k+1)$ such that $f^{-1}[\boldsymbol{S} \cap \alpha(3k+1)] \subseteq \{p\} \times E_{\alpha(3k+2)} \cap \alpha(3k+2)$;

(iii) $\alpha(3k+3) > \alpha(3k+2)$ such that $E_{\alpha(3k+3)}$ slices $\boldsymbol{T}_{\alpha(3k+3)}$.

Now clearly the supremum of the $\alpha(k)$'s belongs to A.

5.8. Completion of the proof

In this subsection we complete the program outlined in 5.2. We fix the Suslin tree constructed in 5.7. Let $\boldsymbol{C}$ be a complete atomless subalgebra of $\boldsymbol{B} = \mathrm{RO}(\langle \boldsymbol{T}, \leq^* \rangle)$, and $\equiv_C$ be the good relation on $\boldsymbol{T}$ associated with $\boldsymbol{C}$. To show that $\boldsymbol{C} = \boldsymbol{B}$, it suffices to show that $\boldsymbol{C}$ is dense in $\boldsymbol{B}$. Thus, it suffices to show that for $p \in \boldsymbol{T}$, there is $q \in \boldsymbol{T}$, such that $\boldsymbol{0} \neq i(q) \leq i(p)$ and $i(q) \in \boldsymbol{C}$. Fix $p \in \boldsymbol{T}$. By Comments 5.7.3, there is $q \geq p$ such that its equivalence class, modulo $\equiv_C$ is the singleton $\{q\}$. By Observation 5.6.8, $i(q) = \Sigma\, \mathrm{cl}_C(q) \in \boldsymbol{C}$, and we are done.

5.9. COMMENT. We assume ($\diamondsuit$). There are at least ω_1 pairwise non-isomorphic simple complete algebras of cardinality ω_1.

PROOF. Let $\boldsymbol{B}$ be a simple complete algebra obtained by ($\diamondsuit$). So $|\boldsymbol{B}| = \omega_1$. For $a \neq \mathbf{0}, \mathbf{1}$ and $a \in \boldsymbol{B}$, the algebra $\boldsymbol{B} \upharpoonright a$ is simple and complete. Now we will prove $\boldsymbol{B} \upharpoonright a$ is not isomorphic to $\boldsymbol{B} \upharpoonright b$ for $a \neq b$. This follows easily from the fact that $\boldsymbol{B}$ is rigid. Assume, without loss of generality, that $a \not\leq b$, and for a contradiction, let f be an isomorphism from $\boldsymbol{B} \upharpoonright a$ onto $\boldsymbol{B} \upharpoonright b$. Setting $c = a \cdot -b$, we remark that the function g from $\boldsymbol{B}$ into itself, defined by

$$g(x) = f(x \cdot c) + f^{-1}(x \cdot f(c)) + x \cdot -c \cdot -f(c)$$

is a non-trivial automorphism of $\boldsymbol{B}$. $\square$

6. Odds and ends on rigid algebras

We begin by mentioning the other places in the Handbook where Rigid algebras are constructed:

(1) the section on tree algebras in Part I, where Brenner's construction is given;

(2) Monk's chapter (Chapter 13) on Endomorphisms of Boolean Algebras, where endo-rigid algebra is constructed;

(3) Monk's chapter (Chapter 12) on The Number of Boolean Algebras, where Shelah's complicated construction is given;

(4) Štěpánek's chapter on Embedding and Automorphisms (Chapter 16), where the Shelah construction of complete rigid algebras is developed; and

(5) Koppelberg's chapter (Chapter 20) on Projective Boolean Algebras, where a rigid projective Boolean algebra of power $\aleph_\omega$ is constructed, assuming $\aleph_\omega < 2^\omega$.

For further information, in particular on the different constructions of rigid Boolean algebras, see the bibliography and the Index of the Handbook.

A rigid Boolean algebra could be obtained by a topological way in 1937, mixing a construction of KURATOWSKI [1926], and a result of ČECH [1937], as that was observed by VAN DOUWEN, MONK and RUBIN [1980]. More precisely, KURATOWSKI [1926] showed that there is a subset $\boldsymbol{P}$ of the real line $\boldsymbol{R}$, which has no non-trivial homeomorphism onto itself. From such a $\boldsymbol{P}$, there follows the existence of rigid Boolean space by considering the Stone–Čech compactification $\beta(\boldsymbol{P})$. To show the rigidity of $\beta(\boldsymbol{P})$, use ČECH's [1937] theorem, which insures that for a completely regular space $\boldsymbol{X}$, no point of $\beta(\boldsymbol{X})\backslash\boldsymbol{X}$ is a G_δ in $\beta(\boldsymbol{X})$ (see chapter 3 of WALKER [1974], or chapter 9 of GILLMAN–JERISON [1960]). Consequently, every homeomorphism of $\beta(\boldsymbol{P})$ is necessarily induced by a homeomorphism of $\boldsymbol{P}$, since if $x \in \boldsymbol{P} \subseteq \beta(\boldsymbol{P})$, then $\{x\}$ is a G_δ-set in $\beta(\boldsymbol{P})$).

Now we recall some well-known results on the real line $\boldsymbol{R}$. First, let us state Bernstein's lemma (see, for instance, section 40 of KURATOWSKI [1940]):

LEMMA. *Let $(X_\alpha)_{\alpha<\kappa}$ be a family of sets of a set X such that each X_α is of cardinality κ. Then there is a subset Z of X of cardinality κ which meets X_α and $X\backslash X_\alpha$, for $\alpha<\kappa$.*

Let us recall that a separable and metrizable space Z is said to be *totally imperfect* whenever there is no homeomorphism from the Cantor space 2^ω into the space Z; this is equivalent to: Z does not contain an uncountable compact subspace. The following two results are due to KURATOWSKI [1926]:

LEMMA. *There is a sequence $(Z_n)_{n<\omega}$ of subsets of the real line such that:*
(i) *each Z_n is a totally imperfect subset of R, and*
(ii) *for $m\neq n$, if U is a non-empty open subset of Z_m then there is no homeomorphism from U into Z_n.*

PROOF. First, let $\mathcal{M}_0$ be the set of all subspaces of R, homeomorphic to the Cantor space. The existence of Z_0 follows from Bernstein's lemma. Now, let us suppose that $Z_0, Z_1, \ldots, Z_{n-1}$ are constructed and satisfy (i) and (ii). Let $\mathcal{M}_n$ be the set of all subspaces of R, defined as follows:
(1) every subspace of R homeomorphic to some $F\cap Z_i$, where F is a closed perfect subset of R and $i<n$, belongs to $\mathcal{M}_n$, and
(2) if H is a subset of R, homeomorphic to $G\cap Z_i$ for some $i<n$ and some G_δ-subset G of R, and P is a closed perfect subset of R; then $P\backslash H$ belongs to $\mathcal{M}_n$.
Obviously, $\mathcal{M}_n$ is of cardinality 2^ω. Clearly, each member of $\mathcal{M}_n$ is of cardinality 2^ω. Now, again, that Z_n exists follows from Bernstein's lemma. By construction, each Z_n is totally imperfect. Now let $m\neq n$ and U be a non-empty open subset of R. Then there is no homeomorphism from $U\cap Z_m$ into Z_n. By contradiction. Let W be a subspace of Z_n homeomorphic to $U\cap Z_m$. Let F be a closed perfect subset of R, contained in U, and V be a subspace of Z_n, homeomorphic to $F\cap Z_m$. First, suppose $m<n$. The set V is a member of $\mathcal{M}_n$ and thus $V\backslash Z_n\neq\emptyset$, contradicting $V\subseteq Z_n$. Now assume $n<m$. Let F be a closed perfect subset of R, contained in U, and f be a homeomorphism from $A=F\cap Z_m$ onto a subspace B of Z_n. Let us suppose:

(L) there are two subsets $\underline{A}$ and $\underline{B}$ of R, containing A and B, respectively, such that $\underline{A}$ and $\underline{B}$ are G_δ-sets, and f is extendable to a homeomorphism $\underline{f}$ from $\underline{A}$ onto $\underline{B}$.

So, we have $B\subseteq\underline{B}\cap Z_n\subseteq Z_n$, and thus $\underline{B}\cap Z_n$ is homeomorphic to some subspace H containing $F\cap Z_m$. Consequently, $F\backslash H$ belongs to $\mathcal{M}_m$, and thus $Z_m\cap(F\backslash H)\neq\emptyset$, contradicting $F\cap Z_m\subseteq H$.
Now it is sufficient to prove (L): that is a direct consequence of the following result of LAVRENTIEV [1924] or KURATOWSKI [1940]:

Let A and B be subspaces of complete metrizable spaces X and Y, respectively, and f be a homeomorphism from A onto B. Then there are subsets $\underline{A}$ and $\underline{B}$ of X and Y, containing A and B, respectively, such that $\underline{A}$ and $\underline{B}$ are G_δ-sets, and f is extendable to a homeomorphism $\underline{f}$ from $\underline{A}$ onto $\underline{B}$.

Notice that the proof of this result can be done by a back-and-forth argument on f and f^{-1}, extending f to A^*, the set of points in the topological closure of A such that the oscillation $\omega(f)$ of f is zero (this is connected with the construction of Bonnet-rigid algebras: see Section 3.7). □

The following theorem (KURATOWSKI [1926]) is similar to SIERPIŃSKI's [1940] result, rediscovered by BONNET [1980], on the existence of strongly-rigid subchain of the real line.

LEMMA. *There is a subspace P of the real line, such that P has no non-trivial homeomorphism onto itself.*

PROOF. We will construct P as a subspace of 2^ω. There is no loss in assuming that P is a subspace of $[0,1]\times[0,1]=K\subseteq R^2$, since $2^\omega\times 2^\omega$ and 2^ω are homeomorphic spaces. Now, let D be a countable dense subset of K. Let (x'_n, x''_n), for $n<\omega$ be an enumeration of the elements of D, with $x'_n, x''_n\in[0,1]$. We suppose that $x'_m\neq x'_n$ for $m\neq n$. For each $n<\omega$, let $J_n=\{x'_n\}\times[a_n,b_n]$, with $a_n<x'_n<b_n$ and $0<b_n-a_n<1/n$. Now, let H_n be a subspace of J_n homeomorphic to Z_n, where the Z_n's are given by the above result. Let P be the union of the H_n's. We remark that:

(1) H_n is not a meager set, and

(2) for $m\neq n$, if U is a non-empty open subset of H_m, then there is no homeomorphism from U into H_n.

As a consequence of (1) and (2), we obtain:

(3) H_n is not homeomorphic to a subspace of $\bigcup\{H_k\colon k<\omega, k\neq n\}$.

Now P has no non-trivial homeomorphism onto itself since (3) is satisfied, which completes the proof. □

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CHAPTER 18

Homogeneous Boolean Algebras

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Contents

HANDBOOK OF BOOLEAN ALGEBRAS
Edited by J.D. Monk, with R. Bonnet

0. Introduction

Homogeneous Boolean algebras constitute one of the important classes of algebras defined by the properties of automorphisms. All interesting examples of homogeneous algebras are atomless. In fact, many complete Boolean algebras used in forcing arguments are homogeneous and, in some cases, this property can be recognized in the underlying partial ordering. Automorphisms of complete algebras can be constructed from infinitely many components and the automorphism groups of complete algebras are susceptible to deeper analysis. For that reason, certain emphasis will be given on automorphisms of complete algebras.

Our discussion will proceed as follows. Sections 1 and 2 start with sample facts about automorphisms and isomorphisms of factors of Boolean algebras. The concepts of homogeneous and weakly homogeneous algebras will be compared. In Section 3 we briefly discuss a well-known theorem of Kripke, stating that there are complete homogeneous algebras that admit complete embeddings of all Boolean algebras up to a certain size. In Section 4 we show that every complete weakly homogeneous algebra is a product of copies of one of its homogeneous factors. This theorem was proved by Solovay and independently by Koppelberg. It shows that forcing with weakly homogeneous algebras gives exactly the same results as forcing with homogeneous algebras and it has many interesting applications to the theory of Boolean algebras, as well. In Section 5 we prove that the automorphism groups of certain homogeneous Boolean algebras are simple. The key to these results is a well-known theorem of Anderson that gives a sufficient condition for simplicity of automorphism groups. In particular, the theorem implies that the automorphism group of every σ-complete homogeneous Boolean algebra is simple. The results of Koppelberg, Kemmerich and Richter, and Rubin, show that every free algebra and every homogeneous subalgebra of an interval algebra have simple automorphism groups. However, it is consistent with the axioms of Zermelo–Fraenkel set theory (ZFC) that there are homogeneous algebras the automorphism groups of which are not simple. This will be illustrated by theorems of Koppelberg and van Douwen. In Section 6 we discuss the possibility of stronger concepts of homogeneity.

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1. Homogeneous algebras

1.1. Definition. A Boolean algebra B is homogeneous if the relative algebra $B \restriction b$ is isomorphic to B whenever b is a non-zero element of B.

It is clear that the trivial algebra 1 and the two-element algebra 2 are homogeneous and that every algebra containing more than one atom is not homogeneous. All interesting examples of homogeneous algebras are atomless.

1.2. EXAMPLES. (a) Every countable atomless Boolean algebras is homogeneous. It follows immediately from the definition and from the well-known fact that all countable atomless algebras are isomorphic. If B is countable and atomless, then every relativized algebra $B \restriction b$ for a non-zero $b \in B$ is countable and atomless, too.

(b) Every infinite free algebra is homogeneous according to Theorem 9.14 of Part I of this Handbook. In fact, every countable atomless algebra is isomorphic to a free algebra with countably many free generators. Hence, Example (a) is a special case of (b).

(c) The Boolean algebra $P(\omega)/fin$ is homogeneous. If a is a non-zero element of B, then there is an infinite subset $A \subseteq \omega$ such that $a = [A]$. Every bijective mapping f of ω onto A determines an isomorphism f^* of B and $B \restriction a$ defined by $f^*([X]) = [f(X)]$ for every $X \subseteq \omega$.

1.3. HOMOGENEOUS ALGEBRAS AND PARTIAL ORDERINGS. If B is a homogeneous algebra and $\leq$ is the corresponding partial ordering, it follows from the definition that the partially ordered set $(B, \leq)$ satisfies the following condition.

If b is a non-zero element of B, then the principal ideal

$$(\leftarrow, b] = \{a \in B\colon a \leq b\}$$

is isomorphic to B with respect to $\leq$. Moreover, $\leq$ is a separative ordering of B. According to Theorem 4.16 of Part I of this Handbook, every separatively ordered set $(\mathbb{P}, \leq)$ determines (up to isomorphism) a complete Boolean algebra $B(\mathbb{P})$ such that $\mathbb{P}$ is a dense subset of $B(\mathbb{P})$ and the canonical ordering of $B(\mathbb{P})$ extends the ordering of $\mathbb{P}$.

1.4. LEMMA. *If $\mathbb{P}$ is a separatively ordered set and if for every $p \in \mathbb{P}$ the principal ideal*

$$(\leftarrow, p] = \{q \in \mathbb{P}\colon q \leq p\}$$

is isomorphic to $\mathbb{P}$ with respect to $\leq$, then $B(\mathbb{P})$ is a complete homogeneous Boolean algebra.

In particular, the completion of every homogeneous Boolean algebra is a complete homogeneous Boolean algebra.

PROOF. Let $B = B(\mathbb{P})$ be the complete Boolean algebra determined by the partially ordered set $(\mathbb{P}, \leq)$. It follows from our assumption on $\mathbb{P}$ that for every two elements $p, q \in \mathbb{P}$, the relativized algebras $B \restriction p$ and $B \restriction q$ are isomorphic and have the same saturation as B. Let b be an arbitrary non-zero element of B. Since $\mathbb{P}$ is a dense subset of B, there exist a cardinal κ and sets

(1) $\quad \{p_\alpha\colon \alpha < \kappa\}$,

(2) $\quad \{q_\alpha\colon \alpha < \kappa\}$,

consisting of pairwise incompatible elements of $\mathbb{P}$ such that (1) is a partition of unity in B in (2) is a partition of b. For every $\alpha < \kappa$, let f_α be an isomorphism of $B \upharpoonright p_\alpha$ and $B \upharpoonright q_\alpha$. If we put

$$f(a) = \sum \{f_\alpha(a \cdot p_\alpha) \colon \alpha < \kappa\} \quad \text{for every } a \in B\ ,$$

then f is the isomorphism of B and $B \upharpoonright b$. Hence, B is a complete homogeneous algebra. □

1.5. EXAMPLES. (a) Let $\mathrm{Fn}(I, 2)$ be the set of all partial functions from finite subsets of an infinite set I to $\{0, 1\}$ ordered by reverse inclusion, i.e. $f \leq g \leftrightarrow g \subseteq f$. It is easy to see that $\mathrm{Fn}(I, 2)$ satisfies the assumption of Lemma 1.4. Hence, the corresponding complete Boolean algebra $B(I)$ is homogeneous. $\mathrm{Fn}(I, 2)$ is the standard set of forcing conditions to add $|I|$ new subsets of ω that are called Cohen reals. Note that $B(I)$ is the completion of a free algebra generated by I free generators.

(b) Let κ be an infinite cardinal and $\mathrm{Fn}(\omega, \kappa)$ be the set of all partial functions from finite subsets of ω to κ ordered by reverse inclusion. According to Lemma 1.4, the corresponding complete Boolean algebra $C(\kappa)$ is homogeneous. $\mathrm{Fn}(\omega, \kappa)$ is the standard set of forcing conditions to add a new bijective mapping of ω onto κ that collapses the cardinal κ to a countable ordinal. Algebras $C(\kappa)$ are often called collapsing algebras. It is not difficult to check that $C(\kappa)$ is isomorphic to the complete algebra $\mathrm{RO}({}^{\omega}\kappa)$ of regular open sets of the topological product of ω copies of κ, where κ is endowed with the discrete topology.

(c) Let κ, Λ be infinite cardinals, $\kappa \leq \Lambda$, and let $\mathrm{Fn}(\Lambda, \kappa, 2)$ be the set of all partial functions from subsets of Λ of cardinality less than κ to $\{0, 1\}$ ordered by reverse inclusion. According to Lemma 1.4, the corresponding complete Boolean algebra $B(\kappa, \Lambda)$ is homogeneous. Moreover, if κ is regular, then every decreasing sequence in $F(\Lambda, \kappa, 2)$ of length less than κ has a lower bound in $\mathrm{Fn}(\Lambda, \kappa, 2)$. According to Proposition 14.7 of Part I of this Handbook, $B(\kappa, \Lambda)$ is (μ, ∞)-distributive for every cardinal $\mu < \kappa$.

2. Weakly homogeneous algebras

The concept of homogeneity can be defined in terms of automorphisms instead of isomorphisms of relativized algebras. It was shown in Lemma 9.13 of Part I of this Handbook that every Boolean algebra B, $|B| \neq 4$, is homogeneous iff for every two elements a, b of B such that $0 < a$, $b < 1$, there is an automorphism φ of B such that $\varphi(a) = b$. The restriction on a and b is necessary since 0 and 1 are left fixed by every automorphism of B. Note that the four-element algebra satisfies the condition of the lemma but it is not homogeneous. Modifying the automorphism condition, we get the following:

2.1. DEFINITION. A Boolean algebra B is weakly homogeneous if for every two non-zero elements a, b of B, there is an automorphism φ of B such that

$\varphi(a) \cdot b \neq 0$. Equivalently, B is weakly homogeneous iff for every two non-zero elements a, b of B, there are non-zero a', b' such that $a' \leq a$, $b' \leq b$ and $B \restriction a'$ is isomorphic to $B \restriction b'$.

Hence, B is weakly homogeneous iff there is no pair of totally different relativized algebras in B.

2.2. EXAMPLES. (a) If A is a set, then the power-set algebra $P(A)$ is weakly homogeneous. In fact, $P(A)$ is isomorphic to the product of A copies of the two-element algebra.

(b) If B is a (weakly) homogeneous algebra, then the product of arbitrary many copies of B is weakly homogeneous.

Moreover, if B is a complete homogeneous algebra and $|I| < \mathrm{sat}(B)$, then the product B^I of I copies of B is isomorphic to B and hence it is homogeneous. On the other hand, the product of more than 2^ω copies of the homogeneous algebra $P(\omega)/fin$ is a weakly homogeneous atomless algebra that is not homogeneous.

(c) (BRENNER [1982]) Let F be the Cartesian product $\times \{\omega_n : n < \omega\}$ and let T be the set of finite mappings defined by

$$T = \{t : t = f \restriction n \text{ for some } f \in F \text{ and } n < \omega\},$$

and let T be partially ordered by inclusion. Then $(T, \subseteq)$ is a tree of height ω and every element t of the nth level T_n of T has ω_n successors in T_{n+1}. It is obvious that the empty mapping is the root of the tree and that $|T_{n+1}| = \omega_n$ for every n.

For every $t \in T$, let b_t denote the cone of all successors of t, hence $b_t = \{s \in T : t \subseteq s\}$. Note that b_t and b_s are isomorphic subsets of T iff t and s belong both to the same level of T. Let B_T be the tree algebras generated by the sets $\{b_t, t \in T\}$ as in Section 16 of Part I of this Handbook. Since the set of generators b_t is dense in B_T, for all non-zero elements u, v of B_T, there exist n and $t, s \in T_n$ such that $b_t \leq u$ and $b_s \leq v$. Hence, B_T is weakly homogeneous but no relativized algebra $B_T \restriction a$, $a > 0$, is homogeneous.

Note that B_T is not complete. We shall see later that every complete weakly homogeneous Boolean algebra is isomorphic to a product of copies of a complete homogeneous Boolean algebra.

2.3. If B is a weakly homogeneous algebra, it follows from the definition that every $a \in B$, $0 < a < 1$, is moved by some automorphism of B. The zero and unit elements are left fixed by every automorphism. It turns out that this fact gives the characterization of complete weakly homogeneous Boolean algebras.

2.4. DEFINITION. Let B be a Boolean algebra. The set of all elements of B that are left fixed by every automorphism of B is called the center of B and denoted by center(B).

Note that the center of B is closed under Boolean operations and hence it is a subalgebra of B. Moreover, if B is complete, the center of B is a complete regular subalgebra of B.

2.5. LEMMA. *Let B be a complete Boolean algebra. Then B is weakly homogeneous iff* $\operatorname{center}(B) = \{0, 1\}$.

PROOF. The condition holds if B is a weakly homogeneous algebra. Suppose that B is complete but not weakly homogeneous. Let $\operatorname{Aut}(B)$ denote the automorphism group of B. There are non-zero elements a, b such that $\varphi(a)$ and b are disjoint for every $\varphi \in \operatorname{Aut}(B)$. If we put

$$c(a) = \sum \{\varphi(a): \varphi \in \operatorname{Aut}(B)\} ,$$

then $c(a) \in \operatorname{center}(B)$ holds for every a. It follows from our assumption that $c(a)$ is non-zero and disjoint from b. Hence, $\operatorname{center}(B) \neq \{0, 1\}$. □

3. κ-universal homogeneous algebras

We shall briefly comment on the following result due to Kripke. It was proved as Theorem 14.18 in Part I of this Handbook.

3.1. THEOREM (KRIPKE [1967]). *Every Boolean algebra A can be completely embedded in a complete countable generated algebra B in such a way that every automorphism of A extends to an automorphism of B.*

The origins of the theorem can be traced back to an apparently unrelated problem of RIEGER (1951) who asked whether or not there exists a free complete Boolean algebra. All free algebras with a finite number of generators are finite and hence complete. Thus, the question is interesting only if at least countably many generators are required. Such an algebra, if it existed, would be a complete Boolean algebra generated by a countable set of free generators that are free for infinite Boolean operations. In such a case, every mapping of the free generators into any other complete Boolean algebra could be extended to a complete homomorphism. Consequently, every complete Boolean algebra with countably many generators would be a homomorphic image of such an algebra.

Gaifman, and independently Hales, gave the negative solution. They showed that there are complete, countably generated Boolean algebras of arbitrary large cardinalities. Later, Solovay gave a simple proof of this result using collapsing algebras. For any infinite cardinal κ, the complete homogeneous algebra $C(\kappa)$, from Example 1.5(b), describes a new mapping f of ω onto κ in the corresponding Boolean extension of the set-theoretic universe. Solovay observed that the countable subset of $C(\kappa)$ that describes the well-ordering of the type ω induced on κ by f, generates the whole algebra $C(\kappa)$. Eventually, Kripke showed that every Boolean algebra can be completely embedded to some $C(\kappa)$ for κ large enough.

3.2. We have already noted that $C(\kappa)$ and $\operatorname{RO}({}^{\omega}\kappa)$ are isomorphic homogeneous complete Boolean algebras. In fact, $C(\kappa)$ has a dense subset consisting of all finite sequences of ordinals less than κ. Every such a sequence f corresponds to a clopen subset $o_f \subset \mathrm{RO}({}^\omega\kappa)$ and the above dense set of $C(\kappa)$ corresponds to a clopen basis for the topology of the product ${}^\omega\kappa$.

According to Theorem 14.18 of Part I, if A is a Boolean algebra of power at most κ, then the completion of the free product $A \oplus \mathrm{RO}({}^\omega\kappa)$ is isomorphic to $\mathrm{RO}({}^\omega\kappa)$. Hence, A is completely embedded in $\mathrm{RO}({}^\omega\kappa)$ and every automorphism of A extends to an automorphism of $\mathrm{RO}({}^\omega\kappa)$. This shows that $C(\kappa)$ is a κ^+-universal Boolean algebra. Note that the proof of Theorem 14.18 can be carried out if A has a dense subset of power at most κ. Hence, Theorem 3.1 can be rephrased as follows.

3.3. THEOREM (Kripke). *If A is a Boolean algebra with a dense subset of power at most κ, then there is a complete embedding e of A in $C(\kappa)$.*

Moreover, $C(\kappa)$ is a complete, homogeneous countably generated Boolean algebra and every automorphism of $e(A)$ extends to an automorphism of $C(\kappa)$.

3.4. Let us recall that $\mathrm{sat}(B)$ denotes the least cardinal κ such that the algebra B satisfies the κ-chain condition. We call it the saturation of B.

Let us compare the saturation of A and $C(\kappa)$ in the previous theorem. It is clear that $\mathrm{sat}(C(\kappa)) = \kappa^+$. In particular, $C(\kappa)$ satisfies CCC iff $\kappa = \omega$. If A satisfies CCC and has no countable dense subset, Theorem 3.3 guarantees an embedding in a homogeneous algebra with a higher saturation than that of A. This is not a peculiarity of Kripke's theorem, the statement "every Boolean algebra can be completely embedded in a complete homogeneous algebra with the same saturation" is not a theorem of ZFC.

3.5. THEOREM ($V = L$). *There exists a Boolean algebra B satisfying the countable chain condition (CCC) which cannot be completely embedded in any weakly homogeneous Boolean algebra satisfying CCC.*

In particular, B cannot be completely embedded in any CCC homogeneous complete Boolean algebra.

The proof of Theorem 3.5 combines a construction of a Suslin ω_1-tree in the constructible universe L with a forcing argument. JENSEN and JOHNSBRATEN [1974] have shown that there is a well-pruned Suslin ω_1-tree T in L with the following property: if $\omega_1^L = \omega_1$, then there is at most one cofinal branch in T. If we denote by B_T the complete Boolean algebra determined by the reversed tree-ordering of T, then B_T satisfies CCC in L. Suppose that there is a complete embedding of B_T in a complete weakly homogeneous Boolean algebra C in L which satisfies CCC. If G is a C-generic ultrafilter over L, then $\omega_1^L = \omega_1$ holds in $L[G]$ since all cardinals are preserved by the countable chain condition of C. On the other hand, $G \cap B_T$ defines a cofinal branch in T which has many copies in T. In fact, every constructible automorphism φ of C maps G onto a C-generic ultrafilter $\varphi[G]$ over L and $\varphi[G] \cap B_T$ defines a cofinal branch in T. There are at least two cofinal branches in T in the universe $L[G]$, because C is weakly homogeneous in L. This contradicts the theorem of Jensen and Johnsbraten since $\omega_1^L = \omega_1^{L[G]}$. Hence, no general theorem on saturation-preserving embeddings into the class of homogeneous algebras can be proved in ZFC. It is still open whether one can prove in ZFC that every CCC Boolean algebra can be completely embedded in a homogeneous complete algebra of saturation ω_2.

On the other hand, if we assume Martin's axiom MA(ω_1), then the free product of any number of CCC Boolean algebras satisfies CCC. In particular, the free product B^ω of countably many copies of a Boolean algebra B satisfying CCC is weakly homogeneous and satisfies CCC as well. It follows from the theorem of Solovay and Koppelberg (see Section 4) that the completion of B^ω is homogeneous. Hence, MA(ω_1) implies that every CCC Boolean algebra can be completely embedded in a complete homogeneous Boolean algebra with the same saturation.

4. Complete weakly homogeneous algebras

The product of any number of copies of a homogeneous Boolean algebra is weakly homogeneous. Surprisingly, the converse holds for every complete weakly homogeneous algebra.

4.1. THEOREM (Solovay, Koppelberg). *Let B be a complete weakly homogeneous Boolean algebra. Then there is a complete homogeneous algebra H such that B is isomorphic to the product H^I for some I.*

4.2. The theorem was proved first by Solovay but the proof has not been published. It was stated without proof in GRIGORIEFF [1972] to illustrate the fact that the forcing with weakly homogeneous Boolean algebras gives the same results as forcing with homogeneous algebras. The theorem was proved independently by KOPPELBERG [1981] in a different context. We shall present here the proof due to Solovay.

Note that all factors $H \upharpoonright h$ of a homogeneous algebra are isomorphic. If B is isomorphic to a product of copies of H, then the set of all elements b of B such that $B \upharpoonright b$ is isomorphic to H is dense in B. To prove the theorem, it is convenient to deal with the isomorphism types of factors $B \upharpoonright b$ rather than with the elements of B themselves.

4.3. DEFINITION. Let B be a Boolean algebra and $a, b \in B$.

(i) We say that a, b have the same isomorphism type, and we write $a \approx b$, if $B \upharpoonright a$ is isomorphic to $B \upharpoonright b$. Then $\approx$ is an equivalence relation on B and the isomorphism type of a can be defined as the equivalence class

$$[a] = \{b \in B\colon a \approx b\}\ .$$

(ii) We say that $[b]$ extends $[a]$, and we write $[a] \leq [b]$, if there is some $c \leq b$, $[a] = [c]$. We say that $[b]$ is a proper extension of $[a]$ and write $[a] < [b]$ if $[a] \leq [b]$ and $[a] \neq [b]$.

Note that if B is σ-complete, then $\leq$ is a partial ordering of the isomorphism types in B. We shall define two partial operations on isomorphism types.

4.4. DEFINITION. For any two elements $a, b \in B$, the sum $[a]+[b]$ of their isomorphism types is the isomorphism type of the product of factors $B \restriction a$ and $B \restriction b$. Hence, an element $[c]$ of B has the type $[a]+[b]$ iff there are disjoint elements c_1, c_2 such that $[c_1]=[a]$, $[c_2]=[b]$ and $c = c_1 + c_2$. Note that such an element c need not exit in B, hence the addition of types is a partial operation.

Similarly, if η is an ordinal, then $\eta \cdot [a]$ is the isomorphism type of the product of η copies of $B \restriction a$. Hence, $c \in B$ has the type $\eta \cdot [a]$ iff there is a partition $\langle c_\alpha : \alpha < \eta \rangle$ of c such that $[c_\alpha]=[a]$ for every $\alpha < \eta$. Clearly, $|\eta| = |\mu|$ implies $\eta \cdot [a] = \mu[a]$ for every a and ordinals η, μ.

4.5. LEMMA. *Let B be a complete atomless weakly homogeneous algebra. Then*

(i) *for every non-zero $b \in B$, there is $a < b$ such that $[b] = 2 \cdot [a]$.*

(ii) *For any two elements $a, b \in B$, we have $[a] \leq [b]$ or $[b] \leq [a]$. Hence, $\leq$ is a linear ordering of isomorphism types in B.*

PROOF. (i) Since B is atomless and weakly homogeneous, for every non-zero element $c < b$, there are two disjoint non-zero elements $a, a' < c$ of the same isomorphism type. Using the completeness of B, we can construct two sequences of pairwise disjoint elements $\langle a_\alpha : \alpha < \beta \rangle$ and $\langle a'_\alpha : \alpha < \beta \rangle$ such that $[a_\alpha] = [a'_\alpha]$ for every α and the union of both sequences makes a partition of b. Then $a = \Sigma\, a_\alpha$ has the same isomorphism type as $\Sigma\, a'_\alpha$ and $[b] = 2 \cdot [a]$.

(ii) By induction on α, we can construct pairwise disjoint elements $a_\alpha < a$, $b_\alpha < b$ such that $[a_\alpha] = [b_\alpha]$ holds for every α. There exists an ordinal β such that $\Sigma\, \{a_\alpha : \alpha < \beta\} = a$ or $\Sigma\, \{b_\alpha : \alpha < \beta\} = b$. We have $[a] \leq [b]$ in the former case and $[b] \leq [a]$ in the latter. $\square$

4.6. ABSORPTION LEMMA. *Let η be an infinite ordinal, B a complete algebra and $a, b, c \in B$.*

If $[a] = \eta \cdot [b] + [c]$ and $[c] \leq [b]$, then $[a] = \eta \cdot [b]$.

PROOF. First, we shall prove the lemma for $\eta = \omega$. Suppose that $[a] = \omega \cdot [b] + [c]$ and $[c] \leq [b]$. Then there is a partition of a consisting of pairwise disjoint elements b_n, $n < \omega$, and c_{-1}, where $[c_{-1}] = [c]$ and $[b_n] = [b]$ for every n. As $[c] \leq [b]$, there exists an element d, $d \leq b$, such that $[b] = [c] + [d]$. Consequently, for every $n < \omega$, there are pairs c_n, d_n of disjoint elements, $b_n = c_n + d_n$, $[c_n] = [c]$ and $[d_n] = [d]$.

Hence,

$$a = \sum_{n=0}^{\infty} (c_n + d_n) + c_{-1}$$

and

$$a = \sum_{n=0} (c_{n-1} + d_n)\,,$$

where all elements on the right-hand side are pairwise disjoint. If we go back to isomorphism types, we get $[a] = \omega \cdot [b]$.

If $\eta > \omega$, then $[a] = (\eta - \omega) \cdot [b] + \omega \cdot [b] + [c]$ and we can apply the Absorption Lemma to the last two terms on the right-hand side of the equality. □

COROLLARY. (i) *If* $[b] = \Lambda \cdot [b]$ *and* $\omega \leq \kappa \leq \Lambda$, *then* $[b] = \kappa \cdot [b]$.
(ii) *If* $[a] = \omega \cdot [b] + [c]$, *then* $[a] = \eta \cdot [b]$ *for some infinite ordinal* η.

PROOF. (i) Clearly, $[b] = \kappa \cdot [b] + \Lambda \cdot [b]$. Since $[b] = \Lambda \cdot [b]$, we have $[b] = (\kappa + 1) \cdot [b]$ and the result follows. (ii) Let $\{b_\alpha : \alpha < \eta\}$ be a maximal disjoint family such that $b_\alpha \leq a$ and $[b_\alpha] = [b]$ for all $\alpha < \eta$; we may assume that η is infinite. If we put $d = a - \Sigma\, b_\alpha$, then $[d] \leq [b]$ by maximality. Hence, the result follows by the Absorption Lemma. □

4.7. Any complete weakly homogeneous Boolean algebra containing at least one atom is isomorphic to a power-set algebra and hence it is isomorphic to a product of two-element homogeneous algebras. It remains to prove the theorem for atomless algebras.

4.8. LEMMA. *Let* B *be an atomless complete weakly homogeneous algebra. If there is a non-zero* $h \in B$ *such that* $B \upharpoonright h$ *is homogeneous, then* B *is isomorphic to a product of copies of* $B \upharpoonright h$.

PROOF. It follows from weak homogeneity of B that for every non-zero $b \in B$, there are non-zero $b_1 < b$ and $h_1 < h$ such that $[b_1] = [h_1] = [h]$. Consequently, there is a partition $\langle b_i : i \in I \rangle$ of the unity in B such that every factor $B \upharpoonright b_i$ is isomorphic to $B \upharpoonright h$. □

4.9. The proof of the theorem reduces to the existence of a homogeneous factor in every complete weakly homogeneous and atomless algebra. We shall show that the homogeneous factor is minimal with respect to a cardinal function.

DEFINITION. Let B be complete and b be a non-zero element of B. We denote by $\alpha(b)$ the least infinite ordinal β such that $[b] \neq \beta \cdot [b]$.

Note that $\alpha(b)$ is the least infinite ordinal such that there is no partition of b to β elements of the same isomorphism type as b. Hence, $\alpha(b)$ is an infinite cardinal, and if B is atomless, then $\omega \leq \alpha(b) \leq \mathrm{sat}(B \upharpoonright b)$ for every non-zero $b \in B$.

4.10. In the rest of the proof, let B denote an atomless complete weakly homogeneous algebra and let c be a non-zero element of B with the minimal value $\alpha(c)$. We shall consider two cases with respect to $\alpha(c)$. If $\alpha(c)$ is uncountable, we shall show that $B \upharpoonright c$ is homogeneous. The remaining case $\alpha(c) = \omega$ leads to a contradiction. It implies that (i) every two elements d, e, $d < e < c$, have different isomorphism types, and (ii) $B \upharpoonright c$ carries a strictly positive measure. Then it follows from a well-known theorem of MAHARAM [1942] that $B \upharpoonright c$ is, up to isomorphism, a product of a finite or countable family of pairwise totally different homogeneous measure algebras. This contradicts (i).

4.11. *Case 1.* $\alpha(c) > \omega$. We shall use the Absorption Lemma to show that $B \restriction c$ is homogeneous. Let d be an arbitrary non-zero element of $B \restriction c$. It follows from the minimality of $\alpha(c)$ that $\alpha(d) \geq \alpha(c) > \omega$ and hence $[d] = \omega \cdot [d]$ by Corollary 4.6(i). If we put $e = c - d$, we get $[c] = \omega \cdot [d] + [e]$. It follows from Corollary 4.6(ii) that $[c] = \eta \cdot [d]$ for an infinite ordinal η. Since $|\eta \times \eta| = |\eta|$, we have

$$(\eta \cdot \eta) \cdot [d] = \eta \cdot [d]$$

and

$$\eta \cdot [c] = [c] \, .$$

Hence, $\eta < \alpha(c) \leq \alpha(d)$ and $[c] = \eta \cdot [d] = [d]$. This shows that $B \restriction c$ is homogeneous.

4.12. *Case 2.* $\alpha(c) = \omega$. First, we shall show that there is no contraction below c, more precisely that $d < e < c$ implies $[d] < [e]$ for every d, $e < c$. Note that any contraction below c, i.e. any pair of elements d, e, $d < e < c$, such that $[d] = [e]$, implies that there is a $d' < c$ with $[d'] = [c]$. It suffices to show that for every $d < c$, we have $[d] < [c]$.

Suppose that there is $d < c$, $[d] = [c]$. We shall show that in this case $\alpha(c)$ must be uncountable. By induction on n, we can define elements d_n, e_n, $n < \omega$, as follows. We put $d_0 = d$, $e_0 = e = c - d$. If all the elements d_k, e_k, $k \leq n$, have been defined such that $[d_k] = [d]$ and $[e_k] = [e]$ for every $k \leq n$, we choose disjoint elements d_{n+1}, e_{n+1} such that $d_n = d_{n+1} + e_{n+1}$, $[d_{n+1}] = [d]$ and $[e_{n+1}] = [e]$. If we put

$$d_\omega = \prod \{d_n \colon n < \omega\} \, ,$$

we have $[c] = \omega \cdot [e] + [d_\omega]$. By Corollary 4.6(ii), we get $[c] = \varepsilon \cdot [e]$ for some $\varepsilon \geq \omega$. By the same argument as above, we can conclude that $[c] = \varepsilon \cdot [c]$. Hence $\alpha(c) > \varepsilon \geq \omega$, which contradicts our assumption. This shows that there is no contraction in $B \restriction c$.

In fact, we have proved the following:

COROLLARY. *If* $[a] = \omega \cdot [b] + [c]$, *then* $\alpha(a) > \omega$.

4.13. LEMMA. $\alpha(c) = \omega$ *implies that there is a strictly positive measure on* $B \restriction c$.

PROOF. Let us recall that a σ-additive measure μ on a Boolean algebra is strictly positive if $\mu(a) > 0$ for every non-zero element a. We shall construct such a measure using Lemma 4.5(i) and non-contractibility of $B \restriction c$. According to 4.5(i), for every non-zero $a < c$, there is $b < a$ such that $[a] = 2 \cdot [b]$. We shall show that the isomorphism type of b is uniquely determined by a. Suppose that $[a] = 2 \cdot [b']$ for some b' such that $[b] \neq [b']$. The isomorphism types of b and b'

are comparable since B is weakly homogeneous. We may assume that $[b] \leq [b']$. Then there is a non-zero e such that $[b'] = [b] + [e]$ and

$$[a] = 2 \cdot [b'] = 2 \cdot [b] + 2 \cdot [e] .$$

Hence, a is contractable, contradicting 4.12.

Now, we can define the isomorphism type $[a]/2$ as the uniquely determined isomorphism type of any $[b]$, $[a] = 2 \cdot [b]$. In particular, we can define $[c]/2$ and $[c]/2^n$ for every natural number n.

It follows from 4.12 that for every n,

$$[c]/2^{n+1} < [c]/2^n ,$$

and we can construct pairwise disjoint elements c_n, $n < \omega$, such that $c_n < c$ and $[c_n] = [c]/2^{n+1}$ holds for every n.

Let c_0, d_0 be disjoint elements such that $c = c_0 + d_0$ and $[c_0] = [d_0]$. Suppose that disjoint $c_0, \ldots, c_n, d_n$ have been constructed so that

$$c = c_0 + \cdots + c_n + d_n \quad \text{and} \quad c_n = d_n = c/2^{n+1} .$$

Let c_{n+1}, $d_{n+1} < d_n$ be disjoint elements such that $d_n = c_{n+1} + d_{n+1}$ and $[c_{n+1}] = [d_{n+1}]$. Then clearly

$$[c_n] = [c]/2^{n+1} \qquad \text{for every } n ,$$

$$[c_n]/2^{m-n} = [c_m] \quad \text{if } m > n .$$

We shall show that $\langle c_n : n < \omega \rangle$ is a partition of c and

$$(6) \qquad [c] = \sum_{n=0}^{\infty} [c_n]$$

by abuse of notation.

Claim 1. For every non-zero $d < c$, there is a natural n such that $[c]/2^n \leq [d]$.

Suppose that the above inequality does not hold for any n. Since any two isomorphism types are comparable, we have

$$[d] \leq [c]/2^n \quad \text{for every } n .$$

It follows from (6) that

$$[c] = \omega \cdot [d] + [e]$$

for some e. According to Corollary 4.12, this contradicts our assumption on $\alpha(c)$. Hence, $[c]/2^n \leq [d]$ for some n.

To prove (6), let

$$d = c - \sum \{c_k : k < \omega\} .$$

It suffices to show that $d=0$. Suppose not. Then for some n we have $[c_n] \leq [d]$ by Claim 1. By the construction of c_n, d_n, we have $d_n > d$ and $[c_n] = [d_n]$. Since $[d] \geq [c_n]$ this contradicts the non-contractibility of c. Hence, $d=0$ and (6) is proved. Since $\langle c_k\colon n<k<\omega\rangle$ is a partition of d_n, we have

(7) $$[c_n] = \sum_{k=n+1}^{\infty} [c_k]\,.$$

If

$$r = \sum_{n=0}^{\infty} r_n/2^{n+1}$$

is the dyadic expansion (with infinitely many non-zero r_n's) of a real number r, $0<r\leq 1$, we define

$$r\cdot[c] = \left[\sum_{n=0}^{\infty} r_n \cdot c_n\right],$$

where $1\cdot c_n$ is c_n and $0\cdot c_n$ is 0. We define $r\cdot[c]=[0]$ if $r=0$.

Claim 2. If r, s are different real numbers, $0\leq r$, $s\leq 1$, then $r\cdot[c]\neq s\cdot[c]$.

Let n be the least number such that $r_n \neq s_n$. Suppose that $r_n=1$, $s_n=0$ and $r\cdot[c]=s\cdot[c]$. By (7), we have

$$[c_n] = \sum_{k=n+1}^{\infty} [c_k]\,,$$

so we can write

$$s\cdot[c] = \left[\sum_{k<n} s_k\cdot c_k + d\right],$$

with $d\leq c_n$ and thus

$$\sum_{k<n} s_k\cdot c_k + d < \sum_{k=0}^{\infty} r_k\cdot c_k\,,$$

which contradicts non-contractibility of c.

Claim 3. For every $d<c$, there is a unique real number r such that $[d]=r\cdot[c]$.

This follows from Claims 1 and 2.

Now, we can define a measure μ on $B\restriction c$. For every $d<c$, let $\mu(d)$ be the unique real number r, $[d]=r\cdot[c]$.

Note that $d\leq e$ implies $\mu(d)\leq\mu(e)$. If $r\leq\mu(d)$ is a non-negative real, then there is $e\leq d$ with $\mu(e)=r$. Moreover, if

$$\mu(d) = \sum_{i=0}^{\infty} r_i\,,$$

where $r_i\geq 0$ holds for every i, then there is a partition $\langle d_i\colon i<\omega\rangle$ of d such that $\mu(d_i)=r_i$ for all i. It follows from Claim 1 that $\mu(d)>0$ iff d is a non-zero element of $B\restriction c$, hence μ is strictly positive. We shall show that μ is σ-additive.

Let $\langle d_n : n < \omega \rangle$ be a partition of some $d < c$. We shall show that $\mu(d) = \Sigma \mu(d_n)$. Otherwise, there exists a positive real number ε such that $|\mu(d) - \Sigma\ \mu(d_n)| = \varepsilon$. Suppose, for example, that $\mu(d) = \varepsilon + \Sigma\ \mu(d_n)$, then there is a non-zero $e < d$ and a sequence $\langle d'_n : n < \omega \rangle$ of pairwise disjoint elements $d'_n < d - e$ such that $d = e + \Sigma\ d'_n$, $\mu(e) = \varepsilon$ and $\mu(d'_n) = \mu(d_n)$ for every n. Hence, $[d'_n] = [d_n]$ for all n and, consequently, $[d] = [d - e]$, which contradicts the fact that there is no contraction below c. We have shown that μ is a strictly positive σ-additive measure on $B \restriction c$.

According to a well-known theorem of Maharam, $B \restriction c$ is then a product of homogeneous algebras. This again implies that there is a contraction below c, contradicting 4.12. Hence, there is no element c of B, $\alpha(c) = \omega$ and, according to the Case 1, there is a homogeneous factor $B \restriction c$ in B. This completes the proof of the theorem of Solovay and Koppelberg. □

4.14. EXAMPLES. (a) Let us consider the algebra B_T from Example 2.2(c). B_T is a weakly homogeneous (non-complete) Boolean algebra. It follows from the properties of the tree T that B_T does not have any homogeneous factors.

Let B denote the completion of B_T, then B is a complete weakly homogeneous algebra, which is isomorphic to a product of copies of a complete homogeneous Boolean algebra H, according to the above theorem. Note that every non-zero element of B has the same cellularity as B_T and, consequently, as H. This implies that B itself is homogeneous. This shows that B_T is a weakly homogeneous algebra with no homogeneous factors which has a homogeneous completion.

(b) Let B be an arbitrary Boolean algebra and I an infinite set. Then it is not difficult to see that the free product $B_I = \bigoplus_{i \in I} B$ of I copies of B is a weakly homogeneous algebra. Hence, the completion of C of B_I is weakly homogeneous and has a homogeneous factor according to the theorem of Solovay and Koppelberg. It is not difficult to see that every non-trivial factor of C has the same saturation as B_I and, consequently, as C itself. This shows that C is a complete homogeneous Boolean algebra. Moreover, B is a regular subalgebra of C and every automorphism of B extends to C. This is a result similar to Kripke's theorem and to Theorem 11.10 of Part I of this Handbook due to Grätzer.

A similar construction gives embeddings that preserve certain types of distributivity. Let us recall that a dense subset D of a Boolean algebra is κ-closed for a cardinal κ, if every decreasing sequence of length less than κ which consists of elements of D has a lower bound in D. If B has a κ^+-closed dense subset, then it is (κ, ∞)-distributive.

4.15. THEOREM. *Let κ be a regular cardinal and let B be a complete Boolean algebra with a κ-closed dense subset D. Then B can be completely embedded in a complete homogeneous Boolean algebra H with a κ-closed dense subset. Moreover, every automorphism of B extends to an automorphism of C.*

PROOF. We may assume that B is the complete algebra determined by the dense subset D and its canonical ordering $\leq$. Let $1_B \in D$ and let E be the subset of the cartesian product of κ copies of D defined as follows:

$$E=\{x\in D^{\kappa}: |\{\alpha<\kappa: x_{\alpha}\neq 1_B\}|<\kappa\}\,.$$

If E is ordered by $\leq_E$, where

$$x\leq_E y \quad \text{iff } x_{\alpha}\leq y_{\alpha} \text{ for every } \alpha<\kappa\,,$$

then E is a κ-closed partially ordered set. If C is the complete Boolean algebra determined by $(E,\leq_E)$, then a similar argument as above shows that C is homogeneous and isomorphic to the free product $B\oplus C$. Again, B is a regular subalgebra of C and every automorphism of B extends to C. □

5. Results and problems concerning the simplicity of automorphism groups of homogeneous BAs

The automorphism groups of homogeneous BAs have raised a significant amount of interest. One line of research which we are about to describe here is concerned with simplicity of such groups. Let us recall that a group is simple if it has no normal subgroup except for itself and the identity group.

The first positive simplicity result was proved by ANDERSON [1958]. As a special case of a theorem on the simplicity of certain homeomorphism groups, Anderson proved that the automorphism group of the countable atomless Boolean algebra is simple. We shall soon see why it was tempting to ask whether the automorphism group of every homogeneous BA is simple.

KOPPELBERG [1985] assuming CH proved that there is a homogeneous BA whose automorphism group is not simple. Soon afterwards VAN DOUWEN [1984] showed that in a certain model of ZFC constructed by Shelah, the automorphism group of the homogeneous BA $P(\omega)/\mathit{fin}$ is not simple.

At the same time some classes of homogeneous BAs have been shown to have simple automorphism groups. KOPPELBERG [1981] proved the above for free Boolean algebras, BRENNER [1982] proved it for tree algebras, KEMMERICH and RICHTER [1978] for interval algebras, FUCHINO [1985] for saturated BAs, and RUBIN [1986] for subalgebras of interval algebras. Simplicity also holds for the automorphism groups of homogeneous σ-complete BAs. This is an immediate consequence of Anderson's theorem.

We now wish to make some easy observations that will demonstrate why in many cases the automorphism group of a homogeneous BA turns out to be simple.

Let $\bar{B}$ denote the completion of B. Every automorphism of B can be uniquely extended to an automorphism of $\bar{B}$, so we regard $\mathrm{Aut}(B)$ as a subgroup of $\mathrm{Aut}(\bar{B})$. For $a\in\bar{B}$, let $B\restriction a\overset{\text{def}}{=}\{b\cdot a \mid b\in B\}$. Let f be a function such that $\mathrm{Dom}(f)\supseteq B\restriction a$; then $f\restriction a$ denotes $f\restriction(B\restriction a)$. For $f\in\mathrm{Aut}(B)$ let $\mathrm{var}(f)=\Sigma\,\{a\in B \mid f(a)\cdot a=0\}$. Hence, $\mathrm{var}(f)\in\bar{B}$.

For a group G and $f,g\in G$ let $f^g\overset{\text{def}}{=} gfg^{-1}$ and $[f,g]\overset{\text{def}}{=}fgf^{-1}g^{-1}$. f^g is called the conjugate of f by g and $[f,g]$ is the commutator of f and g. For $A,B\subseteq G$, let $A^B=\{f^g \mid f\in A,\ g\in B\}$. $[A,B]$, f^A, etc. are defined similarly. Let $\mathrm{NC}(A,G)$ denote the normal closure of A in G, that is $\mathrm{NC}(A,G)$ is the smallest normal subgroup of G containing A. $\mathrm{NC}(f,G)$ denotes $\mathrm{NC}(\{f\},G)$, and when $G=\mathrm{Aut}(B)$ we write $\mathrm{NC}(A)$, $\mathrm{NC}(f)$ to denote $\mathrm{NC}(A,G)$, $\mathrm{NC}(f,G)$, respectively.

Note that $[f, g] = f(f^{-1})^g = g^f \cdot g^{-1}$, hence $[f, g] \in \mathrm{NC}(f, G)$ and $[f, g] \in \mathrm{NC}(g, C)$.

Observation 1. Let L be homogeneous, $a \in B - \{1_B\}$ $h, f \in \mathrm{Aut}(B)$, $\mathrm{var}(h) \le a$ and $f \ne Id$. Then there are conjugates f_1 and f_2 of f such that $f_2 f_1 \restriction a = h \restriction a$. This implies that there is $h' \in \mathrm{NC}(f)$ such that $h' \restriction a = h \restriction a$.

Even though the above observation will not be used exactly as is, it still constitutes a key fact in what follows. So let us indicate how it is proved.

Let $0 < b < -a$. Since B is homogeneous, there is an isomorphism k_1 between $B \restriction a$ and $B \restriction b$. Let $k_2 = (h \restriction a) \circ k_1^{-1}$. Hence, $k_2 k_1 \restriction a = h \restriction a$. It is easy to see that there are conjugates f_1 and f_2 of f such that $f_1 \restriction a = k_1$ and $f_2 \restriction b = k_2$. So $f_2 f_1 \restriction a = h \restriction a$.

We have used the fact that if $a \cdot b = 0$, $a + b \ne 1_B$, k is an isomorphism between $B \restriction a$ and $B \restriction b$ and $f \in \mathrm{Aut}(B) - \{Id\}$, then there is a conjugate f_1 of f such that $f_1 \restriction a = k$. Let us see why this is true. Let $c \ne 0$ be such that $c \cdot f(c) = 0$ and $c + f(c) \ne 1_B$. Let $l \in \mathrm{Aut}(B)$ be such that $l(c + f(c)) < -(a + b)$, and let $f_3 = lfl^{-1}$. Then for $d = l(c)$, $d \cdot f_3(d) = 0$ and $(d + g_d(d)) \cdot (a + b) = 0$. Now let $m \in \mathrm{Aut}(B)$ be such that $m(d) = a$ and $m \restriction f_3(d) = kmf_3^{-1} \restriction f_3(d)$. Then $mf_3m^{-1} \restriction a = k$ as desired.

The following observation is proved in 5.3.

Observation 2. Let $f \in \mathrm{Aut}(B)$, then there are $h_1, h_2 \in \mathrm{Aut}(B)$ such that $h_1 h_2 = f$ and $\mathrm{var}(h_1)$, $\mathrm{var}(h_2) \ne 1_B$.

Let $\mathrm{Aut}'(B)$ denote the commutator group of $\mathrm{Aut}(B)$, that is, $\mathrm{Aut}'(B)$ is the group generated by $\{[f, g] \mid f, g \in \mathrm{Aut}(B)\}$. Observations 1 and 2 will essentially imply that if B is homogeneous, then $\mathrm{Aut}'(B)$ is simple.

The third and last observation indicates how in certain circumstances it is possible to prove that $\mathrm{Aut}'(B) = \mathrm{Aut}(B)$.

Observation 3. Let $a_0 \in B - \{1_B\}$, $h \in \mathrm{Aut}(B)$ and $\mathrm{var}(h) \le a_0$. Suppose that the following situation occurs. There are pairwise disjoint non-zero elements $\{a_z \mid z \in \mathbb{Z}\}$ ($\mathbb{Z}$ denotes the set of integers), and there are $f, g \in \mathrm{Aut}(B)$ having the following properties: (1) for every $z \in \mathbb{Z}$, $f(a_z) = a_{z+1}$, $g(a_z) = a_{z-1}$; (2) for every $z \le 0$, $g \restriction a_z = g^{-1} \restriction a_z$, and for every $z > 0$, $g \restriction a_z = f^{-1}(h^{f^z}) \restriction a_z$; and (3) $fg \restriction (1_B - \Sigma_{z \in \mathbb{Z}}\, a_z) = gf \restriction (1_B - \Sigma_{z \in \mathbb{Z}}\, a_z)$. Then $[g^{-1}, f^{-1}] = h$.

So, by Observation 2, if $f \in \mathrm{Aut}(B)$, then there are h_1, $h_2 \in \mathrm{Aut}(B)$ such that $h_1 h_2 = f$ and $\mathrm{var}(h_1)$, $\mathrm{var}(h_2) < 1_B$. If the situation in Observation 3 happens for both h_1 and h_2, then h_1, $h_2 \in \mathrm{Aut}'(B)$, and so $f \in \mathrm{Aut}'(B)$.

5.1. DEFINITION. Let h, f, g and $\{a_z \mid z \in \mathbb{Z}\}$ satisfy conditions (1), (2), and (3) of Observation 3; then $\langle f, g, \{a_z \mid z \in \mathbb{Z}\}\rangle$ is called an Anderson system for h.

Anderson used a somewhat different and more complicated system to show that an automorphism h is in the normal closure of f. However, the proof presented here is similar to Anderson's proof.

5.2. THEOREM (RUBIN 1986). *Let B be a homogeneous BA, then*

(a) $\mathrm{Aut}'(B)$ *is simple.*

(b) *Every non-trivial normal subgroup of* $\mathrm{Aut}(B)$ *contains* $\mathrm{Aut}'(B)$.

The proof of Theorem 5.2 is divided into sublemmas.

The set of good elements of $\mathrm{Aut}(B)$ is defined as follows: $\mathrm{Gd}(B) = \{h(h^{-1})^g \mid h, g \in \mathrm{Aut}(B)$ and there is $a \in B$ such that $\mathrm{var}(h) \le a$ and $g(a) \cdot a = 0$ and $a + g(a) \ne 1_B\}$. Since $h(h^{-1})^g = [h, g]$, $\mathrm{Gd}(B) \subseteq \mathrm{Aut}'(B)$. Note also that for $h(h^{-1})^g \in \mathrm{Gd}(B)$, $(h(h^{-1})^g)^{-1} = ((h^{-1})^g)^{-1}h^{-1} = h^{-1}(h^{g^{-1}}) \in \mathrm{Gd}(B)$, hence $\mathrm{Gd}(B) = \mathrm{Gd}(B)^{-1}$. It is also trivial that $\mathrm{Gd}(B)^{\mathrm{Aut}(B)} = \mathrm{Gd}(B)$.

We shall prove two facts: (1) for every $f \in \mathrm{Aut}(B) - \{Id\}$, $\mathrm{Gd}(B)$ is contained in the group generated by $f^{\mathrm{Gd}(B)}$; and (2) $\mathrm{Gd}(B)$ generates $\mathrm{Aut}'(B)$. Clearly, facts (1) and (2) imply both (a) and (b) of Theorem 5.2.

Let $\mathrm{Aut}^*(B) = \{h \in \mathrm{Aut}(B) \mid \mathrm{var}(h) \ne 1_B\}$.

5.3. PROPOSITION. *For every $f \in \mathrm{Aut}(B)$ there are $f_1, f_2 \in \mathrm{Aut}^*(B)$ such that $f = f_1 f_2$.*

PROOF. If $f = Id$, then there is nothing to prove. Otherwise, let $a \in B - \{0\}$ be such that $a \cdot f(a) = 0$ and $a + f(a) \ne 1_B$. Let f_2 be the automorphism of B extending $f \restriction a \cup f^{-1} \restriction f(a) \cup Id \restriction -(a + f(a))$, and let $f_1 = ff_2$. Hence, $f_1 f_2 = ff_2^2 = f$. $\mathrm{var}(f_2) = a + f(a) \ne 1_B$, and $f_1 \restriction f(a) = Id$, hence $\mathrm{var}(f_1) \ne 1_B$. So f_1, f_2 are as required. □

5.4. PROPOSITION. *Let B be homogeneous and $f \in \mathrm{Aut}(B) - \{Id\}$, then there is $h \in \mathrm{Gd}(B)$ such that $f(f^{-1})^h \in \mathrm{Aut}^*(B) - \{Id\}$.*

PROOF. Let $a \in B - \{0\}$ be such that $a \cdot f(a) = 0$ and $a + f(a) < \mathrm{var}(f)$. Let $b \in B - \{0\}$ be such that $b < \mathrm{var}(f) - (a + f(a))$ and $b \cdot f(b) = 0$, and let $h \in \mathrm{Gd}(B) - \{Id\}$ be such that $\mathrm{var}(h) \le b$. $\mathrm{var}(f \cdot (f^{-1})^h) = \mathrm{var}(fhf^{-1}h^{-1}) = \mathrm{var}((h^f)h^{-1})$. Note that $\mathrm{var}(h^f) = f(\mathrm{var}(h))$. Hence, $\mathrm{var}(h^f h^{-1}) \le \mathrm{var}(h^{-1}) + f(\mathrm{var}(h)) \le b + f(b)$. $(b + f(b)) \cdot f(a) = 0$, hence $\mathrm{var}(f \cdot (f^{-1})^h) \ne 1_B$. Since $b \cdot f(b) = 0$, $\mathrm{var}(h^{-1}) \le b$ and $\mathrm{var}(h^f) \le f(b)$, it follows that $\mathrm{var}(h^f h^{-1}) = \mathrm{var}(h^f) + \mathrm{var}(h^{-1}) \ne 0$. Hence, h is as required.

Let A and B be subsets of a group G, then $A \cdot B \overset{\mathrm{def}}{=} \{a \cdot b \mid a \in A, b \in B\}$, $A^2 = A \cdot A$; A^n is defined similarly.

5.5. LEMMA. *Let B be homogeneous and $f \in \mathrm{Aut}(B) - \{Id\}$. Then* (a) $\mathrm{Gd}(B) \subseteq f^{\mathrm{Aut}(B)} \cdot (f^{-1})^{\mathrm{Aut}(B)}$, *and* (b) *if, in addition,* $f \in \mathrm{Aut}^*(B)$, *then* $\mathrm{Gd}(B) \subseteq f^{\mathrm{Gd}(B)^3} \cdot (f^{-1})^{\mathrm{Gd}(B)^2}$.

PROOF. (a) Let $f \in \mathrm{Aut}(B) - \{Id\}$ and $g_0 \in \mathrm{Gd}(B)$. By the definition of $\mathrm{Gd}(B)$, there are $h, g \in \mathrm{Aut}(B)$ and $a \in B$ such that $g_0 = h(h^{-1})^g$, and $\mathrm{var}(h) \le a$, $g(a) \cdot a = 0$ and $a + g(a) \overset{\mathrm{def}}{=} a_0 \ne 1_B$. □

We prove $(*)$: There is $k \in \mathrm{Gd}(B)^2$ such that $f^k \restriction a = g \restriction a$.

PROOF. Let $c \ne 0$ be such that $c \cdot f(c) = 0$ and $-a_0 - (c + f(c)) \ne 0$. Let d_1, $d_2 < -a_0 - (c + f(c))$ be pairwise disjoint non-zero elements such that $d_1 + d_2 + c + f(c) + a_0 \ne 1_B$. We construct $k_1 \in \mathrm{Gd}(B)$ such that $k_1(c) = d_1$ and $k_1(f(c)) =$

d_2. c, $f(c)$, d_1, d_2 are pairwise disjoint non-zero elements, hence there is $l_1 \in \mathrm{Aut}(B)$ such that $\mathrm{var}(l_1) = c + f(c) + d_1 + d_2$, $l_1(c) = d_1$ and $l_1(f(c)) = d_2$. Since $\mathrm{var}(l_1) \in B - \{1_B\}$, there is $m_1 \in \mathrm{Aut}(B)$ such that $m_1(\mathrm{var}(l_1)) \cdot \mathrm{var}(l_1) = 0$. Let $k_1 = l_1(l_1^{-1})^{m_1}$. Clearly k_1 is as required.

Let $f_1 = f^{k_1}$. Hence, $f_1(d_1) = d_2$, d_1, d_2, a, $g(a)$ are pairwise disjoint and $d_1 + d_2 + a + g(a) \neq 1$. Let us see that there is $k_2 \in \mathrm{Gd}(B)$ such that $k_2(d_1) = a$ and $k_2 \restriction d_2 = (g \restriction a) \circ (k_2 \restriction d_1) \circ (f_1^{-1} \restriction d_2)$. d_1, d_2, a and $g(a)$ are pairwise disjoint non-zero elements whose sum is $<1_B$. Let l be an isomorphism between $B \restriction d_1$ and $B \restriction a$. Hence, $l' \stackrel{\mathrm{def}}{=} (g \restriction a) \circ l \circ f_1^{-1} \restriction d_2$ is an isomorphism between $B \restriction d_2$ and $B \restriction g(a)$. So there is $l_2 \in \mathrm{Aut}^*(B)$ such that $l_2 \supseteq l \cup l'$. If $e \in B - \{1_B\}$ is such that $\mathrm{var}(l_2) < e$, and $m_2 \in \mathrm{Aut}(B)$ is such that $e \cdot m_2(e) = 0$, then $k_2 \stackrel{\mathrm{def}}{=} l_2(l_2^{-1})^{m_2}$ is as required.

Let $f_2 = f_1^{k_2}$. A direct computation shows that $f_2 \restriction a = g \restriction a$. This proved $(*)$.

We can now easily prove (a). Since $g \restriction \mathrm{var}(h) = f_2 \restriction \mathrm{var}(h)$ $(h^{-1})^g = (h^{-1})^{f_2}$. Hence, $g_0 = h(h^{-1})^g = h(h^{-1})^{f_2} = hf_2h^{-1}f_2^{-1} = (f_2)^h f_2^{-1} \in f^{\mathrm{Aut}(B)} \cdot (f^{-1})^{\mathrm{Aut}(B)}$. This proves (a).

(b) Suppose that $f \in \mathrm{Aut}^*(B)$. Since f_2 is a conjugate of f, $\mathrm{var}(f_2) < 1_B$. Let $b \in B$ be such that $\mathrm{var}(f_2) \leq b < 1_B$. Let $0 < c < 1_B - b$ and $g_1 \in \mathrm{Aut}(B)$ be such that $g_1(a) = c$. Hence, $k \stackrel{\mathrm{def}}{=} h \cdot (h^{-1})^{g_1} \in \mathrm{Gd}(B)$. We compute $k \cdot (k^{-1})^{f_2}$. $k \cdot (k^{-1})^{f_2} = h \cdot (h^{-1})^{g_1} \cdot (h^{g_1})^{f_2} \cdot (h^{-1})^{f_2}$. Since $f_2 \restriction \mathrm{var}(h^{g_1}) = Id$, $(h^{g_1})^{f_2} = h^{g_1}$. Hence, the above is equal to $h \cdot (h^{-1})^{f_2} = g_0$. So $g_0 = k \cdot (k^{-1})^{f_2} = kf_2k^{-1}f_2^{-1} \in f^{\mathrm{Gd}(B)^3} \cdot (f^{-1})^{\mathrm{Gd}(B)^2}$. This proves (b). $\square$

5.6. COROLLARY. *Let B be a homogeneous BA, and $f \in \mathrm{Aut}(B) - \{Id\}$, then* $\mathrm{Gd}(B)$ *is contained in the group generated by* $f^{\mathrm{Aut}'(B)}$.

PROOF. Let $f \in \mathrm{Aut}(B) - \{Id\}$. By 5.4 there is $h \in \mathrm{Gd}(B)$ such that $f_1 \stackrel{\mathrm{def}}{=} f(f^{-1})^h \in \mathrm{Aut}^*(B) - \{Id\}$. By 5.5(b), $\mathrm{Gd}(B) \subseteq f_1^{\mathrm{Aut}'(B)} \cdot (f_1^{-1})^{\mathrm{Aut}'(B)} \subseteq ((\{f, f^{-1}\})^{\mathrm{Aut}'(B)})^4$. This proves 5.6. $\square$

5.7. LEMMA. $\mathrm{Gd}(B)$ *generates* $\mathrm{Aut}'(B)$.

PROOF. It suffices to show that every commutator belongs to $\mathrm{Gd}(B)^{12}$.

Let us first notice that if $g_1 \restriction \mathrm{var}(f) = g_2 \restriction \mathrm{var}(f)$, then $[f, g_1] = [f, g_2]$. $[f, g_1] = f \cdot (f^{-1})^{g_i}$ $i = 1, 2$. Hence, it suffices to show that $(f^{-1})^{g_1} = (f^{-1})^{g_2}$. $(f^{-1})^{g_1}(a) = b$ iff $f^{-1}(g_1^{-1}(a)) = g_1^{-1}(b)$ iff $f^{-1}(g_2^{-1}(a)) = g_2^{-1}(b)$ iff $(f^{-1})^{g_2}(a) = b$.

Claim 1. Let $f, g \in \mathrm{Aut}(B)$ and $\mathrm{var}(f) \neq 1_B$. Then $[f, g] \in \mathrm{Gd}(B)^6$.

Proof. Let us choose $a_1, \ldots, a_4 \in B - \{0\}$ such that $\mathrm{var}(f) < a_1$, $g(\mathrm{var}(f)) < a_4$ and for every $i = 1, 2, 3$, $a_i \cdot a_{i+1} = 0$ and $a_i + a_{i+1} \neq 1_B$. We first choose a_1, $a_4 \in B - \{1_B\}$ such that $\mathrm{var}(f) < a_1$ and $g(\mathrm{var}(f)) < a_4$. Then choose a_2 such that $a_1 \cdot a_2 = 0$ and $a_1 + a_2 \neq 1_B \neq a_2 + a_4$; and then a_3 can be chosen so that $a_2 \cdot a_3 = 0$, $a_3 \cdot a_4 = 0$ and both $a_2 + a_3$ and $a_3 + a_4$ are $<1_B$.

There are $g_1, g_2, g_3 \in \mathrm{Gd}(B)$ such that for $i = 1, 2, 3$, $g_i(a_i) = a_{i+1}$ and $g_3 \restriction a_3 = (g \restriction a_1) \circ ((g_2 g_1)^{-1} \restriction a_3)$. Hence, $g_3 g_2 g_1 \restriction \mathrm{var}(f) \subseteq g_3 g_2 g_1 \restriction a_1 = g \restriction a_1 \supseteq g \restriction \mathrm{var}(f)$, and so $[f, g_3 g_2 g_1] = [f, g]$.

A direct computation shows that the following is an identity in group theory:

$$[f, h_2h_1] = [f, h_2] \cdot [f, h_1]^{h_2}.$$

So $[f, g_3g_2g_1] = [f, g_3g_2] \cdot [f, g_1]^{g_3g_2} = [f, g_3] \cdot [f, g_2]^{g_3} \cdot [f, g_1]^{g_3g_2}$. Since $\mathrm{Gd}(B) = \mathrm{Gd}(B)^{-1}$, $[f, g_i] = (g_i)^f \cdot g_i^{-1} \in \mathrm{Gd}(B)^2$. Since $\mathrm{Gd}(B)^{\mathrm{Aut}(B)} = \mathrm{Gd}(B)$, $[f, g] = [f, g_3g_2g_1] \in \mathrm{Gd}(B)^6$. This proves Claim 1.

We now prove that every commutator belongs to $\mathrm{Gd}(B)^{12}$. Let $[f, g]$ be a commutator, and let $f_1, f_2 \in \mathrm{Aut}^*(B)$ be such that $f_1f_2 = f$. The following is an identity of group theory: $[f_1f_2, g] = [f_2, g]^{f_1} \cdot [f_1, g]$. By the previous case, $[f_2, g]$, $[f_1, g] \in \mathrm{Gd}(B)^6$, hence $[f_2, g]^{f_1} \in \mathrm{Gd}(B)^6$, hence $[f_1f_2, g] = [f_2, g]^{f_1}[f_1, g] \in \mathrm{Gd}(B)^{12}$. Q.E.D.

Proof of Theorem 5.2. (a) Let $f \in \mathrm{Aut}'(B) - \{Id\}$. By 5.6 and 5.7 $\mathrm{NC}(f, \mathrm{Aut}'(B) \supseteq \mathrm{Aut}'(B))$. Hence, $\mathrm{Aut}'(B)$ is simple.

(b) The above argument applied to $f \in \mathrm{Aut}(B) - \{Id\}$ shows that every non-trivial normal subgroup of $\mathrm{Aut}(B)$ contains $\mathrm{Aut}'(B)$.

We shall next deal with those classes of homogeneous BAs whose automorphism groups are known to be simple. We first wish to make some observations concerning Anderson systems.

Recall that an Anderson system for $h \in \mathrm{Aut}^*(B)$ is a triple $\langle f, g, \{a_z \mid z \in \mathbb{Z}\}\rangle$ such that: (1) $f, g \in \mathrm{Aut}(B)$, $\{a_z \mid z \in \mathbb{Z}\}$ is a set of pairwise disjoint non-zero elements of B and $\mathrm{var}(h) \leq a_0$; (2) for every $z \in \mathbb{Z}$ $f(a_z) = a_{z+1}$ and $g(a_z) = a_{z-1}$; (3) for every $z \leq 0$ $g \restriction a_z = f^{-1} \restriction a_z$ and for every $z > 0$ $g \restriction a_z = f^{-1}(h^{f^z}) \restriction a_z$; and (4) $fg \restriction (1 - \Sigma_{z\in\mathbb{Z}}\, a_z) = gf \restriction (1 - \Sigma_{z\in\mathbb{Z}}\, a_z)$.

5.8. LEMMA. *If $\langle f, g, \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h, then $h = [g^{-1}, f^{-1}]$.*

PROOF. The proof amounts to a direct computation which is left to the reader. □

5.9a. COROLLARY. *Let B be homogeneous and suppose that every $h \in \mathrm{Aut}^*(B)$ has an Anderson system. Then*

(a) $\mathrm{Aut}(B)$ *is simple.*

(b) $\mathrm{Aut}^*(B) \subseteq \mathrm{Gd}(B)^2$.

PROOF. (a) Let $h \in \mathrm{Aut}(B)$. By Proposition 5.3 there are $h_1, h_2 \in \mathrm{Aut}^*(B)$ such that $h_1h_2 = h$. Since h_1 and h_2 have Anderson systems they belong to $\mathrm{Aut}'(B)$, hence $h \in \mathrm{Aut}'(B)$. Hence, $\mathrm{Aut}(B) = \mathrm{Aut}'(B)$, and by 5.2 it is simple.

(b) Let $h \in \mathrm{Aut}^*(B)$. Hence, there is an Anderson system $\langle f, g, \{a_z \mid z \in \mathbb{Z}\}\rangle$ for h. Let $a, b \in B - \{1_B\}$ be such that $\mathrm{var}(h) < b < a$. Let $\phi\colon B \to B \restriction a$ be an isomorphism such that $\phi \restriction b = Id$. Clearly, $h \restriction a = h^\phi$, and $\langle f^\phi, g^\phi, \{\phi(a_z) \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h^ϕ. Let $f_1, g_1 \in \mathrm{Aut}(B)$ extend $f^\phi \cup (Id \restriction -a)$ and $g^\phi \cup (Id \restriction -a)$, respectively. It follows that $\langle f_1, g_1, \{\phi(a_z) \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h. Hence, we can assume that there is $a \in B - \{1_B\}$ such that $\mathrm{var}(f) + \mathrm{var}(g) \leq a < 1_B$. Let $k \in \mathrm{Aut}(B)$ be such that

$a \cdot k(a) = 0$ and $a + k(a) \neq 1_B$, and let $f_1 = f(f^{-1})^k$. So $f_1 \in \mathrm{Gd}(B)$. It is easy to see that $(f_1^{-1})^{g^{-1}} \cdot f_1 = [g^{-1}, f_1^{-1}] = [g^{-1}, f^{-1}] = h$. Since $\mathrm{Gd}(B) = (\mathrm{Gd}(B))^{-1} = \mathrm{Gd}(B)^{\mathrm{Aut}(B)}$, $(f_1^{-1})^{g^{-1}} \in \mathrm{Gd}(B)$, hence $h \in \mathrm{Gd}(B)^2$. Q.E.D.

Let us next see what could be the difficulty in constructing an Anderson system for $h \in \mathrm{Aut}^*(B)$. Let $\{a_z \mid z \in \mathbb{Z}\}$ be a family of pairwise disjoint non-zero elements such that $\mathrm{var}(h) \leq a_0$. For every $z \in \mathbb{Z}$ let f_z: $B \restriction a_z \cong B \restriction a_{z+1}$. Clearly, $\bigcup \{f_z \mid z \in \mathbb{Z}\} \cup Id \restriction (1 - \Sigma_{z \in \mathbb{Z}}\, a_z)$ extends to an automorphism f of $\bar{B}$. However, it will not happen in general that $f \in \mathrm{Aut}(B)$. Suppose that we were successful in choosing the a_z's and f_z's in such a way that f turns out to be in $\mathrm{Aut}(B)$. $g \restriction \Sigma_{a \in \mathbb{Z}}\, a_z$ is now completely determined. For let $h_z = (h \restriction a_0)^{f_z}$, then for $z > 0$ $g_z \overset{\mathrm{def}}{=} g \restriction a_z$ has to equal $f^{-1}h_z \restriction a_z$, and for $z \leq 0$ $g_z \overset{\mathrm{def}}{=} g \restriction a_z$ has to equal $f^{-1} \restriction a_z$. Certainly, $\bigcup \{g_z \mid z \in \mathbb{Z}\} \cup Id \restriction (1 - \Sigma_{z \in \mathbb{Z}}\, a_z)$ extends to an automorphism g of $\bar{B}$, but we cannot be sure in general that $g \in \mathrm{Aut}(B)$.

There is no problem of course if B is complete or even if B is just σ-complete, but in other cases we shall have to choose the a_z's and/or the f_z's more carefully.

The following is a special case of ANDERSON's [1958] theorem.

5.9b. THEOREM. *Let B be a homogeneous σ-complete BA. Then* $\mathrm{Aut}(B)$ *is simple.*

PROOF. Let $h \in \mathrm{Aut}^*(B)$. Let $\{a_z \mid z \in \mathbb{Z}\}$ be a family of pairwise disjoint non-zero elements of B such that $\mathrm{var}(h) \leq a_0$; for every $z \in \mathbb{Z}$ let f_z: $B \restriction a_z \to B \restriction a_{z+1}$ be an isomorphism. $\bigcup \{f_z \mid z \in \mathbb{Z}\} \cup Id \restriction (1 - \Sigma_{z \in \mathbb{Z}}\, a_z)$ extends to an automorphism f of B. Let $h_0 = h \restriction a_0$ and for every $z > 0$ let $h_z = (h_0)^{f_z}$. For every $z > 0$ let $g_z = f^{-1}h_z$ and for every $z \leq 0$ let $g_z = f^{-1} \restriction a_z$. $\bigcup \{g_z \mid z \in \mathbb{Z}\} \cup Id\ (1 - \Sigma_{z \in \mathbb{Z}}\, a_z)$ extends to an automorphism g of B. It is easy to see that $\langle f, g, \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h. □

5.10. THEOREM (KOPPELBERG [1981]). *If B is an infinite free algebra, then* $\mathrm{Aut}(B)$ *is simple.*

PROOF. Let $C_1 * C_2$ denote the free product of the algebras C_1 and C_2, and let B denote an infinite free algebra.

Let us first notice that if C is countable, then $B * C \cong B$. Let B_L denote the countable free algebra, then clearly $B * B_L \cong B$. $B_L * C$ and B_L are both countable and atomless, hence they are isomorphic. Hence,

$$B * C \cong (B * B_L) * C \cong B * (B_L * C) \cong B * B_L \cong B\,.$$

Let C_0 denote the interval algebra of $\langle \mathbb{Z}, < \rangle$. Every element of $B * C_0$ can be represented as $\Sigma_{i<n}\, b_i \cdot c_i$, where each b_i is a member of B and every c_i is either a finite subset of $\mathbb{Z}$ or an initial segment of $\mathbb{Z}$ or a final segment of $\mathbb{Z}$.

Let $h \in \mathrm{Aut}^*(B)$, and we wish to construct an Anderson system for h. Let $a_0 \in B - \{1_B\}$ be such that $\mathrm{var}(h) \leq a_0$. W.l.o.g. we may assume that $B = D^* C_0$, where D is a free algebra. We can further assume that $a_0 = 1_D \cdot \{0\}$. For every

$z \in \mathbb{Z}$ let $a_z = 1_D \cdot \{z\}$. The function $F(z) = z + 1$ induces an automorphism $\tilde{F}$ of C_0, and $\tilde{F}$ in turn induces an automorphism f of $D * C_0$, namely $f(\Sigma_{i<n} d_i \cdot c_i) = \Sigma_{i<n} d_i \cdot (c_i + 1)$, where $c + 1 = \{x + 1 \mid x \in c\}$.

We wish to define a function g such that $\langle f, g, \{a_z \mid z \in \mathbb{Z}\} \rangle$ will be an Anderson system for h. Indeed, since $\Sigma_{z\in\mathbb{Z}} a_z = 1_B$, g is completely determined by h, f and $\{a_z \mid z \in \mathbb{Z}\}$. However, a priori we only know that $g \in \mathrm{Aut}(\bar{B})$, and it remains to prove that $g \in \mathrm{Aut}(B)$, that is, for every $b \in B$, $g(b) \in B$. So for every $z \in \mathbb{Z}$ let $g_z = f^{-1} \restriction a_z$ if $z \leq 0$, and let $g_z = f^{-1} \circ (h^{f^z}) \restriction a_z$ if $z > 0$. Clearly, for every $z \in \mathbb{Z}$, g_z is an isomorphism between $B \restriction a_z$ and $B \restriction a_{z-1}$. Hence, $\bigcup \{g_z \mid z \in \mathbb{Z}\}$ is uniquely extendible to an automorphism g of $\bar{B}$. Let us check that for every $b \in B$, $g(b) \in B$. B is generated by the set of all elements of the form $d \cdot c$, where $d \in D$ and c is an initial segment of $\mathbb{Z}$, so it suffices to show that for every b of this form $g(b) \in B$. Let $b = d \cdot (-\infty, z)$. Recall that $\mathrm{var}(h) \leq a_0$ and $a_0 = 1_D \cdot \{0\}$, hence every member of $B \restriction a_0$ has the form $e \cdot \{0\}$, where $e \in D$. Thus, let $d_1 \in D$ be such that $h(d \cdot \{0\}) = d_1 \cdot \{0\}$. A direct computation shows that $g(b) = g(d \cdot (-\infty, z)) = d \cdot (-\infty, z_1) + d_1 \cdot [0, z-1)$, where $z_1 = \min(z - 1, 0)$. (Note that if $z \leq 1$, then $[0, z-1) = \emptyset$.) Hence, $g(b) \in B$, so $g \in \mathrm{Aut}(B)$; and so $\langle f, g, \{a_z \mid z \in \mathbb{Z}\} \rangle$ is an Anderson system for h.

This concludes the proof of the theorem. $\square$

5.11. THEOREM. (RUBIN 1986). *If B is a homogeneous subalgebra of an interval algebra, then* $\mathrm{Aut}(B)$ *is simple.*

PROOF. Let us first prove the following claim.

Claim 1. Let B be a BA and $A \subseteq B$ be a set of pairwise disjoint elements such that for every $b \in B$ either $\{a \in A \mid b \cdot a \neq 0\}$ is finite or $\{a \in A \mid (-b) \cdot a \neq 0\}$ is finite.

Then for every permutation π of A and a set $\{f_a \mid a \in A\}$ such that for every $a \in A$, f_a is an isomorphism between $B \restriction a$ and $B \restriction \pi(a)$ there is $f \in \mathrm{Aut}(B)$ extending $\bigcup \{f_a \mid a \in A\} \cup Id \restriction (1_B - \Sigma A)$.

Proof of Claim 1. Clearly, there is $f \in \mathrm{Aut}(\bar{B})$ which extends $\bigcup \{f_a \mid a \in A\} \cup Id \restriction (1_B - \Sigma A)$. We wish to show that $f \in \mathrm{Aut}(B)$. By $(*)$ B is generated by the set of all elements b such that $\{a \in A \mid a \cdot b \neq 0\}$ is finite, so it suffices to show that for every b as above $f(b) \in B$. Let $b \in B$ and suppose that $\{a \in A \mid a \cdot b \neq 0\} = \{a_i \mid i < n\}$.

$$\begin{aligned} f(b) &= \left(b - \Sigma A\right) + \Sigma \{f_a(b \cdot a) \mid a \in A\} \\ &= b - \sum_{i<n} a_i + \sum_{i<n} f_{a_i}(b \cdot a_i) \in B . \end{aligned}$$

Hence, $g \in \mathrm{Aut}(B)$ and thus Claim 1 is proved.

An infinite set A of pairwise disjoint elements which satisfies $(*)$ of Claim 1 will be called a strong Anderson system.

Claim 2. If B is homogeneous and B contains a strong Anderson system, then every $h \in \mathrm{Aut}^*(B)$ has an Anderson system.

Proof. Let $h \in \mathrm{Aut}^*(B)$. By the homogeneity of B there is a strong Anderson system $\{a_z \mid z \in \mathbb{Z}\}$ such that $\mathrm{var}(h) \leq a_0$. For every $z \in \mathbb{Z}$ let $f_z\colon B \restriction a_z \cong$

$B \upharpoonright a_{z+1}$. By Claim 1 there is $f \in \mathrm{Aut}(B)$ extending $\bigcup \{f_z \mid z \in \mathbb{Z}\} \cup Id \upharpoonright (1_B - \Sigma_{z\in\mathbb{Z}}\, a_z)$. Let $g_z = f_z^{-1} \upharpoonright a_z$ if $z \le 0$, and $g_z = f_z^{-1} \circ (h^{f^z}) \upharpoonright a_z$ if $z > 0$. Again by Claim 1, there is $g \in \mathrm{Aut}(B)$ extending $\bigcup \{g_z \mid z \in \mathbb{Z}\} \cup Id \upharpoonright (1_B - \Sigma_{z\in\mathbb{Z}}\, a_z)$. $\langle f, g, \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h, hence Claim 2 is proved.

Claim 3. If B is an infinite subalgebra of an interval algebra, then B contains a strong Anderson system.

Proof. If $|B| = \aleph_0$, then it is easy to construct inductively a_i, $i \in \omega$, such that $\{a_i \mid i \in \omega\}$ is a strong Anderson system. So we assume that B is uncountable. Let $\langle X, <\rangle$ be a linear ordering, C be the inverval algebra of $\langle X, <\rangle$ and B be an uncountable subalgebra of C. For $Y, Z \subseteq X$ let $Y < Z$ mean that for every $y \in Y$ and every $x \in Z$, $y < z$. Every $c \in C$ can be represented uniquely in the form $c = \bigcup \{I_i(c) \mid i < k(c)\}$, where each $I_i(c)$ is a left closed right open interval (including intervals of the forms $[-\infty, x) \overset{\text{def}}{=} \{y \in X \mid y < x\}$, $[x, \infty)$ and $[-\infty, \infty)$), and for every $i < k(c) - 1$ $I_i(c) < I_{i+1}(c)$ and $I_i(c) \cup I_{k+1}(c)$ is not an interval.

For $c \in C$ let $X(c) = \{x_l(c) \mid l < m(c)\}$ be the set of all $x \in X \cup \{-\infty, \infty\}$ which are endpoints of the intervals $\{I_j(c) \mid j < k(c)\}$. A set $D \subseteq C$ is called semi-uniform if: (1) there is m such that for every $d \in D$ $m(d) = m$; and (2) for every $d_1, d_2 \in D$ $\{-\infty, \infty\} \cap X(d_1) = \{-\infty, \infty\} \cap X(d_2)$. We denote m by $m(D)$. If D is semi-uniform, then there is $k \in \omega$ such that for every $d \in D$, $k(d) = k$. We denote this number by $k(D)$.

$A = \{a_i \mid i \in \omega\}$ is called uniform if: (1) A is semi-uniform, for every $i \in \omega$ $a_i \ne 0$ and for every $i \ne j$ $a_i \cdot a_j = 0$; and (2) for every l, $m < k(A)$ and natural numbers $i_1 < j_1$ and $i_2 < j_2$: $I_l(a_{i_1}) < I_m(a_{j_1})$ iff $I_l(a_{i_2}) < I_m(a_{j_2})$.

We say that A is a minimal uniform set if it is uniform and for every infinite semi-uniform set of pairwise disjoint elements A', $m(A) \le m(A')$.

We divide the claim that B contains a strong Anderson system into two subclaims.

Claim 4. B contains a uniform set.

Claim 5. A minimal uniform set is a strong Anderson system.

Proof of Claim 4. We first show that B contains an infinite semi-uniform set E of pairwise disjoint elements. Since B is uncountable, B contains an infinite semi-uniform set D. Let $\{d_i \mid i \in \omega\}$ be a one-to-one enumeration of D. Let n be such that $2^n > n \cdot m(D) + 1$. We claim that every subset of D of power n is dependent. For assume $e_0, \ldots, e_{n-1}$ are distinct elements of D, then $|\bigcup_{i<n} X(e_i)| \le n \cdot m(D)$, hence the Boolean algebra E generated by $\{e_0, \ldots, e_{n-1}\}$ has at most $n \cdot m(D) + 1$ atoms. However, if $\{e_0, \ldots, e_{n-1}\}$ were independent, then E would have had 2^n atoms. Since $2^n > n \cdot m(D) + 1$, $\{e_0, \ldots, e_{n-1}\}$ cannot be independent. So we have proved that every n elements of D are dependent. (*Remark*: The above argument is due to Koppelberg. It shows that for every semi-uniform set D there is $n \in \omega$ such that D is n-dependent.)

We next show that by possibly renaming n and D, it can be assumed that every distinct $n - 1$ elements of D are independent. We color the n-element subsets of D by the colors $\{2, \ldots, n\}$:

$$\mathrm{color}(\sigma) = \min(\{k \mid \sigma \text{ has a dependent subset of power } k\})\,.$$

By Ramsey's theorem there is an infinite $D' \subseteq D$ and n' such that for every

n-element subset σ of D color$(\sigma) = n'$. It is clear that every $n'-1$ distinct elements of D' are independent. So we rename D' and call it D and denote n' by n.

Let τ be a term of the following form $\tau = \Pi_{l<n} t_l$ and for every $l<n$ either $t_l = x_l$ or $t_l = -x_l$. We call such a term a good term. We know that for every $i_0 < \cdots < i_{n-1}$ there is a good term $\tau_{i_0,\ldots,i_{n-1}}(x_0,\ldots,x_{n-1})$ such that $\tau_{i_0,\ldots,i_{n-1}}(d_{i_0},\ldots,d_{i_{n-1}}) = 0$. So, by Ramsey theorem we can assume that $\tau_{i_0,\ldots,i_{n-1}} \overset{\text{def}}{=} \tau$ does not depend on $i_0,\ldots,i_{n-1}$.

If $\tau = x_0 - x_1$, then D is a strictly increasing chain. In this case let $E' = \{d_{i+1} - d_i \mid i \in \omega\}$. Hence, E' is an infinite set of pairwise disjoint elements and for every $e \in E'$ $m(e) \le 2m(D)$. Hence, E' contains an infinite semi-uniform set E of pairwise disjoint elements.

If $\tau = -x_0 \cdot x_1$, then D is a strictly decreasing chain. E' is then defined as $\{d_i - d_{i+1} \mid i \in \omega\}$ and E' contains an infinite semi-uniform set of pairwise disjoint elements.

Let $\tau = \Pi_{l<n} \varepsilon_l x_l$, where $\varepsilon_l \in \{-,+\}$ and $+x_l$ denotes x_l. If $\tau \ne x_0 - x_1$ and $\tau \ne -x_0 \cdot x_1$, then either there is $l>0$ such that $\varepsilon_0 = \varepsilon_l$ or there is $l<n-1$ such that $\varepsilon_{n-1} = \varepsilon_l$. In the first case let $\tau' = \Pi_{l<n-1} \varepsilon_{l+1} x_l$ and in the second case let $\tau' = \Pi_{l<n-1} \varepsilon_l x_l$.

Let $E' = \{\tau'(d_{i(n-1)},\ldots,d_{i(n-1)+n-2}) \mid i \in \omega\}$. It is easy to see that E' is an infinite set of pairwise disjoint elements, and clearly for every $e \in E'$, $m(e) \le (n-2)m(D)$. Hence, E' contains an infinite semi-uniform set E of pairwise disjoint elements.

We have thus proved the existence of a set E as required.

For every l, $m < k(E)$ and every distinct $e', e'' \in E$ $I_l(e') \cap I_m(e'') = \emptyset$ and so either $I_l(e') < I_m(e'')$ or $I_m(e'') < I_l(e')$. By Ramsey's theorem, E contains a uniform set. This concludes the proof of Claim 4.

The above argument also shows that every infinite semi-uniform set of pairwise disjoint elements contains an infinite uniform set. Hence, if we choose an infinite semi-uniform set A of pairwise disjoint elements with $m(A)$ minimal, and apply the above argument, we get a minimal uniform set.

Proof of Claim 5. Let $A = \{a_i \mid i \in \omega\}$ be a minimal uniform set. Let $b = \bigcup_{l<n} [x_{2l}, x_{2l+1}) \in B$. Since A is a set of pairwise disjoint elements, there is i_0 such that for every $i > i_0$, $\{x_0,\ldots,x_{2n-1}\} \cap a_i = \emptyset$. Hence, for every $i > i_0$ and $l < k(A)$ either $I_l(a_i) \subseteq b$ or $I_l(a_i) \cap b = \emptyset$. If there are infinitely many i's for which both $\{l < k(A) \mid I_l(a_i) \subseteq b\}$ and $\{l < k(A) \mid I_l(a_i) \cap b = \emptyset\}$ are non-empty, then $\{b \cdot a_i \mid i \in \omega\}$ contains an infinite semi-uniform set A' such that $m(A') < m(A)$, which is impossible by the minimality of A.

Hence, there is $i_1 > i_0$ such that for every $i > i_1$ either $a_i \le b$ or $a_i \cdot b = 0$. Since for every $l < k(A)$, $\langle \{I_l(a_i) \mid i \in \omega\}, < \rangle$ is either of order type ω or of order type ω^*, there is $i_2 > i_1$ such that for every $j_1, j_2 > i_2$ every $l < k(A)$ and $m < 2n+1$ $x_m < I_l(a_{j_1})$ iff $x_m < I_l(a_{j_2})$. It follows that either for every $i > i_2$ $a_i \cdot b = 0$ or for every $i > i_2$ $a_i \subseteq b$. Hence, either $\{a \in A \mid b \cdot a \ne 0\}$ is finite or $\{a \in A \mid (-b) \cdot a \ne 0\}$ is finite, so A is a strong Anderson system. This proves Claim 5.

We have thus proved that every infinite subalgebra of an interval algebra contains a strong Anderson system (Claim 3). We have also proved that every homogeneous BA B which contains a strong Anderson system, contains an Anderson system for every $h \in \mathrm{Aut}^*(B)$. Since by Lemma 5.8 the latter implies the simplicity of $\mathrm{Aut}(B)$, Theorem 5.11 is proved.

The last simplicity result is concerned with saturated BAs. □

5.12. THEOREM (FUCHINO [1985]). *If B is an atomless saturated BA, then* $\mathrm{Aut}(B)$ *is simple.*

PROOF. It suffices to prove the claim for uncountable BAs. Hence, we will show that if B is an uncountable atomless saturated BA and $h \in \mathrm{Aut}^*(B)$, then h has an Anderson system.

We shall first state two sublemmas, then show how the theorem follows from the lemmas, and then prove them.

5.13. LEMMA. *Let B be a BA of power $\lambda \geq \aleph_0$. Let $A \subseteq B$ be a set of pairwise disjoint elements such that $|A| < \lambda$ and for every $a \in A$, $B \upharpoonright a$ is atomless and saturated. Let $L \subseteq \mathrm{Aut}(B)$ be such that $|L| \leq \lambda$ and for every $f \in L$ and $a \in A$, $f(a), f^{-1}(a) \in A$. Then there is a BA B^* and for every $f \in L$ there is $f^* \in \mathrm{Aut}(B^*)$ such that:* (1) *B is a subalgebra of B^* and for every $a \in A$ $B^* \upharpoonright a = B \upharpoonright a$, and B^* is atomless saturated and has power λ;* (2) *for every $f \in L$, $f^* \supseteq f$; and* (3) *let $L^* = \{f^* \mid f \in L\}$, $M = (B, L)$ and $M^* = (B^*, L^*)$, that is, the language of both M and M^* contains in addition to the Boolean algebraic operations function symbols $\{\hat{f} \mid f \in L\}$, $\hat{f}$ is interpreted as f in M, and as f^* in M^*. Then for every universal formula ϕ in the language of M and every $a_1, \ldots, a_n \in M$, $M \models \phi[a_1, \ldots, a_n]$ iff $M^* \models \phi[a_1, \ldots, a_n]$.*

5.14. LEMMA. *Let B be a BA and $h \in \mathrm{Aut}(B)$. Then there is a BA B', f', $g' \in \mathrm{Aut}(B')$ and $\{a_z \mid z \in \mathbb{Z}\} \subseteq B'$ such that $B = B' \upharpoonright a_0$ and $\langle f', g', \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for the automorphism h' of B' which extends $h \cup (Id \upharpoonright -a_0)$.*

Continuation of the proof of 5.12. We assume Lemmas 5.13 and 5.14. Let B be a saturated atomless BA of power $\lambda > \aleph_0$, and let $h \in \mathrm{Aut}^*(B)$. Let $a \in B - \{0_B, 1_B\}$ be such that $\mathrm{var}(h) \leq a$. Let $B_1 = B \upharpoonright a$, and let B_1', f', g' and $\{a_z \mid z \in \mathbb{Z}\}$ be as assured by 5.14 for B_1 and $h \upharpoonright B_1$. Let $h' \in \mathrm{Aut}(B')$ be such that $h' \supseteq h \upharpoonright B_1 \cup (Id \upharpoonright -a_0)$. Let $A = \{a_z \mid z \in \mathbb{Z}\}$ and $L = \{h', f', g'\}$. Clearly, B_1', A and L satisfy the conditions of 5.13, so let B_1^* and $L^* = \{h^*, f^*, g^*\}$ be as assured by Lemma 5.13.

Let us first check that $\langle f^*, g^*, \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h^*.

Let $M = (B_1', L)$ and $M^* = (B_1^*, L^*)$. The definition of an Anderson system is stated explicitly before Lemma 5.8. Properties (1) and (2) in that definition follow from the fact that M is a submodel of M^* and that $\langle f', g', \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h' in M. Property (3) in the above definition can be expressed by a universal formula $\phi_-(a_z)$ for $z \leq 0$, and by another universal formula $\phi_+(a_z)$ for $z > 0$. Since for every $z \in \mathbb{Z}$, $a_z \in M$, and since $M \models \phi_\pm[a_z]$ implies that $M^* \models \phi_\pm[a_z]$, property (3) holds also in B_1^*.

(4) M satisfies the universal formula saying that $\mathrm{var}([g^{-1}, f^{-1}]) \leq a_0$, hence so does M^*. Combined with propety (2) this fact implies that $fg \upharpoonright (1 - \Sigma_{z\in\mathbb{Z}} a_z) = gf \upharpoonright (1 - \Sigma_{z\in\mathbb{Z}} a_z)$. Thus, property (4) holds in B_1^*. So $\langle f^*, g^*, \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h^* in B_1^*.

Next, we see that the models (B, h) and (B_1^*, h^*) are isomorphic. This will clearly imply that B contains an Anderson system for h. $(B \upharpoonright a, h \upharpoonright a) = (B_1, h \upharpoonright a) = (B_1' \upharpoonright a_0, h' \upharpoonright a_0) = (B_1^*, h^* \upharpoonright a_0)$. Let ϕ be an isomorphism between $B \upharpoonright -a$ and $B_1^* \upharpoonright -a_0$. Such a ϕ exists since both BAs are atomless saturated and have the same cardinality. Let $\tilde{\phi}$ be an isomorphism between B and B_1^* extending $(Id \upharpoonright a) \cup \phi$. Since $h \upharpoonright -a = Id$ and since $h^* \upharpoonright -a_0 = Id$, $\tilde{\phi}$ is an isomorphism between (B, h) and (B_1^*, h^*). Thus, assuming Lemmas 5.13 and 5.14 we have proved Theorem 5.12.

Before we turn to the proof of 5.13, let us mention without proof some basic facts to be used later. We believe the reader will find it easy to verify these facts.

(1) The theory of atomless BAs admits elimination of quantifiers, that is, for every first-order formula $\phi(x_1, \ldots, x_n)$ in the language of BAs there is a formula $\phi^*(x_1, \ldots, x_n)$ without quantifiers such that for every atomless BA B and $a_1, \ldots, a_n \in B$, $B \models \phi[a_1, \ldots, a_n]$ iff $B \models \phi^*[a_1, \ldots, a_n]$. More explicitly: if B is an atomless BA, $\{a_1, \ldots, a_n\} \subseteq B$ is a partition of unity, and $b_1, b_2 \in B$; then the following are equivalent: (a) for every first-order formula $\phi(x_1, \ldots, x_n, y)$, $B \models \phi[a_1, \ldots, a_n, b_1]$ iff $B \models \phi[a_1, \ldots, a_n, b_2]$; (b) for every $i = 1, \ldots, n$, $b_1 \cdot a_i = 0$ iff $b_2 \cdot a_i = 0$ and $b_1 \geq a_i$ iff $b_2 \geq a_i$.

(2) If B is a saturated BA of power $\lambda \geq \aleph_0$, then $\lambda^{<\lambda} \overset{\text{def}}{=} \Sigma_{\mu<\lambda} \lambda^\mu = \lambda$.

(3) Let B be an atomless BA, D be a subset of B, and Φ be a complete type over D. Let A be the subalgebra of B generated by D. Then Φ is equivalent to a type Ψ of the following form:

$$\Psi = \{v_0 \geq a \mid a \in A_1\} \cup \{v_0 \cdot a = 0 \mid a \in A_2\}$$
$$\cup \{a \cdot v_0 \neq 0 \neq a - v_0 \mid a \in A_3\},$$

where $A_1 \cup A_2 \cup A_3 = A$.

(3) follows easily from the explicit form of (1).

Proof of Lemma 5.13. Let B, A and L be as in Lemma 5.13. Let $M = (B, L)$. Let $\tilde{M} = (\tilde{B}, \tilde{L})$ be a λ-compact elementary extension of M. (λ compactness means that every consistent type with $<\lambda$ formulas is realized.) For every $f \in L$ let $\tilde{f} \in \tilde{L}$ be the automorphism of $\tilde{M}$ extending f.

Since $M < \tilde{M}$, it follows that for every model M^* such that $M \subseteq M^* \subseteq \tilde{M}$ (inclusion stands for being a submodel), for every universal formula ϕ and $a_1, \ldots, a_n \in |M|$, $M \models \phi[a_1, \ldots, a_n]$ iff $M^* \models \phi[a_1, \ldots, a_n]$. Hence, it suffices to find $M^* = (B^*, L^*)$ such that: (1) $M \subseteq M^* \subseteq \tilde{M}$; (2) B^* is saturated; and (3) for every $a \in A$, $B \upharpoonright a = B^* \upharpoonright a$.

We define by induction on $i \leq \lambda$ a continuous increasing chain of models $M_i = (B_i, L_i) \subseteq \tilde{M}$ such that the following hold: (1) $M_0 = M$ and (2) for every i and $a \in A$, $B_i \upharpoonright a = B \upharpoonright a$. Let $\{s_i \mid i < \lambda\}$ be an enumeration of all consistent types in the language of BAs of power less than λ with parameters in $\tilde{B}$ such that for every $i < \lambda$, $\{j \mid s_j = s_i\}$ is unbounded in λ. (The above enumeration exists, since $\lambda^{<\lambda} = \lambda$.)

Let $i < \lambda$ and suppose that M_i has been defined. If the set of parameters of s_i is not contained in $|M_i|$, let $M_{i+1} = M_i$. Otherwise, s_i can be extended to a consistent type s with parameters in M_i such that $|s| < \lambda$, every $a \in A$ is a parameter in s, and s has the following form:

$$s = \{c \le v_0 \mid c \in C_1\} \cup \{v_0 \cdot c = 0 \mid c \in C_2\} \\ \cup \{c \cdot v_0 \ne 0 \ne c - v_0 \mid c \in C_3\}\,.$$

For $l = 1, 2, 3$ let $A_i = A \cap C_i$. For every $a \in A_1$ let $d_a = a$, and for every $a \in A_2$ let $d_a = 0$. For every $a \in A_3$ let s_a be the following type:

$$s_a = \{a \cdot c \le v_0 \mid c \in C_1\} \cup \{v_0 \cdot a \cdot c = 0 \mid c \in C_2\} \\ \cup \{a \cdot c \cdot v_0 \ne 0 \ne a \cdot c - v_0 \mid c \in C_3\}\,.$$

By the induction hypothesis on M_i and since s is a type with parameters in M_i for every $a \in A$ and $c \in C_1 \cup C_2 \cup C_3$ $a \cdot c \in B \restriction a$. Hence, s_a is a type with parameters in $B \restriction a$. If e realizes s in $\tilde{B}$, then $e \cdot a$ realizes s_a in $B \restriction a$, hence s_a is consistent. Since $|s_a| < \lambda$ and $B \restriction a$ is saturated of power λ, then there is $d_a \in B \restriction a$ which realizes s_a. Let $s' = s \cup \{v_0 \cdot a = d_a \mid a \in A\}$. We show that s' is consistent. Let $e \in \tilde{B}$ and e realizes s, and let $a_0, \ldots, a_{n-1} \in A$, then $(e - \Sigma_{i<n} a_i) + \Sigma_{i<n} d_{a_i}$ realizes $s \cup \{v_0 \cdot a_i = d_{a_i} \mid i < n\}$. Hence, s' is consistent. Since $|s'| < \lambda$ and $\tilde{B}$ is saturated of power λ, there is $e \in \tilde{B}$ which realizes s'. Hence, for every $a \in A$, $e \cdot a = d_a \in B \restriction a$. Let B_{i+1} be the smallest subalgebra of $\tilde{B}$ containing $B_i \cup \{e\}$ and closed under every member of $\tilde{L} \cup \tilde{L}^{-1}$, and let M_{i+1} be the submodel of $\tilde{M}$ whose universe is B_{i+1}. We show that induction hypothesis (2) holds. Let B_i' be the subalgebra of $\tilde{B}$ generated by $B_i \cup \{e\}$. Then clearly for every $a \in A$, $B_i' \restriction a = B \restriction a$. B_{i+1} is generated by $B_i' \cup \{g(b) \mid b \in B_i', g \in \tilde{L} \cup \tilde{L}^{-1}\}$. Hence, it suffices to show that for every $a \in A$, $b \in B_i'$ and $g \in \tilde{L} \cup \tilde{L}^{-1}$, $a \cdot g(b) \in B \restriction a$. Let $a \in A$, $b \in B_i'$ and $g \in \tilde{L} \cup \tilde{L}$. If $f \in L^{-1}$, then $\tilde{f}$ denotes $(\widetilde{f^{-1}})^{-1}$. Hence, there is $f \in L \cup L^{-1}$ such that $g = \tilde{f}$. So $\tilde{f} \restriction B = f \restriction B$, and so $a' \overset{\text{def}}{=} \tilde{f}^{-1}(a) \in A$. Hence, $a \cdot g(b) = a \cdot \tilde{f}(b) = \tilde{f}(\tilde{f}^{-1}(a) \cdot b) = \tilde{f}(a' \cdot b)$. Since $b \cdot a' \in B_i'$ $b \cdot a' \in B$. Hence, $\tilde{f}(a' \cdot b) = f(a' \cdot b) \in B$, and hence $a \cdot g(b) \in B \restriction a$. So induction hypothesis (2) holds. Let $B^* = B_\lambda$ and $L^* = L_\lambda$, then B^*, L^* are as required in the lemma. This concludes the proof of Lemma 5.13. □

Proof of Lemma 5.14. Let B and h be as in 5.14, and B' be the countable weak power of B. So w.l.o.g. there is a pairwise disjoint family $\{a_z \mid z \in \mathbb{Z}\} \subseteq B'$ such that $B' \restriction a_0 = B$, for every $z \in \mathbb{Z}$, $B \restriction a_z \cong B$, and B' is generated by $\bigcup \{B' \restriction a_z \mid z \in \mathbb{Z}\}$. These facts imply that $\{a_z \mid z \in \mathbb{Z}\}$ is a strong Anderson system in B'. Let $h' \in \operatorname{Aut}(B')$ extend $h \cup (Id \restriction -a_0)$, hence $\operatorname{var}(h') \le a_0$. Just as in the proof of Claim 2 in Theorem 5.11 this implies that there are $f', g' \in \operatorname{Aut}(B')$ such that $\langle f', g', \{a_z \mid z \in \mathbb{Z}\}\rangle$ is an Anderson system for h'. This proves Lemma 5.14. Hence the proof of Theorem 5.12 is complete. □

There is one type of question that we have neglected mentioning until now, that is, counting the number of conjugates of f and f^{-1} needed in the generation of a member of the normal closure of f. The proofs yield such a number but we do not know if and when it is the minimal possible.

5.15. COROLLARY. *If B is homogeneous and every $h \in \mathrm{Aut}^*(B)$ has an Anderson system, then for every $f \in \mathrm{Aut}(B) - \{Id\}$ and $g \in \mathrm{Aut}(B)$: g is the product of eight members of $(\{f, f^{-1}\})^{\mathrm{Aut}(B)}$, and g is the product of* 16 *conjugates of f.*

PROOF. Let B, f and g be as above and suppose first that $g \in \mathrm{Aut}^*(B)$. By 5.5(b), g is a product of two members g_1 and g_2 of $\mathrm{Gd}(B)$. By 5.5(a) both g_1 and g_2 belong to $f^{\mathrm{Aut}(B)} \cdot (f^{-1})^{\mathrm{Aut}(B)}$. Hence, $g \in (\{f, f^{-1}\}^{\mathrm{Aut}(B)})^4$. Since by 5.3 every $g \in \mathrm{Aut}(B)$ is a product of two members of $\mathrm{Aut}^*(B)$, $\mathrm{Aut}(B) \subseteq (\{f, f^{-1}\}^{\mathrm{Aut}(B)})^8$. This proves the first part of 5.15.

To prove the second part of 5.15, let us notice that if in 5.5(a) there is a non-zero element b such that $b \cdot f(b) = 0$ and $f^2 \restriction b = Id$, then there is $k \in \mathrm{Gd}(B)^2$ such that $f^k \restriction a = g \restriction a$ and $f^k \restriction g(a) = g^{-1} \restriction g(a)$. Hence, $g_0 = h(f^k h^{-1} f^k) = (f^k)^h f^k$.

It is easy to see that for every $f \in \mathrm{Aut}(B) - \{Id\}$ there is a conjugate f_1 of f such that $f_1 f$ has a b as above, that is, $b \neq 0$, $b \cdot f_1 f(b) = 0$ and $(f_1 f)^2 \restriction b = Id$. So, for every $g_0 \in \mathrm{Gd}(B)$ and $f \in \mathrm{Aut}(B) - \{Id\}$, g_0 is a product of four conjugates of f. The rest of the proof of 5.15 remains unchanged. □

We next turn to two non-simplicity results. We first present Koppelberg's theorem that assuming CH there is a homogeneous BA with a non-simple automorphism group, and then we show van Douwen's result that in a model of ZFC constructed by Shelah, $\mathrm{Aut}(P(\omega)/\mathit{fin})$ is not simple.

5.16. THEOREM (KOPPELBERG [1985]). (*CH*) *There is a homogeneous BA C such that* $\mathrm{Aut}(C)$ *is not simple.*

We start with some notations and definitions. Let B_0 denote the countable atomless Boolean algebra, and $\bar{B}_0$ be its completion. The Boolean algebra C that we construct will satisfy $B_0 \subseteq C \subseteq \bar{B}_0$.

DEFINITION. Let A be a subset of $\mathrm{Aut}(B)$ and $g \in \mathrm{Aut}(B)$, we say that g is piecewise in A if there is a partition $\{a_1, \ldots, a_n\}$ of 1_B and $f_1, \ldots, f_n \in A$ such that for every $i = 1, \ldots, n$, $g \restriction a_i = f_i \restriction a_i$. Let $\mathrm{PW}(A) = \{g \in \mathrm{Aut}(B) \mid g$ is piecewise in $A\}$. Clearly, if A is a subgroup of $\mathrm{Aut}(B)$, then so is $\mathrm{PW}(A)$.

Let $\mathrm{Aut}^+(B) = \{f \in \mathrm{Aut}(B) \mid \mathrm{var}(f) = 1_B\}$. A subgroup G of $\mathrm{Aut}(B)$ is called a good subgroup if $G \subseteq \mathrm{Aut}^+(B)$.

5.17. LEMMA. *If F is a good, free subgroup of* $\mathrm{Aut}(B)$, *and $F \neq \{Id\}$, then* $\mathrm{PW}(F)$ *is not simple.*

5.18. LEMMA. (*CH*) *There is a homogeneous BA B and a good, free subgroup F of* $\mathrm{Aut}(B)$ *such that* $\mathrm{Aut}(B) = \mathrm{PW}(F)$.

Clearly 5.17 and 5.18 imply Theorem 5.16.

The question whether Lemma 5.17 holds without assuming that F is free is open.

Proof of Lemma 5.17. Let Q be a set of free generators for F, and let $G = \mathrm{PW}(F)$. If $f \in F$, $a \in B$ and $a \cdot f(a) = 0$, let $t_{f,a}$ denote the automorphism extending $(f \upharpoonright a) \cup (f^{-1} \upharpoonright f(a)) \cup Id \upharpoonright -(a + f(a))$. Let $T \stackrel{\mathrm{def}}{=} \{t_{q,a} \mid q \in Q,\ a \in B$ and $a \cdot q(a) = 0\}$, and N be the normal closure of T in G. Note that $t_{f,a} = (t_{f,a})^{-1}$, hence every member of N is a product of conjugates of members of T.

We prove that $Q \cap N = \emptyset$, thus showing that N is a proper subgroup of G. Suppose by contradiction that $q_0 \in Q \cap N$. Let $q_0 = (t_{q_1,a_1})^{g_1} \circ \cdots \circ (t_{q_n,a_n})^{g_n}$, where $q_1, \ldots, q_n \in Q$ and $g_1, \ldots, g_n \in G$. For every $i = 1, \ldots, n$ there are $f_{i,0}, \ldots, f_{i,l_i-1} \in F$ and a partition of unity $\{a_{i,0}, \ldots, a_{i,l_i-1}\} \subseteq B$ such that g_i extends $\bigcup_{j<l_i} f_{i,j} \upharpoonright a_{i,j}$. Each $f_{i,j}$ can be represented as a product of members of $Q \cup Q^{-1}$. Let $Q_{i,j}$ be the minimal subset of Q such that $f_{i,j}$ is a product of members of $Q_{i,j} \cup (Q_{i,j})^{-1}$. Let $Q_0 = \{q_0\} \cup \bigcup \{Q_{i,j} \mid i = 1, \ldots, n,\ j < l_i\}$, F_0 be the subgroup of F generated by Q_0 and $G_0 = \mathrm{PW}(F_0)$. Clearly, F_0 is countable. Let X denote the Stone space of B. We can regard every $f \in \mathrm{Aut}(B)$ as a homeomorphism of X; for $x \in X$, $f(x)$ is defined to be $\{f(a) \mid a \in x\}$. Since for every $f \in F_0$, $\mathrm{var}(f) = 1_B$, for every distinct $f_1, f_2 \in F_0 \{x \in X \mid f_1(x) \neq f_2(x)\}$ is an open dense subset of X. By the Baire category theorem and the countability of F_0 there is $x_0 \in X$ such that for every distinct $f_1, f_2 \in F_0$, $f_1(x_0) \neq f_2(x_0)$. Let $A_0 = \{f(x_0) \mid f \in F_0\}$. G_0 can be regarded as a permutation group of A_0. For every $f \in F_0$ let w_f be the reduced word which represents f as a product of members of $Q_0 \cup (Q_0)^{-1}$. For every $f \in F_0$ let $x_f = f(x_0)$. Hence, $f \mapsto x_f$ is a one-to-one correspondence between F_0 and A_0 and for every $f, g \in F_0$, $g(x_f) = x_{gf}$. Let $D_0 = \{x_f \mid f \in F_0$ and the last element in w_f is $q_0\}$, and let $D_1 = A_0 - D_0$. For every $q \in Q_0 - \{q_0\}$ $q(D_0) = D_0$ and $q(D_1) = D_1$; the only member x of D_1 such that $q_0(x) \in D_0$ is $x_{Id} = x_0$; also $q_0(D_0) \subseteq D_0$. For every $g \in G_0$ and $l \in \{0, 1\}$ let $m_l(g) = |\{x \in D_l \mid g(x) \in D_l\}|$. Since for every $q \in Q_0$ and $l \in \{0, 1\}$ $m_l(q)$ is finite this also holds for every $f \in F_0$, and since $G_0 = \mathrm{PW}(F_0)$, the same holds for every $g \in G_0$. For every $g \in G_0$ let $m(g) = m_1(g) - m_0(g)$. We shall see that for every $g_1, g_2 \in G_0$, $m(g_2 g_1) = m(g_2) + m(g_1)$ thus showing that m is a homomorphism from G_0 to $\mathbb{Z}$.

For $g \in G$ and $l = 0, 1$, let $M_l(g) = \{x \in D_l \mid g(x) \in D_{1-l}\}$. Let $g_1, g_2 \in G_0$. Let $A^0 = \bigcup_{l=0}^{1} M_l(g_1) \cup g_1^{-1}(\bigcup_{l=0}^{1} m_l(g_2)) \cup \bigcup_{l=0}^{1} M_l(g_2 g_1)$. Note that A^0 is finite, and that $A^1 \stackrel{\mathrm{def}}{=} g_1(A^0) \supseteq \bigcup_{l=0}^{1} M_l(g_2)$. Let $A^2 = g_2(a^1)$. For $l = 0, 1$ and $i = 0, 1, 2$ let $D_l^i = D_l \cap A^i$. Since $A^0 \supseteq \bigcup_{l=0}^{1} m_l(g_1)$, $m(g_1) = m_1(g_1) - m_0(g_1) = |D_1^0| - |D_1^1|$. Similarly, since $A^1 \supseteq \bigcup_{l=0}^{1} M_l(g_2)$, $m(g_2) = |D_1^1| - |D_1^2|$. Again, since $A^0 \supseteq \bigcup_{l=0}^{1} M_l(g_2 g_1)$, $m(g_2 g_1) = |D_1^0| - |g_2 g_1(A^0) \cap D_1| = |D_1^0| - |A^2 \cap D_1| = |D_1^0| - |D_1^2|$. So $m(g_2 g_1) = m(g_2) + m(g_1)$.

If $g \in G_0$ and for some $k > 0$, $g^k = Id$, then $g \in \ker(m)$. Let us now return to the original assumption that $q_0 = (t_{q_1,a_1})^{g_1} \circ \cdots \circ (t_{q_n,a_n})^{g_n}$. The $(t_{q_i,a_i})^{g_i}$'s have order 2, hence they belong to $\ker(m)$. However, by our computation $m(q_0) = 1$, so $q_0 \not\in \ker(m)$. This contradiction proves the lemma.

Proof of Lemma 5.18. We construct by induction on $i \leq \aleph_1$ an increasing continuous chain of BAs $\{B_i \mid i \leq \aleph_1\}$ such that B_0 is the countable atomless BA, and for every $i \leq \aleph_1$, $B_i \subseteq \bar{B}_0$. For every $i < \aleph_1$ we shall define a countable good, free subgroup F_i of $\mathrm{Aut}(B_i)$. The desired group G will be $\bigcup_{i<\aleph_1} F_i$. Also, for every $i < \aleph_1$ we shall define a subset D_i of $\bar{B}_0 - B_i$. Note that by CH for every $i < \aleph_1$, B_i is atomless and countable. We regard $\mathrm{Aut}(B_i)$ as a subgroup of $\mathrm{Aut}(\bar{B}_0)$. Let $\{s_i \mid i < \aleph_1\}$ be an enumeration of $\bar{B}_0 \times \bar{B}_0 \cup \mathrm{Aut}(\bar{B}_0)$, and for every $i < \aleph_1$, $|\{j \mid s_j = s_i\}| = \aleph_1$. If $s_i = \langle a, b\rangle \in B_i \times B_i$ we regard s_i as the task of adding an automorphism g to F_i such that $g(a) = b$. If $s_i \in \mathrm{Aut}(B_i)$ and $s_i \not\in \mathrm{PW}(F_i)$, we regard s_i as the task of extending B_i to BA B such that s_i does not extend to an automorphism of B. So we need two lemmas to assure that the two types of tasks can be fulfilled.

5.19. LEMMA. *Let F be a countable good subgroup of* $\mathrm{Aut}(B_0)$, *and let* $h \in \mathrm{Aut}(B_0) - \mathrm{PW}(F)$. *Let $D \subseteq \bar{B}_0 - B_0$ be countable. Then there is $e \in \bar{B}_0$ such that if B is the smallest subalgebra of $\bar{B}_0$ containing $B_0 \cup \{e\}$ and closed under F, then B is disjoint from $D \cup \{h(e)\}$.*

5.20. LEMMA. *Let F be a countable good subgroup of* $\mathrm{Aut}(B_0)$ *and* $a, b \in B_0 - \{0_B, 1_B\}$. *Then there is $g \in \mathrm{Aut}(B_0)$ such that*: (1) $g(a) = b$, (2) *the subgroup G of* $\mathrm{Aut}(B_0)$, *generated by $F \cup \{g\}$ is good*; *and* (3) *let $F^*\mathbb{Z}$ denote the free product of F and the integers, then the homomorphism ϕ: $F^*\mathbb{Z} \to G$ which is the identity on F and sends $1_{\mathbb{Z}}$ to g is an isomorphism between $F^*\mathbb{Z}$ and G.*

Proof of 5.19. The proof of Lemma 5.19 is again divided into subclaims. Let F, D and h be as in 5.19. Let $I = \{c \in B_0 \mid \text{there are } c_0, \ldots, c_{n-1} \in B_0$ and $f_0, \ldots, f_{n-1} \in F$ such that $\Sigma_{i<n} c_i = c$ and for every $i < n$, $h \restriction c_i = f_i \restriction c_i\}$, and let $R = B_0 - I$. We define by induction on $i \in \omega$, $a_i, b_i \in B_0$ such that for every $i \in \omega$, $a_i \cdot b_i = 0$, $a_i \leq a_{i+1}$, $b_i \leq b_{i+1}$ and $-(a_i + b_i) \in R$. The element e of $\bar{B}_0$ required in 5.19 will be defined to be $\Sigma_{i\in\omega} a_i$. Let $P = \{\langle a, b\rangle \in B_0 \times B_0 \mid a \cdot b = 0$ and $-(a+b) \in R\}$. For $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in P$ we say that $\langle a_2, b_2\rangle$ extends $\langle a_1, b_1\rangle$ ($\langle a_1, b_1\rangle \preccurlyeq \langle a_2, b_2\rangle$) if $a_1 \leq a_2$ and $b_1 \leq b_2$. $c \in \bar{B}_0$ extends $\langle a, b\rangle$ ($\langle a, b\rangle \preccurlyeq c$) if $a \leq c$ and $c \cdot b = 0$.

For $c \in \bar{B}_0$ let $B(c)$ denote the smallest subalgebra of $\bar{B}_0$ containing $B_0 \cup \{c\}$ and closed under F. A generalized term is a term in the language containing an individual constant for every $b \in B_0$, a function symbol for every $f \in F$, and the Boolean operations. Let T denote the set of all generalized terms with one free variable x. Clearly, $B(c) = \{\tau(c) \mid \tau \in T\}$. Let $\langle a, b\rangle \in P$, $\tau(x) \in T$ and $d \in D$; we say that $\langle a, b\rangle$ is τ-good if for every $c \in \bar{B}_0$ extending $\langle a, b\rangle$, $h(c) \neq \tau(c)$; we say that $\langle a, b\rangle$ is $\langle \tau, d\rangle$-good if for every $c \in \bar{B}_0$ extending $\langle a, b\rangle$, $\tau(c) \neq d$.

It is clear, and thus left to the reader, that 5.19 follows from the following two claims.

Claim 1. For every $\langle a, b\rangle \in P$ and $\tau \in T$ there is $\langle a', b'\rangle \succcurlyeq \langle a, b\rangle$ such that $\langle a', b'\rangle$ is τ-good.

Claim 2. For every $\langle a, b\rangle \in P$, $\tau \in T$ and $d \in D$, there is $\langle a', b'\rangle \succcurlyeq \langle a, b\rangle$ such that $\langle a', b'\rangle$ is $\langle \tau, d\rangle$-good.

To prove Claims 1 and 2 we need some additional claims. For $a \leq b$ in $\bar{B}_0$ let $[a, b] \overset{\mathrm{def}}{=} \{x \in \bar{B}_0 \mid a \leq x \leq b\}$.

Claim 3. Let $p = \langle a, b\rangle \in P$ and $\tau \in T$. Let $g_{p,\tau}$: $[a, -b] \to \bar{B}_0$ be defined as $g_{p,\tau}(x) = \tau(x)$, and suppose that $g \in \mathrm{Aut}(\bar{B}_0)$ extends $g_{p,\tau}$. Then $g \restriction -(a+b)$ is piecewise in F, that is, there are $b_0, \ldots, b_{n-1} \in B_0$ and $f_0, \ldots, f_{n-1} \in F$ such that $\Sigma_{a<n} b_i = -(a+b)$ and for every $i<n$, $g \restriction b_i = f_i \restriction b_i$.

Claim 4. Let $\langle a, b\rangle \in P$, $\tau \in T$, $c \in \bar{B}_0$ and $e \in B_0 - \{0\}$ be such that $e \leq \tau(c)$. Then there are $\langle a_1, b_1\rangle \geqslant \langle a, b\rangle$ and a non-zero $e_1 \leq e$ such that $c \geqslant \langle a_1, b_1\rangle$, and for every $x \geqslant \langle a_1, b_1\rangle$, $e_1 \leq \tau(x)$. The analogous claim for $e \leq -\tau(c)$ also holds.

Claim 5. Let $\langle a, b\rangle \in P$, $\tau \in T$ and $d \in D$, then there is $x \geqslant \langle a, b\rangle$ such that $\tau(x) \neq d$.

Assuming Claims 3–5 we now prove Claims 1 and 2.

Proof of Claim 1. Let $p = \langle a, b\rangle$ and τ be as in the claim.

Case 1. There is $g \in \mathrm{Aut}(\bar{B}_0)$ which extends $g_{p,\tau}$. By Claim 3, $g \restriction -(a+b)$ is piecewise in F. Since $-(a+b) \in R$ there is $x_0 \leq -(a+b)$ such that $g(x_0) \neq h(x_0)$. It is easy to see that there is $x \in [a, -b]$ such that $g(x) \neq h(x)$, and hence $\tau(x) \neq h(x)$. W.l.o.g. $e \stackrel{\text{def}}{=} \tau(x) - h(x) \neq 0$. It is easy to see that there is $e_1 \in B - \{0\}$ such that $e_1 \leq e$ and $\langle a + g^{-1}(e_1), b + h^{-1}(e_1)\rangle \in P$. Let $\langle a_1, b_1\rangle = \langle a + g^{-1}(e_1), b + h^{-1}(e_1)\rangle$, then clearly $\langle a_1, b_1\rangle$ is τ-good.

Case 2. $g_{p,\tau}$ is not extendible. Since $h \restriction [a, -b]$ is extendible and $g_{p,\tau}$ is not, there is $x \in [a, -b]$ such that $h(x) \neq \tau(x)$. W.l.o.g. $e \stackrel{\text{def}}{=} \tau(x) - h(x) \neq 0$. There is $e' \in B$ such that $0 < e' \leq e$ and $\langle a, b + h^{-1}(e')\rangle \in P$. Let $b' = b + h^{-1}(e')$. Hence, for every $y \geqslant \langle a, b'\rangle$, $e' \cdot h(y) = 0$. By Claim 4 there is $\langle a_1, b_1\rangle \geqslant \langle a, b'\rangle$ and $e_1 \in B \restriction e' - \{0\}$ such that for every $y \geqslant \langle a_1, b_1\rangle$, $e_1 \leq \tau(y)$. Hence, for every $y \geqslant \langle a_1, b_1\rangle$, $\tau(y) - h(y) \neq 0$, so $\langle a_1, b_1\rangle$ is τ-good. This proves Claim 1.

The proof of Claim 2 is similar to the proof of Claim 1, and uses Claims 4 and 5.

Proof of Claim 3. Let $\langle a, b\rangle$, τ and g be as in Claim 3. For every $x \in [0, -(a+b)]$, $g(x) = \tau(a+x) - \tau(a)$, hence there is $\sigma' \in T$ such that for every $x \in [0, -(a+b)]$, $g(x) = \sigma'(x)$. $\sigma'(x)$ can be brought to the following form: $\sigma'(x) = \Sigma_{i<n} c_i \cdot \sigma_i(x)$, where $c_0, \ldots, c_{n-1}$ are pairwise disjoint, and every σ_i has the form

$$(*) \qquad \sum_{j<m} \prod_{l<m(j)} \varepsilon_{jl} \cdot f_{jl}(x)\,,$$

where $f_{jl} \in F$, $\varepsilon_{jl} \in \{-1, 1\}$, $1 \cdot e$ denotes e and $-1 \cdot e$ denotes $-e$. Let $c_i^* = g^{-1}(c_i) \cdot -(a+b)$. Clearly $\Sigma_{i<n} c_i^* = -(a+b)$. Let $i_0 < n$, $c' = c_{i_0}^*$, $\sigma = \sigma_{i_0}$ and σ have the form $(*)$. We wish to show that $g \restriction c'$ is piecewise in F. W.l.o.g. $c' \neq 0$. There are $c_0', \ldots, c_{k-1}' \in B_0$ such that $\Sigma_{i<k} c_i' = c'$, and for every $i<k$, $j<m$ and $l<m(j)$ either $f_{jl}(c_i') \cdot g(c') = 0$ or $f_{jl}(c_i') \leq g(c')$. Let $x \leq c_i'$, hence $g(x) \leq g(c')$. Hence, $g(x) = g(x) \cdot g(c') = g(x) \cdot c_{i_0} = c_{i_0} \cdot \Sigma_{j<m} \Pi_{l<m(j)}\, \varepsilon_{jl} f_{jl}(x)$. Hence, there is a term $\eta = \Sigma_{j<m_1} \Pi_{l<m_1(j)}\, \varepsilon_{jl} \cdot g_{jl}(x)$ such that every g_{jl} belongs to F and for every $x \leq c_i'$, $g(x) = \eta(x)$. Since F is good, $H \stackrel{\text{def}}{=} \{y \leq c_i' \mid \text{for every distinct } \langle j, l\rangle, \langle j', l'\rangle, g_{jl}(y) \cdot g_{j'l'}(y) = 0\}$ is dense in $\bar{B}_0 \restriction c_1'$. Hence, there is $\eta' = \Sigma_{t<r}\, \varepsilon_t \cdot g_t(x)$ such that for every $t<r$, $g_t \in F$, and for every $y \in H$, $g(y) = \eta'(y)$. Since g is order preserving, every ε_t is equal to 1, and since for every $y \in H$, $g \restriction y$ is onto $\bar{B}_0 \restriction g(y)$ $r = 1$. Hence, $\eta' = g_0(x)$. That is, $g \restriction H = g_0 \restriction H$, and since H is

dense in $\bar{B}_0 \restriction c_i'$, $g \restriction c_i' = g_0 \restriction c_i'$. This proves that for every $i_0 < n$, $g \restriction c_{i_0}^*$ is piecewise in F, so $g \restriction -(a+b)$ is piecewise in F.

We leave the easy proof of Claim 4 and the trivial proof of Claim 5 to the reader.

We have thus concluded the proof of 5.19.

The proof of 5.20 is again an inductive construction in ω steps. However, the argument there is much simpler than in 5.19 and is thus left to the reader.

Continuation of the proof of 5.18. Let B_0 be the countable atomless BA, $F_0 = \{Id\}$ and $D_0 = \emptyset$. For limit δ, let $B_\delta = \bigcup_{i<\delta} B_i$, $F_\delta = \bigcup_{i<\delta} F_i$ and $D_\delta = \bigcup_{i<\delta} D_i$. Suppose B_i has been defined. If $s_i \in B_i \times B_i$, let us apply Lemma 5.20 to $B_0 = B_i$ $F = F_i$ and $\langle a, b\rangle = s_i$. Let $g \in \mathrm{Aut}(B_i)$ be as assured by 5.20, and let F_{i+1} be the subgroup of $\mathrm{Aut}(\bar{B}_0)$ generated by $F \cup \{g\}$. Let $B_{i+1} = B_i$ and $D_{i+1} = D_i$.

If $s_i \in \mathrm{Aut}(B_i)$, let us apply Lemma 5.19 to $B_0 = B_i$, $F = F_i$, $D = D_i$ and $h = s_i$. Hence, there is $x_i \in \bar{B}_0$ such that $B_i(x_i) \cap (D_i \cup \{s_i(x_i)\}) = \emptyset$. Let $B_{i+1} = B_i(x_i)$, $D_{i+1} = D_i \cup \{s_i(x_i)\}$ and $F_{i+1} = F_i$.

In the case $s_i \not\in B_i \times B_i \cup \mathrm{Aut}(B_i)$, let $B_{i+1} = B_i$, $F_{i+1} = F_i$ and $D_{i+1} = D_i$.

It is easy to see that B_{i+1} and F_{i+1} are as required in Lemma 5.18.

This concludes the proof of Theorem 5.16.

We now turn to van Douwen's result that it is consistent that $\mathrm{Aut}(P(\omega)/\mathit{fin})$ is not simple. Let B_1 denote $P(\omega)/\mathit{fin}$. If CH holds, then B_1 is a saturated atomless BA, hence by Theorem 5.12, $\mathrm{Aut}(B_1)$ is simple. Every permutation of ω induces an automorphism of B_1. So if a/fin, $b/\mathit{fin} \in B_1 - \{0_{B_1}, 1_{B_1}\}$ and f is a permutation of ω taking a to b, then the automorphism of B_1 induced by f takes a/fin to b/fin. This means that B_1 is homogeneous.

Let $\mathrm{Sym}(\omega)$ denote the group of permutations of ω. Let $\mathrm{Sym}^+(\omega) = \{f \mid \mathrm{Dom}(f)$ and $\mathrm{Rng}(f)$ are cofinite subsets of ω and f is $1-1\}$. $\langle \mathrm{Sym}^+(\omega), \circ\rangle$ is a semigroup. For every $f \in \mathrm{Sym}^+(\omega)$ let $\bar{f}$: $B_1 \to B_1$ be defined as follows: $\bar{f}(a/\mathit{fin}) = f(a)/\mathit{fin}$. We leave it to the reader to verify that the definition of $\bar{f}(a/\mathit{fin})$ does not depend on the choice of a. It is also trivial to check that $\overline{\mathrm{Sym}}^+(\omega) \overset{\mathrm{def}}{=} \{\bar{f} \mid f \in \mathrm{Sym}^+(\omega)\}$ is a subgroup of $\mathrm{Aut}(B_1)$, and that for every f, $g \in \mathrm{Sym}^+(\omega)$ $\overline{f \circ g} = \bar{f} \circ \bar{g}$.

Next we formulate a difficult consistency result of Shelah which is the basis for the non-simplicity result of van Douwen.

5.21. THEOREM (SHELAH [1982]). *It is consistent with ZFC that* $\mathrm{Aut}(B_1) = \overline{\mathrm{Sym}}^+(\omega)$.

The proof of Shelah's theorem will not be presented here. We proceed to show how 5.21 implies that $\mathrm{Aut}(B_1)$ is not simple.

5.22. THEOREM (VAN DOUWEN [1984]). $\overline{\mathrm{Sym}}^+(\omega)$ *is not simple.*

PROOF. We shall present a non-trivial homomorphism $\bar{d}$ from $\overline{\mathrm{Sym}}^+(\omega)$ to $\langle \mathbb{Z}, +\rangle$. $\ker(\bar{d})$ will turn out to be $\{\bar{f} \mid f \in \mathrm{Sym}(\omega)\}$. This will imply that $\overline{\mathrm{Sym}}^+(\omega)$ has a non-trivial normal subgroup.

For $f \in \mathrm{Sym}^+(\omega)$ let $d(F) = |\omega - \mathrm{Rng}(f)| - |\omega - \mathrm{Dom}(f)|$. Let us note the following easy facts. (a) For every $f_1, f_2 \in \mathrm{Sym}^+(\omega)$, $\bar{f}_1 = \bar{f}_2$ iff $f_1 \cap f_2 \in \mathrm{Sym}^+(\omega)$. (b) If $f, g \in \mathrm{Sym}^+(\omega)$ and $f \subseteq g$, then $d(f) = d(g)$. (c) If $f, g \in \mathrm{Sym}^+(\omega)$ and $\mathrm{Rng}(g) = \mathrm{Dom}(f)$, then $d(f \circ g) = d(f) + d(g)$. (d) If $f, g \in \mathrm{Sym}^+(\omega)$, then there are $f_1, g_1 \in \mathrm{Sym}^+(\omega)$ such that $f_1 \subseteq f$, $g_1 \subseteq g$ and $\mathrm{Rng}(g_1) = \mathrm{Dom}(f_1)$.

Let $\bar{d}$: $\overline{\mathrm{Sym}}^+(\omega) \to \mathbb{Z}$ be defined as follows: $\bar{d}(\bar{f}) = d(f)$. By (a) and (b), $\bar{d}$ is well defined. By (a), (c) and (d), for every $\bar{f}, \bar{g} \in \overline{\mathrm{Sym}}^+(\omega)$, $\bar{d}(\bar{f} \circ \bar{g}) = \bar{d}(\bar{f}) + \bar{d}(\bar{g})$. Hence, $\bar{d}$ is indeed a homomorphism. Clearly, $\ker(\bar{d}) = \{\bar{f} \mid f \in \mathrm{Sym}(\omega)\} \neq \overline{\mathrm{Sym}}^+(\omega)$. Hence, $\overline{\mathrm{Sym}}^+(\omega)$ is not simple.

5.23. COROLLARY (van Douwen). *It is consistent with ZFC that* $\mathrm{Aut}(P(\omega)/\mathit{fin})$ *is not simple.*

PROOF. Combine 5.22 and 5.23. □

Finally we mention some open questions.

5.24. QUESTION. Does it follow from ZFC that there is a homogenous BA B such that $\mathrm{Aut}(B)$ is not simple?

There are three types of questions that were asked for various groups other than the automorphism groups of Boolean algebras, which are also meaningful for the automorphism groups of homogeneous BAs.

5.25. QUESTION 1. We have seen in 5.15 that if B is homogeneous and every $h \in \mathrm{Aut}^*(B)$ has an Anderson system, then for every $f, g \in \mathrm{Aut}(B) - \{Id\}$, g is a product of eight conjugates of f and f^{-1}, and g is a product of 16 conjugates of f.

What are the minimal numbers that can replace 8 and 16 in 5.15? Are there such numbers for every homogeneous BA with a simple automorphism group? In particular, is there a homogeneous BA B with $\mathrm{Aut}(B)$ being simple, and in which not every $h \in \mathrm{Aut}^*(B)$ has an Anderson system?

QUESTION 2. $f \in \mathrm{Aut}(B)$ is called an involution if $f^2 = Id$. Another variant of the questions asked above is the following. Given that B is homogeneous and that every $h \in \mathrm{Aut}^*(B)$ has an Anderson system, what is the minimal number n such that every member of $\mathrm{Aut}(B)$ is a product of $\leq n$ involutions?

It is true that if B is homogeneous, and $\mathrm{Aut}(B)$ is generated by the set of all involutions, then $\mathrm{Aut}(B)$ is simple?

QUESTION 3. By a term of group theory we mean a term in the language $\{\cdot, {}^{-1}\}$. Let G be a group and $t(x_1, \ldots, x_n)$ be a term of group theory. t is universal for G if for every $g \in G$ there are $g_1, \ldots, g_n \in B$ such that $g = t(g_1, \ldots, g_n)$.

(a) Find the terms universal for every $\mathrm{Aut}(B)$, B homogeneous. (b) Find the terms universal for every $\mathrm{Aut}(B)$, B homogeneous and every $h \in \mathrm{Aut}^*(B)$ has an Anderson system. (c) Find the terms universal for $\mathrm{Aut}(B)$, B a specific homogeneous BA.

Koppelberg's construction raises the following interesting question.

5.26. QUESTION. The notation PW(A) and the notion of a good subgroup of Aut(B) were defined in 5.17. Suppose that B is homogeneous and there is a good subgroup G of Aut(B) such that Aut(B) = PW(G). Is it true that Aut(B) is not simple?

6. Stronger forms of homogeneity

Homogeneous Boolean algebras can be characterized by the extendibility of automorphisms of finite subalgebras.

6.1. LEMMA. *If B is a Boolean algebra, $|B|>4$, then B is homogeneous iff every automorphism of any finite subalgebra of B extends to an automorphism of B.*

PROOF. If B is homogeneous and C is a finite subalgebra of B, then every automorphism of C is uniquely determined by a permutation of atoms of C. Every term of C corresponds to a non-trivial factor of B and all such factors are isomorphic. Hence, every automorphism of C can be extended to an automorphism of B.

If B satisfies the condition of the lemma and $|B|>4$, then every two disjoint non-zero $a, b \in B$ constitute isomorphic factors. In particular, for every b, $0<b<1$, $B \restriction b$ and $B \restriction -b$ are isomorphic and B is atomless since $|B|>4$. It remains to show that B is homogeneous. Let b be an arbitrary non-zero element of B. We shall show that $B \cong B \restriction b$. We may assume that $b<1$. Then $B \restriction b \cong B \restriction -b$ and there are disjoint non-zero elements b_1, b_2, $b = b_1 + b_2$. Now both $B \restriction b_1$ and $B \restriction b_2$ are isomorphic to $B \restriction -b$ and consequently, to $B \restriction b$. We have

$$B \cong B \restriction b \times B \restriction -b \cong B \restriction b$$

and B is homogeneous. □

6.2. All finite subalgebras of any homogeneous algebra have the automorphism extension property. To ask the same for more (or possibly all) subalgebras seems to be a promising way to stronger concepts of homogeneity. The following result, due to Weese, shows the limits of this approach.

6.3. THEOREM (Weese). *If $|B|>4$, then there is a subalgebra $C \subseteq B$ and an automorphism of C which does not extend to B.*

PROOF. Suppose that $|B|>4$ and that the automorphisms of all subalgebras of B extend to B. Clearly, B is atomless and it is homogeneous according to Lemma 6.1. Choose three arbitrary non-zero elements a, b, c that make a partition of unity in B. Let φ_1, φ_2 be isomorphisms that map the factor $B \restriction a$ onto $B \restriction b$ and $B \restriction c$, respectively. If we put

$$\psi(x) = \varphi_1(x) + \varphi_2(x) \quad \text{for every } x \leq a ,$$

then

$$C = \{x + \psi(y) : x, y \leq a\}$$

is a subalgebra of B. For every $x, y \leq a$, let

$$\phi(x + \psi(y)) = y + \psi(x) .$$

It is not difficult to check that ϕ is an automorphism of C. Note that the factor $B \upharpoonright a$ is a subset of C and that $\phi(x) = \psi(x)$ for every $x \leq a$. We shall show that ϕ does not extend to B. Suppose that ϕ^* is an automorphism of B which extends ϕ and that $d \in B$ is the preimage of b. Then $d > 0$ and

$$\phi^*(d) = b \leq b + c = \phi^*(a) .$$

Hence, $d \leq a$ and $\phi^*(d) = \psi(d)$ has a non-zero intersection with c which contradicts $\phi^*(d) = b$. This shows that every Boolean algebra consisting of more than four elements has a subalgebra without the automorphism extension property. □

6.4. REMARK. Note that if B is homogeneous and C is defined as in the proof of the previous theorem, then $|B| = |C|$. Moreover, C is a complete regular subalgebra of B provided that B is complete.

The following example shows that we have similar problems if we restrict our attention to (non-complete) countable subalgebras.

6.5. EXAMPLE. Let B be an infinite complete Boolean algebra. Then B is uncountable and there is a countable subalgebra $C \subseteq B$ and an automorphism of C which does not extend to B. To construct C, choose an arbitrary partition $\langle b_\alpha : \alpha < \omega + \omega \rangle$ of unity to countably many non-zero elements. Put

$$c_0 = \sum \{b_n : n < \omega\} , \qquad c_1 = \sum \{b_{\omega+n} : n < \omega\} .$$

Let C be the countable subalgebra of B generated by elements c_0, c_1 and all b_α, $\alpha \neq \omega$. Clearly, C is not complete and the automorphism ψ of C defined as follows:

$$\psi(c_0) = c_1 , \qquad \psi(c_1) = c_0$$

and

$$\psi(b_n) = b_{\omega+1+n} , \qquad \psi(b_{\omega+1+n}) = b_n \quad \text{for every } n ,$$

does not extend to B since $c_1 = \psi(c_0) \neq \sum \{b_{\omega+1+n} : n < \omega\}$.

Weese's theorem and the above example show that we cannot get a reasonable stronger concept of homogeneity if we base it on a simple generalization of the automorphism extension property. We can still prove the following.

6.6. THEOREM. *Let κ be an infinite cardinal. For every regular complete subalgebra B of the collapsing algebra $C(\kappa)$, there is a complete embedding $e: B \to C(\kappa)$ such that every automorphism of $e(B)$ extends to $C(\kappa)$.*

Moreover, e is the identity if B is finite.

PROOF. Given κ, let $C(\kappa)$ be the collapsing algebra defined as in Example 1.5(b). If B is a regular complete subalgebra of $C(\kappa)$, it suffices to show that B has a dense subset of power at most κ. The result then follows from Theorem 3.3 and Lemma 6.1.

We shall define a mapping $\pi: C(\kappa) \to B$ as follows. For every $c \in C(\kappa)$, we put

$$\pi(c) = \prod \{b \in B : c \leq b\} .$$

Then $\pi(c)$ is the least element of B such that $\pi(c) \geq c$. It is obvious that for every dense subset D of $C(\kappa)$, the image $\pi(D)$ is dense in B. Hence, B has a dense subset of power at most κ, for $C(\kappa)$ has one of power κ. This completes the proof. $\square$

6.7. We shall conclude our discussion by two open problems.

PROBLEM 3. Does there exist an infinite complete Boolean algebra B such that every regular complete $C \subseteq B$, $|C| < |B|$ has the automorphism extension property?

Since it might happen that the only regular complete subalgebras of a complete algebra are the finite ones or have the same power as the whole algebra, the following version of the problem seems to be better.

PROBLEM 4. Does there exist an infinite complete Boolean algebra B such that every regular complete subalgebra of B generated by less than $|B|$ elements of B has the automorphism extension property?

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Index of Notation, Volume 2

This index lists the notation in the order that it is introduced. The page on which the notation is defined is given, and in most cases a brief definition of it is also supplied. Notation which is only in force for a page or two is omitted.

Chapter 8: Distributive Laws

Chapter 9: Disjoint Refinement

Chapter 10: Subalgebras

Chapter 11: Cardinal Functions on Boolean Spaces

Chapter 12: The number of Boolean Algebras

Chapter 13: Endomorphisms of Boolean Algebras

Chapter 14: Automorphism Groups

Chapter 15: On the Reconstruction of Boolean Algebras from their Automorphism Groups

Chapter 16: Embeddings and Automorphisms

Chapter 17: Rigid Boolean Algebras

Chapter 18: Homogeneous Boolean Algebras

Index, Volume 2

Page numbers in bold-face give the definition or main references for an item, if there is such, in case there are several pages for an item.

measure algebra, 338, 547, 551