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INDEPENDENCE IN BOOLEAN ALGEBRAS

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Introduction

A subset X of a BA A is *independent* if X freely generates the subalgebra of A which it generates. For any BA A, ind A, the independence of A, is

$$\sup \{|X| : X \text{ is an independent subset of } A\}.$$

Further, A is said to have *free caliber* $\varkappa$ if $\varkappa \leq |A|$ and among any $\varkappa$ elements of A there are $\varkappa$ independent elements.

This paper is devoted to a systematic study of independence is Boolean algebras. We are interested in the behaviour of independence under algebraic operations like products, free products, and homomorphisms, and independence in special algebras, such as complete algebras. Some of the main new results are as follows. Let cell$'A$ = least $\varkappa$ (every pairwise disjoint subset of A has power $<\varkappa$). If cell$'A = \varkappa$, $\omega \leq \lambda \leq \mu < \varkappa$, and A has an independent set of power λ, then A has a subalgebra B with ind $B = \lambda$ and $|B| = \mu$. If A has an independent set of size $\varkappa$, $\lambda \leq \varkappa$, $\lambda^\omega = \lambda$ and $\varkappa^\omega = \varkappa$, then A has a homomorphic image B with ind $B = \lambda$ and $|B| = \varkappa$. If A is the BA of finite and cofinite subsets of $\varkappa$, then ind $({}^\lambda A) = 2^\lambda$ for any λ. If $\varkappa$ is uncountable and regular, then $\varkappa$ is a free caliber of a free product $*_{i\in I}A_i$ iff $\varkappa \leq |*_{i\in I}A_i|$ and $\varkappa$ is a free caliber of each A_i such that $\varkappa \leq |A_i|$. If A is weakly countably complete, then (ind $A)^\omega$ = ind A.

To put our results in perspective, we mention in the course of the paper many related results in the literature. The most striking known results are the result of Balcar and Franěk [3] that any complete BA A has an independent subset of power $|A|$, a characterization of independence in terms of density of ultrafilters due to Šapirovskiĭ [25], and the following strong result of Shelah [26] connecting chain conditions and independence:

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(S) *If $\varkappa$, λ are infinite and regular, $\forall \mu < \lambda\,(\mu^{<\varkappa} < \lambda)$, and A has no pairwise disjoint set of power $\varkappa$, then A has free caliber λ.*

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0. Notation

Our set-theoretic notation is standard, with these possible exceptions. $f[X]$ denotes the f-image of the set X under the function f. $\varkappa^{+n}$ is the n^{th} cardinal successor of $\varkappa$: $\varkappa^{+0} = \varkappa$, $\varkappa^{+(n+1)} = (\varkappa^{+n})^+$: $\mu_\alpha \uparrow \varkappa$ *for* $\alpha < \lambda$ means that $\langle \mu_\alpha : \alpha < \lambda \rangle$ is a strictly increasing sequence of cardinals with supremum $\varkappa$. $f: A \to B$, $f: A \twoheadrightarrow B$, $f: A \rightarrowtail B$, and $f: A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ indicate that f is a function from A into B, resp. onto, one-one, one-one and onto. The cardinals $\beth_\alpha \varkappa$ are defined by induction: $\beth_0 \varkappa = \varkappa$. $\beth_{\alpha+1} \varkappa = 2^{\beth_\alpha \varkappa}$, $\beth_\lambda \varkappa = \bigcup_{\alpha<\lambda} \beth_\alpha \varkappa$ for λ limit. Further, $\beth_\alpha = \beth_\alpha \omega$. reg is the class of all regular cardinals. $[X]^\varkappa$ is the collection of subsets of X of power $\varkappa$. Similarly for $[X]^{\leq\varkappa}$, $[X]^{<\varkappa}$. $[X]^{<\varkappa}_{+co}$ is the collection of subsets of X of power $<\varkappa$ together with their complements, frequently considered as a BA. In particular, finco $\lambda = [\lambda]^{<\omega}_{+co}$ and cblco $\lambda = [\lambda]^{\leq\omega}_{+co}$.

A Boolean algebra is identified with its universe. If A is a BA and $a \in A$, we let $a^1 = a$ and $a^0 = -a$. A satisfies the $\varkappa$-chain condition, or $\varkappa$-cc, if every set of pairwise disjoint elements of A has power $<\varkappa$. We let

$$\text{cell}\, A = \sup \{|X| : X \subseteq A,\ X \text{ pairwise disjoint}\},$$

and

$$\text{cell}'\, A = \min \{\varkappa : A \text{ satisfies } \varkappa\text{-cc}\}.$$

Thus $\text{cell}'\, A = (\text{cell}\, A)^+$ unless cell A is weakly inaccessible and not attained, in which case $\text{cell}'\, A = \text{cell}\, A$. $\text{Sg}^A X$ indicates the subalgebra of A generated by X; for A complete, $\text{Csg}^A X$ is the complete subalgebra of A generated by X. $A_\alpha \uparrow B$ *for* $\alpha < \varkappa$ means that $\langle A_\alpha : \alpha < \varkappa \rangle$ is a strictly increasing system of subalgebras of B with union B. $\Pi^w_{i\in I} A_i$ is the weak product of $\langle A_i : i \in I \rangle$, i.e.,

$$\Pi^w_{i\in I} A_i = \{f \in \Pi_{i\in I} A_i : |\{i \in I : f_i \neq 0\}| < \omega \text{ or } |\{i \in I : f_i \neq 1\}| < \omega\}.$$

Similarly for $\Pi^{<\varkappa}_{i\in I} A_i$, replacing "$\omega$" by "$\varkappa$". A partition of 1 in A is a system $\langle a_i : i \in I \rangle$ of pairwise disjoint non-zero elements of A such that $\Sigma_{i\in I} a_i = 1$. The completion of A is denoted by A^{cmpl}, its σ-completion by $A^{\sigma\,\text{cmpl}}$. cBA means "complete Boolean algebra". Fr $\varkappa$ is the free BA on $\varkappa$ generators.

For any linear order L with first element, intalg L is the interval algebra of L; we apply this notation also to linear orders without a first element, by adjoining one. $A*B$ is the free product of A and B, and $*_{i\in I}A_i$ is the free product of the system $\langle A_i: i\in I\rangle$. A BA A is *weakly countably complete*, wcc, if $\forall X, Y\in [A]^{\leq\omega}$, if $x\leq y$ for all $x\in X$ and $y\in Y$ then there is an $a\in A$ with $x\leq a\leq y$ for all $x\in X$ and $y\in Y$. At A is the set of atoms of A.

$X\subseteq A$ satisfies the *finite intersection property*, fip, if $\forall Y\in [X]^{<\omega}$ $(\Pi Y\neq 0)$. A has *precaliber* $\varkappa$ if

$$\forall x\in {}^{\varkappa}A\ \exists X\in [\varkappa]^{\varkappa}(\langle x_\alpha: \alpha\in \Gamma\rangle \text{ has fip})$$

(i.e.,

$$\forall \Delta\in [\Gamma]^{<\omega}\Pi_{\alpha\in\Delta}x_\alpha\neq 0).$$

X is *independent* if X freely generates Sg X, or, equivalently, if

$$\forall Y\in [X]^{<\omega}\forall \varepsilon\in {}^{Y}2(\Pi_{y\in Y}y^{\varepsilon y}\neq 0).$$

$$\operatorname{ind} A = \sup\{|X|: X\subseteq A,\ X \text{ independent}\}.$$

We say that independence is *attained* in A if

$$\exists X\in [A]^{\operatorname{ind} A}(X \text{ is independent});$$

obviously this always happens when ind A is a successor cardinal. A has *free caliber* $\varkappa$, $\varkappa\in$ freecal A, if $\varkappa\leq |A|$ and

$$\forall X\in [A]^{\varkappa}\exists Y\in [X]^{\varkappa}\ (Y \text{ is independent}).$$

Note that A never has free caliber $\varkappa$ if cf $\varkappa=\omega$. This is obvious if $\varkappa=\omega$. Suppose $\varkappa>\omega$, say $\omega\leq\mu_m\uparrow\varkappa$ for $m\in\omega$, and $x\in {}^{\varkappa}A$ is independent. Let

$$X=\{x_{\mu_{m+1}}\cdot -x_{\mu_m}\cdot x_\alpha: m\in\omega,\ \mu_m<\alpha<\mu_{m+1}\}.$$

Then $X\in [A]^{\varkappa}$, but X has no independent subset of power $\varkappa$.

Cardinals and Boolean algebras are infinite unless otherwise mentioned.

1. Finite products

The basic but easy fact here is:

THEOREM 1.1. *If neither A nor B has an independent set of power $\varkappa$, then $A\times B$ also does not.*

PROOF. Suppose $\langle (a_\alpha, b_\alpha): a<\varkappa\rangle$ is a system of elements of $A\times B$. Choose $\Gamma\in [\varkappa]^{<\omega}$ and $\varepsilon\in {}^{\Gamma}2$ so that $\Pi_{\alpha\in\Gamma}a_\alpha^{\varepsilon\alpha}=0$, and then choose $\Delta\in [\varkappa\backslash\Gamma]^{<\omega}$ and $\delta\in {}^{\Delta}2$ so that $\Pi_{\alpha\in\Delta}b_\alpha^{\delta\alpha}=0$. Then $\Pi_{\alpha\in\Gamma\cup\Delta}(a_\alpha, b_\alpha)^{(\delta\cup\varepsilon)\alpha}=0$, as desired.

Corollary 1.2. ind $(A \times B) = \max(\text{ind } A, \text{ind } B)$.

Corollary 1.3. ind $(A \times B)$ *is attained if and only if one of the following conditions holds*:

(i) ind $A <$ ind B *and* ind B *is attained*; *or similarly with* A *and* B *interchanged*;

(ii) ind $A =$ ind B *and one of them is attained.*

The following theorem characterizes free caliber of a product of two BA's.

Theorem 1.4. $\varkappa \in \text{freecal}(A \times B)$ *if and only if one of the following conditions holds*:

(i) $\varkappa \in \text{freecal } A$ *and* $\varkappa < |B|$;

(ii) $\varkappa \in \text{freecal } B$ *and* $\varkappa < |A|$;

(iii) $\varkappa \in \text{freecal } A \cap \text{freecal } B$.

Proof. Clearly $\varkappa \in \text{freecal}(A \times B)$ implies that one of (i)–(iii) holds. The converse is also clear, except possibly for (iii) when $\varkappa$ is singular. In this case, if $X \in [A \times B]^\varkappa$, then there is a $Y \in [X]^\varkappa$ such that $\forall y, z \in Y\ (y \neq z \Rightarrow \Rightarrow y_0 \neq z_0)$ or $\forall y, z \in Y\ (y \neq z \Rightarrow y_1 \neq z_1)$, so again $\varkappa \in \text{freecal}(A \times B)$ follows.

From this theorem we easily get by induction:

Corollary 1.5. *If* F *is finite, then* $\varkappa \in \text{freecal } \Pi_{i \in F} A_i$ *if and only if* $\exists i \in F\ (\varkappa \leq |A_i|)$ *and*

$$\varkappa \in \bigcap \{\text{freecal } A_i : \varkappa \leq |A_i|\}.$$

2. Weak products

Corresponding to Theorem 1.1 we have:

Theorem 2.1. *If* $\forall i \in I$ *(*A_i *does not have an independent set of power* $\varkappa$*) then neither does* $\Pi^w_{i \in I} A_i$.

Proof. Suppose $X \in [\Pi^w_{i \in I} A_i]^\varkappa$ is independent, and fix $x \in X$. We may, assume that $F = \{i \in I : x_i \neq 0\}$ is finite. Then

$$\langle y \restriction \Pi_{i \in F} A_i : y \in X \setminus \{x\} \rangle$$

is a system of $\varkappa$ independent elements of $\Pi_{i \in F} A_i$, contradicting 1.1.

COROLLARY 2.2. $\operatorname{ind} \Pi^w_{i\in I} A_i = \sup_{i\in I} \operatorname{ind} A_i$.

COROLLARY 2.3. *$\operatorname{ind} \Pi^w_{i\in I} A_i$ is attained if and only if there is an $i \in I$ such that $\operatorname{ind} A_i = \sup_{j\in I} \operatorname{ind} A_j$ and independence is attained in A_i.*

The situation for free caliber is a little involved, but can be completely described.

THEOREM 2.4. *Let $\varkappa$ be regular, or else singular with $|I| < \operatorname{cf} \varkappa$. Then the following conditions are equivalent:*

(i) $\varkappa \in \operatorname{freecal} \Pi^w_{i\in I} A_i$;

(ii) $|I| < \varkappa$, $\exists i \in I\ (\varkappa \leq |A_i|)$, *and*

$$\varkappa \in \bigcap \{\operatorname{freecal} A_i \colon \varkappa \leq |A_i|\}.$$

PROOF. Clearly (i) $\Rightarrow$ (ii). Now assume (ii), and suppose that $X \in [\Pi^w_{i\in I} A_i]^\varkappa$ For each $x \in X$ let $F_x = \{i \in I : x_i \neq 0\}$. We may assume that each F_x is finite. Hence there is a $G \in [I]^{<\omega}$ so that $Y = \{x \in X : F_x = G\}$ has power $\varkappa$. Applying 1.5 to $\{x \restriction G : x \in Y\}$, we get (i).

THEOREM 2.5. *Suppose $\varkappa$ is singular and $\operatorname{cf} \varkappa \leq |I|$. Then the following conditions are equivalent:*

(i) $\varkappa \in \operatorname{freecal} \Pi^w_{i\in I} A_i$;

(ii) $\sup_{i\in I} \{|A_i| : |A_i| < \varkappa\} < \varkappa$, $|I| < \varkappa$, $|\{i \in I : \varkappa \leq |A_i|\}| < \operatorname{cf} \varkappa$, $\exists i \in I(\varkappa \leq |A_i|)$, *and*

$$\varkappa \in \bigcap \{\operatorname{freecal} A_i \colon \varkappa \leq |A_i|\}.$$

PROOF. Clearly (i) $\Rightarrow$ (ii). Now assume (ii), and suppose that $X \in [\Pi^w_{i\in I} A_i]^\varkappa$ Let $J = \{i \in I : |A_i| \geq \varkappa\}$ and $K = \langle i \in I : |A_i| < \varkappa\}$. Thus $|\Pi^w_{i\in K} A_i| < \varkappa$. Let F_x be as in the proof of 2.4, and assume that each F_x is finite. Then there is a $G \in [J]^{<\omega}$ so that

$$Y = \{x \in X \colon F_x \cap J = G\}$$

has power $\varkappa$. Applying 1.5 to $\{x \restriction (G \cup K) : x \in Y\}$, we get (i).

3. Attainment

For each limit cardinal $\varkappa$ there is a BA B with independence $\varkappa$ not attained. For $\varkappa = \omega$ we simply take for B any infinite hereditarily atomic BA. Now assume that $\varkappa > \omega$. Let $\omega \leq \lambda_\alpha \uparrow \varkappa$ for $\alpha < \operatorname{cf} \varkappa$. Then $B = \Pi^w_{\alpha<\operatorname{cf}\varkappa} \operatorname{Fr} \lambda_\alpha$ works, by 2.3.

4. Cardinality

If $\omega \leq \varkappa \leq \lambda$, then there is a BA A with $\operatorname{ind} A = \varkappa$ and $|A| = \lambda$: we can take $A = \operatorname{Fr} \varkappa \times \operatorname{intalg} \lambda$; if $\varkappa$ is a limit cardinal and we want independence not attained, then we can take $A = \operatorname{intalg} \lambda$ if $\varkappa = \omega$, and

$$A = \Pi^w_{\alpha < \operatorname{cf} \varkappa} \operatorname{Fr}_\alpha \mu_\alpha \times \operatorname{intalg} \lambda$$

if $\varkappa > \omega$ and $\omega \leq \mu_\alpha \uparrow \varkappa$.

Concerning free caliber and cardinality we make three simple observations. First, for each $\lambda \geq \omega$ there is a BA A of power λ with $\operatorname{freecal} A = 0$: one can take $A = \operatorname{intalg} \lambda$. Second, if $\operatorname{freecal} A \neq 0$ and $\varkappa = \operatorname{supfreecal} A$, then there is a subalgebra B of A with $|B| = \varkappa$ and $\operatorname{freecal} A = \operatorname{freecal} B$. To see this, for each $\lambda \in [\omega, \varkappa] \cap \operatorname{freecal} A$ let $X_\lambda \in [A]^\lambda$ have no independent subset of power λ, and let B be any subalgebra of A of power $\varkappa$ containing each such set X_λ. Finally if $\operatorname{freecal} A \neq 0$, then there is an upper bound on $|B|$ for B such that $\operatorname{freecal} A = \operatorname{freecal} B$. Namely, let $\varkappa = \operatorname{supfreecal} A$ and let λ be the least element of $\operatorname{freecal} A$. Any such B satisfies the λ-cc, so by (S) $|B| \leq \varkappa^\lambda$.

5. Subalgebras

The relationship between independence of A and of B when $A \subseteq B$ was characterized as follows by Šapirovskiĭ [25]:

(Š) *Suppose that $A \subseteq B$. For each maximal ideal I on A let I^+ be the ideal in B generated by I. Then*

$$\operatorname{ind} B = \operatorname{ind} A \cup \sup \{\operatorname{ind} (B/I^+): I \textit{ a maximal ideal of } A\}.$$

Although this is a strong result, it is still nontrivial to figure out the exact possible relationships between $\operatorname{ind} A$ and $\operatorname{ind} B$. First we consider passage from an algebra A to a superalgebra $B \supseteq A$.

Example 5.1. If $\operatorname{ind} A \leq \varkappa \leq \lambda \geq |A|$, then there is a $B \supseteq A$ with $|B| = \lambda$, $\operatorname{ind} B = \varkappa$ attained. We take for B the BA $A \times \operatorname{Fr} \varkappa \times \operatorname{intalg} \lambda$; to see that this works, see Theorem 1.1.

Example 5.2. Let $\varkappa$ be a limit cardinal, and assume that $\operatorname{ind} A \leq \varkappa \leq \lambda \geq |A|$, and that if $\operatorname{ind} A = \varkappa$, then independence in A is not attained. Then there is a $B \supseteq A$ with $|B| = \lambda$, $\operatorname{ind} B = \varkappa$ not attained. For, if $\operatorname{ind} A = \varkappa$, we let $B = A \times \operatorname{finco} \lambda$; it is easily verified that this works. Assume $\operatorname{ind} A < \varkappa$. Thus $\varkappa > \omega$; say $\omega \leq \mu_\alpha \uparrow \varkappa$ for $\alpha < \operatorname{cf} \varkappa$. Let $B = A \times \Pi^w_{\alpha < \operatorname{cf} \varkappa} \operatorname{Fr} \mu_\alpha \times \operatorname{finco} \lambda$ Then B works by Theorem 2.1.

As to freecaliber, we have already noticed in section 4 that in general it is impossible to go from A to $B \supseteq A$ with freecal A = freecal B. One can still say a little, though:

EXAMPLE 5.3. If A and D are arbitrary, with $|A| < |D|$, then there is a $B \supseteq A$ with

$$\text{freecal } B = (\text{freecal } A \cap \text{freecal } D) \cup (\text{freecal } D \cap (|A|, \infty)),$$

and there is a $C \supseteq A$ with

$$\text{freecal } C = \text{freecal } D \cap (|A|, \infty).$$

For, one can take $B = A \times D$ and $C = A \times D \times \text{intalg } |A|$; see 1.5.

Now we consider the passage from an algebra A to a subalgebra B. Note that (S) establishes some limits on what happens to independence in this case. The following theorem appears to be about the best one can hope for without getting involved with those limits.

THEOREM 5.4. *Suppose* cell′ $A = \varkappa$, $\omega \leq \lambda \leq \mu < \varkappa$, *and* A *has an independent set of power* λ. *Then there is a* $B \subseteq A$ *with* ind $B = \lambda$ *attained and* $|B| = \mu$.

PROOF. Let $X \in [A]^\lambda$ be independent and $Y \in [A]^\mu$ be pairwise disjoint. Set $B = \text{Sg}(X \cup Y)$, $C = \text{Sg}\, X$, and $D = \text{Sg}\, Y$. Then for every ultrafilter I on D, if I^+ is the ideal on B generated by I then clearly $C \twoheadrightarrow B/I^+$. Since ind $D = \omega$, it then follows that ind B = ind $C = \lambda$.

Now we consider the situation in Theorem 5.4 with λ a limit cardinal, and ind $B = \lambda$ not attained; we have a positive and a negative result.

THEOREM 5.5. *Suppose cell′* $A = \varkappa$, $\omega \leq \lambda \leq \mu < \varkappa$, λ *is a limit cardinal*, cf $\lambda = \omega$, *and* $\lambda \leq$ ind A. *Then there is a* $B \subseteq A$ *with* $|B| = \mu$ *and* ind $B = \lambda$ *not attained.*

PROOF. The theorem is trivial if $\lambda = \omega$, so suppose $\lambda > \omega$ and $\omega \leq \mu_n \uparrow \lambda$ for $n \in \omega$. It is also trivial if A has no independent set of power λ, so suppose $\langle x_\alpha : \alpha < \lambda \rangle$ is independent in A. For all $n \in \omega$ and $\alpha < \mu_n$ let $a_{n\alpha} = x_{\mu_{n+1}} \cdot \cdot -x_{\mu_n} \cdot x_\alpha$. Let $Y \in [A]^\mu$ be pairwise disjoint. We consider two cases.

Case 1.

$$\sum_{n \in \omega} |\{y \in Y:\ y \cdot x_{\mu_{n+1}} \cdot -x_{\mu_n} \neq 0\}| = \mu.$$

Let

$$B = \text{Sg}(\{a_{n\alpha}:\ n < \omega, \alpha < \mu_n\} \cup \{y \cdot x_{\mu_{n+1}} \cdot -x_{\mu_n}:\ n < \omega, y \in Y\}).$$

It is easily checked that there is an isomorphism of B into

$$\Pi^w_{n\in\omega} \operatorname{Sg} (\{a_{n\alpha}:\ \alpha < \mu_n\} \cup \{y \cdot x_{\mu_{n+1}} \cdot -x_{\mu_n}:\ y \in Y\}).$$

The n^{th} factor of this weak product has independence μ_n, by the argument for 5.4. So B is as desired, by 2.1.

Case 2. Otherwise, we may assume that $y \cdot x_{\mu_{n+1}} \cdot -x_{\mu_n} = 0$ for all $n \in \omega$ and all $y \in Y$. Let

$$B = \operatorname{Sg} (\{a_{n\alpha}:\ n < \omega, \alpha < \mu_n\} \cup Y).$$

Then there is an ismorphism from B into

$$\Pi^w_{n\in\omega} \operatorname{Sg} \{a_{n\alpha}: \alpha < \mu_n\} \times \operatorname{Sg} Y,$$

and the argument runs as before.

EXAMPLE 5.6. Suppose $\omega \leq \lambda \leq \mu < \varkappa^+$, where λ is a limit cardinal with $\operatorname{cf} \lambda > \omega$. Then there is a BA A with $\operatorname{cell}' A = \varkappa^+$ such that for all $B \subseteq A$, if $|B| = \mu$ and $\operatorname{ind} B = \lambda$ then $\operatorname{ind} B$ is attained. In fact, we simply let $A = \operatorname{Fr} \lambda \times \operatorname{finco} \varkappa$. Suppose $B \subseteq A$ with $|B| = \mu$ and $\operatorname{ind} B = \lambda$. Say $\omega \leq \mu_\alpha \uparrow \lambda$ for $\alpha < \operatorname{cf} \lambda$. For each $\alpha < \operatorname{cf} \lambda$ let $\langle x^\alpha_\gamma: \gamma < \mu_\alpha\rangle$ be a system of independent elements of B. Say $x^\alpha_\gamma = (y^\alpha_\gamma, z^\alpha_\gamma)$, where $y^\alpha_\gamma \in \operatorname{Fr} \lambda$ and $z^\alpha_\gamma \in \operatorname{finco} \varkappa$. If there are λ different y^α_γ's, then by Corollary 10.13 below there are λ independent y^α_γ's, giving λ independent x^α_γ's, as desired. Otherwise there is an $\alpha < \operatorname{cf} \lambda$ for which there is a $\Gamma \in [\mu_\alpha]^{\omega_1}$ with $y^\alpha_\gamma = y^\alpha_\delta$ for all $\gamma, \delta \in \Gamma$. Then a Δ-system argument on the z^α_γ, $\gamma \in \Gamma$, yields three distinct elements $\gamma 0$, $\gamma 1$, $\gamma 2$ of Γ and $\varepsilon 0$, $\varepsilon 1$, $\varepsilon 2$ with

$$(z^\alpha_{\gamma 0})^{\varepsilon 0} \cdot (z^\alpha_{\gamma 1})^{\varepsilon 1} \cdot (z^\alpha_{\gamma 2})^{\varepsilon 2} = 0.$$

This easily implies that $\langle x^\alpha_\gamma: \gamma < \mu_\alpha\rangle$ is not independent, contradiction.

Finally, some simple remarks on free caliber:

(1) If B is a subalgebra of A, then

$$\operatorname{freecal} A \cap [\omega], |B|], \subseteq \operatorname{freecal} B.$$

(2) If $\operatorname{freecal} A \neq 0$, $\operatorname{supfreecal} A = \varkappa \leq \lambda \leq |A|$, then there is a subalgebra B of A with $|B| = \lambda$ and $\operatorname{freecal} A = \operatorname{freecal} B$. (See the argument in section 4.)

(3) If $\operatorname{freecal} A \neq 0$ and $\operatorname{supfreecal} A = \varkappa$ attained, then there is a subalgebra B of A with $|B| = \varkappa$ and

$$\operatorname{freecal} B = \{\lambda: \omega_1 \leq \lambda \leq \varkappa, \operatorname{cf} \lambda > \omega\};$$

see section 10.

Some natural questions that arise here seem to be connected with problems about the free caliber spectrum.

6. Unions

The behavior of independence under unions is expressed in the following simple theorem.

THEOREM 6.1. *Suppose $\varkappa$ is regular, $\varkappa \neq \mathrm{cf}\,\lambda$, $A_\alpha \uparrow B$ for $\alpha < \varkappa$ and no A_α has an independent subset of power λ. Then B has none either.*

COROLLARY 6.2. *If $\varkappa$ is regular, $\lambda = \sup_{\alpha<\varkappa} \mathrm{ind}\, A_\alpha$, and $A_\alpha \uparrow B$, then:*
(i) $\varkappa \neq \lambda^+ \Rightarrow \mathrm{ind}\, B = \lambda$;
(ii) $\varkappa \neq \lambda^+$, $\mathrm{cf}\,\lambda$ *and no A_α has an independent set of power $\lambda \Rightarrow \mathrm{ind}\, B = \lambda$ not attained.*

If $\varkappa = \mathrm{cf}\,\lambda$, it is easy to construct $A_\alpha \uparrow B$ for $\alpha < \varkappa$ where no A_α has an independent subset of power λ but B does.

Concerning free caliber in unions we have the following theorem, also easily established:

THEOREM 6.3. *Suppose $\varkappa$ is regular, $\varkappa \neq \mathrm{cf}\,\lambda$, and $A_\alpha \uparrow B$ for $\alpha < \varkappa$. Then $\lambda \in \mathrm{freecal}\, B$ if and only if $\exists \alpha < \varkappa\, (\lambda \leq |A_\alpha|)$ and*

$$\forall \alpha < \varkappa\, (\lambda \leq |A_\alpha| \Rightarrow \lambda \in \mathrm{freecal}\, A_\alpha).$$

If $\varkappa = \mathrm{cf}\,\lambda > \omega$, then the equivalence in 6.3 no longer holds: one can construct $A_\alpha \uparrow B$ where $\forall \alpha < \varkappa (|A_\alpha| < \lambda)$ while $\lambda \in \mathrm{freecal}\, B$, and one can construct $A_\alpha \uparrow B$ where $\exists \alpha < \varkappa (|A_\alpha| \geq \lambda)$ and

$$\forall \alpha < \varkappa\, (\lambda \leq |A_\alpha| \Rightarrow \lambda \in \mathrm{freecal}\, A_\alpha)$$

but $\lambda \notin \mathrm{freecal}\, B$.

7. Homomorphisms

We begin with the following easy theorem noticed by Sabine Koppelberg.

THEOREM 7.1. *Let I be an ideal in A. If neither A/I nor $I \cup -I$ has an independent set of power λ, then A does not either.*

PROOF. Suppose $X \in [A]^\lambda$ is independent. Then there exist distinct $x_1, \ldots, x_n \in X$ for which there are $\varepsilon_1, \ldots, \varepsilon_n \in 2$ such that $x_1^{\varepsilon 1} \cdot \ldots \cdot x_n^{\varepsilon n} \in I$.

Then

$$\{x_1^{\varepsilon 1} \cdot \ \ldots \ \cdot x_n^{\varepsilon n} \cdot y \colon y \in X \backslash \{x_1, \ldots, x_n\}\}$$

is a collection of λ independent elements in $I \cup -I$, contradiction.

COROLLARY 7.2.

$$\operatorname{ind} A = \max (\operatorname{ind} (A/I), \operatorname{ind} (I \cup -I));$$

it is attained in A if and only if it is attained in one of A/I, $I \cup -I$ which gives the maximum.

Now suppose that A is given, and we want to find B of prescribed independence with $A \twoheadrightarrow B$. There are some limitations on this For example, if A is wcc, then so is B, and $|\operatorname{ind} C|^\omega = |\operatorname{ind} C|$ for any wcc C by Theorem 11.4. Thus the following simple theorem is in a sense best possible.

THEOREM 7.3. *Suppose A has an independent set of power λ, where $\lambda^\omega = \lambda$. Then there is a B with $A \twoheadrightarrow B$ and* ind $B = \lambda$ *attained.*

PROOF. Let C be the subalgebra of A generated by an independent set of power λ. By Sikorski's extension theorem, there is a homomorphism f of A into C^{cmpl} extending $\mathrm{id} \restriction C$; $f[A] = B$ is as desired.

For λ a limit cardinal, the problem naturally arises concerning attainment in 7.3; to answer this question we adapt the proof of a theorem of Kunen, Shelah, Balcar, and Simon (unpublished), that cblco λ is a homomorphic image of $(\mathrm{Fr}\ \lambda)^{\mathrm{cmpl}}$ for $\lambda > \omega$.

THEOREM 7.4. *Suppose A has an independent set of power λ, where $\lambda^\omega = \lambda$ and λ is a limit cardinal, with $\mu^\omega < \lambda$ for all $\mu < \lambda$. Then there is a B with $A \twoheadrightarrow B$ and* ind $B = \lambda$ *not attained.*

PROOF. By Sikorski's extension theorem there is a homomorphism $A \twoheadrightarrow (\mathrm{Fr}\ \lambda)^{\mathrm{cmpl}}$, so we may work with $(\mathrm{Fr}\ \lambda)^{\mathrm{cmpl}}$ rather than A. We treat $(\mathrm{Fr}\ \lambda)^{\mathrm{cmpl}}$ as the class of regular open sets in ${}^\lambda 2$. Fix an arbitrary ultrafilter F on $(\mathrm{Fr}\ \lambda)^{\mathrm{cmpl}}$ and an arbitrary $x \in \bigcap_{a \in F} \bar{a}$. Let $\langle \Gamma_\xi \colon \xi < \mathrm{cf}\ \lambda \rangle$ be a partition of λ into cf λ pairwise disjoint sets of power λ. For all $\xi < \mathrm{cf}\ \lambda$ let $x_\xi \in {}^\lambda 2$ be defined by: $x_\xi \alpha = x\alpha$ for $\alpha \notin \Gamma_\xi$, $x_\xi \alpha = 1 - x\alpha$ for $\alpha \in \Gamma_\xi$. For each $y \in \mathrm{Fr}\ \lambda$ let sy be the support of y: the smallest $\Delta \in [\lambda]^{<\omega}$ such that y is a cylinder on Δ.

(1) If $a \in F$, $a = \Sigma_{i \in \omega} b_i$ with each $b_i \in \mathrm{Fr}\ \lambda$, and $\Gamma_\xi \cap \bigcup_{i \in \omega} sb_i = 0$, then $x_\xi \in \bar{a}$.

For, let $x_\xi \in U$, where $U = \{y \in {}^\lambda 2 \colon z \subseteq y\}$, z finite (a basic neighborhood of x_ξ). Let

$$V = \{y \in {}^\lambda 2 \colon z \restriction (\lambda \backslash \Gamma_\xi) \subseteq y\}.$$

Then $x \in V$, and $x \in \bar{a} = (\bigcup_{i\in\omega} b_i)^-$, so there is an $i \in \omega$ such that $V \cap b_i \neq 0$. Since $\Gamma_\xi \cap \mathrm{s}b_i = 0$, it follows that $U \cap b_i \neq 0$, as desired.

Now for all $\xi < \mathrm{cf}\,\lambda$ and all $\alpha \in \Gamma_\xi$ let

$$\mathscr{F}_{\xi\alpha} = \{a \in F\colon \text{there exists } \langle b_i\colon i \in \omega\rangle \in {}^\omega\mathrm{Fr}\,\lambda \text{ such that } a = \Sigma_{i\in\omega} b_i \text{ and } \Gamma_\xi \cap \bigcup_{i\in\omega} \mathrm{s}b_i = 0\} \cup \{\{y \in {}^\lambda 2\colon y\alpha = x_\xi\alpha\}\}.$$

Note by (1) that $\forall a \in \mathscr{F}_{\xi\alpha}(x_\xi \in \bar{a})$, so $\mathscr{F}_{\xi\alpha}$ has fip. Let

$E_\xi = \{G\colon G \text{ an ultrafilter on } (\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}},\ \exists \alpha \in \Gamma_\xi\ (\mathscr{F}_{\xi\alpha} \subseteq G)\}$,

$D = \bigcup_{\xi<\lambda} E_\xi$.

(2) If $\xi \neq \eta$, then $E_\xi \cap E_\eta = 0$.

For, suppose $G \in E_\xi \cap E_\eta$. Say $\mathscr{F}_{\xi\alpha} \subseteq G$, $\mathscr{F}_{\eta\beta} \subseteq G$. Let

$$a = \{y \in {}^\lambda 2\colon y\beta = x\beta\}.$$

Then a is closed-open and $x \in a$, so $a \in F$. Hence $a \in \mathscr{F}_{\xi\alpha} \subseteq G$. But

$$\{y \in {}^\lambda 2\colon y\beta = 1 - x\beta\} \in G$$

also, contradiction.

Let $s\colon (\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}} \rightarrowtail\!\!\!\!\twoheadrightarrow X$ be the Stone isomorphism onto the space X of ultrafilters of $(\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}}$. For any $x \in (\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}}$ and any $\xi < \mathrm{cf}\,\lambda$ let $g_\xi x = sx \cap E_\xi$. Thus g_ξ a homomorphism of $(\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}}$ onto a BA A_ξ.

(3) A_ξ has an independent set of power λ.

For, for each $\alpha \in \Gamma_\xi$ let

$$y_\alpha = \{z \in {}^\lambda 2\colon z\alpha = 1\}.$$

We claim that $\langle g_\xi y_\alpha\colon \alpha \in \Gamma_\xi\rangle$ is independent. To show this, suppose $\Delta \in [\Gamma_\xi]^{<\omega}$ and $\varepsilon \in {}^\Delta 2$. Choose $\beta \in \Gamma_\xi \backslash \Delta$. Let $z\alpha = x\alpha$ for all $\alpha \in \lambda\backslash\Gamma_\xi$, $z\alpha = \varepsilon\alpha$ for $\alpha \in \Delta$, $z\beta = x_\xi\beta$, and $z\alpha = 0$ otherwise. As in the proof of (1), $z \in \bar{a}$ for all $a \in \mathscr{F}_{\xi\beta}$, and $z \in \Pi_{\alpha\in\Delta} y_\alpha^{\varepsilon\alpha}$. Hence there is an ultrafilter G on $(\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}}$ such that $\mathscr{F}_{\xi\beta} \subseteq G$ and $\Pi_{\alpha\in\Delta} y_\alpha^{\xi\alpha} \in G$. Hence $G \in E_\xi \cap \bigcap_{\alpha\in\Delta} (\mathrm{s}y_\alpha)^{\varepsilon\alpha}$, as desired.

Now assume that $\omega \leq \mu_\xi \uparrow \lambda$ for $\xi < \mathrm{cf}\,\lambda$, with $\mu_\xi^\omega = \mu_\xi$. Let $B = \Pi^{<\omega_1}_{\xi<\mathrm{cf}\,\lambda} (\mathrm{Fr}\,\mu_\xi)^{\mathrm{cmpl}}$. By the argument in 2.1, B has independence λ not attained. For each $\xi < \mathrm{cf}\,\lambda$, let h_ξ be a homomorphism from Δ_ξ onto $(\mathrm{Fr}\,\mu)^{\mathrm{cmpl}}$ (using (3)). Finally, define

$$k\colon (\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}} \to \Pi_{\xi<\mathrm{cf}\,\lambda} (\mathrm{Fr}\,\mu_\xi)^{\mathrm{cmpl}}$$

by $(ky)_\xi = h_\xi g_\xi y$. Thus k is a homomorphism, and clearly the range of k is a BA with independence at least λ. We finish the proof by showing that k maps into B. In fact, take any $y \in (\mathrm{Fr}\,\lambda)^{\mathrm{cmpl}}$, and suppose without loss of generality that $y \in F$. Say $y = \Sigma_{i\in\omega} b_i$, where each $b_i \in \mathrm{Fr}\,\lambda$. If ξ is such that $\Gamma_\xi \cap \bigcup_{i\in\omega} \mathrm{s}b_i = 0$, then $y \in \mathscr{F}_{\xi\alpha}$ for any $\alpha \in \Gamma_\xi$, and so $y \in G$ for each $G \in E_\xi$. Thus $g_\xi y = 1$ in this case. This establishes that

$$\{\xi < \mathrm{cf}\,\lambda\colon (ky)_\xi \neq 1\}$$

is countable, as desired.

One may also try to improve 7.3 by finding a B as in its conclusion, with the additional requirement that $|B| = \varkappa$, $\varkappa$ some cardinal given in advance with $\lambda < \varkappa \leq |A|$. This is possible if A has an independent set of size λ, but not possible in general:

Theorem 7.5. *Suppose A has an independent set of size $\varkappa$, $\lambda \leq \varkappa$ and $\lambda^\omega = \lambda$, $\varkappa^\omega = \varkappa$. Then there is a B with $A \twoheadrightarrow B$,* ind *$B = \lambda$ attained, and $|B| = \varkappa$.*

Proof. First choose B as in the proof of 7.3: ind $B = \lambda$, $|B| = \lambda$, and there is a homomorphism $f: A \twoheadrightarrow B$. Next, by a result of Kunen, Shelah, Balcar, and Simon (unpublished), there is a homomorphism g of $(\mathrm{Fr}\,\varkappa)^{\mathrm{cmpl}}$ onto cbclo $\varkappa$. Hence it easily follows that there is a homomorphism h of A onto cblco $\varkappa$: by Sikorski's extension theorem one can get k: $A \to (\mathrm{Fr}\,\varkappa)^{\mathrm{cmpl}}$ so that $g \circ k$ is the desired h. Finally, let $la = (fa, ha)$ for all $a \in A$; $l[A]$ is as desired in the theorem.

Example 7.6. Let $\omega \leq \lambda \leq \varkappa$ and $A = [\varkappa]^{\leq\lambda}_{+\mathrm{co}}$. Thus $|A| = \varkappa^\lambda$ and ind $A = 2^\lambda$. Suppose f: $A \twoheadrightarrow B$ and $|B| \geq (2^\lambda)^+$; then, we claim, ind $B = 2^\lambda$ also. Thus the independence of A cannot be reduced among homomorphic images of power $\geq (2^\lambda)^+$.

To prove the claim, note that there is a set $C \subseteq [\varkappa]^{\leq\lambda}$ of power $(2^\lambda)^+$ such that $fx \neq fy$ for any two distinct $x, y \in C$. Let D be a subset of C of power $(2^\lambda)^+$ which is a Δ-system, say with kernel Δ. By throwing away at most one element of D we may assume that $fx \neq f\Delta$ for all $x \in D$. Let $\langle x_\alpha$: $\alpha < \lambda\rangle$ be a one-one list of λ of the members of D. Let I be a family of 2^λ independent subsets of λ. For each $\Gamma \in I$ let $b_\Gamma = f(\bigcup_{\alpha\in\Gamma}(x_\alpha\backslash\Delta))$. We claim that $\langle b_\Gamma$: $\Gamma \in I\rangle$ is an independent system of elements of B. In fact, if $\Gamma_0, \ldots, \Gamma_{m-1}$ are distinct elements of I, $\varepsilon_0, \ldots, \varepsilon_{m-1} \in 2$ and

$$\alpha \in \Gamma_0^{\varepsilon 0} \cap \ldots \cap \Gamma_{m-1}^{\varepsilon(m-1)},$$

then it is clear that

$$0 \neq f(x_\alpha\backslash\Delta) \subseteq b_{\Gamma 0}^{\varepsilon 0} \cdot \ldots \cdot b_{\Gamma(m-1)}^{\varepsilon(m-1)},$$

as desired.

Now we turn to free caliber and homomorphisms.

Theorem 7.7. *Let I be an ideal in A, and $\varkappa > |I|$. Then $\varkappa \in$* freecal *A if and only if $\varkappa \in$* freecal *(A/I).*

Proof. Note first that $\varkappa \leq |A|$ iff $\varkappa \leq |A/I|$.

Necessity. Suppose $\langle[x_\alpha]$: $\alpha < \varkappa\rangle$ is a system of distinct elements of A/I. Choose $\Gamma \in [\varkappa]^\varkappa$ so that $\langle x_\alpha$: $\alpha \in \Gamma\rangle$ is an independent system. If $\langle[x_\alpha]$:

$\alpha \in \Gamma\rangle$ is dependent, then there is a $\Delta \in [\Gamma]^{<\omega}$ and an $\varepsilon \in {}^{\Delta}2$ such that $y = \Pi_{\alpha\in\Delta} x_\alpha^{\varepsilon\alpha} \in I$. But then $\{y \cdot x_\beta : \beta \in \Gamma\backslash\Delta\}$ is a system of $\varkappa$ distinct elements of I, contradiction.

Sufficiency. Let $X \in [A]^\varkappa$. Each $[x]$ for $x \in X$ has $|I| < \varkappa$ elements, so if X' has one element of X from each class $[x]$, $x \in X$, then $|X'| = \varkappa$. Let $Y \in [X']^\varkappa$ be such that $\langle [x] : x \in Y\rangle$ is an independent system. Clearly Y is independent.

If $\varkappa \leq |I|$, then the equivalence of 7.9 no longer holds in general. Thus in $\mathrm{Fr}\,\varkappa \twoheadrightarrow \mathrm{finco}\,\varkappa$ we have

$$\mathrm{Freecal}\,(\mathrm{Fr}\,\varkappa) = \{\lambda : \omega_1 \leq \lambda \leq \varkappa, \mathrm{cf}\,\lambda > \omega\}$$

and $\mathrm{Freecal}\,(\mathrm{finco}\,\varkappa) = 0$. In $\mathrm{Fr}\,\varkappa \times \mathrm{finco}\,\varkappa \twoheadrightarrow \mathrm{Fr}\,\varkappa$, the situation is reversed.

8. m-independence

We discuss now a weaker kind of independence that will play a role below in our treatment of independence in infinite products; naturally our discussion of the weaker notion will not be so thorough as for ordinary independence; we do not consider for the weaker notion the algebraic questions like the above.

For $m \in \omega$, a set X is a BA A is *m-independent* if

$$\forall\, Y \in [X]^m \forall\, \varepsilon \in {}^Y 2 (\Pi_{y\in Y} y^{\varepsilon y} \neq 0).$$

Thus X is independent iff it is m-independent for every $m \in \omega$. There are some related notions:
X is *m-dependent* if

$$\forall\, Y \in [X]^m \exists\, \varepsilon \in {}^Y 2 (\Pi_{y\in Y} y^{\varepsilon y} = 0).$$

X is *strongly m-dependent* if

$$\forall\, \varepsilon \in {}^m 2\, \exists y : |X| \twoheadrightarrow X\, \forall \alpha \in {}^m|X|$$

(if α is strictly increasing then $\Pi_{i<m}(y\alpha_i)^{\varepsilon i} = 0$).

X is *very strongly m-dependent* if

$$\exists\, \varepsilon \in {}^m 2\, \forall\, Y \in^m [X]\, \forall y : m \twoheadrightarrow Y (\Pi_{i<m} y^{\varepsilon i} = 0).$$

Clearly X very strongly m-dependent $\Rightarrow X$ strongly m dependent $\Rightarrow X$ m-dependent $\Rightarrow X$ not m-independent.

We list some known results concerning these notions and state some additional facts.

(1) (T. Cramer [6]) For every $m \geq 1$ and every $\varkappa \geq \omega$ there is an hereditarily atomic BA of power $\varkappa$ having a subset X of power $\varkappa$ which is m-independent but very strongly $(m+1)$-dependent.

(2) (R. Bonnet [5]) For every $\varkappa \geq \omega$ with $2^\varkappa = \varkappa^+$ there is a BA A of power $\varkappa^+$ with no 2-independent subset of power $\varkappa^+$; in fact A has no system of pairwise incomparable elements of power $\varkappa^+$. For limit λ the situation is more complicated; see [7] for a description.

(3) (J. Baumgartner [4]) Con (every uncountable BA has an uncountable 2-independent subset). In [4] it is actually shown that Con (ZFC + MA + + $2^\omega > \omega_1$ + "every uncountable BA has an uncountable system X of pairwise incomparable elements"). Working in this theory, let A be any uncountable BA. If A has a system $\langle x_\alpha : \alpha < \omega_1 \rangle$ of pairwise disjoint non-zero elements, then $\{x_0 + x_\alpha : 0 < \alpha < \omega_1\}$ is a 2-independent set. Suppose A has no such system, and let X be an uncountable system of pairwise incomparable elements. By MA (ω_1), let X' be an uncountable subset of X having fip (see, e.g., Kunen [17], 2.23). Applying this also to $\{-x : x \in X'\}$, we obtain the desired 2-independent set $X'' \subseteq X'$.

(4) (S. Koppelberg) For any $\varkappa \geq \aleph_0$ the following conditions are equivalent:

(a) $\mathrm{cf}\, \varkappa > \aleph_0$,

(b) for every linear order L and every $P \in [\mathrm{intalg}\, L]^\varkappa$ there is a partition $\langle Q_i : i \in \omega \rangle$ of P such that $\forall i \in \omega \exists m \in \omega$ (Q_i is m-dependent), and $\mathrm{cf}\, \varkappa > \aleph_0$.

(c) for every linear order L and every $P \in [\mathrm{intalg}\, L]^\varkappa$

$$\exists Q \in [P]^\varkappa \exists m \in \omega \ (Q \text{ is } m\text{-dependent}).$$

The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are proved in [16]; the part of (b) without "$\mathrm{cf}\, \varkappa > \aleph_0$" holds in general. For completeness we give a proof for (c) $\Rightarrow$ (a) here. Suppose $\mathrm{cf}\, \varkappa = \aleph_0$. We prove a little more than $\neg$(c): we show $\exists$ linear order L

$$\exists P \in [\mathrm{intalg}\, L]^\varkappa \forall Q \in [P]^\varkappa \exists T \in [Q]^\omega \ (T \text{ is independent}).$$

Say $\omega \leq \lambda_i \uparrow \varkappa$ for $i < \omega$. Let $L = \eta + \varkappa$, and let $\langle x_i : i < \omega \rangle$ be independent elements of intalg η. For each $i < \omega$ let $M_i = \{x_i \cup [0, \alpha] : \alpha < \lambda_i\}$, and set $P = \bigcup_{i<\omega} M_i$. The above property is clear.

(5) Suppose $\omega_1 \leq \lambda \leq \varkappa$, λ regular. Then an easy Δ-system argument gives:

$$\forall X \in [\mathrm{finco}\, \varkappa]^\lambda \exists Y \in [X]^\lambda \ (Y \text{ is very strongly 3-dependent});$$

here one can take for the required ε the sequence $\langle 1, 1, 0 \rangle$ if X has $\varkappa$ finite members, otherwise $\langle 1, 0, 0 \rangle$. On the other hand,

$$\exists X \in [\mathrm{finco}\, \varkappa]^\lambda \ (X \text{ is 2-independent}).$$

(6) As noticed by Sabine Koppelberg, it easily follows from a result of Rubin [24] that every uncountable subset of an interval algebra contains an infinite set which is strongly 4-dependent. Hence no uncountable subset of an interval algebra is 4-independent. Combined with (1) this shows that there are hereditarily atomic algebras not embeddable in interval algebras. The result of Koppelberg is obtained by making a careful analysis of the homogeneity notion in section 5 of [24], then applying Ramsey's theorem.

(7) For each $\varkappa \geq \omega$ there is a BA of power $\varkappa$ having a subset P of power $\varkappa$ which is strongly 2-dependent, but such that

$$\forall m \geq 2\ \forall Q \in [P]^{\geq m}\ (Q \text{ is not very strongly } m\text{-dependent}).$$

In fact, we take

$$P = \{[0, \alpha) \colon \alpha \in \varkappa \backslash \{0\}\}$$

in intalg $\varkappa$.

(8) For each $\varkappa \geq \omega$ there is an interval algebra of power $\varkappa^+$ having a subset P of power $\varkappa^+$ which is 2-dependent, but such that

$$\forall m \geq 2 \forall Q \in [P]^{\varkappa^+}\ (Q \text{ is not strongly } m\text{-dependent}).$$

To construct such a BA, let L be a linear order of power $\varkappa^+$ which does not embed $\varkappa^+$ or $\varkappa^{+*}$ (if $\varkappa = \omega$ we can take $L \in [\mathbf{R}]^{\omega_1}$; otherwise we take a subset of ${}^{\varkappa}2$ lexicographically ordered); we take $A = \text{intalg}\, L$. Let $x \colon \varkappa^+ \rightarrowtail L$, and for each $\alpha \in \varkappa^+$ let $y_\alpha = (-\infty, x_\alpha)$. Thus $P = \{y_\alpha : \alpha \in \varkappa^+\}$ has power $\varkappa^+$ and is 2-dependent. Now we prove by induction on m, where $\bar{0}^m = \langle 0 \colon i < m\rangle$ and $\bar{1}^m = \langle 1 \colon i < m\rangle$:

$$(*)\qquad \forall m \geq 1\ \forall \Gamma \in [\varkappa^+]^{\varkappa^+} \forall \varepsilon \in {}^m2 \backslash \{\bar{0}^m, \bar{1}^m\}\, \exists \alpha_0 < \ldots < \alpha_{m-1} \text{ in } \Gamma$$
$$(\min \{x_{\alpha i} \colon \varepsilon i = 1\} > \max \{x_{\alpha i} \colon \varepsilon i = 0\})$$

Thus (*) holds for $m = 1$ vacuously, since ${}^12 = \{\bar{0}^1, \bar{1}^1\}$. Assume that (*) holds for m, and suppose that $\Gamma \in [\varkappa^+]^{\varkappa^+}$ and $\varepsilon \in {}^{m+1}2 \backslash \{\bar{0}^{m+1}, \bar{1}^{m+1}\}$.

We consider four cases.

Case 1. $\varepsilon \restriction m = \bar{0}^m$. Then $\varepsilon m = 1$. By $\varkappa^+ \rightarrow (\varkappa^+, \omega)$ there is a $\Delta \in [\Gamma]^\omega$ such that if $\alpha, \beta \in \Delta$ and $\alpha < \beta$ then $x_\alpha < x_\beta$. Any $m + 1$ elements of Δ satisfy the desired conclusion.

Case 2. $\varepsilon \restriction m = \bar{1}^m$. This is similar to Case 1.

Case 3. $\varepsilon \restriction m \notin \{\bar{0}^m, \bar{1}^m\}$ and $\varepsilon m = 1$. By the induction hypothesis there are $\Delta_0, \ldots, \Delta_\alpha, \ldots \subseteq \Gamma$, $\alpha < \varkappa^+$, each of power m, such that for each $\alpha < \varkappa^+$, if $\Delta_\alpha = \{\beta_{\alpha 0}, \ldots, \beta_{\alpha, m-1}\}$ with $\beta_{\alpha 0} < \ldots < \beta_{\alpha, m-1}$ then

$$\min \{x_{\beta \alpha i} \colon \varepsilon i = 1\} > \max \{x_{\beta \alpha i} \colon \varepsilon i = 0\}.$$

For each $\alpha < \varkappa^+$ let $\beta_{\alpha i_\alpha}$ be such that

$$\max \{x_{\beta \alpha i} \colon \varepsilon i = \sigma) = x_{\beta \alpha i_\alpha},$$

and let $\gamma_\alpha = \beta_{\alpha i_\alpha}$. Suppose that the conclusion of (*) fails. Then if $\alpha < \delta < \varkappa^+$ we have

$$\min(\{x_{\beta\alpha i}: \varepsilon i = 1\} \cup \{x_{\gamma\delta}\}) < \max\{x_{\beta\alpha i}: \varepsilon i = 0\},$$

i.e., $x_{\gamma\delta} < x_{\gamma\alpha}$. This embeds $\varkappa^{+*}$ in L, contradiction.

Case 4. $\varepsilon \upharpoonright m \in \{\bar{0}^m, \bar{1}^m\}$ and $\varepsilon m = 0$. Similar to Case 3.

Thus (*) holds. Now suppose $m \geq 2$, $Q \in [P]^{\varkappa+}$, and Q is strongly m-dependent. Choose $\varepsilon \in {}^m2$ and $z: |Q| \rightarrowtail\!\!\!\!\twoheadrightarrow Q$ accordingly. Recall $|Q| = \varkappa^+$. Say $z_\alpha = y_{\beta\alpha}$ for all $\alpha < \varkappa^+$. By $\varkappa^+ \to (\varkappa^+, \omega)$, choose $\Gamma' \in [\varkappa^+]^{\varkappa+}$ so that β is strictly increasing on Γ'. Let $\Gamma = \{\beta\alpha: \alpha \in \Gamma'\}$. Note that $\varepsilon \neq \bar{0}^m, \bar{1}^m$. We apply (*) to Γ and obtain $\alpha_0 < \ldots < \alpha_{m-1}$ in Γ' so that

$$\min\{x_{\beta\alpha i}: \varepsilon i = 1\} > \max\{x_{\beta\alpha i}: \varepsilon i = 0\}.$$

This means that $\Pi_{i<m}(z\alpha_i)^{\varepsilon i} \neq 0$, contradiction.

(9) For each $\varkappa \geq \omega$ there is an interval algebra of power $\varkappa^+$ having a subset P of power $\varkappa^+$ such that

$$\forall Q \in [P]^{\varkappa+} \text{ (Q has two elements which are 2-dependent and two which are 2-independent).}$$

Again we take a linear order L of power $\varkappa^+$ which does not embed $\varkappa^+$ or $\varkappa^{+*}$; we assume that L has a first element $0'$, and let $L' = \varkappa^+ + L$. We work in intalg L'. Let $x: \varkappa^+ \rightarrowtail L$, and for each $\alpha < \varkappa^+$ let $y_\alpha = [0, \alpha) \cup [0', x_\alpha)$. Let $\Gamma \in [\varkappa^+]^{\varkappa+}$. If for all $\alpha, \beta \in \Gamma$ with $\alpha < \beta$ we have $y_\alpha \subset y_\beta$, then $x_\alpha < x_\beta$, and $\varkappa^+$ is embedded in L. Hence there are $\alpha, \beta \in \Gamma$ with $\alpha < \beta$ and $y_\alpha \not\subseteq y_\beta$, and so $\{y_\alpha, y_\beta\}$ is independent. If for all $\alpha, \beta \in \Gamma$ with $\alpha < \beta$ we have $y_\alpha \not\subseteq y_\beta$, then $x_\beta < x_\alpha$, and $\varkappa^{+*}$ is embedded in L. So, there exist $\alpha, \beta \in \Gamma$ with $\alpha < \beta$ and $y_\alpha \subset y_\beta$, so $\{y_\alpha, y_\beta\}$ is dependent.

(10) For every $\varkappa \geq \omega$ there is an interval algebra of power $\varkappa$ with a subset of power $\varkappa$ which is 3-independent. We let $L = \varkappa + \varkappa + \varkappa + \varkappa$, and use 0 to 3 primes to denote elements of the four copies of $\varkappa$. For each $\alpha \in \varkappa\backslash\{0\}$ let

$$x_\alpha = [0, \alpha) \cup [\alpha', \infty) \cup [0'', \alpha'') \cup [(\alpha+1)'', \infty) \cup [\alpha''', (\alpha+1)''').$$

That $\{x_\alpha: \alpha \in \varkappa\backslash\{0\}\}$ is 3-independent is easily checked.

(11) For each $\varkappa \geq \omega$ there is a BA A of power $2^\varkappa$ having a subset P of power $2^\varkappa$ such that $\forall Q \in [P]^{\varkappa+}$ Q contains an infinite chain and also an infinite independent subset. For, let $(S, \leq_S)$ be an partial ordering of power $2^\varkappa$ with no chain of size $\varkappa^+$ and no antichain (pairwise incomparable set) of size $\varkappa^+$. Let A be free on generators a_t, $t \in S$, and let I be the ideal of A generated by all elements $a_t \cdot -a_s$ with $t \leq_S s$. Now we claim that if $s_0, \ldots, s_{m-1}, t_0, \ldots, t_{n-1}$ are distinct elements of S, then in A/I,

$$[a_{s0}] \cdot \ldots \cdot [a_{s(m-1)}] \cdot -[a_{t0}] \cdot \ \ldots \ \cdot [-a_{t(n-1)}] = 0 \text{ iff } \exists i < m \exists j < n\, (s_i \leq t_j).$$

The direction $\Leftarrow$ is obvious. For $\Rightarrow$, the left-hand side implies that there exist $u_0, v_0, \ldots, u_{p-1}, v_{p-1} \in S$ with $u_i \leq v_i$ for all $i < p$ such that

$$(*) \qquad a_{s0} \cdot \ldots \cdot a_{s(m-1)} \cdot -a_{t0} \cdot \ldots - a_{t(n-1)} \leq a_{u0} \cdot -a_{v0} + \ldots + a_{u(p-1)} \cdot -a_{v(p-1)};$$

we suppose that p is minimum so that (*) holds for some distinct $s_0, \ldots, s_{m-1}, t_0, \ldots, t_{n-1}$ such that $\forall i < m \forall j < n(s_i \leq t_j)$. By this minimality, $u_i \neq t_j$ for all i, j. If $\forall i < p \forall j < m\ (u_i \neq s_j)$, then clearly the left-hand side of (*) is 0, contradiction. Without loss of generality assume that $u_0 = s_0$. Then $\forall j < n\ v_0 \nleq t_j$, so

$$a_{v0} \cdot a_{s0} \cdot \ldots \cdot a_{s(m-1)} \cdot -a_{t0} \cdot \ldots \cdot -a_{t(n-1)} \leq$$
$$\leq a_{u1} \cdot -a_{v1} + \ldots + a_{u(p-1)} \cdot -a_{v(p-1)}$$

contradicts the minimality of p.

Now let $P = \{[a_t]: t \in S\}$ and take any $Q \in [P]^{\varkappa+}$, say $Q = \{[a_t]: t \in T\}$ with $T \in [S]^{\varkappa+}$. By $\varkappa^+ \to (\varkappa^+, \omega)$, T contains both an infinite chain and an infinite antichain, which by the claim gives the desired conclusion.

(12) By (4), if A is an uncountable interval algebra, then there is a $Q \in [A]^{\omega_1}$ such that Q has no infinite independent subset; thus there is no interval algebra A satisfying the property in (11) for $\varkappa = \omega$. We now consider a weaker condition on interval algebras:

(IA) For every $m \geq 2$ there is an interval algebra having a subset P of power ω_1 such that $\forall Q \in [P]^{\omega_1}$ Q has m pairwise comparable elements and m independent elements.

To treat this condition we consider the following related condition on linear orderings:

(LO) For every $m \geq 2$ and every permutation π of m there exist a linear ordering L_π and a function $x_\pi: \omega_1 \rightarrowtail L_\pi$ such that the following two conditions hold:

(a) $\forall \Gamma \in [\omega_1]^{\omega_1}\ \exists \alpha_0 < \ldots < \alpha_{m-1}$ in Γ such that for every permutation π of m, $x_\pi\alpha_0 < \ldots < x_\pi\alpha_{m-1}$;

(b) $\forall \Gamma \in [\omega_1]^{\omega_1}\ \exists \alpha_0 < \ldots < \alpha_{m-1}$ in Γ such that for every permutation π of m, $x_\pi\alpha_{\pi 0} < \ldots < x_\pi\alpha_{\pi(m-1)}$.

THEOREM 8.1. (LO) $\Rightarrow$ (IA).

PROOF. Let

$$P = \{\pi: \pi \text{ is a permutation of } m\}.$$

For each $\pi \in P$ we assume given x_π and L_π as in (LO), where L_π has a first element $0_\pi < x_\pi 0$. Let

$$M = \Sigma_{\pi \in P} L_\pi \text{ (with any linear ordering on } P).$$

Let $A = \operatorname{intalg} M$, and for each $\alpha < \omega_1$ let $a_\alpha = \bigcup_{\pi \in P}[0_\pi, x_\pi \alpha]$. Let $P = \{a_\alpha : \alpha < \omega_1\}$. Then (LO) (a) and (b) easily yield the desired result.

It is easily seen that if P is the partial ordering for adding ω_1 Cohen reals, then (LO) holds in V^P. Thus, e.g., (LO) and (IA) are consistent both with CH and with $\neg$ CH. The following problems remain open.

PROBLEM 1. Does (IA) hold in ZFC?

PROBLEM 2. Does (LO) hold in ZFC?

9. Infinite products

If I is an infinite set and $\langle A_i : i \in I\rangle$ is a system of non-trivial BA's, then $\Pi_{i\in I} A_i$ contains an isomorphic copy of PI, and so has an independent set of size $2^{|I|}$. This gives a lower bound on the independence of infinite products. Another, usually bigger, lower bound follows from the following set-theoretical fact found in [16] and [3]:

FACT. *If $\langle X_i : i \in I\rangle$ is a system of infinite sets, then there is an $F \subseteq \Pi_{i\in I} X_i$ with $|F| = |\Pi_{i\in I} X_i|$ such that for every $G \in [F]^{<\omega}$ there is an $i \in I$ with $|G| = |\{fi : f \in G\}|$.*

Thus if X_i is an infinite independent set in A_i for $i \in I$, then $\Pi_{i\in I} A_i$ has an independent set of size $|\Pi_{i\in I} X_i|$. Actually, in [16] the following stronger form of the above fact is proved:

FACT′. *Let $\langle X_\xi : \xi < \alpha\rangle$ be a system of infinite sets, where $|X_\xi| \leq |X_\eta|$ for $\xi < \eta < \alpha$. Assume that $\alpha = \lambda + m$ where λ is a limit ordinal and $m \in \omega$, and suppose that $|\Pi_{\xi<\lambda} X_\xi| \geq |\Pi_{\lambda<\xi<\alpha} X_\xi|$. Then there is an $F \subseteq \Pi_{\xi<\alpha} X_\xi$ with $|F| = |\Pi_{\xi<\alpha} X_\xi|$ such that for every $G \in [F]^{<\omega}$ there are infinitely many $\xi < \alpha$ such that $|G| = |\{f_i\} : f \in G\}|$.*

We can use this fact to obtain an even stronger lower bound, which we state in a special case.

THEOREM 9.1. *Let $\langle A_i : i \in \omega\rangle$ be a system of BA's and suppose that for every $m \geq 2$,*

$$|\{i \in \omega : \exists X \in [A_i]^\varkappa (X \text{ is } m\text{-independent}\}| \geq \omega,$$

where $\varkappa = (2^\omega)^+$. Then $\operatorname{ind} \Pi_{i\in\omega} A_i \geq \varkappa$.

COROLLARY 9.2. *For any* $\varkappa \geq 2^\omega$ *there is an hereditarily atomic BA B such that* ind $({}^\omega B) = \varkappa$.

PROOF. For each $i \in \omega$ let A_i be hereditarily atomic with $|A_i| = \varkappa$ and with a subset of power $\varkappa$ which is $(i+1)$-independent; see (1) in section 8. Then set $B = \Pi^w_{i\in\omega} A_i$.

On the other hand, the very first lower bound which we gave is in a sense best possible, as the following example shows.

EXAMPLE 9.3. ind $({}^\lambda\text{finco}\,\varkappa) = 2^\lambda$ for all infinite $\varkappa$, λ. For $\varkappa \leq 2^\lambda$ this is true on cardinality grounds, so assume that $2^\lambda < \varkappa$, and suppose that $B \in [{}^\lambda\text{finco}\,\varkappa]^\mu$, where $\mu = (2^\lambda)^+$; we show that B is not even 4-independent. Now

$$B = \bigcup_{f\in{}^\lambda 2}\{g \in B\colon \forall \alpha < \lambda(g\alpha \text{ is finite iff } f\alpha = 1)\}.$$

so there is an $f \in {}^\lambda 2$ and a $B' \in [B]^\mu$ such that $\forall g \in B' \forall \alpha \in \lambda$ ($g\alpha$ is finite iff $f\alpha = 1$). Now for all $g \in B'$ let

$$Hg = \{(\alpha, x)\colon g\alpha \text{ is finite and } x \in g\alpha\} \cup$$
$$\cup\, \{(\alpha, x)\colon g\alpha \text{ is cofinite and } x \in \varkappa \backslash g\alpha\}.$$

Thus $Hg \leq \lambda$. By the Δ-system lemma, there is a $B'' \in [B']^\mu$ such that $\langle Hg\colon g \in B'\rangle$ forms a Δ-system, say with kernel K. Now take any four distinct f, g, h, $k \in B''$, and suppose $\alpha \in \lambda$ is such that $(f\cdot g\cdot -h\cdot -k)\alpha \neq 0$; say

$$x \in f\alpha \cap g\alpha \cap (\varkappa\backslash\alpha) \cap (\varkappa\backslash h\alpha).$$

If $f\alpha$ is finite, so are $g\alpha$, $h\alpha$, and $k\alpha$, and

$$(\alpha, x) \in H_f \cap H_g = K \subseteq H_h,$$

contradiction. A similar contradition is reached if $f\alpha$ is cofinite.

The main problem concerning independence in infinite products concerns interval algebras:

PROBLEM 3. If A_i is an interval algebra for each $i \in I$, is ind $\Pi_{i\in I} A_i = 2^{|I|}$?

T. Carlson has shown that $\beth_2|I|$ is an upper bound for ind $\Pi_{i\in I} A_i$ when each A_i is an interval algebra.

THEOREM 9.4. *Let* $\varkappa$ *be an infinite cardinal. If* A_α *is an interval algebra for each* $\alpha < \varkappa$, *then* ind $\Pi_{\alpha<\varkappa} A_\alpha \leq \beth_2\varkappa$.

Proof. Let $\lambda = 2^\varkappa$. If L is a linearly ordered set, then each $x \in$ Intalg can be written in the form

$$x = [a_0^x, a_1^x) \cup \ldots \cup [a_{2n_x-2}^x, a_{2n_x-1}^x).$$

where $-\infty \leq a_0^x < a_1^x < \ldots < a_{2n_x-1}^x \leq \infty$; here $-\infty$ and ∞ are external to L. We call n_x the *length* of x. If x and y have the same length n, then they have *position* φ, where $\varphi \in {}^{2n\times 2n}3$, if $\forall i, j < 2n$ ($\varphi(i, j) = 0, 1$, or 2 according as $a_i^x < a_j^x$, $a_i^x = a_j^y$, or $a_i^x < a_j^y$. Now we claim:

(1) if x_0, x_1, x_2, x_3 have the same length n, and if x_i and x_j have position φ whenever $0 \leq i < j \leq 3$, then $x_0 \cdot -x_1 \cdot x_2 \cdot -x_3 = 0$.

To prove (1) it suffices to take i, j, k, l with $i, k < n$ and $j, l \in n \cup \{-1\}$ and show that

$$(2) \qquad [a_{2i}^{x_0}, a_{2i+1}^{x_0}) \cap [a_{2j+1}^{x_1}, a_{2j+2}^{x_1}) \cap [a_{2k}^{x_2}, a_{2k+1}^{x_2}) \cap [a_{2l+1}^{x_3}, a_{2l+2}^{x_3}) = 0.$$

where in general a_1^Y is $-\infty$ if $a_0^Y \neq -\infty$ and $a_{2n}^Y = \infty$ if $a_{2n-1}^Y \neq \infty$. Suppose (2) fails.

We consider only one typical case: $i \leq j$ and $k \leq l$. Then $a_{2l+1}^{x_3} < a_{2i+1}^{x_0}$. So

$$a_{2k+1}^{x_2} \leq a_{2l+1}^{x_3} < a_{2i+1}^{x_1} \leq a_{2j+1}^{x_1},$$

contradiction.

Hence (1) holds. Now suppose X is an independent subset of $\Pi_{\alpha<\varkappa} A_\alpha$ of power $(2^\lambda)^+$. Using $(2^\lambda)^+ \to (\lambda^+)^2_\lambda$ we can easily find a one-one $x \in {}^{\lambda^+}X$ such that for all $\alpha < \varkappa$ and all $\beta, \gamma, \delta, \varepsilon < \lambda^+$ with $\beta < \gamma$ and $\delta < \varepsilon$, the elements $x_\beta\alpha$ and $x_\gamma\alpha$ of A_α have the same length and position as $x_\delta\alpha$ and $x_\varepsilon\alpha$. But then by (1), $x_\beta \cdot -x_\gamma \cdot x_\delta \cdot -x_\varepsilon = 0$ whenever $\beta < \gamma < \delta < \varepsilon < \lambda^+$, contradiction.

The following is an obvious theorem about free caliber of infinite products.

Theorem 9.5. *Suppose $\varkappa \leq |\Pi_{i\in I} A_i|$, and for every system $\langle \lambda_i : i \in I\rangle$ of cardinals less than $\varkappa$ we have $\Pi_{i\in I}\lambda_i < \varkappa$. Then there is an $i \in I$ such that $\varkappa \leq |A_i|$, and $\varkappa \in$ freecal $\Pi_{i\in I} A_i$ if and only if*

$$\varkappa \in \bigcap \{\text{freecal } A_i : \varkappa \leq |A_i|\}.$$

Corollary 9.6. *If $\varkappa \leq |\Pi_{i\in I} A_i|$, $\varkappa = \lambda^+$, and $\lambda^{|I|} = \lambda$, then the conclusion of Theorem 9.5 holds.*

Note that if $\varkappa \leq 2^{|I|}$ and there is a linear order of power $2^{|I|}$ with a dense subset of power $\leq \varkappa$, then $\varkappa \notin$ freecal $\Pi_{i\in I} A_i$ if all A_i are non-trivial. Thus it is reasonable to assume that $2^{|I|} < \varkappa$ when considering a free caliber $\varkappa$ of an I-indexed product. Hence a natural conjecture would be that if $2^{|I|} < \varkappa$

$\varkappa \leq |A_i|$ for all $i \in I$, and

$$\varkappa \in \bigcap \{\text{freecal } A_i : i \in I\},$$

then $\varkappa \in \text{freecal } \Pi_{i \in I} A_i$, but this is not the case:

EXAMPLE 9.7. Let

$$A = \text{finco}\beth_\omega \times \text{Fr}\,(\beth_\omega)^+, \ \varkappa = (\beth_\omega)^+.$$

Thus $\varkappa \in \text{freecal } A$, but $\varkappa \notin \text{freecal } {}^\omega A$, by 9.3.

Another possible way to improve 9.5 is to consider $\varkappa \leq |\Pi_{i \in I} A_i|$ with $|A_i| < \varkappa$ for all $i \in I$. For example, the following problem arises.

PROBLEM 4. Let $\varkappa = \beth_\omega^+$, and for all $n \in \omega$ let $A_n = \text{Fr } \beth_n$. Then does $\Pi_{n \in \omega} A_n$ have free caliber $\varkappa$?

10. Free products

In [25] (Section 5) it is shown that for any infinite cardinal λ, if neither A nor B has an independent subset of size λ, then $A * B$ also does not. Thus

$$\text{ind } A * B = \max\,(\text{ind } A, \text{ind } B).$$

Hence it follows easily that if $\langle A_i : i \in I\rangle$ is a system of BA's, then

$$\text{ind } (*_{i \in I} A_i) = \max\,(|I|, \sup_{i \in I} \text{ind } A_i).$$

Now we turn to free caliber questions, first for finite free products.

THEOREM 10.1. *Let I be a finite and $\varkappa$ regular and uncountable. Then $\varkappa \in \text{freecal } (*_{i \in I} A_i)$ if and only if $\exists\, i \in I\ (\varkappa \leq |A_i|)$ and*

$$\varkappa \in \bigcap \{\text{freecal } A_i : \varkappa \leq |A_i|\}.$$

PROOF. The *necessity* is obvious.

Sufficiency. It suffices to take two factors A, B. Suppose $X \in [A*B]^\varkappa$. For each $x \in X$ we can write

$$x = \Sigma_{i < m_x} a_i^x \cdot b_i^x$$

where $a_i^x \in A$, $b_i^x \in B$, and $\langle b_i^x : i < m_x\rangle$ is a partition of 1. We may assume that $m_x = m$ does not depend on x.

Case 1. $\forall i < m(|\{a_i^x : x \in X\}| < \varkappa)$. By the regularity of $\varkappa$ we may assume that for all $i < m$, $a_i^x = a_i$ does not depend on x. Then we may assume that $\langle a_i : i < m\rangle$ is a partition of 1 (at the expense of dropping this assumption on $\langle b_i^x : i < m\rangle$). Now there is an $i < m$ such that $|\{b_i^x : x \in X\}| = \varkappa$; otherwise $|X| < \varkappa$. Then we may assume that $\langle b_i^x : x \in X\rangle$ is independent. Now X is independent, since if $Y, Z \subseteq X$ are disjoint and finite, then

$$\Pi_{x\in Y} x \cdot \Pi_{x\in Z} - x = \Pi_{x\in Y}\Sigma_{j<m} a_j \cdot b_j^x \cdot \Pi_{x\in Z}\Sigma_{j<m} a_j \cdot - b_j^x \geq$$
$$\geq \Pi_{x\in Y} a_i \cdot b_i^x \cdot \Pi_{x\in Z} a_i \cdot - b_i^x \neq 0.$$

Case 2. There is an $i < m$ such that $|\{a_i^x : x \in X\}| = \varkappa$. We may assume that $\langle a_i^x : x \in X\rangle$ is independent. If $|\{b_i^x : x \in X\}| < \varkappa$, then we may assume that $b_i^x = b_i$ does not depend on x, and hence X is independent. If $|\{b_i^x : x \in X\}| = \varkappa$, then we may assume that $\langle b_i^x : x \in X\rangle$ is independent, and hence X is independent.

In Case 2 of this proof we have not given the actual argument that X is independent, since it is the same, essentially, as in Case 1. Also in further proofs in this section we shall omit that step. Note that the various steps "we may assume" amount to restricting attention to a subset of X of power $\varkappa$.

For $\varkappa$ singular the situation is more complicated:

THEOREM 10.2. *Let I be finite, $|I| \geq 2$, and $\varkappa$ singular with* $\mathrm{cf}\,\varkappa > \omega$. *Then* $\varkappa \in \mathrm{freecal}\,(*_{i\in I} A_i)$ *if and only if one of the following conditions holds*:

(i) *there is exactly one $i \in I$ such that $\varkappa \leq |A_i|$, and* $\varkappa \in \mathrm{freecal}\, A_i$ *while* $\mathrm{cf}\,\varkappa \in \bigcap_{j\neq i} \mathrm{precal}\, A_j$;

(ii) *there are at least two $i \in I$ such that $\varkappa \leq |A_i|$*,

$$\varkappa \in \bigcap \{\mathrm{freecal}\, A_i : |A_i| \leq \varkappa\},$$

and $\mathrm{cf}\,\varkappa \in \bigcap_{i\in I} \mathrm{precal}\, A_i$.

PROOF.

Necessity. First assume that there is exactly one $i \in I$ such that $\varkappa \leq |A_i|$. Clearly $\varkappa \in \mathrm{freecal}\, A_i$. Suppose $j \neq i$ and $\mathrm{cf}\,\varkappa \notin \mathrm{precal}\, A_i$. Say $a \in {}^{\mathrm{cf}\,\varkappa}A_j$ such that $\forall \Gamma \in [\mathrm{cf}\,\varkappa]^{\mathrm{cf}\,\varkappa}\ \langle a_\alpha : \alpha \in \Gamma\rangle$ does not have fip. Say $\omega \leq \lambda_\alpha \uparrow \varkappa$ for $\alpha < \mathrm{cf}\,\varkappa$. For each $\alpha < \mathrm{cf}\,\varkappa$ let $X_\alpha \in [A_i]^{\lambda_\alpha}$. Then

$$\{a_\alpha \cdot b : \alpha < \mathrm{cf}\,\varkappa, b \in X_\alpha\}$$

is a subset of $*_{k\in I} A_k$ of power $\varkappa$ with no subset of power $\varkappa$ having fip, contradiction. Condition (iii) is treated similarly.

Sufficiency. First assume (i). Suppose $X \in [*_{j\in I} A_j]^\varkappa$. For each $x \in X$ write

$$x = \Sigma_{j<m_x} a_j^x \cdot b_j^x,$$

where $a_j^x \in A_i$ and $b_j^x \in B = *_{j\in I\setminus\{i\}} A_j$, and $\langle b_j^x : j < m_x\rangle$ is a partition of 1. We may assume that $m_x = m$ does not depend on x. Now there is a $j < m$ such that $|\{a_j^x : x \in X\}| = \varkappa$. We may assume that $\langle a_j^x : x \in X\rangle$ is independent.

We can then find $y \in {}^{\mathrm{cf}\,\varkappa}B$ such that

$$|\{x \in X : b_j^x = y_\alpha\}| \geq \lambda_\alpha$$

for all $\alpha < \mathrm{cf}\,\varkappa$, the λ_α's as above. Let $\Gamma \in [\mathrm{cf}\,\varkappa]^{\mathrm{cf}\,\varkappa}$ so that $\langle y_\alpha : \alpha \in \Gamma \rangle$ has fip; this is possible since $\mathrm{cf}\,\varkappa \in \mathrm{precal}\, B$. For each $\alpha \in \Gamma$ choose $Y_\alpha \in [X]^{\lambda_\alpha}$ so that $b_j^x = y_\alpha$ for all $x \in Y_\alpha$, and set $Z = \bigcup_{\alpha < \mathrm{cf}\,\varkappa} Y_\alpha$. It is then easily checked that Z is independent.

Now assume (ii). By an easy induction we may assume that we have just two algebras A and B, with

$$\varkappa \in \mathrm{freecal}\, A \cap \mathrm{freecal}\, B \text{ and } \mathrm{cf}\,\varkappa \in \mathrm{precal}\, A \cap \mathrm{precal}\, B.$$

Again we assume $X \in [A * B]^\varkappa$ with

$$x = \Sigma_{i<m} a_i^x \cdot b_i^x$$

for each $x \in X$, $\langle b_i^x : i < m \rangle$ a partition of 1.

Case 1. $\forall i < m\ |\{b_i^x : x \in X\}| < \varkappa$. Then $\exists i < m\ |\{a_i^x : x \in X\}| = \varkappa$, and we can proceed as for (i).

Case 2. $\exists i < m\ |\{b_i^x : x \in X\}| = \varkappa$. We may assume that $\langle b_i^x : x \in X \rangle$ is independent. For each $x \in X$ let

$$c_i^x = \Sigma_{j \neq i} a_j^x,\ d_i^x = a_i^x \cdot - c_i^x,\ e_i^x = c_i^x \cdot - a_i^x,\ f_i^x = a_i^x \cdot c_i^x.$$

Then

$$x = a_i^x \cdot b_i^x + c_i^x \cdot - b_i^x = d_i^x \cdot b_i^x + e_i^x \cdot - b_i^x + f_i^x.$$

Now if

$$|\{x \in X : d_i^x \neq 0 \text{ or } e_i^x \neq 0\}| < \varkappa,$$

then we may assume that $d_i^x = e_i^x = 0$ for all x, so $x \in A$ for all $x \in X$ and our result follows. So assume, by symmetry that $|\{x \in X : d_i^x \neq 0\}| = \varkappa$ hence assume that $d_i^x \neq 0$ for all $x \in X$. If $|\{d_i^x : x \in X\}| = \varkappa$, we may assume that $\langle d_i^x : x \in X \rangle$ is independent, and the result easily follows. Otherwise, we can again proceed as in (i).

Turning to infinite free products, we first take the case $\varkappa$ regular.

THEOREM 10.3. *Suppose I is infinite and $\varkappa$ is uncountable and regular. Then* $\varkappa \in \mathrm{freecal}\,(*_{i\in I} A_i)$ *if and only if* $\varkappa \leq |*_{i \in I} A_i|$ *and*

$$\forall i \in I (\varkappa \leq |A_i| \Rightarrow \varkappa \in \mathrm{freecal}\, A_i).$$

PROOF. The necessity is obvious. To prove the sufficiency, suppose $X \in [*_{i\in I} A_i]^\varkappa$. For all $x \in X$ choose $F_x \in [I]^{<\omega}$ so that $x \in *_{i \in F_x} A_i$. We may assume that $\langle F_x : x \in X \rangle$ forms a Δ-system, say with kernel G, and that for some $m \in \omega$, $|F_x| = m$ for all $x \in X$. If $F_x = G$ for all $x \in X$, the desired result follows by 10.1. So, assume that $F_x \neq G$ for some, and hence all, $x \in X$. For each $x \in X$ write

$$x = \Sigma_{i<n_x} a_i^x \cdot b_i^x,$$

where $a_i^x \in *_{i\in G} A_i$ and $b_i^x \in *_{i\in F_x/G} A_i$. We may assume that $n_x = n$ does not depend upon x, and that $\langle b_i^x : i < n\rangle$ is a partition of 1 for all $x \in X$. If

$$\forall i \leq n \, |\{a_i^x : x \in X\}| < \varkappa,$$

then we can proceed as in Case 1 of the proof of 10.1 without any reduction on acount of the b's, since their supports are disjoint and hence $\langle b_i^x : x \in X\rangle$ is independent for each $i < n$. So, assume that $|\{a_i^x : x \in X\}| = \varkappa$ for some $i < n$. We may assume that $\langle a_i^x : x \in X\rangle$ is independent; it is then easily checked that X is independent.

COROLLARY 10.4. *Suppose I is infinite, $\varkappa$ is uncountable and regular, and $|A_i| < \varkappa$ for all $i \in I$, and $\varkappa \leq |*_{i\in I} A_i|$. Then $\varkappa \in$* freecal $(*_{i\in I} A_i)$.

COROLLARY 10.5. *If $\varkappa$ is uncountable and regular, and $\varkappa \leq \lambda$, then $\varkappa \in$* freecal (Fr λ).

Concerning the history of this corollary, see the remark following 10.9 below.

To treat the singular case for infinite free products we need a double Δ-system lemma. This lemma is well-known, but for completeness we give a proof here; this proof follows a suggestion of Richard Laver.

THEOREM 10.6. *Let $\varkappa$ be a singular cardinal with* cf $\varkappa > \omega$, *and* cf $\varkappa < \lambda_\alpha \uparrow \varkappa$ *for $\alpha <$* cf $\varkappa$, *each λ_α a successor cardinal. Suppose $\langle A_\xi : \xi < \varkappa\rangle$ is a system of finite sets. Then there exist $\Gamma \subseteq$* cf $\varkappa$, *G, and sequences $\langle \Xi_\alpha : \alpha \in \Gamma\rangle$, $\langle F_\alpha : a \in \Gamma\rangle$ satisfying*

(i) *$\langle \Xi_\alpha : \alpha \in \Gamma\rangle$ is a system of pairwise disjoint subsets of $\varkappa$, with $|\Xi_\alpha| = \lambda_\alpha$ for all $\alpha \in \Gamma$.*

(ii) $|\Gamma| =$ cf $\varkappa$;

(iii) *$\langle A_\xi : \xi \in \Xi_\alpha\rangle$ is a Δ-system with kernel F_α, for every $a \in \Gamma$;*

(iv) *for distinct $\alpha, \beta \in \Gamma$ we have $F_\alpha \cap F_\beta = G$; further, if $\xi \in \Xi_\alpha$ and $\eta \in \Xi_\beta$ then $A_\xi \cap A_\eta = G$.*

PROOF. Write $\varkappa = \bigcup_{\alpha < \mathrm{cf}\,\varkappa} \Xi_\alpha$, where the Ξ_α's are pairwise disjoint and $|\Xi_\alpha| = \lambda_\alpha$ for every $\alpha <$ cf $\varkappa$. For each $\alpha <$ cf $\varkappa$ choose $\Xi'_\alpha \in [\Xi_\alpha]^{\lambda_\alpha}$ so that $\langle A_\xi : \xi \in \Xi'_\alpha\rangle$ is a Δ-system, say with kernel F_α. Choose $\Gamma \in [\mathrm{cf}\,\varkappa]^{\mathrm{cf}\,\varkappa}$ so that $\langle F_\alpha : \alpha \in \Gamma\rangle$ is a Δ-system, say with kernel G. For each $\alpha \in \Gamma$ let

$$B_\alpha = \bigcup \{\bigcup_{\xi\in\Xi_\beta'} A_\xi : \beta \in \Gamma, \beta < \alpha\};$$

thus $|B_\alpha| < \lambda_\alpha$. Hence there is a finite subset C_α of B_α and a $\Xi''_\alpha \in [\Xi'_\alpha]^{\lambda_\alpha}$ such that $A_\xi \cap B_\alpha = C_\alpha$ for all $\xi \in \Xi''_\alpha$. Note that $C_\alpha \subseteq F_\alpha$. Now let

$$\Xi'''_\alpha = \Xi''_\alpha \setminus \{\xi \in \Xi''_\alpha \colon (A_\xi \setminus F_\alpha) \cap F_\beta \neq 0 \text{ for some } \beta \in \Gamma\}.$$

Since $\operatorname{cf} \varkappa < \lambda_\alpha$, we still have $|\Xi'''_\alpha| = \lambda_\alpha$. It is easy to check that $\Gamma, G, \langle \Xi_\alpha \colon \alpha \in \Gamma \rangle$, $\langle F_\alpha \colon x \in \Gamma \rangle$ have the desired properties.

Our main result on free caliber in the singular case is as follows.

THEOREM 10.7. *Let $\varkappa$ be singular with $\operatorname{cf} \varkappa > \omega$, and suppose that $\varkappa \leq \leq |*_{i \in I} A_i|$. Then $\varkappa \in \operatorname{freecal}(*_{i \in I} A_i)$ if and only if* (i) *below holds, and exactly one of* (ii), (iii) *hold.*

(i) *If $G \in {}^{\operatorname{cf} \varkappa}([I]^{<\omega})$ consist of pairwise disjoint sets, and if $\langle X_\alpha \colon \alpha < \operatorname{cf} \varkappa \rangle$ is such that*

$$\forall \alpha < \operatorname{cf} \varkappa (X_\alpha \subseteq *_{j \in G_\alpha} A_j) \text{ and } \sup_{\alpha \in \operatorname{cf} \varkappa} |X_\alpha| = \varkappa,$$

then there is a system $\langle Y_\alpha \colon \alpha < \operatorname{cf} \varkappa \rangle$ such that $\forall \alpha < \operatorname{cf} \varkappa$ ($Y_\alpha \subseteq X_\alpha$ and Y_α is independent) and $\sup_{\alpha < \operatorname{cf} \varkappa} |Y_\alpha| = \varkappa$.

(ii) *There is an $i \in I$ such that $\varkappa \leq |A_i|$, $|*_{j \neq i} A_j| < \varkappa$, and $\operatorname{cf} \varkappa \in \bigcap_{j \neq i}$* precal A_j.

(iii) *For all $i \in I$ we have $|*_{j \neq i} A_j| \geq \varkappa$, $|A_i| \leq \varkappa \Rightarrow \varkappa \in$* freecal A_i, *and* $\operatorname{cf} \varkappa \in$ precal A_i.

PROOF. *Necessity.* (i) is clear. That exactly one of (ii), (iii) holds is seen as in the proof of 10.2, noticing in addition that if there is no $i \in I$ such that $\varkappa \leq |A_i|$ and $|*_{j \neq i} A_j| < \varkappa$, then $|*_{j \neq i} A_j| \geq \varkappa$ for all $i \in I$.

Sufficiency. First we note that it is sufficient to treat these two cases:

(1) For all $i \in I$ we have $\varkappa \leq |A_i|$, $\varkappa \in$ freecal A_i, and $\operatorname{cf} \varkappa \in$ precal A_i.

(2) For all $i \in I$ we have $|A_i| < \varkappa$, and $\operatorname{cf} \varkappa \in$ precal A_i.

For, suppose that we have proved the theorem in these two cases. If (ii) holds, then the desired conclusion follows directly from 10.2 (i). Assume (iii), and let $J = \{i \in I \colon |A_i| \geq \varkappa\}$. By (1), $\varkappa \in$ freecal $*_{j \in J} A_j$ if $J \neq 0$. Now if $|*_{i \in I \setminus J} A_i| < \varkappa$, then 10.2 (i) gives the desired result, while if $|*_{i \in I \setminus J} A_i| \geq \varkappa$ we can use the case (2) and then 10.2 (ii).

We give the proof for (1) and (2) simultaneously. Let $\langle \lambda_\alpha \colon \alpha < \operatorname{cf} \varkappa \rangle$ be as in the double Δ-system theorem. Suppose $X \in [*_{i \in I} A_i]^\varkappa$. For each $x \in X$ choose $F_x \in [I]^{<\omega}$ so that $x \in *_{i \in F_x} A_i$. Applying the double Δ-system theorem we get $\Gamma \in [\operatorname{cf} \varkappa]^{\operatorname{cf} \varkappa}$, $\langle Y_\alpha \colon \alpha \in \Gamma \rangle$, $\langle G_\alpha \colon \alpha \in \Gamma \rangle$, and H so that the Y_α's are pairwise disjoint subsets of X, $|Y_\alpha| = \lambda_\alpha$ for all $\alpha \in \Gamma$, $\langle F_x \colon x \in Y_\alpha \rangle$ is a Δ-system with kernel G_α for all $\alpha \in \Gamma$, and for distinct $\alpha, \beta \in \Gamma$ we have $G_\alpha \cap G_\beta = H$ and $F_x \cap F_y = H$ whenever $x \in Y_\alpha$ and $y \in Y_\beta$. We may assume that $\bigcup_{\alpha \in \Gamma} Y_\alpha = X$, $|F_x| = m$ for all $x \in X$, and $|G_\alpha| = n$ for all $\alpha \in \mathrm{Y}$.

Now we assume that $0 \neq |H| < n < m$; other possibilities are easier to treat than this one. For each $x \in X$ write

$$x = \Sigma_{i<p_x} a_i^x \cdot b_i^x \cdot c_i^x,$$

where $a_i^x \in *_{j\in H} A_j$, $b_i^x \in *_{j\in G_\alpha \setminus H} A_j$, $c_i^x \in *_{j\in F_x \setminus G_\alpha} A_j$, with $x \in Y_\alpha$ and $\langle a_i^x : i < p_x \rangle$ a partition of 1. We may assume that $p_x = p$ does not depend on x. Now we consider two cases.

Case 1. $\forall i < p(|a_i^x : x \in X\}| < \varkappa)$. Then there is an $i < p$ such that $|\{b_i^x \cdot c_i^x : x \in X\}| = \varkappa$. We may assume that $\langle b_i^x \cdot c_i^x : x \in X\rangle$ is one-one. Furthermore, there is a

$$Z \in [\{a_i^x : x \in X\}]^{\leq \mathrm{cf}\, \varkappa}$$

such that $|\{x \in X : a_i^x \in Z\}| = \varkappa$, and so we may assume that $\{a_i^x : x \in X\}$ has fip, since $\mathrm{cf}\, \varkappa \in$ precal $(*_{j\in H} A_j)$.

Subcase 1.1. $|\{b_i^x : x \in X\}| < \varkappa$. Then by the same argument we may assume that $\{b_i^x : x \in X\}$ has fip. Furthermore, $|\{c_i^x : x \in X\}| = \varkappa$ in this subcase, so we may assume that $\langle c_i^x : x \in X\rangle$ is one-one. It is now easy to check that X is independent.

Subcase 1.2. $|\{b_i^x : x \in X\}| = \varkappa$. Let $G'_\alpha = G_\alpha \setminus H$ for all $\alpha \in \Gamma$ and $X' = \{b_i^x : x \in Y_\alpha\}$. Then G' and X' satisfy the hypotheses of (i). It follows that we may assume that $\langle b_i^x : x \in X\rangle$ is independent. Now it is clear that we may assume that $\{c_i^x : x \in X\}$ has fip, and so X is independent.

Case 2. There is an $i < p$ such that $|\{a_i^x : x \in X\}| = \varkappa$. Note that then we are in case (1), and so 10.6 applies and we may assume that $\langle a_i^x : x \in X\rangle$ is independent. If $|\{b_i^x \cdot c_i^x : x \in X\}| = \varkappa$, we can proceed exactly as in the subcases 1.1 and 1.2. Assume that $|\{b_i^x \cdot c_i^x : x \in X\}| < \varkappa$. Note that then $|\{b_i^x : x \in X\}| < \varkappa$ and $|\{c_i^x : x \in X|\} < \varkappa$, by disjointness of supports. For each $x \in X$ let

$$d_i^x = \Sigma_{j\neq i} b_j^x \cdot c_j^x,\ e_i^x = b_i^x \cdot - d_i^x,\ f_i^x = d_i^x \cdot - b_i^x,\ g_i^x = b_i^x \cdot d_i^x.$$

Then

$$x = a_i^x \cdot e_i^x + - a_i^x \cdot f_i^x + g_i^x,$$

and

$$- x = - a_i^x \cdot e_i^x + a_i^x \cdot f_i^x + - e_i^x \cdot - f_i^x \cdot - g_i^x.$$

Since $|\{g_i^x : x \in X\}| < \varkappa$, the set

$$\{x \in X : e_i^x \neq 0 \text{ or } f_i^x \neq 0\}$$

has power $\varkappa$. So we may assume, say, that $|\{x \in X : e_i^x \neq 0\}| = \varkappa$, hence assume that $e_i^x \neq 0$ for all $x \in X$, hence assume that $\{e_i^x : x \in X\}$ has fip. Then X is independent.

COROLLARY 10.8. *Suppose $\varkappa$ is singular with* $\mathrm{cf}\, \varkappa > \omega$, $\forall i \in I$ $(|A_i| < \mathrm{cf}\, \varkappa)$, *and* $\varkappa \leq |*_{i\in I} A_i|$. *Then* $\varkappa \in$ freecal $(*_{i\in I} A_i)$.

COROLLARY 10.9. *Suppose* $\varkappa$ *is singular with* $\mathrm{cf}\,\varkappa > \omega$, *and* $\varkappa \le \lambda$. *Then* $\varkappa \in \mathrm{freecal}\,(\mathrm{Fr}\,\lambda)$.

Corollaries **10.6** and **10.9** generalize in one direction some results in the literature; see Efimov [8], [9], Gerlits [12], Hagler [13] and Haydon [14]. Those results deal with more general topological situations, but specialized to Boolean algebras they give that if $\varkappa$ is regular, or singular with $\mathrm{cf}\,\varkappa > \omega$, and if $\varkappa \le \lambda$ and A is a subalgebra of $\mathrm{Fr}\,\lambda$, of power $\ge \varkappa$, then there is an independent $X \in [A]^{\varkappa}$. So 10.5 and 10.9 generalize "subalgebra" to "subset". Moreover, our proofs are very simple compared to those in the indicated references; note that the proofs of **10.3** and 10.7 simplify further when applied to the situation of 10.5 and 10.9. The two corollaries 10.5 and 10.9 were announced in Monk [21].

REMARK 10.10. The condition (i) in **10.7** cannot be dropped, i.e., it does not follow from (iii), for example. To see this, assume $\varkappa = \aleph_{\omega_1} < 2^{\omega}$. For each $\alpha < \omega_1$ let A_α be the BA of subsets of ω generated by $\{\{n\}: n \in \omega\}$ together with $\aleph_\alpha$ pairwise almost disjoint sets. Then (iii) holds but (i) fails.

11. Wcc algebras

For the basic fact here, that $|\mathrm{ind}\,A|^{\omega} = |\mathrm{ind}\,A|$ for any wcc A, we need some preliminaries.

LEMMA 11.1. *If* A *satisfies ccc and* $A \subseteq B \subseteq A^{\mathrm{cmpl}}$, *where* B *is wcc, then* $B = A^{\mathrm{cmpl}}$.

PROOF. Let $x \in A^{\mathrm{cmpl}}$. Then we may write $x = \Sigma X$ and $-x = \Sigma Y$, where $X, Y \in [A]^{\le\omega}$. Thus $X \cdot Y = 0$, so there is a $b \in B$ with $X \le b$ and $b \cdot Y = 0$. So $x \le b$ and $-x \le -b$, i.e., $x = b$.

THEOREM 11.2. *If* A *is infinite wcc and has an independent set of size* λ, *then it has one of size* λ^{ω}.

PROOF. Let $X \in [A]^{\lambda}$ be independent. Let I be maximal among ideals of A such that $I \cap \mathrm{Sg}\,X = \{0\}$. Then the natural homomorphism $\pi\colon A \twoheadrightarrow A/I$ is one-one on $\mathrm{Sg}\,X$, and $\pi[\mathrm{Sg}\,X]$ is dense in A/I, so $\pi[\mathrm{Sg}\,X] \subseteq A/I \subseteq$ $\subseteq (\pi[\mathrm{Sg}\,X])^{\mathrm{cmpl}}$ and hence by Lemma 11.1, $A/I = (\pi[\mathrm{Sg}\,X])^{\mathrm{cmpl}}$. It follows by Balcar, Franěk, [3] that A/I has an independent set of size λ^{ω}, so A does too.

The following theorem is well-known:

THEOREM 11.3. *If A is wcc and $\langle x_n : n \in \omega\rangle$ is a system of pairwise disjoint elements of A, then for any $f \in \Pi_{n\in\omega} A \upharpoonright x_n$ there is an $a \in A$ such that $a \cdot x_n = fn$ for all $n \in \omega$.*

PROOF. Let $X = \{fn : n \in \omega\}$ and $Y = \{x_n \cdot -fn : n \in \omega\}$. Then $x \cdot y = 0$ for all $x \in X$ and all $y \in Y$. Hence there is an $a \in A$ such that $x \leq a$ for all $x \in X$ and $a \cdot y = 0$ for all $y \in Y$. Hence $fn \leq x_n \cdot a$ and $-f_n \cdot x_n \cdot a = 0$, hence $a \cdot x_n = fn$, for any $n \in \omega$.

Now we can prove the main theorem. For a simplification in the proof we are grateful to P. Simon.

THEOREM 11.4. *If A is wcc, then* $(\mathrm{ind}\, A)^\omega = \mathrm{ind}\, A$.

PROOF. Let $\lambda = \mathrm{ind}\, A$. If $\mathrm{ind}\, A$ is attained, the desired conclusion is clear from Theorem 11.2. Suppose $\mathrm{ind}\, A$ is not attained. By 11.2 we then have $\mu^\omega < \lambda$ whenever $\mu < \lambda$. If $\mathrm{cf}\,\lambda > \omega$, it follows that $\lambda^\omega = \lambda$, as desired.

Assume $\mathrm{cf}\,\lambda = \omega$; we shall get a contradiction. Say $\omega \leq \mu_n \uparrow \lambda$ for $n \in \omega$. Now we define by induction elements $x_0, x_1, \ldots$ of A. Suppose $x_0, \ldots, x_{n-1}$ have been defined so that $\mathrm{ind}\,(A \upharpoonright -x_0 \cdot \ldots \cdot -x_{n-1}) = \lambda$. Let $b = -x_0 \cdot \cdot \ldots \cdot -x_{n-1}$ for brevity. Choose $X \in [A \upharpoonright b]^{\mu_n}$ so that X is independent in $A \upharpoonright b$. For any $y \in X$ it is clear that both $A \upharpoonright (b \cdot y)$ and $A \upharpoonright (b \cdot -y)$ have independent subsets of size μ_n. Now by 1.1 choose $x_n \in X$ so that $\mathrm{ind}\,(A \upharpoonright (b \cdot -x_n)) = \lambda$. This completes the construction.

Note that for all $n \in \omega$. $A \upharpoonright x_n$ has an independent subset Y_n of size μ_n. Now by a main lemma of Balcar. Franěk [3], let Z be a subset of $\Pi_{n\in\omega} Y_n$ of size λ which is finitely distinguished, i.e., such that for any finite $F \subseteq Z$ there is an $n \in \omega$ such that $|\{fn : f \in F\}| = |F|$. For each $f \in Z$ choose $a_f \in A$ by 11.3 so that $a_f \cdot x_n = fn$ for all $n \in \omega$. Now we claim that $\langle a_f : f \in Z\rangle$ is independent (contradiction). Let F be a finite subset of Z and $\varepsilon \in {}^F2$. Choose $n \in \omega$ so that $|\{fn : f \in F\}| = |F|$. Then

$$\Pi_{f\in F} a_f^{\varepsilon f} \cdot x_n = \Pi_{f\in F} (\mathrm{fn})^{\varepsilon f} \cdot x_n \neq 0,$$

as desired.

Concerning attainment of independence in wcc algebras, we have the following result.

EXAMPLE 11.5. Suppose λ is an uncountable limit cardinal, with $\mu^\omega < \lambda$ for all $\mu < \lambda$, and $\mathrm{cf}\,\lambda > \omega$. Then there is a σ-BA A with $\mathrm{ind}\, A = \lambda$ not attained. For, say $\omega \leq \mu_\alpha \uparrow \lambda$ for $\alpha < \mathrm{cf}\,\lambda$. Let $A = \Pi^{<\omega_1}_{\alpha<\mathrm{cf}\,\lambda} (\mathrm{Fr}\,\mu_\alpha)^{\mathrm{cmpl}}$. That A has the desired properties is easily checked (cf. 2.1.).

EXAMPLE 11.6. If $\varkappa < \lambda$ and $\varkappa^\omega = \varkappa$, $\lambda^\omega = \lambda$, then there is a wcc algebra A with ind $A = \varkappa$ and $|A| = \lambda$. We can take $A = (\mathrm{Fr}\,\varkappa)^{\mathrm{cmpl}} \times \mathrm{cblco}\,\lambda$.

Concerning free caliber of wcc algebras, we note first of all that for any wcc A, $\varkappa \in \mathrm{freecal}\, A \Rightarrow 2^\omega < \varkappa$. This is because A contains a copy of $P\omega$, and freecal $P\omega = 0$. Next, note that if $\varkappa \leq \lambda$, $\varkappa^\omega = \varkappa$, $\lambda^\omega = \lambda$, then there is a wcc A with ind $A = \varkappa$, $|A| = \lambda$, and freecal $A = 0$; the algebra of 11.6 works here.

Complete BA's are the most important kind of wcc algebras. The result of Balcar and Franěk mentioned in the introduction is the most important fact about independence in complete Boolean algebras. A related result is found in S. Koppelberg [16]: any complete BA is completely generated by an independent set. The following theorem of Ralph McKenzie generalizes this result:

THEOREM 11.7. *Let A be an infinite complete BA, completely generated by a set with λ elements. Suppose $\lambda \leq \varkappa \leq |A|$. Then A can be completely generated by an independent set with $\varkappa$ elements.*

PROOF. If is easy to see that there is a partition $\langle a_i : i \in \omega \rangle$ of unity in A such that $|A \restriction a_i| = |A|$ for all $i \in \omega$. Now let $\langle b_\eta : \eta < \varkappa \rangle$ be a complete generating set for A, possibly with repetitions. By the Balcar—Franěk theorem, for each $i \in \omega$ let $\langle c^i_{j\delta} : j < \omega, \delta < \varkappa \rangle$ be an independent subset of $A \restriction a_i$. Now the following elements are $\varkappa$ in number and completely generate A: for each $\delta < \varkappa$ and $j \in \omega \backslash \{0\}$.

$$d_{0\delta} = b \cdot (a_0 + a_1) + \Sigma_{1<i} c^i_{0\delta};$$
$$d_{j\delta} = a_j + b_\delta \cdot a_{j+1} + \Sigma_{j+1<i} c^i_{j\delta}.$$

These elements are independent, since if $(j_0, \delta_0), \ldots, (j_n, \delta_n)$ are distinct members of $\omega \times \varkappa$, with $n \in \omega$, and $\varepsilon \in {}^{n+1}2$, we have, with $i = \max\{j_h : h \leq n\} + 2$,

$$a_i \cdot \Pi_{h \leq n} d^{\varepsilon h}_{j_h \delta_h} = \Pi_{h \leq n} (c^i_{j_h \delta_h})^{\varepsilon h} \cdot a_i \neq 0.$$

They completely generate A, since it is clear that for each $i > 0$,

$$\Sigma_{i \leq j} d_{j0} = \Sigma_{i \leq j} a_j,$$

and hence every a_i, $i \geq 0$, is generated. Furthermore, we have

$$\{d_{0\delta} \cdot a_0 : \delta < \varkappa\} = \{b_\delta \cdot a_0 : \delta < \varkappa\},$$
$$\{d_{0\delta} \cdot a_1 : \delta < \varkappa\} = \{b_\delta \cdot \alpha_1 : \delta < \varkappa\}$$

and, for all $i > 1$,

$$\{d_{i-1,\delta} \cdot a_i : \delta < \varkappa\} = \{b_\delta \cdot a_i : \delta < \varkappa\}.$$

Hence for all $i \geq 0$, each element of $A \restriction a_i$ is generated. So, every element of A is generated.

The free caliber of complete BA's is strongly related to chain conditions, and we discuss the situation in the next section. For now we mention only the following problem.

PROBLEM 5. Is $\beth_{\omega+1} \in \text{freecal}\,(\text{Fr}\,\beth_{\omega+1})^{\text{cmpl}}$?

12. Chain conditions

We have already used several times the basic result (S) of Shelah connecting chain conditions with independence. We begin our discussion by indicating some immediate consequences of (S).

COROLLARY 12.1.

(i) *If* $(2^\varkappa)^+ \leq |A|$ *and* A *satisfies the* $\varkappa^+$*-cc, then* A *has an independent set of power* $(2^\varkappa)^+$.

(ii) $|A| \leq 2^{\text{cell}\,A \cup \text{ind}\,A}$.

(iii) $\text{ind}\,A \leq 2^{\text{cell}\,A} \Rightarrow |A| \leq 2^{\text{cell}\,A}$.

(iv) *If* $2^{\text{cell}\,A} < |A|$, *then*

$$\text{ind}\,A \geq \sup\{(2^\varkappa)^{+n} : n \in \omega\backslash\{0\},\ (2^\varkappa)^{+n} \leq |A|\}.$$

(v) $|A|$ *strong limit* $\Rightarrow |A| = \text{cell}\,A \cup \text{ind}\,A$.

(vi) (GCH) $|A| \leq (\text{cell}\,A)^+ \cup \text{ind}\,A$ *unless* $|A| = \varkappa^+$ *with* $\varkappa$ *singular.*

PROOF (i), (ii) and (iii) are immediate from (S). For (iv), let $\lambda = \text{cell}\,A$ and suppose that $2^\lambda < |A|$. Suppose $n \in \omega\backslash\{0\}$ and $(2^\varkappa)^{+n} \leq |A|$. Using the law $(\mu^+)^\nu = \mu^\nu \cdot \mu^+$ we see that $((2^\varkappa)^{+(n-1)})^\lambda = 2^{\lambda \cup \varkappa} \cdot (2^\varkappa)^{+(n-1)}$. Thus if $\lambda \leq \varkappa$ we can apply (S) to $(2^\varkappa)^{+n}$ to get an independent set of power $(2^\varkappa)^{+n}$. If $\varkappa < \lambda$, then $(2^\varkappa)^{+n} \leq (2^\lambda)^{+m} \leq |A|$ for some $m \in \omega\backslash\{0\}$, and we can apply (S) to $(2^\lambda)^{+m}$. Condition (v) follows from (i). For (vi), assume GCH and $(\text{cell}\,A)^+ < |A|$. For $|A|$ limit (v) gives the desired result $|A| = \text{ind}\,A$. For $|A| = \varkappa^+$ with $\varkappa$ regular, (S) gives $|A| = \text{ind}\,A$.

Part (i) of this corollary was first proved by Šapirovskiĭ [25]. The restriction in (vi) will be discussed below.

Now we shall try to indicate to what extent (S) is best possible. We suppose $\varkappa$ and λ are infinite cardinals; given a BA A satisfying the $\varkappa$-cc, when does it have an independent set of power λ? By Erdős, Tarski [10] it is natural to always assume that $\varkappa$ is regular. To start with we shall also assume that λ is regular.

For $\lambda < \varkappa$, note that finco λ satisfies the $\varkappa$-cc but has no infinite independent set.

Now suppose $\lambda = \varkappa$. For $\varkappa$ a successor cardinal we can also find a strong counterexample to a generalization of (S):

Examples 12.2. Suppose $\mu^\omega > \mu$. Then there is an hereditarily atomic BA A of power μ^ω with exactly μ atoms; thus A satisfies the μ^+-cc but has no infinite independent set. In fact, if $\mu^\omega > \mu$ then there are μ^ω pairwise almost disjoint denumerable subsets of μ, and the BA generated by them together with $[\mu]^{<\omega}$ is as desired.

Even with $\mu^\omega = \mu$, there is an hereditarily atomic BA of power $> \mu$ with exactly μ atoms, as has been shown by Susan Dubisch (unpublished). In fact, if γ is minimum such that $\mu^\gamma > \mu$, there is such a BA of power μ^γ.

If $\varkappa$ is a limit cardinal with $\lambda = \varkappa$, we have three cases, since we assume $\varkappa$ is regular. First suppose $\varkappa$ is weakly inaccessible but not strongly ineccessible. Then again we get the strongest possible counterexample to improving (S): there is an hereditarily atomic BA of power $\varkappa$ satisfying $\varkappa$-cc. In fact, we can use the result of Dubisch mentioned in 12.2. Let μ be minimum such that $\varkappa \leq 2^\mu$. If $\mu = 2^\nu$ for some ν, then $2^\nu < \varkappa$, and the least γ such that $(2^\nu)^\gamma > 2^\nu$ is μ, so 12.2 applies. Now suppose μ is a limit cardinal. Say $\varrho_\xi \uparrow \mu$ for $\xi < \operatorname{cf}\mu$. Let $\nu = \sup_{\xi<\operatorname{cf}\mu} 2^{\varrho_\xi}$. Thus $\nu < \varkappa$. Clearly $\xi < \operatorname{cf}\mu \Rightarrow \nu^\xi = \nu$. Now

$$\nu^{\operatorname{cf}\mu} \geq \Pi_{\xi<\operatorname{cf}\mu} 2^{\varrho_\xi} = 2^\mu \geq \varkappa,$$

so 12.2 applies.

For $\varkappa$ strongly inaccessible, we can no longer expect to get an hereditarily atomic counterexample. If $\varkappa$ is strongly inaccessible but not weakly compact, assuming $V = L$ there is a linear order L of power $\varkappa$ such that any family of pairwise disjoint intervals in L has power $< \varkappa$. Thus intalg L has power $\varkappa$, satisfies the $\varkappa$-cc, but has no uncountable independent set.

Problem 6. In ZFC, or assuming only GCH, for $\varkappa$ strongly inaccessible but not weakly compact, is there a BA of power $\varkappa$ satisfying the $\varkappa$-cc which has no uncountable independent set (or no independent set of power $\varkappa$)?

Finally, for $\varkappa$ weakly compact, Shelah [26] proved the following extension of (S):

(S′) *For $\varkappa$ weakly compact, if A satisfies $\varkappa$-cc and $X \in [A]^\varkappa$ then there is a $Y \in [X]^\varkappa$ with Y independent.*

Now we turn to the case $\varkappa < \lambda$. If the hypothesis of (S) fails, then $\exists \mu < \lambda\ (\mu^{<\varkappa} \geq \lambda)$; since we assume that $\varkappa$ and λ are regular, $\exists \mu < \lambda\ \exists \nu <$

$< \varkappa\ (\mu^\nu \geq \lambda)$. Let μ be minimum such that $\exists \nu < \varkappa (\mu^\nu \geq \lambda)$, and let ν be minimum such that $\mu^\nu \geq \lambda$.

First suppose $\mu = 2$. Thus $\nu < \varkappa < \lambda \leq 2^\nu$, so this possibility disappears under GCH. If $\forall \varrho < \nu\ (2^\varrho < \varkappa)$, let $\sigma = \sup_{\varrho<\nu} 2^\varrho$; thus $\sigma < \varkappa$ by the regularity of $\varkappa$. Then there is a linear order L of power λ with a dense subset of power σ, so intalg L has power λ and satisfies the $\varkappa$-cc: it is the desired counterexample. So assume there is a $\varrho < \nu$ such that $\varkappa \leq 2^\varrho$. So we have $\varrho < \nu < \varkappa \leq 2^\varrho < < \lambda \leq 2^\nu$. In this situation we do not know about counterexamples:

PROBLEM 7. Assume that $\varrho < \nu < \varkappa \leq 2^\varrho < \lambda \leq 2^\nu$, with $\varkappa$ and λ regular. Is there a $\varkappa$-cc BA A of power λ with no independent set of power λ?

A positive solution to the following problem would yield a very strong positive solution to Problem 7.

PROBLEM 8. In ZFC, is there for every $\varkappa$ an hereditarily atomic BA of power $2^\varkappa$ with exactly $\varkappa$ atoms?

(Assuming GCH, there always is such a BA.)

A similar positive solution ot Problem 7 using interval algebras is not possible. Thus consider the following conditions.

(a) There is a linear order of power 2^{ω_1} satisfying the ω_2-cc.

(b) There is a linear order of power 2^{ω_1} with a dense subset of power ω_2.

Now (a) $\Rightarrow$ (b) by a well-known result; see Kurepa [19]. On the other hand, in Mitchell [20] it is shown that if $V \models$ GCH and P is the partial ordering for adding $\aleph_{\omega_1}$ Cohen reals, then in V^P, $2^\omega = \aleph_{\omega_1}$, $2^{\omega_1} = \aleph_{\omega_1+1}$, and for every $\varkappa < \aleph_{\omega_1}$ there is no linear order of power 2^{ω_1} with a dense subset of power $\varkappa$. Instead of ω_1 we can take any regular uncountable cardinal.

Second suppose $\mu \neq 2$, in which case $\mu \geq \omega_1$, and $\nu < \mu$. Then for any $\varrho < \varkappa$ such that $\mu^\varrho \geq \lambda$ we have $\forall \sigma < \mu\ (\sigma^\varrho < \mu)$; otherwise, $\sigma^\varrho \geq \mu^\varrho \geq \lambda$, contradicting minimality of μ. It follows that μ is singular and $\nu = \mathrm{cf}\,\mu$; also, $\mu^{<\varkappa} = \mu^{\mathrm{cf}\,\mu}$ and $\varrho^{<\varkappa} < \mu$ for all $\varrho < \mu$. So we have

$$(1) \qquad \mathrm{cf}\,\mu < \varkappa < \mu < \lambda \leq \mu^{\mathrm{cf}\,\mu} = \mu^{<\varkappa},\ \ \forall \varrho < \mu\ \ (\varrho^{<\varkappa} < \mu).$$

(In particular, $2^{\mathrm{cf}\,\mu} < \mu$, so $\mu^{\mathrm{cf}\,\mu} = \mu^+$ under the singular cardinals hypothesis.) Assuming GCH, Argyros [1] has given counterexamples in the case (1). Namely, he showed that if μ is singular strong limit, $2^\mu = \mu^+$, and $(\mathrm{cf}\,\mu)^\alpha = = \mathrm{cf}\,\mu$ for all $\alpha < \mathrm{cf}\,\mu$, then there is a BA A having precaliber $(\mathrm{cf}\,\mu)^+$, $|A| = \mu^+$, with no independent subset of power μ^+. The proof is based on the following lemma, relevant also to our later discussion. We gave a shorter proof than in [1].

Lemma 12.3. *Suppose A is a BA, $a \in {}^{\varkappa}A$, $\varkappa$ uncountable and $a[\varkappa]$ generates A. Suppose there is no $\Gamma \in [\varkappa]^{\varkappa}$ such that $a[\Gamma]$ has fip. Then A has no independent subset of power $\varkappa$.*

Proof. First we note:

(1) if $\langle Z_\alpha : \alpha < \varkappa \rangle$ is a system of pairwise disjoint finite subsets of $a[\varkappa]$, then $\langle \Sigma Z_\alpha : \alpha < \varkappa \rangle$ does not have fip.

For, otherwise we can find an ultrafilter F such that $\Sigma Z_\alpha \in F$ for all $\alpha < \varkappa$. Then for all $\alpha < \varkappa$ choose $y_\alpha \in Z_\alpha \cap F$. Thus $\{y_\alpha : \alpha < \varkappa\} \in [a[\varkappa]]^{\varkappa}$ has fip, contradiction. So, (1) holds.

Now suppose $X \in [A]^{\varkappa}$ is independent. For all $x \in X$ choose $Sx \in [a[\varkappa]]^{<\omega}$ so that $x \in \mathrm{Sg}\,(Sx)$. Choose $Y \in [X]^{\varkappa}$ so that $\langle Sx : x \in Y \rangle$ forms a Δ-system, say with kernel K. By further reductions we may assume that each $x \in Y$ can be written in the form

$$x = \Sigma_{i<m} c_i \cdot b_i^x,$$

where $c_i \cdot b_i^x \neq 0$, c_i is an atom of $\mathrm{Sg}\,K$, and b_i^x is an atom of $\mathrm{Sg}\,(Sx \backslash K)$. Let $\mathrm{AtSg}\,K = \{d_0, \ldots, d_{n-1}\}$. We define finite subsets $L_0, \ldots, L_{n-1}$ of Y by induction. Suppose $p < m$ and L_i has been defined for all $i < p$; let $M = \bigcup_{i<p} L_i$. If

$$\exists x \in Y \backslash M \; (x \cdot dp = 0 \text{ or } -x \cdot dp = 0),$$

let $Lp = \{x\}$ for some such x. Assume that

$$\forall x \in Y \backslash M \; (x \cdot dp \neq 0 \neq -x \cdot dp).$$

Let $\theta = \{i < m : c_i = dp\}$. For each $x \in Y \backslash M$ let

$$\varepsilon x = \begin{cases} 1 & \text{if there is no } i \in \theta \text{ such that } b_i^x = \Pi\{-y : y \in Sx \backslash K\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows that for each $x \in Y \backslash M$ there is a finite $\psi_x \subseteq Sx \backslash K$ such that $x^{\varepsilon x} \cdot dp \leq \Sigma \psi_x$. Since $\langle Sx \backslash K : x \in Y \backslash M \rangle$ is a family of $\varkappa$ pairwise disjoint subsets of $a[\varkappa]$, (1) yields that $\langle \Sigma \psi_x : x \in Y \backslash M \rangle$ does not have fip. This implies that there is a finite set $Lp \subseteq Y \backslash M$ such that $\Pi_{x \in L_p} x^{\varepsilon x} \cdot dp = 0$. This finishes the construction of $L_0, \ldots, L_{n-1}$. Clearly $L_0 \cup \ldots \cup L_{n-1}$ is dependent, contradiction.

The result of Argyros leaves the following problem open.

Problem 9. (In ZFC) Suppose $\mathrm{cf}\,\mu < \varkappa < \mu < \lambda \leq \mu^{\mathrm{cf}\,\mu} = \mu^{<\varkappa}$ and $\forall \varrho < \mu \; (\varrho^{<\varkappa} < \mu)$. Is there a BA of power λ satisfying $\varkappa$-cc with no independent subset of power λ?

Now we consider λ singular. Although (S) does not apply in this case, Shelah [26] proved the following, essentially reducing the independence problem to precaliber questions, except for an initial cardinality conditions.

(S″) *Suppose $\varkappa$ is regular, λ is singular, and $\forall \mu < \lambda (\mu^{<\varkappa} < \lambda)$. Then the following conditions are equivalent*:

(a) *Every BA of power λ satisfying $\varkappa$-cc has an independent set of power λ.*

(b) *Every BA satisfying $\varkappa$-cc has precaliber* $\mathrm{cf}\,\lambda$.

We now consider this cardinality conditions, and then discuss the precaliber notion.

If $\exists \mu < \lambda (\mu^{<\varkappa} \geq \lambda)$, λ singular and $\varkappa$ regular, then the nontrivial case is when $\varkappa < \lambda$. If in addition $\exists \nu < \varkappa (\mu^{\nu} \geq \lambda)$, then the analysis is essentially as before. So assume μ is minimum such that $\mu^{<\varkappa} \geq \lambda$, and $\forall \nu < \varkappa$ $(\mu^{\nu} < \lambda)$. Hence $\mu^{<\varkappa} = \lambda$, so $\varkappa$ is uncountable and weakly inaccessible. For $\mu = 2$ this gives the following problem.

Problem 10. Suppose that $\varkappa$ is uncountable and weakly inaccessible, $2^{\nu} < \lambda$ for all $\nu < \varkappa$, and $2^{<\varkappa} = \lambda$ is singular; thus $\mathrm{cf}\,\lambda = \varkappa$. Is there a $\varkappa$-*cc* BA of power λ with no independent subset of power λ?

A little cardinal arithmetic shows that μ must be 2 in the situation considered:

Theorem 12.4. *Suppose $\varkappa$ is weakly inaccessible, λ is singular μ is minimum such that $\mu^{<\varkappa} \geq \lambda$, and $\forall \nu < \varkappa (\mu^{\nu} < \lambda)$. Then $\mu = 2$.*

Proof. Suppose $\mu \neq 2$. Thus $\mu \geq \omega_1$ and $\mu^{<\varkappa} = \lambda$. Note that $\varkappa \leq \mu$, since $\mu < \varkappa \Rightarrow \mu^{<\varkappa} = 2^{<\varkappa}$, contradiction. Since $\forall \nu < \varkappa$ $(\mu^{\nu} < \lambda)$, we have $\mu < \lambda$. Now

$$\forall \nu < \varkappa \ \ \forall \varrho < \mu (\varrho^{\nu} < \mu), \tag{1}$$

for otherwise, for any $\sigma < \varkappa$ we would have $\varrho^{\nu \cup \sigma} \geq \mu^{\nu \cup \sigma}$ and hence $\varrho^{<\varkappa} \geq \mu^{<\varkappa} = \lambda$, contradicting the minimality of μ. Now (1) implies $\forall \nu < \varkappa$ $(\mu^{\nu} \leq \mu^{\mathrm{cf}\,\mu})$ and $\mathrm{cf}\,\mu < \varkappa$, so $\mu^{<\varkappa} = \mu^{\mathrm{cf}\,\mu}$, contradicting $\forall \nu < \varkappa (\mu^{\nu} < \lambda)$.

It is convenient now to treat chain conditions and free caliber, before returning to (S″). To do this, some special notation is useful. We write

$$\varkappa \xrightarrow{\text{precal}} (\lambda, \mu)$$

if for every BA A, and every $X \in [A]^{\varkappa}$, either $\exists Y \in [\mathrm{Sg}\, X])^{\lambda}$ (Y is pairwise disjoint) or $\exists Z \in [X]^{\mu}$ (Z has fip). Similarly

$$\varkappa \xrightarrow{\text{freecal}} (\lambda, \mu),$$

replacing "Z has fip" by "Z is independent". Using these symbols, many of the facts already mentioned in this section can be briefly formulated.

By way of summary, we shall give some of these formulations, some additional facts from the literature, and a new result. Note that

$$\varkappa \xrightarrow{\text{freecal}} (\lambda, \mu) \Rightarrow k \xrightarrow{\text{precal}} (\lambda, \mu).$$

Condition (S) and examples 12.2 give

(1) $\varkappa$, λ regular,

$$\forall \mu < \lambda \; (\mu^{<\varkappa} < \lambda) \Rightarrow \lambda \xrightarrow{\text{freecal}} (\varkappa, \lambda).$$

(2)

$$\mu^+ \overset{\text{freecal}}{\not\longrightarrow} (\mu^+, \omega).$$

(3) $\varkappa$ weakly inaccessible but not stongly inaccessible $\Rightarrow$

$$\varkappa \overset{\text{freecal}}{\not\longrightarrow} (\varkappa, \omega).$$

Conditions (2) and (3) contrast to the following result of Rosenthal [22]:

(4)

$$\varkappa \text{ regular} \Rightarrow \varkappa \xrightarrow{\text{precal}} (\varkappa, \omega).$$

The following result was mentioned before Problem 7; (6) is shown in [26].

(5) $(V = L)$ $\varkappa$ strongly inaccessible but not weakly compact $\Rightarrow$

$$\varkappa \overset{\text{freecal}}{\not\longrightarrow} (\varkappa, \omega_1).$$

(6) $(V = L)$ $\varkappa$ strongly inaccessible but not weakly compact $\Rightarrow$

$$\varkappa \overset{\text{precal}}{\not\longrightarrow} (\varkappa, \varkappa).$$

PROBLEM 11. $(V = L)$ $\varkappa$ strongly inaccessible but not weakly compact $\Rightarrow$

$$\varkappa \xrightarrow{\text{precal}} (\varkappa, \omega_1)?$$

Condition (S′) and the above result of Argyros give:

(7) $\varkappa$ weakly compact $\Rightarrow$

$$\varkappa \xrightarrow{\text{freecal}} (\varkappa, \varkappa).$$

(8) μ strong limit, $2^\mu = \mu^+$,

$$\forall \alpha < \operatorname{cf} \mu \big((\operatorname{cf} \mu)^\alpha = \operatorname{cf} \mu\big) \Rightarrow \mu^+ \overset{\text{precal}}{\not\longrightarrow} \big((\operatorname{cf} \mu)^+, \mu^+\big).$$

On the other hand, a construction of Erdős given in Kunen–Tall [18] gives a ccc BA A and $a \in {}^{\varkappa}A$ as in Lemma 12.3, assuming MA and $\varkappa = 2^\omega$. Thus

(9) (MA) $$2^\omega \overset{\text{precal}}{\not\longrightarrow} (\omega_1, 2^\omega).$$

The following result is due to Laver; see Galvin [11].

(10) $$\mu^+ = 2^\mu \Rightarrow \mu^+ \overset{\text{precal}}{\not\longrightarrow} (\mu^+, \mu^+).$$

Comparing this with (2) and (4) the following natural problem occurs.

PROBLEM 12. (GCH) $\mu^+ \xrightarrow{\text{precal}} (\mu^+, \omega_1)$?
Two additional result from [20] are:

(11) (MA) $\quad 2^\omega > \lambda \geq \operatorname{cf} \lambda > \omega \Rightarrow \lambda \xrightarrow{\text{precal}} (\omega_1, \lambda)$.

(12) $\varkappa$ regular, λ singular,

$$\forall \mu < \lambda\ (\mu^{<\varkappa} < \lambda) \Rightarrow \lambda^+ \xrightarrow{\text{freecal}} (\varkappa, \lambda)$$

The following theorem generalizes the corresponding precaliber result found in [2].

THEOREM 12.5. $\omega \leq \varkappa < \lambda$, *$\varkappa$ and λ regular, μ minimum such that*

$$\exists \nu < \varkappa\ (\mu^\nu \geq \lambda),\ \mu \geq \omega \Rightarrow \lambda \xrightarrow{\text{freecal}} (\varkappa, \mu).$$

PROOF. If $\mu = \lambda$, this follows from (S), so suppose $\mu < \lambda$. Let $\nu < \varkappa$ be minimum such that $\mu^\nu \geq \lambda$. Then

(1) $$\forall \varrho < \mu\ \forall \sigma < \varkappa\ (\varrho^\sigma < \mu).$$

For, otherwise $\varrho^{\sigma \cup \nu} \geq \mu^\nu \geq \lambda$, contradicting the minimality of μ.

(2) $$\varkappa < \mu.$$

For, $\mu \geq \omega$, so $\forall\, \varrho < \varkappa\ (2^\varrho < \lambda)$, hence for any $\varrho < \varkappa$,

$$\varkappa^\varrho \leq \Sigma_{\alpha < \varkappa} |\alpha|^\varrho < \lambda,$$

using λ regular and $\varkappa < \lambda$. Hence (2) follows. Now by (1) we have μ singular and $\nu = \operatorname{cf} \mu$. So $\operatorname{cf} \mu < \varkappa < \mu$ and hence $\forall\, \varrho < \mu\ (\varrho^{<\varkappa} < \mu)$, using (1) again. The desired result now follows from (12).

Comparing the above results with (S″) we obtain the following corollary:

COROLLARY 12.6. *Suppose $\varkappa$ is regular, λ is singular, $\varkappa < \lambda$, and $\forall\, \mu < \lambda$ $(\mu^{<\varkappa} < \lambda)$. Then under any of the following conditions* (i)–(iii), *every BA of power λ xatisfying the $\varkappa$-cc has an independent set of power λ*:

(i) $\forall\, \mu < \operatorname{cf} \lambda (\mu^{<\varkappa} < \operatorname{cf} \lambda)$.

(ii) (MA) $\varkappa = \omega_1 \leq \operatorname{cf} \lambda < 2^\omega$.

(iii) $k = \operatorname{cf} \lambda$ *is weakly compact.*

Under each of the following conditions (iv)—(viii) *there is a $\varkappa$-cc BA of power λ with no independent subset of power λ*:

(iv) ($V = L$) $\varkappa = \operatorname{cf}\lambda$ *is strongly inaccessible but not weakly compact.*

(v) $\operatorname{cf}\lambda < \varkappa$.

(vi) $\varkappa = (\operatorname{cf}\mu)^+$, $\operatorname{cf}\lambda = \mu^+$, μ *strong limit singular*, $2^\mu = \mu^+$,

$$\forall\alpha < \operatorname{cf}\mu\big((\operatorname{cf}\mu)^\alpha = \operatorname{cf}\mu\big).$$

(vii) (MA) $\varkappa = \omega_1$, $\operatorname{cf}\lambda = 2^\omega$.

(viii) $\varkappa = \operatorname{cf}\lambda = \mu^+ = 2^\mu$.

We thus see that under $V = L$ the situation is resolved. In ZFC the following problem, related to Problem 9, seems to be the most important in this connection.

Problem 13. (In ZFC) For μ singular, does

$$\mu^+ \xrightarrow{\text{precal}} \big((\operatorname{cf}\mu)^+, \mu^+\big)?$$

We conclude this section with some remarks on free calibers of complete BA's. Most questions about this do not lead to any new results or problems. Note that if L is a linear order of power 2^μ with a dense subset of power μ, then

$$\big(\text{freecal (intalg } L)^{\text{cmpl}}\big) = 0.$$

Because of independence results already mentioned, the following problem is natural:

Problem 14. *Is there for every μ a cBA A of power 2^μ with* freecal $A = 0$?

Along these lines the following result is easily established using the material of this section.

Theorem 12.9. *($V = L$) If $\lambda^\omega = \lambda$, then the following conditions are equivalent*:

(i) $\exists A$ (A *is a cBA*, $|A| = \lambda$, freecal $A = 0$).

(ii) λ *is neither singular nor weakly compact.*

13. The free caliber spectrum

It is natural to try to characterize those sets of cardinals which are the free caliber spectrum freecal A of some BA A, in cardinality terms. By remarks in the introduction,

$$\text{freecal } A \subseteq [\text{cell}' A, |A|]\backslash\{\varkappa \colon \operatorname{cf}\varkappa = \omega\}.$$

By section 10, we have equality for free BA's. Also recall that freecal (intalg ω) $= 0$.

Now for simplicity let us assume GCH and deal with regular cardinals only. Then we have:

THEOREM 13.1. (GCH) *If*

$$K = \text{freecal } A \cap \text{reg} \neq 0,$$

then the following conditions hold, where $\mu = \min K$ *and* $\nu = \sup K$:

(i) μ *is uncountable*;

(ii) *for all* $\lambda \in (\mu, \nu]$, *if* λ *is regular and not the successor of a singular cardinal, then* $\lambda \in K$;

(ii) *for all* $\lambda \in (\mu, \nu]$, *if* $\lambda = \sigma^+$ *for some singular* σ *with* $\mu \leq \text{cf}\,\sigma$, *then* $\lambda \in K$.

PROOF: by (S).

For given μ, ν with $\omega < \mu \leq \nu$, μ regular, the largest set K satisfying **13.1** (i)—(iii) is $[\mu, \nu] \cap$ reg, and this set can be realized:

THEOREM 13.2. *Suppose* $\omega < \mu \leq \nu$ *and* μ *is regular. Then there is a BA A such that*

$$\text{freecal } A = [\mu, \nu] \setminus \{\varkappa \colon \text{cf}\,\varkappa = \omega\}.$$

PROOF. If $\mu = \lambda^+$ for some λ, let $A = \text{finco}\,\lambda \times \text{Fr}\,\nu$, and use 1.4. If μ is weakly inaccessible, let

$$A = \text{Fr}\,\nu * (*_{\lambda<\mu}\,\text{finco}\,\lambda),$$

and use 10.3.

The smallest set K satisfying **13.1** (i)—(iii) can almost be realized:

THEOREM 13.3. (GCH) *Suppose* $\omega < \mu \leq \nu$ *and* μ *is regular. Then there is a BA A such that*

$$(\text{freecal } A \cap \text{reg}) \setminus \{\mu\} = \{\varkappa \in (\mu, \nu] \colon \varkappa \textit{ does not have the form } \sigma^+, \textit{ where } \sigma \textit{ is a singular cardinal such that } \text{cf}\,\sigma < \mu, \textit{ but } \varkappa \textit{ is regular}\}.$$

PROOF. First suppose $\mu = \lambda^+$ for some λ. Let

$$L = \{\sigma \colon \sigma^+ \in (\mu, \nu], \sigma \text{ singular}, \text{cf}\,\sigma < \mu\},$$

and for each $\sigma \in L$ let B_σ be the Argyros example (see the remark preceding 12.3): B_σ has precaliber $(\text{cf}\,\sigma)^+$, and is of power σ^+ with no independent subset of power σ^+. Then let

$$A = (\text{finco } \lambda) * (*_{\sigma \in L} B_\sigma) * \text{Fr } \nu.$$

Then use 10.3. The case μ weakly inaccessible is similar (see the proof of 13.2).

PROBLEM 15. If K is a set of regular cardinals with $\mu = \min K$ and $\nu = \sup K$, and if K satisfies 13.1 (i)–(iii), is there a BA A such that

$$\text{freecal } A \cap \text{reg} = K?$$

One can use the above simple constructions to settle this question affirmatively provided that the following generalization of the Argyros examples holds:

PROBLEM 16. If σ is singular, is there a BA A of power σ^+ such that

$$\text{freecal } A \cap \text{reg} = [(\text{cf } \sigma)^+, \sigma) \cap \text{reg}?$$

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