SINGULARY CYLINDRIC AND POLYADIC EQUALITY ALGEBRAS

BY

DONALD MONK[1]

This paper is a contribution to the structure and representation theory of finite-dimensional cylindric and polyadic equality algebras. The main result is that every singulary algebra is representable, where an algebra is *singulary* if it is generated by elements supported by singletons. In particular, all *prime* algebras are representable (a prime algebra has no proper subalgebra, and every cylindric or polyadic equality algebra has a prime subalgebra). Since every polyadic algebra can be embedded in a polyadic equality algebra the results also apply to them. Every infinite-dimensional singulary algebra is locally finite and hence is known to be representable, so we shall be concerned just with the finite-dimensional case, where previously very little was known about representation. It may be mentioned that for any dimension greater than one there are nonrepresentable cylindric and polyadic equality algebras[2]. For some special kinds of singulary algebras we obtain a direct construction of the representation, which brings out the structure of the algebra clearly.

The methods used essentially constitute an algebraic version of Behmann's solution of the decision problem for the singulary predicate calculus with equality[3]. Since only finitely many variables are available, the algebraization has some novel features. The essential ideas can be seen in the case of prime algebras, for which no advanced results about cylindric or polyadic algebras are needed. In the general case, however, use is made of the fact that a cylindric algebra is representable if every finitely generated subalgebra of it is.

In the last section of the paper we discuss the logical counterpart of the methods and results of the algebraic part of the paper.

1. **Introduction.** A *cylindric algebra of dimension* α (a CA_α) is a system

$$\mathfrak{A} = \langle A, +, \cdot, -, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$$

such that $\langle A, +, \cdot, -\rangle$ is a Boolean algebra, α is an ordinal, c_κ is a singulary

Presented to the Society, November 17, 1962 under the title *On finite dimensional cylindric algebras*; received by the editors April 2, 1963.

(1) The research herein reported on was supported by NSF grant G–19286. Most of the material appeared without proof in the abstract [12]. Several conversations with Professors Gebhard Fuhrken and Leon Henkin were stimulating in this research.

(2) See [2; 4; 7; 9] concerning these statements.

(3) See [1; 8].

operation on A and $d_{\kappa\lambda} \in A$ for all $\kappa, \lambda < \alpha$, and the following axioms hold for all $x, y \in A$ and for all $\kappa, \lambda, \mu < \alpha$:

C1. $x \leqq c_\kappa x$,

C2. $c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y$,

C3. $c_\kappa c_\lambda x = c_\lambda c_\kappa x$,

C4. $d_{\kappa\lambda} = d_{\lambda\kappa}$,

C5. $d_{\kappa\kappa} = 1$,

C6. $d_{\kappa\lambda} = c_\mu(d_{\kappa\mu} \cdot d_{\mu\lambda})$ *if* $\mu \neq \kappa, \lambda$,

C7. $c_\kappa(d_{\kappa\lambda} \cdot -x) = -c_\kappa(d_{\kappa\lambda} \cdot x)$ *if* $\kappa \neq \lambda$.

We will use without proof the most elementary properties of cylindric algebras(4). For example, the following statements hold for every CA_α $\mathfrak{A}$, all $\kappa, \lambda, \mu < \alpha$ and all $x, y \in A$:

(1) $c_\kappa 0 = 0$,

(2) $c_\kappa 1 = 1$,

(3) $c_\kappa c_\kappa x = c_\kappa x$,

(4) $c_\kappa - c_\kappa x = -c_\kappa x$,

(5) *if* $x \leqq y$ *then* $c_\kappa x \leqq c_\kappa y$,

(6) $x \cdot c_\kappa y = 0$ *if and only if* $c_\kappa x \cdot y = 0$,

(7) $c_\kappa(x + y) = c_\kappa x + c_\kappa y$,

(8) $c_\kappa(d_{\kappa\lambda} \cdot x \cdot y) = c_\kappa(d_{\kappa\lambda} \cdot x) \cdot c_\kappa(d_{\kappa\lambda} \cdot y)$ *if* $\kappa \neq \lambda$,

(9) $c_\mu d_{\kappa\lambda} = d_{\kappa\lambda}$ *if* $\mu \neq \kappa, \lambda$.

Certain auxiliary concepts are useful in this paper. If $\mathfrak{A}$ is a CA_α and $\kappa < \alpha$, the operation e_κ is defined to be $-c_\kappa-$, the composition of the three operations $-$, c_κ, and $-$. The operation e_κ corresponds to the universal quantifier in logic while c_κ corresponds to the existential quantifier. If $x \in A$, the *dimension set of* x, Δx, is the set $\{\kappa : c_\kappa x \neq x\}$. Thus $\Delta 0$ and $\Delta 1$ are both empty. If $\kappa, \lambda < \alpha$ and $x \in A$, we define

$$S(\lambda/\kappa)x = \begin{cases} x & \textit{if } \kappa = \lambda, \\ c_\kappa(d_{\kappa\lambda} \cdot x) & \textit{if } \kappa \neq \lambda. \end{cases}$$

Some common properties of these concepts which we will use without proof are as follows, where κ, λ, μ, x and y are arbitrary, as before:

(10) $e_\kappa 0 = 0$,

(11) $e_\kappa 1 = 1$,

(12) $e_\kappa x \leqq x$,

(13) $e_\kappa(x + e_\kappa y) = e_\kappa x + e_\kappa y$,

(14) $e_\kappa(x \cdot y) = e_\kappa x \cdot e_\kappa y$,

(15) *if* $x \leqq y$ *then* $e_\kappa x \leqq e_\kappa y$,

(16) $e_\kappa(-d_{\kappa\lambda} + x) = -e_\kappa(-d_{\kappa\lambda} + -x)$,

(17) $\Delta x = \Delta(-x)$,

(18) $\Delta(x \cdot y) \subseteq \Delta x \cup \Delta y$,

(4) For further information on cylindric algebras, as well as for notation not explicitly introduced here, see [7].

(19) $\Delta(c_\kappa x) \subseteq \Delta x \sim \{\kappa\}$,

(20) $\Delta(d_{\kappa\lambda})$ *is empty or equal to* $\{\kappa,\lambda\}$,

(21) $S(\lambda/\kappa)$ *is an endomorphism of* $\langle A, +, \cdot, - \rangle$,

(22) $S(\lambda/\kappa)c_\kappa = c_\kappa$,

(23) $S(\lambda/\kappa)d_{\kappa\mu} = d_{\lambda\mu}$ *if* $\kappa \neq \lambda, \mu$,

(24) $d_{\kappa\lambda} \cdot S(\lambda/\kappa)x = d_{\kappa\lambda} \cdot x$,

(25) $S(\mu/\lambda)S(\lambda/\kappa) = S(\mu/\lambda)S(\mu/\kappa)$.

The following unusual set-theoretic notation will be used later on. We let $K \hat{\times} \Lambda = \{\langle \kappa, \lambda \rangle : \kappa \in K \text{ and } \lambda \in \Lambda \text{ and } \kappa \neq \lambda\}$. Also, $|A|$ is the cardinal number of A. The identity mapping on a set A is denoted by δ_A. If f is a function, we let $f^{-1*}(A) = \{x : f(x) \in A\}$. If κ is an ordinal, then $\kappa = \{\lambda : \lambda < \kappa\}$.

2. **Lemmas.** In this section we shall prove a number of lemmas which revolve about several new concepts which are here introduced. The main technical details of the paper are absorbed in these lemmas. Throughout the section $\mathfrak{A}$ will be a fixed but arbitrary CA_α; we assume that $0 < \alpha < \omega$.

If $K, \Lambda \subseteq \alpha$, we let

$$\mathscr{D}(K,\Lambda) = \sum_{\langle \kappa\lambda \rangle \in K \hat{\times} \Lambda} d_{\kappa\lambda}.$$

In the following theorem we summarize some useful properties of these *generalized diagonal elements.*

THEOREM 1.

(i) $\mathscr{D}(0,\Lambda) = \mathscr{D}(K,0) = 0$,

(ii) $\mathscr{D}(\{\kappa\},\{\kappa\}) = 0$,

(iii) *if* $\kappa \neq \lambda$, *then* $\mathscr{D}(\{\kappa\},\{\lambda\}) = d_{\kappa\lambda}$,

(iv) *if* $|A| > 1$, $K \neq \Lambda$, *and* $K, \Lambda \neq 0$, *then* $\mathscr{D}(K,\Lambda) \neq 0$,

(v) $\mathscr{D}(\{\kappa\},\Lambda) = \mathscr{D}(\{\kappa\},\Lambda \sim \{\kappa\}) = \mathscr{D}(\{\kappa\},\Lambda \cup \{\kappa\})$,

(vi) $\mathscr{D}(K,\Lambda) + \mathscr{D}(\mathrm{M},\Lambda) = \mathscr{D}(K \cup \mathrm{M},\Lambda)$,

(vii) $\mathscr{D}(K,\Lambda) + \mathscr{D}(K,\mathrm{M}) = \mathscr{D}(K,\Lambda \cup \mathrm{M})$,

(viii) $\mathscr{D}(K,\Lambda) = \mathscr{D}(\Lambda,K)$,

(ix) *if* $\mu \notin K \cup \Lambda$, *then* $e_\mu\mathscr{D}(K,\Lambda) = \mathscr{D}(K,\Lambda)$,

(x) $\Delta e_\kappa \mathscr{D}(\{\kappa\},\Lambda) \subseteq \Lambda \sim \{\kappa\}$,

(xi) *if* $\kappa \notin \Lambda$, $\lambda \in \Lambda$, *and* $x \in A$, *then*

$$c_\lambda(\mathscr{D}(\{\kappa\},\Lambda)\cdot x) = c_\lambda(d_{\kappa\lambda}\cdot x) + \mathscr{D}(\{\kappa\},\Lambda \sim \{\lambda\})\cdot c_\lambda x,$$

(xii) *if* $\kappa \in K \cap \Lambda$, *then*

$$e_\kappa\mathscr{D}(K,\Lambda) = e_\kappa(\mathscr{D}(\{\kappa\},\Lambda) + \mathscr{D}(K,\{\kappa\})) + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\}),$$

(xiii) *if* $\kappa \in K \subseteq \Lambda$, *then* $e_\kappa\mathscr{D}(K,\Lambda) = e_\kappa\mathscr{D}(\{\kappa\},\Lambda) + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\})$,

(xiv) *if* $\kappa, \lambda \notin \Lambda$, *then* $S(\lambda/\kappa)\mathscr{D}(\{\kappa\},\Lambda) = \mathscr{D}(\{\lambda\},\Lambda)$,

(xv) *if* $\lambda \notin K \cup \Lambda$ *and* $\kappa \in K \cap \Lambda$, *then*

$$S(\lambda/\kappa)\mathscr{D}(K,\Lambda) = \mathscr{D}((K \sim \{\kappa\}) \cup \{\lambda\},(\Lambda \sim \{\kappa\}) \cup \{\lambda\}),$$

(xvi) *if* $\kappa, \lambda \in \Lambda$, *then*

$$\mathscr{D}(\{\kappa,\lambda\},\Lambda) = \mathscr{D}(\{\kappa\},\Lambda \sim \{\kappa,\lambda\}) + d_{\kappa\lambda} + \mathscr{D}(\{\lambda\},\Lambda \sim \{\kappa,\lambda\}),$$

(xvii) *if* $\kappa,\lambda \in \Lambda$, *then* $\mathscr{D}(\{\kappa,\lambda\},\Lambda) = \mathscr{D}(\{\kappa\},\Lambda \sim \{\lambda\}) + \mathscr{D}(\{\lambda\},\Lambda)$,

(xviii) *if* $\kappa \in K \subseteq \Lambda$ *and* $\lambda \in \Lambda \sim K$, *then*

$$\mathscr{D}(\{\kappa\},\Lambda\sim\{\lambda\})+\mathscr{D}(K\sim\{\kappa\},\Lambda\sim\{\kappa\})=\mathscr{D}(K,\Lambda\sim(K\cup\{\lambda\}))+\mathscr{D}(K\sim\{\kappa\},K\cup\{\lambda\}),$$

(xix) *if* $K \subseteq \Lambda$ *and* $\lambda \in \Lambda \sim K$, *then* $\mathscr{D}(\{\lambda\},\Lambda)+\mathscr{D}(K,\Lambda\sim\{\lambda\})=\mathscr{D}(K\cup\{\lambda\},\Lambda)$,

(xx) *if* $\kappa \notin K$, *then* $\mathscr{D}(K\cup\{\kappa\},K\cup\{\kappa\}) = \mathscr{D}(K,K) + \mathscr{D}(\{\kappa\},K)$.

Proof. We prove only the less trivial parts of this theorem, which are considered to be the following:

(iv) Say $K \sim \Lambda \neq 0$, and choose $\kappa \in K \sim \Lambda$. Choose $\lambda \in \Lambda$. Thus $\kappa \neq \lambda$, and hence $d_{\kappa\lambda} \leqq \mathscr{D}(K,\Lambda)$. Thus $1 = c_\kappa d_{\kappa\lambda} \leqq c_\kappa\mathscr{D}(K,\Lambda)$ from which it follows, since $0 \neq 1$, that $\mathscr{D}(K,\Lambda) \neq 0$.

(xii) Assuming $\kappa \in K \cap \Lambda$, we have

$$\begin{aligned} e_\kappa\mathscr{D}(K,\Lambda) &= e_\kappa(\mathscr{D}(K,\Lambda \sim \{\kappa\}) + \mathscr{D}(K,\{\kappa\})) && \text{(vii)}\\ &= e_\kappa(\mathscr{D}(\{\kappa\},\Lambda \sim \{\kappa\}) + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\}) + \mathscr{D}(K,\{\kappa\}) && \text{(vi)}\\ &= e_\kappa(\mathscr{D}(\{\kappa\},\Lambda) + \mathscr{D}(K,\{\kappa\})) + \mathscr{D}(K \sim \{\kappa\}, \Lambda \sim \{\kappa\}). \end{aligned}$$

(xiii) The statement may be proved by applying (xii) and (viii), noting that under the given assumption one has $\mathscr{D}(K,\{\kappa\}) \leqq \mathscr{D}(\Lambda,\{\kappa\})$.

(xviii) First note that under the given assumptions we have

$$[\Lambda \sim (K\cup\{\lambda\})] \cup (K \sim \{\kappa\}) = \Lambda \sim \{\kappa,\lambda\}.$$

Hence

$$\begin{aligned} &\mathscr{D}(\{\kappa\},\Lambda \sim \{\lambda\}) + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\})\\ &= \mathscr{D}(\{\kappa\},\Lambda \sim \{\kappa,\lambda\}) + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa,\lambda\}) + \mathscr{D}(K \sim \{\kappa\},\{\lambda\})\\ &= \mathscr{D}(K,\Lambda \sim \{\kappa,\lambda\}) + \mathscr{D}(K \sim \{\kappa\},\{\lambda\})\\ &= \mathscr{D}(K,\Lambda \sim (K\cup\{\lambda\})) + \mathscr{D}(K,K \sim \{\kappa\}) + \mathscr{D}(K \sim \{\kappa\},\{\lambda\})\\ &= \mathscr{D}(K,\Lambda \sim (K\cup\{\lambda\})) + \mathscr{D}(K \sim \{\kappa\},K\cup\{\lambda\}). \end{aligned}$$

Q.E.D.

Suppose $x \in A$ and $\Delta x \subseteq \{0\}$. Then for $\kappa < \alpha$ we let $x[\kappa] = S(\kappa/0)x$; if $K \subseteq \alpha$ we let $x(K) = \sum_{\kappa \in K} x[\kappa]$. Thus, in particular, $x(0) = 0$. Elementary properties of this concept are given in the following theorem.

THEOREM 2. *Assume that* $x \in A$ *and* $\Delta x \subseteq \{0\}$. *Then*

(i) *if* $\kappa \notin K$, *then* $c_\kappa(x(K)\cdot y) = x(K)\cdot c_\kappa y$,

(ii) $\Delta x(K) \subseteq K$,

(iii) $S(\lambda/\kappa)x[\kappa] = x[\lambda]$,

(iv) *if* $\kappa \notin K$, *then* $x[\kappa]\cdot\mathscr{D}(\{\kappa\},K) \leqq x(K)$,

(v) *if* $\kappa \in K \cap \Lambda$, *then*

$e_\kappa[\mathscr{D}(K,\Lambda) + x(K)]$

$$= e_\kappa[\mathscr{D}(\{\kappa\},K) + \mathscr{D}(\Lambda,\{\kappa\}) + x[\kappa]] + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\}) + x(K \sim \{\kappa\}),$$

(vi) *if* $\kappa \in K \subseteq \Lambda$, *then*

$$e_\kappa[\mathscr{D}(K,\Delta) + x(K)] = e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa]] + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\}) + x(K \sim \{\kappa\}).$$

Proof. The least trivial part of the theorem is part (v). Under the hypothesis of (v) we have

$e_\kappa[\mathscr{D}(K,\Lambda) + x(K)]$

$= e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + \mathscr{D}(K,\{\kappa\}) + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\}) + x(K \sim \{\kappa\}) + x[\kappa]]$

$= e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + \mathscr{D}(K,\{\kappa\}) + x[\kappa]] + \mathscr{D}(K \sim \{\kappa\},\Lambda \sim \{\kappa\}) + x(K \sim \{\kappa\})$. Q.E.D.

The next two theorems become more transparent if one considers the special case $\alpha = 3$ and $x = 0$. One then obtains as a special case of the second lemma the equation

$$e_0e_1(d_{01} + d_{02} + d_{12}) = e_0e_1e_2(d_{01} + d_{02} + d_{12}).$$

This equation holds in every CA_3. The logical counterpart of this equation is the equivalence

$$\bigwedge v_0 \bigwedge v_1(v_0 = v_1 \vee v_0 = v_2 \vee v_1 = v_2)$$
$$\leftrightarrow \bigwedge v_0 \bigwedge v_1 \bigwedge v_2(v_0 = v_1 \vee v_0 = v_2 \vee v_1 = v_2),$$

which is universally valid. The inequality in Theorem 4 is just a generalized version of this simple equality.

THEOREM 3. *Suppose* $K \subseteq \alpha$, $\kappa \in \alpha$, $\lambda \in K$, $x \in A$, *and* $\Delta x \subseteq \{0\}$. *Then*

$$e_\kappa[\mathscr{D}(\{\kappa\},K) + x[\kappa]] \leqq e_\kappa e_\lambda[\mathscr{D}(\{\kappa,\lambda\},K) + x(\{\kappa,\lambda\})].$$

Proof. For brevity let $y = e_\kappa[\mathscr{D}(\{\kappa\},K) + x[\kappa]]$. We may assume without loss of generality that $\kappa \neq \lambda$. Then

$$\begin{aligned}
y &= e_\kappa[\mathscr{D}(\{\kappa\},K \sim \{\kappa\}) + x[\kappa]] && \text{Theorem 1 (v)}\\
&\leqq c_\lambda(y \cdot [\mathscr{D}(\{\kappa\},K \sim \{\kappa\}) + x[\kappa]]) && \text{§1(12), C1}\\
&= c_\lambda(\mathscr{D}(\{\kappa\},K \sim \{\kappa\}) \cdot y) + x[\kappa] \cdot c_\lambda y && \text{Theorem 2 (i)}\\
&= c_\lambda(d_{\kappa\lambda} \cdot y) + \mathscr{D}(\{\kappa\},K \sim \{\kappa,\lambda\}) \cdot c_\lambda y + x[\kappa] \cdot c_\lambda y && \text{Theorem 1 (xi)}\\
&\leqq c_\lambda(d_{\kappa\lambda} \cdot y) + \mathscr{D}(\{\kappa\},K \sim \{\kappa,\lambda\}) + x[\kappa]\\
&= e_\lambda(-d_{\kappa\lambda} + y) + \mathscr{D}(\{\kappa\},K \sim \{\kappa,\lambda\}) + x[\kappa]\\
&= e_\lambda[-d_{\kappa\lambda} + y + \mathscr{D}(\{\kappa\},K \sim \{\kappa,\lambda\}) + x[\kappa]].
\end{aligned}$$

Hence

$$\begin{aligned}
y &\leqq e_\lambda e_\kappa[-d_{\kappa\lambda} + y + \mathscr{D}(\{\kappa\}, K \sim \{\kappa,\lambda\}) + x[\kappa]] \\
&= e_\lambda(y + e_\kappa[-d_{\kappa\lambda} + \mathscr{D}(\{\kappa\}, K \sim \{\kappa,\lambda\}) + x[\kappa]]) \\
&= e_\lambda(y + S(\lambda/\kappa)[\mathscr{D}(\{\kappa\}, K \sim \{\kappa,\lambda\}) + x[\kappa]]) \\
&= e_\lambda(y + \mathscr{D}(\{\lambda\}, K \sim \{\kappa,\lambda\}) + x[\lambda]) && \text{Theorem 1 (xiv), Theorem 2 (iii)} \\
&= e_\kappa e_\lambda[\mathscr{D}(\{\kappa\}, K \sim \{\kappa\}) + \mathscr{D}(\{\lambda\}, K \sim \{\kappa\}) + x[\kappa] + x[\lambda]] \\
&= e_\kappa e_\lambda[\mathscr{D}(\{\kappa,\lambda\}, K \sim \{\kappa\}) + x(\{\kappa,\lambda\})] && \text{Theorem 1 (vi)} \\
&= e_\kappa e_\lambda[\mathscr{D}(\{\kappa,\lambda\}, K) + x(\{\kappa,\lambda\})] && \text{Q.E.D.}
\end{aligned}$$

For the next theorem we need the notion of a generalized e-operator. If $K \subseteq \alpha$, say with $K = \{\kappa(0), \cdots, \kappa(m)\}$, let $e(K) = e_{\kappa(0)} \cdots e_{\kappa(m)}$; in case $K = 0$, let $e(K)$ be the identity function on A. According to §1,C3, this definition is unambiguous. We shall refrain from enumerating the very easy properties of this generalized e-operator.

THEOREM 4. *Suppose* $K \subseteq \Lambda \subseteq \alpha$, $\kappa \in \Lambda$, $x \in A$, *and* $\Delta x \subseteq \{0\}$. *Then*

$$e(K)[\mathscr{D}(K,\Lambda) + x(K)] \leqq e(K \cup \{\kappa\})[\mathscr{D}(K \cup \{\kappa\}, \Lambda) + x(K \cup \{\kappa\})].$$

Proof. We may assume that $\kappa \notin K$. The inequality is obvious if $K = 0$, and is given by Theorem 3 for $|K| = 1$. Hence assume $|K| > 1$.

For any $\lambda \in K$ we have

$$\begin{aligned}
&e(K)[\mathscr{D}(K,\Lambda) + x(K)] \\
&\quad= e(K \sim \{\lambda\})\, e_\lambda[\mathscr{D}(K,\Lambda) + x(K)] \\
&\quad= e(K \sim \{\lambda\})(e_\lambda[\mathscr{D}(\{\lambda\},\Lambda) + x[\lambda]] + \mathscr{D}(K \sim \{\lambda\}, \Lambda \sim \{\lambda\}) + x(K \sim \{\lambda\})) \\
&\qquad && \text{Theorem 2 (vi)} \\
&\quad\leqq e(K \sim \{\lambda\})(e_\lambda e_\kappa[\mathscr{D}(\{\kappa,\lambda\},\Lambda) + x(\{\kappa,\lambda\})] \\
&\qquad\qquad + \mathscr{D}(K \sim \{\lambda\}, \Lambda \sim \{\lambda\}) + x(K \sim \{\lambda\})) && \text{Theorem 3} \\
&\quad= e(K)(e_\kappa[\mathscr{D}(\{\kappa,\lambda\},\Lambda) + x(\{\kappa,\lambda\})] + \mathscr{D}(K \sim \{\lambda\}, \Lambda \sim \{\lambda\}) + x(K \sim \{\lambda\})) \\
&\quad= e(K)(e_\kappa[\mathscr{D}(\{\lambda\}, \Lambda \sim \{\kappa\}) + \mathscr{D}(\{\kappa\},\Lambda) + x[\kappa] + x[\lambda]] \\
&\qquad\qquad + \mathscr{D}(K \sim \{\lambda\}, \Lambda \sim \{\lambda\}) + x(K \sim \{\lambda\})) && \text{Theorem 1 (xvii)} \\
&\quad= e(K)(e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa]] + \mathscr{D}(\{\lambda\}, \Lambda \sim \{\kappa\}) \\
&\qquad\qquad + \mathscr{D}(K \sim \{\lambda\}, \Lambda \sim \{\lambda\}) + x[\lambda] + x(K \sim \{\lambda\})) \\
&\quad= e(K)(e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa]] + \mathscr{D}(K, \Lambda \sim (K \cup \{\kappa\})) \\
&\qquad\qquad + \mathscr{D}(K \sim \{\lambda\}, K \cup \{\kappa\}) + x(K)).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&e(K)[\mathscr{D}(K,\Lambda) + x(K)] \\
&\quad \leqq \prod_{\lambda \in K} e(K)(e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa]] + \mathscr{D}(K,\Lambda \sim (K \cup \{\kappa\})) \\
&\qquad + \mathscr{D}(K \sim \{\lambda\}, K \cup \{\kappa\}) + x(K)) \\
&\quad = e(K)\prod_{\lambda \in K} (e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa]] + \mathscr{D}(K,\Lambda \sim (K \cup \{\kappa\})) \\
&\qquad + \mathscr{D}(K \sim \{\lambda\}, K \cup \{\kappa\}) + x(K)) \\
&\quad = e(K)(e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa]] + \mathscr{D}(K,\Lambda \sim (K \cup \{\kappa\})) \\
&\qquad + x(K) + \prod_{\lambda \in K} \mathscr{D}(K \sim \{\lambda\}, K \cup \{\kappa\})).
\end{aligned}$$

Now suppose $\langle \mu,\nu \rangle \in (K \sim \{\lambda\}) \hat{\times} (K \cup \{\kappa\})$. If $\nu \neq \kappa$, then $d_{\mu\nu} \leqq \mathscr{D}(K,K)$. Assume on the other hand that $\nu = \kappa$. Then it is clear that $d_{\mu\nu} \cdot \mathscr{D}(K \sim \{\mu\}, K \cup \{\kappa\}) \leqq \mathscr{D}(K,K)$. From these two facts we infer that

$$\prod_{\lambda \in K} \mathscr{D}(K \sim \{\lambda\}, K \cup \{\kappa\}) \leqq \mathscr{D}(K,K).$$

Thus, if we continue the preceding chain of inequalities we get

$$\begin{aligned}
&e(K)[\mathscr{D}(K,\Lambda) + x(K)] \\
&\quad \leqq e(K)(e_\kappa[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa]] + \mathscr{D}(K,\Lambda \sim (K \cup \{\kappa\})) + x(K) + \mathscr{D}(K,K)) \\
&\quad = e(K \cup \{\kappa\})[\mathscr{D}(\{\kappa\},\Lambda) + x[\kappa] + \mathscr{D}(K,\Lambda \sim (K \cup \{\kappa\})) + x(K) + \mathscr{D}(K,K)] \\
&\quad = e(K \cup \{\kappa\})[\mathscr{D}(K \cup \{\kappa\},\Lambda) + x(K \cup \{\kappa\})]. \qquad \text{Q.E.D.}
\end{aligned}$$

Another generalization of the notion of diagonal elements of a cylindric algebra is obtained by setting, for $K \subseteq \alpha$,

$$d(K) = \prod_{\kappa,\lambda \in K} d_{\kappa\lambda}.$$

Thus $d(K) = 1$ if $|K| \leqq 1$. Further elementary properties of this concept will be used without proof.

If $x \in A$, $\Delta x \subseteq \{0\}$, and $K \subseteq \alpha$, we set

$$\pi(x,K) = \prod_{\kappa \in K} x[\kappa].$$

Elementary properties of this concept follow.

THEOREM 5. *Suppose* $x, y \in A$ *and* $\Delta x, \Delta y \subseteq \{0\}$. *Then*

(i) *if* $\kappa \in K$ *and* $\lambda \notin K$, *then* $S(\lambda/\kappa)\pi(x,K) = \pi(x,(K \sim \{\kappa\}) \cup \{\lambda\})$,

(ii) $\pi(x, K \cup \Lambda) = \pi(x,K) \cdot \pi(x,\Lambda)$,

(iii) *if* $K \neq 0$, *then* $\pi(x,K) + x(K) = x(K)$,

(iv) *if* $\kappa \notin K$, *then* $x[\kappa] \cdot \pi(-x,K) \cdot \mathscr{D}(\{\kappa\},K) = 0$,

(v) $-\pi(x,K) = (-x)(K)$,

(vi) *if* $x \cdot y = 0$, $K \cap \Lambda = 0$ *and* $K,\Lambda \neq 0$, *then* $\pi(x,K)\cdot\pi(y,\Lambda)\cdot\mathscr{D}(K,\Lambda) = 0$.

Proof. We consider only the nontrivial parts (iv) and (vi).

(iv) We compute:

$$\begin{aligned} x[\kappa]\cdot\pi(-x,K)\cdot\mathscr{D}(\{\kappa\},K) &= \sum_{\lambda\in K}(x[\kappa]\cdot d_{\kappa\lambda}\cdot\pi(-x,K)) \\ &= \sum_{\lambda\in K}(x[\lambda]\cdot d_{\kappa\lambda}\cdot\pi(-x,K)) \\ &= 0. \end{aligned}$$

(vi) Under the given assumptions, if $\kappa\in K$ and $\lambda\in\Lambda$, then

$$\begin{aligned} \pi(x,K)\cdot\pi(y,\Lambda)\cdot d_{\kappa\lambda} &\leqq x[\kappa]\cdot y[\lambda]\cdot d_{\kappa\lambda} \\ &= x[\kappa]\cdot y[\kappa]\cdot d_{\kappa\lambda} \\ &= (x\cdot y)[\kappa]\cdot d_{\kappa\lambda} \\ &= 0. \end{aligned}$$

THEOREM 6. *If* $0\neq\Lambda\subseteq\alpha$, $x\in A$, *and* $\Delta x\subseteq\{0\}$, *then*

$$\sum\left(-\mathscr{D}(K,K)\cdot\pi(x,\Delta)\cdot\pi(-x,K\sim\Delta)\cdot\sum_{\Gamma\in\mathscr{P}}d(\Gamma)\right) = 1,$$

the sum being taken over the set $\mathscr{A}$ *of all triples* $\langle\mathscr{P},K,\Delta\rangle$ *such that* $\mathscr{P}$ *is a partition of* Λ, $|K\cap\Gamma| = 1$ *for all* $\Gamma\in\mathscr{P}$, *and* $\Delta\subseteq K$.

Proof. The following temporary notation is used in this proof. If $x\in A$, we let $(-1)\cdot x = -x$ and $(+1)\cdot x = x$. Also, we let Λ' be the set of all functions mapping $\Lambda\times\Lambda$ into $\{-1,+1\}$.

We have by the finite distributive law

$$(1)\qquad \begin{aligned} 1 &= \prod_{\kappa,\lambda\in\Lambda}(d_{\kappa\lambda} + -d_{\kappa\lambda}) \\ &= \sum_{f\in\Lambda'}\prod_{\kappa,\lambda\in\Lambda}(f(\kappa,\lambda)\cdot d_{\kappa\lambda}). \end{aligned}$$

Now suppose $f\in\Lambda'$. We shall now verify the following inequality:

$$(2)\qquad \prod_{\kappa,\lambda\in\Lambda}f(\kappa,\lambda)\cdot d_{\kappa\lambda} \leqq \sum\left(-\mathscr{D}(K,K)\cdot\prod_{\Gamma\in\mathscr{P}}d(\Gamma)\right),$$

the sum being taken over the set $\mathscr{B}$ of all pairs $\langle\mathscr{P},K\rangle$ such that $\mathscr{P}$ is a partition of Λ and $|K\cap\Gamma| = 1$ for all $\Gamma\in\mathscr{P}$.

For $\kappa,\lambda\in\Lambda$, define $\kappa\equiv\lambda$ if and only if $f(\kappa,\lambda) = +1$. If $\kappa\not\equiv\kappa$ for some $\kappa\in\Lambda$, then $f(\kappa,\kappa) = -1$, $f(\kappa,\kappa)\cdot d_{\kappa\kappa} = 0$, and (2) obviously holds. Hence we may assume that $\equiv$ is reflexive on Λ. In a similar manner it is seen that we may assume that $\equiv$ is actually an equivalence relation on Λ. Let $\mathscr{P} = \Lambda/\equiv$ be the associated partition of Λ, and let K be any subset of Λ such that $|K \cap \Gamma| = 1$ for all $\Gamma \in \mathscr{P}$. Then clearly

$$\prod_{\kappa,\lambda \in \Lambda} f(\kappa,\lambda) \cdot d_{\kappa\lambda} = -\mathscr{D}(K,K) \cdot \prod_{\Gamma \in \mathscr{P}} d(\Gamma),$$

and (2) again follows. From (1) and (2) we obtain

$$\Sigma \left(-\mathscr{D}(K,K) \cdot \prod_{\Gamma \in \mathscr{P}} d(\Gamma) \right) = 1, \tag{3}$$

the sum being taken over $\mathscr{B}$.

Given $\langle \mathscr{P}, K \rangle \in \mathscr{B}$ we have, again using the finite distributive law, and for brevity letting K' be the set of all functions mapping K into $\{-1, +1\}$,

$$\begin{aligned} 1 &= \prod_{\kappa \in K} (x[\kappa] + -x[\kappa]) \\ &= \sum_{f \in K'} \prod_{\kappa \in K} (f(\kappa) \cdot x[\kappa]) \\ &= \sum_{\Delta \subseteq K} \left(\pi(x,\Delta) \cdot \pi(-x, K \sim \Delta) \right); \end{aligned}$$

combining this with (3) the desired result follows.

The generalized c-operator is defined analogously to the generalized e-operator: for $\Gamma \subseteq \alpha$ we let $c(\Gamma) = c_{\kappa(0)} \cdots c_{\kappa(m)}$, where $\Gamma = \{\kappa(0), \cdots, \kappa(m)\}$; for $\Gamma = 0$, $c(\Gamma)$ is again the identity mapping on A.

If $\kappa \leqq \alpha$ and $x \in A$ with $\Delta x \subseteq \{0\}$, we let $L(x,\kappa) = c(\kappa)(-\mathscr{D}(\kappa,\kappa) \cdot \pi(x,\kappa))$. Intuitively, confusing algebra and logic for simplicity, $L(x,\kappa)$ states that there are at least κ things such that x holds. We want to prove first that $L(x,\kappa)$ does not depend upon the particular variables used in this statement. Note that $L(x,0) = 1$ and $L(x,1) = c_0 x$.

Theorem 7. *If* $K \subseteq \alpha$, $\kappa \in K$, $\lambda \in \alpha \sim K$, $x \in A$, *and* $\Delta x \subseteq \{0\}$, *then* $e(K)(\mathscr{D}(K,K) + -\pi(x,K)) = e(\Lambda)(\mathscr{D}(\Lambda,\Lambda) + -\pi(x,\Lambda))$, *where* $\Lambda = (K \sim \{\kappa\}) \cup \{\lambda\}$.

Proof. We have

$$\begin{aligned} e(K)(\mathscr{D}(K,K) + -\pi(x,K)) &= e(K \sim \{\kappa\}) e_\kappa e_\lambda (-d_{\kappa\lambda} + \mathscr{D}(K,K) + -\pi(x,K)) \\ &= e(\Lambda) e_\kappa (-d_{\kappa\lambda} + \mathscr{D}(K,K) + -\pi(x,K)) \\ &= e(\Lambda) S(\lambda/\kappa)(\mathscr{D}(K,K) + -\pi(x,K)) \\ &= e(\Lambda)(\mathscr{D}(\Lambda,\Lambda) + -\pi(x,\Lambda)) \end{aligned}$$

Theorem 1 (xv), Theorem 5 (i).

Theorem 8. *If* $K \subseteq \alpha$, $x \in A$, *and* $\Delta x \subseteq \{0\}$, *then*

$$e(K)(\mathscr{D}(K,K) + -\pi(x,K)) = -L(x,|K|).$$

Proof. The statement is trivial if $K=\alpha$, so assume $K\neq\alpha$. Let λ be the least element of $\alpha\sim K$. If for every $\kappa\in K$ we have $\kappa<\lambda$, the conclusion is again trivial, so assume that we are given $\kappa\in K$ with $\lambda<\kappa$. Let $\Lambda=(K\sim\{\kappa\})\cup\{\lambda\}$. By Theorem 7 we have $e(K)(\mathscr{D}(K,K)+-\pi(x,K)) = e(\Lambda)(\mathscr{D}(\Lambda,\Lambda)+-\pi(x,\Lambda))$, and the desired conclusion follows by induction.

THEOREM 9. *If* $x,y\in A$, $x\cdot y=0$, $\Delta x,\Delta y\subseteq\{0\}$, *and* $\kappa+\lambda\leqq\alpha$, *then* $L(x,\kappa)\cdot L(y,\lambda)\leqq L(x+y,\kappa+\lambda)$.

Proof. Choose $K,\Lambda\subseteq\alpha$ with $|K|=\kappa$, $|\Lambda|=\lambda$, and $K\cap\Lambda=0$. Then

$$\begin{aligned}
L(x,\kappa)\cdot L(y,\lambda) &= c(K)(-\mathscr{D}(K,K)\cdot\pi(x,K))\cdot c(\Lambda)(-\mathscr{D}(\Lambda,\Lambda)\cdot\pi(y,\Lambda))\\
&= c(K\cup\Lambda)(-\mathscr{D}(K,K)\cdot-\mathscr{D}(\Lambda,\Lambda)\cdot\pi(x,K)\cdot\pi(y,\Lambda))\\
&\leqq c(K\cup\Lambda)(-\mathscr{D}(K,K)\cdot-\mathscr{D}(\Lambda,\Lambda)\cdot-\mathscr{D}(K,\Lambda)\cdot\pi(x+y,K)\\
&\qquad\cdot\pi(x+y,\Lambda))\\
&= c(K\cup\Lambda)(-\mathscr{D}(K\cup\Lambda,\,K\cup\Lambda)\cdot\pi(x+y,K\cup\Lambda))\\
&= L(x+y,\kappa+\lambda).
\end{aligned}$$

Our last lemma combines most of the preceding lemmas to give an expression for the action of a cylindrification in terms of the basic building blocks which have been discussed. This is an algebraic form of the classical logical method of elimination of quantifiers.

THEOREM 10. *If* $0\neq\Lambda\subseteq\alpha$, $\kappa\in\alpha\sim\Lambda$, $x\in A$, *and* $\Delta x\subseteq\{0\}$, *then*

$$c_\kappa[-\mathscr{D}(\{\kappa\},\Lambda)\cdot x[\kappa]] = \prod_{K\subseteq\Lambda}[L(x,|K|+1)+\mathscr{D}(K,K)+-\pi(x,K)].$$

Proof. We prove each inequality separately.

Part 1. $\leqq$. Suppose $K\subseteq\Lambda$. Let $\Delta=K\cup\{\kappa\}$. Then

$$\begin{aligned}
&-L(x,|K|+1)\cdot-\mathscr{D}(K,K)\cdot\pi(x,K)\\
&\quad= e(\Delta)(\mathscr{D}(\Delta,\Delta)+-\pi(x,\Delta))\cdot-\mathscr{D}(K,K)\cdot\pi(x,K) && \text{Theorem 8}\\
&\quad= e_\kappa e(K)(\mathscr{D}(K,K)+\mathscr{D}(\{\kappa\},K)+-x[\kappa]+-\pi(x,K))\cdot-\mathscr{D}(K,K)\cdot\pi(x,K)\\
&&& \text{Theorem 1 (xx)}\\
&\quad\leqq e_\kappa(\mathscr{D}(K,K)+\mathscr{D}(\{\kappa\},K)+-x[\kappa]+-\pi(x,K))\cdot-\mathscr{D}(K,K)\cdot\pi(x,K)\\
&\quad= (\mathscr{D}(K,K)+-\pi(x,K)+e_\kappa(\mathscr{D}(\{\kappa\},K)+-\pi(x,K)))\cdot-\mathscr{D}(K,K)\cdot\pi(x,K)\\
&\quad\leqq e_\kappa(\mathscr{D}(\{\kappa\},K)+-\pi(x,K))\\
&\quad\leqq e_\kappa(\mathscr{D}(\{\kappa\},\Lambda)+-\pi(x,K)).
\end{aligned}$$

Part 2. $\geqq$. Let $\mathscr{P}$ be any partition of Λ, and suppose $K \subseteq \Lambda$, $|K \cap \Gamma| = 1$ for each $\Gamma \in \mathscr{P}$, and suppose $\Delta \subseteq K$. Then

$$-\mathscr{D}(K,K)\cdot\pi(x,\Delta)\cdot\pi(-x,K\sim\Delta)\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma)\cdot\prod_{\Gamma\subseteq\Lambda}[L(x,|\Gamma|+1)+\mathscr{D}(\Gamma,\Gamma)+-\pi(x,\Gamma)]$$

$$\leqq -\mathscr{D}(K,K)\cdot\pi(x,\Delta)\cdot\pi(-x,K\sim\Delta)\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma)\cdot(L(x,|\Delta|+1)+\mathscr{D}(\Delta,\Delta)+-\pi(x,\Delta))$$

$$\leqq L(x,|\Delta|+1)\cdot\pi(-x,K\sim\Delta)\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma)$$

$$= c(\Delta\cup\{\kappa\})(-\mathscr{D}(\Delta\cup\{\kappa\},\Delta\cup\{\kappa\})\cdot\pi(x,\Delta\cup\{\kappa\}))\cdot\pi(-x,K\sim\Delta)\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma)$$

$$\leqq c_\kappa(-\mathscr{D}(\{\kappa\},\Delta)\cdot x[\kappa])\cdot\pi(-x,K\sim\Delta)\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma) \qquad \text{Theorem 4}$$

$$= c_\kappa\left(-\mathscr{D}(\{\kappa\},\Delta)\cdot x[\kappa]\cdot\pi(-x,K\sim\Delta)\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma)\right)$$

$$\leqq c_\kappa\left(-\mathscr{D}(\{\kappa\},\Delta)\cdot x[\kappa]\cdot -\mathscr{D}(\{\kappa\},K\sim\Delta)\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma)\right)$$

$$= c_\kappa\left(-\mathscr{D}(\{\kappa\},K)\cdot x[\kappa]\cdot\prod_{\Gamma\in\mathscr{P}} d(\Gamma)\right)$$

$$\leqq c_\kappa(-\mathscr{D}(\{\kappa\},\Lambda)\cdot x[\kappa]).$$

An application of Theorem 6 now completes the proof.

3. **Singulary cylindric algebras.** A CA_α $\mathfrak{A}$ is *singulary* if it is generated by a set P such that $|\Delta x| \leqq 1$ for every $x \in P$. In the special case in which $P = 0$, $\mathfrak{A}$ is said to be a *prime* cylindric algebra. Obviously every CA_α $\mathfrak{A}$ with $\alpha > 0$ has a prime subalgebra, which may also be characterized as the unique minimum subalgebra of $\mathfrak{A}$. We let SA_α (resp. PA_α) be the class of all singulary (resp. prime) CA_α's. In this section assume $0 < \alpha < \omega$.

First we note a simpler characterization of the singulary algebras.

THEOREM 11. *If $\mathfrak{A} \in SA_\alpha$, then $\mathfrak{A}$ is generated by a set Q such that $\Delta x \subseteq \{0\}$ for all $x \in Q$.*

Proof. Let $\mathfrak{A}$ be generated by a set P such that $|\Delta x| \leqq 1$ for all $x \in P$. For each $x \in P$ choose κ_x such that $\Delta x \subseteq \{\kappa_x\}$. Let $Q = \{S(0/\kappa_x)x : x \in P\}$. Since $S(\kappa_x/0)S(0/\kappa_x)x = x$ for each $x \in P$, Q is the desired set.

From Theorem 10 the following simple property of singulary algebras follows.

THEOREM 12. *If $\mathfrak{A} \in SA_\alpha$ and $\mathfrak{A}$ is generated by a set P such that $\Delta x \subseteq \{0\}$ for each $x \in P$, then $\mathfrak{A}_B = \langle A, +, \cdot, - \rangle$ coincides with the Boolean algebra generated by the following elements: $d_{\kappa\lambda}$ $(\kappa, \lambda < \alpha)$, $x[\kappa]$ $(x \in P$ and $\kappa < \alpha)$, and $L(x,\kappa)$ $(\kappa < \alpha$, x a member of the subalgebra of $\mathfrak{A}_B$ generated by $P)$.*

An easy consequence of this theorem is as follows.

THEOREM 13. *If $\mathfrak{A} \in \mathrm{SA}_\alpha$ and $\mathfrak{B}$ is a finitely generated subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ is finite. Moreover, there is a finite set $P \subseteq A$ such that $\Delta x \subseteq \{0\}$ for each $x \in P$, $\mathfrak{B}$ is a subalgebra of the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by P, and $\mathfrak{C}$ is finite.*

THEOREM 14. *If $\mathfrak{A} \in \mathrm{SA}_\alpha$ and $\mathfrak{A}$ is generated by a finite set P such that $\Delta x \subseteq \{0\}$ for each $x \in P$, then there is a finite set Q such that Q generates $\mathfrak{A}$, $\Delta x \subseteq \{0\}$ for each $x \in Q$, and $x \cdot y = 0$ for distinct $x, y \in Q$.*

Proof. Let $P = \{p_0, \cdots, p_{n-1}\}$ with p biunique. Then $\mathfrak{A}$ is generated by

$$\left\{ \prod_{i<n} f(i) \cdot p_i : f \in \{-1, +1\}^n \right\}$$

(with the notation used in the proof of Theorem 6), and the nonzero elements of this set may be taken for Q.

With this theorem we have arrived at a simple type of object which is easy to classify. By an *α-singulary couple* we mean a pair $\langle \mathfrak{A}, Q \rangle$ such that

(i) $\mathfrak{A} \in \mathrm{SA}_\alpha$,
(ii) *Q generates $\mathfrak{A}$,*
(iii) *for all $x \in Q$ we have $\Delta x = \{0\}$,*
(iv) *Q is finite,*
(v) *$x \cdot y = 0$ for distinct $x, y \in Q$,*
(vi) *$\mathfrak{A}$ is simple.*

Given any such couple, let $Q^+ = Q \cup \{1\}$. We associate with every α-singulary couple $\langle \mathfrak{A}, Q \rangle$ a function $f(\mathfrak{A}, Q)$ mapping Q^+ into $\alpha + 1$, as follows. For $x \in Q^+$, we let $(f(\mathfrak{A}, Q))(x)$ = the greatest $\kappa \leqq \alpha$ such that $L(x, \kappa) = 1$ (recall that $L(x, 0) = 1$). Two monadic pairs $\langle \mathfrak{A}, Q \rangle$ and $\langle \mathfrak{B}, P \rangle$ are said to be *corresponding* if there is a biunique function g mapping Q onto P such that

$$(f(\mathfrak{A}, Q))(x) = (f(\mathfrak{B}, P))(g(x)) \textit{ for all } x \in Q,$$

and

$$(f(\mathfrak{A}, Q))(1) = (f(\mathfrak{B}, P))(1).$$

Then g is called a *correspondence* from $\langle \mathfrak{A}, Q \rangle$ to $\langle \mathfrak{B}, P \rangle$. We shall see that in such a case $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

To reduce possible confusion in the succeeding theorems, it is convenient to introduce for each α such that $0 < \alpha < \omega$ and for each $\kappa < \omega$ a first order logic $\mathscr{F}_{\alpha\kappa}$. The nonlogical constants of $\mathscr{F}_{\alpha\kappa}$ are as follows:

individual constants: $\mathbf{0}$, $\boldsymbol{d}_{\lambda\mu}$ $(\lambda, \mu < \alpha)$, $\boldsymbol{k}_\lambda^\mu$ $(\lambda < \alpha, \mu < \kappa)$.
operation symbols: $\boldsymbol{-}$ (singulary), $\boldsymbol{+}$, $\boldsymbol{\cdot}$ (binary).

Actually we are only interested in *variable-free terms* (VFT) of $\mathscr{F}_{\alpha\kappa}$. If $\langle \mathfrak{A}, Q \rangle$ is an α-singulary couple and f is a biunique function mapping $\kappa = |Q|$ onto Q, and if σ is a VFT in $\mathscr{F}_{\alpha\kappa}$, then by $\mathrm{Val}_f^{\mathfrak{A}}(\sigma)$ we mean that element of A obtained by interpreting $\boldsymbol{d}_{\lambda\mu}$ as $d_{\lambda\mu}$ $(\lambda,\mu<\alpha)$, $\boldsymbol{k}_{\lambda}^{\mu}$ as $(f(\mu))[\lambda]$ $(\lambda<\alpha,\ \mu<\kappa)$, $\boldsymbol{-}$ as $-$, $\boldsymbol{+}$ as $+$, $\boldsymbol{\cdot}$ as $\cdot$, and $\boldsymbol{0}$ as 0.

Theorem 15. *If $\langle\mathfrak{A},Q\rangle$ is an α-singulary couple with $|Q|=\kappa$ and σ is a* VFT *in $\mathscr{F}_{\alpha\kappa}$, and if $\lambda<\alpha$, then there is a* VFT *τ in $\mathscr{F}_{\alpha\kappa}$ such that*

$$c_\lambda \mathrm{Val}^{\mathfrak{B}}_{g\circ f}(\sigma) = \mathrm{Val}^{\mathfrak{B}}_{g\circ f}(\tau)$$

whenever f is a biunique function mapping κ onto Q and g is a correspondence from $\langle\mathfrak{A},Q\rangle$ to a singulary pair $\langle\mathfrak{B},P\rangle$. Moreover, if $\boldsymbol{d}_{\mu\pi}$ or $\boldsymbol{k}_{\mu}^{\xi}$ occurs in τ, then μ and π are distinct from λ and they occur as subscripts of $\boldsymbol{d}$ or $\boldsymbol{k}$ in σ.

This theorem is easily proved using Theorem 10 and the definition of correspondence between singulary pairs. The import of Theorem 15 is that the "elimination of quantifiers" of Theorem 10 can be done uniformly for corresponding pairs.

Theorem 16. *If g is a correspondence between α-singulary pairs $\langle\mathfrak{A},Q\rangle$ and $\langle\mathfrak{B},P\rangle$, f is a biunique mapping of $\kappa=|Q|$ onto Q, and if σ is a* VFT *of $\mathscr{F}_{\alpha\kappa}$, then $\mathrm{Val}^{\mathfrak{A}}_{f}(\sigma)=0$ if and only if $\mathrm{Val}^{\mathfrak{B}}_{g\circ f}(\sigma)=0$.*

Proof. Since for any element x in any CA_α and any $\lambda<\alpha$ we have $x=0$ if and only if $c_\lambda x=0$, Theorem 15 can be used inductively to limit ourselves to the case in which σ does not contain any occurrences of the individual constants $\boldsymbol{d}_{\lambda\mu}$ or $\boldsymbol{k}_{\lambda}^{\mu}$. This case is trivial.

Theorem 17. *If $\langle\mathfrak{A},Q\rangle$ and $\langle\mathfrak{B},P\rangle$ are corresponding, then $\mathfrak{A}\cong\mathfrak{B}$.*

Proof. In accordance with Theorem 12 and the simplicity of $\mathfrak{A}$, $\mathfrak{A}$ coincides with the Boolean algebra generated by the elements $d_{\kappa\lambda}$ and $x[\kappa]$ $(\kappa,\lambda<\alpha,\ x\in Q)$. Similar considerations apply to $\mathfrak{B}$. Hence in accordance with a well-known result in the theory of Boolean algebras([5]), by Theorem 16 the Boolean parts of $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. The isomorphism can be chosen so that $d_{\kappa\lambda}$ (in $\mathfrak{A}$) corresponds to $d_{\kappa\lambda}$ (in $\mathfrak{B}$), for all $\kappa,\lambda<\alpha$. By Theorem 15 the isomorphism preserves the cylindric operations also.

Theorem 18. *If $\langle\mathfrak{A},Q\rangle$ is an α-singulary couple and $(f(\mathfrak{A},Q))(1)<\alpha$, then*

$$\sum_{x\in Q}(f(\mathfrak{A},Q))(x) \leqq (f(\mathfrak{A},Q))(1).$$

Proof. Assume the contrary. For brevity let $f=f(\mathfrak{A},Q)$. Note that $f(x)\leqq f(1)$ for all $x\in Q$. Choose $P\subseteq Q$ and $x\in Q\sim P$ such that $\sum_{y\in P}f(y)\leqq f(1)$ while $\sum_{y\in P}f(y)+f(x)>f(1)$. Let $\sum_{y\in P}f(y)=\kappa$ and let $\lambda=f(1)+1-\kappa$. Then

([5]) See [13, Theorem 12.2].

$$\begin{aligned} 1 &= \prod_{y \in P} L(y,f(y)) \cdot L(x,f(x)) \\ &\leqq \prod_{y \in P} L(y,f(y)) \cdot L(x,\lambda) \\ &\leqq L\left(\sum_{y \in P} y + x,\ \sum_{y \in P} f(y) + \lambda\right) && \text{Theorem 9} \\ &\leqq L(1, f(1) + 1), \end{aligned}$$

which is a contradiction.

THEOREM 19. *If $\langle \mathfrak{A}, Q \rangle$ is an α-singular couple, then there is a corresponding couple $\langle \mathfrak{B}, P \rangle$ such that $\mathfrak{B}$ is a* CS_α.

Proof. We distinguish two cases.

Case 1. $(f(\mathfrak{A}, Q))(1) < \alpha$. Let U be a set with $(f(\mathfrak{A}, Q))(1)$ elements. Let g be a function with domain Q such that $g(x) \subseteq U$ and $|g(x)| = (f(\mathfrak{A}, Q))(x)$ for each $x \in Q$, and $g(x) \cap g(y) = 0$ for distinct $x, y \in Q$; there is such a function g by Theorem 18. For each $x \in Q$, let

$$h(x) = \{y \in U^\alpha : y(0) \in g(x)\}.$$

Let $\mathfrak{B}$ be the subalgebra of the cylindric algebra of all subsets of U^α generated by the range of h, and let P be the range of h. Clearly $\langle \mathfrak{B}, P \rangle$ is an α-singulary couple and h is a correspondence from $\langle \mathfrak{A}, Q \rangle$ to $\langle \mathfrak{B}, P \rangle$.

Case 2. $(f(\mathfrak{A}, Q))(1) = \alpha$. Let U be a set of any cardinality greater than or equal to the maximum of α and $\sum_{x \in Q}(f(\mathfrak{A}, Q))(x)$. The further details are just like in Case 1.

From Theorems 17 and 19 we obtain

THEOREM 20. *If $\langle \mathfrak{A}, Q \rangle$ is an α-singulary couple, then $\mathfrak{A}$ is isomorphic to a* CS_α.

The main result of this section can now be proved.

THEOREM 21. $\mathrm{SA}_\alpha \subseteq \mathrm{RCA}_\alpha$.

Proof. Suppose $\mathfrak{A} \in \mathrm{SA}_\alpha$; then by Theorem 2.5 of [7], $\mathfrak{A}$ is isomorphic to a subdirect product of simple CA_α's $\{\mathfrak{B}_i : i \in I\}$. It suffices to show that for any i, $\mathfrak{B}_i$ is representable. Since $\mathfrak{B}_i$ is a homomorphic image of $\mathfrak{A}$ we have $\mathfrak{B}_i \in \mathrm{SA}_\alpha$. In accordance with Theorem 2.13 of [7] it suffices to show that an arbitrary finitely generated subalgebra $\mathfrak{C}$ of $\mathfrak{B}_i$ is representable. By Theorem 13 there is a pair $\langle \mathfrak{D}, P \rangle$ such that $\mathfrak{D}$ is a subalgebra of $\mathfrak{B}_i$, $\mathfrak{C}$ is a subalgebra of $\mathfrak{D}$, P generates $\mathfrak{D}$, $\Delta x \subseteq \{0\}$ for each $x \in P$, and P is finite. It suffices to show that $\mathfrak{D}$ is representable. By Theorem 2.5 of [7], $\mathfrak{D}$ is simple. Hence by Theorem 14 there is a Q such that $\langle \mathfrak{D}, Q \rangle$ is an α-singulary couple. Hence by Theorem 20, $\mathfrak{D}$ is representable.

Hence, of course, every prime cylindric algebra is representable (take the empty set of generators). Possible improvements in the result of Theorem 21, along the lines of cardinality conditions on dimension sets of elements of a cylindric algebra, seem unlikely but not impossible. In [7] there is mentioned a nonrepresentable CA_α, $\alpha \geqq 3$, generated by three elements x,y,z with $\Delta x, \Delta y, \Delta z \subseteq \{0,\cdots,\alpha-2\}$. Thus for $\alpha = 3$ the most obvious possible improvement of Theorem 21 is false. For $\alpha > 3$ this nonrepresentable algebra at least puts an upper bound on the possible improvements of Theorem 21.

By an *identity* in the theory of CA_α's one means the universal closure of an equation involving formal symbols for $+, \cdot, -, c_\kappa$ and $d_{\kappa\lambda}\,(\kappa,\lambda < \alpha)$ as well as individual variables. Because of Theorem 10 and Theorem 21 the following result is easy to prove.

THEOREM 22. *There is an algorithm for deciding whether or not an identity holds in every* SA_α.

Of course by "algorithm" we mean that after a suitable Gödel numbering the decision method can be expressed by a recursive function.

4. **Singulary polyadic equality algebras.** A *polyadic I-algebra* is a system

$$\mathfrak{A} = \langle A, +, \cdot, -, \exists(J), S(\tau)\rangle_{J \subseteq I, \tau \in I^I}$$

such that $\langle A, +, \cdot, - \rangle$ is a Boolean algebra, $\exists(J)$ and $S(\tau)$ are singulary operations on A for each $J \subseteq I$ and $\tau \in I^I$, and the following axioms hold for all $x, y \in A$, $J, K \subseteq I$, and $\sigma, \tau \in I^I$:

P1. $\exists(J)0 = 0$,
P2. $x \leqq \exists(J)x$,
P3. $\exists(J)(x \cdot \exists(J)y) = \exists(J)x \cdot \exists(J)y$,
P4. $S(\sigma)(x+y) = S(\sigma)x + S(\sigma)y$,
P5. $S(\sigma)(-x) = -S(\sigma)x$,
P6. $S(\delta_I)x = x$,
P7. $S(\sigma \circ \tau)x = S(\sigma)S(\tau)x$,
P8. $\exists(0)x = x$,
P9. $\exists(J \cup K)x = \exists(J)\exists(K)x$,
P10. $S(\sigma)\exists(J)x = S(\tau)\exists(J)x$ *if* $\sigma \restriction I \sim J = \tau \restriction I \sim J$,
P11. $\exists(J)S(\tau)x = S(\tau)\exists(\tau^{-1*}(J))$ *if* $\tau \restriction \tau^{-1*}(J)$ is biunique.

The most elementary properties of polyadic algebras, such as are given in the first part of [4], will be used without proof or specific reference.

A *polyadic equality I-algebra* is a system

$$\mathfrak{A} = \langle A, +, \cdot, -, \exists(J), S(\tau), E(i,j)\rangle_{J \subseteq I, \tau \in I^I, i,j \in I}$$

such that $\langle A, +, \cdot, -, \exists(J), S(\tau)\rangle_{J \subseteq I, \tau \in I^I}$ is a polyadic I-algebra, $E(i,j) \in A$ for each $i,j \in I$, and the following axioms hold for all $x \in A$, $i,j \in I$, and $\tau \in I^I$:

E1. $S(\tau)E(i,j) = E(\tau(i),\tau(j))$,

E2. $E(i,j) = 1$,

E3. $x \cdot E(i,j) \leqq S(j/i)x$, where (j/i) is that element of I^I such that $(j/i)(k) = k$ if $k \neq i$ and $(j/i)(i) = j$.

Again we will assume without proof elementary properties of polyadic equality algebras. If $\mathfrak{A}$ is a polyadic equality α-algebra, α an ordinal, then

$$\langle A, +, \cdot, -, \exists(\{\kappa\}), E(\kappa,\lambda)\rangle_{\kappa,\lambda<\alpha}$$

is a cylindric algebra. Hence we can hope to use the results of the preceding section to obtain information about polyadic equality algebras.

A polyadic equality algebra is *singulary* if it is generated by a set P such that $|\Delta x| \leqq 1$ for each $x \in P$. A *polyadic equality set I-algebra*(6) is a system

$$\langle A, \cup, \cap, \sim, \exists'(J), S'(\tau), E'(i,j)\rangle_{J\subseteq I, \tau \in I^I, i,j \in I}$$

such that for some set U, A is a field of subsets of U^I, $E(i,j) \in A$, and A is closed under the singulary operations $\exists'(J)$ and $S'(\tau)$ $(J \subseteq I, \tau \in I^I)$, where

S1. $\exists'(J)(x) = \{y \in U^I$: *there is a* $z \in x$ *with* $y \restriction I \sim J = z \restriction I \sim J\}$,

S2. $S'(\tau)(x) = \{y \in U^I : y \circ \tau \in x\}$,

S3. $E'(i,j) = \{y \in U^I : y(i) = y(j)\}$.

A *representable* polyadic equality algebra is a polyadic equality algebra which is isomorphic to a subdirect product of polyadic equality set algebras.

THEOREM 23. *Every singulary polyadic equality algebra is representable.*

Proof. Let $\mathfrak{A}$ be a singulary polyadic equality I-algebra; we may assume that $I = \alpha$, an ordinal. We assume $0 < \alpha < \omega$. Let

$$\mathfrak{A}^- = \langle A, +, \cdot, -, \exists(\{\kappa\}), E(\kappa,\lambda)\rangle_{\kappa,\lambda<\alpha}.$$

Then $\mathfrak{A}^-$ is an SA_α. By Theorem 21, $\mathfrak{A}^-$ is isomorphic, say by an isomorphism f, to a subdirect product of CS_α's $\{\mathfrak{B}_t : t \in T\}$. Since each $\mathfrak{B}_t$ is a homomorphic image of $\mathfrak{A}^-$, each $\mathfrak{B}_t$ is singulary. It is easy to verify using Theorem 12 that, for each $\tau \in \alpha^\alpha$, B_t is closed under $S'(\tau)$. Let pr_t be the projection from $\prod_{t\in T} B_t$ into $B_t : pr_t(g) = g(t)$ for all $t \in T$ and g in the product. Then, if in accordance with Theorem 11, Q is a set generating $\mathfrak{A}^-$ such that $\Delta x \subseteq \{0\}$ for all $x \in Q$, and if $\kappa,\lambda < \alpha$, $\tau \in \alpha^\alpha$, and $x \in Q$, then

$$\begin{aligned}(pr_t \circ f)(S(\tau)L(x,\kappa)) &= (pr_t \circ f)(L(x,\kappa)) \\ &= S'(\tau)(pr_t \circ f)(L(x,\kappa)), \\ (pr_t \circ f)(S(\tau)d_{\kappa\lambda}) &= (pr_t \circ f)d_{\tau(\kappa)\tau(\lambda)} \\ &= d_{\tau(\kappa)\tau(\lambda)} \\ &= S'(\tau)((pr_t \circ f)(d_{\kappa\lambda})),\end{aligned}$$

(6) Apart from trivial technical details this is the same as Halmos' concept of an O-value functional I-algebra with functional equality.

$$\begin{aligned}(pr_t \circ f)(S(\tau)x[\kappa]) &= (pr_t \circ f)(x[\tau(\kappa)]) \\ &= ((pr_t \circ f)(x))[\tau(\kappa)] \\ &= S'(\tau)((pr_t \circ f)(x[\kappa])).\end{aligned}$$

Now since each B_t is closed under $S'(t)$ for each $\tau \in \alpha^\alpha$, operations can be defined on B_t to form a polyadic set α-algebra $\mathfrak{B}'_t$ such that $\mathfrak{B}'^{-}_t = \mathfrak{B}_t$. The operations can be extended pointwise to $\prod_{t \in T} \mathfrak{B}_t$. The above equations show that for each $t \in T$, $pr_t \circ f$ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}'_T$. Hence f is an isomorphism of $\mathfrak{A}$ onto a subdirect product of the $\mathfrak{B}_t$'s. Q.E.D.

An independent proof of Theorem 22 can be given by modifying the proof for Theorem 21 suitably. If this is carried out, certain details in the proofs of §2 become much easier. Nonetheless, the complete proof remains rather lengthy.

The notions of a *singulary polyadic algebra* (without equality), a *polyadic set algebra*, and a *representable polyadic algebra* should be clear without specific definitions.

THEOREM 24. *Every singulary polyadic algebra is representable.*

REMARK. This theorem is considerably less "deep" than the preceding results. Hence it seems advisable to give two proofs for the result—one based on the preceding theorems and hence short, and one independent, longer proof.

First Proof. By Theorem (7.14) of [5], every polyadic I-algebra can be embedded in a polyadic equality I-algebra. Clearly if we start with a singulary algebra, by taking an appropriate subalgebra of the equality algebra we can embed in a singulary equality algebra. Hence the desired result follows by Theorem 22 (we assume $0 < |I| < \omega$).

Second Proof. Let

$$\mathfrak{A} = \langle A, +, \cdot, -, \exists(J), S(\tau)\rangle_{J \subseteq I, \tau \in I^I}$$

be a singulary polyadic algebra with I nonempty but otherwise arbitrary. Since $\mathfrak{A}$ is semisimple, it is isomorphic to a subdirect product of simple singulary algebras. In short, we may assume that $\mathfrak{A}$ is simple. Let $B = \{x \in A : \Delta x \subseteq \{i\}\}$, where i is a fixed element of I. As in the proof of Theorems 11 and 12 it can be seen that $\langle A, +, \cdot, - \rangle$ is the Boolean algebra generated by $\{S(j/i)x : j \in I$ and $x \in B\}$. Clearly $\langle B, +, \cdot, - \rangle$ is a subalgebra of $\langle A, +, \cdot, - \rangle$. By Boolean representation there is a set U and an isomorphism g of $\langle B, +, \cdot, - \rangle$ into the Boolean algebra of all subsets of U. It is easy to see that there is a function f such that

$$f(S(j/i)x) = \{y \in U^I : y_j \in g(x)\}$$

for all $j \in I$ and $x \in B$. If

$$S(j_1/i)x_1 \cdot \cdots \cdot S(j_m/i)x_m = 0,$$

with $x_1, \cdots, x_m \in B$, then as is easily seen there is a subsequence $x_{k_0}, \cdots, x_{k_n}$ such that $j_{k_0} = \cdots = j_{k_n}$ and $x_{k_0} \cdot \cdots \cdot x_{k_n} = 0$. Hence $g(x_{k_0} \cdot \cdots \cdot x_{k_n}) = 0$, and so $f(S(j_{k_0}/i)(x_{k_0} \cdot \cdots \cdot x_{k_n})) = 0$.

Since clearly

$$f(S(j_{k_0}/i)x_{k_0}) \cap \cdots \cap f(S(j_{k_0}/i)x_{k_n}) = f(S(j_{k_0}/i)(x_{k_0} \cdot \cdots \cdot x_{k_n})),$$

it follows that

$$f(S(j_1/i)x_1) \cap \cdots \cap f(S(j_m/i)x_m) = 0.$$

Since $\{S(j/i)x : j \in I \text{ and } x \in B\}$ is closed under complements, it follows that f can be extended to a homomorphism of $\langle A, +, \cdot, - \rangle$ into the Boolean algebra of all subsets of U^I. It is easily seen as in §3 that f preserves the polyadic operations as well. This completes the proof.

The situation concerning possible improvements of Theorems 23 and 24 is essentially the same as in the cylindric case.

THEOREM 25. *The decision problem for the identities holding in all singulary polyadic equality algebras (or polyadic algebras without equality) is recursively solvable.*

5. **Applications to logic.** As we have seen in §4, for singulary algebras one can go back and forth from cylindric to polyadic equality structures. Hence in considering applications to logic we may as well restrict ourselves to, say, singulary cylindric algebras.

By an *α-singulary logic* ($1 < \alpha < \omega$) we mean a first order logic with equality, with the usual formulation rules, and with the following primitive symbols:

I. *Logical constants.*

A. Individual variables: $v_0, v_1, \cdots, v_{\alpha-1}$.

B. Connectives: $\vee$, $\wedge$, $\neg$, $\bigwedge$.

C. Equality: $=$.

II. *Nonlogical constants*: arbitrarily many singulary predicate symbols.

As axioms we take the following:

A1. ϕ, *if* ϕ *is tautologous,*

A2. $\bigwedge\alpha(\phi \to \psi) \to (\bigwedge\alpha\phi \to \bigwedge\alpha\psi)$,

A3. $\phi \to \bigwedge\alpha\phi$, *if* α *does not occur freely in* ϕ,

A4. $\bigwedge\alpha\phi \to \phi$,

A5. $\neg\bigwedge\alpha\neg\alpha = \beta$, *if* $\alpha \neq \beta$,

A6. $\alpha = \beta \to (\phi \to \psi)$, *if* ϕ *is atomic and* ψ *is obtained from* ϕ *by replacing one occurrence of* α *by* β.

As rules of inference we take detachment and generalization. The connectives $\to$, $\leftrightarrow$, $\bigvee$ are defined as usual.

This system of logic is just the restriction of a system of Tarski's to finitely many variables(7).

(7) See [14].

If Γ is a set of sentences, ϕ a formula, we write $\Gamma \vdash \phi$, to mean that ϕ is formally derivable from Γ; the corresponding semantic notion is that $\Gamma \models \phi$, meaning that every model of Γ is a model of the universal closure of ϕ.

Given an α-singulary logic $\mathscr{F}$, we form an algebraic system $\mathfrak{A}(\mathscr{F})$ associated with $\mathscr{F}$. Namely, $\mathfrak{A}(\mathscr{F}) = \langle A, +, \cdot, -, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$, where A is the set of all formulas of $\mathscr{F}$, $\phi + \psi = \phi \vee \psi$, $\phi \cdot \psi = \phi \wedge \psi$, $-\phi = \neg\phi$, $c_\kappa \phi = \bigvee v_\kappa \phi$, and $d_{\kappa\lambda} = v_\kappa = v_\lambda$ for all $\phi, \psi \in A$ and $\kappa, \lambda < \alpha$. If Γ is a set of sentences of $\mathscr{F}$, we define $\phi \equiv_\Gamma \psi$ if and only if $\Gamma \vdash \phi \leftrightarrow \psi$, for all $\phi, \psi \in A$. Clearly $\equiv_\Gamma$ is a congruence relation on $\mathfrak{A}(\mathscr{F})$, thus giving rise to an algebra $\mathfrak{A}(\mathscr{F})/\equiv_\Gamma$.

THEOREM 26. *$\mathfrak{A}(\mathscr{F})/\equiv_\Gamma$ is an* SA_α.

Proof. The axioms for cylindric algebras are verified as follows.

C1. This axiom follows from the fact that $\vdash \bigwedge \alpha\phi \rightarrow \phi$ for every variable α and formula ϕ.

C2. We need to show that $\vdash \bigvee \alpha(\phi \wedge \bigvee \alpha\psi) \leftrightarrow \bigvee \alpha\phi \wedge \bigvee \alpha\psi$ for all $\phi, \psi \in A$ and for all variables α. It suffices to prove the dual $\bigwedge \alpha(\phi \rightarrow \bigwedge \alpha\psi) \leftrightarrow (\bigwedge \alpha \neg \phi \vee \bigwedge \alpha\phi)$. We have:

$$\vdash \bigwedge \alpha(\phi \rightarrow \bigwedge \alpha\psi) \rightarrow \bigwedge \alpha(\neg \bigwedge \alpha\psi \rightarrow \neg\phi),$$

$$\vdash \bigwedge \alpha(\phi \rightarrow \bigwedge \alpha\psi) \rightarrow (\neg \bigwedge \alpha\psi \rightarrow \bigwedge \alpha \neg\phi),$$

$$\vdash \bigwedge \alpha(\phi \rightarrow \bigwedge \alpha\psi) \rightarrow (\bigwedge \alpha \neg\phi \vee \bigwedge \alpha\psi).$$

In the other direction,

$$\vdash (\bigwedge \alpha \neg\phi \vee \bigwedge \alpha\psi) \rightarrow (\neg\phi \vee \bigwedge \alpha\psi),$$

$$\vdash (\bigwedge \alpha \neg\phi \vee \bigwedge \alpha\psi) \rightarrow (\phi \rightarrow \bigwedge \alpha\psi),$$

$$\vdash (\bigwedge \alpha \neg\phi \vee \bigwedge \alpha\psi) \rightarrow \bigwedge \alpha(\phi \rightarrow \bigwedge \alpha\psi).$$

C3. The essential statement $\vdash \bigwedge \alpha \bigwedge \beta\phi \rightarrow \bigwedge \beta \bigwedge \alpha\phi$ may be proved as in the proof of Lemma 32 of [14].

C4, C5. See Lemmas 6 and 7 of [14].

C6. It suffices to show that $\vdash \neg v_\kappa = v_\lambda \leftrightarrow \bigwedge v_\mu(\neg v_\kappa = v_\mu \vee v_\mu = v_\lambda)$, where $\mu \neq \kappa, \lambda$.

Since the analog of Lemma 15 of [14] can be proved, we have

$$\vdash \bigwedge v_\mu(\neg v_\kappa = v_\mu \vee \neg v_\mu = v_\lambda) \rightarrow (\neg v_\kappa = v_\kappa \vee \neg v_\kappa = v_\lambda),$$

$$\vdash \bigwedge v_\mu(\neg v_\kappa = v_\mu \vee \neg v_\mu = v_\lambda) \rightarrow \neg v_\kappa = v_\lambda.$$

Conversely, by an analog of Lemma 13 of [14],

$$\vdash \neg v_\kappa = v_\lambda \rightarrow (v_\kappa = v_\mu \rightarrow \neg v_\mu = v_\lambda),$$

$$\vdash \neg v_\kappa = v_\lambda \rightarrow \bigwedge v_\mu(\neg v_\kappa = \vee \neg v_\mu = v_\lambda).$$

C7. It suffices to show that $\vdash \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi) \leftrightarrow \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi)$, where $\kappa \neq \lambda$. We have

$$\vdash \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi) \wedge \neg \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi) \rightarrow \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \neg \phi),$$

$$\vdash \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi) \wedge \neg \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi) \rightarrow (v_\kappa = v_\lambda \rightarrow \phi) \wedge (v_\kappa = v_\lambda \rightarrow \neg \phi),$$

$$\vdash \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi) \wedge \neg \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi) \rightarrow \neg v_\kappa = v_\lambda,$$

$$\vdash \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi) \wedge \neg \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi) \rightarrow \bigwedge v_\kappa \neg v_\kappa = v_\lambda,$$

$$\vdash \neg \bigwedge v_\kappa \neg v_\kappa = v_\lambda \rightarrow (\bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi) \rightarrow \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi)),$$

$$\vdash \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi) \rightarrow \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi).$$

Conversely, from the analog of Lemma 16 of [14] one gets

$$\vdash v_\kappa = v_\lambda \rightarrow (\neg \phi \rightarrow \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \neg \phi)),$$

$$\vdash \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi) \rightarrow (v_\kappa = v_\lambda \rightarrow \phi),$$

$$\vdash \bigvee v_\kappa(v_\kappa = v_\lambda \wedge \phi) \rightarrow \bigwedge v_\kappa(v_\kappa = v_\lambda \rightarrow \phi).$$

Thus we have shown that $\mathfrak{A}(\mathscr{F})/\equiv_\Gamma$ is a CA_α. By A3 and A4 we see that $\mathfrak{A}(\mathscr{F})/\equiv_\Gamma$ is an SA_α.

THEOREM 27. *If Γ is a formally consistent set of sentences of $\mathscr{F}$, then Γ has a model.*

Proof. Since $\mathfrak{A}(\mathscr{F})/\equiv_\Gamma$ is representable, there is a singulary CS_α $\mathfrak{B}$, say with base U, and a homomorphism f of $\mathfrak{A}(\mathscr{F})/\equiv_\Gamma$ onto $\mathfrak{B}$. For each singulary predicate symbol π of $\mathscr{F}$ let

$$g(\pi) = \{x \in U: \textit{there exist } y_1, \cdots, y_{\alpha-1} \in U \textit{ such that } \langle x, y_1, \cdots, y_{\alpha-1} \rangle \in f(\pi v_0/\equiv_\Gamma)\}.$$

Let $\mathfrak{C} = \langle U, g(\pi) \rangle_{\pi \in S}$, where S is the set of singulary predicate symbols of $\mathscr{F}$. Then by induction on ϕ one sees that a sequence of $x \in U^\alpha$ satisfies ϕ in $\mathfrak{C}$ if and only if $x \in f(\phi/\equiv_\Gamma)$. Since $\phi/\equiv_\Gamma = 1$ for each $\phi \in \Gamma$, $\mathfrak{C}$ is a model of Γ.

From Theorem 27 the completeness theorem follows in the usual manner.

THEOREM 28. *$\Gamma \vdash \phi$ if and only if $\Gamma \models \phi$.*

In particular, the equivalence written before Theorem 3 can be proved using only three variables. Such a proof can be obtained from the proofs of Theorems 3 and 4.

The results of this section obviously also apply to logic without equality. The sameproofs also work for substitutionless predicate logic with identity(8).

(8) This kind of logic is mentioned briefly in [7] and discussed in detail in [11].

References

1. H. Behmann, *Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem*, Math. Ann. **86** (1922), 163–229.

2. A. Daigneault and D. Monk, *Representation theory for polyadic algebras*, Fund. Math. **52** (1963), 151–176.

3. P. R. Halmos, *Algebraic logic*. I. *Monadic Boolean algebras*, Compositio Math. **12** (1955), 217–249.

4. ———, *Algebraic logic*. II. *Homogeneous locally finite polyadic Boolean algebras of infinite degree*, Fund. Math. **43** (1956), 255–325.

5. ———, *Algebraic logic*. IV. *Equality in polyadic algebras*, Trans. Amer. Math. Soc. **86** (1957), 1–27.

6. ———, *Algebraic logic*, Chelsea, New York, 1962.

7. L. Henkin and A. Tarski, *Cylindric algebras*, Proc. Sympos. Pure Math. Vol. 2, pp. 83–113, Amer. Math. Soc., Providence, R.I., 1960.

8. D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Vol. 1, Springer, Berlin, 1934.

9. H. J. Keisler, *A complete first order logic with infinitary predicates*, Fund. Math. **52** (1963), 176–203.

10. J. D. Monk, *On the representation theory for cylindric algebras*, Pacific J. Math. **11** (1961), 1447–1457.

11. ———, *Substitutionless predicate logic with identity*, Arch. Math. Logik Grundlagenforsch. (to appear).

12. ———, *On finite dimensional cylindric algebras*, Abstract 595-1, Notices Amer. Math. Soc. **9** (1962), 470.

13. R. Sikorski, *Boolean algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 25, Springer, Berlin, 1960.

14. A. Tarski, *A simplified formalization of predicate logic with identity*, Arch. Math. Logik Grundlagenforsch. (to appear).

University of Colorado,
Boulder, Colorado