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Cardinal Invariants on Boolean Algebras

Second Revised Edition

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Foreword

This book is the successor of **Cardinal functions on Boolean algebras** (Birkhäuser 1990) and **Cardinal invariants on Boolean algebras** (Birkhäuser 1996). It contains most of the material of these books, and adds the following:

(1) Indication of the progress made on the open problems formulated in the earlier versions, with detailed solutions in many cases.
(2) Inclusion of some new cardinal functions, mainly those associated with continuum cardinals.

The material on sheaves, Boolean products, and Boolean powers has been omitted, since these no longer play a role in our discussion of the cardinal invariants.

Although many problems in the earlier versions have been solved, many of them are still open. In this edition we repeat those unsolved problems, and add several new ones.

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0 Introduction

This book is concerned with the theory of certain natural functions k which assign to each infinite Boolean algebra A a cardinal number $k(A)$ or a set $k(A)$ of cardinal numbers. The purpose of the book is to survey this area of the theory of BAs, giving proofs for a large number of results, some of which are new, mentioning most of the known results, and formulating open problems. Some of the open problems are somewhat vague ("Characterize..." or something like that), but frequently these are even more important than the specific problems we state; so we have opted to enumerate problems of both sorts in order to focus attention on them.

The framework that we shall set forth and then follow in investigating cardinal functions seems to us to be important for several reasons. First of all, the functions themselves seem intrinsically interesting. Many of the questions which naturally arise can be easily answered on the basis of our current knowledge of the structure of Boolean algebras, but some of these answers require rather deep arguments of set theory, algebra, or topology. This provides another interest in their study: as a natural source of applications of set-theoretical, algebraic, or topological methods. Some of the unresolved questions are rather obscure and uninteresting, but some of them have a general interest. Altogether, the study of cardinal functions seems to bring a unity and depth to many isolated investigations in the theory of BAs.

There are several surveys of cardinal functions on Boolean algebras, or, more generally, on topological spaces: See Arhangelskiĭ [78], Comfort [71], van Douwen [89], Hodel [84], Juhász [75], Juhász [80], Juhász [84], Monk [84], Monk [90], and Monk [96]. We shall not assume any acquaintance with any of these. On the other hand, we shall frequently refer to results proved in Part I of the Handbook of Boolean Algebras, Koppelberg [89]. One additional bit of terminology: in a weak product $\prod^{\mathrm{w}}_{i\in I} A_i$, we call an element a of *type* 1 iff $\{i\in I : a_i\neq 0\}$, the 1-*support* of a, is finite, and of *type* 2 iff $\{i\in I : a_i\neq 1\}$, called the 2-*support* of a, is finite.

We have not attempted to give a complete history of the results mentioned in this book. The references can be consulted for a detailed background.

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Definition of the main cardinal functions considered

We defer until later the discussion of the existence of some of these functions; they do not all exist for every BA.

Cellularity. A subset X of a BA A is called *disjoint* if its members are pairwise disjoint. The *cellularity* of A, denoted by $\mathrm{c}(A)$, is

$$\sup\{|X| : X \text{ is a disjoint subset of } A\}.$$

Depth. $\mathrm{Depth}(A)$ is

$$\sup\{|X| : X \text{ is a subset of } A \text{ well ordered by the Boolean ordering}\}.$$

Topological density. The *density* of a topological space X, denoted by $\mathrm{d}(X)$, is the smallest cardinal κ such that X has a dense subspace of cardinality κ. The *topological density* of a BA A, also denoted by $\mathrm{d}(A)$, is the density of its Stone space $\mathrm{Ult}(A)$.

π-weight. A subset X of a BA A is *dense* in A if for all $a \in A^+$ there is an $x \in X^+$ such that $x \leq a$. The *π-weight* of a BA A, denoted by $\pi(A)$, is

$$\min\{|X| : X \text{ is dense in } A\}.$$

This could also be called the *algebraic density* of A. (Recall that for any subset X of a BA, X^+ is the collection of nonzero elements of X.)

Length. $\mathrm{Length}(A)$ is

$$\sup\{|X| : X \text{ is a chain in } A\}.$$

Irredundance. A subset X of a BA A is *irredundant* if for all $x \in X$, $x \notin \langle X\backslash\{x\}\rangle$. (Recall that $\langle Y\rangle$ is the subalgebra generated by Y.) The *irredundance* of A, denoted by $\mathrm{Irr}(A)$, is

$$\sup\{|X| : X \text{ is an irredundant subset of } A\}.$$

Cardinality. This is just $|A|$. Sometimes we denote it by $\mathrm{card}(A)$.

Independence. A subset X of A is called *independent* if X is a set of free generators for $\langle X\rangle$. Then the *independence* of A, denoted by $\mathrm{Ind}(A)$, is

$$\sup\{|X| : X \text{ is an independent subset of } A\}.$$

π-character. For any ultrafilter F on A, let $\pi\chi(F) = \min\{|X| : X \text{ is dense in } F\}$. Note here that it is not required that $X \subseteq F$. Then the *π-character* of A, denoted by $\pi\chi(A)$, is

$$\sup\{\pi\chi(F) : F \text{ is an ultrafilter of } A\}.$$

Tightness. For any ultrafilter F on A, let $\mathrm{t}(F) = \min\{\kappa :$ if Y is contained in $\mathrm{Ult}(A)$ and F is contained in $\bigcup Y$, then there is a subset Z of Y of power at most κ such that F is contained in $\bigcup Z\}$. Then the *tightness* of A, denoted by $\mathrm{t}(A)$, is

$$\sup\{\mathrm{t}(F) : F \text{ is an ultrafilter on } A\}.$$

Spread. The *spread* of A, denoted by $\mathrm{s}(A)$, is

$$\sup\{|D| : D \subseteq \mathrm{Ult}(A), \text{ and } D \text{ is discrete in the relative topology}\}.$$

Character. The *character* of A, denoted by $\chi(A)$, is

$$\min\{\kappa : \text{ every ultrafilter on } A \text{ can be generated by at most } \kappa \text{ elements}\}.$$

Hereditary Lindelöf degree. For any topological space X, the *Lindelöf degree* of X is the smallest cardinal $\mathrm{L}(X)$ such that every open cover of X has a subcover with at most $\mathrm{L}(X)$ elements. Then the *hereditary Lindelöf degree* of A, denoted by $\mathrm{hL}(A)$, is

$$\sup\{\mathrm{L}(X) : X \text{ is a subspace of } \mathrm{Ult}(A)\}.$$

Hereditary density. The *hereditary density* of A, $\mathrm{hd}(A)$, is

$$\sup\{\mathrm{d}(S) : S \text{ is a subspace of } \mathrm{Ult}(A)\}.$$

Incomparability. A subset X of A is *incomparable* if for any two distinct elements $x, y \in X$ we have $x \not\leq y$ and $y \not\leq x$. The *incomparability* of A, denoted by $\mathrm{Inc}(A)$, is

$$\sup\{|X| : X \text{ is an incomparable subset of } A\}.$$

Hereditary cofinality. This cardinal function, h-cof(A), is

$$\min\{\kappa : \text{ for all } X \subseteq A \text{ there is a } C \subseteq X \text{ with } |C| \leq \kappa \text{ and } C \text{ cofinal in } X\}.$$

Number of ultrafilters. Of course, this is the same as the cardinality of the Stone space of A, and is denoted by $|\mathrm{Ult}(A)|$.

Number of automorphisms. We denote by $\mathrm{Aut}(A)$ the set of all automorphisms of A. So this cardinal function is $|\mathrm{Aut}(A)|$.

Number of endomorphisms. We denote by $\mathrm{End}(A)$ the set of all endomorphisms of A, and hence this cardinal function is $|\mathrm{End}(A)|$.

Number of ideals of ***A***. We denote by $\mathrm{Id}(A)$ the set of all ideals of A, so here we have the cardinal function $|\mathrm{Id}(A)|$.

Number of subalgebras of ***A***. We denote by $\mathrm{Sub}(A)$ the set of all subalgebras of A; $|\mathrm{Sub}(A)|$ is this cardinal function.

Some classifications of cardinal functions

Some theorems which we shall present, especially some involving unions or ultraproducts, are true for several of our functions, with essentially the same proof. For this reason we introduce some rather ad hoc classifications of the functions. Some of the statements below are proved later in the book.

A cardinal function k is an *ordinary sup-function* with respect to P if P is a function assigning to every infinite BA A a subset $P(A)$ of $\mathscr{P}(A)$ so that the following conditions hold for any infinite BA A:

(1) $k(A) = \sup\{|X| : X \in P(A)\}$;
(2) If B is a subalgebra of A, then $P(B) \subseteq P(A)$ and $X \cap B \in P(B)$ for any $X \in P(A)$.
(3) For each infinite cardinal κ there is a BA C of size κ such that there is an $X \in P(C)$ with $|X| = \kappa$.

Table 0.1 lists some ordinary sup-functions.

Table 0.1: ordinary sup-functions

Function	The subset $P(A)$
$\mathrm{c}(A)$	$\{X : X$ is disjoint$\}$
$\mathrm{Depth}(A)$	$\{X : X$ is well ordered by the Boolean ordering of $A\}$
$\mathrm{Length}(A)$	$\{X : X$ is linearly ordered by the Boolean ordering of $A\}$
$\mathrm{Irr}(A)$	$\{X : X$ is irredundant$\}$
$\mathrm{Ind}(A)$	$\{X : X$ is independent$\}$
$\mathrm{s}(A)$	$\{X : X$ is ideal-independent$\}$
$\mathrm{Inc}(A)$	$\{X : X$ is incomparable$\}$

Given any ordinary sup-function k with respect to a function P and any infinite cardinal κ, we say that A satisfies the *$\kappa - k$-chain condition* provided that $|X| < \kappa$ for all $X \in P(A)$.

A cardinal function k is an *ultra-sup function with respect to P* if P is a function assigning to each infinite BA a subset $P(A)$ of $\mathscr{P}(A)$ such that the following conditions hold:

(1) $k(A) = \sup\{|X| : X \in P(A)\}$.
(2) If $\langle A_i : i \in I\rangle$ is a sequence of BAs, F is an ultrafilter on I, and $X_i \in P(A_i)$ for all $i \in I$, then $\{f/F : f(i) \in X_i$ for all $i \in I\} \in P\left(\prod_{i\in I} A_i/F\right)$.

All of the above ordinary sup-functions except Depth are also ultra-sup functions.

For the next classification, extend the first-order language for BAs by adding two unary relation symbols $\mathbf{F}$ and $\mathbf{P}$. Then we say that k is a *sup-min* function if there are sentences $\varphi(\mathbf{F},\mathbf{P})$ and $\psi(\mathbf{F})$ in this extended language such that:

(1) $k(A) = \sup\{\min\{|P| : (A,F,P) \models \varphi\} : A$ is infinite and $(A,F) \models \psi\}$. In particular, for any BA A there exist $F, P \subseteq A$ such that $(A,F,P) \models \varphi$.

(2) φ has the form $\forall x \in \mathbf{P}(x \neq 0 \wedge \varphi'(\mathbf{F},x)) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}\varphi''(\mathbf{F},x,y)$.

(3) $(A,F) \models \psi(\mathbf{F}) \rightarrow \exists x(x \neq 0 \wedge \varphi'(\mathbf{F},x))$.

Some sup-min functions are listed in Table 0.2, where $\mu(\mathbf{F})$ is the formula saying that $\mathbf{F}$ is an ultrafilter.

Table 0.2: sup-min functions

Function	$\psi(\mathbf{F})$	$\varphi(\mathbf{F},\mathbf{P})$
π	$\forall x \mathbf{F} x$	$\forall x \in \mathbf{P}(x \neq 0) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(x \neq 0 \rightarrow y \leq x)$
$\pi\chi$	$\mu(\mathbf{F})$	$\forall x \in \mathbf{P}(x \neq 0) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(y \leq x)$
χ	$\mu(\mathbf{F})$	$\forall x \in \mathbf{P}(x \neq 0 \wedge x \in \mathbf{F}) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(y \leq x)$
h-cof	$x = x$	$\forall x \in \mathbf{P}(x \neq 0 \wedge x \in \mathbf{F}) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(y \geq x)$

A cardinal function k is an *order-independence* function if there exists a sentence φ in the language of $(\omega, <, \omega, \omega)$ such that the following two conditions hold:

(1) For any infinite BA A we have $k(A) = \sup\{\lambda :$ there exists a sequence $\langle a_\alpha : \alpha < \lambda\rangle$ of elements of A such that for all finite $G, H \subseteq \lambda$ such that $(\lambda, <, G, H) \models \varphi$ we have $\prod_{\alpha\in G} a_\alpha \cdot \prod_{\alpha\in H} -a_\alpha \neq 0\}$.

(2) If λ is an infinite cardinal, $(\lambda, <, G, H) \models \varphi$, $G', H' \subseteq \lambda$, and f is a one-to-one function from $G \cup H$ onto $G' \cup H'$ such that for all $\alpha, \beta \in G \cup H$, if $\alpha < \beta$ then $f(\alpha) < f(\beta)$, then $(\lambda, <, G', H') \models \varphi$.

Some order-independence functions are listed in Table 0.3.

Table 0.3: order-independence functions

Function	φ
t	$\forall x \in \mathbf{G}\,\forall y \in \mathbf{H}\,(x < y)$
hd	$\exists x \in \mathbf{G}\,\forall y \in \mathbf{G}\,(x = y) \wedge \forall x \in \mathbf{G}\,\forall y \in \mathbf{H}\,(x < y)$
hL	$\exists x \in \mathbf{H}\,\forall y \in \mathbf{H}\,(x = y) \wedge \forall x \in \mathbf{G}\,\forall y \in \mathbf{H}\,(x < y)$

Algebraic properties of a single function

Now we go into more detail on the properties of a single function which we shall investigate. From the point of view of general algebra, the main questions are: what happens to the cardinal function k under the passage to subalgebras, homomorphic images, products, and free products? There are natural problems too about more special operations on algebras in general, or on Boolean algebras in particular: what happens to k under weak products, amalgamated free products, unions of well-ordered chains of subalgebras, ultraproducts, dense subalgebras, subdirect products, moderate products, one-point gluing, Alexsandroff duplication, and the exponential? The mentioned operations which are not discussed in the Handbook will be explained in Chapter 1. There are also several special kinds of subalgebras where one can ask what happens to the functions when passing to such a special subalgebra. Many of these special subalgebras are discussed in Heindorf, Shapiro [94]. For ease of reference, we list here ones which we consider to be worthwhile to investigate in this context:

$A \leq_{\text{reg}} B$: A is a regular subalgebra of B. (Handbook, page 21.)

$A \leq_{\text{rc}} B$; A is relatively complete in B. (Handbook, page 123.)

$A \leq_{\pi} B$: A is a dense subalgebra of B.

$A \leq_{s} B$: B is a simple extension of A. (See Chapter 2.)

$A \leq_{m} B$: B is a minimal extension of A. (See Chapter 2.)

$A \leq_{\text{mg}} B$: B is minimally generated over A. (See Chapter 2.)

$A \leq_{\text{free}} B$: B is a free extension of A. This means that $B = A \oplus F$ for some free BA F.

$A \leq_{\sigma} B$: A is σ-embedded in B. This means that $A \leq B$, and for every $b \in B$, the ideal $\{a \in A : a \leq b\}$ of A is countably generated.

$A \leq_{\text{proj}} B$: A is projectively embedded in B. This means that there is a free BA C and homomorphisms $e : B \to A \oplus C$ and $q : A \oplus C \to B$ such that $q \circ e = \text{Id}_B$ and $e \restriction A = q \restriction A = \text{Id}_A$. See Koppelberg [89b], page 752. This is illustrated by the following diagram:

$$\begin{tikzcd}
B \arrow[rr, bend left=15, "e"] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & & \\
A \arrow[r, phantom, "\leq"] & \; & A \oplus C \arrow[ull, bend left=15, "q"]
\end{tikzcd}$$

$A \leq_{\text{u}} B$: every ultrafilter on A has at least two different extensions to ultrafilters on B.

One may notice that several of the above functions, such as depth and spread, are defined as supremums of the cardinalities of sets satisfying some property P. So, a natural question is whether such sups are *attained*, that is, with depth as an example, whether for every BA A there always is a subset X well ordered by the Boolean ordering, with $|X| = \mathrm{Depth}(A)$. Of course, this is only a question in case $\mathrm{Depth}(A)$ is a limit cardinal. For such functions k defined by sups, we can define a closely related function k'; $k'(A)$ is the least cardinal such that there is no subset of A with the property P. So $k'(A) = (k(A))^+$ if k is attained, and $k'(A) = k(A)$ otherwise.

Derived functions

From a given cardinal function one can define several others; part of our work is to see what these new cardinal functions look like; frequently it turns out that they coincide with others of our basic functions, but sometimes we arrive at a new function in this way:

$$
\begin{aligned}
k_{\mathrm{H}+}(A) &= \sup\{k(B) : B \text{ is a homomorphic image of } A\}.\\
k_{\mathrm{H}-}(A) &= \inf\{k(B) : B \text{ is an infinite homomorphic image of } A\}.\\
k_{\mathrm{S}+}(A) &= \sup\{k(B) : B \text{ is a subalgebra of } A\}.\\
k_{\mathrm{S}-}(A) &= \inf\{k(B) : B \text{ is an infinite subalgebra of } A\}.\\
k_{\mathrm{h}+}(A) &= \sup\{k(Y) : Y \text{ is a subspace of } \mathrm{Ult}A\}.\\
k_{\mathrm{h}-}(A) &= \inf\{k(Y) : Y \text{ is an infinite subspace of } \mathrm{Ult}A\}.\\
{}_{\mathrm{d}}k_{\mathrm{S}+}(A) &= \sup\{k(B) : B \text{ is a dense subalgebra of } A\}.\\
{}_{\mathrm{d}}k_{\mathrm{S}-}(A) &= \inf\{k(B) : B \text{ is a dense subalgebra of } A\}.
\end{aligned}
$$

Note that $k_{\mathrm{h}+}(A)$ and $k_{\mathrm{h}-}(A)$ make sense only if k is a function which naturally applies to topological spaces in general as well as BAs. Any infinite Boolean space has a denumerable discrete subspace, and frequently $k_{\mathrm{h}-}$ will take its value on such a subspace. Also note with respect to ${}_{\mathrm{d}}k_{\mathrm{S}+}(A)$ and ${}_{\mathrm{d}}k_{\mathrm{S}-}(A)$ that one could consider other kinds of subalgebras, as in the previous list of them.

Given a function defined in terms of ultrafilters, like character above, there is usually an associated function l assigning a cardinal number to each ultrafilter on A. Then one can introduce two cardinal functions on A itself:

$$
\begin{aligned}
l_{\sup}(A) &= \sup\{l(F) : F \text{ is an ultrafilter on } A\}.\\
l_{\inf}(A) &= \inf\{l(F) : F \text{ is a non-principal ultrafilter on } A\}.
\end{aligned}
$$

Another kind of derived function applies to cases where the function is defined as the sup of cardinalities of sets X with a property P, where P is such that maximal families with the property P exist (usually seen by Zorn's lemma). For such a function k, we define

$$
\begin{aligned}
k_{\mathrm{mm}}(A) &= \min\{|X| : X \text{ is an infinite maximal family satisfying } P\};\\
k_{\mathrm{spect}}(A) &= \{|X| : X \text{ is an infinite maximal family satisfying } P\}.
\end{aligned}
$$

We also consider the following two spectrum functions, which assign to each BA a set of cardinal numbers:

$$\begin{aligned} k_{\mathrm{Hs}}(A) &= \{k(B) : B \text{ is an infinite homomorphic image of } A\} \\ &\quad (\text{the } \textit{homomorphic spectrum} \text{ of } A) \\ k_{\mathrm{Ss}}(A) &= \{k(B) : B \text{ is an infinite subalgebra of } A\} \\ &\quad (\text{the } \textit{subalgebra spectrum} \text{ of } A) \end{aligned}$$

It is also possible to define a *caliber* notion for many of our functions, in analogy to the well-known caliber notion for cellularity. Given a property P associated with a cardinal function, a BA A is said to have $\kappa, \lambda,$ P-*caliber* if among any set of λ elements of A there are κ elements with property P. The property P is not necessarily one used to define the function; thus for cellularity P is the finite intersection property, while for independence it is, indeed, independence.

Comparing two functions

Given two cardinal functions k and l, one can try to determine whether $k(A) \leq l(A)$ for every BA A or $l(A) \leq k(A)$ for every BA A. Given that one of these cases arises, it is natural to consider whether the difference can be arbitrarily large (as with cellularity and spread, for example), or if it is subject to restrictions (as with depth and length). If no general relationship is known, a counterexample is needed, and again one can try to find a counterexample with an arbitrarily large difference between the two functions. Of course, the known inequalities between our functions help in order to limit the number of cases that need to be considered for constructing such counterexamples; here the diagrams at the end of the book are sometimes useful. For example, knowing that $\pi\chi$ can be greater than c, we also know that χ can be greater than c.

Other considerations

In addition to the above systematic goals in discussing cardinal functions, there are some more ideas which we shall not explore in such detail. One can compare several cardinal functions, instead of just two at a time. Several deep theorems of this sort are known, and we shall mention a few of them. There is also a large number of relationships between cardinal functions which involve cardinal arithmetic; for example, $\mathrm{Length}(A) \leq 2^{\mathrm{Depth}(A)}$ for any BA A. We mention a few of these as we go along.

One can compare two cardinal functions while considering algebraic operations; for example, comparing functions k, l with respect to the formation of

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subalgebras. We shall investigate just two of the many possibilities here:

$$\begin{aligned} k_{\mathrm{Sr}}(A) = \{(\kappa,\lambda) :\ & \text{there is an infinite subalgebra } B \text{ of } A \\ & \text{such that } |B| = \lambda \text{ and } k(B) = \kappa\}; \\ k_{\mathrm{Hr}}(A) = \{(\kappa,\lambda) :\ & \text{there is an infinite homomorphic image } B \text{ of } A \\ & \text{such that } |B| = \lambda \text{ and } k(B) = \kappa\}. \end{aligned}$$

These are called, respectively, the *subalgebra k relation* and the *homomorphic k relation.*

For each function k, it would be nice to be able to characterize the possible relations k_{Sr} and k_{Hr} in purely cardinal number terms.

Another general idea applies to several functions that are defined somehow in terms of finite sets; the idea is to take bounded versions of them. For example, independence has bounded versions: for any positive integer n, a subset X of a BA A is called *n-independent* if for every subset Y of X with at most n elements and every $\varepsilon \in {}^Y2$ we have $\prod_{y\in Y} y^{\varepsilon y} \neq 0$. (Here $x^1 = x$, $x^0 = -x$ for any x.) And then we define $\mathrm{Ind}_n(A) = \sup\{|X| : X \text{ is } n\text{-independent}\}$. It is interesting to investigate this notion and its relationship to actual independence; and similar things can be done for various other functions.

Special classes of Boolean algebras

We are interested in all of the above ideas not only for the class of all BAs, but also for various important subclasses: complete BAs, interval algebras, tree algebras, and superatomic algebras, which are discussed in the Handbook. To a lesser extent we give facts about cardinal functions for other subclasses like all atomic BAs, atomless BAs, initial chain algebras, minimally generated algebras, pseudo-tree algebras, semigroup algebras, and tail algebras. In Chapter 2 we describe some properties of the special classes mentioned which are not discussed in the Handbook, partly to establish notation.

1 Special Operations on Boolean Algebras

We give the basic definitions and facts about several operations on Boolean algebras which were not discussed in the Handbook.

We begin with some elementary but useful results concerning products.

Proposition 1.1. *C is a homomorphic image of $A\times B$* iff *C is isomorphic to $A'\times B'$ for some homomorphic images A' and B' of A and B respectively.*

Proof. We may assume that A and B are non-trivial.

$\Leftarrow$: obvious. $\Rightarrow$: suppose that f is a homomorphism from $A\times B$ onto C. It suffices to show that $C\restriction f(1,0)$ is a homomorphic image of A. Let I be a maximal ideal in A, and for any $a\in A$ let $g(a)=f(a,a/I)\cdot f(1,0)$. Clearly g is a homomorphism from A into $C\restriction f(1,0)$. To show that it is onto, let $x\in C\restriction f(1,0)$. Say $f(a,b)=x$. Then

$$g(a)=f(a,a/I)\cdot f(1,0)=f(a,b)\cdot f(1,0)=x. \qquad \square$$

Proposition 1.2. *Let $\langle A_i : i\in I\rangle$ be a system of non-trivial BAs, with I infinite. Then C is a homomorphic image of $\prod^{\mathrm{w}}_{i\in I} A_i$* iff *there is a system $\langle B_i : i\in I\rangle$ of BAs such that $\forall i\in I[B_i$ is a homomorphic image of $A_i]$, and $C\cong\prod^{\mathrm{w}}_{i\in I} B_i$.*

Proof. For brevity let $D=\prod^{\mathrm{w}}_{i\in I} A_i$.

For $\Leftarrow$, suppose that $f_i : A_i\to B_i$ is a surjective homomorphism for each $i\in I$. Define $g : D\to\prod^{\mathrm{w}}_{i\in I} B_i$ by setting, for each $a\in D$, $(g(a))_i=f_i(a_i)$. Clearly g is a homomorphism from D into $\prod^{\mathrm{w}}_{i\in I} B_i$. To show that it is surjective, let $b\in\prod^{\mathrm{w}}_{i\in I} B_i$. For each $i\in I$ choose $a_i\in A_i$ such that $f_i(a_i)=b_i$, with $a_i=0$ if $b_i=0$ and $a_i=1$ if $b_i=1$. Clearly $a\in D$ and $g(a)=b$. Thus $\prod^{\mathrm{w}}_{i\in I} B_i\cong D/\mathrm{ker}(g)$.

For $\Rightarrow$, suppose that K is a proper ideal in D. For each $i\in I$ let $L_i=\{b_i : b\in K\}$. Clearly L_i is an ideal in A_i.

Case 1. There is an $a\in K$ of type 2. Let F be the 2-support of a. Define $f([b]_K)=\langle [b_i]_{L_i} : i\in F\rangle$ for each $b\in D$. Clearly f is well defined, and is a homomorphism from D/K into $\prod_{i\in F} A_i/L_i$. It is one-one, for suppose that $b_i\in L_i$ for all $i\in F$. For each $i\in F$ choose $c^i\in K$ such that $b_i=c^i_i$. Then $b\le a+\sum_{i\in F} c^i\in K$, and so

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$b \in K$. It maps onto $\prod_{i\in F} A_i/L_i$; for suppose that $x_i \in A_i$ for each $i \in F$. Extend to a function $x \in D$. Then $f([x]_K) = \langle [x_i]_{L_i} : i \in F\rangle$, as desired. Let $B_i = A_i/L_i$ for all $i \in F$, and let B_i be trivial for all $i \in I\backslash F$.

Case 2. Every $a \in K$ is of type 1. For each $i \in I$ define $\chi^i \in D$ by setting, for each $j \in I$,

$$\chi^i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $J = \{i \in I : \chi^i \notin K\}$. If $J = \emptyset$, then K is a maximal ideal in D, and $|D/K| = 2$, giving the desired result, letting one B_i be a two element homomorphic image of A_i, the other B_is trivial.

Suppose that $\emptyset \neq J$ and J is finite. Let M be the maximal ideal of D consisting of all elements of type 1. For each $b \in D$ define $f([b]_K) = (\langle [b_i]_{L_i} : i \in J\rangle, [b]_M)$. Clearly f is a well-defined homomorphism from D/K into $(\prod_{i\in J}(A_i/L_i)) \times 2$. f is one-one: suppose that $b_i \in L_i$ for all $i \in J$ and $b \in M$. For each $i \in J$ choose $c^i \in K$ such that $b_i = c^i_i$. Let F be the 1-support of b. Define $d \in D$ by

$$d(i) = \begin{cases} c^i_i & \text{if } i \in J, \\ 1 & \text{if } i \in F\backslash J, \\ 0 & \text{otherwise.} \end{cases}$$

Then $d \in K$ and $b \leq d$, so $b \in K$. Also, f maps onto $(\prod_{i\in J} A_i/L_i) \times 2$. For, suppose that $c \in \prod_{i\in J} A_i$ and $\varepsilon \in 2$. If $\varepsilon = 1$, extend c to $b \in D$ by defining $b(i) = 1$ for all $i \in I\backslash J$. Clearly $f([b]_K) = (\langle [c_i]_{L_i} : i \in J\rangle, 1)$. If $\varepsilon = 0$, extend c to $b \in D$ by defining $b(i) = 0$ for all $i \in I\backslash J$. Clearly $f([b]_K) = (\langle [c_i]_{L_i} : i \in J\rangle, 0)$. Now we can let $B_i = A/L_i$ for all $i \in J$, B_i a two-element homomorphic image of A_i for some $i \in I\backslash J$, and all other B_is trivial.

Suppose that J is infinite. For each $b \in D$ define $f([b]_K) = \langle [b_i]_{L_i} : i \in J\rangle$. Clearly f is a well-defined homomorphism from D/K into $\prod^{\mathrm{w}}_{i\in J} A_i/L_i$. f is one-one: suppose that $b_i \in L_i$ for all $i \in J$. For each $i \in J$ choose $c^i \in K$ such that $b_i = c^i_i$. Since J is infinite, $1 \notin L_i$ for each $i \in J$, and $b_i \in L_i$ for each $i \in J$, it follows that b is of type 1. Let F be the 1-support of b. Define $d \in D$ by

$$d(i) = \begin{cases} c^i_i & \text{if } i \in J, \\ 1 & \text{if } i \in F\backslash J, \\ 0 & \text{otherwise.} \end{cases}$$

Then $d \in K$ and $b \leq d$, so $b \in K$. Also, f maps onto $\prod^{\mathrm{w}}_{i\in J} A_i/L_i$. For, suppose that $c \in \prod^{\mathrm{w}}_{i\in J} A_i$. If c is of type 2, extend c to $b \in D$ by defining $b(i) = 1$ for all $i \in I\backslash J$. Clearly $f([b]_K) = \langle [c_i]_{L_i} : i \in J\rangle$. If c is of type 1, extend c to $b \in D$ by defining $b(i) = 0$ for all $i \in I\backslash J$. Clearly $f([b]_K) = \langle [c_i]_{L_i} : i \in J\rangle$. Now we can define $B_i = A_i/L_i$ for all $i \in J$, with the other B_is trivial. □

Concerning arbitrary products, we have the following simple result.

Proposition 1.3. *Let $\langle A_k : k \in K\rangle$ be a system of BAs, and $\langle I_k : k \in K\rangle$ a system such that I_k is an ideal in A_k for all $k \in K$. Let $J = \{a \in \prod_{k\in K} A_k : \forall k \in K[a_k \in I_k]\}$. Then J is an ideal in $\prod_{k\in K} A_k$, and $(\prod_{k\in K} A_k)/J \cong \prod_{k\in K}(A_k/I_k)$.*

Proof. Clearly J is an ideal in $\prod_{k\in K} A_k$. Now for each $a \in \prod_{k\in K} A_k$ and each $k \in K$ let $(f(a))_k = [a_k]_{I_k}$. Clearly f is a homomorphism from $\prod_{k\in K} A_k$ onto $\prod_{k\in K}(A_k/I_k)$ with kernel J, so the result follows by the homomorphism theorem. $\square$

However, the property of Proposition 1.2 does not extend to arbitrary products. In the following example we use the notation $\mathrm{Finco}(I)$ for the BA of finite and cofinite subsets of I.

Example 1.4. For each $k \in \omega$ let $A_k = \mathrm{Finco}(\omega)$. Then ${}^\omega 2$ is a subalgebra of $\prod_{k\in\omega} A_k$. Let f be a homomorphism from ${}^\omega 2$ onto $\mathscr{P}(\omega)/\mathrm{fin}$, and extend f to a homomorphism g from $\prod_{k\in\omega} A_k$ into the completion of $\mathscr{P}(\omega)/\mathrm{fin}$. Then there is an ideal J on $\prod_{k\in\omega} A_k$ such that $(\prod_{k\in\omega} A_k)/J$ is isomorphic to $\mathrm{rng}(g)$. Note that $\mathrm{rng}(g)$ is atomless. But if $\langle I_k : k \in \omega\rangle$ is any sequence of ideals on $\mathrm{Finco}(\omega)$, then $\prod_{k\in\omega}(A_k/I_k)$ is atomic. Thus $(\prod_{k\in\omega} A_k)/J$ is not isomorphic to $\prod_{k\in\omega}(A_k/I_k)$.

Moderate products

This operation, due to Weese [80] and Gurevich [82] independently, is extensively studied in Heindorf [92], to whom the name is due. Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs; we assume that A_i is a field of subsets of some set J_i, and that the J_i's are pairwise disjoint. Furthermore, let B be an algebra of subsets of I containing all of the finite subsets of I. For each $b \in B$ let $\bar{b} = \bigcup_{i\in b} J_i$. Set $K = \bigcup_{i\in I} J_i$. For each $b \in B$, each finite $F \subset I$, and each $a \in \prod_{i\in F} A_i$, the set

$$\bar{b} \cup \bigcup_{i\in F} a_i$$

will be denoted by $h(b, F, a)$. If $F \cap b = \emptyset$ and $0 \subset a_i \subset J_i$ for every $i \in F$, then we call (b, F, a) *normal.*

Proposition 1.5. *Assume the above notation.*

(i) *For any $b \in B$, $F \in [I]^{<\omega}$, and $a \in \prod_{i\in F} A_i$ we have $h(b, F, a) = h(b', F', a')$, where $b' = b \cup \{i \in F : a_i = J_i\}$, $F' = \{i \in F\backslash b : \emptyset \subset a_i \subset J_i\}$, and $a' = a \restriction F'$; moreover, (b', F', a') is normal.*

(ii) *If (b, F, a) is normal, then $K\backslash h(b, F, a) = h(I\backslash(b\cup F), F, a')$, where $a'_i = J_i\backslash a_i$ for all $i \in F$; and $(I\backslash(b \cup F), F, a')$ is normal.*

(iii) *$h(b, F, a) \cap h(b', F', a') = h(b'', F'', a'')$ for normal (b, F, a) and (b', F', a'), where $b'' = b \cap b'$, $F'' = (F' \cap b) \cup (F \cap b') \cup \{i \in F \cap F' : a_i \cap a'_i \neq \emptyset\}$, and* *for any* $i \in F''$,

$$a''_i = \begin{cases} a'_i & \textit{if } \ i \in F' \cap b, \\ a_i & \textit{if } \ i \in F \cap b', \\ a_i \cap a'_i & \textit{if } \ i \in F \cap F' \textit{ and } a_i \cap a'_i \neq \emptyset; \end{cases}$$

and (b'', F'', a'') *is normal.*

(iv) *If* (b, F, a) *and* (b', F', a') *are normal, then* $h(b, F, a) \subseteq h(b', F', a')$ *iff* $b \subseteq b'$, $F' \cap b = \emptyset$, $F \subseteq b' \cup F'$, *and* $\forall i \in F \cap F'[a_i \subseteq a'_i]$.

(v) *If* (b, F, a) *and* (b', F', a') *are normal and* $h(b, F, a) = h(b', F', a')$, *then* $b = b'$, $F = F'$, *and* $a = a'$.

Proof. It is straightforward to check (i)–(iii). For (iv), suppose that (b, F, a) and (b', F', a') are normal.

First suppose that $h(b, F, a) \subseteq h(b', F', a')$. Clearly $b \subseteq b'$ and $F' \cap b = \emptyset$. Next, suppose that $i \in F \backslash b'$. Then $a_i \subseteq h(b', F', a')$, so $i \in F'$. Now suppose that $i \in F \cap F'$. Then $a_i \subseteq h(b, F, i)$ and $i \notin b'$, so $a_i \subseteq a'_i$.

Second, suppose that $b \subseteq b'$, $F' \cap b = \emptyset$, $F \subseteq b' \cup F'$, and $\forall i \in F \cap F'[a_i \subseteq a'_i]$. Take any $x \in h(b, F, a)$. If $x \in b$, then $x \in b'$ and hence $x \in h(b', F', a')$. Suppose that $i \in F$ and $x \in a_i$. If $i \in b'$, then $x \in h(b', F', a')$. Suppose that $i \notin b'$. So $i \in F \backslash b'$, and so $i \in F'$. Since then $i \in F \cap F'$, we have $a_i \subseteq a'_i$, and so $x \in h(b', F', a')$.

(v) follows from (iv). $\square$

The BA of all sets $h(b, F, a)$ is the *moderate product of the* A_i*'s over* B, and is denoted by $\prod_{i \in I}^B A_i$.

Theorem 1.6. *Suppose that* $\langle A_i : i \in I \rangle$ *is a system of BAs; each* A_i *is a field of subsets of some set* J_i*; the* J_i*'s are pairwise disjoint, and* $\mathrm{Finco}(I) \leq B \leq \mathscr{P}(I)$.

(i) *For every* $i \in I$ *we have* $J_i \in \prod_{i \in I}^B A_i$ *and* $A_i = \left(\prod_{i \in I}^B A_i \right) \restriction J_i$.

(ii) B *is isomorphic to a subalgebra of* $\prod_{i \in I}^B A_i$*; in fact,* $\langle \overline{b} : b \in B \rangle$ *is an isomorphic embedding.*

(iii) *If* $\mathrm{Finco}(I) \leq B \leq C \leq \mathscr{P}(I)$, *then* $\prod_{i \in I}^B A_i \leq \prod_{i \in I}^C A_i$.

(iv) $\prod_{i \in I}^B A_i$ *can be embedded in* $\prod_{i \in I} A_i$.

(v) $\prod_{i \in I}^{\mathrm{w}} A_i \cong \prod_{i \in I}^{\mathrm{Finco}(I)} A_i$.

(vi) *If* I *is finite, then* $B = \mathscr{P}(I)$, *and* $\prod_{i \in I}^B A_i \cong \prod_{i \in I} A_i$.

(vii) *For each* $i \in I$ *and each* $x \in \prod_{i \in I}^B A_i$ *let* $f(x) = x \cap J_i$. *Then* f *is a homomorphism from* $\prod_{i \in I}^B A_i$ *onto* A_i.

(viii) *Let* g *be the natural isomorphism of* $\mathscr{P}(I)$ *onto* ${}^I 2$. *Then*

$$\prod_{i \in I}^B A_i \cong \left\langle \prod_{i \in I}^{\mathrm{w}} A_i \cup g[B] \right\rangle_{\prod_{i \in I} A_i}.$$

(ix) *If* $b \in [I]^{<\omega}$*, then* $\prod_{i \in b} A_i \cong \left(\prod_{i \in I}^B A_i\right) \restriction \overline{b}$.

(x) *Suppose that* $U \subseteq \prod_{i \in I}^B A_i$*. Then* U *is an ultrafilter on* $\prod_{i \in I}^B A_i$ iff *one of the following holds:*

(a) *There exist an* $i \in I$ *and an ultrafilter* V *on* A_i *such that* $U = \{h(b,F,a) : (b,F,a)$ *is normal, and* $i \in b$ *or* $(i \notin b$ *and* $i \in F$ *and* $a_i \in V)\}$.

(b) *There is a nonprincipal ultrafilter* W *on* B *such that* $U = \{h(b,F,a) : (b,F,a)$ *is normal and* $b \in W\}$.

Proof. (i)–(iii) are clear by Proposition 1.5. For (iv), define $(f(x))_i = x \cap J_i$ for any $x \in \prod_{i \in I}^B A_i$ and each $i \in I$. We show that f is the desired embedding. For $\cdot$, let normal $(b,F,a), (b',F',a')$ be given, and let (b'',F'',a'') be as in Proposition 1.5(iii). If $i \in b''$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = J_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

If $i \in F' \cap b$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = J_i \cap a'_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

If $i \in F \cap b'$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = a_i \cap J_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

Finally, if $i \in F \cap F'$ and $a_i \cap a'_i \neq \emptyset$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = a_i \cap a'_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

For $-$, recall Proposition 1.5(ii). If $i \in b$, then

$$[K \backslash h(b,F,a)] \cap J_i = \emptyset = J_i \backslash J_i = J_i \backslash [h(b,F,a) \cap J_i].$$

If $i \in F$, then

$$[K \backslash h(b,F,a)] \cap J_i = J_i \backslash a_i = J_i \backslash [h(b,F,a) \cap J_i].$$

Finally, if $i \in I \backslash (b \cup F)$, then

$$[K \backslash h(b,F,a)] \cap J_i = J_i = J_i \backslash \emptyset = J_i \backslash [h(b,F,a) \cap J_i].$$

Clearly f is one-one; this finishes the proof of (iv).

In case $B = \mathrm{Finco}(I)$, this mapping is easily seen to be onto $\prod_{i \in I}^{\mathrm{w}} A_i$, proving (v).

(vi) clearly follows from (v).

For (vii), clearly f is a homomorphism from $\prod_{i \in I}^B A_i$ into A_i. For each $x \in A_i$ we have $x \in \prod_{i \in I}^B A_i$ and $f(x) = x$. So f maps onto A_i.

For (viii), we use the function f defined in the proof of (iv). Note that if $b \in B$, then $f(\overline{b}) = g(b)$, and if F is a finite subset of I and $a \in \prod_{i\in I} A_i$, then $f(h(\emptyset, F, a) = k$, where

$$k(i) = \begin{cases} a_i & \text{if } \ i \in F, \\ \emptyset & \text{otherwise.} \end{cases}$$

If (b, F, a) is normal, then $h(b, F, a) = \overline{b} \cup h(\emptyset, F, a)$. Hence $\{\overline{b} : b \in B\} \cup \{h(\emptyset, F, a) : F \in [I]^{<\omega}, a \in \prod_{i\in I} A_i\}$ generates $\prod^B_{i\in I} A_i$. Clearly also the image of this set under f generates the right side of the equation in (viii). Hence (viii) follows.

For (ix), define $f(x) = \bigcup_{i\in b} x_i$ for any $x \in \prod_{i\in b} A_i$; clearly this is the desired isomorphism.

Finally, we consider (x). First suppose that U is an ultrafilter on $\prod^B_{i\in I} A_i$.

Case 1. There is an $i \in I$ such that $h(\{i\}, \emptyset, \emptyset) \in U$. Let $V = \{x \in A_i^+ : h(\emptyset, \{i\}, \{(i, x)\}) \in U$. Suppose that $x \in V$ and $x \subseteq y \subseteq J_i$. Then

$$h(\emptyset, \{i\}, \{(i, x)\}) \subseteq h(\emptyset, \{i\}, \{(i, y)\}),$$

so $h(\emptyset, \{i\}, \{(i, y)\}) \in U$. It follows that $y \in V$. Next suppose that $x, y \in V$. Now $h(\emptyset, \{i\}, \{(i, x)\}) \cap h(\emptyset, \{i\}, \{(i, y)\}) = h(\emptyset, \{i\}, \{(i, x \cap y)\})$. Thus $h(\emptyset, \{i\}, \{(i, x \cap y)\}) \in U$, hence $x \cap y \in V$. So V is a filter. Now let $x \in A_i$, and suppose that $x \notin V$. Thus $K \backslash h(\emptyset, \{i\}, \{(i, x)\}) \in U$. Now

$$K \backslash h(\emptyset, \{i\}, \{(i, x)\}) = h(I \backslash \{i\}, \{i\}, \{(i, J_i \backslash x)\}),$$

and

$$h(\{i\}, \emptyset, \emptyset) \cap h(I \backslash \{i\}, \{i\}, \{(i, J_i \backslash x)\}) = h(\emptyset, \{i\}, \{(i, J_i \backslash x)\}),$$

so $h(\emptyset, \{i\}, \{(i, J_i \backslash x)\}) \in U$. It follows that $J_i \backslash x \in V$. So V is an ultrafilter on A_i.

Now suppose that $h(b, F, a) \in U$ with (b, F, a) normal. Suppose that $i \notin b$. Now $h(b, F, a) \cap h(\{i\}, \emptyset, \emptyset) \in U$ and hence this set is nonempty, so it follows that $i \in F$, and hence

$$h(b, F, a) \cap h(\{i\}, \emptyset, \emptyset) = h(\emptyset, \{i\}, \{(i, a_i)\});$$

hence $a_i \in V$.

Conversely, suppose that (b, F, a) is normal. Suppose first that $i \in b$. Then we have $h(\{i\}, \emptyset, \emptyset) \subseteq h(b, F, a)$, so $h(b, F, a) \in U$. Second, suppose that $i \notin b$, $i \in F$, and $a_i \in V$. Then $h(\emptyset, \{i\}, \{(i, a_i)\}) \in U$. Then $h(\emptyset, \{i\}, \{(i, a_i)\}) \subseteq h(b, F, a)$, so $h(b, F, a) \in U$.

Thus we have shown that Case 1 implies (a).

Case 2. There is no $i \in I$ such that $h(\{i\}, \emptyset, \emptyset) \in U$. Let $W = \{b \in B : h(b, \emptyset, \emptyset) \in U\}$. Clearly W is a nonprincipal ultrafilter on B. Suppose that $h(b, F, a) \in U$ with (b, F, a) normal. For each $i \in F$ we have $K \backslash h(\{i\}, \emptyset, \emptyset) = h(I \backslash \{i\}, \emptyset, \emptyset) \in U$, and hence

$$h(b, F, a) \cap \bigcap_{i\in F} h(I \backslash \{i\}, \emptyset, \emptyset) = h(b, \emptyset, \emptyset)$$

is in U. So $b \in W$.

Conversely, suppose that (b, F, a) is normal and $b \in W$. Since $h(b, \emptyset, \emptyset) \subseteq h(b, F, a)$, we have $h(b, F, a) \in U$.

This finishes the proof of $\Rightarrow$.

For $\Leftarrow$, first suppose that (a) holds; we want to show that U is an ultrafilter on $\prod_{i\in I}^{B} A_i$. Suppose that $h(b, F, a) \in U$, $h(b, F, a) \subseteq h(b', F', a')$, with (b, F, a) and (b', F', a') normal.

Case 1. $i \in b$. Then also $i \in b'$, so $h(b', F', a') \in U$.

Case 2. $i \notin b$, $i \in F$, and $a_i \in V$. If $i \in b'$, then $h(b', F', a') \in U$. Suppose that $i \notin b'$. Then $i \in F'$, so $i \in F \cap F'$ and hence $a_i \subseteq a'_i$. Hence $a'_i \in V$ and so $h(b', F', a') \in U$.

Next, suppose that $h(b, F, a), h(b', F', a') \in U$, with $(b, F, a), (b', F', a')$ normal. Then $h(b, F, a) \cap h(b', F', a') = h(b'', F'', a'')$ as in 1.5(iii). We consider some subcases.

Subcase 2.1. $i \in b \cap b'$. Then $h(b'', F'', a'') \in U$.

Subcase 2.2. $i \in b \backslash b'$. Then $i \in F'$ and $a'_i \in V$. Now $i \in F' \cap b \subseteq F''$ and $a''_i = a'_i$, so $h(b'', F'', a'') \in U$.

Subcase 2.3. $i \in b' \backslash b$. Similar to Subcase 2.2.

Subcase 2.4. $i \notin b \cup b'$. Then $i \in F$, $a_i \in V$, $i \in F'$, $a'_i \in V$. So $i \in F \cap F'$ and $a_i \cap a'_i \in V$, hence $a_i \cap a'_i \neq \emptyset$. So $h(b'', F'', a'') \in U$.

Thus U is a filter. Clearly $\emptyset \notin U$. Finally, suppose that a normal (b, F, a) is given. Suppose that $i \notin b$ and ($i \notin F$, or $i \in F$ and $a_i \notin V$). If $i \notin F$, then $i \in I \backslash (b \cup F)$ and hence $K \backslash h(b, F, a) \in U$. Suppose that $i \in F$ and $a_i \notin V$. Then $J_i \backslash a_i \in V$, and hence $K \backslash h(b, F, a) = h(I \backslash (b \cup F), F, a') \in U$, where $a'_j = J_j \backslash a_j$ for all $j \in F$.

Second, suppose that (b) holds. Clearly U is a proper filter. To show that it is an ultrafilter, suppose that (b, F, a) is normal and $h(b, F, a) \notin U$. Now $K \backslash h(b, F, a) = h(I \backslash (b \cup F), F, a')$ with $a'_i = J_i \backslash a_i$ for all $i \in F$. Since $h(b, F, a) \notin U$, we have $b \notin W$, hence $I \backslash b \in W$. Since W is nonprincipal, we have $\{i\} \notin W$ for all $i \in F$, hence $(I \backslash \{i\}) \in W$. Hence $I \backslash (b \cup F) = (I \backslash b) \cap \bigcap_{i \in F} (I \backslash \{i\}) \in W$, and so $(K \backslash h(b, F, a)) \in U$. So U is an ultrafilter. $\square$

Theorem 1.6(viii) suggests an alternative formulation of the notion of moderate products. Let I be an infinite set, $\langle A_i : i \in I \rangle$ a system of BAs, and $\mathrm{Finco}(I) \leq B \leq \mathscr{P}(I)$. Let g be the natural isomorphism from $\mathscr{P}(I)$ onto ${}^I 2$. Then we can take the moderate product to be

$$\left\langle \prod_{i \in I}^{\mathrm{w}} A_i \cup g[B] \right\rangle_{\prod_{i \in I} A_i}.$$

Note that here one does not need to assume that each A_i is an algebra of sets.

Theorem 1.7. *Assume the hypotheses of Theorem 1.6, and suppose that* $L \stackrel{\text{def}}{=} \{i \in I : |A_i| > 2\}$ *is infinite. Then* $\prod_{i\in I}^B A_i$ *is not complete.*

Proof. For each $i \in L$ choose $a_i \in A_i$ such that $0 \subset a_i \subset J_i$. Suppose that $\sum_{i\in L} a_i$ exists in $\prod_{i\in I}^B A_i$; say it is equal to $h(b, F, c)$, where we may assume that (b, F, c) is normal. Fix $i \in L \backslash F$. Then $a_i \subseteq h(b, F, c)$ implies that $i \in b$. But then $h(b\backslash\{i\}, F', c')$ is still an upper bound, where $F' = F \cup \{i\}$ and c' extends c with $c'_i = a_i$. Since $h(b\backslash\{i\}, F', c') \subset h(b, F, c)$, this is a contradiction. □

It is clear that if each A_i is atomless, then so is $\prod_{i\in I}^B A_i$; similarly for each A_i atomic. It is somewhat less trivial to check that the moderate product preserves superatomicity:

Theorem 1.8. *If each* A_i *is superatomic and also* B *is superatomic, then* $\prod_{i\in I}^B A_i$ *is superatomic.*

Proof. For brevity write $C = \prod_{i\in I}^B A_i$. It suffices to show that if f is a homomorphism from C onto a nontrivial BA D, then D has an atom. We consider two cases.

Case 1. $f(J_i) \neq 0$ for some $i \in I$. Let $f' = f \restriction A_i$. Clearly f' is a homomorphism from A_i onto $D \restriction f(J_i)$. Let $f'(u_i)$ be an atom of $D \restriction f(J_i)$; this is possible since A_i is superatomic. Clearly $f'(u_i)$ is also an atom of D.

Case 2. $f(J_i) = 0$ for all $i \in I$. Note that $h(b, F, a) = h(b, \emptyset, \emptyset) \cup h(\emptyset, F, a)$ for any b, F, a. Hence $f(h(b, F, a)) = f(h(b, \emptyset, \emptyset))$. For each $b \in B$, let $k(b) = f(h(b, \emptyset, \emptyset)$. Clearly k is a homomorphism from B onto D, so D has an atom since B is superatomic. □

An important use of moderate products in connection with homomorphisms has been given by Koszmider [99]. We give details on his construction.

Proposition 1.9. *If* K *is an ideal in* $\prod_{i\in I}^B A_i$ *and* $i \in I$, *then* $\{x \cap J_i : x \in K\}$ *is an ideal in* A_i.

Proof. Clearly $\{x \cap J_i : x \in K\}$ is closed under unions. Now suppose that $y \in A_i$ and $y \le x \cap J_i$ with $x \in K$. Clearly $y \in \prod_{i\in I}^B A_i$ and $y = y \cap J_i$, so y is in the set too. □

Note in fact that this ideal is merely $K \cap A_i$.

Proposition 1.10. *Suppose that* C *is an infinite homomorphic image of* $\prod_{i\in I}^B A_i$, *with* I *infinite. Then there is an infinite homomorphic image* D *of* C *which is also a homomorphic image of* B *or of some* A_i.

Proof. Let K be the kernel of a homomorphism of $\prod_{i\in I}^B A_i$ onto C. So $\prod_{i\in I}^B A_i/K$ is isomorphic to C and hence is infinite. For each $i \in I$ let $L_i = K \cap A_i$. So by Proposition 1.9, each L_i is an ideal in A_i. We now consider several cases.

Case 1. $N \stackrel{\text{def}}{=} \{i \in I : J_i \notin K\}$ is finite, and $\forall i \in I[A_i/L_i$ is finite$]$. Let $M = \langle K \cup \{J_i : i \in I, J_i \notin K\}\rangle^{\text{id}}$.

(1) $\prod^B_{i\in I} A_i/M$ is infinite.

In fact, define $f : \prod^B_{i\in I} A_i/K \to \prod^B_{i\in I} A_i/M \times \prod_{i\in N}(A_i/L_i)$ by

$$f([a]_K) = ([a]_M, \langle [a_i]_{L_i} : i \in N\rangle).$$

First of all, f is well defined. For, if $a \in K$, then $a \in M$, and $a_i \in L_i$ for each $i \in N$. Moreover, f is an injection. For, if $a \in M$ and $a_i \in L_i$ for each $i \in N$, then there is a $b \in K$ such that $a \subseteq b \cup \bigcup_{i\in N} J_i$, so

$$\begin{aligned} a &= \left[a \backslash \bigcup_{i\in N} J_i\right] \cup \left[a \cap \bigcup_{i\in N} J_i\right] \\ &= \left[a \backslash \bigcup_{i\in N} J_i\right] \cup \bigcup_{i\in N} a_i \\ &\subseteq b \cup \bigcup_{i\in N} a_i \\ &\in K. \end{aligned}$$

Thus, indeed, f is an injection. Hence (1) follows.

Now define $g(b) = [h(b, \emptyset, \emptyset)]_M$ for any $b \in B$. Then g is a homomorphism of B into $\prod^B_{i\in I} A_i/M$, by Theorem 1.6(ii). For any element $h(b, F, a)$ of $\prod^B_{i\in I} A_i$ we have $[h(b, F, a)]_M = [h(b, \emptyset, \emptyset)]_M$, since $h(b, F, a) \triangle h(b, \emptyset, \emptyset) = \bigcup_{i\in F} a_i$, and for all $i \in F$, if $i \notin N$ then $a_i \subseteq J_i \in K$, while if $i \in N$, then still $a_i \subseteq J_i \in M$. Thus g maps onto $\prod^B_{i\in I} A_i/M$, and so $\prod^B_{i\in I} A_i/M$ is as desired.

Case 2. N is infinite, and $\forall i \in I[A_i/L_i$ is finite$]$. For each $i \in N$ let M_i be a maximal ideal in A_i such that $L_i \subseteq M_i$. Let P be the ideal generated by $K \cup \bigcup_{i\in N} M_i$. For each $i \in N$ we have $J_i \notin P$, and so $\prod^B_{i\in I} A_i/P$ is infinite. Define $g(b) = [h(b, \emptyset, \emptyset)]_P$ for all $b \in B$. Then g is a homomorphism of B into $\prod^B_{i\in I} A_i/M$, by Theorem 1.6(ii). We claim that it is a surjection. Let $h(b, F, a)$ be any element of $\prod^B_{i\in I} A_i$, with (b, F, a) normal, and let $c = b \cup \{i \in F : a_i \notin M_i\}$. Then

$$\begin{aligned} h(b, F, a) \triangle h(c, \emptyset, \emptyset) &= (h(b, F, a) \backslash h(c, \emptyset, \emptyset)) \cup (h(c, \emptyset, \emptyset) \backslash h(b, F, a)) \\ &= \bigcup_{\substack{i\in F \\ a_i \in M_i}} a_i \cup \bigcup_{\substack{i\in F \\ a_i \notin M_i}} (J_i \backslash a_i). \end{aligned}$$

Now if $i \in F$ and $a_i \in M_i$ then $a_i \in P$, while if $i \in F$ and $a_i \notin M_i$, then $(J_i \backslash a_i) \in M_i \subseteq P$. Hence g maps onto $\prod^B_{i\in I} A_i/P$, as desired.

Case 3. There is an $i_0 \in I$ such that A_{i_0}/L_{i_0} is infinite. Let M be the ideal generated by $K \cup \left\{\bigcup_{i\neq i_0} J_i\right\}$, and define

$$g([h(b, F, a)]_M) = [h(b, F, a) \cap J_{i_0}]_{L_{i_0}}.$$

If $h(b, F, a) \in M$, then there is a $c \in K$ such that $h(b, F, a) \subseteq c \cup \bigcup_{i \neq i_0} J_i$, hence $h(b, F, a) \cap J_{i_0} \subseteq c$, and so $h(b, F, a) \cap J_{i_0} \in (K \cap A_{i_0}) = L_{i_0}$. So g is well defined. It clearly maps onto A_{i_0}/L_{i_0}, as desired. $\square$

Proposition 1.11. *Suppose that X is a subset of $\prod_{i \in I}^B A_i$ with $|X| = \kappa$ uncountable and regular, $\kappa > |I|$. Then there exist $Y \in [X]^\kappa$ and a finite $G \subseteq I$ such that $\langle Y \rangle$ is isomorphic to a subalgebra of $(B \restriction (I \backslash G)) \times \prod_{i \in G} A_i$.*

Proof. Let $\langle F_\xi : \xi < \kappa \rangle \in {}^\kappa([I]^{<\omega})$, $b \in {}^\kappa B$ and a with domain κ be such that $a_\xi \in \prod_{i \in F_\xi} A_i$ for all $\xi < \kappa$, (b_ξ, F_ξ, a_ξ) is normal, and

$$X = \{h(b_\xi, F_\xi, a_\xi) : \xi < \kappa\}.$$

Choose $Y \in [\kappa]^\kappa$ and $G \in [I]^{<\omega}$ such that $F_\xi = G$ for all $\xi \in Y$. Let

$$W = \left\{ h(c, G, d) : c \in B,\ c \cap G = \emptyset,\ d \in \prod_{i \in G} A_i \right\}.$$

This is a subalgebra of $\prod_{i \in I}^B A_i$ by the above computation rules, and $Y \subseteq W$. Now define

$$f(h(c, G, d)) = (h(c, \emptyset, \emptyset), d)$$

for each $h(c, G, d) \in W$. We claim that f is an isomorphism from W into $(B \restriction (I \backslash G)) \times \prod_{i \in G} A_i$. In fact, f clearly preserves $\cdot$. For $-$,

$$\begin{aligned} f(-h(c, G, d)) &= f(h(I \backslash (c \cup G), G, -d) \\ &= (h(I \backslash (c \cup G), \emptyset, \emptyset), -d) \\ &= (h(I \backslash G, \emptyset, \emptyset) \cdot h(I \backslash c, \emptyset, \emptyset), -d) \\ &= (h(I \backslash G, \emptyset, \emptyset) \cdot -h(c, \emptyset, \emptyset), -d) \\ &= -f(h(c, G, d)). \end{aligned}$$ $\square$

Let L be the set of all countable limit ordinals, and L_2 the set of all countable limits of elements of L. Let $\langle c^n_\alpha : \alpha \in L_2, n \in \omega \rangle$ be such that for all $\alpha \in L_2$, $\langle c^n_\alpha : n \in \omega \rangle$ is strictly increasing, cofinal in α, with $c^0_\alpha = 0$ and $c^{i+1}_\alpha \in L$ for all $i \in \omega$.

Let A and B be subalgebras of $\mathscr{P}(\omega)$, with $[\omega]^{<\omega} \subseteq B$. We construct $\langle C_\alpha : \alpha \in L \rangle$ by recursion. Each C_α will be a subalgebra of $\mathscr{P}(\alpha)$. For each $\beta \in L$ let f_β be a bijection from ω to $[\beta, \beta + \omega)$.

Define $C_\omega = A$. If C_α has been defined, with $\alpha \in L$, let

$$C_{\alpha+\omega} = \langle C_\alpha \cup \{f_\alpha[a] : a \in A\} \rangle_{\mathscr{P}(\alpha+\omega)}.$$

If $\beta \in L_2$ and C_α has been defined for all $\alpha < \beta$ such that $\alpha \in L$, let $C_\beta = \prod_{i \in \omega}^B D^\beta_i$, with

$$D^\beta_i = C_{c^{i+1}_\beta} \restriction [c^i_\beta, c^{i+1}_\beta).$$

Here in general $G \restriction h = \{g \cdot h : g \in G\}$ for G a subalgebra of H and $h \in H$, without assuming that $h \in G$. Next, for each $\alpha \in L$ let $E_\alpha = C_\alpha \cup \{a : \omega_1 \backslash a \in C_\alpha\}$. Thus E_α is a subalgebra of $\mathscr{P}(\omega_1)$. Finally, let

$$\prod_c^B A = \left\langle \bigcup_{\alpha \in L} C_\alpha \right\rangle_{\mathscr{P}(\omega_1)}.$$

Lemma 1.12. *Assume the above notation. Then*

(i) *If $\alpha, \beta \in L$ and $\alpha < \beta$, then $C_\alpha \subseteq C_\beta$.*
(ii) *If $\alpha, \beta \in L$ and $\alpha \leq \beta$, then $\alpha \in C_\beta$.*
(iii) *If $\alpha_1, \alpha_2, \beta \in L$ and $\alpha_1 < \alpha_2 \leq \beta$, then $[\alpha_1, \alpha_2) \in C_\beta$.*
(iv) *If $\alpha, \beta \in L$, $\alpha < \beta$, define $f_{\beta\alpha}(a) = a \cap \alpha$ for each $a \in C_\beta$. Then $f_{\beta\alpha}$ is a homomorphism from C_β onto C_α which is the identity on C_α.*
(v) *If $\alpha, \beta \in L$ and $\alpha < \beta$, then $E_\alpha \subseteq E_\beta$, and there is a homomorphism from E_β onto E_α which is the identity on E_α.*
(vi) $\prod_c^B A = \bigcup_{\alpha \in L} E_\alpha$.
(vii) *For any $\alpha \in L$, $C_{\alpha+\omega} \cong C_\alpha \times A$.*
(viii) $L \subseteq \prod_c^B A$.

Proof. We prove (i)–(iv) simultaneously by induction on β. The case $\beta = \omega$ holds vacuously. Now assume inductively that $\beta > \omega$.

Case 1. $\beta = \gamma + \omega$ for some $\gamma \in L$. For (i), suppose that $\alpha \in L$ and $\alpha < \beta$. Then $C_\alpha \subseteq C_\gamma$ by the inductive hypothesis, and $C_\gamma \subseteq C_\beta$ by construction.

For (ii), suppose that $\alpha \leq \beta$ with $\alpha \in L$. If $\alpha < \beta$, then $\alpha \in C_\alpha \subseteq C_\beta$ by the inductive hypothesis. Obviously $\beta \in C_\beta$.

For (iii), we consider two subcases.

Subcase 1.1. $\alpha_2 < \beta$. Then $[\alpha_1, \alpha_2) \in C_\gamma$ by the inductive hypothesis, and $C_\gamma \subseteq C_\beta$ by construction.

Subcase 1.2. $\alpha_2 = \beta$. Then $[\alpha_1, \gamma) \in C_\gamma$ by the inductive hypothesis, and $[\gamma, \beta) = f_\gamma[\omega] \in C_\beta$ by construction. So $[\alpha_1, \beta) = [\alpha_1, \gamma) \cup [\gamma, \beta) \in C_\beta$.

For (iv), we have $a \cap \alpha \in C_\alpha$ for each a in the generating set of C_β, and so (iv) holds.

Case 2. $\beta \in L_2$. First we take (ii). Clearly $\beta \in C_\beta$ by construction. Now suppose that $\alpha < \beta$. Choose $i \in \omega$ such that $c_\beta^i \leq \alpha < c_\beta^{i+1}$. Then by the inductive hypothesis, $\alpha \in C_{c_\beta^{i+1}}$, so $[c_\beta^i, \alpha) \in C_{c_\beta^{i+1}} \restriction [c_\beta^i, c_\beta^{i+1}) = D_i^\beta$. Clearly $[0, c_\beta^i) \in C_\beta$. Hence $\alpha \in C_\beta$.

Now assume the hypotheses of (iii). By (ii), $[\alpha_1, \alpha_2) = \alpha_2 \backslash \alpha_1 \in C_\beta$. Thus (iii) holds.

Now assume the hypotheses of (i), and suppose that $a \in C_\alpha$. In particular, $a \subseteq \alpha$. Choose i_0 such that $c_\beta^{i_0} \leq \alpha < c_\beta^{i_0+1}$. If $i + 1 \leq i_0$, then $a \cap c_\beta^{i+1} \in C_{c_\beta^{i+1}}$ by the inductive hypothesis on (iv), so $a \cap [c^i_\beta, c^{i+1}_\beta) \in D^\beta_i$. Also, $[c^{i_0}_\beta, \alpha) \in C_{c^{i_0+1}_\beta}$ by the inductive hypothesis on (iii), and $a \in C_{c^{i_0+1}_\beta}$ by the inductive hypothesis on (i). Hence $a \cap [c^{i_0}_\beta, \alpha) = a \cap [c^{i_0}_\beta, \alpha) \cap [c^{i_0}_\beta, c^{i_0+1}_\beta) \in D^\beta_{i_0}$. Thus a is a sum of elements of D^β_i for $i+1 \leq i_0$ and of an element of $D^\beta_{i_0}$, so $a \in C_\beta$, as desired in (i).

For (iv), suppose that $\alpha < \beta$, and suppose that $a \in C_\beta$. Again take i_0 such that $c^{i_0}_\beta \leq \alpha < c^{i_0+1}_\beta$.

(1) If $i+1 \leq i_0$, then $[c^i_\beta, c^{i+1}_\beta) \cap a \in C_\alpha$.

In fact, clearly $[c^i_\beta, c^{i+1}_\beta) \cap a \in D^\beta_i$. By (iii) for c^{i+1}_β we have $[c^i_\beta, c^{i+1}_\beta) \in C_{c^{i+1}_\beta}$, so $D^\beta_i \subseteq C_{c^{i+1}_\beta}$, and hence $[c^i_\beta, c^{i+1}_\beta) \cap a \in C_{c^{i+1}_\beta} \subseteq C_\alpha$ by the inductive hypothesis on (i) for α. Thus (1) holds.

(2) $[c^{i_0}_\beta, \alpha) \cap a \in C_\alpha$.

For, again clearly $[c^{i_0}_\beta, c^{i_0+1}_\beta) \cap a \in D^\beta_{i_0}$, so as above, $[c^{i_0}_\beta, c^{i_0+1}_\beta) \cap a \in C_{c^{i_0+1}_\beta}$. Then by the inductive hypothesis on (iv) for $c^{i_0+1}_\beta$,

$$[c^{i_0}_\beta, \alpha) \cap a = [c^{i_0}_\beta, c^{i_0+1}_\beta) \cap a \cap \alpha \in C_\alpha,$$

giving (2). Now by (1) and (2), $a \cap \alpha = \bigcup_{i+1 \leq i_0} (a \cap [c^i_\beta, c^{i+1}_\beta)) \cup ([c^{i_0}_\beta, \alpha) \cap a) \in C_\alpha$.

This finishes the proof of (i)–(iv). For (v), from (i) it is clear that $E_\alpha \subseteq E_\beta$. Now for each $a \in E_\beta$, let

$$g(a) = \begin{cases} a \cap \alpha & \text{if } a \in C_\beta, \\ (\omega_1 \backslash \alpha) \cup a & \text{otherwise.} \end{cases}$$

Now g maps into E_α. For, if $a \in C_\beta$, then $g(a) = a \cap \alpha \in C_\alpha \subseteq E_\alpha$ by (iv). If $\omega_1 \backslash a \in C_\beta$, then

$$\omega_1 \backslash g(a) = \omega_1 \backslash ((\omega_1 \backslash \alpha) \cup a) = \omega_1 \cap \alpha \cap (\omega_1 \backslash a) = \alpha \cap (\omega_1 \backslash a),$$

and this is in C_α by (iv) again.

We check that g preserves $\cup$. Suppose that $a, b \in E_\beta$. If $a, b \in C_\beta$, then $a \cup b \in C_\beta$, and $g(a \cup b) = (a \cup b) \cap \alpha = (a \cap \alpha) \cup (b \cap \alpha) = g(a) \cup g(b)$. If $a \in C_\beta$ and $b \notin C_\beta$, then $\omega_1 \backslash b \in C_\beta \subseteq \beta$, hence $\omega_1 \backslash \beta \subseteq b$, and so $a \cup b \notin C_\beta$; it follows that

$$\begin{aligned} g(a) \cup g(b) &= (a \cap \alpha) \cup (\omega_1 \backslash \alpha) \cup b \\ &= (\omega_1 \backslash \alpha) \cup a \cup b \\ &= g(a \cup b). \end{aligned}$$

Similarly if $a \notin C_\beta$ and $b \in C_\beta$. Finally, if $a, b \notin C_\beta$, then

$$g(a) \cup g(b) = (\omega_1 \backslash \alpha) \cup a \cup b = g(a \cup b).$$

So g preserves $\cup$. For complement, if $a \in C_\beta$, then

$$g(\omega_1\backslash a) = (\omega_1\backslash\alpha) \cup (\omega_1\backslash a) = \omega_1\backslash(a \cap \alpha) = \omega_1\backslash g(a),$$

and if $a \notin C_\beta$, then

$$\omega_1\backslash g(a) = \omega_1\backslash((\omega_1\backslash\alpha) \cup a) = \alpha\backslash a = g(\omega_1\backslash a).$$

So g is a homomorphism. If $a \in C_\alpha$, clearly $g(a) = a$. If $\omega_1\backslash a \in C_\alpha$, then $g(a) = (\omega_1\backslash\alpha) \cup a = a$ since $(\omega_1\backslash\alpha) \subseteq a$. Thus (v) holds.

For (vi), note that each E_α is clearly a subalgebra of $\langle\bigcup_{\alpha\in L} C_\alpha\rangle$, and so $\bigcup_{\alpha\in L} E_\alpha \subseteq \langle\bigcup_{\alpha\in L} C_\alpha\rangle$. Since $\bigcup_{\alpha\in L} E_\alpha$ is a subalgebra of $\mathscr{P}(\omega_1)$ containing $\bigcup_{\alpha\in L} C_\alpha$, (vi) follows.

Turning to (vii), for any $(a, b) \in C_\alpha \times A$ we define $g(a, b) = a \cup f_\alpha[b]$. Clearly g preserves $+$. For $-$, we have

$$\begin{aligned}
-g(a,b) &= -(a \cup f_\alpha[b])\\
&= (\alpha+\omega)\backslash(a \cup f_\alpha[b])\\
&= ((\alpha+\omega)\backslash a) \cap ((\alpha+\omega)\backslash f_\alpha[b])\\
&= ((\alpha\backslash a) \cup [\alpha, \alpha+\omega)) \cap (\alpha \cup [\alpha, \alpha+\omega)\backslash f_\alpha[b])\\
&= (\alpha\backslash a) \cup ([\alpha, \alpha+\omega)\backslash f_\alpha[b])\\
&= (\alpha\backslash a) \cup f_\alpha[-b]\\
&= g(-a, -b)\\
&= g(-(a, b)).
\end{aligned}$$

Thus g is a homomorphism. Its range clearly contains $C_\alpha \cup \{f_\alpha[a] : a \in A\}$ and is contained in $\langle C_\alpha \cup \{f_\alpha[a] : a \in A\}\rangle$. It is clearly one-one. So g is the desired isomorphism.

Finally, (viii) is immediate from (ii). $\square$

Proposition 1.13. *Assume the notation above.*

(i) *If $x \in E_\alpha$, then $x \cap \alpha \in C_\alpha$.*
(ii) *C_α is a maximal ideal in E_α.*
(iii) *For every ideal K in E_α, the set $K \cap C_\alpha$ is an ideal in C_α.*
(iv) *For every ideal K in E_α and every $x \in E_\alpha$, let $f([x]_K) = [x \cap \alpha]_{C_\alpha \cap K}$. Then f is well defined, and is a homomorphism from E_α/K onto $C_\alpha/(K \cap C_\alpha)$. Moreover, it is one-one on C_α/K.*
(v) *For every ideal K in E_α, if E_α/K is infinite, then so is $C_\alpha/(K \cap C_\alpha)$.*

Proof. (i): Assume that $x \in E_\alpha$. If $x \in C_\alpha$, then the conclusion is obvious. Suppose that $x \notin C_\alpha$. Then $(\omega_1\backslash x) \in C_\alpha$, and so the set $\alpha\backslash(\omega_1\backslash x)$ is also in C_α. This set is equal to $x \cap \alpha$. So (i) holds.

(ii): Obviously C_α is closed under $\cup$. If $x \in C_\alpha$ and $y \subseteq x$, with $y \in E_\alpha$, then by (i), $y \in C_\alpha$. Finally, C_α is obviously maximal.

(iii): Obvious from (ii).

(iv): If $x \in K$, then $(x \cap \alpha) \in C_\alpha \cap K$ by (i). So f is well defined. It is clearly a homomorphism from E_α/K into $C_\alpha/(K \cap C_\alpha)$. For $a \in C_\alpha$ we have $f([a]_K) = [a]_{C_\alpha \cap K}$, so f is clearly one-one on C_α/K, and f maps onto $C_\alpha/(K \cap C_\alpha)$.

(v): Let K be an ideal of E_α, with E_α/K infinite. Now E_α/K is partitioned into two parts of equal size, C_α/K being one of them. From (iv) it then follows that $C_\alpha/(K \cap C_\alpha)$ is infinite. $\square$

Proposition 1.14. *Let A, B, c be as above. Suppose that E is an infinite homomorphic image of $\prod_c^B A$. Then there exist an infinite subalgebra F of E and an infinite homomorphic image G of F such that G is also a homomorphic image of A or of B.*

Proof. Let J be an ideal of $\prod_c^B A$ such that $\prod_c^B A/J$ is isomorphic to E. Choose $a_i \in \prod_c^B A$ for each $i < \omega$ such that $[a_i]_J \neq [a_j]_J$ for all $i \neq j$. Choose $\alpha < \omega_1$ such that $\{a_i : i < \omega\} \subseteq E_\alpha$. For each $x \in E_\alpha$ let $f([x]_{E_\alpha \cap J}) = [x]_J$. Clearly f is well defined, and is an isomorphism from $E_\alpha/(E_\alpha \cap J)$ into $\prod_c^B A/J$. By Proposition 1.13(iv), $C_\alpha/(C_\alpha \cap J)$ is a homomorphic image of $E_\alpha/(E_\alpha \cap J)$, and by Proposition 1.13(v) it is infinite. Hence it is enough to prove

(1) For every $\alpha \in L$ and every infinite homomorphic image G of C_α there is an infinite homomorphic image F of G such that F is also a homomorphic image of A or of B.

We prove (1) by induction on α. Since $C_\omega = A$, it is obvious for $\alpha = \omega$. Assume it is true for C_α. By Lemma 1.12(vii), $C_{\alpha+\omega} \cong C_\alpha \times A$. If G is an infinite homomorphic image of $C_{\alpha+\omega}$, then G is isomorphic to $H \times K$ for some homomorphic images H, K of C_α, A respectively, by Proposition 1.1. One of H, K is infinite, and so the desired result follows from the inductive assumption on C_α.

Now suppose that $\alpha \in L_2$. By Proposition 1.10 let F be an infinite homomorphic image of G such that F is a homomorphic image of B or of some D_i^α. The desired conclusion has been reached if F is a homomorphic image of B. Suppose that F is a homomorphic image of D_i^α. Obviously D_i^α is a homomorphic image of $C_{c_\alpha^{i+1}}$. Hence the inductive hypothesis applies to finish the proof. $\square$

One-point gluing

Our next algebraic operation is one-point gluing. Suppose we are given a system $\langle A_i : i \in I\rangle$ of BAs, and a corresponding system $\langle F_i : i \in I\rangle$ of ultrafilters: F_i is an ultrafilter on A_i for each $i \in I$. The one-point gluing of the pair $(\langle A_i : i \in I\rangle, \langle F_i : i \in I\rangle)$ is the following subalgebra of the direct product $\prod_{i\in I} A_i$:

$$\left\{x \in \prod_{i\in I} A_i : \text{for all } i, j \in I (x_i \in F_i \text{ iff } x_j \in F_j)\right\}.$$

In the case of two factors A_i and A_j this amounts to identifying the two points F_i and F_j in the disjoint union of the Stone spaces; this is a special case of the following theorem.

Theorem 1.15. *Let $\langle A_i : i \in I\rangle$ be a system of BAs, and let $\langle F_i : i \in I\rangle$ be a system with F_i an ultrafilter on A_i for each $i \in I$. Let $C = \prod_{i\in I} A_i$, and let B be the one-point gluing of the pair $(\langle A_i : i \in I\rangle, \langle F_i : i \in I\rangle)$.*

Then for each $i \in I$ the set $F'_i \stackrel{\text{def}}{=} \{x \in C : x_i \in F_i\}$ is an ultrafilter on C. Further, let $K = \overline{\{F'_i : i \in I\}}$. Let X be the quotient of $\mathrm{Ult}(C)$ obtained by collapsing K to a point. Then $\mathrm{Ult}(B)$ is homeomorphic to X.

Proof. The first assertion of the theorem is obvious. Now let π be the natural continuous mapping of $\mathrm{Ult}(C)$ onto X. We now define g from $\mathrm{Ult}(B)$ into X by setting $g(F \cap B) = \pi(F)$ for any ultrafilter F on C. To see that g is well defined, suppose that $F \cap B = G \cap B$, where F and G are ultrafilters on C. If both F and G are in K, obviously $\pi(F) = \pi(G)$. Now suppose, say, that $F \notin K$. We claim then that $F = G$. Suppose to the contrary that $F \neq G$. Choose $u \in F\backslash G$. Now choose $x \in F$ such that $\mathcal{S}^C(x) \cap K = 0$. Here $\mathcal{S}^C$ is the Stone map associated with C. If $(x \cdot u)_i \in F_i$ for some $i \in I$, then $x \cdot u \in F'_i$, and hence $\mathcal{S}^C(x) \cap K \neq 0$, contradiction. Thus $(x \cdot u)_i \notin F_i$ for all $i \in I$, and consequently $x \cdot u \in B$. But $x \cdot u \in F$, so $x \cdot u \in G$ and so $u \in G$, contradiction.

g is one-one: suppose that F and G are ultrafilters on C and $\pi(F) = \pi(G)$; we want to show that $F \cap B = G \cap B$. We may assume that $F, G \in K$. Let $x \in B$; by symmetry we want to show that if $x \in F$ then $x \in G$. So, assume that $x \in F$. Thus $F \in \mathcal{S}^C(x)$ so, since $F \in K$, we can choose $i \in I$ so that $F'_i \in \mathcal{S}^C(x)$. Thus $x \in F'_i$ and so $x_i \in F_i$. If $x \notin G$, then $-x \in G$, and a similar argument gives $-x_j \in F_j$ for some $j \in I$. This contradicts the assumption that $x \in B$.

g maps onto X: let $F \in X$. If $F \notin K$, then $g(F \cap B) = F$. If F is the point which is the collapse of K, then for any $i \in I$ we have $g(F'_i \cap B) = F$.

g is continuous: suppose that U is open in X, and $F \cap B \in g^{-1}[U]$, where F is an ultrafilter on C. Thus $\pi(F) \in U$, and hence $F \in \pi^{-1}[U]$. So, choose $x \in C$ so that $F \in \mathcal{S}^C(x) \subseteq \pi^{-1}[U]$.

Case 1. $F \notin K$. Choose $y \in C$ so that $F \in \mathcal{S}^C(y)$ and $\mathcal{S}^C(y) \cap K = \emptyset$. Thus $F \in \mathcal{S}^C(x \cdot y)$. Now if $(x \cdot y)_i \in F_i$, then $x \cdot y \in F'_i$, so $y \in F'_i \in K$, contradicting $\mathcal{S}^C(y) \cap K = \emptyset$. This being true for all i, it follows that $x \cdot y \in B$. We claim that $F \cap B \in \mathcal{S}^B(x \cdot y) \subseteq g^{-1}[U]$. Obviously $F \cap B \in \mathcal{S}^B(x \cdot y)$. Suppose that $x \cdot y \in G \cap B$, where G is an ultrafilter on C. Then $G \in \mathcal{S}^C(x)$, and hence $g(G \cap B) = \pi(G) \in U$, as desired.

Case 2. $F \in K$. Thus $K \subseteq \pi^{-1}[U]$. Choose $y \in C$ such that $K \subseteq \mathcal{S}^C(y) \subseteq \pi^{-1}[U]$. Thus $y \in B$. We claim that $F \cap B \in \mathcal{S}^B(y) \subseteq g^{-1}[U]$. Obviously $F \cap B \in \mathcal{S}^B(y)$. Now suppose that $G \cap B \in \mathcal{S}^B(y)$, where G is an ultrafilter on C. Then $G \in \mathcal{S}^C(y)$, and $g(G \cap B) = \pi(G) \in U$, as desired.

Finally, it is clear that X is Hausdorff. □

The Alexsandroff duplicate

Given a BA A, its *Alexsandroff duplicate*, denoted by $\mathrm{Dup}(A)$, is the subalgebra of $A \times \mathscr{P}(\mathrm{Ult}(A))$ whose set of elements is

$$\{(a, X) : a \in A, X \subseteq \mathrm{Ult}(A), \text{ and } \mathcal{S}(a)\triangle X \text{ is finite}\}.$$

(It is easy to check that this is a subalgebra of $A \times \mathscr{P}(\mathrm{Ult}(A))$.)

We show now that this is equivalent to the usual definition. (See, for example, Gardner, Pfeffer [84], page 1010.) That definition runs like this. Let X be a topological space. We put a topology on $X \times 2$ as follows: a base consists of all sets

- $F \times \{1\}$, F any subset of X;
- $(G \times 2)\backslash(F \times \{1\})$, G open in X, F a finite subset of X.

Theorem 1.16. *Let A be a BA. Under the above topology,* $\mathrm{Ult}(A) \times 2$ *is a Boolean space, and* $\mathrm{Dup}(A)$ *is isomorphic to* $\mathrm{Clop}(\mathrm{Ult}(A) \times 2)$.

Proof. Hausdorff: If U and V are ultrafilters on A, then we have disjoint open neighborhoods

$\{(U, 1)\}$ and $\{(V, 1)\}$ for the elements $(U, 1)$ and $(V, 1)$ when these are distinct;

$\{(U, 1)\}$ and $(\mathrm{Ult}(A) \times 2)\backslash\{(U, 1)\}$ for the elements $(U, 1)$ and $(V, 0)$.

$\{(V, 1)\}$ and $(\mathrm{Ult}(A) \times 2)\backslash\{(V, 1)\}$ for the elements $(V, 1)$ and $(U, 0)$.

$\mathcal{S}(a) \times 2$ and $\mathcal{S}(-a) \times 2$ for the elements $(U, 0)$ and $(V, 0)$, for distinct U, V, where $a \in U\backslash V$.

Now we show that $X \times 2$ is compact. Let $\mathscr{O}$ be a cover of $\mathrm{Ult}(A) \times 2$ by members of the given base. Then $\{G : (G \times 2)\backslash(F \times \{1\}) \in \mathscr{O}$ for some open $G \subseteq X$ and some finite $F \subseteq X\}$ is an open cover of $\mathrm{Ult}(A)$, so we can choose

$$(G_1 \times 2)\backslash(F_1 \times \{1\}), \ldots, (G_m \times 2)\backslash(F_m \times \{1\}) \in \mathscr{O}$$

such that $G_1, \ldots, G_m$ is a cover of $\mathrm{Ult}(A)$. There are only finitely many elements of $\mathrm{Ult}(A) \times 2$ remaining – some of the elements of $(F_1 \times \{1\}) \cup \cdots \cup (F_m \times \{1\})$ – so $\mathscr{O}$ has a finite subcover.

Next we determine the clopen subsets of $\mathrm{Ult}(A) \times 2$. Clearly each set $F \times \{1\}$ is clopen, when F is a finite subset of $\mathrm{Ult}(A)$. And if G is a clopen subset of $\mathrm{Ult}(A)$ and F is a finite subset of $\mathrm{Ult}(A)$, then $(G \times 2)\backslash(F \times \{1\})$ is clopen, since its complement is $[(\mathrm{Ult}(A)\backslash G) \times 2] \cup (F \times \{1\})$. These two kinds of clopen subsets form a base for the topology, so every clopen set is a finite join of these two kinds.

So $\mathrm{Ult}(A) \times 2$ is a Boolean space. Now we define a function f which will extend to the desired isomorphism. For any $a \in A$ let $f(a, \mathcal{S}(a)) = \mathcal{S}(a) \times 2$, and for any ultrafilter F on A let $f(0, \{F\}) = \{F\} \times \{1\}$. So f maps a set of generators of $\mathrm{Dup}(A)$ onto a set of generators of $\mathrm{Clop}(\mathrm{Ult}(A) \times 2)$. An easy application of Sikorski's extension criterion shows that f extends to a one-one homomorphism, as desired. We give this argument.

Let $a_0, \ldots, a_{n-1}$ be distinct elements of A, and $F_0, \ldots, F_{m-1}$ distinct ultrafilters on A, with $n, m > 0$. With other natural assumptions,

$$(a_0, \mathcal{S}(a_0))^{\varepsilon(0)} \cdot \ldots \cdot (a_{n-1}, \mathcal{S}(a_{n-1}))^{\varepsilon(n-1)} \cdot (0, \{F_0\})^{\delta(0)} \cdot \ldots \cdot (0, \{F_{m-1}\})^{\delta(m-1)} = 0$$

iff one of the following holds:

(1) There are distinct $i, j < m$ such that $\delta(i) = \delta(j) = 1$.

(2) There is exactly one $i < m$ such that $\delta(i) = 1$, and

$$\prod_{j<n} a_j^{\varepsilon(j)} \notin F_i.$$

(3) Each $\delta(i) = 0$ and $\prod_{i<n} a_i^{\varepsilon(i)} = 0$.

On the other hand,

$$(\mathcal{S}(a_0)\times 2)^{\varepsilon(0)} \cdot \ldots \cdot (\mathcal{S}(a_{n-1})\times 2)^{\varepsilon(n-1)} \cdot (\{F_0\}\times\{1\})^{\delta(0)} \cdot \ldots \cdot (\{F_{m-1}\}\times\{1\})^{\delta(m-1)} = 0$$

holds iff the same conditions hold. □

We give some more simple facts about the Alexsandroff duplicate. Note that the duplicate is always atomic. The following result is easy.

Proposition 1.17. *For any BA A, define $g : A \to \mathrm{Dup}(A)$ by setting $g(a) = (a, \mathcal{S}(a))$ for any $a \in A$. Then g is an isomorphism from A into* $\mathrm{Dup}(A)$.

The special nature of the duplicate is brought out by the following simple theorem. For any BA A, let I_{at}^A be the ideal of A generated by its atoms.

Theorem 1.18. *Let A be any BA. Then $A/I_{\mathrm{at}}^A \cong \mathrm{Dup}(A)/I_{\mathrm{at}}^{\mathrm{Dup}(A)}$.*

Proof. Let π be the natural homomorphism from $\mathrm{Dup}(A)$ onto $\mathrm{Dup}(A)/I_{\mathrm{at}}^{\mathrm{Dup}(A)}$. We show that $\pi \circ g$ is a homomorphism from A onto $\mathrm{Dup}(A)/I_{\mathrm{at}}^{\mathrm{Dup}(A)}$ with kernel I_{at}^A, where g is as in Proposition 1.17. If $(a, X) \in \mathrm{Dup}(A)$, then $(a, \mathcal{S}(a)) \triangle (a, X) = (0, \mathcal{S}(a) \triangle X) \in I_{\mathrm{at}}^{\mathrm{Dup}(A)}$; thus $\pi \circ g$ maps onto $\mathrm{Dup}(A)/I_{\mathrm{at}}^{\mathrm{Dup}(A)}$. Now for any $a \in A$, $(\pi \circ g)(a) = 0$ iff $(a, \mathcal{S}(a)) \in I_{\mathrm{at}}^{\mathrm{Dup}(A)}$ iff $a \in I_{\mathrm{at}}^A$, as desired. □

A corollary of this theorem is that if A is superatomic, then so is $\mathrm{Dup}(A)$. (It is easy to see that for any BA B, B is superatomic iff B/I_{at}^B is superatomic.)

The ultrafilters on $\mathrm{Dup}(A)$ are characterized as follows.

Proposition 1.19. *Let A be an infinite BA and $F \subseteq \mathrm{Dup}(A)$. Then F is an ultrafilter on* $\mathrm{Dup}(A)$ iff *one of the following conditions holds:*

(i) *There is an ultrafilter G on A such that $F = \{(a, X) \in \mathrm{Dup}(A) : (0, \{G\}) \leq (a, X)\}$.*

(ii) *There is an ultrafilter G on A such that $F = \{(a, X) \in \mathrm{Dup}(A) : a \in G\}$.*

Proof. Clearly each of (i), (ii) implies that F is an ultrafilter on $\mathrm{Dup}(A)$. Now suppose that F is an ultrafilter on $\mathrm{Dup}(A)$. If $(0, X) \in F$ for some finite nonempty X, clearly (i) holds. So suppose that $(0, X) \notin F$ for every finite nonempty X. Let $G = \{a \in A : (a, \mathcal{S}(a)) \in F\}$. Clearly G is an ultrafilter on A. Suppose that $a \in G$ and $(a, X) \notin F$. So $(a, \mathcal{S}(a)), (-a, \mathrm{Ult}(A)\backslash X) \in F$, hence $(0, \mathcal{S}(a)\backslash X) \in F$, from which it follows that $\mathcal{S}(a) \subseteq X$. Then $(a, \mathcal{S}(a)) \leq (a, X)$, so $(a, X) \in F$, contradiction. Hence $\forall a \in G \forall X[\mathcal{S}(a) \triangle X \text{ finite} \rightarrow (a, X) \in F]$. Now suppose that $(a, X) \in F$ and $a \notin G$. so $(-a, \mathcal{S}(-a))) \in F$, hence $(0, X\backslash\mathcal{S}(a)) \in F$, hence $X \subseteq \mathcal{S}(a)$. Now $(a, X) \leq (a, \mathcal{S}(a))$, so $(a, \mathcal{S}(a)) \in F$, contradiction. □

The exponential

We describe here for Boolean algebras the notion of a hyperspace – the space of closed subsets of a topological space with the Vietoris topology. Later we consider various cardinal functions on the associated Boolean algebra, but we have not attempted to describe all known results about it.

Let X be any topological space. $\mathrm{Exp}\,(X)$ is the collection of all non-empty closed subspaces of X. We topologize it by taking the collection of sets of the following form as a base:

$$\mathscr{V}(U_1, \ldots, U_m) \stackrel{\text{def}}{=} \{F \in \mathrm{Exp}(X) : F \subseteq U_1 \cup \cdots \cup U_m \text{ and } F \cap U_i \neq 0 \text{ for all } i\},$$

where $U_1, \ldots, U_m$ are open in X. We call $\mathrm{Exp}\,(X)$ the *exponential* of X.

Theorem 1.20. *Let X be a compact Hausdorff space. Then* $\mathrm{Exp}\,(X)$ *is also Hausdorff and compact. Moreover, if X is a Boolean space, then so is* $\mathrm{Exp}\,(X)$*, and the set*

$$\{\mathscr{V}(U_1, \ldots, U_m) : \text{ each } U_i \text{ clopen}\}$$

is a collection of clopen sets forming a base for the topology on $\mathrm{Exp}\,(X)$*.*

Proof. Hausdorff: suppose that F and G are distinct non-empty closed sets. Say $x \in F\backslash G$. Let W and U be disjoint open sets such that $x \in W$ and $G \subseteq U$. Thus $G \in \mathscr{V}(U)$, $F \in \mathscr{V}(X, W)$, and $\mathscr{V}(U) \cap \mathscr{V}(X, W) = 0$.

Compact: First we note some facts. For each open U in X let $T_U = \{F \in \mathrm{Exp}(X) : F \cap U \neq 0\}$. This is an open set, since $T_U = \mathscr{V}(X, U)$. Next,

(1) $\{\mathscr{V}(U) : U \text{ open}\} \cup \{T_U : U \text{ open}\}$ is a subbase for the topology on $\mathrm{Exp}(X)$.

For, $\mathscr{V}(U_1, \ldots, U_m) = \mathscr{V}(U_1 \cup \cdots \cup U_m) \cap T_{U_1} \cap \cdots \cap T_{U_m}$.

Now to prove compactness of $\mathrm{Exp}(X)$, suppose that $\mathscr{O}$ is a cover of $\mathrm{Exp}(X)$ by subbase members in (1).

Case 1. $\{W : T_W \in \mathscr{O}\}$ covers X. Choose $W_1, \ldots, W_m$ with each $T_{W_i} \in \mathscr{O}$ such that $X = W_1 \cup \cdots \cup W_m$. Then $\{T_{W_1}, \ldots, T_{W_m}\}$ covers $\mathrm{Exp}(X)$, as desired.

Case 2. $\{W : T_W \in \mathscr{O}\}$ does not cover X. Let $Y = \bigcup_{T_W \in \mathscr{O}} W$. So Y is a proper open subset of X, and hence $X\backslash Y \in \mathrm{Exp}(X)$. Therefore $X\backslash Y \in \mathscr{V}(U)$ for some $\mathscr{V}(U) \in \mathscr{O}$. Thus $X\backslash Y \subseteq U$, so $X\backslash U \subseteq Y$. Since $X\backslash U$ is compact, there exist $W_1, \ldots, W_m$ with each $T_{W_i} \in \mathscr{O}$ such that $X\backslash U \subseteq W_1 \cup \cdots \cup W_m$. Hence $\{\mathscr{V}(U)\} \cup \{T_{W_1}, \ldots, T_{W_m}\}$ covers $\mathrm{Exp}(X)$, as is easily verified.

Now assume that X is a Boolean space. Then

(2) If $U_1, \ldots, U_m$ are clopen in X, then $\mathscr{V}(U_1, \ldots, U_m)$ is clopen in $\mathrm{Exp}(X)$.

For, suppose that $F \in \mathrm{Exp}(X)\backslash\mathscr{V}(U_1, \ldots, U_m)$.

Case 1. $F \not\subseteq U_1 \cup \cdots \cup U_m$. Then for some $\Gamma \subseteq \{1, \ldots, m\}$ we have $F \in \mathscr{V}(X\backslash(U_1 \cup \cdots \cup U_m), \langle U_i\rangle_{i\in\Gamma})$, and this set is disjoint from $\mathscr{V}(U_1, \ldots, U_m)$.

Case 2. $F \subseteq U_1 \cup \cdots \cup U_m$. Then there is an i such that $F \cap U_i = \emptyset$. Hence $F \in \mathscr{V}(X\backslash U_i)$, and $V(X\backslash U_i)$ is disjoint from $\mathscr{V}(U_1, \ldots, U_m)$.

(3) If $U_1, \ldots, U_m, W_1, \ldots, W_n$ are open and $F \in \mathscr{V}(U_1, \ldots, U_m) \cap \mathscr{V}(W_1, \ldots, W_n)$, then $F \in \mathscr{V}(\langle U_i \cap W_j : F \cap U_i \cap W_j \neq 0\rangle)$.

(4) $\{\mathscr{V}(U_1, \ldots, U_m) : \text{each } U_i \text{ clopen}\}$ is a base for the topology on $\mathrm{Exp}(X)$.

To prove (4), assume that $F \in \mathscr{V}(U_1, \ldots, U_m)$ with each U_i open; we want to find clopen $W_1, \ldots, W_n$ such that $F \in \mathscr{V}(W_1, \ldots, W_n) \subseteq \mathscr{V}(U_1, \ldots, U_m)$. It suffices by (3) to show that the set

$$\begin{gathered}\{\mathrm{Exp}(X)\backslash\mathscr{V}(U_1, \ldots, U_m)\} \cup \{\mathscr{V}(W_1, \ldots, W_n) : \\ \text{each } W_i \text{ clopen and } F \in \mathscr{V}(W_1, \ldots, W_n)\}\end{gathered}$$

has empty intersection. Suppose to the contrary that G is in each member of this set.

Case 1. $G \not\subseteq U_1 \cup \cdots \cup U_m$. Thus $G \not\subseteq F$; say $x \in G\backslash F$. Let W be a clopen set such that $x \in W$ and $W \cap F = 0$. Thus $F \in \mathscr{V}(X\backslash W)$, so $G \in \mathscr{V}(X\backslash W)$, contradiction.

Case 2. $G \subseteq U_1 \cup \cdots \cup U_m$. Since $G \in \mathrm{Exp}(X)\backslash\mathscr{V}(U_1, \ldots, U_m)$, it follows that $G \cap U_i = 0$ for some i. Say $x \in F \cap U_i$. Let W be clopen with $x \in W$ and $W \cap G = 0$. Then $F \in \mathscr{V}(W, X\backslash W)$ or $F \in \mathscr{V}(W)$, so $G \in \mathscr{V}(W, X\backslash W)$ or $G \in \mathscr{V}(W)$, contradiction. □

For any BA A, we denote by $\mathrm{Exp}(A)$ the Boolean algebra $\mathrm{Clop}(\mathrm{Exp}(\mathrm{Ult}(A)))$; this is called the *exponential* of A.

The following somewhat technical result will be useful.

Proposition 1.21. *For any BA A,* $\mathrm{Exp}\,(A)$ *is generated by* $\{\mathscr{V}(\mathcal{S}(a)) : a \in A\}$.

Proof. We already know from Theorem 1.20 that $\mathrm{Exp}(A)$ is generated by the set $\{\mathscr{V}(U_1,\ldots,U_m) : \text{each } U_i \text{ clopen}\}$. So it suffices to see that each element of this set is generated by $\{\mathscr{V}(\mathcal{S}(a)) : a \in A\}$. This follows from:

$$\mathscr{V}(\mathcal{S}(a_1),\ldots,\mathcal{S}(a_m)) = \mathscr{V}(\mathcal{S}(a_1+\cdots+a_m))\backslash(\mathscr{V}(\mathcal{S}(-a_1))\cup\cdots\cup\mathscr{V}(\mathcal{S}(-a_m))). \ \square$$

Lemma 1.22.

(i) *If* $x \neq y$, *then* $\mathscr{V}(\mathcal{S}(x)) \neq \mathscr{V}(\mathcal{S}(y))$.
(ii) $\mathscr{V}(\mathcal{S}(a)) = \{X \subseteq \mathcal{S}(a) : X \text{ is closed and nonempty}\}$.
(iii) $\mathscr{V}(\mathcal{S}(a_0)) \cap\cdots\cap \mathscr{V}(\mathcal{S}(a_{m-1})) = \mathscr{V}(\mathcal{S}(a_0\cdot\ldots\cdot a_{m-1}))$.
(iv) $\mathscr{V}(\mathcal{S}(a),\mathcal{S}(-a)) = -(\mathscr{V}(\mathcal{S}(a)) \cup \mathscr{V}(\mathcal{S}(-a)))$.
(v) $-\mathscr{V}(\mathcal{S}(a)) = \mathscr{V}(\mathcal{S}(a),\mathcal{S}(-a)) \cup \mathscr{V}(\mathcal{S}(-a))$.
(vi) *The following are equivalent:*
  (a) $\mathscr{V}(\mathcal{S}(a_0)) \cap\cdots\cap \mathscr{V}(\mathcal{S}(a_{m-1})) \cap -\mathscr{V}(\mathcal{S}(b_0)) \cap\cdots\cap -\mathscr{V}(\mathcal{S}(b_{n-1})) = \emptyset$.
  (b) $\exists i < n[a_0\cdot\ldots\cdot a_{m-1} \leq b_i]$.

Proof. For (i), say $x \not\leq y$. Let F be an ultrafilter such that $x\cdot -y \in F$. Then $\{F\} \in \mathscr{V}(\mathcal{S}(x))\backslash\mathscr{V}(\mathcal{S}(y))$.

(ii)–(iv) are clear.

For (v), note that $\mathscr{V}(\mathcal{S}(-a)) \subseteq -\mathscr{V}(\mathcal{S}(a))$. Hence

$$\begin{aligned}
&\mathscr{V}(\mathcal{S}(a),\mathcal{S}(-a)) \cup \mathscr{V}(\mathcal{S}(-a))\\
&\quad = -(\mathscr{V}(\mathcal{S}(a)) \cup \mathscr{V}(\mathcal{S}(-a))) \cup (\mathscr{V}(\mathcal{S}(-a)) \cap -\mathscr{V}(\mathcal{S}(a))\\
&\quad = (-\mathscr{V}(\mathcal{S}(a)) \cap -\mathscr{V}(\mathcal{S}(-a))) \cup (\mathscr{V}(\mathcal{S}(-a)) \cap -\mathscr{V}(\mathcal{S}(a))\\
&\quad = -\mathscr{V}(\mathcal{S}(a)).
\end{aligned}$$

For (vi), first assume (a). We may assume that $a_0\cdot\ldots\cdot a_{m-1} \neq 0$. Then $\mathcal{S}(a_0\cdot\ldots\cdot a_{m-1}) \in \mathscr{V}(\mathcal{S}(a_0)) \cap\cdots\cap \mathscr{V}(\mathcal{S}(a_{m-1}))$, and so by (a) there is an $i < n$ such that $\mathcal{S}(a_0\cdot\ldots\cdot a_{m-1}) \in \mathscr{V}(\mathcal{S}(b_i))$. Hence (b) holds. (b)$\Rightarrow$(a) similarly. $\square$

Proposition 1.23. *If A is atomic, then so is* $\mathrm{Exp}(A)$.

Proof. For each atom a of A, let F_a be the principal ultrafilter determined by a. Suppose that $\langle a_0,\ldots,a_{m-1}\rangle$ is a system of distinct atoms of A. Then

$$\mathscr{V}(\mathcal{S}(a_0),\ldots,\mathcal{S}(a_{m-1})) = \{\{F_{a_0},\ldots,F_{a_{m-1}}\}\},$$

and this is hence an atom of $\mathrm{Exp}(A)$. Now take any nonzero $x \in \mathrm{Exp}(A)$. To show that there is an atom below x it suffices to take the case in which x has the form $\mathscr{V}(\mathcal{S}(b_0),\ldots,\mathcal{S}(b_{m-1}))$. Note that each b_i is nonzero, since $x \neq 0$, and each element of x has nonempty intersection with each $\mathcal{S}(b_i)$. Let a_i be an atom below b_i for each $i < m$. Then $\{\{F_{a_0},\ldots,F_{a_{m-1}}\}\}$ is the desired atom. $\square$

Proposition 1.24. *If A is atomless, then so is* $\mathrm{Exp}(A)$.

Proof. Suppose that $0 \neq x \in \mathrm{Exp}(A)$; we want to find $0 < y < x$. We may assume that x has the form $\mathscr{V}(\mathcal{S}(b_0), \ldots, \mathcal{S}(b_{m-1}))$. Note that each b_i is nonzero. And we clearly may assume that all the b_i are distinct from one another. Finally, we may assume that b_0 is minimal among them, i.e., if $b_i \leq b_0$ then $i = 0$. Now choose $0 < a_0 < b_0$. We claim that the desired element is

$$y = \mathscr{V}(\mathcal{S}(a_0), \mathcal{S}(b_1 \cdot -b_0), \ldots, \mathcal{S}(b_{m-1} \cdot -b_0)).$$

Clearly $y \subseteq x$. Let $u = a_0 + b_1 \cdot -b_0 + \cdots + b_{m-1} \cdot -b_0$. Then $\mathcal{S}(u)$ is a member of y, so $y \neq 0$. Let $v = b_0 \cdot -a_0 + b_1 \cdot -b_0 + \cdots + b_{m-1} \cdot -b_0$. Clearly $\mathcal{S}(v)$ is a member of x. Since clearly $v \cdot a_0 = 0$, it is not a member of y. So $y < x$. □

From this proposition it follows that the free BA on ω free generators is isomorphic to its own exponential. Sirota [68] proved that also the free BA on ω_1 free generators is isomorphic to its own exponential. But Shapiro [76a], [76b] showed that this does not extend to higher cardinals.

The definition of $\mathrm{Exp}(A)$ can be given an equivalent formulation which is more algebraic. Let $\mathrm{Id}'(A)$ be the set of all proper ideals of A. For each $a \in A$ let

$$X_a = \{I \in \mathrm{Id}'(A) : \exists b \in I[-b \leq a]\},$$
$$Y_a = \{I \in \mathrm{Id}'(A) : a, -a \notin I\} \cup X_{-a}.$$

Lemma 1.25. $\mathrm{Id}'(A) \backslash X_a = Y_a$.

Proof. First suppose that $I \in \mathrm{Id}'(A) \backslash X_a$, and suppose that $I \notin X_{-a}$. Thus $\forall b \in I[-b \cdot -a \neq 0]$, so $-a \notin I$. Also $\forall b \in I[-b \cdot a \neq 0]$, so $a \notin I$. This proves $\subseteq$.

Second, suppose that $I \in Y_a$, but suppose that $I \in X_a$. Choose $b \in I$ so that $-b \leq a$. Thus $-a \in I$. Hence $I \in X_{-a}$. Choose $c \in I$ such that $-c \leq -a$. Then $a \leq c$, so $1 = a + -a \leq c + b$ and I is not proper, contradiction. □

Let $\mathrm{Exp}'(A)$ be the subalgebra of $\mathscr{P}(\mathrm{Id}'(A))$ generated by all elements X_a.

Theorem 1.26. $\mathrm{Exp}(A) \cong \mathrm{Exp}'(A)$.

Proof. For any $a \in A$ let $f(\mathscr{V}(\mathcal{S}(a))) = X_a$. f is well defined by Lemma 1.22(i). By Proposition 1.21 it suffices to show that f extends to an isomorphism. We use Sikorski's extension criterion. Suppose that $a_1, \ldots, a_m, b_1, \ldots, b_n$ are distinct elements of A with $m, n > 0$. We want to show that the following two conditions are equivalent:

(1) $\mathscr{V}(\mathcal{S}(a_1) \cap \cdots \cap \mathscr{V}(\mathcal{S}(a_m) \cap -\mathscr{V}(\mathcal{S}(b_1) \cap \cdots \cap -\mathscr{V}(\mathcal{S}(b_n) = \emptyset$.

(2) $X_{a_1} \cap \cdots \cap X_{a_m} \cap Y_{b_1} \cap \cdots \cap Y_{b_n} = \emptyset$.

By Lemma 1.22(vi), condition (1) is equivalent to

(3) $\exists i < n[a_1 \cdot \ldots \cdot a_m \leq b_i]$.

Assume (3), and suppose that I is a member of (2). Then for all $j = 1, \ldots, m$ there is a $c_j \in I$ such that $-c_j \le a_j$. Hence $-c_1 \cdot \ldots \cdot -c_m \le a_1 \cdot \ldots \cdot a_m \le b_i$. Moreover, $c_1 + \cdots + c_m \in I$. Hence $-b_i \in I$, so $I \in X_{-b_i}$ since $I \in Y_{b_i}$. Choose $d \in I$ such that $-d \le -b_i$. Hence $b_i \le d$, so $-c_1 \cdot \ldots \cdot -c_m \cdot -d = 0$, hence $c_1 + \cdots + c_m + d = 1$ and so I is not proper, contradiction.

Now assume that (2) holds, while $\forall i < n[a_1 \cdot \ldots \cdot a_m \cdot -b_i \neq 0]$. Let $I = A \upharpoonright (-a_1 + \cdots + -a_m)$. Then for any $i = 1, \ldots, m$ we have $-a_i \in I$, and hence $I \in X_{a_i}$. Hence by (2), there is an i with $1 \le i \le m$ such that $I \notin Y_{b_i}$. We have $-b_i \not\le -a_1 + \cdots + -a_n$, so $-b_i \notin I$. It follows that $b_i \in I$. Hence $b_i \le -a_1 + \cdots + -a_m$, so $a_1 \cdot \ldots \cdot a_m \le -b_i$. It follows that $I \in X_{-b_i}$, hence $I \in Y_{b_i}$, contradiction. □

To make this discussion of the exponential more concrete, we describe the exponential for A the finite-cofinite algebra on an infinite cardinal κ. For each $\alpha < \kappa$ let F_α be the ultrafilter of all $\Gamma \in A$ such that $\alpha \in \Gamma$. Let G be the ultrafilter of all cofinite subsets of κ. Thus $\mathrm{Ult}(A) = \{G\} \cup \{F_\alpha : \alpha < \kappa\}$. Now $\{\mathcal{S}(a) : a \in A\}$ is a basis for $\mathrm{Ult}(A)$. Note:

$$\begin{aligned} a \text{ finite} \quad &\Rightarrow \quad \mathcal{S}(a) = \{F_\alpha : \alpha \in a\}; \\ a \text{ cofinite} \quad &\Rightarrow \quad \mathcal{S}(a) = \{G\} \cup \{F_\alpha : \alpha \in a\}. \end{aligned}$$

Thus each F_α is isolated. The open subsets of $\mathrm{Ult}(A)$ are

$$\begin{aligned} &\{F_\alpha : \alpha \in \Gamma\} \quad \text{for any} \quad \Gamma \subseteq \kappa; \\ &\{G\} \cup \{F_\alpha : \alpha \in \Gamma\} \quad \text{for any cofinite } \Gamma \subseteq \kappa. \end{aligned}$$

Hence the closed sets are

$$\begin{aligned} y_\Gamma &\stackrel{\text{def}}{=} \{G\} \cup \{F_\alpha : \alpha \in \Gamma\} \quad \text{for any} \quad \Gamma \subseteq \kappa, \\ z_\Gamma &\stackrel{\text{def}}{=} \{F_\alpha : \alpha \in \Gamma\} \quad \text{for any finite} \quad \Gamma \subseteq \kappa. \end{aligned}$$

Hence

$$\mathrm{Exp}(\mathrm{Ult}(A)) = \{y_\Gamma : \Gamma \subseteq \kappa\} \cup \{z_\Gamma : 0 \neq \Gamma \text{ a finite subset of } \kappa\}.$$

Now we claim:

(1) Each z_Γ is isolated, Γ a finite nonempty subset of κ.

In fact, write $\Gamma = \{\alpha_0, \ldots, \alpha_{m-1}\}$, $m > 0$. Then $\{z_\Gamma\} = \mathscr{V}(\{F_{\alpha_0}\}, \ldots, \{F_{\alpha_{m-1}}\})$, as desired.

The following two statements are obvious:

(2) If $a_0, \ldots, a_{m-1} \in A$ are all finite and $m > 0$, then

$$\mathscr{V}(\mathcal{S}(a_0), \ldots, \mathcal{S}(a_{m-1})) = \{z_\Gamma : \Gamma \subseteq a_0 \cup \cdots \cup a_{m-1} \text{ and } \Gamma \cap a_i \neq 0 \text{ for all } i < m\}.$$

(3) If $a_0, \ldots, a_{m-1} \in A$ and some a_i is cofinite, then

$$\begin{aligned}\mathscr{V}(\mathcal{S}(a_0), \ldots, \mathcal{S}(a_{m-1})) = \{&y_\Gamma : \Gamma \subseteq a_0 \cup \cdots \cup a_{m-1} \text{ and } \Gamma \cap a_i \neq 0 \\ &\text{for all } i \text{ such that } a_i \text{ is finite}\} \\ \cup \{&z_\Gamma : \Gamma \subseteq a_0 \cup \cdots \cup a_{m-1},\ \Gamma \text{ finite },\ \Gamma \cap a_i \neq 0 \\ &\text{for all } i < m\}.\end{aligned}$$

Next,

(4) No y_Γ is isolated.

For, suppose that $y_\Gamma \in \mathscr{V}(\mathcal{S}(a_0), \ldots, \mathcal{S}(a_{m-1}))$, $m > 0$. Since $G \in y_\Gamma$, some a_i is cofinite. By the above, if

$$\bigcup\{a_i : a_i \text{ finite }\} \subseteq \Delta \subseteq a_0 \cup \cdots \cup a_{m-1},$$

then $y_\Delta \in \mathscr{V}(\mathcal{S}(a_0), \ldots, \mathcal{S}(a_{m-1}))$. Thus $\mathscr{V}(\mathcal{S}(a_0), \ldots, \mathcal{S}(a_{m-1}))$ has infinitely many members, as desired. We also proved:

(5) $\{y_\Gamma : \Gamma \subseteq \kappa\}$, as a subspace of $\mathrm{Ult}(A)$, is closed and has no isolated points.

The following is obvious:

(6) $\{\{z_\Gamma\} : \Gamma$ finite and non-empty$\}$ is the set of all atoms of $\mathrm{Exp}(A)$, which is atomic.

Let $x_\alpha = \mathscr{V}(\mathcal{S}(\kappa\backslash\{\alpha\}), \mathcal{S}(\{\alpha\}))$ for all $\alpha < \kappa$.

(7) $\langle x_\alpha/\mathrm{fin} : \alpha < \kappa\rangle$ is a system of independent elements of $\mathrm{Exp}(A)/\mathrm{fin}$.

To show this, suppose that Γ and Δ are finite disjoint subsets of κ; we want to show that $\bigcap_{\alpha\in\Gamma} x_\alpha \cap \bigcap_{\alpha\in\Delta} -x_\alpha$ is not a finite sum of atoms. Note by (3) that

$$\begin{aligned}x_\alpha = \{&y_\Omega : \alpha \in \Omega\} \\ \cup \{&z_\Omega : \alpha \in \Omega \text{ and } \Omega \neq \{\alpha\},\ \Omega \text{ finite}\}.\end{aligned}$$

It follows that $y_\Gamma \in \bigcap_{\alpha\in\Gamma} x_\alpha \cap \bigcap_{\alpha\in\Delta} -x_\alpha$. Hence (7) holds.

(8) $\langle x_\alpha/\mathrm{fin} : \alpha < \kappa\rangle$ generates $\mathrm{Exp}(A)/\mathrm{fin}$.

To prove this, by Proposition 1.21 it suffices to show that if a is cofinite then $\mathscr{V}(\mathcal{S}(a))/\mathrm{fin}$ is generated by $\langle x_\alpha/\mathrm{fin} : \alpha < \kappa\rangle$. So (8) follows from

(9) $\mathscr{V}(\mathcal{S}(a))/\mathrm{fin} = \bigcap_{\alpha\in\kappa\backslash a} -x_\alpha/\mathrm{fin}$.

To prove this, first note that if $\alpha \in \kappa\backslash a$, then

$$\begin{aligned}\mathscr{V}(\mathcal{S}(a)) \cap x_\alpha = &\ (\{y_\Gamma : \emptyset \neq \Gamma \subseteq a\} \cup \{z_\Gamma : 0 \neq \Gamma \subseteq a,\ \Gamma \text{ finite}\}) \\ &\cap (\{y_\Gamma : \alpha \in \Gamma\} \cup \{z_\Gamma : \alpha \in \Gamma,\ \Gamma \neq \{\alpha\},\ \Gamma \text{ finite}\}) \\ = &\ \emptyset.\end{aligned}$$

This proves $\leq$ in (9). For the other direction, first note that

$$-x_\alpha = \{y_\Gamma : \alpha \notin \Gamma\} \cup \{z_\Gamma : (\alpha \notin \Gamma \text{ or } \Gamma = \{\alpha\}) \text{ and } \Gamma \text{ finite}\}.$$

Hence

$$\begin{aligned}\left(\bigcap_{\alpha\in\kappa\backslash a} -x_\alpha\right)\backslash \mathscr{V}(\mathcal{S}(a)) &= (\{y_\Gamma : \Gamma \cap (\kappa\backslash a) = 0\} \\ &\quad \cup \{z_\Gamma : \forall\alpha \in \kappa\backslash a(\alpha \notin \Gamma \text{ or } \Gamma = \{\alpha\}) \text{ and } \Gamma \text{ finite}\}) \\ &\quad \cap (\{y_\Gamma : \Gamma \not\subseteq a\} \cup \{z_\Gamma : \Gamma \not\subseteq a,\ \Gamma \text{ finite}\}) \\ &= \{z_{\{\alpha\}} : \alpha \in \kappa\backslash a\},\end{aligned}$$

as desired.

Some further properties of the exponential will be developed in the discussion of semigroup algebras in the next chapter.

2 Special Classes of Boolean Algebras

We discuss several special classes of Boolean algebras not mentioned in the Handbook.

Semigroup algebras

The notion of a semigroup algebra is due to Heindorf [89]. We give basic definitions and facts only. A subset H of a BA A is said to be *disjunctive* if $0 \notin H$, and $h, h_1, \ldots, h_n \in H$ and $h \leq h_1 + \cdots + h_n$ $(n > 0)$ imply that $h \leq h_i$ for some i. If P is any partially ordered set, $M \subseteq P$, and $p \in P$, we define

$$M \uparrow p = \{a \in M : p \leq a\};$$
$$M \downarrow p = \{a \in M : a \leq p\}.$$

Proposition 2.1. *Let A be a BA and $H \subseteq A^+$. Then H is disjunctive* iff *for every $M \subseteq H$ there is a homomorphism f from $\langle H \rangle$ into $\mathscr{P}(M)$ such that $f(h) = M \downarrow h$ for all $h \in H$.*

Proof. $\Rightarrow$: In order to apply Sikorski's extension criterion, assume that $h_1, \ldots, h_m$, $k_1, \ldots, k_n \in H$ and $h_1 \cdot \ldots \cdot h_m \leq k_1 + \cdots + k_n$; we want to show that $(M \downarrow h_1) \cap \cdots \cap (M \downarrow h_m) \subseteq (M \downarrow k_1) \cup \cdots \cup (M \downarrow k_n)$. Let $x \in (M \downarrow h_1) \cap \cdots \cap (M \downarrow h_m)$. Then $x \leq h_1 \cdot \ldots \cdot h_m$, so $x \leq k_1 + \cdots + k_n$. Note that $n > 0$, since otherwise $x = 0$, contradicting $M \subseteq H \subseteq A^+$. Hence by disjunctiveness, $x \leq k_i$ for some i; so $x \in (M \downarrow k_i)$, as desired.

$\Leftarrow$: Suppose that $h, h_1, \ldots, h_m \in H$ and $h \leq h_1 + \cdots + h_m$ $(m > 0)$. Let $M = \{h, h_1, \ldots, h_m\}$, and take the function f corresponding to M. Then

$$h \in (M \downarrow h) = f(h) \subseteq f(h_1) \cup \cdots \cup f(h_m) = (M \downarrow h_1) \cup \cdots \cup (M \downarrow h_m),$$

so $h \leq h_i$ for some i. □

A *semigroup* is an algebra $(H, \cdot)$ such that H is a nonempty set and $\cdot$ is an associative binary operation on H. A *zero* of H is an element $z \in H$ such that $z \cdot x = x \cdot z = z$ for all $x \in H$. An *identity* of H is an element $e \in H$ such that $e \cdot x = x = x \cdot e$ for all $x \in H$. Clearly a zero element, if it exists, is unique, and

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similarly for identity elements. Also it is clear that if f is a homomorphism from a semigroup H *onto* a semigroup K, then f preserves the zero and the identity, if they exist.

A BA A is a *semigroup algebra* if it is generated by a subset H with the following properties:

(1) $0, 1 \in H$;
(2) H is closed under the operation $\cdot$ of A;
(3) $H\backslash\{0\}$ is disjunctive.

Here are three important examples of semigroup algebras:

A. *Tree algebras.* Let $A = \mathrm{TreeAlg}(T)$. Without loss of generality T has only one root. Set $H = \{T \uparrow t : t \in T\} \cup \{0, 1\}$. The conditions for a semigroup algebra are easily verified.

B. *Interval algebras.* Let $A = \mathrm{IntAlg}(L)$, where L is a linear ordering with first element 0_L. Let $H = \{[0_L, a) : a \in L\} \cup \{1\}$. Again the indicated conditions are easily checked.

C. *Free algebras.* Let A be freely generated by X, and set $H = \{x \in A : x$ is a finite product of members of $X\} \cup \{0, 1\}$. The indicated conditions clearly hold.

It is also useful to note that if A is a semigroup algebra, then so is $\mathrm{Dup}(A)$. We prove this now. Let H be a generating set for A satisfying conditions (1)–(3). We define

$$H' = \{(a, \mathrm{S}(a)) : a \in H\} \cup \{(0, \{F\}) : F \in \mathrm{Ult}(A)\}.$$

We claim that H' shows that $\mathrm{Dup}(A)$ is a semigroup algebra. Clearly $(0, 0), (1, 1) \in H'$ and H' is closed under $\cdot$. To show that H' is disjunctive, we consider two possibilities:

$$(a, \mathrm{S}(a)) \leq h_0 + \cdots + h_n$$

where $a \neq 0$. Say that h_i has the form $(b_i, \mathrm{S}(b_i))$ for all $i \in M$, and the form $(0, \{F_i\})$ for all $i \in (n+1)\backslash M$. Hence $a \leq \sum_{i \in M} b_i$, hence $a \leq b_i$ for some i, hence $(a, \mathrm{S}(a)) \leq h_i$.

Second possibility:

$$(0, \{F\}) \leq h_0 + \cdots + h_n$$

The desired conclusion is clear.

Thus $H'\backslash\{(0, 0\}$ is disjunctive, so (1)–(3) hold.

Finally, we need to show that H' generates $\mathrm{Dup}(A)$. Take any $(a, X) \in \mathrm{Dup}(A)$. If $a = 0$, then X is a finite set of ultrafilters, and so (a, X) is generated by H'. Suppose that $a \neq 0$. Then since H generates A, we can write $a = \sum_{i<m} \prod_{j<n} b_{ij}^{\varepsilon(i,j)}$ with each $b_{ij} \in H$. Then

$$(a, \mathrm{S}(a)) = \sum_{i<m} \prod_{j<n} (b_{ij}, \mathrm{S}(b_{ij})^{\varepsilon(i,j)},$$

so that $(a, \mathrm{S}(a))$ is generated by H'. Finally, let $F = \mathrm{S}(a) \triangle X$; so F is finite, and $X = \mathrm{S}(a) \triangle F$. So

$$(a, X) = [(a, \mathrm{S}(a)) \backslash (0, F)] \cup [(0, F \backslash \mathrm{S}(a))],$$

as desired.

Proposition 2.2. *Suppose that A is a semigroup algebra with associated semigroup H, B is a BA, and f is a homomorphism from $(H, \cdot)$ into the semigroup $(B, \cdot)$ preserving 0 and 1. Then f has a unique extension to a homomorphism from A into B. Moreover, if f is onto, the extension is too. Finally, if B is a semigroup algebra on $(K, \cdot)$ and f is an isomorphism from $(H, \cdot)$ into $(K, \cdot)$ preserving 0 and 1, then the extension is an isomorphism into.*

Proof. In order to apply Sikorski's criterion, let $b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}$ be distinct elements of H and suppose that

$$b_0 \cdot \ldots \cdot b_{m-1} \cdot -c_0 \cdot \ldots \cdot -c_{n-1} = 0.$$

Without loss of generality, $m > 0$ and each c_i is different from 0. If $n = 0$, then

$$f(b_0) \cdot \ldots \cdot f(b_{m-1}) = f(b_0 \cdot \ldots \cdot b_{m-1}) = f(0) = 0,$$

as desired. Assume that $n > 0$. Then $b_0 \cdot \ldots \cdot b_{m-1} \leq c_0 + \cdots + c_{n-1}$, so $b_0 \cdot \ldots \cdot b_{m-1} \leq c_i$ for some i; hence $b_0 \cdot \ldots \cdot b_{m-1} \cdot c_i = b_0 \cdot \ldots \cdot b_{m-1}$ and

$$\begin{aligned}
&f(b_0) \cdot \ldots \cdot f(b_{m-1}) \cdot -f(c_0) \cdot \ldots \cdot -f(c_{n-1}) \\
&\quad = f(b_0 \cdot \ldots \cdot b_{m-1}) \cdot -f(c_0) \cdot \ldots \cdot -f(c_{n-1}) \\
&\quad = f(b_0 \cdot \ldots \cdot b_{m-1} \cdot c_i) \cdot -f(c_0) \cdot \ldots \cdot -f(c_{n-1}) \\
&\quad = f(b_0) \cdot \ldots \cdot f(b_{m-1}) \cdot f(c_i) \cdot -f(c_0) \cdot \ldots \cdot -f(c_{n-1}) \\
&\quad = 0,
\end{aligned}$$

as desired. Clearly if f is onto, then the extension is onto.

Assume the hypothesis of "Finally...". Let $b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}$ be distinct elements of H such that

$$f(b_0) \cdot \ldots \cdot f(b_{m-1}) \cdot -f(c_0) \cdot \ldots \cdot -f(c_{n-1}) = 0.$$

We want to show that $b_0 \cdot \ldots \cdot b_{m-1} \cdot -c_0 \cdot \ldots \cdot -c_{n-1} = 0$. Wlog $m, n > 0$. Thus $f(b_0 \cdot \ldots \cdot b_{m-1}) \in K$, and $f(b_0 \cdot \ldots \cdot b_{m-1}) \leq f(c_0) + \cdots + f(c_{n-1})$, so there is an $i < n$ such that $f(b_0 \cdot \ldots \cdot b_{m-1}) \leq f(c_i)$. Hence $f(b_0 \cdot \ldots \cdot b_{m-1}) = f(b_0 \cdot \ldots \cdot b_{m-1} \cdot c_i)$, so $b_0 \cdot \ldots \cdot b_{m-1} = b_0 \cdot \ldots \cdot b_{m-1} \cdot c_i$ since f is one-one, and the desired conclusion follows. □

Corollary 2.3. *If A and B are semigroup algebras both with the same associated semigroup $(H, \cdot)$, then there is an isomorphism from A onto B which fixes H pointwise.* □

Now we indicate the connection of the exponential of a BA with semigroup algebras.

Proposition 2.4. *For any BA A,* $\mathrm{Exp}(A)$ *is a semigroup algebra on a semigroup isomorphic to* $(A,\cdot)$.

Proof. For any $a \in A$ let $f(a) = \mathscr{V}(\mathcal{S}(a))$, and let $H = f[A]$. We want to show that $\mathrm{Exp}(A)$ is a semigroup algebra on H and f is an isomorphism from $(A,\cdot)$ onto $(H,\cap)$ preserving 0 and 1. Clearly it preserves 0 and 1. If $a, b \in A$, then

$$f(a\cdot b) = \mathscr{V}(\mathcal{S}(a\cdot b)) = \mathscr{V}(\mathcal{S}(a)\cap\mathcal{S}(b)) = \mathscr{V}(\mathcal{S}(a))\cap\mathscr{V}(\mathcal{S}(b)) = f(a)\cap f(b).$$

If $a \neq b$, say $a \not\leq b$; then $\mathcal{S}(a) \in \mathscr{V}(\mathcal{S}(a))$ but $\mathcal{S}(a) \notin \mathscr{V}(\mathcal{S}(b))$; this shows that f is one-one. So we have checked that f is an isomorphism from $(A,\cdot)$ onto $(H,\cap)$ preserving 0 and 1.

Note that H generates $\mathrm{Exp}(A)$ by Proposition 1.21. Finally, the disjunctive property follows like this: suppose that $\mathscr{V}(\mathcal{S}(a)) \subseteq \mathscr{V}(\mathcal{S}(b_1)) \cup \cdots \cup \mathscr{V}(\mathcal{S}(b_m))$, $m > 0$. Now $\mathcal{S}(a) \in \mathscr{V}(\mathcal{S}(a))$, so $\mathcal{S}(a) \in \mathscr{V}(\mathcal{S}(b_i))$ for some i, and hence $\mathscr{V}(\mathcal{S}(a)) \subseteq \mathscr{V}(\mathcal{S}(b_i))$, as desired. □

The following result will also be useful.

Proposition 2.5. *For any BA A,* $\mathrm{Exp}(A)$ *embeds in* $\prod_{n\geq 1} A^{*n}$*, where A^{*n} denotes the free product of n copies of A.*

Proof. We use the notation of the proof of Proposition 2.4. For each $n \geq 1$ define $g_n : H \to A^{*n}$ as follows:

$$g_n(f(a)) = \prod_{i<n} h_i(a),$$

where h_i is the natural embedding of A into the ith factor of A^{*n}. Clearly g_n is a homomorphism from $(H,\cdot)$ into $(A^{*n},\cdot)$ taking 0 to 0 and 1 to 1. Hence by Proposition 2.2 it extends to a homomorphism, still denoted by g_n, from $\mathrm{Exp}(A)$ into A^{*n}. For any $x \in \mathrm{Exp}(A)$ let $(k(x))_n = g_n(x)$ for all $n \geq 1$. Clearly k is a homomorphism from $\mathrm{Exp}(A)$ into $\prod_{n\geq 1} A^{*n}$, so it suffices to show that k is one-one. We take an arbitrary non-zero member of $\mathrm{Exp}(A)$; we may assume that it has the form $\mathscr{V}(\mathcal{S}(a_0),\ldots,\mathcal{S}(a_{m-1}))$, with each $a_i \neq 0$. Then

$$\begin{aligned}
&k(\mathscr{V}(\mathcal{S}(a_0),\ldots,\mathcal{S}(a_{m-1})))_m\\
&\quad= g_m(\mathscr{V}(\mathcal{S}(a_0),\ldots,\mathcal{S}(a_{m-1})))\\
&\quad= g_m(\mathscr{V}(\mathcal{S}(a_0+\cdots+a_{m-1}))\backslash(\mathscr{V}(\mathcal{S}(-a_0))\cup\cdots\cup\mathscr{V}(\mathcal{S}(-a_{m-1}))))\\
&\quad= \prod_{i<m} h_i(a_0+\cdots+a_{m-1})\cdot -\prod_{i<m} h_i(-a_0)\cdot\ldots\cdot -\prod_{i<m} h_i(-a_{m-1})\\
&\quad= \prod_{i<m} h_i(a_0+\cdots+a_{m-1})\cdot \sum_{i<m} h_i(a_0)\cdot\ldots\cdot \sum_{i<m} h_i(a_{m-1})
\end{aligned}$$

$$\geq \prod_{i<m} h_i(a_0 + \cdots + a_{m-1}) \cdot \prod_{i<m} h_i(a_i)$$
$$= \prod_{i<m} h_i(a_i) \neq 0. \qquad \square$$

Proposition 2.6. *For any BA, there is a homomorphism from* $\mathrm{Exp}(A)$ *onto* A.

Proof. By Proposition 2.4, $\mathrm{Exp}(A)$ is a semigroup algebra on a semigroup H, with an isomorphism f from $(H, \cdot)$ onto $(A, \cdot)$. Then by Proposition 2.2 there is an extension of f to a homomorphism from $\mathrm{Exp}(A)$ onto A. $\square$

Proposition 2.7. *If f is a homomorphism from A into B, then there is a homomorphism g from* $\mathrm{Exp}(A)$ *into* $\mathrm{Exp}(B)$. *Moreover, if f is onto, then g may be taken to be onto.*

Proof. By Proposition 2.4, the algebras $\mathrm{Exp}(A)$ and $\mathrm{Exp}(B)$ are semigroup algebras on semigroups H and K isomorphic to $(A, \cdot)$ and $(B, \cdot)$ respectively; hence f yields a homomorphism from H into K preserving 0 and 1, and the result follows by Proposition 2.2. The last statement of the proposition is obvious. $\square$

Proposition 2.8. *Any semigroup algebra can be isomorphically embedded in its exponential. Hence for any BA A, the algebra* $\mathrm{Exp}(A)$ *can be isomorphically embedded in* $\mathrm{Exp}(\mathrm{Exp}(A))$.

Proof. Let A be a semigroup algebra. By Proposition 2.4, there is an isomorphism f from $(A, \cdot)$ onto a semigroup $(K, \cdot)$ such that $\mathrm{Exp}(A)$ is a semigroup on $(K, \cdot)$. But A is a semigroup algebra on some semigroup $(H, \cdot)$, so there is an isomorphism of $(H, \cdot)$ into $(K, \cdot)$. By the final part of Proposition 2.2, our proposition follows. $\square$

Pseudo-tree algebras

A *pseudo-tree* is a partially ordered system $(T, \leq)$ such that for each $t \in T$ the set $T \downarrow t$ is simply ordered. Thus this notion generalizes that of a tree, where $T \downarrow t$ is required to be well ordered. We define $\mathrm{Treealg}(T)$ to be the subalgebra of $\mathscr{P}(T)$ generated by $\{T \uparrow t : t \in T\}$; such algebras are called *pseudo-tree algebras.* Pseudo-tree algebras are treated thoroughly in Koppelberg, Monk [92].

Much of the theory of tree algebras described in §16 of the BA Handbook, Vol. 1, carries over to pseudo-tree algebras. For the convenience of the reader, we give a self-contained treatment of the fundamentals of the theory of pseudo-tree algebras.

Theorem 2.9. *For any pseudo-tree T there is a pseudo-tree T^* with a smallest element such that* $\mathrm{Treealg}(T) \cong \mathrm{Treealg}(T^*)$.

Proof. We consider two cases.

Case 1. There is a finite $F \subseteq T$ such that $T = \bigcup_{t\in F}(T \uparrow t)$, with the members of F distinct. Fix $x \in F$. We let $T^* = T$, with the new order given by

$$(*) \qquad s \leq_{T^*} t \quad \text{iff} \quad s \leq_T t \text{ or } s = x.$$

Let $R = T\backslash\{x\}$. Note that $(T^* \uparrow t) = (T \uparrow t)$ for any $t \in R$. For any $t \in R$ let $f(T \uparrow t) = (T \uparrow t)$. Note that $T^* \uparrow x = T$. Thus R generates Treealg(T^*). We claim that f extends to an isomorphism from Treealg(T^*) into Treealg(T). This is clear by Sikorski's extension criterion. Since $(T \uparrow x) = -\sum_{s\in F\backslash\{x\}}(T \uparrow x)$, f maps onto Treealg(T).

Case 2. There is no F as in Case 1. Let x be a new element, not in T, and define $T^* = T\cup\{x\}$. We define the order on T^* by $(*)$ again. Since $T^* \uparrow x = T^*$, it follows that $\{(T \uparrow t) : t \in T\}$ generates Treealg(T^*). For each $t \in T$ let $f(T \uparrow t) = (T \uparrow t)$ We claim that f extends to an isomorphism from Treealg(T^*) into Treealg(T). To prove this, by Sikorski's extension criterion we need to show that the following conditions are equivalent:

$$(**) \quad (T \uparrow s_0) \cap \cdots \cap (T \uparrow s_{m-1}) \cap [T^*\backslash(T \uparrow t_0)] \cap \cdots \cap [T^*\backslash(T \uparrow t_{n-1}] = \emptyset;$$

$$(***) \quad (T \uparrow s_0) \cap \cdots \cap (T \uparrow s_{m-1}) \cap [T\backslash(T \uparrow t_0)] \cap \cdots \cap [T\backslash(T \uparrow t_{n-1}] = \emptyset.$$

Since $T \subseteq T^*$, it is obvious that $(**)$ implies $(***)$. Now suppose that $(***)$ holds. By the case assumption, $m > 0$. Hence x is not a member of the set of $(**)$, and so $(**)$ holds.

Obviously the extension of f maps onto Treealg(T). □

The following normal form theorem is very useful.

Theorem 2.10. *Let T be a pseudo-tree with a least element. Then every element of* Treealg(T) *can be written in the form*

$$\bigcup_{t\in M}\left[(T \uparrow t)\backslash \bigcup_{s\in N_t}(T \uparrow s)\right],$$

where:

(i) *M is a finite subset of T.*

(ii) *$\forall t \in M[N_t$ is a finite subset of $(T \uparrow t)\backslash\{t\}$ consisting of pairwise incomparable elements$]$.*

(iii) *For all distinct $t, u \in M$, the elements*

$$(T \uparrow t)\backslash \bigcup_{s\in N_t}(T \uparrow s) \text{ and } (T \uparrow u)\backslash \bigcup_{s\in N_u}(T \uparrow s)$$

are disjoint.

(iv) *For all distinct $t, u \in M$, one of the following holds:*

  (a) *t and u are incomparable.*
  (b) *There is an $s \in N_t$ such that $s < u$.*
  (c) *There is an $s \in N_u$ such that $s < t$.*

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Proof. Let a be any element of Intalg(T). If $a=\emptyset$, we can take $M=\emptyset$. Suppose that $a\neq\emptyset$. Then we can write

$$a=\bigcup_{\varepsilon\in\Gamma}\bigcap_{t\in P}(T\uparrow t)^{\varepsilon(t)},$$

where P is a finite nonempty subset of T, $\Gamma\subseteq {}^P2$, and $\bigcap_{t\in P}(T\uparrow t)^{\varepsilon(t)}\neq\emptyset$ for all $\varepsilon\in\Gamma$. If u is the least element of T, then $(T\uparrow u)=T$; hence we may assume that for each $\varepsilon\in\Gamma$ there is a $t\in P$ such that $\varepsilon(t)=1$. Since $(T\uparrow t)\cap(T\uparrow u)=\emptyset$ for t and u incomparable, it follows that for each $\varepsilon\in\Gamma$, all of the elements $t\in P$ with $\varepsilon(t)=1$ are comparable, and hence they can be replaced by a single element. So we may write

$$a=\bigcup_{t\in M}\left[(T\uparrow t)\cap\bigcap_{s\in N_t}[T\backslash(T\uparrow s)]\right],$$

where M is a finite subset of T, for each $t\in M$ the set N_t is finite, $(T\uparrow t)\cap\bigcap_{s\in N_t}[T\backslash(T\uparrow s)]\neq\emptyset$ for each $t\in M$, and

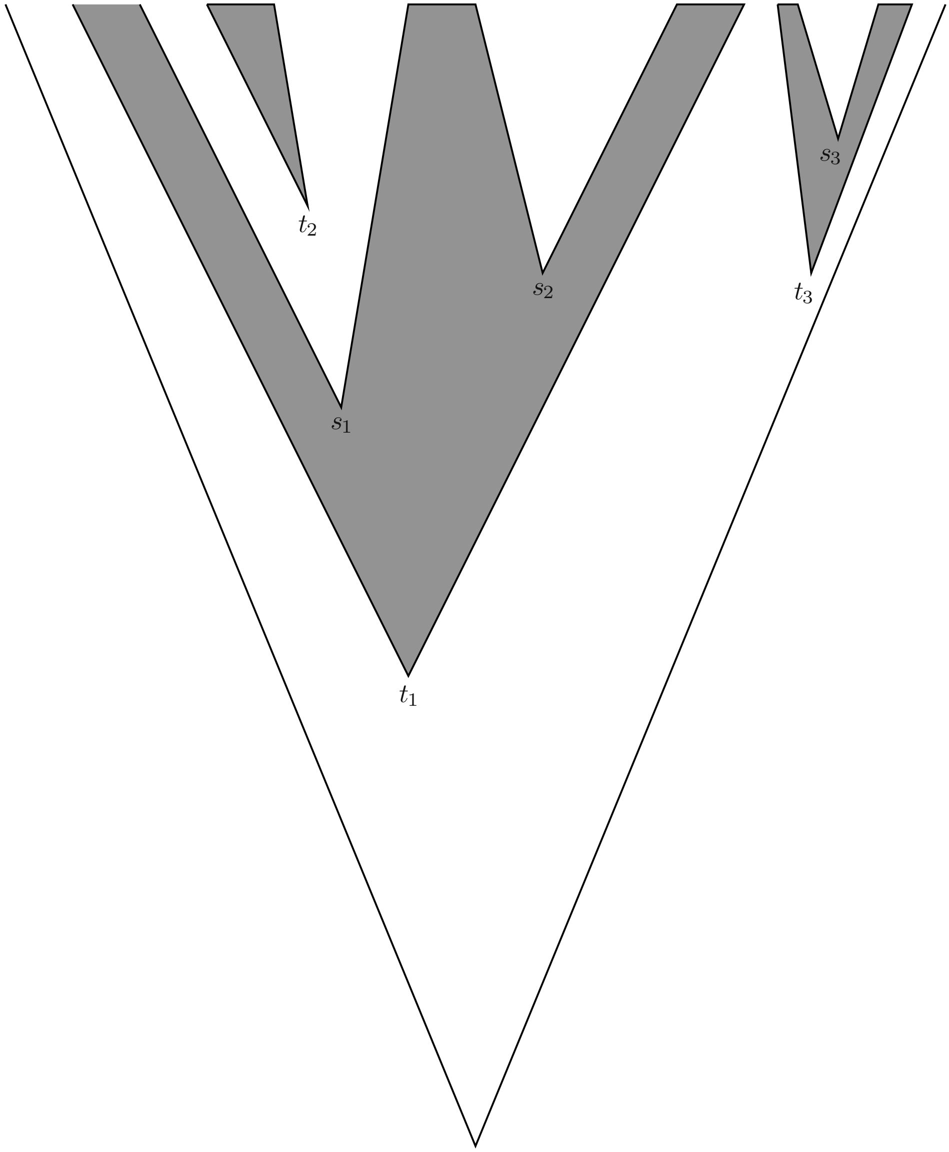
$$(T\uparrow t)\cap\bigcap_{s\in N_t}[T\backslash(T\uparrow s)]\cap(T\uparrow t')\cap\bigcap_{s\in N_{t'}}[T\backslash(T\uparrow s)]=\emptyset$$

for distinct $t,t'\in M$. If $s\in N_t$ is incomparable with t, then $T\uparrow t$ is a subset of $T\backslash(T\uparrow s)$, and s can be omitted from N_t. If $s\leq t$, then $(T\uparrow t)\subseteq(T\uparrow s)$, hence $(T\uparrow t)\cap[T\backslash(T\uparrow s)]=\emptyset$, contradiction. Hence we may assume that $t<s$ for all $s\in N_t$. If $s,w\in N_t$ and $s\leq w$, then $(T\uparrow w)\subseteq(T\uparrow s)$, and hence $[T\backslash(T\uparrow s)]\subseteq[T\backslash(T\uparrow w)]$. So we may assume that all members of N_t are incomparable. We have now arrived at a representation of a satisfying conditions (i)–(iii) of the theorem. Take such a representation, satisfying these three conditions, with $|M|$ as small as possible. We claim that then also condition (iv) holds. To prove this, suppose that t and u are distinct members of M and they are comparable; say $t<u$. If $\forall s\in N_t[s\not\leq u]$, then u is a member of both $(T\uparrow t)\cap\bigcap_{s\in N_t}[T\backslash(T\uparrow s)$ and $(T\uparrow u)\cap\bigcap_{s\in N_u}[T\backslash(T\uparrow s)$, contradiction. So there is an $s\in N_t$ such that $s\leq u$. If $s=u$, then

$$\begin{aligned}a=&\left(\bigcup_{w\in M\backslash\{t,u\}}\left[(T\uparrow w)\cap\bigcap_{s\in N_w}[T\backslash(T\uparrow s)]\right]\right)\\&\cup\left((T\uparrow t)\cap\bigcap_{s\in N_t\backslash\{u\}}[T\backslash(T\uparrow s)]\cap\bigcap_{s\in N_u}[T\backslash(T\uparrow s)]\right),\end{aligned}$$

which contradicts the minimality of $|M|$. Thus (iv)(b) in the theorem holds. $\square$

The normal form is illustrated on the next page.



$$a = [(T \uparrow t_1)\backslash[(T \uparrow s_1) \cup (T \uparrow s_2)]] \cup (T \uparrow t_2) \cup [(T \uparrow t_3)\backslash(T \uparrow s_3)]$$

Now we want to give an abstract characterization of pseudo-tree algebras. For this purpose we need some easy propositions. A subset R of a BA A is a *ramification set* provided that any two elements of R are either comparable or disjoint. Thus R is then a pseudo-tree under the inverse ordering of the BA.

Proposition 2.11. *Let* $(P, \leq)$ *be a partially ordered system. Then* $\{P \uparrow p : p \in P\}$ *is a disjunctive set in* $\mathscr{P}(P)$.

Proof. Obviously $\emptyset$ is not in the indicated set. Now suppose that $p, p_1, \ldots, p_n \in P$, where $n > 0$, and assume that $P \uparrow p \subseteq (P \uparrow p_1) \cup \cdots \cup (P \uparrow p_n)$. Then $p \in (P \uparrow p)$, and hence $p \in (P \uparrow p_i)$ for some i, and hence $(P \uparrow p) \subseteq (P \uparrow p_i)$, as desired. □

Corollary 2.12. *Every pseudo-tree algebra is a semigroup algebra.* □

Proposition 2.13. *Let* R *be a disjunctive ramification set of non-zero elements and let* X, Y *be finite subsets of* R. *Then* $\prod X \leq \sum Y$ iff *one of the following three conditions holds:*

(1) $X = \emptyset$ *and* $\sum Y = 1$;
(2) $x \cdot y = 0$ *for some* $x, y \in X$;
(3) $x \leq y$ *for some* $x \in X$, $y \in Y$.

Proof. Obviously any of (1)–(3) implies that $\prod X \leq \sum Y$. Now suppose that $\prod X \leq \sum Y$ and (1) and (2) do not hold. Note that if $X = \emptyset$ then $\sum Y = 1$; hence $X \neq \emptyset$. From the falsity of (2) it then follows that $\prod X \in X$ and $Y \neq \emptyset$. Then disjunctiveness yields (3). □

Proposition 2.14. *Let* $R \subseteq A^+$ *be a ramification set, and let* S *be a subset of* R *maximal among disjunctive subsets of* R. *Then* $\langle S \rangle = \langle R \rangle$.

Proof. We need only show that $R \subseteq \langle S \rangle$; so let $r \in R \backslash S$. Since $r \notin S$, the set $S \cup \{r\}$ is not disjunctive. There are then two cases:

Case 1. $r \leq s_1 + \cdots + s_n$ for certain $s_1, \ldots, s_n \in S$ $(n > 0)$, but $r \not\leq s_i$ for all i. Let n be minimal such that this can happen. By the minimality, $r \cdot s_i \neq 0$ for all i, so $s_i \leq r$ and hence $r = s_1 + \cdots + s_n \in \langle S \rangle$, as desired.

Case 2. $s_1 \leq r + s_2 + \cdots + s_n$ for certain $s_1, \ldots, s_n \in S$ $(n > 0)$, but $s_1 \not\leq r$ and $s_1 \not\leq s_i$ for all $i > 1$. Again, take n minimal for this situation. Since $s_1 \not\leq r$, we have $n > 1$. By the minimality of n and the fact that R is a ramification set, the elements $r, s_2, \ldots, s_n$ are pairwise disjoint. Also by the minimality of n, $s_1 \cdot r \neq 0 \neq s_1 \cdot s_i$ for all $i > 1$; hence $r \leq s_1$ and $s_i \leq s_1$ for all $i > 1$. Hence $s_1 = r + s_2 + \cdots + s_n$, and it follows that $r = s_1 \cdot -(s_2 + \cdots + s_n) \in \langle S \rangle$, as desired. □

With these preliminaries over, we can now give our abstract characterization of pseudo-tree algebras.

Theorem 2.15. *For any BA A, the following conditions are equivalent:*

(i) *A is isomorphic to* $\mathrm{Treealg}(T)$ *for some pseudo-tree T with a smallest element;*
(ii) *A is isomorphic to* $\mathrm{Treealg}(T)$ *for some pseudo-tree T;*
(iii) *A is generated by a ramification set;*
(iv) *A is generated by a ramification set $S \subseteq A^+$ such that $1 \in S$ and S is disjunctive.*

Proof. Obviously (i)⇒(ii), and it is also clear that (ii)⇒(iii). For (iii)⇒(iv), suppose that R is a ramification set which generates A. We may assume that $0 \notin R$ and $1 \in R$. Then by Proposition 2.14 we get a ramification set S as desired in (iv).

Finally, we prove (iv)⇒(i). Clearly S is a pseudo-tree with smallest element under the converse of the Boolean ordering; so it suffices to show that A is isomorphic to $\mathrm{Treealg}(S)$. By Proposition 2.1, there is a homomorphism f from A into $\mathscr{P}(S)$ such that $f(s) = S \uparrow s$ for all $s \in S$; here $S \uparrow s$ is in the tree sense. Clearly f maps onto $\mathrm{Treealg}(S)$. It is also one-one; we see this by using Sikorski's criterion: assume that

$$t_0 \cdot \ldots \cdot t_{m-1} \cdot -s_0 \cdot \ldots \cdot -s_{n-1} \neq 0,$$

where all t_i and s_i are in S and $m, n \in \omega$. Since $1 \in S$, we may assume that $m > 0$. Then $u \stackrel{\text{def}}{=} t_0 \cdot \ldots \cdot t_{m-1}$ is an element of S, and $u \not\leq s_i$ for all i. Hence

$$u \in f(t_0) \cap \cdots \cap f(t_{m-1}) \cap -f(s_0) \cap \cdots \cap -f(s_{n-1}),$$

as desired. □

Proposition 2.16. *If T is a pseudo-tree, then $\{T \uparrow t : t \in T\}$ is a disjunctive ramification set in* $\mathrm{Treealg}(T)$. □

Sikorski's extension criterion takes the following form for pseudo-tree algebras with minimum element.

Proposition 2.17. *Let T be a pseudo-tree with minimum element r. Suppose that f maps $\{T \uparrow t : t \in T\}$ into a BA A. Then*

(i) *f extends to a homomorphism* iff *the following conditions hold:*
   (a) *$f(T \upharpoonright r) = 1$.*
   (b) *If s and t are incomparable elements of T, then $f(T \uparrow s) \cdot f(T \uparrow t) = 0$.*
   (c) *If $s \leq t$, then $f(T \uparrow t) \leq f(T \uparrow s)$.*
(ii) *f extends to a monomorphism* iff (a)–(c) *hold, along with:*
   (d) *If $w \in T$, F is a finite subset of T, and $\forall x \in F[x \not\leq w]$, then $f(T \uparrow w) \cdot \prod_{x \in F} -f(T \uparrow x) \neq 0$.*

Proof. (i): Clearly ⇒ holds. For ⇐, assume that (a)–(c) hold. To apply Sikorski's extension criterion, suppose that $t_0, \ldots, t_{m-1}, s_0, \ldots, s_{n-1}$ are distinct elements of

T and

$$(*)\qquad (T\uparrow t_0)\cap\cdots\cap(T\uparrow t_{m-1})\cap[T\backslash(T\uparrow s_0)]\cap\cdots\cap[T\backslash(T\uparrow s_{n-1})]=\emptyset.$$

We then want to prove that

$$(**)\qquad f(T\uparrow t_0)\cdot\ldots\cdot f(T\uparrow t_{m-1})\cdot -f(T\uparrow s_0)\cdot\ldots\cdot -f(T\uparrow s_{n-1})=0.$$

By Propositions 2.13 and 2.16 we then have three cases.

Case 1. $m=0$ and $s_j=r$ for some $j<n$. Then by (a), $f(T\uparrow s_j)=1$, and $(**)$ follows.

Case 2. There are $i,j<m$ such that t_i and t_j are incomparable. Then by (b), $f(T\uparrow t_i)\cdot f(T\uparrow t_j)=0$, and $(**)$ follows.

Case 3. There are $i<m$ and $j<n$ such that $s_j\le t_i$. Then by (c), $f(T\uparrow t_i)\le f(T\uparrow s_j)$, and $(**)$ follows.

(ii): $\Rightarrow$: Suppose that $w\in T$, F is a finite subset of T, and $\forall x\in F[x\not\le w]$. Then $w\in(T\restriction w)\cap\bigcap_{x\in X}[T\backslash(T\restriction x)]$, so $(T\restriction w)\cap\bigcap_{x\in X}[T\backslash(T\restriction x)]\neq\emptyset$, hence, with f a monomorphism, $f(T\restriction w)\cdot\prod_{x\in X}-f(T\restriction x)\neq 0$, as desired.

$\Leftarrow$: Assume (a)–(d). It suffices now to show that if $(*)$ fails, then so does $(**)$. Assume that $(*)$ fails. Let w be a member of the set in $(*)$. Thus $t_i\le w$ for all $i<m$, and $s_j\not\le w$ for all $j<n$. By (d), $f(T\uparrow w)\cdot\prod_{j<n}-f(T\restriction s_j)\neq 0$, and by (c), $f(T\uparrow w)\le f(T\restriction t_i)$ for all $i<m$. It follows that $f(T\uparrow w)\cdot\prod_{j<n}-f(T\restriction s_j)$ is $\le$ the left side of $(**)$, and so $(**)$ fails. □

The topological dual of pseudo-tree algebras is given in the next theorem.

Theorem 2.18. *Let T be a pseudo-tree with a smallest element r, and let I be the set of all nonempty initial chains of T, i.e., the collection of all nonempty chains C under the tree ordering such that if $s\le t\in C$ then $s\in C$. For each $F\in\mathrm{Ult}(\mathrm{Treealg}(T))$ let $f(F)=\{t\in T:(T\uparrow t)\in F\}$. Then*

(i) *$f[\mathrm{Ult}(A)]=I$.*
(ii) *f is injective.*
(iii) *I is a closed subset of $\mathscr{P}(T)$.*
(iv) *$f:\mathrm{Ult}(\mathrm{Treealg}(T))\to\mathscr{P}(T)$ is continuous.*

Proof. (i): If $F\in\mathrm{Ult}(A)$, then $(T\uparrow r)=1\in F$, and so $r\in f(F)$. So $f(F)$ is nonempty. If $s\le t\in f(F)$, then $(T\uparrow t)\in F$ and $(T\uparrow t)\subseteq(T\uparrow s)$, so $(T\uparrow s)\in F$ and hence $s\in f(F)$. Thus $f(F)\in I$.

Conversely, let $C\in I$. Let $P=\{T\uparrow t:t\in C\}\cup\{T\backslash(T\uparrow t):t\in T\backslash C\}$. Then P has the f.i.p. For, suppose that

$$(*)\qquad (T\uparrow t_0)\cap\cdots\cap(T\uparrow t_{m-1})\cap[T\backslash(T\uparrow s_0)]\cap\cdots\cap[T\backslash(T\uparrow s_{n-1})]=\emptyset,$$

with $t_i\in C$ and $s_j\notin C$ for each $i<m$ and $j<n$. We may assume that $m>0$ since $C\neq\emptyset$. Let t_k be maximum among all t_i's. Then $t_k\in(T\uparrow t_0)\cap\cdots\cap(T\uparrow t_{m-1})$, so by $(*)$, $t_k \in T \uparrow s_j$ for some $j < n$. Hence $s_j \leq t_k$, so $s_j \in C$, contradiction. So P does have f.i.p., and so it is contained in an ultrafilter F. Clearly $f(F) = C$. This proves (i).

(ii): Suppose that $F, G \in \mathrm{Ult}(A)$ with $F \neq G$. Then there is a $t \in T$ such that $T \uparrow t$ is in one of F, G but not the other. Say $(T \uparrow t) \in F \backslash G$. Then $t \in f(F) \backslash f(g)$. So (ii) holds.

(iii): Suppose that $X \in \mathscr{P}(T) \backslash I$. We want to find an open set U in $\mathscr{P}(T)$ such that $X \in U$ and $U \cap I = \emptyset$.

Case 1. There are incomparable $s, t \in X$. Let $U = \{Y \subseteq T : s, t \in Y\}$. Then U is as desired.

Case 2. All members of X are comparable, but there exist $s, t \in T$ such that $s \leq t \in X$ and $s \notin X$. Let $U = \{Y \subseteq T : t \in Y \text{ and } s \notin Y\}$; U is as desired.

Case 3. $X = \emptyset$. Let $U = \{Y \subseteq T : r \notin Y\}$.

(iv): We take a basic open set U in $\mathscr{P}(T)$. Say $U = \{Y \subseteq T : H \subseteq Y$ and $G \cap Y = \emptyset\}$, where H and G are finite subsets of T. Suppose that $F \in f^{-1}[U]$. Thus $f(F) \in U$, so that $H \subseteq f(F)$ and $G \cap f(F) = \emptyset$. Thus $\bigcap_{t \in H}(T \uparrow t) \cap \bigcap_{s \in G}[T \backslash (T \uparrow s)] \in F$, so

$$F \in \mathcal{S}\left(\bigcap_{t \in H}(T \uparrow t) \cap \bigcap_{s \in G}[T \backslash (T \uparrow s)]\right) \subseteq U. \qquad \square$$

Now we give a theorem which is deeper than the ones presented so far. The hard part, (ii)$\Rightarrow$(i), is due to Purisch and Nikiel, who proved it topologically. The present purely algebraic proof is due to Heindorf [97]. For the easy part we need two simple lemmas.

Lemma 2.19. *If T is a pseudo-tree and $t \in T$, then* $(\mathrm{Treealg}(T)) \restriction (T \uparrow t)$ *is the same as* $\mathrm{Treealg}(T \uparrow t)$. $\square$

Lemma 2.20. *If T is a finite tree with a least element r, then* $\mathrm{Treealg}(T)$ *can be embedded in an interval algebra. In fact, there is an isomorphism f from* $\mathrm{Treealg}(T)$ *into an interval algebra* $\mathrm{Intalg}(L)$ *with the following properties:*

(i) *For each $t \in T$, $f(T \uparrow t)$ has the form $[c_t, d_t)$ with $c_t < d_t \leq \infty$.*

(ii) *If $s < t$, then $c_s < c_t$ and $d_t < d_s$.*

(iii) *If s and t are incomparable, then $d_t < c_s$ or $d_s < c_t$.*

Proof. We prove this by induction on $|T|$. It is clear if $|T| = 1$. Now assume inductively that $|T| > 1$. Let $s_0, \ldots, s_{m-1}$ be all of the immediate successors of r. By the inductive hypothesis, for each $i < m$ let f_i be an isomorphism of $\mathrm{Treealg}(T \uparrow s_i)$ into an interval algebra $\mathrm{Intalg}(L_i)$ satisfying the conditions of the lemma. We may assume that $L_i \cap L_j = \emptyset$ if $i \neq j$. Let $t_0, \ldots, t_m$ be new elements. Let

$$M = \bigcup_{i<m} L_i \cup \{t_i : i \leq m\}$$

with the following ordering:

$$t_0 < L_0 < t_1 < L_1 < \cdots < L_{m-1} < t_m$$

(understood in the natural way), with each L_i being ordered in its original way. Then we define $g : \{T \uparrow u : u \in T\} \to M$ as follows: $g(T) = [t_0, \infty)$ (the 1 of Intalg(M)), while if $s_i \leq u$, then $g(T \uparrow u) = f_i(T \uparrow u)$. Note that $g(T \uparrow s_i) = f_i(T \uparrow s_i) = [0_{L_i}, \infty_i) = [0_{L_i}, t_{i+1})$. Now we want to check the conditions (a)–(d) of Proposition 2.17 as well as conditions (i)–(iii) of our lemma. Conditions (a) and (i) are clear. Conditions (b) and (c) follow from (iii) and (ii) respectively. So we want to check conditions (d), (ii), and (iii).

For (iii), suppose that $u, v \in T$ and u and v are incomparable. Choose $i, j < m$ such that $s_i \leq u$ and $s_j \leq v$. Then $g(T \uparrow u) = f_i(T \uparrow u) = [c_u, d_u)$ and $g(T \uparrow v) = f_j(T \uparrow v) = [c_v, d_v)$. If $i = j$, then $d_u < c_v$ or $d_v < c_u$ by our assumption on f_i. Suppose $i \neq j$; say $i < j$. Then $d_u < c_v$. Thus (iii) holds.

For (ii), suppose that $u < v$. If $u = r$, say $s_i \leq v$. Then $g(T \uparrow u) = [t_0, \infty)$ and $g(T \uparrow v) = f_i(T \uparrow v) = [c_v, d_v)$, and $t_0 < c_v$ and $d_v \leq t_{i+1} < \infty$. If $u \neq r$, the conclusion is clear. So (ii) holds.

For (d), suppose that $w \in T$, F is a finite subset of T, and $\forall x \in F[x \not\leq w]$.

Case 1. $w = r$. Now clearly for all $x \in F$ we have $r \notin g(T \uparrow x)$. Hence $w \in \bigcap_{x \in F}[M \backslash g(T \uparrow x)]$, as desired.

Case 2. $r < w$. Say $s_i \leq w$. Now take any $x \in F$. Thus $g(T \uparrow x) = f_i(T \uparrow x) = [c_x, d_x)$. We claim that $c_w \notin g(T \uparrow x)$.

Subcase 2.1. $w < x$. By (ii) for f_i, $c_w < c_x$, so $c_w \notin g(T \uparrow x)$.

Subcase 2.2. x and w are incomparable. Then by (b), $g(T \uparrow w)$ and $g(T \uparrow x)$ are disjoint. Since $c_w \in g(T \uparrow w)$, it follows that $c_w \notin g(T \uparrow x)$.

This proves our claim. Hence $c_w \in g(T \uparrow w) \cap \bigcap_{x \in F}[M \backslash g(T \uparrow x)]$, as desired for (d). $\square$

Theorem 2.21. *For any BA A the following are equivalent:*

(i) *A is isomorphic to a pseudo-tree algebra.*
(ii) *A can be isomorphically embedded into an interval algebra.*

Proof. (i)⇒(ii): Let T be a pseudo-tree; by Theorem 2.9 we may assume that it has a minimum element 0_T. We consider a first-order language with a binary relation symbol $<$, an individual constant 0, and for each $t \in T \backslash \{0_T\}$ two individual constants $\mathbf{a}_t, \mathbf{b}_t$. Let Σ be a set of first-order sentences expressing that in any model $\overline{A}$ of Σ the following hold:

$(A, <)$ is a linear order with first element $0^{\overline{A}}$.
$0^{\overline{A}} < \mathbf{a}_t^{\overline{A}} < \mathbf{b}_t^{\overline{A}}$ for all $t \in T \backslash \{0_T\}$.
$\mathbf{a}_t < \mathbf{a}_s$ and $\mathbf{b}_s < \mathbf{b}_t$ for $0_T < t < s$ in T.
$\mathbf{b}_t < \mathbf{a}_s$ or $\mathbf{b}_s < \mathbf{a}_t$ if $s, t \in T \backslash \{0_T\}$ and s, t are incomparable.

To show that Σ has a model, take any finite subset Σ_0 of Σ. Then there is a finite subset T_0 of T such that $0_T \in T_0$ and for every $t \in T$, if $\mathbf{a}_t$ or $\mathbf{b}_t$ occurs in some member of Σ_0, then $t \in T_0$. We may assume in fact that if one of $\mathbf{a}_t$ or $\mathbf{b}_t$ occurs in some member of Σ_0, the so does the other.

By Lemma 2.20, let f be an isomorphic embedding of T_0 into an interval algebra Intalg(L) satisfying the additional conditions of the lemma. A model of Σ_0 is formed as follows. The universe is L and the ordering of L is assigned to $<$. For each t for which $\mathbf{a}_t$ occurs in Σ_0 let $\mathbf{a}_t$ have the value c_t and $\mathbf{b}_t$ the value d_t. Then the conditions of Lemma 2.20 shows that we have a model of Σ_0.

Thus by the compactness theorem we obtain a model $\overline{A}$ of Σ. Now we define a function f mapping Treealg(T) into Intalg$(A,<)$. Let $f(T) = A$. For $t \in T\backslash\{0_T\}$ let $f(T \uparrow t) = [a_t^{\overline{A}}, b_t^{\overline{A}})$. Now we check the conditions of Proposition 2.17 to see that f is an isomorphic embedding. In fact, (a)–(c) are clear. Now assume the hypotheses of (d).

Case 1. $w = 0_T$. Then for all $x \in F$ we have $0^{\overline{A}} < \mathbf{a}_x^{\overline{A}}$, and so $0_{\overline{A}} \in [A\backslash[\mathbf{a}_x^{\overline{A}}, \mathbf{b}_x^{\mathfrak{A}})] = [A\backslash f(T \uparrow x)]$, as desired.

Case 2. $w \neq 0_T$. Take any $x \in F$.

Subcase 2.1. $w < x$. Then $\mathbf{a}_w^{\overline{A}} < \mathbf{a}_x^{\overline{A}}$, so $\mathbf{a}_w^{\overline{A}} \notin [\mathbf{a}_x^{\overline{A}}, \mathbf{b}_x^{\overline{A}}) = f(T \uparrow x)$.

Subcase 2.2. w and x are incomparable. Then $\mathbf{b}_w^{\overline{A}} < \mathbf{a}_x^{\overline{A}}$ or $\mathbf{b}_x^{\overline{A}} < \mathbf{a}_w^{\overline{A}}$, so again $\mathbf{a}_w^{\overline{A}} \notin [\mathbf{a}_x^{\overline{A}}, \mathbf{b}_x^{\overline{A}}) = f(T \uparrow x)$.

It follows that $f(T \uparrow w) \cap \bigcap_{x \in F}[A\backslash(T \uparrow x)] \neq \emptyset$.

(ii)⇒(i): Let B be the interval algebra on a linear order L which has first element 0, and let A be a subalgebra of B; we want to show that A is isomorphic to a pseudo-tree algebra. By Theorem 2.15 it suffices to show that A is generated by some ramification set.

The proof uses the symmetric difference operation extensively, and we start with some facts about that operation on elements of B.

(1) If $0 < a_0 < a_1 < \cdots < a_{n-1} \leq \infty$ in L $(n > 0)$, then

$$\begin{aligned}
&[0, a_0)\triangle[0, a_1)\triangle \cdots \triangle[0, a_{n-1}) \\
&\quad = \begin{cases} [a_0, a_1) \cup [a_2, a_3) \cup \cdots \cup [a_{n-2}, a_{n-1}) & \text{if } n \text{ is even,} \\ [0, a_0) \cup [a_1, a_2) \cup \cdots \cup [a_{n-2}, a_{n-1}) & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

We prove (1) by induction on n. It is clear for $n = 1$. Assume it for $n - 1$, with $n > 1$. Then for n even we have

$$\begin{aligned}
&[0, a_0)\triangle[0, a_1)\triangle \cdots \triangle[0, a_{n-1}) \\
&\quad = \Big([0, a_0)\triangle[0, a_1)\triangle \cdots \triangle[0, a_{n-2})\Big)\triangle[0, a_{n-1}) \\
&\quad = \Big([0, a_0) \cup [a_1, a_2) \cup \cdots \cup [a_{n-3}, a_{n-2})\Big)\triangle[0, a_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&= [0, a_{n-1})\backslash\Big([0, a_0) \cup [a_1, a_2) \cup \cdots \cup [a_{n-3}, a_{n-2})\Big)\\
&= [a_0, a_1) \cup [a_2, a_3) \cup \cdots \cup [a_{n-2}, a_{n-1}).
\end{aligned}
$$

For n odd we have

$$
\begin{aligned}
&[0, a_0)\triangle[0, a_1)\triangle \cdots \triangle[0, a_{n-1})\\
&\quad= \Big([0, a_0)\triangle[0, a_1)\triangle \cdots \triangle[0, a_{n-2})\Big)\triangle[0, a_{n-1})\\
&\quad= \Big([a_0, a_1) \cup [a_2, a_3) \cup \cdots \cup [a_{n-3}, a_{n-2})\Big)\triangle[0, a_{n-1})\\
&\quad= [0, a_{n-1})\backslash\Big([a_0, a_1) \cup [a_2, a_3) \cup \cdots \cup [a_{n-3}, a_{n-2})\Big)\\
&\quad= [0, a_0) \cup [a_1, a_2) \cup \cdots \cup [a_{n-2}, a_{n-1}).
\end{aligned}
$$

Thus (1) holds. Now let $K = \{[0, a) : 0 < a \leq \infty \text{ in } L\}$. Thus K generates B.

(2) If $k_0, \ldots, k_{n-1}$ are distinct elements of K, then $k_0\triangle k_1\triangle \cdots \triangle k_{n-1} \neq \emptyset$.

This is clear from (1).

(3) Every element x of B can be written in the form $x = k_0\triangle k_1\triangle \cdots \triangle k_{n-1}$ for distinct elements $k_0, \ldots, k_{n-1}$ of K. This form is unique, in the sense that if also $x = l_0\triangle \cdots \triangle l_{m-1}$ with distinct $l_0, \ldots, l_{m-1} \in K$, then $m = n$ and $\{k_i : i < n\} = \{l_i : i < n\}$.

To prove existence in (3) it suffices to show that the set of elements x which can be written in the indicated form contains K and is closed under $-$ and $\cap$. Of course each $k \in K$ can be written in the indicated form. For closure under $-$, suppose that x is written as in (3). Then

$$
-x = -(k_0\triangle k_1\triangle \cdots \triangle k_{n-1}) = [0, \infty)\triangle k_0\triangle k_1\triangle \cdots \triangle k_{n-1},
$$

and this is of the correct form if each $k_i \neq [0, \infty)$, while if $k_i = [0, \infty)$, then $-x$ is the sum of all the k_j such that $j \neq i$.

For closure under $\cdot$, suppose that x is as in (1), and $y = l_0\triangle l_1\triangle \cdots \triangle l_{m-1}$. Then

$$
\begin{aligned}
x \cap y &= (k_0\triangle k_1\triangle \cdots \triangle k_{n-1}) \cap (l_0\triangle l_1\triangle \cdots \triangle l_{m-1})\\
&= (k_0 \cap l_0)\triangle(k_1 \cap l_1)\triangle \cdots \triangle(k_0 \cap l_{m-1})\\
&\quad \triangle(k_1 \cap l_0)\triangle(k_1 \cap l_1)\triangle \cdots\\
&\quad \cdots \triangle(k_{n-1} \cap l_0)\triangle \cdots \triangle(k_{n-1} \cap l_{m-1}).
\end{aligned}
$$

Now each element $k_i \cap l_j$ is a member of K, or is $\emptyset$. One can omit $\emptyset$ in the above expression, and also omit two equal terms. So doing all these simplifications, we arrive at the desired expression for $x \cap y$. This finishes the proof of existence in (3).

For uniqueness, suppose that x is written in two different ways, as in the formulation of (3). Then

$$0 = x\triangle x = k_0\triangle k_1\triangle\cdots k_{n-1}\triangle l_0\triangle l_1\triangle\cdots\triangle l_{m-1};$$

from (2) it then follows that $\{k_i : i < n\} = \{l_i : i < m\}$ and hence also $m = n$.

By (3), we can associate with each $x \in B$ the set $\tau(x) = \{k_0, \ldots, k_{n-1}\}$ given in (3).

(4) $\tau(x\triangle y) = \tau(x)\triangle\tau(y)$ for all $x, y \in B$.

In fact, it is clear that k is in the representation according to (3) of $x\triangle y$ iff it is in exactly one of the representations of x and y, and this gives (4).

(5) $\tau(x \cap y) \subseteq \tau(x) \cup \tau(y)$.

In fact, the above calculation of the representation of $x \cap y$ essentially shows this, as each $k_i \cap l_j$ is one of k_i or l_j.

Now we can start on the real proof. Let $\mathscr{A} = \{M \subseteq K : M \cap \tau(x) \neq \emptyset$ for all $x \in A^+\}$. Note that $\mathscr{A}$ is nonempty, since for example $\{[0,\infty)\} \in \mathscr{A}$. If $\mathscr{B}$ is a chain of members of $\mathscr{A}$ under $\subseteq$, then $\bigcap\mathscr{B} \in \mathscr{A}$. In fact, suppose not: so there is an $x \in A^+$ such that $\bigcap\mathscr{B} \cap \tau(x) = \emptyset$. It follows that $\mathscr{B}$ does not have a least element. Let $\langle C_\xi : \xi < \kappa\rangle$ be a strictly decreasing sequence of members of $\mathscr{B}$ coinitial in $\mathscr{B}$, κ an infinite cardinal. For each $\xi < \kappa$ there is a $z_\xi \in C_\xi \cap \tau(x)$. Since $\tau(x)$ is finite, there is a w such that $z_\xi = w$ for cofinally many $\xi < \kappa$. But then $w \in \bigcap\mathscr{B} \cap \tau(x)$, contradiction. So, we have checked the hypothesis of Zorn's lemma for $(\mathscr{A}, \supseteq)$. By Zorn's lemma, let M be a minimal member of $\mathscr{A}$ under $\subseteq$.

(6) If $x, y \in A$ and $x \neq y$, then $M \cap \tau(x) \neq M \cap \tau(y)$.

In fact, otherwise $M \cap (\tau(x)\triangle\tau(y)) = \emptyset$, so by (4) $M \cap \tau(x\triangle y) = \emptyset$, and $x\triangle y \neq \emptyset$ since $x \neq y$, contradicting the definition of M.

(7) For each $m \in M$ there is a unique $r_m \in A$ such that $M \cap \tau(r_m) = \{m\}$.

In fact, by the minimality of M we have $(M\backslash\{m\}) \cap \tau(x) = \emptyset$ for some nonempty $x \in A$. On the other hand, $M \cap \tau(x) \neq \emptyset$; so $M \cap \tau(x) = \{m\}$. Thus we can let $r_m = x$. It is unique by (6).

Let $R = \{r_m : m \in M\}$. We claim that R is the desired generating ramification set for A.

To see that R generates A, take any $a \in A^+$. Then $M \cap \tau(a)$ is a nonempty finite subset of M; say $M \cap \tau(a) = \{m_0, \ldots, m_{n-1}\}$ with the m_i's distinct. Now also

$$\begin{aligned}
M \cap \tau(r_{m_0}\triangle\cdots\triangle r_{m_{n-1}}) &= M \cap (\tau(r_{m_0})\triangle\cdots\triangle\tau(r_{m_{n-1}})) \quad \text{by (4)}\\
&= [M \cap \tau(r_{m_0})]\triangle\cdots\triangle[M \cap \tau(r_{m_{n-1}})]\\
&= \{m_0\}\triangle\cdots\triangle\{m_{n-1}\}\\
&= \{m_0, \ldots, m_{n-1}\}.
\end{aligned}$$

Thus $M \cap \tau(a) = M \cap \tau(r_{m_0} \triangle \cdots \triangle r_{m_{n-1}})$, and so by (6), $a = r_{m_0} \triangle \cdots \triangle r_{m_{n-1}}$. This shows that R generates A.

To see that R is a ramification set, take any two distinct $m, n \in M$. By (5), we have

$$M \cap \tau(r_m \cap r_n) \subseteq [M \cap \tau(r_m)] \cup [M \cap \tau(r_n)] = \{m, n\}.$$

Thus it follows that $M \cap \tau(r_m \cap r_n) \in \{\emptyset, \{m\}, \{n\}, \{m, n\}\}$.

Case 1. $M \cap \tau(r_m \cap r_n) = \emptyset$. Then by the definition of M it follows that $r_m \cap r_n = \emptyset$, as desired.

Case 2. $M \cap \tau(r_m \cap r_n) = \{m\}$. Then $M \cap \tau(r_m \cap r_n) = M \cap \tau(r_m)$, so by (6), $r_m \cap r_n = r_m$.

Case 3. $M \cap \tau(r_m \cap r_n) = \{n\}$. Similar to Case 2.

Case 4. $M \cap \tau(r_m \cap r_n) = \{m, n\}$. Then

$$\begin{aligned} M \cap \tau(r_m \cap r_n) &= \{m, n\} = [M \cap \tau(r_m)] \triangle [M \cap \tau(r_n)] \\ &= M \cap [\tau(r_m) \triangle \tau(r_n)] = M \cap \tau(r_m \triangle r_n), \end{aligned}$$

and so by (6), $r_m \cap r_n = r_m \triangle r_n$, which implies that $r_m = r_n = \emptyset$, contradicting (7). □

Theorem 2.22. *If A and B are pseudo-tree algebras, then $A \times B$ is isomorphic to a pseudo-tree algebra.*

Proof. Say $A = \text{Treealg}(T_1)$ and $B = \text{Treealg}(T_2)$, where T_1 and T_2 are pseudo-trees with least elements t_1, t_2 respectively, and $T_1 \cap T_2 = \emptyset$. Then $T_1 \cup T_2$ is a pseudo-tree with the ordering $<_{T_1} \cup <_{T_2}$. Moreover, $\text{Treealg}(T_1 \cup T_2) \restriction (\text{Treealg}(T_1 \cup T_2) \uparrow t_1) = \text{Treealg}(T_1)$, and similarly for t_2 and T_2. □

Theorem 2.23. *Every homomorphic image of a pseudo-tree algebra is isomorphic to a pseudo-tree algebra.*

Proof. Let A be a pseudo-tree algebra, and f a homomorphism from A onto some algebra B. Then by Theorem 2.15, A is generated by a ramification set R. Clearly $f[R]$ is also a ramification set, and it generates B. Hence by Theorem 2.15 again, B is isomorphic to a pseudo-tree algebra. □

Simple extensions of Boolean algebras

Given BAs A and B, we call B a *simple extension* of A provided that A is a subalgebra of B and $B = \langle A \cup \{x\} \rangle$ for some $x \in B$; then we write $B = A(x)$. We recall that each element of $A(x)$ can be written in the form $a \cdot x + b \cdot -x$ with $a, b \in A$; or in the form $c + a \cdot x + b \cdot -x$ with a, b, c pairwise disjoint elements of A. We now introduce some important ideals for studying simple extensions. If A is a subalgebra of B and $x \in B$, we let $A \restriction x = \{a \in A : a \leq x\}$. This is a slight extension of the usual notion; x is not necessarily in A. Under the same assumptions we let

$$\mathrm{Smp}_x^A = \langle (A \restriction x) \cup (A \restriction -x) \rangle^{\mathrm{Id}},$$

the ideal in A generated by $(A \restriction x) \cup (A \restriction -x)$. The three ideals $A \restriction x$, $A \restriction -x$, and Smp_x^A are important for studying simple extensions.

Proposition 2.24. *Let $A(x)$ be a simple extension of A. Then $A = A(x)$ iff $x \in \mathrm{Smp}_x^A$.*

Proof. If $A = A(x)$, then $x \in A \restriction x \subseteq \mathrm{Smp}_x^A$, as desired. Conversely, suppose that $x \in \mathrm{Smp}_x^A$. Write $x = a + b$, with $a \in A \restriction x$ and $b \in A \restriction -x$. Clearly $b = 0$, so $x = a \in A$. □

Proposition 2.25. *Let $A(x)$ be a simple extension of A, and let $a \in A$. Then the following conditions are equivalent:*

(i) *$a \in \mathrm{Smp}_x^A$;*
(ii) *$a = b + c$ for some $b \in A \restriction x$ and $c \in A \restriction -x$;*
(iii) *$a \cdot x \in A$;*
(iv) *$a \cdot -x \in A$;*
(v) *For all $y \in A(x)$, if $y \leq a$ then $y \cdot x \in A$ and $y \cdot -x \in A$, and so $y \in A$ and $A(x) \restriction a = A \restriction a$;*
(vi) *For all $y \in A(x)$, if $y \leq a$ then $y \in \mathrm{Smp}_x^A$.*

Proof. Clearly (i)⇔(ii). Assume (ii). Then $a \cdot x = b \in A$, i.e., (iii) holds. Assume (iii). Then $a \cdot -x = a \cdot -(a \cdot x) \in A$, i.e., (iv) holds. Similarly (iv)⇒(iii). If (iii) holds, then (iv) holds and $a = a \cdot x + a \cdot -x$, so (ii) holds. (ii)⇒(v): Assume (ii), and suppose that $y \in A(x)$ and $y \leq a$. Write $y = u \cdot x + v \cdot -x$ with $u, v \in A$. Then

$$y \cdot x = a \cdot y \cdot x = a \cdot u \cdot x \in A$$

by (iii). Similarly $y \cdot -x \in A$, so $y \in A$. Clearly (v) implies (vi) and (vi) implies (i). □

Corollary 2.26. *If $A(x)$ is a simple extension of A, then Smp_x^A is an ideal of $A(x)$.*

Proposition 2.27. *Suppose that $A(x)$ and $A(y)$ are simple extensions of A. Then the following conditions are equivalent:*

(i) *There is a homomorphism from $A(x)$ into $A(y)$ which is the identity on A and sends x to y.*
(ii) *$A \restriction x \subseteq A \restriction y$ and $A \restriction -x \subseteq A \restriction -y$.*

And also the following two conditions are equivalent:

(iii) *There is an isomorphism from $A(x)$ onto $A(y)$ which is the identity on A and sends x to y.*
(iv) *$A \restriction x = A \restriction y$ and $A \restriction -x = A \restriction -y$.*

Proof. (i)⇒(ii): obvious. (ii)⇒(i): by Sikorski's extension criterion. Since the homomorphism of (i) is unique, the equivalence of (iii) and (iv) is clear. □

Proposition 2.28. *Let A be a BA, and let I_0 and I_1 be two ideals of A such that $I_0 \cap I_1 = \{0\}$. Then there is a simple extension $A(x)$ of A such that $A \upharpoonright x = I_0$ and $A \upharpoonright -x = I_1$.*

Proof. Let $A(y)$ be a free extension of A by y, that is, let $A(y)$ be the free product of A with a four-element BA B, where $0 < y < 1$ in B. Consider the following ideal K of $A(y)$:

$$K = \langle \{a \cdot -y : a \in I_0\} \cup \{a \cdot y : a \in I_1\} \rangle^{\mathrm{Id}}.$$

Let f be the natural mapping from A onto A/K. It suffices to prove the following things:

(1) $A \cap K = \{0\}$.

(2) $(A/K) \upharpoonright (y/K) = f[I_0]$.

(3) $(A/K) \upharpoonright -(y/K) = f[I_1]$.

For (1), if $a \in A \cap K$, then $a \leq b \cdot -y + c \cdot y$ for some $b \in I_0$ and $c \in I_1$. An easy argument using freeness then yields $a \leq b \cdot c = 0$, as desired.

For (2), first suppose that $a \in A$ and $a/K \leq y/K$. Thus $a \cdot -y \in K$, so $a \cdot -y \leq b \cdot -y + c \cdot y$ for some $b \in I_0$ and $c \in I_1$. An easy argument then yields $a \leq b$, and hence $a \in I_0$, as desired. On the other hand, if $a \in I_0$, obviously $a \cdot -y \in K$, and hence $a/K \leq y/K$ and $a/K \in (A/K) \upharpoonright (y/K)$, as desired.

A similar argument proves (3). □

Proposition 2.29. *Let A be a BA, and let I_0 and I_1 be two ideals of A such that $I_0 \cap I_1 = \{0\}$. Suppose that the simple extension $A(x)$ of A is such that $A \upharpoonright x = I_0$ and $A \upharpoonright -x = I_1$.*

Then $A(x) \cong (A/I_0) \times (A/I_1)$.

Proof. For $a, b \in A$ define $f(a \cdot x + b \cdot -x) = ([b]_{I_0}, [a]_{I_1})$. Then f is well defined, since from $a \cdot x + b \cdot -x = a' \cdot x + b' \cdot -x$ it follows that $a \triangle a' \leq -x$, hence $[a]_{I_1} = [a']_{I_1}$; similarly $[b]_{I_0} = [b']_{I_0}$.

Clearly now f is a homomorphism. f is one-one, for if $[a]_{I_1} = 0 = [b]_{I_0}$, then $a \in I_1$ and $b \in I_0$, hence $a \leq -x$ and $b \leq x$, so $a \cdot x + b \cdot -x = 0$. Finally, clearly f is onto. □

Proposition 2.30. *If A is superatomic, then so is $A(x)$.*

Proof. This follows from Proposition 2.29, since a product of two superatomic BAs is superatomic. □

Proposition 2.31. *Suppose that $A(x)$ is a simple extension of A, and K is an ideal in $A(x)$.*

(i) *There is an isomorphism f of $A/(K \cap A)$ into $A(x)/K$ such that $f([a]_{K\cap A}) = [a]_K$ for any $a \in A$.*
(ii) *$A(x)/K$ is a simple extension of* $\mathrm{rng}(f)$ *by the element $[x]_K$.*
(iii) *$A(x)/K$ is isomorphic to a simple extension of $A/(K \cap A)$.*

Proof. (i) is clear. For (ii), note that any element of $A(x)/K$ can be written in the form $[a \cdot x + b \cdot -x]_K$ with $a, b \in A$; and

$$[a \cdot x + b \cdot -x]_K = [a]_K \cdot [x]_K + [b]_K \cdot -[x]_K = f([a]_{K\cap A}) \cdot [x]_K + f([b]_{K\cap A}) \cdot -[x]_K.$$

(iii) follows from (i) and (ii). □

Minimal extensions of Boolean algebras

We say that B is a *minimal extension* of a BA A, in symbols $A \leq_m B$, if B *is* an extension of A and there is no subalgebra C of B such that $A \subset C \subset B$. Clearly then B is a simple extension of A. This notion is studied in Koppelberg [89a]. First we want to see what this means in terms of the ideal Smp^A_x:

Proposition 2.32. *Let $A(x)$ be a simple extension of A. Then the following conditions are equivalent:*

(a) *$A \leq_m A(x)$.*
(b) Smp^A_x *is either equal to A or is a maximal ideal of A.*
(c) *$A = \langle\{a \in A : a$ is comparable with $x\}\rangle$.*
(d) *There is a $G \subseteq A$ which generates A and consists exclusively of elements comparable with x.*
(e) *If $y \in A(x)\backslash A$, then $x \triangle y \in A$.*

Proof. Obviously (c)⇔(d). (a)⇒(b): assume that (b) fails. Then there is an element $a \in A$ such that neither a nor $-a$ is in Smp^A_x. Then, we claim, $A \subset A(a \cdot x) \subset A(x)$. In fact, $a \cdot x \notin A$ by Proposition 2.25. And if $x \in A(a \cdot x)$, then we can write $x = b \cdot a \cdot x + c \cdot -(a \cdot x)$ with $b, c \in A$. But then $x = b \cdot a \cdot x + c \cdot -a + c \cdot -x$, hence $c \cdot -x = 0$ and $x = b \cdot a \cdot x + c \cdot -a$. Therefore $-a \cdot x = c \cdot -a \in A$, and hence by Proposition 2.25, $-a \in \mathrm{Smp}^A_x$, contradiction.

(b)⇒(c): Assume (b). Then Smp^A_x generates A. Let $G = \{a \in A : a$ is comparable with $x\}$. Now $A \upharpoonright x \subseteq G$, and if $a \in A \upharpoonright -x$ then $-a \in G$. It follows that $A = \langle \mathrm{Smp}^A_x \rangle \subseteq \langle G \rangle$, and hence G generates A.

(d)⇒(b): Clearly $G \subseteq \mathrm{Smp}^A_x \cup \{a : -a \in \mathrm{Smp}^A_x\}$, and the latter set is a subalgebra of A; hence it is all of A, which means that (b) holds.

(b)⇒(e): Assume (b), and suppose that $y \in A(x)\backslash A$. Write $y = a + b \cdot x + c \cdot -x$, where a, b, c are pairwise disjoint elements of A. If b and c are both elements of Smp^A_x, then $b \cdot x$ and $c \cdot -x$ are both elements of A by Proposition 2.25, and so

$y \in A$, contradiction. Assume that $b \notin \mathrm{Smp}_x^A$. Hence $-b \in \mathrm{Smp}_x^A$ by (b). Now $y \cdot b = x \cdot b$, so $x \triangle y \leq -b$, and hence $x \triangle y \in A$ by Corollary 2.26. If $c \notin \mathrm{Smp}_x^A$, we obtain $(-x)\triangle y \in A$ similarly; then note that $(-x)\triangle y = -(x \triangle y)$.

(e)⇒(a): If $y \in A(x)\backslash A$, then $x = x\triangle y \triangle y \in A(y)$, so $A(y) = A(x)$. □

Proposition 2.33. *If $A \leq B$, $x \in B$, and $A \upharpoonright x$ is a maximal ideal in A, then $A \leq_m A(x)$.*

Proof. By Proposition 2.32. □

The notion of minimal extension can also be expressed using extensions of ultrafilters. Here the following general result motivates what follows.

Proposition 2.34. *If A is a proper subalgebra of B, then there is an ultrafilter F on A such that F does not generate an ultrafilter on B.*

Proof. Let $b \in B\backslash A$. We claim that

$$(*) \qquad \{a \in A : -a \leq b\} \cup \{a \in A : -a \leq -b\}$$

has fip. Otherwise, there are $a_0, a_1 \in A$ such that $-a_0 \leq b$, $-a_1 \leq -b$, and $a_0 \cdot a_1 = 0$. Then $-b \leq a_0 \leq -a_1 \leq -b$, so $-b = a_0 \in A$, contradiction.

Let F be an ultrafilter on A containing the set $(*)$. Suppose that F generates an ultrafilter G on B.

Case 1. $b \in G$. Say $a \in F$, $a \leq b$. Then $-a \in (*) \subseteq F$, contradiction.

Case 2. $-b \in G$. Say $a \in F$, $a \leq -b$. Then $-a \in (*) \subseteq F$, contradiction. □

Given $A \leq B$, $A \neq B$, $b \in B\backslash A$, and F an ultrafilter on A, we say that b is *minimal for* (A, F) provided that F is the only ultrafilter on A which does not generate an ultrafilter on $A(b)$.

Proposition 2.35. *Suppose that $A(x)$ is a minimal extension of A, $A \neq A(x)$. Then x is minimal for $(A, -\mathrm{Smp}_x^A)$.*

Proof. First let H be the filter on $A(x)$ generated by $-\mathrm{Smp}_x^A$. Suppose that $x \in H$. Say $a \in -\mathrm{Smp}_x^A$ and $a \leq x$. Then $a \in \mathrm{Smp}_x^A$, hence $1 \in \mathrm{Smp}_x^A$, contradicting Proposition 2.24. Suppose $-x \in H$. Say $a \in -\mathrm{Smp}_x^A$ and $a \leq -x$. Then $-a \in \mathrm{Smp}_x^A$, so $-a \leq b + c$ with $b \leq x$ and $c \leq -x$. Then $x \leq -a \leq b + c$ with $c \cdot x = 0$, so $x \leq b \leq x$. This gives $x = b \in A$, contradicting Proposition 2.24 again. Thus $-\mathrm{Smp}_x^A$ does not generate an ultrafilter on $A(x)$.

Now suppose that G is an ultrafilter on A different from $-\mathrm{Smp}_x^A$. Say $a \in G\backslash -\mathrm{Smp}_x^A$. Since $-\mathrm{Smp}_x^A$ is an ultrafilter on A, it follows that $-a \in -\mathrm{Smp}_x^A$. Thus $a \in \mathrm{Smp}_x^A$, and it follows from the definition of Smp_x^A that x or $-x$ is in the filter generated by G in $A(x)$, so that filter is an ultrafilter. □

Proposition 2.36. *Suppose that $A \leq B$, $A \neq B$, $x \in B\backslash A$, and F is an ultrafilter on A. Suppose that x is minimal for (A, F). Then $F = -\mathrm{Smp}_x^A$.*

Proof. By Proposition 2.24, $\mathrm{Smp}_x^A \neq A$, and so also $-\mathrm{Smp}_x^A \neq A$. Hence it is enough to show that $F \subseteq -\mathrm{Smp}_x^A$. Suppose that $a \in F \backslash -\mathrm{Smp}_x^A$. Then $-\mathrm{Smp}_x^A \cup \{-a\}$ has fip, so it is included in an ultrafilter G. Thus $F \neq G$, so G generates an ultrafilter H in $A(x)$. Suppose that $x \in H$. Choose $b \in G$ such that $b \leq x$. Thus $b \in \mathrm{Smp}_x^A$, so $-b \in -\mathrm{Smp}_x^A \subseteq G$, contradiction. If $-x \in H$, a similar contradiction is obtained. □

Proposition 2.37. *If $A \leq B$, $A \neq B$, $x \in B \backslash A$, F is an ultrafilter on A, and x is minimal for (A, F), then $x, -x \notin \langle F\rangle^{\mathrm{fi}}_{A(x)}$.*

Proof. Otherwise $\langle F\rangle^{\mathrm{fi}}_{A(x)}$ would be an ultrafilter on $A(x)$, contradicting the definition of x minimal for (A, F). □

Proposition 2.38. *Suppose that $A \leq B$, $A \neq B$. Then the following conditions are equivalent:*

(i) *B is a minimal extension of A;*
(ii) $\exists x \in B \backslash A[B = A(x)$ *and* $\exists F \in \mathrm{Ult}A(x$ *is minimal for* $(A, F))]$;
(iii) $\forall x \in B \backslash A[B = A(x)$ *and* $\exists F \in \mathrm{Ult}A(x$ *is minimal for* $(A, F))]$;
(iv) $\forall x \in B \backslash A[B = A(x)$ *and* $-\mathrm{Smp}_x^A$ *is an ultrafilter on* $A]$;
(v) $\exists x \in B \backslash A[B = A(x)$ *and* $-\mathrm{Smp}_x^A$ *is an ultrafilter on* $A]$.
(vi) $\forall x \in B \backslash A[B = A(x)$ *and x is minimal for* $(A, -\mathrm{Smp}_x^A)]$.
(vii) $\exists x \in B \backslash A[B = A(x)$ *and x is minimal for* $(A, -\mathrm{Smp}_x^A)]$.

Proof. (i)⇒(iv): Let $x \in B \backslash A$. Then $B = A(x)$ since B is a minimal extension of A. By Proposition 2.32, $-\mathrm{Smp}_x^A$ is an ultrafilter on A.

(iv)⇒(v): obvious.

(v)⇒(i): by Proposition 2.32.

(i)⇒(iii): Propositions 2.32, 2.35.

(iii)⇒(ii): obvious.

(ii)⇒(v): Proposition 2.36.

(i)⇒(vi): Proposition 2.35.

(vi)⇒(vii): obvious.

(vii)⇒(ii): obvious. □

Proposition 2.39. *Suppose that $A \subseteq B$, $A \neq B$, $x \in B \backslash A$, and F is an ultrafilter on A. Then the following conditions are equivalent:*

(i) $\{a \in A : a \cdot x \in A\} = -F$;
(ii) $\{a \in A : a \cdot x \notin A\} = F$;
(iii) *x is minimal for (A, F).*
(iv) $F = -\mathrm{Smp}_x^A$.

Proof. (i)⇒(ii): $\{a \in A : a \cdot x \notin A\} = A \backslash \{a \in A : a \cdot x \in A\} = A \backslash (-F) = F$.

(ii)⇒(i): $\{a \in A : a \cdot x \in A\} = A \backslash \{a \in A : a \cdot x \notin A\} = A \backslash F = -F$.

(ii)⇒(iii): By Proposition 2.34 it suffices to show that every ultrafilter $G \neq F$ on A generates an ultrafilter on $A(x)$. Let G generate the filter H on $A(x)$. Say $a \in G \backslash F$. Thus $a \cdot x \in A$. If $a \cdot x \in G$, then $x \in H$. If $a \cdot x \notin G$, then $a \cdot -(a \cdot x) = a \cdot -x \in G$, so $-x \in H$. So H is an ultrafilter.

(iii)⇒(i): by Proposition 2.36, $F = -\mathrm{Smp}_x^A$, so (i) follows from Proposition 2.25 (the equivalence of (i) and (iii) there).

(iii)⇒(iv): by Proposition 2.36.

(iv)⇒(iii): By Proposition 2.35. □

Proposition 2.40. *Suppose that $A \subseteq B$, $A \neq B$, $x \in B \backslash A$, and F is an ultrafilter on A. Then x is minimal for (A, F)* iff *$-x$ is minimal for (A, F).*

Proof. Note that $\mathrm{Smp}_x^A = \mathrm{Smp}_{-x}^A$. So if x is minimal for (A, F), then by Proposition 2.36, $F = -\mathrm{Smp}_x^A = -\mathrm{Smp}_{-x}^A$. Since $A(x) = A(-x)$, it follows that $-x$ is minimal for (A, F). The converse is similar. □

Proposition 2.41. *Suppose that $A \leq C \leq B$, G is an ultrafilter on A, and G generates a filter F on C which is an ultrafilter. Suppose that $x \in B \backslash C$ and it is minimal for (A, G). Then it is also minimal for (C, F).*

Proof. We have $G = -\mathrm{Smp}_x^A$ by Proposition 2.36. We now show that $F = -\mathrm{Smp}_x^C$. Since $C \neq C(x)$, by Proposition 2.24 $\mathrm{Smp}_x^C \neq C$. Hence $-\mathrm{Smp}_x^C$ is a proper filter of C, so it suffices to show that $F \subseteq -\mathrm{Smp}_x^C$.

So, suppose that $a \in F$. Choose $b \in G$ such that $b \leq a$. then $b \in -\mathrm{Smp}_x^A$, so $-b \in \mathrm{Smp}_x^A$. Thus $-b \cdot x \in A$ by Proposition 2.25. Then $-a \leq -b$, so $-a \cdot x = -a \cdot -b \cdot x \in C$. Thus $-a \in \mathrm{Smp}_x^C$ by Proposition 2.25, so $a \in -\mathrm{Smp}_x^C$. □

Proposition 2.42. *Let $A \leq B$, let $f : B \to Q$ be an epimorphism, and set $P = f[A]$. Then $A \leq_m B$ implies that $P \leq_m Q$.*

If, moreover, $\ker(f) \subseteq A$, *then $A \leq_m B$* iff *$P \leq_m Q$, and $A \neq B$* iff *$P \neq Q$.*

Proof. This is a result of universal-algebraic nonsense: the function assigning to each subalgebra C of Q the subalgebra $f^{-1}[C]$ of B is one-one, and it maps $\{C : P \leq C \leq Q\}$ into $\{D : A \leq D \leq B\}$. In case $\ker(f) \subseteq A$, it maps onto the latter set: the preimage of such a D is $f[D]$, and $P \leq f[D] \leq Q$. For the final statement, clearly $P \neq Q$ implies that $A \neq B$, even without the assumption $\ker(f) \subseteq A$. Now suppose that $A \neq B$; say $b \in B \backslash A$. If $f(b) \in P$, then there is an $a \in A$ such that $f(b) = f(a)$. So $f(b \triangle a) = 0$, hence $b \triangle a \in \ker(f) \subseteq A$ and $b = b \triangle a \triangle a \in A$, contradiction. □

Proposition 2.43. *Assume that $A \leq B \leq M \geq D$. Set $P = A \cap D$ and $Q = B \cap D$. Then $A \leq_m B$ implies that $P \leq_m Q$.*

Proof. Assume all the hypotheses. We may also assume that $P \neq Q$. Now take any $x \in Q \backslash P$; we want to show that $Q = P(x)$. To this end, take any $y \in Q$; we show that $y \in P(x)$. Now $x \in B$ since $x \in Q$. Also $x \notin A$ since $x \in Q \subseteq D$ and $x \notin P$. It follows that $B = A(x)$. We may assume that $y \notin P$; hence $y \in B \backslash A$. Now by

Proposition 2.32(e) we get $x \triangle y \in A$. Also, $x \in D$ and $y \in D$, so $x \triangle y \in D$; hence $x \triangle y \in P$. It follows that $y = (x \triangle y) \triangle x \in P(x)$, as desired. $\square$

Proposition 2.44. *Suppose that $A(x)$ is a proper minimal extension of A. Then the following conditions are equivalent:*

(i) *$A \upharpoonright x$ and $A \upharpoonright -x$ are non-principal ideals.*
(ii) *A is dense in $A(x)$.*

Proof. Assume (i). Take any non-zero element y of $A(x)$. Wlog we may assume that $y = a \cdot x$ for some $a \in A$. By Proposition 2.32 there are two cases.

Case 1. $a \in \mathrm{Smp}^A_x$. Say $a = b + c$ with $b \in A \upharpoonright x$ and $c \in A \upharpoonright -x$. Then $a \cdot x = b \in A$ and there is nothing to prove.

Case 2. $-a \in \mathrm{Smp}^A_x$. Say $-a = b + c$ with $b \in A \upharpoonright x$ and $c \in A \upharpoonright -x$. Choose $d \in A \upharpoonright x$ with $b < d$ (which we can do because $A \upharpoonright x$ is non-principal). Then

$$d \cdot -b \cdot -a = d \cdot -b \cdot (b + c) \leq d \cdot c = 0$$

since $c \in A \upharpoonright -x$. Hence $0 \neq d \cdot -b \leq a \cdot x$, as desired.

Now assume (ii). To show that $A \upharpoonright x$ is non-principal, let $a \in A \upharpoonright x$. Now $x \cdot -a \neq 0$, so we can choose a non-zero $b \in A$ such that $b \leq x \cdot -a$. Then $a < a + b \leq x$, as desired. Similarly, $A \upharpoonright -x$ is non-principal. $\square$

Proposition 2.45. *Suppose that $A(x)$ is a proper minimal extension of A, and $A \upharpoonright x$ is a principal ideal generated by an element a^*. Set $y = -a^* \cdot x$. Then:*

(i) *$y \notin A$, and hence $A(x) = A(y)$;*
(ii) *for all $a \in A$, $y \leq a$ iff $-a \in \mathrm{Smp}^A_x$;*
(iii) *y is an atom of $A(x)$;*
(iv) *if D is dense in A, then $\langle D \cup \{y\} \rangle$ is dense in $A(x)$.*

Proof. (i): Clearly $y \notin A$, since otherwise $y \leq a^*$, $y = 0$, $x \leq a^*$, and $x = a^* \in A$. So $A(x) = A(y)$ by minimality.

(ii): First assume that $y \leq a$. Thus $-a \leq a^* + -x$. Now $-a = -a \cdot a^* + -a \cdot -a^*$, and $-a \cdot -a^* \leq -x$, proving that $-a \in \mathrm{Smp}^A_x$.

Second, assume that $-a \in \mathrm{Smp}^A_x$. Say $-a = b + c$ with $b \in A \upharpoonright x$ and $c \in A \upharpoonright -x$. Then $-a^* \cdot x \cdot -a = -a^* \cdot b = 0$, showing that $y \leq a$.

(iii): Clearly $y \neq 0$, by (i). Suppose that $z \leq y$; we show that $z = 0$ or $z = y$. Say $z = a \cdot x + b \cdot -x$ with $a, b \in A$. Since $y \leq x$, we have $b \cdot -x = 0$. By Proposition 2.32, either $a \in \mathrm{Smp}^A_x$ or $-a \in \mathrm{Smp}^A_x$. If $-a \in \mathrm{Smp}^A_x$, then $y \leq a$ by (ii), hence $y \leq x \cdot a = z$ and so $y = z$, as desired. Now suppose that $a \in \mathrm{Smp}^A_x$. Write $a = c + d$ with $c \in A \upharpoonright x$ and $d \in A \upharpoonright -x$. Then $c \leq a^*$, hence $a \cdot x = c \cdot x \leq a^* \cdot x$, but also $a \cdot x = z \leq y = -a^* \cdot x$, so $z = a \cdot x = 0$, as desired.

(iv): Assume that u is a non-zero element of $A(x)$. If $u \cdot y \neq 0$, then $y \leq u$ by (iii), as desired. So, assume that $u \cdot y = 0$. Then we can write $u = b \cdot -y$ with $b \in A$, by (i). Choose $d \in D^+$ so that $d \leq b$. If $d \cdot -y = 0$, then $d \leq y$, hence $d = y$, from which it follows that $y \in A$, contradicting (i). So $0 \neq d \cdot -y \leq b \cdot -y = u$, as desired. $\square$

Proposition 2.46. *Suppose that B is superatomic and A is a proper subalgebra of B. Then there is an $x \in B \backslash A$ such that $A \leq_m A(x) \leq B$.*

Proof. Assume the hypotheses. We recall the definition of the standard sequence $\langle I_\alpha : \alpha \text{ an ordinal}\rangle$ of ideals in B:

$$\begin{aligned} I_0 &= \{0\};\\ I_{\alpha+1} &= \text{ the ideal generated by } \{a \in B : a/I_\alpha \text{ is an atom of } B/I_\alpha\};\\ I_\gamma &= \bigcup_{\alpha<\gamma} I_\alpha \quad \text{for } \gamma \text{ limit.} \end{aligned}$$

Because B is superatomic, there is an ordinal α such that $I_\alpha = B$; since A is a proper subalgebra of B, there is a smallest ordinal β such that $I_\beta \not\subseteq A$. Clearly β is a successor ordinal $\gamma+1$. Hence there is an atom b/I_γ of B/I_γ such that $b \notin A$. For any $x \in A/I_\gamma$ we have $x \cdot (b/I_\gamma) = 0$ or $(b/I_\gamma) \leq x$, so that $\mathrm{Smp}^{A/I_\gamma}_{b/I_\gamma}$ is a maximal ideal of A/I_γ. It then follows by Proposition 2.32 that $A/I_\gamma \leq_{\mathrm{m}} (A/I_\gamma)(b/I_\gamma)$. Note that $A(b)/I_\gamma = (A/I_\gamma)(b/I_\gamma)$. Let $f : A(b) \to A(b)/I_\gamma$ be the natural mapping. Then $\ker(f) = I_\gamma$. Hence by Proposition 2.42 we get $A \leq_{\mathrm{m}} A(b)$. $\square$

Minimally generated Boolean algebras

Let A and B be BAs. A *representing chain* for B over A is a sequence $\langle C_\alpha : \alpha < \rho\rangle$ of BAs with the following properties:

(1) $\alpha < \beta < \rho$ implies that $C_\alpha \leq C_\beta$.
(2) If λ is a limit ordinal less than ρ, then $C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha$.
(3) $C_0 = A$.
(4) $\bigcup_{\alpha<\rho} C_\alpha = B$.

B is *minimally generated over* A if there is a representing chain for B over A such that $C_\alpha \leq_{\mathrm{m}} C_{\alpha+1}$ whenever $\alpha + 1 < \rho$. And B is *minimally generated* if it is minimally generated over 2. We write $A \leq_{\mathrm{mg}} B$ to abbreviate that B is minimally generated over A. Finally, if $A \leq_{\mathrm{mg}} B$, then $\mathrm{len}(B : A)$ is the smallest ordinal ρ demonstrating the minimal generation of B over A, and if B is minimally generated, then $\mathrm{len}(B) = \mathrm{len}(B : 2)$. The notion of a minimally generated BA is due to S. Koppelberg [89a].

Proposition 2.47.

(i) *Suppose that $A \leq B$ and f is an epimorphism from B onto Q. Let $P = f[A]$. Then:*
  (a) *if $A \leq_{\mathrm{mg}} B$, then $P \leq_{\mathrm{mg}} Q$ and* $\mathrm{len}(Q : P) \leq \mathrm{len}(B : A)$*;*

(b) *if A includes the kernel of f, then $A \leq_{\mathrm{mg}} B$* iff *$P \leq_{\mathrm{mg}} Q$, and if one and hence both of these holds then* $\mathrm{len}(Q : P) = \mathrm{len}(B : A)$.

(ii) *A homomorphic image Q of a minimally generated BA B is minimally generated. Moreover,* $\mathrm{len}(Q) \leq \mathrm{len}(B)$.

Proof. By Proposition 2.42. □

Proposition 2.48.

(i) *Suppose that $A \leq B \leq M \geq D$ and $A \leq_{\mathrm{mg}} B$. Set $P = A \cap D$ and $Q = B \cap D$. Then $P \leq_{\mathrm{mg}} Q$, and* $\mathrm{len}(Q : P) \leq \mathrm{len}(B : A)$.

(ii) *Every subalgebra D of a minimally generated BA B is minimally generated. Moreover,* $\mathrm{len}(D) \leq \mathrm{len}(B)$.

(iii) *If $A \leq_{\mathrm{mg}} B$ and $A \leq C \leq B$, then $A \leq_{\mathrm{mg}} C$.*

Proof. (i) and (ii) are clear from Proposition 2.43. For (iii), let $D = C$ in (i). □

Proposition 2.49. *Suppose that A is an atomless subalgebra of B and there is an element $u \in B$ which is independent over A, i.e., $a \cdot u \neq 0 \neq a \cdot -u$ for all $a \in A^+$. Then B is not minimally generated over A.*

Proof. Otherwise we would have $A \leq_{\mathrm{mg}} A(u)$ by Proposition 2.48(iii). Hence there is an $x \in A(u)$ such that $x \notin A$ and $A \leq_m A(x)$. Say $x = a + b \cdot u + c \cdot -u$ with a, b, c pairwise disjoint elements of A. From the independence of u over A it then follows that $A \upharpoonright x = A \upharpoonright a$, a principal ideal in A. In fact, if $v \in A$ and $v \leq x$, then $v \leq a + b + c$; if $v \cdot b \neq 0$, then $v \cdot b \cdot -u \neq 0$. But $v \cdot b \leq x \cdot b = b \cdot u$, contradiction. So $v \cdot b = 0$, and similarly $v \cdot c = 0$. Similarly, $A \upharpoonright -x$ is a principal ideal in A. So Smp_x^A is a principal ideal in A. By Proposition 2.32, Smp_x^A is a maximal ideal; so this gives us an atom of A, contradiction. □

Proposition 2.50. *If A and B are minimally generated, then so is $A \times B$.*

Proof. Let $\langle C_\alpha : \alpha < \rho \rangle$ be a representing sequence for A's minimal generation (thus with $C_0 = 2$ and $C_\alpha \leq_m C_{\alpha+1}$ for all $\alpha + 1 < \rho$), and let $\langle D_\alpha : \alpha < \sigma \rangle$ be similarly chosen for B. The desired sequence for $A \times B$ is $\langle G \rangle^\frown \langle E_\alpha : \alpha < \rho + \sigma \rangle$, where G is the two-element subalgebra $\{(0, 0), (1, 1)\}$ of $A \times B$ and for $\alpha < \rho$ we set

$$E_\alpha = \{(a, b) : a \in C_\alpha \text{ and } b \in \{0, 1\}\},$$

and for $\alpha < \sigma$ we set

$$E_{\rho+\alpha} = \{(a, b) : a \in A \text{ and } b \in D_\alpha\}.$$

We show that this works. Note that $E_0 = \{(a, b) : a \in \{0, 1\} \text{ and } b \in \{0, 1\}\}$, so $E_0 = G((0, 1))$ and there is no algebra strictly between G and E_0. Now take any α with $\alpha + 1 < \rho$. Let $C_{\alpha+1} = C_\alpha(u)$, with no algebra properly in between. We claim that $E_{\alpha+1} = E_\alpha((u, 1))$ with no algebra properly in between. For, take any

$(x, y) \in E_{\alpha+1}$. Thus $x \in C_{\alpha+1}$ and $y \in \{0, 1\}$. Hence we can write $x = a \cdot u + b \cdot -u$ with $a, b \in C_\alpha$. Then

$$(x, 1) = (a, 1) \cdot (u, 1) + (b, 1) \cdot (-u, 0) \quad \text{and}$$
$$(x, 0) = (a, 0) \cdot (u, 1) + (b, 0) \cdot (-u, 0).$$

So $E_{\alpha+1} = E_\alpha((u, 1))$. Now suppose that $E_\alpha \subseteq F \subseteq E_{\alpha+1}$. Then $C_\alpha \subseteq \mathrm{pr}_0[F] \subseteq C_{\alpha+1}$, which gives two possibilities.

Case 1. $C_\alpha = \mathrm{pr}_0[F]$. We claim then that $E_\alpha = F$. For, suppose that $(u, v) \in F$. Then $u \in C_\alpha$ and $v \in \{0, 1\}$, as desired.

Case 2. $C_{\alpha+1} = \mathrm{pr}_0[F]$. Similarly.

Now consider E_ρ. If $\rho = \xi + 1$, then $A = C_\xi$, $E_\xi = A \times \{0, 1\}$, and $E_\rho = E_\xi$. If ρ is a limit ordinal, then $E_\rho = \{(a, b) : a \in A \text{ and } b \in \{0, 1\}\} = \bigcup_{\xi<\rho} E_\xi$.

Next, take $\alpha < \sigma$ with $\alpha+1 < \sigma$. Let $D_{\alpha+1} = D_\alpha(u)$ with no algebra properly in between. We claim that $E_{\rho+\alpha+1} = E_{\rho+\alpha}((1, u))$ with no algebra properly in between. For, take any $(x, y) \in E_{\rho+\alpha+1}$. Thus $y \in D_{\alpha+1}$ and $x \in A$. Hence we can write $y = a \cdot u + b \cdot -u$ with $a, b \in D_\alpha$. Then for any $x \in A$ we have

$$(x, y) = (x, a) \cdot (1, u) + (0, b) \cdot (0, -u).$$

So $E_{\rho+\alpha+1} = E_{\rho+\alpha}((1, u))$. Now suppose that $E_{\rho+\alpha} \subseteq F \subseteq E_{\rho+\alpha+1}$. Hence $D_\alpha \subseteq \mathrm{pr}_1[F] \subseteq D_{\alpha+1}$. This gives two cases.

Case 1. $D_\alpha = \mathrm{pr}_1[F]$. Then we claim that $E_{\rho+\alpha} = F$. For, suppose that $(u, v) \in F$. Then $v \in D_\alpha$, so $(u, v) \in E_\alpha$.

Case 2. $D_{\alpha+1} = \mathrm{pr}_1[F]$. Similarly.

Finally, clearly $\bigcup_{\alpha<\rho+\sigma} E_\alpha = A \times B$. □

Proposition 2.51. *If $\langle A_i : i \in I\rangle$ is a system of minimally generated BAs, then so is $\prod^{\mathrm{w}}_{i\in I} A_i$.*

Proof. Wlog each A_i is nontrivial, and I is an infinite cardinal κ. For all $\beta < \kappa$ let $\langle C_{\beta\alpha} : 0 < \alpha < \rho_\beta\rangle$ be a representing sequence for A_β (for technical reasons starting at 1 rather than 0) such that $C_{\beta\alpha} \leq_m C_{\beta,\alpha+1}$ and $C_{\beta\alpha} \neq C_{\beta,\alpha+1}$ if $\alpha + 1 < \rho_\beta$. Thus $C_{\beta 1} = 2$, and each ρ_β is at least 2. Let $\sigma = \sum_{\beta<\kappa} \rho_\beta$. For each $\xi < \sigma$ we define a subalgebra E_ξ of $\prod^{\mathrm{w}}_{\beta<\kappa} A_\beta$. Choose $\delta < \kappa$ such that $\mu \overset{\mathrm{def}}{=} \sum_{\beta<\delta} \rho_\beta \leq \xi < \sum_{\beta\leq\delta} \rho_\beta$; say $\xi = \mu + \varepsilon$ with $\varepsilon < \rho_\delta$. If $\varepsilon = 0$ we set

$$E_\xi = \left\{x \in \prod^{\mathrm{w}}_{\beta<\kappa} A_\beta : \forall\theta \in [\delta, \kappa)[x_\theta = 1] \text{ or } \forall\theta \in [\delta, \kappa)[x_\theta = 0]\right\}.$$

If $\varepsilon \neq 0$ we set

$$E_\xi = \left\{x \in \prod^{\mathrm{w}}_{\beta<\kappa} A_\beta : x_\delta \in C_{\delta\varepsilon} \text{ and } \forall\theta \in (\delta, \kappa)[x_\theta = 1] \text{ or } \forall\theta \in (\delta, \kappa)[x_\theta = 0]\right\}.$$

Then $\langle E_\xi : \xi < \sigma\rangle$ is as desired.

To prove this, first note that for $\xi = 0$ we have $\delta = 0$ and so E_0 is the two-element subalgebra of $\prod^{\mathrm{w}}_{i\in I} A_i$. Clearly each E_ξ is a subalgebra of $\prod^{\mathrm{w}}_{\beta<\kappa} A_\beta$.

Now suppose that $\xi < \eta$; we show that $E_\xi \subseteq E_\eta$. Say $\delta < \kappa$, $\mu \stackrel{\mathrm{def}}{=} \sum_{\beta<\delta} \rho_\beta \leq \xi < \sum_{\beta\leq\delta} \rho_\beta$, and $\xi = \mu + \varepsilon$ with $\varepsilon < \rho_\delta$, and $\delta' < \kappa$, $\mu' \stackrel{\mathrm{def}}{=} \sum_{\beta<\delta'} \rho_\beta \leq \eta < \sum_{\beta\leq\delta'} \rho_\beta$, and $\eta = \mu' + \varepsilon'$ with $\varepsilon' < \rho_{\delta'}$. Let $x \in E_\xi$.

Case 1. $\delta = \delta'$. So $\mu = \mu'$ and $\varepsilon < \varepsilon'$.

Subcase 1.1. $\varepsilon = 0$. Then $\forall\theta \in [\delta,\kappa)[x_\theta = 1]$ or $\forall\theta \in [\delta,\kappa)[x_\theta = 0]$. Hence $x_\delta = 1$ or $x_\delta = 0$, so that $x_\delta \in C_{\delta'\varepsilon'}$, and $\forall\theta \in (\delta',\kappa)[x_\theta = 1]$ or $\forall\theta \in (\delta',\kappa)[x_\theta = 0]$. Thus $x \in E_\eta$.

Subcase 1.2. $\varepsilon > 0$. Then $x_\delta \in C_{\delta\varepsilon}$ and $[\forall\theta \in (\delta,\kappa)[x_\theta = 1]$ or $\forall\theta \in (\delta,\kappa)[x_\theta = 0]$. Hence, since $C_{\delta\varepsilon} = C_{\delta'\varepsilon} \subseteq C_{\delta'\varepsilon'}$, we have $x_\delta \in C_{\delta'\varepsilon'}$. Thus $x \in E_\eta$.

Case 2. $\delta < \delta'$.

Subcase 2.1. $\varepsilon' = 0$. Clearly $\forall\theta \in [\delta',\kappa)[x_\theta = 1]$ or $\forall\theta \in [\delta',\kappa)[x_\theta = 0]$. Thus $x \in E_\eta$.

Subcase 2.2. $\varepsilon' > 0$. Clearly $x_{\delta'} = 0$ or $x_{\delta'} = 1$, so $x_{\delta'} \in C_{\delta'\varepsilon'}$. Also clearly $\forall\theta \in (\delta',\kappa)[x_\theta = 1]$ or $\forall\theta \in (\delta',\kappa)[x_\theta = 0]$. Thus $x \in E_\eta$.

Next we show that for ξ limit the set E_ξ is the union of previous E_η's. For, let $x \in E_\xi$. Assuming notation as in the definition of E_ξ, we take two cases.

Case 1. $\varepsilon = 0$. So $\xi = \mu$ is a limit ordinal. This gives two subcases.

Subcase 1.1. For all $\theta \in [\delta,\kappa)$ we have $x_\theta = 1$. Since $[\delta,\kappa)$ is infinite, we must have $\{\nu < \kappa : x_\nu \neq 1\}$ finite. Now we consider two subcases.

Subsubcase 1.1.1. δ is a successor ordinal $\delta'+1$. Now $\xi = \mu = \sum_{\beta<\delta'} \rho_\beta + \rho_{\delta'}$, so $\rho_{\delta'}$ is a limit ordinal. Now $x_{\delta'} \in A_{\delta'}$, so there is a $\sigma < \rho_{\delta'}$ such that $x_{\delta'} \in C_{\delta'\sigma}$ and $0 < \sigma$. Let $\xi' = \sum_{\beta<\delta'} \rho_\beta + \sigma$. Then $x \in E_{\xi'}$ and $\xi' < \xi$, as desired.

Subsubcase 1.1.2. δ is a limit ordinal. Then by the fact that $\{\nu < \kappa : x_\nu \neq 1\}$ is finite, there is a $\delta' < \delta$ such that $x \in E_{\xi'}$, with $\xi' = \sum_{\beta<\delta'} \rho_\beta$. Note that $\xi' < \xi$.

Subcase 1.2. For all $\theta \in [\delta,\kappa)$ we have $x_\theta = 0$. This is similar to Subcase 1.1.

Case 2. ε is a limit ordinal. Since $x_\delta \in C_{\delta\varepsilon}$, it follows that there is an $\eta < \varepsilon$ such that $x_\delta \in C_{\delta\eta}$ and $\eta \neq 0$. Hence $x \in E_{\mu+\eta}$ and $\mu + \eta < \xi$, as desired.

Next, we need to show that $E_\xi \leq_m E_{\xi+1}$ for all $\xi < \sigma$. Let $\xi, \delta, \mu, \varepsilon$ be as in the definition. Take any $y \in E_{\xi+1}\backslash E_\xi$; we want to show that $E_{\xi+1} = E_\xi(y)$. To this end, take any $z \in E_{\xi+1}\backslash E_\xi$; we want to show that $z \in E_\xi(y)$.

Case 1. $\varepsilon = 0$. Note that $C_{\delta 1} = \{0, 1\}$ and $\xi + 1 = \mu + 1$. We have some subcases.

Subcase 1.1. $z_\theta = 1 = y_\theta$ for all $\theta \in (\delta,\kappa)$. Since $y, z \notin E_\xi$, it follows that $z_\delta = 0 = y_\delta$. So $y \triangle z \in E_\xi$, and hence $z \in E_\xi(y)$.

Subcase 1.2. $z_\theta = 1$ for all $\theta \in (\delta, \kappa)$, and $y_\theta = 0$ for all $\tau \in (\delta, \kappa)$. Since $y, z \notin E_\xi$, it follows that $z_\delta = 0$ and $y_\delta = 1$. Then $(y \triangle z)_\theta = 1$ for all $\theta \in [\delta, \kappa)$, so again $y \triangle z \in E_\xi$.

The other subcases are similar.

Case 2. $\varepsilon > 0$. Since $y_\delta, z_\delta \in C_{\delta,\varepsilon+1}$, it follows that there exist $a, b \in C_{\delta\varepsilon}$ such that $z_\delta = a \cdot y_\delta + b \cdot -y_\delta$. Now define $u, v, w \in \prod^{\mathrm{w}}_{\alpha<\kappa} A_\alpha$ as follows: for any $\theta < \kappa$,

$$u_\theta = \begin{cases} 0 & \text{if } \theta \neq \delta, \\ a & \text{otherwise;} \end{cases}$$
$$v_\theta = \begin{cases} 0 & \text{if } \theta \neq \delta, \\ b & \text{otherwise;} \end{cases}$$
$$w_\theta = \begin{cases} z_\theta & \text{if } \theta \neq \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then $u, v, w \in E_\xi$ and $z = u \cdot y + v \cdot -y + w$.

Finally, we show that $\bigcup_{\xi<\sigma} E_\xi = \prod^{\mathrm{w}}_{\beta<\kappa} A_\beta$. Take any $x \in \prod^{\mathrm{w}}_{\beta<\kappa} A_\beta$. Choose $\delta < \kappa$ such that $\forall\theta \in (\delta, \kappa)[x_\theta = 1]$ or $\forall\theta \in (\delta, \kappa)[x_\theta = 0]$. Let $\mu = \sum_{\beta<\delta} \rho_\beta$. Now $x_\delta \in A_\delta$, so there is an $\varepsilon < \rho_\delta$ such that $0 < \varepsilon$ and $x_\delta \in C_{\delta\varepsilon}$. Then with $\xi = \mu + \varepsilon$ we have $x \in E_\xi$. □

Proposition 2.52. *Every interval algebra is minimally generated.*

Proof. Let A be an interval algebra. Then it is generated by a chain C. Enumerate C: $C = \{c_\alpha : \alpha < \rho\}$. For each $\alpha < 1 + \rho$ let $B_\alpha = \langle\{c_\beta : \beta < \alpha\}\rangle$. Then by Proposition 2.32 we have $B_\alpha \leq_m B_{\alpha+1}$ whenever $\alpha + 1 \leq 1 + \rho$, so this shows the minimal generation of A. □

Corollary 2.53. *Every pseudo-tree algebra is minimally generated.*

Proof. By Propositions 2.52, 2.48(ii), and 2.21. □

Proposition 2.54. *Every superatomic BA is minimally generated.*

Proof. This follows from Proposition 2.46 by an easy transfinite construction. □

Theorem 2.55. *A is superatomic* iff *$B \leq_{\mathrm{mg}} A$ for every $B \leq A$.*

Proof. $\Rightarrow$ follows from Proposition 2.46 by an easy transfinite construction. Now suppose that A is not superatomic. Then A has an atomless subalgebra, and hence has a denumerable atomless subalgebra B. B is freely generated by some set X. Take any $x \in X$. Then x is independent over $\langle X \backslash \{x\}\rangle$. It follows then from Proposition 2.49 that A is not minimally generated over $\langle X \backslash \{x\}\rangle$. □

Proposition 2.56. *$A(x)$ is minimally generated over A* iff *A/Smp^A_x is superatomic.*

Proof. First suppose that A/Smp_x^A is superatomic. Note that Smp_x^A is an ideal of both A and $A(x)$, by Corollary 2.26. By Proposition 2.31(iii), $A(x)/\mathrm{Smp}_x^A$ is isomorphic to a simple extension of A/Smp_x^A, and so it is superatomic. By Theorem 2.55, $A/\mathrm{Smp}_x^A \leq_{\mathrm{mg}} A(x)/\mathrm{Smp}_x^A$. Hence by Proposition 2.47, $A \leq_{mg} A(x)$.

Conversely, suppose that A/Smp_x^A is not superatomic. Then there is an ideal K on A such that $\mathrm{Smp}_x^A \subseteq K$ and A/K is atomless. Let $L = \langle K\rangle^{\mathrm{id}}_{A(x)}$. Let $h: A(x)/\mathrm{Smp}_x^A \to A(x)/L$ be the natural homomorphism, taking y/Smp_x^A to y/L.

(1) x/L is independent over $h[A/\mathrm{Smp}_x^A]$.

In fact, take any $a \in A$ such that $h(a/\mathrm{Smp}_x^A) \neq 0$. Thus $a \notin L$. If $(x/L)\cdot(a/L) = 0$, then $x \cdot a \in L$, hence there is a $b \in K$ such that $x \cdot a \leq b$. This gives $a \cdot -b \leq -x$, so that $a \cdot -b \in \mathrm{Smp}_x^A \subseteq K$. Now $a \leq b + a \cdot -b \in K$, so $a \in K$, contradiction. So $(x/L) \cdot (a/L) \neq 0$. Similarly, $-(x/L) \cdot (a/L) \neq 0$. This proves (1).

It now follows from Proposition 2.49 that $h[A/\mathrm{Smp}_x^A] \not\leq_{\mathrm{mg}} A(x)/L$. Let $l : A(x) \to A(x)/L$ be the natural homomorphism. Then $l[A] = h[A/\mathrm{Smp}_x^A]$. so $l[A] \not\leq_{\mathrm{mg}} A(x)/L$. Then by Proposition 2.47, $A \not\leq_{\mathrm{mg}} A(x)$. ☐

Proposition 2.57. *The free BA A on ω_1 free generators is not minimally generated.*

Proof. Suppose it is, and let $\langle B_\alpha : \alpha < \sigma\rangle$ be a representing chain which demonstrates this. Wlog $\sigma = \mathrm{len}(A)$, and $B_\alpha \subset B_{\alpha+1}$ whenever $\alpha + 1 < \sigma$. Clearly $\sigma \geq \omega_1$.

We claim that $\sigma = \omega_1$. Otherwise B_{ω_1} is a subalgebra of A. Now $|B_{\omega_1}| = \omega_1$, so by Theorem 9.16 of the Handbook, it has a subalgebra C isomorphic to A. Hence by Proposition 2.48,

$$\mathrm{len}(A) > \omega_1 \geq \mathrm{len}(B_{\omega_1}) \geq \mathrm{len}(C) = \mathrm{len}(A),$$

contradiction.

Let $\{x_\alpha : \alpha < \omega_1\}$ be the set of free generators of A, and for each $\alpha < \omega_1$ let $C_\alpha = \langle\{x_\beta : \beta < \alpha\}\rangle$. This gives another representing chain for A. Hence $K \stackrel{\mathrm{def}}{=} \{\alpha < \omega_1 : B_\alpha = C_\alpha\}$ is a club in ω_1. Take any infinite member α of K. Now x_α is independent over B_α and A is minimally generated over B_α, which contradicts Proposition 2.49. ☐

Proposition 2.58. *If an infinite BA A satisfies any of the following conditions then it is not minimally generated:*

(1) *A is complete;*
(2) *A is σ-complete;*
(3) *A is ω_1-saturated (in the sense of model theory);*
(4) *A has the countable separation property;*
(5) *A is the product of infinitely many non-trivial algebras.*

Proof. Each of (1), (2), (3) implies (4), and (5) implies that A has an infinite CSP subalgebra. Hence it suffices to show that if A is CSP then it is not minimally generated (using Proposition 2.48(ii)). Now A has $\mathscr{P}(\omega)$ as a homomorphic image, and $\mathscr{P}(\omega)$ has an independent set of size ω_1, so the same is true of A, and the conclusion follows from Proposition 2.57 and Proposition 2.48(ii). $\square$

Theorem 2.59. *For every minimally generated BA B there is a dense subalgebra A of B such that A is isomorphic to a tree algebra and B is minimally generated over A.*

Proof. Fix a subset X of B generating B and a well-ordering $<_X$ of X such that if we let $B_x = \langle\{y : y <_X x\}\rangle$ then the chain $\langle B_x : x \in X\rangle$ demonstrates the minimal generation of B; we assume that $1 \in X$, and, moreover, that $x \notin B_x$ for all $x \in X\backslash\{1\}$. In particular, $0 \notin X$. Define $S = \{x \in X : x \neq 1,\ B_x \text{ is dense in } B_x(x)\}$, and set $T = X\backslash S$. Then:

$(*)$ If $x \in T\backslash\{1\}$, then there is an ultrafilter F on B_x and an element $y \in B_x(x)$ such that $B_x(x) = B_x(y)$ and $\forall a \in B_x(y \leq a \text{ iff } a \in F)$.

In fact, by Proposition 2.44, one of $B_x \restriction x$ and $B_x \restriction -x$ is a principal ideal. Thus $(*)$ follows from Proposition 2.45.

By $(*)$, wlog we may assume:

$(**)$ If $x \in T\backslash\{1\}$ then there is an ultrafilter F on B_x such that $\forall a \in B_x(x \leq a$ iff $a \in F)$.

Next we claim that T is a tree under the inverse of the Boolean ordering. In fact, suppose $x, y, z \in T$ and $x < y$, $x < z$; we want to show that y and z are comparable. Say $y <_X z$. Then $y \in B_z$, so by $(**)$, $z \leq y$ or $z \leq -y$; and $z \leq -y$ is ruled out since $0 \neq x \leq y \cdot z$. Thus z and y are comparable. Moreover, if $x, y \in T$ and $x < y$, then $y <_X x$; otherwise $x <_X y$, hence $x \in B_y$, and so by this same property, $y \leq x$ or $y \leq -x$, both of which are false. So, T is a tree.

Also note that for $u, v \in T$ we have that u and v are incomparable iff $u \cdot v = 0$. In fact, assume that u and v are incomparable; say $u <_X v$. Then by $(**)$ it follows that $u \notin F_v$ and hence $-u \in F_v$ and $v \leq -u$, as desired.

Next, T is disjunctive. For, assume that $t, t_1, \ldots, t_n \in T$, where $n > 0$, and $t \leq t_1 + \cdots + t_n$, but $t \not\leq t_i$ for all i. Now for all i, either t and t_i are comparable, or they are disjoint, by the tree property and the previous paragraph. So we may assume that $t_i < t$ for each i. Similarly, we may assume that the t_i's are pairwise disjoint. So, $t = t_1 + \cdots + t_n$. If t is $<_X$-maximum in $\{t, t_1, \ldots, t_n\}$, then $t \in B_t$, contradiction. Otherwise some t_i is $<_X$-maximum in $\{t, t_1, \ldots, t_n\}$, and $t_i = t \cdot \sum_{j \neq i} -t_j$, hence $t_i \in B_{t_i}$, contradiction. So, T is disjunctive.

Now by the proof of Theorem 2.15$(iv) \Rightarrow (i)$ it follows that $A \stackrel{\text{def}}{=} \langle T\rangle$ is isomorphic to $\operatorname{Treealg} T$.

We claim that A is dense in B. To prove this, we show by $<_X$-induction on $x \in X$ that $\langle\{y \in T : y <_X x\}\rangle$ is dense in B_x for all $x \in X$. For x the smallest element of X the algebra B_x has only two elements, and the conclusion is obvious. Now suppose that the statement holds for $x \in X$, and $x' \in X$ is the immediate successor of x under $<_X$. Now $B_{x'} = B_x(x)$. We consider two cases.

Case 1. $x \in T$. Let F be as in $(**)$. By the inductive hypothesis, the set $D \stackrel{\text{def}}{=} \langle\{y \in T : y <_X x\}\rangle$ is dense in B_x. We claim that $\langle D \cup \{x\}\rangle$ is dense in $B_x(x)$. Suppose that $0 \neq u \in B_x(x)$. Write $u = x \cdot a + -x \cdot b$ with $a, b \in B_x$. First suppose that $x \cdot a \neq 0$. If $-a \in F$, then $x \leq -a$ by $(**)$, contradiction. So $a \in F$, and hence $x \leq a$ and so $x \cdot a = x$ and the desired conclusion holds. Suppose that $-x \cdot b \neq 0$. Choose $d \in D^+$ such that $d \leq b$. Suppose that $d \cdot -x = 0$.

Subcase 1.1. $d \in F$. Then $x \leq d$ by $(**)$, so $x = d \in D$, contradiction.

Subcase 1.2. $-d \in F$. Then $x \leq -d$ by $(**)$, so $x \cdot d = 0$. Now $d \leq x$, so $d = 0$, contradiction.

Thus $d \cdot -x \neq 0$ and the desired conclusion again holds.

Case 2. $x \in S$. Then B_x is dense in $B_x(x)$ and the desired conclusion is obvious.

Next, for x limit under $<_X$, the induction hypothesis clearly implies the desired conclusion. Hence our inductive statement holds, and then it is clear that A is dense in B.

If $T = X$, then $A = B$ and we are through. So assume that $S \neq 0$. We define a new well-ordering $\ll$ on X by putting T before S: for $x, y \in X$, we define $x \ll y$ iff ($x, y \in T$ and $x <_X y$) or ($x, y \in S$ and $x <_X y$) or ($x \in T$ and $y \in S$). For each $x \in S$ let $C_x = \langle\{y \in X : y \ll x\}\rangle$. We claim that $\langle\{C_x : x \in S\}\rangle$ demonstrates that B is minimally generated over A. Since $C_s = A$ for s the least member of S under $\ll$, by Corollary 2.53 all we really need to prove is that $C_x \leq_{\mathrm{m}} C_x(x)$ for all $x \in S$. To do this, it suffices to take any $y \ll x$ and show that $y \cdot x \in C_x$ or $y \cdot -x \in C_x$. In fact, if we can do this, then by Proposition 2.25, y or $-y$ will be in $\mathrm{Smp}_x^{C_x}$, and since this will be true for each generator of C_x, it will follow that $\mathrm{Smp}_x^{C_x}$ is either all of C_x or is a maximal ideal in C_x, so that $C_x \leq_{\mathrm{m}} C_x(x)$ by Proposition 2.32.

If $y <_X x$, then $y \in B_x \leq_{\mathrm{m}} B_x(x)$, and so by Propositions 2.25 and 2.32, one of $y \cdot x$ or $y \cdot -x$ is in B_x. Now $B_x \subseteq C_x$ since $x \in S$, so we obtain the desired conclusion. Assume, on the other hand, that $x <_X y$. This can only happen if $y \in T$. Hence by $(**)$ applied to B_y, either $y \leq x$ and hence $y \cdot x = y \in C_x$, or $y \leq -x$ and $y \cdot x = 0 \in C_x$, as desired. □

Two BAs A and B are *co-complete* iff they have isomorphic completions.

Corollary 2.60. *Any minimally generated BA A is co-complete with an interval algebra.*

Proof. By Theorem 2.59, A has a dense subalgebra B isomorphic to a tree algebra. From Theorem 2.21 we know that B can be embedded in an interval algebra, so let f be an isomorphism from B into an interval algebra C. Let I be an ideal of C

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maximal with respect to the property that $f[B] \cap I = \{0\}$. Then if $g : C \to C/I$ is the natural homomorphism, $g \circ f$ is still an embedding, and C/I is isomorphic to an interval algebra. (See the Handbook, Proposition 15.9.) Now $g[f[B]]$ is dense in C/I. For, suppose that $c \in C$ and $c/I \neq 0$. Then $\langle I \cup \{c\} \rangle^{\text{id}}$, and so there is a $b \in B$ such that $0 \neq f(b) \leq d + c$ for some $d \in I$. Then $0 \neq g(f(b)) \leq c/I$, as desired. Therefore we may assume in the original situation that $f[B]$ is dense in C. Now by the uniqueness of completions (see the remark following 4.14 in the Handbook), it follows that $\overline{B} = \overline{A}$ is isomorphic to $\overline{C}$. □

Our next result on minimal generation requires several elementary but interesting lemmas.

Lemma 2.61. *If I is an ideal in A and C is a subalgebra of A, then $C/(I \cap C) \cong \langle I \cup C \rangle / I$.*

Proof. For any $c \in C$ let $f([c]_{I \cap C}) = [c]_I$. Clearly f is well defined, and is in fact a monomorphism. Now note that

$$\langle I \cup C \rangle = \left\{ a + \sum_{i<m} (c_i \cdot -d_i) : a \in I, m \in \omega, c_i \in C, d_i \in I \right\}.$$

Given an element of $\langle I \cup C \rangle$ of the form indicated, $f([\sum_{i<m} c_i]_{I \cap C}) = [a + \sum_{i<m}(c_i \cdot -d_i)]_I$ is clear. Hence f is the desired isomorphism. □

For the next lemmas, note that if I is an ideal in A, then $\langle I \rangle = (I \cup -I)$, where $-I = \{-a : a \in I\}$.

Lemma 2.62. *Let I be an ideal of A and C be a subalgebra of A. Let $f : A \to A/I$ be the natural homomorphism, taking any $a \in A$ to $[a]_I$. Then $g \stackrel{\text{def}}{=} \langle f[B] : \langle I \rangle \leq B \leq \langle I \cup C \rangle \rangle$ is an isomorphism from $(\{B : \langle I \rangle \leq B \leq \langle I \cup C \rangle, \leq)$ onto $(\{D : D \leq \langle I \cup C \rangle / I\}, \leq)$.*

Proof. Clearly g maps $(\{B : \langle I \rangle \leq B \leq \langle I \cup C \rangle, \leq)$ into $(\{D : D \leq \langle I \cup C \rangle / I\}, \leq)$, and $\langle I \rangle \leq B \leq B' \leq \langle I \cup C \rangle$ implies that $g(B) \leq g(B')$. Now g is one-one. For, suppose that $\langle I \rangle \leq B, B' \leq \langle I \cup C \rangle$, $B \neq B'$, and $g(B) = g(B')$. Say by symmetry that $b \in B \backslash B'$. Then $[b]_I \in g(B) = g(B')$, so there is a $b' \in B'$ such that $[b]_I = [b']_I$. Thus $b \triangle b' \in I \subseteq B'$. Hence $b = (b \triangle b') \triangle b' \in B'$, contradiction. So g is one-one.

Next, if $D \leq \langle I \cup C \rangle / I$, then $\langle I \rangle \leq f^{-1}[D] \leq \langle I \cup C \rangle$, and $g(f^{-1}[D]) = f[f^{-1}[D]] = D$. So g is a surjection.

Finally, if $g(B) \leq g(B') \leq \langle I \cup C \rangle / I$, then

$$B = g^{-1}(g(B)) = f^{-1}[g(B)] \leq f^{-1}[g(B')] = g^{-1}(g(B')) = B'. \qquad \square$$

Proposition 2.63. *If I is an ideal in A, C is a subalgebra of A, and C is minimally generated, then $\langle I \rangle \leq_{\text{mg}} \langle I \cup C \rangle$.*

Proof. Note that $C/(I \cap C)$ is minimally generated, by Proposition 2.47(ii); hence by Proposition 2.61, so is $\langle I \cup C\rangle/I$. Now we apply Proposition 2.47(i) to $\langle I\rangle$, $\langle I \cup C\rangle$, $\langle I \cup C\rangle/I$, f in place of A, B, Q, f, where f is the natural homomorphism from $\langle I \cup C\rangle$ onto $\langle I \cup C\rangle/I$; the conclusion of our proposition follows. □

Lemma 2.64. *Let A be a BA and $a \in A$ with $a \neq 1$. Then $\langle A \restriction a\rangle \cong (A \restriction a) \times 2$.*

Proof. We define $f : \langle A \restriction a\rangle \to (A \restriction a) \times 2$ by setting, for any $x \in \langle A \restriction a\rangle$,

$$f(x) = \begin{cases} (x, 0) & \text{if } x \in (A \restriction a), \\ (x \cdot a, 1) & \text{if } x \in -(A \restriction a). \end{cases}$$

Note that f is well defined because $a \neq 1$.

Now f preserves $+$. In fact,

if $x, y \in (A \restriction a)$, then $x + y \in (A \restriction a)$, and
$$f(x + y) = (x + y, 0) = (x, 0) + (y, 0) = f(x) + f(y);$$
if $x \in (A \restriction a)$ and $y \in -(A \restriction a)$, then $x + y \in -(A \restriction a)$, and
$$f(x + y) = ((x + y) \cdot a, 1) = (x, 0) + (y \cdot a, 1) = f(x) + f(y);$$
if $x \notin (A \restriction a)$ and $y \in (A \restriction a)$, then $x + y \in -(A \restriction a)$, and
$$f(x + y) = ((x + y) \cdot a, 1) = (x \cdot a, 1) + (y, 0) = f(x) + f(y);$$
if $x, y \in -(A \restriction a)$, then $x + y \in -(A \restriction a)$, and
$$f(x + y) = ((x + y) \cdot a, 1) = (x \cdot a, 1) + (y \cdot a, 1) = f(x) + f(y).$$

Similarly, f preserves $-$:

if $x \in (A \restriction a)$, then $-x \notin (A \restriction a)$, and
$$f(-x) = (-x \cdot a, 1) = -(x, 0) = -f(x);$$
if $x \notin (A \restriction a)$, then $-x \in (A \restriction a)$, and
$$f(-x) = (-x, 0) = -(x \cdot a, 1) = -f(x).$$

So f is a homomorphism.

Clearly f is one-one. Finally, f maps onto, since if $x \in (A \restriction a)$, then $f(x) = (x, 0)$, and $f(x + -a) = ((x + -a) \cdot a, 1) = (x, 1)$. □

Proposition 2.65. *If I is an ideal on A and $A \restriction a$ is minimally generated for every $a \in I$, then the subalgebra $\langle I\rangle$ of A is also minimally generated.*

Proof. Let $\langle a_\alpha : \alpha < \kappa\rangle$ enumerate I, κ an infinite cardinal. For each $\alpha < \kappa$ let J_α be the ideal generated by $\{a_\beta : \beta < \alpha\}$. Since $I = \bigcup_{\alpha<\kappa} J_\alpha$, we also have $\langle I\rangle = \bigcup_{\alpha<\kappa}\langle J_\alpha\rangle$. Hence it suffices to show that $\langle J_\alpha\rangle \leq_{\mathrm{m}} \langle J_{\alpha+1}\rangle$ for every $\alpha < \kappa$. We may assume that $a_\alpha \notin \langle J_\alpha\rangle$.

Let $C = \langle A \restriction a_\alpha\rangle$. Now C is minimally generated by Lemma 2.64 and Proposition 2.50; and so $C/(J_\alpha \cap C)$ is minimally generated by Proposition 2.47(ii). Then by Lemma 2.61, $\langle J_a \cup C\rangle/J_\alpha$ is minimally generated. It follows by Lemma 2.63 that $\langle J_\alpha\rangle \leq_{\mathrm{mg}} \langle J_\alpha \cup C\rangle$. Clearly $\langle J_\alpha \cup C\rangle = \langle J_{\alpha+1}\rangle$, so $\langle J_\alpha\rangle \leq_{\mathrm{m}} \langle J_{\alpha+1}\rangle$. □

In conclusion of our treatment of minimally generated BAs we mention the following problem from Koppelberg, Monk [92].

Problem 1. *Is there in ZFC an infinite minimally generated BA with no countably infinite homomorphic image?*

The problem is formulated in this way since Koppelberg [88] showed that $\diamondsuit$ implies that there is a BA of this sort.

Tail algebras

For any partial order P, let the *tail algebra* of P be the subalgebra $\mathrm{Tailalg}(P)$ of $\mathscr{P}(P)$ generated by $\{P \uparrow p : p \in P\}$. Thus these algebras generalize tree algebras and pseudo-tree algebras. The notion is due to Gary Brenner. It is studied in Koppelberg, Monk [92], the main results being due to Koppelberg and Blass.

Theorem 2.66. *Every semigroup algebra is isomorphic to a tail algebra.*

Proof. Let A be a semigroup algebra, and choose a generating set H for A such that $0, 1 \in H$, H is closed under $\cdot$, and $P \stackrel{\text{def}}{=} H\backslash\{0\}$ is disjunctive. Let f be the homomorphism from A onto $\mathrm{Tailalg}\,(P^{-1})$ given by Proposition 2.1: $f(p) = P \downarrow p$ for any $p \in P$. We show that f is one-one, which will finish the proof. By Sikorski's criterion, we have to show that $f(p_1) \cap \cdots \cap f(p_n) \subseteq f(q_1) \cup \cdots \cup f(q_m)$ (where $p_i, q_j \in P$) implies $p_1 \cdot \cdots \cdot p_n \leq q_1 + \cdots + q_m$. Without loss of generality, $p = p_1 \cdot \cdots \cdot p_n$ is nonzero and hence is in P. Now $p \in f(p_1) \cap \cdots \cap f(p_n)$. So $p \in f(q_j)$ for some j, $p \leq q_j$, and $p \leq q_1 + \cdots + q_m$, as desired. □

We also need a set-theoretic lemma:

Lemma 2.67. *Let P be an infinite partially ordered set. Then: either P has a strictly ascending chain of type ω, or P has a strictly descending chain of type ω, or P is well founded (with, say, P_α as its αth level) and there is some $n \in \omega$ such that P_n is infinite.*

Proof. Assume P has no descending chain of type ω (so P is well founded) and no infinite level P_n $(n \in \omega)$. For each $n \in \omega$, let

$$T_n = \{(p_0, \ldots, p_n) : p_i \in P_i \text{ for all } i \leq n, \text{ and } p_0 < \cdots < p_n\}.$$

So $T = \bigcup_{n\in\omega} T_n$ is a tree in which every level is finite and non-empty. But then T has an infinite branch, which yields an increasing chain of type ω in P. □

An algebra which is generated by a disjunctive set is called *disjunctively generated.* Clearly tail algebras are disjunctively generated.

Theorem 2.68. *Every infinite disjunctively generated algebra has a countably infinite homomorphic image.*

Proof. Say $A = \langle P \rangle$, where P is an infinite disjunctive subset of A. We apply Lemma 2.67 to P^{-1}, and have three cases.

Case 1. There is in P an ascending sequence $(p_n : n \in \omega)$. Let then $M = \{p_n : n \in \omega\}$, and consider the homomorphism f_M given by Proposition 2.1. Then f_M maps each $p \in P$ to an initial segment of M, and since $f_M(p_n) = \{p_0, \ldots, p_n\}$ and P generates A, it follows that the image of A under f_M is the finite-cofinite algebra on M, a countable algebra.

Case 2. There is in P a descending sequence of type ω. This is similar to Case 1, again considering $M = \{p_n : n \in \omega\}$.

Case 3. P^{-1} is well founded, and for some (minimal) $n \in \omega$, P_n is infinite. Consider $M = P_n$ and $f = f_M$ as in Proposition 2.1. Note that

1. $f(p) = \emptyset$ if $p \in P_\alpha, \alpha > n$
2. $f(p) = \{p\}$ for $p \in P_n$
3. $\{f(p) : p \in P_k, k < n\}$ is finite.

It follows that the image of A under f is superatomic, since its quotient under the ideal generated by the atoms is finite. It is well known, and easy to check, that every superatomic algebra has a countable homomorphic image, giving the desired result. □

Corollary 2.69. *No infinite Boolean algebra having the countable separation property is disjunctively generated.* □

Theorem 2.70. *Every BA can be embedded into a tail algebra.*

Proof. This is trivial for a finite Boolean algebra B with, say, n atoms – just take a tree with n roots and no other points. So let B be an infinite Boolean algebra; we may assume that it is the algebra of clopen subsets of some Boolean space X.

For each $b \in B$, take two new points p_b, q_b such that the points p_b, q_b $(b \in B)$, are pairwise distinct and not in X. Then put

$$U = \{p_b, q_b : b \in B\}, \quad P = U \cup X$$

and define a partial order on P by setting $p_b < x$ and $q_b < x$ for all $x \in b$. Thus, for $b \in B$, $P \uparrow p_b = \{p_b\} \cup b$, $P \uparrow q_b = \{q_b\} \cup b$ and $b = (P \uparrow p_b) \cap (P \uparrow q_b) \in \mathrm{Tailalg}(P)$.

We define a map e from B into the power set algebra of P by fixing a non-isolated point x^* of X and putting $e(b) = b$ if $x^* \notin b$ and $e(b) = U \cup b$ if $x^* \in b$. It is easily checked that e embeds B into the power set algebra of P and that $e(b) \in \mathrm{Tailalg}(P)$ if $x^* \notin b$; hence e is an embedding from B into $\mathrm{Tailalg}(P)$. □

The following problem is mentioned in Koppelberg, Monk [92].

Problem 2. *Is every disjunctively generated BA isomorphic to a tail algebra?*

Initial chain algebras

Let T be a pseudo-tree. The *initial chain algebra* of T, denoted by $\mathrm{Init}(T)$, is the subalgebra of $\mathscr{P}(T)$ generated by $\{T \downarrow t : t \in T\}$. A thorough treatment of these algebras is found in Baur [00] and Baur, Heindorf [97]. Here we state some of the facts about these algebras and give just a few proofs.

Lemma 2.71. *Let A be a BA generated by a set H with the following properties:*

(i) $0 \notin H$*;*

(ii) $H \cup \{0\}$ *is closed under* $\cdot$*;*

(iii) $H \downarrow h$ *is linearly ordered, for all* $h \in H$*.*

Then H is disjunctive.

Proof. Suppose that $h, h_1, \ldots, h_n \in H$, $n > 0$, and $h \leq h_1 + \cdots + h_n$. We may assume that $h \cdot h_i \neq 0$ for all i. Now $h = h \cdot h_1 + \cdots + h \cdot h_n$, and $h \cdot h_i \in (H \downarrow h)$ for each i, so by (iii) there is an i such that $h \cdot h_i \leq h \cdot h_j$ for all j. Hence $h = h \cdot h_i \leq h_i$, as desired. □

Lemma 2.72. *Let A be a BA generated by a set H with the following properties:*

(i) $0 \notin H$*;*

(ii) $H \cup \{0\}$ *is closed under* $\cdot$*;*

(iii) $H \downarrow h$ *is linearly ordered, for all* $h \in H$*;*

(iv) *for every nonzero* $a \in A$ *there is an* $h \in H$ *such that* $a \cdot h \neq 0$*.*

Then H is a pseudo-tree under the Boolean ordering, and A is isomorphic to $\mathrm{Init}(H)$*.*

If we replace (iii) *by*

(iii′) $H \downarrow h$ *is well ordered, for all* $h \in H$*,*

Then we can conclude that H is a tree under the Boolean ordering.

Proof. Obviously H is a pseudo-tree under the Boolean ordering, and also the final statement of the Proposition holds. By Lemma 2.71 and Proposition 2.1, there is a homomorphism f from A into $\mathscr{P}(H)$ such that $f(h) = H \downarrow h$ for all $h \in H$. Thus f maps onto $\mathrm{Init}(H)$. We need to show that f is one-one. Assume that

$$(H \downarrow h_1) \cap \cdots \cap (H \downarrow h_m) \cap [H \backslash (H \downarrow k_1)] \cap \cdots \cap [H \backslash (H \downarrow k_n)] = 0,$$

but $h_1 \cdot \ldots \cdot h_m \cdot -k_1 \cdot \ldots \cdot -k_n \neq 0$. By (iv) choose $h \in H$ such that $h \cdot h_1 \cdot \ldots \cdot h_m \cdot -k_1 \cdot \ldots \cdot -k_n \neq 0$. Now

$$h \cdot h_1 \cdot \ldots \cdot h_m \in (H \downarrow h_1) \cap \cdots \cap (H \downarrow h_n),$$

so $h \cdot h_1 \cdot \ldots \cdot h_m \in (H \downarrow k_i)$ for some i. Thus $h \cdot h_1 \cdot \ldots \cdot h_m \cdot -k_i = 0$, contradiction. □

Theorem 2.73. *For any BA A the following three conditions are equivalent:*

(i) *A is isomorphic to the initial chain algebra on some pseudo-tree T;*
(ii) *A has a set of generators H such that the conditions of Lemma* 2.71 *hold.*
(iii) *A has a set of generators H such that the conditions of Lemma* 2.72 *hold.*

Proof. (i)⇒(ii): let $U = [T]^{<\omega}\backslash\{\emptyset\}$, and let

$$H = \left\{\bigcap_{t\in F}(T\downarrow t) : F \in U\right\}\backslash\{0\}.$$

Clearly the conditions (i) and (ii) of Lemma 2.71 hold. Condition (iii) requires more work. For each $F \in U$, let $v_F = \bigcap_{t\in F}(T\downarrow t)$. Suppose that $F, G, H \in U$ and $v_G, v_H \subseteq v_F$; we want to show that v_G and v_H are comparable (under $\subseteq$). Suppose that $v_G \not\subseteq v_H$ and $v_H \not\subseteq v_G$. Say $s \in v_G\backslash v_H$ and $t \in v_H\backslash v_G$. Now since F is nonempty, choose $u \in F$. Since $v_G, v_H \subseteq v_F$, we have $s, t \le u$; so s and t are comparable. By symmetry, say $s \le t$. For any $w \in H$ we have $t \le w$, and hence $s \le w$. So $s \in v_H$, contradiction. This verifies (iii) of Lemma 2.71, so (ii) of the present theorem holds.

(ii)⇒(iii): Suppose that (iv) of Lemma 2.72 fails to hold. Then there is some monomial x over H such that $x \cdot h = 0$ for all $h \in H$, with $x \ne 0$. Write $x = h_0 \cdot \ldots \cdot h_{m-1} \cdot -k_0 \cdot \ldots \cdot -k_{n-1}$ with all $h_i, k_j \in H$. Then $m = 0$ since $x \cdot h = 0$ for all $h \in H$. Let $H' = H \cup \{x\}$. Clearly H' satisfies the conditions of Lemma 2.72.

(iii)⇒(i): by Lemma 2.72. □

Theorem 2.74. *For any BA A the following three conditions are equivalent:*

(i) *A is isomorphic to the initial chain algebra on some tree;*
(ii) *A has a set of generators H such that the conditions of Lemma* 2.71 *hold, with (iii) replaced by the condition that $H\downarrow h$ is well ordered for all $h \in H$.*
(iii) *A has a set of generators H such that the conditions of Lemma* 2.72 *hold, with the same replacement as in (ii).*

Proof. We just need to add to the proof of (i)⇒(ii) in the proof of Theorem 2.73. Suppose that $v_{f_0} \supset v_{f_1} \supset \cdots$. Choose $t_i \in v_{F_i}\backslash v_{F_{i+1}}$ for all $i \in \omega$. Then for any $i \in \omega$, $t_i, t_{i+1} \in F_i$, and hence t_i and t_{i+1} are comparable. If $t_i < t_{i+1}$, then $t_i \in v_{F_{i+1}}$, contradiction. So $t_{i+1} < t_i$. Thus we have a strictly decreasing sequence of elements of H, contradiction. □

We say that a pseudo-tree T is *well met* iff for any two $x, y \in T$, if $z \le x, y$ for some $z \in T$, then x and y have a glb in T. Note that for T a tree, this just means that for any two distinct $x, y \in T$ of the same limit level, the sets $T\downarrow x$ and $T\downarrow y$ are different.

Corollary 2.75. *Every initial chain algebra on a pseudo-tree is isomorphic to an initial chain algebra on a well-met pseudo-tree.*

Proof. This is clear from the proof of Theorem 2.73. □

Theorem 2.76. *Let T be a pseudo-tree and $A = \mathrm{Init}(T)$. Then every homomorphic image of A is isomorphic to an initial chain algebra of a pseudo-tree.*

Similarly with "pseudo-tree" replaced by "tree".

Proof. We may assume that T is well met by Corollary 2.75. Suppose that I is an ideal on A. Denote members of A/I by $[a]$. Then with $T' = \{[h] : h \in T\}$, the first part is clear by Theorem 2.73(ii).

For the second part, suppose that $[h_0] > [h_1] > \cdots$ with $h_0, h_1, \ldots$ members of T. For each i, let k_i be the glb of $h_0, \ldots, h_i$. Then each k_i is in T, and $k_0 > k_1 > \cdots$, contradicting the assumption in the second part that T is a tree. □

The following rather obvious normal form theorem for elements is sometimes useful.

Theorem 2.77. *f T is a well-met pseudo-tree and $x \in \mathrm{Init}(T)$, then x is a finite sum of elements of the following forms:*

(i) $T \downarrow s$.
(ii) $(T \downarrow s)\backslash(T \downarrow t)$.
(iii) $T\backslash \bigcup_{s\in F}(T \downarrow s)$, *with F a finite subset of T.*

Proof. Obviously x is a finite sum of elements of the following form:

$$(T \downarrow s_0) \cap \cdots \cap (T \downarrow s_{m-1}) \cap [T\backslash(T \downarrow t_0)] \cap \cdots \cap [T\backslash(T \downarrow t_{n-1})].$$

If $m = 0$, we have condition (iii). If $m > 0$, by the well-met condition we may assume that $m = 1$. Then $n = 0$ gives condition (i). If $n > 0$, then note that

$$\begin{aligned}(T \downarrow s_0)\cap&[T\backslash(T \downarrow t_0)] \cap \cdots \cap [T\backslash(T \downarrow t_{n-1})]\\ &= (T \downarrow s_0) \cap [T\backslash[(T \downarrow s_0) \cap (T \downarrow t_0)]] \cap \cdots \cap [T\backslash[(T \downarrow s_0) \cap (T \downarrow t_{n-1})]];\end{aligned}$$

the elements $(T \downarrow s_0)\cap(T \downarrow t_0), \ldots, (T \downarrow s_0)\cap(T \downarrow t_{n-1})$ are all comparable, by the well-met condition, since they are all below $T \downarrow s_0$. Thus this gives condition (ii). □

Theorem 2.78. *Every initial chain algebra on a tree is superatomic.*

Proof. By Theorem 2.76 it suffices to show that $\mathrm{Init}(T)$ is always atomic for T a tree. Note that if $t \in T$ is not at a limit level, then $\{t\} \in A$. We may assume that T is well met. Let x be a non-zero element of $\mathrm{Init}(T)$. By Theorem 2.77 we may assume that x has one of the three forms mentioned there:

Case 1. $x = (T \downarrow t)$ for some t. Let s be the root such that $s \leq t$. Then $\{s\}$ is the desired atom below x.

Case 2. $x = (T \downarrow t)\backslash(T \downarrow s)$ for some s, t. If $(T \downarrow t) \cap (T \downarrow s) = 0$, then $(T \downarrow t)\backslash(T \downarrow s) = (T \downarrow t)$ and we are back in Case 1. So let $(T \downarrow t) \cap (T \downarrow s) = (T \downarrow r)$. Choose u at a successor level, $r < u \leq t$. Then $\{u\}$ is the desired atom below x.

Case 3. $x = T\backslash\bigcup_{s\in F}(T\downarrow s)$ for some finite subset F of T. Since $x \neq 0$, choose $t \in x$. Then $(T\downarrow t)\backslash\bigcup_{s\in F}(T\downarrow s)$ reduces to Case 1 or Case 2, as desired. □

Theorem 2.79. *Every initial chain algebra on a pseudo-tree is a semigroup algebra.*

Proof. This is immediate from Lemma 2.71 and Theorem 2.73. □

For any BA A we define

$$I^A_{\rm int} = \{a \in A : A \restriction a \text{ is isomorphic to an interval algebra}\}.$$

Corollary 2.80. *$I^A_{\rm int}$ is an ideal of A.*

Proof. If $b \le a \in I^A_{\rm int}$, then $A \restriction b$ is a homomorphic image of $A \restriction a$, and hence $A \restriction b$ is isomorphic to an interval algebra by the Handbook Proposition 15.9. If $a, b \in I^A_{\rm int}$, then $(A \restriction (a+b)) = (A \restriction a)\times(A \restriction (b\cdot -a))$, so $A \restriction (a+b)$ is isomorphic to an interval algebra by the Handbook Proposition 15.9 and 15.11. □

Theorem 2.81. *If A is an initial chain algebra on a pseudo-tree, then $I^A_{\rm int} = A$ or $I^A_{\rm int}$ is a maximal ideal of A.*

Proof. By Theorem 2.73 there is a set H generating A which satisfies the conditions of Lemma 2.72. For each $h \in H$ let $f : A \to (A \restriction h)$ be the natural mapping: $f(a) = a \cdot h$ for all $a \in A$. So f is a surjective homomorphism. Since H generates A, $f[H]$ generates $A \restriction h$. But $f[H] = (H \downarrow h)$ is a chain by Lemma 2.71(iii). Thus $A \restriction h$ is isomorphic to an interval algebra. This is true for any $h \in H$, and H generates A. The conclusion of our theorem follows. □

Theorem 2.82. *Every initial chain algebra is minimally generated.*

Proof. This is immediate from Theorem 2.81, Proposition 2.52, and Proposition 2.65. □

Kinds of subalgebras

We now indicate the relationships between the kinds of subalgebras described in Chapter 0. See Monk [10].

Proposition 2.83. *$A \le_{\rm free} B$ implies that $A \le_{\rm proj} B$.*

Proof. Assume that $A \le_{\rm free} B$; say $B = A \oplus D$, with D free. Let $q = e = {\rm Id}_B$. □

Proposition 2.84. *$A \le_{\rm proj} B$ implies that $A \le_{\rm rc} B$.*

Proof. Suppose that $b \in B$; we want to find the least $u \in A$ such that $b \le u$. Assume the notation in the definition of $A \le_{\rm proj} B$. Write $e(b) = \sum_{i<m}(a_i \cdot c_i)$ with each $a_i \in A$ and each $c_i \in C$. Thus

$$b = q(e(b)) = \sum_{i<m} q(a_i)\cdot q(c_i) = \sum_{i<m}(a_i \cdot q(c_i)) \le \sum_{i<m} a_i \in A.$$

Suppose that $b \leq d \in A$. Then $e(b) \leq e(d) = d$, and so $\sum_{i<m}(a_i \cdot -d \cdot c_i) = 0$, and hence $\sum_{i<m} a_i \leq d$. □

Proposition 2.85. *$A \leq_{\mathrm{rc}} B$ implies that $A \leq_{\mathrm{reg}} B$.*

Proof. Suppose that $X \subseteq A$ and $\sum^A X$ exists. Let $b \in B$ be an upper bound for X; we want to show that $\sum^A X \leq b$. Let $c \in A$ be maximum such that $c \leq b$. Now if $x \in X$, then $x \leq b$; hence $x \leq c$. It follows that $\sum^A X \leq c$, hence $\sum^A X \leq b$. □

The following two propositions are clear.

Proposition 2.86. *$A \leq_{\mathrm{m}} B$ implies that $A \leq_{\mathrm{s}} B$.* □

Proposition 2.87. *$A \leq_{\mathrm{m}} B$ implies that $A \leq_{\mathrm{mg}} B$.* □

Proposition 2.88. *$A \leq_{\pi} B$ implies that $A \leq_{\mathrm{reg}} B$.*

Proof. See 4.17 in the Handbook. □

Proposition 2.89. *$A \leq_{\mathrm{rc}} B$ implies that $A \leq_{\sigma} B$.* □

Proposition 2.90. *$A \leq_{\mathrm{free}} B$ implies that $A \leq_{\mathrm{u}} B$.*

Proof. Suppose that $A \leq_{\mathrm{free}} B$; say $B = A \oplus C$ with C free. Let F be an ultrafilter on A. Let $c \in C \backslash \{0, 1\}$. Clearly $F \cup \{c\}$ and $F \cup \{-c\}$ both have fip. □

Proposition 2.91. *If $A \leq_{\mathrm{u}} B$, A is infinite, and $A \neq B$, then not$(A \leq_{\mathrm{m}} B)$.*

Proof. This is immediate from Proposition 2.35. □

Example 2.92. There exist BAs A, B such that $A \leq_{\mathrm{u}} B$, A infinite, $A \neq B$, and $A \leq_{\mathrm{mg}} B$.

Let $C = \mathrm{finco}(\omega)$. Let $f(\{i\}) = \{2i, 2i+1\}$ for all $i \in \omega$, and extend f to an isomorphism from C into C. Let $A = f[C]$. Then $A \leq_{\mathrm{mg}} C$ by Proposition 2.55. Now the principal ultrafilters on A have the form $\{u \in A : \{2i, 2i+1\} \subseteq u\}$ for some $i \in \omega$, and each of these has two extensions to an ultrafilter on C. A has one more ultrafilter, namely $\{u \in A : u \cap F = \emptyset$ for some finite subset F of ω such that $\forall i \in \omega[2i \in F$ iff $2i+1 \in F]\}$. This ultrafilter extends only to the ultrafilter on C consisting of cofinite subsets of ω. So it suffices to define a minimal extension B of C in which this ultrafilter has two extensions. Let I_0 be the ideal of C consisting of finite sets all of whose members are even, and let I_1 be the ideal of C consisting of finite sets all of whose members are odd. Then $I_0 \cap I_1 = \{\emptyset\}$. By Proposition 2.28 let $B = C(x)$ be such that $C \restriction x = I_0$ and $C\langle -x = I_1$. Now $\mathrm{Smp}^C_x = \langle I_0 \cup I_1 \rangle^{\mathrm{id}}$ is the maximal ideal of all finite subset of ω, so $C \leq_{\mathrm{m}} C(x)$. The ultrafilter in C of cofinite subsets of ω extends to two different ultrafilters on $C(x)$; one with x as a member, and one with $-x$ as a member.

These propositions can be summarized in the diagram below.

To see that no further relations exist, we need several examples. The diagram can be used to check that these examples take care of all possibilities; see also the table below. For the first example we need two lemmas of independent interest.

Proposition 2.93. *Suppose that $A \leq_{\mathrm{proj}} B$, with C, q, e as in the definition. Also suppose that $C \leq D$ with D free. Let $h : D \to C$ be a homomorphism such that $h \restriction C = \mathrm{Id}_C$, and let $q' : A \oplus D \to B$ be such that $q'(a) = a$ for all $a \in A$ and $q'(d) = q(h(d))$ for all $d \in D$. Then D, q', e also witness that $A \leq_{\mathrm{proj}} B$. Namely, $q' \circ e = \mathrm{Id}_B$ and $e \restriction A = q' \restriction A = \mathrm{Id}_A$.*

Proof. This is immediate, upon observing that $q' \restriction (A \oplus C) = q$. $\square$

$$\begin{tikzcd}
 & \leq_{\mathrm{reg}} & \leq_\sigma & & & \\
\leq_\pi \arrow[ur] & \leq_{\mathrm{rc}} \arrow[u] \arrow[ur] & & & & \\
 & \leq_{\mathrm{proj}} \arrow[u] & \leq_{\mathrm{u}} & & \leq_{\mathrm{s}} & \leq_{\mathrm{mg}} \\
 & \leq_{\mathrm{free}} \arrow[u] \arrow[ur] & & & \leq_{\mathrm{m}} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Lemma 2.94. *If $A_1 \leq_{\mathrm{proj}} B_1$ and $A_2 \leq_{\mathrm{proj}} B_2$, then $A_1 \times A_2 \leq_{\mathrm{proj}} B_1 \times B_2$.*

Proof. We first give a different proof for the following fact, Example 11.6(c) in the Handbook.

(1) If B, C, D are BAs, then $(B \times C) \oplus D \cong (B \oplus D) \times (C \oplus D)$.

In fact, by the basic property of free products there is a homomorphism h of $(B \times C) \oplus D$ into $(B \oplus D) \times (C \oplus D)$ such that $h(b, c) = (b, c)$ for any $b \in B$ and $c \in C$, and $h(d) = (d, d)$ for any $d \in D$. If $h((b, c) \cdot d) = (0, 0)$, then $b \cdot d = 0$ and $c \cdot d = 0$, and so by the free product property, $d = 0$ or $b = c = 0$. This shows that h is one-one. Finally, h maps onto since for any $b \in B$, $c \in C$, and $d \in D$ we have $h((b, 0) \cdot d) = (b \cdot d, 0)$, $h((0, c) \cdot d) = (0, c \cdot d)$, and $h(1, 0) = (1, 0)$.

Turning to the proof of our lemma, by hypothesis we get C_i, q_i, e_i for $i \in \{1, 2\}$ such that C_i is free, $q_i : A_i \oplus C_i \to B_i$ is a homomorphism, $e_i : B_i \to A_i \oplus C_i$ is a homomorphism, $q_i \circ e_i = \mathrm{Id}_{B_i}$, and $e_i \restriction A_i = q_i \restriction A_i = \mathrm{Id}_{A_i}$. By Proposition 2.93 we may assume that $C_1 = C_2 = C$, say. Let $h : (A_1 \times A_2) \oplus C \to (A_1 \oplus C) \times (A_2 \oplus C)$ be the isomorphism given in the proof of (1). Define $e' : B_1 \times B_2 \to (A_1 \times A_2) \oplus C$ by setting $e'(b_1, b_2) = h^{-1}(e_1(b_1), e_2(b_2))$. Define $q' : (A_1 \times A_2) \oplus C \to B_1 \times B_2$

by setting $q'(h^{-1}(u,v)) = (q_1(u), q_2(v))$. Clearly e' and q' are homomorphisms. If $b_1 \in B_1$ and $b_2 \in B_2$, then

$$q'(e'(b_1,b_2)) = q'(h^{-1}(e_1(b_1), e_2(b_2))) = (q_1(e_1(b_1)), q_2(e_2(b_2))) = (b_1, b_2).$$

For $a_1 \in A_1$ and $a_2 \in A_2$ we have

$$\begin{aligned} q'(a_1,a_2) &= q'(h^{-1}(h(a_1,a_2))) = q'(h^{-1}(a_1,a_2)) \\ &= (q_1(a_1), q_2(a_2)) = (a_1,a_2) \end{aligned}$$

and

$$e'(a_1,a_2) = h^{-1}(e_1(a_1), e_2(a_2)) = h^{-1}(a_1,a_2) = (a_1,a_2). \qquad \square$$

Example 2.95. Let the free generators of $\mathrm{Fr}(\omega_1)$ be $\langle x_\alpha : \alpha < \omega_1\rangle$. Let $A = \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ and $B = \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega_1)$. Now obviously $\mathrm{Fr}(\omega) \leq_{\mathrm{free}} \mathrm{Fr}(\omega_1)$, and so also $\mathrm{Fr}(\omega) \leq_{\mathrm{proj}} \mathrm{Fr}(\omega_1)$. Hence $A \leq_{\mathrm{proj}} B$ by Lemma 2.94.

We claim that not$(A \leq_{\mathrm{u}} B)$. Let F be any ultrafilter on A such that $(1,0) \in F$. Suppose that G and H are ultrafilters on B such that $F \subseteq G, H$; we want to show that $G = H$. Let (a,b) be any element of G. Since $(1,0) \in F \subseteq G$, we have $(a,0) \in G$. If $(a,0) \notin F$, then $(-a,1) \in F$, hence $(-a,0) = (-a,1)\cdot(1,0) \in F$, and so $(-a,0) \in G$, contradiction. So $(a,0) \in F$. Hence $(a,0) \in H$ and consequently $(a,b) \in H$. This shows that $G \subseteq H$; so $G = H$.

It is also clear that not$(A \leq_\pi B)$ and not$(A \leq_{\mathrm{s}} B)$.

Finally, we claim that not$(A \leq_{\mathrm{mg}} B)$. For, suppose to the contrary that $A \leq_{\mathrm{mg}} B$. Let I be the ideal of B generated by the set

$$\{(1,0)\} \cup \{(0,x_n) : n \in \omega\}.$$

Then $B/I \cong \mathrm{Fr}(\omega_1)$. In fact, let f be the homomorphism of $\mathrm{Fr}(\omega_1)$ into B/I such that $f(x_\alpha) = [(0, x_{\omega+\alpha})]_I$ for every $\alpha < \omega_1$. To show that f is a surjection, consider the generating set

$$\{(x_n,0) : n \in \omega\} \cup \{(1,0)\} \cup \{(0,x_\alpha) : \alpha < \omega_1\}$$

of B. Under the natural homomorphism from B onto B/I, this set goes over into the generating set $\{[(0,x_{\omega+\alpha})]_I : \alpha < \omega_1$ of B/I, showing that f is a surjection. To see that f is an injection, it suffices to take any $y \in \mathrm{Fr}(\omega_1)$ of the form $y = \prod_{\alpha\in F} x_\alpha^{\varepsilon(\alpha)}$ ($F \subseteq \omega_1$ finite, $\varepsilon \in {}^F2$) and show that $f(y) \neq 0$. In fact, clearly $f(y) = [\prod_{\alpha\in F}(0, x_{\omega+\alpha})^{\varepsilon(\alpha)}]_I$, and clearly this last element is nonzero. This proves that $B/I \cong \mathrm{Fr}(\omega_1)$.

It is also clear that $A/(I \cap A)$ is the two element BA. Now since $A \leq_{\mathrm{mg}} B$, by Proposition 2.47 we it follows that B/I is minimally generated. So $\mathrm{Fr}(\omega_1)$ is minimally generated, contradicting Proposition 2.57.

Example 2.96. Let $A = \mathrm{Fr}(\omega)$ and $B = A \oplus \mathrm{Finco}(\omega_1)$. Here $\mathrm{Finco}(\omega_1)$ is the algebra of finite and cofinite subsets of ω_1. So $A \leq_{\mathrm{rc}} B$ by the Handbook 11.8. Suppose that $A \leq_{\mathrm{proj}} B$, with notation as in the definition. Then $A \oplus C$ is free, hence has ccc, but B is isomorphically embeddable in C, contradiction.

We have $A \leq_{\mathrm{u}} B$. In fact, if F is any ultrafilter on A, then the two sets $\{a \cdot \{0\} : a \in F\}$ and $\{a \cdot \{1\} : a \in F\}$ both have fip, and this gives distinct extensions of F to an ultrafilter on B.

Also notice that $A \not\leq_{\mathrm{mg}} B$. In fact, suppose that $A \leq_{\mathrm{mg}} B$. Let $x = \{i\}$. Then by Proposition 2.48(iii), $A \leq_{\mathrm{mg}} A(x)$. This contradicts Proposition 2.49.

Example 2.97. Let $A = \mathrm{Finco}(\omega_1)$ and $B = \mathscr{P}(\omega_1)$. Thus $A \leq_\pi B$. Let $b = \{2\alpha : \alpha < \omega_1\}$. Then

$$\langle A \restriction b\rangle^{\mathrm{id}} = [b]^{<\omega}.$$

Clearly this ideal is not countably generated. So not$(A \leq_\sigma B)$. Also, not$(A \leq_{\mathrm{mg}} B)$. In fact, otherwise with the ideal I of finite sets we would get by Proposition 2.47 that B/I is minimally generated, contradicting Proposition 2.58.

Example 2.98. Let $A = \mathrm{Fr}(\omega)$ and $B = \mathrm{Fr}(\omega) \oplus \mathrm{Fr}(\omega_1)$. Note that not$(A \leq_\pi B)$. Also, not$(A \leq_{\mathrm{mg}} B)$; see the end of Example 2.96.

Example 2.99. Let $A' = \mathrm{Fr}(\omega)$, let f be the Stone isomorphism of A' into $B \stackrel{\mathrm{def}}{=} \mathscr{P}(\mathrm{Ult}(A'))$ ($f(a) = \{F \in \mathrm{Ult}(A') : a \in F\}$ for any $a \in A'$), and let A be the range of f. Let $\langle x_i : i \in \omega\rangle$ be a system of free generators of A'. We claim that $\sum^{A'}_{i\in\omega} x_i = 1$. In fact, if a is a nonzero element of A', write $a = \sum_{\varepsilon\in\Gamma}\prod_{i\in M} x_i^{\varepsilon(i)}$, where M is a finite subset of ω and $\Gamma \subseteq {}^M2$. Then $a \cdot x_j \neq 0$ for any $j \in \omega\backslash M$. This proves the claim.

It follows that $\sum^A_{i\in\omega} f(x_i) = 1$. However, $\bigcup_{i\in\omega} f(x_i) = \sum^B_{i\in\omega} f(x_i) \neq 1$, since for example the ultrafilter generated by $\{-x_i : i \in \omega\}$ is not in $\sum^B_{i\in\omega} f(x_i)$. Hence not$(A \leq_{\mathrm{reg}} B)$.

We claim that $A \leq_{\mathrm{u}} B$. For, suppose that $F \in \mathrm{Ult}(A)$. Then there is an ultrafilter F' on A' such that $f[F'] = F$. For each $a \in F'$ let G'_a be an ultrafilter on A' such that $a \in G'_a$ and $G'_a \neq F'$. This is possible since A' is atomless. Let $X = \{G'_a : a \in F'\}$. Thus $X \subseteq \mathrm{Ult}(A')$, and so $X \in B$. Then $F \cup \{X\}$ has the fip. In fact, if $a \in F$, then $f^{-1}(a) \in F'$, hence $f^{-1}(a) \in G'_{f^{-1}(a)}$, hence $G'_{f^{-1}(a)} \in f(f^{-1}(a)) = a$, so that $G'_{f^{-1}(a)} \in a \cap X$. Let H be an ultrafilter on B containing $F \cup \{X\}$. Thus H extends F. But also the ultrafilter $L \stackrel{\mathrm{def}}{=} \{K \in B : F' \in K\}$ extends F. In fact, if $a \in F$, then $f^{-1}(a) \in F'$, hence $F' \in f(f^{-1}(a)) = a$, as desired. Now $X \in H\backslash L$, so $H \neq L$. This shows that $A \leq_{\mathrm{u}} B$.

Finally, $A \not\leq_{\mathrm{mg}} B$. For, suppose that $A \leq_{\mathrm{mg}} B$. Let I be the ideal in A generated by $\{x_i : i \in \omega\}$. Then $f[I]$ is a maximal ideal in A, so $|A/I| = 2$. Let J be the ideal in B generated by $f[I]$. Then $J = \{X \in B :$ there is a finite $F \subseteq \omega$ such that $X \subseteq f(\sum_{i\in F} x_i)\}$. We claim that B/J is infinite. For, let

$\langle M_i : i \in \omega\rangle$ be a partition of ω into infinite sets. For each $j \in \omega$ let G_j be the ultrafilter on A' such that$x_j \in G_j$ and $-x_k \in G_j$ for all $k \neq j$. For each $i \in \omega$ let $X_i = \{G_j : j \in M_i\}$. Thus $X_i \cap X_k = \emptyset$ for $i \neq k$. We claim that $X_i \notin J$. In fact, otherwise there is a finite $F \subseteq \omega$ such that $X_i \subseteq f(\sum_{k\in F} x_k)$. Take $G_l \in X_i$ such that $l \notin F$. Then $G_l \in f(\sum_{k\in F} x_k)$, so that $\sum_{k\in F} x_k \in G_l$. But $-x_k \in G_l$ for all $k \in F$, contradiction. It follows that B/J is infinite. By Proposition 2.47, B/J is minimally generated. This contradicts Proposition 2.58.

Example 2.100. Let $A = \mathrm{Fr}(\omega)$, let I be a maximal ideal in A, and let $J = \{0\}$. By Proposition 2.28 there is a simple extension $B \stackrel{\text{def}}{=} A(x)$ of A such that $A \restriction x = I$ and $A \restriction -x = J$. By Proposition 2.32, $A \leq_{\mathrm{m}} B$. We have $\sum^A I = 1$ while $\sum^B I = x$; so not($A \leq_{\mathrm{reg}} B$). Since $A \restriction (-x) = \{0\}$, it follows that $A \not\leq_\pi B$.

Example 2.101. Let $A = \mathrm{Finco}(\omega_1)$ and $B = \overline{A \oplus \mathrm{Fr}(\omega)}$. Clearly $A \leq_{\mathrm{u}} B$. For the element $b \stackrel{\text{def}}{=} \sum_{\alpha\in\omega_1}\{2\alpha\}$, the ideal $A \restriction b$ is not countably generated. $A \not\leq_{\mathrm{mg}} B$ by the argument at the end of Example 2.96. $A \not\leq_{\mathrm{s}} B$ since every simple extension of A is superatomic by Proposition 2.30.

Example 2.102. Let $A = \mathrm{Finco}(\omega_1)$, $I = \langle\{\{2\alpha\} : \alpha < \omega_1\}\rangle$, and $J = \langle\{\{2\alpha+1\} : \alpha < \omega_1\}\rangle$. Let $B = A(x)$ with $B \restriction x = I$ and $B \restriction (-x) = J$. Then $A \leq_{\mathrm{m}} B$ and *not*($A \leq_\sigma B$), since $A \restriction x$ is not countably generated. $A \not\leq_\pi B$ by Proposition 2.44. Clearly $A \leq_{\mathrm{reg}} B$.

The properties of these examples are summarized in the following table.

Example	$\leq_{\mathrm{reg}}$	$\leq_\sigma$	$\leq_\pi$	$\leq_{\mathrm{rc}}$	$\leq_{\mathrm{proj}}$	$\leq_{\mathrm{u}}$	$\leq_{\mathrm{free}}$	$\leq_{\mathrm{s}}$	$\leq_{\mathrm{mg}}$	$\leq_{\mathrm{m}}$
2.95	Y	Y	N	Y	Y	N	N	N	N	N
2.96	Y	Y	N	Y	N	Y	N	N	N	N
2.97	Y	N	Y	N	N	N	N	N	N	N
2.98	Y	Y	N	Y	Y	Y	Y	N	N	N
2.99	N	Y	N	N	N	Y	N	N	N	N
2.100	N	Y	N	N	N	N	N	Y	Y	Y
2.101	Y	N	N	N	N	Y	N	N	N	N
2.102	Y	N	Y	N	N	N	N	Y	Y	Y

There are 10 subalgebra notions considered, and there are 13 implications between them, given in the diagram following Proposition 2.93. None of the other 77 implications hold. This is shown by the above examples, or in some cases is obvious:

$\leq_{\rm free}$ and $\not\leq_\pi$	2.98
$\leq_{\rm free}$ and $\not\leq_{\rm m}$	2.98
$\leq_{\rm free}$ and $\not\leq_{\rm mg}$	2.98
$\leq_{\rm free}$ and $\not\leq_{\rm s}$	2.98
$\leq_{\rm proj}$ and $\not\leq_{\rm free}$	2.95
$\leq_{\rm proj}$ and $\not\leq_{\rm u}$	2.95
$\leq_{\rm proj}$ and $\not\leq_\pi$	2.95
$\leq_{\rm proj}$ and $\not\leq_{\rm m}$	2.95
$\leq_{\rm proj}$ and $\not\leq_{\rm s}$	2.95
$\leq_{\rm proj}$ and $\not\leq_{\rm mg}$	2.95

$\leq_{\rm u}$ and $\not\leq_{\rm free}$	2.96
$\leq_{\rm u}$ and $\not\leq_{\rm proj}$	2.96
$\leq_{\rm u}$ and $\not\leq_{\rm rc}$	2.99
$\leq_{\rm u}$ and $\not\leq_\sigma$	2.101
$\leq_{\rm u}$ and $\not\leq_{\rm reg}$	2.99
$\leq_{\rm u}$ and $\not\leq_\pi$	2.98
$\leq_{\rm u}$ and $\not\leq_{\rm m}$	2.98
$\leq_{\rm u}$ and $\not\leq_{\rm s}$	2.98
$\leq_{\rm u}$ and $\not\leq_{\rm mg}$	2.98

$\leq_{\rm rc}$ and $\not\leq_{\rm free}$	2.95
$\leq_{\rm rc}$ and $\not\leq_{\rm proj}$	2.96
$\leq_{\rm rc}$ and $\not\leq_{\rm u}$	2.95
$\leq_{\rm rc}$ and $\not\leq_\pi$	2.96
$\leq_{\rm rc}$ and $\not\leq_{\rm m}$	2.95
$\leq_{\rm rc}$ and $\not\leq_{\rm s}$	2.95
$\leq_{\rm rc}$ and $\not\leq_{\rm mg}$	2.95

$\leq_\pi$ and $\not\leq_{\rm free}$	2.97
$\leq_\pi$ and $\not\leq_{\rm proj}$	2.97
$\leq_\pi$ and $\not\leq_{\rm u}$	2.97
$\leq_\pi$ and $\not\leq_{\rm rc}$	2.97
$\leq_\pi$ and $\not\leq_\sigma$	2.97
$\leq_\pi$ and $\not\leq_{\rm m}$	2.97
$\leq_\pi$ and $\not\leq_{\rm s}$	2.97
$\leq_\pi$ and $\not\leq_{\rm mg}$	2.97

$\leq_{\rm reg}$ and $\not\leq_{\rm u}$	2.95
$\leq_{\rm reg}$ and $\not\leq_\sigma$	2.97
$\leq_{\rm reg}$ and $\not\leq_{\rm m}$	2.95
$\leq_{\rm reg}$ and $\not\leq_{\rm s}$	2.95
$\leq_{\rm reg}$ and $\not\leq_{\rm mg}$	2.95
$\leq_{\rm reg}$ and $\not\leq_{\rm free}$	2.95
$\leq_{\rm reg}$ and $\not\leq_{\rm proj}$	2.96
$\leq_{\rm reg}$ and $\not\leq_{\rm rc}$	2.97
$\leq_{\rm reg}$ and $\not\leq_\pi$	2.96

$\leq \sigma$ and $\not\leq_{\rm u}$	2.95
$\leq \sigma$ and $\not\leq_{\rm reg}$	2.99
$\leq \sigma$ and $\not\leq_\pi$	2.95
$\leq \sigma$ and $\not\leq_{\rm m}$	2.95
$\leq \sigma$ and $\not\leq_{\rm s}$	2.95
$\leq \sigma$ and $\not\leq_{\rm mg}$	2.95
$\leq \sigma$ and $\not\leq_{\rm free}$	2.95
$\leq \sigma$ and $\not\leq_{\rm proj}$	2.96
$\leq \sigma$ and $\not\leq_{\rm rc}$	2.99

$\leq_{\rm m}$ and $\not\leq_{\rm free}$	2.100
$\leq_{\rm m}$ and $\not\leq_{\rm proj}$	2.100
$\leq_{\rm m}$ and $\not\leq_{\rm u}$	2.100
$\leq_{\rm m}$ and $\not\leq_{\rm rc}$	2.100
$\leq_{\rm m}$ and $\not\leq_\pi$	2.100
$\leq_{\rm m}$ and $\not\leq_{\rm reg}$	2.100
$\leq_{\rm m}$ and $\not\leq_\sigma$	2.102

$\leq_{\rm s}$ and $\not\leq_{\rm free}$	2.100
$\leq_{\rm s}$ and $\not\leq_{\rm proj}$	2.100
$\leq_{\rm s}$ and $\not\leq_{\rm u}$	2.100
$\leq_{\rm s}$ and $\not\leq_{\rm rc}$	2.100
$\leq_{\rm s}$ and $\not\leq_\pi$	2.100
$\leq_{\rm s}$ and $\not\leq_{\rm reg}$	2.100
$\leq_{\rm s}$ and $\not\leq_\sigma$	2.102
$\leq_{\rm s}$ and $\not\leq_{\rm m}$	obv.
$\leq_{\rm s}$ and $\not\leq_{\rm mg}$	obv.

$\leq_{\rm mg}$ and $\not\leq_{\rm free}$	2.100
$\leq_{\rm mg}$ and $\not\leq_{\rm proj}$	2.100
$\leq_{\rm mg}$ and $\not\leq_{\rm u}$	2.100
$\leq_{\rm mg}$ and $\not\leq_{\rm rc}$	2.100
$\leq_{\rm mg}$ and $\not\leq_\pi$	2.100
$\leq_{\rm mg}$ and $\not\leq_{\rm reg}$	2.100
$\leq_{\rm mg}$ and $\not\leq_\sigma$	2.102
$\leq_{\rm mg}$ and $\not\leq_{\rm m}$	obv.
$\leq_{\rm mg}$ and $\not\leq_{\rm s}$	obv.

Other classes of Boolean algebras

We briefly mention some other classes of Boolean algebras.

$\perp$-free Boolean algebras. A BA A is $\perp$-freely generated by a subset D iff D generates A, and if $f : D \to B$ is a mapping from D into any BA B such that $\forall a, b \in D(a \cdot b = 0 \Rightarrow f(a) \cdot f(b) = 0)$, then f extends to a homomorphism from A into B. This notion was introduced and studied in Heindorf [94]. Corey Bruns has studied cardinal functions on these algebras, and has generalized the notion; see Bruns [13].

Moderate Boolean algebras. A subset X of a BA A is *moderate* iff $0 \notin X$ and for every $a \in A$, the set $\{x \in X : a \cdot x \neq 0 \neq -a \cdot x\}$ is finite. A BA A is *moderate* iff all of its ideals are generated by moderate sets. This notion was introduced and studied in Heindorf [92]; he also proved some facts about cardinal functions on this class.

Retractive Boolean algebras. A BA A is *retractive* iff for every epimorphism $f : A \to B$ there is a homomorphism $g : B \to A$ such that $f \circ g$ is the identity on B. See Heindorf [86].

A theorem of M. Rubin is that every subalgebra of an interval algebra is retractive; see the Handbook. Thus by Theorem 2.13, every pseudo-tree algebra is retractive.

Projective Boolean algebras. A BA A is *projective* iff for all homomorphisms $f : A \to B$ and $g : C \to B$ with g onto, there is a homorphism $h : A \to C$ such that $f = g \circ h$. See Koppelberg [89b] for an extensive treatment of this notion.

Weakly projective Boolean algebras. A BA A is *weakly projective* iff A has a dense projective subalgebra.

Cohen algebras. A *Cohen algebra* is a BA which is co-complete with a product of free BAs.

Co-interval algebras. A BA A is a *co-interval algebra* iff it is co-complete with an interval algebra.

Rc-filtered Boolean algebras. A BA A is *rc-filtered* iff for all homomorphisms $f : A \to B$ and $g : C \to B$ with g surjective there is an order-preserving map $h : A \to C$ which also preserves disjointness and is such that $g \circ h = f$. See Heindorf, Shapiro [94].

$\boldsymbol{\sigma}$-filtered Boolean algebras. A BA A is *σ-filtered* iff there is a function $I : A \to [A]^{\leq\omega}$ such that for all $a, b \in A$, $a \leq b$ implies that there is an $x \in I(a) \cap I(b)$ such that $a \leq x \leq b$. See Heindorf, Shapiro [94].

Lindelöf Boolean algebras. A BA A is *Lindelöf* iff every ideal in A is countably generated. This notion and the next one were introduced and studied in Heindorf [85].

Paracompact Boolean algebras. A BA A is *paracompact* iff every ideal in A is generated by a set of pairwise disjoint elements.

Model-theoretic classes of Boolean algebras. Many classes of BAs turn out to be model-theoretically definable in the following sense. Suppose that $\mathscr{L}$ is a first-order language. Expand $\mathscr{L}$ to $\mathscr{L}_P$ by adding a unary relation symbol P. A sentence φ of $\mathscr{L}_P$ is *universal over* $\mathscr{L}$ iff it has the form $\forall\overline{x}\varphi$ with φ a Boolean combination of formulas of $\mathscr{L}$ and atomic formulas of the form $P\tau$, τ a term of $\mathscr{L}$. Suppose that Σ is a set of formulas of $\mathscr{L}_P$, each universal over $\mathscr{L}$. Given an $\mathscr{L}$-structure A, we say that $U \subseteq A$ is a Σ-*subset* of A iff $(A, U) \models \Sigma$. We let $X(A, \Sigma)$ be the set of all Σ-subsets of A. Thus $X(A, \Sigma)$ is a subset of $\mathscr{P}(A)$, and under the Tychonoff topology on $\mathscr{P}(A)$ (homeomorphic to ${}^A2$), it is closed. We let $B(\Sigma, A)$ be the BA of clopen subsets of $X(A, \Sigma)$. For any theory T in $\mathscr{L}$ we then define the *Boolean algebra model class of* T *under* Σ to be

$$\{B(A, \Sigma) : A \models T\}.$$

See Koppelberg [93] for this notion.

Free poset algebras. Given a poset P, the *free poset algebra on* P is the BA generated by P freely – subject only to the conditions $p \leq q$ when this holds in P. (The rigorous definition is the quotient of the free BA on P divided by the ideal generated by $\{p \cdot -q : p \leq q\}$.) See Bonnet, Rubin [04].

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3 Cellularity

A BA A is said to satisfy the *κ-chain condition* (κ-cc) if every disjoint subset of A has power $< \kappa$. Thus for κ non-limit, this is the same as saying that the cellularity of A is $< \kappa$. Of most interest is the ω_1-chain condition, called ccc for short (countable chain condition). We shall return to it below.

The attainment problem for cellularity is covered by two classical theorems of Erdős and Tarski: see the Handbook Theorem 3.10 and Example 11.14. Cellularity is attained for any singular cardinal, while for every weakly inaccessible cardinal there are examples of BAs with cellularity not attained.

If A is a subalgebra of B, then obviously $\mathrm{c}(A) \leq \mathrm{c}(B)$ and the difference can be arbitrarily large. We now consider the special kinds of subalgebras defined in Chapter 0; see also Chapter 2 for relationships between them.

- With $A \leq_{\mathrm{rc}} B$ the difference can be large; for example, $\mathrm{Fr}(\omega) \leq_{\mathrm{rc}} \mathrm{Fr}(\omega) \oplus \mathrm{Finco}(\kappa)$; see the Handbook, 11.8. By the known relationships, this example applies also to $\leq_{\mathrm{reg}}$ and $\leq_{\sigma}$.
- With $A \leq_{\pi} B$ it is clear that $\mathrm{c}(A) = \mathrm{c}(B)$.
- With $A \leq_{s} B$ the difference can be large. Namely, let κ be any infinite cardinal, and let A be the free BA on the system $\langle x_\alpha : \alpha < \kappa \rangle$ of free generators. Let I be the ideal in A generated by all elements $x_\alpha \cdot x_\beta$ with $\alpha < \beta < \kappa$. Clearly $x_\alpha \notin I$ for all $\alpha < \kappa$. Let $J = \{0\}$. Then there is a simple extension $B = A(y)$ of A such that $I = \{b \in B : b \leq y\}$ and $J = \{b \in B : b \leq -y\}$. (See Proposition 2.28.) Then $\langle x_\alpha \cdot -y : \alpha < \kappa \rangle$ is a system of pairwise disjoint nonzero elements in B.
- With $A \leq_{\mathrm{m}} B$ and A infinite we have $\mathrm{c}(A) = \mathrm{c}(B)$. In fact, let $B = A(x)$. If $A \restriction x$ and $A \restriction -x$ are both nonprincipal, then $\mathrm{c}(A) = \mathrm{c}(B)$ by Proposition 2.44. If one of these ideals is principal, then by Proposition 2.45 we may assume that x is an atom of B. In this case, A is isomorphic to $B \restriction -x$, and then an easy argument gives $\mathrm{c}(A) = \mathrm{c}(B)$ again.
- With $A \leq_{\mathrm{mg}} B$, since every superatomic BA is minimally generated, it is clear that the difference here can be large.
- With $A \leq_{\mathrm{free}} B$, we claim that $\mathrm{c}(A) = \mathrm{c}(B)$. Write $B = A \oplus F$ with F a free algebra, and suppose that $\langle u_\alpha : \alpha < \mathrm{c}(A)^+ \rangle$ is a system of pairwise disjoint non-zero elements of B; we want to get a contradiction. We may assume

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that each u_α has the form $b_\alpha \cdot c_\alpha$ with $b_\alpha \in A$ and $c_\alpha \in F$. By 9.16 of the Handbook, we may assume that either the c_α's are all equal, or they form an independent set. Then $\langle b_\alpha : \alpha < \mathrm{c}(A)^+\rangle$ is a system of pairwise disjoint nonzero elements of A, contradiction.

- With $A \leq_\sigma B$, the difference in cellularities can be large, by the example for $\leq_{\mathrm{rc}}$.
- With $A \leq_{\mathrm{proj}} B$, since B embeds in a free extension of A, it follows from the remark about $\leq_{\mathrm{free}}$ that $\mathrm{c}(A) = \mathrm{c}(B)$.
- With $A \leq_{\mathrm{u}} B$, we can get an example with A ccc and B of arbitrarily large cellularity. Namely, let κ be an infinite cardinal. Take A countable, let $A(x)$ be the free extension of A by x (the free product of A with the four element algebra $\{0, x, -x, 1\}$), and let $B = A(x) \oplus \mathrm{Finco}(\kappa)$.

If B is a homomorphic image of A, then cellularity can change either way from A to B. For example, if A is a free BA, then it has cellularity ω, while a homomorphic image of A can have very large cellularity. On the other hand, given any infinite BA A, it has a homomorphic image of cellularity ω: take a denumerable subalgebra B of A, and by Sikorski's theorem extend the identity mapping from B into the completion $\overline{B}$ of B to a homomorphism of A into $\overline{B}$.

By an easy argument, $\mathrm{c}\left(\prod_{i\in I} A_i\right) = |I| + \sup_{i\in I}\mathrm{c}(A_i)$, if all the A_i are nontrivial and either I is infinite or some A_i is infinite. The same computation holds for weak products.

Now we turn to chain conditions in free products, where there has been a lot of work done. We formulate some results here using the function c' instead of c. Recall that $\mathrm{c}'(A)$ is the least cardinal greater than $|X|$ for every disjoint subset X of A. Thus $\mathrm{c}'(A) = (\mathrm{c}(A))^+$ if $\mathrm{c}(A)$ is attained, while $\mathrm{c}'(A) = \mathrm{c}(A)$ otherwise. If $\mathrm{c}'(A) = \mathrm{c}(A)$, then $\mathrm{c}'(A)$ is a regular limit cardinal.

The most general question for free products can be formulated as follows.

Problem 3. *For which infinite cardinals κ, λ are there BAs A, B such that $\mathrm{c}'(A) = \kappa$, $\mathrm{c}'(B) = \lambda$, while $\mathrm{c}'(A \oplus B) > \max(\kappa, \lambda)$?*

See Problems 4, 5 below for more specific forms of this question. Before listing some specific results relevant to Problem 3 we make some general remarks.

In considering cellularity, it is frequently useful to work with partially ordered sets rather than BAs. If P is a partially ordered set, then one can take the completion of P; this is a complete BA $\overline{P}$ such that there is a function $e : P \to \overline{P}$ satisfying some natural conditions; see the Handbook, page 55. Elements $p, q \in P$ are *compatible* iff there is an $r \in P$ with $r \leq p, q$. Then p and q are compatible iff $e(p) \cdot e(q) \neq 0$. We say that P *has the* κ-cc iff every compatible family of pairwise incompatible elements of P has size $< \kappa$. Then the countable chain condition, ccc, is by definition the ω_1-cc. Clearly P has the κ-cc iff every pairwise disjoint subset of $\overline{P}$ has size $< \kappa$.

Given partially ordered sets P, Q, we make $P \times Q$ into a partially ordered set by defining $(p_1, q_1) \leq (p_2, q_2)$ iff $p_1 \leq p_2$ and $q_1 \leq q_2$.

Lemma 3.1. *Let P and Q be partially ordered sets. Then*

$$\overline{P \times Q} \cong \overline{\overline{P} \oplus \overline{Q}}.$$

Proof. Let e_P be the embedding of P into $\overline{P}$; and let e_Q be the embedding of Q into $\overline{Q}$. We now define $e : P \times Q \to \overline{\overline{P} \oplus \overline{Q}}$ by setting $e(p,q) = e_P(p) \cdot e_Q(q)$ for any $p \in P$, $q \in Q$. The following conditions are then clear:

(1) If $(p,q) \leq (p',q')$, then $e(p,q) \leq e(p',q')$.

(2) If (p,q) and (p',q') are incompatible (i.e., there is no $(r,s) \in P \times Q$ such that $(r,s) \leq (p,q),(p',q')$, then $e(p,q) \cdot e(p',q') = 0$.

(3) $e[P \times Q]$ is dense in $\overline{\overline{P} \oplus \overline{Q}}$.

Hence the lemma follows from Theorem 4.14 of the Handbook. □

The following simple fact justifies consideration of $\mathrm{c}(A \oplus A)$ instead of the more general $\mathrm{c}(A \oplus B)$.

Lemma 3.2. *The following conditions are equivalent, for any infinite cardinal κ:*

(i) *There are infinite BAs A and B such that* $\mathrm{c}'(A), \mathrm{c}'(B) \leq \kappa$ *and* $\mathrm{c}'(A \oplus B) > \kappa$.
(ii) *There is an infinite BA A such that* $\mathrm{c}'(A) \leq \kappa$ *and* $\mathrm{c}'(A \oplus A) > \kappa$.

In fact, (ii) obviously implies (i). Now assume that (i) holds. Then by the Handbook, Example 11.6(c),

$$(A \times B) \oplus (A \times B) \cong (A \oplus A) \times (A \oplus B) \times (B \oplus A) \times (B \oplus B).$$

By the results on products, $\mathrm{c}'(A \times B) \leq \kappa$, while $\mathrm{c}'(A \oplus B) > \kappa$, so $\mathrm{c}((A \times B) \oplus (A \times B)) > \kappa$. □

Now we list some facts relevant to Problem 3, giving proofs for some of them.

(C1) $\mathrm{c}'(A \oplus B) \leq (2^{\mathrm{c}(A) \cdot \mathrm{c}(B)})^+$.

To see this, suppose that $\langle x_\alpha : \alpha < (2^{\mathrm{c}A \cdot \mathrm{c}B})^+ \rangle$ is a system of disjoint elements of $A \oplus B$. Without loss of generality we may assume that for each $\alpha < (2^{\mathrm{c}A \cdot \mathrm{c}B})^+$, the element x_α has the form $a_\alpha \times b_\alpha$, where $a_\alpha \in A$ and $b_\alpha \in B$ (we use $\times$ to make clear that the indicated product of elements is in the algebra $A \oplus B$). Thus for distinct α, β we have $a_\alpha \cdot a_\beta = 0$ or $b_\alpha \cdot b_\beta = 0$, and hence the Erdős–Rado partition theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ implies that there is a subset Y of $(2^{\mathrm{c}A \cdot \mathrm{c}B})^+$ of power $(\mathrm{c}(A) \cdot \mathrm{c}(B))^+$ such that either $a_\alpha \cdot a_\beta = 0$ for all distinct $\alpha, \beta \in Y$ or $b_\alpha \cdot b_\beta = 0$ for all distinct $\alpha, \beta \in Y$, which is impossible. (For this partition relation, see Erdős, Hajnal, Máté, Rado [84], pp. 98–100.

(C2) Under GCH, $\mathrm{c}'(A \oplus B) \leq \max((\mathrm{c}(A))^{++}, (\mathrm{c}(B))^{++})$.

This follows from (C1).

(C3) If $\mathrm{c}'(A) \leq (2^\kappa)^+$ and $\mathrm{c}'(B) \leq \kappa^+$, then $\mathrm{c}'(A \oplus B) \leq (2^\kappa)^+$.

This holds by the partition theorem $(2^\kappa)^+ \to ((2^\kappa)^+, \kappa^+)^2$ (see the above book, Corollary 17.5).

(C4) Under GCH, if $\mathrm{c}'(A) \le \kappa^{++}$ and $\mathrm{c}'(B) \le \kappa^+$, then $\mathrm{c}'(A \oplus B) \le \kappa^{++}$.

This follows from (C3).

(C5) If κ is strong limit, $\mathrm{c}'(A) \le \kappa^+$, and $\mathrm{c}'(B) \le \mathrm{cf}(\kappa)$, then $\mathrm{c}'(A \oplus B) \le \kappa^+$. This follows from Corollary 17.2 in the above book, upon noting that $L_{\kappa^+}(\kappa^+) = \mathrm{cf}(\kappa)$.

The correspondence between partially ordered sets and BAs is used in the proof of the following result.

(C6) Let M be a countable transitive model of ZFC, κ an uncountable regular cardinal in M, and form $M[G]$ by Cohen forcing using the set of finite functions from a subset of κ into 2. Then in $M[G]$ there is a ccc BA A such that $\mathrm{c}'(A \oplus A) \ge \kappa^+$.

This fact is due to Fleissner [78]; see also Kunen [80], Exercise (C6) on page 292. We give the argument here.

Lemma 3.3. *Let κ be any infinite cardinal. For each subset C of $[\kappa]^2$ we let $P(C)$ be the partially ordered set consisting of*

$$\{s \subseteq \kappa : [s]^2 \subseteq C \textit{ and } s \textit{ is finite}\},$$

with the partial ordering $\supseteq$. Then

(i) *Elements s_1, s_2 of $P(C)$ are compatible* iff $[s_1 \cup s_2]^2 \subseteq C$.
(ii) *For any $C \subseteq [\kappa]^2$, the partially ordered set $P(C) \times P(\kappa\backslash C)$ has a family of κ pairwise incompatible elements.*

Proof. (i) is clear.

For (ii), we let the family be $\{(\{\alpha\}, \{\alpha\}) : \alpha < \kappa\}$. Suppose that $\alpha < \beta < \kappa$ and $(\{\alpha\}, \{\alpha\})$ and $(\{\beta\}, \{\beta\})$ are compatible. By (i) it follows that $\{\alpha, \beta\} \in C \cap (\kappa\backslash C)$, contradiction. □

Theorem 3.4. *Let M be a countable transitive model of ZFC, κ an uncountable regular cardinal in M, and form $M[G]$ by Cohen forcing using the set of finite functions from a subset of κ into 2. Then cardinals are preserved in going from M to $M[G]$, and in $M[G]$ $2^\omega \ge \kappa$ and there are ccc BAs A, B such that* $\mathrm{c}'(A \oplus B) \ge \kappa^+$.

Proof. We use the framework of Kunen [80]. Let R be the partially ordered set of all finite functions $\subseteq \kappa \times 2$ under $\supseteq$, and let S be the partially ordered set of all finite functions $\subseteq [\kappa]^2 \times 2$, under the relation $\supseteq$. If f is a bijection from κ onto $[\kappa]^2$, then R is isomorphic to S via $p \mapsto p \circ f^{-1}$. Hence it suffices to take the generic extension using the partially ordered set S.

Let G be M generic over S. Clearly $\bigcup G$ is a function mapping $[\kappa]^2$ into 2. We define

$$C = \left\{\{\alpha, \beta\} : \left(\bigcup G\right)(\{\alpha, \beta\}) = 1\right\}.$$

Thus $C \subseteq [\kappa]^2$. By Lemma 3.3, $\overline{P(C)} \times \overline{P(\kappa\backslash X)}$ has an incompatible family of size κ. Let $A = \overline{P(C)}$ and $B = \overline{P(\kappa\backslash C)}$. Thus by Lemma 3.1 we have $\mathrm{c}'(A \oplus B) \geq \kappa^+$.

Note that

$$P(C) = \left\{s \subseteq \kappa : s \text{ is finite and for all distinct } \alpha, \beta \in s\left[\left(\bigcup G\right)(\{\alpha,\beta\}) = 1\right]\right\};$$
$$P(\kappa\backslash C) = \left\{s \subseteq \kappa : s \text{ is finite and for all distinct } \alpha, \beta \in s\left[\left(\bigcup G\right)(\{\alpha,\beta\}) = 0\right]\right\}.$$

Thus by symmetry it suffices to show that $P(C)$ is ccc. Suppose not. Let f be a bijection from ω_1 onto an antichain in $P(C)$. We may assume that all members of $\mathrm{rng}(f)$ have the same finite size m. Clearly $m \neq 0$. Thus

(1) If $\alpha \in \omega_1$, then $[f(\alpha)]^2 \subseteq C$.

(2) If $\alpha, \beta \in \omega_1$ and $\alpha \neq \beta$, then there exist $\xi \in f(\alpha)$ and $\eta \in f(\beta)$ such that $\{\xi, \eta\} \notin C$.

Say $f = \tau_G$. Then there is a $p \in G$ such that

(3) $p \Vdash \tau$ is a function $\wedge\, \mathrm{dmn}(\tau) = \check{\omega}_1 \wedge \tau$ is one-one
$$\wedge\, \forall \alpha \in \check{\omega}_1 \forall \xi, \eta \in \tau(\alpha)[\xi \neq \eta \rightarrow \exists q \in \Gamma[\mathrm{op}(\mathrm{up}(\xi,\eta), \check{1}) \in q]]$$
$$\wedge\, \forall \alpha, \beta \in \check{\omega}_1[\alpha \neq \beta \rightarrow \exists \xi \in \tau(\alpha) \exists \eta \in \tau(\beta) \exists q \in \Gamma[\mathrm{op}(\mathrm{up}(\xi,\eta), \check{0}) \in q]].$$

For up and op, see Kunen VII2.16; for Γ, see Kunen VII2.12. Now fix $\alpha \in \omega_1$. From (3) it follows that

$$p \Vdash \exists \xi \in \check{\kappa}[\xi \text{ is the first element of } \tau(\check{\alpha})].$$

Hence there exist an $\xi_0^\alpha \in \kappa$ and a $p_\alpha^0 \leq p$ with $p_\alpha^0 \in G$ such that

$$p_\alpha^0 \Vdash \check{\xi}_0^\alpha \text{ is the first element of } \tau(\check{\alpha}).$$

Continuing by induction, we obtain $\xi_0^\alpha, \ldots, \xi_{m-1}^\alpha$ and $p_\alpha^{m-1} \leq \cdots \leq p_\alpha^0 \leq p$ such that each $p_i^\alpha \in G$ and

$$p_\alpha^i \Vdash \check{\xi}_i^\alpha \text{ is the } (i+1)\text{-st element of } \tau(\check{\alpha})$$

for $i < m$. We write p_α in place of p_α^{m-1}. Let $s_\alpha = \{\xi_0^\alpha, \ldots, \xi_{m-1}^\alpha\}$. Then clearly

$$p_\alpha \Vdash \tau(\check{\alpha}) = \check{s}_\alpha. \tag{4}$$

$$[s_\alpha]^2 \subseteq C. \tag{5}$$

In fact, suppose that $\xi, \eta \in s_\alpha$ and $\xi \neq \eta$. Then by the second line of (3), $p_\alpha \Vdash \exists q \in \Gamma[\mathrm{op}(\mathrm{up}(\check{\xi}, \check{\eta}), \check{1}) \in q]$. Since $p_\alpha \in G$, we get $q \in \Gamma_G = G$ such that $q(\{\xi,\eta\}) = 1$; thus $\{\xi, \eta\} \in C$.

Clearly

(6) For all $\{\xi, \eta\} \in [\kappa]^2$, the set $\{q : \{\xi, \eta\} \in \text{dmn}(q)\}$ is dense in S.

By (6), we may assume that for all $\alpha < \omega_1$, $[s_\alpha]^2 \subseteq \text{dmn}(p_\alpha)$.

(7) If $\xi, \eta \in s_\alpha$ and $\xi \neq \eta$, then $p_\alpha(\{\xi, \eta\}) = 1$.

This is clear from (5) and (6), since $p_\alpha \in G$.

Now let M be an uncountable subset of ω_1 such that $\langle s_\alpha : \alpha \in M \rangle$ is a Δ-system, say with root r. Fix $\alpha \in M$. We claim now that the set

$$\text{(8)} \qquad \{q : \exists \beta \in M \forall \xi \in s_\alpha \backslash r \forall \eta \in s_\beta \backslash r[\{\xi, \eta\} \in \text{dmn}(q) \text{ and } q(\{\xi, \eta\}) = 1]\}$$

is dense in S. For, suppose that t is any member of S. Then the set

$$\{\beta \in M : \exists \xi \in s_\alpha \backslash r \exists \eta \in s_\beta \backslash r[\{\xi, \eta\} \in \text{dmn}(t)\}$$

is finite. Choose $\beta \in M$ not in this set, and let q extend t by letting $q(\{\xi, \eta\}) = 1$ whenever $\xi \in s_\alpha \backslash r$ and $\eta \in s_\beta \backslash r$. Thus q is in the set (8). So that set is dense.

Choose $q \in G$ which is in the set (8). A common extension of p_α and q contradicts the third line of (3). □

We return to our general list of results about cellularity of free products.

(C7) Under CH there are ccc partial orders P, Q such that $P \times Q$ is not ccc.

This result is due to Laver; see Kunen [80], Exercise VIII(C8).

The following result is complementary to (C6) and (C7):

(C8) Under MA + ¬CH, the free product of two ccc BAs is again ccc.

We go through a proof of this.

Lemma 3.5 (MA + ¬CH). *Suppose that $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a system of elements in a ccc BA A. Then there is an uncountable $S \subseteq \omega_1$ such that $\langle x_\alpha : \alpha \in S \rangle$ has the finite intersection property.*

Proof. We may assume that A is complete. For each $\alpha < \omega_1$ let $y_\alpha = \sum_{\gamma > \alpha} x_\gamma$. Then, we claim,

($*$) There is an $\alpha < \omega_1$ such that for all $\beta > \alpha$ we have $y_\beta = y_\alpha$.

Otherwise, since clearly $\alpha < \beta \to y_\alpha \geq y_\beta$, we easily get an increasing sequence $\langle \beta(\xi) : \xi < \omega_1 \rangle$ of ordinals less than ω_1 such that $y_{\beta(\xi)} > y_{\beta(\eta)}$ whenever $\xi < \eta < \omega_1$. But then $\langle y_{\beta(\xi)} \cdot -y_{\beta(\xi+1)} \rangle$ is a disjoint family of power ω_1, contradiction.

Thus ($*$) holds, and we fix an α as indicated there. The partial ordering P that we want to apply Martin's axiom to is $\{x \in A : 0 \neq x \leq y_\alpha\}$ under $\leq$. It is a ccc partial ordering since A is a ccc BA. Now for the dense sets. For each $\beta < \omega_1$ let

$$D_\beta = \{p \in P : \text{there is a } \gamma > \beta \text{ such that } p \leq x_\gamma\}.$$

To see that D_β is dense in P, let $p \in P$ be arbitrary. Choose $\delta \in \omega_1$ with $\delta > \alpha, \beta$. Then $y_\alpha = y_\delta$, so from $0 \neq p \leq y_\alpha$ we infer that there is a $\gamma > \delta$ such that $p \cdot x_\gamma \neq 0$. Thus $p \cdot x_\gamma$ is the desired element of D_β which is $\leq p$.

Now let G be a filter on P intersecting each dense set D_β for $\beta < \omega_1$, by MA + ¬CH. Then it is easy to see that $S \stackrel{\text{def}}{=} \{x_\gamma : \gamma < \omega_1, \text{ and } p \leq x_\gamma \text{ for some } p \in G\}$ is the set desired in the lemma. □

Theorem 3.6 (MA + ¬CH)**.** *The free product of ccc BAs A and B is again ccc.*

Proof. Let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a disjoint system of elements of $A \oplus B$. Without loss of generality we may assume that each x_α has the form $a_\alpha \times b_\alpha$ where $a_\alpha \in A$ and $b_\alpha \in B$. By the lemma, let S be an uncountable subset of ω_1 such that $\langle a_\alpha : \alpha \in S \rangle$ has the finite intersection property. But then, by the free product property, $\langle b_\alpha : \alpha \in S \rangle$ is a disjoint system in B, contradiction. □

Now we turn to higher cellularity.

(C9) For $\mu > \aleph_1$ regular, there is a BA A such that $\mathrm{c}'(A) = \mu^+$ while $\mathrm{c}'(A \oplus A) > \mu^+$.

This is proved in Shelah [91]. We give the construction here. It depends upon a combinatorial result that should be useful in other situations. The proof of this result requires considerable notation, and for the reader's convenience we indicate in boxes the most important notations when they are introduced. We define a general combinatorial principle; see Shelah [94], Appendix 1, for more information on this principle.

$\mathrm{Pr}_1(\lambda, \mu, \kappa, \theta)$means that λ is an infinite cardinal, $\lambda \geq \mu \geq \kappa + \theta$, and

$$\begin{aligned} &\exists c : [\lambda]^2 \to \kappa \forall \xi \in \theta \backslash 1 \forall a \in {}^\mu([\lambda]^\xi) \forall \gamma < \kappa [\forall \alpha, \beta < \mu[\alpha \neq \beta \to a_\alpha \cap a_\beta = \emptyset] \\ &\to \exists \alpha, \beta \in \mu[\alpha \neq \beta \text{ and } \forall \delta \in a_\alpha \forall \varepsilon \in a_\beta [c(\{\delta, \varepsilon\}) = \gamma]]]. \end{aligned}$$

$\boxed{\mathrm{Pr}_1(\lambda, \mu, \kappa, \theta)}$

For any infinite cardinal λ, a *strict club system* for λ is a system $\langle e_\alpha : \alpha < \lambda \rangle$ satisfying the following conditions:

(a) $\forall \alpha < \lambda [e_\alpha$ is club in α of order type $\mathrm{cf}(\alpha)]$.

(b) $C_0 = \emptyset$ and $\forall \alpha < \lambda [C_{\alpha+1} = \{\alpha\}]$

(c) For every limit ordinal $\alpha < \lambda$ and every $\beta \in e_\alpha$, if β is not a supremum of elements of e_α, then β is not a limit ordinal.

A stationary subset S of a regular cardinal λ is *non-reflecting* iff $S \cap \alpha$ is non-stationary in α for every $\alpha < \lambda$ of cofinality greater than ω. For example, if μ is regular, then $\{\alpha < \lambda : \mathrm{cf}(\alpha) = \mu\}$ is clearly a non-reflecting stationary subset of μ^+. This is used to derive (C9) from the more general result.

Theorem 3.7. *Suppose that λ is a regular cardinal $> \omega_2$, and S is a non-reflecting stationary subset of λ such that $\mathrm{cf}(\alpha) > \omega_1$ for every $\alpha \in S$, with every member of S a limit ordinal. Then $\mathrm{Pr}_1(\lambda, \lambda, \omega_1, \omega)$ holds.*

$\boxed{\lambda}$ $\boxed{S}$

Proof. (1) There is a strict club system $\langle e_\alpha : \alpha < \lambda\rangle$ for λ such that $e_\alpha \cap S = \emptyset$ for every limit ordinal $\alpha < \lambda$.

In fact, for each ordinal $\alpha < \lambda$ of uncountable cofinality, let D_α be club in α with $D_\alpha \cap S = \emptyset$, and let e'_α be a cofinal subset of D_α with order type $\mathrm{cf}(\alpha)$. Replace each member β of e'_α which is not a supremum of other members of e'_α by $\beta + 1$, and let e_α be the result. For $\alpha < \lambda$ of cofinality ω, let e_α be the range of a strictly increasing sequence of successor ordinals with supremum α. Further, let $e_0 = \emptyset$ and $e_\alpha = \{\beta\}$ for $\alpha = \beta + 1$. Then (1) holds.

$\boxed{e_\alpha}$

The proof uses Todorčević walks in an essential way. γ is the function defined for all pairs (β, α) such that $\alpha < \beta < \lambda$ by

$\boxed{\gamma(\beta,\alpha)}$
$$\gamma(\beta, \alpha) = \min(e_\beta \backslash \alpha).$$

Thus $\alpha \le \gamma(\beta, \alpha) < \beta$. Now we define

$\boxed{\gamma_l(\beta,\alpha)}$
$$\gamma_0(\beta, \alpha) = \beta;$$
$$\gamma_{l+1}(\beta, \alpha) = \begin{cases} \gamma(\gamma_l(\beta, \alpha), \alpha) & \text{if } \gamma_l(\beta, \alpha) > \alpha \\ & \text{undefined, otherwise.} \end{cases}$$

This gives a strictly decreasing sequence of ordinals, and so there is a greatest $k(\beta, \alpha) < \omega$ such that $\gamma_{k(\beta,\alpha)}$ is defined; then $\gamma_{k(\beta,\alpha)} = \alpha$.

$\boxed{k(\beta,\alpha)}$

Clearly $k(\beta, \alpha) \ge 1$. We define

$\boxed{\rho(\beta,\alpha)}$
$$\rho(\beta, \alpha) = \langle \gamma_0(\beta, \alpha), \ldots, \gamma_{k(\beta,\alpha)}(\beta, \alpha)\rangle.$$

Now let $\langle S^\alpha : \alpha < \omega_1\rangle$ be a partition of S into stationary subsets.

$\boxed{S^\alpha}$

We define $H : \lambda \to \omega_1$ by setting, for any $\alpha \in \lambda$,

$\boxed{H}$
$$H(\alpha) = \begin{cases} \beta & \text{if } \alpha \in S^\beta \text{ with } \beta < \omega_1, \\ 0 & \text{if } \alpha \notin S. \end{cases}$$

Thus $H^{-1}[\{\beta\}] = S^\beta$ for all $\beta < \omega_1$.

Let $\langle R_\alpha : \alpha < \omega_1 \rangle$ be a partition of ω_1 into stationary subsets.

$\boxed{R_\alpha}$

For the next definitions, we omit the arguments α, β from γ_p, k, etc., if these arguments should be clear.

For $\alpha < \beta < \lambda$ define

$\boxed{w_1(\beta,\alpha)}$
$$w_1(\beta,\alpha) = \left\{p \in \left(\frac{k}{2}, k\right] : \forall q < \frac{k}{2}[H(\gamma_p) > H(\gamma_q)]\right\},$$

and

$\boxed{p_1(\beta,\alpha)}$
$$p_1(\beta,\alpha) = \begin{cases} \min(w_1) & \text{if } \ w_1 \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $p_1(\beta,\alpha) \le k(\beta,\alpha)$. Next,

$\boxed{w_2(\beta,\alpha)}$
$$w_2(\beta,\alpha) = \left\{q < \frac{k}{2} : \forall p \in \left(\frac{k}{2}, k\right][p \notin w_1 \to H(\gamma_q) > H(\gamma_p)]\right\}.$$

Let $p_2(\beta,\alpha)$ be the $s \in \omega_1$ such that $\min\{H(\gamma_q) : q \in w_2\} \in R_s$ if $w_2 \neq \emptyset$, and $p_2(\beta,\alpha) = 0$ otherwise.

$\boxed{p_2(\beta,\alpha)}$

Finally, we set

$\boxed{c(\{\alpha,\beta\})}$
$$c(\{\alpha,\beta\}) = \begin{cases} \gamma & \text{if } \ p_1 - p_2 \ge 0 \text{ and } \gamma_{p_1-p_2}(\beta,\alpha) \in S^\gamma, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha < \beta < \lambda$.

The rest of the proof is to check that c is as desired. Before doing this we make some claims.

If $s = \langle s_i : i < m \rangle$ and $t = \langle t_i : i < n \rangle$, with $m, n \ge 2$, and $s_{m-1} = t_0$, then $s^\smile t$ is the sequence $\langle s_0, \ldots, s_{m-1}, t_1, \ldots, t_{n-1} \rangle$.

Claim 1. *Suppose that $\delta \in S$ and $\delta < \beta < \lambda$. Then:*

(i) *If $l < k(\beta,\delta) - 1$, then $e_{\gamma_l(\beta,\delta)} \cap \delta$ is bounded below δ.*

(ii) *Define $\chi(\beta,\delta) = \sup\{\sup\{e_{\gamma_l(\beta,\delta)} \cap \delta\} : l < k(\beta,\delta) - 1\}$, whenever $\delta < \beta < \lambda$.*

$\boxed{\chi(\beta,\delta)}$

Thus $\chi(\beta,\delta) < \delta$ by (i). *Suppose that $\alpha \in (\chi(\beta,\delta), \delta)$. Then*

(a) *If $l < k(\beta,\delta) - 1$, then $e_{\gamma_l(\beta,\delta)} \backslash \delta = e_{\gamma_l(\beta,\delta)} \backslash \alpha$.*

(b) *If $l < k(\beta,\delta)$, then $\gamma_l(\beta,\delta) = \gamma_l(\beta,\alpha)$.*

(c) $\gamma_{k(\beta,\delta)-1} = \delta + 1 = \gamma_{k(\beta,\alpha)-1}$.
(d) $\rho(\beta,\alpha) = \rho(\beta,\delta)^\smile\rho(\delta,\alpha)$.

Proof. (i): The assumption $l < k(\beta,\delta) - 1$ implies that

$$\delta < \gamma_{l+1}(\beta,\delta) = \gamma(\gamma_l(\beta,\delta),\delta) = \min(\gamma_l(\beta,\delta)\backslash\delta),$$

and hence $\delta \notin e_{\gamma_l(\beta,\delta)}$. Hence either $\gamma_l(\beta,\delta)$ is a successor ordinal $\alpha+1$, hence $e_{\gamma_l(\beta,\delta)} = \{\alpha\}$, so that $e_{\gamma_l(\beta,\delta)} \cap \delta \subseteq \{\alpha\}$, and hence $e_{\gamma_l(\beta,\delta)} \cap \delta$ is bounded below δ, or it is a limit ordinal, and so $e_{\gamma_l(\beta,\delta)} \cap \delta$ is bounded below δ because $e_{\gamma_l(\beta,\delta)}$ is club.

(ii)(a): Clearly $e_{\gamma_l(\beta,\delta)} \cap \delta \subseteq \alpha$, so $e_{\gamma_l(\beta,\delta)}\backslash\alpha \subseteq e_{\gamma_l(\beta,\delta)}\backslash\delta \subseteq e_{\gamma_l(\beta,\delta)}\backslash\alpha$, giving (ii)(a).

(ii)(b): We prove this by induction on l. First, $\gamma_0(\beta,\delta) = \beta = \gamma_0(\beta,\alpha)$. Now assume that $\gamma_l(\beta,\delta) = \gamma_l(\beta,\alpha)$, with $l < k(\beta,\delta) - 1$. Then

$$\begin{aligned}
\gamma_{l+1}(\beta,\delta) &= \min(e_{\gamma_l(\beta,\delta)}\backslash\delta)\\
&= \min(e_{\gamma_l(\beta,\delta)}\backslash\alpha) \quad \text{by (ii)(a)}\\
&= \min(e_{\gamma_l(\beta,\alpha)}\backslash\alpha) \quad \text{induction hypothesis}\\
&= \gamma_{l+1}(\beta,\alpha).
\end{aligned}$$

(ii)(c): Note that $\delta < \gamma_{k(\beta,\delta)-1}(\beta,\delta)$. If $\gamma_{k(\beta,\delta)-1}(\beta,\delta)$ is a limit ordinal, then $e_{\gamma_{k(\beta,\delta)-1}(\beta,\delta)}$ is unbounded in $\gamma_{k(\beta,\delta)-1}(\beta,\delta)$, contradicting

$$\delta = \min(e_{\gamma_{k(\beta,\delta)-1}(\beta,\delta)}\backslash\delta).$$

It follows that $\gamma_{k(\beta,\delta)-1}(\beta,\delta)$ is a successor ordinal, and since

$$\delta = \min(e_{\gamma_{k(\beta,\delta)-1}(\beta,\delta)}\backslash\delta),$$

we must have $\gamma_{k(\beta,\delta)-1}(\beta,\delta) = \delta + 1$. Hence also $\gamma_{k(\beta,\alpha)-1} = \delta + 1$ by (ii)(b).

(ii)(d): Immediate from (ii)(c).

This finishes the proof of Claim 1. □

For any $A \subseteq \lambda$ we let $A' = \{\alpha < \lambda : \alpha \text{ is limit and } \alpha = \sup(A \cap \alpha)\}$.

$\boxed{A'}$

Clearly if A is unbounded, then A' is a club in λ. Note that all members of A' are limit ordinals.

Claim 2. *If $A, B \in [\lambda]^\lambda$ and $l \in \omega$, then there exist $\alpha \in A$ and $\beta \in B$ such that $\alpha < \beta$ and $k(\beta,\alpha) > l$.*

Proof. Let $C_0 = A'$. If C_i has been defined and is a club in λ, note that $S \cap C_i$ is unbounded. Let $C_{i+1} = (S \cap C_i)'$; it is a club. We now define $\gamma_l, \gamma_{l-1}, \ldots, \gamma_0$ and $\chi_l, \chi_{l-1}, \ldots, \chi_0$ by recursion. Let $\gamma_l \in C_l \cap S$. Then choose $\beta \in B$ such that $\beta > \gamma_l$, and let $\chi_l = \chi(\beta, \gamma_l)$. Now suppose that γ_{i+1} and χ_{i+1} have been defined so that $\gamma_{i+1} \in C_{i+1} \cap S$ and $\chi_{i+1} < \gamma_{i+1}$. Choose $\gamma_i \in S \cap C_i$ such that $\chi_{i+1} < \gamma_i < \gamma_{i+1}$. Then $\chi(\gamma_{i+1}, \gamma_i), \chi_{i+1} < \gamma_i$. Choose χ_i so that $\chi(\gamma_{i+1}, \gamma_i), \chi_{i+1} < \chi_i < \gamma_i$. This finishes the construction.

Note that $k(\beta, \gamma_l) \geq 1$. For $i < l$ we have $\chi(\beta, \gamma_{i+1}) = \chi_{i+1} < \gamma_i < \gamma_{i+1}$, and hence by Claim 1, $\rho(\beta, \gamma_i)$ extends $\rho(\beta, \gamma_{i+1})$, and so $k(\beta, \gamma_i) \geq k(\beta, \gamma_{i+1}) + 1$. It follows that $k(\beta, \gamma_i) \geq 1 + l - i$ for each $i \leq l$. Hence $k(\beta, \gamma_0) \geq 1 + l$. Now $\chi_0 < \gamma_0$. Clearly $C_i \subseteq A'$ for all $i \in \omega$; so $\gamma_0 \in A'$. Choose $\alpha \in A$ such that $\chi_0 < \alpha < \gamma_0$. Since $\chi_0 = \chi(\beta, \gamma_0)$, it follows by Claim 1 that $\rho(\beta, \alpha)$ extends $\rho(\beta, \gamma_0)$, and hence $k(\beta, \alpha) \geq 2 + l$.

This ends the proof of Claim 2. □

Now we define $\rho_H(\beta, \alpha) = \langle H(\gamma_l(\beta, \alpha)) : l \leq k(\beta, \alpha) \rangle$.

$\boxed{\rho_H(\beta, \alpha)}$

If $\sigma \in {}^{<\omega}\omega_1$ and $i < \omega_1$, then σ^i is the sequence of the same length as σ such that

$\boxed{\sigma^i}$

$$\sigma^i(l) = \begin{cases} \sigma(l) & \text{if } \sigma(l) < i, \\ \omega_1 & \text{otherwise.} \end{cases}$$

If $T \subseteq \lambda$, $\delta < \lambda$, and R is a stationary subset of ω_1, then we define

$$\begin{aligned} U(\delta, T, R) = \{\eta \in {}^{<\omega}(\omega_1 + 1) \backslash {}^{<\omega}\omega_1 : \forall i < \omega_1 \exists \beta \in T \backslash (\delta + 1) \\ [\rho_H(\beta, \delta)^i = \eta \text{ and } \min\{(\rho_H(\beta, \delta))(l) : \eta^i(l) = \omega_1\} \in R]\}. \end{aligned}$$

$\boxed{U(\delta, T, R)}$

For $\chi < \lambda$, $U(\delta, T, R, \chi)$ is defined similarly, with the added stipulation that $\chi(\beta, \delta) < \chi$.

$\boxed{U(\delta, T, R, \chi)}$

Claim 3. *Suppose that R is a stationary subset of ω_1, and $\langle \beta_i : i \in R \rangle$ and $\langle \delta_i : i \in R \rangle$ are systems of ordinals less than λ such that $\beta_i > \delta_i$ for all $i \in R$. Then there exist $k \in \omega$, $\eta \in {}^{<\omega}(\omega_1 + 1)$, and a stationary subset $R' \subseteq R$ such that the following conditions hold:*

(i) $\forall i \in R'[|\operatorname{rng}(\rho_H(\beta_i, \delta_i))| = k]$.
(ii) $\forall i, j \in R' \forall l < k[H(\gamma_l(\beta_i, \delta_i)) < i \leftrightarrow H(\gamma_l(\beta_j, \delta_j)) < j]$.
(iii) $\forall i \in R'[\rho_H(\beta_i, \delta_i)^i = \eta]$.

Proof.

$$R = \bigcup_{k\in\omega}\{i \in R : |\operatorname{rng}(\rho_H(\beta_i, \delta_i))| = k\}.$$

Hence there is a $k \in \omega$ such that $R_1 \stackrel{\text{def}}{=} \{i \in R : |\operatorname{rng}(\rho_H(\beta_i, \delta_i))| = k\}$ is stationary. Also, for each $i \in R_1$ define $f_i \in {}^k2$ by setting

$$f_i(l) = \begin{cases} 1 & \text{if } H(\gamma_l(\beta_i, \delta_i)) < i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $R_1 = \bigcup_{s\in {}^k2}\{i \in R_1 : f_i = s\}$. So there exist a stationary subset R_2 of R and a function $s \in {}^k2$ such that $\forall i \in R_2[f_i = s]$. Thus

$$\forall i, j \in R_2 \forall l < k[H(\gamma_l(\beta_i, \delta_i)) < i \leftrightarrow H(\gamma_l(\beta_j, \delta_i)) < j].$$

Now let $B = \{l < k : H(\gamma_l(\beta_i, \delta_i)) < i\}$, for any (hence all) $i \in R_2$. Say $B = \{l_1, \ldots, l_m\}$. Now $\forall i \in R_2[H(\gamma_{l_1}(\beta_i, \delta_i)) < i]$. So by Fodor's theorem there exist a stationary set $R_{3,1} \subseteq R_2$ and an ordinal μ_1 such that $H(\gamma_{l_1}(\beta_i, \delta_i)) = \mu_1$ for all $i \in R_{3,1}$. Continuing, we end up with a stationary set $R_{3,m} \subseteq R$ and a function $\eta \in {}^{<\omega}(\omega_1 + 1)$ such that $\forall i \in R_{3,m}[\rho_H(\beta_i, \delta_i)^i = \eta]$.

This ends the proof of Claim 3. □

Claim 4. *If $T \in [\lambda]^\lambda$ and R is a stationary subset of ω_1, then there is a $\delta(T) < \lambda$ such that*

$$\boxed{\delta(T)} \qquad \begin{array}{l} \forall\delta \in [\delta(T), \lambda)[U(\delta, T, R) \neq \emptyset] \quad \textit{and} \\ \forall\delta \in [\delta(T), \lambda)[\operatorname{cf}(\delta) > \omega_1 \rightarrow \exists\chi < \delta[U(\delta, T, R, \chi) \neq \emptyset]]. \end{array}$$

Proof. For each $i < \omega_1$ let $A_i = \{\delta < \lambda : \forall\beta \in T \cap (\delta, \lambda)[i \notin \operatorname{rng}(\rho_H(\beta, \delta))]\}$. □

Subclaim. $\forall i < \omega_1[|A_i| < \lambda]$.

Proof of the Subclaim. Suppose that $i < \omega_1$ and $|A_i| = \lambda$. Pick $\delta \in S^i \cap A_i'$, and then pick $\beta \in T$ such that $\delta < \beta$. Choose $\gamma \in A_i \cap (\chi(\beta, \delta), \delta)$. Then δ is an entry of $\rho(\beta, \gamma)$ by Claim 1, and so $i = H(\delta) \in \operatorname{rng}(\rho_H(\beta, \gamma))$, contradiction. This proves the subclaim.

By the subclaim, there is an ordinal $\delta(T) < \lambda$ such that $\bigcup_{i<\omega_1} A_i \subseteq \delta(T)$. Now suppose that $\delta \in [\delta(T), \lambda)$.

(2) $\forall i < \omega_1 \exists\beta_i \in (\delta, \lambda) \cap T[i \in \operatorname{rng}(\rho_H(\beta_i, \delta))]$.

This is immediate from the definition of A_i. Now we apply Claim 3 and get $k \in \omega$, $\eta \in {}^{<\omega}(\omega_1 + 1)$, and a stationary subset R' of R such that

(3) $\forall i \in R'[|\operatorname{rng}(\rho_H(\beta_i, \delta)| = k]$.

(4) $\forall i, j \in R' \forall l < k[H(\gamma_l(\beta_i, \delta)) < i \leftrightarrow H(\gamma_l(\beta_j, \delta)) < j]$.

(5) $\forall i \in R'[\rho_H(\beta_i,\delta)^i = \eta]$.

Since $i \in \mathrm{rng}(\rho_H(\beta_i,\delta))$, we have $\eta \notin {}^{\omega}\omega_1$. If $i \in R'$, say that $i = H(\gamma_l(\beta_i,\delta))$. Thus $\eta(l) = (\rho_H(\beta_i,\delta)^i)(l) = \omega_1$, and $(\rho_H(\beta_i,\delta))(l) = H(\gamma_l(\beta_i,\delta) = i$. If $m < k$ and $\eta(m) = \omega_1$, then $(\rho_H(\beta_i,\delta))(m) \geq i$. Thus $\min\{(\rho_H(\beta_i,\delta))(m) : \eta(m) = \omega_1\} = i \in R' \subseteq R$. So $\eta \in U(\delta,T,R)$, as desired.

Now suppose that $\mathrm{cf}(\delta) > \omega_1$. Then $\{\chi(\beta_i,\delta) : i \in R'\}$ is bounded below δ, say by θ, Hence $\eta \in U(\delta,T,R,\theta)$.

This finishes the proof of Claim 4.

Now suppose that $T \subseteq \lambda$ and $\delta < \lambda$. We define

$L(\delta,T) = \{\rho \in {}^{<\omega}(\omega_1+1)\backslash{}^{<\omega}\omega_1$: there is a stationary $R' \subseteq \omega_1$ such that $\forall i \in R' \forall\alpha < \delta \exists\beta \in T \cap (\alpha,\delta)[\rho_H(\delta,\beta)^i = \rho]\}$.

$\boxed{L(\delta,T)}$

For $T \in [\lambda]^\lambda$ we define $C(T) = \bigcap_{i<\omega_1}(S_i \cap T')'$. Note that this is club in λ.

$\boxed{C(T)}$

Claim 5. *If $T \in [\lambda]^\lambda$, $\delta \in C(T)$, and $\mathrm{cf}(\delta) \geq \omega_1$, then $L(\delta,T) \neq \emptyset$.*

Proof. Case 1. $\mathrm{cf}(\delta) = \omega_1$. Let $\langle\delta_i : i < \omega_1\rangle$ be strictly increasing with supremum δ. Since $\delta \in C(T)$, we can choose $\alpha_i \in S_i \cap T' \cap (\delta_i,\delta)$ for each $i < \omega_1$. Then for each $i < \omega_1$ choose $\beta_i \in T \cap (\delta_i,\alpha_i)$ with $\chi(\delta,\alpha_i) < \beta_i$. Then $i = H(\alpha_i) \in \mathrm{rng}(\rho_H(\delta,\beta_i))$ by claim 1. We now apply claim 3 and get k, η, and a stationary subset R' of ω_1 such that the following condition holds:

(6) $\forall i \in R'[\rho_H(\delta,\beta_i)^i = \eta]$.

Since $i \in \mathrm{rng}(\rho_H(\delta,\beta_i))$ we have $\eta \notin {}^{<\omega}\omega_1$. Hence $\eta \in L(\delta,T)$, as desired.

Case 2. $\mathrm{cf}(\delta) > \omega_1$. Let $\langle\delta_\alpha : \alpha < \mathrm{cf}(\delta)\rangle$ be strictly increasing with supremum δ. Since $\delta \in C(T)$, for each $\alpha < \mathrm{cf}(\delta)$ and each $i < \omega_1$ we can choose $\gamma_{i\alpha} \in S_i \cap T'$ such that $\delta_\alpha < \gamma_{i\alpha} < \delta$. By Claim 1 we have $\chi(\delta,\gamma_{i\alpha}) < \gamma_{i\alpha}$, so we can choose $\beta_{i\alpha} \in T \cap (\chi(\delta,\gamma_{i\alpha}),\gamma_{i\alpha})$. Now by Claim 1 applied to $\gamma_{i\alpha} < \delta < \lambda$, the ordinal $\gamma_{i\alpha}$ is a term of $\rho(\delta,\beta_{i\alpha})$; hence $i = H(\gamma_{i\alpha}) \in \mathrm{rng}(\rho_H(\delta,\beta_{i\alpha}))$. Now if we apply Claim 3 to δ and $\beta_{i\alpha}$ for $i < \omega_1$ we get a stationary subset R' of ω_1 and an $\eta^\alpha \in {}^{<\omega}(\omega_1+1)$ such that $\forall i \in R'[\rho_H(\delta,\beta_{i\alpha})^i = \eta^\alpha]$. Since $i = H(\gamma_{i\alpha}) \in \mathrm{rng}(\rho_H(\delta,\beta_{i\alpha}))$, we have $\eta^\alpha \notin {}^{<\omega}\omega_1$. Since $\omega_1 < \mathrm{cf}(\delta)$, there is an $\eta \in {}^{<\omega}(\omega_1+1)$ such that $\eta^\alpha = \eta$ for $\mathrm{cf}(\delta)$ many α. Hence $\eta \in L(\delta,T)$, as desired.

This finishes the proof of Claim 5. □

Now we begin the actual checking that c, defined above, works. Thus assume that $n \in \omega\backslash 1$, $\gamma < \omega_1$, and $a \in {}^\lambda([\lambda]^n)$ is a system of pairwise disjoint sets.

$\boxed{n}$ $\boxed{\gamma}$ $\boxed{a}$

(7) We may assume that $\alpha < \min(a_\alpha)$, for all $\alpha < \lambda$.

In fact, define $\langle \alpha_\eta : \eta < \lambda \rangle$ by recursion. If α_ξ has been defined for all $\xi < \eta$, then the set $\{\alpha_\xi : \xi < \eta\} \cup \{\beta < \lambda : \min(a_\beta) \le \eta\}$ has size less than λ. Let α_η be any ordinal less than λ which is not in this set. Let $a'_\eta = a_{\alpha_\eta}$ for all $\eta < \lambda$. These sets are clearly pairwise disjoint, and $\eta < \min(a_{\alpha_\eta}) = \min(a'_\eta)$. So (7) holds.

Write $a_\alpha = \{\zeta^0_\alpha, \ldots, \zeta^{n-1}_\alpha\}$ for each $\alpha < \lambda$, with $\zeta^0_\alpha < \cdots < \zeta^{n-1}_\alpha$.

$\boxed{\zeta^l_\alpha}$

Thus

(8) $\forall \alpha < \lambda \forall l < n[\alpha < \zeta^l_\alpha]$.

Now define $\delta_0 = 0$, $\delta_{\alpha+1} = \left(\sup_{\beta \le \delta_\alpha} \zeta^{n-1}_\beta\right) + 1$, and $\delta_\lambda = \bigcup_{\alpha<\lambda} \delta_\alpha$ for λ limit. Let $C = \{\delta_\lambda : \lambda \text{ limit}\}$.

$\boxed{\delta_\alpha}$ $\boxed{C}$

(9) C is club, and $\forall \alpha, \delta[\alpha < \delta \in C \to \zeta^{n-1}_\alpha < \delta]$.

In fact, assume that $\alpha < \delta \in C$. Say $\delta = \delta_\lambda$, λ limit. Choose $\beta < \lambda$ such that $\alpha < \delta_\beta$. Then $\zeta^{n-1}_\alpha \le \delta_{\beta+1} < \delta_\lambda$, as desired.

Now if $\delta \in S \cap C$, then

$$\delta = \bigcup_{s \in U} \{\alpha < \delta : \langle \rho_H(\delta, \zeta^l_\alpha) : l < n \rangle = s\},$$

where $U = {}^n\left({}^{<\omega}\omega_1\right)$. Since $\mathrm{cf}(\delta) > \omega_1$, there is an $s \in U$ such that the indicated set is cofinal in δ. Thus there is a system $\langle \nu^\delta_l : l < n \rangle$ such that

(10) $$\sup\{\alpha < \delta : \forall l < n[\rho_H(\delta, \zeta^l_\alpha) = \nu^\delta_l]\} = \delta.$$

Now

$$S \cap C = \bigcup_{t \in U} \{\delta \in S \cap C : \langle \nu^\delta_l : l < n \rangle = t\}.$$

Hence there is a $t \in U$ such that $T_1 \stackrel{\text{def}}{=} \{\delta \in S \cap C : \langle \nu^\delta_l : l < n \rangle = t\}$ is stationary. We can write $t = \langle \nu_l : l < n \rangle$. So (10) becomes

(11) $$\forall \delta \in T_1[\sup\{\alpha < \delta : \forall l < n[\rho_H(\delta, \zeta^l_\alpha) = \nu_l]\} = \delta.$$

$\boxed{T_1}$ $\boxed{\nu_l}$

Next, by Claim 5,

$$S \cap C(T_1) \cap C = \bigcup_{\rho \in W} \{\delta \in S \cap C(T_1) \cap C : \rho \in L(\delta, T_1)\},$$

where $W = {}^{<\omega}(\omega_1+1)\backslash{}^{<\omega}\omega_1$. Hence there is a stationary $T_2 \subseteq S \cap C(T_1) \cap C$ and a $\tau \in {}^{<\omega}(\omega_1+1)\backslash{}^{<\omega}\omega_1$ such that $\tau \in L(\delta, T_1)$ for all $\delta \in T_2$. Let $l^* = \min\{l \in \omega : \tau(l) = \omega_1\}$.

$\boxed{T_2}$ $\boxed{\tau}$ $\boxed{l^*}$

Next, by Claim 5 again we have

$$S^\gamma \cap C(T_2) = \bigcup_{\rho \in W} \{\delta \in S^\gamma \cap C(T_2) : \rho \in L(\delta, T_2)\},$$

with $W = {}^{<\omega}(\omega_1+1)\backslash{}^{<\omega}\omega_1$, and so there exist a stationary $T_3 \subseteq S^\gamma \cap C(T_2)$ and a $\rho \in {}^{<\omega}(\omega_1+1)\backslash{}^{<\omega}\omega_1$ such that $\rho \in L(\delta, T_2)$ for all $\delta \in T_3$.

$\boxed{T_3}$ $\boxed{\rho}$

Now

$$S = \bigcup_{\sigma \in X} \{\delta \in S : \rho_H(\zeta^0_\delta, \delta) = \sigma\},$$

where $X = {}^{<\omega}\omega_1$. Hence there exist a stationary subset $S' \subseteq S$ and a $\sigma \in X$ such that the indicated set is stationary. Continuing, we arrive at a stationary subset $T^1 \subseteq S$ and a system $\langle \nu^l : l < n \rangle$ such that

(12) $$\forall \delta \in T^1 \forall l < n[\rho_H(\zeta^l_\delta, \delta) = \nu^l].$$

$\boxed{T^1}$ $\boxed{\nu^l}$

Now for each $\delta \in T^1$ we have $\chi(\zeta^0_\delta, \delta) < \delta$. Hence there exist a stationary subset T^1_0 of T^1 and an ordinal θ_0 such that $\chi(\zeta^0_\delta, \delta) = \theta_0$ for all $\delta \in T^1_0$. Continuing, we arrive at a stationary subset $T^2 \subseteq T^1$ and a function $\theta : n \to \lambda$ such that $\chi(\zeta^l_\delta, \delta) = \theta(l)$ for all $\delta \in T^2$ and all $l < n$. Letting $\chi^2 = (\sup_{l<n} \theta(l)) + 1$, we then have

$\boxed{T^2}$ $\boxed{\chi^2}$

(13) $\forall \delta \in T^2 \forall l < n[\chi(\zeta^l_\delta, \delta) < \chi^2]$.

(14) There are $\delta(T^2) < \lambda$, $\eta, \chi^3 \in (\chi^2, \lambda)$, and a stationary $T^3 \subseteq (S \cap C)\backslash\delta(T^2)$ such that $\forall \delta \in T^3[\eta \in U(\delta, T^2, R_{l^*+|\rho|}, \chi^3)]$.

$\boxed{\eta}$ $\boxed{\chi^3}$ $\boxed{T^3}$

In fact, by Claim 4 there is a $\delta(T^2) < \lambda$ such that $\forall \delta \in [\delta(T^2), \lambda)[\mathrm{cf}(\delta) > \omega_1$ implies that $\exists \chi < \delta[U(\delta, T^2, R_{l^*+|\rho|}, \chi) \neq \emptyset]$. Hence

$$\forall \delta \in (S \cap C)\backslash\delta(T^2) \exists \chi < \delta[U(\delta, T^2, R_{l^*+|\rho|}, \chi) \neq \emptyset],$$

so by Fodor's theorem there exist a stationary $T' \subseteq (S \cap C) \backslash \delta(T^2)$ and $\chi^3 < \lambda$ such that $\forall \delta \in T'[U(\delta, T^2, R_{l^*+|\rho|}, \chi^3) \neq \emptyset]$. Now

$$T' = \bigcup_{\eta \in V} \{\delta \in T' : \eta \in U(\delta, T^2, R_{l^*+|\rho|}, \chi^3),$$

where $V = {}^{<\omega}(\omega_1+1) \backslash {}^{<\omega}\omega_1$. Hence there exist a stationary $T^3 \subseteq T'$ and an $\eta \in V$ such that the condition in (14) holds.

Now we apply Claim 2 with $A = T_3 \backslash (\chi^3 + 1)$ and $B = T^3$ to obtain $\beta_3 \in T_3 \backslash (\chi^3 + 1)$ and $\beta^3 \in T^3$ such that $\beta_3 < \beta^3$ and

$\boxed{\beta_3}$ $\boxed{\beta^3}$

(15) $k(\beta^3, \beta_3) > \max\{|\nu_l| : l < n\} + |\tau| + |\rho| + |\eta| + \max\{|\nu^l| : l < n\}$.

Now $\rho \in L(\beta_3, T_2)$, so for all $\alpha < \beta_3$ there are uncountably many $i < \omega_1$ such that $\exists \beta \in T_2 \cap (\alpha, \beta_3)[\rho_H(\beta_3, \beta)^i = \rho]$. We apply this to $\alpha = \max(\chi^3, \chi(\beta^3, \beta_3))$ and with $i_0 < \omega_1$ greater than all countable ordinals in $\rho_H(\beta^3, \beta_3)$, η, $\langle \nu^l : l < n \rangle$, $\langle \nu_l : l < n \rangle$ to get a $\beta_2 \in T_2$ such that $\chi^3, \chi(\beta^3, \beta_3) < \beta_2 < \beta_3$ and $\rho_H(\beta_3, \beta_2)^{i_0} = \rho$.

$\boxed{i_0}$ $\boxed{\beta_2}$

Since $\chi^3 < \beta_2$, $\chi(\beta^3, \beta_3) < \beta_2$, and $\chi(\beta_3, \beta_2) < \beta_2$, choose $\chi_2 \in (\chi^3, \beta_2)$ such that $\chi(\beta^3, \beta_3) < \chi_2$ and $\chi(\beta_3, \beta_2) < \chi_2$.

$\boxed{\chi_2}$

Now let R' be a stationary subset of ω_1 given by the definition of $L(\beta_2, T_1)$.

$\boxed{R'}$

Fix $i_1 \in R'$ greater than i_0 and each ordinal in $\rho_H(\beta_3, \beta_2)$.

$\boxed{i_1}$

Now $\beta^3 \in T^3$, so $\eta \in U(\beta^3, T^2, R_{l^*+|\rho|}, \chi^3)$. Applying this to i_1, we get $\beta \in T^2 \backslash (\beta^3 + 1)$ such that $\rho_H(\beta, \beta^3)^{i_1} = \eta$ and $\chi(\beta, \beta^3) < \chi^3$.

$\boxed{\beta}$

Since $\beta \in T^2 \subseteq T^1$, we have by (13)

$$\forall l < n[\rho_H(\zeta^l_\beta, \beta) = \nu^l \text{ and } \chi(\zeta^l_\beta, \beta) < \chi^2 < \chi^3]. \tag{16}$$

Now since $\beta_2 \in T_2$, we have $\tau \in L(\beta_2, T_1)$. Applying the definition of $L(\beta_2, T_1)$ to $\chi_2 < \beta_2$ we see that for $\forall i \in R' \exists \theta \in T_1 \cap (\chi_2, \beta_2)[\rho_H(\beta_2, \theta)^i = \tau]$. We apply this to an $i_2 \in R'$ such that i_2 is greater than i_1 and all ordinals in $\rho_H(\beta, \beta^3)$.

$\boxed{i_2}$

This gives $\beta_1 \in T_1 \cap (\chi_2, \beta_2)$ such that $\rho_H(\beta_2, \beta_1)^{i_2} = \tau$.

$\boxed{\beta_1}$

Now $\chi(\beta_2, \beta_1), \chi_2 < \beta_1$. Fix $\chi_1 < \beta_1$ with $\chi(\beta_2, \beta_1), \chi_2 < \chi_1$.

$\boxed{\chi_1}$

Since $\beta_1 \in T_1$, by (11) there is an $\alpha \in (\chi_1, \beta_1)$ such that $\forall l < n[\rho_H(\beta_1, \zeta^l_\alpha) = \nu_l]$.

$\boxed{\alpha}$

Now note that

(17) $\alpha < \beta_1 < \beta_2 < \beta_3 < \beta$.

(18) If $l, m < n$, then $\rho(\zeta^l_\beta, \zeta^m_\alpha) = \rho(\zeta^l_\beta, \beta)^\smile \rho(\beta, \zeta^m_\alpha)$.

For, $\beta \in T^2 \subseteq T^1 \subseteq S$, $\beta < \zeta^l_\beta$ by (8), and $\chi(\zeta^l_\beta, \beta) < \chi^2 < \chi^3 < \chi_2 < \chi_1 < \alpha$, so the conclusion follows by Claim 1.

(19) If $m < n$, then $\rho(\beta, \zeta^m_\alpha) = \rho(\beta, \beta^3)^\smile \rho(\beta^3, \zeta^m_\alpha)$.

In fact, we apply Claim 1 with δ, β, α replaced by $\beta^3, \beta, \zeta^m_\alpha$: we have $\beta^3 < \beta < \lambda$ by the definition of β; by the definitions of β, χ_2, χ_1, and α we have $\chi(\beta, \beta^3) < \chi^3 < \chi_2 < \chi_1 < \alpha$, and then (8) gives $\chi(\beta, \beta^3) < \zeta^m_\alpha$; by (17) and the definition of β^3 we have $\alpha < \beta^3 \in C$, and then (9) gives $\zeta^m_\alpha < \beta^3$.

(20) If $m < n$, then $\rho(\beta^3, \zeta^m_\alpha) = \rho(\beta^3, \beta_3)^\smile \rho(\beta_3, \zeta^m_\alpha)$.

For, $\beta_3 < \beta^3$. Also, $\alpha < \beta_3 \in T_2 \subseteq C$, so by (9), $\zeta^m_\alpha < \beta_3$. Finally, since $\chi(\beta^3, \beta_3) < \chi_2 < \chi_1 < \alpha < \zeta^m_\alpha$ by (8), Claim 1 applies again, with β_3, β^3, ζ^m_α in place of δ, β, α.

(21) If $m < n$, then $\rho(\beta_3, \zeta^m_\alpha) = \rho(\beta_3, \beta_2)^\smile \rho(\beta_2, \zeta^m_\alpha)$.

For, $\beta_2 < \beta_3$. Also, $\alpha < \beta_2 \in T_2 \subseteq C$, so $\zeta^m_\alpha < \beta_2$ by (9). Also, $\chi(\beta_3, \beta_2) < \chi_2 < \chi_1 < \alpha < \zeta^m_\alpha$, so Claim 1 again applies with δ, β, α replaced by β_2, β_3, ζ^m_α.

(22) If $m < n$, then $\rho(\beta_2, \zeta^m_\alpha) = \rho(\beta_2, \beta_1)^\smile \rho(\beta_1, \zeta^m_\alpha)$.

For, $\beta_1 < \beta_2$. Also, $\alpha < \beta_1 \in T_1 \subseteq C$, so by (9), $\zeta^m_\alpha < \beta_1$. Since $\chi(\beta_2, \beta_1) < \chi_1 < \alpha < \zeta^m_\alpha$, Claim 1 applies again, with δ, β, α replaced by β_1, β_2, ζ^m_α.

Putting (18)–(22) together, we get the following two statements; for all $l, m < n$,

(23)
$$\begin{aligned}\rho(\zeta^l_\beta, \zeta^m_\alpha) = \rho(\zeta^l_\beta, \beta)^\smile \rho(\beta, \beta^3)^\smile \rho(\beta^3, \beta_3)^\smile \rho(\beta_3, \beta_2)^\smile \\ \rho(\beta_2, \beta_1)^\smile \rho(\beta_1, \zeta^m_\alpha);\end{aligned}$$

(24)
$$\begin{aligned}\rho_H(\zeta^l_\beta, \zeta^m_\alpha) = \rho_H(\zeta^l_\beta, \beta)^\smile \rho_H(\beta, \beta^3)^\smile \rho_H(\beta^3, \beta_3)^\smile \rho_H(\beta_3, \beta_2)^\smile \\ \rho_H(\beta_2, \beta_1)^\smile \rho_H(\beta_1, \zeta^m_\alpha).\end{aligned}$$

Now note that the lengths of the six components of this long sequence are, respectively, $|\nu^l|$, $|\eta|$, s, $|\rho|$, $|\tau|$, and $|\nu_m|$, where $s > \max\{|\nu_l : l < n\} + |\tau| + |\rho| + |\eta| + \max\{|\nu^l| : l < n\}$. Let k be the length of the long sequence. Then for any p, if $p < \frac{k}{2}$ then p comes before the component $\rho(\beta_3, \beta_2)$.

Now

(25) All entries of $\rho_H(\zeta^l_\beta, \beta)$ are less than i_0.

This holds by (12), since $\beta \in T^2 \subseteq T^1$.

(26) Some entries of $\rho_H(\beta, \beta^3)$ are $\geq i_1$, but all entries are $< i_2$.

This holds since $\rho_H(\beta, \beta^3)^{i_1} = \eta \notin {}^{<\omega}\omega_1$.

(27) All entries of $\rho_H(\beta^3, \beta_3)$ are less than i_0.

(28) Some entries of $\rho_H(\beta_3, \beta_2)$ are $\geq i_0$, but all entries are less than i_1.

This holds since $\rho_H(\beta_3, \beta_2)^{i_0} = \rho \notin {}^{<\omega}\omega_1$.

(29) Some entries of $\rho_H(\beta_2, \beta_1)$ are $\geq i_2$, while other entries are less than i_1.

This holds by the definition of i_2 and the fact that $i_1 \in R'$.

(30) All entries of $\rho_H(\beta_1, \zeta^m_\alpha)$ are less than i_0.

It now follows that $w_1(\zeta^l_\beta, \zeta^m_\alpha)$ consists of all p in the $\rho(\beta_2, \beta_1)$ portion with value $\geq i_2$. In particular, $w_1(\zeta^l_\beta, \zeta^m_\alpha) \neq \emptyset$. Hence $p_1(\zeta^l_\beta, \zeta^m_\alpha) = |\rho(\zeta^l_\beta, \beta_3)| + |\rho| + l^*$. Now $w_2(\zeta^l_\beta, \zeta^m_\alpha)$ consists of those q in the $\rho_H(\beta, \beta^3)$ portion whose value is $\geq i_1$. Now $\rho_H(\beta, \beta^3)^{i_1} = \eta$, $\beta^3 \in T^3$, and $\eta \in U(\beta^3, T^2, R_{l*+|\rho|}, \chi^3)$. Thus $\min\{(\rho_H(\beta, \beta^3))(l) : \eta^{i_1}(l) = \omega_1\} \in R_{l^*+|\rho|}$, i.e., $p_2(\zeta^l_\beta, \zeta^m_\alpha) = l^* + |\rho|$. So $\gamma_{p_1 - p_2} = \gamma_s = \beta_3 \in S^\gamma$, so $c(\zeta^l_\beta, \zeta^m_\alpha) = \gamma$, as desired. □

Theorem 3.8. *Suppose that λ is a regular cardinal, and S is a non-reflecting stationary subset of λ such that $\mathrm{cf}(\alpha) > \omega_1$ for every $\alpha \in S$, with every member of S a limit ordinal. Then $\mathrm{Pr}_1(\lambda, \lambda, 2, \omega)$ holds.*

Proof. Let c be obtained by Theorem 3.7. Define

$$c'(\beta, \alpha) = \begin{cases} 0 & \text{if } c(\beta, \alpha) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly c' is as desired. □

Theorem 3.9. *Suppose that $\mu > \aleph_1$ is a regular cardinal. Then $\mathrm{Pr}_1(\mu^+, \mu^+, 2, \omega)$ holds.*

Proof. See the remark before Theorem 3.7. □

Theorem 3.10. *For any regular cardinal λ, $\mathrm{Pr}_1(\lambda, \lambda, 2, \omega)$ implies that there exist BAs B_0, B_1 such that B_0 and B_1 satisfy the λ-cc but $B_0 \oplus B_1$ does not.*

Proof. Choose c by the definition of $\mathrm{Pr}_1(\lambda, \lambda, 2, \omega)$.

Let $A = \mathrm{Fr}(\langle x_\alpha : \alpha < \lambda \rangle)$, and for each $\varepsilon \in 2$ let

$$I_\varepsilon = \langle \{x_\alpha \cdot x_\beta : \alpha \neq \beta \text{ and } c(\alpha, \beta) = \varepsilon\} \rangle^{\mathrm{id}} \quad \text{and} \quad B_\varepsilon = A/I_\varepsilon.$$

(1) $[x_\alpha]_{I_\varepsilon} \neq 0$.

In fact, otherwise we get

$$x_\alpha \leq x_{\beta(0)} \cdot x_{\gamma(0)} + \cdots + x_{\beta(m-1)} \cdot x_{\gamma(m-1)}$$

with $\beta(i) \neq \gamma(i)$ for all $i < m$. Mapping x_α to 1 and all other generators to 0, then extending to a homomorphism from A into 2, gives a contradiction.

(2) For F and G disjoint finite subsets of λ we have

$$\prod_{\alpha \in F} [x_\alpha]_{I_\varepsilon} \cdot \prod_{\alpha \in G} -[x_\alpha]_{I_\varepsilon} = 0 \text{ iff } \text{ there are distinct } \alpha, \beta \in F \text{ such that } c(\alpha, \beta) = \varepsilon.$$

In fact, $\Leftarrow$ is clear. Now suppose that $\prod_{\alpha \in F} [x_\alpha]_{I_\varepsilon} \cdot \prod_{\alpha \in G} -[x_\alpha]_{I_\varepsilon} = 0$ while $c(\alpha, \beta) = 1 - \varepsilon$ for all distinct $\alpha, \beta \in F$. It follows that

$$\prod_{\alpha \in F} x_\alpha \cdot \prod_{\alpha \in G} -x_\alpha \leq x_{\beta(0)} \cdot x_{\gamma(0)} + \cdots + x_{\beta(m-1)} \cdot x_{\gamma(m-1)}$$

for some $\beta(0), \gamma(0), \ldots, \beta(m-1), \gamma(m-1)$, where $\beta(i) \neq \gamma(i)$ and $c(\beta(i), \gamma(i)) = \varepsilon$ for each $i < m$. Mapping x_α to 1 for each $\alpha \in F$ and all other generators to 0, and extending to a homomorphism from A into 2, we get a contradiction. So (2) holds.

Now clearly $\langle [x_\alpha]_{I_0} \cdot [x_\alpha]_{I_1} : \alpha < \lambda \rangle$ is a system of pairwise disjoint nonzero elements of $B_0 \oplus B_1$, so $B_0 \oplus B_1$ does not have the λ-cc.

Now fix $\varepsilon \in 2$. We want to get a contradiction from assuming that $\langle b_\zeta : \zeta < \lambda \rangle$ is a system of pairwise disjoint nonzero elements of B_ε. Wlog each b_ζ has the form

$$b_\zeta = \prod_{\alpha \in M_\zeta} [x_\alpha]^{\delta(\alpha, \zeta)}$$

with M_ζ a finite subset of λ and $\delta(\alpha, \zeta) \in 2$ for all $\alpha \in M_\zeta$. Wlog $\langle M_\zeta : \zeta < \lambda \rangle$ is a Δ-system with kernel N. Wlog we can write each b_ζ as follows:

$$b_\zeta = \prod_{\alpha \in N} [x_\alpha]^{\delta(\alpha)} \cdot \prod_{\alpha \in P_\zeta} [x_\alpha] \cdot \prod_{\alpha \in Q_\zeta} -[x_\alpha],$$

with $N \cap P_\zeta = \emptyset = N \cap Q_\zeta$, $P_\zeta \cap Q_\zeta = \emptyset$, and P_ζ, Q_ζ finite. Since $b_\zeta \neq 0$, we have $\forall \alpha, \beta \in P_\zeta [\alpha \neq \beta \rightarrow c(\alpha, \beta) = 1 - \varepsilon$. Now wlog $|P_\zeta| = n$ for all $\zeta < \lambda$. Now by the condition $\mathrm{Pr}_1(\lambda, \lambda, 2, \omega)$ it follows that there are $\xi < \eta < \lambda$ such that $c(\alpha, \beta) = 1 - \varepsilon$ for all $\alpha \in P_\xi$ and $\beta \in P_\eta$. Now by (2) it follows that $b_\xi \cdot b_\eta \neq 0$, contradiction. $\square$

Corollary 3.11. *For any regular cardinal $\mu > \aleph_1$, there are BAs B_0, B_1 such that B_0 and B_1 satisfy the μ^+-cc but $B_0 \oplus B_1$ does not.*

Proof. By Theorems 3.9 and 3.10. □

Now we continue with our list of results about free products.

(C10) There is a BA A such that $\mathrm{c}'(A) = \aleph_2$ while $\mathrm{c}'(A \oplus A) > \aleph_2$.

This is proved in Shelah [97].

Note that the results given so far answer the productivity question except for cellularity a limit cardinal.

(C11) For singular λ, there is a BA A such that $\mathrm{c}'(A) = \lambda^+$ while $\mathrm{c}'(A \oplus A) > \lambda^+$.

For this, see Shelah [94], Conclusion 4.1 in Chapter II, and Appendix 1, Diagram 1.8. We give this proof here. It involves a part of pcf theory, and also uses an interesting elementary substructure argument which is an instance of a proof method widely used at present.

It is useful to recall an argument proving the downward Löwenheim–Skolem theorem. We suppose that H is a first-order structure which has among its relations a well-order $<$ of H itself. Then with each formula $\exists x\varphi(x, \overline{y})$ of the given language we associate itsSkolem function f defined as follows:

$$\begin{aligned} f_{\exists x\varphi(x,\overline{y})}(\overline{a}) &= \text{the } <\text{-least element } b \text{ such that } H \models \varphi[b, \overline{a}], \\ &\quad\ \text{if there is such an element,} \\ &= a_0, \text{ otherwise.} \end{aligned}$$

Here we assume that $\overline{a} = \langle a_0, a_1, \ldots, a_{n-1}\rangle$ is a sequence of elements of H, with n positive, $\overline{y}$ being a sequence of distinct variables of length n. Note that any Skolem function is definable in the given language.

The following is a version of the well-known lemma of Tarski concerning elementary substructures. A first-order language is relational iff it does not have any operation symbols.

Theorem 3.12. *Suppose that H is a first-order structure over a relational language which has among its relations a well-order $<$ of H itself. Suppose also that X is a nonempty subset of H. Let Y be the union of the ranges of all of the Skolem functions over H, each one restricted to X. Let the relations of Y be the ones of H restricted to Y. Then Y is an elementary substructure of H.*

Proof. We prove by induction on the formula ψ that for any $\overline{a} \subseteq Y$, $H \models \psi[\overline{a}]$ iff $Y \models \psi[\overline{a}]$. Atomic ψ works by the definition of the relations on Y. The inductive steps using sentential connectives are straightforward. If $Y \models \exists x\varphi(x, \overline{a})$, then the inductive step easily applies. Suppose that $H \models \exists x\varphi(x, \overline{a})$. Let χ be the following formula, where $\overline{a}$ has length n, and $a_i = f_i(\overline{b}_i)$, each f_i a Skolem function and each $\overline{b}_i \subseteq X$:

$$\exists z_0, \ldots, z_{n-1} \left(\bigwedge_{i<n} (z_i = f_i(\overline{y}_i)) \wedge \varphi(x, \overline{z}) \right).$$

Then with g the Skolem function associated with $\exists x\chi$ we have

$$H \models \varphi(g(\overline{b}_0, \ldots, \overline{b}_{n-1}), \overline{a}),$$

and the inductive hypothesis gives the desired conclusion. □

The restriction to a relational language in this theorem is made just to shorten the proof, since we will apply the theorem only to such a language.

Lemma 3.13. *Let $\mathscr{L}$ be a relational language extending the language of set theory, with in particular an additional binary relation symbol $<$. Suppose that χ is an uncountable regular cardinal, M is an elementary substructure of $H(\chi)$, and $\theta \in M$ is a cardinal. Assume that $<$ well-orders $H(\chi)$. Let N be the elementary substructure of $H(\chi)$ determined by Theorem* 3.12*, using the set $X = M \cup \theta$. Then for any regular cardinal $\sigma \in M \backslash \theta^+$ we have*

$$\sup(M \cap \sigma) = \sup(N \cap \sigma).$$

Proof. Since $M \subseteq N$, we have $\sup(M \cap \sigma) \subseteq \sup(N \cap \sigma)$. Now suppose that $\alpha \in N \cap \sigma$; we want to find $\beta \in M \cap \alpha$ such that $\alpha \leq \beta$. By Theorem 3.12 we can write $\alpha = f(\overline{b}, \overline{c})$, with f a Skolem function, $\overline{b} \subseteq \theta$, and $\overline{c} \subseteq M \backslash \theta$. Say that n is the length of $\overline{b}$. We define a function F with domain ${}^n\theta$ by setting

$$F(\overline{d}) = \begin{cases} f(\overline{d}, \overline{c}) & \text{if this is an ordinal less than } \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly F is elementarily definable with parameters from M. Let $\beta = \sup(\text{rng}(F))$. Then $\beta \in M$ by the assumption that M is an elementary substructure of $H(\chi)$. Since $\alpha \in \text{rng}(F)$, we have $\alpha \leq \beta$. Also, $\text{rng}(F)$ is a set of at most θ ordinals less than σ, so $\beta < \sigma$ by regularity. □

Theorem 3.14. *If μ is a singular cardinal, then* $\text{Pr}_1(\mu^+, \mu^+, 2, \omega)$ *holds.*

Proof. By Theorems 2.23 and 2.26 of Abraham, Magidor [10] let $\langle \lambda_\alpha : \alpha < \text{cf}(\mu) \rangle$ and $\langle f_\alpha : \alpha < \mu^+ \rangle$ be such that the following conditions hold:

(1) $\langle \lambda_\alpha < \alpha < \text{cf}(\mu) \rangle$ is a strictly increasing sequence of regular cardinals with supremum μ such that $\text{cf}(\mu) < \lambda_0$.

(2) For $f, g \in \prod_{\alpha<\text{cf}(\mu)} \lambda_\alpha$ define $f <_{\text{cf}(\mu)} g$ iff $\exists \beta < \text{cf}(\mu) \forall \alpha \in (\beta, \text{cf}(\mu))[f(\alpha) < g(\alpha)]$. Then $\forall \alpha, \beta \in \text{cf}(\mu)[\alpha < \beta \rightarrow f_\alpha <_{\text{cf}(\mu)} f_\beta]$.

(3) $\forall g \in \prod_{\alpha<\mathrm{cf}(\mu)} \lambda_\alpha \exists \alpha < \mu^+[g <_{\mathrm{cf}(\mu)} f_\alpha]$.

Now we define, for $\alpha < \beta < \mu^+$, $d(\alpha, \beta)$ to be the maximum $i < \mathrm{cf}(\mu)$ such that $f_\beta(i) < f_\alpha(i)$ if there is such an i, and 0 otherwise. Let $h : \mathrm{cf}(\mu) \to 2$ be such that $h^{-1}[\{0\}]$ and $h^{-1}[\{1\}]$ have size $\mathrm{cf}(\mu)$. Define $c : [\mu^+]^2 \to 2$ by setting $c(\{\alpha, \beta\}) = h(d(\alpha, \beta))$ for $\alpha < \beta$.

The rest of the proof is to show that c works. Suppose that $n \in \omega\backslash 1$ and $a \in {}^{\mu^+}([\mu^+]^n)$, with the a_α's pairwise disjoint, and $\varphi < 2$.

(4) We may assume that $\forall \alpha < \mu^+[\alpha < \min(a_\alpha]$ and $\forall \alpha, \beta < \mu^+[\alpha < \beta \to \max(a_\alpha) < \min(a_\beta)]$.

To prove this, we define by recursion a sequence $\langle \gamma_\beta : \beta < \mu^+\rangle$ of ordinals less than μ^+. Suppose that γ_α has been defined for all $\alpha < \beta$. Then for each $\alpha < \beta$, the set $\{\delta < \mu^+ : \min(a_\delta) \leq \max(a_{\gamma_\alpha})\}$ has at most $\max(a_{\gamma_\alpha}) + 1 < \mu^+$ elements. Also, $\{\gamma < \mu^+ : \min a_\gamma \leq \beta\}$ has size less than μ^+. Hence we can choose $\gamma_\beta \in \mu^+$ greater than

$$\sup\left(\bigcup_{\alpha<\beta} \{\delta < \mu^+ : \min(a_\delta) \leq \max(a_{\gamma_\alpha})\}\right) \cup \sup(\{\gamma < \mu^+ : \min a_\gamma \leq \beta\}).$$

Let $a'_\beta = a_{\gamma_\beta}$ for all $\beta < \mu^+$. If $\alpha < \beta < \mu^+$, then $\max(a'_\alpha) = \max(a_{\gamma_\alpha}) < \min(a_{\gamma_\beta}) = \min(a'_\beta)$. If $\beta < \mu^+$, then $\beta < \min a_{\gamma_\beta} = \min a'_\beta$. Thus (4) holds.

Now for $\beta < \mu^+$ we define for $i < \mathrm{cf}(\mu)$

$$f^{\inf}_\beta(i) = \min\{f_\alpha(i) : \alpha \in a_\beta\};$$
$$f^{\sup}_\beta(i) = \max\{f_\alpha(i) : \alpha \in a_\beta\}.$$

(5) $\forall \beta \in \mathrm{cf}(\mu) \exists \gamma \in \mathrm{cf}(\mu) \forall \delta \in (\gamma, \mathrm{cf}(\mu))[f^{\inf}_\beta(\delta) = f_{\min(a_\beta)}(\delta)]$.

In fact, choose $\gamma \in \mathrm{cf}(\mu)$ so that $\forall \varepsilon \in a_\beta\backslash\{\min a_\beta\} \forall \delta \in (\gamma, \mathrm{cf}(\mu))[f_{\min(a_\beta)}(\delta) < f_\varepsilon(\delta)]$. Clearly this proves (5).

(6) $\exists \alpha < \mathrm{cf}(\delta) \forall i \in (\alpha, \mathrm{cf}(\delta))[\lambda_i = \sup\{f^{\inf}_\beta(i) : \beta < \mu^+\}]$.

In fact, suppose not. Then there is an $M \in [\mathrm{cf}(\mu)]^{\mathrm{cf}(\mu)}$ such that for all $i \in M$ we have $\sup\{f^{\inf}_\beta(i) : \beta < \mu^+\} < \lambda_i$. Let $g(i) = \sup\{f^{\inf}_\beta(i) : \beta < \mu^+\} + 1$ for all $i \in M$, and $g(i) = 0$ otherwise. Choose $\beta < \mu^+$ such that $g <_{\mathrm{cf}(\mu)} f_\beta$. Then by (4), also $g <_{\mathrm{cf}(\mu)} f_{\min a_\beta}$. By (5), choose $\gamma < \mathrm{cf}(\mu)$ such that $\forall \delta \in (\gamma, \mathrm{cf}(\mu))[f^{\inf}_\beta(\delta) = f_{\min a_\beta}(\delta)]$. We may also assume that $\forall \delta \in (\gamma, \mathrm{cf}(\mu))[g(\delta) < f_{\min a_\beta}(\delta)]$. Choose $\delta \in M \cap (\gamma, \mathrm{cf}(\mu))$. Then $g(\delta) < f_{\min(a_\beta)}(\delta) = f^{\inf}_\beta(\delta) < g(\delta)$, contradiction. So (6) holds.

Now let M be an elementary submodel of $H(\chi)$ with $|M| = \mathrm{cf}(\mu)$, $\mathrm{cf}(\mu) \subseteq M$, and $\mu, \mu^+, f, a \in M$. Note that if $i < \mathrm{cf}(\mu)$, then $\sup(M \cap \lambda_i) < \lambda_i$, since

$|M| = \mathrm{cf}(\mu) < \lambda_i$ and λ_i is regular. Hence if we define, for each $i < \mathrm{cf}(\mu)$,

$$\mathrm{Ch}_M(i) = \sup(M \cap \lambda_i),$$

we get an element of $\prod_{i<\mathrm{cf}(\mu)} \lambda_i$. Thus we can choose $\alpha < \mu^+$ such that $\mathrm{Ch}_M <_{\mathrm{cf}(\mu)} f_\alpha$; hence by (4) also $\mathrm{Ch}_M <_{\mathrm{cf}(\mu)} f_{\min(a_\alpha)}$.

(7) There is an $i^* < \mathrm{cf}(\mu)$ such that the following conditions hold:
  (a) $h(i^*) = \varphi$.
  (b) $\lambda_{i^*} = \sup\{f^{\inf}_\beta(i^*) : \beta < \mu^+\}$.
  (c) $\forall i \in [i^*, \mathrm{cf}(\mu))[\mathrm{Ch}_M(i) < f^{\inf}_\alpha(i)]$.

This is clear from (6). By (b), choose $\beta < \mu^+$ such that $f^{\sup}_\alpha(i^*) < f^{\inf}_\beta(i^*)$.

Let N be the elementary substructure of $H(\chi)$ obtained from Lemma 3.13 using the set $X = M \cup \lambda_{i^*}$, with $|N| = \lambda_{i^*}$. By Lemma 3.13,

(8) $\sup(M \cap \sigma) = \sup(N \cap \sigma)$ if σ is a regular cardinal greater than λ_{i^*}.

Thus

(9) $\mathrm{Ch}_M \restriction [i^* + 1, \mathrm{cf}(\mu)) = \mathrm{Ch}_N \restriction [i^* + 1, \mathrm{cf}(\mu))$.

Let $\delta = f^{\inf}_\beta(i^*)$. Then $\delta \in \lambda_{i^*} \subseteq N$, $i^* < \mathrm{cf}(\mu) \subseteq M \subseteq N$, and $\mu^+ \in M \subseteq N$. Moreover, $\exists \varepsilon < \mu^+[f^{\inf}_\varepsilon(i^*) = \delta]$. Hence by elementarity, there is an $\varepsilon \in N$ with $\varepsilon < \mu^+$ such that $f^{\inf}_\varepsilon(i^*) = \delta$. So $f^{\sup}_\alpha(i^*) < f^{\inf}_\varepsilon(i^*)$. But for any $i \in (i^*, \mathrm{cf}(\mu))$ we have

$$f^{\sup}_\varepsilon(i) < \mathrm{Ch}_N(i) = \mathrm{Ch}_M(i) < f^{\inf}_\alpha(i).$$

Hence if $\gamma \in a_\alpha$ and $\delta \in a_\varepsilon$, then $d(\gamma, \delta) = i^*$, and so $c(\gamma, \delta) = h(d(\gamma, \delta)) = \varphi$. □

Theorem 3.15. *For any singular λ, there is a BA A such that $\mathrm{c}'(A) = \lambda^+$ while $\mathrm{c}'(A \oplus A) > \lambda^+$.* □

Now we have discussed $\mathrm{c}'(A \oplus A)$ in all cases except that with $\mathrm{c}'(A)$ a regular limit cardinal. Here we have, first of all:

(C12) If κ is weakly compact and $\mathrm{c}'(A) = \kappa$, then $\mathrm{c}'(A \oplus A) = \kappa$.

In fact, assume the hypothesis, and suppose that $\langle x_\alpha : \alpha < \kappa \rangle$ is a system of pairwise disjoint nonzero elements of $A \oplus A$. Wlog we may assume that each x_α has the form $a_\alpha \cdot b_\alpha$, with a_α in the first copy of A and b_α in the second copy. Then define $f : [\kappa]^2 \to 2$ by setting, for any distinct $\alpha, \beta < \kappa$,

$$f(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } a_\alpha \cdot a_\beta = 0, \\ 1 & \text{if } b_\alpha \cdot b_\beta = 0. \end{cases}$$

Then a homogeneous set of size κ gives a system of κ nonzero pairwise disjoint elements of A, contradiction.

On the other hand, under $V = L$ for every regular limit cardinal λ which is not weakly compact, there is a non-reflecting stationary subset S of λ such that every member of S is a limit ordinal of cofinality $> \omega_1$; see Devlin [84], Theorem 1.2′ on page 304. Hence by Theorem 3.8 $V = L$ implies the existence of a BA A with $\mathrm{c}'(A) = \lambda$ while $\mathrm{c}'(A \oplus A) > \lambda$.

This completes our exposition of results concerning the cellularity of free products of two Boolean algebras. The following problems remain open; they are versions of Problem 1 in Monk [96].

Problem 4. *Can one construct BAs A, B such that $\mathrm{c}'(B) < \mathrm{c}'(A) < \mathrm{c}'(A \oplus B)$ with $\mathrm{c}'(B)$ regular limit?*

Problem 5. *Can one prove in ZFC that if κ is inaccessible but not weakly compact, then there a BA A such that $\mathrm{c}'(A) = \kappa$ while $\mathrm{c}'(A \oplus A) > \kappa$?*

Another important and quite elementary fact about free products is that

$$\mathrm{c}(\oplus_{i \in I} A_i) = \sup\{\mathrm{c}(\oplus_{i \in F} A_i) : F \in [I]^{<\omega}\}.$$

In fact, $\geq$ is clear. Now let $\kappa = \sup\{\mathrm{c}(\oplus_{i \in F} A_i) : F \in [I]^{<\omega}\}$, and suppose that X is a disjoint subset of $\oplus_{i \in I} A_i$ of size κ^+. For each $x \in X$ choose a finite $F(x) \subseteq I$ such that $x \in \oplus_{i \in F(x)} A_i$. We may assume that each $x \in X$ has the form $x = \prod_{i \in F(x)} y_i^x$, where $y_i^x \in A_i$ for each $i \in F(x)$. Without loss of generality, $\langle F(x) : x \in X\rangle$ forms a Δ-system, say with kernel G. But then, by the free product property, $\langle \prod_{i \in G} y_i^x : x \in X\rangle$ is a disjoint system of elements of $\oplus_{i \in G} A_i$, contradiction.

Next, we consider amalgamated free products where the following fact is basic:

$$\mathrm{c}(A \oplus_C B) \leq 2^{\mathrm{c}(A) \cdot \mathrm{c}(B) \cdot |C|}.$$

To prove this, let $\kappa = \mathrm{c}(A) \cdot \mathrm{c}(B) \cdot |C|$, and suppose that $\langle c_\alpha : \alpha < (2^\kappa)^+\rangle$ is a disjoint system in $A \oplus_C B$. We may assume that each c_α is non-zero, and has the form $a_\alpha \cdot b_\alpha$, with $a_\alpha \in A$ and $b_\alpha \in B$. Thus for all distinct $\alpha, \beta < (2^\kappa)^+$ there is a $c \in C$ such that $a_\alpha \cdot a_\beta \leq c$ and $b_\alpha \cdot b_\beta \leq -c$. Hence by the Erdős–Rado theorem there is a $\Gamma \in [(2^\kappa)^+]^{\kappa^+}$ and a $c \in C$ such that $a_\alpha \cdot a_\beta \leq c$ and $b_\alpha \cdot b_\beta \leq -c$ for all distinct $\alpha, \beta \in \Gamma$. Thus $(a_\alpha \cdot -c) \cdot (a_\beta \cdot -c) = 0$ and $(b_\alpha \cdot c) \cdot (b_\beta \cdot c) = 0$ for all distinct $\alpha, \beta \in (2^\kappa)^+$. Since $\mathrm{c}A < \kappa^+$, it follows that there is a $\Delta \in [\Gamma]^\kappa$ such that $a_\alpha \cdot -c = 0$ for all $\alpha \in \Gamma \backslash \Delta$; and there is a $\Theta \in [\Gamma \backslash \Delta]^\kappa$ such that $b_\alpha \cdot c = 0$ for all $\alpha \in (\Gamma \backslash \Delta) \backslash \Theta$. But then for any $\alpha \in (\Gamma \backslash \Delta) \backslash \Theta$ we have $a_\alpha \cdot b_\alpha = 0$, contradiction.

The above inequality is best-possible, in a sense. To see this, consider $\mathscr{P}(\omega) \oplus_C \mathscr{P}(\omega)$, where C is the BA of finite and cofinite subsets of ω. Let $\langle \Gamma_\alpha : \alpha < 2^\omega\rangle$ be a system of infinite almost disjoint subsets of ω; and also assume that each Γ_α is not cofinite. For each $\alpha < 2^\omega$ let y_α be the element $\Gamma_\alpha \cdot (\omega \backslash \Gamma_\alpha)$ of $\mathscr{P}(\omega) \oplus_C \mathscr{P}(\omega)$.

These elements are clearly non-zero. For distinct $\alpha, \beta < 2^\omega$ let $F = \Gamma_\alpha \cap \Gamma_\beta$. Then $\Gamma_\alpha \cap \Gamma_\beta = F$ and $(\omega\backslash\Gamma_\alpha) \cap (\omega\backslash\Gamma_\beta) \subseteq (\omega\backslash F)$, which shows that the system is disjoint. This demonstrates equality above, with $A = B = \mathscr{P}(\omega)$ and $C = \mathrm{Finco}(\omega)$.

For free amalgamated products with infinitely many factors we have

$$\mathrm{c}\left(\oplus^C_{i\in I} A_i\right) \le 2^{|C|} \cdot 2^{\sup_{i\in I} \mathrm{c}(A_i)}.$$

To prove this, let κ be the cardinal on the right, and suppose that $\langle y_\alpha : \alpha < \kappa^+\rangle$ is a disjoint system of elements of $\oplus^C_{i\in I}(A_i)$. We may assume that each y_α has the form

$$y_\alpha = \prod_{i\in F_\alpha} a^\alpha_i,$$

where F_α is a finite subset of I and $a^\alpha_i \in A_i$ for all $i \in F_\alpha$. We may assume, in fact, that the F_α's form a Δ-system, say with kernel G; and that they all have the same size. Thus by a change of notation we may write

$$y_\alpha = \prod_{j<m} a^\alpha_{i^\alpha_j} \cdot \prod_{j<n} a^\alpha_{k_j},$$

where $F_\alpha\backslash G = \{i^\alpha_j : j < m\}$ and $G = \{k_j : j < n\}$. For distinct $\alpha, \beta < \kappa^+$ there then exist $c_j \in C$ for $j < m$, $d_j \in C$ for $j < m$, and $e_j \in C$ for $j < n$ such that $a^\alpha_{i^\alpha_j} \le c_j$ for all $j < m$, $a^\beta_{i^\beta_j} \le d_j$ for all $j < m$, and $a^\alpha_{k_j} \cdot a^\beta_{k_j} \le e_j$ for all $j < n$, such that

$$\prod_{j<m} c_j \cdot \prod_{j<m} d_j \cdot \prod_{j<n} e_j = 0.$$

Using the Erdős–Rado theorem again, we get $\Gamma \in [\kappa^+]^\lambda$ and $c, d \in {}^mC$, $e \in {}^nC$ such that the above holds for all distinct $\alpha, \beta \in \Gamma$, where $\lambda = (|C| \cdot \sup_{i\in I} \mathrm{c}(A_i))^+$. If $j < n$ and α, β are distinct members of Γ, then $a^\alpha_{k_j} \cdot -e_j \cdot a^\beta \cdot -e_j = 0$, so since $\mathrm{c}(A_{k_j}) < \kappa^+$ there is a $\Delta_j \in \Gamma^\kappa$ such that $a^\alpha_{k_j} \cdot -e_j = 0$ for all $\alpha \in \Gamma\backslash\Delta_j$. Choose $\alpha \in \Gamma\backslash \bigcup_{j<n} \Delta_j$. Then $y_\alpha \le \prod_{j<m} c_j \cdot \prod_{j<m} d_j \cdot \prod_{j<n} e_j = 0$, contradiction.

Since $\mathscr{P}(\omega) \oplus_C \mathscr{P}(\omega)$ can be considered as a subalgebra of $\oplus^C_{i\in\omega}\mathscr{P}(\omega)$, with C as in the example for the free product of two factors, it follows that the above inequality is again best possible.

The behaviour of cellularity under unions of well-ordered chains is clear on the basis of cardinal arithmetic. We restrict ourselves, without loss of generality, to well-ordered chains of regular type. Actually, we can formulate a more general fact about increasing chains of BAs; this fact will apply to several of our cardinal functions, namely to the ordinary sup-functions (see the introduction).

Theorem 3.16. *Let κ and λ be infinite cardinals, with λ regular. Suppose that k is an ordinary sup-function with respect to P. Then the following conditions are equivalent:*

(i) $\mathrm{cf}(\kappa) = \lambda$.

(ii) *There is a strictly increasing sequence $\langle A_\alpha : \alpha < \lambda\rangle$ of BAs each satisfying the $\kappa - k$-chain condition such that $\bigcup_{\alpha<\lambda} A_\alpha$ does not satisfy this condition.*

Proof. (i)⇒(ii): Assume (i). Let $\langle \mu_\xi : \xi < \lambda\rangle$ be a strictly increasing sequence of *ordinals* with sup κ (maybe κ is a successor cardinal, so that we cannot take the μ_ξ to be *cardinals*). Let A be a BA of size κ with a set $X \in P(A)$ such that $|X| = \kappa$. Write $A = \{a_\alpha : \alpha < \kappa\}$. For each $\xi < \lambda$ let $B_\xi = \langle\{a_\alpha : \alpha < \mu_\xi\}\rangle$. Thus $B_\xi \subseteq B_\eta$ if $\xi < \eta$, and $|B_\xi| < \kappa$ for all $\xi < \lambda$. Hence a strictly increasing subsequence is as desired (since λ is regular).

(ii)⇒(i). Assume that (ii) holds but (i) fails. Let X be a subset of $\bigcup_{\alpha<\lambda} A_\alpha$ of power κ which is in $P\left(\bigcup_{\alpha<\lambda} A_\alpha\right)$. If $\lambda < \mathrm{cf}(\kappa)$, then the facts that $X = \bigcup_{\alpha<\lambda}(X \cap A_\alpha)$, $|X| = \kappa$, and $|X \cap A_\alpha| < \kappa$ for all $\alpha < \lambda$, give a contradiction.

So, assume that $\mathrm{cf}(\kappa) < \lambda$. Now for all $\alpha < \lambda$ there is a $\beta > \alpha$ such that $X \cap A_\alpha \subset X \cap A_\beta$, since otherwise some A_α would contain X. It follows that $\lambda \leq \kappa$, and so κ is singular in the case we are considering. Let $\langle \mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with sup κ. Since $\sup_{\alpha<\lambda}|X \cap A_\alpha| = \kappa$, for each $\alpha < \mathrm{cf}(\kappa)$ choose $\nu(\alpha) < \lambda$ such that $|X \cap A_{\nu(\alpha)}| \geq \mu_\alpha$. Let $\rho = \sup_{\alpha<\mathrm{cf}(\kappa)}\nu(\alpha)$. Then $\rho < \lambda$ since $\mathrm{cf}(\kappa) < \lambda$ and λ is regular. But $|X \cap A_\rho| = \kappa$, contradiction. □

With regard to Theorem 3.16, see also Theorem 3.57.

Now we turn to ultraproducts. Many of the things which we mention hold for other cardinal functions as well. First we consider countably complete ultrafilters; the main result we want to give here is that if F is a countably complete ultrafilter on an infinite set I and A_i is a ccc BA for each $i \in I$, then $\prod_{i\in I} A_i/F$ also satisfies ccc. An analogous statement holds for many of our other functions. The result follows from the following standard facts. If F is countably complete and non-principal, then there is an uncountable measurable cardinal, and $|I|$ is at least as big as the first such – call it κ. (See Comfort, Negrepontis [74], p. 196.) Also, F is κ-complete. To see this, suppose not, and let λ be the least cardinal such that F is not λ-complete. Thus $\omega_1 < \lambda \leq \kappa$. Then there exist a cardinal $\mu < \lambda$ and disjoint $a_\alpha \subseteq I$ for $\alpha < \mu$ such that $I \backslash a_\alpha \in F$ for all $\alpha < \mu$, while $\bigcup_{\alpha<\mu} a_\alpha \in F$. Let $G = \{S \subseteq \mu : \bigcup_{\alpha\in S} a_\alpha \notin F\}$. Then it is easy to check that G is a σ-complete non-principal maximal ideal on μ, which is a contradiction, since μ is less than κ.

Now we can give the simple BA argument from these set-theoretical facts. Suppose $\prod_{i\in I} A_i/F$ does not satisfy ccc. Let $\langle [a_\alpha] : \alpha < \omega_1\rangle$ be a system of non-zero disjoint elements of the product; $[x]$ denotes the equivalence class of x under F. Since F is ω_2-complete, the sets $J_{\alpha\beta} \stackrel{\text{def}}{=} \{i \in I : (a_\alpha)_i \cdot (a_\beta)_i = 0\}$ for $\alpha \neq \beta$ and the sets $K_\alpha \stackrel{\text{def}}{=} \{i \in I : (a_\alpha)_i \neq 0\}$ have a non-zero intersection, since that intersection is in F. But this is obviously a contradiction.

Thus countably complete ultrafilters tend to preserve chain conditions; we skip trying to give a more general version of the above argument.

Problem 6. *Describe the possibilities for chain conditions in ultraproducts with respect to countably complete ultrafilters.*

Next, if F is a countably incomplete ultrafilter on I and each algebra A_i is infinite, then $\prod_{i\in I} A_i/F$ never has ccc. This follows from the fact that the ultraproduct is ω_1-saturated in the model-theoretic sense; see Chang, Keisler [73], p. 305.

Now we present some results of Douglas Peterson; see Peterson [97]. They depend on some well-known notions and results. An ultrafilter F on an infinite set I is *regular* if there is a system $\langle a_i : i \in I\rangle$ of elements of F such that $\bigcap_{j\in J} a_j = 0$ for every infinite subset J of I. The following concept is useful. Let F be an ultrafilter on I, and let $\langle \alpha_i : i \in I\rangle$ be a system of ordinals. We define the *essential supremum* of $\langle \alpha_i : i \in I\rangle$ over F to be

$$\text{ess.sup}^F_{i\in I}\alpha_i = \min\left\{\sup_{i\in b}\alpha_i : b \in F\right\}.$$

Keisler, Prikry [74] show that if F is a regular ultrafilter on an infinite set I and $\langle \kappa_i : i \in I\rangle$ is a system of infinite cardinals, then $|\prod_{i\in I}\kappa_i/F| = (\text{ess.sup}^F_{i\in I}\kappa_i)^{|I|}$.

Given an infinite cardinal κ and an ultrafilter F on some set I, we call F *κ-descendingly incomplete* provided that there is a system $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of F such that $a_\alpha \supseteq a_\beta$ whenever $\alpha < \beta < \kappa$, and $\bigcap_{\alpha<\kappa} a_\alpha = 0$. We need the following well-known fact:

$(*)$ If F is a regular ultrafilter on an infinite set I and κ is an infinite cardinal such that $\kappa \le |I|$, then F is κ-descendingly incomplete.

To prove this fact, let $\langle a_\alpha : \alpha < |I|\rangle$ be a system of elements showing the regularity of F. We may assume that $\kappa \subseteq I$. For each $\alpha < \kappa$, let $b_\alpha = \bigcup_{\kappa>\beta>\alpha} a_\beta$. Thus $\langle b_\alpha : \alpha < \kappa\rangle$ is descending. Now

$$\bigcap_{\alpha<\kappa} b_\alpha = \bigcap_{\alpha<\kappa}\bigcup_{\kappa>\gamma>\alpha} a_\gamma = \bigcup_{f\in F}\bigcap_{\alpha<\kappa} a_{f(\alpha)},$$

where $F = \prod_{\alpha<\kappa}(\alpha,\kappa)$. Given $f \in F$, we have

$$\bigcap_{\alpha<\kappa} a_{f(\alpha)} \subseteq \bigcap_{n\in\omega\setminus 1} a_{f^n(0)} = \emptyset,$$

as desired.

Now we begin Peterson's results, with two useful lemmas.

Lemma 3.17. *Suppose that κ is an uncountable limit cardinal, I is a set such that $\text{cf}(\kappa) \le |I|$, and F is a $\text{cf}(\kappa)$-descendingly incomplete ultrafilter on I. Then there is a sequence $\langle \lambda_i : i \in I\rangle$ of infinite cardinals such that $\lambda_i < \kappa$ for all $i \in I$ and $\text{ess.sup}^F_{i\in I}\lambda_i = \kappa$.*

Proof. By the $\mathrm{cf}(\kappa)$-descending incompleteness of F let $\langle a_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a system of elements of F such that $a_\alpha \supseteq a_\beta$ whenever $\alpha < \beta < \mathrm{cf}(\kappa)$, and $\bigcap_{\alpha<\mathrm{cf}(\kappa)} a_\alpha = 0$. We may assume that $a_0 = I$ and $a_\lambda = \bigcap_{\alpha<\lambda} a_\alpha$ for λ limit. Let $\langle \mu_\delta : \delta < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum κ. Now we define the sequence $\langle \lambda_i : i \in I\rangle$. Let $i \in I$. Then there is a unique $\gamma < \mathrm{cf}(\kappa)$ such that $i \in a_\gamma \backslash a_{\gamma+1}$, and we define $\lambda_i = \mu_\gamma$. Now to show that $\mathrm{ess.sup}^F_{i\in I}\lambda_i = \kappa$, take $a \in F$ and $\delta < \mathrm{cf}(\kappa)$; we shall show that $\sup\{\lambda_i : i \in a\} \ge \mu_\delta$. For any $i \in a_\delta$ we have $\lambda_i \ge \mu_\delta$. Now $a \cap a_\delta \neq 0$ since $a \cap a_\delta \in F$, and if we choose $i \in a \cap a_\delta$ we have $\mu_\delta \le \lambda_i$, and so $\mu_\delta \le \sup\{\lambda_i : i \in a\}$, as desired. □

Lemma 3.18. *Suppose that F is a regular ultrafilter on an infinite set I, κ is a limit cardinal, $\langle \kappa_i : i \in I\rangle$ is a system of infinite cardinals, $\kappa = \mathrm{ess.sup}^F_{i\in I}\kappa_i$, $\mathrm{cf}(\kappa) \le |I|$, and $\omega < \kappa_i < \kappa$ for all $i \in I$. Then there is a system $\langle \lambda_i : i \in I\rangle$ of infinite cardinals such that $\lambda_i < \kappa_i$ for all $i \in I$ and $\mathrm{ess.sup}^F_{i\in I}\lambda_i = \kappa$.*

Proof. Let $\langle a_\alpha : \alpha < |I|\rangle$ be a system of elements of F showing the regularity of F, with $a_0 = I$. Let $\langle \delta_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ such that $\delta_0 = \omega$. Let $G = \{i \in I : \kappa_i = \delta_\alpha$ for some limit α, or for $\alpha = 0\}$, and for each $i \in G$ let $\alpha(i)$ be such that $\kappa_i = \delta_{\alpha(i)}$. Set $a^i_\xi = \bigcup_{\xi\le\alpha<\alpha(i)} a_\alpha$ for each $i \in G$ and $\xi < \alpha(i)$. Then we have:

(1) $\bigcap_{\xi<\alpha(i)} a^i_\xi = \emptyset$ for each $i \in G$.

(2) If $i, j \in G$ are such that $\alpha(i) < \alpha(j)$, then for each $\xi < \alpha(i)$ we have $a^i_\xi \subseteq a^j_\xi$.

(3) If $i \in G$ and ξ is a limit ordinal $< \alpha(i)$, then $\bigcap_{\gamma<\xi} a^i_\gamma = a^i_\xi$.

(1) is proved like $(*)$ above. (2) is clear. For (3), suppose the hypotheses of (3) hold. Note that $a^i_\xi \subseteq a^i_\gamma$ for all $\gamma < \xi$, hence $a^i_\xi \subseteq \bigcap_{\gamma<\xi} a^i_\gamma$.

Suppose that $k \in \left(\bigcap_{\gamma<\xi} a^i_\gamma\right) \backslash a^i_\xi$. Then $k \notin \bigcup_{\xi\le\alpha<\alpha(i)} a_\alpha$, so for each $\gamma < \xi$, $k \in \bigcup_{\gamma\le\alpha<\alpha(i)} a_\alpha \backslash \bigcup_{\xi\le\alpha<\alpha(i)} a_\alpha$, hence k is in some a_α with $\gamma \le \alpha < \xi$. This clearly implies that k is in infinitely many a_α's, contradiction.

Now let $i \in I$ be arbitrary. We will define λ_i by cases.

Case 1. $i \notin G$. There is an $\alpha < \mathrm{cf}(\kappa)$ such that $\delta_\alpha \le \kappa_i < \delta_{\alpha+1}$. If α is a successor ordinal $\beta + 1$, let $\lambda_i = \delta_\beta$. Otherwise α is a limit ordinal or 0, and by $i \notin G$ we have $\delta_\alpha < \kappa_i < \delta_{\alpha+1}$, and we let $\lambda_i = \delta_\alpha$. Under either possibility we then have $\lambda_i < \kappa_i$.

Case 2. $i \in G$. Since $a_0 = I$, we have $i \in a^i_0$. Let $\xi = \sup\{\eta < \alpha(i) : i \in a^i_\eta\}$. Then by (1), $\xi < \alpha(i)$, and by (3), $i \in a^i_\xi \backslash a^i_{\xi+1}$; let $\lambda_i = \delta_\xi$. Thus $\lambda_i = \delta_\xi < \delta_{\alpha(i)} = \kappa_i$.

In order to show that $\mathrm{ess.sup}^F_{i\in I}\lambda_i = \kappa$, suppose that $a \in F$ and $\omega \le \rho < \kappa$; we show that $\sup\{\lambda_i : i \in a\} \ge \rho$. Choose $\alpha < \mathrm{cf}(\kappa)$ such that $\delta_\alpha \le \rho < \delta_{\alpha+1}$. We consider two cases.

Case 1. $G \notin F$. Let $a' = \{i \in I : \kappa_i > \delta_{\alpha+2}\}$. Then $a' \in F$ since $\text{ess.sup}^F_{i\in I}\kappa_i = \kappa$. If $i \in a'\backslash G$, then $\lambda_i \geq \delta_{\alpha+1} > \rho$. Since $I\backslash G \in F$, there is a $j \in (a \cap a')\backslash G$. Then $\sup\{\lambda_i : i \in a\} \geq \lambda_j > \rho$.

Case 2. $G \in F$. Since $\text{ess.sup}_{i\in I}\kappa_i = \kappa$, for each $\gamma < \kappa$ we have $M_\gamma \stackrel{\text{def}}{=} \{i \in G : \delta_{\alpha(i)} > \gamma\} \in F$. Hence choose $k \in G$ such that $\delta_{\alpha(k)} > \delta_{\alpha+1}$. Then choose $j \in M_{\delta_{\alpha(k)}} \cap a^k_{\alpha+1} \cap a$. Then $j \in a^j_{\alpha+1}$ since $a^j_{\alpha+1} \supseteq a^k_{\alpha+1}$ by (2). Choose ε such that $j \in a^j_\varepsilon \backslash a^j_{\varepsilon+1}$. Then $\varepsilon \geq \alpha+1$, and so $\lambda_j = \delta_\varepsilon \geq \delta_{\alpha+1} > \rho$. Hence $\sup\{\lambda_i : i \in a\} \geq \lambda_j > \rho$. □

We need three more simple results.

Theorem 3.19. *If F is a regular ultrafilter over a set I then there is a system $\langle n_i : i \in I\rangle$ of natural numbers such that $\left|\prod_{i\in I} n_i/F\right| = 2^{|I|}$.*

Proof. Let $\langle a_i : i \in I\rangle$ be a system showing that F is regular. For each $i \in I$ let $M_i = \{j \in I : i \in a_j\}$. Thus $|M_i| < \omega$. We will show that $2^{|I|} \leq \left|\prod_{i\in I} {}^{M_i}2/F\right|$, proving the theorem. For each $g \in {}^I2$ define $g' \in \prod_{i\in I} {}^{M_i}2$ by $g'(i) = g \restriction M_i$. If $g, h \in {}^I2$ and $g \neq h$, pick $j \in I$ such that $g(j) \neq h(j)$; then for any $i \in a_j$ we have $j \in M_i$, and hence $g'(i) \neq h'(i)$. This shows that $a_j \subseteq \{i \in I : g'(i) \neq h'(i)\}$, and hence $g'/F \neq h'/F$. □

Recall that if k is a cardinal function defined by supremums with respect to a function P, then

$$k'(A) = \min\{\kappa : |X| < \kappa \text{ for all } X \in P(A)\}.$$

Proposition 3.20. *If k is an ultra-sup function, $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, F is an ultrafilter on I, and $\kappa_i < k'(A_i)$ for all $i \in I$. then $k\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \kappa_i/F\right|$.* □

Corollary 3.21. *If k is an ultra-sup function, $\langle A_i : i \in I\rangle$ is a system of BAs with I infinite, and F is an ultrafilter on I, then $k\left(\prod_{i\in I} A_i/F\right) \geq \text{ess.sup}^F_{i\in I}k(A_i)$.*

Proof. Let $\lambda = \text{ess.sup}^F_{i\in I}k(A_i)$. If λ is a successor cardinal, then we may assume that $k(A_i) = \lambda$ for all $i \in I$, and we are done by Proposition 3.20. If λ is a limit cardinal, then for each regular cardinal $\gamma < \lambda$ we have $k\left(\prod_{i\in I} A_i/F\right) \geq |{}^I\gamma/F| \geq \gamma$, and the result follows. □

Now we are ready for the main result of Peterson:

Theorem 3.22. *Let k be an ultra-sup function with respect to P such that if A is an infinite BA then $P(A)$ contains arbitrarily large finite sets. Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs with I infinite and F is a regular ultrafilter on I. Then $k\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} k(A_i)/F\right|$.*

Proof. Let $\lambda = \text{ess.sup}^F_{i\in I} k(A_i)$, and recall that $\lambda^{|I|} = \left|\prod_{i\in I} k(A_i)/F\right|$ (the result of Keisler and Prikry mentioned above). We now consider several cases.

Case 1. $\lambda = \omega$. Then $\lambda^{|I|} = 2^{|I|}$, and by Theorem 3.19, $k\left(\prod_{i\in I} A_i/F\right) \geq 2^{|I|}$.

Case 2. $\omega < \lambda \leq |I|$. Then $k\left(\prod_{i\in I} A_i/F\right) \geq |^I\omega/F| = 2^{|I|} = \lambda^{|I|}$.

Case 3. $\text{cf}(\lambda) \leq |I| < \lambda$, and $\{i \in I : k(A_i) = \lambda\} \in F$. Then we may assume that $k(A_i) = \lambda$ for all $i \in I$. By Lemma 3.17 and $(*)$ before it, let $\langle \kappa_i : i \in I\rangle$ be a system of infinite cardinals such that $\kappa_i < \lambda$ for all $i \in I$ and $\text{ess.sup}^F_{i\in I}\kappa_i = \lambda$. Then by Proposition 3.20, $k\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} k(A_i)/F\right|$.

Case 4. $\text{cf}(\lambda) \leq |I| < \lambda$, and $\{i \in I : k(A_i) < \lambda\} \in F$. This is like Case 3, except Lemma 3.18 is used.

Case 5. $|I| < \text{cf}(\lambda)$. Then $\lambda^{|I|} = \sup\{\kappa^{|I|} : \kappa < \lambda\}$. If $\omega \leq \kappa < \lambda$, then $k\left(\prod_{i\in I} A_i/F\right) \geq |^I\kappa/F| = \kappa^{|I|}$. Hence $k\left(\prod_{i\in I} A_i/F\right) \geq \lambda^{|I|}$. □

Recall that c is an ultra-sup function. Thus Theorem 3.22 gives a lower bound for $\text{c}\left(\prod_{i\in I} A_i/F\right)$, at least for regular F. The following simple result gives an upper bound.

Theorem 3.23. *Let $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and suppose that F is a uniform ultrafilter on I. Let $\kappa = \max(|I|, \text{ess.sup}^F_{i\in I}\text{c}(A_i))$. Then $\text{c}\left(\prod_{i\in I} A_i/F\right) \leq 2^\kappa$.*

Proof. Let $\lambda = \text{ess.sup}^F_{i\in I}\text{c}(A_i)$. We may assume that $\text{c}(A_i) \leq \lambda$ for all $i \in I$. In order to get a contradiction, suppose that $\langle f_\alpha/F : \alpha < (2^\kappa)^+\rangle$ is a system of disjoint elements. We may assume that $f_\alpha(i) \neq 0$ for all $i \in I$ and $\alpha < (2^\kappa)^+$. Thus $[(2^\kappa)^+]^2 = \bigcup_{i\in I}\{\{\alpha,\beta\} : f_\alpha(i)\cdot f_\beta(i) = 0\}$, so by the Erdős–Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ we get a homogeneous set which gives a contradiction. □

Concerning the inequality in Theorem 3.22, an example with strict inequality is given in Shelah [90], in fact, with $|\prod_{i\in\omega}\text{c}(A_i)/F| < \text{c}\left(\prod_{i\in\omega} A_i/F\right)$ for any uniform ultrafilter F on ω. We give a form of this result, using Theorem 3.7.

Theorem 3.24. (Shelah). *Let μ be a regular cardinal greater than $\aleph_1$. Then there is a system $\langle B_n : n \in \omega\rangle$ of BAs each satisfying the μ^+-cc such that for any non-principal ultrafilter D on ω, the ultraproduct $\prod_{n\in\omega} B_n/D$ does not satisfy the μ^+-cc.*

Proof. By Theorem 3.7, $\text{Pr}_1(\mu^+, \mu^+, \omega_1, \omega)$ holds. Let $c : [\mu^+] \to \omega_1$ be as in the definition of $\text{Pr}_1(\mu^+, \mu^+, \omega_1, \omega)$.

Temporarily fix $n \in \omega$. Let C_n be freely generated by $\langle x^n_\alpha : \alpha < \mu^+\rangle$. Let I_n be the ideal of C_n generated by the set

$$\{x^n_\alpha \cdot x^n_\beta : \alpha < \beta < \mu^+ \text{ and } c(\{\alpha,\beta\}) < n\}.$$

Let $y^n_\alpha = x^n_\alpha/I_n$ for each $\alpha < \mu^+$. Set $B_n = C_n/I_n$. Then for $\alpha < \beta < \mu^+$ we have $\{n \in \omega : y^n_\alpha \cdot y^n_\beta = 0\} \supseteq \{n \in \omega : c(\{\alpha, \beta\}) < n\}$; the latter set is cofinite, so $(y^n_\alpha/D) \cdot (y^n_\beta/D) = 0$. We claim that $y^n_\alpha \neq 0$ for all $n \in \omega$. Otherwise we would get

$$x^n_\alpha \leq x^n_{\gamma_1} \cdot x^n_{\delta_1} + \cdots + x^n_{\gamma_m} \cdot x^n_{\delta_m}$$

with $\gamma_i \neq \delta_i$ for all i. Mapping x^n_α to 1 and all other generators to 0 then extending to a homomorphism, we get a contradiction.

To show that each B_n satisfies μ^+-cc, assume that $\langle b_\alpha : \alpha < \mu^+ \rangle \in {}^{\mu^+}C_n$ is such that $b_\alpha \cdot b_\beta \in I_n$ for all distinct $\alpha, \beta < \mu^+$, while each $b_\alpha \notin I_n$; we want to get a contradiction. Without loss of generality, we may assume that each b_α has the following form:

$$b_\alpha = \prod_{\beta \in F_\alpha} (x^n_\beta)^{\varepsilon_\alpha(\beta)},$$

where F_α is a finite subset of μ^+ and $\varepsilon_\alpha \in {}^{F_\alpha}2$. Without loss of generality we may assume that: $\langle F_\alpha : \alpha < \mu^+ \rangle$ forms a Δ-system, say with kernel G; $\varepsilon_\alpha \restriction G$ is the same for all $\alpha < \mu^+$; and $|F_\alpha \backslash G| = |F_\beta \backslash G|$ for all $\alpha, \beta < \mu^+$. Cutting down further, we may assume that $F_\alpha \backslash G < F_\beta \backslash G$ if $\alpha < \beta$ (that is, $\gamma < \delta$ if $\gamma \in F_\alpha \backslash G$ and $\delta \in F_\beta \backslash G$). Now by the definition of $\mathrm{Pr}_1(\mu^+, \mu^+, \omega_1, \omega)$, choose $\alpha < \beta < \mu^+$ so that $c(\{\varepsilon, \zeta\}) = n$ for all $\varepsilon \in F_\alpha \backslash G$ and $\zeta \in F_\beta \backslash G$. Now we can write

$$b_\alpha \cdot b_\beta \leq x^n_{\gamma_1} \cdot x^n_{\delta_1} + \cdots + x^n_{\gamma_m} \cdot x^n_{\delta_m} \tag{1}$$

with $\gamma_i < \delta_i$ and $c(\{\gamma_i, \delta_i\}) < n$; moreover, we assume that m is minimal so that such an inequality holds. It follows that $\gamma_i, \delta_i \in F_\alpha \cup F_\beta$ for all i. If $\gamma_i \in F_\alpha$, then $\varepsilon_\alpha(\gamma_i) = 1$, since otherwise the summand $x^n_{\gamma_i} \cdot x^n_{\delta_i}$ could be dropped. Similarly $\gamma_i \in F_\beta$ implies that $\varepsilon_\beta(\gamma_i) = 1$, and similarly for the δ_i's. Now it follows, since $b_\alpha \notin I_n$, that we cannot have $\gamma_1, \delta_1 \in F_\alpha$. Similarly, $\gamma_1, \delta_1 \notin F_\beta$. It follows, then, that $\gamma_1 \in F_\alpha \backslash G$ and $\delta_1 \in F_\beta \backslash G$. But then $c(\{\gamma_1, \delta_1\}) = n$, contradiction. □

Corollary 3.25. *For any infinite cardinal κ there is a system $\langle B_n : n \in \omega \rangle$ such that for any non-principal ideal D on ω, $|\prod_{n\in\omega} \mathrm{c}(B_n)/D| \leq (2^\kappa)^+$ while $\mathrm{c}(\prod_{n\in\omega} B_n/D) \geq (2^\kappa)^{++}$.*

Proof. Apply Theorem 3.24 to $\mu = (2^\kappa)^+$. □

According to a result of Donder [88], it is consistent that the lower bound in Theorem 3.22 always holds (since his result says that it is consistent that every uniform ultrafilter is regular). It is also consistent that the bound fails (for cellularity); see Magidor, Shelah [98]. The example they give uses some large cardinals. This answers Problem 2 in Monk [96].

Now we consider the upper bound given in Theorem 3.23. By the following theorem, that bound can be attained. This theorem is a consequence of Lemma 1.5.3 in McKenzie, Monk [82].

Theorem 3.26. *Suppose that κ is an infinite cardinal and $\langle A_\alpha : \alpha < \kappa\rangle$ is a system of infinite BAs. Then there is a non-principal filter F on κ such that for every non-principal ultrafilter $G \supseteq F$, the BA $\prod_{\alpha<\kappa} A_\alpha/G$ has a chain of order type 2^κ, and hence has a pairwise disjoint set of size 2^κ.*

Proof. By Comfort, Negrepontis [74] Corollary 3.17, there is an $S \subseteq {}^\kappa\omega$ with the following properties:

(1) $|S| = 2^\kappa$.

(2) For every finite $M \subseteq S$ and every $a \in {}^M\omega$ there is an $\alpha < \kappa$ such that $f(\alpha) = a(f)$ for every $f \in M$.

Now for each $\alpha < \kappa$ let $\langle x_{\alpha i} : i \in \omega\rangle$ be a system of elements of A_α such that $0 < x_{\alpha 0} < x_{\alpha 1} < \cdots$. Let $\langle f_\xi : \xi < 2^\kappa\rangle$ be a one-one enumeration of S. For each $\xi < 2^\kappa$ we define $g_\xi \in \prod_{\alpha<\kappa} A_\alpha$ by setting $g_\xi(\alpha) = x_{\alpha f_\xi(\alpha)}$. Now for $\xi < \eta < 2^\kappa$ we define

$$J_{\xi\eta} = \{\alpha < \kappa : g_\xi(\alpha) < g_\eta(\alpha)\}.$$

We claim:

(3) $\{J_{\xi\eta} : \xi < \eta < 2^\kappa\}$ has fip.

For, suppose that N is a finite set of pairs (ξ, η) with $\xi < \eta < 2^\kappa$. Let M be the set of all f_ξ such that ξ is the first or second coordinate of some member of N. We can write $M = \{f_{\xi(i)} : i < m\}$ for some natural number m, such that $\xi(i) < \xi(j)$ if $i < j < m$. Then let $a(f_{\xi(i)}) = i$ for all $i < m$. By (2) choose $\alpha < \kappa$ such that $f_{\xi(i)}(\alpha) = a(f_{\xi(i)})$ for every $i < m$. Then for $i < j < m$,

$$g_{\xi(i)}(\alpha) = x_{\alpha f_{\xi(i)}(\alpha)} = x_{\alpha a(f_{\xi(i)})} = x_{\alpha i} < x_{\alpha j} = x_{\alpha a(f_{\xi(j)})} = x_{\alpha f_{\xi(j)}(\alpha)} = g_{\xi(j)}(\alpha),$$

and hence $\alpha \in \bigcap_{(\xi,\eta)\in N} J_{\xi\eta}$. This proves (3).

So, let F be the filter generated by all sets $J_{\xi\eta}$ for $\xi < \eta < 2^\kappa$. Clearly this gives the desired conclusion. □

There are trivial examples where the bound in Theorem 3.23 is not attained. For example, one can take $I = \omega$ and $A_i = \mathrm{Finco}(2^\omega)$ for every $i \in \omega$. Thus $\mathrm{c}(A_i) = 2^\omega$ for each $i \in \omega$, while for any ultrafilter F on ω, the ultraproduct $\prod_{i\in\omega} A_i/F$ has size 2^ω, and hence has cellularity 2^ω.

We turn to the examination of cellularity under other operations on Boolean algebras.

We describe the situation with subdirect products. Suppose that B is a subdirect product of BAs $\langle A_i : i \in I\rangle$; what is the cellularity of B in terms of the cellularity of the A_i's? Well, since a direct product is a special case of a subdirect product, we have the upper bound $\mathrm{c}(B) \le \sup_{i\in I}\mathrm{c}(A_i) \cup |I|$. The lower bound ω is obvious. And that lower bound can be attained, even if the algebras A_i have high cellularity. In fact, consider the following example. Let κ be any infinite cardinal, let A be the free BA on κ free generators, and let B be the algebra of finite and cofinite subsets of κ. We show that A is isomorphic to a subdirect product of copies of B. To do this, it suffices to take any non-zero element $a \in A$ and find a homomorphism of A onto B which takes $-a$ to 0. In fact, $A \restriction a$ is still free on κ free generators, and so there is a homomorphism of it onto B. So our desired homomorphism is obtained as follows:

$$A \to (A \restriction -a) \times (A \restriction a) \to A \restriction a \to B.$$

For moderate products we have $\mathrm{c}\left(\prod_{i \in I}^B A_i\right) = |I| + \sup_{i \in I} \mathrm{c}(A_i)$. In fact, $\geq$ is clear. Now suppose that $\kappa = |I| + \sup_{i \in I} \mathrm{c}(A_i)$ and $\langle h(b^\alpha, F^\alpha, a^\alpha) : \alpha < \kappa^+ \rangle$ is a system of disjoint elements of $\prod_{i \in I}^B A_i$. Then $\langle b^\alpha : \alpha < \kappa^+$ is a system of disjoint elements of B, and $B \subseteq \mathscr{P}(I)$, so fewer than κ^+ of the b^α's are nonzero. So we may assume that $b^\alpha = \emptyset$ for all $\alpha < \kappa^+$. Now let $\langle F^\alpha : \alpha \in M \rangle$ be a Δ-system, say with kernel G, with $|M| = \kappa^+$. Then $\langle F^\alpha \backslash G : \alpha \in M \rangle$ is a system of disjoint subsets of I, so at most κ of these sets are nonempty. Hence we may assume that $F^\alpha \backslash G = \emptyset$ for all $\alpha \in M$. Now $\langle a^\alpha : \alpha \in M \rangle$ is a system of disjoint nonempty elements of $\prod_{i \in G} A_i$, and $|M| = \kappa^+$, contradiction.

Next, let B be obtained from algebras $\langle A_i : i \in I \rangle$ by one-point gluing, as described in Chapter 1. With respect to cellularity, B behaves like the full direct product: If B is infinite and all algebras A_i have at least four elements, then $\mathrm{c}(B) = |I| + \sup_{i \in I} \mathrm{c}(A_i)$. In fact, $\leq$ holds since $B \leq \prod_{i \in I} A_i$. Now suppose that $i \in I$ and $X \subseteq A_i$ is disjoint. At most one member of X is in F_i, so we may assume that all members of X are not in F_i. For each $x \in X$ let f_x be the member of $\prod_{j \in I} A_j$ such that $f_x(i) = x$ and $f_x(j) = 0$ for $j \neq i$. Then $f_x \in B$, and $\langle f_x : x \in X \rangle$ is a disjoint system in B. Hence $\mathrm{c}(A_i) \leq \mathrm{c}(B)$. Next, let $i \in I$. Choose $x \in A_i$ such that $x \notin F_i$; this is possible since $|A_i| \geq 4$. Let $g_i(i) = x$ and $g_i(j) = 0$ for $j \neq i$. Then $\langle g_i : i \in I \rangle$ is a system of nonzero elements of B, and so $|I| \leq \mathrm{c}(B)$.

Our next algebraic operation is the Alexsandroff duplicate.

Note that $|\mathrm{Dup}(A)| = |\mathrm{Ult}(A)|$, and $\mathrm{Dup}(A)$ is atomic with $|\mathrm{Ult}(A)|$ atoms. Hence $\mathrm{c}(\mathrm{Dup}(A)) = |\mathrm{Ult}(A)|$.

We consider now the exponential of a given BA A. The main result here is due to Fedorchuk and Todorčević [97]: the cellularity of $\mathrm{Exp}(A)$ is equal to the cellularity of the ωth free power of A. The proof depends on the following lemma. Let n be a positive integer. A system $\langle\langle a_\xi^0, \ldots, a_\xi^{n-1} \rangle : \xi \in I \rangle$ of n-tuples of members of A^+ *refines* another sequence $\langle\langle b_\xi^0, \ldots, b_\xi^{n-1} \rangle : \xi \in I \rangle$ iff $a_\xi^i \leq b_\xi^i$ for all $i < n$ and $\xi \in I$.

Lemma 3.27. *Let n be a positive integer, τ an uncountable regular cardinal, and A a BA. Suppose that $\langle\langle a_\xi^0, \ldots, a_\xi^{n-1} \rangle : \xi \in \tau \rangle$ is a system of elements of A^+. Then there exist an $I \in [\tau]^\tau$, a refinement $\langle\langle c_\xi^0, \ldots, c_\xi^{n-1} \rangle : \xi \in I \rangle$ of $\langle\langle a_\xi^0, \ldots, a_\xi^{n-1} \rangle : \xi \in I \rangle$, a system $\langle b^0, \ldots, b^{k-1} \rangle$ of pairwise disjoint elements of A^+, with k a positive* *integer, and a partition* $\langle M^0, \ldots, M^{k-1}\rangle$ *of* n *into nonempty sets, such that the following conditions hold:*

(i) *For each* $j < k$, $c^i_\xi \leq b^j$ *for all* $i \in M^j$ *and all* $\xi \in I$.

(ii) *If* $j < k$, $|M^j| > 1$, *and* $\xi, \eta \in I$ *with* $\xi < \eta$, *then either* $(\sum_{i\in M^j} c^i_\xi) \cdot (\sum_{i\in M^j} c^i_\eta) = 0$ *or there is an* $l \in M^j$ *such that* $\sum_{i\in M^j} c^i_\eta \leq c^l_\xi$.

Proof. Induction on n. For $n = 1$ we can take $I = \tau$, $c^i_\xi = a^i_\xi$ for all $i < n$ and $\xi < \tau$, $k = 1$ with $b^0 = 1$ and $M^0 = n$. Now assume that $n > 1$ and the lemma holds for positive integers less than n.

Case 1. There exist a $u \in A^+$ and an $i < n$ such that the set

$$J^i_u \stackrel{\text{def}}{=} \{\xi < \tau : a^i_\xi \cdot u \neq 0 \neq a^i_\xi \cdot -u\}$$

has size τ.

Subcase 1.1. There exist an $I \subseteq J^i_u$ and a $j < n$ such that $|I| = \tau$ and $a^j_\xi \cdot u = 0$ for all $\xi \in I$. Let $N = \{k < n : |\{\xi \in J^i_u : a^k_\xi \cdot u \neq 0\}| < \tau\}$, and let $I = \{J^i_u \backslash \bigcup_{k\in N}\{\xi \in J^i_u : a^k_\xi \cdot u \neq 0\}$. So $|I| = \tau$, and $a^i_\xi \cdot u = 0$ for all $k \in N$ and all $x \in I$. Note that $j \in N$ and $i \notin N$. Let $M = n\backslash N$. So $a^k_\xi \cdot u \neq 0$ for all $k \in M$ and $\xi \in I$. We apply the induction hypothesis to $\langle\langle a^k_\xi \cdot u : k \in M\rangle : \xi \in I\rangle$ to obtain a $K \in [I]^\tau$, a refinement $\langle\langle d^k_\xi : k \in M\rangle : \xi \in K\rangle$ of $\langle\langle a^k_\xi \cdot u : k \in M\rangle : \xi \in K\rangle$, a system $\langle b^k : k < l\rangle$ of pairwise disjoint elements of A^+, with l a positive integer, and a partition $\langle P^k : k < l\rangle$ of M into nonempty sets, such that the following conditions hold:

(1) For each $k < l$, $d^i_\xi \leq b^k$ for all $i \in P^k$ and all $\xi \in I$.

(2) If $j < l$, $|P^j| > 1$, and $\xi, \eta \in I$ with $\xi < \eta$, then either $(\sum_{i\in P^j} d^i_\xi)\cdot(\sum_{i\in P^j} d^i_\eta) = 0$ or there is an $s \in P^j$ such that $\sum_{i\in P^j} d^i_\eta \leq d^s_\xi$.

Then apply the induction hypothesis to $\langle\langle a^k_\xi : k \in N\rangle : \xi \in K\rangle$ to obtain an $L \in [K]^\tau$, a refinement $\langle\langle e^k_\xi : k \in N\rangle : \xi \in L\rangle$ of $\langle\langle a^k_\xi : k \in N\rangle : \xi \in L\rangle$, a system $\langle f^k : k < m\rangle$ of pairwise disjoint elements of A^+, with m a positive integer, and a partition $\langle Q^k : k < m\rangle$ of N into nonempty sets, such that the following conditions hold:

(3) For each $k < m$, $e^i_\xi \leq f^k$ for all $i \in Q^k$ and all $\xi \in L$.

(4) If $j < m$, $|Q^j| > 1$, and $\xi, \eta \in L$ with $\xi < \eta$, then either $(\sum_{i\in Q^j} e^i_\xi) \cdot (\sum_{i\in Q^j} e^i_\eta) = 0$ or there is an $s \in Q^j$ such that $\sum_{i\in Q^j} e^i_\eta \leq e^s_\xi$.

Then $\langle\langle d^k_\xi : k \in M\rangle \cup \langle e^k_\xi : k \in N\rangle : \xi \in L\rangle$ is a refinement of $\langle\langle a^0_\xi, \ldots, a^{n-1}_\xi\rangle : \xi \in I\rangle$, $\langle b^0 \cdot u, \ldots, b^{l-1} \cdot u, f^0 \cdot -u, \ldots, f^{m-1} \cdot -u\rangle$ is a system of pairwise disjoint elements of A^+, and $\langle P^k : k < l\rangle \cup \langle Q^k : k < m\rangle$ is a partition of n into nonempty sets, such that the conditions of the lemma hold.

Subcase 1.2. Subcase 1.1 does not hold. Thus

$$\forall I \in [J^i_u]^\tau \forall j < n \exists \xi \in I[a^j_\xi \cdot u \neq 0]. \tag{5}$$

Let $J^i_u = \{\alpha_\nu : \nu < \tau\}$ without repetitions. Then by (5), $\forall \rho < \tau \exists \xi \in \{\alpha_\nu : \rho \leq \nu < \tau\}[a^0_\xi \cdot u \neq 0]$. Hence there is an $I^0 \in [J^i_u]^\tau$ such that for all $\xi \in I^0$ we have $a^0_\xi \cdot u \neq 0$. Similarly we find $I^0 \supseteq I^1 \supseteq \cdots \supseteq I^{n-1} = I$ such that $I \in [J^i_u]^\tau$ and $\forall \xi \in I \forall j < n[a^j_\xi \cdot u \neq 0]$. We apply the induction hypothesis to $\langle\langle a^j_\xi \cdot u : j < n, j \neq i\rangle : \xi \in I\rangle$ to obtain a $K \in [I]^\tau$, a refinement $\langle\langle d^j_\xi : j < n, j \neq i\rangle : \xi \in K\rangle$ of $\langle\langle a^j_\xi \cdot u : j < n, j \neq i\rangle : \xi \in K\rangle$, a system $\langle b^k : k < l\rangle$ of pairwise disjoint elements of A^+, with l a positive integer, and a partition $\langle P^k : k < l\rangle$ of $n\backslash\{i\}$ into nonempty sets, such that the following conditions hold:

(6) For each $k < l$, $d^i_\xi \leq b^k$ for all $i \in P^k$ and all $\xi \in I$.

(7) If $j < l$, $|P^j| > 1$, and $\xi, \eta \in I$ with $\xi < \eta$, then either $(\sum_{k \in P^j} d^i_\xi) \cdot (\sum_{k \in P^j} d^i_\eta) = 0$ or there is an $s \in P^j$ such that $\sum_{k \in P^j} d^k_\eta \leq d^s_\xi$.

We may assume that $b^k \leq u$ for all $k < l$. Now $\langle\langle d^j_\xi : j < n, j \neq i\rangle^\frown\langle a^i_\xi \cdot -u\rangle : \xi \in K\rangle$ is a refinement of $\langle\langle a^j_\xi \cdot u : j < n\rangle : \xi \in K\rangle$, $\langle b^k : k < l\rangle^\frown\langle -u\rangle$ is a system of pairwise disjoint elements of A^+, $\langle P^k : k < l\rangle^\frown\{i\}\rangle$ is a partition of n, and the conditions of the lemma hold.

Case 2. Case 1 fails. We apply this to $b = a^j_\eta$. Thus for all $i < n$ the set $\{\xi < \tau : a^i_\xi \cdot a^j_\eta \neq 0 \neq a^i_\xi \cdot -a^j_\eta\}$ has size less than τ; so the same is true of

$$\bigcup_{i,j<n} \{\xi < \tau : a^i_\xi \cdot a^j_\eta \neq 0 \neq a^i_\xi \cdot -a^j_\eta\}.$$

Let A_η be the complement of this set. Thus

$$\begin{aligned} A_\eta &= \bigcap_{i,j<n} \{\xi < \tau : a^i_\xi \cdot a^j_\eta = 0 \text{ or } a^i_\xi \cdot -a^j_\eta = 0\} \\ &= \{\xi < \tau : \forall i, j < n[a^i_\xi \cdot a^j_\eta = 0 \text{ or } a^i_\xi \cdot -a^j_\eta = 0]\}. \end{aligned}$$

Let $B = \triangle_{\eta<\tau} A_\eta$. So B is club, and

$$\begin{aligned} B &= \{\xi < \tau : \forall \eta < \xi[\xi \in A_\eta]\} \\ &= \{\xi < \tau : \forall \eta < \xi \forall i, j < n[[a^i_\xi \cdot a^j_\eta = 0 \text{ or } a^i_\xi \cdot -a^j_\eta = 0]\}. \end{aligned}$$

Subcase 2.1. There is a $u \in A^+$ such that $|C| = \tau$, where

$$C = \{\xi \in B : \exists i, j < n[i \neq j \text{ and } a^i_\xi \cdot u \neq 0 \neq a^j_\xi \cdot -u]\}.$$

Now

$$C=\bigcup_{\substack{i,j<n\\ i\neq j}}\{\xi\in C: a^i_\xi\cdot u\neq 0\neq a^j_\xi\cdot -u\},$$

so there are distinct $i,j<n$ such that $D\stackrel{\rm def}{=}\{\xi\in C: a^i_\xi\cdot u\neq 0\neq a^j_\xi\cdot -u\}$ has size τ. Now let

$$E_0=\begin{cases} D\backslash\{\xi\in D: a^0_\xi\cdot u\neq 0\} & \text{if } |\{\xi\in D: a^0_\xi\cdot u\neq 0\}|<\tau,\\ \{\xi\in D: a^0_\xi\cdot u\neq 0\} & \text{otherwise.}\end{cases}$$

Thus either $a^0_\xi\cdot u=0$ for all $\xi\in E_0$, or $a^0_\xi\cdot u\neq 0$ for all $\xi\in E_0$; and $|E_0|=\tau$. Continuing analogously, we arrive at E_{n-1} such that for all $k<n$, either $a^k_\xi\cdot u=0$ for all $\xi\in E_{n-1}$, or $a^k_\xi\cdot u\neq 0$ for all $\xi\in E_{n-1}$; and $|E_{n-1}|=\tau$. Working analogously with $-u$, we end up with $F\subseteq E_{n-1}$ such that $|F|=\tau$ and for all $k<n$, either $a^k_\xi\cdot -u=0$ for all $\xi\in F$, or $a^k_\xi\cdot -u\neq 0$ for all F. Now for $k<n$ and $\xi\in F$ we define

$$c^k_\xi=\begin{cases} a^k_\xi\cdot u & \text{if } \forall\eta\in F[a^k_\eta\cdot u\neq 0] \text{ and } k\neq j,\\ a^k_\xi\cdot -u & \text{otherwise.}\end{cases}$$

Let $M=\{k<n:\forall\eta\in F[a^k_\eta\cdot u\neq 0]\text{ and }k\neq j\}$ and $N=n\backslash M$. So $M\neq\emptyset\neq N$, so we can apply the induction hypothesis to M and N; putting the results together as above, we again get the conditions of the lemma.

Subcase 2.2. Subcase 2.1 fails. Thus

$$\forall u\in A^+|\{\xi\in B:\forall i,j<n[i\neq j\rightarrow a^i_\xi\cdot u=0\text{ or }a^j_\xi\cdot -u=0\}|=\tau. \tag{8}$$

Now we claim

$$|\{\xi\in B:\exists i<n[a^i_\xi\cdot u\neq 0]\rightarrow\forall j<n[a^j_\xi\cdot u\neq 0]\}|=\tau. \tag{9}$$

In fact, take ξ in the set of (8), and suppose that $i<n$ and $a^i_\xi\cdot u\neq 0$. Then by the condition of (8), $a^j_\xi\cdot -u=0$ for all $j\neq i$, hence $a^j_\xi\cdot u=a^j_\xi\neq 0$. So (9) holds. Similarly,

$$|\{\xi\in B:\exists i<n[a^i_\xi\cdot -u\neq 0]\rightarrow\forall j<n[a^j_\xi\cdot -u\neq 0]\}|=\tau. \tag{10}$$

Suppose that $\xi,\eta\in B$ and $\eta<\xi$. We consider two possibilities.

Subsubcase 2.2.1. $\exists i,j<n[a^i_\eta\cdot a^j_\xi\neq 0]$. Then by (9) for η with a^j_ξ for u, $\forall k<n[a^k_\eta\cdot a^j_\xi\neq 0]$. Then by the definition of B, $a^k_\eta\leq a^j_\xi$ for all $k<n$,

Subsubcase 2.2.2. $\forall i,j<n[a^i_\eta\cdot a^j_\xi=0]$.

These two subsubcases give the desired conclusion. □

Theorem 3.28. *For any infinite BA A, $\mathrm{c}(\mathrm{Exp}(A)) = \mathrm{c}(A^{*\omega})$, where $A^{*\omega}$ is the ωth free power of A.*

Proof. By Proposition 2.5 and the remark following Problem 5 we have

$$\mathrm{c}(\mathrm{Exp}(A)) \leq \mathrm{c}(A^{*\omega}).$$

For the other direction it suffices by that same remark to show that $\mathrm{c}(A^{*n}) \leq \mathrm{c}(\mathrm{Exp}(A))$ for any positive integer n. We may assume that $n > 1$. Suppose that $\langle e_\xi : \xi < \tau\rangle$ is a system of pairwise disjoint nonzero elements of A^{*n}. We may assume that each e_ξ has the form $a^0_\xi \cdot \ldots \cdot a^{n-1}_\xi$, where the a^i come from different free factors of A^{*n}. Now we apply Lemma 3.27 to get a set $I \in [\tau]^\tau$, a refinement $\langle\langle c^0_\xi, \ldots, c^{n-1}_\xi\rangle : \xi \in I\rangle$ of $\langle\langle a^0_\xi, \ldots, a^{n-1}_\xi\rangle : \xi \in I\rangle$, a system $\langle b^0, \ldots, b^{k-1}\rangle$ of pairwise disjoint elements of $(A^{*n})^+$, with k a positive integer, and a partition $\langle M^0, \ldots, M^{k-1}\rangle$ of n into nonempty sets, such that the following conditions hold:

(1) For each $j < k$, $c^i_\xi \leq b^j$ for all $i \in M^j$ and all $\xi \in I$.

(2) If $j < k$, $|M^j| > 1$, and $\xi, \eta \in I$ with $\xi < \eta$, then either $(\sum_{i\in M^j} c^i_\xi) \cdot (\sum_{i\in M^j} c^i_\eta) = 0$ or there is an $l \in M^j$ such that $\sum_{i\in M^j} c^i_\eta \leq c^l_\xi$.

Now for each $\xi \in I$ we let d_ξ be the following element of $\mathrm{Exp}(A)$:

$$\mathscr{V}(\mathcal{S}(c^0_\xi + \cdots + c^{n-1}_\xi)) \cap \bigcap\{-\mathscr{V}(\mathcal{S}(-b^j)) : |M^j| = 1\} \cap$$
$$\bigcap\{-\mathscr{V}(\mathcal{S}(-b_j + c^i_\xi)) : |M^j| > 1, i \in M^j\}.$$

First we check that each d_ξ is nonzero. Suppose that $d_\xi = 0$. Then by Lemma 1.22(vi) there are two cases:

Case 1. $c^0_\xi + \cdots + c^{n-1}_\xi \leq -b_j$ for some j with $|M^j| = 1$. Say $M^j = \{i\}$. then $(c^0_\xi + \cdots + c^{n-1}_\xi) \cap b_j = c^i_\xi$, contradiction.

Case 2. $c^0_\xi + \cdots + c^{n-1}_\xi \leq -b^j + c^i_\xi$ for some j with $|M^j| > 1$ and some $i \in M^j$. Then $(c^0_\xi + \cdots + c^{n-1}) \cap b^j \leq c^i_\xi$, a contradiction since $|M^j| > 1$.

Now suppose that $\xi < \eta$; we show that $d_\xi \cap d_\eta = 0$.

Case 1. There is a j such that $|M^j| > 1$ and $\left(\sum_{i\in M^j} c^i_\xi\right) \cap \left(\sum_{i\in M^j} c^i_\eta\right) = 0$. Then

$$\left(\sum_{i<n} c^i_\xi\right) \cap \left(\sum_{i<n} c^i_\eta\right) \cap b^j = \left(\sum_{i\in M^j} c^i_\xi\right) \cap \left(\sum_{i\in M^j} c^i_\eta\right) = 0,$$

hence $d_\xi \cap d_\eta = 0$ by Lemma 1.22(vi).

Case 2. There is a j such that $|M^j| > 1$ and $\left(\sum_{i\in M^j} c^i_\eta\right) \leq c^l_\xi$ for some $l \in M^j$. Then

$$\left(\sum_{i<n} c^i_\xi\right) \cap \left(\sum_{i<n} c^i_\eta\right) \cap b^j = \left(\sum_{i\in M^j} c^i_\xi\right) \cap \left(\sum_{i\in M^j} c^i_\eta\right) \leq c^l_\xi,$$

hence $d_\xi \cap d_\eta = 0$ by Lemma 1.22(vi).

Case 3. $|M^j| = 1$ for all j. Write $M^j = \{i_j\}$ for all j. Choose j, k such that $c_\xi^{i_j} \cdot c_\eta^{i_k} = 0$. By the free product property we must have $i_j = i_k$, hence $j = k$. Then

$$\left(\sum_{i<n} c_\xi^i\right) \cap \left(\sum_{i<n} c_\eta^i\right) \cap b^j = c_\xi^{i_j} \cdot c_\eta^{i_j} = 0,$$

and again Lemma 1.22(vi) applies. □

Now we proceed to discuss the derived functions associated with cellularity. First we show that $\mathrm{c_{H+}}$ is the same as spread. For this, it is convenient to have an equivalent definition of spread. A subset X of a BA A is *ideal independent* if $x \notin \langle X \backslash \{x\}\rangle^{\mathrm{Id}}$ for every $x \in X$; recall that $\langle Y \rangle^{\mathrm{Id}}$ denotes the ideal generated by Y, for any $Y \subseteq A$.

Theorem 3.29. *For any infinite BA A, $\mathrm{s}(A) = \sup\{|X| : X$ is an ideal independent subset of $A\}$.*

Proof. First suppose that D is a discrete subspace of $\mathrm{Ult}(A)$. For each $F \in D$, let $a_F \in A$ be such that $\mathcal{S}(a_F) \cap D = \{F\}$. Then $\langle a_F : F \in D\rangle$ is one-one and $\{a_F : F \in D\}$ is ideal independent. In fact, suppose that $F, G_0, \ldots, G_{n-1}$ are distinct members of D such that $a_F \leq a_{G_0} + \cdots + a_{G_{n-1}}$. Then $\mathcal{S}(a_F) \subseteq \mathcal{S}(a_{G_0}) \cup \cdots \cup \mathcal{S}(a_{G_{n-1}})$, and so $F \in \mathcal{S}(a_{G_0}) \cup \cdots \cup \mathcal{S}(a_{G_{n-1}})$, which is clearly impossible.

Conversely, suppose that X is an ideal independent subset of A. Then for each $x \in X$, $\{x\} \cup \{-y : y \in X\backslash\{x\}\}$ has the finite intersection property, and so is included in an ultrafilter F_x. Let $D = \{F_x : x \in X\}$. Then $\mathcal{S}(x) \cap D = \{F_x\}$ for each $x \in X$, so D is discrete and $|D| = |X|$, as desired. □

The proof of Theorem 3.29 shows that $\mathrm{s}(A)$ is attained in the discrete subspace sense iff it is attained in the ideal independence sense.

Theorem 3.30. *For any infinite BA A, $\mathrm{c_{H+}}(A)$ is equal to $\mathrm{s}(A)$, the spread of A.*

Proof. First let f be a homomorphism from A onto a BA B, and let X be a disjoint subset of B^+. We show that $|X| \leq \mathrm{s}(A)$; this will show that $\mathrm{c_{H+}}(A) \leq \mathrm{s}(A)$. For each $x \in X$ choose $a_x \in X$ such that $f(a_x) = x$. Then $\langle a_x : x \in X\rangle$ is one-one and $\{a_x : x \in X\}$ is ideal independent. In fact, suppose that $x, y(0), \ldots, y(n-1)$ are distinct elements of X, and $a_x \leq a_{y(0)} + \cdots + a_{y(n-1)}$. Applying the homomorphism f to this inequality we get $x \leq y(0) + \cdots + y(n-1)$. Since the elements $x, y(0), \ldots, y(n-1)$ are pairwise disjoint, this is impossible.

For the converse, suppose that X is an ideal independent subset of A; we want to find a homomorphic image B of A having a disjoint subset of size $|X|$. Let $I = \langle\{x \cdot y : x, y \in X, x \neq y\}\rangle^{\mathrm{Id}}$. It suffices now to show that $[x] \neq 0$ for each $x \in X$. ($[u]$ is the equivalence class of u under the equivalence relation naturally associated with the ideal I). Suppose that $[x] = 0$. Then x is in the ideal I, and hence there exist elements $y_0, z_0, \ldots, y_{n-1}, z_{n-1}$ of X such that $y_i \neq z_i$ for all

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$i < n$, and $x \leq y_0 \cdot z_0 + \cdots + y_{n-1} \cdot z_{n-1}$. Without loss of generality, $x \neq y_i$ for all $i < n$. But then $x \leq y_0 + \cdots + y_{n-1}$, contradicting the ideal independence of X. $\square$

For later purposes it is convenient to note the following corollary to the proof of the previous two theorems.

Corollary 3.31. *$\mathrm{c_{H+}}(A)$ and $\mathrm{s}(A)$ have the same attainment properties, in the sense that $\mathrm{s}(A)$ is attained (in either the discrete subspace or ideal independence sense) iff there exist a homomorphic image B of A and a disjoint subset X of B such that $|X| = \mathrm{c_{H+}}(A)$.* $\square$

Note in this corollary that attainment of $\mathrm{c_{H+}}(A)$ involves two sups, while attainment of $\mathrm{s}(A)$ involves only one. Thus if $\mathrm{s}(A)$ is not attained, there are still two possibilities according to Corollary 3.31: there can exist a homomorphic image B of A with $\mathrm{s}(A) = \mathrm{c}(B)$ but $\mathrm{c}(B)$ is not attained, or there is no homomorphic image B of A with $\mathrm{s}A = \mathrm{c}(B)$. Both possibilities are consistent with ZFC; we shall return to this shortly and indicate the examples.

It is easy to see that $\mathrm{c_{H-}}(A) = \omega$ for any infinite BA A: let B be a denumerable subalgebra of A, and extend the identity homomorphism h of B into $\overline{B}$ to a homomorphism from A into $\overline{B}$; the image of A under h is a ccc BA. (We are using here Sikorski's extension theorem; recall that $\overline{B}$ is the completion of B.) It is obvious that $\mathrm{c_{S+}}(A) = \mathrm{c}(A)$ and $\mathrm{c_{S-}}(A) = \omega$ for any infinite BA A. $\mathrm{c_{h+}}(A)$ is equal to $\mathrm{s}(A)$, since a disjoint family of open subsets of a subspace Y of $\mathrm{Ult}(A)$ gives a discrete subset of $\mathrm{Ult}(A)$ of the same size, so that $\mathrm{c_{h+}}(A) \leq \mathrm{s}(A) = \mathrm{c_{H+}}(A) \leq \mathrm{c_{h+}}(A)$. It is obvious that $\mathrm{c_{h-}}(A) = \omega$, and an easy argument gives that ${}_{\mathrm{d}}\mathrm{c_{S+}}(A) = \mathrm{c}(A) = {}_{\mathrm{d}}\mathrm{c_{S-}}(A)$.

The function $\mathrm{c_{mm}}(A)$ has been studied for the algebra $\mathscr{P}(\omega)/\mathrm{fin}$, using the notation $\mathfrak{a}$. For this reason, we use $\mathfrak{a}(A)$ rather than $\mathrm{c_{mm}}(A)$. Thus for any BA A,

$$\mathfrak{a}(A) = \min\{|P| : P \text{ is an infinite partition of } A\};$$
$$\mathfrak{a}_{\mathrm{spect}}(A) = \{|P| : P \text{ is an infinite partition of } A\}.$$

For general BAs, $\mathfrak{a}(A)$ is discussed in Monk [96], [96a], [01], [01a], and [02], and in Mckenzie, Monk [04].

We consider what happens to $\mathfrak{a}$ in subalgebras. It is possible to have $A \leq B$ with $\mathfrak{a}_{\mathrm{spect}}(A) \cap \mathfrak{a}_{\mathrm{spect}}(B) = \emptyset$. Namely let A be arbitrary and extend A to an atomless BA with high saturation. One can have A a subalgebra of B with $\mathfrak{a}(A)$ much smaller than $\mathfrak{a}(B)$; for example, with B κ-saturated and A countable. On the other hand, if B is κ-saturated and $C = B \times D$ with D countable, then B can be considered as a subalgebra of C, and $\mathfrak{a}(C) = \omega < \kappa \leq \mathfrak{a}(B)$. That $\mathfrak{a}(C) = \omega$ is clear, but is stated and proved in Proposition 3.36 below. We now consider our special kinds of subalgebras with respect to $\mathfrak{a}$.

Proposition 3.32. *If $A \leq_{\mathrm{reg}} B$, then $\mathfrak{a}_{\mathrm{spect}}(A) \subseteq \mathfrak{a}_{\mathrm{spect}}(B)$ and $\mathfrak{a}(B) \leq \mathfrak{a}(A)$. Here $\leq_{\mathrm{reg}}$ can be replaced by $\leq_{\mathrm{free}}$, $\leq_{\mathrm{proj}}$, $\leq_{\mathrm{rc}}$, or $\leq_{\pi}$.* $\square$

That the inequality here cannot in general be replaced by equality follows from the next proposition.

Proposition 3.33. *Suppose that $A \leq_{\rm free} B$ with $B = A \oplus C$, C an infinite free BA. Then $\omega \in \mathfrak{a}_{\rm spect}(B)$. In particular, $\mathfrak{a}(B) = \omega$.*

Proof. Since C satisfies ccc. □

Applying Proposition 3.33 with $\mathfrak{a}(A) > \omega$ gives $\mathfrak{a}(B) < \mathfrak{a}(A)$. Here we have $A \leq_{\rm free} B$, and hence also $A \leq_{\rm proj} B$, $A \leq_{\rm u} B$, $A \leq_{\rm rc} B$, $A \leq_\sigma B$, and $A \leq_{\rm reg} B$. For an example with $A \leq_\pi B$ and $\mathfrak{a}(B) < \mathfrak{a}(A)$ one can take any A with $\mathfrak{a}(A) > \omega$ and let B be the completion of A.

If C is finite in Proposition 3.33, then equality does hold; this follows from Proposition 3.37 below.

Proposition 3.34. *If $A \leq_m B$, then $\mathfrak{a}_{\rm spect}(A) \subseteq \mathfrak{a}_{\rm spect}(B)$, and $\mathfrak{a}(B) \leq \mathfrak{a}(A)$.*

Proof. We assume that $A \neq B$. Let X be a partition of unity in A, with $|X| = \mathfrak{a}(A)$. Write $B = A(x)$.

Case 1. $A \restriction x$ and $A \restriction -x$ are nonprincipal. Then by Proposition 2.44, A is dense in $A(x)$ and our result follows from Proposition 3.32.

Case 2. (By symmetry) $A \restriction x$ is principal. By Proposition 2.45 we may assume that x is an atom in $A(x)$. Now the mapping $a \mapsto a \cdot -x$ is clearly a homomorphism from A onto $A \restriction -x$. It is one-one since $a \cdot -x = 0$ implies that $a \leq x$, hence $a = 0$ since x is an atom in $A(x)$ and $x \notin A$. Thus $A(x)$ is isomorphic to $A \times 2$. It is easy to check that $\mathfrak{a}_{\rm spect}(A) = \mathfrak{a}_{\rm spect}(A \times 2)$. □

On the other hand, there are BAs A, B with $A \leq_{\rm mg} B$, $\mathfrak{a}(A) = \omega$, and $\mathfrak{a}(B) > \omega$. In fact, let $B = {\rm Finco}(\omega_1)$, let A be a countably infinite subalgebra of B, and apply Theorem 2.55.

Proposition 3.35. *Suppose that $\mathfrak{a}(A) > \omega$ and $A \leq_\sigma B$. Then $\mathfrak{a}_{\rm spect}(A) \subseteq \mathfrak{a}_{\rm spect}(B)$ and $\mathfrak{a}(B) \leq \mathfrak{a}(A)$.*

Proof. Assume the hypotheses, and suppose that X is an uncountable partition of unity in A. Suppose that $b \in B$, $b < 1$, and $\forall a \in X[a \leq b]$; we want to get a contradiction. Now $A \restriction b$ is a countably generated ideal; say that it is generated by the countable set Y. Note that $\sum F \leq b < 1$ for every finite $F \subseteq Y$. We may assume that Y is closed under $+$. For each $a \in X$ choose $y_a \in Y$ such that $a \leq y_a$. Then $\sum_{a \in X} y_a = 1$; since $\{y_a : a \in X\}$ is countable, this gives a countable partition of unity in A, contradiction. □

One can have $A \leq_\sigma B$ with $\mathfrak{a}(A) = \omega < \mathfrak{a}(B)$. For example, let $B = {\rm finco}(\omega_1)$, and let A be the subalgebra of B generated by $\{\{n\} : n \in \omega\}$.

There are BAs A, B with $A \leq_{\rm u} B$ and $\mathfrak{a}(A) < \mathfrak{a}(B)$. In fact, one can take A to be a denumerable atomless BA. Any ultrafilter on A has two extensions to an ultrafilter on $A(x)$, the free extension of A by adding one element x. $A(x)$ is still denumerable and atomless, and can be extended to an ω_1-saturated BA B.

There exist BAs A, B such that $A \leq_{\mathrm{m}} B$ and $\mathfrak{a}(B) < \mathfrak{a}(A)$. In fact, let $A = \mathrm{Finco}(\kappa)$, where κ is an uncountable cardinal. Thus $\mathfrak{a}(A) = \kappa$. Let

$$I = \langle\{\{n\} : n \in \omega\}\rangle^{\mathrm{Id}},$$
$$J = \langle\{\{\alpha\} : \omega \leq \alpha < \kappa\}\rangle^{\mathrm{Id}}.$$

Let $B = A(x)$ be the simple extension of A such that $B \restriction x = I$ and $B \restriction -x = J$. Clearly $I \cup J$ generates a maximal ideal, so we have $A \leq_{\mathrm{m}} B$. We claim that

$$\{\{n\} : n \in \omega\} \cup \{-x\}$$

is a partition of unity in B. Clearly it is a collection of pairwise disjoint elements. Suppose that $a \cdot x + b \cdot -x$ is disjoint from each member of this set. Then $b \cdot -x = 0$. For each $n \in \omega$, $a \cdot \{n\} \cdot x = 0$; since $\{n\} \leq x$, this implies that a is a finite subset of $\kappa \backslash \omega$. Hence $a \in J$, and so $a \leq -x$. Thus $a \cdot x = 0$. So we have shown that $a \cdot x + b \cdot -x = 0$, proving the claim.

Problem 7. *Does $A \leq_{\mathrm{s}} B$ imply that $\mathfrak{a}(B) \leq \mathfrak{a}(A)$?*

Proposition 3.36. *Let A and B be infinite BAs.*

(i) $\mathfrak{a}_{\mathrm{spect}}(A \times B) = \mathfrak{a}_{\mathrm{spect}}(A) \cup \mathfrak{a}_{\mathrm{spect}}(B)$.
(ii) $\mathfrak{a}(A \times B) = \min(\mathfrak{a}(A), \mathfrak{a}(B))$.

Proof. Clearly (ii) follows from (i). For (i), suppose that X is an infinite partition of unity of A; then

$$\{(x, 0) : x \in X\} \cup \{(0, 1)\}$$

is an infinite partition of unity in $A \times B$. A similar argument works for B, so $\supseteq$ holds. Now suppose that X is an infinite partition of unity in $A \times B$. Let

$$Y = \{a \in A^+ : \exists b \in B[(a, b) \in X]\},$$
$$Z = \{b \in B^+ : \exists a \in A[(a, b) \in X]\}.$$

Then Y is a partition of unity in A and Z is a partition of unity in B. At least one of them is of size $|X|$, and this gives $\subseteq$. □

Proposition 3.37. *If A is infinite and B is finite with at least 4 elements, then $\mathfrak{a}_{\mathrm{spect}}(A) = \mathfrak{a}_{\mathrm{spect}}(A \oplus B)$.*

Proof. By the Handbook 11.6(d), $A \oplus B$ is isomorphic to ${}^nA$ for some positive integer n, so our conclusion is immediate from Proposition 3.36(i). □

Proposition 3.38. *Suppose that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite. Then*

$$\begin{aligned}\mathfrak{a}_{\mathrm{spect}}\left(\prod\nolimits^w_{i\in I} A_i\right) &= \{|I|\} \cup \bigcup_{i\in I} \mathfrak{a}_{\mathrm{spect}}(A_i) \\ &\cup \Big\{\kappa : \kappa > |I|,\ \kappa \text{ is singular, and} \\ &\qquad \exists J \subseteq I \exists \lambda \in \prod\nolimits_{j\in J} \mathfrak{a}_{\mathrm{spect}}(A_j)\left(J \neq \emptyset \text{ and } \kappa = \sup\nolimits_{j\in J}\lambda_j\right)\Big\}.\end{aligned} \tag{i}$$

(ii) $\mathfrak{a}(\prod^w_{i\in I} A_i) = \min(|I|, \min_{i\in I}\mathfrak{a}(A_i))$.

Proof. For brevity let $B = \prod^w_{i\in I} A_i$.

Clearly (i) implies (ii). For (i), by Proposition 3.36 it is clear that $\bigcup_{i\in I}\mathfrak{a}_{\mathrm{spect}}(A_i) \subseteq \mathfrak{a}_{\mathrm{spect}}(B)$, and an easy argument shows that $|I| \in \mathfrak{a}_{\mathrm{spect}}(B)$. Now suppose that J is a nonempty subset of I, $\lambda \in \prod_{j\in J}\mathfrak{a}_{\mathrm{spect}}(A_j)$, and $|I| < \kappa = \sup_{j\in J}\lambda_j$, with κ singular. For each $j \in J$ let X_j be a partition of unity in A_j such that $|X_j| = \lambda_j$. Now let $K = \{(j,x) : j \in J,\ x \in X_j\} \cup \{(i,0) : i \in I\backslash J\}$. For each $j \in J$ and $x \in X_j$, let $y_{(j,x)}$ be the member of $\prod_{i\in I} A_i$ which takes the value x at j and the value 0 elsewhere. For each $j \in I\backslash J$ let $y_{(j,0)}$ be the member of $\prod_{i\in I} A_i$ which takes the value 1 at j and the value 0 elsewhere. Clearly $\langle y_k : k \in K\rangle$ is a partition of unity of size κ. This finishes the proof of $\supseteq$.

For $\subseteq$, suppose that $|X| \notin \{|I|\}\cup\bigcup_{i\in I}\mathfrak{a}_{\mathrm{spect}}(A_i)$, with X an infinite partition of unity of B.

Case 1. There is an $x \in X$ such that $J \overset{\mathrm{def}}{=} \{i \in I : x_i \neq 1\}$ is finite. If $y \in X\backslash\{x\}$ and $i \in I\backslash J$, then $y_i = 0$, since $x \cdot y = 0$. In particular, $y \restriction J \neq z \restriction J$ for all distinct $y, z \in X$. If w is a nonzero element of $\prod_{i\in J} A_i$, extend it to an element w' of $\prod_{i\in I} A_I$ by letting $w'_i = 0$ for all $i \in I\backslash J$. Choose $y \in X$ such that $w' \cdot y \neq 0$. Then $w \cdot (y \restriction Y) \neq 0$. Thus $\{y \restriction J : y \in X\}\backslash\{0\}$ is an infinite partition of unity in $\prod_{i\in J} A_i$, and so by Proposition 3.36, $|X| \in \bigcup_{i\in J}\mathfrak{a}_{\mathrm{spect}}(A_i)$, contradiction.

Case 2. For every $x \in X$ the set $\{i \in I : x_i \neq 0\}$ is finite. Since $\sum X = 1$, it is clear that $|I| \leq |X|$, so $|I| < |X|$. Let $J = \{i \in I : \{x_i : x \in X\} \text{ is infinite}\}$. For each $i \in I$ let $K_i = \{x \in X : x_i \neq 0\}$. Then $X = \bigcup_{i\in I} K_i$. If $J = \emptyset$, then each K_i is finite, and so clearly $|X| = |I|$, contradiction. So $J \neq \emptyset$. For each $j \in J$, the set $\{x_j : x \in X\}\backslash\{\emptyset\}$ is clearly an infinite partition of A_j. Let λ_j be the cardinality if this set. Clearly $\lambda_j = |K_j|$ for each $j \in J$. Since $|X| \notin \mathfrak{a}_{\mathrm{spect}}(A_j)$, we also have $\lambda_j < |X|$. Moreover, $|J| \leq |I| < |X|$. Now

$$\begin{aligned}|X| = \left|\bigcup_{i\in I\backslash J} K_i \cup \bigcup_{j\in J} K_j\right| &\leq \sum_{i\in I\backslash J}|K_i| + \sum_{j\in J}|K_j| \\ &\leq \omega \cdot |I\backslash J| + \sum_{j\in J}\lambda_j \leq |X|,\end{aligned}$$

so $|X| = \sup_{j\in J}\lambda_j$ and $|X|$ is singular, as desired. □

Kevin Selker has generalized Proposition 3.38 to moderate products. The generalization is somewhat complicated, but it has as a corollary the following simple generalization of Proposition 3.38(ii):

For a moderate product, $\mathfrak{a}(\prod_{i\in I}^B A_i) = \min(\mathfrak{a}(B), \min_{i\in I}\mathfrak{a}(A_i))$.

Proposition 3.39. *Let* $\langle A_i : i \in I\rangle$ *be a system of infinite BAs, with* I *infinite. Then:*

$$\begin{aligned}\mathfrak{a}_{\mathrm{spect}}\left(\prod_{i\in I} A_i\right) = [\omega, |I|] \cup \bigcup_{i\in I}\mathfrak{a}_{\mathrm{spect}}(A_i) \\ \cup \Bigg\{\kappa : \kappa > |I|,\ \kappa \textit{ is singular, and} \\ \exists J \subseteq I \exists \lambda \in \prod_{j\in J} a_{\mathrm{spect}}(A_j)\left[J \neq \emptyset \textit{ and } \kappa = \sup_{j\in J}\lambda_j\right]\Bigg\}.\end{aligned} \tag{i}$$

(ii) $$\mathfrak{a}\left(\prod_{i\in I} A_i\right) = \omega.$$

Proof. Let $B = \prod_{i\in I} A_i$. (ii) obviously follows from (i). For $\supseteq$ in (i), first suppose that $\kappa \in [\omega, |I|]$. Write $I = \bigcup_{\alpha<\kappa} J_\alpha$ with each $J_\alpha \neq \emptyset$, and the J_α's pairwise disjoint. For each $\alpha < \kappa$ let f_α be the member of B which takes the value 1 on J_α and the value 0 elsewhere. Clearly this gives a partition of B of size κ.

Each $\mathfrak{a}_{\mathrm{spect}}(A_i)$ is a subset of $\mathfrak{a}_{\mathrm{spect}}(B)$ by Proposition 3.36.

For κ, J, λ as indicated in the statement, one can follow the procedure described in the proof of Proposition 3.38. So $\supseteq$ holds.

For $\subseteq$ the proof follows that of Proposition 3.38. In fact, one can skip Case 1, and start Case 2 with the condition $|I| < |X|$. □

Proposition 3.40. *Let* A *and* B *be infinite BAs. Then*

(i) $\mathfrak{a}_{\mathrm{spect}}(A) \cup \mathfrak{a}_{\mathrm{spect}}(B) \subseteq \mathfrak{a}_{\mathrm{spect}}(A \oplus B)$.
(ii) $\mathfrak{a}(A \oplus B) \leq \min(\mathfrak{a}(A), \mathfrak{a}(B))$ *for any infinite BAs* A, B. □

We will see in Corollary 4.50 that if $\mathfrak{a}(A), \mathfrak{a}(B) > \omega$, then also $\mathfrak{a}(A \oplus B) > \omega$. The following more general question is open.

Problem 8. *Is it true that for all infinite BAs* A, B *one has*

$$\mathfrak{a}(A \oplus B) = \min(\mathfrak{a}(A), \mathfrak{a}(B))?$$

Note that a consequence of Corollary 3.11 is that there are BAs A, B such that $\mathfrak{a}_{\mathrm{spect}}(A) \cup \mathfrak{a}_{\mathrm{spect}}(B) \subset \mathfrak{a}_{\mathrm{spect}}(A \oplus B)$.

Proposition 3.41. *If* $\langle A_i : i \in I\rangle$ *is an infinite system of infinite BAs, then* $\mathfrak{a}(\oplus_{i\in I} A_i) = \omega$.

Proof. Let $f : \omega \to I$ be an injection. For each $i \in \omega$ let a_i be a member of $A_{f(i)}$ such that $0 < a_i < 1$. Now for each $j \in \omega$ let $b_j = a_j \cdot \prod_{i<j} -a_i$. Thus $\langle b_j : j \in \omega\rangle$ is a system of pairwise disjoint elements of $\oplus_{i\in I} A_i$. Clearly $\sum_{i\in\omega} a_i = 1$, and $\sum_{j\in\omega} b_j = \sum_{i\in\omega} a_i = 1$, so we have an infinite partition of size ω. □

Proposition 3.42. *If K is a set of infinite cardinals with a largest element, then there is an atomic BA A such that $\mathfrak{a}_{\mathrm{spect}}(A) = K$.*

Proof. Let κ be the largest element of K. Clearly $|K| \le \kappa$. Let f be a function mapping κ onto K, and for each $\alpha < \kappa$ let $B_\alpha = \mathrm{Finco}(f(\alpha))$. Note that $\mathfrak{a}_{\mathrm{spect}}(B_\alpha) = \{f(\alpha)\}$. Let $A = \prod^{\mathrm{w}}_{\alpha<\kappa} B_\alpha$. Then A is as desired, by Proposition 3.38. □

Note that if A is atomic, then $\mathfrak{a}_{\mathrm{spect}}(A)$ has a largest element, namely the number of atoms of A. Also, if $\sup(K)$ is singular, then any BA A with $\mathfrak{a}_{\mathrm{spect}}(A) = K$ must have $\sup(K) \in \mathfrak{a}_{\mathrm{spect}}(A)$, by the Erdős–Tarski theorem.

We would now like to generalize Theorem 3.42 to the case of two BAs $A \le B$. We adapt some arguments from Monk [07].

Lemma 3.43. *If A is an infinite BA, then there is a BA B such that $A \le B$ and no infinite partition in A remains a partition in B.*

Proof. We take $B = \mathscr{P}(\mathrm{Ult}(A))$ and the Stone isomorphism $\mathcal{S}$ from A into B. Suppose that X is an infinite partition in A. Then by compactness, $\bigcup \mathcal{S}[X] \ne \mathrm{Ult}(A)$. □

Lemma 3.44. *If A is an infinite BA and κ is a regular cardinal, then there is a BA B such that $A \le B$ and every partition in B has size at least κ.*

Proof. We construct $\langle C_\alpha : \alpha < \kappa\rangle$ by recursion. Let $C_0 = A$. Having defined C_α, let $C_{\alpha+1}$ be obtained by Lemma 3.43: $C_\alpha \le C_{\alpha+1}$ and no infinite partition in C_α is a partition in $C_{\alpha+1}$. For α limit $< \kappa$, let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. Now let $B = \bigcup_{\alpha<\kappa} C_\alpha$. Suppose that X is a partition in B of size less than κ. Then $X \subseteq C_\alpha$ for some $\alpha < \kappa$, and hence X is not a partition in $C_{\alpha+1}$, hence also not a partition in B, contradiction. □

Proposition 3.45. *Suppose that K is a set of infinite cardinals with a largest member. Also suppose that κ is a regular cardinal $> \sup(K)$, and L is a set of cardinals less than κ such that L has a largest element.*

Then there exist BAs A, B such that $A \le B$, $\mathfrak{a}_{\mathrm{spect}}(A) = K$, and $\mathfrak{a}_{\mathrm{spect}}(B) \cap \kappa = L$.

Proof. By Proposition 3.42, let A be a BA with $\mathfrak{a}_{\mathrm{spect}}(A) = K$. Then by Lemma 3.44 let C be a BA such that $A \le C$ and every partition of C has size at least κ. Suppose that L has a largest element λ. Clearly $|L| \le \lambda$. Let f map λ onto L. Then set $B = C \times \prod^{\mathrm{w}}_{\alpha<\lambda} \mathrm{Finco}(f(\alpha))$. B is as desired, by Proposition 3.38. □

These results on $\mathfrak{a}$ leave several problems open in addition to Problem 8 above.

Problem 9. *Suppose that K is a set of infinite cardinals without greatest element, with $\sup(K)$ inaccessible. Is there a BA A such that $\mathfrak{a}_{\mathrm{spect}}(A) = K$?*

Problem 10. *Describe in terms not involving BAs the sets K, L such that there exist BAs A, B with $A \leq_{\mathrm{reg}} B$, $\mathfrak{a}_{\mathrm{spect}}(A) = K$, and $\mathfrak{a}_{\mathrm{spect}}(B) = L$.*

Problem 11. *Describe $\mathfrak{a}_{\mathrm{spect}}(\oplus_{i\in I} A_i)$ in terms of the sets $\mathfrak{a}_{\mathrm{spect}}(\oplus_{i\in F} A_i)$ for F a finite subset of I.*

Problem 12. *Describe the behaviour of $\mathfrak{a}(A)$ under ultraproducts.*

In connection with this problem, recall that if $\langle A_i : i \in I\rangle$ is a system of infinite BAs and F is a countably incomplete ultrafilter on I, then $\prod_{i\in I} A_i/F$ is ω_1-saturated in the model-theoretic sense; see Chang, Keisler [73], Theorem 6.1.1. In particular, $\mathfrak{a}(\prod_{i\in I} A_i/F) \geq \omega_1$ under these assumptions.

The homomorphic spectrum of cellularity is interesting. First, we can easily see that $[\omega, \mathrm{s}(A)) \subseteq \mathrm{c_{Hs}}A \subseteq [\omega, \mathrm{s}(A)]$ (for cardinals $\kappa < \lambda$, $[\kappa, \lambda)$ denotes the set of all cardinals μ such that $\kappa \leq \mu < \lambda$; similarly for $[\kappa, \lambda]$). This follows from the fact already proved that $\mathrm{s}(A) = \mathrm{c_{H+}}(A)$: given a homomorphic image B of A and a disjoint subset X of B, one can use Sikorski's extension theorem to get a homomorphic image C of B such that $\mathrm{c}(C) = |X|$.

It is more difficult to decide whether $\mathrm{s}(A) \in \mathrm{c_{Hs}}A$. This amounts to the following question: is there always a homomorphic image B of A such that $\mathrm{c}(B) = \mathrm{s}(A)$? In case $\mathrm{s}(A)$ is attained, this is true by Corollary 3.31. For $\mathrm{s}(A)$ not attained, there are three consistency results which clarify things here and with respect to the question raised above after Corollary 3.31. First, example 11.14 of the BA handbook shows that for each weakly inaccessible cardinal κ there is a BA A such that $|A| = \mathrm{c}(A) = \mathrm{s}(A) = \kappa$ and $\mathrm{c}(A)$ is not attained but $\mathrm{s}(A)$ is attained (as is easily checked). Second, the interval algebra of a κ-Suslin line, for κ strongly inaccessible but not weakly compact, gives an example of a BA A such that $|A| = \mathrm{c}(A) = \mathrm{s}(A)$, with neither $\mathrm{c}(A)$ nor $\mathrm{s}(A)$ attained; see Juhász [75], example 6.6 (V=L or something beyond ZFC is needed for the existence of a κ-Suslin line). Third, an example of Todorčević [86], Theorem 12, shows that it is consistent to have a BA A in which $\mathrm{s}(A)$ is not attained, while there is no homomorphic image B of A with $\mathrm{c}(B)$=$\mathrm{s}(A)$. This example involves some interesting ideas, and we shall now give it. It depends on the following lemma about the real numbers.

Lemma 3.46. *There exist disjoint subsets E_0 and E_1 of $[0, 1]$ which are of cardinality 2^ω, are dense in $[0, 1]$, and satisfy the following two conditions:*

(i) *For any $\kappa < 2^\omega$ there is a strictly increasing function from some subset of E_0 of size κ into E_1.*

(ii) *There is no strictly monotone function from a subset of E_0 of size 2^ω into E_1.*

Proof. The idea of the proof is to construct E_0 and E_1 in steps, "killing" all of the possible big strictly monotone functions as we go along. The very first thing to do is to see that we can list out in a sequence of length 2^ω all of the functions to be "killed".

For the empty set $\emptyset$ we let $\sup\emptyset = 0$, $\inf\emptyset = 1$. For any subset W of [0,1] we let $\mathrm{cl}(W)$ be its topological closure in [0,1], and we let $C_1(W) = \{f : f : W \to [0,1]$, and f is strictly monotone$\}$. For $W \subseteq [0,1]$ and $f \in C_1(W)$ (say f strictly increasing) we define $f_{\mathrm{cl}} : \mathrm{cl}(W) \to [0,1]$ by

$$f_{\mathrm{cl}}(x) = \begin{cases} f(x) & \text{if } x \in W, \\ \sup\{f(y) : x > y \in W\} & \text{if } x \notin W \text{ and } x = \sup\{y \in W : y < x\}, \\ \inf\{f(y) : x < y \in W\} & \text{if } x \notin W \text{ and } x \neq \sup\{y \in W : y < x\}. \end{cases}$$

(A similar definition is given if f is strictly decreasing.) Note that if $x \in \mathrm{cl}(W)\backslash W$ then $x = \sup\{y \in W : y < x\}$ or $x = \inf\{y \in W : x < y\}$. Now f_{cl} is increasing. For, suppose that $x, x' \in \mathrm{cl}(W)$ and $x < x'$. If $x, x' \in W$, then $f_{\mathrm{cl}}(x) = f(x) < f(x') = f_{\mathrm{cl}}(x')$. Suppose that $x \notin W$ and $x' \in W$. If $x = \sup\{y \in W : y < x\}$, then $f(y) < f(x')$ for all $y \in W$ with $y < x$, and so $f_{\mathrm{cl}}(x) \leq f(x') = f_{\mathrm{cl}}(x')$. If $x \neq \sup\{y \in W : y < x\}$, then $x = \inf\{y \in W : x < y\}$, hence $f_{\mathrm{cl}}(x) \leq f(x') = f_{\mathrm{cl}}(x')$. Other possibilities for x and x' are treated similarly. Now if $x, y \in \mathrm{cl}(W)\backslash W$, $x < y$, and $f_{\mathrm{cl}}(x) = f_{\mathrm{cl}}(y)$, then $x = \sup\{z \in W : z < x\}$, $y = \inf\{z \in W : y < z\}$, and $|(x, y) \cap W| \leq 1$. For, if $x \neq \sup\{z \in W : z < x\}$, then $x = \inf\{z \in W : x < z\}$, and so there are $u, v \in W$ with $x < u < v < y$, hence

$$f_{\mathrm{cl}}(x) \leq f_{\mathrm{cl}}(u) = f(u) < f(v) = f_{\mathrm{cl}}(v) \leq f_{\mathrm{cl}}(y),$$

contradiction. A similar contradiction is reached if $y \neq \inf\{z \in W : y < z\}$. And if $|(x, y) \cap W| > 1$ a contradiction is easily reached.

Next, note that for each $z \in [0,1]$ the set $f_{\mathrm{cl}}^{-1}[\{z\}]$ has at most three elements. For, if it has four or more, at most one of them is in W; so this gives three elements $w < x < y$ of $\mathrm{cl}(W)\backslash W$ all with the same value under f_{cl}. If $x = \sup\{u \in W : u < x\}$, this gives infinitely many elements of W between w and x, contradiction; similarly if $x = \inf\{u \in W : x < u$.

For any $W \subseteq [0,1]$ let $C_2(W)$ be the set of all functions $f : W \to I$ such that

(1) f is monotone.

(2) $f^{-1}[\{y\}]$ is finite for all $y \in [0,1]$,

(3) $|\{x \in W : f(x) \neq x\}| = 2^\omega$.

Thus by the above, $f_{\mathrm{cl}} \in C_2(\mathrm{cl}(W))$ whenever $f \in C_1(W)$, $|W| = 2^\omega$, and $f(x) \neq x$ for all $x \in W$.

Now let W be a closed subset of $[0,1]$. Choose a countable dense subset F_1 of W (pick $w_{rs} \in (r, s) \cap W$ for each pair $r < s$ of rationals such that $(r, s) \cap W \neq 0$, and let F_1 be the set of all such elements w_{rs}). Now take any member f of $C_2(W)$. Let

$$F_2^f = \{x \in W : \sup\{f(y) : x \geq y \in F_1\} < \inf\{f(y) : x \leq y \in F_1\}\}.$$

If $x \in F_2^f$, then $x \notin F_1$. Now for each $x \in F_2^f$ let $U_x = (\sup\{f(y) : x \geq y \in F_1\}, \inf\{f(y) : x \leq y \in F_1\})$. This is a nonempty open interval, and if $x, y \in F_2^f$ with $x < y$ we can choose $z \in F_1$ such that $x < z < y$, and then $\inf\{f(u) : x \leq u \in F_1\} \leq f(z) \leq \sup\{f(u) : y \geq u \in F_1\}$, showing that $U_x \cap U_y = \emptyset$. so that $U_x \cap U_y = 0$ for $x \neq y$. Therefore, F_2^f is countable.

Now if $f, g \in C_2(W)$, $F_2^f = F_2^g$, and $f \restriction (F_1 \cup F_2^f) = g \restriction (F_1 \cup F_2^g)$, then $f = g$. For, suppose that $x \in W \backslash (F_1 \cup F_2^f)$. Then

$$f(x) = \sup\{f(y) : x \geq y \in F_1\} = \sup\{g(y) : x \geq y \in F_1\} = g(x).$$

From these considerations it follows that $|C_2(W)| \leq 2^\omega$. Also recall that there are just 2^ω closed sets, since every closed set is the closure of a countable dense subset. Hence the set

$$C = \bigcup\{C_2(F) : F \subseteq [0, 1], F \text{ closed}, |F| = 2^\omega\}$$

has cardinality $\leq 2^\omega$. Let $\langle f_\alpha : \alpha < 2^\omega\rangle$ be an enumeration of C. Let h be a strictly decreasing function from $\mathbb{R}$ onto (0,1); thus h^{-1} is also strictly decreasing. (For example, let $h(x) = 1/(e^x + 1)$ for all $x \in \mathbb{R}$.) Moreover, fix a well-ordering of $\mathbb{R}$.

Now we construct by induction pairwise disjoint subsets A_α of $[0, 1]$ for $\alpha < 2^\omega$. At the end we will let E_0 be the union of the A_α with even α and E_1 be the union of the rest. We will carry along the inductive hypothesis that $|\alpha| \leq |A_\alpha| \leq |\alpha| + \omega$. Let A_0 and A_1 be denumerable disjoint subsets of [0,1] which are dense in [0,1].

Now suppose that A_α has been constructed for all $\alpha < \beta$, where $\beta \geq 2$. Let $B_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and $B_\beta^* = B_\beta \cup \bigcup_{\alpha<\beta} f_\alpha[B_\beta] \cup \bigcup_{\alpha<\beta} f_\alpha^{-1}[B_\beta]$. Note by our assumptions that $|\beta| \leq |B_\beta^*| \leq |\beta| + \omega$. For every real number r, let

$$C_r = h[\{r + h^{-1}(b) : b \in B_\beta^*\}].$$

We claim that there is an $r \in \mathbb{R}$ such that $C_r \cap B_\beta^* = 0$. Suppose not. For every $r \in \mathbb{R}$ choose $b_r \in C_r \cap B_\beta^*$. Since $|B_\beta^*| < 2^\omega$, there exist a set $S \subseteq \mathbb{R}$ and an element $c \in B_\beta^*$ such that $|S| > B_\beta^*$ and $b_r = c$ for all $r \in S$. Say $c = h(d)$ with $d \in \{r + h^{-1}(x) : x \in B_\beta^*\}$ for all $r \in S$. Thus $h(d - r) \in B_\beta^*$ for all $r \in S$. So there exist distinct $r, s \in S$ such that $h(d-r) = h(d-s)$. This contradicts h being one-one.

Finally, let r be the least real number (in the well-ordering fixed above) such that $C_r \cap B_\beta^* = 0$, and let $A_\beta = C_r$. Clearly the inductive hypothesis remains true. This finishes the construction of the sets A_α, $\alpha < 2^\omega$.

Let $E_0 = \bigcup_{\alpha \text{ even}} A_\alpha$ and $E_1 = \bigcup_{\alpha \text{ odd}} A_\alpha$. So E_0 and E_1 are disjoint subsets of [0,1], and both of them are dense in [0,1]. Since all of the sets A_α are non-empty, it is clear that both E_0 and E_1 are of power 2^ω.

Now suppose that g is a strictly monotone function from a subset of E_0 of power 2^ω into E_1. Say $g_{\mathrm{cl}} = f_\alpha$. Now $|\bigcup_{\beta\leq\alpha} A_\alpha| < 2^\omega$, so choose $y \in \mathrm{rng}(g)$ such that $y \in A_\gamma$ for some $\gamma > \alpha$. Say $g(x) = y$ with $x \in A_\delta$. Now δ is even and γ is odd. If $\delta < \gamma$, then $y \in B^*_\gamma$, so $y \in A_\gamma$ is a contradiction. If $\gamma < \delta$, then $x \in B^*_\delta$, so $x \in A_\delta$ is a contradiction. Thus (ii) of the lemma has been verified.

If $\omega \leq \kappa < 2^\omega$, choose $\beta < 2^\omega$ odd with $\beta > \kappa$. Say $A_\beta = C_r$, as in the definition. Now $b \mapsto h(r + h^{-1}b)$ is an increasing mapping from B^*_β into A_β, and $E_0 \cap B^*_\beta$ has at least κ elements. This verifies (i). □

The example also depends upon the following lemma, which will also be useful later on.

Lemma 3.47. *Let A be the interval algebra on $\mathbb{R}$. Then there does not exist in A a strictly increasing sequence $\langle I_\alpha : \alpha < \omega_1\rangle$ of ideals.*

Proof. Suppose that there is such a sequence. For each $\alpha < \omega_1$ define $r \equiv_\alpha s$ iff $r, s \in \mathbb{R}$ and either $r = s$ or else if, say, $r < s$, then $[r, s) \in I_\alpha$. Then $\equiv_\alpha$ is an equivalence relation on $\mathbb{R}$ and the equivalence classes are intervals. For each $r \in \mathbb{R}$ the left endpoints of the intervals $[r]_\alpha$ are decreasing for increasing α, and the right endpoints, increasing ($[r]_\alpha$ denotes the equivalence class of r under the equivalence relation $\equiv_\alpha$). Since there is no strictly monotone sequence of real numbers of type ω_1, there is an ordinal $\beta_r < \omega_1$ such that both the left and right endpoints of $[r]_\alpha$ are constant for $\alpha > \beta_r$. Let $\gamma = \sup\{\beta_r : r \text{ rational}\}$. Then all of the equivalence classes are constant for $\alpha > \gamma$, contradiction. □

Corollary 3.48. *Let A be a subalgebra of the interval algebra on $\mathbb{R}$. Then A does not have an uncountable ideal independent subset.*

Proof. Suppose that X is an uncountable ideal independent subset of A. Let $\langle a_\alpha : \alpha < \omega_1\rangle$ be a one-one enumeration of some elements of X. For each $\alpha < \omega_1$ let $I_\alpha = \langle\{a_\beta : \beta < \alpha\}\rangle^{\mathrm{Id}}$. Clearly then $\langle I_\alpha : \alpha < \omega_1\rangle$ is a strictly increasing sequence of ideals in B, contradicting 3.47. □

Finally, we are ready for the example. The main content of the example is from Todorčević [86], Theorem 12, as we mentioned.

Theorem 3.49. *There is a BA A of power 2^ω such that:*

(i) $\mathrm{Ult}(A)$ *has, for each $\kappa < 2^\omega$, a discrete subspace of power κ, and A has an atomic homomorphic image B with κ atoms;*

(ii) $\mathrm{Ult}(A)$ *has no discrete subspace of power 2^ω;*

(iii) *If B is any homomorphic image of A, then there is a dense subset X of B such that we can write $X = W \cup \bigcup_{i\in\omega} Z_i$ with W the set of all atoms of B and for each $i \in \omega$, the set Z_i has the finite intersection property.*

Proof. Let E_0 and E_1 be as in Lemma 3.46. Without loss of generality, $0, 1 \notin E_0 \cup E_1$. For $i < 2$ let K_i be the linearly ordered set obtained from [0,1] by replacing each element $r \in E_i$ by two new points $r^- < r^+$. Taking the order topology on K_i, we obtain a Boolean space. In fact, K_i is homeomorphic to the Stone space of the interval algebra on $E_i \cup \{0\}$. Namely, the following function

f from $\mathrm{Ult}(\mathrm{Intalg}(E_i \cup \{0\}))$ into K_i is the desired homeomorphism. Take any $F \in \mathrm{Ult}(\mathrm{Intalg}(E_i \cup \{0\}))$. Let $r = \inf\{a \in E_i : [0,a) \in F\}$; so $r \in [0,1]$. If $r \in E_i$ and $[0,r) \in F$, let $f(F) = r^-$; if $r \in E_i$ and $[0,r) \notin F$, let $f(F) = r^+$; and if $r \notin E_i$ let $f(F) = r$. Clearly f is one-one and maps onto K_i. To show that it is continuous, we first note that the following clopen subsets of K_i constitute a subbase for its topology:

$$\{[0,s^+) : s \in E_i\} \cup \{(r^-,1] : r \in E_i\}. \tag{$*$}$$

In fact, it suffices to show that each nonempty open interval of K_i is a union of finite intersections of elements of $(*)$. Let $N = \{r^- : r \in E_i\} \cup \{r^+ : r \in E_i\}$. We consider several cases:

$$\begin{aligned}
&[0,s^+) \text{ is in } (*);\\
[0,s^-) &= \bigcup_{s>t\in E_i} [0,t^+);\\
[0,b) &= \bigcup_{b>s\in E_i} [0,s^+) \quad \text{for } b \notin N;\\
(r^+,1] &= \bigcup_{r<s<1, s\in E_i} (s^-,1];\\
&(r^-,1] \text{ is in } (*);\\
(b,1] &= \bigcup_{b<s<1, s\in E_i} (s^-,1] \quad \text{for} b \notin N;\\
(r^-,s^-) &= \bigcup_{r<t<s, t\in E_i} ([0,t^+) \cap (r^-,1]);\\
(r^-,s^+) &= (r^-,1] \cap [0,s^+);\\
(r^+,s^-) &= \bigcup_{\substack{r<t<u<s,\\ t,u\in E_i}} ((t^+,1] \cap [0,u^+);\\
(r^+,s^+) &= \bigcup_{r<t<s, t\in E_i} ((t^-,1] \cap [0,s^+));\\
(a,r^-) &= \bigcup_{\substack{a<t<u<r\\ t,u\in E_i}} ([0,u^+) \cap (t^-,1]) \quad \text{for } a \notin N;\\
(a,r^+) &= \bigcup_{a<t<r, t\in E_i} ((t^-,1] \cap [0,r^+)) \quad \text{for } a \notin N;\\
(r^-,a) &= \bigcup_{r<t<a, t\in E_i} ((r^-,1] \cap [0,t^+)) \quad \text{for } a \notin N;\\
(r^+,a) &= \bigcup_{\substack{r<t<u<a\\ t,u\in E_i}} ([0,u^+) \cap (t^-,1]) \quad \text{for } a \notin N;
\end{aligned}$$

$$(a,b)=\bigcup_{\substack{a<t<u<b\\ t,u\in E_i}}([0,u^+)\cap(t^-,1])\quad \text{for } a,b\notin N.$$

Next we claim that

$$\begin{aligned}f^{-1}[[0,s^+)]&=\{F:[0,s)\in F\};\\ f^{-1}[(r^-,1)]&=\{F:[r,1]\in F\}.\end{aligned}$$

To prove these equations, let F be any ultrafilter on $\mathrm{Intalg}(E_i\cup\{0\})$. Define $t=\inf\{a\in E_i:[0,a)\in F\}$. We now consider three cases.

Case 1. $t\in E_i$ and $[0,t)\in F$. Hence $f(F)=t^-$, and

$$\begin{aligned}F\in f^{-1}[[0,s^+)]\quad&\text{iff}\quad t^-\in[0,s^+)\\ &\text{iff}\quad t\le s\\ &\text{iff}\quad [0,s)\in F;\\ F\in f^{-1}[(s^-,1]]\quad&\text{iff}\quad t^-\in(s^-,1]\\ &\text{iff}\quad s<t\\ &\text{iff}\quad [s,1)\in F.\end{aligned}$$

Case 2. $t\in E_i$ and $[0,t)\notin F$. Hence $f(F)=t^+$, and

$$\begin{aligned}F\in f^{-1}[[0,s^+)]\quad&\text{iff}\quad t^+\in[0,s^+)\\ &\text{iff}\quad t< s\\ &\text{iff}\quad [0,s)\in F;\\ F\in f^{-1}[(s^-,1]]\quad&\text{iff}\quad t^+\in(s^-,1]\\ &\text{iff}\quad s\le t\\ &\text{iff}\quad [s,1)\in F.\end{aligned}$$

Case 3. $t\notin E_i$. Hence $f(F)=t$, and

$$\begin{aligned}F\in f^{-1}[[0,s^+)]\quad&\text{iff}\quad t\in[0,s^+)\\ &\text{iff}\quad t< s\\ &\text{iff}\quad [0,s)\in F;\\ F\in f^{-1}[(s^-,1]]\quad&\text{iff}\quad t\in(s^-,1]\\ &\text{iff}\quad s<t\\ &\text{iff}\quad [s,1)\in F.\end{aligned}$$

This completes the proof that f is a homeomorphism from $\mathrm{Ult}(\mathrm{Intalg}(E_i\cup\{0\}))$ onto K_i.

By Corollary 3.48, neither K_0 nor K_1 has an uncountable discrete subspace. Also, $K_0\times K_1$ is a Boolean space, and we let A be the BA of closed-open subsets of it.

First we check that for any $\kappa < 2^\omega$, $K_0 \times K_1$ has a discrete subset of power κ. Let f be a strictly increasing function from a subset of E_0 of power κ into E_1. Then we claim that $D \stackrel{\text{def}}{=} \{(r^-, (f(r))^+) : r \in \mathrm{dmn}(f)\}$ is discrete. To show this, for each $r \in \mathrm{dmn}(f)$ let $a_r = [0, r^+) \times ((f(r))^-, 1]$. Suppose $(s^-, (f(s))^+) \in a_r$ and $s \neq r$. Thus $s^- < r^-$ and $(f(r))^+ < (f(s))^+$, contradiction.

From the proofs of 3.29 and 3.30 it now follows that A has a homomorphic image C which has a disjoint subset of power κ. By an easy application of the Sikorski extension theorem, A has an atomic homomorphic image B with κ atoms.

Next we prove (ii). Suppose that D is a discrete subspace of $K_0 \times K_1$ of size 2^ω. Now K_1 has no uncountable discrete subspace, so for each $x \in \mathrm{dmn}(D)$, the set $\{y : (x, y) \in D\}$ is countable. It follows that we may assume that D is a function. Similarly, we may assume that D is one-one.

For $(r, s) \in D$ let a_{rs} and b_{rs} be open intervals in K_0 and K_1 respectively such that $(a_{rs} \times b_{rs}) \cap D = \{(r, s)\}$. Let F_0 and F_1 be countable dense subsets of K_0 and K_1 respectively (in the sense that if $a < b$ in K_0 and $(a, b) \neq 0$ then there is a $c \in F_0$ such that $a < c < b$; similarly for K_1). Suppose that $\mathrm{dmn}(D) \backslash (\{r^- : r \in E_0\} \cup \{r^+ : r \in E_0\})$ has power 2^ω. Then we may successively assume that $\mathrm{dmn}(D) \cap (\{r^- : r \in E_0\} \cup \{r^+ : r \in E_0\}) = 0$, that each a_{rs} is an open interval with endpoints in F_0, and that all of the a_{rs} are equal, which implies that D has only one element, contradiction.

Thus we may assume that $\mathrm{dmn}(D) \subseteq \{r^- : r \in E_0\} \cup \{r^+ : r \in E_0\}$, and similarly for $\mathrm{rng}(D)$. Hence we may assume that there are $\varepsilon, \delta \in \{-, +\}$ such that $\mathrm{dmn}(D) \subseteq \{r^\varepsilon : r \in E_0\}$ and $\mathrm{rng}(D) \subseteq \{r^\delta : r \in E_1\}$. Further, we may assume that there exist $q_i \in F_i$, $i = 0, 1$, such that $q_0 \in a_{\varepsilon\delta}$ and $q_1 \in b_{\varepsilon\delta}$ for each $(r^\varepsilon, s^\delta) \in D$. Finally, we may assume that one of the following four cases holds:

(1) $\varepsilon = -$, $\delta = -$, and for each $(r^-, s^-) \in D$, the right endpoint of $a_{r^-s^-}$ is r^+, and the right endpoint of $b_{r^-s^-}$ is s^+.

(2) $\varepsilon = -$, $\delta = +$, and for each $(r^-, s^+) \in D$, the right endpoint of $a_{r^-s^+}$ is r^+, and the left endpoint of $b_{r^-s^+}$ is s^-.

(3) $\varepsilon = +$, $\delta = -$, and for each $(r^+, s^-) \in D$, the left endpoint of $a_{r^+s^-}$ is r^-, and the right endpoint of $b_{r^+s^-}$ is s^+.

(4) $\varepsilon = +$, $\delta = +$, and for each $(r^+, s^+) \in D$, the left endpoint of $a_{r^+s^+}$ is r^-, and the left endpoint of $b_{r^+s^+}$ is s^-.

Next,

(5) If (1) or (2) holds, $(r^-, s^\delta), (u^-, v^\delta) \in D$, and $r < u$, then $r^- \in a_{u^-v^\delta}$.

In fact, since $q_0 \in a_{r^-s^\delta}$ and $a_{r^-s^\delta}$ has right endpoint r^+, we have $q_0 \leq r^- < u^-$, and then $q_0 \in a_{u^-v^\delta}$ implies that $r^- \in a_{u^-v^\delta}$.

By symmetry we have

(6) If (1) or (3) holds $(r^\varepsilon, s^-), (u^\varepsilon, v^-) \in D$, and $s < v$, then $s^- \in b_{u^\varepsilon v^-}$.

(7) If (3) or (4) holds, $(r^+,s^\delta),(u^+,v^\delta)\in D$, and $r<u$, then $u^+\in a_{r^+s^\delta}$.

In fact, since $q_0\in a_{u^+v^\delta}$ and $a_{u^+v^\delta}$ has left endpoint u^-, we have $r^+<u^+\le q_0$, and then $q_0\in a_{r^+s^\delta}$ implies that $u^+\in a_{r^+s^\delta}$.

Again by symmetry we have

(8) If (2) or (4) holds, $(r^\varepsilon,s^+),(u^\varepsilon,v^+)\in D$, and $s<v$, then $v^+\in b_{r^\varepsilon s^+}$.

Now we obtain a contradiction, as follows. If (1) holds, then the mapping $r\mapsto s$ iff $(r^-,s^-)\in D$ is strictly decreasing, as otherwise we would get $(r^-,s^-),(u^-,v^-)\in D$ with $r<s$ and $u<v$, and then (5) and (6) would give $(r^-,s^-)\in(a_{u^-v^-}\times b_{u^-v^-})$, contradiction. But then Lemma 3.46(ii) is contradicted. Similarly, (4) gives a strictly decreasing function, while (2) and (3) give strictly increasing functions.

Now we turn to the last part of the theorem. Suppose that B is a homomorphic image of A. Let F_i be a countable dense subset of E_i for $i=0,1$. Let $E_i^+=\{r^+: r\in E_i\}$, $E_i^-=\{r^-:r\in E_i\}$ for $i=0,1$. Now we are going to define some subsets $X^{\cdots}_{\cdots}$ of B indexed by various objects in countable sets; each subset will satisfy the finite intersection property, and this will be obvious in each case. What is not so obvious is what these sets are good for. We show after defining them that their union with the set of atoms of B is dense in B, which is the desired conclusion of the theorem. It is convenient to work with the dual of B, which is some closed subspace Y of $K_0\times K_1$.

Suppose that $p,q\in F_0$, $r,s\in F_1$, $p<q$, $r<s$, and $([p^+,q^-]\times[r^+,s^-])\cap Y\neq 0$; then we set

$$X^1_{pqrs}=\{([p^+,q^-]\times[r^+,s^-])\cap Y\}.$$

Next, suppose that $q\in F_0$, $r,s\in F_1$, and $r<s$. Then we set

$$\begin{aligned}X^2_{qrs}=\{([x,q^-]\times[r^+,s^-])\cap Y: x\in E_0^+,\ x<q^-,\ \text{and}\\ \exists y(r^+<y<s^-\text{ and }(x,y)\in Y)\}.\end{aligned}$$

The next three sets are similar to X^2_{qrs}. Suppose that $p\in F_0$, $r,s\in F_1$, and $r<s$. Set

$$\begin{aligned}X^3_{prs}=\{([p^+,x]\times[r^+,s^-])\cap Y: x\in E_0^-,\ p^+<x,\\ \text{and }\exists y(r^+<y<s^-\text{ and }(x,y)\in Y)\}.\end{aligned}$$

Suppose that $p,q\in F_0$, $s\in F_1$, and $p<q$. Set

$$\begin{aligned}X^4_{pqs}=\{([p^+,q^-]\times[y,s^-])\cap Y: y\in E_1^+,\ y<s^-,\\ \text{and }\exists x(p^+<x<q^-\text{ and }(x,y)\in Y)\}.\end{aligned}$$

Suppose that $p,q\in F_0$, $r\in F_1$, and $p<q$. Set

$$\begin{aligned}X^5_{pqr}=\{([p^+,q^-]\times[r^+,y])\cap Y: y\in E_1^-,\ r^+<y,\\ \text{and }\exists x(p^+<x<q^-\text{ and }(x,y)\in Y)\}.\end{aligned}$$

Now suppose that $p, q \in F_0$, $r, s \in F_1$, $p < q$, and $r < s$. Set

$$X^6_{pqrs} = \{([x, q^-] \times [y, s^-]) \cap Y : x \in E_0^+,\ y \in E_1^+,\ x < p^-,\\ y < r^-,\ \text{and } ([p^+, q^-] \times [r^+, s^-]) \cap Y \neq 0\}.$$

The next three sets are similar to X^6_{pqrs}. For each of them we suppose that $p, q \in F_0$, $r, s \in F_1$, $p < q$, and $r < s$.

$$X^7_{pqrs} = \{([x, q^-] \times [r^+, y]) \cap Y : x \in E_0^+,\ y \in E_1^-,\ x < p^-,\\ s^+ < y,\ \text{and } ([p^+, q^-] \times [r^+, s^-]) \cap Y \neq 0\}.$$

$$X^8_{pqrs} = \{([p^+, x] \times [y, s^-]) \cap Y : x \in E_0^-,\ y \in E_1^+,\ q^+ < x,\\ y < r^-,\ \text{and } ([p^+, q^-] \times [r^+, s^-]) \cap Y \neq 0\}.$$

$$X^9_{pqrs} = \{([p^+, x] \times [r^+, y]) \cap Y : x \in E_0^-,\ y \in E_1^-,\ q^+ < x,\\ s^+ < y,\ \text{and } ([p^+, q^-] \times [r^+, s^-]) \cap Y \neq 0\}.$$

Next, if $p \in F_0$ and $r, s \in F_1$ with $r < s$, we set

$$X^{10}_{prs} = \{([p^+, x] \times [r^+, y]) \cap Y : x \in E_0^-,\ y \in E_1^-,\ s^+ < y,\ p < x,\\ \text{and there is a } v \text{ such that } (x, v) \in Y \text{ and } r^+ < v < s^-\}.$$

The other sets are similar to this one; with obvious assumptions,

$$X^{11}_{prs} = \{([p^+, x] \times [y, s^-]) \cap Y : x \in E_0^-,\ y \in E_1^+,\ y < r^-,\ p < x,\\ \text{and there is a } v \text{ such that } (x, v) \in Y \text{ and } r^+ < v < s^-\}.$$

$$X^{12}_{qrs} = \{([x, q^-] \times [r^+, y]) \cap Y : x \in E_0^+,\ y \in E_1^-,\ s^+ < y,\ x < q,\\ \text{and there is a } v \text{ such that } (x, v) \in Y \text{ and } r^+ < v < s^-\}.$$

$$X^{13}_{qrs} = \{([x, q^-] \times [y, s^-]) \cap Y : x \in E_0^+,\ y \in E_1^+,\ y < r^-,\ x < q,\\ \text{and there is a } v \text{ such that } (x, v) \in Y \text{ and } r^+ < v < s^-\}.$$

$$X^{14}_{pqs} = \{([x, q^-] \times [y, s^-]) \cap Y : x \in E_0^+,\ y \in E_1^+,\ x < p^-,\ y < s,\\ \text{and there is a } u \text{ such that } (u, y) \in Y \text{ and } p^+ < u < q^-\}.$$

$$X^{15}_{pqr} = \{([x, q^-] \times [r^+, y]) \cap Y : x \in E_0^+,\ y \in E_1^-,\ x < r^-,\ r < y,\\ \text{and there is a } u \text{ such that } (u, y) \in Y \text{ and } p^+ < u < q^-\}.$$

$$X^{16}_{pqs} = \{([p^+, x] \times [y, s^-]) \cap Y : x \in E_0^-,\ y \in E_1^+,\ q^+ < x,\ y < s,\\ \text{and there is a } u \text{ such that } (u, y) \in Y \text{ and } p^+ < u < q^-\}.$$

$$X^{17}_{pqr} = \{([p^+, x] \times [r^+, y]) \cap Y : x \in E_0^-,\ y \in E_1^-,\ q^+ < x,\ r < y,\\ \text{and there is a } u \text{ such that } (u, y) \in Y \text{ and } p^+ < u < q^-\}.$$

Now we show that the union of these sets with the set of atoms of B is dense in B. Suppose that U is a non-zero element of B; we may assume that U has the form $((a,b)\times(c,d))\cap Y$, and that it is not $\geq$ any atom of B. Fix an element (x,y) of U. We consider various possibilities.

Case 1. $x\notin E_0^-\cup E_0^+$ and $y\notin E_1^-\cup E_1^+$. Then clearly there exist p,q,r,s such that $(x,y)\in Z\subseteq U$ with $Z\in X^1_{pqrs}$.

Case 2. $x\in E_0^-$ and $y\notin E_1^-\cup E_1^+$. There are p,r,s such that $(x,y)\in Z\subseteq U$ with $Z\in X^3_{prs}$.

Case 3. $x\in E_0^+$ and $y\notin E_1^-\cup E_1^+$. There are q,r,s such that $(x,y)\in Z\subseteq U$ with $Z\in X^2_{qrs}$.

Case 4. $x\notin E_0^-\cup E_0^+$. Similar to above cases, using $X^1_{\ldots}$, $X^4_{\ldots}$, or $X^5_{\ldots}$.

Case 5. $x\in E_0^-$ and $y\in E_1^-$. Then it is easy to find $p\in F_0$, $r\in F_1$ so that $(x,y)\in[p^+,x]\times[r^+,y]\cap Y\subseteq U$. Now there are two subcases.

Subcase 5.1. There is a $(u,v)\in Y$ such that $p^+<u<x$ and $r^+<v<y$. Then it is easy to find q,s so that $(x,y)\in Z\subseteq U$ and $Z\in X^9_{pqrs}$.

Subcase 5.2. Otherwise, since we are assuming that U is not $\geq$ any atom of B, either there is a v such that $(x,v)\in Y$ and $r^+<v<y$, or there is a u such that $(u,y)\in Y$ and $p^+<u<x$. In the first instance there is an s such that $(x,y)\in Z\subseteq U$ with $Z\in X^{10}_{prs}$. In the second instance we use X^{17}.

Case 6. $x\in E_0^-$ and $y\in E_1^+$. This is like Case 5. We use X^8, X^{16}, and X^{11}.

Case 7. $x\in E_0^+$ and $y\in E_1^-$. This is like Case 5. We use X^7, X^{15}, and X^{12}.

Case 8. $x\in E_0^+$ and $y\in E_1^+$. This is like Case 5. We use X^6, X^{14}, and X^{13}. □

Corollary 3.50. *Assume that 2^ω is a limit cardinal. Then there is a BA A of power 2^ω with spread 2^ω not attained, such that A has no homomorphic image B such that* $\mathrm{c}(B)=\mathrm{s}(A)$.

Proof. The first part of the conclusion follows immediately from the theorem. Now suppose that B is a homomorphic image of A such that $\mathrm{c}(B)=\mathrm{s}(A)$. Since $\mathrm{s}(A)$ is not attained, it follows from Corollary 3.31 that $\mathrm{c}(B)$ is not attained. Now let X, W, etc., be as in (iii) of the theorem. Then $|W|<2^\omega$ since $\mathrm{c}(B)$ is not attained. Let Z be a disjoint subset of B of power $|W|^+$. For each $z\in Z$ choose $x_z\in X$ such that $x_z\leq z$, and let $X'=\{x_z:z\in Z\}$. Then there has to exist an $i\in\omega$ such that $|X'\cap Z_i|>2$, which is a contradiction, since X' is disjoint and Z_i has the finite intersection property. □

Returning to the program described in the introduction, we note that it is obvious that $\mathrm{c}_{\mathrm{Ss}}(A)=[\omega,\mathrm{c}(A)]$. The caliber notion associated with cellularity has been worked on a lot. There are several variants of this notion. For a survey of results and problems, see Comfort, Negrepontis [82].

We shall compare c with other cardinal functions one-by-one in the discussion of those functions.

We turn to the relation c_{Sr}; see the end of the introduction. We do not have a purely cardinal number characterization of this relation; this is Problem 3 in Monk [96].

Problem 13. *Give a purely cardinal number characterization of* c_{Sr}.

Some restrictions to put on c_{Sr} are given in the following simple theorem:

Theorem 3.51. *For any infinite BA A the following conditions hold:*

(i) *If $(\kappa,\lambda)\in c_{Sr}(A)$, then $\kappa\le\lambda\le|A|$ and $\kappa\le c(A)$.*
(ii) *For each $\kappa\in[\omega, c(A)]$ we have $(\kappa,\kappa)\in c_{Sr}(A)$.*
(iii) *If $(\kappa,\lambda)\in c_{Sr}(A)$ and $\kappa\le\mu\le\lambda$, then $(\kappa,\mu)\in c_{Sr}(A)$.*
(iv) *If $(\lambda,(2^\kappa)^+)\in c_{Sr}(A)$ for some $\lambda\le\kappa$, then $(\omega,(2^\kappa)^+)\in c_{Sr}(A)$.*
(v) *$(c(A),|A|)\in c_{Sr}(A)$.*
(vi) *If $\omega\le\lambda\le|A|$ then $(\kappa,\lambda)\in c_{Sr}(A)$ for some κ.* □

The proof of this theorem is easy; for (iv), use Theorem 10.1 of Part I of the Handbook. Another general theorem about c_{Sr} is as follows; this is Theorem 14 of Monk, Nyikos [97].

Theorem 3.52. *For every infinite cardinal κ, and every BA A, if $c(A)\ge\kappa^{++}$ and $(\kappa,\kappa^{++})\in c_{Sr}(A)$, then $(\kappa^+,\kappa^{++})\in c_{Sr}(A)$.*

Proof. Suppose not. Let B be a subalgebra of size κ^{++} with cellularity κ.

(1) There is an $a\in A$ such that $B\restriction a$ has cellularity κ^{++}.

(Recall here that $B\restriction a=\{b\cdot a : b\in B\}$; we do not assume that $a\in B$.) To prove (1), let $X\subseteq A$ be pairwise disjoint of size κ^+. Then $\langle B\cup X\rangle$ is of size κ^{++} and has cellularity greater than κ, so by supposition its cellularity is κ^{++}; let Y be a pairwise disjoint subset of size κ^{++}. We may assume that each element $y\in Y$ has the form $y=b_y\cdot a_y$ with $b_y\in B$ and $a_y\in\langle X\rangle$. Since $|X|<\kappa^{++}$, we may in fact suppose that each a_y is equal to some element a, as desired in (1).

Choose such an a, and let $X\in[B]^{\kappa^{++}}$ be such that $\langle x\cdot a : x\in X\rangle$ is a system of nonzero pairwise disjoint elements. Let Y be a subset of X of size κ^+, and let

$$C=\langle\{x\cdot a : x\in Y\}\cup\{x\cdot -a : x\in X\backslash Y\}\rangle.$$

Now define $x\equiv y$ iff $x,y\in X\backslash Y$ and $x\cdot -a=y\cdot -a$. Then

(2) Every $\equiv$-class has size at most κ.

For, suppose that $|x/\equiv|>\kappa$. For any $y\in(x/\equiv)\backslash\{x\}$ we have

$$\begin{aligned} y\cdot -x &= y\cdot -x\cdot a + y\cdot -x\cdot -a\\ &= y\cdot a\cdot -(x\cdot a) + x\cdot -x\cdot -a\\ &= y\cdot a. \end{aligned}$$

This means that B has a pairwise disjoint subset of size greater than κ, contradiction. So (2) holds.

From (2) it follows that $|C| = \kappa^{++}$. Since clearly $\mathrm{c}(C) \geq \kappa^+$, by supposition we must have $\mathrm{c}(C) = \kappa^{++}$. Now each element of C has the form $\sum_{i<m} u_i \cdot v_i$ with each $u_i \in \langle\{x \cdot a : x \in Y\}\rangle$ and each $v_i \in \langle\{x \cdot -a : x \in X\backslash Y\}\rangle$. Hence we may assume that in a disjoint subset of C of size κ^{++} each element has the form $u \cdot v$, with $u \in \langle\{x \cdot a : x \in Y\}\rangle$ and $v \in \langle\{x \cdot -a : x \in X\backslash Y\}\rangle$. Hence clearly there is a $d \in \langle\{x \cdot a : x \in Y\}\rangle$ and a

$$Z \in [\langle\{x \cdot -a : x \in X\backslash Y\}\rangle]^{\kappa^{++}}$$

such that $\langle z \cdot d : z \in Z\rangle$ is a system of nonzero pairwise disjoint elements. We may assume that each $z \in Z$ has the form

$$\begin{aligned} &x_{z,0} \cdot -a \cdot \ldots \cdot x_{z,m-1} \cdot -a \\ &\cdot (-y_{z,0} + a) \cdot \ldots \cdot (-y_{z,n-1} + a), \end{aligned}$$

where each $x_{z,i}$ and $y_{z,j}$ is in $X\backslash Y$, and m and n do not depend on z.

Now since $\langle\{x \cdot a : x \in Y\}\rangle$ is isomorphic to $\mathrm{Finco}(\kappa^+)$, there are two cases.

Case 1. $d = \sum_{x\in F}(x \cdot a)$ for some finite $F \subseteq Y$. In this case we have $m = 0$, and then each $z \cdot d$ is just equal to d, contradiction.

Case 2. $d = -\sum_{x\in F}(x \cdot a)$ for some finite $F \subseteq Y$. Thus $d = -a + a \cdot -\sum_{x\in F} x$. Note that

(3) $a \cdot -\sum_{x\in F} x \neq 0$.

In fact, choose $y \in Y\backslash F$. Then

$$a \cdot y \cdot \sum_{x\in F} x = a \cdot y \cdot \sum_{x\in F} a \cdot x = 0;$$

hence $0 \neq a \cdot y \leq a \cdot -\sum_{x\in F} x \neq 0$, proving (3).

If $m = 0$, then each $z \cdot d$ is $\geq a \cdot -\sum_{x\in F} x$, so the elements $z \cdot d$ are not disjoint by (3), contradiction. Thus $m > 0$. Hence $z \cdot d = z \leq -a$ for each $z \in Z$. For each $z \in Z$ write $e_z = x_{z,0} \cdot \ldots \cdot x_{z,m-1}$ and $c_z = e_z \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1}$. So $z = c_z \cdot -a$ for each $z \in Z$. Note that if $z, w \in Z$ and $e_z \neq e_w$ then $e_z \cdot e_w \cdot a = 0$. Define $z \cong w$ iff $z, w \in Z$ and $e_z = e_w$. If $z \not\cong w$, then $e_z \neq e_w$, hence $z \neq w$ and

$$c_z \cdot c_w = c_z \cdot c_w \cdot a + c_z \cdot c_w \cdot -a = z \cdot w = 0.$$

Since $c_z \in B$ for each $z \in Z$, it follows that there are at most κ $\cong$-classes. So, some $\cong$-class has κ^{++} members. Thus we may assume that all of the e_z's are the same. Thus for any $z \in Z$ we have

$$\begin{aligned} z &= x_0 \cdot \ldots \cdot x_{m-1} \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1} \cdot -a, \\ c_z &= x_0 \cdot \ldots \cdot x_{m-1} \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1}. \end{aligned}$$

Note that

$$c_z \cdot a = (x_0 \cdot a) \cdot \ldots \cdot (x_{m-1} \cdot a) \cdot -(y_{z,0} \cdot a) \cdot \ldots \cdot -(y_{z,n-1} \cdot a).$$

It follows that either $c_z \cdot a = 0$, or else $m = 1$ and $c_z \cdot a = x_0 \cdot a$. Thus we may assume that either $c_z \cdot a = 0$ for all $z \in Z$, or $c_z \cdot a = x_0 \cdot a$ for all $z \in Z$. Hence $c_z \cdot a \cdot -(c_w \cdot a) = 0$ for all distinct $z, w \in Z$. So if $z \neq w$, then

$$\begin{aligned} c_z \cdot -c_w &= c_z \cdot -c_w \cdot a + c_z \cdot -c_w \cdot -a \\ &= c_z \cdot a \cdot -(c_w \cdot a) + c_z \cdot -a \cdot -(c_w \cdot -a) \\ &= z \cdot -w = z. \end{aligned}$$

So if we fix $w \in Z$, then $\langle c_z \cdot -c_w : z \in Z \backslash \{w\} \rangle$ is a system of κ^{++} nonzero pairwise disjoint elements of B, a contradiction. □

To understand more about the possibilities for the relation $\mathrm{c}_{\mathrm{Sr}}(A)$, consider the following examples. If κ is an infinite cardinal and A is the finite-cofinite algebra on κ, then $\mathrm{c}_{\mathrm{Sr}}(A) = \{(\lambda, \lambda) : \lambda \in [\omega, \kappa]\}$. If A is the free algebra on κ free generators, then $\mathrm{c}_{\mathrm{Sr}}(A) = \{(\omega, \lambda) : \lambda \in [\omega, \kappa]\}$. If A is an infinite interval algebra and we assume GCH, then $\mathrm{c}_{\mathrm{Sr}}(A)$ does not have any gaps of size 2 or greater. That is, if $(\kappa, \lambda) \in \mathrm{c}_{\mathrm{Sr}}(A)$, then $\lambda = \kappa$ or $\lambda = \kappa^+$. This is seen by using Theorem 10.1 of the Handbook again: such a gap would imply the existence in A of an uncountable independent subset, which does not exist in an interval algebra. Without CH there can be gaps; for example with $\mathrm{Intalg}(\mathbb{R})$.

There are two deeper results:

(1) Todorčević in [87a] shows that it is consistent (namely, it follows from V=L) to have for each regular non-weakly compact cardinal κ a κ-cc interval algebra A of size κ such that any subalgebra or homomorphic image B of A of size $< \kappa$ has a disjoint family of size $|B|$. His proof is based on the following result from Todorčević [81] (Theorem 4.9 there):

(∗) *Assume $V = L$. Let κ be a regular uncountable non-weakly compact cardinal. Then there exists a κ-Suslin tree with no λ-Aronszajn nor ν-Cantor subtrees, for every $\lambda \neq \kappa$ and every ν.*

Here "subtree" merely means a subset considered as a tree under the induced ordering; it is not assumed to be closed downwards.

Applying this result to subalgebras and to non-limit cardinals, this means in our terminology that it is consistent to have an algebra A with $\mathrm{c}_{\mathrm{Sr}}(A) = \{(\lambda, \lambda) : \lambda \in [\omega, \kappa]\} \cup \{(\kappa, \kappa^+)\}$.

(2) In models of Kunen [78] and Foreman, Laver [88], every ω_2-cc algebra of size ω_2 contains an ω_1-cc subalgebra of size ω_1. Thus in these models certain relations c_{Sr} are ruled out; cf. (1).

Now we survey what we know about $\mathrm{c_{Sr}}$ for BAs of size $\leq \omega_2$. Note that there are six pairs here: (ω,ω), (ω,ω_1), (ω,ω_2), (ω_1,ω_1), (ω_1,ω_2), and (ω_2,ω_2). The range of any relation $\mathrm{c_{Sr}}(A)$ consists of all infinite cardinals $\leq |A|$. We go through all possibilities by the number of pairs, and for equal numbers of pairs, lexicographically, skipping those possibilities ruled out by simple considerations.

(S1) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1)\}$ for $A = \mathrm{Fr}(\omega_1)$.

(S2) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega_1,\omega_1)\}$ for $A = \mathrm{Finco}(\omega_1)$.

(S3) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2)\}$ for $A = \mathrm{Fr}(\omega_2)$.

(S4) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1)\}$ for $\mathrm{Fr}(\omega_1) \times \mathrm{Finco}(\omega_1)$.

(S5) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$ for the algebra A of (1). Note that in the models mentioned in (2), such a value for $\mathrm{c_{Sr}}$ is not possible.

(S6) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$ for $A = \mathrm{Finco}(\omega_2)$.

(S7) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$ for the following BA A, assuming CH. Let

$$L = {}^{\omega_1}2 \backslash \{f \in {}^{\omega_1}2 : \exists\alpha < \omega_1(f\alpha = 0 \text{ and } \forall\beta > \alpha(f\beta = 1))\},$$
$$M = \{f \in {}^{\omega_1}2 : \exists\alpha < \omega_1(f\alpha = 1 \text{ and } \forall\beta > \alpha(f\beta = 0))\}.$$

We take the lexicographic order on L and M. Clearly M is dense in L, and $|M| = \omega_1$ by CH. Let L' be a subset of L of size ω_2 which contains M. Then $A \stackrel{\text{def}}{=} \mathrm{Intalg}\,(L')$ is as desired. For, by the denseness of M it has ω_2-cc, and it clearly has depth ω_1, and hence cellularity ω_1; so $(\omega_1,\omega_2) \in \mathrm{c_{Sr}}(A)$. We have $(\omega,\omega_2) \notin \mathrm{c_{Sr}}(A)$ by Theorem 10.1 of Part I of the Handbook. Obviously $(\omega_1,\omega_1) \in \mathrm{c_{Sr}}(A)$. The ordered set L'' constructed from ${}^{\omega}2$ similarly to L' from ${}^{\omega_1}2$ has size ω_1 and a dense subset of size ω. Then $\mathrm{Intalg}\,(L'')$ is isomorphic to a subalgebra of $\mathrm{Intalg}\,(L')$ by Remark 15.2 of the BA Handbook, so $(\omega,\omega_1) \in \mathrm{c_{Sr}}(A)$. It is a problem to get an example as in (S7) without CH. This is Problem 5(i) in Monk [96].

Problem 14. *Can one prove in ZFC that there is a BA A with*

$$\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),\ (\omega_1,\omega_1),(\omega_1,\omega_2)\}?$$

(S8) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$ for $A = \mathrm{Finco}(\omega_2) \times \mathrm{Fr}(\omega_1)$. To prove this, it suffices to show that any subalgebra B of A of size ω_2 has cellularity ω_2. Now $C \stackrel{\text{def}}{=} \{x \in \mathrm{Finco}(\omega_2) : \exists y(x,y) \in B\}$ is a subalgebra of $\mathrm{Finco}(\omega_2)$ of size ω_2, and hence there is a system $\langle c_\alpha : \alpha < \omega_2\rangle$ of nonzero disjoint elements of C. Say $(c_\alpha, d_\alpha) \in B$ for all $\alpha < \omega_2$. Now there are only ω_1 possibilities for the d_α's, so wlog we may assume that they are all equal. Then $(-c_0,-d_0)\cdot(c_\alpha,d_\alpha) = (c_\alpha,0)$ for all $\alpha > 0$, giving a disjoint family of size ω_2.

(S9) $\mathrm{c_{Sr}}(A) = \{(\omega,\omega),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$ for $A = B \times \mathrm{Finco}(\omega_2)$, where B is the algebra of (1), assuming $V = L$. For, $(\omega,\omega_2) \notin \mathrm{c_{Sr}}(A)$ by CH and Theorem

10.1 of the Handbook, Part I. So we just need to show that $(\omega,\omega_1) \notin \mathrm{c}_{\mathrm{Sr}}(A)$. Suppose that D is a subalgebra of A of size ω_1. Let $E = \{b \in B : (b,c) \in D$ for some $c\}$ and $F = \{c \in \mathrm{Finco}(\omega_2) : (b,c) \in D$ for some $b\}$.

Case 1. $|E| = \omega_1$. By the basic property of B, let $\langle e_\alpha : \alpha < \omega_1\rangle$ be a system of nonzero disjoint elements of E. Say $(e_\alpha, f_\alpha) \in D$ for all $\alpha < \omega_1$. If some f_β is cofinite, replace $\langle (e_\alpha, f_\alpha) : \alpha < \omega_1\rangle$ by $\langle (e_\alpha, f_\alpha \cdot -f_\beta) : \alpha < \omega_1, \alpha \neq \beta\rangle$; so wlog all f_α are finite. Wlog the f_α's form a Δ-system, say with kernel g. Pick distinct $\beta,\gamma < \omega_1$. Note that for $\alpha \neq \beta,\gamma$ we have $(e_\alpha, f_\alpha \backslash g) = (e_\alpha, f_\alpha) \cdot -[(e_\beta, f_\beta) \cap (e_\gamma, f_\gamma)]$; hence

$$\langle (e_\alpha, f_\alpha \backslash g) : \alpha < \omega_1, \alpha \neq \beta,\gamma\rangle$$

is a system of disjoint, nonzero elements of D, as desired.

Case 2. $|F| = \omega_1$ and $|E| < \omega_1$. We may assume that each member of F is finite. Let $F = \{c_\alpha : \alpha < \omega_1\}$, and choose $b_\alpha \in B$ so that $(b_\alpha, c_\alpha) \in D$. We may assume that all b_α's are the same. Let $\langle c_\alpha : \alpha \in M\rangle$ be a Δ-system, with $|M| = \omega_1$. Fix $\alpha \in M$. Then $\langle (b_\beta \cdot -b_\alpha, c_\beta \cdot -c_\alpha) : \beta \in M\backslash\{\alpha\}\rangle$ is a system of ω_1 disjoint elements.

Note that in the models of (2), an algebra A of the sort just described is not possible.

(S10) $\mathrm{c}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1), (\omega,\omega_2), (\omega_1,\omega_2)\}$ for $A = \mathrm{Finco}(\omega_1) \times \mathrm{Fr}(\omega_2)$.

(S11) $\mathrm{c}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1), (\omega,\omega_2), (\omega_2,\omega_2)\}$ is ruled out by Theorem 3.52. This answers Problem 4 in Monk [96].

(S12) $\mathrm{c}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1), (\omega_1,\omega_2), (\omega_2,\omega_2)\}$ for $B = A \times \mathrm{Finco}\,\omega_2$, with A as in (S7), assuming CH; the argument for this is easy.

We do not know whether CH is really needed here; this is Problem 5(ii) in Monk [96].

Problem 15. *Can one prove in ZFC that there is a BA A with*

$$\mathrm{c}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1),\ (\omega_1,\omega_1), (\omega_1,\omega_2), (\omega_2,\omega_2)\}?$$

(S13) $\mathrm{c}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1), (\omega,\omega_2), (\omega_1,\omega_2), (\omega_2,\omega_2)\}$, where A is the algebra $B \times \mathrm{Finco}(\omega_2)$, B the subalgebra of $\mathscr{P}(\omega_1)$ generated by the singletons and a set of ω_2 independent elements.

We turn to the relation $\mathrm{c}_{\mathrm{Hr}}(A)$.

Problem 16. *Give a cardinal number characterization of* $\mathrm{c}_{\mathrm{Hr}}(A)$*.*

This is Problem 6 in Monk [96].

We begin the discussion of $\mathrm{c}_{\mathrm{Hr}}(A)$ with a general theorem. Part (v) of this theorem is due to Piotr Koszmider.

Theorem 3.53. *For any infinite BA A the following conditions hold:*

(i) *If* $(\kappa, \lambda) \in \mathrm{c_{Hr}}(A)$, *then* $\kappa \leq \lambda \leq |A|$ *and* $\kappa \leq \mathrm{s}(A)$.

(ii) *For each* $\kappa \in [\omega, \mathrm{s}(A))$ *there is a* $\lambda \leq 2^\kappa$ *such that* $(\kappa, \lambda) \in \mathrm{c_{Hr}}(A)$.

(iii) *If* $(\lambda, (2^\kappa)^+) \in \mathrm{c_{Hr}}(A)$ *for some* $\lambda \leq \kappa$, *then* $(\omega, (2^\kappa)^+) \in \mathrm{c_{Hr}}(A)$.

(iv) $(\mathrm{c}(A), |A|) \in \mathrm{c_{Hr}}(A)$.

(v) *If* $(\kappa', \lambda') \in \mathrm{c_{Hr}}(A)$, *where* κ' *is a successor cardinal or a singular cardinal and* $\kappa' < \mathrm{cf}(|A|)$, *then there is a* $\kappa'' \geq \kappa'$ *such that* $(\kappa'', |A|) \in \mathrm{c_{Hr}}(A)$.

Proof. Only (ii) and (v) need need proofs. For (ii), let $\kappa \in [\omega, \mathrm{s}(A))$. Take a homomorphic image B of A such that $\mathrm{c}(B) > \kappa$; let C be a subalgebra of B generated by a disjoint set of power κ, and extend the identity on C to a homomorphism from B onto a subalgebra D of $\overline{C}$; then D is as desired.

Now we prove (v). For brevity let $\lambda = |A|$. There is nothing to prove if $\kappa' = \lambda$, so assume that $\kappa' < \lambda$. Let f be a homomorphism from A onto a BA B with $|B| = \lambda'$ and $\mathrm{c}(B) = \kappa'$. By the Erdős–Tarski theorem, there is a system of $\langle b_\xi : \xi < \kappa' \rangle$ of nonzero disjoint elements in B. For each $\xi < \kappa'$ choose $a_\xi \in A$ such that $f(a_\xi) = b_\xi$. We now consider two cases.

Case 1. $|A \upharpoonright a_\xi| < \lambda$ for all $\xi < \kappa'$. Let J be the ideal in A generated by $\{a_\xi \cdot a_\eta : \xi < \eta < \kappa'\}$. Then

$(*)$ $|J| < \lambda$.

In fact, $a \in J$ if and only if there is a finite set Γ of ordered pairs (ξ, η) with $\xi < \eta < \kappa'$ such that

$$a \leq \sum_{(\xi,\eta)\in\Gamma} (a_\xi \cdot a_\eta),$$

and the number of such sets Γ is κ'. Take any such set Γ. Write $\Gamma = \{(\xi_i, \eta_i) : i < n\}$. Define $c_i = a_{\xi_i} \cdot -\sum_{j<i} a_{\xi_j}$ for each $i < n$. Then $\sum_{i<n} c_i = \sum_{i<n} a_{\xi_i}$, and hence

$$\left|\left\{a \in A : a \leq \sum_{i<n} (a_{\xi_i} \cdot a_{\eta_i})\right\}\right| \leq \left|\prod_{i<n} (A \upharpoonright c_i)\right| < \lambda,$$

which proves $(*)$, since $\kappa' < \lambda$.

It is also clear that $a_\xi \notin J$ for all $\xi < \kappa'$. It follows that in A/J there is a system of κ' nonzero disjoint elements, and $|A/J| = \lambda$, as desired.

Case 2. There is a $\xi_0 < \kappa'$ such that $|A \upharpoonright a_{\xi_0}| = \lambda$. Then if we take the homomorphism

$$A \cong (A \upharpoonright a_{\xi_0}) \times (A \upharpoonright -a_{\xi_0}) \to (A \upharpoonright a_{\xi_0}) \times (B \upharpoonright -b_{\xi_0})$$

determined by the identity and $f \upharpoonright (A \upharpoonright -a_{\xi_0})$, we get a homomorphism from A onto an algebra C of size λ and with κ' disjoint elements. $\square$

We now list some additional facts concerning $\mathrm{c_{Hr}}$.

(1) (Theorem 4 of Monk, Nyikos [97]) Suppose that $\omega \le \rho \le \kappa$. Let $A = \langle[\kappa]^{\le\rho}\rangle_{\mathscr{P}(\kappa)}$. Then $\mathrm{c_{Hr}}(A) = S \cup T \cup U$, where

$$\begin{aligned} S &= \{(\mu,\nu) : \omega \le \mu \le \nu \le 2^\rho, \nu^\omega = \nu\};\\ T &= \{(\mu,\mu^\rho) : 2^\rho < \mu \le \kappa\};\\ U &= \{(\mu,\kappa^\rho) : 2^\rho < \mu, \mu^\rho = \kappa^\rho, \kappa < \mu\}. \end{aligned}$$

Note that S,T,U are all needed here. This can be seen as follows. For $\rho = \omega$ and $\kappa = (2^\omega)^+$ we have $(\omega, 2^\omega) \in S\backslash(T \cup U)$. For $\rho = \omega$ and $\kappa = (2^\omega)^{++}$ we have $((2^\omega)^+, (2^\omega)^+) \in T\backslash(S \cup U)$. Finally, for $\rho = \omega_1$ and $\kappa = \omega_1$ we have $(2^{\omega_1}, 2^{\omega_1}) \in U\backslash(S \cup T)$.

(2) As a corollary of (1), assume CH, and let $A = \langle[\omega_2]^{\le\omega}\rangle_{\mathscr{P}(\omega_2)}$. Then

$$\mathrm{c_{Hr}}(A) = \{(\omega,\omega_1), (\omega_1,\omega_1), (\omega_2,\omega_2)\}.$$

(3) (Part of Theorem 7 of Monk, Nyikos [97]) Suppose that λ is uncountable and strongly inaccessible, and $\lambda \le \kappa$. Let $A = \langle[\kappa]^{<\lambda}\rangle_{\mathscr{P}(\kappa)}$. Then $\mathrm{c_{Hr}}(A) = S \cup T \cup U \cup \{(\lambda,\lambda)\}$, where

$$\begin{aligned} S &= \{(\mu,\nu) : \omega \le \mu \le \nu < \lambda, \nu^\omega = \nu\};\\ T &= \{(\mu,\mu^{<\lambda}) : \lambda < \mu \le \kappa\};\\ U &= \{(\mu,\kappa^{<\lambda}) : \lambda < \mu, \mu^{<\lambda} = \kappa^{<\lambda}, \kappa < \mu\}. \end{aligned}$$

The sets here all needed. In fact, $(\omega, 2^\omega) \in S\backslash(T \cup U \cup \{(\lambda,\lambda)\})$. For $\kappa = \lambda^+$ we have $(\lambda^+,\lambda^+) \in T\backslash(S \cup U \cup \{(\lambda,\lambda)\}$. For U, let $\kappa = \lambda^{+\omega}$ and $\mu = \lambda^{+\omega+1}$.

(4) (Theorem 10 of Monk, Nyikos [97]) Assume GCH, and suppose that $\mathscr{A} \in [[\kappa^+]^{\kappa^+}]^{\kappa^{++}}$, with any two distinct members of $\mathscr{A}$ having intersection of size at most κ. Let A be the κ^+-complete subalgebra of $\mathscr{P}(\kappa^+)$ generated by $\mathscr{A} \cup \{\{\alpha\} : \alpha < \kappa^+\}$. then

$$\mathrm{c_{Hr}}(A) = \{(\mu,\nu) : \omega \le \mu \le \nu \le \kappa^+, \mathrm{cf}(\nu) > \omega\} \cup \{(\kappa^+,\kappa^{++}), (\kappa^{++},\kappa^{++})\}.$$

(5) (A corollary of (4)) Under GCH there is a BA A such that

$$\mathrm{c_{Hr}}(A) = \{(\omega,\omega_1), (\omega_1,\omega_1), (\omega_1,\omega_2), (\omega_2,\omega_2)\}.$$

(6) (A result of Nyikos, based upon a consistency result of Laver; see Monk, Nyikos [97], Theorems 11, 12) It is consistent to have a BA A such that

$$\mathrm{c_{Hr}}(A) = \{(\omega,\omega), (\omega_1,\omega_1), (\omega,\omega_2), (\omega_2,\omega_2)\}.$$

(7) (A result of Nyikos; see Monk, Nyikos [97], Theorem 13) If $(\kappa^+,\kappa^{++}) \in \mathrm{c_{Hr}}(A)$ and $(\kappa,\kappa^{++}) \notin \mathrm{c_{Hr}}(A)$, then $(\kappa^+,\kappa^+) \in \mathrm{c_{Hr}}(A)$.

(8) By Corollary 3.50, $\kappa = \mathrm{s}(A)$ is not in general possible in Theorem 3.53(ii).

(9) If A is complete and $(\kappa,\lambda) \in \mathrm{c}_{\mathrm{Hr}}(A)$, then $\lambda^\omega = \lambda$.

(10) If A is the finite-cofinite algebra on an infinite cardinal κ, then $\mathrm{c}_{\mathrm{Hr}}(A) = \{(\lambda,\lambda) : \omega \le \lambda \le \kappa\}$.

(11) If A is the free BA on κ free generators, κ infinite, then $\mathrm{c}_{\mathrm{Hr}}(A) = \{(\lambda,\mu) : \omega \le \lambda \le \mu \le \kappa\}$.

(12) If A an infinite interval algebra and GCH is assumed, then there do not exist cardinals κ,λ such that $\kappa^{++} \le \lambda$, $(\kappa,\mu),(\lambda,\mu) \in \mathrm{c}_{\mathrm{Hr}}(A)$, while $(\rho,\mu) \notin \mathrm{c}_{\mathrm{Hr}}(A)$ for all $\rho \in (\kappa,\lambda)_{\mathrm{card}}$.

(13) The algebra A of Todorčević [87a] (assuming $V = L$) has $\mathrm{c}_{\mathrm{Hr}}(A) = \{(\lambda,\lambda) : \lambda \in [\omega,\kappa]\} \cup \{(\kappa,\kappa^+)\}$. See (1) in the discussion of c_{Sr}.

(14) Under CH, $\mathrm{c}_{\mathrm{Hr}}(\mathscr{P}(\omega)) = \{(\omega,\omega_1),(\omega_1,\omega_1)\}$.

(15) If $2^\omega = \omega_2$ then $\mathrm{c}_{\mathrm{Hr}}(\mathscr{P}(\omega)) = \{(\omega,\omega_2),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$.

(16) (See Koppelberg [77]) Assuming MA, if A is an infinite BA with $|A| < 2^\omega$, then A has a countable homomorphic image. This is proved in Theorem 9.9.

(17) A special case of a result of Just, Koszmider [91] is that it is consistent to have $2^\omega = \omega_2$ with an algebra A having homomorphic cellularity relation $\{(\omega,\omega_1),(\omega_1,\omega_1)\}$.

(18) In Juhász [92] it is shown that if $\kappa > \omega$ and $|A| \ge \kappa$, then A has a homomorphic image of size λ for some λ with $\kappa \le \lambda \le 2^{<\kappa}$. In particular, if CH holds and $|A| = \omega_2$, then A has a homomorphic image of size ω_1.

(19) Fedorchuk [75] constructed, assuming $\diamondsuit$, a BA A such that $\mathrm{c}_{\mathrm{Hr}}A = \{(\omega,\omega_1)\}$.

(20) Koszmider [99] has modified Fedorchuk's construction as follows. If M is a c.t.m. of ZFC and λ is a cardinal of M of uncountable cofinality with $\lambda > \omega_1$, then there is a generic extension $M[G]$ of M preserving cardinalities and cofinalities such that in $M[G]$ we have $2^\omega = \lambda$ and there are BAs A, B, C, and D with the following properties:

(a) $\mathrm{hd}(A) = \omega$, $|A| = \omega_1$, and A does not have a countably infinite homomorphic image. (Theorem 5.5 of Koszmider [99])

(b) $\mathrm{hd}(B) = \omega$, $|A| = \lambda$, and all homomorphic images of A have size λ. (Theorem 5.5 of Koszmider [99])

(c) $\mathrm{c}_{\mathrm{Hr}}(C) = \{(\omega,\lambda),(\omega_1,\lambda)\}$. (Theorem 6.11 of Koszmider [99])

(d) $\mathrm{c}_{\mathrm{Hr}}(D) = \{(\omega,\omega_1),(\omega_1,\omega_1)\}$. (Theorem 6.11 of Koszmider [99])

Later we show that $\mathrm{c}(D) \le \mathrm{hd}(D)$ for any BA D.

Using Proposition 1.1 we then get

(e) $\mathrm{c}_{\mathrm{Hr}}(A \times B) = \{(\omega,\omega_1),(\omega,\lambda)\}$.

(f) $\mathrm{c_{Hr}}(C \times D) = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega, \lambda), (\omega_1, \lambda)\}$.
(g) $\mathrm{c_{Hr}}(A \times C) = \{(\omega, \omega_1), (\omega, \lambda), (\omega_1, \lambda)\}$.

The following interesting result of Koszmider was used to construct the algebra C above.

Proposition 3.54. *Suppose that* $\mathrm{c_{Hr}}(A) = \{(\omega, \lambda)\}$ *with* $\lambda \geq \omega_2$. *Then there is a BA E such that* $\mathrm{c_{Hr}}(E) = \mathrm{Depth_{Hr}}(E) = \{(\omega, \lambda), (\omega_1, \lambda)\}$.

Proof. By Sikorski's extension theorem we may assume that $\mathrm{Finco}(\omega) \leq A \leq \mathscr{P}(\omega)$. Let $E = \prod_c^A A$, using the notation concerning moderate products in Chapter 1. It is easy to see that $|E| = \lambda$. Hence by Proposition 1.10, every homomorphic image of E has size λ. By Lemma 1.12(i), $\mathrm{Depth}(E) \geq \omega_1$. So it remains only to prove that E does not have an ideal L such that E/L has a disjoint set of size greater than ω_1. Suppose that L is such an ideal, with X a subset of E such that $\langle [x]_L : s \in X \rangle$ is a one-one system of nonzero pairwise disjoint elements of E/L, with $|X| = \omega_2$. By Lemma 1.11, let $Y \in [X]^{\omega_2}$ and $n \in \omega$ be such that $\langle Y \rangle_E$ can be isomorphically embedded in ${}^nA$. By Sikorski's extension theorem, $\langle Y \rangle / L$ can be isomorphically embedded in a homomorphic image F of ${}^nA$. By Proposition 1.1, F is isomorphic to a finite product H of homomorphic images of A. Hence $\mathrm{c}(H) \geq \omega_2$, contradiction. □

Now we consider small cardinals, like we did for $\mathrm{c_{Sr}}$. There are many more problems here. The problems are of two sorts: cases in which we know that the existence of the appropriate BA is consistent but have no construction in ZFC, and cases in which we know that the existence of the appropriate BA is inconsistent, but have no proof of non-existence in ZFC. Even if we know both the consistency of the existence of a BA with a specified $\mathrm{c_{Hr}}$ and the consistency of the nonexistence of such, there may be problems remaining, but we do not formulate them.

Possibilities for $\mathbf{c_{Hr}}$. For the convenience of the reader we mention all of the 63 a priori possibilities. As a guide through these, we arrange the six possible pairs lexicographically and go through them in order of the number of pairs.

(H1) $\{(\omega, \omega)\}$. Any countably infinite BA works.

(H2) $\{(\omega, \omega_1)\}$. The Fedorchuk example (19) gives this, assuming $\diamondsuit$. If MA+$2^\omega > \omega_1$, it is ruled out by Koppelberg (16).

(H3) $\{(\omega, \omega_2)\}$. This is impossible under CH, by Juhász (18). If MA + $2^\omega > \omega_2$, it is ruled out by Koppelberg (16). A consistent example is given by Koszmider's algebra B in (20).

(H4) $\{(\omega_1, \omega_1)\}$. Any BA has a homomorphic image of countable cellularity, so this relation is impossible.

(H5) $\{(\omega_1, \omega_2)\}$. Impossible; see (H4).

(H6) $\{(\omega_2, \omega_2)\}$. Impossible; see (H4).

(H7) $\{(\omega, \omega), (\omega, \omega_1)\}$. A subalgebra of Intalg($\mathbb{R}$) of size ω_1 gives an example. In fact let L be an unbounded subset of $\mathbb{R}$ of size ω_1. Let A be the subalgebra of Intalg($\mathbb{R}$) generated by $\{(-\infty, r) : r \in L\}$. For each $r \in L$ let $f((-\infty, r)) = [0, n)$, with $n \in \omega$ minimum such that $r < n$. Then f extends to a homomorphism of Intalg(L) into an infinite subalgebra of Intalg(ω); see the Handbook, 15.2.

(H8) $\{(\omega, \omega), (\omega, \omega_2)\}$. This is impossible under CH, by Juhász (18). Assuming $2^\omega = \omega_2$, the algebra Intalg($\mathbb{R}$) works; see Theorem 9.13.

(H9) $\{\omega, \omega), (\omega_1, \omega_1)\}$. Finco($\omega_1$) works.

(H10) $\{(\omega, \omega), (\omega_1, \omega_2)\}$. This is impossible, by the result of Nyikos (7).

(H11) $\{(\omega, \omega), (\omega_2, \omega_2)\}$. Any BA of cellularity ω_2 has a homomorphic image of cellularity ω_1, so this relation is impossible.

(H12) $\{(\omega, \omega_1), (\omega, \omega_2)\}$. Not possible under CH, by Theorem 10.1 of the Handbook. If MA$+2^\omega > \omega_1$, it is ruled out by Koppelberg's Theorem (16). A consistent example is given by $A \times B$, where A and B are Koszmider's algebras; see (20)(e).

(H13) $\{(\omega, \omega_1), (\omega_1, \omega_1)\}$. Under CH, $\mathscr{P}(\omega)$ works; and also the Ju example (17) works. Koppelberg's Theorem (16) indicates that it is not possible to have such an example in ZFC.

(H14) $\{(\omega, \omega_1), (\omega_1, \omega_2)\}$. This is ruled out by the result of Nyikos (7).

(H15) $\{(\omega, \omega_1), (\omega_2, \omega_2)\}$. Not possible: see (H11).

(H16) $\{(\omega, \omega_2), (\omega_1, \omega_1)\}$. This is not possible, since the homomorphic image of size ω_1 should have a homomorphic image which is ccc.

(H17) $\{(\omega, \omega_2), (\omega_1, \omega_2)\}$. This is impossible under CH, by Juhász (18). If MA + $2^\omega > \omega_2$, it is ruled out by Koppelberg's Theorem (16). A consistent example is given by Koszmider's algebra C; see (20).

(H18) $\{(\omega, \omega_2), (\omega_2, \omega_2)\}$. Not possible; see (H11).

(H19) $\{(\omega_1, \omega_1), (\omega_1, \omega_2)\}$. Not possible; see (H4).

(H20) $\{(\omega_1, \omega_1), (\omega_2, \omega_2)\}$. Not possible; see (H4).

(H21) $\{(\omega_1, \omega_2), (\omega_2, \omega_2)\}$. Not possible; see (H4).

(H22) $\{(\omega, \omega), (\omega, \omega_1), (\omega, \omega_2)\}$. Not possible under CH, since then there is an uncountable independent set. Under $\neg$CH, a product of certain interval algebras works. Namely, with L_0, L_1, L_2 subsets of $\mathbb{R}$ of sizes $\omega, \omega_1, \omega_2$ respectively, each containing $\mathbb{Q}$, Intalg(L_0) $\times$ Intalg(L_1) $\times$ Intalg(L_2) works.

(H23) $\{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1)\}$. A free algebra of size ω_1 works.

(H24) $\{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_2)\}$. This is not possible, by the result of Nyikos (7).

(H25) $\{(\omega,\omega),(\omega,\omega_1),(\omega_2,\omega_2)\}$. Not possible; see (H11).

(H26) $\{(\omega,\omega),(\omega,\omega_2),(\omega_1,\omega_1)\}$. Ruled out by Theorem 3.53(v).

(H27) $\{(\omega,\omega),(\omega,\omega_2),(\omega_1,\omega_2)\}$. This is impossible under CH, by Juhász (18). A consistent example is given by $C\times\mathrm{Finco}(\omega)$, where C is Koszmider's algebra; see (20).

(H28) $\{(\omega,\omega),(\omega,\omega_2),(\omega_2,\omega_2)\}$. Not possible; see (H11).

(H29) $\{(\omega,\omega),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$. Under V=L this is possible, by the result of Todorčević (13). This is not possible in the models of Kunen [78] and of Foreman and Laver; see (2) in the discussion of c_{Sr}. Namely, suppose that $\mathrm{c}_{\mathrm{Hr}}A$ is the indicated relation in one of the indicated models A. Let B be an ω_1-cc subalgebra of A of size ω_1. By the Sikorski extension theorem, there is a homomorphism from A onto some BA C such that $B\le C\le\overline{B}$. Thus C has ccc and size ω_1 or ω_2, contradiction.

(H30) $\{(\omega,\omega),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$. $\mathrm{Finco}(\omega_2)$ works.

(H31) $\{(\omega,\omega),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. This is ruled out by the result of Nyikos (7).

(H32) $\{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1)\}$. This is ruled out by Theorem 3.53(v).

(H33) $\{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_2)\}$. This is impossible under CH, by Juhász (18). A consistent example is given by $A\times C$, both Koszmider's algebras; see (20).

(H34) $\{(\omega,\omega_1),(\omega,\omega_2),(\omega_2,\omega_2)\}$. This is not possible; see (H11).

(H35) $\{(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$. If MA $+\ 2^\omega>\omega_2$, this is ruled out by Koppelberg's Theorem (16). It is open whether an example is consistent; this is Problem 8(ii) in Monk [96].

Problem 17. *Is it consistent that* $\mathrm{c}_{\mathrm{Hr}}(A)=\{(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$ *for some BA A?*

(H36) $\{(\omega,\omega_1),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$. If MA$+2^\omega>\omega_2$, this is ruled out by Koppelberg's theorem (16). Under CH, an example exists; see (2). This solves Problem 8(i) in Monk [96].

(H37) $\{(\omega,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. This is ruled out by the result of Nyikos (7).

(H38) $\{(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$. The homomorphic image of size ω_1 and cellularity ω_1 must have a homomorphic image of cellularity ω, contradiction.

(H39) $\{(\omega,\omega_2),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$. Impossible; see (H38).

(H40) $\{(\omega,\omega_2),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. This is impossible under CH, by Juhász (18). Assuming $2^\omega=\omega_2$, $\mathscr{P}(\omega)$ has this relation. If MA $+\ 2^\omega>\omega_2$, it is ruled out by Koppelberg's Theorem (16).

(H41) $\{(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. Not possible; see (H4).

(H42) $\{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1)\}$. This is ruled out by Theorem 3.53(v).

(H43) $\{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_2)\}$. Not possible under CH, since then there must be an independent subset of size ω_2, and one of the pairs must be (ω_2,ω_2). A consistent example with this relation is $C\times E$, where C is Koszmider's example and E is a subalgebra of $\mathrm{Intalg}(\mathbb{R})$ of size ω_1.

(H44) $\{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_2,\omega_2)\}$. Not possible: see (H11).

(H45) $\{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$. Under GCH the following algebra works: the standard interval algebra constructed from ${}^{\omega_1}2$; see (S7). It is open whether there is an example in ZFC; this is Problem 7(i) in Monk [96].

Problem 18. *Can one prove in ZFC that there is a BA A such that*

$$\mathrm{c_{Hr}}(A)=\{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2)\}?$$

(H46) $\{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$. Let B be a subalgebra of $\mathrm{Intalg}(\mathbb{R})$ of size ω_1, and set $A=B\times\mathrm{Finco}(\omega_2)$. Proposition 1.1 implies that $\mathrm{c_{Hr}}(A)$ is as desired.

(H47) $\{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. By the result of Nyikos (7) this is impossible.

(H48) $\{(\omega,\omega),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$. Not possible under CH, since then there must be an independent subset of size ω_2, and one of the pairs must be (ω_2,ω_2). A consistent example is provided by $C\times\mathrm{Finco}(\omega_1)$, where C is Koszmider's algebra; see (20).

(H49) $\{(\omega,\omega),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$. Not possible under GCH, since then there must be an independent subset of size ω_2, and one of the pairs must be (ω_1,ω_2). The Nyikos, Laver example (6) gives a consistent example.

(H50) $\{(\omega,\omega),(\omega,\omega_2),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. This is impossible under CH, by Juhász. Assuming that $2^\omega=\omega_2$, the algebra $\mathrm{Finco}(\omega)\times\mathscr{P}(\omega)$ gives an example.

(H51) $\{(\omega,\omega),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. This is possible under V=L, namely with the algebra $\mathrm{Finco}(\omega_2)\times A$, with A the algebra of Todorčević (13) with $\kappa=\omega_1$.

This relation is not possible in the models of Kunen and of Foreman and Laver mentioned in (2), in the discussion of $\mathrm{c_{Sr}}$. In fact, in those models, if A is a BA of size ω_2 with cellularity ω_1, then A has a subalgebra of size ω_1 which is ccc. Hence by Sikorski's extension theorem A has an uncountable homomorphic image which is ccc, so that the relation (H51) is not possible.

(H52) $\{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$. Not possible under CH, since then there is an independent subset of size ω_2, and one of the ordered pairs must be (ω_2,ω_2).

Also ruled out by Koppelberg's theorem (16) if $\mathrm{MA} + 2^\omega > \omega_2$. A consistent example is provided by $C \times D$, where C and D are Koszmider's algebras (20).

(H53) $\{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$. Not possible under GCH, since then there is an independent set of size ω_2, and one of the ordered pairs must be (ω_1,ω_2). Also ruled out by Koppelberg's theorem (16) if $\mathrm{MA} + 2^\omega > \omega_2$. The consistency of the existence of such an algebra is open; this is problem 8(iv) in Monk [96]:

Problem 19. *Is it consistent that* $\mathrm{c_{Hr}}(A) = \{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$ *for some BA A?*

(H54) $\{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. Not possible under CH since then there is an independent set of size ω_2, and hence (ω_1,ω_1) would have to be present. If $\mathrm{MA} + 2^\omega > \omega_2$, this is ruled out by Koppelberg's Theorem (16). A consistent example is given by $A \times \mathscr{P}(\omega)$, where A is Koszmider's algebra (20).

(H55) $\{(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. If $\mathrm{MA} + 2^\omega > \omega_2$, this is ruled out by Koppelberg's Theorem (16). Under GCH there is such an algebra; see (5). This answers Problem 8(iii) in Monk [96].

(H56) $\{(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. This is impossible; a BA of size ω_1 has a ccc homomorphic image.

(H57) $\{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2)\}$. This is not possible under CH, since then there must be an independent subset of size ω_2, and one of the pairs must be (ω_2,ω_2). Assuming that $2^\omega > \omega_1$, we can take the standard linear order which is a subset of ${}^\omega 2$, take a subset L of size ω_2, containing a dense subset of size ω, and let $B = \mathrm{Intalg}(L)$ and $A = B \times \mathrm{Finco}(\omega_1)$.

(H58) $\{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$. This is not possible under CH, since then there must be an independent subset of size ω_2, and one of the pairs must be (ω_1,ω_2). A consistent example is given by $A \times B$, where A is the algebra of Laver and Nyikos (6) and B is a subalgebra of $\mathrm{Intalg}(\mathbb{R})$ of size ω_1.

(H59) $\{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. Assuming $2^\omega = \omega_2$, one can take $A \times \mathscr{P}(\omega)$, where A is the interval algebra on a subset of $\mathbb{R}$ of size ω_1. Under CH, no algebra with this $\mathrm{c_{Hr}}$ exists; for suppose that A is such an algebra. Then A has a ccc homomorphic image B of size ω_2. Then B has an independent set of size ω_2, and it follows that $\mathscr{P}(\omega)$ is a homomorphic image of B. In turn, $\mathscr{P}(\omega)$ has a homomorphic image of size and cellularity ω_1. Thus $(\omega_1,\omega_1) \in \mathrm{c_{Hr}}(A)$, contradiction.

(H60) $\{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. This is possible under GCH: take

$$\mathrm{Intalg}({}^{\omega_1}2) \times \mathrm{Intalg}(\mathbb{R}) \times \mathrm{Finco}(\omega_2),$$

where ${}^{\omega_1}2$ has the lexicographic order. It is open to give an example in ZFC; this is Problem 7(ii) in Monk [96].

Problem 20. *Can one construct in ZFC a BA A such that*

$$\mathrm{c_{Hr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}?$$

(H61) $\{(\omega,\omega),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. Not possible under CH, since then there must be an independent subset of size ω_2, and one of the pairs must be (ω,ω_1). Assuming $2^\omega=\omega_2$, $\mathscr{P}(\omega)\times\mathrm{Finco}(\omega_2)$ works.

(H62) $\{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. $\mathscr{P}(\omega_1)$ works, assuming GCH. Ruled out by Koppelberg's theorem (16) if $\mathrm{MA}+2^\omega>\omega_2$.

(H63) $\{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$. Many examples.

The following problem is related to some of the above problems, but is simpler to state.

Problem 21. *Can one prove in ZFC that there is a BA A of size ω_2 such that $(\omega_1,\omega_2)\in\mathrm{c_{Hr}}(A)$ but $(\omega,\omega_2)\notin\mathrm{c_{Hr}}(A)$?*

To conclude this chapter, we consider cellularity for special classes of BAs. For an atomic BA A, $\mathrm{c}(A)$ coincides with the number of atoms of A. Also note that some of the free product questions are trivial for atomic algebras; in particular, $\mathrm{c}(A\oplus B)=\max\{\mathrm{c}A,\mathrm{c}B\}$ if A and B are atomic.

Recall from Propositions 3.42 our partial characterization of the sets $\mathfrak{a}_{\mathrm{spect}}$. Atomic BAs were used there. It is natural to ask for a similar result with atomless BAs. We now give such a theorem.

Lemma 3.55. *For any regular cardinal κ there is an atomless interval algebra A such that $\mathfrak{a}_{\mathrm{spect}}(A)=\{\kappa\}$.*

Proof. The case $\kappa=\omega$ is taken care of by $\mathrm{Intalg}(\mathbb{Q})$, so assume that κ is uncountable.

By Hausdorff [1908] there is a dense linear order L such that every element of L has character (κ,κ^*), and L has coinitiality and cofinality κ. We claim that we may assume that $|L|=\kappa$. To see this we define by recursion a sequence $\langle M_i : i\in\omega\rangle$ of subsets of L. Let M_0 be a subset of L whose order type is $\kappa^*+\kappa$, consisting of a coinitial subset of type κ^* and a cofinal subset of type κ. Having defined $M_{2i+1}\subseteq L$ of size κ, let $\langle(a_\alpha,b_\alpha):\alpha<\kappa\rangle$ list all pairs (c,d) with $c,d\in M_{2i+1}$ and $c<d$. For each $\alpha<\kappa$ choose $c_\alpha\in L$ such that $a_\alpha<c_\alpha<b_\alpha$, and let $M_{2i+2}=M_{2i+1}\cup\{c_\alpha:\alpha<\kappa\}$. Now list out the elements of M_{2i+2}: $\langle d_\alpha:\alpha<\kappa\rangle$. For each $\alpha<\kappa$ let $\langle e_{\alpha\xi}:\xi<\kappa\rangle$ be strictly increasing in L with supremum d_α, and let $\langle f_{\alpha\xi}:\xi<\kappa\rangle$ be strictly decreasing in L with infimum d_α. Let

$$M_{2i+3}=M_{2i+2}\cup\{e_{\alpha\xi}:\alpha,\xi<\kappa\}\cup\{f_{\alpha\xi}:\alpha,\xi<\kappa\}.$$

Thus $|M_{2i+3}|=\kappa$. Let $N=\bigcup_{i\in\omega}M_i$. Clearly $|N|=\kappa$, N is dense, and each element of N has character (κ,κ^*).

This proves our initial claim in this proof. Now let $A = \text{Intalg}(L)$. Now L does not have a first or last element, since all its elements have character (κ, κ^*). So in forming $\text{Intalg}(L)$ we introduce a first element $-\infty$ to L. External to L we also have a largest element ∞. Since L has a subset of order type κ, clearly A has a partition of size κ, and on cardinality grounds it does not have any partitions of larger size. Suppose now that X is an infinite partition of A of size less than κ; we want to get a contradiction. We may assume that each element of X has the form $[a, b)$ with $a < b$.

(1) There is an element of X of the form $[-\infty, b)$.

Suppose not. Let $Y = \{a : [a, b) \in X\}$. Since $|X| < \kappa$, it follows that there is a $c \in L$ with $-\infty < c$ and $c < a$ for all $a \in Y$. Then $[-\infty, c) \cap [a, b) = \emptyset$ for all $[a, b) \in X$, contradiction.

So let b be such that $[-\infty, b) \in X$. By an argument similar to that proving (1), there is an element of X of the form $[b, c)$ with $b < c$. In fact, by induction we get elements $[d_i, d_{i+1}) \in X$ with $d_0 = b$, for each $i \in \omega$. Let $Y = \{[e, f) \in X : d_i < e$ for all $i \in \omega\}$. Since $\omega, |X| < \kappa$, it follows that there are elements $u < v$ in L such that $d_i < u$ for all $i \in \omega$ and $v < e$ for all $[e, f) \in Y$. Then $[u, v) \cap [e, f) = \emptyset$ for all $[e, f) \in X$, contradiction. □

Now if we apply Proposition 3.38 as in the proof of Proposition 3.42 we obtain the following:

Proposition 3.56. *If K is a set of regular cardinals with a largest element, then there is an atomless BA A such that $\mathfrak{a}_{\text{spect}}(A) = K$.*

There is an interesting result which comes up in considering cellularity and unions for complete BAs; this result is evidently due to Solovay, Tennenbaum [71]:

Theorem 3.57. *Let κ and λ be uncountable regular cardinals, and suppose that $\langle A_\alpha : \alpha < \lambda \rangle$ is an increasing sequence of complete BAs satisfying the κ-chain condition, such that A_α is a complete subalgebra of A_β for $\alpha < \beta < \lambda$, and for γ limit $< \lambda$, $\bigcup_{\alpha<\gamma} A_\alpha$ is dense in A_γ. Then $\bigcup_{\alpha<\lambda} A_\alpha$ also satisfies the κ-chain condition.*

Proof. By Theorem 3.16 we may assume that $\kappa = \lambda$. Let $B = \bigcup_{\alpha<\kappa} A_\alpha$. For each $\alpha < \kappa$ we define c_α mapping B into A_α by setting

$$c_\alpha(x) = \prod_{x \le a \in A_\alpha}^{A_\alpha} a.$$

Now, in order to get a contradiction, assume that X is a disjoint subset of B of size $\geq \kappa$. We may assume that X is maximal disjoint. Temporarily fix $\alpha < \kappa$. Now $\sum^B X = 1$, and hence $\sum^B \{c_\alpha(x) : x \in X\} = 1$. Hence clearly $\sum^{A_\alpha} \{c_\alpha(x) :$

$x \in X\} = 1$ too. Since A_α satisfies the κ-chain condition, choose $X_\alpha \subseteq X$ of size $< \kappa$ such that

(1) $\sum^{A_\alpha}\{c_\alpha(x) : x \in X_\alpha\} = 1$.

(See Lemma 3.12 of the Handbook.) Choose $\beta_\alpha < \kappa$ such that $X_\alpha \subseteq A_{\beta_\alpha}$; the ordinal β_α exists since $|X_\alpha| < \kappa$ and κ is regular.

Now unfix α. Define $\delta_0 = 0$ and if δ_n has been defined, let δ_{n+1} be an ordinal less than κ such that $\delta_n < \delta_{n+1}$ and $\beta_\alpha < \delta_{n+1}$ for all $\alpha \leq \delta_n$. Finally, let $\gamma = \sup_{n\in\omega} \delta_n$. Thus if $\alpha < \gamma$, then there is an $n \in \omega$ such that $\alpha < \delta_n$, and so $\beta_\alpha < \delta_{n+1} < \gamma$. Thus

(2) γ is a limit ordinal, and for any $\alpha < \gamma$ we have $\beta_\alpha < \gamma$.

We shall now prove that $X \subseteq A_\gamma$ (contradiction!).

Let $x \in X$ be arbitrary. Since $\bigcup_{\alpha<\gamma} A_\alpha$ is dense in A_γ, choose a non-zero $b \in \bigcup_{\alpha<\gamma} A_\alpha$ such that $b \leq c_\gamma(x)$. Say $b \in A_\alpha$ with $\alpha < \gamma$. By (1), choose $a \in X_\alpha$ such that $c_\alpha(a) \cdot b \neq 0$. If $b \cdot a = 0$, then $a \leq -b$ and hence $c_\alpha a \leq -b$ and so $c_\alpha a \cdot b = 0$, contradiction. Thus $b \cdot a \neq 0$, and so $c_\gamma(x) \cdot a \neq 0$. It follows that $x \cdot a \neq 0$, by the same argument as above. But both x and a are in X, so $x = a$. Thus $x \in X_\alpha \subseteq A_\gamma$, as desired. $\square$

Small cellularity is rather trivial for complete BAs: $\mathfrak{a}(A) = \omega$ for A complete, and

$$\mathfrak{a}_{\mathrm{spect}}(A) = \begin{cases} [\omega, \mathrm{c}(A)] & \text{if } \mathrm{c}(A) \text{ is attained,} \\ [\omega, \mathrm{c}(A)) & \text{otherwise.} \end{cases}$$

There is a large literature on cellularity for BAs of the form $\mathscr{P}(\kappa)/I$; for a start, see Baumgartner, J., Taylor, A., Wagon, S. [82]. Usually BA terminology is not used in such investigations; *saturation* of ideals is the term used.

Note that $\mathrm{c}(\mathscr{P}(\omega)/\mathrm{fin}) = 2^\omega$, and this value is attained; see the Handbook, Example 5.28. The function $\mathfrak{a}$ for this algebra has been carefully studied. The main results are that $\omega_1 \leq \mathfrak{a}(\mathscr{P}(\omega)/\mathrm{fin})$, Martin's axiom implies that $\mathfrak{a}(\mathscr{P}(\omega)/\mathrm{fin}) = 2^\omega$, and it is relatively consistent that $\mathfrak{a}(\mathscr{P}(\omega)/\mathrm{fin}) < 2^\omega$. The first two of these results will be proved at the end of Chapter 4.

If L is a linearly ordered set with first element, then clearly $\mathrm{c}(\mathrm{Intalg}(L))$ is the supremum of the cardinalities of sets of pairwise disjoint open intervals in L.

Theorem 3.58. *If A is an interval algebra, then $|A| \leq 2^{\mathrm{c}(A)}$.*

Proof. Let $A = \mathrm{Intalg}(L)$, and suppose that $|A| > 2^{\mathrm{c}(A)}$. Let $\prec$ be a well-order of L. Then

$$[L]^2 = \{\{a,b\} : a, b \in L, a <_L b, a \prec b\} \cup \{\{a,b\} : a, b \in L, a <_L b, b \prec a\}.$$

Applying the Erdős–Rado theorem we get a subset of L of size $(\mathrm{c}(A))^+$ which is well ordered or inversely well ordered. This yields a pairwise disjoint set of that size also, contradiction. $\square$

An interval algebra A such that $\mathfrak{a}(A) < \mathrm{c}(A)$ is provided by $\mathrm{Intalg}(\kappa \times \mathbb{Q})$, where κ is an uncountable cardinal and the product $\times$ uses the lexicographic order.

Clearly $\mathfrak{a}(A) \geq \kappa$ if $A = \mathrm{Intalg}(L)$ with L κ-saturated. This gives rise to the following problem.

Problem 22. *Are the following conditions equivalent, for any linearly ordered set L?*

(i) $\mathfrak{a}(\mathrm{Intalg}(L)) \geq \kappa$.
(ii) *L is κ-saturated.*

Another related problem, somewhat vague, is as follows.

Problem 23. *For each infinite cardinal κ, give a criterion, purely in terms of the linear order L, for there to exist a partition of size κ in* $\mathrm{Intalg}(L)$.

The cellularity of tree algebras has been described in Brenner [82]:

Theorem 3.59. *For $A = \mathrm{Treealg}(T)$, T a tree, $\mathrm{c}(A)$ is the maximum of $|\{t \in T : t$ has finitely many immediate successors$\}|$ and $\sup\{|X| : X$ is a collection of pairwise incomparable elements of $T\}$.*

Proof. If t has finitely many immediate successors, then $\{t\} \in A$. And if s and t are incomparable, then $(T \uparrow s) \cap (T \uparrow t) = 0$. Hence $\geq$ is clear. Now suppose that X is a collection of pairwise disjoint elements of A; we want to show that $|X|$ is $\leq$ the indicated maximum. Without loss of generality we may assume that each element $x \in X$ has the form $(T \uparrow t_x) \setminus \bigcup_{s \in F_x}(T \uparrow s)$, where F_x is a finite set of $s > t_x$. And we may assume that if t_x has only finitely many immediate successors, then $x = \{t_x\}$. Write $X = X_0 \cup X_1$, where X_0 is the set of singletons in X and $X_1 = X \setminus X_0$. Thus if $x \in X_1$, then t_x has infinitely many immediate successors. Therefore, if $x, y \in X_1$, $x \neq y$, then either t_x and t_y are incomparable, or $s \leq t_y$ for some $s \in F_x$, or $s \leq t_x$ for some $s \in F_y$. For each $x \in X_1$ let u_x be an immediate successor of t_x such that $u_x \not\leq s$ for all $s \in F_x$. Then it is easy to check that if x and y are distinct elements of X_1, then u_x and u_y are incomparable. This proves that $|X_1|$ is $\leq$ the sup mentioned in the theorem. □

This characterization does not work for pseudo-tree algebras: for example, if L is a dense linear order of size ω_1 with an increasing subset of order type ω_1, then $\mathrm{c}(\mathrm{Treealg}(L)) = \omega_1$ (recall that for L a linear order, $\mathrm{Treealg}(L) = \mathrm{Intalg}(L)$).

There are problems for tree algebras similar to Problems 10 and 24:

Problem 24. *If K is a nonempty set of infinite cardinals, is there a tree algebra A such that $\mathfrak{a}_{\mathrm{spect}}(A) = K$?*

Problem 25. *For each infinite cardinal κ, give a criterion, purely in terms of the tree T, for there to exist a partition of size κ in* $\mathrm{Treealg}(T)$.

A characterization of cellularity for pseudo-tree algebras was obtained by Brown [06], answering Problem 9 in Monk [96]. To describe it, we need some notation.

Let T be a pseudo-tree. A *fan element* of T is an element t such that there is a finite set F of pairwise incomparable elements of T greater than t, with $|F| \geq 2$, such that every element greater than t is comparable with exactly one $s \in F$. A *pure chain* of T is a chain C of size at least 2 such that for all $a, b \in C$ with $a < b$ and for all $c > a$, c is comparable with b. Then the characterization is as follows:

The cellularity of a pseudo-tree algebra is equal to the maximum of the following four cardinals:

(i) the supremum of the cellularities of $\mathrm{Intalg}(L)$ for chains $L \subseteq T$.
(ii) the supremum of the cardinalities of sets of pairwise incomparable elements of T.
(iii) the number of fan elements of T.
(iv) the number of maximal pure chains of T.

Moreover, there are examples showing that none of these four conditions is redundant; two of the examples involve consistency results, and such consistency results are really necessary.

Similarly to the above, we have the following problems.

Problem 26. *If K is a nonempty set of infinite cardinals, is there a pseudo-tree algebra A such that $\mathfrak{a}_{\mathrm{spect}}(A) = K$?*

Problem 27. *For each infinite cardinal κ, give a criterion, purely in terms of the pseudo-tree T, for there to exist a partition of size κ in* $\mathrm{Treealg}(t)$.

4 Depth

Recall that Depth(A) is the supremum of cardinalities of subsets of A which are well ordered by the Boolean ordering. There are two main references for results about this notion: McKenzie, Monk [82] and (implicitly) Grätzer, Lakser [69].

If A is a subalgebra of B, then clearly $\mathrm{Depth}(A) \leq \mathrm{Depth}(B)$. The difference can be arbitrarily large: consider $B = A \oplus C$. This same example shows that the difference can be arbitrarily large for $A \leq_{\mathrm{reg}} B$, $A \leq_{\mathrm{rc}} B$ or $A \leq_{\sigma} B$; see Proposition 11.8 in the Handbook, and the diagram concerning special subalgebras in Chapter 2.

Concerning $A \leq_{\pi} B$, note that if B is the interval algebra on κ and $A \leq_{\pi} B$ is the finite-cofinite algebra on κ, then $\mathrm{Depth}(B) = \kappa$, while $\mathrm{Depth}(A) = \omega$. It is possible to get atomless examples here. Namely, let $B = {}^{\kappa}\mathrm{Fr}(\omega)$ and let A be the weak power which is a subalgebra of B. It is easy to see, and is proved below, that $\mathrm{Depth}(B) = \kappa$, while $\mathrm{Depth}(A) = \omega$.

For $A \leq_{\mathrm{s}} B$, we can use the same idea as for cellularity to obtain a big difference.

For $\leq_m$ we have the following result.

Proposition 4.1. *If* $A \leq_m B$, *then* $\mathrm{Depth}(A) = \mathrm{Depth}(B)$.

Proof. Let $B = A(x)$. Since $\mathrm{Depth}(B) = \max(\mathrm{Depth}(B \upharpoonright x), \mathrm{Depth}(B \upharpoonright -x))$ (proved below), by symmetry it suffices to show that $\mathrm{Depth}(B \upharpoonright x) \leq \mathrm{Depth}(A)$. So, suppose that $\langle a_\alpha \cdot x : \alpha < \kappa \rangle$ is strictly increasing, where each $a_\alpha \in A$. Now we consider two cases.

Case 1. $|\{\alpha < \kappa : a_\alpha \in \mathrm{Smp}_x^A\}| = \kappa$. Then we may assume that $a_\alpha \in \mathrm{Smp}_x^A$ for all $\alpha < \kappa$. Write $a_\alpha = b_\alpha + c_\alpha$ with $b_\alpha \leq x$ and $c_\alpha \leq -x$. Then $a_\alpha \cdot x = b_\alpha$, so $\langle b_\alpha : \alpha < \kappa \rangle$ is a strictly increasing sequence of element os A.

Case 2. $|\{\alpha < \kappa : a_\alpha \in \mathrm{Smp}_x^A\}| < \kappa$. Then we may assume that $-a_\alpha \in \mathrm{Smp}_x^A$ for all $\alpha < \kappa$. For each $\alpha < \kappa$ write $-a_\alpha = u_\alpha + v_\alpha$ with $u_\alpha, v_\alpha \in A$ and $u_\alpha \leq x$, $v_\alpha \leq -x$. Then $-(a_\alpha \cdot x) = -a_\alpha + -x = u_\alpha + -x$, and it follows that $\langle u_\alpha : \alpha < \kappa \rangle$ is strictly decreasing. $\square$

Since every interval algebra is minimally generated (Proposition 2.52), it is clear that one can have $A \leq_{\mathrm{mg}} B$ with $\mathrm{Depth}(A)$ much less than $\mathrm{Depth}(B)$.

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Clearly $\mathrm{Depth}(A) = \mathrm{Depth}(B)$ if $A \leq_{\mathrm{free}} B$. Hence easily also $\mathrm{Depth}(A) = \mathrm{Depth}(B)$ if $A \leq_{\mathrm{proj}} B$.

Depth can increase arbitrarily when $A \leq_u B$; see the argument for cellularity.

If A is a homomorphic image of B, then depth can change in either way in going from B to A; see the argument for cellularity.

Now we turn to products. Some of the results which we shall present depend on the following simple lemma.

Lemma 4.2. *Let A and B be BAs, and let X be a chain in $A \times B$ of infinite cardinality κ. Then the projections of X are chains, and at least one of them has cardinality κ. Furthermore, if X has order type κ, then X has a subset of order type κ on which one of the two projections is one-one.*

Proof. For any $z \in X$ write $z = (z_0, z_1)$. For $i = 0, 1$ write $z \equiv_i w$ iff $z, w \in X$ and $z_i = w_i$. Now note that

$$\{\{x\} : x \in X\} \subseteq \{a \cap b : a \in X/\equiv_0,\ b \in X/\equiv_1\}\backslash\{0\};$$

hence one of the two equivalence relations $\equiv_0, \equiv_1$ has κ equivalence classes, and the lemma follows. □

Theorem 4.3. $\mathrm{Depth}(\prod_{i\in I} A_i) = \max(|I|, \sup_{i\in I}\mathrm{Depth}(A_i))$ *if each A_i is nontrivial and either I is infinite or some A_i is infinite.*

Proof. Clearly $\geq$ holds. Suppose $=$ fails to hold, and let f be an order isomorphism of κ^+ into $\prod_{i\in I} A_i$, where $\kappa = \max(|I|, \sup_{i\in I}\mathrm{Depth}(A_i))$. For each $i \in I$ there is an ordinal $\alpha_i < \kappa^+$ such that $(f(\alpha))_i = (f(\beta))_i$ for all $\beta > \alpha_i$. Let $\gamma = \sup_{i\in I}\alpha_i$. Then for all $\delta > \gamma$ we have $f(\delta) = f(\gamma)$, contradiction. □

Theorem 4.4. *Let $\kappa = \sup_{i\in I}\mathrm{Depth}(A_i)$, and suppose that κ is regular. Then the following conditions are equivalent:*

(i) $\mathrm{Depth}(\prod_{i\in I} A_i)$ *is not attained.*
(ii) $|I| < \kappa$, *and for all $i \in I$, A_i has no chain of order type κ.* □

The proof of this theorem is very similar to that of Theorem 4.3. The case of singular cardinals is a little more involved:

Theorem 4.5. *Let $\kappa = \sup_{i\in I}\mathrm{Depth}(A_i)$, and suppose that κ is singular. Then the following conditions are equivalent:*

(i) $\mathrm{Depth}(\prod_{i\in I} A_i)$ *is not attained.*
(ii) *These four conditions hold:*
  (a) $|I| < \kappa$.
  (b) *For all $i \in I$, A_i has no chain of type κ.*
  (c) $|\{i \in I : \mathrm{Depth}(A_i) = \kappa\}| < \mathrm{cf}(\kappa)$.
  (d) $\sup\{\mathrm{Depth}(A_i) : i \in I,\ \mathrm{Depth}(A_i) < \kappa\} < \kappa$.

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Proof. Let $\langle \mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ, with $\mu_0 = 0$.

(i)⇒(ii): (a) and (b) are clear. Suppose that (c) fails to hold; we show that (i) fails. Let i be a one-one function from $\mathrm{cf}(\kappa)$ into $\{i \in I : \mathrm{Depth}(A_i) = \kappa\}$. For each $\alpha < \mathrm{cf}(\kappa)$ let $\langle a_{i\beta} : \mu_\alpha \le \beta < \mu_{\alpha+1}\rangle$ be a strictly increasing sequence of elements of A_{i_α}. Now we define a sequence $\langle x_\beta : \beta < \kappa\rangle$ of elements of $\prod_{i\in I} A_i$. For each $\beta < \kappa$ choose $\alpha < \mathrm{cf}(\kappa)$ so that $\mu_\alpha \le \beta < \mu_{\alpha+1}$, and for any $j \in I$ set

$$x_\beta(j) = \begin{cases} 1 & \text{if } \ j = i_\gamma \text{ for some } \gamma < \alpha; \\ a_{i\beta} & \text{if } \ j = i_\alpha; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly this sequence is as desired.

Next we show that if (d) fails then (i) fails. By induction we can define i_α for $\alpha < \mathrm{cf}(\kappa)$ so that

$$\sup_{\beta<\alpha}(\mathrm{Depth}(A_{i_\beta}) \cup \mu_\beta) < \mathrm{Depth}(A_{i_\alpha}) < \kappa,$$

and then we can proceed as for (c).

(ii)⇒(i): Assume (ii), and suppose that $\langle x_\alpha : \alpha < \kappa\rangle$ is strictly increasing in $\prod_{i\in I} A_i$. Define

$$\begin{aligned} J_i &= \{\alpha < \kappa : x_\alpha(i) < x_{\alpha+1}(i)\} \text{ for } i \in I; \\ K &= \{i \in I : \mathrm{Depth}(A_i) = \kappa\}; \\ \lambda &= \sup\{\mathrm{Depth}(A_i) : i \in I,\ \mathrm{Depth}(A_i) < \kappa\}. \end{aligned}$$

Then by the above assumptions we have $\lambda < \kappa$, $|J_i| \le \lambda$ for all $i \in I\backslash K$, $|K| < \mathrm{cf}(\kappa)$, and $|J_i| < \kappa$ for all $i \in K$. It follows that $|\bigcup_{i\in I} J_i| < \kappa$. But for any $\alpha \in \kappa\backslash\bigcup_{i\in I} J_i$ we have $x_\alpha = x_{\alpha+1}$, contradiction. □

The above theorems completely describe the depth of products. The case of weak products is even simpler:

Theorem 4.6. *Let $\kappa = \sup_{i\in I}\mathrm{Depth}(A_i)$, and suppose that $\mathrm{cf}(\kappa) > \omega$. Then the following conditions are equivalent:*

(i) *$\prod^{\mathrm{w}}_{i\in I} A_i$ has no chain of order type κ.*
(ii) *For all $i \in I$, A_i has no chain of order type κ.*

Proof. (i)⇒(ii) is clear. (ii)⇒(i): Suppose that $\langle x_\alpha : \alpha < \kappa\rangle$ is strictly increasing in $\prod^{\mathrm{w}}_{i\in I} A_i$. For any $y \in \prod^{\mathrm{w}}_{i\in I} A_i$ let $S(y) = \{i \in I : y_i \ne 0\}$.

Case 1. $S(x_\alpha)$ is finite for all $\alpha < \kappa$. Since $\mathrm{cf}(\kappa) > \omega$, it follows that there is an $\alpha < \kappa$ such that $S(x_\alpha) = S(x_\beta)$ whenever $\alpha < \beta < \kappa$. But then Lemma 4.2 easily gives a contradiction.

Case 2. Otherwise we may assume that $\{i \in I : x_\alpha(i) \ne 1\}$ is finite for all $\alpha < \kappa$, and a contradiction is reached as in Case 1. □

Corollary 4.7. $\text{Depth}(\prod_{i\in I}^{\text{w}} A_i) = \sup_{i\in I}\text{Depth}(A_i)$. □

Now we shall show that $\text{Depth}(A)$ is attained if $\text{Depth}(A)$ is a successor cardinal or a cardinal of cofinality ω; otherwise, there are counterexamples.

Theorem 4.8. *If* $\text{cf}(\text{Depth}(A)) = \omega$, *then* $\text{Depth}(A)$ *is attained.*

Proof. Let $\kappa = \text{Depth}(A)$. We may assume that κ is an uncountable limit cardinal. Let $\langle \lambda_i : i < \omega\rangle$ be a strictly increasing sequence of cardinals with supremum κ, and with $\lambda_0 = 0$ and λ_1 infinite. Now we call an element a of A an ∞-*element* if λ_i is embeddable in $A \restriction a$ for all $i < \omega$. We claim

(*) If a is an ∞-element, and $a = b + c$ with $b \cdot c = 0$, then b is an ∞-element or c is an ∞-element.

In fact, by Lemma 4.2, for each $i < \omega$, λ_i is embeddable in $A \restriction b$ or $A \restriction c$, so (*) follows.

Using (*), we construct a sequence $\langle a_i : i < \omega\rangle$ of elements of A by induction. Suppose that a_j has been constructed for all $j < i$ so that $b \stackrel{\text{def}}{=} \prod_{j<i} -a_j$ is an ∞-element. Let $\langle c(\alpha) : \alpha < \lambda_{i+1}\rangle$ be an isomorphic embedding of λ_{i+1} into b. By (*), one of the elements $c(\lambda_i)$ and $b \cdot -c(\lambda_i)$ is an ∞-element, while clearly λ_i is embeddable in both of these elements. So we can choose $a_i \le b$ so that λ_i is embeddable in a_i, and $\prod_{j\le i} -a_j$ is an ∞-element. This finishes the construction.

For each $i < \omega$ let $\langle b_{i\alpha} : \alpha < \lambda_i\rangle$ be an embedding of λ_i into a_i. Note that $a_i \cdot a_j = 0$ for $i < j < \omega$. Hence the following sequence $\langle d_\alpha : \alpha < \kappa\rangle$ is clearly the desired embedding of κ into A. Given $\alpha < \kappa$, there is a unique $i < \omega$ such that $\lambda_i \le \alpha < \lambda_{i+1}$. We let $d_\alpha = a_0 + \cdots + a_i + b_{i+1,\alpha}$. □

Theorem 4.6 enables us to easily show that Theorem 4.8 is best possible: if κ is a limit cardinal with $\text{cf}(\kappa) > \omega$, then it is easy to construct a weak product B such that $\text{Depth}(B) = \kappa$ but depth is not attained in B. Namely suppose that $\langle \lambda_\xi : \xi < \text{cf}(\kappa)\rangle$ is strictly increasing with supremum κ. For each $\xi < \text{cf}(\kappa)$ let $A_\xi = \text{Intalg}(\lambda_\xi)$. Then $\prod_{\xi<\text{cf}(\kappa)}^{\text{w}} A_\xi$ is as desired.

Depth in free products is discussed thoroughly in McKenzie, Monk [82]. The main results of that paper are as follows.

- If $\text{cf}(\kappa) > \omega$, A has no chain of order type $\text{cf}(\kappa)$, and B has no chain of order type κ, then $A \oplus B$ has no chain of order type κ.
- If $\text{Depth}(B) = \kappa$ and A has a chain of order type $\text{cf}(\kappa)$, then $A \oplus B$ has a chain of order type κ.
- If κ is regular and uncountable, and for every $i \in I$ the BA A_i has no chain of order type κ, then also $\oplus_{i\in I} A_i$ has no chain of order type κ.
- $\text{Depth}(\oplus_{i\in I} A_i) = \sup_{i\in I} \text{Depth}(A_i)$ for any system $\langle A_i : i \in I\rangle$ of infinite BAs.

We give a proof of a special case of the first of these results, as it will be needed later.

Theorem 4.9. *If κ is an uncountable regular cardinal and neither A nor B has a chain of order type κ, then $A \oplus B$ also does not have such a chain.*

Proof. An element x of $A \oplus B$ is of *length* n if we can write $x = \sum_{i<n} a_i \cdot b_i$ where $\forall i < n[0 \neq a_i \in A$ and $0 \neq b_i \in B]$, and $b_i \cdot b_j = 0$ for distinct $i, j < n$, and x cannot be written in this way for any $m < n$. Now it suffices to show that for all n, $A \oplus B$ does not have a chain of order type κ in which all entries have length n. Suppose that n is minimum such that this is not true; we want to get a contradiction. Say $\langle x(\alpha) : \alpha < \kappa\rangle$ is a strictly increasing sequence of elements of $A \oplus B$, each $x(\alpha)$ being of length n, say $x(\alpha) = \sum_{i<n} a_i^\alpha \cdot b_i^\alpha$ with $\forall i < n[0 \neq a_i^\alpha \in A$ and $0 \neq b_i^\alpha \in B]$, and $b_i^\alpha \cdot b_j^\alpha = 0$ for distinct $i, j < n$.

If $n = 1$, then $\alpha < \beta < \kappa$ implies that $a_0^\alpha \cdot b_0^\alpha < a_0^\beta \cdot b_0^\beta$, hence $a_0^\alpha \leq a_0^\beta$ and $b_0^\alpha \leq b_0^\beta$. Since A does not have a chain of order type κ, it follows that there is a $\gamma < \kappa$ such that $a_0^\alpha = a_0^\gamma$ for all $\alpha > \gamma$. Similarly there is a $\delta < \kappa$ such that $b_0^\alpha = b_0^\delta$ for all $\alpha > \delta$. Then $x(\alpha) = x(\varepsilon)$ for all $\alpha, \varepsilon \geq \gamma, \delta$, contradiction.

Thus $n > 1$. Now for $\alpha < \beta < \kappa$ and $i < n$ we have $a_i^\alpha \cdot b_i^\alpha \leq \sum_{i<n} a_i^\beta \cdot b_i^\beta \leq \sum_{i<n} b_i^\beta$, and hence:

(1) If $\alpha < \beta < \kappa$ and $i < n$, then $b_i^\alpha \leq \sum_{i<n} b_i^\beta$.

Next we claim

(2) If $\alpha < \beta < \kappa$, $i, j < n$, and $b_i^\alpha \cdot b_j^\beta \neq 0$, then $a_i^\alpha \leq a_j^\beta$.

In fact, $\alpha < \beta < \kappa$ implies that $a_i^\alpha \cdot b_i^\alpha \cdot \prod_{k<n}(-a_k^\beta + -b_k^\beta) = 0$; multiplying by b_j^β gives $a_i^\alpha \cdot b_i^\alpha \cdot b_j^\beta \cdot -a_j^\beta = 0$, and the conclusion of (2) follows.

Now we claim

(3) There is a $g \in {}^\kappa n$ such that $a_{g(\alpha)}^\alpha \leq a_{g(\beta)}^\beta$ whenever $\alpha < \beta < \kappa$.

In fact, if $\alpha < \beta < \kappa$, then the set $\{g \in {}^\kappa n : a_{g(\alpha)}^\alpha \leq a_{g(\beta)}^\beta\}$ is closed in the space ${}^\kappa n$ (with the discrete topology on n and the product topology on ${}^\kappa n$). Thus by compactness of ${}^\kappa n$ it suffices to show that if $\alpha_1 < \cdots < \alpha_m < \kappa$ then there is a $g \in {}^\kappa n$ such that $a_{g(\alpha_1)}^{\alpha_1} \leq \cdots \leq a_{g(\alpha_m)}^{\alpha_m}$. This is clear by induction from (1) and (2).

Choose g as in (3). Then there is an $\alpha < \kappa$ such that $a_{g(\alpha)}^\alpha = a_{g(\beta)}^\beta$ for all $\beta > \alpha$. Now for each $\beta < \kappa$ let h_β be a permutation of n such that $h_\beta(0) = g(\alpha+\beta)$. Define for any $\beta < \kappa$ and $i < n$

$$c_i^\beta = a_{h_\beta(i)}^{\alpha+\beta} \quad \text{and} \quad d_i^\beta = b_{h_\beta(i)}^{\alpha+\beta};$$
$$y(\beta) = \sum_{i<n} c_i^\beta \cdot d_i^\beta.$$

Then for any $\beta < \kappa$ we have

$$y(\beta) = \sum_{i<n} c_i^\beta \cdot d_i^\beta = \sum_{i<n} a_{h_\beta(i)}^{\alpha+\beta} \cdot b_{h_\beta(i)}^{\alpha+\beta} = \sum_{i<n} a_i^{\alpha+\beta} \cdot b_i^{\alpha+\beta} = x(\alpha+\beta).$$

Moreover,

$$c_0^\beta = a_{h_\beta(0)}^{\alpha+\beta} = a_{g(\alpha+\beta)}^{\alpha+\beta} = a_{g(\alpha+\gamma)}^{\alpha+\gamma} = a_{h_\gamma(0)}^{\alpha+\gamma} = c_0^\gamma$$

for any $\beta < \gamma < \kappa$.

Now for each $\beta < \kappa$ the element $y(\beta) \cdot -c_0^0$ has length less than n, and $y(\beta) \cdot -c_0^0 \leq y(\gamma) \cdot -c_0^0$ for $\beta < \gamma$. Hence by the choice of n there is a $\beta < \kappa$ such that $y(\beta) \cdot -c_0^0 = y(\gamma) \cdot -c_0^0$ for all $\gamma > \beta$. Consequently, $y(\gamma) \cdot c_0^0 < y(\delta) \cdot c_0^0$ whenever $\beta \leq \gamma < \delta < \kappa$. Now for each $\gamma < \kappa$ and $i < n$ we define

$$e_i^\gamma = c_i^{\beta+\gamma} \cdot c_0^0 \quad \text{and} \quad z(\gamma) = \sum_{i<n} e_i^\gamma \cdot d_i^{\beta+\gamma}.$$

Note that $e_0^\gamma = c_0^0$ for all $\gamma < \kappa$. Now $\langle z(\gamma) : \gamma < \kappa \rangle$ is strictly increasing, and $e_i^\gamma \leq e_0^0$ for all $\gamma < \kappa$ and $i < n$. Each $z(\gamma)$ has length at most n, and so by the choice of n there are κ of them which do have length n. This gives a strictly increasing sequence $\langle \gamma_\xi : \xi < \kappa \rangle$ of ordinals less than κ such that each $z(\gamma_\xi)$ has length n.

Now we claim

(4) If $\xi < \eta < \kappa$ and $0 < i < n$, then $d_0^{\beta+\gamma_\xi} \cdot d_i^{\beta+\gamma_\eta} = 0$.

In fact, otherwise we have $b_{h_{\beta+\gamma_\xi}(0)}^{\alpha+\beta+\gamma_\xi} \cdot b_{h_{\beta+\gamma_\eta}(i)}^{\alpha+\beta+\gamma_\eta} \neq 0$, and hence by (2), $a_{h_{\beta+\gamma_\xi}(0)}^{\alpha+\beta+\gamma_\xi} \leq a_{h_{\beta+\gamma_\eta}(i)}^{\alpha+\beta+\gamma_\eta}$. It follows that $c_0^{\beta+\gamma_\xi} \leq c_i^{\beta+\gamma_\eta}$, so $c_0^0 = e_0^{\gamma_\xi} \leq e_i^{\gamma_\eta} \leq c_0^0$. It follows that $e_i^{\gamma_\eta} = e_0^{\gamma_\eta}$. But then $z(\gamma - \eta)$ has length less than n, since $e_0^{\gamma_\eta} \cdot d_0^{\beta+\gamma_\eta}$ and $e_i^{\gamma_\eta} \cdot d_i^{\beta+\gamma_\eta}$ can be combined. This contradiction proves (4).

(5) If $\xi < \eta < \kappa$, then $d_0^{\beta+\gamma_\xi} \leq d_0^{\beta+\gamma_\eta}$.

We have $d_0^{\beta+\gamma_\xi} = b_{h_\beta(0)}^{\alpha+\beta+\gamma_\xi}$ and $d_0^{\beta+\gamma_\eta} = b_{h_\beta(0)}^{\alpha+\beta+\gamma_\eta}$, so (5) follows from (1) and (4).

Now by (5) there is a $\theta < \kappa$ such that $d_0^{\beta+\gamma_\xi} = d_0^{\beta+\gamma_\theta}$ for all $\xi > \theta$. Thus for $\xi \geq \theta$ we have

$$z(\gamma_\xi) = c_0^0 \cdot d_0^{\beta+\gamma_\theta} + \sum_{0<i<n} e_i^{\gamma_\xi} \cdot d_i^{\beta+\gamma_\xi},$$

and $d_0^{\beta+\gamma_\theta} \cdot \sum_{0<i<n} e_i^{\gamma_\xi} \cdot d_i^{\beta+\gamma_\xi} = 0$, so

$$\left\langle \sum_{0<i<n} e_i^{\gamma_\xi} \cdot d_i^{\beta+\gamma_\xi} : \theta \leq \xi < \kappa \right\rangle$$

is strictly increasing with each entry of length less than n, contradiction. $\square$

Proposition 4.10. *There is a BA A such that* $\mathrm{c}(A) = \mathrm{Depth}(A) = \mathrm{Depth}(A \oplus A) < \mathrm{c}(A \oplus A)$.

Proof. By Lemma 3.2 and Corollary 3.11, let B be a BA such that $\mathrm{c}(B) < \mathrm{c}(B \oplus B)$. Say $\mathrm{c}(B) = \kappa$. Let $A = \mathrm{Intalg}(\kappa) \times B$. Since $B \oplus B$ can be isomorphically embedded in $A \oplus A$, it is clear that A satisfies the conditions of the proposition. □

We now briefly discuss depth and amalgamated free products. For the first result we need the following well-known set theoretical fact.

Lemma. *There is a system $\langle a_\xi : \xi < \omega_1 \rangle$ of infinite subsets of ω such that for all $\xi, \eta < \omega_1$, if $\xi < \eta$ then $a_\xi \backslash a_\eta$ is finite and $a_\eta \backslash a_\xi$ is infinite.*

Proof. We define $\langle a_\xi : \xi < \omega_1 \rangle$ by recursion. Let a_0 be the set of all even natural numbers. Now suppose that $\alpha < \omega_1$ and a_ξ has been defined for all $\xi < \alpha$ so that the following conditions hold:

(1) If $\xi < \eta < \alpha$, then $a_\xi \backslash a_\eta$ is finite and $a_\eta \backslash a_\xi$ is infinite.

(2) If F is a finite subset of α, then $\omega \backslash \bigcup_{x \in F} a_\xi$ is infinite.

Let $\langle b_i : i < \omega \rangle$ enumerate $\{a_\xi : \xi < \alpha\}$, possibly with repetitions. Now for each $i < \omega$ we define j_i, k_i recursively to be any two distinct members of the set

$$\omega \backslash \left(\bigcup_{s<i} b_s \cup \{j_s, k_s : s < i\} \right).$$

Then let $a_\alpha = \omega \backslash \{j_i : i \in \omega\}$. Now if $s < i$ then $j_i \notin b_s$, and hence $b_s \backslash a_\alpha \subseteq \{j_i : i \leq s\}$, so that $b_s \backslash a_\alpha$ is finite. Also, if $s < i$ then $k_i \notin b_s$, so $a_\alpha \backslash b_s \supseteq \{k_i : s < i\}$, so that $a_\alpha \backslash b_s$ is infinite. Finally, for any s we have $\{j_i : i > s\} \subseteq \omega \backslash \left(\bigcup_{i<s} b_i \cup a_\alpha \right)$. Hence the conditions (1) and (2) hold for $\alpha + 1$. Inductively, they also hold for limit α. □

The following theorem is a special case of a theorem in McKenzie, Monk [82].

Theorem 4.11. *Let A be the BA of finite and cofinite subsets of ω. Then there exist $B, C \geq A$ both satisfying ccc such that* $\mathrm{Depth}(B \oplus_A C) = \omega_1$.

Proof. Let M be the collection of all even integers, N the set of all odd integers. Then we take two sequences $\langle a_\alpha : \alpha < \omega_1 \rangle$ and $\langle b_\alpha : \alpha < \omega_1 \rangle$ such that

(1) Each a_α is an infinite subset of M, and for $\alpha < \beta < \omega_1$ we have $a_\alpha \backslash a_\beta$ finite and $a_\beta \backslash a_\alpha$ infinite.

(2) Similarly for the b_α's, subsets of N.

These sequences exist by the lemma. Now let $B = C = A \times \mathscr{P}(\omega)$. For each $a \in A$ let $g(a) = (a, a)$. Then g is an isomorphism of A into $B = C$. So it is enough to prove that $\mathrm{Depth}(B \oplus_{g[A]} C) = \omega_1$. For each $\alpha < \omega_1$ let $c_\alpha = (0, a_\alpha) \times (0, b_\alpha)$ [using

$\times$ rather than $\cdot$ to indicate which one is in B and which one in C]. We claim that $\langle c_\alpha : \alpha < \omega_1 \rangle$ is as desired. Let $\alpha < \beta < \omega_1$. Then

$$\begin{aligned} c_\alpha \cdot -c_\beta &= [((0, a_\alpha) \cdot (1, -a_\beta)) \times (0, b_\alpha)] + [(0, a_\alpha) \times ((0, b_\alpha) \cdot (1, -b_\beta))] \\ &= [(0, a_\alpha \backslash a_\beta) \times (0, b_\alpha)] + [(0, a_\alpha) \times (0, b_\alpha \backslash b_\beta)]. \end{aligned}$$

Now $d \stackrel{\text{def}}{=} a_\alpha \backslash a_\beta$ is a finite subset of M, so $d \in A$. And $(0, a_\alpha \backslash a_\beta) \leq (d, d)$, while $(0, b_\alpha) \cdot (d, d) = (0, 0)$. Using a similar argument for the second summand, this shows that $c_\alpha \cdot -c_\beta = 0$.

Now suppose that $\alpha < \beta$ and $c_\beta \cdot -c_\alpha = 0$. So $(0, a_\beta \backslash a_\alpha) \times (0, b_\beta) = 0$, hence there is a $d \in A$ such that $(0, a_\beta \backslash a_\alpha) \leq (d, d)$ and $(0, b_\beta) \cdot (d, d) = (0, 0)$. Now $a_\beta \backslash a_\alpha$ is infinite, so d is cofinite. Hence $b_\beta \cap d \neq 0$, contradiction. □

The following theorem solves Problem 2 of McKenzie, Monk [82]:

Theorem 4.12. *Let A be the BA of finite and cofinite subsets of ω, and let κ be an uncountable cardinal. Then there exist $B, C \geq A$ such that $|B| = |C| = \kappa$,* $\mathrm{Depth}(B) = \mathrm{Depth}(C) = \omega$, *and* $\mathrm{Depth}(B \oplus_A C) \geq \omega_1$.

Proof. First we choose B and C as in the proof of Theorem 4.11. In particular, $B \oplus_A C$ has a chain of the form $\langle b_\alpha \times c_\alpha : \alpha < \omega_1 \rangle$, while B and C have size 2^ω. Clearly we may assume that $|B| = |C| = \omega_1$. Let $B' = B \times \mathrm{Finco}(\kappa)$ and $C' = C \times \mathrm{Finco}(\kappa)$. Thus $|B'| = |C'| = \kappa$. Set

$$A' = \{(a, 0) : a \in [\omega]^{<\omega}\} \cup \{(a, 1) : \omega \backslash a \in [\omega]^{<\omega}\}.$$

Clearly A' is isomorphic to A. To prove the theorem it suffices to show that $B' \oplus_{A'} C'$ has depth ω_1. Let $b'_\alpha = (b_\alpha, 0)$, $c'_\alpha = (c_\alpha, 0)$, and $d_\alpha = b'_\alpha \times c'_\alpha$ for all $\alpha < \omega_1$. We claim that $\langle d_\alpha : \alpha < \omega_1 \rangle$ is a chain in $B' \oplus_{A'} C'$, as desired. To prove this, suppose that $\alpha < \beta$. Choose $u, v \in A$ such that $b_\alpha \cdot -b_\beta \leq u$, $c_\alpha \cap u = 0$, $b_\alpha \cdot v = 0$, and $c_\alpha \cdot -c_\beta \leq v$. Now

$$d_\alpha \cdot -d_\beta = (b'_\alpha \cdot -b'_\beta) \times c'_\alpha + b'_\alpha \times (c'_\alpha \cdot -c'_\beta).$$

Note that $(b'_\alpha \cdot -b'_\beta) \times c'_\alpha = (b_\alpha \cdot -b_\beta, 0) \times (c_\alpha, 0)$. Then for some $\varepsilon \in \{0, 1\}$ we have $(u, \varepsilon) \in A'$, $(b_\alpha \cdot -b_\beta, 0) \leq (u, \varepsilon)$, and $(u, \varepsilon) \cdot (c_\alpha, 0) = (0, 0)$. Therefore $(b'_\alpha \cdot -b'_\beta) \times c'_\alpha = 0$. Similarly for the other summand, so $d_\alpha \cdot -d_\beta = 0$. Suppose that also $d_\beta \cdot -d_\alpha = 0$. This easily gives $(b_\beta \cdot -b_\alpha, 0) \times (c_\beta, 0) = (0, 0)$. Hence there is a $(w, \varepsilon) \in A'$ such that $(b_\beta \cdot -b_\alpha, 0) \leq (w, \varepsilon)$ and $(c_\beta, 0) \cdot (w, \varepsilon) = (0, 0)$. Hence $b_\beta \cdot -b_\alpha \leq w$ and $c_\beta \cdot w = 0$, contradiction. □

The following result of Shelah [02] (Remark 1.2, 4) solves Problem 10 in Monk [96]:

There is a countable BA A such that for every strong limit cardinal μ of cofinality $\aleph_0$ there exist $B, C \geq A$ such that $\mathrm{Depth}(B), \mathrm{Depth}(C) \leq \mu$ *while* $\mathrm{Depth}(B \oplus_A C) \geq \mu^+$.

Some results in Shelah [02] partially solve Problem 11 of Monk [96]. The easiest to state is the following part of Observation 1.8:

If λ is weakly compact, $A \leq B, C$, $|A| < \lambda$, and $\mathrm{Depth}'(B \oplus_A C) = \lambda^+$, *then* $\mathrm{Depth}'(B) = \lambda^+$ *or* $\mathrm{Depth}'(C) = \lambda^+$.

We give a proof of this.

Theorem 4.13. *If κ is weakly compact, $A \leq B, C$, $|A| < \kappa$, and $B \oplus_A C$ has a chain with order type κ, then B or C has such a chain.*

Proof. Say $\langle d_\alpha : \alpha < \kappa\rangle$ is strictly increasing in $D \overset{\text{def}}{=} B \oplus_A C$. Now for all $\alpha < \kappa$ the set $\{-d_\beta : \beta < \alpha\} \cup \{d_\alpha\}$ has fip, and so is contained in an ultrafilter E_α of D. Since $|\mathscr{P}(A)| < \kappa$, there exist a subset Γ of κ of size κ and an ultrafilter E^* on A such that $E_\alpha \cap A = E^*$ for all $\alpha \in \Gamma$. Let $I^* = \{a \in A : -a \in E^*\}$; this is a maximal ideal of A. Let $J = \langle I^*\rangle^{\mathrm{Id}}_B$, $K = \langle I^*\rangle^{\mathrm{Id}}_C$, and $K = \langle I^*\rangle^{\mathrm{Id}}_D$.

(1) $\langle [d_\alpha]_L : \alpha \in \Gamma\rangle$ is strictly increasing.

In fact, clearly $[d_\alpha]_L \leq [d_\beta]_L$ if $\alpha < \beta$, both in Γ. Now suppose that $\alpha, \beta \in \Gamma$, $\alpha < \beta$, and $[d_\beta]_L \leq [d_\alpha]_L$. It follows that there is an $a \in I^*$ such that $d_\beta \cdot -d_\alpha \leq a$. Hence $d_\beta \cdot -a \cdot -d_\alpha = 0$. So there is an $f \in A$ such that $d_\beta \cdot -a \leq f$ and $-d_\alpha \leq -f$.

Case 1. $f \in I^*$. Then $d_\beta \leq a + f \in I^*$. Hence $-a \cdot -f \in E^* = E_\beta \cap A$. Also $d_\beta \in E_\beta$, so $d_\beta \cdot -a \cdot -f \neq 0$, contradiction.

Case 2. $-f \in I^*$. Then $f \in E^* = E_\beta \cap A$. Also $-d_\alpha \in E_\beta$, so $-d_\alpha \cdot f \neq 0$, contradiction.

Thus (1) holds.

(2) $D/L \cong B/J \oplus C/K$.

In fact, it suffices to show that if $b \in B\backslash J$ and $c \in C\backslash K$ then $b \cdot c \notin L$. Suppose to the contrary that $b \cdot c \in L$. Say $b \cdot c \leq e \in I^*$. Then $b \cdot -e \cdot c = 0$, so there is an $f \in A$ such that $b \cdot -e \leq f$ and $c \leq -f$.

Case 1. $f \in I^*$. Then $b \leq e + f \in I^*$, so $b \in J$, contradiction.

Case 2. $-f \in I^*$. Then $c \in K$, contradiction.

Thus (2) holds.

Now by (1), (2), and Theorem 4.9, B/J or C/K has a chain of order type κ. Say by symmetry that $\langle [e_\alpha]_J : \alpha < \kappa\rangle$ is strictly increasing in B/J. Thus for $\alpha < \beta$ we have $e_\alpha \cdot -e_\beta \in J$ while $e_\beta \cdot -a_\alpha \notin J$. Say $e_\alpha \cdot -e_\beta \leq a_{\alpha\beta} \in I^*$. Thus $e_\alpha \cdot -a_{\alpha\beta} \leq e_\beta \cdot -a_{\alpha\beta}$ while $e_\beta \cdot -e_\alpha \cdot -a_{\alpha\beta} \neq 0$. Let $f(\{\alpha, \beta\}) = a_{\alpha\beta}$. Since κ is weakly compact, there exist a $\Gamma \in [\kappa]^\kappa$ and an $a \in A$ such that $a_{\alpha\beta} = a$ for all $\alpha, \beta \in \Gamma$ such that $\alpha < \beta$. Thus $\langle e_\alpha \cdot -a : \alpha \in \Gamma\rangle$ is strictly increasing. $\square$

The following form of Problem 11 of Monk [96] remains open.

Problem 28. *Find necessary and sufficient conditions on a BA A for there to exist extensions B, C of A such that* $\mathrm{Depth}'(B \oplus_A C) > \max(\mathrm{Depth}'(B), \mathrm{Depth}'(C))$.

Concerning unions, we note that Depth is an ordinary sup function with respect to the function P, where $P(A) = \{X \subseteq A : X \text{ is a well-ordered chain in } A\}$, and so Theorem 3.16 applies.

For ultraproducts the situation is similar to that for cellularity. It is easy to see that if F is a countably complete ultrafilter on an infinite set I and A_i is a BA with depth ω for each $i \in I$, then $\prod_{i\in I} A_i/F$ has depth ω. The following problem has not been considered:

Problem 29. *Determine the possibilities for* $\mathrm{Depth}(\prod_{i\in I} A_i/F)$ *in terms of* $\langle \mathrm{Depth}(A_i) : i \in I\rangle$ *with F countably complete.*

If F is a countably incomplete ultrafilter on I and each algebra is infinite, then $\prod_{i\in I} A_i/F$ has depth $> \omega$. This is easiest to see by recalling that $\prod_{i\in I} A_i/F$ is ω_1-saturated, and noting

$(*)$ *If an infinite BA A is κ-saturated, then A has a chain of order type κ.*

To prove $(*)$, we construct $a \in {}^\kappa A$ by recursion. Suppose that a_β has been defined for all $\beta < \alpha$, so that if β is a successor ordinal $\gamma + 1$, then $A \restriction -a_\gamma$ is infinite. If β is a successor ordinal, it is clear how to proceed in order to still have the indicated condition. If β is limit, consider the set

$$\{\mathbf{c}_{x_\alpha} < v_0 : \alpha < \beta\} \cup \{\text{ “there are at least } n\text{”} v_1(v_0 < v_1) : n \in \omega\}.$$

This set is finitely satisfiable in A, and so an element satisfying all of these formulas gives the desired element a_β.

Now we consider regular ultrafilters. The first result follows easily from a theorem of W. Hodges, that if F is a regular ultrafilter on I then in ${}^I\langle\omega, >\rangle/F$ there is a chain of order type $|I|^+$. We give a direct BA proof of the BA result:

Theorem 4.14. *Let F be a $|I|$-regular ultrafilter on I, and suppose that A_i is an infinite BA for every $i \in I$. Then in $\prod_{i\in I} A_i/F$ there is a chain of order type $|I|^+$.*

Proof. For brevity set $\kappa = |I|$. By the definition of regularity choose $E \subseteq F$ such that $|E| = \kappa$ and for all $i \in I$ the set $\{e \in E : i \in e\}$ is finite. Let G be a one-one function from E onto κ. For each $i \in I$ choose a strictly increasing sequence $\langle x_{ij} : j < \omega\rangle$ in A_i, and let $X_i = \{x_{ij} : j < \omega\}$. Then it suffices to show:

$(*)$ If $g_\alpha \in \prod_{i\in I} X_i$ for all $\alpha < \kappa$, then there is an $f \in \prod_{i\in I} X_i$ such that $g_\alpha/F < f/F < 1$ for all $\alpha < \kappa$.

To define f, let $i \in I$. Let $e(1), \ldots, e(m)$ be all of the elements u of E such that $i \in u$. Then let $f(i)$ be any element of X_i greater than all of the elements $g_{G(e(1))}(i), \ldots, g_{G(e(m))}(i)$. This defines f. Now if $\alpha < \kappa$ and $i \in G^{-1}(\alpha)$, we have $g_\alpha(i) < f(i) < 1$, as desired. $\square$

Theorem 4.15. *Let I be an infinite set, and suppose that A_i is an infinite BA for every $i \in I$. Then there is a proper filter G on I such that G contains all cofinite sets, and $\prod_{i\in I} A_i/F$ has a chain of order type $2^{|I|}$ for every ultrafilter F including G.*

Proof. Again let $\kappa = |I|$. Let $S \subseteq {}^\kappa\omega$ satisfy the following condition:

(1) $|S| = 2^\kappa$, and for every finite sequence $i_0, \ldots, i_{k-1}$ of natural numbers and every sequence $f_0, \ldots, f_{k-1}$ of distinct members of S of length k, there is an $\alpha < \kappa$ such that $f_t(\alpha) = i_t$ for all $t < k$.

For the existence of such a set, see Comfort, Negrepontis [74], pp. 75–77. Let $\langle f_\alpha : \alpha < 2^\kappa\rangle$ enumerate S without repetitions. For $\alpha < \beta < 2^\kappa$, let $J_{\alpha\beta} = \{\gamma < \kappa : f_\alpha(\gamma) < f_\beta(\gamma)\}$. From (1) it is clear that the intersection of any finite number of the sets $J_{\alpha\beta}$ is infinite. Hence

$$\{J_{\alpha\beta} : \alpha < \beta < 2^\kappa\} \cup \{\Gamma \subseteq \kappa : |\kappa\backslash\Gamma| < \omega\}$$

generates a proper filter G containing all cofinite sets. Clearly G is as desired. □

Now we give some results of Douglas Peterson; see Peterson [97]. Recall the notion of essential supremum, ess.sup., from just before Lemma 3.17.

Theorem 4.16. *Suppose that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and F is an ultrafilter on I. Then* $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \mathrm{ess.sup}^F_{i\in I}(\mathrm{Depth}(A_i))$.

Proof. For any linearly ordered set L let $\mathrm{Depth}(L)$ be the supremum of the size of well-ordered subsets of L. Let $\lambda = \mathrm{ess.sup}^F_{i\in I}(\mathrm{Depth}(A_i))$. If λ is a successor cardinal, then $\{i \in I : \mathrm{Depth}(A_i) = \lambda\} \in F$, and hence clearly $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \mathrm{Depth}({}^I\lambda/F) \geq \lambda$, as desired. If λ is a limit ordinal, then by similar reasoning, $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \kappa$ for every successor cardinal $\kappa < \lambda$, and so also $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \lambda$. □

Theorem 4.17. *Suppose that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and that F is a regular ultrafilter on I. Let $\lambda = \mathrm{ess.sup}^F_{i\in I}(\mathrm{Depth}(A_i))$, and assume that* $\mathrm{cf}(\lambda) \leq |I| < \lambda$. *Then* $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \lambda^+$.

Proof. Case 1. $\{i \in I : \mathrm{Depth}(A_i) = \lambda\} \in F$. We may assume that $\mathrm{Depth}(A_i) = \lambda$ for all $i \in I$. By Lemma 3.18 we get a system $\langle \kappa_i : i \in I\rangle$ of infinite cardinals such that $\kappa_i < \lambda$ for all $i \in I$, and $\mathrm{ess.sup}^F_{i\in I}(\kappa_i) = \lambda$. Let $\delta_i = \kappa_i^+$ for all $i \in I$. Then $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \mathrm{Depth}\left(\prod_{i\in I} \delta_i/F\right)$, so it suffices to show that $\mathrm{Depth}\left(\prod_{i\in I} \delta_i/F\right) \geq \lambda^+$. Suppose that $\{f_\alpha/F : \alpha < \lambda\}$ is a set of elements of $\prod_{i\in I} \delta_i/F$; we shall find an element $f \in \prod_{i\in I} \delta_i$ such that $f/F > f_\alpha/F$ for all $\alpha < \lambda$, and this will clearly finish the proof. Let $i \in I$. Then $\{f_\alpha(i) : \alpha < \kappa_i\}$ is not cofinal in δ_i, so we can let $f(i)$ be an element of δ_i greater than each $f_\alpha(i)$, $\alpha < \kappa_i$. Then for any $\alpha < \lambda$ we have $\{i \in I : f(i) > f_\alpha(i)\} \supseteq \{i \in I : \kappa_i > \alpha\} \in F$, so $f/F > f_\alpha/F$, as desired.

Case 2. $\{i \in I : \mathrm{Depth}(A_i) < \lambda\} \in F$. Then we can assume that $\mathrm{Depth}(A_i) < \lambda$ for all $i \in I$. Then by Lemma 3.18 there is a system $\langle \kappa_i : i \in I \rangle$ of infinite cardinals such that $\kappa_i < \mathrm{Depth}(A_i)$ for all $i \in I$, and $\mathrm{ess.sup}^F_{i\in I}(\kappa_i) = \lambda$. Let $\delta_i = \kappa_i^+$. Note that A_i has a well-ordered subset of size δ_i. Hence the procedure of Case 1 can be used. □

Theorem 4.18 (GCH)**.** *Suppose that $\langle A_i : i \in I \rangle$ is a system of infinite BAs, with I infinite, and F is a regular ultrafilter on I. Then* $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Depth}(A_i)/F\right|$.

Proof. Let $\lambda = \mathrm{ess.sup}^F_{i\in I}\mathrm{Depth}(A_i)$. Then we consider three cases:

Case 1. $\lambda \leq |I|$. Then $\lambda^{|I|} = 2^{|I|} = |I|^+$, and this case follows from Theorem 4.15.

Case 2. $\mathrm{cf}(\lambda) \leq |I| < \lambda$. Then $\lambda^{|I|} = \lambda^+$, and the result follows from Theorem 4.17.

Case 3. $|I| < \mathrm{cf}(\lambda)$. Then $\lambda^{|I|} = \lambda$ and we are through by Theorem 4.16. □

By a result of Donder [88], every uniform ultrafilter is regular in the core model. Thus by Theorem 4.18 it is consistent that

$$\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Depth}(A_i)/F\right|$$

always holds. The question arises to find an example with $>$ here. This was done in Shelah [02] (publication 652), assuming $V = L$ or actually something weaker.

The following is Problem 12 of Monk [96]; but see Shelah [05].

Problem 30. *Is an example with* $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) > \left|\prod_{i\in I} \mathrm{Depth}(A_i)/F\right|$ *possible in ZFC?*

We turn to the other direction. First we have:

Theorem 4.19. *Suppose that $\langle A_i : i \in I \rangle$ is a system of infinite BAs, I infinite, F is a uniform ultrafilter on I, and $\kappa = \max(|I|, \mathrm{ess.sup}^F_{i\in I}(\mathrm{Depth}(A_i)))$.*

Then $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) \leq 2^\kappa$.

Proof. Suppose that $\mathrm{Depth}\left(\prod_{i\in I} A_i/F\right) > 2^\kappa$. Choose $b \in F$ so that

$$\sup_{i\in b} \mathrm{Depth}(A_i) = \mathrm{ess.sup}^F_{i\in I}(\mathrm{Depth}(A_i)).$$

Suppose that $\langle [f_\alpha] : \alpha < (2^\kappa)^+ \rangle$ is strictly increasing in $\prod_{i\in I} A_i/F$. For $\alpha < \beta$ let $F(\{\alpha, \beta\})$ be the least $i \in b$ such that $f_\alpha(i) < f_\beta(i)$. By the Erdős–Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ let M be a subset of $(2^\kappa)^+$ of size κ^+ such that for some $i \in I$, $F(\{\alpha, \beta\}) = i$ for all $\alpha < \beta$, both in M. Since $\mathrm{Depth}(A_i) \leq \kappa$, this is a contradiction. □

Concerning Theorem 4.18, it is consistent to have inequality in the other direction; this is a result of Shelah [90] which also solves Problem 4 of Monk [90]. The proof is very similar to the proof of Theorem 1.5.8 in McKenzie, Monk [82] (also due to Shelah). Note that the theorem says that it is consistent to have a BA A such that $\mathrm{Depth}({}^{\omega}A/F) < |{}^{\omega}\mathrm{Depth}(A)/F|$. We give a complete proof here, except for an essential use of a result of Laver. For forcing notation we follow Kunen [80].

Theorem 4.20. *Suppose M is a countable transitive model of* ZFC+CH*, let κ be any regular uncountable cardinal of M, let P be the partial order for adding κ Sacks reals side-by-side (explained below), and let G be P-generic over M. Then in $M[G]$ cardinals are preserved, $2^{\omega} \geq \kappa$, and there is a nonprincipal ultrafilter F on ω such that* $\mathrm{Depth}({}^{\omega}A/F) = \omega_1$*, where A is the BA of finite and cofinite subsets of ω.*

Note by Theorem 3.19 that $|{}^{\omega}\mathrm{Depth}(A)/F| \geq 2^{\omega}$, so this theorem gives the desired result.

Proof. A *perfect tree* is a nonempty subset T of ${}^{<\omega}2$ such that if $t \in T$ and m is smaller than the domain of t then $t \restriction m \in T$, and such that for any $t \in T$ there is some $s \in T$ with $t \subseteq s$ such that $s0, s1 \in T$. We write $p \leq q$ in place of $p \subseteq q$ for perfect trees p, q. A *branching point* of p is a point $t \in p$ such that $t0 \in p$ and $t1 \in p$. An nth *branching point* is a branching point t such that there are exactly n branching points $< t$. Thus

(1) A 0th branching point is a branching point $t \in p$ such that there is no branching at all from the root $\emptyset$ up to t.

(2) For any $s \in p$ with at most n branching points below s there is an nth branching point t of p such that $s \leq t$.

For perfect trees p and q, $p \leq_n q$ means that $p \subseteq q$ and every nth branching point of q is a branching point of p. Clearly then

(3) If $p \leq_n q$, then $F \subseteq p$, where

$$F = \{t \in q : t \subseteq s\varepsilon \text{ for some } n\text{th branching point } s \text{ of } q \text{ and some } \varepsilon \in 2\}.$$

(4) If $p \leq_n q$, then $p \leq_i q$ for every $i \leq n$.

We also note:

(5) If $p \leq q$ and $n \in \omega$, then there is an nth branching point t of q such that $t \in p$.

For, let s be an nth branching point of p. Then it is an mth branching point of q for some $m \geq n$. Let $t \leq s$ be an nth branching point of q. Thus $t \in p$, as desired.

(6) If $p \leq_n q$, then every nth branching point of q is an nth branching point of p.

Thus $p \leq_n q$ means that $p \subseteq q$, and any points of q thrown away to get p have more than n branching points strictly below them.

A *fusion sequence* is a sequence such that

$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \cdots \geq_{n-1} p_n \geq_n \cdots$$

Lemma 4.21 (Fusion lemma). *If $\langle p_n : n \in \omega\rangle$ is a fusion sequence, then $p \stackrel{\text{def}}{=} \bigcap_{n\in\omega} p_n$ is a perfect tree, and $p \leq_n p_n$ for all $n \in \omega$.*

Proof. Let $n \in \omega$, and let s be an nth branching point of p_n. If $n \leq m$, then $p_n \geq_n p_m$, and so s is a branching point of p_m, so that $s, s0, s1 \in p_m$. Hence $s, s0, s1 \in p$, and s is a branching point of p.

Thus we just need to see that p is a perfect tree. If $t \in p$ and $m < \mathrm{dmn}(t)$, then obviously $t \restriction m \in p$.

Now suppose that $s \in p$; we want to find $t \geq s$ such that $t0, t1 \in p$. Let $m = \mathrm{dmn}(s)$. Choose an mth branching point t of p_m with $s \leq t$. By the first paragraph of this proof we have $t, t0, t1 \in p$, as desired. □

If p is a perfect tree and $t \in p$, we define

$$p \restriction t = \{u \in p : u \text{ and } t \text{ are comparable}\}.$$

So $p \restriction t$ is a perfect tree; it does not branch up to the height of t, and from t and above it is the tree p.

Now let p be a perfect tree, s an nth branching point of p, and t one of the immediate successors of s in p. Suppose that $q \leq p \restriction t$. Then

$$r \stackrel{\text{def}}{=} q \cup \{u \in p : u \text{ and } t \text{ are incomparable}\}$$

is a perfect tree called the *amalgamation* of q into p at t. It is obtained from p by replacing the part of p above t by the part of q above t. Clearly $r \leq_n p$. Also note that $(r \restriction t) = q$.

Let Q be the collection of all perfect trees. Q is called *Sacks forcing*. Its greatest element is ${}^{<\omega}2$, the full binary tree of height ω. The partial order P that we are concerned with is the σ-product of κ copies of Q; it consists of all $p \in {}^{\kappa}Q$ such that $p(\alpha) = 1$ for all but countably many $\alpha \in \kappa$, where 1 is the full binary tree of height ω. The *support* of an element $p \in P$ is the set of all $\alpha \in \kappa$ such that $p(\alpha) \neq 1$; it is denoted by $\mathrm{supp}(p)$. For $p, q \in P$ we define $p \leq q$ iff $p(\alpha) \subseteq q(\alpha)$ for all $\alpha < \kappa$.

Lemma 4.22. *Assuming CH, P has the ω_2-chain condition.*

Proof. Let $\langle p_\alpha : \alpha < \omega_2\rangle$ be a sequence of members of P; we want to find distinct $\alpha, \beta < \omega_2$ such that p_α and p_β are compatible. By CH, we may assume that $\langle \mathrm{supp}(p_\alpha) : \alpha < \omega_2\rangle$ forms a Δ-system, say with kernel M. Now by CH there are at most ω_1 perfect trees, so there are at most ω_1 functions from M into the collection of all perfect trees. Hence we may assume that $p_\alpha \restriction M = p_\beta \restriction M$ for all $\alpha, \beta < \omega_2$. Then any distinct $\alpha, \beta < \omega_2$ work. □

The following lemma is essentially Theorem 5.12 of Jech [86], but we give a proof of it. The notation introduced in the proof will be used later.

Lemma 4.23. *Let M be a countable transitive model of* ZFC*, and in M let P be the σ-product of κ-many Sacks forcings, where κ is any infinite cardinal. Suppose that $B\in M$, $p\in P$, $\dot X$ is a P-name, and $p\Vdash\dot X:\omega\to\check B$. Then there is a countable $A\in M$ and a $p_\infty\le p$, $p_\infty\in P$, such that $p_\infty\Vdash\dot X:\omega\to\check A$.*

Proof. We assume given a well-ordering of all objects that play a role in this proof. This is so we can make the construction very definite, implicitly choosing the "first" object when we make an arbitrary choice. We need an auxiliary function $g:\omega\to\omega\times\omega$; it is the standard bijection. Let $g(0)=(0,0)$. If $g(n)$ has been defined, say $g(n)=(i,j)$, let

$$g(n+1)=\begin{cases}(i+1,j-1), & \text{if } j\ne 0;\\ (0,i+1), & \text{otherwise.}\end{cases}$$

Then g maps onto $\omega\times\omega$, and if $g(n)=(i,j)$, then $i\le n$.

We construct a sequence $p=p_0\ge p_1\ge p_2\ge\cdots$ of members of P, and finite sets A_i,S_i,F_i,G_{ij} for $i,j\in\omega$. Let $p_0=p$, $A_0=\emptyset$, $S_0=\mathrm{supp}(p)$, $\langle G_{0j}:j<\omega\rangle$ the first system of finite sets such that $\bigcup_{j\in\omega}G_{0j}=S_0$, and $F_0=G_{00}$.

Now suppose that p_i,A_i,S_i,F_i,G_{ij} have been defined for all $i<n$ and all $j\in\omega$. For each $i\in F_{n-1}$ let E_{in} be the set of all successors of all nth branching points of the tree $p_{n-1}(i)$. Let $\sigma_{0n},\ldots,\sigma_{l_nn}$ be all of the functions σ with domain F_{n-1} such that $\sigma(i)\in E_{in}$ for all $i\in F_{n-1}$. We construct $q_{l_n+1,n}\le\cdots\le q_{1n}\le q_{0n}$ in P and $A_n=\{a_{0n},\ldots,a_{l_nn}\}$ as follows. Let $q_{0n}=p_{n-1}$. Assume that q_{kn} has been defined so that $q_{kn}\le q_{(k-1)n}\le\cdots\le q_{0n}$, and $q_{kn}(i)\le_n q_{(k-1)n}(i)\le_n\cdots\le_n q_{0n}(i)$ if $i\in F_{n-1}$. Now $\sigma_{kn}(i)$ is a successor of an nth branching point of $q_{kn}(i)$ if $i\in F_{n-1}$. Let

$$q'_{kn}(i)=\begin{cases}q_{kn}(i)\restriction\sigma_{kn}(i), & \text{if } i\in F_{n-1},\\ q_{kn}(i), & \text{otherwise.}\end{cases}$$

So $q'_{kn}\le q_{kn}$. Hence there is an $r_{kn}\le q'_{kn}$ and an $a_{kn}\in B$ such that $r_{kn}\Vdash\dot X(n)=\check a_{kn}$. Let

$$q_{(k+1)n}(i)=\begin{cases}\text{amalgamation of } r_{kn}(i) \text{ into } q_{kn}(i) & \text{if } i\in F_{n-1},\\ r_{kn}(i), & \text{otherwise.}\end{cases}$$

Thus $q_{(k+1)n}(i)\le_n q_{kn}(i)$ if $i\in F_{n-1}$. Let $p_n=q_{l_n+1,n}$. Thus $p_n(i)\le_n p_{n-1}(i)$ for all $i\in F_{n-1}$.

Let $S_n=\mathrm{supp}(p_n)$, and let $\langle G_{nj}:j\in\omega\rangle$ be the first system of finite sets with union S_n. Note that if $i\le n$ and $g(i)=(k,l)$, then $k\le i\le n$, so that $G_{g(i)}$ has been defined. Let $F_n=\bigcup_{i\le n}G_{g(i)}$. This finishes the construction.

Let $S = \bigcup_{n\in\omega} S_n$. Hence for all $i \in S$, $p_\infty(i) \stackrel{\text{def}}{=} \bigcap_{n\in\omega} p_n(i)$ is a perfect tree, by the fusion lemma, since for $i \in F_n$ we have

$$p_0(i) \geq \cdots \geq p_{n-1}(i) \geq_n p_n(i) \geq_{n+1} p_{n+1}(i) \geq \cdots .$$

Let $p_\infty(i) = 1$ for $i \notin S$. Define $A = \bigcup_{n\in\omega} A_n$. Now we prove that $p_\infty \Vdash \dot{X} : \omega \to A$, which will finish the proof. And to do this it suffices to show that, for any $n \in \omega$,

$$p_\infty \Vdash \dot{X}(n) = a_{0n} \vee \cdots \vee \dot{X}(n) = a_{l_n n}.$$

In turn, to do this it suffices to take an arbitrary $q \leq p_\infty$ and find $\tilde{q} \leq q$ and k such that $\tilde{q} \Vdash \dot{X}(n) = a_{kn}$. Choose $\overline{q} \leq q$ and $b \in B$ such that $\overline{q} \Vdash \dot{X}(n) = b$. Consider F_{n-1} and $\sigma_{1n} \ldots \sigma_{l_n n}$ as above. For each $i \in F_{n-1}$ let $\tau(i) \in E_{in} \cap \overline{q}(i)$; it exists since $\overline{q}(i) \leq q(i) \leq p_\infty(i) \leq_n p_{n-1}(i)$. Say $\tau = \sigma_{kn}$. Then if r_{kn} is as above, we have $\overline{q}(i) \restriction \sigma_{kn}(i) \leq q_{(k+1)n}(i) \restriction \sigma_{kn}(i) = r_{kn}(i)$ for $i \in F_{n-1}$. Thus if $\tilde{q}(i) = \overline{q}(i) \restriction \sigma_{kn}(i)$ for $i \in F_{n-1}$ and $\tilde{q}(i) = \overline{q}(i)$ otherwise, then $\tilde{q} \leq r_{kn}, \overline{q}$. In fact, clearly $\tilde{q} \leq \overline{q}$, and $\tilde{q}(i) \leq r_{kn}(i)$ for $i \in F_{n-1}$. For $i \notin F_{n-1}$,

$$\tilde{q}(i) = \overline{q}(i) \leq q(i) \leq p_\infty(i) \leq p_n(i) \leq q_{(k+1)n}(i) = r_{kn}(i),$$

as desired. $\tilde{q} \Vdash \dot{X}(n) = a_{kn}$, as desired. □

We now begin the proof of Theorem 4.20 itself. Let G be P-generic over M. By Lemma 4.22, cardinals $\geq \omega_2$ are preserved in $M[G]$. Applying Lemma 4.23 with $B = \omega_1$ we see that ω_1 is also preserved. So cardinals are preserved in $M[G]$.

Next, for $\alpha < \kappa$ and $n \in \omega$ let

$M_{\alpha n} = \{p \in P : p_\alpha$ has no branching up the nth level$\}$.

Clearly each such set $M_{\alpha n}$ is dense in P. Letting $g_\alpha = \bigcap_{p\in G} p_\alpha$, it follows that $g_\alpha \in {}^\omega 2$. For $\alpha \neq \beta$, let

$N_{\alpha\beta} = \{p \in P : \exists n \exists s \in {}^n 2[s \in p_\alpha \triangle p_\beta]\}$.

We claim that each set $N_{\alpha\beta}$ is dense. In fact, let $p \in P$ be arbitrary, and choose s so that $s0, s1 \in p_\alpha$. If $s0 \notin p_\beta$ or $s_1 \notin p_\beta$, then $p \in N_{\alpha\beta}$, as desired. Suppose that $s0, s1 \in p_\beta$. Choose $q \leq p$ such that $s1 \notin q_\alpha$ but $s1 \in q_\beta$. Then $q \in N_{\alpha\beta}$, as desired.

It follows that $\kappa \leq 2^\omega$ in $M[G]$.

For each $p \in P$ and each subset Γ of κ, let $p \restriction \Gamma$ be the function which agrees with p on Γ and is the 1 of P otherwise. By a result of Laver [84], let F' be a Ramsey ultrafilter in M which generates a Ramsey ultrafilter F in $M[G]$. By Theorem 4.14, we only need to show that ${}^\omega A/F$ has no chain of type ω_2, in $M[G]$. Note that A is still the BA of finite and cofinite subsets of ω in $M[G]$, by absoluteness. We argue by contradiction. Thus there is a sequence $f \stackrel{\text{def}}{=} \langle f_\alpha : \alpha < \omega_2 \rangle \in M[G]$ of members of ${}^{\omega}A$ such that $\langle f_\alpha/F : \alpha < \omega_2\rangle$ is strictly increasing. Let $\dot{f}$ be a name for f. Then there is a $p \in P$ such that

$$\begin{aligned} p \Vdash &\forall\alpha < \omega_2(\dot{f}_\alpha \in {}^{\omega}A) \\ &\wedge \forall\alpha,\beta < \omega_2 \exists a \in \check{F}'[\alpha < \beta \to a \subseteq \{n \in \omega : \dot{f}_\alpha(n) < \dot{f}_\beta(n)\}]. \end{aligned}$$

Thus for each $\alpha < \omega_2$ we have $p \Vdash \dot{f}_\alpha : \omega \to A$, so we can apply the *proof* of Lemma 4.23 to p. We thus obtain for each α certain *constructed objects* in V; with an obvious correspondence with that proof, they are, for all $n, j \in \omega$,

$$\begin{array}{ll} p^\alpha_n, & G^\alpha_{nj}, \\ F^\alpha_n, & E^\alpha_{in} \text{ for all } i \in F^\alpha_{n-1}, \\ A^\alpha_n, & l^\alpha_n, \end{array}$$

and, for all $i = 1, \ldots, l^\alpha_n$,

$$\begin{array}{ll} \sigma^\alpha_{in}, & q^\alpha_{in}, \\ (q^\alpha_{in})', & r^\alpha_{in}, \\ \text{and } a^\alpha_{in}; & \end{array}$$

finally, we have p^α_∞. Now we claim

(1) $\forall\alpha < \omega_2 \forall u \leq p^\alpha_\infty \forall n \in \omega \forall j \in \omega[[u \Vdash j \in \dot{f}_\alpha(n)$ iff $u \restriction \operatorname{supp}(p^\alpha_\infty) \Vdash j \in \dot{f}_\alpha(n)]$ and $[(u \Vdash j \notin \dot{f}_\alpha(n)$ iff $u \restriction \operatorname{supp}(p^\alpha_\infty) \Vdash j \notin \dot{f}_\alpha(n)]]$.

Suppose that $\alpha < \omega_2$, $u \leq p^\alpha_\infty$, $n \in \omega$, $j \in \omega$, $u \Vdash j \in \dot{f}_\alpha(n)$, and $u \restriction \operatorname{supp}(p^\alpha_\infty) \nVdash j \in \dot{f}_\alpha(n)$; we want to get a contradiction. Choose $v \leq u \restriction \operatorname{supp}(p^\alpha_\infty)$ such that $v \Vdash j \notin \dot{f}_\alpha(n)$. For each $i \in F^\alpha_{n-1}$ let $\tau(i) \in E^\alpha_{in} \cap v(i)$. Such a $\tau(i)$ exists since $v(i) \leq u(i) \leq p^\alpha_\infty(i) \leq p^\alpha_{n-1}(i)$; then (5) gives an nth branching point w of $p^\alpha_{n-1}(i)$ such that $w \in v(i)$, and a successor of w in $v(i)$ is as desired. Say $\tau = \sigma^\alpha_{kn}$. Thus $r^\alpha_{kn} \Vdash \dot{f}_\alpha(n) = a^\alpha_{kn}$, and $r^\alpha_{kn} \leq (q^\alpha_{kn})'$. Also, if $i \in F^\alpha_{n-1}$, then $v(i) \restriction \sigma^\alpha_{kn}(i) \leq q^\alpha_{k+1,n} \restriction \sigma^\alpha_{kn}(i) = r^\alpha_{kn}(i)$. Let $\tilde{v}(i) = v(i) \restriction \sigma^\alpha_{kn}(i)$ for $i \in F^\alpha_{n-1}$ and $\tilde{v}(i) = v(i)$ otherwise. Then $\tilde{v} \leq r^\alpha_{kn}, v$, so $j \notin a^\alpha_k$. On the other hand, $u(i) \restriction \sigma^\alpha_{kn} \leq r^\alpha_{kn}(i)$ for $i \in F^\alpha_{n-1}$. Let $\tilde{u}(i) = u(i) \restriction \sigma^\alpha_{kn}(i)$ for $i \in F^\alpha_{n-1}$ and $\tilde{u}(i) = u(i)$ otherwise. Then $\tilde{u} \leq r^\alpha_{kn}, u$. So $j \in a^\alpha_k$, contradiction. The other part of (1) is similar.

Now we may assume that $\langle \operatorname{supp}(p^\alpha_\infty) : \alpha < \omega_2\rangle$ forms a Δ-system, say with kernel Δ. Note that for each $\alpha < \omega_2$, $p^\alpha_\infty \restriction \Delta : \Delta \to \mathscr{P}({}^{<\omega}2)$; the set of all such functions has, by CH in V, ω_1 elements. Hence we may assume that for all $\alpha, \beta < \omega_2$ we have $p^\alpha_\infty \restriction \Delta = p^\beta_\infty \restriction \Delta$. Next, for each $\alpha < \omega_2$, the set $\operatorname{supp}(p^\alpha_\infty)\backslash\Delta$ has a certain countable order type. There are ω_1 countable order types, so we may assume that all such order types are the same. Thus for any $\alpha, \beta < \omega_2$ there is a unique order isomorphism $\pi_{\alpha\beta}$ from $\operatorname{supp}(p^\alpha_\infty)\backslash\Delta$ onto $\operatorname{supp}(p^\beta_\infty)\backslash\Delta$. We extend $\pi_{\alpha\beta}$ to a permutation of κ, still denoted by $\pi_{\alpha\beta}$, by letting it be $\pi_{\beta\alpha}$ $(= \pi^{-1}_{\alpha\beta})$ on $\operatorname{supp}(p^\beta_\infty)\backslash\Delta$ and the identity elsewhere. Thus $\pi_{\alpha\beta} = \pi_{\beta\alpha}$. And this permutation $\pi_{\alpha\beta}$ extends to other objects; for example, if $p \in P$, then $\pi_{\alpha\beta}(p)$ is the member q

of P such that $q(i) = p(\pi_{\alpha\beta}(i))$ for all $i \in \kappa$. Note here that if p has support Γ, then $\pi_{\alpha\beta}(p)$ has support $\pi_{\alpha\beta}[\Gamma]$. Now consider the objects

$$\begin{array}{ll} \pi_{\alpha 0}(p^\alpha_n), & \pi_{\alpha 0}[G^\alpha_{nj}], \\ \pi_{\alpha 0}[F^\alpha_n], & \langle E^\alpha_{(\pi_{\alpha 0}(i))n} : i \in \pi_{\alpha 0}[F^\alpha_{n-1}]\rangle, \\ A^\alpha_n, & l^\alpha_n, \end{array}$$

and, for all $i = 1, \ldots, l^\alpha_n$,

$$\begin{array}{ll} \sigma^\alpha_{in} \circ \pi_{0\alpha}, & \pi_{\alpha 0}(q^\alpha_{in}), \\ \pi_{\alpha 0}((q^\alpha_{in})'), & \pi_{\alpha 0}(r^\alpha_{in}), \\ \text{and } a^\alpha_{in}; & \end{array}$$

and, finally, $\pi_{\alpha 0}(p^\alpha_\infty)$. By CH, there are only ω_1 of these things, so we may assume that they are the same for all $\alpha \in \omega_2\backslash\{0\}$. Now take any two distinct $\alpha, \beta \in \omega_2\backslash\{0\}$. Thus, for example,

$$\begin{aligned} (\pi_{\alpha\beta}(p^\alpha_\infty))(i) &= (\pi_{0\beta}(\pi_{\alpha 0}(p^\alpha_\infty)))(i) \\ &= (\pi_{\alpha 0}(p^\alpha_\infty))(\pi_{0\beta}(i)) \\ &= (\pi_{\beta 0}(p^\beta_\infty))(\pi_{0\beta}(i)) \\ &= p^\beta_\infty(i). \end{aligned}$$

Hence $\pi_{\alpha\beta}(p^\alpha_\infty) = p^\beta_\infty$. Another useful fact now is that if $i \in F^\alpha_{n-1}$ then $E_{(\pi_{\alpha\beta}(i))n} = E^\alpha_{in}$ for all $i \in F^\alpha_{n-1}$. In fact, $\pi_{\alpha 0}(i) \in \pi_{\alpha 0}[F^\alpha_{n-1}] = \pi_{\beta 0}[F^\beta_{n-1}]$ and $E^\beta_{(\pi_{\alpha\beta}(i))n} = E^\beta_{(\pi_{\beta 0}(\pi_{\alpha 0}(i)))n} = E^\alpha_{(\pi_{\alpha 0}(\pi_{\alpha 0}(i)))n} = E^\alpha_{in}$.

Next,

(2) $\forall u \le p^\alpha_\infty \forall n \in \omega \forall j \in \omega (u \Vdash j \in \dot{f}_\alpha(n)$ iff $\pi_{\alpha\beta}(u) \Vdash j \in \dot{f}_\beta(n))$.

For, suppose that $u \Vdash j \in \dot{f}_\alpha(n)$ but $\pi_{\alpha\beta}(u) \nVdash j \in \dot{f}_\beta(n)$. By (1) we may assume that $\mathrm{supp}(u) \subseteq \mathrm{supp}(p^\alpha_\infty)$. Choose $v \le \pi_{\alpha\beta}(u)$ such that $v \Vdash j \notin \dot{f}_\beta(n)$. Now if $i \in F^\beta_{n-1}$, then

$$v(i) \le (\pi_{\alpha\beta}(u))(i) \le (\pi_{\alpha\beta}(p^\alpha_\infty))(i) = p^\beta_\infty(i),$$

so there is a $\tau(i) \in E^\beta_{in} \cap v(i)$. Say $\tau = \sigma^\beta_{kn}$. Thus $r^\beta_{kn} \Vdash \dot{f}_\beta(n) = a^\beta_{kn}$. As above, let $\tilde{v}(i) = v(i) \restriction \sigma^\beta_{kn}(i)$ for $i \in F^\beta_{n-1}$ and $\tilde{v}(i) = v(i)$ otherwise. Then $\tilde{v} \le r^\beta_{kn}, v$, so $j \notin a^\beta_{kn}$. Now $\pi_{\beta\alpha} v \le u$. Furthermore, if $i \in F^\alpha_{n-1}$ then $\pi_{\alpha\beta} i \in F^\beta_{n-1}$, and so

$$\sigma^\alpha_{kn}(i) = (\pi_{\beta\alpha}(\sigma^\beta_{kn}))(i) = \sigma^\beta_{kn}(\pi_{\alpha\beta}(i)) \in E^\beta_{(\pi_{\alpha\beta}(i))n} \cap v(\pi_{\alpha\beta}(i)) = E^\alpha_{in} \cap (\pi_{\beta\alpha}(v))(i).$$

Now let $\tilde{w}(i) = (\pi_{\beta\alpha} v)(i) \restriction \sigma^\alpha_{kn}(i)$ for $i \in F^\alpha_{n-1}$ and $\tilde{w}(i) = (\pi_{\beta\alpha} v)(i)$ otherwise. Then $\tilde{w} \le r^\alpha_{kn}$ and $\tilde{w} \le \pi_{\beta\alpha} v \le u$, so $j \in a_k$, contradiction. This proves (2).

Now let s be the member of P which agrees with p^α_∞ and p^β_∞ on their supports and is 1 otherwise. Clearly $\pi_{\alpha\beta}(s) = s$. We may assume that $\alpha < \beta$. Then

(3) $s \Vdash \exists X \in F' \forall i \in X \forall j \in \omega (j \in \dot{f}_\alpha(i) \to j \in \dot{f}_\beta(i))$.

We claim that

(4) $s \Vdash \exists X \in F' \forall i \in X \forall j \in \omega (j \in \dot{f}_\beta(i) \to j \in \dot{f}_\alpha(i))$.

This is a clear contradiction. So, it suffices to prove (4). By (3), there is a $u \leq s$ and an $X \in F'$ such that

(5) $u \Vdash \forall i \in X \forall j \in \omega (j \in \dot{f}_\alpha(i) \to j \in \dot{f}_\beta(i))$.

It suffices now to show

$\pi_{\alpha\beta}(u) \Vdash \forall i \in X \forall j \in \omega (j \in \dot{f}_\beta(i) \to j \in \dot{f}_\alpha(i))$.

So, let $v \leq \pi_{\alpha\beta}(u)$, $i \in X$, $j \in \omega$, and assume that $v \Vdash j \in \dot{f}_\beta(i)$. Since $v \leq s \leq p^\beta_\infty$, from (2) we get $\pi_{\alpha\beta}(v) \Vdash j \in \dot{f}_\alpha(i)$. And $\pi_{\alpha\beta}(v) \leq u$, so by (5) we get $\pi_{\alpha\beta}(v) \Vdash j \in \dot{f}_\beta(i)$. But $\pi_{\alpha\beta}(v) \leq s \leq p^\alpha_\infty$, so by (2) again, $v \Vdash j \in \dot{f}_\alpha(i)$, as desired. $\square$

Shelah has a more recent construction providing examples with

$$\mathrm{Depth}\left(\prod_{i\in I} A_i/D\right) < \prod_{i\in I} \mathrm{Depth}(A_i)/D,$$

and this construction applies to some other functions too. To formulate this result, we need a definition.

Suppose that $\mathbf{O}$ is an operation on sequences of BAs, and inv is a cardinal invariant on BAs such that $|\mathrm{inv}(B)| \leq |B|$ for every infinite BA B. Then we say that the property $\Box_{\mathbf{O}}$ holds provided that if μ is a cardinal and B_i is a BA for each $i < \mu^+$, then

$$\sup_{i<\mu^+} \mathrm{inv}(B_i) \leq \mathrm{inv}\left(\mathbf{O}_{i<\mu^+} B_i\right) \leq \mu + \sup_{i<\mu^+} \mathrm{inv} B_i.$$

Rosłanowski, Shelah [98] proved the following result:

Suppose that inv *is a cardinal invariant on BAs satisfying* $\Box_\oplus$ *or* $\Box_{\prod^{\mathrm{w}}}$*; suppose that* $\mathrm{inv}(B) \leq |B|$ *for any BA* B*. Suppose that for each infinite cardinal* χ *there is a BA* B *such that* $\chi < \mathrm{inv}(B)$ *and there is no inaccessible cardinal in the interval* $(\chi, |B|]$*. Assume further that*

$\odot$ $\langle \lambda_i : i < \kappa \rangle$ *is a sequence of weakly inaccessible cardinals* $\lambda_i > \kappa^+$*,* D *is an* $\aleph_1$*-complete ultrafilter on* κ*, and* $\prod_{i<\kappa}(\lambda_i, <)/D$ *is* μ^+*-like.*

Then there exist BAs B_i *for* $i < \kappa$ *such that*

$$\mathrm{inv}(B_i) = \lambda_i \quad \text{and} \quad \mathrm{inv}\left(\prod_{i<\kappa} B_i/D\right) \leq \mu.$$

As a corollary of this and Magidor, Shelah [98] one has:

Suppose that inv *is a cardinal invariant on BAs such that the following three conditions hold:*

(i) $\mathrm{inv}(B) \leq |B|$ *for all infinite BAs* B.

(ii) $\sup_{i<\mu^+} \mathrm{inv}(B_i) \leq \mathrm{inv}\left(\prod^{\mathrm{w}}_{i<\mu^+} B_i\right) \leq \mu + \sup_{i<\mu^+} \mathrm{inv}(B_i)$ *for every system* $\langle B_i : i < \mu^+\rangle$ *of BAs.*

Then it is consistent to have a system $\langle B_i : i < \kappa\rangle$ *of BAs such that* $\mathrm{inv}\left(\prod_{i<\kappa} B_i/D\right) < \prod_{i<\kappa} \mathrm{inv}(B_i)/D$.

This corollary applies not only to depth, but also to length, independence, π-character, and tightness.

For subdirect products the situation is similar to that for cellularity, with essentially the same proof: there is a BA with depth ω which is a subdirect product of BAs having high depth.

Depth of moderate products can easily be described using the arguments for products:

Theorem 4.24. $\mathrm{Depth}(\prod^B_{i\in I} A_i) = \max\{\mathrm{Depth}(B), \sup_{i\in I} \mathrm{Depth}(A_i)\}$.

Proof. We use the notation introduced in Chapter 1 for moderate products. Clearly $\geq$ holds. Now let $\kappa = \max\{\mathrm{Depth}(B), \sup_{i\in I} \mathrm{Depth}(A_i)\}$, and, to get a contradiction, suppose that $\langle h(b_\alpha, F_\alpha, a^\alpha) : \alpha < \kappa^+\rangle$ is a strictly increasing sequence. Without loss of generality each $h(b_\alpha, F_\alpha, a^\alpha)$ is normal. Then $b_\alpha \subseteq b_\beta$ for $\alpha < \beta < \kappa^+$. Hence there is a $\Gamma \in [\kappa^+]^{\kappa^+}$ such that $b_\alpha = b_\beta$ for all $\alpha, \beta \in \Gamma$. Then $a^\alpha < a^\beta$ for $\alpha < \beta$, both in Γ, and this easily gives a contradiction. □

Depth for one-point gluing behaves like arbitrary products if all algebras have more than two elements and I is infinite or some A_i is infinite. In fact, assume the notation from Chapter 1, and let B be the one-point gluing. We claim that $\mathrm{Depth}(B) = |I| + \sup_{i\in I} \mathrm{Depth}(A_i)$. $\leq$ holds since B is a subalgebra of the product. Now suppose that $i \in I$ and $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing in A_i, κ an infinite cardinal. If some $a_\alpha \in F_i$, we may assume that all $a_\alpha \in F_i$, and then we can define

$$b_\alpha(j) = \begin{cases} a_\alpha & \text{if } \ j = i, \\ 1 & \text{otherwise,} \end{cases}$$

and we obtain a strictly increasing sequence of length κ in B. If all a_α are not in F_i we can work similarly, with 0 instead of 1. This shows that $\mathrm{Depth}(A_i) \leq \mathrm{Depth}(B)$. To show that $|I| \leq \mathrm{Depth}(B)$ when I is infinite, let $\langle f_\alpha : \alpha < \kappa\rangle$ be a one-one enumeration of I. For each $\alpha < \kappa$ choose $a_\alpha \in A^+_{f(\alpha)} \backslash F_{f(\alpha)}$. Now for each $\beta < \kappa$ define

$$b_\beta(\alpha) = \begin{cases} a_\alpha & \text{if } \ \alpha < \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle b_\beta : \beta < \kappa\rangle$ is a strictly increasing sequence of elements of B.

Note that a one-point gluing of a product of two-element algebras still has just two elements.

For the Alexsandroff duplicate we have $\mathrm{Depth}(\mathrm{Dup}(A)) = \mathrm{Depth}(A)$. In fact, if $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing in A, then $\langle (a_\alpha, \mathcal{S}(a_\alpha)) : \alpha < \kappa\rangle$ is strictly increasing in $\mathrm{Dup}(A)$. Now suppose that $\langle (a_\alpha, X_\alpha) : \alpha < \kappa\rangle$ is strictly increasing in $\mathrm{Dup}(A)$, but A does not have a strictly increasing sequence of length κ. Note that $\kappa > \omega$. Define $\alpha \equiv \beta$ iff $\alpha, \beta < \kappa$ and $a_\alpha = a_\beta$. Then there are fewer than κ equivalence classes under $\equiv$, so some equivalence class M is uncountable. Let $\langle \alpha_\xi : \xi < \delta\rangle$ be a strictly increasing enumeration of M. Now if $\xi < \delta$, then $X_{\alpha_\xi}\backslash\mathcal{S}(a_{\alpha_0})$ is finite, so there is an $\eta < \omega_1$ such that $X_{\alpha_\xi}\backslash\mathcal{S}(a_{\alpha_0}) = X_{\alpha_\eta}\backslash\mathcal{S}(a_{\alpha_0})$ for all $\xi \in [\eta, \omega_1)$. Similarly there is a $\rho \in [\eta, \omega_1)$ such that $\mathcal{S}(a_{\alpha_0})\backslash X_{\alpha_\xi} = \mathcal{S}(a_{\alpha_0})\backslash X_{\alpha_\rho}$ for all $\xi \in [\rho, \omega_1)$. Now $X_{\alpha_\rho} \subset X_{\alpha_{\rho+1}}$. Take any $b \in X_{\alpha_{\rho+1}}\backslash X_{\alpha_\rho}$. Thus

$$b \in \mathcal{S}(a_{\alpha_0}) \to b \in (\mathcal{S}(a_{\alpha_0})\backslash X_{\alpha_\rho})\backslash(\mathcal{S}(a_{\alpha_0})\backslash X_{\alpha_{\rho+1}});$$
$$b \notin \mathcal{S}(a_{\alpha_0}) \to b \in (X_{\alpha_{\rho+1}}\backslash\mathcal{S}(a_{\alpha_0}))\backslash(X_{\alpha_\rho}\backslash\mathcal{S}(a_{\alpha_0}));$$

contradiction.

Concerning the exponential, from Lemma 1.22(i),(vi) it is clear that if $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing in A, then $\langle \mathscr{V}(\mathcal{S}(a_\alpha)) : \alpha < \kappa\rangle$ is strictly increasing in $\mathrm{Exp}(A)$. Hence $\mathrm{Depth}(A) \le \mathrm{Depth}(\mathrm{Exp}(A))$. We do not know whether there is a BA A with $<$ here.

Next we discuss derived functions with respect to depth. The first result is that $\mathrm{Depth}_{\mathrm{H}+}$ is the same as tightness. To prove this, we need an equivalent form of tightness due to Arhangelskiĭ and Shapirovskiĭ. It involves the notion of a free sequence in a topological space. Let X be a topological space. A *free sequence in* X is a sequence $\langle x_\xi : \xi < \alpha\rangle$ (α an ordinal) of elements of X such that for all $\xi < \alpha$ we have $\overline{\{x_\eta : \eta < \xi\}} \cap \overline{\{x_\eta : \xi \le \eta < \alpha\}} = \emptyset$. For an arbitrary topological space X and a point $x \in X$, the *tightness* $\mathrm{t}(x)$ of x in X is, by definition, the least cardinal κ such that if $Y \subseteq X$ and $x \in \overline{Y}$, then there is a subset $Z \subseteq Y$ such that $|Z| \le \kappa$ and $x \in \overline{Z}$. And the tightness $\mathrm{t}(X)$ of X itself is $\sup_{x\in X} \mathrm{t}(x)$. Clearly this means that $\mathrm{t}(A) = \mathrm{t}(\mathrm{Ult}(A))$ for any BA A. The equivalent form of tightness due to Arhangelskiĭ (based on proofs of Shapirovskiĭ) is given in the following theorem.

Theorem 4.25. *Let X be a compact Hausdorff space, and let κ be a regular cardinal.*

(i) *If there is a free sequence of length κ, then there is a $y \in X$ such that* $\mathrm{t}(y) \ge \kappa$.

(ii) *If there exist $y \in X$ and $Y \subseteq X$ such that $y \in \overline{Y}$ and $\forall Z \in [Y]^{<\kappa}[y \notin \overline{Z}]$, then there is a free sequence of length κ.*

(iii) $\mathrm{t}(X) = \sup\{\lambda :$ *there is a free sequence of length* $\lambda\}$.

Proof. First we note that (i) gives $\ge$ in (iii). For $\le$ in (iii), suppose that $\mu < \mathrm{t}(X)$. Then the hypothesis of (ii) holds for $\kappa = \mu^+$, and so (ii) yields a free sequence of length μ^+. This gives $\le$ in (iii). So it suffices to prove (i) and (ii).

(i): Suppose that $\langle x_\xi : \xi < \kappa\rangle$ is a free sequence, where κ is regular; we shall find a point $y \in X$ such that $\mathrm{t}(y) \geq \kappa$. First note:

(1) There is a $y \in X$ such that $|U \cap \{x_\xi : \xi < \kappa\}| = \kappa$ for each neighborhood U of y.

In fact, otherwise for every $y \in X$ let $U(y)$ be an open neighborhood of y such that $|U(y) \cap \{x_\xi : \xi < \kappa\}| < \kappa$. Thus $\{U(y) : y \in X\}$ is an open cover of X. Let $U(y_0), \ldots, U(y_{n-1})$ be a finite subcover. Then

$$\{x_\xi : \xi < \kappa\} = \bigcup_{i<n}(U(y_i) \cap \{x_\xi : \xi < \kappa\}),$$

and the right side has cardinality $< \kappa$, contradiction. So (1) holds.

Take y as in (1). Assume that $\mathrm{t}(y) < \kappa$. Now $y \in \overline{\{x_\xi : \xi < \kappa\}}$. Hence by the definition of tightness, choose a subset Γ of κ of power $< \kappa$ such that $y \in \overline{\{x_\xi : \xi \in \Gamma\}}$. Let $\eta = \sup(\Gamma) + 1$. Hence $y \in \overline{\{x_\xi : \xi < \eta\}}$, so by freeness $y \notin \overline{\{x_\xi : \eta \leq \xi\}}$. So there is a neighborhood U of y such that $U \cap \{x_\xi : \eta \leq \xi\} = 0$. This contradicts (1). Thus (i) holds.

(ii): Suppose that κ is regular and $\exists y \in X \exists Y \subseteq X[y \in \overline{Y}$ and $\forall Z \in [Y]^{<\kappa}[y \notin \overline{Z}]]$. We will construct a free sequence of length κ. Set

$$Y' = \{x : \text{there is a } Z \in [Y]^{<\kappa} \text{ such that } x \in \overline{Z}\}.$$

Thus $Y \subseteq Y'$, so $y \in \overline{Y'}$. Note

(2) If $Z \in [Y']^{<\kappa}$, then $y \notin \overline{Z}$.

In fact, suppose that $Z \in [Y']^{<\kappa}$. For each $x \in Z$ choose $W_x \in [Y]^{<\kappa}$ such that $x \in \overline{W_x}$. Let $V = \bigcup_{x\in Z} W_x$. Then $V \in [Y]^{<\kappa}$ and $\overline{Z} \subseteq \overline{V}$. So $y \notin \overline{Z}$.

By the same proof we have

(3) If $Z \in [Y']^{<\kappa}$ and $z \in \overline{Z}$, then $z \in Y'$.

We now construct x_ξ, F_ξ, U_ξ for $\xi < \kappa$ such that $x_\xi \in Y'$, $y \in F_\xi \subseteq U_\xi$ with U_ξ open and F_ξ a closed neighborhood of y, by recursion. Suppose these have been constructed for all $\eta < \xi$, where $\xi < \kappa$. Since $y \notin \overline{\{x_\eta : \eta < \xi\}}$, let U_ξ be an open neighborhood of y such that $U_\xi \cap \{x_\eta : \eta < \xi\} = 0$. Let F_ξ be a closed neighborhood of y such that $F_\xi \subseteq U_\xi$. Then we claim

(4) $Y' \not\subseteq \bigcup_{\eta\leq\xi}(X\backslash F_\eta) \cup \overline{\{x_\eta : \eta < \xi\}}$.

For, suppose not; then we show that $y \in \overline{\{x_\eta : \eta < \xi\}}$ (contradiction). For, let U be an open neighborhood of y and let F' be a closed neighborhood of y which is included in U. Let W be the closure of the set $\{F_\eta : \eta \leq \xi\} \cup \{F'\}$ under finite intersections. Since $y \in \overline{Y'}$, for all $H \in W$ choose $z_H \in Y' \cap H$. Then $H' \cap \{z_H : H \in W\} \neq 0$ for all $H' \in W$. Choose

$$t \in \bigcap_{H' \in W} H' \cap \overline{\{z_H : H \in W\}}.$$

By (3), $t \in Y'$. Now $t \in F_\eta$ for all $\eta \leq \xi$, so by the "suppose not" for (4), $t \in \overline{\{x_\eta : \eta < \xi\}}$. Since $t \in F' \subseteq U$, it follows that $U \cap \{x_\eta : \eta < \xi\} \neq 0$, as desired.

So (4) holds; choose x_ξ in the left side of (4) but not in the right side. This completes the construction.

Suppose $\xi < \kappa^+$ and $s \in \overline{\{x_\eta : \eta < \xi\}} \cap \overline{\{x_\eta : \xi \leq \eta < \kappa\}}$. Then $s \notin U_\xi$, so $s \notin F_\xi$. Thus $s \in X \backslash F_\xi$, which is open, so there is an η with $\xi \leq \eta < \kappa$ such that $x_\eta \in X \backslash F_\xi$, contradiction. □

By Theorem 4.25, if $\mathrm{t}(X)$ is regular and is attained in the free sequence sense then it is attained in the defined sense, i.e., there is a point y with tightness $\mathrm{t}(X)$.

Theorem 4.26. *For any infinite BA A we have* $\mathrm{Depth}_{\mathrm{H}+}(A) = \mathrm{t}(A)$.

Proof. For $\geq$, let $\langle F_\xi : \xi < \kappa \rangle$ be a free sequence; we produce a quotient A/I of A having a strictly increasing sequence of order type κ. For brevity let $Y = \{F_\xi : \xi < \kappa\}$. For every $\xi < \kappa$ there is an element a_ξ of A such that $\{F_\eta : \eta < \xi\} \subseteq \mathcal{S}(a_\xi)$ and $\mathcal{S}(a_\xi) \cap \{F_\eta : \xi \leq \eta < \kappa\} = 0$.

Consider the following ideal on A: $I = \{x \in A : Y \subseteq \mathcal{S}(-x)\}$. Suppose $\xi < \eta < \kappa$. Then $\mathcal{S}(a_\xi \cdot -a_\eta) \cap Y = 0$: if $F_\nu \in \mathcal{S}(a_\xi \cdot -a_\eta)$, then $F_\nu \in \mathcal{S}(a_\xi)$, hence $\nu < \xi$, and $-a_\eta \in F_\nu$, hence $\eta \leq \nu$, so $\eta < \xi$, contradiction. This shows that $[a_\xi] \leq [a_\eta]$ for $\xi < \eta < \kappa$. Still suppose that $\xi < \eta < \kappa$. Then $F_\xi \in \mathcal{S}(a_\eta) \backslash \mathcal{S}(a_\xi) = \mathcal{S}(a_\eta \cdot -a_\xi)$. Thus $Y \not\subseteq \mathcal{S}(-a_\eta + a_\xi)$, so $a_\eta \cdot -a_\xi \notin I$, which means that $[a_\eta] < [a_\xi]$, as desired.

For $\leq$, let I be an ideal in A, and let $\langle [a_\xi] : \xi < \kappa \rangle$ be a strictly increasing sequence in A/I. For each $\xi < \kappa$, the set $\{x : -x \in I\} \cup \{a_{\xi+1}, -a_\xi\}$ has the finite intersection property, since $a_{\xi+1} \cdot -a_\xi \notin I$. Let F_ξ be an ultrafilter including this set. Then, we claim, $\langle F_\xi : \xi < \kappa \rangle$ is a free sequence. To prove this it suffices to show that for any $\xi < \kappa$ we have

(1) $\{F_\eta : \eta < \xi\} \subseteq \mathcal{S}(a_\xi)$ and $\mathcal{S}(a_\xi) \cap \{F_\eta : \xi \leq \eta < \kappa\} = 0$.

If $\eta < \xi < \kappa$, then $a_{\eta+1} \cdot -a_\xi \in I$, and hence $-a_{\eta+1} + a_\xi \in F_\eta$; but also $a_{\eta+1} \in F_\eta$, so $a_\xi \in F_\eta$ and so $F_\eta \in \mathcal{S}(a_\xi)$, proving the first part of (1). For the second part, suppose that $\xi \leq \eta < \kappa$ and $F_\eta \in \mathcal{S}(a_\xi)$. Now $a_\xi \cdot -a_\eta \in I$, so $-a_\xi + a_\eta \in F_\eta$; but also $-a_\eta \in F_\eta$, so $-a_\xi \in F_\eta$, contradiction. □

Corollary 4.27. $\mathrm{Depth}_{\mathrm{H}+}$ *and* t (*for free sequences*) *have the same attainment properties, i.e., for any BA A and any infinite cardinal κ, A has a homomorphic image with a chain of order type κ iff* $\mathrm{Ult}(A)$ *has a free sequence of type κ.* □

Note that, as in the relation between spread and cellularity, $\mathrm{Depth}_{\mathrm{H}+}$ involves two sups, while t for free sequences involves only one; we return to this below.

Since $\mathrm{Depth}(A) \le \mathrm{c}(A)$, it is clear that $\mathrm{Depth}_{\mathrm{H}-}(A) = \omega$. It is also easy to see that $\mathrm{Depth}_{\mathrm{S}+}(A) = \mathrm{Depth}(A)$ and $\mathrm{Depth}_{\mathrm{S}-}(A) = \omega$. $\mathrm{Depth}_{\mathrm{h}+}$ is a little more interesting:

Theorem 4.28. $\mathrm{Depth}_{\mathrm{h}+}(A) = \mathrm{s}(A)$ *for any infinite BA A.*

Proof. For $\ge$, suppose that Y is a discrete subspace of $\mathrm{Ult}(A)$; clearly Y, since it is discrete, has an increasing sequence of closed-open sets of order type $|Y|$. For $\le$, suppose that Y is a subspace of $\mathrm{Ult}(A)$ and $\langle U_\kappa : \kappa < \kappa\rangle$ is a strictly increasing system of clopen subsets of Y. For each $\kappa < \kappa$ choose $y_\kappa \in U_{\kappa+1} \backslash U_\kappa$. Clearly $\{y_\kappa : \kappa < \kappa\}$ is a discrete subspace of $\mathrm{Ult}(A)$. □

The proof shows that $\mathrm{Depth}_{\mathrm{h}+}(A)$ and $\mathrm{s}(A)$ have the same attainment properties.

Since $\mathrm{Depth}_{\mathrm{h}-}(A) \le \mathrm{Depth}_{\mathrm{H}-}(A)$, we have $\mathrm{Depth}_{\mathrm{h}-}(A) = \omega$ for any infinite BA A. Obviously ${}_{\mathrm{d}}\mathrm{Depth}_{\mathrm{S}+}(A) = \mathrm{Depth}(A)$ for any BA A.

The status of the derived function ${}_{\mathrm{d}}\mathrm{Depth}_{\mathrm{S}-}$ is not clear. Note that for A the interval algebra on a cardinal κ we have ${}_{\mathrm{d}}\mathrm{Depth}_{\mathrm{S}-}(A) = \omega$: this follows upon considering the subalgebra of A generated by $\{\{\alpha\} : \alpha$ a non-limit ordinal $< \kappa\}$. Also, Koppelberg and Shelah have independently observed that if A is atomless and λ-saturated (in the model-theoretic sense), then ${}_{\mathrm{d}}\mathrm{Depth}_{\mathrm{S}-}(A) \ge \lambda$. To show this, suppose that B is a dense subalgebra of A. By induction choose elements $a_\alpha \in A$ and $b_\alpha \in B$ for $\alpha < \lambda$ so that $\alpha < \beta$ implies that $a_\alpha > b_\alpha > a_\beta > 0$; the a_α's can be chosen by λ-saturation, and the b_α's by denseness. So the sequence $\langle b_\alpha : \alpha < \lambda\rangle$ shows that the depth of B is at least λ.

Problem 31. *Is it true that for all infinite cardinals $\kappa \le \lambda$ there is a BA A of size λ with ${}_{\mathrm{d}}\mathrm{Depth}_{\mathrm{S}-}(A) = \kappa$?*

We now discuss small depth $\mathrm{tow}_{\mathrm{spect}}$ and tow. These are defined as follows. A *tower* in a BA A is a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of $A\backslash\{1\}$ which is strictly increasing and has sum 1. We define

$$\mathrm{tow}(A) = \min\{\kappa : A \text{ has a tower of length } \kappa\};$$
$$\mathrm{tow}_{\mathrm{spect}}(A) = \{\kappa : \kappa \text{ is regular and } A \text{ has a tower of length } \kappa\}.$$

For convenience we define $\mathrm{tow}(A) = \infty$ if A does not have a tower. Note that $\mathrm{tow}_{\mathrm{spect}}$ is a set of regular cardinals. tow and $\mathrm{tow}_{\mathrm{spect}}$ are discussed in Monk [01a], Monk [02], Monk [07], and in several articles concerning the special case $A = \mathscr{P}(\omega)/\mathrm{fin}$. We indicate some of the general results about BAs.

In some cases we consider towers in a reversed sense: a strictly decreasing sequence of nonzero elements with product 0; we call these $\ge$-towers. Obviously towers in one sense exist iff they exist in the other sense.

Note that $\mathrm{tow}(A) < \infty$ for all atomless BAs, but not for all BAs. For example, the finite-cofinite algebra on an uncountable set does not have any towers. There

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are atomic BAs with towers, though, for example an interval algebra on a cardinal. The following simple characterization of BAs that have towers is sometimes useful.

Proposition 4.29. *For any infinite BA A the following conditions are equivalent:*

(i) *A has a tower.*
(ii) *Every maximal ideal I on A has one of the following properties:*
   (a) *There is a strictly increasing sequence of elements of I with sum 1.*
   (b) *There is a strictly decreasing sequence of elements in $I\backslash\{0\}$ with product 0.*

Proof. (i)⇒(ii): suppose that $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing with sum 1, κ an infinite cardinal. Let I be any maximal ideal in A. If all $a_\alpha \in I$, then (a) holds. If $a_\alpha \notin I$ for some α, then $\langle -a_\beta : \alpha \le \beta < \kappa\rangle$ is a strictly decreasing sequence of elements of $I\backslash\{0\}$ with product 0.

(ii)⇒(i): obvious. □

Corollary 4.30. *Each of the following BAs fails to have a tower:*

(i) $\mathrm{Finco}(\kappa)$*, with κ an uncountable cardinal;*
(ii) $\prod^{\mathrm{w}}_{i\in I} A_i$*, with I uncountable, and such that each A_i fails to have a tower.*

Proof. (i): Consider the maximal ideal of all finite subsets of κ.

(ii): Consider the maximal ideal consisting of all $a \in \prod^{\mathrm{w}}_{i\in I} A_i$ such that $\{i \in I : a_i \neq 0\}$ is finite. □

Corollary 4.31.

(i) *Every atomless BA has a tower.*
(ii) *If $\mathfrak{a}(A) = \omega$, in particular if $\mathrm{c}(A) = \omega$ or $|A| = \omega$, then A has a tower.*
(iii) *If A has a tower, then so do $A \times B$ and $A \oplus B$.* □

A *weak tower* is a strictly increasing sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of A different from 1, with $\sum_{\alpha<\kappa} a_\alpha = 1$, where κ is a cardinal, not necessarily regular. For brevity let

$$\mathrm{tow}^w_{\mathrm{spect}}(A) = \{\kappa : \kappa \text{ is the length of a weak tower in } A\}.$$

Clearly $\mathrm{tow}_{\mathrm{spect}}(A) \subseteq \mathrm{tow}^w_{\mathrm{spect}}(A)$.

The following general set theoretic fact is useful in discussing this function.

Lemma 4.32. *If κ and λ are infinite cardinals, with $\lambda < \kappa$ and $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$, then there is a strictly increasing function $f : \lambda \to \kappa$ with $\mathrm{rng}(f)$ cofinal in κ.*

Proof. Let $\langle \mu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of ordinals with supremum λ, with $\mu_0 = 0$, and let $\langle \nu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of ordinals with supremum κ, with $\lambda \le |\nu_{\xi+1}\backslash\nu_\xi|$ for each $\xi < \mathrm{cf}(\kappa)$. Then f can be taken as the union of strictly increasing functions from $\mu_{\xi+1}\backslash\mu_\xi$ into $\nu_{\xi+1}\backslash\nu_\xi$ for $\xi < \mathrm{cf}(\kappa)$. □

Corollary 4.33. *If A has a weak tower of order type κ, and λ is a cardinal such that $\lambda < \kappa$ and $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$, then A has a weak tower of order type λ.*

Proof. Let $\langle a_\xi : \xi < \kappa\rangle$ be a weak tower in A. Let f be as in Proposition 4.32. Then $\langle a_{f(\xi)} : \xi < \lambda\rangle$ is also a weak tower in A. □

Problem 32. *Let C be a nonempty set of infinite cardinals such that the following condition holds:*

(∗) *If $\kappa \in C$, $\lambda < \kappa$, and $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$, then $\lambda \in C$.*

Is there an atomless BA A such that $\mathrm{tow}^w_{\mathrm{spect}}(A) = C$?

For this problem, see Theorem 4.44.

A *very weak tower* is a strictly increasing sequence $\langle a_\alpha : \alpha < \beta\rangle$ of elements of A different from 1, with $\sum_{\alpha<\beta} a_\alpha = 1$, where β is a limit ordinal, not necessarily a cardinal.

Problem 33. *Characterize those sets K of limit ordinals for which there is an atomless BA A such that K is the collection of all order types of very weak towers of A.*

Now we consider our special subalgebras and tow.

Proposition 4.34. *If $A \le_{\mathrm{reg}} B$, then $\mathrm{tow}_{\mathrm{spect}}(A) \subseteq \mathrm{tow}_{\mathrm{spect}}(B)$ and $\mathrm{tow}(B) \le \mathrm{tow}(A)$. The conclusion also holds if $A \le_{\mathrm{free}} B$, $A \le_{\mathrm{proj}} B$, $A \le_{\mathrm{rc}} B$, $A \le_\pi B$.* □

This proposition suggests the following question.

Problem 34. *Given nonempty sets $K \subseteq L$ of cardinals, are there BAs $A \le_{\mathrm{reg}} B$ such that $\mathrm{tow}_{\mathrm{spect}}(A) = K$ and $\mathrm{tow}_{\mathrm{spect}}(B) = L$? There is a similar question for $A \le_{\mathrm{free}} B$, $A \le_{\mathrm{proj}} B$, $A \le_{\mathrm{rc}} B$, $A \le_\pi B$.*

See also Proposition 4.45.

Proposition 4.35. *If $A \le_\sigma B$ and $\mathrm{tow}(A) > \omega$, then $\mathrm{tow}_{\mathrm{spect}}(A) \subseteq \mathrm{tow}_{\mathrm{spect}}(B)$ and $\mathrm{tow}(B) \le \mathrm{tow}(A)$.*

Proof. The proof is much like that of Proposition 3.35. Thus let T be a tower in A, but suppose that it is not one in B. Then there is a $b \in B\backslash\{1\}$ such that $a \le b$ for all $a \in T$. By hypothesis, $A \restriction b$ is a countably generated ideal of A; say it is generated by the countable set Y. Let $\langle a_\alpha : \alpha < \kappa\rangle$ enumerate T in strictly increasing order; thus κ is a regular uncountable cardinal by hypothesis. For each $\alpha < \kappa$ let $b_\alpha \in Y$ be such that $a_\alpha \le b_\alpha$. Since κ is regular uncountable and Y is countable, there is a $c \in Y$ such that $b_\alpha = c$ for κ many $\alpha < \kappa$. Hence c is above each member of T, contradiction. □

Problem 35. *Given nonempty sets $K \subseteq L$ of cardinals with $\omega \notin K$, are there BAs $A \leq_\sigma B$ such that* $\mathrm{tow}_{\mathrm{spect}}(A) = K$ *and* $\mathrm{tow}_{\mathrm{spect}}(B) = L$*?*

If $\mathrm{tow}(B) > \omega$, then for any countably infinite subalgebra of B we have $A \leq_\sigma B$ and $\mathrm{tow}(A) < \mathrm{tow}(B)$.

As in the case of $\mathfrak{a}$, there are BAs A, B with $A \leq_\mathrm{u} B$ and $\mathrm{tow}(A) < \mathrm{tow}(B)$.

There are BAs A, B with $A \leq_{\mathrm{mg}} B$ and $\mathrm{tow}(A) < \mathrm{tow}(B)$. Namely, let $B = \mathrm{Intalg}(\omega_1)$ and let A be a countably infinite subalgebra of B appearing in a chain showing that A is minimally generated; see Theorem 2.52.

It is possible, in the other direction, to have BAs A, B with $A \leq_{\mathrm{mg}} B$ and $\mathrm{tow}(A) > \mathrm{tow}(B)$. Namely let L be an η_1 set, and let $M = L + \omega$. Set $B = \mathrm{Intalg}(M)$ and $A = \mathrm{Intalg}(L)$. Then $\mathrm{tow}(B) = \omega$, since $\langle [-\infty, i) : i \in \omega \rangle$ is a tower in B. Let $C = \{[-\infty, u) : u \in L\}$. Then C generates A, and the function f which is the identity on C and maps 1_A to 1_B extends to an isomorphism of A into B by Remark 15.2 of the handbook. By the proof of Proposition 2.52 we have $A \leq_{\mathrm{mg}} B$. Finally, we claim that $\mathrm{tow}(A) > \omega$. For, suppose that $\langle t_i : i < \omega \rangle$ is a tower in A, with $t_0 = 0$. Let E be the set of all intervals $[a, b)$ occurring in the standard representations of members of $\{t_i : i < \omega\}$. Note that E is infinite. If $a \neq -\infty$ for all $[a, b) \in E$, then by the η_1 property there exist $c < d$ with $d < a$ for all $[a, b) \in E$. Hence $[c, d)$ is disjoint from each member of D, contradiction. So $[-\infty, b) \in E$ for some b. Then by the same argument, there is some element $[b, c) \in E$. Continuing, we get elements $[-\infty, b_0), [b_0, b_1), \ldots$ of E. Then by the η_1 property there exist $[c, d)$ with each $b_i < c < d$ and $d < e$ for any element $[e, f) \in E$ such that $b_i < e$ for all $i \in \omega$. Hence again $[c, d)$ is disjoint from each member of D, contradiction. This proves the claim.

Proposition 4.36. *If $A \leq_\mathrm{m} B$, then* $\mathrm{tow}_{\mathrm{spect}}(A) \subseteq \mathrm{tow}_{\mathrm{spect}}(B)$ *and* $\mathrm{tow}(B) \leq \mathrm{tow}(A)$*.*

Proof. See the proof of Proposition 3.34. □

Again we have a problem in connection with this proposition:

Problem 36. *Given nonempty sets $K \subseteq L$ of cardinals, are there BAs $A \leq_\mathrm{m} B$ such that* $\mathrm{tow}_{\mathrm{spect}}(A) = K$ *and* $\mathrm{tow}_{\mathrm{spect}}(B) = L$*?*

There are BAs A, B with $A \leq_{\mathrm{free}} B$, $\mathrm{tow}(B) = \omega$, and $\mathrm{tow}(A)$ arbitrarily large. Similarly for $\leq_{\mathrm{proj}}$, $\leq_{\mathrm{rc}}$, $\leq_\sigma$, $\leq_{\mathrm{reg}}$, and $\leq_u$.

One can have $A \leq_\pi B$, hence also $A \leq_{\mathrm{reg}} B$, with $\mathrm{tow}(A)$ arbitrary and $\mathrm{tow}(B) = \omega$. Simply take $B = \overline{A}$.

There are BAs A, B with $A \leq_\mathrm{s} B$ and $\mathrm{tow}(B) < \mathrm{tow}(A)$. In fact, let A be ω_1-saturated. Clearly $\mathrm{tow}(A) \geq \omega_1$. Let $\langle a_\xi : \xi < \omega_1 \rangle$ be strictly increasing in A. Let $I = \{b \in A : a_n \cdot b = 0 \text{ for all } n \in \omega\}$. Then I is an ideal in A. Let $J = \{0\}$. Then let $B = A(x)$ with $A \upharpoonright x = I$ and $A \upharpoonright -x = J$; see Proposition 2.28. We claim that

$\langle a_n\cdot -x+x : n\in\omega\rangle$ is a tower in B. If $m<n$, then clearly $a_m\cdot -x+x\le a_n\cdot -x+x$. Suppose that $(a_n\cdot -x+x)\cdot -(a_m\cdot -x+x)=0$. Thus $a_n\cdot -a_m\cdot -x=0$, so $a_n\cdot a_m\in I$. Hence $a_n\cdot -a_m=a_n\cdot a_n\cdot -a_m=0$, contradiction. Thus $\langle a_n\cdot -x+x : n\in\omega\rangle$ is strictly increasing. Suppose that $(a_n\cdot -x+x)\cdot(b\cdot x+c\cdot -x)=0$ for all $n\in\omega$; we want to show that $b\cdot x+c\cdot -x=0$. We have

$$0=(a_n\cdot -x+x)\cdot(b\cdot x+c\cdot -x)=b\cdot x+a_n\cdot c\cdot -x,$$

so $b\cdot x=0$ and $a_n\cdot c\cdot -x=0$ for all $n\in\omega$. Hence $b\le -x$ and so $b=0$. Also $a_n\cdot c\in I$ for all $n\in\omega$; so $a_n\cdot c=a_n\cdot a_n\cdot c=0$ for all $n\in\omega$, so that $c\in I$. Hence $c\le x$, so $c\cdot -x=0$.

Problem 37. *Does* $A\le_{\mathrm{s}} B$ *imply that* $\mathrm{tow}_{\mathrm{spect}}(A)\subseteq\mathrm{tow}_{\mathrm{spect}}(B)$ *or* $\mathrm{tow}(B)\le\mathfrak{t}(A)$*?*

Problem 38. *Are there BAs* A,B *such that* $A\le_{\mathrm{m}} B$ *and* $\mathrm{tow}(B)<\mathrm{tow}(A)$*?*

Proposition 4.37. *Let* A *and* B *be infinite BAs. Then*

(i) $\mathrm{tow}_{\mathrm{spect}}(A\times B)=\mathrm{tow}_{\mathrm{spect}}(A)\cup\mathrm{tow}_{\mathrm{spect}}(B)$.
(ii) $\mathrm{tow}(A\times B)=\min(\mathrm{tow}(A),\mathrm{tow}(B))$.

(Note that since we allow tow to have ∞ as a value, (ii) includes the statement that if neither $\mathrm{tow}(A)$ nor $\mathrm{tow}(B)$ have towers, then also $A\times B$ does not have a tower.)

Proof. (i): Suppose that $\langle a_\alpha:\alpha<\kappa\rangle$ is a tower in A. We may assume that $a_0\ne 0$. Then

$$\langle(0,1)\rangle^\frown\langle(a_\alpha,1):\alpha<\kappa\rangle$$

is a tower in $A\times B$. Similarly, a tower in B yields a tower in $A\times B$. Thus $\supseteq$ in (i) holds.

Now suppose that $\langle(a_\alpha,b_\alpha):\alpha<\kappa\rangle$ is a tower in $A\times B$, κ regular. Define $\alpha\equiv\beta$ iff $a_\alpha=a_\beta$. Thus $\equiv$ is an equivalence relation on κ. If there are κ equivalence classes, this gives a tower in A of order type κ. If there are fewer than κ equivalence classes, then some equivalence class has κ elements, and this gives a tower in B of order type κ.

(ii): Note that the argument for (i) shows that $A\times B$ has a tower iff A or B has a tower; so (ii) follows from (i). □

Proposition 4.38. *Suppose that* $\langle A_i:i\in I\rangle$ *is a system of infinite BAs, with* I *infinite.*

(i) *If* $|I|=\omega$, *then* $\mathrm{tow}_{\mathrm{spect}}(\prod^{\mathrm{w}}_{i\in I}A_i)=\{\omega\}\cup\bigcup_{i\in I}\mathrm{tow}_{\mathrm{spect}}(A_i)$.
(ii) *If* $|I|>\omega$, *then* $\mathrm{tow}_{\mathrm{spect}}(\prod^{\mathrm{w}}_{i\in I}A_i)=\bigcup_{i\in I}\mathrm{tow}_{\mathrm{spect}}(A_i)$.
(iii) *If* $|I|=\omega$, *then* $\mathrm{tow}(\prod^{\mathrm{w}}_{i\in I}A_i)=\omega$.
(iv) *If* $|I|>\omega$, *then* $\mathrm{tow}(\prod^{\mathrm{w}}_{i\in I}A_i)=\min_{i\in I}\mathrm{tow}(A_i)$.

Proof. Clearly (iii) and (iv) follow from (i) and (ii). From Proposition 4.37 it follows that $\mathrm{tow}_{\mathrm{spect}}(A_i) \subseteq \mathrm{tow}_{\mathrm{spect}}(\prod^{\mathrm{w}}_{i\in I} A_i)$ for every $i \in I$. Now suppose that $\langle x_\alpha : \alpha < \kappa\rangle$ is a tower in $\prod^{\mathrm{w}}_{i\in I} A_i$, κ regular. We want to show that either $|I| = \omega = \kappa$, or $\kappa \in \bigcup_{i\in I} \mathrm{tow}_{\mathrm{spect}}(A_i)$.

Case 1. There is an $\alpha < \kappa$ such that $\{i \in I : x_\alpha(i) \neq 1\}$ is finite. Then $\kappa \in \bigcup_{i\in I} \mathrm{tow}_{\mathrm{spect}}(A_i)$ by Proposition 4.37.

Case 2. $M_\alpha \stackrel{\mathrm{def}}{=} \{i \in I : x_\alpha(i) \neq 0\}$ is finite for all $\alpha < \kappa$. If $\kappa < |I|$, then there is an $i \in I$ such that $x_\alpha(i) = 0$ for all $\alpha < \kappa$, contradicting $\sum_{\alpha<\kappa} x_\alpha = 1$. So $|I| \leq \kappa$. We may assume that κ is uncountable and regular. Clearly $M_\alpha \subseteq M_\beta$ if $\alpha < \beta < \kappa$. Hence there is an $\alpha < \kappa$ such that $M_\alpha = M_\beta$ for all $\beta \in [\alpha, \kappa)$. This again gives an $i \in I$ such that $x_\alpha(i) = 0$ for all $\alpha < \kappa$, contradicting $\sum_{\alpha<\kappa} x_\alpha = 1$. □

Kevin Selker has generalized Proposition 4.38 to moderate products:

Let $\langle A_i : i \in I\rangle$ be a system of BAs, and B a BA, subject to the conditions for a moderate product. Then $\mathrm{tow}(\prod^B_{i\in I} A_i) = \min(\mathrm{tow}(B), \min_{i\in I} \mathrm{tow}(A_i))$.

Proposition 4.39. *If I is infinite, then* $\mathrm{tow}(\prod_{i\in I} A_i) = \omega$ *for infinite BAs A_i.* □

Concerning towers and free products we have the following result from Monk [07].

Theorem 4.40. $\mathrm{tow}_{\mathrm{spect}}(A \oplus B) = \mathrm{tow}_{\mathrm{spect}}(A) \cup \mathrm{tow}_{\mathrm{spect}}(B)$ *for any infinite BAs A, B. Hence* $\mathrm{tow}(A \oplus B) = \min\{\mathrm{tow}(A), \mathrm{tow}(B)\}$.

Proof. Clearly every tower in A is a tower in $A \oplus B$; and similarly for B. So $\supseteq$ holds. Now suppose that $\langle c_\alpha : \alpha < \kappa\rangle$ is a tower in $A \oplus B$. Thus κ is a regular cardinal, and we want to show that $\kappa \in \mathrm{tow}_{\mathrm{spect}}(A) \cup \mathrm{tow}_{\mathrm{spect}}(B)$. For each $\alpha < \kappa$ write

$$c_\alpha = \sum_{i<n_\alpha} (a_i^\alpha \cdot b_i^\alpha), \tag{1}$$

where $\forall i < n_\alpha [0 \neq a_i \in A$ and $0 \neq b_i \in B]$, and for all distinct $i, j < n_\alpha$, $b_i^\alpha \cdot b_j^\alpha = 0$. We call n_α the *length* of c_α.

First we take the case $\kappa > \omega$, where we adapt arguments of McKenzie, Monk [82]. Here we first take the case in which $n_\alpha = n$ does not depend on α, and we proceed by induction on n. The statement that we prove by induction is as follows:

(2) For every positive integer n and every pair $(w, x) \in A \times B$, if $\langle c_\alpha : \alpha < \kappa\rangle$ is a tower in $(A \oplus B) \restriction (w \cdot x)$ and each c_α is given by (1), with $n_\alpha = n$ for all $\alpha < \kappa$, $a_i^\alpha \leq w$ and $b_i^\alpha \leq x$ for all $i < n$ and $\alpha < \kappa$, then A or B has a tower of order type κ.

For $n = 1$ we have $c_\alpha = a_0^\alpha \cdot b_0^\alpha$ for each $\alpha < \kappa$. Clearly $\alpha < \beta$ implies that $a_0^\alpha \leq a_0^\beta$, $b_0^\alpha \leq b_0^\beta$, and $a_0^\alpha < a_0^\beta$ or $b_0^\alpha < b_0^\beta$. Define $\alpha \equiv \beta$ iff $a_0^\alpha = a_0^\beta$. This is an equivalence relation on κ. If there are κ many equivalence classes, then there is a strictly increasing subsequence $\langle a_0^{\alpha(\xi)} : \xi < \kappa\rangle$ of order type κ. It is easily checked that $\langle a_0^{\alpha(\xi)} + -w : \xi < \kappa\rangle$ is a tower in A. If there are fewer than κ equivalence classes, then one of them has size κ, and we get a strictly increasing sequence $\langle b_0^{\alpha(\xi)} + -x : \xi < \kappa\rangle$ which is a tower in B. This takes care of the case $n = 1$.

Now we assume inductively that $n > 1$. Obviously we have

(3) If $\alpha < \beta$ and $i < n$, then $a_i^\alpha \cdot b_i^\alpha \cdot \prod_{j<n}(-a_j^\beta + -b_j^\beta) = 0$.

(4) If $\alpha < \beta$ and $i < n$, then $b_i^\alpha \leq \sum_{j<n} b_j^\beta$.

In fact, from (3) we have $b_i^\alpha \cdot \prod_{j<n}(-b_j^\beta) = 0$, giving (4).

(5) If $\alpha < \beta$, $i, k < n$, and $b_i^\alpha \cdot b_k^\beta \neq 0$, then $a_i^\alpha \leq a_k^\beta$.

For, multiplying (3) by b_k^β we get $a_i^\alpha \cdot b_i^\alpha \cdot b_k^\beta \cdot -a_k^\beta = 0$, giving (5).

(6) If M is a finite subset of κ, then there is a $\gamma \in {}^M n$ such that $a_{\gamma(\alpha)}^\alpha \leq a_{\gamma(\beta)}^\beta$ whenever $\alpha, \beta \in M$ and $\alpha < \beta$.

This follows by an easy induction from (4) and (5).

(7) There is a $\gamma \in {}^\kappa n$ such that $\forall \alpha, \beta \in \kappa[\alpha < \beta \Rightarrow a_{\gamma(\alpha)}^\alpha \leq a_{\gamma(\beta)}^\beta]$.

This follows from (6) and the compactness of the space ${}^\kappa n$, since for any $\alpha < \beta$ the set $\{\gamma \in {}^\kappa n : a_{\gamma(\alpha)}^\alpha \leq a_{\gamma(\beta)}^\beta\}$ is clearly closed.

Define $\alpha \equiv \beta$ iff $a_{\gamma(\alpha)}^\alpha = a_{\gamma(\beta)}^\beta$. If there are κ equivalence classes, we get a tower in A of order type κ, as in the argument for $n = 1$. So, assume otherwise, let M be an equivalence class with κ elements, and let $u = a_{\gamma(\alpha)}^\alpha$ for any $\alpha \in M$. Then $c_\alpha = c_\alpha \cdot -u + c_\alpha \cdot u$. By Lemma 4.2 we now have two cases.

Case 1. There is an $N \in [M]^\kappa$ such that $\langle c_\alpha \cdot -u : \alpha \in N\rangle$ is strictly increasing. Each $c_\alpha \cdot -u$ clearly has length less than n; so there is a $P \in [N]^\kappa$ such that each $c_\alpha \cdot -u$ has the same length less than n for each $\alpha \in P$. Let $\langle \alpha(\xi) : \xi < \kappa\rangle$ be the strictly increasing enumeration of P. We claim that $\langle c_{\alpha(\xi)} \cdot -u : \xi < \kappa\rangle$ is a tower in $(A \oplus B) \restriction (w \cdot -u \cdot x)$. To show this, it suffices to take $d \in A \restriction (w \cdot -u)$, $e \in B \restriction x$, assume that $c_{\alpha(\xi)} \cdot -u \cdot d \cdot e = 0$ for every $\xi < \kappa$ and show that $d = 0$ or $e = 0$. Since $\langle c_{\alpha(\xi)} : \xi < \kappa\rangle$ is a tower in $(A \oplus B) \restriction (w \cdot x)$ we have $-u \cdot d \cdot e = 0$. Hence either $e = 0$ or $-u \cdot d = 0$. In the latter case $d = 0$ since $d \leq -u$. This proves the claim. Now the induction hypothesis gives the desired result.

Case 2. There is an $N \in [M]^\kappa$ such that $\langle c_\alpha \cdot u : \alpha \in N\rangle$ is strictly increasing. If $\{\alpha \in N : c_\alpha \cdot u$ can be written as in (1) with a smaller $n\}$ has size κ, then we get our result as in Case 1. So, suppose that this set has size less than κ. Let $P = \{\alpha \in N : c_\alpha \cdot u$ cannot be written as in (1) with a smaller $n\}$; so $|P| = \kappa$.

(8) For all $\alpha, \beta \in P$, if $\alpha < \beta$ then $b_{\gamma(\alpha)}^\alpha \leq b_{\gamma(\beta)}^\beta$.

For, by (4) it suffices to show that $b^\alpha_{\gamma(\alpha)}\cdot b^\beta_i=0$ for all $i\neq\gamma(\beta)$. Suppose that this product is nonzero for some $i\neq\gamma(\beta)$. Then by (5), $a^\alpha_{\gamma(\alpha)}\le a^\beta_i$, i.e., $u\le a^\beta_i$. Hence

$$\begin{aligned}c_\beta\cdot u&=u\cdot\sum_{j<n}(a^\beta_j\cdot b^\beta_j)\\&=u\cdot a^\beta_{\gamma(\beta)}\cdot b^\beta_{\gamma(\beta)}+u\cdot a^\beta_i\cdot b^\beta_i+u\cdot\sum_{j\neq i,\gamma(\beta)}(a^\beta_j\cdot b^\beta_j)\\&=u\cdot(b^\beta_{\gamma(\beta)}+b^\beta_i)+u\cdot\sum_{j\neq i,\gamma(\beta)}(a^\beta_j\cdot b^\beta_j),\end{aligned}$$

which contradicts $\beta\in P$. Thus (8) holds.

Again, define $\alpha\equiv\beta$ iff $\alpha,\beta\in P$ and $b^\alpha_{\gamma(\alpha)}=b^\beta_{\gamma(\beta)}$. This is an equivalence relation on P. If there are κ equivalence classes, this yields a tower in B of order type κ. So, suppose that there are fewer than κ equivalence classes, and let Q be a class with κ elements. Say $b^\alpha_{\gamma(\alpha)}=v$ for all $\alpha\in Q$. Thus for any $\alpha\in Q$ we have $c_\alpha=u\cdot v+\sum_{i\neq\gamma(\alpha)}(a^\alpha_i\cdot b^\alpha_i)$. It follows that the sequence $\langle\sum_{i\neq\gamma(\alpha)}(a^\alpha_i\cdot b^\alpha_i):\alpha\in Q\rangle$ is strictly increasing. Let $\langle\alpha(\xi):\xi<\kappa\rangle$ be the strictly increasing enumeration of Q. We claim that $\langle\sum_{i\neq\gamma(\alpha(\xi))}(a^{\alpha(\xi)}_i\cdot b^{\alpha(\xi)}_i):\xi<\kappa\rangle$ is a tower in $(A\oplus B)\restriction(w\cdot x\cdot -v)$. To prove this it suffices to take any $d\in A\restriction w$ and $e\in B\restriction(x\cdot -v)$, assume that $d\cdot e\cdot\sum_{i\neq\gamma(\alpha(\xi))}(a^{\alpha(\xi)}_i\cdot b^{\alpha(\xi)}_i)=0$ for all $\xi<\kappa$, and show that $d=0$ or $e=0$. But $e\cdot v=0$, so $d\cdot e\cdot c_{\alpha(\xi)}=0$ for all $\xi<\kappa$, so $d=0$ or $e=0$. Now the desired conclusion follows by the inductive hypothesis.

This finishes the induction on n. Now we consider the general case, where n_α depends upon α. But since κ is regular, there is an $m\in\omega$ such that $\{\alpha<\kappa: n_\alpha=m\}$ has κ elements, and so our inductive result applies.

So this completely finishes the case κ uncountable and regular.

Now we consider the case $\kappa=\omega$. It is convenient to consider the dual of towers for this proof. So, suppose that $\langle c_i:i<\omega\rangle$ is a strictly decreasing sequence of elements of $A\oplus B$ with meet 0; we want to get a contradiction. For each $i<\omega$ write

$$c_i=\sum_{k<m_i}a_{i,k}\cdot b_{i,k}$$

with $0\neq a_{i,k}\in A$, $0\neq b_{i,k}\in B$, and $b_{i,k}\cdot b_{i,l}=0$ if $k\neq l$; we do not assume that m_i is minimal. Now we define by recursion on i an element $m'_i\in\omega$, $a'_{ik}\in A$ for $k<m'_i$, and $b'_{ik}\in B$ for $k<m'_i$ so that $c_i=\sum_{k<m'_i}a'_{ik}\cdot b'_{ik}$. Let $m'_0=m_0$, $a'_{0k}=a_{0k}$, and $b'_{0k}=b_{0k}$ for $k<m_0$. Suppose that the definition has been made for i. Note that if $k<m_{i+1}$ then $a_{i+1,k}\cdot b_{i+1,k}\cdot\prod_{l<m'_i}(-a'_{il}+-b'_{il})=0$, and hence $b_{i+1,k}\le\sum_{l<m'_i}b'_{il}$. It follows that

$$c_{i+1}=\sum_{k<m_{i+1}}a_{i+1,k}\cdot\sum\{b_{i+1,k}\cdot b'_{il}:l<m'_i,b_{i+1,k}\cdot b'_{il}\neq0\}.$$

Rearranging this sum, we get

$$c_{i+1} = \sum_{k<m'_{i+1}} a'_{i+1,k} \cdot b'_{i+1,k}, \tag{9}$$

where $b'_{i+1,k} \cdot b'_{i+1,l} = 0$ if $k \neq l$, and for each $k < m'_{i+1}$ there is an $l < m'_i$ such that $b'_{i+1,k} \leq b'_{il}$.

Next we claim:

(10) $\exists k < m_0 \forall i > 0 \exists l < m_i (b'_{i,l} \leq b'_{0,k})$.

In fact, otherwise for every $k < m_0$ there is an $i(k) > 0$ such that for every $l < m_{i(k)}$ we have $b'_{i(k),l} \not\leq b'_{0,k}$; by (9) we have $b'_{i(k),l} \leq b'_{0s}$ for some $s < m_0$, hence $s \neq k$ and so $b'_{i(k),l} \cdot b'_{0k} = 0$. So $c_{i(k)} \cdot b'_{0,k} = 0$. If $j > i(k)$ for all $k < m_0$, we then get $c_j \cdot c_0 = 0$, contradiction. So (10) holds. We fix such a k.

(11) For every $i > 0$ there is an $l \in \prod_{0<j\leq i} m_j$ such that $b'_{0,k} \geq b'_{1,l(1)} \geq \cdots \geq b'_{i,l(i)}$.

In fact, let $l(0) = k$. By (10), choose $l(i) < m_i$ such that $b'_{i,l(i)} \leq b'_{0,k}$. Then for $0 < j < i$, by (9) choose $l(j) < m_j$ such that $b'_{i,l(i)} \leq b'_{j,l(j)}$. Now suppose that $0 \leq j < n \leq i$. Then $b'_{i,l(i)} \leq b'_{j,l(j)} \cdot b'_{n,l(n)}$, so by (9) we get $b'_{n,l(n)} \leq b'_{j,l(j)}$. So (11) holds.

By (11) and König's tree lemma we get $l \in \prod_{0<j<\omega} m_j$ such that $b'_{0,k} \geq b'_{1,l(1)} \geq \cdots \geq b'_{i,l(i)} \geq \cdots$. Now assume that B has no tower of length ω. It follows that the meet of all $b'_{i,l(i)}$'s is not zero; say that $d \neq 0$ and $d \leq b'_{i,l(i)}$ for every positive integer i. Now if $0 < i < j$, then $0 \neq c_j \cdot d \leq c_i \cdot d$, and for any $i > 0$ we have $c_i \cdot d = a_{l(i)} \cdot d$. Hence $0 = \prod_{0<i<\omega}(c_i \cdot d) = \prod_{0<i<\omega}(a_{l(i)} \cdot d)$, and it follows that $\langle a_{l(i)} : 0 < i < \omega\rangle$ is a decreasing sequence with meet 0, and hence some subsequence is a (dual) tower in A. $\square$

Proposition 4.41. *If $\langle A_i : i \in I\rangle$ is a system of BAs each with at least 4 elements, with I infinite, then* $\mathrm{tow}(\oplus_{i\in I} A_i) = \omega$.

Proof. Choose $j \in {}^\omega I$ one-one, and for each $k \in \omega$ choose $c_k \in A_{j_k}$ with $0 < c_k < 1$. Let $b'_i = \sum_{k\leq i} c_k$ for each $i \in \omega$. Clearly $\langle b'_i : i \in \omega\rangle$ is strictly increasing. We claim that $\sum_{i\in\omega} b'_i = 1$. For, suppose that $x \in \oplus_{i\in I} A_i$ with $b'_i \leq x < 1$ for all $i \in \omega$. We can write $x = \sum_{k\in M} d_k$ with M a finite subset of I and $d_k \in A^+_k$ for all $k \in M$. Choose $l \in \omega$ such that $j_l \notin M$. Let f be a homomorphism of $\oplus_{i\in I} A_i$ into 2 such that $f(d_k) = 0$ for all $k \in M$ and $f(c_{j_l}) = 1$. Then $f(b'_l) = 1$ while $f(x) = 0$, contradiction. $\square$

Problem 39. *Describe the behaviour of* tow *under ultraproducts.*

Now we work towards a characterization of $\mathrm{tow}_{\mathrm{spect}}$, taken from Monk [07].

Lemma 4.42. *Let A be any infinite BA, and let $\mathcal{S}$ be the Stone isomorphism. Then no tower in $\mathcal{S}[A]$ remains a tower in $\mathscr{P}(\mathrm{Ult}(A))$.*

Proof. Suppose that $\langle a_\alpha : \alpha < \kappa\rangle$ is a tower in A, with κ a regular cardinal. Let $X = \bigcup_{\alpha<\kappa} \mathcal{S}(a_\alpha)$. By compactness, $X \neq \mathrm{Ult}(A)$. $\square$

Lemma 4.43. *Let A be an atomless BA, and let κ be a regular cardinal. Then $A \leq B$ for some atomless BA B such that* $\mathrm{tow}_{\mathrm{spect}}(B) = \{\kappa\}$.

Proof. We define a sequence $\langle C_\alpha : \alpha < \kappa\rangle$ of BAs as follows. Let $C_0 = A$. If C_α has been defined, let $C_\alpha \subseteq C_{\alpha+1}$, where $C_{\alpha+1}$ is obtained from C_α by applying Lemma 4.42 and then extending to an atomless BA. For $\alpha < \kappa$ limit, let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. Finally, let $B = \bigcup_{\alpha<\kappa} C_\alpha$. We claim that B is as desired.

Every atomless BA has a tower, so it suffices to show that B has no tower of regular length different from κ. Suppose that $\langle b_\alpha : \alpha < \lambda\rangle$ is a tower in B, where $\lambda \neq \kappa$ is a regular cardinal. For each $\alpha < \lambda$ there is a $\xi_\alpha < \kappa$ such that $b_\alpha \in C_{\xi_\alpha}$.

First suppose that $\lambda < \kappa$. Choose $\beta < \kappa$ such that $\xi_\alpha < \beta$ for all $\alpha < \lambda$. Then $\langle b_\alpha : \alpha < \lambda\rangle$ is a tower in C_β. By construction and Lemma 4.42 it is not a tower in $C_{\beta+1}$ and hence also not in B, contradiction.

Second suppose that $\kappa < \lambda$. There is an $\beta < \kappa$ such that $\xi_\alpha = \beta$ for λ many $\alpha < \lambda$. Then $\langle b_\alpha : \alpha < \lambda\rangle$ has a cofinal subsequence in C_β; this gives a tower in C_β, and a contradiction follows as above. $\square$

According to the following theorem, there are no restrictions on the possibilities for $\mathrm{tow}_{\mathrm{spect}}$.

Theorem 4.44. *If M is a nonempty set of regular cardinals, then there is a BA A such that* $\mathrm{tow}_{\mathrm{spect}}(A) = M$.

Proof. Let f be a function mapping some uncountable set I onto M. For each $i \in I$ let A_i be a BA such that $\mathrm{tow}_{\mathrm{spect}}(A_i) = \{f(i)\}$, using Lemma 4.43. Then $\prod^{\mathrm{w}}_{i\in I} A_i$ is as desired, by Proposition 4.38(ii). $\square$

Theorem 4.45. *Let K and L be nonempty sets of regular cardinals. Then there are atomless BAs A, B such that $A \leq B$,* $\mathrm{tow}_{\mathrm{spect}}(A) = K$, *and* $\mathrm{tow}_{\mathrm{spect}}(B) = L$.

Proof. Choose A such that $\mathrm{tow}_{\mathrm{spect}}(A) = K$, by Theorem 4.44. Let κ be the least member of L. let C be such that $A \leq C$ and $\mathrm{tow}_{\mathrm{spect}}(C) = \{\kappa\}$, by Lemma 4.43. Choose D such that $\mathrm{tow}_{\mathrm{spect}}(D) = L$, by Theorem 4.44. Now $B \stackrel{\mathrm{def}}{=} C \times D$ is as desired. $\square$

Proposition 4.46. *For any atomless BA A, if $\omega \leq \kappa < \mathrm{tow}(A)$, then $2^\kappa \leq \mathrm{c}(A)$.*

Proof. We define elements a_t for each $t \in \bigcup_{\alpha\leq\kappa} {}^\alpha 2$ by recursion on α. for $\alpha = 0$ and $t \in {}^0 2$ we have $t = \emptyset$, and we set $a_0 = 1$. If a_τ has been defined for a certain $t \in {}^\alpha 2$ with $\alpha < \kappa$, we split a_t into two nonzero disjoint elements $a_{t^\frown\langle 0\rangle}$ and $a_{t^\frown\langle 1\rangle}$. If α is a limit ordinal (including the possibility that $\alpha = \kappa$, and a_t has been defined for all $t \in \bigcup_{\beta<\alpha} {}^{\beta}2$, and if $s \in {}^{\alpha}2$, let a_s be a nonzero element $\leq a_{s\restriction\beta}$ for all $\beta < \alpha$; such an element exists since $\kappa < \mathrm{tow}(A)$.

This finishes the construction. Clearly the elements $\langle a_t : t \in {}^{\kappa}2\rangle$ are nonzero and pairwise disjoint. □

We now consider some other "small" cardinal functions related to tow: $\mathfrak{a}$ (already discussed in Chapter 3),

spl, $\mathfrak{h}$. and $\mathfrak{p}$. The last three are defined as follows. A subset X of a BA A is *splitting* iff for all $a \in A$ there is an $x \in X$ such that $a \cdot x \neq 0 \neq a \cdot -x$.

$$\begin{aligned}
\mathrm{spl}(A) &= \min\{|X| : X \text{ is a splitting subset of } A\};\\
\mathfrak{h}(A) &= \min\{\kappa : A \text{ is not } (\kappa,\infty)\text{-distributive}\};\\
\mathfrak{p}(A) &= \min\left\{|X| : \sum X = 1 \text{ and } \sum Y \neq 1 \text{ for every finite subset } Y \text{ of } X\right\};\\
\mathfrak{p}_{\mathrm{spect}}(A) &= \left\{|X| : \sum X = 1 \text{ and } \sum Y \neq 1 \text{ for every finite subset } Y \text{ of } X\right\}.
\end{aligned}$$

These are well-known functions in the case $A = \mathscr{P}(\omega)/\mathrm{fin}$, where $\mathrm{spl}(A)$ is denoted by $\mathfrak{s}$. In the case of general BAs they are discussed in Monk [01a].

Proposition 4.47.

(i) $\mathfrak{a}_{\mathrm{spect}}(A) \subseteq \mathfrak{p}_{\mathrm{spect}}(A)$.
(ii) $\mathrm{tow}_{\mathrm{spect}}(A) \subseteq \mathfrak{p}_{\mathrm{spect}}(A)$.
(iii) $\mathfrak{p}(A) \leq \mathfrak{a}(A), \mathrm{tow}(A)$.

Proof. Clearly every infinite partition of unity is a $\mathfrak{p}$-set for A, and similarly for towers. □

Proposition 4.48. *If* $\mathfrak{p}(A) = \omega$, *then* $\mathrm{tow}(A) = \omega = \mathfrak{a}(A)$.

Proof. Let $Y = \{a_i : i \in \omega\}$ satisfy the condition defining $\mathfrak{p}(A)$. Then clearly $\langle \sum_{j\leq i} a_j : i \in \omega\rangle$ has a subsequence which is a tower; so $\mathrm{tow}(A) = \omega$. Furthermore, let $b_i = a_i \cdot \prod_{j<i} -a_j$ for each $i \in \omega$. Then $\{b_i : i \in \omega\}\backslash\{0\}$ is clearly an infinite partition of unity; so $\mathfrak{a}(A) = \omega$. □

Corollary 4.49. $\mathfrak{a}(A) = \omega$ iff $\mathfrak{p}(A) = \omega$ iff $\mathrm{tow}(A) = \omega$. □

Corollary 4.50. *If* $\mathfrak{a}(A), \mathfrak{a}(B) > \omega$, *then* $\mathfrak{a}(A \oplus B) > \omega$.

Proof. If $\mathfrak{a}(A\oplus B) = \omega$, then $\mathrm{tow}(A\oplus B) = \omega$, by Corollary 4.49. Hence $\mathrm{tow}(A) = \omega$ or $\mathrm{tow}(B) = \omega$, by Theorem 4.40. So $\mathfrak{a}(A) = \omega$ or $\mathfrak{a}(B) = \omega$ by Corollary 4.49 again. □

This corollary is relevant to Problem 8.

Now we consider the relationship between $\mathfrak{a}$ and tow when they are uncountable.

Theorem 4.51. *If $\omega < \kappa$ and M is a nonempty collection of regular cardinals each greater than κ, then there is a BA A such that* $\mathfrak{a}(A) = \kappa$ *and* $\mathrm{tow}_{\mathrm{spect}}(A) = M$.

Proof. By Theorem 4.44 there is a BA B such that $\mathrm{tow}_{\mathrm{spect}}(B) = M$. Note that the proofs of Lemma 4.43 and Theorem 4.44 also give $\mathfrak{a}(B)$ greater than each member of M. Now the weak product of κ many copies of B gives the algebra A desired, by Proposition 3.38(ii) and Proposition 4.38(ii). □

Note that one cannot hope to have $\mathfrak{a}_{\mathrm{spect}}(A) = \{\kappa\}$ here, since a tower of size greater than κ clearly gives rise to a partition of size greater than κ.

It is more difficult to get examples with $\mathfrak{t}$ less than $\mathfrak{a}$. See McKenzie, Monk [04] for the following result:

If κ and λ are regular cardinals with $\omega < \kappa < \lambda$, then there is a BA A such that $\mathrm{tow}(A) = \mathrm{spl}(A) = \kappa$ *and* $\mathfrak{a}(A) = \lambda$.

Problem 40. *Describe in cardinal number terms the pairs M, N of cardinals such that there is a BA A with* $\mathrm{tow}_{\mathrm{spect}}(A) = M$ *and* $\mathfrak{a}_{\mathrm{spect}}(A) = N$.

For our special kinds of subalgebras, the facts and problems concerning $\mathfrak{p}$ are very similar to those for $\mathfrak{a}$ and tow.

Proposition 4.52. *If $A \leq_{\mathrm{reg}} B$, then $\mathfrak{p}_{\mathrm{spect}}(A) \subseteq \mathfrak{p}_{\mathrm{spect}}(B)$ and hence $\mathfrak{p}(B) \leq \mathfrak{p}(A)$. The conclusion also holds if $A \leq_{\mathrm{free}} B$, $A \leq_{\mathrm{proj}} B$, $A \leq_{\mathrm{rc}} B$, $A \leq_{\pi} B$.* □

Problem 41. *Given nonempty sets $K \subseteq L$ of cardinals, are there BAs $A \leq_{\mathrm{reg}} B$ such that $\mathfrak{p}_{\mathrm{spect}}(A) = K$ and $\mathfrak{p}_{\mathrm{spect}}(B) = L$? There is a similar question for $A \leq_{\mathrm{free}} B$, $A \leq_{\mathrm{proj}} B$, $A \leq_{\mathrm{rc}} B$, $A \leq_{\pi} B$.*

Proposition 4.53. *If $A \leq_{\sigma} B$ and $\mathfrak{p}(A) > \omega$, then $\mathfrak{p}_{\mathrm{spect}}(A) \subseteq \mathfrak{p}_{\mathrm{spect}}(B)$, and $\mathfrak{p}(B) \leq \mathfrak{p}(A)$.*

Proof. See the proof of Proposition 4.35. □

Problem 42. *Given nonempty sets $K \subseteq L$ of cardinals with $\omega \notin K$, are there BAs $A \leq_{\sigma} B$ such that $\mathfrak{p}_{\mathrm{spect}}(A) = K$ and $\mathfrak{p}_{\mathrm{spect}}(B) = L$?*

One can have $A \leq_{\sigma} B$ with $\mathfrak{p}(A) = \omega < \mathfrak{p}(B)$. For example, let B have a large pseudo-intersection number, and let A be a countable subalgebra.

There are BAs A, B with $A \leq_{\mathrm{u}} B$ and $\mathfrak{p}(A) < \mathfrak{p}(B)$; see the argument for $\mathfrak{a}$.

There are BAs A, B with $A \leq_{\mathrm{mg}} B$ and $\mathfrak{p}(A) < \mathfrak{p}(B)$.

Proposition 4.54. *If $A \leq_{\mathrm{m}} B$, then $\mathfrak{p}_{\mathrm{spect}}(A) \subseteq \mathfrak{p}_{\mathrm{spect}}(B)$, and hence $\mathfrak{p}(B) \leq \mathfrak{p}(A)$.* □

Problem 43. *Given nonempty sets $K \subseteq L$ of cardinals with $\omega \notin K$, are there BAs $A \leq_{\mathrm{mm}} B$ such that $\mathfrak{p}_{\mathrm{spect}}(A) = K$ and $\mathfrak{p}_{\mathrm{spect}}(B) = L$?*

There are BAs A, B with $A \leq_{\mathrm{free}} B$, $\mathfrak{p}(B) = \omega$, and $\mathfrak{p}(A)$ arbitrarily large. Similarly for $\leq_{\mathrm{proj}}$, $\leq_{\mathrm{rc}}$, $\leq_{\sigma}$, $\leq_{\mathrm{reg}}$, and $\leq_{u}$.

One can have $A \leq_\pi B$, hence also $A \leq_{\rm reg} B$, with $\mathfrak{p}(A)$ arbitrary and $\mathfrak{p}(B) = \omega$. Simply take $B = \overline{A}$.

One can have $A \leq_{\rm s} B$ with $\mathfrak{p}(B) < \mathfrak{p}(A)$. See the corresponding example for tow.

Problem 44. *Does $A \leq_{\rm s} B$ imply that $\mathfrak{p}_{\rm spect}(A) \subseteq \mathfrak{p}_{\rm spect}(B)$?*

There are BAs A, B such that $A \leq_{\rm mg} B$ and $\mathfrak{p}(B) < \mathfrak{p}(A)$. See the construction following Problem 35.

Problem 45. *Are there BAs A, B such that $A \leq_{\rm m} B$ and $\mathfrak{p}(B) < \mathfrak{p}(A)$?*

Proposition 4.55. $\mathfrak{p}(A \times B) = \min(\mathfrak{p}(A), \mathfrak{p}(B))$.

Proof. If $X \subseteq A$, $\sum X = 1$, and $\sum F < 1$ for every finite subset F of A, then $\sum(\{(x,0) : x \in X\} \cup \{(0,1)\}) = 1$ and $\sum(\{(x,0) : x \in F\} \cup \{(0,1)\}) < (1,1)$ for every finite subset F of X. Similarly for B, so $\leq$ holds.

Now suppose that $Y \subseteq A \times B$, $\sum Y = (1,1)$, and $\sum F < (1,1)$ for every finite subset F of Y. Let $U = \{a \in A : \exists b \in B[(a,b) \in Y]\}$ and $V = \{b \in B : \exists a \in A[(a,b) \in Y]\}$. Then $\sum U = 1$ and $\sum V = 1$. If $\sum F = 1$ and $\sum G = 1$ for some finite subsets F of U and G of V, this clearly yields a finite subset H of Y such that $\sum H = (1,1)$, contradiction. $\square$

The following two propositions have proofs just like the proofs of Propositions 3.38 and 3.39.

Proposition 4.56. *Suppose that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite. Then*

(i)
$$\mathfrak{p}_{\rm spect}\left(\prod_{i\in I}^{w} A_i\right) = \{|I|\} \cup \bigcup_{i\in I} \mathfrak{p}_{\rm spect}(A_i) \cup \Bigg\{\kappa : \kappa > |I|,\ \kappa \textit{ is singular, and } \exists J \subseteq I \exists \lambda \in \prod_{j\in J} \mathfrak{p}_{\rm spect}(A_j)\left(J \neq \emptyset \textit{ and } \kappa = \sup_{j\in J} \lambda_j\right)\Bigg\}.$$

(ii)
$$\mathfrak{p}\left(\prod_{i\in I}^{w} A_i\right) = \min(|I|, \min_{i\in I} \mathfrak{p}(A_i)). \qquad \square$$

Proposition 4.57. *Let $\langle A_i : i \in I\rangle$ be a system of infinite BAs, with I infinite.*

Then:

(i)
$$\mathfrak{p}_{\rm spect}\left(\prod_{i\in I}A_i\right) = [\omega, |I|] \cup \bigcup_{i\in I}\mathfrak{p}_{\rm spect}(A_i)$$
$$\cup \left\{\kappa : \kappa > |I|,\ \kappa \textit{ is singular, and}\right.$$
$$\left. \exists J \subseteq I \exists \lambda \in \prod_{j\in J} p_{\rm spect}(A_j)\left[J \neq \emptyset \textit{ and } \kappa = \sup_{j\in J}\lambda_j\right]\right\}.$$

(ii)
$$\mathfrak{a}\left(\prod_{i\in I}A_i\right) = \omega.$$
□

Kevin Selker has generalized Proposition 4.56 to moderate products. In particular, he showed:

For a moderate product, $\mathfrak{p}(\prod_{i\in I}^B A_i) = \min(\mathfrak{p}(B), \min_{i\in I}\mathfrak{p}(A_i))$.

Proposition 4.58. *If* $\mathfrak{p}(A), \mathfrak{p}(B) > \omega$, *then* $\mathfrak{p}(A \oplus B) > \omega$, *for infinite BAs* A, B.

Proof. Suppose that $\mathfrak{p}(A \oplus B) = \omega$. Then $\mathfrak{a}(A \oplus B) = \omega$ by Corollary 4.48. So $\mathfrak{a}(A) = \omega$ or $\mathfrak{a}(B) = \omega$ by Corollary 4.50. Then $\mathfrak{p}(A) = \omega$ or $\mathfrak{p}(B) = \omega$ by Corollary 4.49. □

Problem 46. *Is it true that for all infinite BAs* A, B *we have* $\mathfrak{p}(A \oplus B) = \min(\mathfrak{p}(A), \mathfrak{p}(B))$*?*

Proposition 4.59. $\mathfrak{p}(\oplus_{i\in I}A_i) = \omega$ *for any infinite system* $\langle A_i : i \in I\rangle$ *of infinite BAs.*

Proof. By Propositions 4.41 and 4.49. □

Concerning the following theorem, see Proposition 4.47(iii).

Theorem 4.60. *Let* κ *and* λ *be regular cardinals, with* $\omega_1 < \kappa \leq \lambda$. *Then there is a linear order* L *such that for* $A = \text{Intalg}(L)$ *we have* $\mathfrak{p}(A) = \kappa$, $\text{tow}(A) = \lambda$, *and* $\mathfrak{a}(A) \geq \lambda$.

Proof. By Hausdorff's theorem (Hausdorff [1908]) let L have the following properties:

(1) Every element of L has character (λ, λ^*).

(2) The gaps of L have the following characters:
 (a) (ω_1, κ^*) and (κ, ω_1^*).
 (b) (ν, λ^*) and (λ, ν^*) for all $\nu \in [\omega, \lambda)$.

(3) L has coinitiality λ^* and cofinality λ.

Let $A = \text{Intalg}(L)$. Later in this section we characterize those μ such that an interval algebra has a tower of length μ. Applying this to A, we get $\text{tow}(A) = \lambda$. In Chapter 14 we prove that $\mathfrak{p}(A) = \kappa$. It remains to show that $\mathfrak{a}(A) \geq \lambda$. Let X be any infinite partition of A. Without changing $|X|$ we may assume that each element of X has the form $[a, b)$ with $a, b \in L \cup \{-\infty, \infty\}$.

If $[a, b) \in X$ with $b < \infty$, and $\forall [c, d) \in X[b \neq c]$, then there is a sequence $\langle [x_\xi, y_\xi) : \xi < \lambda$ of elements of X, strictly decreasing under $<$ (between disjoint subsets of L), with inf b. Thus we may assume that

(1) $\forall [a, b) \in X \exists c \in L \cup \{\infty\}[[b, c) \in X]$.

If $\forall [a, b) \in X[a \neq -\infty]$, then there is a sequence $\langle [x_\xi, y_\xi) : \xi < \lambda$ of elements of X, strictly decreasing under $<$, with inf $-\infty$. Hence we may assume that $[-\infty, a) \in X$ for some $a \in L$. Then by (1) we get an increasing sequence $\langle [b_i, c_i) : i \in \omega \rangle$ of members of X such that $a = b_0$ and $c_i = b_{i+1}$ for all $i \in \omega$. Then $\{x \in L : x \leq b_i$ for some $i \in \omega\}$ determines a gap in L with lower character ω. Its upper character is λ^*, and hence there is a sequence $\langle [x_\xi, y_\xi) : \xi < \lambda$ of elements of X, strictly decreasing under $<$, coinitial in this gap. □

However, the following two questions arise.

Problem 47. *Given cardinals κ, λ, μ with $\omega < \kappa \leq \lambda, \mu$, is there a BA A such that $\mathfrak{p}(A) = \kappa$, $\text{tow}(A) = \lambda$, and $\mathfrak{a}(A) = \mu$?*

Problem 48. *Is*

$$\mathfrak{p}(A) = \min\{|X| : X \textit{ is a maximal ramification set in } A\}?$$

Recall here the notion of a ramification set from Chapter 2.

We also mention the following well-known problem.

Problem 49. *Is it consistent that $\mathfrak{p}(\mathscr{P}(\omega)/\text{fin}) < \text{tow}(\mathscr{P}(\omega)/\text{fin})$?*

For some results on this problem, see Shelah [09]. The following simple lemma is found in this paper specialized to $\mathscr{P}(\omega)/\text{fin}$. A *strong $\mathfrak{p}$-set for A* is a subset X of A^+ closed under $\cdot$ such that $\prod X = 0$ and $|X| = \mathfrak{p}(A)$.

Proposition 4.61. *Suppose that $\mathfrak{p}(A) < \text{tow}(A)$, and X is a strong $\mathfrak{p}$-set for A. Then there exist a regular $\kappa < \mathfrak{p}(A)$ and a system $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of A such that the following conditions hold:*

(i) *If $\alpha < \beta < \kappa$, then $a_\beta \leq a_\alpha$.*
(ii) *$a_\alpha \cdot b \neq 0$ for all $\alpha < \kappa$ and $b \in X$.*
(iii) *if c is a nonzero lower bound for $\{a_\alpha : \alpha < \kappa\}$, then there is a $b \in X$ such that $b \cdot c = 0$.*

Proof. Write $X = \{b_\alpha : \alpha < \mathfrak{p}(A)\}$. We define $\langle a_\alpha : \alpha < \beta \rangle$ by recursion, where $\beta \leq \mathfrak{p}(A)$ is to be determined. Let $a_0 = 1$. Suppose that a_α has been defined so that $a_\alpha \cdot c \neq 0$ for all $c \in X$. Let $a_{\alpha+1} = a_\alpha \cdot b_\alpha$. Clearly $a_{\alpha+1} \cdot c \neq 0$ for all $c \in X$. Now suppose that a_α has been defined for all $\alpha < \gamma$, with γ a limit ordinal. If there is a nonzero x such that $x \leq a_\alpha$ for all $\alpha < \gamma$ and $x \cdot c \neq 0$ for all $c \in X$, we let a_γ be such an x. Otherwise the construction stops, with $\beta = \gamma$.

If the construction continues all the way to $\mathfrak{p}(A)$, then clearly $\{a_\alpha : \alpha < \mathfrak{p}(A)\}$ is a downwards tower, contradicting $\mathfrak{p}(A) < \mathrm{tow}(A)$. Thus it ends at some limit ordinal $\beta < \mathfrak{p}(A)$. Clearly the conclusion of the proposition holds with $\kappa = \mathrm{cf}(\alpha)$ and a subsequence of $\langle a_\alpha : \alpha < \beta \rangle$. □

Turning to $\mathfrak{h}$, we recall some facts about distributivity; see the Handbook. $\mathfrak{h}(A)$ is defined for any non-atomic BA A. An equivalent condition for (κ, ∞)-distributivity is that every set of at most κ partitions of A has a common refinement.

Proposition 4.62. $\mathrm{tow}(A) \leq \mathfrak{h}(A)$ *for every atomless BA A.*

Proof. Let $\kappa = \mathfrak{h}(A)$. Let $\langle P_\alpha : \alpha < \kappa \rangle$ be a system of partitions of A which do not have a common refinement. We define by recursion another system $\langle Q_\alpha : \alpha < \kappa \rangle$ of refinements of A. Let $Q_0 = P_0$. If Q_α has been defined for all $\alpha < \beta$, where $\beta < \kappa$, let Q_β be a common refinement of all the Q_α's for $\alpha < \beta$ together with P_β. Clearly a common refinement of all the Q_α's with $\alpha < \kappa$ would be a common refinement of all the P_α's; so there is no such refinement of the Q_α's.

Let R be a maximal family of pairwise disjoint nonzero elements of A such that $\forall x \in R \forall \alpha < \kappa \exists y \in Q_\alpha[x \leq y]$. Then R cannot be a partition, so there is some $z \in A^+$ such that $x \cdot z = 0$ for all $x \in R$. Now we define a decreasing sequence $w_0, w_1, \ldots$ of elements of A. Choose $w_0 \in Q_0$ such that $z \cdot w_0 \neq 0$ Suppose that $w_\alpha \in Q_\alpha$ has been defined so that $w_\alpha \cdot z \neq 0$. Choose $w_{\alpha+1} \in Q_{\alpha+1}$ such that $w_\alpha \cdot z \cdot w_{\alpha+1} \neq 0$. Then $w_{\alpha+1} \leq w_\alpha$ since $Q_{\alpha+1}$ refines Q_α. Now suppose that β is a limit ordinal less than κ and w_α has been defined for all $\alpha < \beta$. If there is a nonzero $y \leq w_\alpha \cdot z$ for all $\alpha < \beta$, then there is a $v \in P_\beta$ such that $v \cdot y \neq 0$, and we let w_β be such a v. Note that $w_\beta \cdot z \neq 0$, and $w_\beta \leq w_\alpha$ for all $\alpha < \beta$ since Q_β refines Q_α. If there is no such y, the construction stops, and we have a $\geq$-tower which is a cofinal subsequence of $\langle w_\alpha \cdot z : \alpha < \beta$ of length $\beta < \kappa$.

Suppose that the construction does not stop for any $\beta < \kappa$. Then $\prod_{\alpha<\kappa} w_\alpha \cdot z = 0$, since otherwise this product would contradict the maximality of R. Again this yields a $\geq$-tower of length at most κ. □

Proposition 4.63. $\mathfrak{h}(A) \leq \mathrm{spl}(A)$ *for any atomless BA A.*

Proof. Suppose that $\mathrm{spl}(A) < \mathfrak{h}(A)$, and let S be a splitting set in A of size $\mathrm{spl}(A)$. We may assume that $0, 1 \notin S$. For each $s \in S$ let $P_s = \{s, -s\}$. So P_s is a partition. Let Q be a common refinement of all the P_s's. Take any $q \in Q$, and choose $s \in S$ such that $q \cdot s \neq 0 \neq q \cdot -s$. But Q refines P_s, contradiction. □

The above facts describe all of the relationships between the functions $\mathfrak{p}$, $\mathfrak{a}$, tow, $\mathfrak{h}$, and spl; see the diagrams at the end of the book. To see that no other relations hold, and that the differences can be large, we need some examples. Recall here the result of McKenzie, Monk [04] mentioned earlier, showing that $\mathrm{spl}(A) < \mathfrak{a}(A)$ is possible, with an arbitrarily large difference. For $\mathfrak{a}(A) < \mathrm{tow}(A)$, see Proposition 4.51. Examples with $\mathrm{tow}(A) < \mathfrak{h}(A)$ and $\mathfrak{h}(B) < \mathrm{spl}(B)$ will be given shortly.

Proposition 4.64. *$\mathfrak{h}(A)$ is regular, for any atomless BA A.*

Proof. Suppose that $\kappa \stackrel{\text{def}}{=} \mathfrak{h}(A)$ is singular. Let $\mathscr{P}$ be a collection of κ many partitions with no common refinement. Write $\mathscr{P} = \bigcup_{\alpha<\mathrm{cf}(\kappa)} \mathscr{Q}_\alpha$, with each Q_α of size less than κ. Then each $\mathscr{Q}_\alpha$ has a common refinement Q_α, and $\{Q_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ has a common refinement P. Clearly P is a common refinement of $\mathscr{P}$, contradiction. □

Proposition 4.65. *Suppose that κ is regular. Assume one of the following:*

(i) *A is a κ-saturated atomless BA.*
(ii) *With $\kappa = \aleph_\alpha$, there is an η_α-set L such that $A = \mathrm{Intalg}(L)$.*

Then $\mathfrak{h}(A) \geq \kappa$.

Proof. It suffices to show that any system of partitions $\langle P_\xi : \xi < \lambda\rangle$ has a refinement, where λ is an infinite cardinal less than κ.

In case (ii), the set $X \stackrel{\text{def}}{=} \{[u, v) : u, v \in L,\ u < v\}$ is dense in A, and so there is a partition with elements from X. Taking a refinement of X with P_ξ for each $\xi < \lambda$, we may assume that each P_ξ refines X.

We construct partitions $\langle Q_\xi : \xi \leq \lambda\rangle$ such that for each $\xi \leq \lambda$, Q_ξ refines all Q_η for $\eta < \xi$, and for each $\xi < \lambda$, Q_ξ refines P_ξ. Let $Q_0 = P_0$. If Q_ξ has been defined, with $\xi < \lambda$, let $Q_{\xi+1}$ be a refinement of Q_ξ and $P_{\xi+1}$. Suppose that Q_ξ has been defined for all $\xi < \eta$ satisfying these conditions, where η is a limit ordinal $\leq \lambda$. Let R be maximal disjoint subject to the condition $\forall\xi < \eta\forall x \in R\exists y \in Q_\xi[x \leq y]$. We claim that $\sum R = 1$. For, take any $z \neq 0$.

Now we split the proof into the cases corresponding to the assumptions (i) or (ii).

(i): We construct $\langle a_\xi : \xi \leq \eta\rangle$. Choose $a_0 \in Q_0$ so that $a_0 \cdot z \neq 0$. If a_ξ has been defined, $\xi < \eta$, so that $a_\xi \cdot z \neq 0$, choose $a_{\xi+1} \in Q_{\xi+1}$ such that $a_{\xi+1} \cdot a_\xi \cdot z \neq 0$. Since $Q_{\xi+1}$ refines Q_ξ, it follows that $a_{\xi+1} \leq a_\xi$. Now suppose that $\xi \leq \eta$ is limit and a_σ has been defined for all $\sigma < \xi$ so that $\sigma < \tau < \xi$ implies that $a_\tau \leq a_\sigma$, and so that $a_\sigma \cdot z \neq 0$ for all $\sigma < \xi$. Consider the following set of formulas in one free variable v:

$$(*) \qquad \{v \neq 0\} \cup \{v \leq z\} \cup \{v \leq a_\sigma : \sigma < \xi\}.$$

A finite subset of this set is contained in a finite subset of the form

$$(**) \qquad \{v \neq 0\} \cup \{v \leq z\} \cup \{v \leq a_\sigma : \sigma \in F\}$$

for some finite nonempty subset F of ξ. Let σ be the largest member of F. Then by assumption, $a_\sigma \cdot z$ satisfies $(**)$.

So by κ-saturatedness, we can take a $b \in A$ such that $a \neq 0$, $a \leq z$, and $a \leq a_\xi$ for all $\sigma < \xi$. Choose $a_\xi \in Q_\xi$ so that $a_\xi \cdot b \neq 0$. Then $a_\xi \cdot z \neq 0$. This finishes the construction of $\langle a_\xi : \xi \leq \eta\rangle$. We have $a_\eta \in R$ and $a_\eta \cdot z \neq 0$. So $\sum R = 1$, as claimed. If $\eta < \lambda$ we let Q_η be a refinement of R and P_η. If $\eta = \lambda$, then $Q_\eta \stackrel{\text{def}}{=} R$ is the desired refinement of each P_ξ for $\xi < \lambda$.

(ii): We may assume that $z = [s,t)$ for some $s,t \in L$ with $s < t$. We want to find $[x,y) \in R$ such that $[s,t) \cap [x,y) \neq \emptyset$. To do this we define two sequences $\langle v_\xi : \xi < \eta\rangle$ and $\langle w_\xi : \xi < \eta\rangle$ of elements of L such that $\langle v_\xi : \xi < \eta\rangle$ is increasing, $\langle w_\xi : \xi < \eta\rangle$ is decreasing, $v_\xi < w_\gamma$ for all $\xi, \gamma < \eta$, $[v_\xi, w_\xi) \in Q_\xi$, and $\max(s, v_\xi) < \min(t, w_\xi)$ for all $\xi < \eta$. Choose $v_0 < w_0$ so that $[v_0, w_0) \in Q_0$ and $[v_0, w_0) \cap [s,t) \neq \emptyset$. Thus $\max(s, v_0) < \min(t, w_0)$. Now suppose that v_ξ and w_ξ have been constructed for all $\xi < \gamma$, where $\gamma < \eta$, so that $\langle v_\xi : \xi < \gamma\rangle$ is increasing, $\langle w_\xi : \xi < \gamma\rangle$ is decreasing, and $\forall \xi < \gamma[\max(s, v_\xi) < \min(t, w_\xi)]$. Then by the η_α property there are elements x, y of L such that $\max(s, v_\xi) < x < y < \min(t, w_\delta)$ for all $\xi, \delta < \gamma$. Choose $[v_\gamma, w_\gamma) \in Q_\gamma$ such that $[v_\gamma, w_\gamma) \cap [x,y) \neq \emptyset$. Now if $\xi < \gamma$, then $[v_\xi, w_\xi) \cap [v_\gamma, w_\gamma) \neq \emptyset$ so, since Q_γ refines Q_ξ, we must have $[v_\gamma, w_\gamma) \subseteq [v_\xi, w_\xi)$, and so $v_\xi \leq v_\gamma$ and $w_\gamma \leq w_\xi$. Now choose $u \in [v_\gamma, w_\gamma) \cap [x,y)$. Then $s < x < u < y < t$ and $v_\gamma \leq u < w_\gamma$, so $\max(s, v_\gamma) \leq u < \min(t, w_\gamma)$. This finishes the construction of $\langle v_\xi : \xi < \eta\rangle$ and $\langle w_\xi : \xi < \eta\rangle$.

By the η_α property choose $p, r \in L$ such that $\max(s, v_\xi) < p < r < \min(t, w_\xi)$ for all $\xi < \eta$. Thus $[p,r) \subseteq [v_\xi, w_\xi)$ for all $\xi < \eta$. By the maximality of R it follows that there is a $q \in R$ such that $q \cap [p,r) \neq \emptyset$. If f is in this set, then $f \in q \cap [s,t)$, as desired.

If $\eta < \lambda$ we let Q_η be a refinement of R and P_η. If $\eta = \lambda$, then $Q_\eta \stackrel{\text{def}}{=} R$ is the desired refinement of each P_ξ for $\xi < \lambda$. $\square$

Corollary 4.66. (GCH) *For each regular cardinal there is a BA A such that* $\mathfrak{h}(A)=\kappa$. $\square$

Problem 50. *Can one prove in ZFC that for each regular cardinal κ there is a BA A such that $\mathfrak{h}(A) = \kappa$?*

Proposition 4.67. *Suppose that κ and λ are cardinals, κ regular, with $\kappa \leq \lambda$. Then there is an interval algebra A such that* $\mathrm{tow}(A) \leq \kappa$ *and* $\mathfrak{h}(A) \geq \lambda$.

Proof. Let M be a dense linear order in which every element has character (κ, κ^*), while the gaps have character (μ, κ^*) and (κ, μ^*) for $\omega \leq \mu < \kappa$, μ regular; and M has coinitiality κ^* and cofinality κ.

Write $\lambda = \aleph_\varepsilon$, and let L be an η_ε-set. Finally, let $N = M \times L$, lexicographically ordered, and set $A = \mathrm{Intalg}(N)$. We claim that $\mathrm{tow}(A) = \kappa$ and $\mathfrak{h}(A) \geq \lambda$.

Let $\langle m_\beta : \beta < \kappa\rangle$ be strictly increasing and cofinal in M, and let u be any member of L. Then $\langle [-\infty, (m_\beta, u)) : \beta < \lambda\rangle$ is a tower in A of order type κ. So $\mathrm{tow}(A) \leq \kappa$.

To prove that $\mathfrak{h}(A) \geq \lambda$, take a system $\langle P_\alpha : \alpha < \mu\rangle$ of partitions, with $\mu < \lambda$; we want to find a refinement of them. Now the set

$$R \stackrel{\text{def}}{=} \{[(m,s),(m,t)) : m \in M, s,t \in L, s < t\}$$

is dense in A, and hence there is a partition of A containing only elements of this set. Taking a refinement of each P_α with such a partition, we may assume that each P_α consists of elements of R.

Now we define partitions $\langle Q_\alpha : \alpha \leq \mu\rangle$ by recursion. Let $Q_0 = P_0$. If Q_α has been defined, let $Q_{\alpha+1}$ be a refinement of Q_α and P_α. Now suppose that β is a limit ordinal $\leq \mu$ and $\langle Q_\alpha : \alpha < \beta\rangle$ has been defined so that Q_γ is a refinement of Q_α if $\alpha < \gamma < \beta$. Let S be maximal disjoint such that $\forall v \in S \forall \alpha < \beta \exists q \in Q_\alpha[v \leq q]$. We claim that $\sum S = 1$.

To prove this it suffices to take any $m \in M$ and $s,t \in L$ with $s < t$ and find $q \in S$ such that $q \cap [(m,s),(m,t)) \neq \emptyset$. To do this we define two sequences $\langle v_\alpha : \alpha < \beta\rangle$ and $\langle w_\alpha : \alpha < \beta\rangle$ of elements of L such that $\langle v_\alpha : \alpha < \beta\rangle$ is increasing, $\langle w_\alpha : \alpha < \beta\rangle$ is decreasing, $v_\alpha < w_\gamma$ for all $\alpha, \gamma < \beta$, $[(m,v_\alpha),(m,w_\alpha)) \in Q_\alpha$, and $\max(s,v_\alpha) < \min(t,w_\alpha)$ for all $\alpha < \beta$. Choose $v_0 < w_0$ so that $[(m,v_0),(m,w_0)) \in Q_0$ and $[(m,v_0),(m,w_0)) \cap [(m,s),(n,t)) \neq \emptyset$. Thus $\max(s,v_0) < \min(t,w_0)$. Now suppose that v_α and w_α have been constructed for all $\alpha < \gamma$, where $\gamma < \beta$ so that $\langle v_\alpha : \alpha < \gamma\rangle$ is increasing, $\langle w_\alpha : \alpha < \gamma\rangle$ is decreasing, and $\forall \alpha < \gamma[\max(s,v_\alpha) < \min(t,w_\alpha)]$. Then by the η_ε property there are elements x,y of L such that $\max(s,v_\alpha) < x < y < \min(t,w_\delta)$ for all $\alpha, \delta < \gamma$. Choose $[(m,v_\gamma),(m,w_\gamma)) \in Q_\gamma$ such that $[(m,v_\gamma),(m,w_\gamma)) \cap [(m,x),(m,y)) \neq \emptyset$. Now if $\alpha < \gamma$, then $[(m,v_\alpha),(m,w_\alpha)) \cap [(m,v_\gamma),(m,w_\gamma)) \neq \emptyset$ so, since Q_γ refines Q_α, we must have $[(m,v_\gamma),(m,w_\gamma)) \subseteq [(m,v_\alpha),(m,w_\alpha))$, and so $v_\alpha \leq v_\gamma$ and $w_\gamma \leq w_\alpha$. Now choose $(m,z) \in [(m,v_\gamma),(m,w_\gamma)) \cap [(m,x),(m,y))$. Then $s < x < z < y < t$ and $v_\gamma \leq z < w_\gamma$, so $\max(s,v_\gamma) \leq z < \min(t,w_\gamma)$. This finishes the construction of $\langle v_\alpha : \alpha < \beta\rangle$ and $\langle w_\alpha : \alpha < \beta\rangle$.

By the η_ε property choose $p,r \in L$ such that $\max(s,v_\alpha) < p < r < \min(t,w_\alpha)$ for all $\alpha < \beta$. Thus $[(m,p),(m,r)) \subseteq [(m,v_\alpha),(m,w_\alpha))$ for all $\alpha < \beta$. By the maximality of S it follows that there is a $q \in S$ such that $q \cap [(m,p),(m,r)) \neq \emptyset$. If f is in this set, then $f \in q \cap [(m,s),m,t))$, as desired.

Hence we can let $Q_\beta = S$, finishing the recursive definition; Q_μ is as desired. □

We postpone giving an example with $\mathfrak{h}(A) < \mathrm{spl}(A)$ until after discussing algebraic operations concerning these two functions.

Proposition 4.68. *For any atomless BAs A and B we have*

(i) $\mathfrak{h}(A \times B) = \min(\mathfrak{h}(A), \mathfrak{h}(B))$.

(ii) $\mathrm{spl}(A \times B) = \max(\mathrm{spl}(A), \mathrm{spl}(B))$.

Proof. For (i), suppose that $\langle P_\alpha : \alpha < \mathfrak{h}(A)\rangle$ is a system of partitions of A with no common refinement. For each $\alpha < \mathfrak{h}(A)$ let

$$Q_\alpha = \{(x,0) : x \in P_\alpha\} \cup \{(0,1)\}.$$

Thus Q_α is a partition of $A \times B$. Clearly a refinement of all the Q_α's would yield a refinement of the P_α's. This shows that $\mathfrak{h}(A \times B) \leq \mathfrak{h}(A)$. similarly $\mathfrak{h}(A \times B) \leq \mathfrak{h}(B)$.

Now suppose that $\langle R_\alpha : \alpha < \kappa\rangle$ is a system of partitions of $A \times B$, where $\kappa < \min(\mathfrak{h}(A), \mathfrak{h}(B))$. We will find a common refinement of them, proving the equality. For each $\alpha < \kappa$ let

$$S_\alpha = \{a \in A^+ : \exists b \in B[(a,b) \in R_\alpha]\};$$
$$T_\alpha = \{b \in B^+ : \exists a \in A[(a,b) \in R_\alpha]\}.$$

Clearly each S_α is a partition of A, and each T_α is a partition of B. Let U be a common refinement of the S_α's, and V a common refinement of the T_α's. Let

$$W = \{(x,0) : x \in U\} \cup \{(0,y) : y \in V\}.$$

Then W is a partition of $A \times B$ which refines each R_α.

(ii): If X splits A and Y splits B, then $X \times Y$ splits $A \times B$. Hence $\mathrm{spl}(A \times B) \leq \max(\mathrm{spl}(A), \mathrm{spl}(B))$. Now suppose that $Z \subseteq A \times B$ splits $A \times B$ and $|Z| < \max(\mathrm{spl}(A), \mathrm{spl}(B))$. Say by symmetry that $\max(\mathrm{spl}(A), \mathrm{spl}(B)) = \mathrm{spl}(A)$. Then $\{a \in A : \exists b[(a,b) \in Z]\}$ has size less than $\mathrm{spl}(A)$, so there is an $a \in A^+$ such that for all $(u,v) \in Z$, $a \cdot u = 0$ or $a \cdot -u = 0$. Then $(a,0) \in (A \times B)^+$ and for all $(u,v) \in Z$, $(a,0) \cdot (u,v) = (0,0)$ or $(a,0) \cdot -(u,v) = (0,0$, contradiction. ☐

Proposition 4.69. *If $\langle A_i : i \in I\rangle$ is a system of atomless BAs, with I infinite, then*

(i) $\mathfrak{h}\left(\prod^{\mathrm{w}}_{i \in I} A_i\right) = \min_{i \in I} \mathfrak{h}(A_i)$.

(ii) $\mathrm{spl}\left(\prod^{\mathrm{w}}_{i \in I} A_i\right) = \max(|I|, \sup_{i \in I} \mathrm{spl}(A_i))$.

Proof. For brevity let $B = \prod^{\mathrm{w}}_{i \in I} A_i$. For each $i \in I$ and $a \in A_i$ define $\mathrm{up}^i(a) \in B$ by setting, for any $j \in I$,

$$\mathrm{up}^i(a)_j = \begin{cases} a & \text{if } \ i = j, \\ 0 & \text{otherwise.} \end{cases}$$

(i): By Proposition 4.68, $\leq$ holds. Now suppose that $\kappa < \min_{i \in I} \mathfrak{h}(A_i)$ and $\langle P_\alpha : \alpha < \kappa\rangle$ is a system of partitions of B. For each $i \in I$ and $\alpha < \kappa$ let $Q^i_\alpha = \{x_i : x \in P_\alpha\} \backslash \{0\}$. Then Q^i_α is a partition of A_i, and so there is a refinement R^i of all Q^i_α's. Let

$$S = \{\mathrm{up}^i(x) : i \in I, \ x \in R^i\}.$$

Then S is a partition of B which refines each P_α. Thus (i) holds.

(ii): For each $i \in I$ let X_i be a splitting set for A_i of size $\mathrm{spl}(A_i)$. Then

$$\{\mathrm{up}^i(x) : i \in I,\ x \in X_i\}$$

is a splitting set for B, and its size is $\max(|I|, \sup_{i\in I} \mathrm{spl}(A_i))$.

Now suppose that Z splits B and $|Z| < \max(|I|, \sup_{i\in I} \mathrm{spl}(A_i))$. If $|Z| < |I|$, then there is an $i \in I$ such that $z_i = 0$ or $z_i = 1$ for all $z \in Z$, contradiction. So $|I| \leq |Z|$. Choose $i \in I$ such that $|Z| < \mathrm{spl}(A_i)$. Then $\{z_i : z \in Z\}$ splits A_i, contradiction. □

The proof of Proposition 4.69 also works for moderate products.

Proposition 4.70. *If $\langle A_i : i \in I\rangle$ is a system of atomless BAs, with I infinite, then*

(i) $\mathfrak{h}\left(\prod_{i\in I} A_i\right) = \min_{i\in I} \mathfrak{h}(A_i)$.
(ii) $\mathrm{spl}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right) = \sup_{i\in I} \mathrm{spl}(A_i)$.

Proof. Again let $B = \prod_{i\in I} A_i$ for brevity. (i): as in the proof of Proposition 4.69. (ii): For each $i \in I$ let X_i be a splitting set for A_i with $|X_i| = \mathrm{spl}(A_i)$. Let $\kappa = \sup_{i\in I} \mathrm{spl}(A_i)$, and for each $i \in I$ let $\langle x^i_\alpha : \alpha < \kappa\rangle$ be an enumeration of X_i, repetitions allowed. For each $\alpha < \kappa$ define $y^\alpha \in B$ by setting $y^\alpha_i = x^i_\alpha$ for any $i \in I$. Clearly $\{y^\alpha : \alpha < \kappa\}$ splits B. So $\leq$ holds.

Now suppose that Z splits B and $|Z| < \kappa$. Choose $i \in I$ such that $|Z| < \mathrm{spl}(A_i)$. Then $\{z_i : z \in Z\}$ splits A_i, contradiction. □

Concerning free products we have the following problem.

Problem 51. *What is the relationship, if any, between $\mathfrak{h}(A \oplus B)$ and $\mathfrak{h}(A)$, $\mathfrak{h}(B)$?*

Clearly if X splits A, then also X splits $A \oplus B$. Thus we have

Proposition 4.71. $\mathrm{spl}(A \oplus B) \leq \min(\mathrm{spl}(A), \mathrm{spl}(B))$ *for atomless A, B.* □

Problem 52. *Is* $\mathrm{spl}(A \oplus B) = \min(\mathrm{spl}(A), \mathrm{spl}(B))$ *for atomless A, B?*

Problem 53. *What is the relationship between $\mathfrak{h}$ of algebras and their ultraproduct?*

Problem 54. *What is the relationship between* spl *of algebras and their ultraproduct?*

Proposition 4.72. *If κ is regular and $2^\kappa \leq \lambda$, then there is an atomless BA A such that $\kappa \leq \mathfrak{h}(A) \leq 2^\kappa$ and* $\mathrm{spl}(A) = \lambda$.

Proof. Let B be κ-saturated of size at most 2^κ. and let $C_\alpha = B$ for all $\alpha < \lambda$. We let $A = \prod^{\mathrm{w}}_{\alpha<\lambda} C_\alpha$. Then $\kappa \leq \mathfrak{h}(A) \leq 2^\kappa$ and $\mathrm{spl}(A) = \lambda$ by Proposition 4.69. □

Problem 55. *Suppose that κ is regular and $\kappa \leq \lambda$. Is there an atomless BA A such that $\mathfrak{h}(A) = \kappa$ and* $\mathrm{spl}(A) = \lambda$*?*

We return to the discussion of other derived functions for depth. Clearly $[\omega, \mathrm{t}(A)) \subseteq \mathrm{Depth}_{\mathrm{Hs}}(A) \subseteq [\omega, \mathrm{t}(A)]$, by an argument very similar to that used for the functions c and s. Like for cellularity, there is a problem whether $\mathrm{t}(A) \in \mathrm{Depth}_{\mathrm{Hs}}(A)$. This is trivially true if $\mathrm{t}(A)$ is a successor cardinal or a limit cardinal of cofinality ω by Corollary 4.27 and Theorem 12.2 below. For each singular cardinal κ with $\mathrm{cf}(\kappa) > \omega$ there is a BA A such that $|A| = \mathrm{Depth}(A) = \mathrm{t}(A)$ and $\mathrm{Ult}(A)$ has no free sequence of length κ, hence by Corollary 4.27 A has no homomorphic image B such that $\mathrm{Depth}(B) = \mathrm{t}(A)$ and $\mathrm{Depth}(B)$ is attained. Namely, let $\langle \mu_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing sequence of infinite cardinals with sup κ, let $A = \prod^{\mathrm{w}}_{\alpha < \mathrm{cf}\kappa} \mathrm{Intalg}(\mu_\alpha)$, and use Theorem 12.1.

The following result of Rosłanowski, Shelah [00] is relevant here (see Conclusion 7.6 in this paper):

It is consistent that there is a Boolean algebra A of size λ such that there is an ultrafilter of A of tightness λ, there is no free sequence of length λ in A, and $\mathrm{t}(A) = \lambda \notin \mathrm{Depth}_{\mathrm{Hs}}(A)$.

Of course it would be of interest to construct such an example in ZFC; this is a version of Problem 13 in Monk [96].

Problem 56. *Can one construct in ZFC a BA A such that* $\mathrm{t}(A) \notin \mathrm{Depth}_{\mathrm{Hs}}(A)$*?*

Clearly $\mathrm{Depth}_{\mathrm{Ss}} = [\omega, \mathrm{Depth} A]$ for any infinite BA A.

Next comes the relation $\mathrm{Depth}_{\mathrm{Sr}}$. It is easy to see that parts (i)–(ii) and (iv)–(v) of Theorem 3.51 hold with cellularity replaced by depth. This is expressed in part of the following proposition.

Proposition 4.73. *For any infinite BA A the following conditions hold:*

(i) *If* $(\kappa, \lambda) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$*, then* $\kappa \leq \lambda \leq |A|$ *and* $\kappa \leq \mathrm{Depth}(A)$.
(ii) *For each* $\kappa \in [\omega, \mathrm{Depth}(A)]$ *we have* $(\kappa, \kappa) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$.
(iii) *If* $(\kappa, \lambda) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$ *and* $\kappa \leq \mu \leq \lambda$*, then* $(\kappa, \mu) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$.
(iv) $(\mathrm{Depth} A, |A|) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$.
(v) *If* $\omega \leq \lambda \leq |A|$ *then* $(\kappa, \lambda) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$ *for some* κ.
(vi) *If* $(\kappa^+, \lambda) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$*, then* $(\omega, \kappa^+) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$.
(vii) *If* $(\kappa^+, \lambda) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$ *and* $\omega \leq \mu < \kappa^+$*, then* $(\mu, \kappa^+) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$.

Proof. (i)–(v) are easy. For (vi), assume that $(\kappa^+, \lambda) \in \mathrm{Depth}_{\mathrm{Sr}}(A)$. Then A has a subalgebra B isomorphic to $\mathrm{Finco}(\kappa^+)$, and $\mathrm{Depth}(B) = \omega$.

For (vii), assume the hypotheses. Let B be a subalgebra of A such that $|B| = \lambda$ and $\mathrm{Depth}(B) = \kappa^+$. Let $\langle b_\alpha : \alpha < \kappa^+ \rangle$ be a strictly increasing sequence in B. Then $B \cong (B \restriction b_\mu) \times (B \restriction -b_\mu)$. Since $b_{\alpha+1} \cdot -b_\alpha \leq -b_\mu$ for all $\alpha \in [\mu, \kappa^+)$, it follows that $B \restriction -b_\mu$ has a subalgebra isomorphic to $\mathrm{Finco}(\kappa^+)$. Hence B has a subalgebra of size κ^+ with depth μ. □

Concerning (iii) of Theorem 3.51, the following result of Rosłanowski, Shelah [01] (Conclusion 18) is relevant:

It is consistent that there is a cardinal κ *such that there is a BA* A *of size* $(2^\kappa)^+$ *with* $\text{Depth}(A) = \kappa$ *while* $(\omega, (2^\kappa)^+) \notin \text{Depth}_{\text{Sr}}(A)$.

Again it would be of interest to construct such an example in ZFC. This is a version of Problem 14 of Monk [96].

Problem 57. *Can one show in ZFC that there is a cardinal* κ *such that there is a BA* A *of size* $(2^\kappa)^+$ *with* $\text{Depth}(A) = \kappa$ *while* $(\omega, (2^\kappa)^+) \notin \text{Depth}_{\text{Sr}}(A)$*?*

Problem 58. *Characterize the relation* Depth_{Sr}.

This is Problem 15 of Monk [96].

The following theorem and corollary are relevant to this problem:

Theorem 4.74. *Let* κ *be an infinite cardinal, and let* μ *be minimum such that* $\omega^\mu > \kappa$. *Thus* $\mu \le \kappa$. *Let* L *be the linearly ordered set* ${}^\mu Q$ *ordered lexicographically, where* Q *is the set of all rationals in the interval* $[0, 1)$. *Then*

(i) *L does not have any strictly increasing sequence of order type* μ^+ *or any strictly decreasing sequence of order type* μ^+.

Further, set $D = \{f \in {}^\mu Q :$ *there is an* $\alpha < \mu$ *such that* $f(\beta) = 0$ *for all* $\beta > \alpha\}$. *It is clear that* $|D| \le \kappa$ *and* D *is dense in* L. *Let* M *be a subset of* L *of size* κ^+ *which includes* D, *and let* A *be the interval algebra on* M. *Then*

(ii) *Any subset* N *of* A *of size* κ^+ *contains a linearly ordered subset of size* κ^+ *which can be isomorphically embedded in* M *or* M^*.

Proof. For (i), assume the contrary. Assume that $\langle f_\alpha : \alpha < \mu^+\rangle$ is strictly increasing. (The strictly decreasing case is treated similarly.) Now a contradiction will follow rather easily from the following statement:

(1) If $\gamma \le \mu$, $\Gamma \in [\mu^+]^{\mu^+}$, and $f_\alpha \restriction \gamma < f_\beta \restriction \gamma$ for all $\alpha, \beta \in \Gamma$ for which $\alpha < \beta$, then there exist $\delta < \gamma$ and $\Delta \in [\Gamma]^{\mu^+}$ such that $f_\alpha \restriction \delta < f_\beta \restriction \delta$ for all $\alpha, \beta \in \Delta$ for which $\alpha < \beta$.

Assume the hypothesis of (1). For each $\alpha \in \Gamma$ let $f'_\alpha = f_\alpha \restriction \gamma$. Clearly Γ does not have a largest element. For each $\alpha \in \Gamma$ let $\alpha' \in \Gamma$ be minimum with $\alpha < \alpha'$, and let $\chi(\alpha) < \gamma$ be minimum such that $f_\alpha(\chi(\alpha)) \ne f_{\alpha'}(\chi(\alpha))$. Now

$$\Gamma = \bigcup_{\delta<\gamma} \{\alpha \in \Gamma : \chi(\alpha) = \delta\},$$

so there exist $\delta < \gamma$ and $\Theta \in [\Gamma]^{\mu^+}$ such that $\chi(\alpha) = \delta$ for all $\alpha \in \Theta$. Define $\alpha \equiv \beta$ iff $\alpha, \beta \in \Theta$ and $f_\alpha \restriction \delta = f_\beta \restriction \delta$. Note that $f_\alpha(\delta) < f_\beta(\delta)$ if $\alpha \equiv \beta$ and $\alpha < \beta$.

Hence each equivalence class under $\equiv$ is countable. Let $\Delta \subseteq \Theta$ have exactly one element from each equivalence class. Clearly the conclusion of (1) now holds.

From (1) we get a strictly decreasing sequence $\mu > \delta(0) > \delta(1) > \cdots$ of ordinals, contradiction. So (i) holds.

For (ii), let $N \in [A]^{\kappa^+}$. For each $x \in A$ write

$$x = [a(1,x), b(1,x)) \cup \cdots \cup [a(m_x,x), b(m_x,x)),$$

where $a(1,x), b(1,x), \ldots, a(m_x,x), b(m_x,x)$ are in $M \cup \{+\infty\}$ and $a(1,x) < b(1,x) < \cdots < a(m_x,x) < b(m_x,x)$. Note that if $x \in A$ and $a(1,x) > 0$, then $a(1,-x) = 0$. If $|\{x \in N : a(1,x) = 0\}| < \kappa^+$, let $N_1 = \{x : -x \in N, a(1,-x) > 0\}$; otherwise let $N_1 = \{x \in N : a(1,x) = 0\}$. So $|N_1| = \kappa^+$ and $\forall x \in N_1[a(1,x) = 0]$.

Next,

$$N_1 = \bigcup_{n \in \omega} \{x \in N_1 : m_x = n\}.$$

It follows that there is an $n \in \omega$ such that $N_2 \stackrel{\text{def}}{=} \{x \in N_1 : m_x = n\}$ has size κ^+. Now for each $x \in N_2$ we choose $c(1,x), \ldots, c(n,x), d(1,x), \ldots, d(n,x) \in D$ so that $a(i,x) < c(i,x) < b(i,x) < d(i,x) < a(i+1,x)$ for all $i = 1, \ldots, n$ (omitting the term $a(i+1,x)$ for $i = n$, and also omitting $d(n,x)$ if $b(n,x) = \infty$). Now

$$N_2 = \bigcup_{e,f \in E} \{x \in N_2 : \forall i = 1, \ldots, n[c(i,x) = e(i) \text{ and } d(i,x) = f(i)]\},$$

where E is the set of all functions mapping $\{1, \ldots, n\}$ into D. Since $|D| \le \kappa$, it follows that there exist $e, f \in E$ so that

$$N_3 \stackrel{\text{def}}{=} \{x \in N_2 : \forall i = 1, \ldots, n[c(i,x) = e(i) \text{ and } d(i,x) = f(i)]\}$$

has size κ^+.

Now we construct subsets $P_0, \ldots, P_s$ of N_3 by recursion, where s is to be determined. Let $P_0 = N_3$. Suppose that P_k has been defined, so that $P_k \subseteq N_3$ and $|P_k| = \kappa^+$. If $k < n$, define $x \equiv_k y$ iff $x, y \in P_k$ and $a(k+1,x) = a(k+1,y)$. This is an equivalence relation on P_k. If there are κ^+ equivalence classes under $\equiv_k$, we let P_{k+1} be a subset of P_k containing exactly one element from each $\equiv_k$ class, and we let $s = k+1$. If there are less than κ^+ equivalence classes, then some equivalence class P_{k+1} has κ^+ elements. If $n \le k < 2n$, we work with b instead of a, in the same fashion. Namely, define $x \equiv_k y$ iff $x, y \in P_k$ and $b(k-n+1,x) = b(k-n+1,y)$. This is an equivalence relation on P_k. If there are κ^+ equivalence classes under $\equiv_k$, we let P_{k+1} be a subset of P_k containing exactly one element from each $\equiv_k$ class, and we let $s = k+1$. If there are less than κ^+ equivalence classes, then some equivalence class P_{k+1} has κ^+ elements. This construction must stop for some $s \le 2n$. Otherwise $\forall k = 1, \ldots, n \forall x, y \in P_{2n}[a(k,x) = a(k,y) \text{ and } b(k,x) = b(k,y)]$, contradiction.

If $1 \le s \le n$, then the elements $a(s,x)$ for $x \in P_s$ are all distinct, while if $n < s \le 2n$, then the elements $b(s-n,x)$ for $x \in P_s$ are all distinct.

Case 1. $1 \le s \le n$. We define a homomorphism f of $\langle P_s\rangle_A$ into the BA of all subsets of $L \cap [d_{s-1}, c_s)$ by setting $f(x) = x \cap [d_{s-1}, c_s)$ for all $x \in \langle P_s\rangle_A$. Now if $x, y \in P_s$ and $x < y$, then $f(x) = [a(s,x), c_s) \subseteq f(y) = [a(s,y), c_s)$, hence $a(s,x) > a(s,y)$. Thus P_s is isomorphic to a subset of M^*.

Case 2. $n < s \le 2n$. Similarly: P_s is isomorphic to a subset of M. □

Corollary 4.75. (GCH) *For every infinite cardinal λ there is an interval algebra A of size λ^{++} such that every subalgebra of A of size λ^{++} has depth λ^+.*

Proof. We can apply Theorem 4.74 with $\kappa = \lambda^+$; then $\mu = \lambda^+$. If B is a subalgebra of A of size λ^{++} and X is a chain in B of order type λ^{++}, then by (ii), X has a linearly ordered subset of size λ^{++} which can be isomorphically embedded in M or M^*, contradicting D dense in M. By the Erdős–Rado theorem B has a chain of order type λ^+. □

Problem 59. *Can one prove in ZFC that for every infinite cardinal λ there is an interval algebra A such that $|A| = \lambda^+$ and every subalgebra of A of size λ^+ has depth λ?*

Now we consider systematically the possibilities for $\mathrm{Depth}_{\mathrm{Sr}}$ for algebras of size at most ω_2, following the lexicographic order used in discussing c_{Sr}, and omitting most easy cases. Note in particular that (ω, ω) is always a member of $\mathrm{Depth}_{\mathrm{Sr}}$.

(S1) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1)\}$ for $A = \mathrm{Fr}(\omega_1)$.

(S2) There does not exist a BA A such that $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega_1,\omega_1)\}$, by Proposition 4.73(vi).

(S3) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega,\omega_2)\}$ for $A = \mathrm{Fr}(\omega_2)$.

(S4) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1)\}$ for $A = \mathrm{Intalg}(\omega_1)$.

(S5) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1), (\omega_1,\omega_2)\}$ for the algebra A of Corollary 4.75, assuming GCH.

Problem 60. *Can one construct in ZFC a BA A such that*

$$\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1), (\omega_1,\omega_2)\}?$$

(S6) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega_1,\omega_1), (\omega_2,\omega_2)\}$ is ruled out by Theorem 4.73(vi).

(S7) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega), (\omega,\omega_1), (\omega,\omega_2), (\omega_1,\omega_1), (\omega_1,\omega_2)\}$ for

$$\mathrm{Fr}(\omega_2) \times \mathrm{Intalg}\,(\omega_1).$$

(S8) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$ is ruled out by Theorem 4.73(vii).

(S9) $\mathrm{Depth}_{\mathrm{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}$ for $A = \mathrm{Fr}(\omega_2) \times \mathrm{Intalg}(\omega_2)$.

We turn to $\mathrm{Depth}_{\mathrm{Hr}}$.

Theorem 4.76.

(i) *If* $(\kappa,\lambda) \in \mathrm{Depth}_{\mathrm{Hr}}A$, *then* $\kappa \le \lambda \le |A|$ *and* $\kappa \le \mathrm{t}(A)$.
(ii) *If* $\kappa \in [\omega, \mathrm{t}(A))$ *then there is a* $\lambda \le 2^\kappa$ *such that* $(\kappa,\lambda) \in \mathrm{Depth}_{\mathrm{Hr}}A$.
(iii) $(\mathrm{Depth}(A), |A|) \in \mathrm{Depth}_{\mathrm{Hr}}(A)$.
(iv) *If* $(\kappa',\lambda') \in \mathrm{Depth}_{\mathrm{Hr}}(A)$ *where* κ' *is a successor cardinal or a limit cardinal of cofinality* ω, *then there is a* $\kappa'' \ge \kappa'$ *such that* $(\kappa'', |A|) \in \mathrm{Depth}_{\mathrm{Hr}}(A)$.

Proof. (i)–(iii) are clear. For (iv), we follow the proof of Theorem 3.51(v). For brevity let $\lambda = |A|$. There is nothing to prove if $\kappa' = \lambda$, so suppose that $\kappa' < \lambda$. Let f be a homomorphism from A onto a BA B such that $\mathrm{Depth}(B) = \kappa'$. By Theorem 4.8 $\mathrm{Depth}(B)$ is attained. Let $\langle b_\xi : \xi < \kappa'\rangle$ be a strictly increasing sequence in B. For each $\xi < \kappa'$ choose $a_\xi \in A$ such that $f(a_\xi) = b_\xi$. We now consider two cases.

Case 1. $|A \restriction a_\xi| < \lambda$ for all $\xi < \kappa'$. Let J be the ideal in A generated by $\{a_\xi \cdot -a_\eta : \xi < \eta < \kappa'\}$. Then

$(*)$ $|J| < \lambda$.

In fact, $a \in J$ if and only if there is a finite set Γ of ordered pairs (ξ,η) with $\xi < \eta < \kappa'$ such that

$$a \le \sum_{(\xi,\eta)\in\Gamma} (a_\xi \cdot -a_\eta). \tag{$**$}$$

Take any such set Γ. For any a satisfying $(**)$ let $g(a) = \langle a \cdot a_\xi \cdot -a_\eta : (\xi,\eta) \in \Gamma\rangle$. Clearly g is a one-one function from the set of a satisfying $(**)$ into $\prod_{(\xi,\eta)\in\Gamma}(A \restriction a_\xi)$. Hence by our case condition, there are fewer than λ such a's. Since the number of sets Γ is $\kappa' < \lambda$, $(*)$ follows.

From $(*)$ we see that $|A/J| = \lambda$.

Now we claim that if $\alpha < \beta < \kappa'$, then $a_\beta \cdot -a_\alpha \notin J$. [Hence $\langle a_\alpha/J : \alpha < \kappa'\rangle$ is a strictly increasing sequence in A/J, as desired.] For, suppose that $\alpha < \beta < \kappa'$ and $a_\beta \cdot -a_\alpha \in J$. Then there is a finite set Γ of ordered pairs (ξ,η) with $\xi < \eta < \kappa'$ such that

$$a_\beta \cdot -a_\alpha \le \sum_{(\xi,\eta)\in\Gamma} (a_\xi \cdot -a_\eta). \tag{$***$}$$

Applying f, we get $b_\beta \cdot -b_\alpha = 0$, contradiction. So the claim holds, and this case is finished.

Case 2. There is a $\xi_0 < \kappa'$ such that $|A \restriction a_{\xi_0}| = \lambda$. Then if we take the homomorphism

$$g : A \cong (A \restriction a_{\xi_0}) \times (A \restriction -a_{\xi_0}) \to (A \restriction a_{\xi_0}) \times (B \restriction -b_{\xi_0})$$

where the second homomorphism is determined by the identity and $f \restriction (A \restriction -a_{\xi_0})$, we get a homomorphism from A onto the algebra $C \stackrel{\text{def}}{=} (A \restriction a_{\xi_0}) \times (B \restriction -b_{\xi_0})$, which has size λ. The depth of C is at least κ', since $\langle b_\alpha \cdot -b_{\xi_0} : \xi_0 \le \alpha < \kappa' \rangle$ is clearly strictly increasing in $B \restriction -b_{\xi_0}$. $\square$

Also, the following examples are relevant:

(1) If A is the finite-cofinite algebra on κ, then $\text{Depth}_{\text{Hr}} A = \{(\omega, \lambda) : \omega \le \lambda \le \kappa\}$.

(2) If A is free on κ, then $\text{Depth}_{\text{Hr}} A = \{(\lambda, \mu) : \omega \le \lambda \le \mu \le \kappa\}$.

(3) If L is a linear order, then every homomorphic image A of $\text{Intalg}(L)$ of size $(2^\kappa)^+$ has a chain of order type κ^+. This is true by the Erdős–Rado theorem, since A is isomorphic to an interval algebra. This fact gives a trivial solution of Problem 16 of Monk [96].

We give full details on the next examples.

Theorem 4.77. *Suppose that $\omega \le \rho \le \kappa$. Let $A = \langle [\kappa]^{\le \rho} \rangle_{\mathscr{P}(\kappa)}$. Let*

$$S = \{(\mu, \nu) : \omega \le \mu \le \nu \le 2^\rho, \nu^\omega = \nu\};$$
$$T = \{(\mu, \nu^\rho) : \rho^+ \le \mu \le 2^\rho, 2^\rho < \nu^\rho, \nu \le \kappa\}.$$

Then $\text{Depth}_{\text{Hr}}(A) = S \cup T$.

Proof. First suppose that $(\mu, \nu) \in S$. The mapping $a \mapsto a \cap \rho$ gives a homomorphism of A onto $\mathscr{P}(\rho)$. Now by the theorem of Fichtenholz, Kantorovich and Hausdorff, $\mathscr{P}(\rho)$ has an free subalgebra of size 2^ρ; hence by Sikorski's extension theorem there is a homomorphism of $\mathscr{P}(\rho)$ onto an algebra B such that $\text{Fr}(\nu) \le B \le \overline{\text{Fr}(\nu)}$. Since $\nu^\omega = \nu$ and free algebras have ccc, it follows that $|B| = \nu$. Now by Sikorski's extension theorem again, there is a homomorphism of B onto an algebra C such that $\text{Fr}(\nu) \times \text{Intalg}(\mu) \le C \le \overline{\text{Fr}(\nu) \times \text{Intalg}(\mu)}$. Thus $|C| = \nu$ and $\text{Depth}(C) = \mu$, so $(\mu, \nu) \in \text{Depth}_{\text{Hr}}(A)$.

Second, suppose that $\rho^+ \le \mu \le 2^\rho$, $2^\rho < \nu^\rho$, and $\nu \le \kappa$. The mapping $a \mapsto a \cap [\nu]^{\le \rho}$ is a homomorphism from A onto $\langle [\nu]^{\le \rho} \rangle_{\mathscr{P}(\nu)}$. Then the mapping $a \mapsto (a \cap \rho, a \backslash \rho)$ is an isomorphism from $\langle [\nu]^{\le \rho} \rangle_{\mathscr{P}(\nu)}$ onto $\mathscr{P}(\rho) \times \langle [\nu \backslash \rho]^{\le \rho} \rangle_{\mathscr{P}(\nu \backslash \rho)}$. Now $\rho < \nu$, as otherwise $\nu^\rho \le \rho^\rho = 2^\rho$, contradiction. It follows that $|\nu| = |\nu \backslash \rho|$, and so $\langle [\nu \backslash \rho]^{\le \rho} \rangle_{\mathscr{P}(\nu \backslash \rho)} \cong \langle [\nu]^{\le \rho} \rangle_{\mathscr{P}(\nu)}$. As above there is a homomorphism from $\mathscr{P}(\rho)$ onto an algebra with depth μ. Putting this together with the identity on $\langle [\nu \backslash \rho]^{\le \rho} \rangle_{\mathscr{P}(\nu \backslash \rho)}$, we get a homomorphism from $\mathscr{P}(\rho) \times \langle [\nu \backslash \rho]^{\le \rho} \rangle_{\mathscr{P}(\nu \backslash \rho)}$ onto an algebra B with depth $\max(\mu, \rho^+) = \mu$ and size ν^ρ. This shows that $(\mu, \nu^\rho) \in \text{Depth}_{\text{Hr}}(A)$.

Now suppose conversely that $(\mu, \nu) \in \mathrm{c_{Hr}}(A)$. Since A is σ-complete, by a theorem of Koppelberg we have $\nu^\omega = \nu$. So if $\nu \le 2^\rho$, then $(\mu, \nu) \in S$.

Suppose that $2^\rho < \nu$. By Juhász, Shelah [98], $\nu^\rho = \nu$. Let σ be minimum such that $\sigma^\rho = \nu$. If $\kappa < \sigma$, then $\kappa^\rho \le \sigma^\rho = \nu \le \kappa^\rho$, so $\kappa^\rho = \nu$, contradiction. Hence $\sigma \le \kappa$. Thus we have $\nu = \sigma^\rho$ with $\sigma \le \kappa$. So now it suffices to show that $\rho^+ \le \mu \le 2^\rho$. Let I be an ideal on A such that $|A/I| = \nu$ and $\mathrm{Depth}(A/I) = \mu$. Let $B = A/I$. Define by recursion $a_\xi \in [\kappa]^{\le\rho}$ such that $\xi < \eta$ implies that $[a_\xi]_I < [a_\eta]_I$, continuing as long as possible. Say that this produces a sequence $\langle a_\xi : \xi < \alpha\rangle$. We claim that $\rho^+ \le \alpha$. For, suppose that $\alpha < \rho^+$. Then $b \stackrel{\text{def}}{=} \bigcup_{\xi<\alpha} a_\xi$ has size $\le \rho$. Now $B \cong (B \upharpoonright [b]_I) \times (B \upharpoonright -[b]_I)$, and $|B \upharpoonright [b]_I| \le 2^\rho < \nu$. It follows that there is a $c \in [\kappa]^{\le\rho}$ such that $0 < [c]_I < -[b]_I$. Let $a_\alpha = b \cup (c\backslash b)$. So $[a_\xi]_I \le [a_\alpha]_I$ for all $\xi < \alpha$. Moreover, $a_\alpha\backslash a_\xi \supseteq c\backslash b$ and $c\backslash b \notin I$, so we could have extended our sequence further, contradiction. Hence $\rho^+ \le \alpha$. This proves that $\rho^+ \le \mu$.

Now we show that $\mu \le 2^\rho$. In fact, suppose that $2^\rho < \mu$. Let $\langle [a_\xi]_I : \xi < \mu\rangle$ be strictly increasing.

Case 1. $\{\xi < (2^\rho)^+ : |a_\xi| \le \rho\}$ has size $(2^\rho)^+$. We may assume that $\forall \xi < (2^\rho)^+[|a_\xi| \le \rho]$. By the general Δ-system lemma, let $M \in [(2^\rho)^+]^{(2^\rho)^+}$ be such that $\langle a_\xi : \xi \in M\rangle$ is a Δ-system, say with kernel c. Take $\xi < \eta < \theta$ in M. Then

$$0 \ne [a_\eta\backslash a_\xi] = [a_\eta\backslash c] = [a_\eta\backslash a_\theta] = 0,$$

contradiction.

Case 2. $\{\xi < (2^\rho)^+ : |a_\xi| \le \rho\}$ has size less than $(2^\rho)^+$. This case is treated similarly to Case 1. □

Both sets S and T are needed here. Thus for $\rho = \omega$ and $\kappa = (2^\omega)^+$ we have $(\omega, 2^\omega) \in S\backslash T$ and $(\omega_1, (2^\omega)^+) \in T\backslash S$.

The theorem simplifies as follows assuming that $2^\rho = \rho^+$.

Corollary 4.78. *Suppose that $\omega \le \rho \le \kappa$ and $2^\rho = \rho^+$. Let $A = \langle [\kappa]^{\le\rho}\rangle_{\mathscr{P}(\kappa)}$. Let*

$$S = \{(\mu, \nu) : \omega \le \mu \le \nu \le \rho^+, \omega < \mathrm{cf}(\nu)\};$$
$$T = \{(\rho^+, \nu^\rho) : \rho^+ < \nu^\rho, \nu \le \kappa\}.$$

Then $\mathrm{Depth_{Hr}}(A) = S \cup T$.

Corollary 4.79. (CH) $\mathrm{Depth_{Hr}}([\omega_2]^{\le\omega}) = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$.

Corollary 4.80. *Suppose that $\omega \le \rho \le \kappa$. Then $\langle [\kappa]^{\le\rho}\rangle_{\mathscr{P}(\kappa)}$ has an independent subset of size 2^ρ, but none of size $(2^\rho)^+$.*

Proof. Clearly $\langle [\kappa]^{\le\rho}\rangle_{\mathscr{P}(\kappa)}$ has an independent subset of size 2^ρ. Suppose that X is an independent subset of size $(2^\rho)^+$. Then by Sikorski's extension theorem $\langle [\kappa]^{\le\rho}\rangle_{\mathscr{P}(\kappa)}$ has a homomorphic image with depth $(2^\rho)^+$, contradicting Theorem 4.77. □

Theorem 4.81. *Suppose that λ is an uncountable regular limit cardinal and $\lambda \le \kappa$. Let $A = \langle [\kappa]^{<\lambda} \rangle_{\mathscr{P}(\kappa)}$. Define*

$$\begin{aligned} S &= \{(\mu,\nu) : \omega \le \mu \le \nu < 2^{<\lambda}, \nu^\omega = \nu\};\\ T &= \{(\mu,\nu^{<\lambda}) : \lambda \le \mu \le 2^{<\lambda} \le \nu \le \kappa\};\\ U &= \{(\mu, 2^{<\lambda}) : \omega \le \mu \le 2^{<\lambda}\}. \end{aligned}$$

Then:

(i) $S \cup T \subseteq \mathrm{Depth}_{\mathrm{Hr}}(A)$*;*
(ii) *If there is a $\rho < \lambda$ such that $2^\rho = 2^{<\lambda}$, then* $\mathrm{Depth}_{\mathrm{Hr}}(A) = S \cup T \cup U$*.*
(iii) *If $2^\rho < 2^{<\lambda}$ for all $\rho < \lambda$, then* $\mathrm{Depth}_{\mathrm{Hr}}(A) = S \cup T$*.*

Proof. To prove that $S \subseteq \mathrm{Depth}_{\mathrm{Hr}}(A)$, suppose that $\omega \le \mu \le \nu < 2^{<\lambda}$ and $\nu^\omega = \nu$. Choose $\rho < \lambda$ such that $\nu < 2^\rho$. The mapping $a \mapsto a \cap \rho$ gives a homomorphism of A onto $\mathscr{P}(\rho)$, and by Theorem 4.77 with $\kappa = \rho$ we get $(\mu,\nu) \in \mathrm{Depth}_{\mathrm{Hr}}(A)$.

To prove that $T \subseteq \mathrm{Depth}_{\mathrm{Hr}}(A)$, suppose that $\lambda \le \mu \le 2^{<\lambda} \le \nu \le \kappa$. Now first suppose also that $\mu < 2^{<\lambda}$. Choose $\rho < \lambda$ such that $\mu < 2^\rho$. By Theorem 4.77 there is a homomorphic image B of $\mathscr{P}(\rho)$ with depth μ. Then the desired homomorphic image of A is obtained from the following obvious composition of homomorphisms:

$$\begin{aligned} A &\cong \mathscr{P}(\rho) \times \langle [\kappa\backslash\rho]^{<\lambda} \rangle\\ &\to B \times A\\ &\to B \times \langle [\nu]^{<\lambda} \rangle. \end{aligned}$$

If $\mu = 2^{<\lambda}$ a more complicated construction is needed. Let $\langle M_\alpha : \alpha < \lambda \rangle$ be a disjoint sequence of infinite members of $[\kappa]^{<\lambda}$ such that $\langle |M_\alpha| : \alpha < \lambda \rangle$ is strictly increasing, with supremum λ. For each $\alpha < \lambda$ let f_α be a homomorphism of $\mathscr{P}(M_\alpha)$ onto an algebra B_α whose size and depth are both equal to $2^{|M_\alpha|}$; this is possible by Theorem 4.77. Let $N = \bigcup_{\alpha<\lambda} M_\alpha$, and let $g : A \to C \stackrel{\mathrm{def}}{=} \langle [N]^{<\lambda} \rangle$ be the homomorphism (onto) given by $g(a) = a \cap N$ for all $a \in A$. Next, define $h : C \to \prod_{\alpha<\lambda} \mathscr{P}(M_\alpha)$ by setting $(h(a))_\alpha = a \cap M_\alpha$ for all $a \in C$ and $\alpha < \lambda$. Clearly h is a homomorphism. Let $D = \mathrm{rng}(h)$.

(1) $|D| \le 2^{<\lambda}$.

For, note that if $a \in [N]^{<\lambda}$, then $a \subseteq \bigcup_{\beta<\alpha} M_\beta$ for some $\alpha < \lambda$; then $(h(a))_\beta = \emptyset$ for all $\beta \in (\alpha, \lambda)$. Hence

$$|D| \le \sum_{\alpha<\lambda} \prod_{\beta<\alpha} 2^{|M_\beta|} = \sum_{\alpha<\lambda} 2^{\sum_{\beta<\alpha}|M_\beta|} \le 2^{<\lambda}.$$

So (1) holds.

Now define $k : \prod_{\alpha<\lambda} \mathscr{P}(M_\alpha) \to \prod_{\alpha<\lambda} B_\alpha$ by setting $(k(x))_\alpha = f_\alpha(x_\alpha)$ for every $x \in \prod_{\alpha<\lambda} \mathscr{P}(M_\alpha)$. So k is a homomorphism. Let $E = \mathrm{rng}(k \circ h \circ g)$. So by (1) we have $|E| \le 2^{<\lambda}$.

(2) $\mathrm{Depth}(E) = 2^{<\lambda}$.

In fact, we construct a strictly increasing sequence of members of E, as follows. For $\alpha < \lambda$, let $\langle a_{\alpha\xi} : \xi < 2^{|M_\alpha|}\rangle$ be a sequence of subsets of M_α such that $\langle f_\alpha(a_{\alpha\xi}) : \xi < 2^{|M_\alpha|}\rangle$ is a strictly increasing sequence of elements of B_α. Then for $\alpha < \lambda$ and $\xi < 2^{|M_\alpha|}$ let

$$c_{\alpha\xi} = \bigcup_{\beta<\alpha} M_\beta \cup a_{\alpha\xi}.$$

Clearly if $(\alpha, \xi) < (\beta, \eta)$ lexicographically, then $k(h(g(c_{\alpha\xi}))) < k(h(g(c_{\beta\eta})))$. Thus (2) holds.

Next, for each $a \in A$ let $l(a) = a \cap \nu$. So l is a homomorphism from A onto $\langle[\nu]^{<\lambda}\rangle_{\mathscr{P}(\nu)}$. Finally, for each $a \in A$ let $s(a) = (k(h(g(a))), l(a))$. Clearly s is a homomorphism from A onto a BA with depth μ and size $\nu^{<\lambda}$. Thus (i) holds.

For (ii), suppose that $\rho < \lambda$ such that $2^\rho = 2^{<\lambda}$. For $\supseteq$, by (i) we just need to show that $U \subseteq \mathrm{Depth}_{\mathrm{Hr}}(A)$. The mapping $a \mapsto a \cap \rho$ is a homomorphism from A onto $\mathscr{P}(\rho)$, and so $U \subseteq \mathrm{Depth}_{\mathrm{Hr}}(A)$ by Theorem 4.77 with $\kappa = \rho$.

For $\subseteq$, suppose that $(\mu, \nu) \in \mathrm{Depth}_{\mathrm{Hr}}(A)$. Let I be an ideal of A such that $|A/I| = \nu$ and $\mathrm{Depth}(A/I) = \mu$. To reach a contradiction, suppose that $(\mu, \nu) \notin S \cup T \cup U$. Since $(\mu, \nu) \notin U$, we have $2^{<\lambda} < \nu$. Hence by Juhász, Shelah [98] we have $\nu^{<\lambda} = \nu$. Let σ be minimum such that $\sigma^{<\lambda} = \nu$. As above it follows that $\sigma \le \kappa$. So now since $(\mu, \nu) \notin T$ we get $\mu < \lambda$ or $2^{<\lambda} < \mu$.

Suppose that $\mu < \lambda$. Define by recursion $a_\xi \in [\kappa]^{<\lambda}$ such that $\xi < \eta$ implies that $[a_\xi]_I < [a_\eta]_I$, continuing as long as possible. Say that this produces a sequence $\langle a_\xi : \xi < \alpha\rangle$. Thus by our supposition, $\alpha < \lambda$. Then $b \overset{\mathrm{def}}{=} \bigcup_{\xi<\alpha} a_\xi$ has size less than λ. Now $B \cong (B \restriction [b]_I) \times (B \restriction -[b]_I)$, and $|B \restriction [b]_I| \le 2^{<\lambda} < \nu$. It follows that there is a $c \in [\kappa]^{\le\rho}$ such that $0 < [c]_I < -[b]_I$. Let $a_\alpha = b \cup (c\backslash b)$. So $[a_\xi]_I \le [a_\alpha]_I$ for all $\xi < \alpha$. Moreover, $a_\alpha\backslash a_\xi \supseteq c\backslash b$ and $c\backslash b \notin I$, so we could have extended our sequence further, contradiction.

The proof that $\mu \le 2^{<\lambda}$ is very similar to the argument in the proof of Theorem 4.77.

For (iii), $\supseteq$ is given by (i). Now suppose that $(\mu, \nu) \in \mathrm{Depth}_{\mathrm{Hr}}(A)\backslash(S \cup T)$. Then $2^{<\lambda} \le \nu \le \kappa$ and $\mu < \lambda$ or $2^{<\lambda} < \mu$. The above arguments then give a contradiction. □

Corollary 4.82. *If λ is an uncountable strongly inaccessible cardinal $\le \kappa$, then*

$$\mathrm{Depth}_{\mathrm{Hr}}(\langle[\kappa]^{<\lambda}\rangle) = \{(\mu, \nu) : \omega \le \mu \le \nu < \lambda, \nu^\omega = \nu\} \cup \{(\lambda, \nu^{<\lambda}) : \lambda \le \nu \le \kappa\}. \ \square$$

Problem 61. *Characterize the relation* $\mathrm{Depth}_{\mathrm{Hr}}$.

This is Problem 17 of Monk [96].

For small cardinals there are many open questions, and we will not survey the situation here.

Now we turn to the depth and small depth in special classes of BAs. Depth is the same as cellularity for complete BAs, while every infinite complete BA has a tower of length ω. For any complete BA A, $\mathrm{tow_{spect}}(A)$ contains all regular cardinals in the interval $[\omega, \mathrm{c}(A))$, and also has $\mathrm{c}(A)$ as a member if it is attained.

It is possible to have $\mathrm{Depth}(A) < \mathrm{c}(A)$ for an interval algebra. For example, let τ be the order type of the real numbers, let L be an ordered set of type $0 + (\omega + \omega^*) \cdot \tau$, and let A be the interval algebra on L. It is easily seen that $\mathrm{Depth}(A) = \omega$ while $\mathrm{c}(A) = 2^\omega$.

Concerning small depth for interval algebras we have the following result, which was given in Monk [01] in a somewhat weaker form.

Theorem 4.83. *Suppose that L is an infinite linear order with first element 0, and κ is a regular cardinal. Then the following conditions are equivalent.*

(i) $\mathrm{Intalg}(L)$ *has a tower of order type κ.*

(ii) *One of the following holds:*

  (a) *There is a $c \in L$ and a strictly decreasing sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of (c, ∞) coinitial with c.*

  (b) *There is a $c \in L \cup \{\infty\}$ and a strictly increasing sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of $[0, c)$ cofinal in c.*

  (c) *There exist a strictly increasing sequence $\langle b_\alpha : \alpha < \kappa\rangle$ of elements of L and a strictly decreasing sequence $\langle c_\alpha : \alpha < \kappa\rangle$ of elements of L such that $b_\alpha < c_\beta$ for all $\alpha, \beta < \kappa$, and there is no element $d \in L$ such that $b_\alpha < d < c_\beta$ for all $\alpha, \beta < \kappa$.*

Proof. (ii)⇒(i): Assume (ii)(a). Then $\langle [0, c) \cup [a_\alpha, \infty) : \alpha < \kappa\rangle$ is a tower in $\mathrm{Intalg}(L)$.

Assume (ii)(b). Then $\langle [0, a_\alpha) \cup [c, \infty) : \xi < \kappa\rangle$ is a tower in $\mathrm{Intalg}(L)$.

Assume (ii)(c). Then $\langle [0, b_\alpha) \cup [c_\alpha, \infty) : \alpha < \kappa\rangle$ is a tower in $\mathrm{Intalg}(L)$.

(i)⇒(ii): Let $\langle a_\alpha : \alpha < \kappa\rangle$ be a tower in $\mathrm{Intalg}(L)$. Write

$$a_\alpha = [b_0^\alpha, c_0^\alpha) \cup \cdots \cup [b_{m_\alpha-1}^\alpha, c_{m_\alpha-1}^\sigma),$$

where $0 \le b_0^\alpha < c_0^\alpha < \cdots < b_{m_\alpha-1}^\alpha < c_{m_\alpha-1}^\alpha \le \infty$. Clearly $b_0^\alpha \ge b_0^\beta$ if $\alpha < \beta < \kappa$. If each $b_0^\alpha \ne 0$, then clearly 0 is the g.l.b. of $\langle b_0^\alpha : \alpha < \kappa\rangle$, and (ii)(a) holds with $c = 0$. Hence we may assume that $b_0^\alpha = 0$ for all $\alpha < \kappa$. Clearly $c_0^\alpha \le c_0^\beta$ if $\alpha < \beta < \kappa$. If $\sup_{\alpha<\kappa} c_0^\alpha = \infty$, then (ii)(b) holds for ∞ and some subsequence of $\langle c_0^\alpha : \alpha < \kappa\rangle$. So we may assume that $\sup_{\alpha<\kappa} c_0^\alpha \ne \infty$. We now consider several cases.

Case 1. There is an $\alpha < \kappa$ such that $c_0^\beta = c_0^\alpha$ for all $\beta \in [\alpha, \kappa)$. Clearly then each $m_\alpha > 1$ and there is a subsequence of $\langle b_1^\alpha : \alpha < \kappa\rangle$ which gives (ii)(a).

Case 2. There is no α as in Case 1, but $\sup_{\alpha<\kappa} c_0^\alpha$ exists. This gives (ii)(b).

Case 3. There is no α as in Case 1, but $\sup_{\alpha<\kappa} c_0^\alpha$ does not exist. Clearly then there is a subsequence of $\langle b_1^\alpha : \alpha < \kappa\rangle$ which gives (ii)(c). □

As an application we give an example of an infinite linear order whose interval algebra does not have any towers. The construction depends on some more-or-less standard notation, which we now introduce.

A *gap* in a linear order L is a pair (A, B) of nonempty subsets of L such that $L = A \cup B$, $\forall a \in A \forall b \in B(a < b)$, A has no largest element, and B has no smallest element. The *lower character* of such a gap is the least cardinality of a subset of A cofinal in A, and similarly for *upper character*; these are both regular cardinals. The *character* of (M, N) is the pair of these characters.

Theorem 4.84. *There is an infinite linear order L such that* $\mathrm{Intalg}(L)$ *does not have a tower.*

Proof. We start out with a dense linear order M such that every point of M has character (ω_1, ω_1), the gaps of M have characters (ω, ω_1) and (ω_1, ω), M has no first or last element, and M has cofinality and coinitiality ω_1. The existence of M follows from a theorem in Hausdorff [1908]. We replace each element of M by $\omega^* + \omega$, put ω to the left of the result, and ω^* to the right. For definiteness let a and b be one-one functions such that $\mathrm{rng}(a) \cap \mathrm{rng}(b) = \emptyset$ and $(\mathrm{rng}(a) \cup \mathrm{rng}(b)) \cap (M \times \mathbb{Z}) = \emptyset$. Let

$$L = \mathrm{rng}(a) \cup \mathrm{rng}(b) \cup (M \times \mathbb{Z}),$$

and for $m, n \in \omega$, $p, q \in \mathbb{Z}$, and $c, d \in M$ define

$$\begin{aligned}
a_m < a_n &\quad \text{iff} \quad m < n; \\
b_m < b_n &\quad \text{iff} \quad n < m; \\
a_m < (c, p)&; \\
(c, p) < (c, q) &\quad \text{iff} \quad p < q; \\
(c, p) < (d, q) &\quad \text{iff} \quad c < d, \quad \text{when } c \neq d; \\
a_m < b_n&; \\
(c, p) < b_m&.
\end{aligned}$$

Clearly this gives a linear order, and the elements do not have infinite characters. Thus it suffices by the preceding theorem to show that the characters of the gaps of L are (ω, ω_1) or (ω_1, ω).

Suppose that (A, B) is a gap in L. Then there are these possibilities:

Case 1. $A = \{a_n : n \in \omega\}$. Then the character of (A, B) is (ω, ω_1).

Case 2. There is a $c \in M$ such that $A = \{x \in L : x < (c, m)$ for every $m \in \mathbb{Z}\}$. Then the character of (A, B) is (ω_1, ω).

Case 3. There is a $c \in M$ such that $A = \{x \in L : x < (d, m)$ for every $d > c$ and every $m \in \mathbb{Z}\}$. Then the character of (A, B) is (ω, ω_1).

Case 4. There is a gap (C, D) in M such that $A = \{(c, m) : c \in C, m \in \mathbb{Z}\} \cup \mathrm{rng}(a)$. Then the character of (A, B) is the same as the character of (C, D), and thus is (ω, ω_1) or (ω_1, ω).

Case 5. $A = \{x \in L : x < b_m \text{ for all } m \in \omega\}$. Then the character of (A, B) is (ω_1, ω). □

Another application of Theorem 4.82 is to the tower spectrum.

Theorem 4.85. *If K is a nonempty set of regular cardinals, then there is a dense linear order L such that* $\mathrm{tow}_{\mathrm{spect}}(\mathrm{Intalg}(L)) = K$.

Proof. Let μ be any successor cardinal greater than each member of K, and also fix $\lambda \in K$. By the main theorem of Hausdorff [08], let L be a dense linear order such that L has an element with character (κ, κ) for each $\kappa \in K$, L has coinitiality λ^* and cofinality λ, and the set of gap characters of L is

$$\{(\rho, \mu) : \rho \text{ regular}, \rho < \mu\} \cup \{(\mu, \rho) : \rho \text{ regular}, \rho < \mu\}.$$

Then L is as desired. □

Now we turn to tree algebras. By Proposition 16.20 in the Handbook, if T is an infinite tree then $\mathrm{Depth}(\mathrm{Treealg}(T))$ is equal to $\max(\sup\{|C| : C \text{ is a chain in } T\}, \omega)$.

We turn to towers in tree algebras.

Lemma 4.86. *Suppose that T is a tree with a single root r. Suppose that κ is a regular cardinal, and T has a chain of order type κ with finitely many immediate successors.*

Then $\mathrm{Treealg}(T)$ *has a tower of order type κ.*

Proof. Let $\langle t_\alpha : \alpha < \kappa\rangle$ be a chain in T with the finite set M of immediate successors. For each $\alpha < \kappa$ let

$$a_\alpha = [(T \uparrow r) \backslash (T \uparrow t_\alpha)] \cup \bigcup_{s \in M} (T \uparrow s).$$

Clearly $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing. To show that it is a tower, it suffices to take an element b of the form $b = (T \uparrow s) \backslash \bigcup_{u \in N} (T \uparrow u)$ with N a finite set of elements of $(T \uparrow s) \backslash \{s\}$ and find $\alpha < \kappa$ such that $b \cap a_\alpha \neq \emptyset$. If $t_\alpha \not\leq s$ for some $\alpha < \kappa$, then $s \in b \cap [(T \uparrow r) \backslash (T \uparrow t_\alpha)] \subseteq b \cap a_\alpha$, as desired. If $t_\alpha \leq s$ for all $\alpha < \kappa$, then there is an $u \in M$ such that $u \leq s$, and $a_0 \cap b \neq \emptyset$. □

Theorem 4.87. *Let T be an infinite tree with a single root r. Then the following conditions are equivalent:*

(i) $\mathrm{Treealg}(T)$ *has a tower of order type ω.*
(ii) *One of the following conditions holds:*
  (a) *T has an element with exactly ω immediate successors.*
  (b) *T has a chain of countable limit length with at most ω immediate successors.*

Proof. (ii)⇒(i): Assume (ii)(a); let t be an element with $\langle s_i : i \in \omega\rangle$ an enumeration of all distinct immediate successors. For each $i \in \omega$ let

$$a_i = [(T \uparrow r)\backslash(T \uparrow t)] \cup \bigcup_{j \le i}(T \uparrow s_j).$$

Clearly $\langle a_i : i \in \omega\rangle$ is a strictly increasing sequence of elements of Treealg(T). To show that it is a tower, it suffices to take an element b of Treealg(T) of the form $b = (T \uparrow u)\backslash \bigcup_{v\in M}(T \uparrow v)$, where M is a collection of pairwise incomparable elements of $(T \uparrow u)\backslash\{u\}$ and show that $b \cap a_i \ne \emptyset$ for some $i \in \omega$. If $t \not\le u$, then $i = 0$ works. If $t = u$, then $i = 0$ still works, since $(T \uparrow s_i) \subseteq b$ for any s_i which is not below any member of M. If $t < u$, then there is an i such that $s_i \le u$, and hence $b \le a_i$, as desired.

Now assume (ii)(b); let $\langle t_i : i \in \omega\rangle$ be strictly increasing with a set M of immediate successors, $|M| \le \omega$. For M finite, Lemma 4.85 gives the desired result. For M infinite, say with $M = \{s_i : i \in \omega\}$, define for each $i \in \omega$

$$a_i = [(T \uparrow r)\backslash(T \uparrow t_i)] \cup \bigcup_{j \le i}(T \uparrow s_i).$$

The argument for Lemma 4.85 just needs a slight change to work for this case.

Now suppose that both conditions (ii)(a) and (ii)(b) fail, but Treealg(T) has a tower of order type ω; we want to get a contradiction. Say that $\langle a_i : i \in \omega\rangle$ is a tower in Treealg(T). For each $i < \omega$ write a_i in full normal form:

$$a_i = \bigcup_{t \in M_i} e_{it};$$
$$e_{it} = (T \uparrow t)\backslash \bigcup_{s \in N_{it}} (T \uparrow s);$$

where M_i is a finite subset of T, N_{it} is a finite set of pairwise incomparable elements of $(T \uparrow t)\backslash\{t\}$, $e_{it} \cap e_{iu} = \emptyset$ for $t \ne u$, and $t \notin N_{iu}$ for $t \ne u$.

(1) If $i < \omega$, $t \in M_i$, and $s \in N_{it}$, then $s \notin a_i$.

In fact, suppose that $i < \omega$, $t \in M_i$, $s \in N_{it}$, and $s \in a_i$. Choose $u \in M_i$ such that $s \in e_{iu}$. Since $s \notin e_{it}$, it follows that $u \ne t$. Moreover, $s \ne u$ by normality. Now $u < s$ and $t < s$, giving two cases:

Case 1. $t < u$. Then $u \in e_{it}$ (since $u < s$) and $u \in e_{iu}$, contradicting disjointness.

Case 2. $u < t$. Then $t \in e_{iu}$ (since $t < s \in e_{iu}$) and $t \in e_{it}$, contradicting disjointness.

Thus (1) holds.

(2) $T = \bigcup_{i<\omega} a_i$.

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In fact, suppose that $t \in T\setminus\bigcup_{i<\omega} a_i$. If t has a finite set M of immediate successors, then $\{t\} = (T \uparrow t)\setminus\bigcup_{s\in M}(T \uparrow s)$, so $\{t\}\cap a_i \neq \emptyset$ for some $i < \omega$, contradiction. So t has infinitely many immediate successors. Since (ii)(a) fails, it has uncountably many.

Let Q be the set of immediate successors of t below some member of $\bigcup_{i<\omega} M_i$. So Q is countable. Let u be an immediate successor of t not in Q. Choose $i < \omega$ such that $a_i \cap (T \uparrow u) \neq \emptyset$. Say $s \in a_i \cap (T \uparrow u)$. Choose $v \in M_i$ such that $v \leq s$. Then $u \leq s$ and $v \leq s$, so u and v are comparable. If $u \leq v$, then $u \in Q$, contradiction. Hence $v < u$, and so $v \leq t$. So $t \in [v, s]$ and $v, s \in a_i$, hence $t \in a_i$, contradiction. Thus (2) holds.

By (2) we may assume that $r \in a_i$ for all $i < \omega$. Hence $r \in M_i$ for all $i < \omega$.

(3) If $i < j$ and $s \in N_{jr}$, then there is a unique $t \in N_{ir}$ such that $t \leq s$.

In fact, under the hypotheses of (3), if $s \notin \bigcup_{t\in N_{ir}}(T \uparrow t)$, then $s \in e_{ir} \subseteq a_i \subseteq a_j$. Since $s \notin e_{jr}$, it follows that $s \in e_{jt}$ for some $t \in M_j\setminus\{r\}$. So $t \leq s$. Since $t \in e_{jt}$, it follows that $t \notin e_{jr}$. Hence $t = s$, contradicting normality of a_j. So it follows that there is a $t \in N_{ir}$ with $t \leq s$. This t is unique, since the elements of N_{ir} are pairwise incomparable. Thus (3) holds.

(4) If $i < j$, then $e_{ir} \subseteq e_{jr}$.

For, suppose that $s \notin e_{jr}$. Then there is a $t \in N_{jr}$ such that $t \leq s$. By (3) we get $u \in N_{ir}$ such that $u \leq t$, and hence $s \notin e_{ir}$.

Now let $T' = \{(s, i) : i < \omega \text{ and } s \in N_{ir}\}$. We define $(s, i) < (t, j)$ iff $i < j$ and $s \leq t$. This makes T' into a tree of height ω, by (3). For each $i < \omega$, the set $N'_i = \{(s, i) : s \in N_{ir}\}$ is the set of all elements of level i, and it is finite. Hence by König's tree lemma, there is a branch $(s_0, 0) < (s_1, 1) < \cdots$ through T'. By (4), there is a subsequence $(s_{i(0)}, i(0)) < (s_{i(1)}, i(1))$ such that $i(0) < i(1) < \cdots$ and $s_{i(0)} < s_{i(1)} < \cdots$. Hence by the assumption that (ii)(b) fails, the sequence $\langle s_{i(k)} : k < \omega\rangle$ has uncountably many immediate successors. Let Q be the collection of all immediate successors t of $\langle s_{i(k)} : k < \omega\rangle$ such that $t \leq u$ for some $u \in \bigcup_{\gamma<\omega} M_\gamma$. So Q is countable. Let t be an immediate successor of $\langle s_{i(k)} : k < \omega\rangle$ which is not in Q. Choose $k < \omega$ such that $(T \uparrow t) \cap a_{i(k)} \neq \emptyset$. Choose $u \in (T \uparrow t) \cap a_{i(k)}$. Since $u \in a_{i(k)}$, choose $v \in M_{i(k)}$ such that $u \in e_{i(k)v}$. Thus $v, t \leq u$, so v and t are comparable. If $t \leq v$, the choice of t is contradicted. So $v < t$. Choose $h < \omega$ such that $v < s_{i(h)}$. We may assume that $k < h$. Now $s_{i(h)} \in a_{i(k)}$ since $v, u \in e_{i(k)v}$ and $v < s_{i(h)} < t \leq u$. But $a_{i(k)} \subseteq a_{i(h)}$, so $s_{i(h)} \in a_{i(h)}$, contradicting (1). □

Theorem 4.88. *Suppose that T is an infinite tree with a single root r, and let κ be an uncountable regular cardinal. Then the following conditions are equivalent.*

(i) Treealg(T) *has a tower of order type κ.*

(ii) *There is a strictly increasing sequence $\langle t_\alpha : \alpha < \kappa\rangle$ of elements of T which has only finitely many immediate successors.*

Proof. (ii)⇒(i) holds by Lemma 4.85. Now suppose that (i) holds but (ii) fails. Let $\langle a_\alpha : \alpha < \kappa\rangle$ be a tower in Intalg(T). For each $\alpha < \kappa$ write a_α in full normal form:

$$a_\alpha = \bigcup_{t\in M_\alpha} e_{\alpha t};$$
$$e_{\alpha t} = (T \uparrow t)\backslash \bigcup_{s\in N_{\alpha t}} (T \uparrow s);$$

where M_α is a finite subset of T, $N_{\alpha t}$ is a finite set of pairwise incomparable elements of $(T \uparrow t)\backslash\{t\}$, $e_{\alpha t} \cap e_{\alpha u} = \emptyset$ for $t \neq u$, and $t \notin N_{\alpha u}$ for $t \neq u$.

(1) $T = \bigcup_{\alpha<\kappa} a_\alpha$.

For, suppose that $w \in T\backslash \bigcup_{\alpha<\kappa} a_\alpha$. Then as in the previous proof, w must have infinitely many immediate successors. For each $\alpha < \kappa$ let $P(\alpha)$ consist of all immediate successors of w which are below some $t \in M_\alpha$. Suppose that $\alpha < \beta < \kappa$. Let $s \in P(\alpha)$; say $s \leq t \in M_\alpha$. Then $t \in a_\alpha \subseteq a_\beta$, so there is a $u \in M_\beta$ such that $t \in (T \uparrow u)$. Thus $u \leq t$. So $s, u \leq t$ and hence s and u are comparable. If $u < s$, then $u \leq w < t$ and $u, t \in a_\beta$, so $w \in a_\beta$, contradiction. It follows that $s \leq u$. This proves that $P(\alpha) \subseteq P(\beta)$. Now κ uncountable implies that there is an $\alpha < \kappa$ such that $P(\alpha) = P(\beta)$ for all $\beta \in [\alpha, \kappa)$. Choose $\beta < \kappa$ with some $t \in a_\beta \cap (T \uparrow w)\backslash \bigcup_{s\in P(\alpha)}(T \uparrow s)$. Then $w < t$ since $w \notin a_\beta$, so there is an immediate successor s of w such that $s \leq t$. Choose $u \in M_\beta$ such that $u \leq t$. Since $s, u \leq t$, they are comparable. If $u < s$, then $u \leq w$ and hence $w \in [u, t] \subseteq a_\beta$, contradiction. So $s \leq u$, hence $s \in P(\beta)$ and $s \leq t$, contradiction. This proves (1).

By (1) we may assume that $r \in a_\alpha$ for all $\alpha < \kappa$.

(2) If $\alpha < \beta$ and $s \in N_{\beta r}$, then there is a unique $t \in N_{\alpha r}$ such that $t \leq s$.

This is clear.

(3) If $\alpha < \beta$, then $e_{\alpha r} \subseteq e_{\beta r}$.

For, suppose that $s \notin e_{\beta r}$. Then there is a $t \in N_{\beta r}$ such that $t \leq s$. By (2) we get $u \in N_{\alpha r}$ such that $u \leq t$, and hence $s \notin e_{\alpha r}$.

Now let $T' = \{(s, \alpha) : \alpha < \kappa \text{ and } s \in N_{\alpha r}\}$. We define $(s, \alpha) < (t, \beta)$ iff $\alpha < \beta$ and $s \leq t$. This makes T' into a tree of height κ. Now for each finite subset F of κ let

$$M_F = \{f \in P_{\alpha<\kappa} N_{\alpha r} : \forall \alpha, \beta \in F[\alpha < \beta \to f(\alpha) < f(\beta)]\}.$$

Clearly M_F is a closed subset of $P_{\alpha<\kappa} N_{\alpha r}$. By (2), each M_F is nonempty. Hence by the Tychonoff product theorem we obtain a function $f \in P_{\alpha<\kappa} N_{\alpha r}$ such that $f(\alpha) < f(\beta)$ whenever $\alpha < \beta$.

(4) There do not exist $\alpha < \kappa$ and $s \in T$ such that $f(\beta) = (s, \beta)$ for all $\beta \geq \alpha$.

This is true by (1). It follows that there are strictly increasing sequences $\langle \alpha_\xi : \xi < \kappa\rangle$ of ordinals and $\langle s_\xi : \xi < \kappa\rangle$ of elements of T such that $s_\xi \in N_{\alpha_\xi r}$ for all

$\xi < \kappa$. Let $\beta = \sup_{\xi<\kappa} \alpha_\xi$. So β is a limit ordinal of cofinality κ. Moreover, by the hypothesis that (ii) fails, it follows that $\langle s_\xi : \xi < \kappa\rangle$ has infinitely many immediate successors. Let S be the set of all immediate successors of $\langle s_\xi : \xi < \kappa\rangle$.

(5) If $w \in S$, then there is a $\xi < \kappa$ such that $w \in a_{\alpha_\xi}$, and for any such ξ we have $w \in M_{\alpha_\xi}$.

In fact, by (1) $T = \bigcup_{\xi<\kappa} a_{\alpha_\xi}$, so there is a $\xi < \kappa$ such that $w \in a_{\alpha_\xi}$. Take any such ξ. Then there is a $t(w) \in M_{\alpha_\xi}$ such that $t(w) \le w$. If $t(w) < w$, choose $\eta < \kappa$ such that $\alpha_\xi < \alpha_\eta$ and $t(w) < s_\eta$. Now $s_\eta \in [t(w), w] \subseteq a_{\alpha_\xi} \subseteq a_{\alpha_\eta}$, contradiction. It follows that $t(w) = w$, and so $w \in M_{\alpha_\xi}$.

Now let $\langle w(n) : n \in \omega\rangle$ be a one-one enumeration of some elements of S. For each $n \in \omega$ choose by (5) a $\xi(n) < \kappa$ such that $\forall \eta \in [\xi(n), \kappa)[w(n) \in M_{\alpha_\eta}]$. Let $\eta = \sup_{n\in\omega} \xi(n)$. Then M_{α_η} is infinite, contradiction. □

Theorem 4.89. *Let T be an infinite tree. Then the following conditions are equivalent:*

(i) $\mathrm{Treealg}(T)$ *has a tower.*
(ii) *One of the following conditions holds:*
  (a) *T has an element with exactly ω immediate successors.*
  (b) *T has a chain of countable limit length with at most ω immediate successors.*
  (c) *For some uncountable regular cardinal κ, T has an chain of order type κ with only finitely many immediate successors.* □

Theorem 4.90. *If K is a nonempty set of regular cardinals, then there is an atomless tree algebra A such that $\mathrm{tow}_{\mathrm{spect}}(A) = K$.*

Proof. Let λ be the smallest member of K. Let T_ω be the tree ${}^{<\omega}\omega_1$. For each uncountable regular cardinal, let T_κ be the tree determined by the following conditions. T has a unique root. T has height κ. Each element of T, and each initial chain of T of limit ordinal type less than κ has exactly ω_1 immediate successors. Now the tree desired in the theorem is formed by adjoining a new root beneath the following trees: ω_1 copies of T_λ and, for each $\kappa \in K\backslash\{\lambda\}$, a copy of T_κ. By the above theorems, T is as desired.

Although these theorems are fairly definitive, they use unusual trees – trees that have many elements directly above certain infinite chains. Let us call a tree T *limit-normal* iff every initial chain of T of limit ordinal length has at most one immediate successor. Concerning towers in such trees we have the following results.

Theorem 4.91. *Suppose that T is a limit-normal tree with a single root and* $\mathrm{Treealg}(T)$ *is atomless. Then the following conditions are equivalent:*

(i) $\mathrm{Treealg}(T)$ *has a tower.*
(ii) $\mathrm{Treealg}(T)$ *has a tower of length ω.*

(iii) *T has an element with exactly ω immediate successors or T has infinite height.* □

Theorem 4.92. *Suppose that T is a limit-normal tree with a single root, with* $\mathrm{Treealg}(T)$ *atomless. Suppose that κ is an uncountable regular cardinal. Then the following conditions are equivalent:*

(i) $\mathrm{Treealg}(T)$ *has a tower of order type κ.*
(ii) *T has height at least κ.* □

Theorem 4.93. *Suppose that T is a limit-normal tree with a single root, with* $\mathrm{Treealg}(T)$ *atomless. Then* $tow_{\mathrm{spect}}(\mathrm{Treealg}(T))$ *has the form* $[\omega,\lambda)_{\mathrm{reg}}$ *for some* $\lambda > \omega$. □

Theorems 4.85 and 4.86 generalize to pseudo-trees; see Monk [09].

Concerning superatomic BAs we mention a result of Dow, Monk [97]: if λ is a Ramsey cardinal, then every superatomic BA with depth at least λ^+ has depth at least λ. This result has been strengthened by Shelah, Spinas [99].

We finish this chapter by giving two theorems concerning depth and related functions in the algebra $\mathscr{P}(\omega)/\mathrm{fin}$.

Theorem 4.94. *Under MA,* $\mathfrak{p}(\mathscr{P}(\omega)/\mathrm{fin}) = 2^\omega$.

Proof. It clearly suffices to prove the following statement:

(∗) If $X \subseteq [\omega]^\omega$, $|X| < 2^\omega$, and $\omega\backslash\bigcup F$ is infinite for every finite $F \subseteq X$, then there is an infinite set $d \subseteq \omega$ such that $x \cap d$ is finite for all $x \in X$ and $\omega\backslash(d \cup \bigcup F)$ is infinite for every finite $F \subseteq X$.

This is immediate from Kunen [80], II.2.15. □

Corollary 4.95. *Under MA, each of the following functions, applied to* $\mathscr{P}(\omega)/\mathrm{fin}$, *is equal to* 2^ω*:* c, $\mathfrak{a}$, Depth, tow, $\mathfrak{p}$, $\mathfrak{h}$, spl. □

Actually the proof of Theorem 4.93 shows that if MA(κ), then $\mathfrak{p}(\mathscr{P}(\omega)/\mathrm{fin}) > \kappa$. Hence we get the following additional corollary, by MA(ω):

Corollary 4.96. (In ZFC) *Each of the following functions, applied to* $\mathscr{P}(\omega)/\mathrm{fin}$, *is greater than ω:* c, $\mathfrak{a}$, Depth, tow, $\mathfrak{p}$, $\mathfrak{h}$, spl. □

For both corollaries, recall that $\mathrm{c}(\mathscr{P}(\omega)/\mathrm{fin}) = 2^\omega$ in ZFC.

The second result is of a folklore nature.

Theorem 4.97. *There is a model of ZFC in which $2^\omega > \omega_1$ while* $\mathscr{P}(\omega)/\mathrm{fin}$ *has depth ω_1. In fact, we can take $M[G]$, where M satisfies CH and G adds Cohen reals.*

Proof. We follow Kunen [80] for the forcing framework. Assume that (in M) CH holds, and κ is a regular uncountable cardinal. Let $\mathscr{P} = \{p : p$ is a finite function with domain contained in κ and range contained in $2\}$. And let G be M-generic over $\mathscr{P}$. Let φ be the formula

$$\begin{aligned}&\langle T_\alpha : \alpha < \omega_2\rangle \text{ is a sequence of subsets of } \omega,\\ &\text{and } \forall\alpha,\beta \in \omega_2[\alpha < \beta \Rightarrow T_\alpha\backslash T_\beta \text{ is finite and } T_\beta\backslash T_\alpha \text{ is infinite}],\end{aligned}$$

and suppose that $M[G] \models \varphi$; we want to get a contradiction. Choose $p \in G$ so that $p \Vdash \varphi$. From now on we work in M.

Temporarily fix $\alpha < \omega_2$ and $n \in \omega$. Now $p \Vdash n \in T_\alpha \vee n \notin T_\alpha$, so $\forall q \leq p \exists r \leq q(r \Vdash n \in T_\alpha \vee r \Vdash n \notin T_\alpha)$. Hence there is a maximal pairwise incompatible set $A_{\alpha n} \subseteq \{q : q \leq p\}$ such that $\forall q \in A_{\alpha n}(q \Vdash n \in T_\alpha \vee q \Vdash n \notin T_\alpha)$.

Now for any $\alpha \in \omega_2$ let

$$C_\alpha = \mathrm{dmn}(p) \cup \bigcup_{n\in\omega} \bigcup_{q\in A_{\alpha n}} \mathrm{dmn}(q).$$

Thus C_α is countable. By CH there is an $X \in [\omega_2]^{\omega_2}$ such that $\langle C_\alpha : \alpha \in X\rangle$ is a Δ-system, say with kernel C. We may also assume that there is a $\gamma < \omega_1$ such that $C_\alpha\backslash C$ has order type γ for all $\alpha \in X$. For all $\alpha, \beta \in X$, let $j_{\alpha\beta}$ be the permutation of $C_\alpha \cup C_\beta$ such that $j_{\alpha\beta}$ is the identity on C, and is the unique order preserving map from $C_\alpha\backslash C$ onto $C_\beta\backslash C$ and from $C_\beta\backslash C$ onto $C_\alpha\backslash C$. Thus $j_{\alpha\alpha}$ is the identity on C_α, and $j_{\alpha\beta} = j_{\beta\alpha}$. Extend each $j_{\alpha\beta}$ to a permutation $j'_{\alpha\beta}$ of κ by letting $j'_{\alpha\beta}$ be the identity outside of $C_\alpha \cup C_\beta$. Note that $j'_{\alpha\beta} \circ j'_{\alpha\beta}$ is the identity mapping on κ. Obviously also

(1) $(j'_{\beta\gamma} \circ j'_{\alpha\beta}) \restriction C_\alpha = j'_{\alpha\gamma} \restriction C_\alpha$ for any $\alpha, \beta, \gamma \in X$.

Now $j'_{\alpha\beta}$ naturally induces an automorphism $j''_{\alpha\beta}$ of $\mathscr{P}$: for any $r \in \mathscr{P}$, the domain of $j''_{\alpha\beta}(r)$ is $j'_{\alpha\beta}[\mathrm{dmn}(r)]$ and for any $\gamma \in \mathrm{dmn}(r)$, $(j''_{\alpha\beta}(r))(j'_{\alpha\beta}(\gamma)) = r(\gamma)$. Clearly

(2) $j''_{\alpha\beta} = j''_{\beta\alpha}$.

(3) $j''_{\alpha\beta} \circ j''_{\alpha\beta}$ is the identity on $\mathscr{P}$.

We will use the notation $(j''_{\alpha\beta})_*$ introduced in Kunen [80] VII.7.12; the property VII.7.13 is important. Now for each $\alpha \in X$ define $\mathscr{P}_\alpha = \{q \in \mathscr{P} : \mathrm{dmn}(q) \subseteq C_\alpha$ and $q \leq p\}$. Note that $\mathscr{P}_\alpha$ is countable.

(4) $j''_{\alpha\beta}[\mathscr{P}_\alpha] = \mathscr{P}_\beta$ for any $\alpha, \beta \in X$.

In fact, suppose that $q \in \mathscr{P}_\alpha$. Then $\mathrm{dmn}(j''_{\alpha\beta}(q)) = j'_{\alpha\beta}[\mathrm{dmn}(q)] \subseteq j'_{\alpha\beta}[C_\alpha] = C_\beta$. Next, $\mathrm{dmn}(p) \subseteq C \subseteq C_\beta$. If $\gamma \in \mathrm{dmn}(p)$, then $\gamma \in \mathrm{dmn}(q) \cap C$, so $\gamma = j'_{\alpha\beta}(\gamma) \in \mathrm{dmn}(j''_{\alpha\beta}(q))$ and $(j''_{\alpha\beta}(q))(\gamma) = j''_{\alpha\beta}(q))(j'_{\alpha\beta}(\gamma)) = q(\gamma) = p(\gamma)$. This shows that $j''_{\alpha\beta}(q) \in \mathscr{P}_\beta$.

Hence $j''_{\alpha\beta}[\mathscr{P}_\alpha] \subseteq \mathscr{P}_\beta$. By symmetry, $j''_{\alpha\beta}[\mathscr{P}_\beta] \subseteq \mathscr{P}_\alpha$. So

$$\mathscr{P}_\beta = j''_{\alpha\beta}[j''_{\alpha\beta}[\mathscr{P}_\beta]] \subseteq j''_{\alpha\beta}[\mathscr{P}_\alpha].$$

So (4) holds.

(5) $(j''_{\beta\gamma} \restriction \mathscr{P}_\beta) \circ (j''_{\alpha\beta} \restriction \mathscr{P}_\alpha) = (j''_{\alpha\gamma} \restriction \mathscr{P}_\alpha)$.

To prove this, take any $q \in \mathscr{P}_\alpha$. Then

$$\mathrm{dmn}(j''_{\beta\gamma}(j''_{\alpha\beta}(q))) = j'_{\beta\gamma}[\mathrm{dmn}(j''_{\alpha\beta}(q))] = j'_{\beta\gamma}[j'_{\alpha\beta}[\mathrm{dmn}(q)]] = j'_{\alpha\gamma}[\mathrm{dmn}(q)],$$

using (1). Now take any $\delta \in \mathrm{dmn}(q)$. Then

$$\begin{aligned}(j''_{\beta\gamma}(j''_{\alpha\beta}(q)))(j'_{\alpha\gamma}(\delta)) &= (j''_{\beta\gamma}(j''_{\alpha\beta}(q)))(j'_{\beta\gamma}(j'_{\alpha\beta}(\delta)))\\ &= (j''_{\alpha\beta}(q))(j'_{\alpha\beta}(\delta)) = q(\delta) = j''_{\alpha\gamma}(j'_{\alpha\gamma}(\delta)),\end{aligned}$$

proving (5).

We define $h_\alpha : \mathscr{P}_\alpha \to {}^\omega 3$ by

$$(h_\alpha(q))(n) = \begin{cases} 0, & \text{if } \ q \Vdash n \notin T_\alpha,\\ 1, & \text{if } \ q \Vdash n \in T_\alpha,\\ 2, & \text{otherwise.}\end{cases}$$

Define $\alpha \equiv \beta$ iff $\alpha, \beta \in X$ and $h_\beta \circ (j''_{\alpha\beta} \restriction \mathscr{P}_\alpha) = h_\alpha$. It is straightforward to check that $\equiv$ is an equivalence relation on X. Then:

(6) There are at most ω_1 equivalence classes under $\equiv$. $\square$

In fact, suppose that $\Gamma \in [X]^{\omega_2}$ consists of pairwise inequivalent ordinals. Fix $\alpha \in \Gamma$. Then for any distinct $\beta, \gamma \in \Gamma$ we have $h_\beta \circ (j''_{\alpha\beta} \restriction \mathscr{P}_\alpha) \neq h_\gamma \circ (j''_{\alpha\gamma} \restriction \mathscr{P}_\alpha)$, since otherwise

$$h_\beta \circ (j''_{\gamma\beta} \restriction \mathscr{P}_\gamma) = h_\beta \circ (j''_{\alpha\beta} \restriction \mathscr{P}_\alpha) \circ (j''_{\gamma\alpha} \restriction \mathscr{P}_\gamma) = h_\gamma \circ (j''_{\alpha\gamma} \restriction \mathscr{P}_\alpha) \circ (j''_{\gamma\alpha} \restriction \mathscr{P}_\gamma) = h_\gamma,$$

contradiction. But this gives $\aleph_2$ members of ${}^{\mathscr{P}_\alpha}({}^\omega 3)$, which by CH has cardinality $\aleph_1$, contradiction. Thus (6) holds.

By (6), let X' be an equivalence class with $\aleph_2$ elements. Fix $\alpha, \beta \in X'$ with $\alpha < \beta$. Now $p \Vdash (T_\beta \backslash T_\alpha$is infinite), so, since $\mathrm{dmn}(p) \subseteq C$,

(7) $p \Vdash ((j''_{\alpha\beta})_*(T))_\beta \backslash (j''_{\alpha\beta})_*(T))_\alpha$ is infinite.

Next we claim:

(8) $p \Vdash \forall n \in \omega(n \notin T_\alpha \to n \notin ((j''_{\alpha\beta})_*(T))_\beta)$.

To prove this, suppose that $q \leq p$ and $q \Vdash n \notin T_\alpha$; we want to show that $q \Vdash n \notin ((j''_{\alpha\beta})_*(T))_\beta$. Suppose that this is not true. Then there is an $r \leq q$ such that

$r \Vdash n \in ((j''_{\alpha\beta})_*(T))_\beta$. Since $r \le q$, we also have $r \Vdash n \notin T_\alpha$, so there is a $t \in A_{\alpha n}$ such that r and t are compatible. Thus $t \in \mathscr{P}_\alpha$ and $(h_\alpha(t))(n) = 0$. Let $s = j''_{\alpha\beta}(t)$. Since $\alpha \equiv \beta$, we have $h_\beta(s) = h_\beta(j''_{\alpha\beta}(t)) = h_\alpha(t)$. Hence $(h_\beta(s))(n) = 0$, so $s \Vdash n \notin T_\beta$. Hence $t \Vdash n \notin ((j''_{\alpha\beta})_*(T))_\beta$. Since r and t are compatible and $r \Vdash n \in (j''_{\alpha\beta})_*(T))_\beta$, this is a contradiction. So, (8) holds.

Similarly:

(9) $p \Vdash \forall n \in \omega(n \in T_\beta \to n \in ((j''_{\alpha\beta})_*(T))_\alpha)$.

Next, since $p \Vdash (T_\alpha \backslash T_\beta$ is finite), choose $m \in \omega$ and $q \le p$ so that

(10) $q \Vdash \forall n \ge m(n \in T_\alpha \to n \in T_\beta)$.

By (7), $q \Vdash \exists n \ge m(n \in ((j''_{\alpha\beta})_*(T))_\beta \wedge n \notin ((j''_{\alpha\beta})_*(T))_\alpha)$, so choose $n \ge m$ and $r \le q$ so that $r \Vdash n \in ((j''_{\alpha\beta})_*(T))_\beta \wedge n \notin ((j''_{\alpha\beta})_*(T))_\alpha)$. By (9), $r \Vdash n \in T_\alpha$, so by (10), $r \Vdash n \in T_\beta$. Then by (9), $r \Vdash n \in (j''_{\alpha\beta})_*(T))_\alpha$, contradiction. $\square$

5 Topological Density

We begin with some equivalents of this notion. A set X of non-zero elements of a BA A is said to be *centered* provided that it satisfies the finite intersection property. And A is called *κ-centered* if $A\backslash\{0\}$ is the union of κ centered sets.

Theorem 5.1. *For any infinite BA A, $\mathrm{d}(A)$ is equal to each of the following cardinals:*

$\min\{\kappa : A$ *is isomorphic to a subalgebra of* $\mathscr{P}(\kappa)\}$;
$\min\{\kappa : A$ *is* κ*-centered*$\}$;
$\min\{\kappa :$ *there is a system of* κ *centered subsets of* A *whose union is dense in* $A\}$.
$\min\{\kappa : A\backslash\{0\}$ *is a union of* κ *proper filters*$\}$;
$\min\{\kappa : A\backslash\{0\}$ *is a union of* κ *ultrafilters*$\}$.

Proof. Call the six cardinals mentioned $\kappa_0, \ldots, \kappa_5$ respectively, starting with $\mathrm{d}(A)$ itself. $\kappa_0 \leq \kappa_1$: Let g be an isomorphism of A into $\mathscr{P}(\kappa)$. For each $\alpha < \kappa$ let $F_\alpha = \{a \in A : \alpha \in g(a)\}$. Then, as is easily checked, F_α is an ultrafilter on A. Let $Y = \{F_\alpha : \alpha < \kappa\}$. We claim that Y is dense in $\mathrm{Ult}(A)$. For, let U be a non-empty open set in $\mathrm{Ult}(A)$. We may assume that $U = \mathcal{S}(a)$ for some $a \in A$. Thus $a \neq 0$, so choose $\alpha \in g(a)$. Then $a \in F_\alpha$, and so $F_\alpha \in Y \cap U$, as desired.

$\kappa_1 \leq \kappa_2$: Suppose that $A\backslash\{0\} = \bigcup_{\alpha<\kappa} X_\alpha$, where each X_α is centered. Extend each X_α to an ultrafilter F_α. For each $a \in A$ let $f(a) = \{\alpha < \kappa : a \in F_\alpha\}$. Clearly f is an isomorphism of A into $\mathscr{P}(\kappa)$, as desired.

$\kappa_2 \leq \kappa_3$: Suppose that $\langle X_\alpha : \alpha < \kappa\rangle$ is a system of centered sets whose union is dense in A. For each $\alpha < \kappa$ let Y_α be the set of all elements above some member of X_α. Clearly Y_α is centered, and $\bigcup_{\alpha<\kappa} Y_\alpha = A\backslash\{0\}$.

Obviously $\kappa_3 \leq \kappa_4 \leq \kappa_5$.

$\kappa_5 \leq \kappa_0$: Let X be a dense subset of $\mathrm{Ult}(A)$. Then obviously $A\backslash\{0\} = \bigcup_{F\in X} F$, as desired. □

Corollary 5.2. $\mathrm{c}(A) \leq \mathrm{d}(A)$ *and* $|A| \leq 2^{\mathrm{d}(A)}$ *for any infinite BA A.*

Proof. The first inequality is clear from Theorem 5.1(i). For the second inequality, let D be a dense subset of $\mathrm{Ult}(A)$ of size $\mathrm{d}(A)$. Define $f(a) = \mathcal{S}(a) \cap D$ for any

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$a \in A$, where $\mathcal{S}$ is the Stone isomorphism. If $a \neq b$, then $\mathcal{S}(a \triangle b) \neq \emptyset$, hence $D \cap \mathcal{S}(a \triangle b) \neq \emptyset$ and so $D \cap \mathcal{S}(a) \neq D \cap \mathcal{S}(b)$, showing that f is one-one. □

We begin the discussion of algebraic operations for d. If A is a subalgebra of B, then $\mathrm{d}(A) \leq \mathrm{d}(B)$, and the difference can be arbitrarily large. If $A \leq_\pi B$, then $d(A) = d(B)$. In fact, let f be an isomorphism from A into $\mathscr{P}(\kappa)$. By Sikorski's extension theorem, f extends to a homomorphism from B into $\mathscr{P}(\kappa)$, and this homomorphism is one-one since $A \leq_\pi B$. If $A \leq_s B$, then $d(A) = d(B)$. In fact, if $B \backslash \{0\}$ is the union of a set $\mathscr{F}$ of ultrafilters, then $A \backslash \{0\} = \bigcup_{F \in \mathscr{F}} (F \cap A$. Clearly the difference can be large when $A \leq_{\mathrm{mg}} B$ or $A \leq_{\mathrm{free}} B$, and so the same applies to $\leq_{\mathrm{proj}}$, $\leq_{\mathrm{rc}}$, $\leq_\sigma$, $\leq_{\mathrm{reg}}$, and $\leq_{\mathrm{u}}$.

If A is a homomorphic image of B, then d can change either direction in going from B to A. Thus if B is a large free BA and A is a countable homomorphic image of B, then d goes down. On the other hand, if $B = \mathscr{P}(\omega)$, and $A = \mathscr{P}(\omega)/\mathrm{fin}$, then $\mathrm{d}(B) = \omega$ while $\mathrm{d}(A) = 2^\omega$, since in A there is a disjoint set of size 2^ω.

Next, $\mathrm{d}(A \times B) = \max(\mathrm{d}(A), \mathrm{d}(B))$ for infinite BAs A, B. To see this, note that $\geq$ is clear, since A and B are isomorphic to subalgebras of $A \times B$. For the other inequality, suppose that f (resp. g) is an isomorphism of A (resp. B) into $\mathscr{P}(\kappa)$ (resp. $\mathscr{P}(\lambda)$). Let

$$X = \{(0, \alpha) : \alpha < \kappa\} \cup \{(1, \alpha) : \alpha < \lambda\}.$$

We define h mapping $A \times B$ into $\mathscr{P}(X)$ by setting

$$h(a, b) = \{(0, \alpha) : \alpha \in f(a)\} \cup \{(1, \alpha) : \alpha \in g(b)\}$$

for all $(a, b) \in A \times B$. It is easily verified that h is an isomorphism of $A \times B$ into $\mathscr{P}(X)$, and this proves $\leq$. A similar idea works for products and weak products in general:

Theorem 5.3. *If $\langle A_i : i \in I \rangle$ is a system of non-trivial BAs and $\prod^{\mathrm{w}}_{i \in I} A_i \leq B \leq \prod_{i \in I} A_i$, then $\mathrm{d}(B) = \sum_{i \in I} \mathrm{d}(A_i)$.*

Proof. Clearly $|I| \leq \mathrm{d}(B)$. Each A_i can be isomorphically embedded in $\prod^{\mathrm{w}}_{i \in I} A_i$, hence in B, so $\mathrm{d}(A_i) \leq \mathrm{d}(B)$. Thus $\geq$ holds.

Now for each $i \in I$ let f_i be an isomorphism from A_i into $\mathscr{P}(\mathrm{d}(A_i))$. We define $g : B \to \mathscr{P}(\{(i, \alpha) : i \in I$ and $\alpha \in \mathrm{d}(A_i)\}$ by setting $g(b) = \{(i, \alpha) : \alpha \in f_i(a_i)\}$ for each $b \in B$. Clearly g is an isomorphic embedding, and this gives $\leq$. □

Clearly $\mathrm{d}(A \oplus B) = \max(\mathrm{d}(A), \mathrm{d}(B))$: if f is an isomorphism of A into $\mathscr{P}(\kappa)$ and g is an isomorphism of B into $\mathscr{P}(\lambda)$, then the following function clearly extends to an isomorphism of $A \oplus B$ into $\mathscr{P}(\kappa \times \lambda)$: for $a \in A$ and $b \in B$, $h(a) = f(a) \times \lambda$ and $h(b) = \kappa \times g(b)$. For free products of several algebras there is a much more general topological result. To prove it, we need the following lemma.

Lemma 5.4. *Let κ be an infinite cardinal. Then the product space ${}^{({}^{\kappa}2)}\kappa$ has density $\leq \kappa$ (where κ has the discrete topology).*

Proof. Let $D = \{f \in {}^{({}^{\kappa}2)}\kappa$: there is a finite subset M of κ such that for all $x, y \in {}^{\kappa}2$, if $x \upharpoonright M = y \upharpoonright M$, then $f(x) = f(y)\}$. We show that $|D| \leq \kappa$. First,

$$D = \bigcup_{M \in [\kappa]^{<\omega}} \{f \in {}^{({}^{\kappa}2)}\kappa : \text{ for all } x, y \in {}^{\kappa}2(\text{ if } x \upharpoonright M = y \upharpoonright M, \text{ then } f(x) = f(y))\}.$$

So, it suffices to take any finite $M \subseteq \kappa$ and show that $N \stackrel{\text{def}}{=} \{f \in {}^{({}^{\kappa}2)}\kappa$: for all $x, y \in {}^{\kappa}2$, if $x \upharpoonright M = y \upharpoonright M$ then $f(x) = f(y)\}$ has power at most κ. For any $f \in N$, let $f' \in {}^{({}^{M}2)}\kappa$ be defined as follows: for any $x \in {}^{M}2$, choose any $y \in {}^{\kappa}2$ such that $x \subseteq y$ and let $f'(x) = f(y)$. Clearly the assignment $f \mapsto f'$ is one-one. So $|N| \leq \kappa$, as desired.

To show that D is dense in ${}^{({}^{\kappa}2)}\kappa$, let U be a nonempty open set in ${}^{({}^{\kappa}2)}\kappa$. We may assume that U has a very special form, namely that there is a finite subset F of ${}^{\kappa}2$ and a function g mapping F into κ such that

$$U = \{f \in {}^{({}^{\kappa}2)}\kappa : g \subseteq f\}.$$

Now let G be a finite subset of κ such that $f \upharpoonright G \neq h \upharpoonright G$ for distinct $f, h \in F$. Define $k \in {}^{({}^{\kappa}2)}\kappa$ in the following way: for any $x \in {}^{\kappa}2$, set $k(x) = g(f)$ if $x \upharpoonright G = f \upharpoonright G$ for some $f \in F$, otherwise let $k(x)$ be 0. Clearly $k \in D \cap U$, as desired. $\square$

Note that if f is a continuous function from X onto Y, then $\mathrm{d}(Y) \leq \mathrm{d}(X)$.

Theorem 5.5. *Let $\langle X_i : i \in I\rangle$ be a system of topological spaces each having at least two disjoint non-empty open sets. Then $\mathrm{d}(\prod_{i\in I} X_i) = \max(\lambda, \sup_{i\in I}\mathrm{d}(X_i))$, where λ is the least cardinal such that $|I| \leq 2^{\lambda}$.*

Proof. Clearly $\mathrm{d}(X_i) \leq \mathrm{d}(\prod_{i\in I} X_i)$ for each $i \in I$. Suppose that D is dense in $\prod_{i\in I} X_i$ but $|D| < \lambda$. Then $2^{|D|} < |I|$. Let U_i^0 and U_i^1 disjoint non-empty open sets in X_i for all $i \in I$. For each $i \in I$ let

$$V_i = \left\{x \in \prod_{j\in I} X_j : x_i \in U_i^0\right\}.$$

Then our supposition implies that there are distinct $i, j \in I$ such that $V_i \cap D = V_j \cap D$. Let $W = \{x : x_i \in U_i^0 \text{ and } x_j \in U_j^1\}$. Choose $x \in W \cap D$. Then $x \in V_i$ but $x \notin V_j$, contradiction.

Up to this point we have proved the inequality $\geq$. Now for each $i \in I$, let D_i be dense in X_i with $|D_i| = \mathrm{d}(X_i)$. Set $\kappa = \max(\lambda, \sup_{i\in I}|D_i|)$. Then for each $i \in I$ there is a function f_i mapping κ onto D_i. Since $|I| \leq 2^{\lambda}$, we then get a continuous function from ${}^{({}^{\kappa}2)}\kappa$ onto $\prod_{i\in I} D_i$. Namely, let g be a one-one function from I into ${}^{\kappa}2$. For each $x \in {}^{({}^{\kappa}2)}\kappa$ and each $i \in I$ let $(h(x))_i = f_i(x_{g(i)})$. Then h is the desired continuous function. To see that h is continuous, let U be basic open in $\prod_{i\in I} D_i$. Then there is a finite $F \subseteq I$ such that $\mathrm{pr}_i[U]$ is open in D_i for all $i \in F$ and $\mathrm{pr}_i[U] = D_i$ for all $i \in I \backslash F$. Let $L = \{l \in {}^{g[F]}\kappa : \forall i \in F(f_i(l_{g(i)}) \in \mathrm{pr}_i[U])\}$. For each $l \in L$ the set $W_l \stackrel{\text{def}}{=} \{k \in {}^{(\kappa 2)}\kappa : l \subseteq k\}$ is open in ${}^{(\kappa 2)}\kappa$, and $h^{-1}[U] = \bigcup_{l \in L} W_l$. So h is continuous. Clearly h maps onto $\prod_{i \in I} D_i$.

Now Lemma 5.4 yields the desired result. $\square$

The second part of the following corollary was observed by Sabine Koppelberg.

Corollary 5.6. *Let A be a free BA on κ free generators. Then* $\mathrm{d}(A)$ *is the smallest cardinal λ such that $\kappa \le 2^\lambda$. More generally, if B is an infinite subalgebra of A, then* $\mathrm{d}(B)$ *is the least cardinal μ such that $|B| \le 2^\mu$.*

Proof. For each infinite cardinal ν let $\log_2\nu$ be the least cardinal μ such that $\nu \le 2^\mu$. For A itself the corollary is true directly by Theorem 5.5. Now let B be an infinite subalgebra of A. Note that B is a subalgebra of a subalgebra of A generated by $|B|$ free generators, and so $\mathrm{d}(B) \le \log_2|B|$. If $|B| = \omega$, the desired conclusion is obvious. If $\omega < |B|$ and $|B|$ is regular, the conclusion follows from Theorem 9.16 of the BA handbook. Finally, suppose that $|B|$ is a singular cardinal. Then for each regular $\nu < |B|$ we have $\log_2\nu \le \mathrm{d}(B)$ by Theorem 9.16. Since clearly $\log_2|B| = \sup_{\nu < |B|} \log_2\nu$, this case now follows too. $\square$

Theorem 5.5 characterizes the topological density of free products of BAs. We do not have a characterization of the topological density of amalgamated free products of BAs. Note by the example $\mathscr{P}(\omega) \oplus_{\mathrm{Finco}(\omega)} \mathscr{P}(\omega)$ that the topological density can be larger than that of the constituent algebras; see the discussion of cellularity of amalgamated free products.

Problem 62. *Describe the topological density of amalgamated free products of BAs.*

Next we treat the topological density of the union of a well-ordered chain.

Proposition 5.7. *Let $\langle B_\alpha : \alpha < \kappa \rangle$ be a strictly increasing sequence of BAs with union A. Then:*

(i) $\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha) \le \mathrm{d}(A) \le \sum_{\alpha<\kappa} \mathrm{d}(B_\alpha) \le \kappa \cdot \sup_{\alpha<\kappa} \mathrm{d}(B_\alpha) \le (2^{\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha)})^+$.
(ii) $\kappa \le |A| \le 2^{\mathrm{d}(A)}$.

Proof. If $A \backslash \{0\} = \bigcup K$ with K a set of ultrafilters, and if $\beta < \kappa$, then $B_\beta \backslash \{0\} = \bigcup_{F \in K} (B_\alpha \cap F)$. This shows that $\mathrm{d}(B_\alpha) \le \mathrm{d}(A)$. Now for each $\alpha < \kappa$ let X_α be a set of ultrafilters on A such that $|X_\alpha| = \mathrm{d}(B_\alpha)$ and $\{F \cap B_\alpha : F \in X_\alpha\}$ is dense in $\mathrm{Ult}(B_\alpha)$. Now $\bigcup_{\alpha<\kappa} X_\alpha$ is dense in $\mathrm{Ult}(A)$. In fact, given $a \in A^+$, choose $\alpha < \kappa$ such that $a \in B_\alpha$. Then there is an $F \in X_\alpha$ such that $F \cap B_\alpha \in \mathcal{S}^{B_\alpha}(a)$, hence $a \in F$ and $F \in \mathcal{S}^A(a)$. So

$$\mathrm{d}(A) \le \left| \bigcup_{\alpha<\kappa} X_\alpha \right| \le \sum_{\alpha<\kappa} \mathrm{d}(B_\alpha) \le \kappa \cdot \sup_{\alpha<\kappa} \mathrm{d}(B_\alpha).$$

Since

$$|B_\beta| \le 2^{\mathrm{d}(B_\beta)} \le 2^{\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha)}$$

for each $\beta < \kappa$, we must have $\kappa \leq (2^{\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha)})^+$, since otherwise

$$(2^{\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha)})^+ \leq |B_\beta| \leq 2^{\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha)}$$

with $\beta = (2^{\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha)})^+$, contradiction. Since $\langle B_\alpha : \alpha < \kappa\rangle$ is strictly increasing, clearly $\kappa \leq |A|$. $|A| \leq 2^{\mathrm{d}(A)}$ by Corollary 5.6. $\square$

Proposition 5.7 is clarified somewhat by the following example. Let λ be an uncountable regular cardinal, and let $B = \mathrm{Finco}(\lambda)$. For each $\alpha < \lambda$ let $A_\alpha = \langle\{\{\xi\} : \xi < \alpha\}\rangle_B$. Then $\langle A_\alpha : \alpha < \lambda\rangle$ is a strictly increasing system of subalgebras with union B. We have $\mathrm{d}(A_\alpha) < \lambda$ for each $\alpha < \lambda$, while $\mathrm{d}(B) = \lambda$.

Corollary 5.8. *Let $\langle B_\alpha : \alpha < \kappa\rangle$ be a strictly increasing sequence of BAs with union A. Suppose that $\sup_{\alpha<\kappa} \mathrm{d}(B_\alpha) < \mathrm{d}(A)$. Then $\mathrm{d}(A) \leq \kappa$.* $\square$

A connection between cellularity and topological density is given in Shelah [80]; we can use it to get another result about unions. Shelah's Theorem depends on a combinatorial result which we prove first. For this combinatorial result, see Comfort, Negrepontis [74], Theorem 3.16.

Theorem 5.9. *Suppose that $\lambda = \lambda^{<\kappa}$. Then there is a system $\langle f_\alpha : \alpha < 2^\lambda\rangle$ of members of ${}^\lambda\lambda$ such that*

$$\forall M \in \left[2^\lambda\right]^{<\kappa} \forall g \in {}^M\lambda \exists \beta < \lambda \forall \alpha \in M[f_\alpha(\beta) = g(\alpha)].$$

Proof. Let

$$\mathscr{F} = \left\{(F, G, s) : F \in [\lambda]^{<\kappa},\ G \in [\mathscr{P}(F)]^{<\kappa},\ \text{and } s \in {}^G\lambda\right\}.$$

Now if $F \in [\lambda]^{<\kappa}$, say $|F| = \mu$, then

$$\left|[\mathscr{P}(F)]^{<\kappa}\right| \leq (2^\mu)^{<\kappa}| \leq (\lambda^\mu)^{<\kappa} \leq \lambda^{<\kappa} = \lambda,$$

and if $G \in [\mathscr{P}(F)]^{<\kappa}$ then $|{}^G\lambda| \leq \lambda^{<\kappa} = \lambda$. It follows that $|\mathscr{F}| = \lambda$. Let h be a bijection from λ onto $\mathscr{F}$, and let k be a bijection from 2^λ onto $\mathscr{P}(\lambda)$. Now for each $\alpha < 2^\lambda$ we define $f_\alpha \in {}^\lambda\lambda$ by setting, for each $\beta < \lambda$, with $h(\beta) = (F, G, s)$,

$$f_\alpha(\beta) = \begin{cases} s(k(\alpha) \cap F) & \text{if } \ k(\alpha) \cap F \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Now to prove that $\langle f_\alpha : \alpha < 2^\lambda\rangle$ is as desired, suppose that $M \in [2^\lambda]^{<\kappa}$ and $g \in {}^M\lambda$. For distinct members α, β of M choose $\gamma(\alpha, \beta) \in k(\alpha) \triangle k(\beta)$. Then let

$$F = \{\gamma(\alpha, \beta) : \alpha, \beta \in M, \alpha \neq \beta\} \quad \text{and} \quad G = \{k(\alpha) \cap F : \alpha \in M\}.$$

Moreover, define $s : G \to \lambda$ by setting $s(k(\alpha) \cap F) = g(\alpha)$ for any $\alpha \in M$. This is possible since $k(\alpha) \cap F) \neq k(\beta) \cap F)$ for distinct $\alpha, \beta \in M$. Finally, let $\beta = h^{-1}(F, G, s)$. Then for any $\alpha \in M$ we have

$$f_\alpha(\beta) = s(k(\alpha) \cap F) = g(\alpha). \qquad \square$$

Corollary 5.10. *Suppose that* $\lambda^{<\kappa}=\lambda$. *Then there is a system* $\langle f_\alpha:\alpha<\lambda\rangle$ *of functions mapping* λ^+ *into* $[\lambda]^{<\kappa}$ *such that* $\forall M\in[\lambda^+]^{<\kappa}\forall g\in {}^M([\lambda]^{<\kappa})\exists\beta<\lambda[g\subseteq f_\beta]$.

Proof. Let $k:\lambda\to[\lambda]^{<\kappa}$ be a bijection, and let $\langle f'_\alpha:\alpha<2^\lambda\rangle$ be obtained from Theorem 5.9. For each $\beta<\lambda$ define $f_\beta:\lambda^+\to[\lambda]^{<\kappa}$ by setting $f_\beta(\alpha)=k(f'_\alpha(\beta))$ for any $\alpha<\lambda^+$. Now suppose that $M\in[\lambda^+]^{<\kappa}$ and $g:M\to[\lambda]^{<\kappa}$. Then $k^{-1}\circ g\in {}^M\lambda$, so there is a $\beta<\lambda$ such that for all $\alpha\in M$ we have $f'_\alpha(\beta)=(k^{-1}\circ g)(\alpha)$. Hence for all $\alpha\in M$, $g(\alpha)=k(f'_\alpha(\beta))=f_\beta(\alpha$. Thus $g\subseteq f_\beta$. □

Theorem 5.11. *Suppose that* κ *is uncountable and regular, and* $\lambda^{<\kappa}=\lambda$. *Let* A *be a BA of size* λ^+ *satisfying the* κ*-cc. Then* $\mathrm{d}(A)\le\lambda$.

In more detail, $A\backslash\{0\}$ *is the union of a family* $\mathscr{F}$ *of proper filters with* $|\mathscr{F}|\le\lambda$, *and every proper filter generated by* $<\kappa$ *elements is contained in some member of* $\mathscr{F}$.

Proof. Note that the final condition of the theorem implies that $A\backslash\{0\}$ is the union of $\mathscr{F}$, so we really just need to establish that condition.

By Corollary 10.5 of the BA handbook we have $|\overline{A}|=\lambda^+$, so it clearly suffices to assume that A is complete. For any subset X of A and any $a\in A$ let

$$\mathrm{high}(a,X)=\inf\{b\in\langle X\rangle_A^{\mathrm{cm}}:a\le b\}.$$

Write $A=\{b_i:i<\lambda^+\}$ without repetitions. We now define $\langle B_i:i<\lambda^+\rangle$. Let $B_0=2$. For i limit, let $B_i=\bigcup_{j<i}B_j$. Now suppose that B_i has been defined, with $|B_i|\le\lambda$. Choose $M_i\subseteq A\backslash B_i$ with $b_i\in B_i\cup M_i$ and $|M_i|=\lambda$. Let $B_{i+1}=\langle B_i\cup M_i\rangle^{\mathrm{cm}}$. Then $|B_i|=\lambda$ by the Handbook Corollary 10.5. Write $B_{i+1}\backslash B_i=\{a^i_\alpha:\alpha<\lambda\}$ without repetitions. Clearly

(1) $\bigcup_{i<\lambda^+}B_i=A$, and if $\delta<\lambda^+$ and $\mathrm{cf}(\delta)\ge\kappa$ then B_δ is complete.

Let $\langle f_\alpha:\alpha<\lambda\rangle$ be as in Corollary 5.10.

Now we define ultrafilters D^i_ζ on B_i for $i\le\lambda^+$ and $\zeta<\lambda$, by recursion on i. Let $D^0_\zeta=\{1\}$ for all $\zeta<\lambda$. For i limit and any $\zeta<\lambda$ let $D^i_\zeta=\bigcup_{j<i}D^j_\zeta$. Now suppose that D^i_ζ has been defined. If $D^i_\zeta\cup\{a^i_\alpha:\alpha\in f_\zeta(i)\}$ has fip, extend it to an ultrafilter D^{i+1}_ζ on B_{i+1}. Otherwise let D^{i+1}_ζ be any extension of D^i_ζ to an ultrafilter on B_{i+1}. This completes the definition of the D^i_ζ.

For each $\zeta<\lambda$ let $E_\zeta=D^{\lambda^+}_\zeta$. For each $i<\lambda^+$ define

$$S_i=\begin{cases}\{i\} & \text{if } i=0,\ i \text{ is a successor, or } i \text{ is a limit ordinal}\\ & \text{with } \mathrm{cf}(i)\ge\kappa,\\ \{\alpha(i,j):j<\mathrm{cf}(i)\} & \text{if } i \text{ is limit, } \mathrm{cf}(i)<\kappa, \text{ and } \langle\alpha(i,j):j<\mathrm{cf}(i)\rangle\\ & \text{is a strictly increasing sequence of successor}\\ & \text{ordinals with supremum } i\end{cases}$$

Also, for each $a \in A$ let

$$i(a) = \begin{cases} \text{the unique } i \text{ such that } a \in B_{i+1} \backslash B_i & \text{if } a \neq 0, 1, \\ 0 & \text{if } \ a = 0 \text{ or } a = 1. \end{cases}$$

Now suppose that $C \in [A]^{<\kappa}$ has fip; we want to show that $C \subseteq E_\zeta$ for some $\zeta < \lambda$. Let $C = \{c_\beta : \beta < \gamma\}$ for some $\gamma < \kappa$. Now we define $\langle F_n : n \in \omega\rangle$ by recursion. Let $F_0 = C$. Then let

$$F_{n+1} = \langle F_n \cup \{\text{high}(a, B_i) : a \in F_n,\ i \in S_{i(a)}\}\rangle_A.$$

Now each set S_i has size less than κ, so by induction, also each F_n has size less than κ. Let $G = \bigcup_{n\in\omega} F_n$. So also $|G| < \kappa$. Let H be an ultrafilter on G such that $C \subseteq H$. Let $M = \{i(a) : a \in G\}$ and define g with domain M by setting $g(i) = \{\alpha < \lambda : a^i_\alpha \in H\}$. Since $|H| < \kappa$, we have $|g(i)| < \kappa$. Thus $g : M \to [\lambda]^{<\kappa}$. Moreover, $|M| < \kappa$. Hence by the choice of $\langle f_\alpha : \alpha < \lambda\rangle$ above (using Corollary 5.10), choose $\xi < \lambda$ such that $g \subseteq f_\xi$. Now the rest of the proof consists in proving by induction on $i \leq \lambda^+$ that $H \cap B_i \subseteq D^i_\xi$. (Hence $C \subseteq H = H \cap A = \bigcup_{i<\lambda^+}(H \cap B_i) \subseteq D^{\lambda^+}_\xi = E_\xi$.)

Trivially $H \cap B_0 \subseteq D^0_\xi$. The inductive step to a limit ordinal i is obvious. So now assume that $H \cap B_k \subseteq D^k_\xi$ for all $k \leq i$; we want to prove that $H \cap B_{i+1} \subseteq D^{i+1}_\xi$. If $B_{i+1} \cap G = B_i$, then $H \cap B_{i+1} = H \cap B_i \subseteq D^i_\xi \subseteq D^{i+1}_\xi$, as desired. So, suppose that $G \cap B_{i+1} \backslash B_i \neq \emptyset$. It follows that $i \in M$, and hence $g(i) = f_\xi(i)$. Also, if $\alpha \in f_\xi(i)$, then $a^i_\alpha \in H$.

(2) $D^i_\xi \cup \{a^i_\alpha : \alpha \in f_\xi(i)\}$ has fip.

To prove (2), suppose that w is a finite subset of $f_\xi(i)$ and let $b = \prod_{\alpha\in w} a^i_\alpha$; we want to show that $b \cdot c \neq 0$ for every member c of D^i_ξ. Clearly $b \in H$. If $b \in B_i$, then $b \in B_i \cap H \subseteq D^i_\xi$, so obviously $b \cdot c \neq 0$ for every member c of D^i_ξ. Now each a^i_α is in B_{i+1}, so if $b \notin B_i$ then $b = a^i_\beta$ for some β. Since $b \in H$ it follows that $\beta \in f_\xi(i)$. Suppose that $c \in D^i_\xi$ and $b \cdot c = 0$.

Suppose that i is a successor, or i is limit and $\text{cf}(i) \geq \kappa$. Then $i \in S_{i(b)}$, and hence $\text{high}(b, B_i) \in G$. Moreover, B_i is complete. Now from $b \cdot c = 0$ we get $b \leq -c$, hence $\text{high}(b, B_i) \leq -c$, so $\text{high}(b, B_i) \cdot c = 0$. But $b \leq \text{high}(b, B_i) \in B_i$, so $\text{high}(b, B_i) \in H \cap B_i \subseteq D^i_\xi$, contradiction.

Now suppose that i is limit and $\text{cf}(i) < \kappa$. Then there is a $j < \text{cf}(i)$ such that $c \in B_{\alpha(i,j)}$. Since $\alpha(i, j)$ is a successor ordinal, it follows that $B_{\alpha(i,j)}$ is complete. Hence $\text{high}(b, B_{\alpha(i,j)}) \in B_{\alpha(i,j)}$. As above we get $\text{high}(b, B_{\alpha(i,j)}) \cdot c = 0$. Hence $\text{high}(b, B_{\alpha(i,j)}) \in H \cap B_{\alpha(i,j)} \subseteq D^{\alpha(i,j)}_\xi$, contradiction.

This finishes the proof of (2).

Now suppose that $u \in H \cap B_{i+1}$. If $u \in B_i$, then $u \in D^i_\xi \subseteq D^{i+1}_\xi$, as desired. Suppose that $u \notin B_i$. So $u \in B_{i+1} \backslash B_i$, and hence $u = a^i_\alpha$ for some $\alpha < \lambda$. Since $u \in H$, it follows that $\alpha \in g(i) = f_\xi(i)$. Hence $u = a^i_\alpha \in D^{i+1}_\xi$, as desired. $\square$

Corollary 5.12. *Let $\langle B_\alpha : \alpha < \kappa\rangle$ be a strictly increasing sequence of BAs whose union is A.*

Suppose that $\mathrm{d}(B_\alpha) \le \mu$ *for all* $\alpha < \kappa$, $\nu^\mu = \nu$, $\kappa = \nu^+$, *and* $2^\mu = \mu^+$. *Then A satisfies the μ^+-cc,* $|A| = \nu^+$, *and* $\mathrm{d}(A) \le \nu$.

In particular, if $\kappa = \omega_2$, $\mathrm{d}(B_\alpha) = \omega$ *for all* $\alpha < \omega_2$ *and CH holds, then A has ccc,* $|A| = \omega_2$, *and* $\mathrm{d}(A) = \omega_1$.

Proof. Let $\alpha < \kappa$. Since $2^\mu = \mu^+$ and $\mathrm{d}(B_\alpha) \le \mu$ we have $|B_\alpha| \le \mu^+$. Also, since $\nu^\mu = \nu$ we have $\mu^+ \le \nu$. So $\mu^+ < \nu^+ = \kappa$. Thus if $X \subseteq A$, $|X| = \mu^+$, and X is pairwise disjoint, then $X \subseteq B_\alpha$ for some $\alpha < \kappa$, hence $\mu^+ \le \mathrm{c}(B_\alpha) \le \mathrm{d}(B_\alpha) \le \mu$, contradiction. So A satisfies the μ^+-cc. Also, $\kappa = \nu^+ \le |A| \le \mu^+ \cdot \nu^+ = \nu^+$, so $|A| = \nu^+$. Now $\nu^{<\mu^+} = \nu^\mu = \nu$, so by Theorem 5.11, $\mathrm{d}(A) \le \nu$.

The second part of the corollary follows from the first part, with $\kappa = \omega_2$, $\mu = \omega$, $\nu = \omega_1$. □

We turn to ultraproducts.

Proposition 5.13. *If $\langle A_i : i \in I\rangle$ is a system of infinite BAs, and F an ultrafilter on I. Then* $\mathrm{d}(\prod_{i\in I} A_i/F) \le |\prod_{i\in I} \mathrm{d}(A_i)/F|$.

Proof. Let f_i be an isomorphism of A_i into $\mathscr{P}(\mathrm{d}(A_i))$ for each $i \in I$. It is easy to see that there is a function g mapping $\prod_{i\in I} A_i/F$ into $\mathscr{P}(\prod_{i\in I} \mathrm{d}(A_i)/F)$ such that for any $w \in \prod_{i\in I} A_i$,

$$g(w/F) = \left\{ y/F : y \in \prod_{i\in I} \mathrm{d}(A_i) \text{ and } \{i \in I : y_i \in f_i(w_i)\} \in F \right\}.$$

To see that g is one-one, suppose that $w, w' \in \prod_{i\in I} A_i$ with $w/F \ne w'/F$. So $M \stackrel{\text{def}}{=} \{i \in I : w_i \ne w'_i\} \in F$. For each $i \in M$ choose $y_i \in f_i(w_i) \triangle f_i(w'_i)$. Define $y_i = 0$ for all $i \in I \backslash M$. Note that $M \subseteq \{i \in I : y_i \in f_i(w_i)\} \cup \{i \in I : y_i \in f_i(w'_i)\}$, so $\{i \in I : y_i \in f_i(w_i)\} \in F$ or $\{i \in I : y_i \in f_i(w'_i)\} \in F$.

Case 1. $N \stackrel{\text{def}}{=} \{i \in I : y_i \in f_i(w_i)\} \in F$. Thus $y/F \in g(w/F)$. Suppose that $y/F \in g(w'/F)$. Then there exists $y' \in \prod_{i\in I} \mathrm{d}(A_i)$ such that $y/F = y'/F$ and $P \stackrel{\text{def}}{=} \{i \in I : y'_i \in f_i(w'_i)\} \in F$. Also, $Q \stackrel{\text{def}}{=} \{i \in I : y_i = y'_i\} \in F$. Thus $N \cap P \cap Q \in F$, and so $N \cap P \cap Q \ne \emptyset$. Take any $i \in N \cap P \cap Q$. Then $y_i \in f_i(w_i)$, $y'_i \in f_i(w'_i)$, and $y_i = y'_i$, contradiction. Thus $y/F \notin g(z')$, so $g(z) \ne g(z')$.

Case 2. $\{i \in I : y_i \in f_i(x'_i)\} \in F$. This is treated similarly. □

Rosłanowski, Shelah [98] constructed a system $\langle B_i : i \in I\rangle$ of BAs in ZFC such that $\mathrm{d}\left(\prod_{i\in I} B_i/F\right) < \left|\prod_{i\in I} \mathrm{d}(B_i)/F\right|$; this is a positive solution of Problem 9 of Monk [90].

We give this construction here. It depends on a finite version of topological density which is interesting in itself. For any positive integer n, a subset X of a BA A *satisfies the n-intersection property* iff the product of any n or fewer members of X is always nonzero. We define $\mathrm{d}_n(A)$ to be the least cardinal κ such that $A\backslash\{0\}$ is the union of κ sets each satisfying the n-intersection property. Note that $\mathrm{d}_n(A) \leq \mathrm{d}(A)$.

The following proposition generalizes Proposition 5.13.

Proposition 5.14. *Suppose that κ is an infinite cardinal, $\langle B_i : i < \kappa\rangle$ is a system of infinite BAs, D is an ultrafilter on κ, $f : \kappa \to \omega$, and $\forall i \in \omega[\{j < \kappa : f(j) \geq i\} \in D]$. Then*

$$\mathrm{d}\left(\prod_{i<\kappa} B_i/D\right) \leq \left|\prod_{i<\kappa} \mathrm{d}_{f(i)}(B)/D\right|.$$

Proof. For each $i \in \omega$. let $\langle M_{ij} : j < \mathrm{d}_{f(i)}(B_i)\rangle$ be a system of sets satisfying the $f(i)$-intersection property such that $B_i^+ = \bigcup_{j<\mathrm{d}_{f(i)}} M_{ij}$. Now for each $z \in \prod_{i<\kappa} \mathrm{d}_{f(i)}(B)/D$ let

$$N_z = \left\{y \in \prod_{i<\kappa} B_i/D : \exists x \in y \exists u \in z \forall i < \kappa[x_i \in M_{iu(i)}]\right\}.$$

(1) $\forall z \in \prod_{i<\kappa} \mathrm{d}_{f(i)}(B)/D[N_z$ satisfies the fip].

To prove (1), let $z \in \prod_{i<\kappa} \mathrm{d}_{f(i)}(B)/D$ and $F \in [N_z]^{<\omega}$. Let $|F| = m$, and define $M = \{i < \kappa : f(i) \geq m\}$; so $M \in D$. For each $y \in F$ choose $x^y \in y$ and $u^y \in z$ such that $\forall i < \kappa[x_i^y \in M_{iu^y(i)}]$. Now if $y, y' \in F$, then $u^y, u^{y'} \in z$, and so $\{i < \kappa : u^y(i) = u^{y'}(i)\} \in D$. Thus the set

$$P \stackrel{\text{def}}{=} M \cap \bigcap_{y,y' \in F} \{i < \kappa : u^y(i) = u^{y'}(i)\} \in D$$

is in D. Take any $i \in P$. Then for any $y \in F$, $x_i^y \in M_{iu^y(i)}$, and hence $\prod_{y\in F} x_i^y \neq 0$. It follows that $\prod F \neq 0$. So (1) holds.

(2) $\bigcup\left\{N_z : z \in \prod_{i<\kappa} \mathrm{d}_{f(i)}(B)/D\right\} = A^+$, where $A = \prod_{i<\kappa} B_i/D$.

To prove (2), take any $y \in A^+$. Choose $x \in y$. Then $M \stackrel{\text{def}}{=} \{i < \kappa : x_i \neq 0\} \in D$. For any $i \in M$ choose $u(i) < \mathrm{d}_{f(i)}$ so that $x_i \in M_{iu(i)}$, and let $u(i) = 0$ for $i \in \kappa\backslash M$. Let $z = u/D$. Then $y \in N_z$, as desired in (2). □

Proposition 5.15. *Suppose that κ is an infinite cardinal, $\langle \lambda_i : i < \kappa\rangle$ is a system of cardinals, $2^\kappa < \prod_{i<\kappa} \lambda_i$, and $2 < n < \omega$.*

Then there is a BA B such that

$$\mathrm{d}_{n-1}(B) \leq \sum_{\alpha<\kappa} \prod_{i<\alpha} \lambda_i \quad \text{and} \quad \mathrm{d}_n(B) = |B| = \prod_{i<\kappa} \lambda_i.$$

Proof. Let A be freely generated by $\langle x_\eta : \eta \in \prod_{i<\kappa} \lambda_i \rangle$, and let I be the ideal in A generated by

$$\left\{ x_{\eta_0} \cdot \ldots \cdot x_{\eta_{n-1}} : \exists \alpha < \kappa \left[\forall l < n \left[\eta_l \in \prod_{i<\kappa} \lambda_i \right], \forall l, m < n [\eta_l \restriction \alpha = \eta_m \restriction \alpha], \right.\right.$$
$$\left.\left. \langle \eta_l(\alpha) : l < n \rangle \text{ is one-one} \right] \right\}.$$

Let $B = A/I$. The basic property of B is as follows:

(1) For any $l < m < \omega$ and any distinct $\eta_0, \ldots, \eta_m \in \prod_{i<\kappa} \lambda_i$ the following conditions are equivalent:

(a) $[x_{\eta_0}] \cdot \ldots \cdot [x_{\eta_l}] \cdot -[x_{\eta_{l+1}}] \cdot \ldots \cdot -[x_{\eta_m}] = 0$.
(b) There exist $\alpha < \kappa$ and $\nu_0, \ldots, \nu_{n-1} \in \prod_{i<\kappa} \lambda_i$ such that $\forall s, t < n[\nu_s \restriction \alpha = \nu_t \restriction \alpha]$, $\langle \nu_s(\alpha) : s < n \rangle$ is one-one, and $\{\nu_s : s < n\} \subseteq \{\eta_0, \ldots, \eta_l\}$.

In fact, obviously (b) implies (a), and an easy argument using freeness gives the other implication.

Now define

$$X = \big\{ [x_{\eta_0}] \cdot \ldots \cdot [x_{\eta_l}] \cdot -[x_{\eta_{l+1}}] \cdot \ldots \cdot -[x_{\eta_m}] : 0 \le l < m < \omega,$$
$$\forall k \le m \left[\eta_k \in \textstyle\prod_{i<\kappa} \lambda_i \right], \text{and } \langle \eta_0, \ldots, \eta_m \rangle \text{ is one-one} \big\} \backslash \{0\}.$$

Clearly X is dense in B. Now if $0 \le l < m < \omega$, $\alpha < \kappa$, $\forall k \le m[\varphi_k \in \prod_{i<\alpha} \lambda_i]$, and $\langle \varphi_k : k \le m \rangle$ is one-one, let

$$D^{lm\alpha}_{\langle \varphi_k : k \le m \rangle} = \big\{ [x_{\eta_0}] \cdot \ldots \cdot [x_{\eta_l}] \cdot -[x_{\eta_{l+1}}] \cdot \ldots \cdot -[x_{\eta_m}] :$$
$$\forall k \le m \left[\varphi_k \subseteq \eta_k \in \textstyle\prod_{i<\kappa} \lambda_i \right], \text{ and } \langle \eta_0, \ldots, \eta_m \rangle \text{ is one-one} \big\} \backslash \{0\}.$$

Note that there are at most $\sum_{\alpha<\kappa} \prod_{i<\alpha} \lambda_i$ of the sets $D^{lm\alpha}_{\langle \varphi_k : k \le m \rangle}$. Hence for the first conclusion of the proposition it suffices to show that the union of all such sets is X, and each such set satisfies the $(n-1)$-intersection property.

To show that the union of all such sets is X, let $0 \le l < m < \omega$ and let $\langle \eta_k : k \le m \rangle$ be a sequence of distinct members of $\prod_{i<\kappa} \lambda_i$ such that

$$[x_{\eta_0}] \cdot \ldots \cdot [x_{\eta_l}] \cdot -[x_{\eta_{l+1}}] \cdot \ldots \cdot -[x_{\eta_m}] \neq 0.$$

Then clearly there is an $\alpha < \kappa$ such that the functions $\eta_k \restriction \alpha$ are distinct for $k \le m$. Let $\varphi_k = \eta_k \restriction \alpha$ for all $k \le m$. Then

$$[x_{\eta_0}] \cdot \ldots \cdot [x_{\eta_l}] \cdot -[x_{\eta_{l+1}}] \cdot \ldots \cdot -[x_{\eta_m}] \in D^{lm\alpha}_{\langle \varphi_k : k \le m \rangle},$$

as desired.

Now we take any set $D^{lm\alpha}_{\langle \varphi_k : k \le m \rangle}$ and show that it satisfies the $(n-1)$-intersection property. So, suppose we are given $\langle \eta^t_k : k \le m, t < n-1 \rangle$ such that $\forall t < n-1 \forall k \le m[\varphi_k \subseteq \eta^t_k \in \prod_{i<\kappa}\lambda_i]$, $\forall t < n-1[\langle \eta^t_k : k \le m\rangle$ is one-one$]$, and

$$\forall t < n-1\big[[x_{\eta^t_0}]\cdot\ldots\cdot[x_{\eta^t_l}]\cdot -[x_{\eta^t_{l+1}}]\cdot\ldots\cdot -[x_{\eta^t_m}] \ne 0\big].$$

We want to show that

$$\prod_{t<n-1}[x_{\eta^t_0}]\cdot\ldots\cdot[x_{\eta^t_l}]\cdot -[x_{\eta^t_{l+1}}]\cdot\ldots\cdot -[x_{\eta^t_m}] \ne 0.$$

Suppose that this product is 0. Then by (1) choose $\beta < \kappa$ and $\nu_0,\ldots,\nu_{n-1} \in \prod_{i<\kappa}\lambda_i$ such that $\forall s,t < n[\nu_s \restriction \beta = \nu_t \restriction \beta]$, $\langle \nu_s(\beta) : s < n\rangle$ is one-one, and $\{\nu_s : s < n\} \subseteq \{\eta^s_k : k \le l, s < n-1\}$. Then there is an $s < n-1$ and distinct $u, v < n$ and distinct $a, b \le l$ such that $\nu_u = \eta^s_a$ and $\nu_v = \eta^s_b$.

Case 1. $\alpha \le \beta$. Now $\varphi_a \subseteq \eta^s_a = \nu_u$ and $\varphi_b \subseteq \eta^s_b = \nu_v$ and $\nu_u \restriction \alpha = \nu_v \restriction \alpha$, so $\varphi_a = \varphi_b$, contradiction.

Case 2. $\beta < \alpha$. For each $s < n$ choose $t(s) < n-1$ and $k(s) \le l$ such that $\nu_s = \eta^{t(s)}_{k(s)}$. Since $\varphi_{k(s)} \subseteq \eta^{t(s)}_{k(s)}$ and $\beta < \alpha$, it follows that $\nu_s \restriction (\beta+1) \subseteq \varphi_{k(s)}$. Now for $s \ne u$ we have $\nu_s \restriction (\beta+1) \ne \nu_u \restriction (\beta+1)$, so it follows that $k(s) \ne k(u)$. This being true for all $s < n$, it follows from (1) that $D^{lm\alpha}_{\langle \varphi_k : k \le m\rangle} = \emptyset$, contradiction.

It remains only to show that $\mathrm{d}_n(B) = \prod_{i<\kappa}\lambda_i$. Suppose to the contrary that $B^+ = \bigcup_{j<\theta} C_j$, where $\theta < \prod_{i<\kappa}\lambda_i$ and each C_i satisfies the n-intersection property. Fix $j < \theta$, and let $D_j = \left\{\eta \in \prod_{i<\kappa}\lambda_i : [x_\eta] \in C_j\right\}$. We claim that $|D_j| \le 2^\kappa$. Suppose not. Define $f : [D_j]^2 \to \kappa$ by setting

$$f(\{\sigma,\tau\}) = \min_{\alpha<\kappa}[\sigma(\alpha) \ne \tau(\alpha)]$$

for any two distinct $\sigma, \tau \in D_j$. By the Erdős–Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ we get a subset E of D of size κ^+ and an $\alpha < \kappa$ such that $\sigma \restriction \alpha = \tau \restriction \alpha$ and $\sigma(\alpha) \ne \tau(\alpha)$ for all distinct $\sigma, \tau \in E$. But then by (1), any set of n or more elements $[x_\sigma]$, $\sigma \in E$, has zero product, contradiction. Thus $|C_j| \le 2^\kappa$, so $\prod_{i<\kappa}\lambda_i = \bigcup_{j<\theta} D_j$ has size at most $\theta \cdot 2^\kappa < \prod_{i<\kappa}\lambda_i$, contradiction. □

Corollary 5.16. *If λ is a strong limit singular cardinal and $2 < n < \omega$, then there is a BA B such that $\mathrm{d}_n(B) = \mathrm{d}(B) = |B| = 2^\lambda$ and $\mathrm{d}_{n-1}(B) \le \lambda$.*

Proof. Let $\langle \mu_i : i < \mathrm{cf}(\lambda)\rangle$ be a strictly increasing sequence of infinite cardinals with supremum λ. Then

$$2^\lambda = 2^{\sum_{i<\mathrm{cf}(\lambda)}\mu_i} = \prod_{i<\mathrm{cf}(\lambda)} 2^{\mu_i} = \prod_{i<\mathrm{cf}(\lambda)}\mu_i \le \prod_{i<\mathrm{cf}(\lambda)}\lambda = \lambda^{\mathrm{cf}(\lambda)} \le \lambda^\lambda = 2^\lambda;$$

Hence $\prod_{i<\mathrm{cf}(\lambda)}\mu_i = 2^\lambda$. Hence Proposition 5.15 applies to give the desired result. □

Theorem 5.17. *Let λ be a strong limit singular cardinal, and $\kappa < \mathrm{cf}(\lambda)$. Suppose that D is an ultrafilter on κ which is not countably complete. Then there is a system $\langle B_i : i < \kappa\rangle$ of BAs such that*

$$\mathrm{d}\left(\prod_{i<\kappa} B_i/D\right) \leq \lambda < 2^\lambda = \left|\prod_{i<\kappa} \mathrm{d}(B_i)/D\right|.$$

Proof. Let $\langle M_i : i \in \omega\rangle$ be a sequence of members of D such that $\bigcap_{i\in\omega} M_i \notin D$, with $M_0 \neq \kappa$. Let $N_i = \bigcap_{j\leq i} M_j$ for every $i \in \omega$. Then $\langle N_i : i \in \omega\rangle$ is a decreasing sequence of sets but it is not eventually constant. So there is a function $g : \omega \to \omega$ such that $g(0) = 0$ and $N_{g(i)} \supset N_{g(i+1)}$ for all $i \in \omega$. Define $P_0 = (\kappa\backslash N_0)\cup\bigcap_{i\in\omega} N_i$ and $P_{i+1} = N_{g(i)}\backslash N_{g(i+1)}$ for all $i \in \omega$. Thus $\langle P_i : i \in \omega\rangle$ is a partition of κ with all members not in D. For each $\alpha < \kappa$ let $f(\alpha) = i + 2$, where i is minimum such that $\alpha \in P_i$. Then

(1) For all $i \in \omega$ we have $\{\alpha \in \kappa : f(\alpha) \geq i\} \in D$.

Now by Corollary 5.16, for each $i < \kappa$ let B_i be a BA such that $\mathrm{d}_{f(i)}(B_i) \leq \lambda$ while $|B_i| = \mathrm{d}(B_i) = \mathrm{d}_{f(i)+1}(B_i) = 2^\lambda$. Then by Proposition 5.14

$$\mathrm{d}\left(\prod\nolimits_{i<\kappa} B_i\right) \leq \left|\prod\nolimits_{i<\kappa} \mathrm{d}_{f(i)}(B_i)/D\right| \leq \lambda^\kappa = \lambda.$$

On the other hand, since $\mathrm{d}(B_i) = 2^\lambda$ for all $i < \kappa$, it is clear that

$$\left|\prod\nolimits_{i<\kappa} \mathrm{d}(B_i)/D\right| = 2^\lambda. \qquad \square$$

If B is a subdirect product of $\langle A_i : i \in I\rangle$, then $d(B) \leq \sum_{i\in I} d(A_i)$, by the argument used for products. Unlike full products and weak direct products, however, it is not necessary that $|I| \leq d(B)$, or that $d(A_i) \leq d(B)$ for all $i \in I$. For example, the subalgebra $B \stackrel{\text{def}}{=} \{f \in {}^{\omega_1}A : f(\alpha) = f(\beta)$ for all $\alpha, \beta < \omega_1\}$ of ${}^{\omega_1}A$, where A is countable, is isomorphic to A and $\omega_1 \not\leq d(B)$. The subalgebra $B \stackrel{\text{def}}{=} \{([a], a) : a \in \mathscr{P}(\omega)\}$ of $\mathscr{P}(\omega)/\mathrm{fin} \times \mathscr{P}(\omega)$ is a subdirect product isomorpic to $\mathscr{P}(\omega)$, and $d(\mathscr{P}(\omega)/\mathrm{fin}) = 2^\omega$ while $d(B) = \omega$.

Moderate products are taken care of by Theorem 5.3.

For the one-point gluing B of the pair $(\langle A_i : i \in I\rangle, \langle F_i : i \in I\rangle)$ we have $\mathrm{d}(B) = \sum_{i\in I} \mathrm{d}(A_i)$. Here we use the notation of Chapter 1, and we assume that each A_i is infinite. In fact, $\leq$ holds since B is a subalgebra of $\prod_{i\in I} A_i$. To show that $|I| \leq \mathrm{d}(B)$, for each $i \in I$ let a^i_i be a nonzero element of A_i not in F_i, and let $a^i_j = 0$ for $j \neq i$. This gives a system $\langle a^i : i \in I\rangle$ of nonzero pairwise disjoint elements of B, and so $|I| \leq \mathrm{d}(B)$. Finally, we show that A_i is isomorphic to a subalgebra of B for every $i \in I$: for each $a \in A_i$ let

$$(f_i(a))(j) = \begin{cases} a & \text{if } j = i, \\ a/F_i & \text{otherwise.} \end{cases}$$

Clearly $\mathrm{d}(\mathrm{Dup}(A)) = |\mathrm{Ult}(A)|$.

A special case of a result in Mizokami [79] is that $\mathrm{d}(\mathrm{Exp}(A)) = \mathrm{d}(A)$ for any infinite BA A. In fact, $\mathrm{d}(\mathrm{Exp}(A)) \leq \mathrm{d}(A)$ by Proposition 2.5 and the above remarks on products and free products. On the other hand, let $X \subseteq \mathrm{Ult}(A)$ be dense, with $|X| = \mathrm{d}(A)$. We claim that $[X]^{<\omega}$ is dense in $\mathrm{Exp}(A)$. For, take a basis element $\mathscr{V}(\mathcal{S}(a_1), \ldots, \mathcal{S}(a_m))$ of $\mathrm{Exp}(A)$. For each $i = 1, \ldots, m$ choose $F_i \in X \cap \mathcal{S}(a_i)$. Then $\{F_i : 1 \leq i \leq m\} \in \mathscr{V}(\mathcal{S}(a_1), \ldots, \mathcal{S}(a_m))$.

We turn to derived functions for topological density. We shall show that $\mathrm{d}_{\mathrm{H}+} = \mathrm{hd}$, but to do this we need two results about tightness and spread which are corollaries of Theorems 4.26 and 3.30.

Theorem 5.18 (Shapirovskiĭ)**.** $\mathrm{t}(A) \leq \mathrm{s}(A)$ *for any infinite BA A.*

Proof. By Theorem 4.25 it suffices to note that if $\langle F_\xi : \xi < \kappa\rangle$ is a free sequence, with κ regular, then $\langle F_\xi : \xi < \kappa\rangle$ is one-one and $\{F_\xi : \xi < \kappa\}$ is discrete. Let $\xi < \kappa$. There exist clopen sets $\mathcal{S}(a), \mathcal{S}(b)$ such that $\{F_\eta : \eta < \xi\} \cap \mathcal{S}(a) = 0$, $\{F_\eta : \xi \leq \eta < \kappa\} \subseteq \mathcal{S}(a)$, $\{F_\eta : \eta < \xi + 1\} \subseteq \mathcal{S}(b)$, and $\{F_\eta : \xi + 1 \leq \eta < \kappa\} \cap \mathcal{S}(b) = 0$. Clearly then $\mathcal{S}(a \cdot b) \cap \{F_\eta : \eta < \kappa\} = \{F_\xi\}$, as desired. □

Theorem 5.19. $\mathrm{s}(A) \leq \mathrm{d}_{\mathrm{H}+}(A)$ *for any infinite BA A.*

Proof. By Theorem 3.30 and Corollary 5.2, $\mathrm{s}(A) = \mathrm{c}_{\mathrm{H}+}(A) \leq \mathrm{d}_{\mathrm{H}+}(A)$. □

Theorem 5.20. $\mathrm{d}_{\mathrm{H}+}(A) = \mathrm{d}_{\mathrm{h}+}(A) = \mathrm{hd}(A)$ *for any infinite BA A.*

Proof. $\mathrm{d}_{\mathrm{h}+}(A) = \mathrm{hd}(A)$ by definition, and $\mathrm{d}_{\mathrm{H}+}(A) \leq \mathrm{d}_{\mathrm{h}+}(A)$ since homomorphic images correspond to closed sets in $\mathrm{Ult}(A)$. So it remains to show that $\mathrm{hd}(A) \leq \mathrm{d}_{\mathrm{H}+}(A)$. Let $\kappa = \mathrm{d}_{\mathrm{H}+}(A)$, and suppose that $\kappa < \mathrm{hd}(A)$. Choose $Y \subseteq \mathrm{Ult}(A)$ such that $\kappa < \mathrm{d}(Y)$. Let Z be a dense subset of $\overline{Y}$ of size $\leq \kappa$. For each $z \in Z$ we have $z \in \overline{Y}$, and so $z \in \overline{W_z}$ for some $W_z \in [Y]^{\leq\kappa}$ by Theorems 5.18 and 5.19. We claim now that $\bigcup_{z\in Z} W_z$ is dense in Y; since $\left|\bigcup_{z\in Z} W_z\right| \leq \kappa$, this will be a contradiction. Let U be an open set in $\mathrm{Ult}(A)$ such that $U \cap Y \neq 0$. Then $U \cap \overline{Y} \neq 0$, so choose $z \in U \cap Z$. Since $z \in \overline{W_z}$, we get $U \cap W_z \neq 0$, as desired. □

Notice that attainment in the $\mathrm{d}_{\mathrm{H}+}$ sense obviously implies attainment in the hd sense.

We have $\mathrm{d}_{\mathrm{H}-}(A) = \mathrm{d}_{\mathrm{h}-}(A) = \omega$ for infinite A, since by Sikorski's extension theorem there is a homomorphism of A onto an infinite subalgebra of $\mathscr{P}(\omega)$. Clearly $\mathrm{d}_{\mathrm{S}+}(A) = \mathrm{d}(A)$, $\mathrm{d}_{\mathrm{S}-}(A) = \omega$, and ${}_{\mathrm{d}}\mathrm{d}_{\mathrm{S}+}(A) = \mathrm{d}(A)$ for any infinite BA A. Furthermore, ${}_{\mathrm{d}}\mathrm{d}_{\mathrm{S}-}(A) = \mathrm{d}(A)$: if B is a dense subalgebra of A and f is an isomorphism of B into $\mathscr{P}(\kappa)$, then f can be extended to an isomorphism of A into $\mathscr{P}(\kappa)$, as desired.

Concerning the spectrum function d_{Hs} we mention the following problem, Problem 18 in Monk [96].

Problem 63. *Is it true that $[\omega, \mathrm{hd}(A)) \subseteq \mathrm{d}_{\mathrm{Hs}}(A)$ for every infinite BA A?*

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Problem 64. *Completely describe* d_{Hs}.

This is problem 19 in Monk [96],

Concerning d_{Ss} we have the following theorem and example, due to S. Koppelberg, solving Problem 11 in Monk [90].

Theorem 5.21. (GCH) $\mathrm{d}_{\mathrm{Ss}}(A) = [\omega, \mathrm{d}(A)]$ *for every infinite BA A.*

Proof. Suppose that $\omega \leq \kappa < \mathrm{d}(A)$; we want to find a subalgebra B of A such that $\mathrm{d}(B) = \kappa$. If κ is a limit cardinal, then any subalgebra of A of size κ will do, by GCH. So we may assume that κ is a successor cardinal, and hence is regular. If A has a disjoint subset of size κ, then the subalgebra generated by such a subset is isomorphic to $\mathrm{Finco}(\kappa)$, which has topological density κ. So we may assume that A satisfies the κ-cc. Now $\mu^{<\kappa} < \kappa^+$ for every $\mu < \kappa^+$ by GCH. Hence by Theorem 10.1 of the BA Handbook, Part I, A has a free subalgebra B of size κ^+. By GCH, $\mathrm{d}(B) = \kappa$, as desired. $\square$

The equality in Theorem 5.21 cannot be proved in ZFC. Namely, if for example $2^\omega = 2^{\omega_1} = \omega_2$ and $2^{\omega_2} = \omega_4$, then for A the free BA on ω_4 free generators we have $\mathrm{d}_{\mathrm{Ss}}A = \{\omega, \omega_2\}$ by Corollary 5.6.

We now consider d_{Sr}. First we have the following vague problem.

Problem 65. *Characterize in cardinal number terms the sets* d_{Sr}.

Here are some general remarks about d_{Sr}:

(1) (ω, ω) is in every d_{Sr}.

(2) $[\omega, |A|] \subseteq \mathrm{rng}(\mathrm{d}_{\mathrm{Sr}}(A))$

Proposition 5.22.

(i) *If* $(\kappa, \kappa) \notin \mathrm{d}_{\mathrm{Sr}}(A)$, *then* $\mathrm{c}'(A) \leq \kappa$.

(ii) *If in addition* $(2^\kappa)^+ \leq |A|$, *then for every infinite* $\lambda \leq (2^\kappa)^+$ *we have* $(\mu, \lambda) \in \mathrm{d}_{\mathrm{Sr}}(A)$, *where* μ *is the least cardinal such that* $\lambda \leq 2^\mu$.

Proof. (i): If $\mathrm{c}'(A) > \kappa$, then A has a disjoint subset of size κ, and hence a subalgebra isomorphic to $\mathrm{Finco}(\kappa)$. This subalgebra has topological density κ.

(ii): If in addition $(2^\kappa)^+ \leq |A|$, then by Corollary 10.9 of the Handbook, A has an independent subset of size $(2^\kappa)^+$. Then the conclusion follows by Corollary 5.6. $\square$

There are many open questions concerning $\mathrm{d}_{\mathrm{Sr}}(A)$ for $|A| \leq \omega_2$, so we do not survey them. For example, the following question is open.

Problem 66. *Is there a BA A such that* $\mathrm{d}_{\mathrm{Sr}}(A) = \{(\omega, \omega), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$*?*

For d_{Hr} we have as usual the following vague problem.

Problem 67. *Characterize in cardinal number terms the sets* d_{Hr}.

Here are some general remarks concerning $\mathrm{d_{Hr}}$.

(1) For every infinite BA A there is a $\kappa \in [\omega, |A|]$ such that $(\omega, \kappa) \in \mathrm{d_{Hr}}(A)$.

In fact, A has a subalgebra isomorphic to $\mathrm{Finco}(\omega)$, and so gives a homomorphic image B of A such that $\mathrm{d}(B) = \omega$. By this argument we have the following more general result:

(2) $[\omega, \mathrm{c}'(A)) \subseteq \mathrm{dmn}(\mathrm{d_{Hr}}(A))$ for every infinite BA A.

(3) Assuming CH, if A is ccc and $|A| = \omega_2$, then $\mathrm{d}(A) = \omega_1$.

This holds by Theorem 5.11.

(4) Since subalgebras of interval algebras are retractive, if A is a subalgebra of an interval algebra and B is a homomorphic image of A, then $\mathrm{d}(B) \leq \mathrm{d}(A)$.

(5) Every infinite homomorphic image of $\mathrm{Finco}(\kappa)$ (κ an infinite cardinal) is isomorphic to $\mathrm{Finco}(\lambda)$ for some infinite cardinal λ.

Again there are many problems concerning $\mathrm{d_{Hr}}(A)$ for $|A| \leq \omega_2$, so we do not survey them. For example,

Problem 68. *Is it consistent to have a BA A such that* $\mathrm{d_{Hr}}(A) = \{(\omega, \omega), (\omega_1, \omega_2)\}$*?*

The difference between $\mathrm{c}(A)$ and $\mathrm{d}(A)$ can be arbitrarily large, for example in free BAs. In fact, Rabus and Shelah [99] have shown that for each uncountable cardinal κ there is a ccc BA A such that $\mathrm{d}(A) = \kappa$. Hence for $\omega \leq \lambda \leq \kappa$ there is a BA B such that $\mathrm{c}(B) = \lambda$ and $\mathrm{d}(B) = \kappa$. Namely, the case $\kappa = \omega$ is easy, and for $\kappa > \omega$ one can take $B = \mathrm{Finco}(\lambda) \times A$, with A as above.

Turning to topological density for special classes of BAs, note first that if A is atomic, then $\mathrm{d}(A)$ is the number of atoms of A. Hence if A is the finite-cofinite algebra on κ, then $\mathrm{d_{Sr}}(A) = \{(\lambda, \lambda) : \omega \leq \lambda \leq \kappa\} = \mathrm{d_{Hr}}(A)$.

For any infinite BA A we have $\mathrm{d}(A) = \mathrm{d}(\overline{A})$. In fact, let $\kappa = \mathrm{d}(A)$. By Theorem 5.1, let f be an isomorphism from A into $\mathscr{P}(\kappa)$. By Sikorski's extension theorem, there is an extension g of f to a homomorphism from $\overline{A}$ into $\mathscr{P}(\kappa)$. If $x \in \overline{A}^+$, choose $a \in A^+$ such that $a \leq x$. Then $0 \neq f(a) = g(a) \leq g(x)$. So g is one-one, as desired.

For interval algebras, we have one interesting inequality not true for BAs in general. It is actually true for linearly ordered spaces in general, and we give that general form, due to Kurepa [35]. This result has evidently been rediscovered by many people independently; see, e.g., Juhász [75].

Theorem 5.23. *If L is an infinite linearly ordered space, then* $\mathrm{d}(L) \leq (\mathrm{c}(L))^+$.

Proof. Assume the contrary. Set $\kappa = (\mathrm{c}(L))^+$. Let $\prec$ be a well-ordering of L. Now we set

$$N = \{p \in L : p \text{ is the } \prec\text{-least element of some neighborhood of } p\}.$$

Clearly N is dense in L. Hence $|N| > \kappa$. Now for each $p \in N$ let I_p be the union of all open intervals having p as their $\prec$-first element. Then, we claim,

(1) If $p, p' \in N$ and $p \prec p'$, then $I_p \cap I_{p'} = 0$ or $I_{p'} \subset I_p$.

In fact, suppose $p \prec p'$ and $I_p \cap I_{p'} \neq 0$. This means that there exist an open interval U with $\prec$-first element p and an open interval U' with $\prec$-first element p' such that $U \cap U' \neq 0$; hence $U \cup U'$ is an open interval with both p and p' as members, and with p as $\prec$-first member. So, if V is any open interval with $\prec$-first element p', then $V \cup U \cup U'$ is an open interval with $\prec$-first element p, and hence $V \subseteq I_p$. This shows that $I_{p'} \subseteq I_p$. Since $p \in I_p \backslash I_{p'}$, (1) then follows.

Next, set

$$N_0 = \{p \in N : I_p \text{ is not contained in any other } I_{p'}\}.$$

Now $I_p \cap I_{p'} = 0$ for all distinct $p, p' \in N_0$, so $|N_0| < \kappa$. We continue inductively for all $\xi < \kappa$:

$$H_\xi = N \backslash \bigcup_{\eta < \xi} N_\eta;$$

$$N_\xi = \{p \in H_\xi : I_p \text{ is not contained in any other } I_{p'} \text{ for } p' \in H_\xi\}.$$

Note that $|N_\xi| < \kappa$, and hence always $H_\xi \neq 0$. Hence $|\bigcup_{\xi<\kappa} N_\xi| \leq \kappa$, so there is a $p \in N \backslash \bigcup_{\xi<\kappa} N_\xi$. Thus $p \in H_\xi$ for all $\xi < \kappa$. But then for each $\xi < \kappa$ there is a $p(\xi) \in N_\xi$ such that $I_p \subset I_{p(\xi)}$. In fact, there is a $q \in H_\xi$ such that $I_p \subset I_q$. Taking the smallest such q under $\prec$, we get the desired $p(\xi)$. Hence for all $\xi, \eta < \kappa$ we have $I_{p(\xi)} \subset I_{p(\eta)}$ or $I_{p(\eta)} \subset I_{p(\xi)}$. By the partition relation $\kappa \to (\kappa, \omega)^2$ we may assume that $p(\xi) \prec p(\eta)$ whenever $\xi < \eta < \kappa$, and hence the sequence $\langle I_{p(\xi)} : \xi < \kappa \rangle$ is strictly decreasing. For each $\xi < \kappa$ choose $x_\xi \in I_{p(\xi)} \backslash I_{p(\xi+1)}$. Let

$$K^l = \{x_\xi : x_\xi < p(\xi+1)\},\ K^r = \{x_\xi : x_\xi > p(\xi+1)\}.$$

Now

(2) if $\xi < \eta$ and $x_\eta, x_\xi \in K^l$, then $x_\xi < x_\eta$.

In fact, suppose that $x_\eta \leq x_\xi$. Now $x_\xi < p(\xi+1) \in I_{p(\xi+1)}$, $x_\xi \notin I_{p(\xi+1)}$, and $I_{p(\xi+1)}$ is convex. Hence x_ξ is less than all members of $I_{p(\xi+1)}$. Since $\xi < \eta$, we have $\xi + 1 \leq \eta$, hence $p(\xi+1) \preceq p(\eta)$, and so $I_{p(\eta)} \subseteq I_{p(\xi+1)}$. It follows that $x_\xi < p(\eta)$. Thus $x_\eta \leq x_\xi < p(\eta)$. Since x_η and $p(\eta)$ are in $I_{p(\eta)}$ and $I_{p(\eta)}$ is convex, it follows that $x_\xi \in I_{p(\eta)}$. Now $\xi + 1 \leq \eta$, so $I_{p(\eta)} \subseteq I_{p(\xi+1)}$. So $x_\xi \in I_{p(\xi+1)}$, contradiction. Thus (2) holds

Similarly, if $\xi < \eta$ and $x_\eta, x_\xi \in K^r$, then $x_\xi > x_\eta$. Now one of K^1, K^2 has κ elements, so we obtain a strictly increasing or strictly decreasing sequence of length κ, and hence κ disjoint open intervals, contradiction. $\square$

Some care must be taken in translating from topology to Boolean algebras for interval algebras. For example, the linear order $2 \times \mathbf{R}$ with the anti-lexicographic order has density ω as a linearly ordered space, but $\mathrm{Intalg}(2 \times \mathbf{R})$ has a family of size 2^ω of pairwise disjoint elements, and so has topological density 2^ω. However, the following fact should be noted.

Proposition 5.24. *If L is a dense linear order, then* $\mathrm{d}(L) = \mathrm{d}(\mathrm{Intalg}(L)$.

Proof. First suppose that $X \subseteq L$ is dense of size $\mathrm{d}(L)$. Let $Y = \{[a, b) : a, b \in X$ and $a < b$. We claim that Y is dense in $\mathrm{Intalg}(L)$. For, suppose that $[c, d) \in \mathrm{Intalg}(L)$. Choose $e \in X \cap (c, d)$ and $f \in X \cap (e, d)$. Then $[e, f) \in Y$ and $[e, f) \subseteq [c, d)$. This proves the claim. So $\mathrm{d}(\mathrm{Intalg}(L)) \leq \mathrm{d}(L)$.

Second suppose that Y is dense in $\mathrm{Intalg}(L)$. We may assume that the members of Y have the form $[a, b)$. Let $X = \{b : [a, b) \in Y$ for some $a\}$. We claim that X is dense in L. Suppose that $c < d$. Choose $[a, b) \in Y$ such that $[a, b) \subseteq [c, d)$. Then $b \in (c, d) \cap X$, as desired. □

The interval algebra of a dense Suslin line gives an example of an interval algebra A in which $\mathrm{c}(A) < \mathrm{d}(A)$; on the other hand, Martin's axiom plus $\neg$CH implies that for an interval algebra, $\mathrm{c}(A) = \omega \Rightarrow \mathrm{d}(A) = \omega$ (see Kunen [80], Theorem II4.2).

Since interval algebras are retractive, it follows that if B is a homomorphic image of an interval algebra A, then $\mathrm{d}(B) \leq \mathrm{d}(A)$. Hence $\mathrm{d}(A) = \mathrm{hd}(A)$ for every interval algebra A.

For a minimally generated BA A we also have $\mathrm{d}(A) \leq (\mathrm{c}(A))^+$. In fact, by Corollary 2.60 A is co-complete with an interval algebra B. Hence

$$\mathrm{d}(A) = \mathrm{d}(\overline{A}) = \mathrm{d}(\overline{B}) = \mathrm{d}(B) \leq (\mathrm{c}(B))^+ = (\mathrm{c}(A))^+.$$

An example of a complete BA A for which $\mathrm{c}(A) < \mathrm{d}(A)$ can be obtained by taking A to be the completion of a large free BA.

Theorem 5.25. *If T is a tree, then* $\mathrm{d}(\mathrm{Treealg}(T)) = |T|$.

Proof. For brevity let $\kappa = |T|$ and $A = \mathrm{Treealg}(T)$. Suppose that $\mathrm{d}(A) < \kappa$. Each level of T gives rise to a disjoint set in A, and so has size at most $\mathrm{d}(A)$, since $\mathrm{c}(A) \leq \mathrm{d}(A)$. Also, each chain in T has size at most $\mathrm{d}(A)$, since a chain gives rise to a disjoint subset of A of its size. Hence κ is regular, and T has κ levels. Let $\mathscr{F}$ be a collection of ultrafilters on A such that $|\mathscr{F}| = \mathrm{d}(A)$ and $A \backslash \{0\} = \bigcup \mathscr{F}$. For each $F \in \mathscr{F}$ let $s(F) = \{t \in T : T \uparrow t \in F\}$. Then $s(F)$ is an initial chain of T and so has length $\leq \mathrm{d}(A)$. Let α be a level greater than the lengths of all $s(F)$ for $F \in \mathscr{F}$. Take any $t \in T$ of height α. Then $T \uparrow t \notin \bigcup \mathscr{F}$, contradiction. □

We do not have a clear description of $\mathrm{d}(A)$ for A a pseudo-tree algebra:

Problem 69. *Describe $d(A)$ for A a pseudo-tree algebra.*

6 π-Weight

If A is a subalgebra of B, then $\pi(A)$ can vary either way from $\pi(B)$; for clearly one can have $\pi(A) < \pi(B)$, while if we take $B = \mathscr{P}(\omega)$ and A the subalgebra of B generated by an independent subset of size 2^ω, then we have $\pi(B) = \omega$ and $\pi(A) = 2^\omega$. Similarly, if A is a homomorphic image of B: it is easy to get such A and B with $\pi(A) < \pi(B)$, and if we take $B = \mathscr{P}(\omega)$ and $A = B/\text{fin}$, then $\pi(B) = \omega$ while $\pi(A) = 2^\omega$ since A has a disjoint subset of size 2^ω.

Now we discuss special subalgebras.

Proposition 6.1. *If $A \leq_\pi B$, then $\pi(A) = \pi(B)$.*

Proof. Assume that $A \leq_\pi B$. If X is dense in A, then it is also dense in B; so $\pi(B) \leq \pi(A)$. Now suppose that Y is dense in B. For each $y \in Y^+$ choose $a_y \in A^+$ such that $a_y \leq y$. Then $\{a_y : y \in Y\}$ is dense in A; so $\pi(A) \leq \pi(B)$. $\square$

Proposition 6.2. *If $A \leq_{\mathrm{m}} B$, then $\pi(A) = \pi(B)$.*

Proof. Write $B = A(x)$. If $A \restriction x$ and $A \restriction -x$ are both nonprincipal, then A is dense in B by Proposition 2.44, and so $\pi(A) = \pi(B)$ by Proposition 6.1. If one of these ideals is principal, then by Proposition 2.45 we can write $B = A(y)$ for some atom y of B. Then A is isomorphic to $B \restriction -y$, and it is easy to see that $\pi(A) = \pi(B)$. $\square$

Proposition 6.3. *If $A \leq_{\mathrm{reg}} B$, $A \leq_{\mathrm{free}} B$, $A \leq_{\mathrm{proj}} B$, or $A \leq_{\mathrm{rc}} B$, then $\pi(A) \leq \pi(B)$.*

Proof. By general results it suffices to take the case $A \leq_{\mathrm{reg}} B$. Let X be dense in B, with $|X| = \pi(B)$. For each nonzero $x \in X$ there is a lower bound $0 \neq a_x \in A$ of the set $\{a \in A : x \leq a\}$. In fact, otherwise we have $\prod^A\{a \in A : x \leq a\} = 0$, hence $\prod^B\{a \in A : x \leq a\} = 0$; but $0 \neq x \leq \prod^B\{a \in A : x \leq a\}$, contradiction. Clearly $\{a_x : x \in X\}$ is dense in A. $\square$

Clearly one can have $\pi(A)$ much smaller than $\pi(B)$ if $A \leq_{\mathrm{free}} B$; similarly for $A \leq_{\mathrm{reg}} B$, $A \leq_{\mathrm{proj}} B$, $A \leq_{\mathrm{rc}} B$, $A \leq_{\mathrm{u}} B$, or $A \leq_\sigma B$.

An example with $A \leq_{\mathrm{s}} B$ and $\pi(A) < \pi(B)$ is as follows. Let $A = \mathscr{P}(\omega)$, $I = [\omega]^{<\omega}$, $J = \{\emptyset\}$, and let $B = A(x)$ be the simple extension of A associated with I, J. (See Proposition 2.28.) Thus $A \restriction x = I$ and $A \restriction -x = J$. Clearly $\pi(A) = \omega$.

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Let $\langle y_\alpha : \alpha < 2^\omega \rangle$ be a system of infinite pairwise almost disjoint subsets of ω. Then $\langle y_\alpha \cdot -x : \alpha < 2^\omega \rangle$ is a system of 2^ω nonzero pairwise disjoint elements of B.

Since every superatomic BA is minimally generated, one can have $A \leq_{mg} B$ with $\pi(A) < \pi(B)$; it suffices to take B superatomic with uncountably many atoms and A a countable subalgebra which is the ωth algebra in some representing chain for B over 2.

On the other hand, one can have $A \leq_{mg} B$ with $\pi(A) > \pi(B)$. Let X be an uncountable independent set in $\mathscr{P}(\omega)$, and let $A = \langle X \rangle$. Thus $\pi(A) > \omega$. We define a sequence $\langle C_\alpha : \alpha < \omega \cdot \omega \rangle$ of subalgebras of $\mathscr{P}(\omega)$. Let $C_0 = A$. Suppose that C_α has been defined, where $\alpha = \omega \cdot m + n$ with $m, n \in \omega$. Let $C_\alpha(x)$ be the simple extension of C_α such that $C_\alpha \restriction x = \{c \in C_\alpha : n \notin c\}$ and $C_\alpha \restriction -x = \{0\}$. This extension exists since $\{c \in C_\alpha : n \notin c\}$ is a maximal ideal in C_α, using Proposition 2.28; and $C_\alpha \leq_{\mathrm{m}} C_\alpha(x)$ by Proposition 2.32. Now by Sikorski's extension criterion, there is a homomorphism f of $C_\alpha(x)$ into $\mathscr{P}(\omega)$ such that $f(c) = c$ for all $c \in C_\alpha$, and $f(x) = \bigcup\{c \in C_\alpha : n \notin c\}$. Let $C_{\alpha+1} = f[C_\alpha(x)]$. Then $C_\alpha \leq_{\mathrm{m}} C_{\alpha+1}$ by Proposition 2.42. For α a limit ordinal $\leq \omega \cdot \omega$, let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$.

Then $A \preceq_{\mathrm{mg}} C_{\omega\cdot\omega}$. For $m, n \in \omega$ let

$$d_{mn} = \bigcup\{c \in C_{\omega\cdot m+n} : n \notin c\}.$$

Thus by construction $d_{mn} \in C_{\omega\cdot m+n+1} \subseteq C_{\omega\cdot\omega}$. Clearly $d_{mn} \neq \omega$, so $-d_{mn} \neq 0$. We claim that $\{-d_{mn} : m, n \in \omega\}$ is dense in $C_{\omega\cdot\omega}$. For, suppose that $c \in C^+_{\omega\cdot\omega}$. Say $c \in C_{\omega\cdot m+n}$ with $m, n \in \omega$. Since $c \neq 0$ we have $-c \neq \omega$. Say $p \in \omega \backslash c$. Now $c \in C_{\omega\cdot(m+1)+p}$, so $-c \subseteq d_{m+1,p}$. Hence $-d_{m+1,p} \subseteq c$, as desired.

These facts concerning special subalgebras leave the following questions open.

Problem 70. *Are there BAs A, B such that $A \leq_\sigma B$ and $\pi(A) > \pi(B)$?*

Problem 71. *Are there BAs A, B such that $A \leq_{\mathrm{s}} B$ and $\pi(A) > \pi(B)$?*

Problem 72. *Are there BAs A, B such that $A \leq_{\mathrm{u}} B$ and $\pi(A) > \pi(B)$?*

Turning to products, we have $\pi(\prod_{i\in I} A_i) = \max(|I|, \sup_{i\in I}\pi(A_i))$ for any system $\langle A_i : i \in I \rangle$ of infinite BAs. For, $\geq$ is clear; now suppose D_i is a dense subset of A_i for each $i \in I$. Let

$$E = \left\{ f \in \prod_{i\in I}(D_i \cup \{0\}) : f(i) \neq 0 \text{ for only finitely many } i \in I \right\}.$$

Clearly E is dense in $\prod_{i\in I} A_i$, and $|E| = \max(|I|, \sup_{i\in I}\pi(A_i))$, as desired. The equation $\pi(\prod^{\mathrm{w}}_{i\in I} A_i) = \max(|I|, \sup_{i\in I}\pi(A_i))$ is proved by the same argument.

Next, we discuss free products.

An easy argument shows that $\pi(\oplus_{i\in I}A_i) = \max(|I|, \sup_{i\in I}\pi(A_i))$ for any system $\langle A_i : i \in I\rangle$ of Boolean algebras. In fact, if D_i is dense in A_i for each $i \in I$, then

$$E \stackrel{\text{def}}{=} \{d_0 \cdot \ldots \cdot d_{n-1} : \exists \text{ distinct } i_0, \ldots, i_{n-1} \in I \text{ such that } \forall j < n(d_j \in D_{i_j})\}$$

is clearly dense in $\oplus_{i\in I}A_i$, and it has the indicated cardinality. On the other hand, suppose X is dense in $\oplus_{i\in I}A_i$. We may assume that each element of X is a product of members of $\bigcup_{i\in I} A_i$, with distinct factors coming from distinct A_i's. For each $i \in I$ let $Y_i = \{x \in X : x \le a \text{ for some } a \in A_i\}$. For each $x \in Y_i$, let x_i^+ be obtained from x by replacing each factor of x which is not in A_i by 1. Clearly then $\{x_i^+ : x \in Y_i\} \subseteq A_i$ and this set is dense in A_i, so $\pi(A_i) \le |\{x_i^+ : x \in Y_i\}| \le |Y_i| \le |X|$. It is also clear that $|I| \le |X|$; so $|X| \ge \max(|I|, \sup_{i\in I}\pi(A_i))$, as desired.

We do not have a characterization of the density of amalgamated free products of BAs; see the corresponding discussions for cellularity and topological density.

Problem 73. *Characterize the density of amalgamated free products of BAs.*

We turn to the discussion of unions. The following theorem takes care of some possibilities.

Theorem 6.4. *Suppose that $\langle A_\alpha : \alpha < \kappa\rangle$ is a strictly increasing sequence of BAs, with union B, where κ is regular. Let $\lambda = \sup_{\alpha<\kappa}\pi(A_\alpha)$. Then*

(i) $\pi(B) \le \sum_{\alpha<\kappa}\pi(A_\alpha) \le \max(\kappa, \lambda)$.

Now assume that, in addition, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit $\alpha < \kappa$. Then

(ii) $\kappa \le 2^\lambda$,

(iii) $\pi(B) \le \lambda^+$.

Proof. For (i), if X_α is dense in A_α for each $\alpha < \kappa$, then $\bigcup_{\alpha<\kappa} X_\alpha$ is dense in B. Now we make the additional assumption indicated, and prove (ii). Assume that $\kappa \ge (2^\lambda)^+$. Let $\mu = (2^\lambda)^+$. Since clearly $|A_\alpha| \le 2^\lambda$ for all $\alpha < \kappa$, we have $\kappa = \mu$. Let $S = \{\alpha < \mu : \mathrm{cf}(\alpha) = \lambda^+\}$. Thus S is stationary in μ. For each $\alpha \in S$, A_α has a dense subset D_α of size $\le \lambda$. Since $\mathrm{cf}(\alpha) = \lambda^+$, it follows that there is an $f(\alpha) < \alpha$ such that $D_\alpha \subseteq A_{f(\alpha)}$. Now f is regressive on S, so f is constant on a stationary subset S' of S. Let β be the constant value of f on S'. Then D_β is dense in B. But $|B| \ge (2^\lambda)^+$, contradiction. So, (ii) holds. For (iii), if $\pi(B) > \lambda^+$, then by (i) we have $\kappa > \lambda^+$, and we can use an argument similar to that for (ii), taking $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda^+\}$. □

Note that the upper bound λ^+ mentioned in Theorem 6.4 can be attained: use a free algebra.

Turning to ultraproducts, it is clear that $\pi(\prod_{i\in I} A_i/F) \le |\prod_{i\in I}\pi(A_i)/F|$. In Koppelberg, Shelah [95] there is a forcing construction in which $<$ holds; this answers Problem 12 of Monk [90]. Several results about ultraproducts and π hold more generally for the sup-min functions defined in the introduction. These results are due to Douglas Peterson [97].

Theorem 6.5. *Let k be a sup-min function, $\langle A_i : i \in I\rangle$ a sequence of infinite BAs with I infinite, and F a regular ultrafilter on I. Suppose that $k(A_i) \geq \omega$ for all $i \in I$, $\lambda = \text{ess.sup}^F_{i\in I} k(A_i)$, and $\text{cf}(\lambda) \leq |I| < \lambda$. Then $k\left(\prod_{i\in I} A_i/F\right) \geq \lambda^+$.*

Proof. For brevity let $B = \prod_{i\in I} A_i/F$. Let $b = \{i \in I : k(A_i) > \omega\}$. Now $b \in F$, as otherwise $\text{ess.sup}^F_{i\in I} k(A_i) = \omega$.

(1) There is a system $\langle \kappa_i : i \in I\rangle$ such that $\{i \in I : \omega < \kappa_i < k(A_i)\} \in F$ and $\text{ess.sup}^F_{i\in I}\kappa_i = \lambda$.

To prove (1) we apply Lemma 3.18 with κ replaced by λ to obtain a sequence $\langle \kappa_i : i \in I\rangle$ such that $\forall i \in I[\kappa_i < \lambda]$ and $\text{ess.sup}^F_{i\in I}\kappa_i = \lambda$. Choose $d \in F$ so that $\sup_{i\in d} \kappa_i = \text{ess.sup}^F_{i\in I}\kappa_i$. Then

$$\sup_{i\in b\cap d} \kappa_i \leq \sup_{i\in d} \kappa_i$$

and $\forall i \in b \cap d[\omega < \kappa_i < k(A_i)]$, as desired in (1).

Now fix $i \in I$. By (1) and (1) in the definition of sup-min functions, we can find $G_i \subseteq A_i$ such that $(A_i, G_i) \models \psi$ and $\min\{|P| : (A_i, G_i, P) \models \varphi\} > \kappa_i$. Let $H = \prod_{i\in I} G_i/F$. For each $i \in I$ there is a $P_i \subseteq A_i$ such that $(A_i, G_i, P_i) \models \varphi$. Hence $(B, H, \prod_{i\in I} P_i/F) \models \varphi$. Also, $(B, H) \models \psi$. We claim that $\min\{|P| : (B, H, P) \models \varphi\} \geq \lambda^+$; this will prove the theorem, by (1) in the definition of sup-min functions. To prove the claim, suppose that $P = \{f_\alpha/F : \alpha < \lambda\} \subseteq B$ and $(B, H, P) \models \forall x \in \mathbf{P}(x \neq 0 \wedge \varphi'(\mathbf{F}, x))$; we will show

$$(*) \qquad (B, H, P) \models \neg\forall x \in \mathbf{F}\, \exists y \in \mathbf{P}\varphi''(\mathbf{F}, x, y).$$

We may assume that $f_\alpha(i) \neq 0$ for all $\alpha < \lambda$ and $i \in I$. Now for any $\alpha < \lambda$ we have $(B, H) \models \varphi'(\mathbf{F}, f_\alpha/F)$, and hence $\{i \in I : (A_i, G_i) \models \varphi'(\mathbf{F}, f_\alpha(i))\} \in F$. Hence we can assume that $(A_i, G_i) \models \varphi'(\mathbf{F}, f_\alpha(i))$ for all $\alpha < \lambda$ and $i \in I$. (If $(A_i, G_i) \not\models \varphi'(\mathbf{F}, f_\alpha(i))$, replace $f_\alpha(i)$ by a nonzero element a_i such that $(A_i, G_i) \models \varphi'(\mathbf{F}, a_i)$; a_i exists by (3) of the definition of sup-min function.) Now fix $i \in I$ again. Then

$$(A_i, G_i, \{f_\alpha(i) : \alpha < \kappa_i\}) \models \forall x \in \mathbf{P}(x \neq 0 \wedge \varphi'(\mathbf{F}, x)),$$

so, since $(A_i, G_i, \{f_\alpha(i) : \alpha < \kappa_i\}) \not\models \varphi$, we can choose $a_i \in G_i$ such that $(A_i, G_i) \models \neg\varphi''(\mathbf{F}, a_i, f_\alpha i)$ for all $\alpha < \kappa_i$. Now for any $\alpha < \lambda$ we have $\{i \in I : \alpha < \kappa_i\} \in F$, and hence $(B, H) \models \neg\varphi''(\mathbf{F}, a_0/F, f_\alpha/F]$, and this proves $(*)$. $\square$

Theorem 6.6. *Suppose that k is a sup-min function, $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and F is an ultrafilter on I. Then $k\left(\prod_{i\in I} A_i/F\right) \geq \text{ess.sup}^F_{i\in I} k(A_i)$.*

Proof. Let $\lambda = \text{ess.sup}^F_{i\in I} k(A_i)$ and $B = \prod_{i\in I} A_i/F$. Take any $\kappa < \lambda$. Then the set $K \stackrel{\text{def}}{=} \{i \in I : k(A_i) > \kappa\}$ is in F. For each $i \in K$ choose $G_i \subseteq A_i$ such that $(A_i, G_i) \models \psi$ and $\min\{|P| : (A_i, G_i, P) \models \varphi\} > \kappa$. For $i \in I\backslash K$ let $G_i = A_i$. Set $H = \prod_{i\in I} G_i/F$. Then $(B, H) \models \psi$. We claim that $\min\{|P| : (B, H, P) \models \varphi\} > \kappa$; this will prove the theorem. Suppose that $P = \{f_\alpha/F : \alpha < \kappa\} \subseteq B$ and $(B, H, P) \models \forall x \in \mathbf{P}(x \neq 0 \wedge \varphi'(\mathbf{F}, x))$. As in the proof of Theorem 6.5 we can assume that $(A_i, G_i) \models \varphi'(\mathbf{F}, f_\alpha(i))$ and $f_\alpha(i) \neq 0$ for all $\alpha < \lambda$ and $i \in I$. Fix $i \in K$. Then

$$(A_i, G_i, \{f_\alpha(i) : \alpha < \kappa\}) \models \forall x \in \mathbf{P}\,(x \neq 0 \wedge \varphi'(\mathbf{F}, x)),$$

so, since $(A_i, G_i, \{f_\alpha(i) : \alpha < \kappa\}) \not\models \varphi$, we can choose $a_i \in G_i$ such that $(A_i, G_i) \models \neg\varphi''(\mathbf{F}, a_i, f_\alpha(i))$ for all $\alpha < \kappa$. Then $(B, H) \models \neg\varphi''(\mathbf{F}, a/F, f_\alpha/F]$, as desired. $\square$

Theorem 6.7. *Suppose that k is a sup-min function such that for every infinite BA A there is a $G \subseteq A$ such that $(A, G) \models \psi$ and $\min\{|P| : (A, G, P) \models \varphi\} \geq \omega$. Assume that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and F is a regular ultrafilter on I. Then $k\left(\prod_{i\in I} A_i/F\right) \geq |I|^+$.*

Proof. Let $B = \prod_{i\in I} A_i/F$. For each $i \in I$ choose $G_i \subseteq F_i$ such that $(A_i, G_i) \models \psi$ and $\min\{|P| : (A_i, G_i, P) \models \varphi\} \geq \omega$. Let $H = \prod_{i\in I} G_i/F$. Thus $(B, H) \models \psi$. We want to show that $\min\{|P| : (B, H, P) \models \varphi\} \geq |I|^+$; this will prove the theorem. To this end, suppose that $P = \{f_\alpha/F : \alpha < |I|\} \subseteq B$ and $(B, H, P) \models \forall x \in \mathbf{P}\,(x \neq 0 \wedge \varphi'(\mathbf{F}, x))$. As in the proof of Theorem 6.5 we may assume that $(A_i, G_i) \models \varphi'(\mathbf{F}, f_\alpha(i))$ and $f_\alpha(i) \neq 0$ for all $\alpha < |I|$ and $i \in I$. Let $\{a_\alpha : \alpha < |I|\}$ be a regular family for F. Now fix $i \in I$. Then the set $K_i \stackrel{\text{def}}{=} \{\alpha < |I| : i \in a_\alpha\}$ is finite. Since $\min\{|Q| : (A_i, G_i, Q) \models \varphi\} \geq \omega$ and $P_i \stackrel{\text{def}}{=} \{f_\alpha(i) : \alpha \in K_i\}$ is finite, it follows that $(A_i, G_i, P_i) \models \neg\varphi$. But $(A_i, G_i, P_i) \models \forall x \in \mathbf{P}\,(x \neq 0 \wedge \varphi'\mathbf{F}, x))$, so we can choose $a_i \in G_i$ such that $(A_i, G_i) \models \neg\varphi''(\mathbf{F}, a_i, f_\alpha(i)]$ for all $\alpha \in K_i$. Therefore $\{i \in I : (A_i, G_i) \models \neg\varphi''(\mathbf{F}, a_i, f_\alpha(i))\} \supseteq a_\alpha \in F$, so $(B, H) \models \neg\varphi''(\mathbf{F}, a/F, f_\alpha/F)$, as desired. $\square$

Theorem 6.8. *Suppose that k is a sup-min function such that the formula ψ in the definition is $\forall x\mathbf{P}x$ and the formula φ' is $x = x$. Assume that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and F is an ultrafilter on I. Then $k\left(\prod_{i\in I} A_i/F\right) \leq \left|\prod_{i\in I} k(A_i)/F\right|$.*

Proof. For each $i \in I$ choose $P_i \subseteq A_i$ such that $|P_i| = k(A_i)$ and $(A_i, A_i, P_i) \models \varphi$. Then, we claim, $\left(\prod_{i\in I} A_i/F, \prod_{i\in I} A_i/F, \prod_{i\in I} P_i/F\right) \models \varphi$, which will prove the theorem. So, suppose that $x \in \prod_{i\in I} A_i$. For each $i \in I$ choose $y_i \in P_i$ such that $(A_i, A_i) \models \varphi''(\mathbf{F}, x(i), y_i)$. Then

$$\left(\prod_{i\in I} A_i/F, \prod_{i\in I} A_i/F\right) \models \varphi''[\mathbf{F}, x/F, y/F] \qquad \square$$

Theorem 6.9 (GCH)**.** *Suppose that k is a sup-min function such that the formula ψ in the definition is $\forall x\mathbf{P}x$ and the formula φ' is $x=x$. We also suppose that for every infinite BA A there is a $G \subseteq A$ such that $(A,G) \models \psi$ and $\min\{|P| : (A,G,P) \models \varphi\} \geq \omega$. If $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and F is a regular ultrafilter on I, then $k\left(\prod_{i\in I} A_i/F\right) = \left|\prod_{i\in I} k(A_i)/F\right|$.*

Proof. Let $\lambda = \text{ess.sup}^F_{i\in I} k(A_i)$. We consider several cases.

Case 1. $\lambda \leq |I|$. Then, using Keisler, Prikry [74] and Theorems 6.7 and 6.8 we have

$$2^{|I|} = |I|^+ \leq k\left(\prod_{i\in I} A_i/F\right) \leq \left|\prod_{i\in I} k(A_i)/F\right| \leq \lambda^{|I|} = 2^{|I|}.$$

Case 2. $\operatorname{cf}\lambda \leq |I| < \lambda$. Then by Theorem 6.5 we get $\lambda^+ \leq k\left(\prod_{i\in I} A_i/F\right) \leq \left|\prod_{i\in I} k(A_i)/F\right| \leq \lambda^{|I|} = \lambda^+$.

Case 3. $|I| < \operatorname{cf}\lambda$. Then by Theorem 6.6 we have

$$\lambda \leq k\left(\prod_{i\in I} A_i/F\right) \leq \left|\prod_{i\in I} k(A_i)/F\right| \leq \lambda^{|I|} = \lambda. \qquad \square$$

By a result of Donder [88], $V = L$ implies that every uniform ultrafilter is regular, and hence that the equality in Theorem 6.9 always holds; thus this answers Problem 12 of Monk [90] in a different way from the solution of Koppelberg and Shelah mentioned at the outset.

The formula for the algebraic density of products and weak products given above no longer works for subdirect products in general. For example, let $f : \mathscr{P}(\omega) \to (\mathscr{P}(\omega)/\text{fin}) \times \mathscr{P}(\omega)$ be defined by $f(x) = ([x], x)$. Then $\text{rng}(f)$ is a subdirect product of $\mathscr{P}(\omega)/\text{fin}$ and $\mathscr{P}(\omega)$, and $\pi(\mathscr{P}(\omega)/\text{fin}) = 2^\omega$ while $\pi(\mathscr{P}(\omega)) = \omega$.

That formula does hold for moderate products, by the same argument given for products.

Concerning one-point gluing we have the following result.

Proposition 6.10. *If B is the one-point gluing of the pair $(\langle A_i : i \in I\rangle, \langle F_i : i \in I\rangle)$, then $\pi(B) = \pi(\prod_{i\in I} A_i)$.*

Proof. For each $i \in I$ let $X_i \subseteq A_i^+$ be dense in A_i and of size $\pi(A_i)$. Clearly $u \in X_i$ for each atom u of A_i. If F_i is the principal ultrafilter generated by $\{u\}$, let $X_i' = \{x \in X_i : x \cdot u = 0\}$; otherwise let $X_i' = X_i$.

We now consider two cases.

Case 1. Every F_i is principal, say generated by $\{u_i\}$, with u_i an atom in A_i. For each $i \in I$ and each $x \in X_i'$, define $z_{xi} \in B$ by

$$z_{xi}(j) = \begin{cases} x & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\{u\} \cup \{z_{xi} : i \in I, x \in X_i, x \cdot u_i = 0\}$ is dense in B and it is of the desired size.

Case 2. Some F_i is nonprincipal.

(1) If F_i is nonprincipal, then for each $x \in X_i$ there is a nonzero $y_x \leq x$ such that $y_x \notin F_i$.

In fact, assume the hypothesis. If $x \notin F_i$, we can take $y_x = x$. Otherwise, with $x \in F_i$ there is a $z < x$ with $z \in F_i$, and we can take $y_x = x \cdot -z$.

Now for each $i \in I$ and $x \in X'_i$, define $z_{xi} \in B$ by

$$z_{xi}(j) = \begin{cases} 0 & \text{if } \ j \neq i, \\ y_x & \text{if } \ j = i \text{ and } F_i \text{ is nonprincipal} \\ x & \text{if } \ j = i \text{ and } F_i \text{ is principal.} \end{cases}$$

We claim that $\{z_{xi} : i \in I, x \in X'_i\}$ is dense in B. For, suppose that $b \in B^+$. Then there are two possibilities:

Subcase 2.1. $\forall i \in I[b_i \in F_i]$. Take $i \in I$ such that F_i is nonprincipal. Now $b_i \in F_i$, and so $b_i \neq 0$. Choose $x \in X_i$ such that $x \leq b_i$. Since F_i is nonprincipal, we have $X_i = X'_i$. Then $z_{xi} \leq b$, as desired.
Subcase 2.2. $\forall i \in I[b_i \notin F_i]$. Choose $i \in I$ such that $b_i \neq 0$, and choose $x \in X_i$ such that $x \leq b_i$. Then $x \in X'_i$ and $z_{xi} \leq b$, as desired.

Thus we have now shown that B has a dense subset of size $\max\{|I|, \sup_{i \in I} \pi(A_i)\}$. Now let X be dense in B. To show that $|I| \leq |X|$, suppose that $i \in I$. Choose $a \in A_i \backslash F_i$, and let x be the member of B which is equal to a at the ith place and is 0 elsewhere. There is some nonzero element of X below x. This shows that $|I| \leq |X|$. To show that $\pi(A_i) \leq |X|$, it suffices to show that $\{x_i : x \in X\}$ is dense in A_i. Suppose that $a \in A_i^+$. There is an element y of B such that $y_i = a$, and the desired conclusion follows. $\square$

Clearly $\pi(\mathrm{Dup}(A)) = |\mathrm{Ult}(A)|$ for any infinite BA A.

Proposition 6.11. $\pi(\mathrm{Exp}(A)) = \pi(A)$ *for any infinite BA* A.

Proof. First we show that $\pi(\mathrm{Exp}(A)) \leq \pi(A)$. Let X be dense in A of size $\pi(A)$. Define

$$Y = \{\mathscr{V}(\mathcal{S}(x_0), \ldots, \mathcal{S}(x_{m-1})) : m \in \omega, x \in {}^mX\}.$$

We claim that Y is dense in $\mathrm{Exp}(A)$, giving the desired inequality. For, by Theorem 1.20 it suffices to take a nonzero element of $\mathrm{Exp}(A)$ of the form $\mathscr{V}(\mathcal{S}(b_0), \ldots, \mathcal{S}(b_{m-1}))$ and find a nonzero element of Y below it. For each $i < m$ let x_i be a nonzero element of X below b_i. Then $\mathscr{V}(\mathcal{S}(x_0), \ldots, \mathcal{S}(x_{m-1})) \subseteq \mathscr{V}(\mathcal{S}(b_0), \ldots, \mathcal{S}(b_{m-1}))$, as desired.

To show that $\pi(A) \leq \pi(\mathrm{Exp}(A))$, let Y be dense in $\mathrm{Exp}(A)$, of size $\pi(\mathrm{Exp}(A))$. By Theorem 1.20 we may assume that for each element $y \in Y$ we can write

$$y = \mathscr{V}(\mathcal{S}(a_0^y), \ldots, \mathcal{S}(a_{m_y-1}^y)).$$

Let $X = \{a_i^y : y \in Y, i < m_y\}$. We claim that X is dense in A; this will prove the above inequality. Suppose that b is a nonzero element of A. Choose $y \in Y$ such that $y \subseteq \mathscr{V}(\mathcal{S}(b))$. We claim that $a_i^y \leq b$ for some $i < m_y$. Suppose that this is not true. For each $i < m_y$ let U_i be an ultrafilter such that $a_i^y \cdot -b \in U_i$. Let $F = \{U_i : i < m\}$. Then $F \in y \backslash \mathscr{V}(\mathcal{S}(b))$, contradiction. So the claim holds, giving the desired inequality. $\square$

derived functionsTurning to the functions derived from π, we first work toward proving that $\pi_{\mathrm{H}+} = \pi_{\mathrm{h}+} = \mathrm{hd}$.

We call a sequence $\langle x_\xi : \xi < \kappa\rangle$ of elements of a space X *left-separated* provided that κ is a cardinal, and for every $\xi < \kappa$ there is an open set U in X such that $U \cap \{x_\eta : \eta < \kappa\} = \{x_\eta : \xi \leq \eta\}$. The definition of hd given in the introduction obviously extends as follows to arbitrary spaces X:

$$\mathrm{hd}(X) = \sup\{\mathrm{d}(S) : S \text{ a subspace of } X\}.$$

We now need the following important fact relating left separation to hd:

Theorem 6.12. *For any infinite Hausdorff space X, $\mathrm{hd}(X)$ is the supremum of all cardinals κ such that there is a left-separated sequence in X of type κ.*

Proof. If $\langle x_\xi : \xi < \kappa\rangle$ is a left-separated sequence in X and κ is infinite and regular, then clearly the density of $\{x_\xi : \xi < \kappa\}$ is κ. Hence the inequality $\geq$ holds. Now suppose that Y is a subspace of X, and set $\mathrm{d}(Y) = \kappa$. We construct a left-separated sequence $\langle x_\xi : \xi < \kappa\rangle$ as follows: having constructed $x_\eta \in Y$ for all $\eta < \xi$, where $\xi < \kappa$, it follows that $\{x_\eta : \eta < \xi\}$ is not dense in Y, and so we can choose $x_\xi \in Y \backslash \overline{\{x_\eta : \eta < \xi\}}$. This proves the other inequality. $\square$

Note that the proof of Theorem 6.12 shows that if $\mathrm{hd}(X)$ is attained, then it is also attained in the left-separated sense, and conversely if $\mathrm{hd}(X)$ is regular.

There is an algebraic version of left-separated sequences. We call a sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A *left-separated* if for all $\xi < \alpha$ and all finite $F \subseteq \alpha$ such that $\xi < \beta$ for all $\beta \in F$ we have $a_\xi \cdot \prod_{\eta \in F} -a_\eta \neq 0$.

Lemma 6.13. *Let κ be an infinite cardinal. Then a BA A has a left-separated sequence of length κ* iff $\mathrm{Ult}(A)$ *has a left-separated sequence of length κ.*

Proof. $\Rightarrow$: Let $\langle a_\xi : \xi < \kappa\rangle$ be a left-separated sequence of elements of A. For each $\xi < \kappa$, let F_ξ be an ultrafilter on A containing the set $\{a_\xi\} \cup \{-a_\eta : \xi < \eta\}$. For any $\xi < \kappa$ we have $\{F_\eta : \eta < \kappa\} \cap \bigcup_{\xi \leq \eta} \mathcal{S}(a_\eta) = \{F_\eta : \xi \leq \eta\}$. In fact, if $F_\eta \in \bigcup_{\xi \leq \rho} \mathcal{S}(a_\rho)$, choose ρ with $\xi \leq \rho$ and $F_\eta \in \mathcal{S}(a_\rho)$. Then $a_\rho \in F_\eta$, so $\eta \not< \rho$, so $\xi \leq \rho \leq \eta$. If $\xi \leq \eta$, then $a_\eta \in F_\eta$, hence $F_\eta \in \mathcal{S}(a_\eta)$ and so $F_\eta \in \bigcup_{\xi \leq \rho} \mathcal{S}(a_\rho)$. Thus $\langle F_\xi : \xi < \kappa\rangle$ is a left-separated sequence in $\mathrm{Ult}(A)$.

$\Leftarrow$: Let $\langle F_\xi : \xi < \kappa\rangle$ be a left-separated sequence in $\mathrm{Ult}(A)$. For each $\xi < \kappa$ choose an element a_ξ such that $F_\xi \in \mathcal{S}(a_\xi)$ and $\mathcal{S}(a_\xi) \cap \{F_\eta : \eta < \kappa\} \subseteq \{F_\eta : \xi \leq \eta < \kappa\}$. To check that $\langle a_\xi : \xi < \kappa\rangle$ is a left-separated sequence, suppose on the contrary that $a_\xi \le \sum_{\eta\in F} x_\eta$, where $\xi < \eta$ for all $\eta \in F$, F a finite subset of κ. Since $a_\xi \in F_\xi$, it follows that $a_\eta \in F_\xi$ for some $\eta \in F$. This contradicts the choice of a_η. □

The essential step in proving that $\pi_{\mathrm{H}+} = \mathrm{hd}$ is as follows; we follow the proof of van Douwen [89], 10.1.

Lemma 6.14. *Let A be an infinite BA. Then there exists a left-separated sequence $\langle x_\xi : \xi < \pi(A)\rangle$ such that $\{x_\xi : \xi < \pi(A)\}$ is dense in A.*

Proof. The major part of the proof consists in proving

(1) There is a sequence $\langle a_\xi : \xi < \pi(A)\rangle$ of non-zero members of A such that $\{a_\xi : \xi < \pi(A)\}$ is dense in A and for each $\eta < \pi(A)$, $|\{\xi < \eta : a_\xi \cdot a_\eta \neq 0\}| < \pi(A \upharpoonright a_\eta)$.

To prove this, call an element $b \in A$ *π-homogeneous* provided that $\pi(A \upharpoonright c) = \pi(A \upharpoonright b)$ for every non-zero $c \le b$. Clearly the collection of all π-homogeneous elements of A is dense in A. Let $\mathscr{A}$ be a maximal disjoint collection of π-homogeneous elements of A. Let $\kappa = |\mathscr{A}|$; then $\kappa \le \mathrm{c}(A) \le \pi(A)$. For each $b \in \mathscr{A}$ let M_b be a dense subset of $A \upharpoonright b$ of cardinality $\pi(A \upharpoonright b)$. Then $\bigcup\{M_b : b \in \mathscr{A}\}$ is dense in A. Now let $\langle N_b : b \in \mathscr{A}\rangle$ be a partition of $\pi(A)$ into disjoint subsets of power $\pi(A)$. For each $b \in \mathscr{A}$ let f_b be a one-one function from M_b onto a subset of N_b of order type $\pi(A \upharpoonright b)$. Now for each $\xi < \pi(A)$, let

$$a_\xi = \begin{cases} 0, & \text{if } \ \xi \notin \bigcup_{b\in\mathscr{A}} \mathrm{rng}(f_b); \\ f_b^{-1}(\xi), & \text{if } \ \xi \in \mathrm{rng}(f_b),\ b \in \mathscr{A}. \end{cases}$$

Note that if $a_\xi \neq 0$, then $a_\xi \in M_b$ for a unique b. If $a_\xi \cdot a_\eta \neq 0$, then there is a unique b such that $a_\xi, a_\eta \in M_b$. Now suppose that $\eta < \pi(A)$ and $a_\eta \neq 0$. Say $\eta \in \mathrm{rng}(f_b)$. Then

$$|\{\xi < \eta : a_\xi \cdot a_\eta \neq 0\}| \le |\{\xi \in \mathrm{rng}(f_b) : \xi < \eta\}| < \pi(A \upharpoonright b).$$

Thus we have (1), except that some of the a_η's are zero. If we renumerate the non-zero a_η's in increasing order of their indices, we really get (1).

Now we construct a sequence $\langle b_\alpha : \alpha < \pi(A)\rangle$ of non-zero elements of A so that the following conditions hold: (2_α) $b_\alpha \le a_\alpha$ for all $\alpha < \pi(A)$,

and

(3_α) for all $\xi < \alpha$ and every finite $F \subseteq (\xi, \alpha]$ we have $b_\xi \cdot \prod_{\eta\in F} -b_\eta \neq 0$ for all $\alpha < \pi(A)$.

Assume that $\beta < \pi(A)$, and b_α has been constructed for all $\alpha < \beta$ so that (2_α) and (3_α) hold. Then by (1) and the assumption that (2_α) holds for all $\alpha < \beta$ we see that the set $\Gamma \stackrel{\text{def}}{=} \{\alpha < \beta : b_\alpha \cdot a_\beta \neq 0\}$ has power $< \pi(A \upharpoonright a_\beta)$. Hence there is a non-zero b_β in A such that $b_\beta \le a_\beta$ and for all $\varphi \in \Gamma$ and all finite $G \subseteq \Gamma$, if

$b_\varphi \cdot \prod_{\gamma\in G} -b_\gamma \neq 0$, then $b_\varphi \cdot \prod_{\gamma\in G} -b_\gamma \not\subseteq b_\beta$. Thus (2_β) and (3_β) hold, and the construction is complete.

It is clear from (2_α) and (3_α) that $\langle b_\alpha : \alpha < \pi(A)\rangle$ is the desired dense sequence. $\square$

The following theorem is due to Shapirovskiĭ.

Theorem 6.15. $\pi_{\mathrm{H}+}(A) = \pi_{\mathrm{h}+}(A) = \mathrm{hd}(A)$ *for any infinite BA A.*

Proof. It is obvious that $\pi_{\mathrm{H}+}(A) \leq \pi_{\mathrm{h}+}(A)$. Now if $\mathcal{O}$ is a π-base for $Y \subseteq \mathrm{Ult}(A)$ with $|\mathcal{O}| = \pi(Y)$, without loss of generality $\mathcal{O}$ has the form $\{\mathcal{S}(a) \cap Y : a \in \mathscr{A}\}$ for some $\mathcal{A} \subseteq A$. Let $f(x) = \mathcal{S}(x) \cap Y$ for any $x \in A$. Then f is a homomorphism onto some algebra B of subsets of Y, and $\mathcal{O}$ is dense in B. This shows that $\pi_{\mathrm{h}+}(A) \leq \pi_{\mathrm{H}+}(A)$. It is also trivial that $\mathrm{hd}(A) \leq \pi_{\mathrm{h}+}(A)$, since if $S \subseteq \mathrm{Ult}(A)$ then $\mathrm{d}(S) \leq \pi(S)$. It remains just to show that $\pi_{\mathrm{H}+}(A) \leq \mathrm{hd}(A)$. Suppose that f is a homomorphism of A onto B, where B is infinite. Apply 6.14 to B to get a system $\langle b_\xi : \xi < \pi(B)\rangle$ of elements of B such that for any $\xi < \pi(B)$ and any finite subset G of $(\xi, \pi(B)$ we have $b_\xi \cdot \prod_{\eta\in G} -b_\eta \neq 0$. For each $\xi < \pi(B)$ choose a_ξ so that $f(a_\xi) = b_\xi$. Clearly $\langle a_\xi : \xi < \pi(B)\rangle$ is a left-separated sequence of elements of A, which by 6.12 and 6.13 is as desired. $\square$

The proof of Theorem 6.15 shows that $\pi_{\mathrm{H}+}$ and $\pi_{\mathrm{h}+}$ have the same attainment properties; also, if $\pi_{\mathrm{H}+}$ is attained, then hd is attained in the left-separated sense. Also, if hd is attained in the defined sense then it is attained in the $\pi_{\mathrm{h}+}$ sense.

Clearly $\pi_{\mathrm{H}-}(A) = \pi_{\mathrm{h}-}(A) = \omega$ for any infinite BA A. The cardinal function $\pi_{\mathrm{S}+}$ is of some interest, since it does not coincide with any of our standard ones. Obviously $\pi A \leq \pi_{\mathrm{S}+}(A)$ for any infinite BA A. Moreover, $\pi_{\mathrm{S}+}(A) \leq \pi_{\mathrm{H}+}(A)$; this follows from the following fact: for every subalgebra B of A there is a homomorphic image C of A such that $\pi(B) = \pi(C)$. To see this, by the Sikorski extension theorem extend the identity function from B into $\overline{B}$ to a homomorphism from A onto a subalgebra C of $\overline{B}$. Since $B \subseteq C \subseteq \overline{B}$, it is clear that $\pi(B) = \pi(C)$. Thus we have shown that $\pi(A) \leq \pi_{\mathrm{S}+}(A) \leq \pi_{\mathrm{H}+}(A)$ for any infinite BA A. It is possible to have $\pi(A) < \pi_{\mathrm{S}+}(A)$: let $A = \mathscr{P}(\kappa)$; then $\pi(A) = \kappa$, while $\pi_{\mathrm{S}+}(A) = 2^\kappa$, since A has a free subalgebra B of power 2^κ, and clearly $\pi(B) = 2^\kappa$. It is more difficult to come up with an example of an algebra where the other inequality is proper (this example is due to Monk):

Example 6.16. There is an infinite BA A such that $\pi_{\mathrm{S}+}(A) < \pi_{\mathrm{H}+}(A)$.

To see this, let B be the interval algebra on the real numbers, and let $A = B \oplus B$. Now, we claim, $\pi_{\mathrm{S}+}(A) = \omega$, while $\pi_{\mathrm{H}+}(A) = 2^\omega$. To prove that $\pi_{\mathrm{H}+}(A) = 2^\omega$, by 5.19, 5.20, and 6.15 it suffices to show that $\mathrm{s}(A) = 2^\omega$. For each real number r let $c_r = b_r \cdot b'_r$, where $b_r = [-\infty, r)$ (as a member of the first factor of $B \oplus B$) and $b'_r = [r, \infty)$ (as a member of the second factor of $B \oplus B$). Note that we have adjoined $-\infty$ as a member of $\mathbb{R}$ in order to fulfill the requirement for interval algebras that the ordered set in question always has a first element. To show that $\langle c_r : r \in \mathbb{R}\rangle$ is an ideal independent system of elements, suppose that $c_r \le c_{s_1} + \cdots + c_{s_m}$ with $r \notin \{s_1, \ldots, s_m\}$. Let $\Gamma = \{i : s_i < r\}$ and $\Delta = \{i : r < s_i\}$. Then

$$c_r \cdot -c_{s_1} \cdot \ldots \cdot -c_{s_m} \ge b_r \cdot b'_r \cdot \prod_{i\in\Gamma} -b_{s_i} \cdot \prod_{i\in\Delta} -b'_{s_i} \ne 0,$$

contradiction. To prove that $\pi_{\mathrm{S}+}(A) = \omega$, we proceed as follows. Let C be any subalgebra of A. We want to show that $\pi(C) = \omega$. Now for each element c of C we choose a representation of c of the form

$$\sum_{i<m(c)} x_{0ic} \cdot x_{1ic},$$

where x_{0ic} and x_{1ic} are half-open intervals in B; x_{0ic} is in the first factor of $B \oplus B$ and x_{1ic} in the second. Let $T = \{(m, r, s) : m \in \omega\backslash\{0\}$ and $r, s \in {}^m\mathbb{Q}\}$. An element (m, r, s) of T is a *frame* for $c \in C$ provided that $m(c) = m$, and $r_i \in x_{0ic}$, $s_i \in x_{1ic}$ for all $i \in m$. For each $(m, r, s) \in T$ let D_{mrs} be the set of all $c \in C$ with frame (m, r, s). Since C is the union of all sets D_{mrs}, it suffices to take an arbitrary $(m, r, s) \in T$ and find a countable subset of C dense in D_{mrs}.

For each $c \in D_{mrs}$ and each $i < m$ write

$$x_{0ic} = [a_{ic}, b_{ic}) \quad \text{and} \quad x_{1ic} = [d_{ic}, e_{ic}).$$

Thus $a_{ic} \le r_i < b_{ic}$ and $d_{ic} \le s_i < e_{ic}$ for all $c \in D_{mrs}$ and $i < m$. For each $i < m$ let N_{0i} be a countable subset of $\{a_{ic} : c \in D_{mrs}\}$ cofinal in that set. Similarly choose N_{1i} coinitial for the b_{ic}'s, N_{2i} cofinal for the d_{ic}'s, and N_{3i} coinitial for the e_{ic}'s. Let M be the set of all products

$$\prod_{i<m, j<4} u_{ij}$$

with $u \in {}^{m\times 4}\bigcup_{i<m}(N_{0i} \cup N_{1i} \cup N_{2i} \cup N_{3i})$. Clearly all such products are nonzero. We claim that M is dense in D_{mrs}. For, let $c \in D_{mrs}$. For each $i < m$ choose $u_{i0} \in N_{0i}$ such that $a_{ic} \le u_{i0}$; similarly for u_{ij}, $j = 1, 2, 3$. Then clearly $\prod_{i<m,j<4} u_{ij} \le c$, as desired. □

Shelah [96] showed that it is consistent to have a BA A with $\pi_{\mathrm{S}+}(A)$ not attained; this answers Problem 13 in Monk [90]. On the other hand, under GCH the cardinal $\pi_{\mathrm{S}+}(A)$ is always attained. In fact, if $|A|$ is a limit cardinal, then $\pi(A) = |A|$ and so of course $\pi_{\mathrm{S}+}(A) = |A|$ is attained. If $|A| = \kappa^+$, then $\pi(A)$ is either κ or κ^+, and it follows that $\pi_{\mathrm{S}+}(A)$ is attained. This answers Problem 20 in Monk [96].

Clearly $\pi_{\mathrm{S}-}A = \omega$ for any infinite BA A. Furthermore, ${}_{\mathrm{d}}\pi_{\mathrm{S}+}(A) = {}_{\mathrm{d}}\pi_{\mathrm{S}-}(A) = \pi(A)$ for any infinite BA A.

We now consider *weak density* and the cardinal function $\mathfrak{r}$. Several generalizations of $\mathfrak{r}$ have been intensively studied in the literature, and we survey this area. We present various results from Dow, Steprāns, Watson [96], Balcar, Simon [92], Peterson [98], Bozeman [91], and Brown [05]. A subset X of A^+ is *weakly dense* iff for any $a \in A$ there is an $x \in X$ such that $x \leq a$ or $x \leq -a$. The *weak density* or *reaping number* of A is the smallest size $\mathfrak{r}(A)$ of a weakly dense subset of A.

We generalize this notion as follows. A *weak partition* of a BA A is a system $\langle x_i : i \in I\rangle$ of pairwise disjoint elements of A, with sum 1. We call it "weak" because it is not assumed that each x_i is nonzero. Of course there is at least one nonzero x_i. For any integer $m \geq 2$, we call a subset X of A^+ *m-dense* iff for every weak partition $\langle a_i : i < m\rangle$ of A, $\exists x \in X \exists i < m[x \leq a_i]$. Let

$$\mathfrak{r}_m(A) = \min\{|X| : X \text{ is } m\text{-dense in } A\}.$$

The following proposition should be noted, although we will not use it.

Proposition 6.17. *Let A be an infinite BA, m an integer, and $X \subseteq A$. Then the following are equivalent:*

(i) *X is m-dense.*

(ii) *For every partition $\langle a_i : i < m\rangle$ of A there exist an $x \in X$ and an $i < m$ such that $x \leq a_i$.*

Proof. Obviously (i)⇒(ii). Now suppose that (ii) holds, and $\langle a_i : i < m\rangle$ is a weak partition. Let $G = \{i < m : a_i = 0\}$. We may assume that G is nonempty. Fix $j < m$ such that $A \upharpoonright a_j$ is infinite. Let $\langle b_i : i < |G| + 1\rangle$ be a system of nonzero pairwise disjoint elements of $A \upharpoonright a_j$ with sum a_j, and let f be a bijection from G onto $|G|$. Define

$$a'_k = \begin{cases} a_k & \text{if } k \notin G \cup \{j\}, \\ b_{f(k)} & \text{if } k \in G, \\ b_{|G|} & \text{if } k = j. \end{cases}$$

Clearly $\langle a'_k : k < m\rangle$ is a partition, and an application of (ii) yields $x \in X$ and $i < m$ such that $x \leq a_i$. □

Proposition 6.18.

(i) *If a is an atom of A, then $\{a\}$ is weakly dense, and hence $\mathfrak{r}(A) = 1$.*

(ii) *If A is atomless, then $\mathfrak{r}(A)$ is infinite.*

(iii) *$\mathfrak{r}_2(A) \leq \mathfrak{r}_3(A) \leq \cdots$ for any BA A.*

Proof. (i) is obvious. For (ii), suppose that A is atomless and $X \subseteq A$ is finite and weakly dense. Let $a_0, \ldots, a_{n-1}$ list all of the atoms of $\langle X\rangle$, and for each $i < n$ choose b_i with $0 < b_i < a_i$. Let $c = \sum_{i<n} b_i$. Choose a nonzero $x \in X$ such that $x \leq c$ or $x \leq -c$. Say $x \cdot a_i \neq 0$. Then since a_i is an atom of $\langle X\rangle$, we have $a_i \leq x$. This contradicts $x \leq c$ or $x \leq -c$.

(iii) is clear. □

A finer analysis of weak density can be made as follows. Suppose that m and n are integers, with $2 \le n \le m$. A subset X of A^+ is (m,n)-*dense* provided that for every weak partition $\langle a_i : i < m \rangle$ there is an $x \in X$ such that $|\{i < m : x \cdot a_i \neq 0\}| < n$. Then we define

$$\mathfrak{r}_{mn}(A) = \min\{|X| : X \text{ is } (m,n)\text{-dense}\}.$$

We now work towards a result of Dow, Steprāns, Watson [96] to the effect that $\mathfrak{r}_n(A) \le \mathfrak{r}_2(A)^+$ for every integer $n \ge 2$.

Some simple relationships between these notions of weak density are as follows.

Proposition 6.19. *Suppose that $2 \le n \le m$. Then for any BA A,*

(i) *If also $n+1 \le m$, then $\mathfrak{r}_{m,n+1}(A) \le \mathfrak{r}_{mn}(A)$.*
(ii) $\mathfrak{r}_{mn}(A) \le \mathfrak{r}_{m+1,n}(A)$.
(iii) $\mathfrak{r}_{m+1,n+1}(A) \le \mathfrak{r}_{mn}(A)$.
(iv) *If $\exists k[m \le i \le mk$ and $(n-1)k < j]$, then $\mathfrak{r}_{ij}(A) \le \mathfrak{r}_{mn}(A)$.*
(v) $\mathfrak{r}_2(A) \le \mathfrak{r}_{mn}(A)$.
(vi) $\mathfrak{r}_n(A) = \mathfrak{r}_{n,2}(A)$.
(vii) *If $m/(n-1) \le 2$, then $\mathfrak{r}_{mn}(A) = \mathfrak{r}_2(A)$.*

Proof. (i) is clear, since every (m,n)-dense subset is also $(m,n+1)$-dense. (ii) holds since we can adjoin 0 to any weak partition of length m to obtain one of length $m+1$; so every $(m+1,n)$-dense set is also (m,n)-dense.

For (iii), let X be (m,n)-dense, with $|X| = \mathfrak{r}_{mn}(A)$. Take any weak partition $\langle a_s : s < m+1 \rangle$. Define for $s < m$

$$a'_s = \begin{cases} a_s & \text{if } s < m-1, \\ a_{m-1} + a_m & \text{if } s = m-1. \end{cases}$$

Choose $x \in X$ such that $\{s < m : x \cdot a'_s \neq 0\}| < n$. Then $|\{s < m+1 : x \cdot a_s \neq 0\}| < n+1$.

For (iv), suppose that X is (m,n)-dense, with $|X| = \mathfrak{r}_{mn}(A)$. Take any weak partition $\langle a_s : s < i \rangle$. Since $m \le i \le mk$, let $\langle u_l : l < m \rangle$ be a partition of i into nonempty subsets, each of size at most k. For each $l < m$ let $b_l = \bigcup_{s \in u_l} a_s$. Thus $\langle b_l : l < m \rangle$ is a weak partition. Choose $x \in X$ such that $|\{l < m : x \cdot b_l \neq 0\}| < n$. Then

$$\begin{aligned} |\{s < i : x \cdot a_s \neq 0\}| &= \sum_{\substack{l<m \\ x \cdot b_l \neq 0}} |\{s \in u_l : x \cdot a_s \neq 0\}| \\ &\le (n-1)k < j. \end{aligned}$$

For (v), suppose that $|X| < \mathfrak{r}_2(A)$. Then

$(*)$ For each $i \ge 2$ there is a (non-weak) partition $\langle a_s : s < i \rangle$ such that $\forall x \in X \forall s < i[x \cdot a_s \neq 0]$.

We prove $(*)$ by induction. Take a such that $\forall x \in X[x \cdot a \neq 0 \neq x \cdot -a]$. Then the partition $\langle a, -a\rangle$ proves $(*)$ for $i = 2$. Now assume the conclusion of $(*)$ for i. Now the set $\{x \cdot a_0 : x \in X\}$ has size less than $\mathfrak{r}(A)$, and so there is a b such that $\forall x \in X[x \cdot a_0 \cdot b \neq 0 \neq x \cdot a_0 \cdot -b]$. Replacing a_0 by $\langle a_0 \cdot b, a_0 \cdot -b\rangle$, we get $(*)$ for $i + 1$.

Now take $i = n$. If X is (m, n)-dense, then $|\{x \in X : x \cdot a_s \neq 0\}| < n$, contradiction.

(vi) is obvious.

For (vii), note by (v) that $\mathfrak{r}_2(A) \leq \mathfrak{r}_{mn}(A)$. Now we apply (iv) with m, n, i, j, k replaced by $2, 2, m, n, n-1$ to get $\mathfrak{r}_{mn}(A) \leq \mathfrak{r}_{22}(A) = \mathfrak{r}_2(A)$. □

To proceed we need some finite combinatorics. First we prove a simple fact about elementary algebra.

Proposition 6.20. *If n is a prime and $\mathbb{Z}_n$ is the group of integers modulo n, then for any $x, k \in \mathbb{Z}_n$ with $k \neq 0$, the elements $x, x+k, x+2k, \ldots, x+(n-1)k$ are all distinct (addition and multiplication $\mod n$).*

Proof. Suppose that there are i, j with $0 \leq i < j < n$ and $x + ik = x + jk$. Then $(j-i)k = 0$. This is mod n, and with ordinary multiplication it says that $(j-i)k$ is divisible by n. Since $j - i, k < n$ and n is prime, this is not possible. □

Now for positive integers m, n, i, j, k, q we write

$$\binom{m}{n} \not\to \binom{j}{i}^{1,1}_{k,q}$$

to mean that there is a function $h : m \times n \to k$ such that $\forall a \in [n]^i \forall b \in [m]^j [h \restriction (a \times b)$ takes on more than q values]. Note that this implies that $q + 1 \leq k$.

Lemma 6.21. *For every prime n and positive integer $k < n$,*

$$\binom{n}{n} \not\to \binom{2}{k}^{1,1}_{n,k}.$$

Proof. Define $h : n \times n \to n$ by $h(i, j) = i + j \pmod n$. Suppose that $a \in [n]^k$ and $b \in [n]^2$. Say $b = \{s, t\}$ with $s < t$, and $a = \{x_i : i < k\}$. Now the integers $x_i + t$ are all distinct for $i < k$. (Again, addition mod n). Similarly the integers $x_i + s$ are all distinct for $i < k$. Suppose that $h \restriction (a \times b)$ takes on at most k values. Then each integer $x_i + t$ is equal to $x_j + s$ for some j. By Proposition 6.20, the integers $x_0 + u(t-s)$ are all distinct for $u < n$. Hence there is a smallest u such that $x_0 + u(t-s) \notin a$. Obviously $u > 0$. Say $x_0 + (u-1)(t-s) = x_j$. Choose v such that $x_j + t = x_v + s$. Then $x_j + (t-s) = x_v \in a$. But $x_j + (t-s) = x_0 + u(t-s)$, contradiction. □

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Lemma 6.22. *Suppose that i, j, k, m, n, q are positive integers such that*

$$\binom{m}{n} \not\to \binom{j}{i}^{1,1}_{k,q}.$$

Also assume that A is a BA with $\mathfrak{r}_{k,q+1}(A) < \mathfrak{r}_{n,i}(A)$. Then $\mathfrak{r}_{m,j}(A) \leq \mathfrak{r}_{k,q+1}(A)^+$.

Proof. Let h be as in the definition of $\binom{m}{n} \not\to \binom{j}{i}^{1,1}_{k,q}$. Suppose that A is a BA with $\mathfrak{r}_{k,q+1}(A) < \mathfrak{r}_{n,i}(A)$. Let X be a subset of A of size $\mathfrak{r}_{k,q+1}(A)$ which is $(k, q+1)$-dense. Let $\kappa = \mathfrak{r}_{k,q+1}(A)^+$. We now define a sequence $\langle B_\alpha : \alpha < \kappa\rangle$ of subalgebras of A, each of size $\mathfrak{r}_{k,q+1}(A)$. Let $B_0 = \langle X\rangle$. Suppose that B_α has been defined. Then $|B_\alpha| < \mathfrak{r}_{n,i}(A)$, so there is a weak partition $\langle b(\alpha, l) : l < n\rangle$ of A such that $\forall x \in B_\alpha[|\{l < n : x \cdot b(\alpha, l) \neq 0\}| \geq i]$. Let $B_{\alpha+1} = \langle B_\alpha \cup \{b(\alpha, l) : l < n\}\rangle$. For β limit $< \kappa$ let $B_\beta = \bigcup_{\alpha<\beta} B_\beta$.

Let $B_\kappa = \bigcup_{\alpha<\kappa} B_\alpha$. Assume now, to get a contradiction, that $\kappa < \mathfrak{r}_{m,j}(A)$. Let $\langle b(\kappa, l) : l < m\rangle$ be a weak partition such that $\forall x \in B_\kappa[|\{l < m : x \cdot b(\kappa, l) \neq 0\}| \geq j]$.

For all $\alpha < \kappa$ and $l < k$ let

$$c(\alpha, l) = \sum_{h(\zeta,\xi)=l} (b(\alpha, \xi) \cdot b(\kappa, \zeta)).$$

Then for each $\alpha < \kappa$, the sequence $\langle c(\alpha, l) : l < k\rangle$ is a weak partition of A. In fact, suppose that $l < s < k$. If $h(\zeta, \xi) = l$ and $h(\zeta', \xi') = s$, then $(\zeta, \xi) \neq (\zeta', \xi')$, and hence $b(\alpha, \xi) \cdot b(\kappa, \zeta) \cdot b(\alpha, \xi') \cdot b(\kappa, \zeta') = 0$. Also,

$$\sum_{l<k} c(\alpha, l) = \sum_{\substack{\xi<n \\ \zeta<m}} (b(\alpha, \xi) \cdot b(\kappa, \zeta)) = 1.$$

Now X is $(k, q+1)$-dense in A, so for each $\alpha < \kappa$ there is an $a_\alpha \in X$ such that $|\{l < k : a_\alpha \cdot c(\alpha, l) \neq 0\}| < q+1$. Let $\langle l^e_\alpha : e < q\rangle$ be such that $\{l < k : a_\alpha \cdot c(\alpha, l) \neq 0\} \subseteq \{l^e_\alpha : e < q\}$. Thus $a_\alpha \leq \sum_{e<q} c(\alpha, l^e_\alpha)$. Now

$$\kappa = \bigcup \{\alpha < \kappa : \langle l^e_\alpha : e < q\rangle = s, a_\alpha = b\},$$

the union being taken over all pairs (s, b) such that $s \in {}^q k$ and $b \in X$.

Hence one of the sets in the union has κ elements. In particular, there are elements $\alpha_1 < \alpha_2 < \cdots < \alpha_i < \kappa$ and integers l^e for $e < q$ such that

$$l^e_{\alpha_1} = l^e_{\alpha_2} = \cdots = l^e_{\alpha_i} = l^e \text{ and } a_{\alpha_1} = a_{\alpha_2} = \cdots = a_{\alpha_i} = a$$

for each $e < q$. Now $\langle b(\alpha_1, l) : l < n\rangle$ is a weak partition, so there is an $t_1 < n$ such that $a \cdot b(\alpha_1, t_1) \neq 0$. Suppose now that $s < i$ and $b(\alpha_1, t_1), \ldots, b(\alpha_s, t_s)$ have been defined so that $a \cdot b(\alpha_1, t_1) \cdot \ldots \cdot b(\alpha_s, t_s) \neq 0$. Since $\langle b(\alpha_{s+1}, l) : l < n\rangle$ is a weak partition, there is then a $t_{s+1} < n$ such that $a \cdot b(\alpha_1, t_1) \cdot \ldots \cdot b(\alpha_{s+1}, t_{s+1}) \neq 0$. This finishes the definition of $t_1, \ldots, t_i$.

Now by the definition of $\langle b(\kappa, l) : l < m \rangle$, the set

$$u \stackrel{\text{def}}{=} \{l < m : a \cdot b(\alpha_1, t_1) \cdot \ldots \cdot b(\alpha_i, t_i) \cdot b(\kappa, l) \neq 0\}$$

has at least j elements. Let u' be a subset of u with exactly j elements. Now we claim

$(*)$ For any $s \in \{1, \ldots, i\}$ and any $y \in u'$ we have $h(t_s, y) \in \{l^e : e < q\}$.

To prove $(*)$, let s and y be given as indicated. Then

$$a \cdot b(\alpha_s, t_s) \cdot b(\kappa, y) \geq a \cdot b(\alpha_1, t_1) \cdot \ldots \cdot b(\alpha_i, t_i) \cdot b(\kappa, y) \neq 0.$$

Now $a \leq \sum_{e<q} c(\alpha_s, l^e)$, and it follows that $h(t^s, y) = l^e$ for some $e < q$. This proves $(*)$.

By $(*)$, h maps $\{t_s : 1 \leq s \leq i\} \times u'$ into $\{l^e : e < q\}$. This contradicts the basic property of h. $\square$

Theorem 6.23. $\mathfrak{r}_n(A) \leq \mathfrak{r}_2(A)^+$ *for every integer* $n \geq 2$ *and every infinite BA* A.

Proof. Since $\mathfrak{r}_2(A) \leq \mathfrak{r}_3(A) \leq \cdots$, it suffices to take the case in which n is prime. If $\mathfrak{r}_n(A) = \mathfrak{r}_2(A)$ the conclusion is obvious, so assume that $\mathfrak{r}_n(A) > \mathfrak{r}_2(A)$. Thus by Proposition 6.19(vi) we have $\mathfrak{r}_{n2}(A) > \mathfrak{r}_2(A)$. Choose $k \leq n$ maximum such that $\mathfrak{r}_{nk}(A) > \mathfrak{r}_2(A)$; see Proposition 6.19(i). Now $\frac{n}{n-1} \leq 2$, so by Proposition 6.19(vii) we have $k < n$. Hence by Proposition 6.20 we have

$$\binom{n}{n} \not\to \binom{2}{k}^{1,1}_{n,k}.$$

Now the maximality of k implies by Proposition 6.19(i) that $\mathfrak{r}_{n,k+1}(A) = \mathfrak{r}_2(A) < \mathfrak{r}_{nk}(A)$. Hence Proposition 6.22 applies to give $\mathfrak{r}_{n2}(A) \leq \mathfrak{r}_{n,k+1}(A)^+ = \mathfrak{r}_2(A)^+$. $\square$

An example with $\mathfrak{r}_u(A) < \mathfrak{r}_{u+1}(A)$ was given by Balcar, Simon [92]. To describe their example we need two lemmas, the second one being of general interest.

Lemma 6.24. *Let* κ *be an infinite cardinal. Then there is a system* $\langle P_n : n \in \omega \rangle$ *of partitions of* κ *with the following properties:*

(i) $\forall n \in \omega[|P_n| = \kappa]$.
(ii) $\forall n \in \omega \forall X \in P_n[|X| = \kappa]$.
(iii) $\forall n \in \omega \forall f \in \prod_{i \leq n} P_i \left[\left|\bigcap_{i \leq n} f(i)\right| = \kappa\right]$.

Proof. Let P_0 be a partition of κ into κ sets, each of size κ; so (i)–(iii) hold. Now suppose that $P_0, \ldots, P_n$ have been defined so that (i)–(iii) hold. For each $f \in \prod_{i \le n} P_n$ let $\langle X_{f\alpha} : \alpha < \kappa \rangle$ be a partition of $\bigcap_{i \le n} f(i)$ into κ sets, each of size κ. Let

$$P_{n+1} = \left\{ \bigcup_{f \in \prod_{i \le n} P_i} X_{f\alpha} : \alpha < \kappa \right\}.$$

Clearly each member of P_{n+1} has size κ, and P_{n+1} itself has size κ. If $\beta < \kappa$, define $f \in \prod_{i \le n} P_i$ by letting $f(i)$ be the member X of P_i such that $\beta \in X$. Then $\beta \in \bigcap_{i \le n} f(i)$, and hence $\beta \in X_{f\alpha}$ for some $\alpha < \kappa$. Thus $\bigcup P_{n+1} = \kappa$. Clearly the members of P_{n+1} are pairwise disjoint. Finally, if $f \in \prod_{i \le n+1} P_i$, then $\bigcap_{i \le n+1} f(i) = X_{(f \restriction (n+1))\alpha}$ for some $\alpha < \kappa$, so that $\left| \bigcap_{i \le n+1} f(i) \right| = \kappa$. $\square$

Lemma 6.25. *If κ is an infinite cardinal, then there is a set $\mathscr{A}$ of independent partitions of κ, with each member of $\mathscr{A}$ of size κ, and with $|\mathscr{A}| = 2^\kappa$.*

Proof. Let $\langle P_n : n \in \omega \rangle$ be as in Lemma 6.24, and let $A = \{P_n : 0 < n \in \omega\}$. By Lemma 13.9 of the Handbook, let $X \subseteq {}^{P_0}A$ be finitely distinguished, of size $\omega^\kappa = 2^\kappa$. For $f \in X$ and $u \in P_0$ we have $f(u) \in A$, and hence we can write $f(u) = \{w(f, u, \alpha) : \alpha < \kappa\}$, without repetitions. Now for any $f \in X$ and $\alpha < \kappa$ let $R(f, \alpha) = \bigcup_{u \in P_0} (u \cap w(f, u, \alpha))$.

(1) If $\alpha < \beta < \kappa$, then $R(f, \alpha) \cap R(f, \beta) = \emptyset$.

In fact,

$$\begin{aligned} R(f, \alpha) \cap R(f, \beta) &= \left(\bigcup_{u \in P_0} (u \cap w(f, u, \alpha)) \right) \cap \left(\bigcup_{v \in P_0} (v \cap w(f, v, \beta)) \right) \\ &= \bigcup_{u, v \in P_0} (u \cap v \cap w(f, u, \alpha) \cap w(f, v, \beta)); \end{aligned}$$

and $u \cap v \cap w(f, u, \alpha) \cap w(f, v, \beta) = \emptyset$ if $u \ne v$ and also if $u = v$.

(2) $\bigcup_{\alpha < \kappa} R(f, \alpha) = \kappa$.

In fact,

$$\begin{aligned} \bigcup_{\alpha < \kappa} R(f, \alpha) &= \bigcup_{\alpha < \kappa} \bigcup_{u \in P_0} (u \cap w(f, u, \alpha)) \\ &= \bigcup_{u \in P_0} \bigcup_{\alpha < \kappa} (u \cap w(f, u, \alpha)) \\ &= \bigcup_{u \in P_0} u = \kappa. \end{aligned}$$

Thus $S(f) \stackrel{\text{def}}{=} \{R(f, \alpha) : \alpha < \kappa\}$ is a partition. Clearly each member of $S(f)$ has size κ Let $\mathscr{A} = \{S(f) : f \in X\}$. Thus $|\mathscr{A}| = 2^\kappa$.

It remains only to show that $\mathscr{A}$ is independent. To this end, let F be a finite subset of X, and for each $f \in F$ let $\alpha_f < \kappa$; we want to show that $\bigcap_{f \in F} R(f, \alpha_f) \neq \emptyset$. Since X is finitely distinguished, choose $v \in P_0$ so that $\langle f(v) : f \in F \rangle$ is one-one. Then

$$\begin{aligned} \bigcap_{f \in F} R(f, \alpha_f) &= \bigcap_{f \in F} \bigcup_{u \in P_0} (u \cap w(f, u, \alpha_f)) \\ &\supseteq v \cap \bigcap_{f \in F} w(f, v, \alpha_f); \end{aligned}$$

since $f(v) \neq g(v)$ for distinct $f, g \in F$, it follows that this last intersection is nonempty. $\square$

Independent partitions in arbitrary BAs have been considered by Robin Chestnut [12].

Now we give the example from Balcar, Simon [92].

Theorem 6.26. *For every infinite cardinal κ and every integer $u \geq 2$ there is a BA B such that $\mathfrak{r}_u(B) = \kappa$ and $\mathfrak{r}_{u+1}(B) = \kappa^+$.*

Proof. Let $\mathscr{F}$ be freely generated by $\langle a(\alpha, 0) : \alpha < \kappa^+ \rangle$, and set $a(\alpha, 1) = -a(\alpha, 0)$ for all $\alpha < \kappa^+$. Let $C = {}^{\kappa}\mathscr{F}^{\mathrm{w}}$ (the weak κ-power of $\mathscr{F}$). The desired algebra B will be a certain algebra with $C \leq B \leq \overline{C}$. If $b \in \mathscr{F}$ and $\alpha < \kappa$ let δ^α_b be defined by

$$\delta^\alpha_b(\beta) = \begin{cases} b & \text{if } \ \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\beta < \kappa$.

Now by Lemma 6.25 we have:

(1) There is a family $\langle A_{\alpha\beta} : \alpha < \kappa^+, \beta < \kappa \rangle$ with the following two properties:

(a) $\forall \alpha < \kappa^+ [\langle A_{\alpha\beta} : \beta < \kappa \rangle$ is a partition of κ.

(b) For every finite $J \subseteq \kappa^+$ and every $f \in {}^J\kappa$, the set $\bigcap_{\alpha \in J} A_{\alpha f(\alpha)}$ is infinite.

For each $\alpha < \kappa^+$ let Ψ_α be the set of all functions ψ having the following two properties:

(2) There is a finite $K \subseteq \alpha$ such that $\mathrm{dmn}(\psi) = {}^K(u+1)$.

(3) $\mathrm{rng}(\psi) \subseteq {}^2(u+1) \backslash \{(0,0), (1,1), \ldots, (u,u)\}$.

Note that Ψ_0 has exactly $(u+1)^2 - (u+1)$ members. Clearly $|\Psi_\alpha| \leq \kappa$ for all $\alpha < \kappa^+$. Let χ^α be a mapping of κ onto Ψ_α.

Now we define by recursion on $\alpha < \kappa^+$ elements $b(\alpha, 0), b(\alpha, 1), \ldots, b(\alpha, u)$ of $\overline{C}$. Suppose that they have been defined for all $\beta < \alpha$ (with $\alpha = 0$ possible) so that the following conditions hold:

(4) $\forall \beta < \alpha [\langle b(\beta, i) : i \leq u \rangle$ is a partition of $\overline{C}$].

(5) $\forall\beta<\alpha\forall i\leq u\forall\gamma<\kappa[b(\beta,i)\cdot\delta^\gamma_1\in\langle\{\delta^\gamma_{a(\xi,0)}:\xi\leq\beta\}\rangle]$.

We define $b(\alpha,i)$ by defining its product with each δ^γ_1. Say $\gamma\in A_{\alpha\beta}$. Let K be the finite set such that $\mathrm{dmn}(\chi^\alpha_\beta)={}^K(u+1)$. We define

$$b(\alpha,i)\cdot\delta^\gamma_1=\sum\left\{\delta^\gamma_{a(\alpha,j)}\cdot\prod_{\beta\in K}b(\beta,h(\beta)):j<2,\ h\in{}^K(u+1),\ (\chi^\alpha_\beta(h))(j)=i\right\}.$$

We check (4) and (5) for α. For distinct $i,k\leq u$ we consider $j,l<2$ and $h,r\in{}^K(u+1)$ such that $(\chi^\alpha_\beta(h))(j)=i$ and $(\chi^\alpha_\beta(r))(l)=k$; we want to show that

$$\delta^\gamma_{a(\alpha,j)}\cdot\prod_{\beta\in K}b(\beta,h(\beta))\cdot\delta^\gamma_{a(\alpha,l)}\cdot\prod_{\beta\in K}b(\beta,r(\beta))=0. \tag{6}$$

If $j\neq l$ this is clear. If $j=l$, then $h\neq r$ and again (6) follows.

Next, we want to show that $\sum_{i\leq u}(b(\alpha,i)\cdot\delta^\gamma_1)=\delta^\gamma_1$. To this end, note that

$$1=\prod_{\alpha\in K}\sum_{i\leq u}b(\alpha,i)=\sum_{h\in{}^K(u+1)}\prod_{\beta\in K}b(\beta,h(\beta)). \tag{7}$$

Hence

$$\begin{aligned}\sum_{i\leq u}(b(\alpha,i)\cdot\delta^\gamma_1)&=\sum\Big\{\delta^\gamma_{a(\alpha,j)}\cdot\prod_{\beta\in K}b(\beta,h(\beta)):j<2,\\&\qquad h\in{}^K(u+1),(\chi^\alpha_\beta(h))(j)=i,i\leq u\Big\}\\&=\sum\left\{\delta^\gamma_{a(\alpha,j)}\cdot\prod_{\beta\in K}b(\beta,h(\beta)):j<2,h\in{}^K(u+1)\right\}\\&=\delta^\gamma_1.\end{aligned}$$

This proves (4). (5) holds using the inductive hypothesis. This finishes the definition of $\langle b(\alpha,i):\alpha<\kappa^+,i\leq u\rangle$.

Let B be the subalgebra of $\overline{C}$ generated by $C\cup\{b(\alpha,i):\alpha<\kappa^+,i\leq u\}$.

We show that $\mathfrak{r}_{u+1}(B)\geq\kappa^+$. Suppose to the contrary that X is a $(u+1)$-dense subset of B of size $\leq\kappa$. Now $\{\delta^\gamma_1\cdot x:\gamma<\kappa,x\in X\}\backslash\{0\}$ is still $(u+1)$-dense and of size $\leq\kappa$. So we may assume that $\forall x\in X\exists\gamma<\kappa[x\leq\delta^\gamma_1]$. Temporarily fix $x\in X$. There is a unique $\gamma<\kappa$ such that $x\leq\delta^\gamma_1$. Then there is an $\alpha(x)<\kappa^+$ such that $x\in\langle\{\delta^\gamma_{a(\xi,0)}:\xi<\alpha(x)\rangle^{C\restriction\delta^\gamma_1}$. Now unfix x. Choose $\varepsilon<\kappa^+$ with $\varepsilon>\sup_{x\in X}\alpha(x)$. We claim that the weak partition $\langle b(\varepsilon,i):i\leq u\rangle$ contradicts X being $(u+1)$-dense. To show this, we take any $x\in X$ and find distinct $i_0,i_1\leq u$ such that $x\cdot b(\varepsilon,i_0)\neq0\neq x\cdot b(\varepsilon,i_1)$. To do this, first choose $\gamma<\kappa$ such that $x\leq\delta^\gamma_1$. Choose

$\beta < \kappa$ such that $\gamma \in A_{\varepsilon\beta}$. Say $\mathrm{dmn}(\chi^\varepsilon_\beta) = {}^K(u+1)$ with $K \in [\varepsilon]^{<\omega}$. Now, as observed above,

$$\sum_{h \in {}^K(u+1)} \prod_{\xi \in K} b(\xi, h(\xi)) = 1,$$

so we can choose $h \in {}^K(u+1)$ such that $x \cdot \prod_{\xi \in K} b(\xi, h(\xi)) \neq 0$. Let $\chi^\varepsilon_\beta(h) = (i_0, i_1)$, with $i_0, i_1 \leq u$ and $i_0 \neq i_1$. Now for any $\xi \in K$ we have $\delta^\gamma_1 \cdot b(\xi, h(\xi)) \in \langle \delta^\gamma_{a(\eta,0)} : \eta \leq \xi \rangle$, so $\delta^\gamma_1 \cdot \prod_{\xi \in K} b(\xi, h(\xi)) \in \langle \delta^\gamma_{a(\xi,0)} : \eta < \xi \rangle$. Hence by freeness,

$$\delta^\gamma_{a(\varepsilon,0)} \cdot x \cdot \prod_{\xi \in K} b(\xi, h(\xi)) \neq 0 \neq \delta^\gamma_{a(\varepsilon,1)} \cdot x \cdot \prod_{\xi \in K} b(\xi, h(\xi)).$$

Hence by definition, $x \cdot b(\varepsilon, i_0) \neq 0 \neq x \cdot b(\varepsilon, i_1)$. This proves that $\mathfrak{r}_{u+1}(B) \geq u+1$.

To show that $\mathfrak{r}_u(B) \leq \kappa$ it suffices to prove

(8) For every weak partition $\langle p_i : i < u \rangle$ of B there exist $\alpha < \kappa$ and $i < u$ such that $\delta^\alpha_1 \leq p_i$.

In fact, it suffices to prove the following more complicated statement:

(9) For every weak partition $\langle p_i : i < u \rangle$ of B and every finite nonempty $K \subseteq \kappa^+$, if $\left\langle \prod_{\alpha \in K} b(\alpha, h(\alpha)) : h \in {}^K(u+1) \right\rangle$ refines $\langle p_i : i < u \rangle$, then there is a $j < u$ such that

$$\forall J \in [(\max(K), \kappa^+)]^{<\omega} \forall f \in {}^J\kappa \left[\bigcap_{\varepsilon \in J} A_{\varepsilon f(\varepsilon)} \cap \{\gamma < \kappa : \delta^\gamma_1 \leq p_j\} \text{ is infinite.} \right]$$

To see that (9) implies (8), assume (9), and let $\langle p_i : i < u \rangle$ be any weak partition of B. By the definition of B, there is a finite subset K of κ^+ such that $p_i \in \langle C \cup \{b(\alpha, j) : \alpha \in K, j \leq u\} \rangle$ for all $i < u$. Let $F = {}^K(u+1)$. Now we claim

(10) $$\langle C \cup \{b(\alpha, j) : \alpha \in K, j \leq u\} \rangle = \Bigg\{ x \in \overline{C} : \text{there is an } d \in {}^F C \text{ such that } x = \sum_{h \in F} \left(d(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) \right) \Bigg\}.$$

In fact, obviously the right side of (10) is a subset of the left, and the right side is a subalgebra (using (7)). So it suffices to show that each $b(\alpha, j)$ is in the right side. Let $F' = \{h \in F : h(\alpha) = j\}$. If $h \in F \backslash F'$, then $b(\alpha, j) \cdot b(\alpha, h(\alpha)) = 0$; hence by (7), $b(\alpha, j) = \sum_{h \in F'} \prod_{\alpha \in K} b(\alpha, h(\alpha))$. So if we define for $h \in F$

$$d(h) = \begin{cases} 1 & \text{if } h \in F', \\ 0 & \text{otherwise,} \end{cases}$$

we get $b(\alpha, j) = \sum_{h \in F} \left(d(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) \right)$. Thus (10) holds.

By (10), for each $i < u$ there is a $d_i \in {}^F C$ such that

$$p_i = \sum_{h \in F} \left(d_i(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) \right).$$

Now for any $h \in F$ let $c_0(h) = d_0(h) + \prod_{j<u} -d_j(h)$, and for $0 < i < u$ let $c_i(h) = d_i(h) \cdot \prod_{j<i} -d_j(h)$.

(11) For any $i < u$ and $h \in F$ we have $p_i \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) = c_i(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha))$.

To prove (11), note that if $h, h' \in F$ with $h \neq h'$, then

$$\prod_{\alpha \in K} b(\alpha, h(\alpha)) \cdot \prod_{\alpha \in K} b(\alpha, h'(\alpha)) = 0,$$

and hence

$$p_i \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) = d_i(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha))$$

for any $i < u$. Hence if $i, j < u$ and $i \neq j$, then

$$0 = p_i \cdot p_j \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) = d_i(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) \cdot d_j(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)).$$

Hence $p_i \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) = d_i(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) \leq d_i(h) \cdot \prod_{j \neq i} -d_j(h)$; hence (11) holds for $0 < i$. Clearly $p_0 \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) \cdot \prod_{j<u} -d_j(h) = 0$, so (11) holds for $i = 0$ too.

By (11) and (7) we have

$$p_i = \sum_{h \in F} \left(c_i(h) \cdot \prod_{\alpha \in K} b(\alpha, h(\alpha)) \right),$$

and for each $h \in F$, the system $\langle c_i(h) : i < u \rangle$ is a weak partition of unity of C.

Now for each $i < u$ let $H_i = \{h \in F : (c_i(h))(\alpha) \neq 1$ for only finitely many $\alpha < \kappa\}$. Note that if $h \in F \backslash H_i$, then $\{\alpha < \kappa : (c_i(h))(\alpha) \neq 1\}$ is infinite, and hence $\{\alpha < \kappa : (c_i(h))\alpha) \neq 0\}$ is finite, since $c_i(h) \in C$. Hence there is at least one i such that $h \in H_i$, as otherwise $\sum_{i<u} c_i(h) \neq 1$. There is at most one, since the $c_i(h)$'s are pairwise disjoint. For each $i < u$ let $p'_i = \sum_{h \in H_i} \prod_{\alpha \in K} b(\alpha, h(\alpha))$.

(12) $\langle p'_i : i < u \rangle$ is a weak partition of B.

In fact, if $i < j < u$, $h \in H_i$, and $k \in H_j$, then $h \neq k$ by the remark preceding (12), and so $\prod_{\alpha \in K} b(\alpha, h(\alpha)) \cdot \prod_{\alpha \in K} b(\alpha, k(\alpha)) = 0$; hence $p'_i \cdot p'_j = 0$. Furthermore, using (7),

$$\begin{aligned} \sum_{i<u} p'_i &= \sum_{i<u} \sum_{h \in H_i} \prod_{\alpha \in K} b(\alpha, h(\alpha)) \\ &= \sum_{h \in F} \prod_{\alpha \in K} b(\alpha, h(\alpha)) = 1. \end{aligned}$$

So (12) holds.

Now by (7), $\langle\prod_{\alpha\in K} b(\alpha, h(\alpha)) : h \in F\rangle$ is a weak partition of B. It clearly refines $\langle p'_i : i < u\rangle$. Hence by (9) we can choose $j < u$ such that

$$\forall J \in [(\max(K), \kappa^+)]^{<\omega}\forall f \in {}^J\kappa \left[\bigcap_{\varepsilon\in J} A_{\varepsilon f(\varepsilon)} \cap \{\gamma < \kappa : \delta_1^\gamma \leq p'_j\} \text{ is infinite.}\right]$$

In particular, $\{\gamma < \kappa : \delta_1^\gamma \leq p'_j\}$ is infinite. For each $h \in H_j$ let $F_h = \{\gamma < \kappa : (c_j(h))(\gamma) \neq 1\}$; so F_h is finite. Choose γ with $\delta_1^\gamma \leq p'_j$ and $\gamma \notin \bigcup_{h\in H_j} F_h$. Thus $(c_j(h))(\gamma) = 1$ for all $h \in H_j$. It follows that

$$\begin{aligned}1 = p'_j(\gamma) &= \sum_{h\in H_j}\left(\prod_{\alpha\in K} b(\alpha, h(\alpha))\right)(\gamma)\\ &= \sum_{h\in H_j}\left((c_j(h)\cdot\prod_{\alpha\in K} b(\alpha, h(\alpha))\right)(\gamma)\\ &\leq p_j(\gamma).\end{aligned}$$

Hence $\delta_1^\gamma \leq p_j$. This is as desired in (8).

Thus it remains only to prove (9), which we do by induction on $|K|$. First suppose that $K = \{\alpha\}$. Now the hypothesis of (9) says in this case that $\langle b(\alpha, i) : i \leq u\rangle$ refines $\langle p_i : i < u\rangle$. Hence there is an $i < u$ such that $b(\alpha, j) + b(\alpha, k) \leq p_i$ for two distinct $j, k \leq u$. Define $\psi : {}^0(u + 1) \to {}^2(u + 1)$ by setting $\psi(0) = (j, k)$. Thus $\psi \in \Psi_\alpha$; say $\chi^\alpha_\beta = \psi$. Take any $\gamma \in A_{\alpha\beta}$. Then

$$\begin{aligned}b(\alpha, j)\cdot\delta_1^\gamma &= \sum\{\delta^\gamma_{a(\alpha,s)} : s < 2, (\chi^\alpha_\beta(0))(s) = j\}\\ &= \sum\{\delta^\gamma_{a(\alpha,s)} : s < 2, (\psi(0))(s) = j\}\\ &= \delta^\gamma_{a(\alpha,0)},\end{aligned}$$

and so $\delta^\gamma_{a(\alpha,0)} \leq b(\alpha, j)$. Similarly, $\delta^\gamma_{a(\alpha,1)} \leq b(\alpha, k)$. It follows that $\delta_1^\gamma = \delta^\gamma_{a(\alpha,0)} + \delta^\gamma_{a(\alpha,1)} \leq p_i$. This is true for any $\gamma \in A_{\alpha\beta}$. Thus if we take any finite $J \subseteq (\alpha, \kappa^+)$ and $f \in {}^J\kappa$, we have that $A_{\alpha\beta} \cap \bigcap_{\varepsilon\in J} A_{\varepsilon f(\varepsilon)}$ is infinite and $\delta_1^\gamma \leq p_i$ for every $\gamma \in A_{\alpha\beta} \cap \bigcap_{\varepsilon\in J} A_{\varepsilon f(\varepsilon)}$, as desired in (9).

For the inductive step, suppose that $|K| = s + 1$ and $\langle p_i : i < u\rangle$ satisfies the hypothesis of (9). Let $K = \{\beta_0, \beta_1, \ldots, \beta_{s-1}, \alpha\}$ with $\beta_0 < \beta_1 < \cdots < \beta_{s-1} < \alpha$, and let $L = \{\beta_0, \beta_1, \ldots, \beta_{s-1}\}$. We now define subsets H_j of ${}^L(u+1)$ for $j < u$ by recursion:

$$\begin{aligned}H_j = \Big\{&h \in {}^L(u + 1) : h \notin \bigcup\nolimits_{k<j} H_k\\ &\text{and } \Big|\Big\{i \leq u : \prod\nolimits_{\beta\in L}(b(\beta, h(\beta)) \cdot b(\alpha, i)) \leq p_j\Big\}\Big| \geq 2\Big\}.\end{aligned}$$

(13) $\bigcup_{j<u} H_j = {}^L(u+1)$.

In fact, by the hypothesis of (9) we have

$$\forall h \in {}^L(u+1)\forall i \leq u \exists j < u \left(\prod_{\beta\in L}(b(\beta, h(\beta)) \cdot b(\alpha, i)) \leq p_j\right);$$

it follows that

$$\forall h \in {}^L(u+1)\exists j < u \left[\left|\left\{i \leq u : \prod_{\beta\in L}(b(\beta, h(\beta)) \cdot b(\alpha, i)) \leq p_j\right\}\right| \geq 2\right],$$

and (13) follows.

Now for each $j < u$ let $q_j = \sum_{h\in H_j} \prod_{\beta\in L} b(\beta, h(\beta))$. Clearly $\langle q_j : j < u\rangle$ is a weak partition of B and $\langle\prod_{\delta\in L} b(\delta, h(\delta)) : h \in {}^L(u+1)\rangle$ refines it. Hence by the inductive hypothesis there is a $j_0 < u$ such that $S \stackrel{\text{def}}{=} \{\gamma < \kappa^+ : \delta_1^\gamma \leq q_{j_0}\}$ meets every member of

$$\left\{\bigcap_{\varepsilon\in J} A_{\varepsilon f(\varepsilon)} : J \in [(\beta_{s-1}, \kappa^+)]^{<\omega}, f \in {}^J\kappa\right\}$$

in an infinite set. Now we define $\psi : {}^L(u+1) \to {}^2(u+1)\backslash\{(0,0), (1,1), \ldots, (u,u)\}$ by setting, for each $h \in {}^L(u+1)$,

$$\psi(h) = \begin{cases} \text{some pair } (j,k) \text{ such that } j \neq k \text{ and} & \\ [b(\alpha, j) + b(\alpha, k)] \cdot \prod_{\beta\in L} b(\beta, h(\beta)) \leq p_{j_0} & \text{if } \prod_{\beta\in L} b(\beta, h(\beta)) \leq q_{j_0}, \\ & \\ (0,1) & \text{otherwise.} \end{cases}$$

This makes sense by the definition of q_{j_0}. Now there is a $\varphi < \kappa$ such that $\chi_\varphi^\alpha = \psi$. Now let J be a finite subset of (α, κ^+), and $f \in {}^J\kappa$. By the definition of S, the set $S \cap A_{\alpha\varphi} \cap \bigcap_{\delta\in J} A_{\delta f(\delta)}$ is infinite. Hence to finish the induction it suffices to take any $\gamma \in S \cap A_{\alpha\varphi} \cap \bigcap_{\delta\in J} A_{\delta f(\delta)}$ and show that $\delta_1^\gamma \leq p_{j_0}$. Since $\sum_{h\in {}^L(u+1)} \prod_{\beta\in L} b(\beta, h(\beta)) = 1$, it suffices to take any $h \in {}^L(u+1)$ such that $\delta_1^\gamma \cdot \prod_{\beta\in L} b(\beta, h(\beta)) \neq 0$ and show that this element is $\leq p_{j_0}$. Now since $\delta_1^\gamma \leq q_{j_0}$, it follows that $\prod_{\beta\in L} b(\beta, h(\beta)) \cap q_{j_0} \neq \emptyset$, hence $\prod_{\beta\in L} b(\beta, h(\beta)) \leq q_{j_0}$, and hence

$$(14) \qquad [b(\alpha, (\psi(h))(0)) + b(\alpha, (\psi(h))(1))] \cdot \prod_{\beta\in L} b(\beta, h(\beta)) \leq p_{j_0}.$$

By the definition of the b's we have

$$\begin{aligned}\delta^\gamma_{a(\alpha,0)} \cdot \prod_{\beta\in L} b(\beta,h(\beta)) &\le b(\alpha,((\chi^\alpha_\beta)(h))(0)) \cdot \prod_{\beta\in L} b(\beta,h(\beta))\\ &= b(\alpha,(\psi(h))(0)) \cdot \prod_{\beta\in L} b(\beta,h(\beta))\\ &\le p_{j_0}.\end{aligned}$$

Similarly, $\delta^\gamma_{a(\alpha,1)} \cdot \prod_{\beta\in L} b(\beta,h(\beta)) \le p_{j_0}$. Hence $\delta^\gamma_1 \cdot \prod_{\beta\in L} b(\beta,h(\beta)) \le p_{j_0}$, as desired.

Thus we have proved that $\mathfrak{r}_u(B) \le \kappa$ and $\mathfrak{r}_{u+1}(B) \ge \kappa^+$. By Proposition 6.18(iii) and Theorem 6.23, $\mathfrak{r}_2(B) = \cdots = \mathfrak{r}_u(B) = \kappa < \kappa^+ = \mathfrak{r}_{u+1}(B) = \mathfrak{r}_{u+2}(B) = \cdots = \mathfrak{r}_2(B)^+$. □

We now give another important theorem from Balcar, Simon [92]. We call a set $X \subseteq A^+$ *finitely weakly dense* iff for every finite weak partition P of A there exist an $x \in X$ and a $p \in P$ such that $x \le p$.

Proposition 6.27. *Let A be an infinite BA and $X \subseteq A$. Then X is finitely weakly dense in A* iff *X is dense in some ultrafilter on A.*

Proof. $\Leftarrow$: Suppose that F is any ultrafilter on A and X is dense in F. We claim that X is finitely weakly dense. For, suppose that P is a finite weak partition. Then there is a $p \in P$ such that $p \in F$, and then there is an $x \in X$ such that $x \le p$, as desired.

$\Rightarrow$: Suppose that X is finitely weakly dense. We will find an ultrafilter F such that X is dense in F, thereby finishing the proof of the proposition. For each finite weak partition P let $w_P = \sum\{p \in P : \exists x \in X^+[x \le p]\}$. We claim that $\{w_P : P$ a finite weak partition of $A\}$ has the fip. For, suppose that $P_0, \ldots, P_{n-1}$ are finite weak partitions, $n > 0$. Let

$$Q = \left\{\prod_{i<n} x_i : x \in \prod_{i<n} P_i\right\} \backslash \{\emptyset\}.$$

Thus Q is a finite weak partition which refines each P_i. Choose $x \in X^+$ and $q \in Q$ such that $x \le q$. Clearly $x \le \prod_{i<n} w_{P_i}$. This proves the claim. Let F be an ultrafilter containing $\{w_P : P$ a finite weak partition of $A\}$. To show that X is dense in F, take any $a \in F$. Then $w_{a,-a} \in F$, and so also $a \cdot w_{a,-a} \in F$. By the definition of $w_{a,-a}$ this means that there is an $x \in X$ such that $x \le a$, as desired. □

Theorem 6.28. *For any infinite BA A we have*

$$\begin{aligned}\sup_{\substack{n\in\omega,\\ n\ge 2}} \mathfrak{r}_n(A) &= \min\{|X| : X \subseteq A^+, \text{ and } X \text{ is finitely weakly dense in } A\}\\ &= \pi\chi_{\inf}(A).\end{aligned}$$

Proof. Suppose that X is finitely weakly dense in A of smallest size. Then for any $n \in \omega$ with $n \geq 2$, the set X is n-dense in A. Hence $\mathfrak{r}_n(A) \leq |X|$.

Now for each $n \in \omega$ with $n \geq 2$, let X_n be n dense with $|X_n| = \mathfrak{r}_n(A)$. Let $Y = \bigcup_{n\in\omega, n\geq 2} X_n$. Clearly Y is finitely weakly dense in A. Together with the preceding paragraph, this proves the first equality of the theorem. The second equality is immediate from Proposition 6.27. □

Corollary 6.29. *For any infinite BA A there is an $m \in \omega$ such that $\pi\chi_{\text{inf}}(A) = \mathfrak{r}_m(A)$.*

Proof. Immediate from Theorems 6.23 and 6.28. □

It is natural to consider arbitrary weak partitions as well as finite ones, in generalizing denseness. We call $X \subseteq A$ *infinitely weakly dense* iff for every weak partition Y in A there exist $x \in X^+$ and $y \in Y$ such that $x \leq y$.

Proposition 6.30.

$$\min\{|X| : X \textit{ is infinitely weakly dense in } A\} = \min\{\pi(A \restriction a) : a \in A^+\}.$$

Proof. Suppose that X is infinitely weakly dense in A, of smallest size, and suppose that $|X| < \min\{\pi(A \restriction a) : a \in A^+\}$. Then for all $a \in A^+$, $X \cap (A \restriction a)$ is not dense in $A \restriction a$; so there is a nonzero $b \leq a$ such that there is no $x \in X$ such that $0 \neq x \leq b$. Thus the set $\{b \in A :$ there is no $x \in X$ such that $0 \neq x \leq b\}$ is dense in A. Let P be a weak partition with elements in this set. Choose $b \in P$ and $x \in X^+$ such that $x \leq b$; this contradicts the choice of P. Hence $\geq$ holds in our proposition.

Conversely, let $\kappa = \min\{\pi(A \restriction a) : a \in A^+\}$, choose $a \in A^+$ such that $\kappa = \pi(A \restriction a)$, and choose $X \subseteq (A \restriction a)$ dense in $A \restriction a$ and of size κ. Suppose that Y is a weak partition of A. Choose $y \in Y$ such that $y \cdot a \neq 0$, and choose $x \in X^+$ with $x \leq y \cdot a$. This shows that X is infinitely weakly dense in A. □

We now consider algebraic operations and the reaping numbers $\mathfrak{r}_m$. As with density, one can have $A \leq B$ with $\mathfrak{r}_m(A) < \mathfrak{r}_m(B)$ or $\mathfrak{r}_m(B) < \mathfrak{r}_m(A)$. If A is an infinite homomorphic image of B, then $\mathfrak{r}_m(A) < \mathfrak{r}_m(B)$ is obviously possible. The other inequality is possible also: for $B = \mathscr{P}(\omega)$ and $A = B/\text{fin}$ we have $\mathfrak{r}_m(B) = 1$ while $\mathfrak{r}_m(A) \geq \omega_1$, as we now prove.

Proposition 6.31. *Let $A = \mathscr{P}(\omega)/\text{fin}$. Then $\mathfrak{r}(A) \geq \omega_1$.*

Proof. Let $I = \text{fin}$. Suppose that $X \subseteq A^+$ is weakly dense and countable. Say $X = \{[x(i)]_I : i < \omega\}$. Now we define $a_i, b_i \in \omega$ for $i < \omega$ by recursion. Let $f : \omega \to \omega \times \omega$ be a bijection. Then pick

$$a_i \in x(1^{st}(f(i)))\backslash(\{a_j : j < i\} \cup \{b_j : j < i\});$$
$$b_i \in x(1^{st}(f(i)))\backslash(\{a_j : j \leq i\} \cup \{b_j : j < i\}).$$

Let $y = \{a_i : i < \omega\}$. Choose $j < \omega$ such that $[x(j)]_I \leq [y]_I$ or $[x(j)]_I \leq -[y]_I$.

Case 1. $[x(j)]_I \leq [y]_I$. So $x(j)\backslash y$ is finite. But for any $k \in \omega$, if $i = f^{-1}(j, k)$, then $b_i \in x(j)\backslash y$, so $x(j)\backslash y$ is infinite, contradiction.

Case 2. $[x(j)]_I \leq -[y]_I$. so $x(j) \cap y$ is finite, and we obtain a contradiction as in Case 1, using a_i's. $\square$

One might want algebras A, B with B atomless, A a homomorphic image of B, and $\mathfrak{r}(B) < \mathfrak{r}(A)$. We can modify the proof of Proposition 6.31 to obtain such algebras:

Proposition 6.32. *Let $B = {}^{\omega}\mathrm{Fr}(\omega)$, and let*

$$I = \{b \in B : \{i \in \omega : b(i) \neq 0\} \textit{ is finite}\}.$$

Let $A = B/I$. Then $\omega = \mathfrak{r}(B) < \mathfrak{r}(A)$.

Proof. Clearly $\mathfrak{r}(B) = \omega$. Now suppose that $X \subseteq A^+$ is weakly dense and countable. Say $X = \{[x(i)]_I : i < \omega\}$. Now we define $a_i, b_i \in \omega$ for $i < \omega$ by recursion. Let $f : \omega \to \omega \times \omega$ be a bijection. Now $x(1^{st}(f(i))) \notin I$, and so we can choose $a_i \in \omega\backslash(\{a_j : j < i\} \cup \{b_j : j < i\})$ so that $(x(1^{st}(f(i))))(a_i) \neq 0$. Then choose $b_i \in \omega\backslash(\{a_j : j \leq i\} \cup \{b_j : j < i\})$ so that $(x(1^{st}(f(i))))(b_i) \neq 0$. Now we define $y \in B$ by

$$y(j) = \begin{cases} 1 & \text{if } \ j = a_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Choose $j < \omega$ such that $[x(j)]_I \leq [y]_I$ or $[x(j)]_I \leq -[y]_I$.

Case 1. $[x(j)]_I \leq [y]_I$. Thus $x(j) \cdot -y \in I$. But for any $k \in \omega$, if $i = f^{-1}(j, k)$, then $(x(j) \cdot -y)(b_i) \neq 0$, so $x(j) \cdot -y \notin I$, contradiction.

Case 2. $[x(j)]_I \leq -[y]_I$. Similarly, using a_i's. $\square$

Proposition 6.33. *For any $m \in \omega\backslash 2$ and any system $\langle A_i : i \in I\rangle$ of infinite BAs we have $\mathfrak{r}_m(\prod_{i\in I} A_i) = \min_{i\in I} \mathfrak{r}_m(A_i)$.*

Proof. Let $\min_{i\in I} \mathfrak{r}_m(A_i) = \mathfrak{r}_m(A_j)$, and let X be m-dense in A_j of size $\mathfrak{r}_m(A_j)$. Define $x_a \in \prod_{i\in I} A_i$ for $a \in X$ by setting

$$x_a(i) = \begin{cases} a & \text{if } \ i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\langle b_k : k < m\rangle$ is a weak partition of $\prod_{i\in I} A_i$. Then $\langle b_k(j) : k < m\rangle$ is a weak partition of A_j. Choose $a \in X^+$ and $k < m$ such that $a \leq b_k(j)$. Then $x_a \leq b_k$. Thus $\{x_a : a \in X\}$ is m-dense in $\prod_{i\in I} A_i$, proving $\leq$.

Now suppose that $X \subseteq \prod_{i\in I} A_i$, $|X| < \mathfrak{r}_m(A_j)$, and X is m-dense in $\prod_{i\in I} A_i$. Then for each $i \in I$ we have $|\{a(i) : a \in X\}| < \mathfrak{r}_m(A_i)$, and so $\{a(i) : a \in X\}$ is not m-dense. So there is a weak m-partition $\langle b_{ik} : k < m\rangle$ of A_i such that there do not exist $a \in X$ and $k < m$ such that $0 \neq a(i) \leq b_{ik}$. Define $c_k = \langle b_{ik} : i \in I\rangle$ for each $k < m$. Clearly $\langle c_k : k < m\rangle$ is a weak partition of $\prod_{i\in I} A_i$ Choose $k < m$ and $a \in X^+$ such that $a \leq c_k$. Say $a(i) \neq 0$. Then $0 \neq a(i) \leq b_{ik}$, contradiction. $\square$

Proposition 6.34. *If $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, then*

$$\pi\chi_{\rm inf}\left(\prod_{i\in I} A_i\right) \le \min_{i\in I}\pi\chi_{\rm inf}(A_i) \le \left(\pi\chi_{\rm inf}\left(\prod_{i\in I} A_i\right)\right)^+.$$

Proof. For all $n \in \omega\backslash 2$ we have $\mathfrak{r}_n(\prod_{i\in I} A_i) = \min_{i\in I}\mathfrak{r}_n(A_i) \le \min_{i\in I}\pi\chi_{\rm inf}(A_i)$, and so the first inequality follows.

Next, choose $m \in \omega\backslash 2$ such that $\pi\chi_{\rm inf}(\prod_{i\in I} A_i) = \mathfrak{r}_m(\prod_{i\in I} A_i)$, and then choose i_0 so that $\mathfrak{r}_m(\prod_{i\in I} A_i) = \mathfrak{r}_m(A_{i_0})$. Thus

(1) $\mathfrak{r}_m(A_{i_0}) = \pi\chi_{\rm inf}(\prod_{i\in I} A_i)$.

Then

$$\min_{i\in I}\pi\chi_{\rm inf}(A_i) \le \pi\chi_{\rm inf}(A_{i_0}) \le (\mathfrak{r}_m(A_{i_0}))^+ = \left(\pi\chi_{\rm inf}\left(\prod\nolimits_{i\in I} A_i\right)\right)^+. \qquad \square$$

According to Proposition 6.34, $\min_{i\in I}\pi\chi_{\rm inf}(A_i)$ is either equal to $\pi\chi_{\rm inf}\left(\prod_{i\in I} A_i\right)$ or to $\left(\pi\chi_{\rm inf}\left(\prod_{i\in I} A_i\right)\right)^+$. Both cases are possible. Thus if all A_i are equal to some atomless algebra B, then $\pi\chi_{\rm inf}\left(\prod_{i\in I} A_i\right) = \min_{i\in I}\pi\chi_{\rm inf}(A_i)$. This is true since by Corollary 6.29 there is an $m \ge 2$ such that $\pi\chi_{\rm inf}(B) = \mathfrak{r}_m(B)$, by Proposition 6.33 $\mathfrak{r}_m(\prod_{i\in I} A_i) = \mathfrak{r}_m(B)$, and by Proposition 6.34 $\pi\chi_{\rm inf}(\prod_{i\in I} A_i) \le \pi\chi_{\rm inf}(B) = \mathfrak{r}_m(B) = \mathfrak{r}_m(\prod_{i\in I} A_i) \le \pi\chi_{\rm inf}(\prod_{i\in I} A_i)$. On the other hand, for each $n \in \omega\backslash 2$ let A_n be a BA with $\mathfrak{r}_n(A_n) = \omega$ while $\mathfrak{r}_{n+1}(A_n) = \omega_1$. Then $\min_{n\in\omega\backslash 2}\pi\chi_{\rm inf}(A_n) = \left(\pi\chi_{\rm inf}\left(\prod_{n\in\omega\backslash 2} A_n\right)\right)^+$. In fact, for any $n \in \omega\backslash 2$ we have $\pi\chi_{\rm inf}(A_n) \ge \mathfrak{r}_{n+1}(A_n) = \omega_1$, so that $\min_{n\in\omega\backslash 2}\pi\chi_{\rm inf}(A_n) = \omega_1$. Also, there is an $m \in \omega\backslash 2$ such that

$$\pi\chi_{\rm inf}\left(\prod\nolimits_{n\in\omega\backslash 2} A_n\right) = \mathfrak{r}_m\left(\prod\nolimits_{n\in\omega\backslash 2} A_n\right),$$

and

$$\mathfrak{r}_m\left(\prod\nolimits_{n\in\omega\backslash 2} A_n\right) = \min\nolimits_{n\in\omega\backslash 2}\mathfrak{r}_m(A_n) = \mathfrak{r}_m(A_m) = \omega.$$

Proposition 6.35. *If $\langle A_i : i \in I\rangle$ is a system of atomless BAs, with I infinite, then $\pi\chi_{\rm inf}(\prod^{\rm w}_{i\in I} A_i) = \omega$.*

Proof. Let J be a countably infinite subset of I, and for each $j \in J$ let a^j be the element of $\prod^{\rm w}_{i\in I} A_i)$ which is 1 at the jth position and 0 otherwise, and let $X = \{a^j : j \in J\}$. Let F be the ultrafilter on $\prod^{\rm w}_{i\in I} A_i)$ consisting of all elements x such that $\{k \in I : x(k) \ne 1\}$ is finite. Clearly X is dense in F. $\square$

Kevin Selker has shown that Proposition 6.35 also holds for moderate products.

Peterson [98] has determined how reaping numbers act under free products; we state these results without proof. $\pi\chi_{\rm inf}(A \oplus B) = \max(\pi\chi_{\rm inf}(A), \pi\chi_{\rm inf}(B))$. If $\pi\chi_{\rm inf}(A) = \pi\chi_{\rm inf}(B)$, then $\pi\chi_{\rm inf}(A \oplus B) = \mathfrak{r}_2(A \oplus B)$.

In the same article, Peterson also investigates derived functions of the reaping numbers.

We now consider another variation of the reaping number. A subset X of a BA A is *homogeneously weakly dense* iff for any two distinct $a, b \in A$ with $a \cdot b = 0$ there is an $x \in X$ such that $(x \cdot a \neq 0 = x \cdot b)$ or $(x \cdot a = 0 \neq x \cdot b)$. Note that neither a nor b is required to be nonzero. We define

$$\mathrm{hwd}(A) = \min\{|X| : X \subseteq A \text{ is homogeneously weakly dense in } A\}.$$

We also define $\mathrm{rel}(X, a) = \{x \in X : x \cdot a \neq 0\}$, the *relevant points* of X for a.

Proposition 6.36. $\mathrm{hwd}(A)$ *is infinite, for every infinite BA A.*

Proof. Suppose that X is finite and homogeneously weakly dense in A. Let $B = \langle X \rangle$. Then B is finite. Let a be an atom of B such that $A \restriction a$ is infinite. Let b and c be nonzero disjoint elements of $A \restriction a$. Choose $x \in X$ so that $x \cdot b \neq 0 = x \cdot c$ or $x \cdot c \neq 0 = x \cdot b$. This contradicts a being an atom of B. $\square$

Proposition 6.37. *For any infinite BA A and any $X \subseteq A$, X is weakly dense in A iff $\forall a \in A[\mathrm{rel}(X, a) \neq \mathrm{rel}(X, -a)]$.*

Proof. First suppose that X is weakly dense in A. Take any $a \in A$. Choose $x \in X^+$ such that $x \leq a$ or $x \leq -a$. Then $x \in \mathrm{rel}(X, a) \triangle \mathrm{rel}(X, -a)$.

Conversely assume the condition of the proposition, and suppose that $a \in A$. Choose $x \in \mathrm{rel}(X, a) \triangle \mathrm{rel}(X, -a)$. Then $x \in X^+$, and $x \leq a$ or $x \leq -a$. $\square$

Proposition 6.38. *For any infinite BA A and any $X \subseteq A$ the following conditions are equivalent:*

(i) *X is homogeneously weakly dense in A.*

(ii) *$X \restriction a$ is weakly dense in $A \restriction a$ for every $a \in A^+$.*

(iii) *For every $a \in A^+$ and every $b \leq a$ there is an $x \in X$ such that $x \cdot a \neq 0$, and $x \cdot a \leq b$ or $x \cdot a \leq -b$.*

(iv) $\mathrm{rel}(X, a) \neq \mathrm{rel}(X, b)$ *for all disjoint $a, b \in A$ with $a + b \neq 0$.*

Proof. (i)⇒(ii): Assume that X is homogeneously weakly dense in A, and $a \in A^+$. Take any $b \leq a$. If $b = a$, we apply the definition of homogeneous weak density to the pair $0, a$ and get $x \in X$ such that $x \cdot 0 \neq 0 = x \cdot a$ or $x \cdot 0 = 0 \neq x \cdot a$. Thus $x \cdot a \neq 0$. So $x \cdot a$ is a nonzero element of $X \restriction a$ which is below b, since $b = a$. If $b \neq a$, choose $x \in X$ such that $x \cdot b \neq 0 = x \cdot a \cdot -b$ or $x \cdot b = 0 \neq x \cdot a \cdot -b$. Then $x \cdot a \neq 0$, and $x \cdot a \leq b$ or $x \cdot a \leq a \cdot -b = -^{A \restriction a} b$.

(ii)⇒(iii): Assume (ii), and suppose that $a \in A^+$ and $b \leq a$. Choose $x \in X$ such that $x \cdot a \neq 0$, and $x \cdot a \leq b$ or $x \cdot a \leq -^{A \restriction a} b$. thus $x \cdot a \leq b$ or $x \cdot a \leq -b$.

(iii)⇒(iv): Assume (iii), and suppose that $a \cdot b = 0$ with $a + b \neq 0$. Choose $x \in X$ such that $x \cdot (a + b) \neq 0$ and $x \cdot (a + b) \leq a$ or $x \cdot (a + b) \leq -a$. So $(x \cdot b = 0$ and $x \cdot a \neq 0)$ or $(x \cdot a = 0$ and $x \cdot b \neq 0)$. Thus $x \in \mathrm{rel}(X, a) \triangle \mathrm{rel}(X, b)$.

(iv)$\Rightarrow$(i): Assume (iv), and suppose that $a \neq b$ with $a \cdot b = 0$. Thus $a+b \neq 0$. By (iv) we have $\mathrm{rel}(X,a) \neq \mathrm{rel}(X,b)$. By symmetry say that $x \in \mathrm{rel}(X,a) \backslash \mathrm{rel}(X,b)$. Thus $x \cdot a \neq 0 = x \cdot b$. $\square$

It would be natural to generalize hwd like ordinary reaping was generalized to the numbers $\mathfrak{r}_n$. By the following result, nothing new is obtained in the case of hwd. Also, the value of hwd is not changed if we require the elements to be nonzero (a result of Jennifer Brown [05]).

Proposition 6.39. *Suppose that $n \in \omega \backslash 2$, A is an infinite BA, and $X \subseteq A$. Let $\varphi(n,X)$ and $\psi(n,X)$ be the following two statements:*

(1) $\varphi(n,X)$: *For every system $\langle a_i : i < n \rangle$ of pairwise disjoint elements with at least one nonzero, there exist $i < n$ and $x \in X$ such that $x \cdot a_i \neq 0$ and $\forall j \neq i[x \cdot a_j = 0]$.*

(2) $\psi(n,X)$: *For every system $\langle a_i : i < n \rangle$ of nonzero pairwise disjoint elements, there exist $i < n$ and $x \in X$ such that $x \cdot a_i \neq 0$ and $\forall j \neq i[x \cdot a_j = 0]$.*

Then for each integer $n \geq 2$, $\mathrm{hwd}(A) = \min\{|X| : \varphi(n,X)\} = \min\{|X| : \psi(n,X)\}$.

Proof. Let X be homogeneously weakly dense. So X is infinite, by Proposition 6.36. Let Y be the closure of X under finite products. We prove $\varphi(n,Y)$ by induction on n. The case $n = 2$ is given. Now suppose that the condition holds for n, and $\langle a_i : i \leq n \rangle$ is a system of pairwise disjoint elements with at least one nonzero. let $b = \sum_{i<n} a_i$. Since X is homogeneously weakly dense, choose $x \in X$ such that one of the following conditions holds: (1) $x \cdot a_n \neq 0 = x \cdot b$. (2) $x \cdot a_n = 0 \neq x \cdot b$. Now (1) gives the desired conclusion. So suppose that (2) holds.

Then $\langle x \cdot a_i : i < n \rangle$ is a system of pairwise disjoint elements, with at least one nonzero. So by the inductive hypothesis choose $y \in X$ and $i < n$ such that $y \cdot x \cdot a_i \neq 0 = y \cdot x \cdot a_j$ for all $j < n$. Clearly this is as desired. This proves that $\mathrm{hwd}(A) \geq \min\{|X| : \varphi(n,X)\}$.

Clearly $\varphi(n,X)$ implies $\psi(n,X)$. Hence $\min\{|X| : \varphi(n,X)\} \geq \min\{|X| : \psi(n,X)\}$.

Now we prove that $\psi(n,X)$ implies that Y is homogeneously weakly dense for some Y with $|Y| = |X|$, by induction on n.

First we note that X is infinite, by the proof of Proposition 6.36. Suppose that $n = 2$. Set $Y = X \cup \{1\}$. Let a and b be different disjoint elements of A. If both are nonzero, then the desired conclusion is clear. If one is 0, the element 1 of Y can be used.

Now suppose that $\varphi(n-1,X)$ implies that Y is homogeneously weakly dense for some Y with $|Y| = |X|$, and assume $\varphi(n,X)$. Suppose that $a_0, \ldots, a_{n-2}$ are $n-1$ nonzero pairwise disjoint elements of A.

Case 1. There is an $i < n-1$ such that $A \restriction a_i$ has more than one element. Write $a_i = b + c$ with b and c nonzero and disjoint. Applying $\varphi(n,X)$ to the sequence $a_0, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_{n-2}$, we get an element which clearly works for $a_0, \ldots, a_{n-2}$ as well.

Case 2. Suppose that each a_i is an atom for $i < n-1$. Suppose

$$\neg \exists i < n-1 \exists x \in X[x \cdot a_i \neq 0 \wedge \forall j \neq i[x \cdot a_j = 0]].$$

Thus $\varphi(n-1, X)$ fails to hold. Now we claim

(1) Either $\varphi(n-1, Y)$ holds for some Y with $|Y| = |X|$, or there are $n-1$ atoms $b_0, \ldots, b_{n-2}$ such that $\{a_0, \ldots, a_{n-2}\} \cap \{b_0, \ldots, b_{n-2}\} = \emptyset$ and

$$\neg \exists i < n-1 \exists x \in X[x \cdot b_i \neq 0 \wedge \forall j \neq i[x \cdot b_j = 0]].$$

In fact, suppose that such atoms do not exist. Let $X' = X \cup \{a_0, \ldots, a_{n-1}\}$. Suppose now that $b_0, \ldots, b_{n-2}$ are $n-1$ nonzero pairwise disjoint elements of A. If one of them is not an atom, then a desired element is obtained as in Case 1. Suppose that all of them are atoms. If $\{a_0, \ldots, a_{n-2}\} \cap \{b_0, \ldots, b_{n-2}\} = \emptyset$, then the desired conclusion follows from the assumption concerning (1). Suppose that $b_i = a_j$. Then $b_i \in X'$ and it is a desired element for $b_0, \ldots, b_{n-2}$. Thus $\varphi(n-1, X')$ holds, as desired.

Thus (1) holds.

Assume that $\varphi(n-1, Y)$ does not hold, for any Y such that $|Y| = |X|$. Choose $b_0, \ldots, b_{n-2}$ by (1). Consider the n nonzero pairwise disjoint elements $c_0 = a_0 + b_0$, $c_1 = a_1 + b_1$, $\ldots$, $c_{n-3} = a_{n-3} + b_{n-3}$, $c_{n-2} = a_{n-2}$, $c_{n-1} = b_{n-2}$. Choose $i < n$ and $x \in X$ such that $x \cdot c_i \neq 0$ while $x \cdot c_j = 0$ for all $j \neq i$. If $i = n-2$, then $x \cdot a_{n-2} \neq 0$ while $x \cdot a_j = 0$ for all $j \neq n-2$, contradicting our assumption on the a_k's. A similar contradiction is reached if $i = n-1$. So $i < n-2$. Thus $x \cdot (a_i + b_i) \neq 0$, while $x \cdot c_j = 0$ for all $j \neq i$. Hence $x \cdot a_j = 0$ for all $j \neq i$, and $x \cdot b_j = 0$ for all $j \neq i$. So our assumption on the a_k's, or on the b_k's, is contradicted.

What this means is that the initial assumption of this case is false, and this completes the proof. □

Corollary 6.40. $\pi\chi_{\text{inf}}(A) \leq \text{hwd}(A)$ *for every infinite BA A.*

Proof. Let $n \in \omega \backslash 2$. By Proposition 6.39, let $X \subseteq A$ be of size $\text{hwd}(A)$ which is n-dense. Thus $\mathfrak{r}_n(A) \leq \text{hwd}(A)$. Hence Proposition 6.28 gives the desired result. □

Proposition 6.41. $\text{Depth}(A) \leq \text{hwd}(A)$ *for every infinite BA A.*

Proof. Suppose that $X \subseteq A$ with $|X| < \text{Depth}(A)$; we show that X is not homogeneously weakly dense. Let $\langle a_\alpha : \alpha < |X|^+ \rangle$ be a strictly increasing sequence of elements of A. Let

$$Y = \{x \in X : \forall \alpha < |X|^+ \exists \beta \in (\alpha, |X|^+)[x \cdot a_\beta \cdot -a_\alpha \neq 0]\},$$

and let $Z = X\backslash Y$. Then for each $x \in Z$ there is an $\alpha_x < |X|^+$ such that $\forall\beta \in (\alpha_x, |X|^+)[x \cdot a_\beta \cdot -a_{\alpha_x} = 0]$. Let $\beta_0 = \sup_{x\in Z} \alpha_x$. Then

(1) For all $x \in Z$, if $\beta_0 \le \beta < \gamma < |X|^+$, then $x \cdot a_\gamma \cdot -a_\beta = 0$.

In fact, $x \cdot a_\gamma \cdot -a_\beta \le x \cdot a_\gamma \cdot -a_{\alpha_x} = 0$.

(2) $\forall\alpha < |X|^+ \exists\beta \in (\alpha, |X|^+)\forall x \in Y[x \cdot a_\beta \cdot -a_\alpha \ne 0]$.

In fact, for any $x \in Y$ choose $\gamma_x \in (\alpha, |X|^+)$ such that $x \cdot a_{\gamma_x} \cdot -a_\alpha \ne 0$. Let $\beta = \sup_{x\in Y} \gamma_x$. Clearly (2) holds for this β.

Now apply (2) with α replaced by β_0 to obtain $\beta_1 \in (\beta_0, |X|^+)$ such that $\forall x \in Y[x \cdot a_{\beta_1} \cdot -a_{\beta_0} \ne 0]$; then apply (2) with α replaced by β_1 to obtain $\beta_2 \in (\beta_1, |X|^+)$ such that $\forall x \in Y[x \cdot a_{\beta_2} \cdot -a_{\beta_1} \ne 0]$. Let $c = a_{\beta_1} \cdot -a_{\beta_0}$ and $d = a_{\beta_2} \cdot -a_{\beta_1}$. Then $c \cdot d = 0$, and $c \ne 0 \ne d$. Then by (1), $\forall x \in Z[c \cdot x = 0 = d \cdot x]$, and by the choice of β_1 and β_2, $\forall x \in Y[c \cdot x \ne 0 \ne d \cdot x]$. Hence X is not homogeneously weakly dense. □

Proposition 6.42. *If κ is regular and there is an isomorphic embedding of $\mathscr{P}(\kappa)$ into A preserving all sums, then $\kappa \le \mathrm{hwd}(A)$.*

Proof. Suppose that $X \in [A]^{<\kappa}$; we want to show that X is not homogeneously weakly dense. Let f be the assumed isomorphic embedding. For each $x \in X$ let $D_x = \{\alpha < \kappa : f(\{\alpha\}) \cdot x \ne 0\}$. Now define

$$Y = \{x \in X : |D_x| < \kappa\}; \quad Z = X\backslash Y; \quad J = \bigcup_{x\in Y} D_x.$$

Thus $|J| < \kappa$ and $\forall x \in Z[|D_x| = \kappa]$. Write $Z = \{z_\gamma : \gamma < \delta\}$, where $\delta < \kappa$. Now we can choose by induction, for each $\gamma < \delta$,

$$\begin{aligned} &\alpha^0_\gamma \in D_{z_\gamma}\backslash(J \cup \{\alpha^1_\beta : \beta < \gamma\}),\\ &\alpha^1_\gamma \in D_{z_\gamma}\backslash(J \cup \{\alpha^0_\beta : \beta \le \gamma\}). \end{aligned}$$

Now let $I_0 = \{\alpha^0_\gamma : \gamma < \delta\}$ and $I_1 = \{\alpha^1_\gamma : \gamma < \delta\}$. Then $I_0 \cap I_1 = \emptyset = I_0 \cap J = I_1 \cap J$. Let $a_0 = f(I_0)$ and $a_1 = f(I_1)$. Since $I_0 \cap I_1 = \emptyset$, we have $a_0 \cdot a_1 = 0$. If $x \in Y$, then $D_x \subseteq J$, hence $I_0 \cap D_x = \emptyset$, so $I_0 \subseteq \{\alpha < \kappa : f(\{\alpha\}) \cdot x = 0\}$. Hence

$$\begin{aligned} x \cdot a_0 &= x \cdot f(I_0)\\ &\le x \cdot f(\{\alpha < \kappa : f(\{\alpha\}) \cdot x = 0\})\\ &= x \cdot f\left(\bigcup\{\{\alpha\} : \alpha < \kappa, f(\{\alpha\}) \cdot x = 0\}\right)\\ &= x \cdot \sum\{f(\{\alpha\}) : \alpha < \kappa, f(\{\alpha\}) \cdot x = 0\}\\ &= \sum\{x \cdot f(\{\alpha\}) : \alpha < \kappa, f(\{\alpha\}) \cdot x = 0\}\\ &= 0. \end{aligned}$$

Similarly, $x \cdot a_1 = 0$.

On the other hand, if $x \in Z$, then $x = z_\gamma$ for some $\gamma < \delta$, and so $\alpha^0_\gamma \in I_0 \cap D_x$. Hence $0 \neq f(\{\alpha^0_\gamma\}) \cdot x \leq f(I_0) \cdot x = a_0 \cdot x$. So $a_0 \cdot x \neq 0$, and similarly $a_1 \cdot x \neq 0$.

This proves that X is not homogeneously weakly dense. $\square$

Proposition 6.43. $\text{hwd}(A \times B) = \max(\text{hwd}(A), \text{hwd}(B))$ *for any infinite BAs* A, B.

Proof. Suppose that $X \subseteq A \times B$ with $|X| < \text{hwd}(A)$. Then $\{1^{st}(x) : x \in X\}$ is a subset of A of size less than $\text{hwd}(A)$, so there exist disjoint $a_0, a_1 \in A$ with $a_0 \neq a_1$ such that $\forall x \in X[1^{st}(x) \cdot a_0 = 0 \text{ iff } 1^{st}(x) \cdot a_1 = 0]$. Hence $(a_0, 0)$ and $(a_1, 0)$ are distinct disjoint elements of $A \times B$ such that $\forall x \in X[x \cdot a_0 = 0 \text{ iff } x \cdot a_1 = 0]$. This proves that $\text{hwd}(A) \leq \text{hwd}(A \times B)$. Similarly for B, so $\geq$ in the proposition holds.

Now let $X \subseteq A$ be homogeneously weakly dense in A, with $|X| = \text{hwd}(A)$, and let $Y \subseteq B$ be homogeneously weakly dense in B, with $|Y| = \text{hwd}(B)$. Let $Z = \{(x, 0) : x \in X\} \cup \{(0, y) : y \in Y\}$. For $\leq$ of the proposition it suffices to show that Z is homogeneously weakly dense in $A \times B$. So, suppose that $(a_0, b_0), (a_1, b_1)$ are distinct disjoint elements of $A \times B$.

Case 1. $a_0 \neq 0$. Then a_0, a_1 are distinct and disjoint, so we can choose $x \in X$ such that $x \cdot a_0 \neq 0 = x \cdot a_1$ or $x \cdot a_0 = 0 \neq x \cdot a_1$. Hence $(x, 0) \cdot (a_0, b_0) \neq (0, 0) = (x, 0) \cdot (a_1, b_1)$ or $(x, 0) \cdot (a_0, b_0) = (0, 0) \neq (x, 0) \cdot (a_1, b_1)$, as desired.

Case 2. $a_1 \neq 0$. Symmetric to Case 1.

Case 3. $a_0 = 0 = a_1$. Then $b_0 \neq 0$ or $b_1 \neq 0$, and the arguments of Cases 1, 2 apply. $\square$

Proposition 6.44. *Suppose that* I *is infinite and* $\langle A_i : i \in I \rangle$ *is a system of infinite BAs. Then* $\text{hwd}(\prod_{i \in I} A_i) = \max\{|I|, \sup_{i \in I} \text{hwd}(A_i)\}$.

Proof. $\text{hwd}(\prod_{i \in I} A_i) \geq \sup_{i \in I} \text{hwd}(A_i)$ by Proposition 6.43. Also, $\text{hwd}(\prod_{i \in I} A_i) \geq |I|$ by Proposition 6.42. Thus $\geq$ holds.

For the other direction, for each $i \in I$ let $X_i \subseteq A_i$ be homogeneously weakly dense in A_i, with $|X_i| = \text{hwd}(A_i)$. For each $i \in I$ and $x \in X_i$ define $y^x_i \in \prod_{j \in I} A_j$ by

$$y^x_i(j) = \begin{cases} x & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y \stackrel{\text{def}}{=} \{y^x_i : i \in I, \ x \in X_i\}$ has size $\sum_{i \in I} |X_i| = \max\{|I|, \sup_{i \in I} \text{hwd}(A_i)\}$, and it is homogeneously weakly dense. In fact, if u, v are distinct disjoint elements of $\prod_{i \in I} A_i$, say by symmetry that $i \in I$ with $u_i \neq 0$. Then u_i and v_i are distinct disjoint elements of A_i, so there is an $x \in X_i$ such that $u_i \cdot x \neq 0 = v_i \cdot x$ or $u_i \cdot x = 0 \neq v_i \cdot x$. Then $u \cdot y^x_i \neq 0 = v \cdot y^x_i$ or $u \cdot y^x_i = 0 \neq v \cdot y^x_i$. $\square$

We state some further results from Peterson [98] without proof.

- If $\langle A_i : i \in I \rangle$ is a system of infinite BAs, with I infinite, then $\text{hwd}(\prod^{\text{w}}_{i \in I} A_i) = \max\{|I|, \sup_{i \in I} \text{hwd}(A_i)\}$.
- If A and B are infinite BAs, then $\text{hwd}(A \oplus B) = \max\{\text{hwd}(A), \text{hwd}(B)\}$.

- If $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, then $\mathrm{hwd}(\oplus_{i\in I}A_i) = \max\{|I|, \sup_{i\in I}\mathrm{hwd}(A_i)\}$.

We now consider density and the above related notions for complete BAs.

Proposition 6.45. *Let A be a an infinite complete BA, and X a subset of A closed under products. Then the following conditions are equivalent:*

(i) *X is dense in some ultrafilter on A.*
(ii) *X is weakly dense in A.*

Proof. (i)$\Rightarrow$(ii) is obvious; it is also part of Proposition 6.27.

For (ii)$\rightarrow$(i), suppose that X is not dense in any ultrafilter; we show that X is not weakly dense. By Proposition 6.27, X is not finitely weakly dense. Hence there is an $m \in \omega\backslash 2$ and a weak partition $\langle a_i : i < m\rangle$ such that no member of X^+ is below any a_i. For each $x \in X^+$ let $I_x = \{i < m : a_i \cdot x \neq 0\}$. Thus each set I_x has at least two elements, and $I_y \leq I_x$ if $y \leq x$. Let

$$Y = \{y \in X^+ : \forall z \in X^+[z \leq y \rightarrow [I_z = I_y]]\}.$$

Now Y is dense in X. For, suppose that $x \in X^+$. Let $y \in X^+$ be such that $y \leq x$, and $|I_y|$ is smallest among all $|I_z|$ with $z \in X^+$ and $z \leq x$. Then $y \in Y$, as desired.

Let D be a maximal disjoint set contained in Y. Then

$(*)$ $\sum D = \sum X$.

In fact, $D \subseteq X$, so $\leq$ holds. Suppose that $<$ holds. Then $\sum_{x\in X}(x \cdot -\sum D) = \sum X \cdot -\sum D \neq 0$, so there is an $x \in X$ such that $x \cdot -\sum D \neq 0$. Choose $y \in Y^+$ such that $y \leq x$. Then $y \cdot d = 0$ for all $d \in D$, contradicting the maximality of D. Hence $(*)$ holds.

Now for each $d \in D$ let $\varphi(d) < m$ be smallest such that $d \cdot a_{\varphi(d)} \neq 0$. Let $u = \sum_{d\in D}(d \cdot a_{\varphi(d)})$. We claim that there does not exist an $x \in X^+$ such that $x \leq u$ or $x \leq -u$, thereby showing that X is not weakly dense.

So, take any $x \in X^+$. By $(*)$, choose $d \in D$ such that $x \cdot d \neq 0$. Now $d \in Y$, so $I_{x\cdot d} = I_d$. Since $d \cdot a_{\varphi(d)} \neq 0$, it follows that $x \cdot d \cdot a_{\varphi(d)} \neq 0$, and so $x \cdot u \neq 0$. But d is not below $a_{\varphi(d)}$, so there is an $i < m$ with $i \neq \varphi(d)$ such that $d \cdot a_i \neq 0$. Hence from $I_{x\cdot d} = I_d$ we get $x \cdot d \cdot a_i \neq 0$. Now the disjointness of D and of $\langle a_j : j < m\rangle$ implies that $d \cdot a_i \cdot u = 0$. Hence $x \cdot -u \neq 0$. □

Theorem 6.46. *If A is an infinite complete BA, then $\mathfrak{r}(A) = \pi\chi_{\mathrm{inf}}(A)$.*

Proof. Let X be weakly dense in A, with $|X| = \mathfrak{r}(A)$. Let Y be the closure of X under products. By Proposition 6.45, Y is dense in some ultrafilter on A, so by Proposition 6.27, Y is finitely weakly dense in A. Then by Theorem 6.28, $\pi\chi_{\mathrm{inf}}(A) \leq |Y|$. Thus our theorem follows from Theorem 6.28. □

We now work towards a result of Bozeman [91] which relates hwd to π.

A subset X of a BA A is *nowhere relatively dense* in A iff for all $a \in A^+$, $X \restriction a$ is not dense in $A \restriction a$.

Proposition 6.47. *X is nowhere relatively dense in B iff for all $a \in B^+$ there is a nonzero $d \leq a$ such that $\mathrm{rel}(X,d) \subseteq \mathrm{rel}(X, a \cdot -d)$.*

Proof. $\Rightarrow$: Given $a \in B^+$, we know that $X \restriction a$ is not dense in $B \restriction a$. Hence there is a nonzero $d \leq a$ such that for all $x \in X$, if $x \cdot a \neq 0$ then $x \cdot a \not\leq d$. Suppose that $x \in \mathrm{rel}(X,d)$. Thus $x \cdot d \neq 0$. Hence $x \cdot a \neq 0$, so $x \cdot a \not\leq d$, hence $x \cdot a \cdot -d \neq 0$, and $x \in \mathrm{rel}(X, a \cdot -d)$.

$\Leftarrow$. Let $a \in B^+$. Choose d as indicated. We claim that $X \restriction a$ is not dense in $B \restriction a$, in fact, $\forall x \in X(x \cdot a \neq 0 \Rightarrow x \cdot a \not\leq d)$. For, suppose that $x \in X$, $x \cdot a \neq 0$, while $x \cdot a \leq d$. Thus $x \in \mathrm{rel}(X,d) \backslash \mathrm{rel}(X, a \cdot -d)$, contradiction. □

Proposition 6.48. *Suppose that B is complete and X is weakly dense in B. Then there is a $c \in B^+$ such that $X \restriction c$ is homogeneously weakly dense in $B \restriction c$.*

Proof. Suppose, on the contrary, that $\forall c \in B^+(X \restriction c$ is not homogeneously weakly dense in $B \restriction c)$. Hence by Proposition 6.38, $Y \stackrel{\text{def}}{=} \{a \in B^+ : X \restriction a$ is not weakly dense in $B \restriction a\}$ is dense in B. Let A be a maximal disjoint set $\subseteq Y$. Thus $\sum A = 1$. For each $a \in A$ we have $X \restriction a$ not weakly dense in $B \restriction a$. Thus by Proposition 6.37, for every $a \in A$ there is a $b_a \leq a$ such that $\mathrm{rel}(X \restriction a, b_a) = \mathrm{rel}(X \restriction a, -^{B\restriction a} b_a)$. Hence

$(*)$ $\mathrm{rel}(X, b_a) = \mathrm{rel}(X, a \cdot -b_a)$.

In fact, let $x \in X$. Then

$$\begin{aligned}
x \in \mathrm{rel}(x, b_a) \quad &\text{iff} \quad x \cdot b_a \neq 0 \\
&\text{iff} \quad x \cdot a \cdot b_a \neq 0 \\
&\text{iff} \quad x \cdot a \in \mathrm{rel}(X \restriction a, b_a) \\
&\text{iff} \quad x \cdot a \in \mathrm{rel}(X \restriction a, -^{B\restriction a} b_a) \\
&\text{iff} \quad x \cdot a \cdot -^{B\restriction a} b_a \neq 0 \\
&\text{iff} \quad x \cdot a \cdot -b_a \neq 0 \\
&\text{iff} \quad x \in \mathrm{rel}(X, a \cdot -b_a).
\end{aligned}$$

It follows that for any $x \in X$,

$$\begin{aligned}
x \cdot \sum_{a \in A} b_a \neq 0 \quad &\text{iff} \quad \exists a \in A[x \cdot b_a \neq 0] \\
&\text{iff} \quad \exists a \in A[x \cdot a \cdot -b_a \neq 0] \\
&\text{iff} \quad x \cdot \sum_{a \in A} (a \cdot -b_a) \neq 0.
\end{aligned}$$

But $\sum_{a \in A} b_a + \sum_{a \in A}(a \cdot -b_a) = \sum A = 1$ and $\left(\sum_{a \in A} b_a\right) \cdot \left(\sum_{a \in A}(a \cdot -b_a)\right) = 0$, so that $\sum_{a \in A} b_a = -\sum_{a \in A}(a \cdot -b_a)$. This contradicts X being weakly dense in B. □

A BA B is hwd-*homogeneous* iff $\mathrm{hwd}(B \restriction a) = \mathrm{hwd}(B)$ for every $a \in B^+$; cf. the Handbook, page 198.

Proposition 6.49.

(i) *If* $a \in B$, *then* $\mathrm{hwd}(B \restriction a) \leq \mathrm{hwd}(B)$.

(ii) $\{a \in B : B \restriction a$ *is* hwd-*homogeneous*$\}$ *is dense in* B.

(iii) *Any complete BA is isomorphic to a product of complete* hwd-*homogeneous BAs.*

Proof. By Proposition 6.43. □

Proposition 6.50. *If B is complete and* hwd-*homogeneous, then* $\mathrm{hwd}(B) = \mathfrak{r}(B)$.

Proof. Let X be weakly dense in B with $|X| = \mathfrak{r}(B)$. By Proposition 6.48 choose $c \in B^+$ such that $X \restriction c$ is homogeneously weakly dense in $B \restriction c$. Then $\mathrm{hwd}(B) = \mathrm{hwd}(B \restriction c) \leq |X \restriction c| \leq |X| = \mathfrak{r}(B) \leq \mathrm{hwd}(B)$ using Proposition 6,38, and so $\mathrm{hwd}(B) = \mathfrak{r}(B)$. □

Theorem 6.51. *Let B be a complete hwd-homogeneous BA. Suppose that $X = \{x_\tau : \tau < \kappa\}$ is homogeneously weakly dense in B, with each x_τ nonzero, and $|X| = \mathrm{hwd}(B) = \kappa$. For all $\rho < \kappa$ let $Y_\rho = \{x_\tau : \tau \leq \rho\}$. We call $W \subseteq X$* **bounded** *if W is included in some Y_ρ. Let $E = \{\sum W : W$ is a bounded subset of $X\}$. Let D be the subalgebra of B generated by E.*

The conclusion is that there is an $a \in B^+$ such that $D \restriction a$ is dense in $B \restriction a$.

Proof. Suppose not. Thus D is nowhere relatively dense in B. Now, we claim,

(1) For all $\rho < \kappa$ and all $a \in B^+$ there exist c, A, f such that c is a nonzero element $\leq a$, A is a disjoint subset of $B \restriction (a \cdot -c)$, $f : A \to \kappa$, and the following conditions hold:

(i) $\sup f[\mathrm{rel}(A, p)] = \kappa$ for all $p \in \mathrm{rel}(X, c)$.

(ii) $\mathrm{rel}(Y_\rho, c) = \mathrm{rel}(Y_\rho, \sum A)$.

(iii) $\sum A = a \cdot -c$.

To prove (1), fix $\rho < \kappa$ and $a \in B^+$. By Proposition 6.47 choose a nonzero $s \leq a$ such that $\mathrm{rel}(D, s) \subseteq \mathrm{rel}(D, a \cdot -s)$. Let $U = \{x \in X : x \cdot s = 0\}$, and let $u = a \cdot \sum U$. Now $s \cdot \sum U = 0$, so $s \leq a \cdot -u$. If $x \in U$, then $x \cdot a \leq a \cdot \sum U = u$, hence $x \cdot a \cdot -u \cdot -s = 0$ and $x \notin \mathrm{rel}(X, a \cdot -u \cdot -s)$. Thus

(2) $\mathrm{rel}(X, -u \cdot a \cdot -s) \subseteq \mathrm{rel}(X, s)$.

Now by the homogeneous weak density of X, $X \restriction (a \cdot -u)$ is weakly dense in $B \restriction (a \cdot -u)$. Hence there is a $q \in X$ such that $q \cdot a \cdot -u \neq 0$ and $q \cdot a \cdot -u \leq s$ or $q \cdot a \cdot -u \leq a \cdot -u \cdot -s$. But the latter is impossible, since otherwise $q \in \mathrm{rel}(X, s)$ by (2), hence $q \cdot s \neq 0$; since $s \leq a \cdot -u$, this implies that $a \cdot -u \cdot -s \geq q \cdot a \cdot -u \geq q \cdot s \neq 0$, which is impossible. Thus

(3) $q \cdot a \cdot -u \leq s$.

Now we make some definitions. Let $c = q \cdot a \cdot -u$, $T = \{x \in X : x \cdot c = 0\}$, $t = a \cdot \sum T$. For all $\alpha < \kappa$ let $T_\alpha = Y_\alpha \cap T$ and $t_\alpha = a \cdot \sum T_\alpha$. Since T_α is bounded, we have $t_\alpha \in (D \restriction a)$. Note that $U \subseteq T$ since $c \leq s$, and hence $u \leq t$, so

(4) $a \cdot -t \leq a \cdot -u$.

Define

$$t'_\alpha = \begin{cases} t_\alpha \cdot -\sum_{\tau<\alpha} t_\tau & \text{if } t_\alpha \cdot -\sum_{\tau<\alpha} t_\tau \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

$$A = \{t'_\alpha : \alpha < \kappa\}$$

$$f(t'_\alpha) = \begin{cases} \alpha & \text{if } \ t'_\alpha \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have defined some objects c, A, and f, and we now show that they satisfy (i) and (ii) of (1). Clearly $0 \neq c \leq a$, A is a disjoint subset of $B \restriction (a \cdot -c)$, and $f : A \to \kappa$.

We prove (1)(i) by contradiction. Suppose that $p \in \text{rel}(X, c)$ and $\sup f[\text{rel}(A, p)] < \kappa$. Thus $p \cdot c \neq 0$ and there is an $\alpha < \kappa$ such that for all $\beta > \alpha$, $p \cdot t_\beta \cdot -\sum_{\tau<\beta} t_\tau = 0$. Then

(5) $p \cdot t_\beta \leq t_\alpha$ for all $\beta < \kappa$.

We prove (5) by induction on β. Since clearly $t_\beta \leq t_\alpha$ when $\beta \leq \alpha$, we only need to consider $\beta > \alpha$. Assume that $p \cdot t_\gamma \leq t_\alpha$ for all $\gamma < \beta$, where $\alpha < \beta < \kappa$. Then $p \cdot t_\beta \leq \sum_{\tau<\beta}(p \cdot t_\tau) \leq t_\alpha$, as desired. So (5) holds. Hence

(6) $p \cdot t = p \cdot t_\alpha$.

Now, using (3) and (4),

$$\begin{aligned} p \cdot q \cdot a &= p \cdot q \cdot a \cdot \sum T + p \cdot q \cdot a \cdot -\sum T \\ &= p \cdot q \cdot t + p \cdot q \cdot a \cdot -t \\ &\leq p \cdot q \cdot t_\alpha + p \cdot q \cdot a \cdot -u \\ &\leq p \cdot q \cdot t_\alpha + s. \end{aligned}$$

Let $w = p \cdot q \cdot a \cdot -t_\alpha$. Since $p, q \in D$ and $t_\alpha \in (D \restriction a)$, we have $w \in (D \restriction a)$. By the displayed calculation above we have $w \leq s$. Now $t \cdot c = 0$, and $0 \neq p \cdot c = p \cdot q \cdot a \cdot -u$, so $p \cdot q \cdot a \not\leq t$. Hence, using (6), $p \cdot q \cdot a \cdot -t = p \cdot q \cdot a \cdot -t_\alpha \neq 0$, i.e., $w \neq 0$. So $p \cdot q \cdot -t_\alpha \in \text{rel}(D, s)$ but $p \cdot q \cdot -t_\alpha \notin \text{rel}(D, a \cdot -s)$, contradicting the choice of s. This proves (1)(i).

For (1)(ii), first suppose that $\tau \leq \rho$ and $x_\tau \in \text{rel}(Y_\rho, c)$. Then $x_\tau \in \text{rel}(X, c)$, so by (1)(i) we get $x_\tau \cdot \sum A \neq 0$, and hence $x_\tau \in \text{rel}(Y_\rho, \sum A)$. Conversely, suppose that $\pi \leq \rho$ and $x_\pi \in \text{rel}(Y_\rho, \sum A)$. Then by the definition of A there is an $\alpha > \rho$ such that $x_\pi \cdot t_\alpha \cdot -\sum_{\tau<\alpha} t_\tau \neq 0$. Now $x_\pi \notin T_\pi$, since otherwise $x_\pi \cdot t_\alpha \leq a \cdot \sum T_\pi = t_\pi$, a contradiction since $\pi < \alpha$. Since $Y_\pi \cap T = T_\pi$ and $x_\pi \in Y_\pi$, it follows that $x_\pi \notin T$, hence $x_\pi \cdot c \neq 0$, i.e., $x_\pi \in \text{rel}(Y_\rho, c)$. So (1)(ii) holds.

Thus we have constructed c, A, and f so that (1)(i) and (1)(ii) hold. We will obtain objects satisfying all three conditions (1)(i)–(iii) by applying Zorn's lemma. Let $\mathscr{P}$ consist of all triples (c, A, f) such that (1)(i) and (1)(ii) hold. Partially order $\mathscr{P}$ by setting $(c, A, f) \leq (c', A', f')$ iff $c \leq c'$ and $\forall t \in A \exists t' \in A'[t \leq t'$ and $f(t) = f'(t')]$. Suppose that Q is a chain in $\mathscr{P}$. Let $c' = \sum_{(c,A,f)\in Q} c$. For each $\alpha < \kappa$ let $h_\alpha = \sum_{(c,A,f)\in Q} \sum f^{-1}[\{\alpha\}]$. If α and β are distinct ordinals less than κ, then $h_\alpha \cdot h_\beta = 0$: for assume that $(c, A, f) \in Q$ and $(c_1, A_1, f_1) \in Q$; we check that $\sum f^{-1}[\{\alpha\}] \cdot \sum f_1^{-1}[\{\beta\}] = 0$. To do this, suppose that $x \in f^{-1}[\{\alpha\}]$ and $y \in f_1^{-1}[\{\beta\}]$; we check that $x \cdot y = 0$. By symmetry say $(c, A, f) \leq (c_1, A_1, f_1)$. Choose $z \in A_1$ so that $x \leq z$ and $f(x) = f_1(z)$. Since $f(x) = \alpha$ and $f_1(y) = \beta$ we have $z \neq y$, and hence $z \cdot y = 0$. So $x \cdot y = 0$, as desired. Hence $A' \stackrel{\text{def}}{=} \{h_\alpha : \alpha < \kappa\}$ is disjoint. Define $f' : A' \to \kappa$ by

$$f'(h_\alpha) = \begin{cases} \alpha & \text{if } h(\alpha) \neq 0, \\ 0 & \text{if } \ h(\alpha) = 0. \end{cases}$$

We check (1)(i) for (c', A', f'): Suppose that $p \in \text{rel}(X, c')$. Choose $(c, A, f) \in Q$ so that $p \in \text{rel}(X, c)$. Thus by (1)(i) for (c, A, f) we have $\sup f[\text{rel}(A, p)] = \kappa$. We claim that $\sup f'[\text{rel}(A', p)] = \kappa$. For, given $\alpha < \kappa$, choose $t \in \text{rel}(A, p)$ such that $\alpha < f(t)$. Let $\beta = f(t)$. Thus $t \in f^{-1}[\{\beta\}]$, and hence $t \leq h_\beta$. So $h_\beta \in \text{rel}(A', p)$, and $f'(h_\beta) = \beta$, as desired.

Next we check (1)(ii) for (c', A', f'): Suppose that $\tau \leq \rho$ and $x_\tau \in \text{rel}(Y_\rho, c')$. Choose $(c, A, f) \in Q$ so that $x_\tau \in \text{rel}(Y_\rho, c)$. By (1)(ii) for (c, A, f) we have $x_\tau \in \text{rel}(Y_\rho, \sum A)$. So there is a $t \in A$ such that $x_\tau \cdot t \neq 0$. Say $f(t) = \alpha$. Then $t \leq h_\alpha$, so $x_\tau \in \text{rel}(Y_\rho, \sum A')$. The converse is similar, so we have proved (1)(ii).

Clearly $(c, A, f) \leq (c', A', f')$ for all $(c, A, f) \in Q$. So at this point we have checked the hypotheses of Zorn's lemma; the first part of the proof of (1) was to show that $\mathscr{P} \neq 0$. By Zorn's lemma, let (c, A, f) be a maximal element of $\mathscr{P}$. We will now show that (1)(iii) holds, completing the proof of (1). In fact, if it does not hold, then $a_1 \stackrel{\text{def}}{=} a \cdot -c \cdot -\sum A$ is nonzero. Now we apply the first part of the proof of (1) to ρ and a_1 to get a nonzero $c_1 \leq a_1$, a disjoint subset A_1 of $B \upharpoonright (a_1 \cdot -c_1)$, and a function $f_1 : A_1 \to \kappa$ such that $\forall p \in \text{rel}(X, c_1)[\sup f_1[\text{rel}(A_1, p)] = \kappa]$ and $\text{rel}(Y_\rho, c_1) = \text{rel}(Y_\rho, \sum A_1)$. Then $(c, A, f) < (c + c_1, A \cup A_1, f \cup f_1)$, contradiction. This finishes the proof of (1).

We now construct by recursion a sequence $\langle c_i : i < \omega\rangle$ of nonzero elements of B such that the following conditions hold for all $n < \omega$:

(7) $c_n \cdot \sum_{k<n} c_k = 0$.

(8) $\text{rel}\left(X, \sum_{k<n} c_k\right) \subseteq \text{rel}(X, c_n)$.

(9) There exist a system A of nonzero pairwise disjoint elements of $B \upharpoonright -\sum_{k\leq n} c_k$ and a function $f : A \to \kappa$ such that $\forall x \in \text{rel}\left(X, \sum_{k\leq n} c_k\right) [\sup f[\text{rel}(A, x)] = \kappa]$.

Applying (1) with anything for ρ and 1 for a, we get a nonzero c_0 so that (9) holds, while (7) and (8) vacuously hold. Now suppose that c_k has been constructed for every $k \leq n$. In particular, by (9) we have A and f as indicated there. Now for each nonzero $a \in A$ we apply (1) to $f(a)$ in place of ρ to get a nonzero $s_a \leq a$, a system C_a of nonzero pairwise disjoint elements of $B \upharpoonright (a \cdot -s_a)$, and a function $g_a : C_a \to \kappa$ so that the following conditions hold:

(10) $\sup g_a[\mathrm{rel}(C_a, p)] = \kappa$ for all $p \in \mathrm{rel}(X, s_a)$.

(11) $\mathrm{rel}(Y_{f(a)}, s_a) = \mathrm{rel}(Y_{f(a)}, \sum C_a)$.

(12) $\sum C_a = a \cdot -s_a$.

Note:

(13) If $a \in A$, $\tau \leq f(a)$, and $x_\tau \cdot a \neq 0$, then $x_\tau \cdot s_a \neq 0$.

This follows from (11) and (12).

Now let $c_{n+1} = \sum_{a \in A} s_a$. Then (7) clearly holds for $n+1$. To prove (8) for $n+1$, suppose that $x_\tau \cdot \sum_{k \leq n} c_k \neq 0$. By (9) for n, choose $a \in A$ such that $a \cdot x_\tau \neq 0$ and $\tau \leq f(a)$. By (13) we have $x_\tau \cdot s_a \neq 0$. Hence $x_\tau \cdot c_{n+1} \neq 0$, as desired: (8) holds for $n+1$.

Now let $D = \bigcup_{a \in A} C_a$, and $h = \bigcup_{a \in A} g_a$. Thus D is a collection of nonzero pairwise disjoint elements of B, and $h : B \to \kappa$. Clearly in fact $D \subseteq B \upharpoonright -\sum_{k \leq n+1} c_k$. Now suppose that $x_\tau \in \mathrm{rel}\left(X, \sum_{k \leq n+1} c_k\right)$.

(14) There is an $a \in A$ such that $x_\tau \cdot s_a \neq 0$.

In fact,

$$\sum_{k \leq n+1} c_k = \sum_{a \in A} s_a + \sum_{k \leq n} c_k,$$

so from $x_\tau \in \mathrm{rel}\left(X, \sum_{k \leq n+1} c_k\right)$ we see that we have two cases.

Case 1. There is an $a \in A$ such that $x_\tau \cdot s_a \neq 0$. So (14) holds.

Case 2. $x_\tau \cdot \sum_{k \leq n} c_k \neq 0$. Then by the choice of A and f, there is an $a \in A$ such that $a \cdot x_\tau \neq 0$ and $f(a) \geq \tau$. Hence by (13), condition (14) again holds.

Now by (14), $x_\tau \in \mathrm{rel}(X, s_a)$, and so by (10), $\sup g_a[\mathrm{rel}(C_a, x_\tau)] = \kappa$. Since

$$g_a[\mathrm{rel}(C_a, x_\tau)] = h[\mathrm{rel}(C_a, x_\tau)] \subseteq h[\mathrm{rel}(D, x_\tau)],$$

it follows that $\sup h[\mathrm{rel}(D, x_\tau)] = \kappa$. Hence (9) holds, and we have finished the construction of $\langle c_i : i < \omega \rangle$.

Note that (8) implies that $\mathrm{rel}(X, c_n) \subseteq \mathrm{rel}(X, c_m)$ whenever $n < m$. It follows that $\mathrm{rel}\left(X, \sum_{n<\omega} c_{2n}\right) = \mathrm{rel}\left(X, \sum_{n<\omega} c_{2n+1}\right)$. But this contradicts the assumption that X is homogeneously weakly dense. $\square$

Proposition 6.52. *If B is an infinite complete BA which is* hwd- *and π-homogeneous, then $\pi(B) \leq \min\{2^{<\mathfrak{r}(B)}, \sup\{\lambda^\kappa : \kappa < \mathrm{c}'(B),\ \lambda < \mathfrak{r}(B)\}\}$.*

Proof. Let D and a be as in Theorem 6.51. Thus $\pi(B) = \pi(B \restriction a) \leq |D|$. Then the conclusion of our proposition follows by making two computations of $|D|$. First, for each $\rho < \mathfrak{r}(B)$ we take subsets of Y_ρ; this gives $|D| \leq 2^{<\mathfrak{r}(B)}$, using Proposition 6.50. Second, we note that for each bounded subset W of X we have $\sum W = \sum Y$, where Y is maximal disjoint such that $\forall y \in Y \exists w \in W[y \leq w]$. □

Proposition 6.53. (GCH) *If B is an infinite complete BA which is* hwd- *and π-homogeneous, then $\mathfrak{r}(B) = \mathrm{hwd}(B) = \pi\chi_{\mathrm{inf}}(B) = \pi\chi(B) = \pi(B)$.*

Proof. We use Propositions 6.50 and 6.52:

$$\mathfrak{r}(B) \leq \pi\chi_{\mathrm{inf}}(B) \leq \pi\chi(B) \leq \pi(B) \leq \mathfrak{r}(B) = \mathrm{hwd}(B).$$ □

Proposition 6.54. (GCH) *If B is an atomless complete BA, then $\pi\chi(B) = \pi(B)$.*

Proof. Let $B \cong \prod_{i\in I} C_i$, where each C_i is hwd- and π-homogeneous and I is infinite. Then, using the remark after Problem 74 and Proposition 6.53,

$$\pi(B) = \max(|I|, \sup_{i\in I} \pi(C_i)) = \max(|I|, \sup_{i\in I} \pi\chi(C_i)) \leq \pi\chi(B) \leq \pi(B);$$

we also used the easily verified fact that $\max(|I|, \sup_{i\in I} \pi\chi(C_i)) \leq \pi\chi(B)$. □

The following problem remains; this was problem 40 in Monk [96].

Problem 74. *Can one prove in ZFC that $\pi\chi(B) = \pi(B)$ for any atomless complete BA?*

This finishes our treatment of weak density and related notions.

Concerning the function π_{Ss} we mention the following result from Shelah [96]:

Theorem 6.55. *Let B be an infinite BA, and suppose that θ is an infinite regular cardinal less than $\pi_{\mathrm{S}+}(B)$. Then there is a subalgebra A of B such that $|A| = \pi(A) = \theta$.*

Proof. Passing to a subalgebra of B if necessary, we may assume that $\omega < \theta \leq \pi(B)$. We define a sequence $\langle A_\alpha : \alpha < \theta\rangle$ of subalgebras of B, each of power less than θ, as follows. Let A_0 be any denumerable subalgebra of B. For α a limit ordinal $< \theta$, let A_α be the union of preceding algebras. If A_α has been defined, then, since it has fewer than $\pi(B)$ elements, there is a nonzero element $b \in B$ such that for all $x \in A_\alpha^+$ we have $x \not\leq b$. Let A_α be the subalgebra of B generated by $A_\alpha \cup \{b\}$. Clearly $\bigcup_{\alpha<\theta} A_\alpha$ is as desired. □

Thus $\pi_{\mathrm{Ss}}(A)$ contains all regular cardinals in the interval $[\omega, \pi_{\mathrm{S+}}(A))$. But Shelah also showed in that paper that it is consistent to have a BA A with some of the singular cardinals in that interval not in $\pi_{\mathrm{Ss}}(A)$; and some special singular cardinals in that interval are always in $\pi_{\mathrm{Ss}}(A)$. These results answer problem 15 in Monk [90].

We turn to π_{Sr}. First we note some easy facts:

(1) In general, $|A| \le 2^{\pi(A)}$, since every member of A is the join of a subset X when X is dense in A. Hence $(\kappa, \lambda) \in \pi_{\mathrm{Sr}}(A)$ implies that $\lambda \le 2^\kappa$.

(2) If $(\kappa, \lambda) \in \pi_{\mathrm{Sr}}(A)$ and $\omega \le \mu \le \lambda$, then there is some ν such that $\omega \le \nu \le \mu$ and $(\nu, \mu) \in \pi_{\mathrm{Sr}}(A)$.

(3) If $(\kappa, \lambda) \in \pi_{\mathrm{Sr}}(A)$, then $(\kappa, \kappa) \in \pi_{\mathrm{Sr}}(A)$. In fact, if X is dense in B with $|X| = \pi(B)$, and if $Y \subseteq \langle X \rangle$ is dense in $\langle X \rangle$, then Y is also dense in B, hence $|Y| = |X|$.

(4) If $(\kappa, \lambda) \in \pi_{\mathrm{Sr}}(A)$ and $\omega \le \mu \le \kappa$ with μ regular, then $(\mu, \mu) \in \pi_{\mathrm{Sr}}(A)$. This is clear from Proposition 6.63 and its proof.

(5) $(\omega, \omega) \in \pi_{\mathrm{Sr}}(A)$ for any infinite BA A.

As for other functions, we have the following vague question.

Problem 75. *Characterize π_{Sr} in cardinal number terms.*

Many problems exist concerning the π_{Sr} relation for small algebras, so we do not survey this area.

Concerning π_{Hr}, we first mention the following general facts. We begin with an extension of Theorem 6.55.

Theorem 6.56. *Let θ be an infinite regular cardinal less than $\pi_{S+}(B)$. Then B has a homomorphic image C with $\pi(C) = \theta$ and $|C| \le \theta^{\le \mathrm{c}(B)}$.*

Proof. By Theorem 6.55, let D be a subalgebra of B such that $|D| = \pi(D) = \theta$. By Sikorski's extension theorem there is a homomorphism from B onto an algebra C such that $D \le C \le \overline{D}$. Thus $\pi(C) \le \theta$ and $|C| \le \theta^{\le \mathrm{c}(B)}$. Suppose that X is dense in C, with $|X| < \theta$. For each $x \in X^+$ we have $x \in \overline{D}$, and so there is a $d_x \in D^+$ such that $d_x \le x$. If $c \in D^+$, choose $x \in X^+$ such that $x \le c$. Then $d_x \le c$. Thus $\{d_x : x \in x\}$ is dense in D and has size less than θ, contradiction. □

(1) If $(\kappa, \lambda) \in \pi_{\mathrm{Hr}}(A)$, then $\lambda \le 2^\kappa$.

(2) For any infinite BA A there is a κ such that $(\omega, \kappa) \in \pi_{\mathrm{Hr}}(A)$.

(3) (See Koppelberg [77]) Assuming MA, if A is an infinite BA and $|A| < 2^\omega$ then A has a countable homomorphic image.

We have the usual problem concerning homomorphic spectrum:

Problem 76. *Characterize π_{Hr} in cardinal number terms.*

Many problems arise in considering $\pi_{\mathrm{Hr}}(A)$ for small A, so we will not try to survey them.

One can see that $\mathrm{d} \leq \pi$ by the following argument. Let A be a BA, and let $X \subseteq A^+$ be dense in A with $|X| = \pi(A)$. For each $x \in X$ let F_x be an ultrafilter on A such that $x \in F_x$. For each $a \in A$ let $f(a) = \{F_x : a \in F_x\}$. It is clear that f is an isomorphism from A into $\mathscr{P}(Y)$, where $Y = \{F_x : x \in X\}$. Thus $\mathrm{d}(A) \leq \pi(A)$ by Theorem 5.1.

The difference between $\mathrm{d}(A)$ and $\pi(A)$ is small, however, since $\mathrm{d}(A) \leq \pi(A) \leq |A| \leq 2^{\mathrm{d}(A)}$ for any infinite BA A.

About π for special classes of BAs, note that $\pi(A) = \mathrm{d}(A)$ for any interval algebra A; in fact, $\pi(A)$ is also equal to $\mathrm{hd}(A)$. To see this, note that $\mathrm{d}(A) = \mathrm{hd}(A)$ for A an interval algebra, since any interval algebra is retractive. Now $\pi(A) \leq \pi_{\mathrm{H}+}(A) = \mathrm{hd}(A)$ by Theorem 6.15, and $d(A) \leq \pi(A)$, so $d(A) = \pi(A)$.

Another interesting fact about π and interval algebras was observed by Douglas Peterson: $\pi(\mathrm{Intalg}(L)) = \mathrm{d}(L) \cdot |M|$, where $\mathrm{d}(L)$ is the density of L as a topological space and M is the set of atoms of $\mathrm{Intalg}(L)$. To prove $\leq$, let X be dense in L with $|X| = \mathrm{d}(L)$; we show that $\{[x, y) : x, y \in X, x < y\} \cup M$ is dense in $\mathrm{Intalg}(L)$. Take any nonzero $a \in \mathrm{Intalg}(L)$. Wlog a has the form $[u, v)$. If $[u, v)$ is finite, then $b \leq [u, v)$ for some atom b. If $[u, v)$ is infinite, then there exist $x, y \in X$ with $u < x < y < v$, and so $[x, y) \subseteq [u, v)$. For $\geq$, suppose to the contrary that R is dense in $\mathrm{Intalg}(L)$ and $|R| < \mathrm{d}(L) \cdot |M|$. Clearly $M \subseteq R$. Wlog each member of R has the form $[a, b)$. Since $|M| \leq |R|$, we have $|R| < \mathrm{d}(L)$. Let $R' = \{a \in L : \exists b([a, b) \in R)\}$. Thus $L \backslash \overline{R'}$ is a non-empty open set. Say $w \in (u, v) \subseteq L \backslash \overline{R'}$. Then $[w, v) \in \mathrm{Intalg}(L)$, so $[a, b) \subseteq [w, v)$ for some $[a, b) \in R$; but then $a \in R'$, contradiction.

If A is a minimally generated algebra, then $\pi(A) = \mathrm{d}(A)$. In fact, A is co-complete with an interval algebra B, $\pi(B) = \pi(\overline{B})$, and $\mathrm{d}(B) = \mathrm{d}(\overline{B})$ for any BA B, so this follows from the interval algebra result.

For A atomic, clearly $\pi(A)$ is the number of atoms of A. Also note that $\pi_{\mathrm{S}+}(A) = |A|$ for A complete, and $\pi_{\mathrm{S}+}(A) = \mathrm{hd}(A)$ for A retractive.

If A is the completion of the free BA on ω_1 free generators, then $\mathrm{d}(A) < \pi(A)$: clearly $\pi(A) = \omega_1$. The identity mapping from the free BA on ω_1 free generators into $\mathscr{P}(\omega)$ can be extended to a homomorphism f from A into $\mathscr{P}(\omega)$, and f must be one-one; so $\mathrm{d}(A) = \omega$.

For tree algebras we recall from Theorem 5.25 that $\pi(A) = |A|$. (Using $d(A) \leq \pi(A)$.)

7 Length

Recall that Length(A) is the sup of cardinalities of subsets of A which are simply ordered by the Boolean ordering. For references see the beginning of Chapter 4. The analysis of Length is similar to that for Depth. We begin by discussing of Length, for which we need two small lemmas about orderings:

Lemma 7.1. *Let L be a linear ordering of regular cardinality λ which has no strictly increasing or strictly decreasing sequences of length λ. Then there exist $a < b$ in L such that $|[a, b)| = \lambda$.*

Proof. If L has both a first and a last element, this is trivial. Suppose that L has a first element a, but no last element. Let $\langle b_\xi : \xi < \kappa\rangle$ be strictly increasing and cofinal in L. Then $\kappa < \lambda$ by assumption, and $L = \bigcup_{\xi<\kappa}[a, b_\xi)$, so there is a $\xi < \kappa$ such that $|[a, b_\xi)| = \lambda$; then a, b_ξ are as desired. Other possibilities – last but no first element, neither a first nor a last element – are treated similarly. $\square$

Lemma 7.2. *Let L be a linear ordering with first element 0, and with cardinality κ^+, where κ is infinite. Then there exist $a < b$ in L such that $|[a, b)| \geq \kappa$ and $|L\backslash[a, b)| \geq \kappa$.*

Proof. Suppose not. Then clearly

(1) in L there is no strictly increasing or strictly decreasing sequence of length κ^+.

We define by induction a sequence $\langle[a_\xi, b_\xi) : \xi < \kappa^+\rangle$ of half-open intervals in L such that $[a_\eta, b_\eta) \subset [a_\xi, b_\xi)$ for $\xi < \eta$, $|[a_\xi, b_\xi)| = \kappa^+$, and $|L\backslash[a_\xi, b_\xi)| < \kappa$ for all $\xi < \alpha$. By Lemma 7.1, choose $a_0 < b_0$ such that $|[a_0, b_0)| = \kappa^+$. Then $|L\backslash[a_0, b_0)| < \kappa$ by our assumption that the Lemma fails. If $[a_\xi, b_\xi)$ has been defined, then $[a_{\xi+1}, b_{\xi+1})$ can be defined: choose c with $a_\xi < c < b_\xi$; then $|[a_\xi, c)| = \kappa^+$ or $|[c, b_\xi)| = \kappa^+$, giving $[a_{\xi+1}, b_{\xi+1})$ in an obvious way, again using our assumption. Suppose that $[a_\xi, b_\xi)$ has been defined for all $\xi < \beta$, where β is a limit ordinal $< \kappa^+$. Then

$$\left|L\backslash\bigcap_{\xi<\beta}[a_\xi, b_\xi)\right| = \left|\bigcup_{\xi<\beta} L\backslash[a_\xi, b_\xi)\right| < \kappa^+,$$

so by (1) and Lemma 7.1 applied to $\bigcap_{\xi<\beta}[a_\xi, b_\xi)$, the interval $[a_\beta, b_\beta)$ can be defined. This finishes the construction.

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Now $a_\xi \le a_\eta$ and $b_\xi \ge b_\eta$ for $\xi < \eta$, so one of $\{a_\xi : \xi < \kappa^+\}$ and $\{b_\xi : \xi < \kappa^+\}$ contains a suborder of L of size κ^+. This contradicts (1). $\square$

Theorem 7.3. *If* $\mathrm{cf}(\mathrm{Length} A) = \omega$, *then* $\mathrm{Length} A$ *is attained.*

Proof. The proof should be fairly clear, following the lines of the proof of 4.8. Some modifications: a is an ∞-*element* provided that for each $i \in \omega$, some ordering of size λ_i is embeddable in $A \restriction a$. When constructing a_i, Lemma 7.2 is used to obtain elements c, d such that $b = c + d$, $c \cdot d = 0$, and both $A \restriction c$ and $A \restriction d$ contain chains of size λ_i; then the new $(*)$ is applied. $\square$

Later we will see that Theorem 7.3 is best possible.

Now we turn to products. The analog of the basic Theorem 4.3 for depth does not hold for length. For example, if A is any denumerable BA, then ${}^\omega A$ has length 2^ω. This is because $\mathscr{P}(\mathbb{Q})$ can be embedded in ${}^\omega A$, and $\mathbb{R}$ can be embedded in $\mathscr{P}(\mathbb{Q})$: for each $r \in \mathbb{R}$, let $f(r) = \{q \in \mathbb{Q} : q < r\}$. To generalize this example, let us call a subset D of a linear order L *weakly dense in* L provided that if $a, b \in L$ and $a < b$, then there is a $d \in D$ such that $a \le d \le b$. Now for any infinite cardinal κ let $\mathrm{Ded}(\kappa) = \sup\{\lambda$: there is an ordering of size λ with a weakly dense subset of size $\kappa\}$. The following theorem from Kurepa [57] shows the connection of this notion with length in $\mathscr{P}(\kappa)$:

Theorem 7.4. *Let κ and λ be cardinals such that $\omega \le \kappa \le \lambda$. Then the following two conditions are equivalent:*

(i) *There is an ordering L of size λ with a weakly dense subset of size κ.*
(ii) *In $\mathscr{P}(\kappa)$ there is a chain of size λ.*

Proof. (i)$\Rightarrow$(ii). We may assume that $\kappa < \lambda$. Let D be weakly dense in L, with $|D| = \kappa$. Thus $|L \backslash D| = \lambda$. Let f be a one-one function from κ onto D. For each $a \in L \backslash D$ let $g(a) = \{\alpha < \kappa : f(\alpha) < a\}$. Clearly $a < b$ implies that $g(a) \subseteq g(b)$. Suppose $a < b$ with $a, b \in L \backslash D$; choose $x \in D$ so that $a \le x \le b$. Hence $a < x < b$, and so $f^{-1}(x) \in g(b) \backslash g(a)$ and $g(a) \ne g(b)$, as desired.

(ii)$\Rightarrow$(i). Let L be a chain in $\mathscr{P}(\kappa)$ of size λ. Define $\alpha \equiv \beta$ iff $\forall c \in L[\alpha \in c$ iff $\beta \in c]$. Let Γ have one member from each $\equiv$-class. For each $\alpha \in \Gamma$ let $x_\alpha = \bigcup\{a \in L : \alpha \notin a\}$. Thus $L \backslash x_\alpha = \bigcap\{a \in L : \alpha \in a\}$. Then

(1) For all $c \in L$ and $\alpha \in \Gamma$, if $\alpha \notin c$ then $c \subseteq x_\alpha$.

(2) For all $c \in L$ and $\alpha \in \Gamma$, if $\alpha \in c$ then $x_\alpha \subseteq c$.

In fact, (1) is clear. Under the assumptions of (2), if $\gamma \in x_\alpha$ then there is a $d \in L$ such that $\alpha \notin d$ and $\gamma \in d$, hence $c \not\subseteq d$, so $d \subseteq c$ and $\gamma \in c$.

For any $\alpha, \beta \in \Gamma$ we have $x_\alpha \subseteq x_\beta$ or $x_\beta \subseteq x_\alpha$. In fact, suppose that $\alpha, \beta \in \Gamma$ and $\alpha \ne \beta$. Then $\alpha \not\equiv \beta$, so there is an $a \in L$ such that not($\alpha \in a$ iff $\beta \in a$). Say $\alpha \in a$ and $\beta \notin a$. We claim then that $x_\alpha \subseteq x_\beta$. For, suppose that $\gamma \in x_\alpha$. Choose

$b \in L$ such that $\alpha \notin b$ and $\gamma \in b$. Since $\alpha \in a$, we have $a \not\subseteq b$, so $b \subseteq a$. Hence $\gamma \in a$. Since $\beta \notin a$, it follows that $\gamma \in x_\beta$. This proves the claim.

Now by (1) and (2), the set $L' = L \cup \{x_\alpha : \alpha \in \Gamma\}$ is a chain. Clearly $|L'| = \lambda$. Let D be a subset of L' of size κ containing $\{x_\alpha : \alpha \in \Gamma\}$. We claim that D is weakly dense in L'. To prove this it suffices to take $a, b \in L$ such that $a < b$ and find $\alpha \in \Gamma$ such that $a \subseteq x_\alpha \subseteq b$. Take any $\alpha \in b \backslash a$. We may assume that $\alpha \in \Gamma$. Then x_α is as desired, by (1) and (2). $\square$

Because of this theorem, about all that we can say about the length of products is this:

$$\max(\mathrm{Ded}(|I|), \sup_{i\in I} \mathrm{Length}(A_i)) \leq \mathrm{Length}\left(\prod_{i\in I} A_i\right) \leq \prod_{i\in I} \mathrm{Length}(A_i).$$

Shelah [90] has shown that $\mathrm{Length}(\prod_{i\in I} A_i)$ cannot be calculated purely from $|I|$ and $\langle \mathrm{Length}(A_i) : i \in I\rangle$.

For weak products, we have the following analogs of 4.6 and 4.7:

Theorem 7.5. *Let $\kappa = \sup_{i\in I}\mathrm{Length}(A_i)$, and suppose that $\mathrm{cf}(\kappa) > \omega$. Then the following conditions are equivalent:*

(i) *$\prod^{\mathrm{w}}_{i\in I} A_i$ has no chain of size κ.*

(ii) *For all $i \in I$, A_i has no chain of size κ.*

Proof. For the non-trivial direction (ii)$\Rightarrow$(i), suppose that X is a chain in $\prod^{\mathrm{w}}_{i\in I} A_i$ of size κ. Wlog assume that for each $x \in X$, the 1-support M_x of x is finite. Define $x \equiv y$ iff $M_x = M_y$. Then it is easy to see that $\equiv$ is a convex equivalence relation on X; there is an order induced on $X/\equiv$, and clearly that order is isomorphic to an interval of the ordered set ω. It follows from $\mathrm{cf}(\kappa) > \omega$ that some equivalence class has cardinality κ. Then Lemma 4.2 gives a contradiction. $\square$

Corollary 7.6. $\mathrm{Length}\big(\prod^{\mathrm{w}}_{i\in I} A_i\big) = \sup_{i\in I}\mathrm{Length}A_i$. $\square$

By 7.5 we see that 7.3 is best possible: if κ is a limit cardinal with $\mathrm{cf}(\kappa) > \omega$, then it is easy to construct a weak product B such that $\mathrm{Length}(B) = \kappa$ but the length of B is not attained.

Now we discuss free products. First we mention some results whose proofs were sketched in McKenzie, Monk [82]; the proofs are similar to the corresponding results about depth.

- If $\mathrm{cf}(\kappa) > \omega$, A has no chain of size $\mathrm{cf}(\kappa)$, and B has no chain of size κ, then $A \oplus B$ has no chain of size κ.
- If κ is uncountable and regular, and for every $i \in I$ the BA A_i has no chain of size κ, then $\oplus_{i\in I} A_i$ has no chain of size κ.
- $\mathrm{Length}\,(\oplus_{i\in I} A_i) = \sup_{i\in I} \mathrm{Length}(A_i)$ when each A_i is infinite.

One of the results for depth does not carry over for length. Namely, it is possible to have $\mathrm{cf}(\kappa) < \kappa$ with BAs A, B such that A has a chain of size $\mathrm{cf}(\kappa)$, B has length κ but no chain of size κ, with $A \oplus B$ also having no chain of size κ. Namely, let $\kappa = \beth_{\omega_1}$, $A = \mathscr{P}(\omega)$, and $B = \prod^w_{\alpha<\omega_1} \mathrm{Intalg}(\beth_\alpha)$. By remarks above, A has a chain of size $2^\omega \geq \omega_1 = \mathrm{cf}(\beth_{\omega_1})$, while by Theorem 7.5, B has length $\beth_{\omega_1}$ but no chain of size $\beth_{\omega_1}$. Finally, to show that $A \oplus B$ also has no chain of size $\beth_{\omega_1}$, suppose that X is a chain in $A \oplus B$. For each $x \in X$ write $x = \sum_{i<m_x} a^x_i \cdot b^x_i$, where $a^x_i \cdot a^x_j = 0$ for $i \neq j$, $a^x_i \in A$, and $b^x_i \in B$. For each $n \in \omega$ let

$$T_n = \{b^x_i : x \in X, i < m_x, n \in a^x_i\}.$$

We claim that T_n is a chain in B. In fact, suppose that $x \in X$, $i < m_x$, $n \in a^x_i$, $y \in X$, $j < m_y$, and $n \in a^y_j$. By symmetry say $x \leq y$. Then

$$a^x_i \cdot b^x_i \prod_{k<m_y} (-a^y_k + -b^y_k) = 0;$$

multiplying by a^y_j we get $a^x_i \cdot a^y_j \cdot b^x_i \cdot -b^y_j = 0$. Since $n \in a^x_i \cap a^y_j$, it follows that $b^x_i \cdot -b^y_j = 0$, i.e., $b^x_i \leq b^y_j$. This proves the claim. Hence $|T_n| < \beth_{\omega_1}$. Let $T = \bigcup_{n\in\omega} T_n$. So also $|T| < \beth_{\omega_1}$. For any $x \in X$ we have $b^x_i \in T$ for all $i < m_x$. It follows that $|X| < \beth_{\omega_1}$, as desired.

The following problem, mentioned in McKenzie, Monk [82], is still open. See that article for additional relevant information.

Problem 77. *Let $\omega < \mathrm{cf}(\kappa) < \kappa$, and let L be a dense linear ordering of size $\mathrm{cf}(\kappa)$ with no dense subset of size less than $\mathrm{cf}(\kappa)$, and with no family of $\mathrm{cf}(\kappa)$ pairwise disjoint open intervals. Let A be the interval algebra on L, and suppose that B is a BA with $\mathrm{Length}(B) = \kappa$. Does $A \oplus B$ have a chain of size κ?*

We have not investigated the length of amalgamated free products.

Problem 78. *Investigate the length of amalgamated free products.*

Concerning unions, Length is an ordinary sup function, and so Theorem 3.16 applies.

Now we turn to ultraproducts, giving some results of Douglas Peterson. Since length is an ultra-sup function, Theorems 3.20–3.22 apply. Thus by Theorem 3.22, for F regular we have $\mathrm{Length}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Length}(A_i)/F\right|$. By Donder's theorem it is consistent that $\geq$ always holds. It is possible in ZFC to have $>$ here, according to the following result of Shelah [99] (Conclusion 15.13 (2)), which answers Problem 22 in Monk [96]:

If D is a uniform ultrafilter on κ, then for a class of cardinals λ such that $\lambda^\kappa = \lambda$, there is a system $\langle B_i : i < \kappa\rangle$ of Boolean algebras such that $\mathrm{Length}(B_i) \leq \lambda$ for each $i < \kappa$, hence $\prod_{i<\kappa} \mathrm{Length}(B_i)/D) \leq \lambda$, while $\mathrm{Length}(\prod_{i<\kappa} B_i/D) = \lambda^+$.

Magidor and Shelah have shown that it is consistent to have an example in which the length of an ultraproduct is strictly less than the size of the ultraproduct of their lengths; see Chapter 4 for details.

The following version of Theorem 3.23 holds:

Theorem 7.7. *Let $\langle A_i : i \in I\rangle$ be a system of infinite BAs, with I infinite. Let F be a uniform ultrafilter on I, and let $\kappa = \max(|I|, \text{ess.sup}^F_{i\in I}\text{Length}(A_i))$.*

Then $\text{Length}\left(\prod_{i\in I} A_i/F\right) \le 2^\kappa$.

Proof. Let $\lambda = \text{ess.sup}^F_{i\in I}\text{Length}(A_i)$. We may assume that $\text{Length}(A_i) \le \lambda$ for all $i \in I$. In order to get a contradiction, suppose that $\langle f_\alpha/F : \alpha < (2^\kappa)^+\rangle$ is a system of distinct comparable elements. Thus $[(2^\kappa)^+]^2 = \bigcup_{i\in I}\{\{\alpha,\beta\} : f_\alpha(i)$ and $f_\beta(i)$ are distinct comparable elements$\}$, so by the Erdős–Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ we get a homogeneous set which gives a contradiction. □

Concerning equality in Theorem 7.7, we note that it holds if F is regular and $\text{ess.sup}\,|A_i| \le |I|$, since then

$$\begin{aligned}
2^{|I|} &= (\text{ess.sup}\,\text{Length}(A_i))^{|I|}\\
&= \left|\prod_{i\in I}\text{Length}(A_i)/F\right| \le \text{Length}\left(\prod_{i\in I} A_i/F\right)\\
&\le \left|\prod_{i\in I} A_i/F\right| \le 2^{|I|}.
\end{aligned}$$

On the other hand, if $|A| = \text{Length}A = \kappa$ and $\kappa^\omega = \kappa$, then $\text{Length}\left({}^\omega A/F\right) < 2^\kappa$ for any nonprincipal ultrafilter F on ω.

The situation for length and subdirect products is similar to that for cellularity and depth: there is a BA with length ω which is a subdirect product of algebras with large length.

For moderate products we have the following result, with essentially the same proof as for depth:

$$\text{Length}\left(\prod_{i\in I}^B A_i\right) = \max\{\text{Length}(B), \sup_{i\in I}\text{Length}(A_i)\}.$$

Clearly $\text{Length}(A) = \text{Length}(\text{Dup}(A))$ for any infinite BA A.

Turning to the exponential, we first note that $\text{Length}(A) \le \text{Length}(\text{Exp}(A))$. In fact, if $a, b \in A$ and $a \le b$, then clearly $\mathscr{V}(\mathcal{S}(a)) \subseteq \mathscr{V}(\mathcal{S}(b))$. Suppose that $a < b$. Let U be an ultrafilter on A such that $b \cdot -a \in U$. Then $\{U\} \in \mathscr{V}(\mathcal{S}(b))\backslash\mathscr{V}(\mathcal{S}(a))$. Hence $\mathscr{V}(\mathcal{S}(a)) \subset \mathscr{V}(\mathcal{S}(b))$.

These elementary facts show that $\text{Length}(A) \le \text{Length}(\text{Exp}(A))$.

Problem 79. *Is there a BA A such that* $\text{Length}(A) < \text{Length}(\text{Exp}(A))$*?*

We turn to derived functions for length. The function $\mathrm{Length}_{\mathrm{H+}}(A)$ seems to be new. Note that $\mathrm{t}(A) = \mathrm{Depth}_{\mathrm{H+}}(A) \le \mathrm{Length}_{\mathrm{H+}}(A)$, using 4.26. It is possible to have $\mathrm{t}(A) < \mathrm{Length}_{\mathrm{H+}}(A)$; this is true when A is the interval algebra on $\mathbb{R}$, since $\mathrm{t}(A) = \omega$, while obviously $\mathrm{Length}_{\mathrm{H+}}(A) = \mathrm{Length}(A) = 2^\omega$. To see that $\mathrm{t}(A) = \omega$, use Theorem 4.26. An elementary characterization of $\mathrm{Length}_{\mathrm{H+}}$ can be given using the following notion of free chain from Monk [11]. A *free chain* for a BA A is an ordered pair (L, a) such that L is a linear order, $a \in {}^L A$, and for any $F, G \in [L]^{<\omega}$, if $F < G$ then $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta \neq 0$. We say that (L, a) is a free chain *over* L.

We define a related topological notion. Let X be a topological space. A *free chain* for X is an ordered pair (L, x) such that L is a linear order, $x \in {}^L X$, and for any $\xi \in L$,

$$\overline{\{x_\eta : \eta < \xi\}} \cap \overline{\{x_\eta : \xi \le \eta\}} = \emptyset.$$

As in the case of free sequences, a BA A has a free chain (L, a) iff $\mathrm{Ult}(A)$ has a free chain (L, x).

Proposition 7.8. *For any infinite BA A we have*

$$\mathrm{Length}_{\mathrm{H+}}(A) = \sup\{|L| : A \textit{ has a free chain } (L, a)\}.$$

Proof. The proof is just a modification of the proof of Theorem 4.26. For $\ge$, suppose that (L, a) is a free chain in A; we will find an ideal I of A such that A/I has a chain of size $|L|$. For each $\xi \in L$ let F_ξ be an ultrafilter on A such that $\{a_\eta : \eta < \xi\} \cup \{-a_\eta : \xi \le \eta \in L\} \subseteq F_\xi$. Let $Y = \{F_\xi : \xi \in L\}$, and let $I = \{x \in A : Y \subseteq \mathcal{S}(-x)\}$. Clearly I is an ideal in A. We claim that

(1) $\forall \xi, \eta \in L[\xi < \eta \to a_\eta/I < a_\xi/I]$.

To prove this, suppose that $\xi < \eta$. To show that $a_\eta \cdot -a_\xi \in I$, take any $\rho \in I$. If $\eta < \rho$, then also $\xi < \rho$ and so $a_\xi \in F_\rho$, and it follows that $-a_\eta + a_\xi \in F_\rho$, so that $F_\rho \in \mathcal{S}(-a_\eta + a_\xi)$. If $\rho \le \eta$, then $-a_\eta \in F_\rho$ and again $F_\rho \in \mathcal{S}(-a_\eta + a_\xi)$. So $a_\eta \cdot -a_\xi \in I$. Thus $a_\eta \le I < a_\xi/I]$. Also, $a_\xi \in F_\eta$ and $-a_\eta \in F_\eta$, so it follows that $a_\eta/I \neq a_\xi/I$. Thus (1) holds.

Conversely, suppose that I is an ideal in A and $\langle a_\alpha/I : \alpha \in L\rangle$ is a chain in A/I. Let $\alpha <_L \beta$ iff $a_\alpha/I < a_\beta/I$. This makes L into a linear order. We may assume that $a_0/I \neq 0$ and no a_α/I is equal to 1. We claim then that $\langle -a_\alpha : \alpha \in L\rangle$ is a free chain. For, suppose that $F, G \in [L]^{<\omega}$ with $F < G$. Then if both F and G are nonempty, we have

$$\left(\prod_{\alpha\in F} -a_\alpha \cdot \prod_{\beta\in G} a_\beta\right)/I = \prod_{\alpha\in F}(-(a_\alpha/I)) \cdot \prod_{\beta\in G}(a_\beta/I) = -(a_\alpha/I)\cdot(a_\beta/I),$$

where α is the largest element of F and β is the smallest element of G. So $-(a_\alpha/I)\cdot (a_\beta/I \neq 0$, and hence $\prod_{\alpha\in F} -a_\alpha \cdot \prod_{\beta\in G} a_\beta \neq 0$. The case when one of F, G is empty is treated similarly. ∎

Problem 80. *Is* $\text{Length}_{\text{H+}}(A) = \text{t}(A) \cdot \text{Length}(A)$ *for every infinite BA A?*

Shelah has constructed an algebra A such that $\omega < \text{Length}(A) < |A|$ while A has no homomorphic image of power smaller that $|A|$, assuming $\neg$CH (email message of December 1990). This answers Problem 17 in Monk [90]. Since an infinite BA always has a homomorphic image of size $\leq 2^\omega$, the assumption $\neg$CH is needed here. We present this result here. It depends on the following notation. If $\langle a_n : n \in \omega\rangle$ is a system of elements of a BA A and $Y \subseteq \omega$, then $\{[x, a_n]^{\text{if } n\in Y} : n \in \omega\}$ denotes the following set of formulas:

$$\{a_n \leq x : n \in Y\} \cup \{a_n \cdot x = 0 : n \in \omega \backslash Y\}.$$

If L is a chain, then a *Dedekind cut* of L is a pair (M, N) such that $L = M \cup N$ and $u < v$ for all $u \in M$ and $v \in N$. If in addition L is a subset of a BA A, then an element $a \in A$ *realizes* the Dedekind cut (M, N) if $u \leq a \leq v$ for all $u \in M$ and $v \in N$.

For any BA A, $\text{Length}'(A)$ is the smallest infinite cardinal κ such that every chain in A has size less than κ.

Shelah's result will follow easily from the following lemma:

Lemma 7.9. *Let* $\aleph_0 \leq \mu < \lambda \stackrel{\text{def}}{=} 2^{\aleph_0}$*, and let A be a subalgebra of* $\mathscr{P}(\omega)$ *containing all singletons* $\{i\}$*, with* $|A| = \mu$*. Then there is a BA B of size* $2^{\aleph_0}$*, a subalgebra of* $\mathscr{P}(\omega)$*, satisfying*

$\otimes_0$ *A is a dense subalgebra of B.*

$\otimes_1$ *If* $\langle a_n : n < \omega\rangle$ *is a system of pairwise disjoint elements of* B^+*, then for* $2^{\aleph_0}$ *subsets Y of* ω *there is an element* $a_Y \in B$ *realizing* $\{[x, a_n]^{\text{if } n\in Y} : n \in \omega\}$*.*

$\otimes_2$ *If* $\langle a_n : n < \omega\}$ *is a chain of members of B, then the number of Dedekind cuts of it realized in B is less than* $2^{\aleph_0}$*.*

Proof. First we obviously have:

(1) there is an enumeration $\langle\langle a_{\zeta n} : n < \omega\rangle : \zeta < \lambda\rangle$ of all of the ω-tuples of subsets of ω, each one repeated λ times.

Next we claim:

(2) There is a function $h : \lambda\backslash\{0\} \to \lambda$ such that for every nonzero $\varepsilon < \lambda$ we have $h(\varepsilon) < \varepsilon$, and for each $\zeta < \lambda$ the set

$$S_\zeta \stackrel{\text{def}}{=} \{\varepsilon \in \lambda\backslash\{0\} : h(\varepsilon) = \zeta\}$$

has power λ.

To see this, first choose a system $\langle D_\alpha : \alpha < \lambda\rangle$ of pairwise disjoint sets whose union is λ, each of power λ, with D_0 the set consisting of 0 and all limit ordinals less than λ. Define

$$E_\alpha = (D_\alpha \cup \{\alpha + 1\}) \backslash \bigcup_{\beta<\alpha} E_\beta.$$

Then

(3) $D_\alpha \backslash E_\alpha \subseteq \alpha + 1$;
(4) $E_\alpha \backslash D_\alpha \subseteq \{\alpha + 1\}$.

For, (4) is obvious. For (3), suppose that $\zeta \in D_\alpha \backslash E_\alpha$. Then there is a $\beta < \alpha$ such that $\zeta \in E_\beta$. Now $\zeta \notin D_\beta$, so $\zeta = \beta + 1$ by (4). Thus (3) holds.

By (3), $|E_\alpha| = \lambda$ for all $\alpha < \lambda$. Next,

(5) $\bigcup_{\alpha<\lambda} E_\alpha = \lambda$.

For, suppose that $\zeta \notin \bigcup_{\alpha<\lambda} E_\alpha$. Since $E_0 = D_0 \cup \{1\}$, ζ is a successor ordinal $\alpha + 1$. Then $\zeta \in E_\alpha$, contradiction.

For each $\varepsilon \in \lambda \backslash \{0\}$ let $h(\varepsilon)$ be the α such that $\varepsilon \in E_\alpha$. If $h(\varepsilon) = 0$, obviously $h(\varepsilon) < \varepsilon$. Suppose that $h(\varepsilon) \neq 0$, so ε is a successor ordinal $\gamma + 1$. Clearly $\varepsilon \in \bigcup_{\beta\leq\gamma} E_\beta$, so $h(\varepsilon) < \gamma + 1 = \varepsilon$. This proves (2).

Now we define a BA B_ε by induction on $\varepsilon < \lambda$ such that:

a) B_ε is a subalgebra of $\mathscr{P}(\omega)$ containing all singletons, of cardinality $< \mu^+ + |\varepsilon|^+$.
b) B_ε is increasing and continuous in ε.
c) $B_0 = A$.
d) If $\rho < \varepsilon$ and $\langle a_{\rho n} : n < \omega \rangle$ is a linearly ordered system of elements of B_ρ (no two equal), then every Dedekind cut of it which is not realized in B_ρ is also not realized in B_ε.
e) Let Evens be the set of all even natural numbers. If $h(\varepsilon) = \zeta$ and $\langle a_{\zeta n} : n < \omega \rangle$ is a system of pairwise disjoint members of B_ε (some possibly zero) with union ω, then for some $Y_\varepsilon \subseteq$ Evens we have
   (i) $B_{\varepsilon+1}$ is generated by $B_\varepsilon \cup \{x_\varepsilon\}$, where $x_\varepsilon = \bigcup_{n\in Y_\varepsilon} a_{\zeta n}$.
   (ii) If $\psi < \varepsilon$ and $h(\psi) = \zeta$, then $Y_\psi \neq Y_\varepsilon$.

If $\langle a_{\zeta n} : n < \omega \rangle$ is not such a system, then $B_{\varepsilon+1} = B_\varepsilon$.

The construction is determined for $\varepsilon = 0$ and for limit ε. At stage $\varepsilon \to \varepsilon + 1$, it the hypothesis of e) does not hold, let $B_{\varepsilon+1} = B_\varepsilon$. Now assume that hypothesis holds. Let $\kappa_\varepsilon = (\mu + |\varepsilon|)^+$, and let $\langle Y_i^\varepsilon : i < \kappa_\varepsilon \rangle$ be a sequence of almost disjoint infinite subsets of Evens. Now for each $i < \kappa_\varepsilon$ let $x_{\varepsilon i} = \bigcup_{n\in Y_i^\varepsilon} a_{\zeta n}$ and let $B_{\varepsilon i} = \langle B \cup \{x_{\varepsilon i}\} \rangle$. Let $P = \{i < \kappa_\varepsilon : \forall \psi < \varepsilon[h(\psi) = \zeta \rightarrow Y_\psi \neq Y_i^\varepsilon]\}$. So $|P| = \kappa_\varepsilon$. Suppose that d) fails for $\varepsilon + 1$ with $B_{\varepsilon i}$ for $B_{\varepsilon+1}$, for each $i \in P$; we want to get a contradiction. Then for each $i \in P$ there is a $\rho_i \leq \varepsilon$ such that d) fails; say that $\langle a^i_{\rho n} : n < \omega \rangle$ is a linearly ordered system of elements of B_{ρ_i} and (M_i, N_i) is a Dedekind cut of it which is not realized in B_{ρ_i} but is realized in $B_{\varepsilon i}$. We may assume that ρ does not depend on i. By d) for ε, (M_i, N_i) is not realized in $B_{\varepsilon i}$. Say c_i realizes (M_i, N_i) in $B_{\varepsilon i}$. We can write $c_i = b_{0i} + b_{1i} \cdot x_{\varepsilon i} + b_{2i} \cdot -x_{\varepsilon i}$, with b_{0i}, b_{1i}, b_{2i} pairwise disjoint elements of B_ε. We may assume that b_{0i}, b_{1i}, b_{2i} do not depend on i.

If $(M_i, N_i) = (M_j, N_j)$ for two distinct $i, j \in P$, we get a contradiction, as follows. Note that $x_{\varepsilon i} \cdot x_{\varepsilon j} \in B_\varepsilon$, since

$$x_{\varepsilon i} \cdot x_{\varepsilon j} = \left(\bigcup_{m \in Y_i^\varepsilon} a_{\zeta m}\right) \cap \left(\bigcup_{m \in Y_j^\varepsilon} a_{\zeta n}\right) = \bigcup_{m \in Y_i^\varepsilon \cap Y_j^\varepsilon} a_{\zeta m}$$

and $Y_i^\varepsilon \cap Y_j^\varepsilon$ is finite. Now let

$$d = b_0 + b_1 \cdot x_{\varepsilon i} \cdot x_{\varepsilon j} + b_2 \cdot (-x_{\varepsilon i} + -x_{\varepsilon j}).$$

So $d \in B_\varepsilon$. We claim that it realizes (M_i, N_i), which is a contradiction. For, if $e \in M_i$ and $f \in N_i$, then

$$\begin{aligned}
e &\leq c_i = b_0 + b_1 \cdot x_{\varepsilon i} + b_2 \cdot -x_{\varepsilon i};\\
e &\leq c_j = b_0 + b_1 \cdot x_{\varepsilon j} + b_2 \cdot -x_{\varepsilon j};\\
e &\leq (b_0 + b_1 \cdot x_{\varepsilon i} + b_2 \cdot -x_{\varepsilon i}) \cdot (b_0 + b_1 \cdot x_{\varepsilon j} + b_2 \cdot -x_{\varepsilon j})\\
&\leq b_0 + b_1 \cdot x_{\varepsilon i} \cdot x_{\varepsilon j} + b_2 \cdot -x_{\varepsilon i} + b_2 \cdot -x_{\varepsilon j}\\
&= d;
\end{aligned}$$

and

$$\begin{gathered}
b_0 \leq c_i \leq f;\\
b_1 \cdot x_{\varepsilon i} \cdot x_{\varepsilon j} \leq b_1 \cdot x_{\varepsilon i} \leq c_i \leq f;\\
b_2 \cdot -x_{\varepsilon i} \leq c_i \leq f\\
b_2 \cdot -x_{\varepsilon j} \leq c_j \leq f\\
d \leq f.
\end{gathered}$$

Thus we have shown that distinct $i, j \in P$ give different Dedekind cuts. Wlog the truth values of the following statements do not depend on i:

(6) $S_i^1 \stackrel{\text{def}}{=} \{n \in Y_i^\varepsilon : a_{\zeta n} \cdot b_1 \neq 0\}$ is infinite;

(7) $S_i^2 \stackrel{\text{def}}{=} \{n \in Y_i^\varepsilon : a_{\zeta n} \cdot b_2 \neq 0\}$ is infinite.

If both of these are false (for all i), take any i, and consider (M_i, N_i). Then, with $c \in M_i$ and $d \in N_i$,

$$c \cdot b_1 \leq b_1 \cdot x_{\varepsilon i} = b_1 \cdot \sum_{n \in S_i^1} a_{\zeta n} \leq d \cdot b_1$$

and $b_2 \cdot x_{\varepsilon i} = b_2 \cdot \sum_{n \in S_i^2} a_{\zeta n}$ and hence

$$c \cdot b_2 \leq b_2 \cdot -x_{\varepsilon i} = b_2 \cdot -\sum_{n \in S_i^2} a_{\zeta n} \leq d \cdot b_2;$$

so (M_i, N_i) is realized by

$$b_0 + b_1 \cdot \sum_{n \in S_i^1} a_{\zeta n} + b_2 \cdot - \sum_{n \in S_i^2} a_{\zeta n}$$

in B_ε, contradiction.

Thus either (6) or (7) is true. Take distinct Dedekind cuts (M_i, N_i) and (M_j, N_j); say $M_i \subset M_j$. Choose $e \in M_j \backslash M_i$.

Case 1. (6) is true. Then in $B_{\varepsilon i}$ we have $b_0 + b_1 \cdot x_{\varepsilon i} + b_2 \cdot -x_{\varepsilon i} \le e$, so

$(*)$ $b_1 \cdot x_{\varepsilon i} \le e \cdot b_1$.

And in $B_{\varepsilon j}$ we have $e \le b_0 + b_1 \cdot x_{\varepsilon j} + b_2 \cdot -x_{\varepsilon j}$, so

$(\star)$ $e \cdot b_1 \le x_{\varepsilon j} \cdot b_1$.

Now if $n \in S_i^1$, then $0 \ne a_{\zeta n} \cdot b_1 \le x_{\varepsilon i} \cdot b_1 \le e \cdot b_1$ (by $(*)$) $\le x_{\varepsilon j} \cdot b_1$ (by $(\star)$), so $n \in Y_j^\varepsilon$, hence $n \in S_j^1$. Thus $S_i^1 \subseteq S_j^1$, contradicting $S_i^1 \cap S_j^1 \subseteq Y_i^\varepsilon \cap Y_j^\varepsilon$ finite.

Case 2. (7) is true. Then in $B_{\varepsilon i}$ we have $b_0 + b_1 \cdot x_{\varepsilon i} + b_2 \cdot -x_{\varepsilon i} \le e$, so

$(**)$ $-x_{\varepsilon i} \cdot b_2 \le e \cdot b_2$.

And

$(\star\star)$ $e \cdot b_2 \le -x_{\varepsilon j} \cdot b_2$.

If $n \in S_j^2$, then $a_{\zeta n} \cdot b_2 \cdot -x_{\varepsilon j} = 0$, so by $(\star\star)$, $a_{\zeta n} \cdot b_2 \cdot e = 0$, hence by $(**)$ $a_{\zeta n} \cdot b_2 \cdot -x_{\varepsilon i} = 0$, so $n \in Y_i^\varepsilon$, hence $n \in S_i^2$. So $S_j^2 \subseteq S_i^2$, again giving a contradiction.

Thus there is an $i \in P$ such that d) holds for $B_{\varepsilon i}$ as $B_{\varepsilon+1}$. Clearly then the first part of e) also holds.

Thus the construction can be carried through. Let $B = \bigcup_{\varepsilon<\lambda} B_\varepsilon$. Clearly $\otimes_0$ holds, by c). Now suppose that $\langle a_n : n < \omega\rangle$ is a system of pairwise disjoint elements of B^+. Choose $\varepsilon < \lambda$ such that $a_n \in B_\varepsilon$ for all $n \in \omega$, extend $\{a_n : n \in \omega\}$ to a maximal disjoint set X in B_ε, and enumerate X as $\langle b_n : n < \omega\rangle$ so that $\{a_n : n < \omega\} = \{b_{2n} : n < \omega\}$. By (1), there is a $\zeta < \lambda$ such that $\langle b_n : n < \omega\rangle = \langle a_{\zeta n} : n < \omega\rangle$. Then by (2) and e) we get $2^{\aleph_0}$ subsets Y of ω such that $\{[x, a_n]^{\text{if } n \in Y} : n \in \omega\}$ is realized in B. Next, suppose that $\langle a_n : n < \omega\rangle$ is a chain of members of B^+. By (1), say $\langle a_n : n < \omega\rangle = \langle a_{\zeta n} : n < \omega\rangle$. Choose $\varepsilon < \lambda$ such that all a_n are in B_ε. Then by d), B realizes at most $|B_\varepsilon|$ Dedekind cuts of $\langle a_n : n < \omega\rangle$. $\square$

Theorem 7.10. *If $\aleph_0 \le \mu < 2^{\aleph_0}$ then there is a BA B such that:*

(i) *B is a subalgebra of $\mathscr{P}(\omega)$ containing all singletons, and hence $\pi(B) = \aleph_0$;*
(ii) $\mu^+ \le \mathrm{Length}'(B) \le 2^{\aleph_0}$*;*
(iii) *every infinite homomorphic image of B has size $2^{\aleph_0}$.*

Proof. We apply Lemma 7.9 to a subalgebra A of $\mathcal{P}(\omega)$ containing all singletons, of size μ and with length μ. We obtain a BA B as a result.

We check that every infinite homomorphic image of B has size $2^{\aleph_0}$. Let f be a homomorphism from B onto C with C infinite. Let $\langle c_n : n < \omega\rangle$ be a system of nonzero disjoint elements of C. Then there is a system $\langle b_n : n < \omega\rangle$ of non-zero disjoint elements of B such that $f(b_n) = c_n$ for all $n < \omega$. Let $\mathscr{D}$ be a collection of $2^{\aleph_0}$ subsets of ω such that for each $Y \in \mathscr{D}$ there is an element b_Y realizing $\{[x, b_n]^{\text{if } n\in Y} : n < \omega\}$. Clearly $\{f(b_Y) : Y \in \mathscr{D}\}$ is a subset of C of size $2^{\aleph_0}$, as desired.

Now suppose that J is a chain in B of size $2^{\aleph_0}$; we shall get a contradiction. For each $i \in \omega$ let $M_i = \{b \in J : i \notin b\}$. Clearly there is a countable subset K_i of M_i cofinal in M_i. Let $I = \bigcup_{i\in\omega} K_i$. So, I is countable. Each element of J realizes a Dedekind cut of I, so we can contradict $\otimes_2$ of the Lemma by showing that any two distinct $u, v \in J$ realize distinct Dedekind cuts of I. Suppose that $u \subset v$ but $\{w \in I : u \subseteq w\} = \{w \in I : v \subseteq w\}$. Choose $i \in v\backslash u$. Choose $w \in K_i$ with $u \subseteq w$. Then $v \subseteq w$ and hence $i \in w$, contradiction. $\square$

The function $\text{Length}_{\text{H}-}$ is also new. Note that $\omega \leq \text{Length}_{\text{H}-}(A) \leq 2^\omega$, by an easy argument using the Sikorski extension theorem. It is obviously possible to have $\omega = \text{Length}_{\text{H}-}(A)$. If $2^\omega = \omega_2$, the algebra B of Theorem 7.10 is such that $\text{Length}_{\text{H}-}(B) < \text{Card}_{\text{H}-}(B)$. This answers Problem 18 in Monk [90]. Obviously $\text{Length}_{\text{S}+}(A) = \text{Length}(A)$ and $\text{Length}_{\text{S}-}A = \omega$. The following is Problem 23 in Monk [96]:

Problem 81. *Is always* $\text{Length}_{\text{h}-}(A) = \omega$?

Here $\text{Length}_{\text{h}-}(A)$ is defined as follows:

$$\text{Length}_{\text{h}-}(A) = \inf\{\sup\{|X| : X \text{ is a chain of clopen subsets of } Y\} : Y \subseteq \text{Ult}(A)\}.$$

Clearly $\text{Length}_{\text{h}+}(A) \geq \text{Depth}_{\text{h}+}(A) = \text{s}(A)$ by 4.28.

Also, $\text{Length}_{\text{h}+}A \geq \text{Length}_{\text{H}+}A$; but it is possible to have $\text{Length}_{\text{h}+}A > \text{Length}_{\text{H}+}A$. This is true, for example, if A is the finite-cofinite algebra on an uncountable cardinal κ. For then $\text{Length}_{\text{h}+}A = \text{Ded}(\kappa)$, while $\text{Length}_{\text{H}+}A = \omega$. That $\text{Length}_{\text{h}+}A = \text{Ded}(\kappa)$ is seen like this: $\text{Ult}(A)$ has a discrete subspace S of size κ, and so Theorem 7.4 applies for the chains of subsets of S, since every subset is clopen.

Clearly ${}_{\text{d}}\text{Length}_{\text{S}+}(A) = \text{Length}(A)$. By the discussion of ${}_{\text{d}}\text{Depth}_{\text{S}-}$ in Chapter 4 we see that ${}_{\text{d}}\text{Length}_{\text{S}-}(A) \geq \lambda$ if A is atomless and λ-saturated (in the model-theoretic sense). Thus Problem 20 of Monk [90] is answered.

Concerning $\text{Length}_{\text{Hs}}$, note the following two examples. For A the free algebra of size κ, $\text{Length}_{\text{Hs}}(A) = [\omega, \kappa]$. If A is an interval algebra on an ordered set L, then $\text{Length}_{\text{Hs}}(A) = \text{Card}_{\text{Hs}}(A)$. In particular, $\text{Length}_{\text{Hs}}(\text{Intalg}(\mathbb{R})) = \{\omega, 2^\omega\}$; see Chapter 9. These examples suggest the following problem.

Problem 82. *What are the possibilities for* $\text{Length}_{\text{Hs}}(A)$*?*

Clearly $\mathrm{Length}_{\mathrm{Ss}}(A) = [\omega, \mathrm{Length}(A)]$. The functions $\mathrm{Length}_{\mathrm{spect}}$ and $\mathrm{Length}_{\mathrm{mm}}$ are studied in Monk [07]. The main result there is that under GCH, if K is a nonempty set of regular cardinals, then there is a BA A such that $\mathrm{Length}_{\mathrm{spect}}(A) = K$. This leaves the following question open.

Problem 83. *Can one prove in ZFC that if K is a nonempty set of infinite cardinals then there is a BA A such that $\mathrm{Length}_{\mathrm{spect}}(A) = K$?*

Proposition 7.11. $\mathrm{tow}(A) \le \mathrm{Length}_{\mathrm{mm}}(A)$ *for every atomless BA A.*

Proof. If X is a maximal chain, then $X\backslash\{1\}$ has no last element, and a cofinal subset of it is a tower. □

Before considering $\mathrm{Length}_{\mathrm{Sr}}$ and $\mathrm{Length}_{\mathrm{Hr}}$, we note an important connection of length with depth. Obviously $\mathrm{Depth}A \le \mathrm{Length}A$ for any infinite BA A. Another clear relationship is that $\mathrm{Length}(A) \le 2^{\mathrm{Depth}(A)}$: if L is an ordered subset of A of power $(2^{\mathrm{Depth}(A)})^+$, let $\prec$ be a well-ordering of L; then by the Erdős–Rado partition relation $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ we get a well-ordered or inversely well-ordered subset of L of power $(\mathrm{Depth}A)^+$, contradiction.

Now for $\mathrm{Length}_{\mathrm{Sr}}$ we have the following adaptation of Theorem 4.73:

Theorem 7.12. *For any infinite BA A the following conditions hold:*

(i) *If $(\kappa, \lambda) \in \mathrm{Length}_{\mathrm{Sr}}(A)$, then $\kappa \le \lambda \le |A|$ and $\kappa \le \mathrm{Length}(A)$.*
(ii) *For each $\kappa \in [\omega, \mathrm{Length}(A)]$ we have $(\kappa, \kappa) \in \mathrm{Length}_{\mathrm{Sr}}(A)$.*
(iii) *If $(\kappa, \lambda) \in \mathrm{Length}_{\mathrm{Sr}}(A)$ and $\kappa \le \mu \le \lambda$, then $(\kappa, \mu) \in \mathrm{Length}_{\mathrm{Sr}}(A)$.*
(iv) $(\mathrm{Length}A, |A|) \in \mathrm{Length}_{\mathrm{Sr}}(A)$.
(v) *If $\omega \le \lambda \le |A|$ then $(\kappa, \lambda) \in \mathrm{Length}_{\mathrm{Sr}}(A)$ for some κ.*
(vi) *If $((2^\kappa)^+, \lambda) \in \mathrm{Length}_{\mathrm{Sr}}(A)$, then $(\omega, \kappa^+) \in \mathrm{Length}_{\mathrm{Sr}}(A)$.*

Proof. (i)–(v) are obvious. For (vi), if B is a subalgebra of A with $\mathrm{Length}(B) = (2^\kappa)^+$, then $\mathrm{Depth}(B) \ge \kappa^+$ by the above, and so $(\omega, \kappa^+) \in \mathrm{Length}_{\mathrm{Sr}}(A)$ by the argument for 4.73(vi). □

As usual we have a vague question about $\mathrm{Length}_{\mathrm{Sr}}$:

Problem 84. *Characterize in cardinal number terminology the sets* $\mathrm{Length}_{\mathrm{Sr}}(A)$.

We also have a vague question about $\mathrm{Length}_{\mathrm{Hr}}$:

Problem 85. *Characterize in cardinal number terminology the sets* $\mathrm{Length}_{\mathrm{Hr}}(A)$.

Note that $\mathrm{Length}(A) > \pi(A)$ for $A = \mathscr{P}(\omega)$; and $\mathrm{c}(A) > \mathrm{Length}(A)$ for A the finite-cofinite algebra on κ. If A is a tree algebra, then $\mathrm{Length}(A) = \mathrm{Depth}(A)$ by Proposition 16.20 of the Handbook.

8 Irredundance

Clearly $\mathrm{Irr}(A) \leq |A|$. If A is a subalgebra of B, then $\mathrm{Irr}(A) \leq \mathrm{Irr}(B)$, and Irr can change to any extent from B to A (along with cardinality). The same is true for A a homomorphic image of B.

Concerning our special kinds of subalgebras, recall that $\pi(A) \leq \mathrm{Irr}(A)$ for every infinite BA, by a theorem of McKenzie (Proposition 4.23 of the Handbook). Hence when B can have much larger size than $A \leq B$, the irredundance can increase. This applies to $\leq_{\mathrm{rc}}$, $\leq_{\mathrm{free}}$, $\leq_{\sigma}$, $\leq_{\mathrm{proj}}$, $\leq_{\mathrm{u}}$, $\leq_{\mathrm{reg}}$, and $\leq_{\mathrm{mg}}$. Also, clearly $\kappa = \mathrm{Irr}(\mathrm{Finco}(\kappa) < 2^{\kappa} = \mathrm{Irr}(\mathscr{P}(\kappa))$.

Whether one can get $<$ for the other subalgebra notions is open.

Problem 86. *Can one have* $\mathrm{Irr}(A) < \mathrm{Irr}(B)$ *for* $A \leq_{\mathrm{s}} B$ *or* $A \leq_{\mathrm{m}} B$*?*

The following equivalent formulation of irredundance will be useful.

Theorem 8.1. *Let A be an infinite BA and $X \subseteq A$. Then the following conditions are equivalent:*

(i) *X is irredundant.*
(ii) *For every $x \in X$ there are ultrafilters F, G on A such that $F \cap (X \backslash \{x\}) = G \cap (X \backslash \{x\})$, $x \in F$, and $-x \in G$.*
(iii) *For every $x \in X$ there are homomorphisms $f, g : A \to 2$ such that $f \restriction (X \backslash \{x\}) = g \restriction (X \backslash \{x\})$ while $f(x) \neq g(x)$.*

Proof. (ii) and (iii) are obviously equivalent. Now assume (ii). Then $F \cap \langle X \backslash \{x\} \rangle = G \cap \langle X \backslash \{x\} \rangle$. It follows that $x \notin \langle X \backslash \{x\} \rangle$.

Now assume that (i) holds, and suppose that $x \in X$. Now the set

$$(*) \qquad \{a \in \langle X \backslash \{x\} \rangle : -a \leq x \text{ or } -a \leq -x\}$$

has fip. For, otherwise we get

$$a_0 \cdot \ldots \cdot a_{m-1} \cdot b_0 \cdot \ldots \cdot b_{n-1} = 0$$

with each $a_i, b_i \in \langle X \backslash \{x\} \rangle$ and $-a_i \leq x$, $-b_i \leq -x$. Then we may assume that $m, n > 0$, then that $m = n = 1$. Hence $a_0 \leq -b_0 \leq -x \leq a_0$, so $x = a_0 \in \langle X \backslash \{x\} \rangle$, contradiction.

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Let H be an ultrafilter on $\langle X\backslash\{x\}\rangle$ containing the set $(*)$. Then $H\cup\{x\}$ has f.i.p. In fact, if $a\in H$ and $a\cdot x=0$, then $a\le -x$, hence $-a\in H$, contradiction. Hence let F be an ultrafilter on A containing $H\cup\{x\}$. Similarly, let G be an ultrafilter on A containing $H\cup\{-x\}$. This proves (ii). $\square$

Rosłanowski, Shelah [00] proved that $\mathrm{Irr}(A\times B)=\max(\mathrm{Irr}(A),\mathrm{Irr}(B))$ if $A\times B$ is infinite. This solves Problem 24 of Monk [96]. We give their proof here. The following construction will be used.

Fix an infinite cardinal λ. Let $\langle x_\alpha:\alpha<\lambda\rangle$ be free generators of $\mathrm{Fr}(\lambda)$. For any $F\subseteq {}^\lambda 2$ let I_F be the ideal in $\mathrm{Fr}(\lambda)$ generated by all elements

$$\prod_{\alpha\in u} x_\alpha^{\varepsilon(\alpha)},$$

where $u\in[\lambda]^{<\omega}$, $\varepsilon\in {}^u2$, and $\forall f\in F[\varepsilon\not\subseteq f]$. Then $B[F]$ is $\mathrm{Fr}(\lambda)/I_F$.

Proposition 8.2. *If $F\subseteq {}^\lambda 2$ and $f\in F$, then there is a homomorphism f^{hom} of $B[F]$ into 2 such that $f^{\mathrm{hom}}([x_\alpha]_F)=f(\alpha)$ for every $\alpha\in\lambda$.*

Proof. If not, by Sikorski's extension criterion we get finite disjoint $u,v\subseteq\lambda$ such that $\prod_{\alpha\in u}[x_\alpha]_F\cdot\prod_{\alpha\in v}-[x_\alpha]=0$ while $\prod_{\alpha\in u}f(\alpha)\cdot\prod_{\alpha\in v}-f(\alpha)\neq 0$. This means that $\prod_{\alpha\in u}x_\alpha\cdot\prod_{\alpha\in v}-x_\alpha\in I_F$ and $u\subseteq\{\alpha<\lambda:f(\alpha)=1\}$ and $v\subseteq\{\alpha<\lambda:f(\alpha)=0\}$. So there exist a finite set $E\subseteq\{(w,\varepsilon):w\in[\lambda]^{<\omega},\varepsilon\in {}^w2\}$ such that $\varepsilon\not\subseteq g$ for all $(\varepsilon,w)\in E$ and all $g\in F$, and

$$\prod_{\alpha\in u}x_\alpha\cdot\prod_{\alpha\in v}-x_\alpha\le\sum_{(w,\varepsilon)\in E}\prod_{\alpha\in w}x_\alpha^{\varepsilon(\alpha)},$$

Extending the mapping $x_\alpha\mapsto f(\alpha)$ to a homomorphism gives a contradiction. $\square$

Proposition 8.3. *Suppose that $F\subseteq {}^\lambda 2$, $u\in[\lambda]^{<\omega}$, and $\varepsilon\in {}^u2$. Then the following conditions are equivalent:*

(i) $\prod_{\alpha\in u}[x_\alpha]^{\varepsilon(\alpha)}=0$.
(ii) $\forall f\in F[\varepsilon\not\subseteq f]$.

Proof. (i)⇒(ii): If $\varepsilon\subseteq f$ for some $f\in F$, then the proof of Proposition 8.2 shows that (i) fails to hold.

(ii)⇒(i): true by definition. $\square$

Theorem 8.4. $\mathrm{Irr}(B_0\times B_1)=\max(\mathrm{Irr}(B_0),\mathrm{Irr}(B_1))$ *for infinite $B_0\times B_1$.*

Proof. Suppose that $\langle(y^0_\alpha,y^1_\alpha):\alpha<\lambda\rangle$ is an irredundant system in $B_0\times B_1$. Hence by Theorem 8.1, for each $\alpha<\lambda$ we get homomorphisms $f^0_\alpha,f^1_\alpha:B_0\times B_1\to 2$ with the following properties:

(1) $f^0_\alpha(y^0_\alpha,y^1_\alpha)\neq f^1_\alpha(y^0_\alpha,y^1_\alpha)$.

(2) $\forall\beta\in\lambda\backslash\{\alpha\}[f^0_\alpha(y^0_\beta,y^1_\beta)=f^1_\alpha(y^0_\beta,y^1_\beta)]$.

Now

$$\begin{aligned}\lambda = &\{\alpha < \lambda : f^0_\alpha(1,0) = f^1_\alpha(1,0) = 0\}\\ &\cup \{\alpha < \lambda : f^0_\alpha(1,0) = f^1_\alpha(1,0) = 1\}\\ &\cup \{\alpha < \lambda : f^0_\alpha(1,0) = 0 \text{ and } f^1_\alpha(1,0) = 1\}\\ &\cup \{\alpha < \lambda : f^0_\alpha(1,0) = 1 \text{ and } f^1_\alpha(1,0) = 0\}.\end{aligned}$$

Thus at least one of these four sets has size λ, so we have four cases:

Case 1. $\{\alpha < \lambda : f^0_\alpha(1,0) = f^1_\alpha(1,0) = 1\}$ has size λ. We may assume that $f^0_\alpha(1,0) = f^1_\alpha(1,0) = 1$ for all $\alpha < \lambda$. Now we claim:

(3) There exist homomorphisms g^l_α of B_0 into 2 such that $g^l_\alpha(y^0_\beta) = f^l_\alpha(y^0_\beta, y^1_\beta)$ for all $\beta < \lambda$.

To prove (3), by Sikorski's extension criterion it suffices to assume that u and v are disjoint finite subsets of λ such that $\prod_{\beta\in u} y^0_\beta \cdot \prod_{\beta\in v} -y^0_\beta = 0$ and show that $\prod_{\beta\in u} f^l_\alpha(y^0_\beta, y^1_\beta) \cdot \prod_{\beta\in v} -f^l_\alpha(y^0_\beta, y^1_\beta) = 0$. We have

$$\begin{aligned}\prod_{\beta\in u} f^l_\alpha(y^0_\beta . y^1_\beta) \cdot \prod_{\beta\in v} -f^l_\alpha(y^0_\beta, y^1_\beta) &= f^l_\alpha(1,0) \cdot \prod_{\beta\in u} f^l_\alpha(y^0_\beta, y^1_\beta) \cdot \prod_{\beta\in v} -f^l_\alpha(y^0_\beta, y^1_\beta)\\ &= f^l_\alpha\left((1,0) \cdot \left(\prod_{\beta\in u}(y^0_\beta, y^1_\beta) \cdot \prod_{\beta\in v} -(y^0_\beta, y^1_\beta)\right)\right)\\ &= f^l_\alpha\left(\prod_{\beta\in u} y^0_\beta \cdot \prod_{\beta\in v} -y^0_\beta, 0\right)\\ &= f^l_\alpha((0,0)) = 0.\end{aligned}$$

Thus (3) holds.

By (1)–(3) we have $g^0_\alpha(y^0_\alpha) \neq g^1_\alpha(y^0_\alpha)$, while $\forall\beta \in \lambda\backslash\{\alpha\}[g^0_\alpha(y^0_\beta) = g^1_\alpha(y^1_\beta)]$. Hence by Theorem 1, $\langle y^0_\beta : \beta < \lambda\rangle$ is irredundant.

Case 2. $\{\alpha < \lambda : f^0_\alpha(1,0) = f^1_\alpha(1,0) = 0\}$ has size λ. This is similar to Case 1, working with $(0,1)$ rather than $(1,0)$.

Case 3. $\{\alpha < \lambda : f^0_\alpha(1,0) = 1 \text{ and } f^1_\alpha(1,0) = 0\}$ has size λ. Hence we may assume that

(4) $\forall\alpha < \lambda[f^0_\alpha(1,0) = 1 \text{ and } f^1_\alpha(1,0) = 0]$.

Now for each $l \in 2$ and $\alpha < \lambda$ define $g^l_\alpha : \lambda \to 2$ by setting $g^l_\alpha(\beta) = f^l_\alpha(y^0_\beta, y^1_\beta)$.

(5) If $\alpha \neq \beta$ then $g^0_\alpha(\beta) = g^1_\alpha(\beta)$.

(6) $g^0_\alpha(\alpha) \neq g^1_\alpha(\alpha)$.

These are immediate from (1)–(2).

For $l \in 2$ let $F_l = \{g^l_\alpha : \alpha < \lambda\}$. By Proposition 8.2 we have:

(7) For each $l \in 2$, the function $(g^l_\alpha)^{\rm hom}$ is a homomorphism from $B[F_l]$ into 2 such that $(g^l_\alpha)^{\rm hom}([x_\beta]_{F_l}) = g^l_\alpha(\beta)$ for every $\beta < \lambda$.

(8) For each $l \in 2$ there is a homomorphism k_l from $\langle\{y^l_\alpha : \alpha < \lambda\}\rangle$ into $B[F_l]$ such that $k_l(y^l_\alpha) = [x_\alpha]_{F_l}$ for all $\alpha < \lambda$.

To prove this it suffices to assume that $M \in [\lambda]^{<\omega}$, $\varepsilon \in {}^M 2$, and $\prod_{\beta\in M}[x_\beta]^{\varepsilon(\beta)}_{F_l} \neq 0$ and show that $\prod_{\beta\in M}(y^l_\beta)^{\varepsilon(\beta)} \neq 0$. By Proposition 8.3 there is an $\alpha < \lambda$ such that $\varepsilon \subseteq g^l_\alpha$. Now

$$f^l_\alpha\left(\prod_{\beta\in M}(y^0_\beta, y^1_\beta)^{\varepsilon(\beta)}\right) = \prod_{\beta\in M}(f^l_\alpha(y^0_\beta, y^1_\beta))^{\varepsilon(\beta)} = \prod_{\beta\in M}(g^l_\alpha(\beta))^{\varepsilon(\beta)} = 1.$$

If $l = 0$, then

$$1 = f^0_\alpha\left((1,0)\cdot\left(\prod_{\beta\in M}(y^0_\beta, y^1_\beta)^{\varepsilon(\beta)}\right)\right) = f^0_\alpha\left(\prod_{\beta\in M}(y^0_\beta)^{\varepsilon(\beta)}, 0\right),$$

and so $\prod_{\beta\in M}(y^0_\beta)^{\varepsilon(\beta)} \neq 0$. Similarly if $l = 1$. Hence (8) holds.

(9) Let $l \in 2$ and $A \subseteq \lambda$, and suppose that for every $\alpha \in A$ there is a homomorphism $h_\alpha : \langle[x_\beta]_{F_l} : \beta \in A\rangle \to 2$ such that $h_\alpha([x_\beta]_{F_l}) = g^{1-l}_\alpha(\beta)$ for every $\beta \in A$.

Then $\langle y^l_\beta : \beta \in A\rangle$ is irredundant in B_l.

To prove (9), first note that for any $\alpha \in A$ we have

$$h_\alpha([x_\alpha]_{F_l}) = g^{1-l}_\alpha(\alpha) \neq g^l_\alpha(\alpha) = (g^l_\alpha)^{\rm hom}([x_\alpha]_{F_l}),$$

while if $\beta \in A\backslash\{\alpha\}$ we have

$$h_\alpha([x_\beta]_{F_l}) = g^{1-l}_\alpha(\beta) = g^l_\alpha(\beta) = (g^l_\alpha)^{\rm hom}([x_\beta]_{F_l}).$$

It follows that $\langle[x_\alpha]_{F_l} : \alpha \in A\rangle$ is irredundant. Hence by (8), also $\langle y^l_\beta : \beta \in A\rangle$ is irredundant in B_l.

Now by (9) we may assume:

(10) For all $A \in [\lambda]^\lambda \neg\forall\alpha \in A\exists h_\alpha : \langle[x_\beta]_{F_0} : \beta \in A\rangle \to 2$ such that $h_\alpha([x_\beta]_{F_0}) = g^1_\alpha(\beta)$ for every $\beta \in A$.

We now construct by recursion sequences $\langle\alpha_\xi : \xi < \lambda\rangle$ and $\langle(u_\xi, v_\xi) : \xi < \lambda\rangle$ so that the following conditions hold:

(11) $u_\xi, v_\xi \in [\lambda]^{<\omega}$ are disjoint.

(12) $(u_\xi \cup v_\xi) \cap \bigcup_{\zeta<\xi}(u_\zeta \cup v_\zeta) = \emptyset$.

(13) $\prod_{\gamma\in u_\xi}[x_\gamma]_{F_0} \cdot \prod_{\gamma\in v_\xi} -[x_\gamma]_{F_0} = 0$.

(14) $\prod_{\gamma\in u_\xi}[x_\gamma]_{F_1} \cdot \prod_{\gamma\in v_\xi} -[x_\gamma]_{F_1} \neq 0$.

(15) $\forall\beta \in u_\xi[g^1_{\alpha_\xi}(\beta) = 1]$ and $\forall\beta \in v_\xi[g^1_{\alpha_\xi}(\beta) = 0]$.

(16) $\alpha_\xi \in u_\xi \cup v_\xi$.

Suppose that we have constructed u_ζ and v_ζ for all $\zeta < \xi$ so that (11)–(16) hold. Then $A \stackrel{\text{def}}{=} \lambda\setminus\bigcup_{\zeta<\xi}(u_\zeta \cup v_\zeta)$ has size λ. Then by (10), there is an $\alpha_\xi \in A$ such that the relation

$$\{([x_\beta]_{F_0}, g^1_{\alpha_\xi}(\beta)) : \beta \in A\}$$

does not extend to a homomorphism. By Sikorski's extension criterion, this means that there are finite disjoint $u'_\xi, v'_\xi \subseteq A$ such that $\prod_{\gamma\in u'_\xi}[x_\gamma]_{F_0} \cdot \prod_{\gamma\in v'_\xi} -[x_\gamma]_{F_0} = 0$ while $g^1_{\alpha_\xi}[u'_\xi] \subseteq \{1\}$ and $g^1_{\alpha_\xi}[v'_\xi] \subseteq \{0\}$. Define

$$u_\xi = \begin{cases} u'_\xi \cup \{\alpha_\xi\} & \text{if } \ g^1_{\alpha_\xi}(\alpha_\xi) = 1, \\ u'_\xi & \text{otherwise;} \end{cases}$$
$$v_\xi = \begin{cases} v'_\xi \cup \{\alpha_\xi\} & \text{if } \ g^1_{\alpha_\xi}(\alpha_\xi) = 0, \\ v'_\xi & \text{otherwise.} \end{cases}$$

Clearly all conditions except possibly (14) hold; (14) follows from (7).

(17) For any $\xi, \zeta < \lambda$, $(\forall\beta \in u_\xi[g^1_{\alpha_\zeta}(\beta) = 1]$ and $\forall\beta \in v_\xi[g^1_{\alpha_\zeta}(\beta) = 0])$ iff $\xi = \zeta$.

In fact, if $\xi = \zeta$ then $(\forall\beta \in u_\xi[g^1_{\alpha_\zeta}(\beta) = 1]$ and $\forall\beta \in v_\xi[g^1_{\alpha_\zeta}(\beta) = 0])$ by (15). If $\xi \neq \zeta$, then $(\exists\beta \in u_\xi[g^0_{\alpha_\zeta}(\beta) \neq 1]$ or $\exists\beta \in v_\xi[g^0_{\alpha_\zeta}(\beta) \neq 0])$ by (13) and (7); and $\alpha_\zeta \notin u_\xi \cup v_\xi$ and hence by (5) $g^1_{\alpha_\zeta} \restriction (u_\xi \cup v_\xi) = g^0_{\alpha_\zeta} \restriction (u_\xi \cup v_\xi)$, so it follows that $(\exists\beta \in u_\xi[g^1_{\alpha_\zeta}(\beta) \neq 1]$ or $\exists\beta \in v_\xi[g^1_{\alpha_\zeta}(\beta) \neq 0])$. So (17) holds.

Now for each $\xi < \lambda$ let

$$z_\xi = \prod_{\gamma\in u_\xi} y^1_\gamma \cdot \prod_{\gamma\in v_\xi} -y^1_\gamma.$$

(18) For each $\xi < \lambda$ there is a homomorphism $h_\xi : \langle\{y^1_\beta : \beta < \lambda\rangle \to 2$ such that $h_\xi(y^1_\beta) = g^1_{\alpha_\xi}(\beta)$ for every $\xi < \lambda$.

In fact, suppose that $a, b \in [\lambda]^{<\omega}$ are disjoint and $\prod_{\beta\in a} g^1_{\alpha_\xi}(\beta) \cdot \prod_{\beta\in b} -g^1_{\alpha_\xi}(\beta) \neq 0$.

Then

$$\begin{aligned}1 &= f^1_{\alpha_\xi}(0,1)\cdot\prod_{\beta\in a} f^1_{\alpha_\xi}(y^0_\beta,y^1_\beta)\cdot\prod_{\beta\in b} -f^1_{\alpha_\xi}(y^0_\beta,y^1_\beta)\\ &= f^1_{\alpha_\xi}\left((0,1)\cdot\prod_{\beta\in a}(y^0_\beta,y^1_\beta)\cdot\prod_{\beta\in b}-(y^0_\beta,y^1_\beta)\right)\\ &= f^1_{\alpha_\xi}\left(0,\prod_{\beta\in a}y^1_\beta\cdot\prod_{\beta\in b}-y^1_\beta\right),\end{aligned}$$

and it follows that $\prod_{\beta\in a} y^1_\beta\cdot\prod_{\beta\in b}-y^1_\beta\neq 0$, proving (18).

Now we have, for any $\xi,\zeta<\lambda$, $h_\zeta(z_\xi)=\prod_{\gamma\in u_\xi} g^1_{\alpha_\zeta}(\gamma)\cdot\prod_{\gamma\in v_\xi}-g^1_{\alpha_\zeta}(\gamma)$, and hence by (17), $h_\zeta(z_\xi)=1$ iff ($\forall\beta\in u_\xi[g^1_{\alpha_\zeta}(\beta)=1]$ and $\forall\beta\in v_\xi[g^1_{\alpha_\zeta}(\beta)=0]$) iff $\zeta=\xi$. So $\langle z_\xi:\xi<\lambda\rangle$ is irredundant.

Case 4. $\{\alpha<\lambda: f^0_\alpha(1,0)=0$ and $f^1_\alpha(1,0)=1\}$ has size λ. Let $k^0_\alpha=f^1_\alpha$ and $k^1_\alpha=f^0_\alpha$ for all $\alpha<\lambda$. Then $\{\alpha<\lambda: k^0_\alpha(1,0)=1$ and $k^1_\alpha(1,0)=0\}$ has size λ, and we can apply Case 3 to get the desired result. □

Corollary 8.5. *If A is infinite and B is finite, then* $\mathrm{Irr}(A)=\mathrm{Irr}(A\times B)$. □

Corollary 8.6. *If* $|B|\le\mathrm{Irr}(A)$, *then* $\mathrm{Irr}(A\times B)=\mathrm{Irr}(A)$. □

Corollary 8.7. *If $\langle A_i:i\in I\rangle$ is a system of infinite BAs, with I infinite, then*

$$\mathrm{Irr}\left(\prod_{i\in I}^{\mathrm{w}}A_i\right)=\max\{|I|,\sup_{i\in I}\mathrm{Irr}(A_i)\}.$$

Proof. For brevity let $B=\prod^{\mathrm{w}}_{i\in I}A_i$. Clearly $\ge$ holds. Now suppose that $X\subseteq B$ is irredundant and $|X|=(\max\{|I|,\sup_{i\in I}\mathrm{Irr}(A_i)\})^+$; we want to get a contradiction. For each $F\in[I]^{<\omega}$ let C_F be the subalgebra of B consisting of all elements with support contained in F. Clearly

$$X=\bigcup_{F\in[I]^{<\omega}}(X\cap C_F).$$

Clearly each set $X\cap C_F$ is irredundant. Now there is an $F\in[I]^{<\omega}$ such that $|X|=|X\cap C_F|$. Clearly $C_F\cong\prod_{i\in F}A_i$, so $|X\cap C_F|>\mathrm{Irr}(C_F)$, contradicting Theorem 8.4. □

Note that if $\langle A_i:i\in I\rangle$ is a system of infinite BAs and $X_i\subseteq A_i$ is irredundant for each $i\in I$, then $\prod_{i\in I}A_i$ has an irredundant subset of size $|\prod_{i\in I}X_i|$. For, by Lemma 13.9 of the Handbook, let $Y\subseteq\prod_{i\in I}X_i$ be finitely distinguished and of size $|\prod_{i\in I}X_i|$. Clearly Y is irredundant. This leads to the following problem.

Problem 87. *For $\langle A_i:i\in I\rangle$ a system of infinite BAs with I infinite, determine, if possible,* $\mathrm{Irr}\left(\prod_{i\in I}A_i\right)$ *in terms of* $\langle\mathrm{Irr}(A_i):i\in I\rangle$.

Obviously $\max(\mathrm{Irr}(A), \mathrm{Irr}(B)) \leq \mathrm{Irr}(A \oplus B)$. We do not know whether equality holds here.

Problem 88. *For infinite BAs A, B is it true that*

$$\max(\mathrm{Irr}(A), \mathrm{Irr}(B)) = \mathrm{Irr}(A \oplus B)?$$

Irredundance is an ultra-sup function, so Theorems 3.20–3.22 of Peterson apply. By Theorem 3.22, $\mathrm{Irr}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Irr}(A_i)/F\right|$ for regular F; hence by Donder's result it is consistent that this inequality always holds. Shelah in [03] showed that it is consistent to have $\mathrm{Irr}\left(\prod_{i\in I} A_i/F\right) < \left|\prod_{i\in I} \mathrm{Irr}(A_i)/F\right|$. On the other hand, Shelah in [99] shows (in 15.10) that it is consistent to have $\mathrm{Irr}\left(\prod_{i\in I} A_i/F\right) > \left|\prod_{i\in I} \mathrm{Irr}(A_i)/F\right|$. These results answer problems 25 and 26 in Monk [96].

Concerning the derived functions, we note just the obvious facts that $\mathrm{Irr}_{\mathrm{S}+}(A) = \mathrm{Irr}(A)$, $\mathrm{Irr}_{\mathrm{S}-}(A) = \omega$, and ${}_{\mathrm{d}}\mathrm{Irr}_{\mathrm{S}+}(A) = \mathrm{Irr}(A)$.

Obviously any chain is irredundant; so $\mathrm{Length}(A) \leq \mathrm{Irr}(A)$. The difference can be large, e.g., in a free BA. By Theorem 4.25 of Part I of the BA handbook, $\pi(A) \leq \mathrm{Irr}(A)$. In particular, if $|A|$ is strong limit, then $|A| = \mathrm{Irr}(A)$, since then $\pi(A) = |A|$.

It is consistent that there is a BA with irredundance less than cardinality, and it is also consistent that every uncountable BA has uncountable irredundance (see Todorčević [93]). Now we prove the first fact, in the form that under CH there is a BA of power ω_1 with countable irredundance. We give two examples for this. The first example is a compact Kunen line. We say "a" since there are various Kunen lines, and we say "compact" since the standard Kunen lines are only locally compact. For the Kunen lines, see Juhász, Kunen, Rudin [76]. The second construction uses considerably less than CH, and can be found in Todorčević [89]. For a forcing construction of an uncountable BA with countable irredundance, see Bell, Ginsburg, Todorčević [82]. A generalization of the main results about irredundance (to other varieties of universal algebras) can be found in Heindorf [89a].

The history of these results is complicated. I think that the first example of an uncountable BA with countable irredundance is due to Rubin [83] (the result was obtained several years before 1983). The papers with the constructions we give do not mention irredundance; their relevance for our purposes is due to a simple theorem of Heindorf [89a]. So, modulo the simple theorem of Heindorf, the first example with irredundance different from cardinality is a Kunen line.

Before beginning the examples we need two topological lemmas. A topological space is *hereditarily separable* iff every subset of it has a countable dense subset.

Lemma 8.8. *Suppose that X is a non-compact, locally compact Hausdorff space, and Y is its one-point compactification. Then:*

(i) *If the compact-open sets of X form a base, then Y is a Boolean space.*
(ii) *For every integer $k > 0$, if ${}^kX$ is hereditarily separable, then so is ${}^kY$.*

Proof. Recall that Y is obtained from X by adding one new point y, and declaring the topology on Y to consist of all open sets U of X together with all sets $\{y\}\cup U$ such that $U\subseteq X$ and $X\backslash U$ is compact in X. Note that if $U\subseteq X$ and $X\backslash U$ is compact in X then U is open in X.

(i): We want to show that the clopen subsets of Y form a base. Any subset of X which is compact and open in X is clopen in Y. So it suffices to show that each "new" basic open set contains a new basic open set which is clopen. So, let W be a new basic open set – say $W=\{y\}\cup U$ where $U\subseteq X$ and $X\backslash U$ is compact in X. Then $X\backslash U\subseteq V$ for some compact open subset V of X, by the compactness of $X\backslash U$ and the hypothesis of (i). Thus $\{y\}\cup(X\backslash V)\subseteq W$ and $\{y\}\cup(X\backslash V)$ is clopen in Y, as desired.

(ii): Fix $x\in X$. Assume that k is a positive integer and ${}^kX$ is hereditarily separable. Now suppose that S is a non-empty subspace of ${}^kY$. For each $\Gamma\subseteq k$ let

$$S_\Gamma=\{z\in S:\forall i<k(z_i=y \text{ iff } i\in\Gamma)\};$$
$$S'_\Gamma=\{w\in {}^kX:\exists z\in S_\Gamma\forall i<k[(i\in\Gamma\Rightarrow w_i=x) \text{ and } (i\notin\Gamma\Rightarrow w_i=z_i)]\}.$$

Then for each $\Gamma\subseteq k$ let C'_Γ be a countable dense subset of S'_Γ. Next, let

$$C_\Gamma=\{z\in {}^kY:\exists w\in C'_\Gamma\forall i<k[(i\in\Gamma\Rightarrow z_i=y) \text{ and } (i\notin\Gamma\rightarrow z_i=w_i)]\}.$$

We claim that $D\stackrel{\text{def}}{=}\bigcup_{\Gamma\subseteq k}C_\Gamma$ is dense in S (as desired). To this end, take an open set U such that $U\cap S\neq 0$. We may assume that U has the form $V_0\times\cdots\times V_{k-1}$, where each V_i is open in Y. Say $U\cap S_\Gamma\neq 0$. Define, for $i<k$,

$$V'_i=\begin{cases}X, & \text{if } i\in\Gamma,\\ V_i\cap X, & \text{if } i\notin\Gamma.\end{cases}$$

Set $U'=V'_0\times\cdots\times V'_{k-1}$. Then $U'\cap S'_\Gamma\neq 0$. In fact, if $z\in U\cap S_\Gamma$, define

$$w_i=\begin{cases}z_i & \text{if } z_i\neq y,\\ x & \text{otherwise.}\end{cases}$$

Then $w\in U'\cap S'_\Gamma$. Hence $U'\cap C'_\Gamma\neq 0$. Take $w\in U'\cap C'_\Gamma$. Define, for $i<k$,

$$z_i=\begin{cases}y, & \text{if } i\in\Gamma,\\ w_i, & \text{if } i\notin\Gamma.\end{cases}$$

Then $z\in U\cap C_\Gamma$, as desired. □

Lemma 8.9. *If X and Y are topological spaces, X is hereditarily separable, and Y is second countable, then $X\times Y$ is hereditarily separable.*

Proof. Let A be a nonempty subset of $X\times Y$. Let $\mathscr{A}$ be a countable base for Y. For each $U\in\mathscr{A}$ let $B_U=\{x\in X:(\{x\}\times U)\cap A\neq\emptyset\}$. Since X is hereditarily separable, we can find a countable dense subset C_U of each nonempty set B_U.

For each U with B_U nonempty and each $x \in C_U$ choose $y_{xU} \in U$ such that $(x, y_{xU}) \in A$. We claim now that $D \stackrel{\text{def}}{=} \{(x, y_{xU}) : U \in \mathscr{A}, B_U \neq \emptyset, x \in C_U\}$ is dense in A. Clearly D is countable. To prove the claim, suppose that V and U are open in X and Y respectively, and $A \cap (V \times U) \neq \emptyset$. We want to show that $D \cap (V \times U) \neq \emptyset$. We may assume that $U \in \mathscr{A}$. Note that $V \cap B_U \neq \emptyset$. Choose $x \in C_U \cap V$. Then $y_{xU} \in U$, and hence $(x, y_{xU}) \in D$. □

Theorem 8.10 (CH)**.** *There is a topological space X satisfying the following conditions:*

(i) *The compact open sets form a base for the topology on X.*
(ii) *X is first-countable and Hausdorff.*
(iii) *${}^kX$ is hereditarily separable for each $k \in \omega\backslash 1$.*
(iv) *X has ω_1 compact open sets.*

Proof. In this construction we follow de la Vega, Kunen [04]. Let r be a bijection from ω_1 onto $\mathbb{R}$, and let ρ be the topology $\{U \subseteq \omega_1 : r[U] \text{ open in } \mathbb{R}\}$ on ω_1. For each $\xi < \omega_1$ we denote by ρ_ξ the subspace topology on ξ determined by ρ. If $n \in \omega\backslash 1$ and $\xi < \omega_1$, then $\mathrm{Iseq}(n, \xi)$ is the set of all $f \in {}^n\omega_1$ such that $f(0) < f(1) < \cdots < f(n-1) = \xi$.

Now we claim

(1) There is an enumeration $\langle S_\mu : \mu \in \omega_1\backslash 1\rangle$ of $\bigcup_{k\in\omega\backslash 1}[{}^k\omega_1]^{\leq\omega}$ such that for each $\mu \in \omega_1\backslash 1$ there is an $n(\mu) \in \omega\backslash 1$ such that $S_\mu \subseteq {}^{n(\mu)}\mu$.

To see this, first let $\langle S'_\mu : \mu < \omega_1\rangle$ be any enumeration of $\bigcup_{k\in\omega\backslash 1}[{}^k\omega_1]^{\leq\omega}$. Suppose that $\mu \in \omega_1\backslash 1$ and S_ν has been defined for all positive $\nu < \mu$. Let $S_\mu = S'_\rho$, where ρ is minimum such that $S'_\rho \subseteq {}^k\mu$ for some $k \in \omega\backslash 1$ and $S'_\rho \notin \{S_\nu : \nu < \mu\}$. To see that this is the desired enumeration, we just need to see that any S'_ρ appears as some S_μ. Suppose that S'_ρ is not in $\{S_\mu : \mu < \omega_1\}$, and choose ρ minimum with this property. Say $S'_\rho \subseteq {}^k\tau$ for some $k \in \omega\backslash 1$ and $\tau < \omega_1$. For each $\sigma < \rho$ let $\nu(\sigma)$ be such that $S'_\sigma = S_{\nu(\sigma)}$. Let $\mu < \omega_1$ be such that $\nu(\sigma) < \mu$ for all $\sigma < \rho$ and $\tau \leq \mu$. Thus $S'_\rho \subseteq {}^k\mu$ and $S'_\rho \notin \{S_\nu : \nu < \mu\}$. If $\sigma < \rho$, then $S'_\sigma \in \{S_\nu : \nu < \mu\}$. Hence ρ is minimum such that $S'_\rho \subseteq {}^k\mu$ and $S'_\rho \notin \{S_\nu : \nu < \mu\}$. So $S_\mu = S'_\rho$, contradiction. This proves (1).

Now we define by recursion $\langle \tau_\xi : 1 \leq \xi \in \omega_1\rangle$ so that the following conditions hold for each $\eta \leq \omega_1$ with $\eta \neq 0$:

(2) τ_η is a topology on η.

(3) If $0 < \xi < \eta$, then $\tau_\xi = \tau_\eta \cap \mathscr{P}(\xi)$.

(4) τ_η is first countable and Hausdorff.

(5) The compact open sets form a base for τ_η.

(6) $\rho_\eta \subseteq \tau_\eta$.

(7) If $\eta = \xi + 1$, $0 < \mu < \xi$, $f \in Iseq(n(\mu), \xi)$, and $f \in \mathrm{cl}(S_\mu, {}^{n(\mu)-1}\tau_\eta \times \rho_\eta)$, then $f \in \mathrm{cl}(S_\mu, {}^{n(\mu)}\tau_\eta)$.

For $0 < \xi \le \omega$ we let $\tau_\xi = \mathscr{P}(\xi)$. Clearly (2)–(7) hold for ξ. For $\eta > \omega$ limit, define $\tau_\eta = \{U \subseteq \eta : \forall \xi < \eta[U \cap \xi \in \tau_\xi]\}$. Again it is clear that (2)–(7) hold.

Now suppose that $\eta = \xi + 1$, with η infinite. We claim:

(8) If $\mathscr{A}$ is a countable collection of subsets of ξ, then there is a topology τ'_η on η such that (2)–(6) hold for τ'_η, and also $\forall E \in \mathscr{A}[\xi \in \mathrm{cl}(E, \rho) \to \xi \in \mathrm{cl}(E, \tau'_\eta)]$.

To prove (8), let $\mathscr{A}' = \{E \in \mathscr{A} : \xi \in \mathrm{cl}(E, \rho)\}$. If $\mathscr{A}' = \emptyset$, we can let $\tau'_\eta = \tau_\xi \cup \{\{\xi\}\} \cup \{\eta\}$, and the desired conclusions follow. Now assume that $\mathscr{A}' \neq \emptyset$. Let $\langle U_n : n \in \omega\rangle$ be a nested countable base for ξ in the topology ρ_ξ. Let $\langle V_m : m < \omega\rangle$ enumerate $\mathscr{A}'$, with each member of $\mathscr{A}'$ appearing ω times. For each $m \in \omega$ choose $p_m \in V_m \cap U_m$, and by (5) and (6) let K_m be compact open in τ_ξ with $p_m \in K_m \subseteq U_m$. Then we define τ'_η to be the topology on η with base $\tau_\xi \cup \{\{\xi\} \cup \bigcup_{m>n} K_m : n \in \omega\}$.

Clearly (2)–(3) hold, and τ'_η is first countable. To show that it is Hausdorff, it suffices to find disjoint neighborhoods of ξ and any $\eta < \xi$. Let V, W be open in ρ with $\eta \in V$, $\xi \in W$, $V \cap W = \emptyset$. Choose n such that $\xi \in U_n \subseteq W$. Then V and $\{\xi\} \cup \bigcup_{m>n} K_m$ are the desired disjoint neighborhoods. So (4) holds. To prove (5) it suffices to show that for any $n \in \omega$ the set $\{\xi\} \cup \bigcup_{m>n} K_m$ is compact. Let $\mathscr{B}$ be an open cover of it by base elements. Then there is a $p \in \omega$ such that $\{\xi\} \cup \bigcup_{m>p} K_m \in \mathscr{B}$. Wlog $n < p$. Then there is a finite $\mathscr{B}' \subseteq \mathscr{B}$ such that $\bigcup_{p<m\le n} K_M \subseteq \bigcup \mathscr{B}'$. Thus

$$\{\xi\} \cup \bigcup_{m>n} K_m \subseteq \{\xi\} \cup \bigcup_{m>p} K_m \cup \bigcup \mathscr{B}',$$

as desired; (5) holds.

For (6), suppose that $V \in \rho$. Then $V \cap \xi \in \rho_\xi \subseteq \tau_\alpha \subseteq \tau'_\eta$. Suppose that $\xi \in V$. Choose $m \in \omega$ such that $\xi \in U_m \subseteq V$. Then

$$V \cap \eta = (V \cap \xi) \cup \{\xi\} \cup \bigcup_{n>m} K_n \in \tau'_\eta,$$

and (6) holds.

Now for the final condition of (8), suppose that $E \in \mathscr{A}$ and $\xi \in \mathrm{cl}(E, \rho)$. Thus $E \in \mathscr{A}'$. Suppose that W is a basic neighborhood of ξ in τ'_η. Say $W = \{\xi\} \cup \bigcup_{m>n} K_m$. Choose $m > n$ with $E = V_m$. Then $p_m \in K_m \cap V_m$, and so $W \cap E \neq \emptyset$. This proves (8).

Now to actually define τ_η we proceed as follows. For each μ with $0 < \mu < \xi$ and $f \in \mathrm{Iseq}(n(\mu), \xi)$ we associate a countable subset $\mathscr{B}_{\mu f}$ of ξ, as follows. If $n(\mu) = 1$, we set

$$\mathscr{B}_\mu = \{\rho < \omega_1 : \{(0, \rho)\} \in S_\mu\}.$$

Now suppose that $n(\mu) > 1$. By (4), let $\mathscr{C}_f$ be a countable neighborhood base for $f \restriction (n(\mu)-1)$ in ${}^{n(\mu)-1}\xi$ under the topology τ_ξ, and suppose that $U \in \mathscr{C}_f$. Define

$$\mathscr{B}_{\mu f U} = \{\sigma < \xi : \exists g \in S_\mu[g \restriction (n(\mu)-1) \in U \text{ and } g(n(\mu)-1) = \sigma]\}.$$

Finally, let

$$\mathscr{C}_f = \bigcup\{\mathscr{B}_\mu : \mu < \xi, n(\mu) = 1\} \cup \bigcup\{\mathscr{B}_{\mu f U} : \mu < \xi, n(\mu) > 1, \text{ and } U \in \mathscr{C}_f\};$$
$$\mathscr{A} = \bigcup\{\mathscr{C}_f : n \in \omega\backslash 1, f \in Iseq(n,\xi)\}.$$

Thus $\mathscr{A}$ is a countable collection of subsets of ξ. We then apply (8) to get our topology τ_η. We now check (7) for η. Suppose that $0 < \mu < \xi$, $f \in Iseq((n(\mu),\xi)$, and $f \in \mathrm{cl}(S_\mu, {}^{n(\mu)-1}\tau_\eta \times \rho_\eta)$. To show that $f \in \mathrm{cl}(S_\mu, {}^{n(\mu)}\tau_\eta)$, let $V \in {}^{n(\mu)}\tau_\eta$ be a system of basic open sets such that $f \in V$. Choose $U \in \mathscr{C}_f$ such that $U_i \subseteq V_i$ for all $i < n(\mu)-1$.

Case 1. $n(\mu) = 1$. Thus $f = \{0,\xi)\}$, and so the assumption that $f \in \mathrm{cl}(S_\mu, {}^{n(\mu)-1}\tau_\eta \times \rho_\eta)$ means that $\xi \in \mathrm{cl}(\mathscr{B}_\mu, \rho_\eta)$. Hence by (8) we get $\xi \in \mathrm{cl}(\mathscr{B}_\mu, \tau_\eta)$, so that $f \in \mathrm{cl}(S_\mu, \tau_\eta)$.

Case 2. $n(\mu) > 1$. We claim

(9) $\xi \in \mathrm{cl}(\mathscr{B}_{\mu f U}, \rho_\eta)$.

To prove this, let W be any neighborhood of ξ in ρ_η. Then $U \times W$ is a neighborhood of f in ${}^{n(\mu)-1}\tau_\eta \times \rho_\eta$, so we can choose $g \in S_\mu \cap {}^{n(\mu)-1}\tau_\eta \times \rho_\eta$. Hence $g(n(\mu)-1) \in \mathscr{B}_{\mu f U} \cap W$, as desired for (9),

Now by (8) we get $\xi \in \mathrm{cl}(\mathscr{B}_{\mu f U}, \tau_\eta)$. Now $\xi \in V_{n(\mu)-1}$, so it follows that there is a $\sigma \in \mathscr{B}_{\mu f U} \cap V_{n(\mu)-1}$. By the definition of $\mathscr{B}_{\mu f U}$, there is a $g \in S_\mu$ such that $f \restriction (n(\mu)-1) \in U$ and $g(n(\mu)-1) = \sigma$. So $g \in S_\mu \cap V$, as desired.

This proves (7) for η, and the construction is finished.

Now we let X be ω_1 with the topology τ_{ω_1}. So (i) and (ii) of the lemma hold. For (iii), we first show that X itself is separable. So, let $\emptyset \neq A \subseteq X$. Now A is separable in the topology ρ, so let D be a countable dense subset of A in the ρ sense. Fix $\mu < \omega_1$ such that $n(\mu) = 1$ and $\mathscr{B}_\mu = D$, where as above $\mathscr{B}_\mu = \{\sigma : \{(0,\sigma)\} \in S_\mu\}$. Suppose that $\mu < \xi < \omega_1$. Let $f = \{(0,\xi)\}$. Then $f \in Iseq(n(\mu),\xi)$ and $f \in \mathrm{cl}(S_\mu, \rho_{\xi+1})$, so by (7), $f \in \mathrm{cl}(S_\mu, \tau_{\xi+1})$. This shows that $A\backslash\mathrm{cl}(S_\mu, \tau_{\xi+1}) \subseteq (\mu+1)$. Hence

$$S_\mu \cup (A\backslash\mathrm{cl}(S_\mu, \tau_{\xi+1}))$$

is a countable dense subset of A in the τ_{ω_1} sense.

Now suppose that we have shown that ${}^m X$ is hereditarily separable for every positive $m < n$, where $n > 1$; we want to show that ${}^n X$ is hereditarily separable. Let $\emptyset \neq A \subseteq {}^n X$. For each $f \in A$ let $k(f) = \{(i,j) \in n \times n : f(i) = f(j)\}$.

Thus $k(f)$ is an equivalence relation on n. For each m with $1 \leq m \leq n$ let $B_m = \{f \in A : |\operatorname{rng}(f)| = m\}$. So $A = \bigcup_{1\leq m\leq n} B_m$. Hence it suffices to show that each B_m is separable. First assume that $1 \leq m < n$. For each $f \in B_m$, let $\langle K_i : i < m\rangle$ enumerate the $k(f)$-equivalence classes, so that

(10) $0 \in K_0$, and for each i with $0 < i < m$, the least element of K_i is the least $j < n$ such that $j \notin \bigcup_{k<i} K_k$.

Now for each $f \in B_m$, define $g_f \in {}^m\alpha_1$ by setting $g_f(i) = f(j)$ for some, and hence every, $j \in K_i$, for each $i < m$. Note that if $f, h \in B_m$ with $f \neq h$, then $g_f \neq g_h$. Let $C_m = \{g_f : f \in B_m\}$. So $C_m \subseteq {}^mX$, and hence by the inductive hypothesis C_m has a countable dense subset D_m. Let $E_m = \{f \in B_m : g_f \in D_m\}$. Since $g_f \neq g_h$ for $f \neq h$, it follows that E_m is countable. We claim that it is dense in B_m. For, let $U \in {}^n\tau_{\omega_1}$ be such that $U \cap B_m \neq \emptyset$. Say $f \in U \cap B_m$. For each $i < m$ let $V_i = \bigcap_{j\in K_i} U_j$. Then $V \in {}^m\tau_{\omega_1}$ and $g_f \in V \cap C_m$. Hence there is an $h \in V \cap D_m$. Say $h = g_k$ with $k \in B_m$. So $k \in E_m$. Clearly also $k \in U$. So $U \cap E_m \neq \emptyset$, as desired. So we have shown that B_m is separable.

It remains only to show that B_n is separable. Now each permutation π of n induces an autohomeomorphism π_e of ${}^nX$, defined by $\pi_e(f) = f \circ \pi$ for every $f \in {}^nX$. Let P be the set of all permutations of n. Now by the inductive hypothesis and Lemma 1, ${}^{n-1}\tau_{\omega_1} \times \rho$ is hereditarily separable. For each $\pi \in P$ let $\mu(\pi)$ be such that $\nu(\mu(\pi)) = n$ and $S_{\mu(\pi)}$ is $({}^{n-1}\tau_{\omega_1} \times \rho)$-dense in $\pi_e[A]$. Note that $\pi_e^{-1}[S_{\mu(\pi)}] \subseteq A$.

We claim that the countable set

$$\{f \in A \cap {}^n[0, \max\{\mu(\pi) : \pi \in P\}]\} \cup \bigcup_{\pi\in P} \pi_e^{-1}[S_{\mu(\pi)}]$$

is dense in A in the topology ${}^n\tau_{\omega_1}$. For, suppose that $f \in A$ and $\max(\operatorname{rng}(f)) > \max\{\mu(\pi) : \pi \in P\}$. Choose $\pi \in P$ so that $f \circ \pi$ is strictly increasing, and let $\xi = f(\pi(n-1))$. Then $\xi > \mu(\pi)$ and $f \circ \pi \in \pi_e[A]$, and hence

$$f \circ \pi \in \operatorname{cl}(S_{\mu(\pi)}, {}^{n-1}\tau_{\omega_1} \times \rho).$$

Moreover, $f \circ \pi \in \operatorname{Iseq}(n(\mu(\pi)), \xi)$. Hence by (7),

$$f \circ \pi \in \operatorname{cl}(S_{\mu(\pi)}, {}^n\tau_{\omega_1}).$$

It follows that

$$f \in \operatorname{cl}\pi_e^{-1}(S_{\mu(\pi)}, {}^n\tau_{\omega_1}).$$

For condition (iv) of the theorem, note that in each nontrivial step from ξ to $\xi + 1$ countably many new compact open sets were introduced: all of the sets $\{\xi\} \cup \bigcup_{m>n} K_m$. □

The theorem of Heindorf mentioned above is as follows.

Theorem 8.11. *Let X be a Boolean space, and A its BA of closed-open sets. Then* $\mathrm{Irr}(A) \le \mathrm{s}(X \times X)$.

Proof. Suppose that I is an infinite irredundant subset of A; we will produce an ideal independent subset of $A \oplus A$ of power $|I|$ (as desired). Namely, take the set $\{a \times -a : a \in I\}$; it is as desired, for suppose that

$$a \times -a \subseteq (b_0 \times -b_0) \cup \cdots \cup (b_{m-1} \times -b_{m-1}),$$

where $a, b_0, \ldots, b_{m-1}$ are distinct elements of I. Now a is not in $\langle\{b_i : i < m\}\rangle$, so it follows that in that subalgebra, a splits some atom; this means that there is an $\varepsilon \in {}^m 2$ such that, if we set $d = \bigcap_{i<m} b_i^{\varepsilon_i}$ then we have $d \cap a \ne 0 \ne d \cap -a$. Choosing $x \in d \cap a$ and $y \in d \cap -a$ it follows that $(x, y) \in a \times -a$ but $(x, y) \notin b_i \times -b_i$ for each $i < m$, giving the desired contradiction. $\square$

Rosłanowski, Shelah [00] show that it is consistent to have $\mathrm{Irr}(A) < \mathrm{s}(A \oplus A)$, answering Problem 27 of Monk [96].

Theorem 8.12. (CH) *There is a BA A of size ω_1 with countable irredundance.*

Proof. Recall that for any BA A, $\mathrm{s}(A) = \mathrm{c_{H+}}(A) \le \mathrm{d_{H+}}(A) = \mathrm{hd}(A)$. $\square$

Example 8.13. This example, which as we mentioned is from Todorčević [89], constructs a topology on a certain subset of ${}^\omega\omega$. First, some notation: If A is a set with a linear order $<$ on it, and if $k \in \omega$, then $\langle A\rangle^k$ denotes the set of all $f \in {}^k A$ such that $f_i < f_j$ for all $i < j < k$. For $f, g \in {}^\omega\omega$ define $f <^* g$ if $\exists m \forall n \ge m(f(n) < g(n))$. The BA we want will be constructed under the assumption that there is a subset A of ${}^\omega\omega$ of power ω_1 which is unbounded under $<^*$. This means that $\mathfrak{b} = \omega_1$, where $\mathfrak{b}$ is a well-known cardinal. It is easy to see that $\omega_1 \le \mathfrak{b} \le 2^\omega$, and it is consistent to have $\mathfrak{b} = \omega_1 < 2^\omega$.

Without loss of generality A has order type ω_1 under $<^*$ and all members of A are strictly increasing. In fact, take the A originally given, and write $A = \{f_\alpha : \alpha < \omega_1\}$. Then one can inductively define $\bar{f}_\alpha$ for $\alpha < \omega_1$ so that $\bar{f}_\beta <^* \bar{f}_\alpha$ for $\beta < \alpha$, $f_\alpha <^* \bar{f}_\alpha$, and $\bar{f}_\alpha$ is strictly increasing. Namely, let $\bar{f}_0$ be arbitrary. If $\bar{f}_\beta$ has been constructed for all $\beta < \alpha$, let $\langle g_n : n < \omega\rangle$ enumerate $\langle \bar{f}_\beta : \beta < \alpha\rangle$. Define $\bar{f}_\alpha(n)$ to be $> \bar{f}_\alpha(m)$ for all $m < n$, also $> f_\alpha(n)$, and also $> g_m(n)$ for all $m < n$. Clearly this works. The new set $\{\bar{f}_\alpha : \alpha < \omega_1\}$ (still denoted by A below) has the desired properties.

We will apply the above notation $\langle A\rangle^k$ to A under the ordering $<^*$. Let T be an Aronszajn subtree of $\{s \in {}^{<\omega_1}\omega : s \text{ is one-one}\}$. (See Kunen [80], p. 70.) For each $\alpha < \omega_1$ let t_α be a member of T with domain α. Define $e : \langle A\rangle^2 \to \omega$ by $e(\bar{f}_\alpha, \bar{f}_\beta) = t_\beta(\alpha)$ for $\alpha < \beta$. Then clearly

(1) For all $b \in A$, the function $e_b \overset{\text{def}}{=} e(\cdot, b)$ is a one-one map from $A_b \overset{\text{def}}{=} \{a \in A : a <^* b\}$ into ω.

Also,

(2) For all $a \in A$, the set $\{e_b \restriction A_a : b \in A, a <^* b\}$ is countable.

In fact, for each $e \in A$ write $e = f_{\beta(e)}$. Then for any b, d such that $a <^* b$ and $a <^* d$ we have

$$\begin{aligned}(e_b \restriction A_a) \neq (e_d \restriction A_a) \quad &\text{iff} \quad \langle e_b(c) : c \in A_a\rangle \neq \langle e_d(c) : c \in A_a\rangle \\ &\text{iff} \quad \langle e(c,b) : c <^* a\rangle \neq \langle e(c,d) : c <^* a\rangle \\ &\text{iff} \quad \langle t_{\beta(b)}(\beta(c)) : c <^* a\rangle \neq \langle t_{\beta(d)}(\beta(c)) : c <^* a\rangle \\ &\text{iff} \quad \text{there is a } \gamma < \beta(a) \text{ such that } t_{\beta(b)}(\gamma) \neq t_{\beta(d)}(\gamma);\end{aligned}$$

hence $(e_b \restriction A_a) \neq (e_d \restriction A_a)$ implies that $t_{\beta(b)} \restriction \beta(a)$ and $t_{\beta(d)} \restriction \beta(a)$ are incomparable. Since level $\beta(a)$ of the tree T is countable, (2) follows.

For distinct $a, b \in A$ let $\Delta(a,b)$ be the least $n < \omega$ such that $a(n) \neq b(n)$. And let $\Delta(a,a) = \infty$. The following fact about this notation will be useful:

($\star$) If $\Delta(a,b) < \Delta(c,b)$ then $\Delta(a,c) = \Delta(a,b)$.

To see this, note that $a \restriction \Delta(a,b) = b \restriction \Delta(a,b) = c \restriction \Delta(a,b)$, and $a(\Delta(a,b)) \neq b(\Delta(a,b)) = c(\Delta(a,b))$, so $\Delta(a,c) = \Delta(a,b)$.

Now we define $H : A \to \mathscr{P}(A)$ by

$$H(b) = \{a \in A : a <^* b \text{ and } e(a,b) \leq b(\Delta(a,b))\}.$$

Now

(3) for all $l < \omega$ and $b \in A$ the set $\{a \in H(b) : \Delta(a,b) = l\}$ is finite.

In fact, if $a \in H(b)$ and $\Delta(a,b) = l$, then $t_{\beta(b)}(\beta(a)) = e(f_{\beta(a)}, f_{\beta(b)}) = e(a,b) \leq b(l)$. Since $b(l) \in \omega$ and $t_{\beta(b)}$ is one-one, (3) follows.

Next we define $C(b)$ for $b \in A$ by recursion on b: $a \in C(b)$ iff $a = b$ or

(4) $\exists c \in H(b)(a \in C(c) \text{ and } \forall d \in H(b)(d \neq a \text{ and } d \neq c \Rightarrow \Delta(a,d) < \Delta(a,c)))$.

Note that

$$H(b) \subseteq C(b) \tag{5}$$

for all $b \in A$ (if $a \in H(b)$, take $c = a$ and note that $\Delta(a,a) = \infty$). Also we have

(6) If $a \in C(b)$, then $a <^* b$ or $a = b$.

We prove (6) by induction on b. Assume that it is true for all $c <^* b$, and suppose that $a \in C(b)$ and $a \neq b$. Then there is a $c \in H(b)$ such that $a \in C(c)$. We have $c <^* b$ since $c \in H(b)$, and $a <^* c$ by the inductive hypothesis. So $a <^* b$.

For each $n \in \omega$ and $b \in A$ let $C_n(b) = \{a \in C(b) : \Delta(a,b) \geq n\}$. Then

$$c \in H(b) \Rightarrow \exists l(C_l(c) \subseteq C(b)). \tag{7}$$

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In fact, $\{x \in H(b) : \Delta(x,b) = \Delta(c,b)\}$ is finite by (3). Choose $l > \Delta(x,c)$ for any $x \neq c$ which is in this set, and with $l > \Delta(c,b)$. Suppose that $d \in C_l(c)$. We claim that $d \in C(b)$, and that the element c works to show this in (4). Indeed, suppose that $x \in H(b)$, $x \neq d$, $x \neq c$, and $\Delta(d,x) \geq \Delta(d,c)$. (In (4), replace a,b,c,d by d,b,c,x respectively.) Now $\Delta(c,b) < \Delta(d,c)$, so $\Delta(b,d) = \Delta(c,b)$ by $(\star)$, and $\Delta(b,d) < \Delta(d,x)$, so $\Delta(x,b) = \Delta(c,b)$ by $(\star)$. So x is in the indicated finite set, which gives $\Delta(x,c) < l \leq \Delta(c,d) \leq \Delta(d,x)$, so $\Delta(c,d) = \Delta(x,c)$ by $(\star)$, contradiction.

$$(8) \qquad a \in C(b) \Rightarrow \exists l(C_l(a) \subseteq C(b)).$$

For, we may assume that $a \notin H(b)$ by (7), and we proceed by induction on b. The conclusion is clear if $a = b$, so suppose that $a \neq b$. Choose c in accordance with (4). Then $a \neq c$ since $a \notin H(b)$. By the induction hypothesis, choose l such that $C_l(a) \subseteq C(c)$. Without loss of generality, $l > \Delta(a,c)$. We claim that $C_l(a) \subseteq C(b)$. To prove this, let $d \in C_l(a)$. So, $d \in C(c)$. Suppose $x \in H(b)$, $x \neq d$, and $x \neq c$. (We are going to show that $d \in C(b)$ by replacing a,b,c,d in (4) by d,b,c,x respectively.) Then $x \neq a$ since $a \notin H(b)$. So $\Delta(a,x) < \Delta(a,c)$ by the choice of c. Now $\Delta(a,c) < l \leq \Delta(a,d)$, so $\Delta(c,d) = \Delta(a,c)$ by $(\star)$. Also, $\Delta(a,x) < \Delta(a,d)$, so $\Delta(d,x) = \Delta(a,x) < \Delta(a,c) = \Delta(c,d)$, showing that $d \in C(b)$.

Now

$$(9) \qquad a \in C_m(b) \Rightarrow \exists l(C_l(a) \subseteq C_m(b)).$$

In fact, suppose that $a \in C_m(b)$. By (8), choose l such that $C_l(a) \subseteq C(b)$. Let $p = \max\{l,m\}$. We claim that $C_p(a) \subseteq C_m(b)$. For, suppose that $c \in C_p(a)$. Thus $c \in C(a)$ and $\Delta(a,c) \geq p$. Since $c \in C_p(a) \subseteq C_l(a) \subseteq C(b)$, we have $c \in C(b)$. Also, $c \restriction m = a \restriction m = b \restriction m$, so $\Delta(b,c) \geq m$, as desired.

From (9) it follows that the collection of sets $\{C_m(b) : b \in A, m \in \omega\}$ forms a base for a topology on A. It is Hausdorff, since, given $a \neq b$, let $l = \Delta(a,b)+1$; clearly $C_l(a) \cap C_l(b) = 0$. Also note that each set $C(b) = C_0(b)$ is open. Next,

$$(10) \qquad C_l(b) \text{ is closed in } C(b).$$

For, suppose that $x \in C(b) \backslash C_l(b)$, and let $m = \Delta(x,b)+1$. Now $C_m(x) \cap C_l(b) = \emptyset$. For suppose that $d \in C_m(x) \cap C_l(b)$. Then $\Delta(x,d) \geq m$, $\Delta(d,b) \geq l$, and $\Delta(b,x) < l$. So $\Delta(b,x) < \Delta(d,b)$, hence by $(\star)$ $m \leq \Delta(d,x) = \Delta(b,x) < m$, contradiction. Then $C_m(x) \cap C(b) \subseteq C(b) \backslash C_l(b)$, as desired.

$$(11) \qquad C(b) \text{ is compact.}$$

We prove this by induction on b. So, assume that it is true for all $c <^* b$, and suppose that $C(b) \subseteq \bigcup_{x \in X} C_{m(x)}(x)$. Then choose $y \in X$ such that $b \in C_{m(y)}(y)$. There is an l such that $C_l(b) \subseteq C_{m(y)}(y)$. Now we consider two cases:

Case 1. $H(b)$ is finite. In this case, we can easily show that $C(b)$ is closed: suppose that $a \in A \backslash C(b)$. Hence $a \neq b$ and

$$(*) \quad \forall c \in H(b)[a \notin C(c) \text{ or } \exists d \in H(b)[d \neq a \text{ and } d \neq c \text{ and } \Delta(a,d) \geq \Delta(a,c)]].$$

If $c \in H(b)$ and $a \notin C(c)$, choose an open neighborhood U_c of a with the property that $U_c \cap C(c) = 0$, using the inductive hypothesis. For $c \in H(b)$ and $a \in C(c)$, choose $d = d(a,c) \in H(b)$ such that $d \neq a$, $d \neq c$, and $\Delta(a,d) \geq \Delta(a,c)$. Let

$$V = C_{\Delta(a,b)+1}(a) \cap \bigcap_{\substack{c \in H(b) \\ a \notin C(c)}} U_c \cap \bigcap_{\substack{c \in H(b) \\ a \in C(c)}} C_{\Delta(a,d(a,c))+1}(a).$$

We claim that $V \cap C(b) = 0$ (as desired, showing that $C(b)$ is closed). For, suppose that $x \in V \cap C(b)$. Since $x \in C_{\Delta(a,b)+1}(a)$, we have $x \neq b$. Choose, then, $c \in H(b)$ such that $x \in C(c)$ and for all $d \in H(b)$, if $d \neq x$ and $d \neq c$ then $\Delta(x,d) < \Delta(x,c)$. If $a \notin C(c)$, then $x \in U_c \cap C(c)$, contradiction. So $a \in C(c)$. Set $d = d(a,c)$. Now $x \in C_{\Delta(a,d)+1}(a)$, so $x \neq d$ and $\Delta(a,x) > \Delta(a,d)$. Hence $\Delta(d,x) = \Delta(a,d)$ by $(\star)$. Now $\Delta(a,d) \geq \Delta(a,c)$, so $\Delta(a,c) < \Delta(a,x)$. Hence by $(\star)$, $\Delta(c,x) = \Delta(a,c) \leq \Delta(a,d) = \Delta(d,x)$, contradicting the choice of c.

Now for each $c \in H(b)$ we have that $C(c) \cap C(b)$ is a closed subset of $C(c)$. Note that $H(b) = \{b\} \cup \bigcup_{c \in H(b)}(C(c) \cap C(b))$. Hence the inductive hypothesis finishes this case.

Case 2. $H(b)$ is infinite. For all $c \in H(b)$ let

$$\begin{aligned} D_c = \{a : a \notin H(b), \Delta(a,b) < l,\ a \in C(c),\ \text{and} \\ \forall d \in H(b)(d \neq a \text{ and } d \neq c \Rightarrow \Delta(a,d) < \Delta(a,c))\}. \end{aligned}$$

Now

$$(**) \qquad \text{If } c \in H(b) \text{ and } a \in D_c, \text{ then } \Delta(c,b) < l.$$

For, assume otherwise. Now $\Delta(a,b) < \Delta(c,b)$, so $\Delta(a,c) = \Delta(a,b)$ by $(\star)$. Also, for all $d \in H(b) \backslash \{a,c\}$ we have $\Delta(d,a) < \Delta(a,c) = \Delta(a,b)$, hence by $(\star)$ we get $\Delta(d,b) = \Delta(d,a) < \Delta(a,c)$. So $H(b)$ is finite by (3), contradiction. Thus $(**)$ holds.

Let $Y = \{c \in H(b) : \Delta(c,b) < l\}$. Note that Y is finite by (3). Now

$$(***) \qquad \text{If } c \in Y,\ a \in D_c, \text{ and } m = \Delta(a,c), \text{ then } C_m(c) \subseteq C(b).$$

For, assume the hypotheses. If $\Delta(a,c) \leq \Delta(a,b)$, then $d \in H(b) \backslash \{a,c\}$ implies that $\Delta(a,d) < \Delta(a,c) \leq \Delta(a,b)$, so $\Delta(d,b) = \Delta(a,d) < \Delta(a,c)$ by $(\star)$, hence $H(b)$ is finite by (3), contradiction. Thus $\Delta(a,c) > \Delta(a,b)$. Now suppose that $u \in C_m(c)$. Thus $\Delta(a,c) \leq \Delta(u,c)$. Suppose that $d \in H(b) \backslash \{u,c\}$. Then $d \neq a$ since $a \notin H(b)$. So $\Delta(a,d) < \Delta(a,c)$ since $a \in D_c$, so $\Delta(d,c) = \Delta(a,d) < \Delta(a,c) \leq \Delta(u,c)$ by

($\star$), hence by ($\star$) again, $\Delta(d,u) = \Delta(d,c) < \Delta(u,c)$. This shows that $u \in C(b)$, and it proves ($***$).

For $c \in Y$ with $D_c \neq \emptyset$ let $n(c) = \min\{\Delta(a,c) : a \in D_c\}$. Then

$$(****) \qquad C(b) = C_l(b) \cup Y \cup \bigcup_{c\in Y, D_c\neq\emptyset} C_{n(c)}(c).$$

For, $\supseteq$ holds by (5) and ($***$). For $\subseteq$, suppose that $a \in C(b)$, $a \notin C_l(b)$, $a \notin Y$. Since $a \notin C_l(b)$, we have $a \neq b$. Since $a \notin C_l(b)$ and $a \notin Y$, we have $a \notin H(b)$. Since $a \in C(b)$, choose $c \in H(b)$ such that $a \in C(c)$ and $\forall d \in H(b)(d \neq a$ and $d \neq c \Rightarrow \Delta(a,d) < \Delta(a,c))$. So $a \in D_c$. By ($**$) we get $c \in Y$. Thus $a \in C_{n(c)}(c)$, as desired for $\subseteq$; ($****$) has been proved.

By the inductive hypothesis and (10), each $C_{n(c)}(c)$ is compact, so it follows that $C(b)$ is compact in Case 2.

So, we have proved (11).

From (10) and (11) we get

$$(12) \qquad C_l(b) \text{ is compact; so } A \text{ is locally compact.}$$

$$(13) \qquad a <^* b \Rightarrow C(a) \neq C(b).$$

This is true because $b \in C(b)\backslash C(a)$, using (6). So there are uncountably many compact open sets.

A subset $F \subseteq \langle A\rangle^k$ is *cofinal in A* provided that for all $a \in A$ there is an $f \in F$ such that $a <^* f_i$ for all $i < k$. Next we prove

$$(14) \quad \forall \text{ finite } k \geq 1 \text{ and } \forall \text{ cofinal } F \subseteq \langle A\rangle^k \exists f, g \in F(f_i \in H(g_i) \text{ for all } i < k).$$

This will take a while to prove, but it leads us close to the end of the proof. Since F is cofinal in A, we may assume that there is an enumeration $\langle h^\alpha : \alpha < \omega_1\rangle$ of F such that $h^\alpha_i <^* h^\beta_j$ whenever $\alpha < \beta < \omega_1$ and $i, j < k$. Hence every uncountable subset of F is also cofinal in A. Let $D \subseteq F$ be countable dense in F in the ordinary topology (${}^\omega\omega$ has the product topology with ω having the discrete topology; ${}^k({}^\omega\omega)$ gets the product topology too). Choose $c \in A$ such that $f_i <^* c$ for all $f \in D$ and $i < k$. Now

$$\forall f \in F \exists m \in \omega \forall n \geq m \forall i < j < k(f_i(n) < f_j(n)).$$

Hence there is an uncountable $F_0 \subseteq F$ and an m_0 such that

$$(15) \qquad \forall f \in F_0 \forall n \geq m_0 \forall i < j < k(f_i(n) < f_j(n)).$$

Next, let $F_1 = \{f \in F_0 : c <^* f_0\}$. Then

$$\forall f \in F_1 \exists m > m_0 \forall n \geq m[c(n) < f_0(n)].$$

Hence there is an uncountable $F_2 \subseteq F_1$ and an $m > m_0$ such that

$$(16) \qquad \forall f \in F_2 \forall n \geq m[c(n) < f_0(n)].$$

Now

$$F_2 = \bigcup\{\{f \in F_2 : \forall i < k(f_i \restriction m = s_i)\} : s \in {}^k({}^m\omega)\}.$$

Hence there is an uncountable subset F_3 of F_2 and an $s \in {}^k({}^m\omega)$ such that

(17) $$\forall f \in F_3 \forall i < k(f_i \restriction m = s_i).$$

Let $C = \{e_b \restriction A_c : c <^* b\}$. Then

$$F_3 = \bigcup\{\{f \in F_3 : \forall i < k(e_{f_i} \restriction A_c = u^i)\} : u \in {}^kC\},$$

so, using (2), there is an uncountable $F_4 \subseteq F_3$ and a $u \in {}^kC$ such that

(18) $$\forall f \in F_4 \forall i < k(e_{f_i} \restriction A_c = u^i).$$

Now there is an $n \in \omega$ such that $\{f_0(n) : f \in F_4\}$ is unbounded in ω, since otherwise, for each $n \in \omega$ let $g(n)$ be greater than each $f_0(n)$ for $f \in F_4$. Since F_4 is cofinal in A, it follows that g is an upper bound for A, contradiction. Take m_1 to be the least such n. Let F_5 be an infinite subset of F_4 such that $\langle f_0(m_1) : f \in F_5\rangle$ is one-one. Then there is a $p_0 \in \omega$ such that $\{f_0 \restriction m_1 : f \in F_5\} \subseteq {}^{m_1}p_0$; so there exist a $t_0 \in {}^{m_1}p_0$ and an infinite subset F_6 of F_5 such that $f_0 \restriction m_1 = t_0$ for all $f \in F_6$, $\{f_0(m_1) : f \in F_6\}$ is unbounded in ω, and $\langle f_0(m_1) : f \in F_6\rangle$ is one-one. Now suppose that $i + 1 < k$ and $p_i, t_i, m_{i+1}, F_{6+i}$ have been defined so that $p_i \in \omega$, $m_{i+1} \in \omega$, $t_i \in {}^{m_{i+1}}p_i$, $F_{6+i} \subseteq F_5$, F_{6+i} is infinite, $\{f_i(m_{i+1}) : f \in F_{6+i}\}$ is unbounded in ω, and $\langle f_i(m_{i+1}) : f \in F_{6+i}\rangle$ is one-one. For any $j \in \omega$ choose $f \in F_{6+i}$ such that $j < f_i(m_{i+1})$. Then $j < f_{i+1}(m_{i+1})$ also. So $\{f_{i+1}(m_{i+1}) : f \in F_{6+i}\}$ is unbounded. Let m_{i+2} be minimum such that $\{f_{i+1}(m_{i+2}) : f \in F_{6+i}\}$ is unbounded. Let F'_{6+i} be an infinite subset of F_{6+i} such that $\langle f_{i+1}(m_{i+2}) : f \in F'_{6+i}\rangle$ is strictly increasing. Now there is a p_{i+1} such that $\{f_{i+1} \restriction m_{i+2} : f \in F'_{6+i}\} \subseteq {}^{m_{i+2}}p_{i+1}$. So there exist $t_{i+1} \in {}^{m_{i+2}}p_{i+1}$ and $F_{6+i+1} \subseteq F'_{6+i}$ such that F_{6+i+1} is infinite and $\forall f \in F_{6+i+1}[f_{i+1} \restriction m_{i+2} = t_{i+1}]$. So we obtain $m_0, \ldots, m_{k+1}$, $t_0, \ldots, t_{k-1}$, and $F_6 \supseteq \cdots \supseteq F_{6+k-1}$ such that the following conditions hold:

(19) $$\forall i < k \forall f \in F_{6+k-1}[t_i \subseteq f_i) \text{ and } t_i \in {}^{m_{i+1}}p_i.$$

(20) $$\forall i < k[\langle f_i(m_{i+1}) : f \in F_{6+k-1}\rangle \text{ is one-one }].$$

Now the open set in ${}^k({}^\omega\omega)$ determined by $\langle t_0, \ldots, t_{k-1}\rangle$ meets F, since F_{6+k-1} is contained in it; so by the denseness of D, choose $d \in D$ in this open set: $t_i \subseteq d_i$ for all $i < k$. Now $d_j < *c$ for all $j < k$ by the choice of c. So $d_j \in A_c = \mathrm{dmn}(u^j)$ for all $j < k$. By (20) there is an $f \in F_{6+k-1}$ such that

$$\forall i < k[f_i(m_{i+1}) > \max\{u^j(d_j) : j < k\}].$$

Now for all $i < k$ we have $f_i \restriction m_{i+1} = t_i = d_i \restriction m_{i+1}$, so $m_{i+1} \leq \Delta(d_i, f_i)$. Moreover, $d_i <^* c <^* f_i$, and hence

$$e(d_i, f_i) = e_{f_i}(d_i) = u^i(d_i) < f_i(m_{i+1}) \leq f_i(\Delta(d_i, f_i)),$$

so d_i is in $H(f_i)$ for all $i < k$, as desired; we have proved (14).

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Next,

(21) For every positive integer k, the space ${}^kA$ does not have an uncountable discrete subspace.

We prove this by induction on k; suppose that it is true for all $k' < k$. Suppose that F is an uncountable discrete subspace of ${}^kA$. We may assume that there is an integer m such that $(C_m(f_0) \times \cdots \times C_m(f_{k-1})) \cap F = \{f\}$ for all $f \in F$. For all $f \in F$ define $\equiv_f$ on k by $i \equiv_f j$ iff $f_i = f_j$. Without loss of generality, $\equiv_f$ is the same for all $f \in F$. Hence by the induction hypothesis, $\equiv$ is the identity relation, so that each $f \in F$ is one-one. And then by a similar argument with permutations of k we may assume that there is a permutation π of k such that $f_{\pi(i)} <^* f_{\pi(j)}$ whenever $f \in F$ and $i < j < k$. Next, we may assume that $\langle \mathrm{rng}(f) : f \in F\rangle$ is a Δ-system, say with kernel G. Say $G = \{a_i : i < n\}$ with $a_0 <^* \cdots <^* a_{n-1}$. For each $f \in F$ there is a strictly increasing sequence $\langle i(j,f) : j < n\rangle$ such that $f_{\pi(i(j,f))} = a_j$ for each $j < n$. We may assume that $i(j,f)$ does not depend on f; so we just write $i(j)$. Thus $f_{\pi(i(j))} = g_{\pi(i(j))}$ for all $f, g \in F$ and all $j < n$. Let $\Gamma = k\backslash\{i(j) : j < n\}$. Then $F' \stackrel{\text{def}}{=} \langle \mathrm{rng}(f \restriction \Gamma) : f \in F\rangle$ is a pairwise disjoint system and it is cofinal in A in the sense defined before (14). Now

$$F' = \bigcup\{\{f \in F' : \forall i \in \Gamma(f_i \restriction m = s_i)\} : s \in {}^k({}^m\omega)\},$$

so we may assume that $f_i \restriction m = g_i \restriction m$ for all $f, g \in F'$ and $i \in \Gamma$. Now we apply (14) to get distinct $f, g \in F'$ such that $g_i \in H(f_i)$ for all $i \in \Gamma$. Since $H(a) \subseteq C(a)$ for all a, this means that $g \in (C_m(f_0) \times \cdots \times C_m(f_{k-1})) \cap F$, contradiction. So (21) holds.

Now $A = \bigcup_{b\in A} C(b)$. By (6), this cover does not have a finite subcover. So A is not compact. By (10) and (11), it is locally compact. Take the one-point compactification A' of A. Lemma 8.8(i) says that A' is a Boolean space. Suppose that k is a positive integer and X is an uncountable discrete subset of ${}^kA'$. We may assume that the "new" point of A' is not in X. For each $x \in X$ let $\langle U_i^x : i < k\rangle$ be open sets such that $U_0^x \cap \cdots \cap U_{k-1}^x \cap X = \{x\}$. We may assume that each U_i^x is open in A. This shows that X is discrete in A, contradiction.

It now follows by Theorem 8.11 that clop(A) is an uncountable BA with countable irredundance. This finishes this example.

Problem 89. *Can one construct in ZFC a BA A such that* $\mathrm{Irr}A < |A|$*?*

This is Problem 28 in Monk [96].

Concerning the min-max function $\mathrm{Irr}_{\mathrm{mm}}$, we first note the following from Monk [08]:

Theorem 8.14. *If κ is an infinite cardinal and A is a subalgebra of $\mathscr{P}(\kappa)$ containing* $\mathrm{Intalg}(\kappa)$*, then* $\mathrm{Irr}_{\mathrm{mm}}(A) = \kappa$.

Proof. $\geq$ holds by Proposition 4.23 of the handbook. Now we claim that $X \stackrel{\text{def}}{=} \{[0,\alpha) : \alpha < \kappa\}$ is a maximal irredundant subset of A. In fact, if $\alpha < \kappa$, then every element of $\langle[0,\beta) : \beta \in \kappa\backslash\{\alpha\}\rangle$ is a union of intervals of the form $[a,b)$ with $a, b \in (\kappa\backslash\{\alpha\}) \cup \{\infty\}$, and so $[0,\alpha) \notin \langle[0,\beta) : \beta \in \kappa\backslash\{\alpha\}\rangle$. So X is irredundant. Now suppose that $a \in A\backslash\langle X\rangle$; we want to show that $X \cup \{a\}$ is redundant. We may assume that $a \neq \emptyset, \kappa$. If $0 \notin a$, let α be the least member of a. Then $[0,\alpha+1)\backslash a = [0,\alpha)$, so that $[0,\alpha) \in \langle(X \cup \{a\})\backslash\{[0,\alpha)\}\rangle$. If $0 \in a$, let α be the least member of $\kappa\backslash a$. Then $[0,\alpha+1) \cap a = [0,\alpha)$, giving the same conclusion. $\square$

Monk [08] gave an example of an atomless BA A such that $\mathrm{Irr}_{\mathrm{mm}}(A) = \omega < |A| = 2^\omega$. In fact, the following conjecture seems possible.

Problem 90. *Is* $\mathrm{Irr}_{\mathrm{mm}}(A) = \pi(A)$ *for every infinite BA A?*

We do not know whether this assertion is true for the interval algebra on $\mathbf{R}$, or for the completion of $\mathrm{Fr}(\omega)$.

There are reasonable finite versions of irredundance. For positive integers m, n, call a subset X of A *mn-irredundant* if for all $x \in X$ one cannot write

$$x = \sum_{i<m} \prod_{j<n} a_{ij},$$

with $a_{ij} \in X\backslash\{x\}$ or $-a_{ij} \in X\backslash\{x\}$ for all $i < m$, $j < n$. So, X is irredundant iff it is mn-irredundant for all m, n. And define

$$\mathrm{Irr}_{\mathrm{mn}}(A) = \sup\{|X| : X \subseteq A, X\ mn - \text{irredundant}\}.$$

These functions have not been studied.

Since $\pi(A) = |A|$ for A a tree algebra, we also have $\mathrm{Irr}(A) = |A|$ for A a tree algebra.

9 Cardinality

We denote $|A|$ also by $\mathrm{Card}(A)$. The behaviour of this function under algebraic operations is for the most part obvious. Note, though, that questions about its behaviour under ultraproducts are the same as the well-known and difficult problems concerning the cardinality of ultraproducts in general.

$\mathrm{Card}_{\mathrm{H-}}$ is a non-obvious function. It is related to some other "small" functions:

cofinality of A:

$$\mathrm{cf}(A) = \min\Big\{\kappa : \kappa \geq \omega \text{ and there is a strictly increasing sequence } \langle B_\alpha : \alpha < \kappa\rangle \text{ of subalgebras of } A \text{ such that } A = \bigcup_{\alpha<\kappa} B_\alpha\Big\};$$

altitude of A:

$$\mathrm{alt}(A) = \min\{\kappa : \kappa \geq \omega \text{ and there is a strictly increasing sequence of length } \kappa \text{ of filters on } A \text{ whose union is an ultrafilter}\};$$

pseudo-altitude of A:

$$\text{p-alt}(A) = \min\{\kappa : \kappa \geq \omega \text{ and there is an infinite homomorphic image of } A \text{ with an ultafilter generated by } \kappa \text{ elements}\}.$$

Proposition 9.1. $\mathrm{cf}(A) \leq \mathrm{alt}(A) \leq \text{p-alt}(A) \leq \mathrm{Card}_{\mathrm{H-}}(A) \leq 2^\omega$ *for any infinite BA* A.

Proof. $\mathrm{cf}(A) \leq \mathrm{alt}(A)$: If $\langle F_\alpha : \alpha < \kappa\rangle$ is a strictly increasing sequence of filters whose union is an ultrafilter G, then $\langle\langle F_\alpha\rangle : \alpha < \kappa\rangle$ is a strictly increasing sequence of proper subalgebras of A with union A; the inequality follows.

$\mathrm{alt}(A) \leq \text{p-alt}(A)$: Let f be a homomorphism from A onto a BA B, and let G be an ultrafilter on B generated by a set $\{x_\alpha : \alpha < \kappa\}$, with κ minimum. Then $\langle\langle f^{-1}[\{x_\beta : \beta < \alpha\}]\rangle^{\mathrm{fi}} : \alpha < \kappa\rangle$ is an increasing sequence of proper filters whose union is the ultrafilter $f^{-1}[G]$; a strictly increasing subsequence proves this inequality.

$\text{p-alt}(A) \leq \mathrm{Card}_{\mathrm{H-}}(A)$: Obvious.

$\mathrm{Card}_{\mathrm{H-}}(A) \leq 2^\omega$: take a denumerable subalgebra B of A, and extend the identity on B to a homomorphism from A onto a subalgebra of $\overline{B}$. $\square$

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We now give several results concerning cf and Card due to Sabine Koppelberg [77].

Lemma 9.2. *Suppose that f is a homomorphism from A onto a denumerable algebra B, and $\langle C_n : n \in \omega\rangle$ is an increasing sequence of finite BAs with union B. (Not necessarily strictly increasing.) Let F be a nonprincipal ultrafilter on B. Then $\langle f^{-1}[F \cap C_n]\rangle_A^{\mathrm{fi}} \neq f^{-1}[F]$, for all $n \in \omega$.*

Proof. Take any $n \in \omega$. Then $F \cap C_n$ is an ultrafilter on C_n, and so there is an atom b of C_n with $b \in F$. Take $c \in B$ with $0 < c < b$; c exists since F is nonprincipal. Take $a \in A$ with $f(a) = c$. Now $\langle f^{-1}[F \cap C_n]\rangle_A^{\mathrm{fi}} \cup \{a\}$ has fip, as otherwise we get some $e \in f^{-1}[F \cap C_n]$ such that $e \cdot a = 0$, hence $f(e) \cdot c = 0$. Since $f(e) \in F \cap C_n$ we have $b \leq f(e)$, so $b \cdot c = 0$, contradiction. Also $\langle f^{-1}[F \cap C_n]\rangle_A^{\mathrm{fi}} \cup \{-a\}$ has fip, as otherwise we get some $e \in f^{-1}[F \cap C_n]$ such that $e \cdot -a = 0$, hence $f(e) \cdot -c = 0$. Since $f(e) \in F$ we have $b \leq f(e)$, so $b \cdot -c = 0$, contradiction. It follows that $\langle f^{-1}[F \cap C_n]\rangle_A^{\mathrm{fi}}$ is not an ultrafilter. Since it is obviously contained in $f^{-1}[F]$, the lemma is proved. $\square$

Theorem 9.3. *For any infinite BA A the following conditions are equivalent:*

(i) $\mathrm{Card}_{\mathrm{H}-}(A) = \omega$.

(ii) *There exist a strictly increasing sequence $\langle B_n : n \in \omega\rangle$ of subalgebras of A and an ultrafilter F on A such that $A = \bigcup_{n\in\omega} B_n$ and $\langle F \cap B_n\rangle_A^{\mathrm{fi}} \neq F$, for all $n \in \omega$.*

Proof. (i)⇒(ii): Assume (i), and let f be a homomorphism from A onto a denumerable BA B. Let $\langle C_n : n \in \omega\rangle$ be a strictly increasing sequence of finite subalgebras of B with union B. Let F be any nonprincipal ultrafilter on B. Then $\langle f^{-1}[C_n] : n \in \omega\rangle$ is a strictly increasing sequence of subalgebras of A with union A, $f^{-1}[F]$ is an ultrafilter on A, and for any $n \in \omega$, $\langle f^{-1}[F \cap C_n]\rangle_A^{\mathrm{fi}} \neq f^{-1}[F]$ by Lemma 9.2. This proves (ii).

Now assume (ii). Then

(1) For each $n \in \omega$ there is an ultrafilter G_n on A such that $F \neq G_n$ but $G_n \cap B_n = F \cap B_n$.

In fact, let $n \in \omega$ and choose $b \in F \backslash \langle F \cap B_n\rangle_A^{\mathrm{fi}}$. We claim that $\langle F \cap B_n\rangle_A^{\mathrm{fi}} \cup \{-b\}$ has fip. Otherwise, we get $c \in \langle F \cap B_n\rangle_A^{\mathrm{fi}}$ such that $c \cdot -b = 0$, hence $c \leq b$, hence $b \in \langle F \cap B_n\rangle_A^{\mathrm{fi}}$, contradiction. Let G_n be an ultrafilter containing this set. Clearly (1) holds.

Now let $X = \{F\} \cup \{G_n : n \in \omega\}$. For each $a \in A$ let $f(a) = \{H \in X : a \in H\}$. Clearly f is a homomorphism from A into $\mathscr{P}(X)$.

(2) If $n \in \omega$ and $a \in B_n \cap F$, then $\{G_m : m \geq n\} \subseteq f(a)$.

For, assume the hypothesis and suppose that $m \geq n$. Then $a \in B_n \cap F \subseteq B_m \cap F = G_m \cap B_m$, so $a \in G_m$.

By (2), f maps into Finco(X). Suppose that $f[F]$ is finite. Each member of $f[F]$ is a cofinite subset of X, so there is a $G_n \in \bigcap f[F]$. Choose $a \in F\backslash G_n$. Then $G_n \notin f(a)$, contradiction. Hence $f[F]$ is infinite, and so f maps onto an algebra of size ω. □

Proposition 9.4. $\mathrm{alt}(A) = \omega$ iff $\text{p-alt}(A) = \omega$ iff $\mathrm{Card}_{\mathrm{H-}}(A) = \omega$, *for any infinite BA A.*

Proof. By Proposition 9.1 it suffices to show that $\mathrm{Card}_{\mathrm{H-}}(A) = \omega$ under the assumption that $\mathrm{alt}(A) = \omega$. So, suppose that $\langle F_n : n \in \omega\rangle$ is a strictly increasing sequence of filters on A whose union is an ultrafilter G. For each $n \in \omega$ let $B_n = \langle F_n\rangle$. Note that $B_n = F_n \cup \{a : -a \in F_n\}$. Hence if $a \in F_{n+1}\backslash F_n$ then also $a \in B_{n+1}\backslash B_n$. For each $n \in \omega$ we have $G \cap B_n = F_n$, and hence $\langle G \cap B_n\rangle^{\mathrm{fi}}_A \neq G$. Hence $\mathrm{Card}_{\mathrm{H-}}(A) = \omega$ by Theorem 9.3. □

The following theorem is a generalization of a result of Koppelberg; it is due independently to Balcar and Simon, Kunen, and Shelah (all evidently unpublished); we follow van Douwen [89].

Theorem 9.5. *If A is a BA such that* $\mathrm{Ind}(A) \geq \omega_1$, *then* $\text{p-alt}(A) \leq \omega_1$.

Proof. Since $\mathrm{Ind}(A) \geq \omega_1$, there is a homomorphism f from A onto a BA B such that $\mathrm{Fr}(\omega_1) \leq B \leq \overline{\mathrm{Fr}(\omega_1)}$. Now we claim that it suffices to find ultrafilters F and G_η on B for $\eta < \omega_1$ such that the following two conditions hold.

(1) $\forall\xi < \omega_1 \exists a \in F \forall\eta < \omega_1[a \in G_\eta \leftrightarrow \xi \leq \eta]$.

(2) $\forall a \in F \exists\xi < \omega_1 \forall\eta < \omega_1[\xi \leq \eta \rightarrow a \in G_\eta]$.

In fact, suppose that (1) and (2) hold. Let $X = \{F\} \cup \{G_\xi : \xi < \omega_1\}$. Define $g : B \to \mathscr{P}(X)$ by setting $g(a) = \{H \in X : a \in H\}$. Thus g is a homomorphism. For each $\xi < \omega_1$ choose $a_\xi \in F$ by (1).

(3) $g[F]$ is an ultrafilter on rng(g).

In fact, clearly $g[F]$ is closed under multiplication. Suppose that $a \in F$ and $g(a) \leq g(b)$. Then $g(a \cdot -b) = 0$, so in particular $F \notin g(a \cdot -b)$, i.e., $a \cdot -b \notin F$, hence $-b \notin F$, hence $b \in F$. So $g[F]$ is closed upwards. Clearly $0 \notin g[F]$. For any $a \in A$ we have $a \in F$ or $-a \in F$, hence $g(a) \in g[F]$ or $-g(a) = g(-a) \in g[F]$. Thus (3) holds.

Now $\{g(a_\xi) : \xi < \omega_1\}$ generates $g[F]$. In fact let $b \in F$. By (2), choose $\xi < \omega_1$ such that

(4) $\forall\eta < \omega_1[\xi \leq \eta \rightarrow b \in G_\eta]$.

We claim that $g(a_\xi) \leq g(b)$. In fact, suppose that $G_\eta \in g(a_\xi)$. Thus $a_\xi \in G_\eta$, so by (1), $\xi \leq \eta$. Hence $b \in G_\eta$ by (4), so $G_\eta \in g(b)$.

Hence $\{g(a_\xi) : \xi < \omega_1\}$ generates $g[F]$.

Finally, $\mathrm{rng}(g)$ is infinite, since if $\xi < \eta$ we have $a_\xi \in G_\xi$ and $a_\eta \notin G_\xi$ by (1), so $G_\xi \in g(a_\xi)\backslash g(a_\eta)$.

Now to construct ultrafilters satisfying (1) and (2), let $\langle x_\xi : \xi < \omega_1\rangle$ be a system of free generators of $\mathrm{Fr}(\omega_1)$. Let H be an ultrafilter on $\overline{\mathrm{Fr}(\omega_1)}$ containing the set $\{x_\xi : \xi < \omega_1\}$. For each $\eta < \omega_1$ let h_η be the automorphism of $\mathrm{Fr}(\omega_1)$ such that

$$h_\eta(x_\xi) = \begin{cases} x_\xi & \text{if } \ \xi \le \eta, \\ -x_\xi & \text{if } \ \eta < \xi. \end{cases}$$

Then h_η extends to an automorphism of $\overline{\mathrm{Fr}(\omega_1)}$ which we still denote by h_η. For each $\eta < \omega_1$ let $K_\eta = h_\eta[H]$. So K_η is an ultrafilter on $\overline{\mathrm{Fr}(\omega_1)}$. Define $F = H \cap B$ and $G_\eta = K_\eta \cap B$.

To check (1), suppose that $\xi < \omega_1$. We try $a = x_\xi$. Suppose that $\eta < \omega_1$. Then

$$x_\xi \in G_\eta \quad \text{iff} \quad h_\eta^{-1}(x_\xi) \in H \quad \text{iff} \quad \xi \le \eta,$$

as desired.

To check (2), suppose that $a \in F$. Write $a = \sum M$, where M is a countable subset of $\mathrm{Fr}(\omega_1)$. Choose $\xi < \omega_1$ such that $\forall b \in M \forall x_\eta \in \mathrm{supp}(b)[\eta < \xi]$. Now suppose that $\xi \le \eta < \omega_1$. Then $h_\eta(a) = a \in F$, so $a \in G_\eta$, as desired. □

Example 9.6. There is a BA A such that $\mathrm{cf}(A) = \omega$, $\mathrm{alt}(A) = \text{p-alt}(A) = \omega_1$, and $\mathrm{Card}_{\mathrm{H}-}(A) = 2^\omega$.

Proof. Let $\mathbb{Z}' = \mathbb{Z}\backslash\{0\}$. For each $n \in \omega$ let $B_n = \{a \subseteq \mathbb{Z}' : \forall i \in \omega\backslash(n+1)[i \in a \text{ iff } -i \in a]\}$. Thus each B_n is a subalgebra of $\mathscr{P}(\mathbb{Z}')$, and $B_m \le B_n$ if $m < n$. Let $A = \bigcup_{m\in\omega} B_m$. So $\mathrm{cf}(A) = \omega$. Note that each B_m is isomorphic to $\mathscr{P}(\omega)$. In fact, B_m is complete and has as its set of atoms the set

$$\{\{i\} : 1 \le i \le m\} \cup \{\{-i\} : 1 \le i \le m\} \cup \{\{i, -i\} : m < i < \omega\}.$$

Now suppose that f is a homomorphism from A onto an infinite BA C. Then we can write $C = \bigcup_{m\in\omega} f[B_m]$. If some $f[B_m]$ is infinite, then $|f[B_m]| = 2^\omega$ since $f[B_m]$ satisfies CSP; hence $|C| = 2^\omega$ in this case. So assume that each $f[B_m]$ is finite; we will get a contradiction in this case. Let F be an ultrafilter on C. Now $\langle f[B_m] : m \in \omega\rangle$ is an increasing sequence of finite BAs with union C. Hence by Lemma 9.2,

(∗) $\langle f^{-1}[F \cap f[B_m]]\rangle^{\mathrm{fi}}_A \ne f^{-1}[F]$, for all $m \in \omega$.

Now we have two possibilities for the ultrafilter $B_0 \cap f^{-1}[F]$ on B_0:

Case 1. There is an atom $\{-m, m\} \in B_0 \cap f^{-1}[F]$, where m is a positive integer. Thus $f^{-1}[F]$ is a principal ideal of A; say $\{m\} \in f^{-1}[F]$. By (∗), choose $a \in f^{-1}[F]\backslash\langle f^{-1}[F \cap f[B_{m+1}]]^{\mathrm{fi}}_A$. Then $\{m\} \subseteq a$. But $\{m\} \in f^{-1}[F] \cap B_{m+1} \subseteq f^{-1}[F \cap f[B_{m+1}]]$, contradiction.

Case 2. $B_0 \cap f^{-1}[F]$ is a nonprincipal ultrafilter on B_0. By (∗), choose an element $a \in f^{-1}[F] \backslash \langle f^{-1}[F \cap f[B_0]]^{\text{fi}}_A$. Say $a \in B_m$. Then

$$a \supseteq a \backslash \bigcup_{0<i<m} \{i, -i\} \in B_0 \cap f^{-1}[F],$$

contradiction.

From Theorem 9.4 it follows that $\text{alt}(A) \geq \omega_1$.

Since, as observed above, each B_n is isomorphic to $\mathscr{P}(\omega)$, it follows that $\text{Ind}(A) \geq \omega_1$. So $\text{p-alt}(A) \leq \omega_1$ by Theorem 9.5. □

The following is an old problem, implicit in Koppelberg [77]:

Problem 91. *Can one prove in ZFC that* $\text{cf}(A) \leq \omega_1$ *for every infinite BA A?*

The following is a problem stated in van Douwen [89]

Problem 92. *Is* $\text{alt}(A) = \text{p-alt}(A)$ *for every infinite BA A?*

Theorem 9.7. *If A is an infinite superatomic BA, then* $\text{Card}_{\text{H}-}(A) = \omega$.

Proof. Let $a \in A$ be such that $[a]$ is an atom, where $[a]$ is the equivalence class of a with respect to the ideal of atoms. Let M be a denumerable set of atoms of A below a. For each $x \in A$ let $f(x) = \{b \in M : b \leq x\}$. Clearly f is a homomorphism of A into $\mathscr{P}(M)$, and the range of f is infinite. We claim that in fact f maps into $\text{Finco}(M)$. For, take any $x \in A$.

Case 1. $[x \cdot a] = 0$. Then $f(x)$ is finite.

Case 2. $[x \cdot a] = [a]$. Then $[a \cdot -x] = 0$, and so $f(-x)$ is finite. □

For our final result on $\text{Card}_{\text{H}-}$ we need a standard fact about Martin's axiom.

Theorem 9.8 (MA)**.** *Suppose that A is an infinite ccc BA, and* $A \leq B \leq \overline{A}$ *with* $|B| < 2^\omega$*. Then there is an ultrafilter F on B such that for every* $b \in F \cap B$ *there is an* $a \in F \cap A$ *such that* $a \leq b$.

Proof. We consider the following collection of dense subsets of B^+:

$$\{\{a \in A^+ : a \leq b \text{ or } a \leq -b\} : b \in B\}.$$

Let F be a generic filter on B intersecting all of these dense subsets. Clearly F is as desired. □

Theorem 9.9 (MA)**.** *If A is an infinite BA of size less than* 2^ω*, then* $\text{Card}_{\text{H}-}(A) = \omega$.

Proof. By Theorem 9.7 we may assume that A is not superatomic. Hence it has a subalgebra isomorphic to $\text{Fr}(\omega)$, and hence by Sikorski's extension theorem it has a homomorphic image B such that $\text{Fr}(\omega) \leq B \leq \overline{\text{Fr}(\omega)}$. By Theorem 9.8 let F be an ultrafilter on B such that

(∗) For every $b \in F \cap B$ there is an a such that $a \in F \cap \text{Fr}(\omega)$ and $a \leq b$.

Now $F \cap \mathrm{Fr}(\omega)$ is an ultrafilter in a denumerable algebra, and hence it has a strictly decreasing sequence $\langle b_n : n \in \omega\rangle$ of generators. For each $n \in \omega$ let $C_n = \{c \in B : b_n \leq c \text{ or } b_n \leq -c\}$. Clearly C_n is a subalgebra of B. Moreover, $b_{n+1} \in C_{n+1} \backslash C_n$ for each $n \in \omega$.

(1) $\bigcup_{n\in\omega} C_n = B$.

In fact, take any $c \in B$. Say $c \in F$. By $(*)$, choose $d \in F \cap \mathrm{Fr}(\omega)$ such that $d \leq c$. Choose $n \in \omega$ such that $b_n \leq d$. Then $b_n \leq c$, so that $c \in C_n$. This proves (1).

Let G be an ultrafilter on B containing F.

(2) $\langle G \cap C_n\rangle^{\mathrm{fi}}_B \neq G$, for all $n \in \omega$.

In fact, take any $n \in \omega$. Clearly $b_{n+1} \in G \backslash \langle G \cap C_n\rangle^{\mathrm{fi}}_B$.

Now Theorem 9.3 gives the desired conclusion. □

W. Just and P. Koszmider [91] have shown that there is a model M in which $2^\omega = \kappa$ arbitrarily large, in M there is a BA A such that $\mathrm{cf}(A) = \omega_1$, and for every $\lambda \leq \kappa$ of uncountable cofinality there is a BA B such that $\mathrm{Card_{H-}}(B) = \lambda$. Koszmider [90] shows that it is consistent to have $2^\omega > \omega_1$ while every BA has pseudo-altitude $\leq \omega_1$.

The cardinal function $\mathrm{Card_{h-}}$ coincides with $\mathrm{Card_{H-}}$: obviously $\mathrm{Card_{h-}}(A) \leq \mathrm{Card_{H-}}(A)$, and if X is any infinite subset of $\mathrm{Ult}(A)$, then the function f such that $f(a) = \mathcal{S}(a) \cap X$ is a homomorphism from A onto an algebra B such that $|B| \leq \mathrm{Clop}(X)$; so $\mathrm{Card_{H-}}(A) \leq \mathrm{Card}_{h-}(A)$. To see that $\mathrm{rng}(f)$ is infinite, given $m \in \omega$ choose distinct $F_i \in X$ for all $i < m$. For each $i < m$ there is an element $a_i \in F_i \backslash \bigcup_{j\neq i} F_j$, and it follows that $f(a_i) \neq f(a_j)$ for $i \neq j$.

The cardinal function $\mathrm{Card_{h+}}$ is defined as follows:

$$\mathrm{Card_{h+}}(A) = \sup\{|\mathrm{Clop}(X)| : X \subseteq \mathrm{Ult}A\}.$$

It is possible to have $\mathrm{Card_{h+}}(A) > |\mathrm{Ult}(A)|$: this is true, for example, with A the finite-cofinite algebra on an infinite cardinal κ, taking X to be the set of all principal ultrafilters on A, so that X is discrete and hence $\mathrm{Clop}(X) = \mathscr{P}(X)$ and $\mathrm{Card_{h+}}(A) = 2^\kappa$. Clearly ${}_{\mathrm{d}}\mathrm{Card_{S+}}(A) = |A|$, and ${}_{\mathrm{d}}\mathrm{Card_{S-}}(A) = \pi(A)$.

We shall now go into some detail concerning the spectrum function $\mathrm{Card_{Hs}}$, which seems to be another interesting derived function associated with cardinality. First we note some more-or-less obvious facts: (1) If A is an infinite free BA, then $\mathrm{Card_{Hs}}(A) = [\omega, |A|]$; (2) If A is infinite and complete, then $\mathrm{Card_{Hs}}(A) = [\omega, |A|] \cap \{\kappa : \kappa^\omega = \kappa\}$ (using in an essential way the Balcar, Franěk theorem); (3) if $\omega \leq \kappa \leq |A|$, then $\mathrm{Card_{Hs}}(A) \cap [\kappa, 2^\kappa] \neq 0$; (4) if A has a free subalgebra of power $\kappa \geq \omega$, then $\mathrm{Card_{Hs}}(A) \cap [\kappa, \kappa^\omega] \neq 0$. Now we prove a few more involved things.

Lemma 9.10. *If κ is an infinite cardinal, L is a linear ordering, the sequence $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing in L, and A is the interval algebra on L, then $[\omega, \kappa] \subseteq \mathrm{Card_{Hs}}(A)$.*

Proof. It suffices to show that $\kappa \in \mathrm{Card}_{\mathrm{Hs}}(A)$. Define $x \equiv y$ iff $x, y \in L$ and $\forall \alpha < \kappa[(a_\alpha < x$ iff $a_\alpha < y)$ and $(x < a_\alpha$ iff $y < a_\alpha)]$. Then $\equiv$ is a convex equivalence relation on L with the equivalence classes of order type κ or $\kappa + 1$, and the desired homomorphism is easy to define. □

Corollary 9.11. *If κ is an infinite cardinal and A is the interval algebra on κ, then* $\mathrm{Card}_{\mathrm{Hs}}(A) = [\omega, \kappa]$. □

Corollary 9.12. *If κ is an infinite cardinal, L is a linear ordering of power $\geq (2^\kappa)^+$, and A is the interval algebra on L, then $[\omega, \kappa^+] \subseteq \mathrm{Card}_{\mathrm{Hs}}(A)$.*

Proof. One can apply the Erdős–Rado theorem$(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ to get a chain in L of order type κ^+ or $(\kappa^+)^*$. □

Theorem 9.13. *Let A be the interval algebra on $\mathbb{R}$. Then* $\mathrm{Card}_{\mathrm{Hs}}(A) = \{\omega, 2^\omega\}$.

Proof. The inclusion $\supseteq$ is obvious. Now suppose that f is a homomorphism of A onto an uncountable BA B; we want to show that $|B| = 2^\omega$. Notice that f is determined by a convex equivalence relation E on $\mathbb{R}$, where the number of E-equivalence classes is $|B|$. Now $L' \overset{\text{def}}{=} \bigcup\{k : k$ is an E-equivalence class with $|k| > 1\}$ is Borel, so $L'' \overset{\text{def}}{=} \mathbb{R}\backslash L'$ is also. There are only countably many E-equivalence classes k such that $|k| > 1$, so clearly $|L''| = |B|$. Hence $|B| = 2^\omega$ by the Alexsandroff–Hausdorff theorem (see Kuratowski [58] Theorem 3, p. 355). □

Theorem 9.14. $\mathrm{Card}_{\mathrm{Hs}}(A \times B) = \mathrm{Card}_{\mathrm{Hs}}(A) \cup \mathrm{Card}_{\mathrm{Hs}}(B)$.

Proof. The inclusion $\supseteq$ is obvious. For $\subseteq$, use Proposition 1.1. □

Theorem 9.15. $\mathrm{Card}_{\mathrm{Hs}}(A \oplus B) = \mathrm{Card}_{\mathrm{Hs}}(A) \cup \mathrm{Card}_{\mathrm{Hs}}(B)$.

Proof. The inclusion $\supseteq$ is obvious. If f is a homomorphism from $A \oplus B$ onto C, then $f[A] \cup f[B]$ generates C, so the other inclusion follows. □

Corollary 9.16. *If $\omega \leq \kappa \leq 2^\omega$, then there is a BA A such that* $\mathrm{Card}_{\mathrm{Hs}}(A) = [\omega, \kappa] \cup \{2^\omega\}$.

Proof. Apply Theorem 9.14 to $A \times \mathscr{P}(\omega)$, where A is the free BA on κ free generators. □

The strongest result known about $\mathrm{Card}_{\mathrm{Hs}}$ is a special case of the following theorem of Juhász [93]:

Let κ be a regular uncountable cardinal and let X be a compact Hausdorff space of weight at least κ. Then there is a closed subspace $F \subseteq X$ such that the weight of F is in $[\kappa, 2^{<\kappa}]$ and

$$|F| \leq \sum_{\lambda<\kappa} 2^{2^\lambda}.$$

As a corollary, under GCH for every BA A the set $\mathrm{Card}_{\mathrm{Hs}}(A)$ contains all regular uncountable cardinals $\leq |A|$. As a special case, this solves Problem 22 of Monk [90].

We now give the proof of this theorem, for BAs. It depends on a result about free sequences which is independently interesting, and we first prove this result. Both proofs are highly topological, but we will try to make our treatment self-contained, appealing to Engelking [89] for simple topological facts.

We introduce some notation and terminology from duality theory.

A subset D of a topological space is *regular closed* iff $D = \overline{\mathrm{int}(D)}$. Engelking uses the term *closed domain.*

Suppose that $X \subseteq \mathrm{Ult}(A)$ and $M \subseteq A$. We defne

$$\begin{aligned} X^{\mathrm{fi}} &= \{a \in A : X \subseteq \mathcal{S}(a)\};\\ X^{\mathrm{id}} &= \{a \in A : \mathcal{S}(a) \subseteq X\};\\ M^c &= \bigcap_{a\in M} \mathcal{S}(a);\\ M^o &= \bigcup_{a\in M} \mathcal{S}(a);\\ M^r &= \{b : \forall a[\forall x \in M(a \le x) \Rightarrow a \le b]\}. \end{aligned}$$

Note that M^r is a filter, and $M \subseteq M^r$. A filter F is *regular* iff $F = F^r$.

Let κ be an infinite cardinal and $S \subseteq \kappa$, $X \subseteq {}^\kappa 2$. We say that X is S-*determined* iff for all $x, y \in {}^\kappa 2$, if $x \restriction S = y \restriction S$ and $x \in X$, then $y \in X$.

For κ an infinite cardinal, let $\langle x^\kappa_\alpha : \alpha < \kappa\rangle$ be a system of free generators of $\mathrm{Fr}(\kappa)$.

Given $Y \subseteq \mathrm{Ult}(\mathrm{Fr}(\kappa))$ and $S \subseteq \kappa$, we say that Y is *determined* by S iff for all $F, G \in \mathrm{Ult}(\mathrm{Fr}(\kappa))$, if $\forall\alpha \in S[x^\kappa_\alpha \in F$ iff $x^\kappa_\alpha \in G]$, then $F \in Y$ iff $G \in Y$.

Suppose that $F \in \mathrm{Ult}(\mathrm{Fr}(\kappa))$, and $S \subseteq \kappa$. Then we define $\Phi(F, S) = \{G \in \mathrm{Ult}(\mathrm{Fr}(\kappa)) : \forall\alpha \in S[x^\kappa_\alpha \in G$ iff $x^\kappa_\alpha \in F]\}$. Suppose that $B \stackrel{\mathrm{def}}{=} \mathrm{Fr}(\kappa) \le A$, $F \in \mathrm{Ult}(A)$, and $S \subseteq \kappa$. Then we define $\mathscr{A}(F, S) = \{a \in F : \{G : G \in \mathrm{Ult}(B)$ and $G \cup \{a\}$ has fip$\}$ is determined by $S\}$. Suppose that κ is an infinite cardinal. With each $f \in {}^\kappa 2$ we associate the ultrafilter ultr_f on $\mathrm{Fr}(\kappa)$ generated by $\{x^\kappa_\alpha : f(\alpha) = 1\} \cup \{-x^\kappa_\alpha : f(\alpha) = 0\}$. Conversely, with each $U \in \mathrm{Ult}(\mathrm{Fr}(\kappa))$ we associate the function $k_U \in {}^\kappa 2$ defined by

$$k_U(\alpha) = \begin{cases} 1 & \text{if } x^\kappa_\alpha \in U,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 9.17. *For any $X \subseteq \mathrm{Ult}(A)$, the set X^{fi} is a filter in A, and $\overline{X} = \bigcap\{\mathcal{S}(a) : a \in X^{\mathrm{fi}}\}$.*

Proof. Clearly X^{fi} is a filter in A. Now we work on the equality.

$\subseteq$: Suppose that $F \in \overline{X}$ and $a \in X^{\mathrm{fi}}$. If $F \notin \mathcal{S}(a)$, then $a \notin F$, hence $-a \in F$, so there is a $G \in X$ such that $-a \in G$. But $X \subseteq \mathcal{S}(a)$, contradiction.

$\supseteq$: suppose that $F \in \bigcap\{\mathcal{S}(a) : a \in X^{\mathrm{fi}}\}$. Let $a \in F$, and suppose that $\mathcal{S}(a) \cap X = \emptyset$. Then $X \subseteq \mathcal{S}(-a)$, and hence $-a \in X^{\mathrm{fi}}$. So $F \in \mathcal{S}(-a)$, contradiction. $\square$

Proposition 9.18. *For any* $X \subseteq \mathrm{Ult}(A)$, $\mathrm{int}(X) = \bigcup\{\mathcal{S}(a) : a \in X^{\mathrm{id}}\}$ *and* X^{id} *is an ideal in* A. $\square$

Proposition 9.19. *For any* $X \subseteq \mathrm{Ult}(A)$, $\overline{\mathrm{int}(X)}^{\mathrm{fi}} = \{a \in A : \forall b \in A[\mathcal{S}(b) \subseteq X \Rightarrow b \leq a]\}$.

Proof. First suppose that $a \in \overline{\mathrm{int}(X)}^{\mathrm{fi}}$. Thus $\overline{\mathrm{int}(X)} \subseteq \mathcal{S}(a)$. Suppose that $\mathcal{S}(b) \subseteq X$. Then $\mathcal{S}(b) \subseteq \mathrm{int}(X) \subseteq \overline{\mathrm{int}(X)} \subseteq \mathcal{S}(a)$, and hence $b \leq a$.

Second, suppose that $\forall b \in A[\mathcal{S}(b) \subseteq X \Rightarrow b \leq a]$; we want to show that $a \in \overline{\mathrm{int}(X))}^{\mathrm{fi}}$. Thus by definition we want to show that $\overline{\mathrm{int}(X)} \subseteq \mathcal{S}(a)$. It suffices to show that $\mathrm{int}(X)) \subseteq \mathcal{S}(a)$. By Proposition 9.18 we take any $b \in X^{\mathrm{id}}$ and try to show that $\mathcal{S}(b) \subseteq \mathcal{S}(a)$, i.e., that $b \leq a$. Now by definition, $\mathcal{S}(b) \subseteq X$. Hence by supposition, $b \leq a$, as desired. $\square$

Proposition 9.20. *For any filter* F *on* A *we have*

$$\overline{\mathrm{int}(F^c)}^{\mathrm{fi}} = \{a \in A : \forall b \in A[\forall c \in F(b \leq c) \Rightarrow b \leq a]\}.$$

Proof. First suppose that a is in the left-hand side. Suppose that $b \in A$ and $\forall c \in F(b \leq c)$. Thus $\mathcal{S}(b) \subseteq \bigcap_{c \in F} \mathcal{S}(c) = F^c$. Hence by Proposition 9.19, $b \leq a$.

Second, suppose that a is in the right-hand side. To apply Proposition 9.19 again, suppose that $b \in A$ and $\mathcal{S}(b) \subseteq \bigcap_{c \in F} \mathcal{S}(c)$. Then $\forall c \in F(b \leq c)$. Hence $b \leq a$, as desired. $\square$

Lemma 9.21. *If* B *is a dense subalgebra of* A *and* F *is a regular filter on* A*, then* $F \cap B$ *is a regular filter on* B.

Proof. Clearly $F \cap B$ is a filter on B. Now suppose that $b \in B$, and

$$(1) \qquad \forall a \in B[\forall x \in F \cap B(a \leq x) \Rightarrow a \leq b].$$

We want to show that $b \in F$. To do this, assume that $a \in A$ and

$$(2) \qquad \forall x \in F(a \leq x).$$

If we show that $a \leq b$, then $b \in F$ from the regularity of F. Suppose that $a \not\leq b$. Thus $a \cdot -b \neq 0$, so we can choose $0 \neq c \leq a \cdot -b$ with $c \in B$. If $x \in F \cap B$, then $a \leq x$ by (2). Hence $c \leq x$ for all $x \in F \cap B$. Hence by (1) we get $c \leq b$. But $c \neq 0$ and $c \leq -b$, contradiction. $\square$

Lemma 9.22. *If* F *is a regular filter on a BA* A*, then* F^c *is regular closed.*

Proof. By Proposition 9.20 we have $\overline{\mathrm{int}(F^c)}^{\mathrm{fi}} = F^r = F$. Hence by Proposition 9.17,

$$\overline{\mathrm{int}(F^c)} = \bigcap\{\mathcal{S}(a) : a \in \overline{\mathrm{int}(F^c)}^{\mathrm{fi}}\} = \bigcap\{\mathcal{S}(a) : a \in F\} = F^c. \qquad \square$$

Lemma 9.23. *Let A be any BA, and let $a \in A$. Then $\{x \in A : a \leq x\}$ is a regular filter on A.*

Proof. Let $F = \{x \in A : a \leq x\}$. Clearly F is a filter on A. Suppose that $b \in A$ and $\forall c[\forall x \in F(c \leq x) \Rightarrow c \leq b]$; we want to show that $b \in F$. Obviously $\forall x \in F(a \leq x)$, so $a \leq b$, as desired. □

Lemma 9.24. *Suppose that $B \leq A$ and $a \in A$. Then*

$$\{G \in \mathrm{Ult}(B) : \{x \in B : a \leq x\} \subseteq G\} = \{G \in \mathrm{Ult}(B) : G \cup \{a\} \textit{ has fip}\}.$$

Proof. Let G be in the left side, and suppose that $x \in G$ and $x \cdot a = 0$. Then $a \leq -x$, so $-x \in G$, contradiction.

Now suppose that G is in the right side and $a \leq x \in B$. If $x \notin G$, then $-x \in G$ and so $G \cup \{a\}$ does not have fip, contradiction. □

Lemma 9.25. *Suppose that B is a dense subalgebra of A, and $a \in A$. Then $\{G \in \mathrm{Ult}(B) : G \cup \{a\}$ has fip$\}$ is regular closed.*

Proof. Let $F = \{x \in A : a \leq x\}$. Thus F is a regular filter on A by Lemma 9.23. Hence by Lemma 9.21, $\{x \in B : a \leq x\}$ is a regular filter on B. Now $\{x \in B : a \leq x\}^c = \bigcap\{\mathcal{S}(x) : x \in B, a \leq x\} = \{G \in \mathrm{Ult}(B) : \{x \in B, a \leq x\} \subseteq G\} = \{G \in \mathrm{Ult}(B) : G \cup \{a\}$ has fip$\}$ by Lemma 9.24. Thus the desired conclusion follows by Lemma 9.22. □

Lemma 9.26. *Suppose that κ is an infinite cardinal, $S \subseteq \kappa$, and $X \subseteq {}^{\kappa}2$ is S-determined. Then also $\overline{X}$ is S-determined.*

Proof. Suppose that $x, y \in {}^{\kappa}2$, $x \restriction S = y \restriction S$, and $x \in \overline{X}$; we want to show that $y \in \overline{X}$. To this end, let $\{z : z \restriction F = y \restriction F\}$, $F \subseteq \kappa$ finite, be a basic open neighborhood of y. Then $x \in \{z : z \restriction (F \cap S) = x \restriction (F \cap S)\}$, so we can choose $w \in X$ such that $w \restriction (F \cap S) = x \restriction (F \cap S)$. Now let w' be such that $w' \restriction S = w \restriction S$ while $w' \restriction (F \backslash S) = y \restriction (F \backslash S)$. Then also $w' \in X$, since X is S-determined. Also, $w' \in \{z : z \restriction F = y \restriction F\}$, as desired. □

Lemma 9.27. *Suppose that B is a dense subalgebra of A, and $a, b \in A$. Then*

$$\begin{aligned}&\mathrm{int}(\{G \in \mathrm{Ult}(B) : G \cup \{a\} \textit{ has fip }\}) \cap \mathrm{int}(\{G \in \mathrm{Ult}(B) : G \cup \{b\} \textit{ has fip }\}) = \\ &\mathrm{int}(\{G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \textit{ has fip }\}).\end{aligned}$$

Proof. Clearly rhs$\subseteq$lhs. Suppose that

$$\begin{aligned}C \stackrel{\mathrm{def}}{=} &(\mathrm{int}(\{G \in \mathrm{Ult}(B) : G \cup \{a\} \text{ has fip}\}) \cap \mathrm{int}(\{G \in \mathrm{Ult}(B) : G \cup \{b\} \text{ has fip}\})) \\ &\backslash \{G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\} \neq \emptyset.\end{aligned}$$

Note that C is open by Lemma 9.25. Then $D \stackrel{\mathrm{def}}{=} \mathcal{S}(a) \cap \{G \in \mathrm{Ult}(A) : G \cap B \in C\} \neq \emptyset$. In fact, take any $H \in C$. Then $H \cup \{a\}$ has fip, and so is contained in an ultrafilter G; so $G \in D$. Now $\{G \in \mathrm{Ult}(A) : G \cap B \in C\}$ is open since it is the inverse image of the open set C under a continuous mapping. It follows that we can choose $0 \neq x \in B$ such that $\mathcal{S}(x) \subseteq D$. Thus $x \leq a$. Let $x \in G \in \mathrm{Ult}(A)$. Then $G \cap B \in C$, and hence $(G \cap B) \cup \{b\}$ has fip; so it is contained in an ultrafilter H on A. Since $x \in G \cap B \subseteq H$, we also have $x \in H$, hence $a \in H$. So $a \cdot b \in H$. Thus $(G \cap B) \cup \{a \cdot b\}$ has fip, contradicting $G \cap B \in C$. □

Lemma 9.28. *Suppose that $M \subseteq {}^{\kappa}2$, M is S-determined, $f, g \in {}^{\kappa}2$, $f \restriction S = g \restriction S$, and $f \in \mathrm{int}(M)$. Then $g \in \mathrm{int}(M)$. Thus $\mathrm{int}(M)$ is S-determined.*

Proof. Let N be a finite subset of κ such that $P \stackrel{\text{def}}{=} \{h \in {}^{\kappa}2 : h \restriction N = f \restriction N\} \subseteq M$. Let $k = g \restriction N$. Let $Q = \{l \in {}^{\kappa}2 : k \subseteq l\}$. So $g \in Q$. We claim that $Q \subseteq M$. (Hence $q \in \mathrm{int}(M)$, as desired.) For, suppose that $l \in Q$. We define $l' \in {}^{\kappa}2$:

$$l'(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in N, \\ l(\alpha) & \text{otherwise.} \end{cases}$$

Then $l' \in P$, so $l' \in M$. We claim that $l \restriction S = l' \restriction S$. (Hence $l \in M$, as desired.) If fact, suppose that $\alpha \in S$. If $\alpha \in N$, then $l'(\alpha) = f(\alpha) = g(\alpha) = l(\alpha)$. If $\alpha \notin N$, then $l'(\alpha) = l(\alpha)$ by definition. □

Lemma 9.29.

(i) *For any ultrafilter U on $\mathrm{Fr}(\kappa)$ we have $U = \mathrm{ultr}_{k_U}$.*
(ii) *For any $f \in {}^{\kappa}2$ we have $f = k_{\mathrm{ultr}_f}$.*

Proof. (i): ultr_{k_U} is generated by

$$\{x^{\kappa}_{\alpha} : k_U(\alpha) = 1\} \cup \{-x^{\kappa}_{\alpha} : k_U(\alpha) = 0\} = \{x^{\kappa}_{\alpha} : x^{\kappa}_{\alpha} \in U\} \cup \{-x^{\kappa}_{\alpha} : x^{\kappa}_{\alpha} \notin U\},$$

so (i) follows.

(ii): For any $\alpha < \kappa$,

$$\begin{aligned} k_{\mathrm{ultr}_f}(\alpha) &= \begin{cases} 1 & \text{if } x^{\kappa}_{\alpha} \in \mathrm{ultr}_f, \\ 0 & \text{otherwise} \end{cases} \\ &= f(\alpha). \end{aligned}$$ □

Lemma 9.30. *k is a homeomorphism from* $\mathrm{Ult}(\mathrm{Fr}(\kappa))$ *onto ${}^{\kappa}2$, and* ultr *is a homeomorphism from ${}^{\kappa}2$ onto* $\mathrm{Ult}(\mathrm{Fr}(\kappa))$.

Proof. Let M be a finite subset of κ and $\varepsilon \in {}^{M}2$. Then

$$k^{-1}[\{f : \varepsilon \subseteq f\}] = \{G : \varepsilon \subseteq k_G\} = \mathcal{S}\left(\prod_{\alpha \in M} x^{\varepsilon(\alpha)}_{\alpha}\right).$$

Also, k is one-one and onto by Lemma 9.29, so it is a homeomorphism.

Similarly, if M be a finite subset of κ, $\varepsilon \in {}^M2$, and $a = \prod_{\alpha\in M}(x^\kappa_\alpha)^{\varepsilon(\alpha)}$, then

$$\begin{aligned}\mathrm{ultr}^{-1}[\mathcal{S}(a)] = \{f \in {}^\kappa 2 : \mathrm{ultr}_f \in \mathcal{S}(a)\} &= \left\{f \in {}^\kappa 2 : \prod_{\alpha\in M}(x^\kappa_\alpha)^{\varepsilon(\alpha)} \in \mathrm{ultr}_f\right\}\\ &= \{f \in {}^\kappa 2 : \varepsilon \subseteq f\}.\end{aligned}$$

Again ultr is one-one and onto by Lemma 9.29, and so it is a homeomorphism. $\square$

Lemma 9.31. *Suppose that κ is an infinite cardinal and $S \subseteq \kappa$. Also suppose that $X \subseteq {}^\kappa 2$. Then X is S-determined* iff $\{\mathrm{ultr}_f : f \in X\}$ *is determined by S.*

Proof. $\Rightarrow$: Suppose that X is S-determined, $F, G \in \mathrm{Ult}(\mathrm{Fr}(\kappa))$, and $\forall\alpha \in S[x^\kappa_\alpha \in F$ iff $x^\kappa_\alpha \in G]$. Let $\alpha \in S$. Then

$$k_F(\alpha) = 1 \text{ iff } x^\kappa_\alpha \in F \text{ iff } x^\kappa_\alpha \in G \text{ iff } k_G(\alpha) = 1,$$

so $k_F \restriction S = k_G \restriction S$. Hence $k_F \in X$ iff $k_G \in X$. So

$$\begin{aligned}F \in \{\mathrm{ultr}_f : f \in X\} &\text{ iff } \mathrm{ultr}_{k_F} \in \{\mathrm{ultr}_f : f \in X\}\\ &\text{ iff } k_F \in X\\ &\text{ iff } k_G \in X\\ &\text{ iff } \mathrm{ultr}_{k_G} \in \{\mathrm{ultr}_f : f \in X\}\\ &\text{ iff } G \in \{\mathrm{ultr}_f : f \in X\}.\end{aligned}$$

$\Leftarrow$: Suppose that $\{\mathrm{ultr}_f : f \in X\}$ is determined by S, $f, g \in {}^\kappa 2$, $f \restriction S = g \restriction S$, and $f \in X$. For $\alpha \in S$ we have $x^\kappa_\alpha \in \mathrm{ultr}_f$ iff $f(\alpha) = 1$ iff $g(a) = 1$ iff $x^\kappa_\alpha \in \mathrm{ultr}_g$. Moreover, $\mathrm{ultr}_f \in \{\mathrm{ultr}_h : h \in X\}$. Hence $\mathrm{ultr}_g \in \{\mathrm{ultr}_h : h \in X\}$, so $g \in X$. The converse is similar. $\square$

Lemma 9.32. *Suppose that κ is an infinite cardinal and $S \subseteq \kappa$. Also suppose that $Y \subseteq \mathrm{Ult}(\mathrm{Fr}(\kappa))$. Then Y is determined by S* iff $\{k_U : U \in Y\}$ *is S-determined.*

Proof.

$$\begin{aligned}Y \text{ is determined by } S &\text{ iff } \{\mathrm{ultr}_{k_U} : U \in Y\} \text{ is determined by } S\\ &\text{ iff } \{k_U : U \in Y\} \text{ is } S\text{-determined, by Lemma 9.31.}\end{aligned}$$ $\square$

Lemma 9.33. *Suppose that $B \overset{\text{def}}{=} \mathrm{Fr}(\kappa)$ is a dense subalgebra of A, $F \in \mathrm{Ult}(A)$, and $S \subseteq \kappa$. Then $\mathscr{A}(F, S)$ is closed under $\cdot$.*

Proof. Suppose $a, b \in \mathscr{A}(F, S)$. Thus $a, b \in F$ and:

$$\{G \in \mathrm{Ult}(B) : G \cup \{a\} \text{ has fip}\} \text{ is determined by } S;$$
$$\{G \in \mathrm{Ult}(B) : G \cup \{b\} \text{ has fip}\} \text{ is determined by } S;$$

hence by Lemma 9.32,

$$\{k_G : G \in \mathrm{Ult}(B) : G \cup \{a\} \text{ has fip}\} \text{ is } S\text{-determined;}$$
$$\{k_G : G \in \mathrm{Ult}(B) : G \cup \{b\} \text{ has fip}\} \text{ is } S\text{-determined;}$$

hence by Lemma 9.28,

$$\mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{a\} \text{ has fip}\}) \text{ is } S\text{-determined;}$$
$$\mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{b\} \text{ has fip}\}) \text{ is } S\text{-determined;}$$

hence clearly

$$\mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{a\} \text{ has fip}\}) \cap \mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{b\} \text{ has fip}\})$$

is S-determined, and so by Lemma 9.27,

$$\mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\})$$

is S-determined. Hence by Lemma 9.26, also

$$\overline{\mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\})}$$

is S-determined. By Lemma 9.31,

$$\mathrm{ultr}\left[\overline{\mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\})}\right]$$

is determined by S; and by Lemma 9.30,

$$\begin{aligned}
&\mathrm{ultr}\left[\overline{\mathrm{int}(\{k_G : G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\})}\right]\\
&= \overline{\mathrm{int}(\{\mathrm{ultr}(k_G) : G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\})}\\
&= \overline{\mathrm{int}(\{G : G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\})}.
\end{aligned}$$

But by Lemma 9.25, $\{G \in \mathrm{Ult}(B) : G \cup \{a \cdot b\} \text{ has fip}\}$ is regular closed, so we conclude that it is determined by S. $\square$

Lemma 9.34. *Suppose that I is an ideal in $\mathrm{Fr}(\kappa)$. Then there is a countably generated ideal $J \subseteq I$ such that $(I^o)^{\mathrm{fi}} = (J^o)^{\mathrm{fi}}$.*

Proof. Let X be maximal pairwise disjoint with $X \subseteq I$, and let $J = \langle X \rangle^{\mathrm{id}}$. Thus X is countable, and so J is countably generated. We claim that $(I^o)^{\mathrm{fi}} = (J^o)^{\mathrm{fi}}$. Clearly $J \subseteq I$, and so $J^o \subseteq I^o$, hence $(I^o)^{\mathrm{fi}} \subseteq (J^o)^{\mathrm{fi}}$. Now suppose that $b \in (J^o)^{\mathrm{fi}}$. Thus $J^o \subseteq \mathcal{S}(b)$, so $a \leq b$ for all $a \in J$. Suppose that $a \in I$ and $a \not\leq b$. Then $0 \neq a \cdot -b \in I$, so there is an $x \in X$ such that $a \cdot -b \cdot x \neq 0$. But $x \in J$, so $x \leq b$, contradiction. Hence $\forall a \in I(a \leq b)$. So $I^o \subseteq \mathcal{S}(b)$, and hence $b \in (I^o)^{\mathrm{fi}}$. $\square$

Lemma 9.35. *If I is an ideal in* $\mathrm{Fr}(\kappa)$ *and* $S \subseteq \kappa$, *then the following conditions are equivalent:*

(i) *I is generated by some set* $K \subseteq \langle\{x^\kappa_\alpha : \alpha \in S\}\rangle$.

(ii) *I^o is determined by S.*

Proof. (i)⇒(ii): Assume (i), and suppose that $F, G \in \mathrm{Ult}(\mathrm{Fr}(\kappa))$ and for all $\alpha \in S$, $x^\kappa_\alpha \in F$ iff $x^\kappa_\alpha \in G$. Then for all $a \in K$, $a \in F$ iff $a \in G$. Suppose that $F \in I^o$. Say $F \in \mathcal{S}(a)$ with $a \in I$. Let M be a finite subset of K such that $a \le \sum M$. Now $a \in F$, so there is a $b \in M$ such that $b \in F$. Since $b \in K$, we also have $b \in G$. So $G \in \mathcal{S}(b) \subseteq I^o$.

(ii)⇒(i): Assume (ii). Let $K = I \cap \langle\{x^\kappa_\alpha : \alpha \in S\}\rangle$. We claim that K generates I. Suppose that $b \in I$, but $b \not\le a$ for all $a \in K$. Then $\{b \cdot -a : a \in K\}$ has the fip, and so is contained in an ultrafilter F. We claim that also

$$(*) \qquad \{(x^\kappa_\alpha)^{\varepsilon(\alpha)} : \alpha \in S \text{ and } (x^\kappa_\alpha)^{\varepsilon(\alpha)} \in F\} \cup \{-a : a \in I\}$$

has fip. Otherwise there is a finite $M \subseteq S$ such that $\prod_{\alpha \in M}(x^\kappa_\alpha)^{\varepsilon(\alpha)} \in F \cap I$; it follows that $\prod_{\alpha \in M}(x^\kappa_\alpha)^{\varepsilon(\alpha)} \in K$, contradicting $-\prod_{\alpha \in M}(x^\kappa_\alpha)^{\varepsilon(\alpha)} \in F$. So $(*)$ has fip, and we extend it to an ultrafilter G. Clearly $\forall \alpha \in S[x^\kappa_\alpha \in F \text{ iff } x^\kappa_\alpha \in G]$, $F \in I^o$, and $G \notin I^o$, contradicting (ii). □

Lemma 9.36. *Suppose that* $B \stackrel{\text{def}}{=} \mathrm{Fr}(\kappa)$ *is a dense subalgebra of A,* $F \in \mathrm{Ult}(A)$, *and* $a \in F$. *Then there is a countable* $S \subseteq \kappa$ *such that* $a \in \mathscr{A}(F, S)$.

Proof. Let $I = \{b \in \mathrm{Fr}(\kappa) : b \le a\}$. Thus I is an ideal in $\mathrm{Fr}(\kappa)$. By Lemma 9.34, there is a countably generated ideal $J \subseteq I$ in $\mathrm{Fr}(\kappa)$ such that $(I^o)^{\mathrm{fi}} = (J^o)^{\mathrm{fi}}$. Say J is generated by $K \subseteq \langle\{x^\kappa_\alpha : \alpha \in S\}\rangle$ with S countable. By Lemma 9.35, J^o is determined by S. Hence by Lemma 9.32, $\{k_U : U \in J^o\}$ is S-determined. Thus by Lemma 9.26, $\overline{\{k_U : U \in J^o\}}$ is S-determined, so by Lemma 9.31, $\mathrm{ultr}\left[\overline{\{k_U : U \in J^o\}}\right]$ is determined by S. Now by Lemma 9.30, $\mathrm{ultr}\left[\overline{\{k_U : U \in J^o\}}\right] = \overline{\mathrm{ultr}[k[J^o]]} = \overline{J^o}$, so $\overline{J^o}$ is determined by S. Now by Proposition 9.17,

$$\overline{J^o} = \bigcap_{a \in (J^o)^{\mathrm{fi}}} \mathcal{S}(a) = \bigcap_{a \in (I^o)^{\mathrm{fi}}} \mathcal{S}(a) = \overline{I^o}.$$

So $\overline{I^o}$ is determined by S. Hence it suffices to prove

$(*)$ $\overline{I^o} = \{G : G \in \mathrm{Ult}(B) \text{ and } G \cup \{a\} \text{ has fip}\}$.

To do this, first suppose that $G \in \overline{I^o}$. Suppose that $G \cup \{a\}$ does not have fip. Then there is a $b \in G$ such that $b \cdot a = 0$. Then $a \le -b$, $\mathcal{S}(a) \subseteq \mathcal{S}(-b)$, and so $-b \in (I^o)^{\mathrm{fi}}$. Hence $G \in \mathcal{S}(-b)$ by Lemma 9.17, so $-b \in G$, contradiction.

Second, suppose that $G \in \mathrm{Ult}(B)$ and $G \cup \{a\}$ has fip. In order to apply Lemma 9.17, suppose that $x \in (I^o)^{\mathrm{fi}}$; we want to show that $G \in \mathcal{S}(x)$, i.e., that $x \in G$. Now $\mathcal{S}(a) \subseteq I^o$, so $\mathcal{S}(a) \subseteq \mathcal{S}(x)$ and hence $a \le x$. If $-x \in G$, then $G \cup \{a\}$ does not have fip, contradiction. So $x \in G$. □

Lemma 9.37. *Suppose that* $\mathrm{Fr}(\kappa)$ *is a dense subalgebra of* A, $F \in \mathrm{Ult}(A)$, *and* $S \subseteq T \subseteq \kappa$. *Then* $(\mathscr{A}(F,T))^c \subseteq (\mathscr{A}(F,S))^c$.

Proof. Clearly $\mathscr{A}(F,S) \subseteq \mathscr{A}(F,T)$, so the conclusion is clear. □

Lemma 9.38. *Suppose that* $B \stackrel{\mathrm{def}}{=} \mathrm{Fr}(\kappa)$ *is a dense subalgebra of* A, $F \in \mathrm{Ult}(A)$, *and* $S \subseteq \kappa$. *Then* $\Phi(F \cap B, S) = \{G \cap B : G \in (\mathscr{A}(F,S))^c\}$.

Proof. First suppose that $G \in (\mathscr{A}(F,S))^c$. Suppose that $\alpha \in S$, $\varepsilon \in 2$, and $(x^\kappa_\alpha)^\varepsilon \in F \cap B$. Clearly $(x^\kappa_\alpha)^\varepsilon \in \mathscr{A}(F,S)$. It follows that $(x^\kappa_\alpha)^\varepsilon \in G$. This shows that $G \cap B \in \Phi(F \cap B, S)$.

Second, suppose that $H \in \Phi(F \cap B, S)$. We claim that $H \cup \mathscr{A}(F,S)$ has fip. Otherwise, by Lemma 9.33 there exist $b \in H$ and $a \in \mathscr{A}(F,S)$ such that $b \cdot a = 0$. Since $a \in \mathscr{A}(F,S)$, we have $a \in F$ and $\{G \in \mathrm{Ult}(B) : G \cup \{a\}$ has fip$\}$ is determined by S. Now $F \cap B$ is in this set, so from $H \in \Phi(F \cap B, S)$ we infer that H is in the set also. So $H \cup \{a\}$ has fip. But $b \in H$ and $b \cdot a = 0$, contradiction. This proves the claim. Let G be an ultrafilter on A such that $H \cup \mathscr{A}(F,S) \subseteq G$. Then $G \in (\mathscr{A}(F,S))^c$ and $G \cap B = H$, as desired. □

Theorem 9.39. *Suppose that* κ *is an uncountable regular cardinal,* $B \stackrel{\mathrm{def}}{=} \mathrm{Fr}(\kappa)$ *is a dense subalgebra of* A, *and* $F \in \mathrm{Ult}(A)$. *Then there is a free sequence* $\langle H_\alpha : \alpha < \kappa \rangle$ *in* $\mathrm{Ult}(A)$ *which converges to* F.

Proof. Choose $\varepsilon \in {}^\kappa 2$ such that $x_\alpha^{\varepsilon(\alpha)} \in F$ for each $\alpha \in \kappa$. For each $\alpha < \kappa$ let G_α be an ultrafilter on $\mathrm{Fr}(\kappa)$ which contains the set

$$\{x_\beta^{\varepsilon(\beta)} : \beta \leq \alpha\} \cup \{x_\beta^{1-\varepsilon(\beta)} : \alpha < \beta < \kappa\}.$$

Then

(1) $\langle G_\alpha : \alpha < \kappa \rangle$ is a free sequence in $\mathrm{Ult}(\mathrm{Fr}(\kappa))$.

In fact, suppose that $\beta < \kappa$. Then $\{G_\alpha : \alpha < \beta\} \subseteq \mathcal{S}(x_\beta^{1-\varepsilon(\beta)})$ and $\{G_\alpha : \beta \leq \alpha < \kappa\} \subseteq \mathcal{S}(x_\beta^{\varepsilon(\beta})$, as desired.

Now clearly $G_\alpha \in \Phi(F \cap B, \alpha+1)$, so by Lemma 9.38 we can choose $H_\alpha \in (\mathscr{A}(F,\alpha+1))^c$ such that $H_\alpha \cap B = G_\alpha$. Hence by the proof of (1),

(2) $\langle H_\alpha : \alpha < \kappa \rangle$ is a free sequence in $\mathrm{Ult}(A)$.

(3) $\bigcap_{\alpha<\kappa} (\mathscr{A}(F,\alpha+1))^c = \{F\}$.

In fact, if $a \in F$, choose by Lemma 9.36 a countable $S \subseteq \kappa$ such that $a \in \mathscr{A}(F,S)$. Then choose $\beta < \kappa$ such that $S \subseteq \beta + 1$. Then by Lemma 9.37,

$$\bigcap_{a<\kappa} (\mathscr{A}(F,\alpha+1))^c \subseteq (\mathscr{A}(F,\beta+1))^c \subseteq (\mathscr{A}(F,S))^c \subseteq \mathcal{S}(a);$$

hence (3) follows.

(4) $\langle H_\alpha : \alpha < \kappa\rangle$ converges to F.

In fact, suppose that $a \in F$. Then $F \in \mathcal{S}(a)$, so $\bigcap_{\alpha<\kappa}(\mathscr{A}(F, \alpha+1))^c \subseteq \mathcal{S}(a)$ by (3). Hence by compactness there is an $\alpha < \kappa$ such that $(\mathscr{A}(F, \alpha+1))^c \subseteq \mathcal{S}(a)$. By Lemma 9.37 it follows that $(\mathscr{A}(F, \beta+1))^c \subseteq \mathcal{S}(a)$ for all $\beta > \alpha$, and hence $H_\beta \in \mathcal{S}(a)$ for all $\beta > \alpha$. □

Proposition 9.40. *If $f : A \to B$ is a surjective homomorphism and $\langle F_\alpha : \alpha < \kappa\rangle$ is a free sequence in* $\mathrm{Ult}(B)$*, then $\langle f^{-1}[F_\alpha] : \alpha < \kappa\rangle$ is a free sequence in* $\mathrm{Ult}(A)$*.*

Proof. Suppose that $\beta < \kappa$. Choose $b_\beta \in B$ such that $\{F_\alpha : \alpha < \beta\} \subseteq \mathcal{S}(b_\beta)$ and $\{F_\alpha : \beta \le \alpha < \kappa\} \subseteq \mathcal{S}(-b_\beta)$. Choose $a_\beta \in A$ such that $f(a_\beta) = b_\beta$. Then $\{f^{-1}[F_\alpha] : \alpha < \beta\} \subseteq \mathcal{S}(a_\beta)$ and $\{f^{-1}[F_\alpha] : \beta \le \alpha < \kappa\} \subseteq \mathcal{S}(-a_\beta)$. □

Proposition 9.41. *If $f : A \to B$ is a surjective homomorphism and $\langle F_\alpha : \alpha < \kappa\rangle$ is a sequence in* $\mathrm{Ult}(B)$ *which converges to G, then $\langle f^{-1}[F_\alpha] : \alpha < \kappa\rangle$ converges to $f^{-1}[G]$.*

Proof. Let $a \in f^{-1}[G]$. Then $f(a) \in G$, so there is a $\beta < \kappa$ such that $f(a) \in F_\alpha$ for all $\alpha \in [\beta, \kappa)$. So $a \in f^{-1}[F_\alpha]$ for all $\alpha \in [\beta, \kappa)$. □

Theorem 9.42. (Juhász, Szentmiklóssy [92]) *Suppose that κ is an uncountable regular cardinal and* $\mathrm{Ult}(A)$ *has a free sequence of length κ. Then* $\mathrm{Ult}(A)$ *has a convergent free sequence of length κ.*

Proof. By the proof of Theorem 4.26, there is a homomorphism f of A onto a BA B such that B has a strictly increasing sequence of order type κ. Then using Sikorski's extension theorem, there is a homomorphism g of B onto a BA C such that $\mathrm{Intalg}(\kappa)$ is a dense subalgebra of C.

(1) We may assume that C does not have an independent subset of size κ.

In fact, suppose that it does have such a subset. Then by Sikorski's extension theorem, there is a homomorphism h of C onto a BA D such that $\mathrm{Fr}(\kappa)$ is a dense subalgebra of D. By Theorem 9.39, $\mathrm{Ult}(D)$ has a convergent free sequence of order type κ. Hence by Propositions 9.40 and 9.41, so does A, as desired. Thus (1) holds.

Now let $I = \langle\{[0, \alpha) : \alpha < \kappa\}\rangle^{\mathrm{id}}_C$, and let $h : C \to C/I$ be the natural homomorphism. Let $k = h \circ g \circ f$.

(2) If $F \in \mathrm{Ult}(C/I)$ and $\alpha < \kappa$, then $[\alpha, \kappa) \in h^{-1}[F]$.

In fact, $h([\alpha, \kappa)) = 1$, so (2) follows.

By (1), C/I does not have an independent subset of size κ. Hence by Handbook 10.21, $\pi\chi_{\mathrm{inf}}(C/I) < \kappa$. Let F be an ultrafilter on C/I such that $\pi\chi(F) < \kappa$. Choose $D \subseteq (C/I)^+$ dense in F with $|D| < \kappa$.

Now suppose that $G \in \mathrm{Ult}(C)$; we define $\varphi(G) \in \kappa + 1$ as follows. If G is the principal ultrafilter generated by $\{\alpha\}$ with $\alpha < \kappa$, we set $\varphi(G) = \alpha + 1$. If G is nonprincipal and $[\alpha, \kappa) \in G$ for all $\alpha < \kappa$, we set $\varphi(G) = \kappa$. Now suppose that

G is nonprincipal and $[0, \alpha) \in G$ for some $\alpha < \kappa$. Then there is a limit ordinal λ such that $[0, \lambda) \in G$, and we let $\varphi(G)$ be the least such λ. Then we have

(3) For any $G \in \mathrm{Ult}(C)$ and any $\gamma < \varphi(G)$ we have $[\gamma, \varphi(G)) \in G$.

In fact, (3) is clear if G is principal, or nonprincipal with $[\alpha, \kappa) \in G$ for all $\alpha < \kappa$. So suppose that λ is the least limit ordinal such that $[0, \lambda) \in G$. Suppose that $\gamma < \lambda$ and $[\gamma, \lambda) \notin G$. Then $[0, \gamma) \in G$ or $[\lambda, \kappa) \in G$. The latter is not possible, as otherwise $\emptyset = [0, \lambda) \cap [\lambda, \kappa) \in G$. So $[0, \gamma) \in G$. Since G is nonprincipal, γ is infinite. If δ is the greatest limit ordinal $\leq \gamma$, then $[0, \delta) \in G$ since $\{\alpha\} \notin G$ for all $\alpha \in [\delta, \gamma)$. This contradicts the choice of λ.

(4) If $a \in C \backslash I$, then $\{\alpha < \kappa : \exists G \in \mathrm{Ult}(C)[a \in G \text{ and } \varphi(G) = \alpha]\}$ is club in κ.

In fact, let $E = \{\alpha < \kappa : \exists G \in \mathrm{Ult}(C)[a \in G \text{ and } \varphi(G) = \alpha]\}$. If E is bounded in κ, then there is a $\beta < \kappa$ such that $\forall G \in \mathrm{Ult}(C)[a \in G \Rightarrow \varphi(G) < \beta]$. Now $a \not\leq [0, \beta)$, so $a \cdot [\beta, \kappa) \neq 0$. Note that $[\beta, \kappa) = \sum^C_{\gamma<\kappa}[\beta, \gamma)$. Hence there is a $\gamma < \kappa$ such that $a \cdot [\beta, \gamma) \neq 0$. Let G is an ultrafilter such that $a \cdot [\beta, \gamma) \in G$, then $\varphi(G) \geq \beta$, contradiction. So E is unbounded in κ. To show that E is closed, suppose that $\beta < \kappa$ is a limit ordinal and $E \cap \beta$ is unbounded in β. Then $a \cup \{[\gamma, \beta) : \gamma < \beta\}$ has fip, and so is contained in an ultrafilter G. Clearly $\varphi(G) = \beta$, as desired. This proves (4).

Now for each $x \in D$ choose $a_x \in C$ such that $[a_x] = x$, and let

$$C_x = \{\alpha < \kappa : \exists G \in \mathrm{Ult}(C)[a_x \in G \text{ and } \varphi(G) = \alpha]\},$$

and let $E = \bigcap_{x \in D} C_x$. Thus E is club in κ.

Now we define

$$V_\alpha = \{b \in h^{-1}[F] : \forall x \in D[x \leq h(b) \Rightarrow a_x \cdot [\alpha, \kappa) \leq b]\}.$$

Note that V_α is closed under $\cdot$. Also, let $F_\alpha = \bigcap_{b \in V_\alpha} \mathcal{S}(b)$.

(5) $\varphi[F_\alpha] = \bigcap_{b \in V_\alpha} \varphi[\mathcal{S}(b)]$.

For, first suppose that $\beta \in \varphi[F_\alpha]$, and suppose that $b \in V_\alpha$. Choose $G \in F_\alpha$ such that $\varphi(G) = \beta$. Then $b \in G$, and so $\beta \in \varphi[\mathcal{S}(b)]$, as desired.

Second, suppose that $\beta \in \bigcap_{b \in V_\alpha} \varphi[\mathcal{S}(b)]$. We claim that $V_\alpha \cup \{[\gamma, \beta) : \gamma < \beta\}$ has fip. Otherwise, there exist $b \in V_\alpha$ and $\gamma < \beta$ such that $b \cdot [\gamma, \beta) = 0$. Now $\beta \in \varphi[\mathcal{S}(b)]$, so we can choose an ultrafilter G on C such that $\varphi(G) = \beta$ and $b \in G$. Then by (3) we get $[\gamma, \beta) \in G$. Since also $b \in G$, this is a contradiction. So our claim holds. Let G be an ultrafilter on C such that $V_\alpha \cup \{[\gamma, \beta) : \gamma < \beta\} \subseteq G$. Then $G \in F_\alpha$ and $\varphi(G) = \beta$. This proves (5).

(6) $E \backslash (\alpha + 1) \subseteq \varphi[F_\alpha]$.

For, suppose that $b \in V_\alpha$; we show that $E \backslash (\alpha+1) \subseteq \varphi[\mathcal{S}(b)]$, and so (6) will follow by (5).

Choose $x \in D$ such that $x \le h(b)$. Then by the definition of V_α, $a_x \cdot [\alpha, \kappa) \le b$. Then we claim

(7) $C_x \backslash (\alpha + 1) \subseteq \varphi[\mathcal{S}(a_x \cdot [\alpha, \kappa))]$.

In fact, suppose that $\beta \in C_x \backslash (\alpha + 1)$. Choose $G \in \mathrm{Ult}(C)$ such that $a_x \in G$ and $\varphi(G) = \beta$. Then $\forall \gamma < \beta [[\gamma, \beta) \in G]$, so $[\alpha, \beta) \in G$. Hence $a_x \cdot [\alpha, \kappa) \in G$, so that $\beta \in \varphi[\mathcal{S}(a_x \cdot [\alpha, \kappa))]$, as desired in (7).

By (7)

$$E \backslash (\alpha + 1) \subseteq C_x \backslash (\alpha + 1) \subseteq \varphi[\mathcal{S}(a_x \cdot [\alpha, \kappa))] \subseteq \varphi[\mathcal{S}(b)].$$

This finishes the proof of (6).

(8) $h^{-1}[F] \in F_\alpha$ but $F_\alpha \ne \{h^{-1}[F]\}$.

Clearly $V_\alpha \subseteq h^{-1}[F]$, so $h^{-1}[F] \in F_\alpha$. Since $h([\beta, \kappa)) = 1$ for each $\beta < \kappa$, we have $\varphi(h^{-1}[F]) = \kappa$. Now choose $\beta \in E \backslash (\alpha + 1)$. Then by (6), $\beta \in \varphi[F_\alpha]$, so there is a $G \in F_\alpha$ such that $\varphi(G) = \beta$. Hence $G \ne h^{-1}[F]$, and (8) holds.

Now for each $\alpha < \kappa$ let $Z_\alpha = \varphi^{-1}[(\kappa + 1) \backslash (\alpha + 1)]$. Then

(9) $\varphi^{-1}[\{\kappa\}] = \bigcap_{\alpha < \kappa} Z_\alpha$.

In fact,

$$\bigcap_{\alpha < \kappa} Z_\alpha = \bigcap_{\alpha < \kappa} \varphi^{-1}[(\kappa + 1) \backslash (\alpha + 1)] = \varphi^{-1}\left[\bigcap_{\alpha < \kappa} ((\kappa + 1) \backslash (\alpha + 1))\right] = \varphi^{-1}[\{\kappa\}].$$

Now for α limit, Z_α is closed. For, suppose that $G \in \overline{Z_\alpha}$. If $[0, \alpha) \in G$, then there is an $H \in Z_\alpha$ such that $[0, \alpha) \in H$, which is clearly impossible. So $[\alpha, \kappa) \in G$, and so $\varphi(G) \ge \alpha + 1$, as desired.

(10) For every $b \in h^{-1}[F]$ there is an $\alpha < \kappa$ such that $b \in V_\alpha$.

In fact, fix $b \in h^{-1}[F]$. Then we claim

(11) For all $x \in D$, if $x \le h(b)$ then $\mathcal{S}(a_x) \cap \varphi^{-1}[\{\kappa\}] \subseteq \mathcal{S}(b)$.

In fact, take any $x \in D$ such that $x \le h(b)$. Suppose that $G \in \mathcal{S}(a_x) \cap \varphi^{-1}[\{\kappa\}]$. Now $\varphi(G) = \kappa$ implies that $G \cap I = \emptyset$, and so $h[G]$ is an ultrafilter on C/I. Since $x \in h[G]$, it follows that $h(b) \in h[G]$, and hence $b \in G$, as otherwise $-b \in G$ and so $-h(b) \in h[G]$. This proves (11).

Now note that $\alpha < \beta < \kappa$ implies that $Z_\beta \subseteq Z_\alpha$. Hence from (9), (11) and compactness, for each $x \in D$ with $x \le h(b)$ we get an $\alpha_x < \kappa$ such that $\mathcal{S}(a_x) \cap Z_{\alpha_x} \subseteq \mathcal{S}(b)$. Since $|D| < \kappa$ and κ is regular, choose $\beta < \kappa$ such that $\alpha_x < \beta$ for all $x \in D$ with $x \le h(b)$. We claim now that $b \in V_\beta$, as desired in (10). For, suppose that $x \in D$ and $x \le h(b)$. If $a_x \cdot [\beta, \kappa) \cdot -b \ne 0$, we can choose an ultrafilter G on C such that $a_x \cdot [\beta, \kappa) \cdot -b \in G$. Thus $\varphi(G) > \beta > \alpha_x$, so $b \in G$, contradiction. This proves (10).

(12) $\{h^{-1}[F]\} = \bigcap_{\alpha<\kappa} F_\alpha$.

For, suppose that G is an ultrafilter on C different from $h^{-1}[F]$. Choose $b \in h^{-1}[F] \backslash G$. By (10), choose $\alpha \in \kappa$ such that $b \in V_\alpha$. Since $b \notin G$, we have $G \notin \bigcap_{c \in V_\alpha} \mathcal{S}(c) = F_\alpha$, and (12) follows, using (8).

Now if $\alpha < \beta$ then $V_\alpha \subseteq V_\beta$ and hence $F_\beta \subseteq F_\alpha$. By (12), $\bigcap_{\alpha<\kappa} F_\alpha = \{h^{-1}[F]\}$, and by (8), $F_\alpha \neq \{h^{-1}[F]\}$. Hence there is a strictly increasing sequence $\langle \beta(\gamma) : \gamma < \kappa \rangle$ of ordinals less than κ such that for all γ, δ, if $\gamma < \delta < \kappa$ then $F_{\beta(\delta)} \subset F_{\beta(\gamma)}$. Now choose $G_\alpha \in F_{\beta(\alpha)} \backslash F_{\beta(\alpha+1)}$ for all $\alpha < \kappa$.

(13) The sequence $\langle G_\alpha : \alpha < \kappa \rangle$ converges to $h^{-1}[F]$.

In fact, let $b \in h^{-1}[F]$. Then $h^{-1}[F] \in \mathcal{S}(b)$, so $\bigcap_{\alpha<\kappa} F_\alpha = \{h^{-1}[F]\} \subseteq \mathcal{S}(b)$. By compactness, there is an $\alpha < \kappa$ such that $F_{\beta(\alpha)} \subseteq \mathcal{S}(b)$. Hence $G_\gamma \in \mathcal{S}(b)$ for all $\gamma \geq \alpha$, proving (13).

(14) $\forall \gamma < \kappa \exists \alpha < \kappa([\gamma, \kappa) \in G_\alpha)$.

In fact, given $\gamma < \kappa$, by (10) there is an $\alpha < \kappa$ such that $[\gamma, \kappa) \in V_\alpha \subseteq V_{\beta(\alpha)}$ and so $F_{\beta(\alpha)} \subseteq \mathcal{S}([\gamma, \kappa))$ and hence $[\gamma, \kappa) \in G_\alpha$.

(15) $\forall \gamma < \kappa \exists \alpha < \kappa[\varphi(G_\alpha) > \gamma]$.

This is immediate from (14).

By (15), there is a strictly increasing sequence $\langle \gamma(\alpha) : \alpha < \kappa \rangle$ such that $\langle \varphi(G_{\gamma(\alpha)}) : \alpha < \kappa \rangle$ is also strictly increasing. We claim, finally, that $\langle G_{\gamma(2\alpha+1)} : \alpha < \kappa \rangle$ is a free sequence converging to $h^{-1}[F]$. Clearly it converges to $h^{-1}[F]$. Now suppose that $\xi < \kappa$. Let $\varphi(G_{\gamma(2\xi)}) = \eta$. Then $[0, \eta) \in G_{\gamma(2\alpha+1)}$ for all $\alpha < \xi$, and $[\eta, \kappa) \in G_{\gamma(2\alpha+1)}$ for all $\alpha \geq \xi$. □

For the next two lemmas, see Engelking [89], page 17.

Lemma 9.43. *If κ is an infinite cardinal, X is a topological space, $w(X) \leq \kappa$, and if $\mathscr{A}$ is a collection of open subsets of X, then there is an $\mathscr{A}' \subseteq \mathscr{A}$ such that $|\mathscr{A}'| \leq \kappa$ and $\bigcup \mathscr{A} = \bigcup \mathscr{A}'$.*

Proof. Let $\mathscr{B}$ be a base for X such that $|\mathscr{B}| \leq \kappa$. Define $\mathscr{B}' = \{U \in \mathscr{B} : \exists V \in \mathscr{A}[U \subseteq V]\}$. For each $U \in \mathscr{B}'$ choose $V_U \in \mathscr{A}$ such that $U \subseteq V_U$. Let $\mathscr{A}' = \{V_U : U \in \mathscr{B}'\}$. Thus $|\mathscr{A}'| \leq \kappa$. Now suppose that $x \in \bigcup \mathscr{A}$. Choose $W \in \mathscr{A}$ such that $x \in W$, and then choose $U \in \mathscr{B}$ such that $x \in U \subseteq W$. Thus $U \in \mathscr{B}'$. So $x \in U \subseteq V_U \in \mathscr{A}'$, hence $x \in \bigcup \mathscr{A}'$. So $\bigcup \mathscr{A} = \bigcup \mathscr{A}'$. □

Lemma 9.44. *If κ is an infinite cardinal, X is a topological space, $w(X) \leq \kappa$, and if $\mathscr{A}$ is a base for X, then there is a base $\mathscr{A}' \subseteq \mathscr{A}$ such that $|\mathscr{A}'| \leq \kappa$.*

Proof. Let $\mathscr{B}$ be a base for X such that $|\mathscr{B}| \le \kappa$. For each $U \in \mathscr{B}$ let $S(U) = \{V \in \mathscr{A} : V \subseteq U\}$. Since $\mathscr{A}$ is a base for X, we have $U = \bigcup S(U)$ for every $U \in \mathscr{B}$; by Lemma 9.43 there is an $S'(U) \in [S(U)]^{\le\kappa}$ such that $\bigcup S'(U) = \bigcup S(U) = U$. Let $\mathscr{A}' = \bigcup_{U\in\mathscr{B}} S'(U)$. Then $|\mathscr{A}'| \le \kappa$. Since $\mathscr{B}$ is a base and $U = \bigcup S'(U)$ for every $U \in \mathscr{B}$, it follows that $\mathscr{A}'$ is a base. □

For any topological space X, $w(X)$ is the smallest size of a base for X.

Lemma 9.45. *If $\langle x_\alpha : \alpha < \kappa\rangle$ is a left separated sequence in a space X, then $w(\{x_\alpha : \alpha < \kappa\}) = \kappa$.*

Proof. For each $\alpha < \kappa$ let U_α be an open set such that $U_\alpha \cap \{x_\beta : \beta < \kappa\} = \{x_\beta : \alpha \le \beta\}$. Let $\mathscr{B}$ be a basis for $\{x_\alpha : \alpha < \kappa\}$ of size $w(\{x_\alpha : \alpha < \kappa\})$. For each $\alpha < \kappa$ let $V_\alpha \in \mathscr{B}$ be such that $x_\alpha \in V_\alpha \subseteq U_\alpha$. If $\alpha < \gamma$, then $x_\alpha \in V_\alpha \backslash V_\gamma$. Hence $|\mathscr{B}| = \kappa$. □

For the next two results, see Juhász [80].

Lemma 9.46. *Suppose that κ is regular, X is a space, X does not have a left separated sequence of length κ, $\langle Y_\alpha : \alpha < \kappa\rangle$ is an increasing sequence of subsets of X, $Z = \bigcup_{\alpha<\kappa} Y_\alpha$, and $\mathscr{B}$ is a family of open sets of X such that for each $\alpha < \kappa$, the set $\{B \cap Y_\alpha : B \in \mathscr{B}\}$ is a base for Y_α. Then $\{B \cap Z : B \in \mathscr{B}\}$ is a base for Z.*

Proof. Suppose not. Then there is a $z \in Z$ and an open set U of X such that $z \in U$ and for all $B \in \mathscr{B}$, if $z \in B$ then $B \cap Z \not\subseteq U$. We now define by induction a sequence $\langle \nu_\alpha : \alpha < \kappa\rangle$ of ordinals less than κ, a sequence $\langle y_\alpha : \alpha < \kappa\rangle$ of members of Z, and a sequence $\langle B_\alpha : \alpha < \kappa\rangle$ of members of $\mathscr{B}$ such that $y_\alpha \in Y_{\nu_\alpha}$ for all $\alpha < \kappa$. Let ν_0 be such that $z \in Y_{\nu_0}$. Now $z \in U \cap Y_{\nu_0}$ and $\{B \cap Y_{\nu_0} : B \in \mathscr{B}\}$ is a base for Y_{ν_0}, so we can choose $B_0 \in \mathscr{B}$ such that $z \in B_0 \cap Y_{\nu_0} \subseteq U \cap Y_{\nu_0}$. Then by assumption $B_0 \cap Z\backslash U \ne \emptyset$, so we choose $y_0 \in B_0 \cap Z\backslash U$.

Now suppose that $\alpha < \kappa$ and we have defined ν_β, y_β and B_β for all $\beta < \alpha$. Choose $\nu_\alpha < \kappa$ such that $\nu_\beta < \nu_\alpha$ and $\{y_\beta : \beta < \alpha\} \subseteq Y_{\nu_\alpha}$ for all $\beta < \alpha$. This is possible since κ is regular. Now $z \in U \cap Y_{\nu_\alpha}$, so by hypothesis there is a $B_\alpha \in \mathscr{B}$ such that $z \in B_\alpha \cap Y_{\nu_\alpha} \subseteq U \cap Y_{\nu_\alpha}$. Then choose $y_\alpha \in B_\alpha \cap Z\backslash U$. This finishes the construction.

We claim that $\langle y_\alpha : \alpha < \kappa\rangle$ is left-separated. (Contradiction.) For, suppose that $\alpha < \kappa$. Then, we claim,

$$(*) \qquad \{y_\beta : \beta < \kappa\} \cap \bigcup_{\alpha\le\gamma} B_\gamma = \{y_\beta : \alpha \le \beta\}.$$

For, if $\beta < \alpha \le \gamma$ and $y_\beta \in B_\gamma$, then $y_\beta \in B_\gamma \cap Y_{\nu_\gamma} \subseteq U$, contradiction. Thus $\subseteq$ holds in $(*)$. If $\alpha \le \beta$, then $y_\beta \in B_\beta$; so $\supseteq$ holds too. □

Theorem 9.47. *If κ is an infinite cardinal, X is a topological space, and $w(Y) < \kappa$ for all $Y \in [X]^{\le\kappa}$, then $w(X) < \kappa$.*

Proof. First we assume that κ is regular. Suppose that the conclusion fails, so that there is no base for X of size less than κ. We now define by recursion two sequences $\langle Y_\alpha : \alpha < \kappa\rangle$ and $\langle \mathscr{B}_\alpha : \alpha < \kappa\rangle$. Let $Y_0 = \emptyset$ and $\mathscr{B}_0 = \emptyset$. Suppose that Y_β and $\mathscr{B}_\beta$ have been defined for all $\beta < \alpha$, where $\alpha < \kappa$ so that $|Y_\beta|, |\mathscr{B}_\beta| < \kappa$ for all $\beta < \alpha$. Let $Z_\alpha = \bigcup_{\beta<\alpha} Y_\beta$. Then $|Z_\alpha| < \kappa$ since κ is regular. By assumption, there is a family $\mathscr{B}_\alpha$ of open sets such that $\bigcup_{\beta<\alpha}\mathscr{B}_\beta \subseteq \mathscr{B}_\alpha$, $|\mathscr{B}_\alpha| < \kappa$, and $\{U \cap Z_\alpha : U \in \mathscr{B}_\alpha\}$ is a base for Z_α. Now by assumption $\mathscr{B}_\alpha$ is not a base for X. Hence there is a $p_\alpha \in X$ and an open set U_α such that $p_\alpha \in U_\alpha$, and if $p_\alpha \in V \in \mathscr{B}_\alpha$ then $V\backslash U_\alpha \neq \emptyset$. Let $C_\alpha = \{V \in \mathscr{B}_\alpha : p_\alpha \in V\}$ and for each $V \in C_\alpha$ pick $q(V) \in V\backslash U_\alpha$. Let

$$Y_\alpha = Z_\alpha \cup \{p_\alpha\} \cup \{q(V) : V \in C_\alpha\}.$$

Clearly $|Y_\alpha| < \kappa$. This finishes the construction.

Let $\mathscr{C} = \bigcup_{\alpha<\kappa}\mathscr{B}_\alpha$. Now by Lemma 9.45, X does not have a left-separated sequence of length κ. Let $W = \bigcup_{\alpha<\kappa} Y_\alpha$. We apply Lemma 9.46 to conclude that $\{D \cap W : D \in \mathscr{C}\}$ is a base for W. Clearly $|\mathscr{C}| \leq \kappa$, so $w(W) < \kappa$. By Lemma 9.44 there is a $\mathscr{C}' \in [\mathscr{C}]^{<\kappa}$ such that $\{D \cap W : D \in \mathscr{C}'\}$ is a base for W. By the regularity of κ, there is an $\alpha < \kappa$ such that $\mathscr{C}' \subseteq B_\alpha$, so $\{D \cap W : D \in B_\alpha\}$ is a base for W. Now $p_\alpha \in U_\alpha$, so there is a $D \in B_\alpha$ such that $p_\alpha \in D \cap W \subseteq U_\alpha$. Hence $D \in C_\alpha$, so $q(D) \in D\backslash U_\alpha$, contradiction. This finishes the proof for κ regular.

Now assume that κ is singular. Then we claim

$(*)$ There is a $\lambda < \kappa$ such that $w(Y) < \lambda$ whenever $Y \in [X]^{\leq\kappa}$.

In fact, if there is no such κ, then for every $\lambda < \kappa$ there is a $Y_\lambda \in [C]^{\leq\kappa}$ such that $w(Y_\lambda) \geq \lambda$. Let $Z = \bigcup_{\lambda<\kappa} Y_\lambda$. Then $Z \in [X]^{\leq\kappa}$ and for each $\lambda < \kappa$ we have $w(Z) \geq w(Y_\lambda) \geq \lambda$, hence $w(Z) \geq \kappa$, contradiction. So $(*)$ holds.

Choose a λ as in $(*)$. Then $w(Y) < \lambda^+$ for all $Y \in [X]^{\leq\lambda^+}$, so by the first part of this proof, $w(X) < \lambda^+ < \kappa$. □

Lemma 9.48. *If $X \subseteq \mathrm{Ult}(A)$, then $w(\overline{X}) \leq 2^{|X|}$.*

Proof. The set $\{\mathcal{S}(a) \cap \overline{X} : a \in A\}$ is a base for the topology on $\overline{X}$. Now if $a, b \in A$, then

$$\begin{aligned} \mathcal{S}(a) \cap \overline{X} \neq \mathcal{S}(b) \cap \overline{X} \quad &\text{iff} \quad \mathcal{S}(a\triangle b) \cap \overline{X} \neq \emptyset \\ &\text{iff} \quad \mathcal{S}(a\triangle b) \cap X \neq \emptyset \\ &\text{iff} \quad \mathcal{S}(a) \cap X \neq \mathcal{S}(b) \cap X; \end{aligned}$$

hence there is a one-one map from $\{\mathcal{S}(a) \cap \overline{X} : a \in A\}$ into $\mathscr{P}(X)$. □

Theorem 9.49. *If X is a Hausdorff space and $Y \subseteq X$, then $|\overline{Y}| \leq 2^{2^{|Y|}}$.*

Proof. For any $p \in \overline{Y}$ let $f(p) = \{U \cap Y : p \in U, U \text{ open}\}$. Thus $f(p) \subseteq \mathscr{P}(Y)$, so $f(p) \in \mathscr{P}(\mathscr{P}(Y))$. Hence it suffices to show that f is one-one. Let $p, q \in \overline{Y}$, $p \neq q$. Choose U, V open and disjoint with $p \in U$ and $q \in V$. Thus $U \cap Y \in f(p)$.

Suppose that $U \cap Y \in f(q)$. Let W be open with $q \in W$ and $U \cap Y = W \cap Y$. Then $q \in V \cap W$, hence $V \cap W \neq \emptyset$, hence $V \cap W \cap Y \neq \emptyset$. But $V \cap W \cap Y = U \cap V \cap Y = \emptyset$, contradiction. □

For any topological space X, the *net weight* of X is

$$\begin{aligned}\mathrm{nw}(X) = \min\{|\mathscr{B}| : {}&\mathscr{B} \text{ is a family of subsets of } X \text{ and}\\ &\forall x \in X \forall U[U \text{ is open and } x \in U \Rightarrow \exists B \in \mathscr{B}[x \in B \subseteq U]]\}.\end{aligned}$$

A set $\mathscr{B}$ as in this definition is called a *network* for X.

Theorem 9.50. *If X is an infinite compact Hausdorff space, then* $\mathrm{nw}(X) = \mathrm{w}(X)$.

Proof. Obviously $\mathrm{nw}(X) \leq \mathrm{w}(X)$. Now let $\mathscr{B}$ be a network for X. For any pair (C, D) of elements of $\mathscr{B}$ such that there are disjoint open U, V with $C \subseteq U$ and $D \subseteq V$, let U_{CD} and V_{CD} be such a pair U, V. We claim that $\{U_{CD} : C, D \in \mathscr{B}$ and such U, V exist$\}$ is a subbase for X. This will give the equality of the theorem.

Suppose that $x \in X$ and $x \in W$, W open. For any $y \in X \backslash W$ let S, T be disjoint open sets with $x \in S$ and $y \in T$. Then choose $E_{xy}, F_{xy} \in \mathscr{B}$ such that $x \in E_{xy} \subseteq S$ and $y \in F_{xy} \subseteq T$. Then $x \in U_{E_{xy}F_{xy}}$, $y \in V_{E_{xy}F_{xy}}$, $E_{xy} \subseteq U_{E_{xy}F_{xy}}$, and $F_{xy} \subseteq V_{E_{xy}F_{xy}}$. Thus $\{V_{E_{xy}F_{xy}} : y \in X \backslash W\}$ covers $X \backslash W$, so there is a finite subcover $\{V_{E_{xy(0)}F_{xy(0)}}, \ldots, V_{E_{xy(n-1)}F_{xy(n-1)}}\}$. It follows that

$$x \in \bigcap_{i<n} U_{E_{xy(i)}F_{xy(i)}} \subseteq W,$$

as desired. □

Theorem 9.51 (Juhász [93]**.** *] If A is a BA and $\omega < \kappa \leq |A|$ with κ regular, then A has a homomorphic image B such that $\kappa \leq |B| \leq 2^{<\kappa}$ and $|\mathrm{Ult}(B)| \leq \sum\{2^{2^\lambda} : \lambda < \kappa\}$.*

Proof. We consider two cases.

Case 1. There exist $y \in \mathrm{Ult}(A)$ and $Y \subseteq \mathrm{Ult}(A)$ such that $y \in \overline{Y}$ and $\forall Z \in [Y]^{<\kappa}[y \notin \overline{Z}]]$. Then by Theorem 4.25, $\mathrm{Ult}(A)$ has a free sequence $\langle F_\xi : \xi < \kappa\rangle$ of length κ. By Theorem 9.42 we may assume that this sequence converges to some ultrafilter G. For each $\alpha < \kappa$ let $Y_\alpha = \{F_\xi : \xi < \alpha\}$. If there is an $\alpha < \kappa$ such that $w(\overline{Y}_\alpha) \geq \kappa$, then by Lemma 9.48, $w(\overline{Y}_\alpha) \leq 2^{|\alpha|} \leq 2^{<\kappa}$. Moreover, $|Y_\alpha| < \kappa$, so by Theorem 9.49, $|\overline{Y_\alpha}| \leq \sum\{2^{2^\lambda} : \lambda < \kappa\}$.

Thus assume that $w(\overline{Y}_\alpha) < \kappa$ for all $\alpha < \kappa$. Let $M = \{F_\xi : \xi < \kappa\}$.

(1) $\overline{M} = \{G\} \cup \bigcup_{\alpha<\kappa} \overline{Y}_\alpha$.

In fact, $\supseteq$ is obvious. Now assume that $H \in \overline{M} \backslash \bigcup_{\alpha<\kappa} \overline{Y}_\alpha$ and $G \neq H$. Choose $a \in H \backslash G$. Then $-a \in G$, so there is a $\xi < \kappa$ such that $\forall \eta \in [\xi, \kappa)[-a \in F_\eta]$. Take any $b \in H$. Then $a \cdot b \in H$, so there is a $\xi < \kappa$ such that $a \cdot b \in F_\xi$. Clearly $\xi < \eta$. This shows that $H \in \overline{Y}_\eta$, contradiction. So (1) holds.

(2) $nw(\overline{M}) \leq \kappa$.

To prove this, choose for each $\alpha < \kappa$ a subset B_α of A of size less than κ such that $\{\mathcal{S}(b) \cap \overline{Y}_\alpha : b \in B_\alpha\}$ is a base for $\overline{Y}_\alpha$. Let

$$P = \bigcup_{\alpha<\kappa} \{\mathcal{S}(b) \cap \overline{Y}_\alpha : b \in B_\alpha\} \cup \{\overline{M} \backslash \overline{Y}_\alpha : \alpha < \kappa\}.$$

We claim that P is a network for $\overline{M}$. Clearly $|P| \leq \kappa$, so this will prove (2). So, suppose that $H \in \mathcal{S}(a) \cap \overline{M}$.

Subcase 1.1. $H \in \overline{Y}_\alpha$ for some $\alpha < \kappa$. Choose $b \in B_\alpha$ such that $H \in \mathcal{S}(b) \cap \overline{Y}_\alpha \subseteq \mathcal{S}(a) \cap \overline{Y}_\alpha$; this is as desired.

Subcase 1.2. $H = G$. Since $a \in H = G$, there is a $\xi < \kappa$ such that $\forall \eta \in [\xi, \kappa)[a \in F_\eta]$. Now $G \in \overline{\{F_\eta : \xi \leq \eta < \kappa\}}$, so by the freeness of $\langle F_\rho : \rho < \kappa \rangle$ it follows that $G \notin \overline{F_\xi}$, i.e., $G \in \overline{M} \backslash \overline{F_\xi}$. We claim that $\overline{M} \backslash \overline{Y_\xi} \subseteq \mathcal{S}(a)$ (as desired). For, suppose that $K \in \overline{M} \backslash \overline{Y_\xi}$. We may assume that $K \neq G$. Say $K \in \overline{Y}_\eta$; so $\xi \leq \eta$. Since $K \notin \overline{Y}_\xi$, we can choose $b \in K$ such that $\mathcal{S}(b) \cap \overline{Y}_\xi = \emptyset$. Suppose that $-a \in K$. Choose $F_\rho \in Y_\eta$ so that $b \cdot -a \in F_\rho$. Since $b \in F_\rho$, it follows that $\xi \leq \rho$. But then $a \in F_\rho$, contradiction. Hence $a \in K$, as desired.

This finishes the proof of (2). Hence by Theorem 9.50 we get $w(\overline{M}) \leq \kappa$.

(3) M is discrete.

See Theorem 5.18 and its proof.

Thus $\kappa = w(M) \leq w(\overline{M})$, so by the above, $w(\overline{M}) = \kappa$. Moreover, $|\overline{M}| \leq \sum_{\alpha<\kappa} |\overline{Y}_\alpha| \leq \sum_{\lambda<\kappa} 2^\lambda \leq \sum_{\lambda<\kappa} 2^{2^\lambda}$.

Case 2. $\forall y \in \mathrm{Ult}(A) \forall Y \subseteq \mathrm{Ult}(A)[y \in \overline{Y}$ implies that $\exists Z \in [Y]^{<\kappa}[y \in \overline{Z}]]$. Now $\mathrm{w}(\mathrm{Ult}(A)) = |A| \geq \kappa$, so by Theorem 9.47 there is an $M \in [\mathrm{Ult}(A)]^{\leq\kappa}$ such that $\mathrm{w}(M) \geq \kappa$. If $|M| < \kappa$, define $f(a) = \mathcal{S}(a) \cap M$ for all $a \in A$. Then f is a homomorphism, and $\kappa \leq |f[A]| \leq 2^{|M|} \leq 2^{<\kappa}$; by Theorem 5.1, $\mathrm{d}(\mathrm{rng}(f)) \leq |M|$ and hence $|\mathrm{Ult}(\mathrm{rng}(f))| \leq 2^{|M|} \leq \sum_{\lambda<\kappa} 2^{2^\lambda}$.

So assume that $|M| = \kappa$; write $M = \{F_\xi : \xi < \kappa\}$ with F one-one. Define $Y_\alpha = \{F_\xi : \xi < \alpha\}$ for every $\alpha < \kappa$. Now since κ is regular, the case assumption yields $\overline{M} = \bigcup_{\alpha<\kappa} \overline{Y_\alpha}$.

Subcase 2.1. There is an $\alpha < \kappa$ such that $w(\overline{Y}_\alpha) \geq \kappa$. Define $f(a) = \mathcal{S}(a) \cap \overline{Y}_\alpha$ for any $a \in A$. Then f is a homomorphism, and $\kappa \leq |f[A]| \leq 2^{|\alpha|} \leq 2^{<\kappa}$; and by Theorem 9.49, $|\overline{Y}_\alpha| \leq 2^{2^{|Y_\alpha|}} \leq \sum_{\lambda<\kappa} 2^{2^\lambda}$.

Subcase 2.2. $w(\overline{Y}_\alpha) < \kappa$ for all $\alpha < \kappa$. For each $\alpha < \kappa$ let $B_\alpha \subseteq A$ be such that $|B_\alpha| < \kappa$ and $\{\mathcal{S}(b) \cap \overline{Y}_\alpha\}$ is a base for $\overline{Y}_\alpha$.

(4) $nw(\overline{M}) \leq \kappa$.

In fact, clearly $\bigcup_{\alpha<\kappa} \{\mathcal{S}(a) \cap \overline{Y}_\alpha : a \in B_\alpha\}$ is a network for $\overline{M}$ of size at most κ.

By Theorem 9.50 we have $w(\overline{M}) \leq \kappa$. Since $w(\overline{M} \geq w(M) \geq \kappa$, it follows that $w(\overline{M}) = \kappa$. Finally, $|\overline{M}| \leq \sum_{\alpha<\kappa} |\overline{Y}_\alpha| \leq \sum_{\lambda<\kappa} 2^\lambda \leq \sum_{\lambda<\kappa} 2^{2^\lambda}$. $\square$

As remarked above, Theorem 9.51 implies under GCH that for every BA A and every uncountable regular cardinal $\kappa \le |A|$, A has a homomorphic image of size κ. This does not extend to $\kappa = \omega$, as any CSP algebra has only uncountable infinite homomorphic image. It also does not extend to all singular cardinals, by a result of Juhász, Shelah [98]. That result, for BAs, is a consequence of Corollary 4.78. For example, under CH we have

$$\mathrm{Card_{Hs}}(\langle[\aleph_\omega]^{\le\omega}_{\mathscr{P}(\aleph_\omega)}) = \{\aleph_n : 1 \le n < \omega\} \cup \{\aleph_{\omega+1}\}.$$

It is an open question to what extent Theorem 9.51 is best possible. Recall from Chapter 3, (20) in the treatment of $\mathrm{c_{Hr}}$, that Koszmider [99] has shown the consistency of the existence of a BA A such that $\mathrm{Card_{Hr}}(A) = \{\omega_1, \lambda\}$, where $\lambda = 2^\omega$ can be arbitrarily large.

Problem 93. *Is Theorem* 9.51 *best possible? In particular, is it consistent that there is a BA A such that* $\mathrm{Card_{Hs}}(A) = \{\omega_2, \omega_4\}$*?*

In comparing cardinality with the cardinal functions so far introduced, we now note explicitly that $2^{\mathrm{d}(A)} \ge |A|$ for any infinite BA A. Finally, recall from Part I of the BA Handbook, Theorem 12.2, that $|A|^\omega = |A|$ for any infinite CSP algebra A, in particular for any (countably) complete infinite BA A.

10 Independence

There is a lot of information about independence in Part I of the Handbook. An even more extensive account is in Monk [83].

If $A \leq B$, then $\mathrm{Ind}(A) \leq \mathrm{Ind}(B)$, and the independence can increase arbitrarily. Now we consider our special subalgebras.

- $A \leq_{\mathrm{free}} B$: the independence obviously can increase arbitrarily. Hence by the general results of Chapter 2, the same applies to $\leq_{\mathrm{proj}}$, $\leq_{\mathrm{u}}$, $\leq_{\mathrm{rc}}$, $\leq_{\mathrm{reg}}$, and $\leq_{\sigma}$.
- $A \leq_{\pi} B$: the independence can increase; for example, in $\mathrm{Fr}(\omega) \leq_{\pi} \overline{\mathrm{Fr}(\omega)}$, using the Balcar, Franěk theorem. For an even larger increase, take $A = \mathrm{Finco}(\kappa)$ and $B = \overline{A}$.

Proposition 10.1. *If A and B are infinite BAs and $A \leq_{\mathrm{s}} B$, then* $\mathrm{Ind}(A) = \mathrm{Ind}(B)$.

Proof. By Proposition 2.29 there are ideals I_0 and I_1 such that $B \cong (A/I_0) \times (A/I_1)$. By Corollary 10.3 (easily proved directly), $\mathrm{Ind}(C \times D) = \max(\mathrm{Ind}(C), \mathrm{Ind}(D))$ for any infinite BAs C, D. Clearly $\mathrm{Ind}(C/J) \leq \mathrm{Ind}(C)$ for any infinite BA C and any ideal J of C. So our proposition follows. □

Thus also if $A \leq_{\mathrm{m}} B$ then $\mathrm{Ind}(A) = \mathrm{Ind}(B)$.

Problem 94. *Do there exist infinite BAs A, B such that $A \leq_{\mathrm{mg}} B$ and* $\mathrm{Ind}(A) < \mathrm{Ind}(B)$*?*

If B is a homomorphic image of A, then $\mathrm{Ind}(B) \leq \mathrm{Ind}(A)$; and clearly the difference here can be arbitrary.

To treat the attainment problem, it is again convenient to first talk about independence in products.

Theorem 10.2. *If neither A nor B has an independent set of power $\kappa \geq \omega$ then $A \times B$ also does not.*

Proof. Let $\langle (a_\alpha, b_\alpha) : \alpha < \kappa \rangle$ be a system of elements of $A \times B$; we want to show that this system is dependent. Choose a finite subset Γ of κ and $\varepsilon \in {}^{\Gamma}2$ so that $\prod_{\alpha\in\Gamma} a_\alpha^{\varepsilon(\alpha)} = 0$, and then choose a finite subset Δ of $\kappa\backslash\Gamma$ and $\delta \in {}^{\Delta}2$ so that $\prod_{\alpha\in\Delta} b_\alpha^{\delta(\alpha)} = 0$. Let $\Theta = \Gamma \cup \Delta$ and $\theta = \delta \cup \varepsilon$. Then $\prod_{\alpha\in\Theta}(a_\alpha, b_\alpha)^{\theta(\alpha)} = 0$, as desired. □

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Corollary 10.3. $\mathrm{Ind}(A \times B) = \max(Ind(A), \mathrm{Ind}(B))$ *for infinite BAs* A, B. □

Corollary 10.4. *If* $\langle A_i : i \in I\rangle$ *is a system of BAs,* κ *is an infinite cardinal, and for every* $i \in I$*, the set* A_i *does not have an independent subset of power* κ*, then* $\prod^{\mathrm{w}}_{i\in I} A_i$ *also has no such subset.*

Proof. Suppose that X is an independent subset of $\prod^{\mathrm{w}}_{i\in I} A_i$ of power κ. Fix $x \in X$. We may assume x is of type 1, with 1-support F. Then

$$\left\langle y \restriction \prod_{i\in F} A_i : y \in X\backslash\{x\}\right\rangle$$

gives κ independent elements of $\prod_{i\in F} A_i$, contradicting Theorem 10.2. □

Corollary 10.5. $\mathrm{Ind}(\prod^{\mathrm{w}}_{i\in I} A_i) = \sup_{i\in I}\mathrm{Ind}(A_i)$. □

Corollary 10.5 enables us to take care of the attainment problem for independence. For each limit cardinal κ there is a BA A with independence κ not attained. For $\kappa = \omega$ we simply take for A any infinite superatomic BA. Now assume that κ is an uncountable limit cardinal. Let I be the set of all infinite cardinals $< \kappa$, and for each $\lambda \in I$ let B_λ be the free BA with λ free generators. Then $A \stackrel{\mathrm{def}}{=} \prod^{\mathrm{w}}_{\lambda\in I} B_\lambda$ is as desired, by Corollary 10.5.

It is perhaps surprising that the analog of Corollary 10.5 for arbitrary products is false. This follows from a theorem of L. Heindorf (unpublished); it was known earlier – see Cramer [74], but the construction there is rather ad hoc. Heindorf proves that $|A| \le \mathrm{Ind}(\prod_{n\in\omega\backslash 1} A^{*n})$ for any infinite BA A, where A^{*n} is the free product of A with itself n times.

Theorem 10.6. *If* A *is an infinite BA, then* $|A| \le \mathrm{Ind}\left(\prod_{n\in\omega\backslash 1} A^{*n}\right)$.

Proof. Let F be the set of all functions f such that f maps ${}^m 2$ into 2 for some $m \in \omega\backslash 1$; m is denoted by $\rho(f)$. Let B be a free BA with free generators x_a for $a \in A$. It suffices to isomorphically embed B into $\prod_{f\in F} A^{*\rho(f)}$. For each $f \in F$ and each $i < \rho(f)$ let g^f_i be the natural embedding of A into the ith free factor of $A^{*\rho(f)}$. We define $G : B \to \prod_{f\in F} A^{*\rho(f)}$ by setting, for each $a \in A$ and $f \in F$

$$(G(x_a))_f = \sum_{\substack{\varepsilon\in{}^{\rho(f)}2,\\ f(\varepsilon)=1}} \left(\prod_{j<\rho(f)} (g^f_j(a))^{\varepsilon(j)}\right),$$

extending G to a homomorphism. We want to show that G is one-one. To this end, let $a_0, \ldots, a_{n-1}$ be distinct elements of A and suppose that $\varepsilon \in {}^n 2$; we want to show that

$$y \stackrel{\mathrm{def}}{=} (G(x_{a_0}))^{\varepsilon(0)} \cdot \ldots \cdot (G(x_{a_{n-1}}))^{\varepsilon(n-1)} \ne 0.$$

Let $\Gamma = \{(i,j) : i < j < n\}$, and choose m and h so that h is a one-one function from m onto Γ. For each $k < m$ write $h(k) = (i,j)$, and let F_k be any ultrafilter on A such that $a_i \triangle a_j \in F_k$. For each $l < n$ we define $\delta_l \in {}^m2$ by setting, for any $k < m$,

$$\delta_l(k) = \begin{cases} 1 & \text{if } \ a_l \in F_k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $i < j < n$ then $\delta_i \neq \delta_j$, since if $k = h^{-1}(i,j)$ we have $\delta_i(k) \neq \delta_j(k)$. Hence there is an $f : {}^m2 \to 2$ such that $f(\delta_i) = \varepsilon(i)$ for all $i < n$. Now we claim that $y_f \neq 0$, as desired. If $l < n$, then for $\varepsilon(l) = 1$ we have $f(\delta_l) = 1$, and so $\prod_{k<m}(g_k^f(a_l))^{\delta_l(k)} \leq (G(x_{a_l}))_f$; and for $\varepsilon(l) = 0$ we have $f(\varepsilon_l) = 0$ and so $\prod_{k<m}(g_k^f(a_l))^{\delta_l(k)} \leq ((G(x_{a_l}))_f)^0$; so in either case we have $\prod_{k<m}(g_k^f(a_l))^{\delta_l(k)} \leq (G(x_{a_l}))_f)^{\varepsilon(l)}$. It follows that $\prod_{l<n}\prod_{k<m}(g_k^f(a_l))^{\delta_l(k)} \leq y_f$. Suppose that $l < n$ and $k < m$. Then $a_i^{\delta_l(k)} \in F_k$, so $\prod_{l<n} a_l^{\delta_l(k)} \in F_k$, so $\prod_{l<n} a_l^{\delta_l(k)} \neq 0$. Hence

$$\prod_{l<n}\prod_{k<m}(g_k^f(a_l))^{\delta_l(k)} = \prod_{k<m} g_k^f\left(\prod_{l<n} a_l^{\delta_l(k)}\right) \neq 0,$$

as desired. □

For the next remark we need the following result from Day [67].

Theorem 10.7. *If A and B are superatomic BAs, $A, B \leq C$, and $C = \langle A \cup B\rangle$, then C is superatomic.*

Proof. Let f be a homomorphism from C onto a BA D. Assume that D is nontrivial. It suffices to show that D has an atom. Note that the elements of C are finite joins of elements of the form $a \cdot b$ with $a \in A$ and $b \in B$. Hence there exist elements $u \in A$ and $v \in B$ such that $f(u \cdot v) \neq 0$. Hence $f[A]$ is nontrivial and so there is an $a \in A$ such that $f(a)$ is an atom of $f[A]$. Now for each $b \in B$ let $g(b) = f(a) \cdot f(b)$. Then g is a homomorphism, so there is a $b \in B$ such that $g(b)$ is an atom of $g[B]$. We claim that $g(b)$ is an atom of D. To prove this, take $u \in A$ and $v \in B$; we show that $f(u \cdot v) \cdot g(b) = 0$ or $f(u \cdot v) \cdot g(b) = g(b)$.

Case 1. $f(u) \cdot f(a) = 0$. Then $f(u \cdot v) \cdot g(b) = f(u) \cdot f(v) \cdot f(a) \cdot f(b) = 0$.

Case 2. $f(a) \leq f(u)$. Since $g(v) \in f[B]$ we have two cases.
Subcase 2.1. $g(v) \cdot g(b) = 0$. Then $f(u \cdot v) \cdot g(b) = f(u) \cdot f(v) \cdot f(a) \cdot f(b) = f(u) \cdot g(v) \cdot g(b) = 0$.
Subcase 2.2. $g(b) \leq g(v)$. Then $g(b) \leq f(a) \cdot f(v) \leq f(u) \cdot f(v) = f(u \cdot v)$. □

Now, given an infinite cardinal κ, let A be the finite-cofinite algebra on κ. Then each algebra A^{*n} is superatomic, hence has no infinite independent set, but the product $\prod_{n\in\omega\setminus 1} A^{*n}$ has independence at least κ. This shows a total failure of Corollary 10.5 for full direct products. Although this example takes care of the most obvious question about independence in products, there is another related question, namely whether an example of this sort can be done with an interval algebra (they always have independence ω too, just like superatomic algebras, although independence is attained for some interval algebras). The answer is no, and after several partial results by several mathematicians a complete solution was given by Shelah in December 1992; see Shelah [94a]:

If A_i is a non-trivial interval algebra for each $i \in I$, where I is infinite, then $\mathrm{Ind}(\prod_{i\in I} A_i) = 2^{|I|}$.

This answers Problem 23 in Monk [90]. We give this result here. The following elementary lemma will be useful.

Lemma 10.8. *Suppose that $\langle a_i : i < 24\rangle$ is a system of elements of a BA A, $\varepsilon \in {}^3 2$, and $a_{3i}^{\varepsilon(0)} \cdot a_{3i+1}^{\varepsilon(1)} \cdot a_{3i+2}^{\varepsilon(2)} = 0$ for all $i < 8$. Let $\langle \delta_i : i < 8\rangle$ enumerate ${}^3 2$. Then*

$$\prod_{i<8} a_{3i}^{\delta_i(0)} \cdot a_{3i+1}^{\delta_i(1)} \cdot a_{3i+2}^{\delta_i(2)} = 0. \qquad \square$$

Now we make some general definitions concerning interval algebras. Let L be a linear order with first element 0. An element ∞ not in L is assumed to be greater than each element of L. Then the interval algebra of L, denoted by $\mathrm{Intalg}(L)$, is the algebra of subsets of L generated by all the half-open intervals $[a, b)$ with $a, b \in L \cup \{\infty\}$ and $a < b$. Every element a of $\mathrm{Intalg}(L)$ can be written uniquely in the form $a = [s_0^a, s_1^a) \cup \cdots \cup [s_{2n(a)-2}^a, s_{2n(a)-1}^a)$ for some $n(a) \in \omega$, with $s_0^a < s_1^a < \cdots < s_{2n(a)-1}^a \le \infty$. We define

$$\begin{aligned} \mathrm{ends}(a) &= \{s_i^a : i < 2n(a)\}; \\ \mathrm{ends}'(a) &= \{s_i^a : i < 2n(a)\} \cup \{0, \infty\}; \\ n'(a) &= |\mathrm{ends}'(a)|; \end{aligned}$$

and we write $\mathrm{ends}'(a) = \{s_i'^a : i < n'(a)\}$ with $0 = s_0'^a < \cdots < s_{n'(a)-1}'^a = \infty$. Note:

$$\text{(1)} \qquad \begin{aligned} &\forall i < n'(a) - 1[[0 \in \mathrm{ends}(a) \to [s_i^a, s_{i+1}^a) \subseteq a \text{ iff } i \text{ is even}]], \text{ and} \\ &\qquad [0 \notin \mathrm{ends}(a) \to [s_i^a, s_{i+1}^a) \subseteq a \text{ iff } i \text{ is odd}]]. \end{aligned}$$

If $A \subseteq L$ and $a \in \mathrm{Intalg}(L)$, an A-partition of a is a subset C of $A \cup \{0, \infty\}$ such that the following conditions hold:

(A) $\{0, \infty\} \subseteq C$.

(B) $\mathrm{ends}'(a) \cap A \subseteq C$.

(C) For every $i < n'(a) - 1$, if $(s_i'^a, s_{i+1}'^a) \cap A \neq \emptyset$, then $(s_i'^a, s_{i+1}'^a) \cap C \neq \emptyset$.

If L is a linear order with first element 0 and $a < b \le \infty$, then $\mathrm{Intalg}(L) \restriction [a, b)$ is the interval algebra on the linear order $[a, b)$.

Now if S is a set of ordinals and $a = \langle a_\alpha : \alpha \in S\rangle$ is a system of elements of $\mathrm{Intalg}(L)$, we say that

(a) a is *homogeneous* if the following conditions hold:

(i) There is a $k \in \omega$ such that for every $\alpha \in S$, $n'(a_\alpha) = k$.

(ii) For all $\alpha, \beta \in S$, $\text{ends}(a_\alpha) \cap \{0, \infty\} = \text{ends}(a_\beta) \cap \{0, \infty\}$.

(iii) For all $\alpha < \beta$ in S there is an $l_{\alpha\beta} < n'(\alpha)-1$ such that $s'^{a_\alpha}_{l_{\alpha\beta}} < t < s'^{a_\alpha}_{l_{\alpha\beta}+1}$ for all $t \in \text{ends}'(a_\beta)\backslash\{0, \infty\}$.

(b) a is *weakly homogeneous* iff there are $0 = t_0 < t_1 < \cdots < t_k = \infty$ such that for every $l < k$ the sequence $\langle a_\alpha \cap [t_l, t_{l+1}) : \alpha \in S\rangle$ is homogeneous in $\text{Intalg}(L) \restriction [t_l, t_{l+1})$. The sequence $\langle t_0, t_1, \ldots, t_k\rangle$ is called a *partitioning sequence* for a.

A 3-*signature* is a sequence $\langle A, m, s_{01}, s_{02}, s_{12}, \varepsilon_{01}, \varepsilon_{02}, \varepsilon_{12}\rangle$ such that $A \subseteq \{0, \infty\}$, m is an integer greater than 1, $s_{01}, s_{02}, s_{12} < m - 1$, and $\varepsilon_{01}, \varepsilon_{02}, \varepsilon_{12} \in 2$. If $\langle a_0, a_1, a_2\rangle$ is homogeneous with notation as above, then its 3-signature is the following sequence: $\langle a_0 \cap \{0, \infty\}, n'(a_0), l_{01}, l_{02}, l_{12}, \varepsilon_0, \varepsilon_1, \varepsilon_2\rangle$, where $s'^{a_0}_{l_{01}} < t < s'^{a_0}_{l_{01}+1}$ for all $t \in \text{ends}'(a_1)\backslash\{0, \infty\}$, and similarly for l_{02} and l_{12}, and $\varepsilon_{01} = 1$ iff $[s'^{a_0}_{l_{01}}, s'^{a_i}_{l_{01}+1}) \subseteq a_0$, and similarly for ε_{02} and ε_{12}.

Lemma 10.9. *With each* 3*-signature* σ *as above we can associate* $\varepsilon_\sigma \in {}^32$ *such that if* A *is an interval algebra and* $\langle a_i : i < 3\rangle$ *is homogeneous with* 3*-signature* σ*, and with the integer* k *of* (a)(i) *at least* 2*, then* $a_0^{\varepsilon_\sigma(0)} \cdot a_1^{\varepsilon_\sigma(1)} \cdot a_2^{\varepsilon_\sigma(2)} = 0$.

Proof. Note that $k \geq 2$ implies that $\{a_0, a_1, a_2\} \cap \{0, 1\} = \emptyset$. Note:

Case 1. $A = \emptyset$, $\varepsilon_{01} = 0$. Let ε_σ be $\langle 1, 1, 1\rangle$.

Case 2. $A = \emptyset$, $\varepsilon_{01} = 1$. Let ε_σ be $\langle 0, 1, 1\rangle$.

Case 3. $A = \{0, \infty\}$, $\varepsilon_{12} = 0$. Let ε_σ be $\langle 1, 1, 0\rangle$.

Case 4. $A = \{0, \infty\}$, $\varepsilon_{12} = 1$. Let ε_σ be $\langle 1, 0, 0\rangle$.

Now in cases with $A = \{0\}$, note:

(1) $a_1 = [0, s'^{a_1}_1) \cup \cdots s'^{a_1}_{n'(a_1)-2})$;

(2) $-a_2 = [s'^{a_2}_1 \cdots \cup [s'^{a_2}_{n'(a_2)-2}, \infty)$;

Case 5. $A = \{0\}$, $\varepsilon_{01} = 0$, $0 < l_{12}$. Let ε_σ be $\langle 1, 1, 0\rangle$. By (1) and (2), $a_1 \cdot -a_2 \subseteq [s'^{a_1}_{l_{12}}, s^{a_1}_{n'(a_1)-2})$.

Case 6. $A = \{0\}$ and $l_{12} = 0$. Let $\varepsilon_\sigma = \langle 1, 0, 1\rangle$. Clearly $a_2 \subseteq a_1$ in this case.

Case 7. $A = \{0\}$, $\varepsilon_{01} = 1$, $0 < l_{12}$. Let ε_σ be $\langle 0, 1, 0\rangle$. See Case 5.

In cases with $A = \{\infty\}$ the following hold.

(3) $a_1 = [s'^{a_1}_1 \cdots \cup [s'^{a_1}_{n(a_1)-2}, \infty)$.

(4) $-a_2 = [0, s'^{a_2}_1) \cup \cdots s'^{a_2}_{n(a_2)-2})$.

Case 8. $A = \{\infty\}$, $\varepsilon_{01} = 0$, $0 < l_{12}$. Let ε_σ be $\langle 1, 0, 1\rangle$. By (3) and (4) we have $a_1 \cdot -a_2 \subseteq [s'^{a_1}_1 \cdots s'^{a_1}_{n(a_1)-2})$.

Case 9. $A = \{\infty\}$, $\varepsilon_{01} = 1$, $0 < l_{12}$. Let ε_σ be $\langle 0,0,1\rangle$. See Case 8.

Case 10. $A = \{\infty\}$, $l_{12} = 0$. ε_σ be $\langle 1,1,0\rangle$. Clearly $a_1 \cdot -a_2 = 0$.

Case 11. in any other case let $\varepsilon_\sigma = \langle 1,1,1\rangle$.

Clearly if $\langle a_i : i < 3\rangle$ is homogeneous with 3-signature σ, notation as before this lemma, then one of Cases 1–10 holds. □

Theorem 10.10. *If* $\langle A_\zeta : \zeta < \kappa\rangle$ *is a system of infinite interval algebras,* κ *an infinite cardinal, then* $\mathrm{Ind}\left(\prod_{\zeta<\kappa} A_\zeta\right) = 2^\kappa$.

Proof. Note that ${}^\kappa 2$ is a subalgebra of $\prod_{\zeta<\kappa} A_\zeta$, and ${}^\kappa 2$ is isomorphic to $\mathscr{P}(\kappa)$. Hence $\prod_{\zeta<\kappa} A_\zeta$ has an independent subset of size 2^κ by the theorem of Fichtenholz, Kantorovich, Hausdorff. So our task is to show that there does not exist a larger independent subset.

Suppose that $A_\zeta = \mathrm{Intalg}(L_\zeta)$ for each $\zeta < \kappa$, each L_ζ a linear order with first element. Suppose that $a : (2^\kappa)^+ \to \prod_{\zeta<\kappa} A_\zeta$. For brevity let $\lambda = (2^\kappa)^+$. We want to show that a is not independent. For each $\zeta < \kappa$ let

$$L'_\zeta = \{s_i^{(a(\alpha))_\zeta} : \alpha < \lambda, i < 2n(a(\alpha)), s_i^{(a(\alpha))_\zeta} \neq \infty\}.$$

Thus $L'_\zeta \subseteq L_\zeta$ and $|L'_\zeta| \leq \lambda$. Let $A'_\zeta = \mathrm{Intalg}(L'_\zeta)$. If F is a finite subset of λ and $\varepsilon \in {}^F 2$, then

$$\prod_{\alpha\in F} (a(\alpha))^{\varepsilon(\alpha)} \neq 0 \text{ in } \prod_{\zeta<\kappa} A_\zeta \text{ iff } \prod_{\alpha\in F} (a(\alpha))^{\varepsilon(\alpha)} \neq 0 \text{ in } \prod_{\zeta<\kappa} A'_\zeta.$$

Hence it suffices to work with $\langle A'_\zeta : \zeta < \kappa\rangle$. Now for each $\zeta < \kappa$ isomorphically embed L'_ζ into a linear order L''_ζ of size λ, and let $A''_\zeta = \mathrm{Intalg}(L''_\zeta)$. Clearly if F is a finite subset of λ and $\varepsilon \in {}^F 2$, then

$$\prod_{\alpha\in F} (a(\alpha))^{\varepsilon(\alpha)} \neq 0 \text{ in } \prod_{\zeta<\kappa} A'_\zeta \text{ iff } \prod_{\alpha\in F} (a(\alpha))^{\varepsilon(\alpha)} \neq 0 \text{ in } \prod_{\zeta<\kappa} A''_\zeta.$$

Now each L''_ζ is isomorphic to a linear algebra $(L''', <_\zeta)$ whose universe is λ, with least element the ordinal 0. Let $A'''_\zeta = \mathrm{Intalg}(L'''_\zeta)$. Now by a change of notation we may assume that we are given linear orders $L_\zeta = (\lambda, <_\zeta)$ with least element the ordinal 0, $A_\zeta = \mathrm{Intalg}(L_\zeta)$, and $a : \lambda \to \prod_{\zeta<\kappa} A_\zeta$.

The following claim gives the main part of the proof.

Claim 1. There is a stationary subset S of λ such that $\langle (a(\alpha))_\zeta : \alpha \in S\rangle$ is weakly homogeneous for every $\zeta < \kappa$.

Proof of Claim 1. Let $S_0 = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa^+\}$. Thus S_0 is stationary in λ. Now for each $\alpha \in S_0$ define $f_\alpha : \kappa \to 4$ by setting, for any $\zeta \in \kappa$,

$$f_\alpha(\zeta) = \begin{cases} 0 & \text{if } 0, \infty \in \mathrm{ends}((a(\alpha))_\zeta), \\ 1 & \text{if } 0 \in \mathrm{ends}((a(\alpha))_\zeta) \text{ and } \infty \notin \mathrm{ends}((a(\alpha))_\zeta), \\ 2 & \text{if } 0 \notin \mathrm{ends}((a(\alpha))_\zeta) \text{ and } \infty \in \mathrm{ends}((a(\alpha))_\zeta), \\ 3 & \text{if } 0, \infty \notin \mathrm{ends}((a(\alpha))_\zeta). \end{cases}$$

Then there exist a $g \in {}^{\kappa}4$ and a stationary $S_1 \subseteq S_0$ such that $f_\alpha = g$ for all $\alpha \in S_1$. Thus

$$(2) \qquad \forall \alpha, \beta \in S_1 \forall \zeta \in \kappa[\text{ends}((a(\alpha))_\zeta) \cap \{0, \infty\} = \text{ends}((a(\beta))_\zeta) \cap \{0, \infty\}].$$

Now for each $\alpha < \lambda$ let $g_\alpha \in {}^{\kappa}\omega$ be defined by $g_\alpha(\zeta) = n'((a(\alpha))_\zeta)$. Then there exist a stationary $S_2 \subseteq S_1$ and an $h \in {}^{\kappa}\omega$ such that $g_\alpha = h$ for all $\alpha \in S_2$. Thus

$$(3) \qquad \forall \alpha, \beta \in S_2 \forall \zeta \in \kappa[n'((a(\alpha))_\zeta) = n'((a(\beta))_\zeta)]$$

Now for each $\alpha \in S_2$ let

$$C_\alpha = \{0, \infty\} \cup \left(\alpha \cap \bigcup_{\zeta \in \kappa} \text{ends}'((a(\alpha))_\zeta)\right) \cup \{\min((s_i'^{(a(\alpha))_\zeta}, s_{i+1}'^{(a(\alpha))_\zeta}) \cap \alpha) :$$
$$\zeta < \kappa, i < n'((a(\alpha))_\zeta) - 1, (s_i'^{(a(\alpha))_\zeta}, s_{i+1}'^{(a(\alpha))_\zeta}) \cap \alpha \neq \emptyset\}.$$

Clearly $|C_\alpha| \leq \kappa$ and C_α is an α-partition of $(a(\alpha))_\zeta$ for each $\zeta < \kappa$. Since $\text{cf}(\alpha) = \kappa^+$ for each $\alpha \in S_2$, it follows that $\sup(C_\alpha \backslash \{\infty\}) < \alpha$. Hence by Fodor's theorem there exist a stationary $S_3 \subseteq S_2$ and a $\beta < \lambda$ such that $\sup(C_\alpha \backslash \{\infty\}) = \beta$ for all $\alpha \in S_3$.

Now $|[\beta]^{\leq\kappa}| \leq (2^\kappa)^\kappa = 2^\kappa < \lambda$, so there exist a stationary $S_4 \subseteq S_3$ and a $D \in [\beta]^{\leq\kappa}$ such that $C_\alpha = D \cup \{\infty\}$ for all $\alpha \in S_4$.

Note that for any $\alpha \in S_4$ and $\zeta < \kappa$ we have

$$\alpha \cap \text{ends}'((a(\alpha))_\zeta) \subseteq D \cap \text{ends}'((a(\alpha))_\zeta) = C_\alpha \cap \text{ends}'((a(\alpha))_\zeta) \subseteq \alpha \cap \text{ends}'((a(\alpha))_\zeta);$$

so $\alpha \cap \text{ends}'((a(\alpha))_\zeta) = D \cap \text{ends}'((a(\alpha))_\zeta)$.

Now for any $\alpha \in S_4$ and $\zeta < \kappa$, let $T_{\alpha\zeta}$ be a finite subset of D such that if $i < n'^{(a(\alpha))_\zeta} - 1$ and $(s_i'^{(a(\alpha))_\zeta}, s_{i+1}'^{(a(\alpha))_\zeta}) \cap D \neq \emptyset$, then $(s_i'^{(a(\alpha))_\zeta}, s_{i+1}'^{(a(\alpha))_\zeta}) \cap T_{\alpha\zeta} \neq \emptyset$. Define

$$F_\alpha(\zeta) = (D \cap \text{ends}'((a(\alpha))_\zeta), \langle (s_i'^{(a(\alpha))_\zeta}, s_{i+1}'^{(a(\alpha))_\zeta}) \cap T_{\alpha\zeta} : i < n'^{(a(\alpha))_\zeta} - 1 \rangle).$$

Let $S_5 \subseteq S_4$ be stationary such that $F_\alpha = F_\beta$ for all $\alpha, \beta \in S_5$. For any $\zeta < \kappa$, take any $\alpha \in S_5$ and let

$$T_\zeta = (D \cap \text{ends}'((a(\alpha))_\zeta)) \cup \bigcup \{(s_i'^{(a(\alpha))_\zeta}, s_{i+1}'^{(a(\alpha))_\zeta}) \cap T_{\alpha\zeta} : i < n'^{(a(\alpha))_\zeta}\}.$$

Then for any $\alpha, \beta \in S_5$ and any $\zeta < \kappa$ we have

(4) (a) T_ζ is an α-partition of $(a(\alpha))_\zeta$.

(b) For every $l < n'^{((a(\alpha))_\zeta}$, if $s_l'^{((a(\alpha))_\zeta} \in T_\zeta$, then $s_l'^{((a(\alpha))_\zeta} = s_l'^{((a(\beta))_\zeta}$.

(c) For every $l < n'^{((a(\alpha))_\zeta} - 1$,

$$T_\zeta \cap (s_i'^{((a(\alpha))_\zeta}, s_{i+1}'^{((a(\alpha))_\zeta}) = T_\zeta \cap (s_i'^{((a(\beta))_\zeta}, s_{i+1}'^{((a(\alpha))_\zeta}).$$

For each $\zeta < \kappa$ let $T_\zeta = \{t_{\zeta 0}, \ldots, t_{\zeta m_\zeta}\}$ with $0 = t_{\zeta 0} < \cdots < t_{\zeta m_\zeta} = \infty$. Now let $F = \{\gamma < \lambda :$ for all $\alpha < \gamma$ and all $\zeta < \kappa$, $\text{ends}'((a(\alpha))_\zeta)\backslash\{\infty\} \subseteq \gamma\}$. Clearly F is club in λ. Let $S_6 = S_5 \cap F$. Thus S_6 is stationary in λ. The following claim will finish the proof of Claim 1.

Claim 2. For each $\zeta < \kappa$, the sequence $\langle (a(\alpha))_\zeta : \alpha \in S_6\rangle$ is weakly homogeneous, with partitioning sequence $\langle t_{\zeta 0}, \ldots, t_{\zeta m_\zeta}\rangle$.

To prove this claim, fix $\zeta < \kappa$ and $l < m_\zeta$. We want to show that $\langle (a(\alpha))_\zeta \cap [t_{\zeta l}, t_{\zeta(l+1)}) : \alpha \in S_6\rangle$ is homogeneous in $\text{Intalg}(L_\zeta) \restriction [t_{\zeta l}, t_{\zeta(l+1)})$. For brevity let $a(\alpha))_\zeta \cap [t_{\zeta l}, t_{\zeta(l+1)}) = b(\alpha)_\zeta$ for each $\alpha \in S_6$.

Case 1. $t_{\zeta(l+1)} \neq \infty$. For any $\alpha \in S_6$ choose $m(\alpha), p(\alpha) < n'^{(a(\alpha))_\zeta} - 1$ such that $t_{\zeta l} \in \left[s'^{(a(\alpha))_\zeta}_{m(\alpha)}, s'^{(a(\alpha))_\zeta}_{m(\alpha)+1}\right)$ and $t_{\zeta(l+1)} \in \left[s'^{(a(\alpha))_\zeta}_{p(\alpha)}, s'^{(a(\alpha))_\zeta}_{p(\alpha)+1}\right)$.

Subcase 1.1. $m(\alpha) = p(\alpha)$ for some $\alpha \in S_6$. Then $(b(\alpha))_\zeta$ is either $[t_{\zeta l}, t_{\zeta(l+1)})$ or $\emptyset$. In either case, $n'^{(b(\alpha))_\zeta} = 2$. Moreover, by (4)(b) and (4)(c), $m(\beta) = p(\beta)$ for all $\beta \in S_6$, so $n'^{(b(\beta))_\zeta} = 2$ for all $\beta \in S_6$. Thus (a)(i) holds.

Now if $0 \in \text{ends}((a(\alpha))_\zeta)$ and $m(\alpha)$ is even, then $(b(\alpha))_\zeta$ is $[t_{\zeta l}, t_{\zeta(l+1)})$; it follows that $\text{ends}((b(\alpha))_\zeta) = \{0, \infty\}$ in the sense of $\text{Intalg}(A_\zeta) \restriction [t_{\zeta l}, t_{\zeta(l+1)})$. By (4)(c) the same is true for $(b(\beta))_\zeta$ for any $\beta \in S_6$. If $0 \in \text{ends}((a(\alpha))_\zeta)$ and $m(\alpha)$ is odd, then $(b(\alpha))_\zeta = \emptyset$, and this is also true for any $(b(\beta))_\zeta$ for $\beta \in S_6$. Similar arguments work for $0 \notin \text{ends}((a(\alpha))_\zeta)$. This shows that (a)(ii) holds.

(a)(iii) holds vacuously, since $\text{ends}'((b(\beta))_\zeta) = \{0, \infty\}$ in the sense of

$$\text{Intalg}(A_\zeta) \restriction [t_{\zeta l}, t_{\zeta(l+1)}).$$

Subcase 1.2. $p(\alpha) = m(\alpha) + 1$. If $0 \in \text{ends}((a(\alpha))_\zeta)$ and $m(\alpha)$ is even, then $(b(\alpha))_\zeta$ is $[t_{\zeta l}, s'^{(a(\alpha))_\zeta}_{m(\alpha)+1})$. Hence $n'^{(b(\alpha))_\zeta} = 3$. Similarly, $n'^{(b(\beta))_\zeta} = 3$ for any $\beta \in S_6$. So (a)(i) holds. Clearly $\text{ends}((b(\beta))_\zeta) \cap \{0, \infty\} = \{0\}$ for any $\beta \in S_6$. Thus (a)(ii) holds. Now assume that $\alpha < \beta$ in S_6. Then $s'^{(a(\alpha))_\zeta}_{m(\alpha)+1} \neq s'^{(a(\beta))_\zeta}_{m(\alpha)+1}$, as otherwise $s'^{(a(\alpha))_\zeta}_{m(\alpha)+1} = s'^{(a(\beta))_\zeta}_{m(\alpha)+1} \in \beta \cap \text{ends}((a(\beta))_\zeta) \subseteq T_\zeta$. Hence (a)(iii) holds.

On the other hand, if $0 \in \text{ends}((a(\alpha))_\zeta)$ and $m(\alpha)$ is odd, then $(b(\alpha))_\zeta$ is the interval $[s'^{(a(\alpha))_\zeta}_{m(\alpha)+1}, t_{\zeta(l+1)})$, and a similar argument works.

The case $0 \notin \text{ends}((a(\alpha))_\zeta)$ is also similar.

Subcase 1.3. $p(\alpha) > m(\alpha) + 1$. Now the n' sequence for $(b(a))_\zeta$ as a member of $\text{Intalg}(A_\zeta) \restriction [t_{\zeta l}, t_{\zeta(l+1)})$ is

$$\langle t_{\zeta l}, s'^{(b(\alpha))_\zeta}_{m(\alpha)+1}, \ldots, s'^{(b(\alpha))_\zeta}_{p(\alpha)}, t_{\zeta(l+1)}\rangle,$$

where there are at least two s terms. Thus $n'((b(\alpha))_\zeta) = p(\alpha) - m(\alpha) + 2$. By (4)(b) and (4)(c), this is true for any $\beta \in S_6$. So (a)(i) holds. Clearly $\text{ends}((b(\alpha))_\zeta) \cap \{0, \infty\} = \{0, \infty\}$ for any $\alpha \in S_6$, so (a)(ii) holds. To prove (a)(iii) it suffices to note that for $\alpha < \beta$ there do not exist i, j such that $s_i'^{(b(\beta))_\zeta} < s_j'^{(b(\alpha))_\zeta} < s_{i+1}'^{(b(\beta))_\zeta}$. This is true since $s_j'^{(b(\alpha))_\zeta} < \beta$, using (4)(a). Thus (a)(iii) holds.

Case 2. $t_{\zeta(l+1)} = \infty$. This is similar to Case 1.

This finishes the proof of Claims 1 and 2.

Now let $S_6 = \bigcup_{\beta<\lambda} T_\beta$, where $|T_\beta| = 3$ for all $\beta < \lambda$ and $T_\beta < T_\gamma$ for $\beta < \gamma$. For each $\zeta < \kappa$ choose m_ζ and $0 = t_{\zeta 0} < \cdots < t_{\alpha m_\zeta} = \infty$ so that for all $n < m_\zeta$ the sequence $\langle (a(\alpha))_\zeta \cap [t_{\zeta n}, t_{\zeta(n+1)}) : \alpha \in S_6 \rangle$ is homogeneous. For each $\beta < \lambda$ let σ_β be the function with domain κ such that for each $\zeta < \kappa$,

$$\begin{aligned}\sigma_\beta(\zeta) = \langle (n, \tau) : n < m_\zeta \text{ and } \tau \text{ is the 3-signature of}\\ \langle (a(\alpha_i))_\zeta \cap [t_{\zeta n}, t_{\zeta(n+1)}) : i < 3 \rangle \rangle,\end{aligned}$$

where $T_\beta = \{\alpha_0, \alpha_1, \alpha_2\}$ with $\alpha < 0 < \alpha_1 < \alpha_2$. Then there is an infinite $U \subseteq \lambda$ such that $\sigma_\beta = \sigma_\gamma$ for all $\beta, \gamma \in U$. Let $\beta_0, \ldots, \beta_7$ be members of U such that $T_{\beta_0} < \cdots < T_{\beta_7}$, and let $\langle \alpha_0, \ldots, \alpha_{23} \rangle$ be the strictly increasing sequence of ordinals such that $T_{\beta_i} = \{\alpha_{3i}, \alpha_{3i+1}, a_{3i+2}\}$ for each $i < 8$.

Claim 3. $\prod_{i<8,k<3} [a(\alpha_{3i+k})]^{\delta_i(k)} = 0$, where $\langle \delta_i : i < 8 \rangle$ enumerates ${}^3 2$.

This claim finishes the proof. To prove the claim, suppose that $\zeta < \kappa$ and $n < m_\zeta$; it suffices to show that

$$(*) \qquad \prod_{i<8,k<3} [(a(\alpha_{3i+k}))_\zeta \cap [t_{\zeta n}, t_{\zeta(n+1)})]^{\delta_i(k)} = 0$$

Now for $i, j < 8$, the 3-signature of

$$\langle (a(\alpha_{3i+k}))_\zeta \cap [t_{\zeta n}, t_{\zeta(n+1)}) : k < 3 \rangle$$

is the same as the 3-signature of

$$\langle (a(\alpha_{3j+k}))_\zeta \cap [t_{\zeta n}, t_{\zeta(n+1)}) : k < 3 \rangle.$$

Thus by Lemma 10.9 there is an $\varepsilon \in {}^3 2$ such that

$$\prod_{k<3} [(a(\alpha_{3i+k}))_\zeta \cap [t_{\zeta n}, t_{\zeta(n+1)})]^{\varepsilon(k)} = 0$$

for all $i < 8$. Then by Lemma 10.8 we obtain $(*)$. □

Now we discuss independence in free products.

Theorem 10.11. $\mathrm{Ind}(A \oplus B) = \max(\mathrm{Ind}(A), \mathrm{Ind}(B))$ *for any BAs* A, B, *at least one of which is infinite.*

Proof. If both A and B are superatomic, then so is $A \oplus B$ by Theorem 10.8 or the Handbook 10.19. If both are not superatomic, then $\mathrm{Ind}(A \oplus B) = \max(\mathrm{Ind}(A), \mathrm{Ind}(B))$ by the Handbook 10.16 and 11.15. Finally, suppose for example that A is superatomic but B is not. By the Handbook 11.17 we have $(A \oplus B)/\overline{p} \cong A$ for every ultrafilter p on B. Then by the Handbook 10.16, 10.17,10.19 we have

$$\begin{aligned}\mathrm{Ind}(A \oplus B) &= \mathrm{Ind}^*(A \oplus B)\\ &= \max(\mathrm{Ind}^*(B), \sup\{\mathrm{Ind}^*(A \oplus B)/\overline{p} : p \in \mathrm{Ult}(B)\})\\ &= \mathrm{Ind}^*(B) = \mathrm{Ind}(B)\end{aligned}$$ □

Theorem 10.12. *If I is infinite and $\langle A_i : i \in I\rangle$ is a system of BAs each with at least 4 elements, then* $\mathrm{Ind}(\oplus_{i\in I} A_i) = \max(|I|, \sup\{\mathrm{Ind}(A_i) : i \in I\})$.

Proof. This is immediate from the Handbook 11.15, 10.16, and 10.19. □

We turn to independence in ultraproducts. As in the case of cellularity, it is easy to see that if F is a countably complete ultrafilter on an index set I and each A_i has countable independence, then so does $\prod_{i\in I} A_i/F$. Namely, suppose that $\langle f_\alpha/F : \alpha < \omega_1\rangle$ is a system of independent elements of $\prod_{i\in I} A_i/F$. Now for every $i \in I$ there exist finite disjoint subsets $M(i), N(i)$ of ω_1 such that

$$\prod_{\alpha\in M(i)} f_\alpha(i) \cdot \prod_{\alpha\in N(i)} -f_\alpha(i) = 0.$$

Hence

$$I = \bigcup\nolimits_{M,N} \{i \in I : M = M(i) \text{ and } N = N(i)\},$$

with M and N ranging over finite subsets of ω_1, so, since F is ω_2-complete, there exist finite disjoint $M, N \subseteq \omega_1$ such that $\{i \in I : M = M(i) \text{ and } N = N(i)\} \in F$. But then $\prod_{\alpha\in M} f_\alpha/F \cdot \prod_{\alpha\in N} -f_\alpha/F = 0$, contradiction.

Further, if F is countably incomplete and each algebra A_i is infinite, then $\prod_{i\in I} A_i/F$ is ω_1-saturated, hence is CSP, from which it follows that $\prod_{i\in I} A_i/F$ has independence $\geq 2^\omega$. (See the Handbook, Theorem 13.20.) Like with cellularity, if I is infinite, F is a $|I|$-regular ultrafilter on I, and A_i is an infinite BA for each $i \in I$, then $\mathrm{Ind}(\prod_{i\in I} A_i/I) \geq 2^{|I|}$. The proof is similar to that for cellularity: let E be a subset of F such that $|E| = |I|$ and each $i \in I$ belongs to only finitely many members of E; let $G(i)$ be the set of all $e \in E$ such that $i \in e$. With each $g \in {}^E2$ we associate $g' \in \prod_{i\in I} A_i$ as follows. Let $\langle x_h : h \in {}^{G(i)}2\rangle$ be a system of independent elements of A_i. Then for any $i \in I$ we set $g'(i) = x_{g\restriction G(i)}$. We claim that $\langle [g'] : g \in {}^E2\rangle$ is an independent system of elements of $\prod_{i\in I} A_i/I$. To see this, let $[(g(0))'], \ldots, [(g(m-1))']$ be distinct elements of $\prod_{i\in I} A_i/I$ and let $\varepsilon \in {}^m2$. Let H be a finite subset of E such that $(g(0)) \restriction H, \ldots, (g(m-1)) \restriction H$ are all distinct. Let $i \in \bigcap H$ be arbitrary. Now $H \subseteq G(i)$, so $(g(0)) \restriction G(i), \ldots, (g(m-1)) \restriction G(i)$ are all distinct. Hence

$$((g(0))'(i))^{\varepsilon(0)} \cdot \ldots \cdot ((g(m-1))'(i))^{\varepsilon(m-1)} = x^{\varepsilon(0)}_{(g(0))\restriction G(i)} \cdot \ldots \cdot x^{\varepsilon(m-1)}_{(g(m-1))\restriction G(i)} \neq 0,$$

as desired.

Independence is an ultra-sup function, so Theorems 3.20–3.22 of Peterson apply, Theorem 3.22 saying that $\mathrm{Ind}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Ind}A_i/F\right|$ for F regular. So by Donder's theorem it is consistent that $\geq$ always holds. The inequality can be strict, as is seen by the following adaptation of Theorem 10.6.

Theorem 10.13. *If A is an infinite BA and X is an infinite disjoint subset of A^+, then for any nonprincipal ultrafilter F on $\omega\backslash 1$ we have $|X| \leq \mathrm{Ind}(\prod_{n\in\omega\backslash 1} A^{*n}/F)$, where A^{*n} is the free nth power of A.*

Proof. For each $a \in X$ we define $f_a \in \prod_{n\in\omega\backslash 1} A^{*n}$ by setting $f_a(n) = g_0(-a) \cdot \ldots \cdot g_{n-1}(-a)$ for each $n \in \omega\backslash 1$, where g_i is the natural isomorphism of A onto the ith free factor of A^{*n} for each $i < n$. Given finite disjoint subsets M and N of X, let $m = |N| + 1$. We claim that $(\prod_{a\in M} f_a(n) \cdot \prod_{a\in N} -f_a(n) \neq 0$ for all $n \geq m$. For, extend N to a subset N' of X, still disjoint from M, of size n. Write $N' = \{a_0, \ldots, a_{n-1}\}$. Then

$$\begin{aligned}\left(\prod_{a\in M} f_a \cdot \prod_{a\in N'} -f_a\right)(n) &= \prod_{a\in M} g_0(-a) \cdot \ldots \cdot \prod_{a\in M} g_{n-1}(-a) \cdot \\ &\quad \prod_{a\in N'} (g_0(a) + \cdots + g_{n-1}(a)) \\ &\geq \prod_{a\in M} g_0(-a) \cdot \ldots \cdot \prod_{a\in M} g_{n-1}(-a) \cdot \\ &\quad g_0(a_0) \cdot \ldots \cdot g_{n-1}(a_{n-1}) \neq 0,\end{aligned}$$

as desired. □

On the other hand, Magidor and Shelah have shown that it is consistent that there is an infinite set I, a system $\langle A_i : i \in I\rangle$ of infinite BAs, and an ultrafilter F on I such that $\mathrm{Ind}\left(\prod_{i\in I} A_i/F\right) < \left|\prod_{i\in I} \mathrm{Ind}A_i/F\right|$. See Rosłanowski, Shelah [98], Proposition 1.14.

Next, note that independence is an ordinary sup-function, and so its behaviour with respect to unions of well-ordered chains is given by Theorem 3.16.

The following theorem is due to Kevin Selker.

Let I, B, $\langle J_i : i \in I\rangle$, and $\langle A_i : i \in I\rangle$ be as in the definition of moderate products. Then

$$\mathrm{Ind}\left(\prod_{i\in I}^{B} A_i\right) = \max\{\mathrm{Ind}(B), \sup_{i\in I} \mathrm{Ind}(A_i)\}.$$

Problem 95. *Describe the independence of the one-point gluing of a system of BAs. In particular, does the analog of Theorem* 10.6 *hold?*

Proposition 10.14. *If A is an infinite BA, then* $\mathrm{Ind}(\mathrm{Dup}(A)) = \mathrm{Ind}(A)$.

Proof. Clearly $\geq$ holds. Now suppose that $X \subseteq \mathrm{Dup}(A)$ and $|X| > \mathrm{Ind}(A)$; we want to show that X is dependent. For each $x \in X$ write $x = (a_x, Y_x)$. Let F, G be finite disjoint subsets of X such that $\prod_{x\in F} a_x \cdot \prod_{y\in G} -a_y = 0$. Then there is a finite $Z \subseteq \mathrm{Ult}(A)$ such that $\prod_{x\in F} x \cdot \prod_{y\in G} -y = (0, Z)$. Let $|Z| < 2^k$. Let $W \subseteq X\backslash(F\cup G)$ with $|W| = k$. Then there is an $\varepsilon \in {}^W 2$ such that $Z \cap \bigcap_{w\in W} Y_w^{\varepsilon(w)} = \emptyset$. Hence $\prod_{w\in W} w^{\varepsilon(w)} \cdot \prod_{x\in F} x \cdot \prod_{y\in G} -y = (0,0)$, as desired. □

Concerning the exponential, the following result is of some interest.

Proposition 10.15. *If A is an infinite BA, then* $\mathrm{c}(A) \leq \mathrm{Ind}(\mathrm{Exp}(A))$.

Proof. Let X be an infinite disjoint subset of A; we produce an independent subset of $\mathrm{Exp}(A)$ of size $|X|$. Namely, for each $x \in X$ let $a_x = \mathscr{V}(\mathcal{S}(x), \mathcal{S}(-x))$. By Lemma 1.22(iv) we have $a_x = -(\mathscr{V}(\mathcal{S}(x)) \cup \mathscr{V}(\mathcal{S}(-x)))$. Now suppose that $x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1}$ are distinct members of X. Then

$$
\begin{aligned}
(1) \qquad & a_{x_0} \cap \cdots \cap a_{x_{m-1}} \cap -a_{y_0} \cap \cdots \cap -a_{y_n-1} \\
&= -\left(\bigcup_{i<m} \mathscr{V}(\mathcal{S}(x_i)) \cup \bigcup_{i<m} \mathscr{V}(\mathcal{S}(-x_i))\right) \\
&\quad \cap \bigcap_{j<n} (\mathscr{V}(\mathcal{S}(y_j)) \cup \mathscr{V}(\mathcal{S}(-y_j)).
\end{aligned}
$$

Here we may assume that $n > 0$. The nonempty closed set $\mathcal{S}(-y_0 \cdot \ldots \cdot -y_{n-1})$ is a member of (1). □

We turn to the functions derived from independence. $\mathrm{Ind}_{\mathrm{H}+}$, $\mathrm{Ind}_{\mathrm{S}+}$, and ${}_{\mathrm{d}}\mathrm{Ind}_{\mathrm{S}+}$ all coincide with Ind itself. $\mathrm{Ind}_{\mathrm{H}-}$ appears to be a new function. Fedorchuk [75] has constructed, using $\diamondsuit$, a BA A such that $\mathrm{Ind}_{\mathrm{H}-}(A) = \mathrm{Ind}(A) = \omega$ and $\mathrm{Card}_{\mathrm{H}-}(A) = \omega_1$.

Problem 96. *Can one construct in ZFC a BA A with the property that* $\mathrm{Ind}_{\mathrm{H}-}A < \mathrm{Card}_{\mathrm{H}-}A$?

This is Problem 29 in Monk [96]. Clearly $\mathrm{Ind}_{\mathrm{S}-}(A) = \omega$ for any infinite BA A. We define

$$\mathrm{Ind}_{\mathrm{h}+}A = \sup\{|X| : X \subseteq \mathrm{Clop}Y,\ X \text{ is independent},\ Y \subseteq \mathrm{Ult}A\}.$$

Then it is possible to have A superatomic, hence with $\mathrm{Ind}A = \omega$, while $\mathrm{Ind}_{\mathrm{h}+}A > |\mathrm{UltA}|$; see the argument for Card. In fact, maybe it is always true that $\mathrm{Ind}_{\mathrm{h}+}A = \mathrm{Card}_{\mathrm{h}+}A$:

Problem 97. *Is* $\mathrm{Ind}_{\mathrm{h}+}A = \mathrm{Card}_{\mathrm{h}+}A$ *for every infinite BA A?*

This is Problem 30 in Monk [96]. $\mathrm{Ind}_{\mathrm{h}-}$ is defined analogously. Again we do not know anything about this cardinal function; for example:

Problem 98. *Is* $\mathrm{Ind}_{\mathrm{h}-}A = \mathrm{Ind}_{\mathrm{H}-}A$ *for every infinite BA A?*

This is Problem 26 in Monk [90]. The function ${}_{\mathrm{d}}\mathrm{Ind}_{\mathrm{S}-}$ appears to be interesting; if A is $\mathscr{P}(X)$ for some infinite X, then we have ${}_{\mathrm{d}}\mathrm{Ind}_{\mathrm{S}-}A = \omega$ since the BA of finite and cofinite subsets of X is dense in A. On the other hand, if A is an infinite free BA of regular cardinality, then ${}_{\mathrm{d}}\mathrm{Ind}_{\mathrm{S}-}A = |A|$; see the Handbook, Theorem 9.16.

We now go into the notions $\mathrm{Ind}_{\mathrm{mm}}$ and $\mathrm{Ind}_{\mathrm{spect}}$. Because of usage in the special case $\mathscr{P}(\omega)/\mathrm{fin}$, we use the notation $\mathfrak{i}$ and $\mathfrak{i}_{\mathrm{sp}}$ instead:

$$\begin{aligned}\mathfrak{i}_{\mathrm{sp}}(A) &= \{|X| : X \text{ is an infinite maximal independent subset of } A\};\\ \mathfrak{i}(A) &= \min(\mathfrak{i}_{\mathrm{sp}}(A)).\end{aligned}$$

The following theorem is from Monk [04].

Theorem 10.16. *For any set X of infinite cardinals there is a BA A such that $\mathfrak{i}_{\mathrm{sp}}(A) = X$.*

Proof. For X empty we just take any infinite superatomic BA. So suppose that X is nonempty. Let κ be the least member of X. The desired algebra is then

$$A \stackrel{\mathrm{def}}{=} \left(\prod_{\lambda\in X}^{\mathrm{w}} \mathrm{Fr}(\lambda)\right) \oplus \mathrm{Fr}(\kappa).$$

To prove this, first note that each element of A can be written in the form

$$\begin{aligned}(1)\qquad z = \sum_{i<m_z} (u_i^z \cdot v_i^z) &\text{ with each } u_i^z \in \prod_{\mu\in X}^{\mathrm{w}} \mathrm{Fr}(\mu),\\ &\text{each } v_i^z \in \mathrm{Fr}(\kappa), \text{ and } v_i^z \cdot v_j^z = 0 \text{ for } i \neq j,\\ &u_i^z \neq 0 \neq v_i^z \text{ for all } i.\end{aligned}$$

Now take any $\lambda \in X$; we show that $\lambda \in \mathfrak{i}_{\mathrm{sp}}(A)$. Let $\langle x_\alpha : \alpha < \lambda\rangle$ be a one-one enumeration of free generators of $\mathrm{Fr}(\lambda)$. For each $\alpha < \lambda$ define $y_\alpha \in \prod_{\mu\in X}^{\mathrm{w}} \mathrm{Fr}(\mu)$ by setting, for each $\mu \in X$,

$$y_\alpha(\mu) = \begin{cases} x_\alpha & \text{if } \mu = \lambda,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $\{y_\alpha : \alpha < \lambda\}$ is an independent system of elements of A; extend it to a maximal independent family Y. To show that $\lambda \in \mathfrak{i}_{\mathrm{sp}}(A)$ it suffices to show:

$$(2)\qquad |Y| = \lambda.$$

Suppose not; so $|Y| > \lambda$. Since $|Y| > \lambda$, we can choose $Y' \in [Y]^{\lambda^+}$ such that there is a positive integer n such that $m_z = n$ for all $z \in Y'$ and the two sequences

$$\langle u_i^z(\lambda) : i < n\rangle \quad\text{and}\quad \langle v_i^z : i < n\rangle$$

do not depend on the particular $z \in Y'$. Take any distinct $w, z \in Y'$, and choose $\alpha < \lambda$ such that $y_\alpha \neq w, z$. Then $w \cdot -z \cdot y_\alpha = 0$, contradiction. in fact,

$$\begin{aligned} w \cdot -z \cdot y_\alpha &= \left(\sum_{i<n}(u_i^w \cdot v_i^w)\right) \cdot \prod_{i<n}((-u^z)_i + -v_i^z) \cdot y_\alpha \\ &= \left(\sum_{\substack{i<n,\\ J\subseteq n}} \left(u_i^w \cdot v_i^w \cdot \prod_{j\in J} -u_j^z \cdot \prod_{j\in n\setminus J} -v_j^z\right)\right) \cdot y_\alpha. \end{aligned}$$

Now take any $i < n$ and $J \subseteq n$. If $i \in J$, then $u_i^w \cdot \prod_{j\in J} -u_i^z \cdot y_\alpha = 0$, as desired. If $i \notin J$, then $v_i^w \cdot \prod_{j\in n\setminus J} -v_i^z = 0$, as desired.

So (2) holds.

Now suppose that $\mu \notin X$ but Y is a maximal independent subset of A of size μ; we want to get a contradiction. Let $\langle x_\alpha : \alpha < \kappa\rangle$ be a one-one system of free generators of $\mathrm{Fr}(\kappa)$.

$$\kappa < |Y|. \tag{3}$$

For, suppose that $|Y| < \kappa$. Choose $\alpha < \kappa$ with x_α not in the support of any v_i^z for $z \in Y$. Clearly each element of $\langle Y\rangle$ can be written in the form $\sum_{i<n}(s_i \cdot t_i)$ with $s_i \in \prod^{\mathrm{w}}_{\lambda\in X} \mathrm{Fr}(\lambda)$ and t_i in the subalgebra of $\mathrm{Fr}(\kappa)$ generated by $\{v_i^z : z \in Y, i < n\}$. Hence $Y \cup \{x_\alpha\}$ is independent, contradiction. So (3) holds.

Note by (3) that Y is uncountable.

(4) Suppose that $\langle y(i) : i < s\rangle$ and $\langle z(j) : j < t\rangle$ are systems of distinct elements of Y with $s, t > 0$, $m_{y(i)} = m_{z(j)} = n$ and $v_k^{y(i)} = v_k^{z(j)} = w_k$ for all $i < s$, $j < t$, and $k < n$. Then

$$\prod_{i<s} y(i) \cdot \prod_{j<t} -z(j) = \sum_{k<n}\left(\prod_{i<s} u_k^{y(i)} \cdot \prod_{j<t} -u_k^{z(j)} \cdot w_k\right).$$

To prove (4), first note that

$$\prod_{i<s} y(i) = \prod_{i<s}\sum_{k<n}(u_k^{y(i)} \cdot w_k) = \sum_{k<n}\left(\prod_{i<s} u_k^{y(i)} \cdot w_k\right)$$

since the w_k's are pairwise disjoint. Hence

$$\begin{aligned} \prod_{i<s} y(i) \cdot \prod_{j<t} -z(j) &= \left(\sum_{k<n}\left(\prod_{i<s} u_k^{y(i)} \cdot w_k\right)\right) \cdot \prod_{j<t}\prod_{k<n}(-u_k^{z(j)} + -w_k) \\ &= \sum_{k<n}\left(\prod_{i<s} u_k^{y(i)} \cdot w_k \cdot \prod_{j<t}\prod_{k<n}(-u_k^{z(j)} + -w_k)\right) = \sum_{k<n}\left(\prod_{a<s} u_k^{y(a)} \cdot \prod_{b<t} -u_k^{z(b)} \cdot w_k\right), \end{aligned}$$

proving (4).

Now for each $y \in Y$ and $i < m_i$ let

$$\delta_{iy} = \begin{cases} 0 & \text{if } \{\lambda \in X : u_i^y(\lambda) \neq 0\} \text{ is finite,} \\ 1 & \text{if } \{\lambda \in X : u_i^y(\lambda) \neq 1\} \text{ is finite,} \end{cases}$$
$$F_{iy} = \{\lambda \in X : u_i^y(\lambda) \neq \delta_{iy}\}.$$

Now clearly there is an uncountable subset Z of Y such that the following conditions hold:

(5) There is an $n \in \omega$ such that $m_y = n$ for all $y \in Z$.

(6) $\delta_{iy} = \delta_i$ for all $y \in Z$.

(7) $v_i^y = v_i$ for all $y \in Z$.

(8) For each $i < n$, $\langle F_{iy} : y \in Z\rangle$ is a Δ-system, say with kernel G_i.

Now let $H = \bigcup_{i<n} G_i$. Then

(9) If z, s, t, w are distinct members of Z, $i < n$, and $\lambda \in X \backslash H$, then $(u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^w)(\lambda) = 0$.

For, if $\delta_i = 0$, then since $F_{iz} \cap F_{is} = G_i$, we get $\lambda \notin F_{iz}$ or $\lambda \notin F_{is}$, hence $u_i^z(\lambda) = 0$ or $u_i^s(\lambda) = 0$; so $(u_i^z \cdot u_i^s)(\lambda) = 0$, and similarly if $\delta_i = 1$ then $(-u_i^t \cdot -u_i^w)(\lambda) = 0$, so (9) holds.

Note that by (4) we have

(10) If z, s, t, w are distinct members of Z, then

$$z \cdot s \cdot -t \cdot -w = \sum_{i<n} (u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^w \cdot v_i).$$

Now let $L = \{\lambda \in X : \lambda < \mu\}$ and $K = \{\lambda \in X : \mu < \lambda\}$.

(11) Suppose that $a, b \in Y$, $a \neq b$, $m_a = m_b$, $u_i^a(\lambda) = u_i^b(\lambda)$ for all $i < m_a$ and all $\lambda \in H \cap L$, and $v_i^a = v_i^b$ for all $i < m_a$. Then $(u_i^a \cdot -u_i^b)(\lambda) = 0$ for all $i < m_a$ and $\lambda \in H \cap L$, and

$$a \cdot -b = \sum_{i<m_a} (u_i^a \cdot -u_i^b \cdot v_i^a).$$

This follows from (4). Hence by (1) we get

(12) If z, s, t, w are distinct members of Z, $a, b \in Y$, $a \neq b$, $\{a, b\} \cap \{z, s, t, w\} = \emptyset$, $m_a = m_b$, $u_i^a(\lambda) = u_i^b(\lambda)$ for all $i < m_a$ and all $\lambda \in H \cap L$, and $v_i^a = v_i^b$ for all $i < m_a$, then

$$z \cdot s \cdot -t \cdot -w \cdot a \cdot -b = \sum_{\substack{i<n \\ j<m_a}} (u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^w \cdot u_j^a \cdot -u_j^b \cdot v_i \cdot v_j^a).$$

Case 1. $H \cap K = \emptyset$. Let z, s, t, w be distinct members of Z. Let

$$M = \left\{(f, p, g) : p \in \omega, f \in {}^p \prod_{\lambda \in H \cap L} \mathrm{Fr}(\lambda), g \in {}^p\mathrm{Fr}(\kappa)\right\}.$$

Clearly $|M| < \mu$. Now with each $a \in Y$ we can associate the triple

$$(\langle u_i^a \restriction (H \cap L) : i < m_a\rangle, m_a, \langle v_i^a : i < m_a\rangle)$$

which is a member of M. It follows that there are distinct $a, b \in Y\backslash\{z, s, t, w\}$ such that $m_a = m_b$, $u_i^a(\lambda) = u_i^b(\lambda)$ for all $i < m_a$ and all $\lambda \in H \cap L$, and $v_i^a = v_i^b$ for all $i < m_a$. Then we obtain the equation in (12). But by (9) and (11) we have $u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^w \cdot u_j^a \cdot -u_j^b = 0$ for all $i < n$ and $j < m_a$, contradiction.

Case 2. $H \cap K \neq \emptyset$. For every $\lambda \in H \cap K$ let $w(\lambda)$ be a free generator of $\mathrm{Fr}(\lambda)$ not in the support of any element $u_i^y(\lambda)$ for $y \in Y$ and $i < m_y$. Clearly $w \notin Y$, so by the maximality of Y there exist a finite $N \subseteq Y$, an $\varepsilon \in {}^N 2$, and a $\delta \in 2$, such that $w^\delta \cdot \prod_{y \in N} y^{\varepsilon(y)} = 0$. Now take distinct $z, s, t, w \in Z\backslash N$. By the argument in Case 1 there are distinct $a, b \in Y\backslash(N \cup H \cup \{z, s, t, w\})$ such that $m_a = m_b$, $v_i^a = v_i^b$ for all $i < m_a$, and $u_i^a(\lambda) = u_i^b(\lambda)$ for all $i < m_a$ and all $\lambda \in H \cap L$. Now if $\lambda \in X\backslash(H \cap K)$, then either $\lambda \in X\backslash H$ or $\lambda \in H \cap L$. Hence by (9) and (11),

(13) $(u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^p \cdot u_j^a \cdot -u_j^b) \restriction (X\backslash(H \cap K)) = 0$ for all $i < n$ and $j < m_a$.

Let $\prod_{y \in N} y^{\varepsilon(y)} = \sum_{k<p} c_k \cdot d_k$, with $c_k \in \prod^{\mathrm{w}}_{\nu \in X} \mathrm{Fr}(\nu)$ and $d_k \in \mathrm{Fr}(\kappa)$. Then by (12),

$$\begin{aligned}&\prod_{y \in N} y^{\varepsilon(y)} \cdot z \cdot s \cdot -t \cdot -w \cdot a \cdot -b\\ &= \sum\{u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^w \cdot u_j^a \cdot -u_j^b \cdot c_k \cdot v_i \cdot v_j^a \cdot d_k :\\ &\qquad i < n, j < m_a, k_p\}.\end{aligned}$$

Hence by (13) there is a $\lambda \in H \cap K$ such that

$$(u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^w \cdot u_j^a \cdot -u_j^b \cdot c_k)(\lambda) \neq 0.$$

By the choice of w it follows that

$$w(\lambda) \cdot (u_i^z \cdot u_i^s \cdot -u_i^t \cdot -u_i^w \cdot u_j^a \cdot -u_j^b \cdot c_k)(\lambda) \neq 0.$$

This contradicts $w \cdot \prod_{y \in N} y^{\varepsilon(y)} = 0$. $\square$

For our next result concerning $\mathfrak{i}$ we need the following lemma, which is of general interest.

Lemma 10.17. *If A is a countable BA and F is an infinite free BA, then $F \cong A \oplus F$.*

Proof. Let X be a set of free generators of F and write $X = Y_0 \cup Y_1$, where $Y_0 \cap Y_1 = \emptyset$ and $|Y_0| = \omega$. Then $A \oplus \langle Y_0 \rangle$ is denumerable and atomless, and so is isomorphic to $\langle Y_0 \rangle$. Hence $A \oplus F \cong A \oplus \langle Y_0 \rangle \oplus \langle Y_1 \rangle \cong \langle Y_0 \rangle \oplus \langle Y_1 \rangle \cong F$. □

The following result is due to McKenzie, Monk [04].

Theorem 10.18. *If A_0 and A_1 are atomless BAs, then $\mathfrak{i}_{\mathrm{sp}}(A_0 \times A_1) = \mathfrak{i}_{\mathrm{sp}}(A_0) \cup \mathfrak{i}_{\mathrm{sp}}(A_1)$.*

Proof. First suppose that $\kappa \in \mathfrak{i}_{\mathrm{sp}}(A_0)$; say $X \subseteq A_0$ is maximal independent with $|X| = \kappa$. Let $Y = \{(x, 0) : x \in X\}$. Clearly Y is an independent subset of $A_0 \times A_1$. Suppose that $(a, b) \in A_0 \times A_1$ with $a \notin X$ or $b \neq 0$. Choose a finite subset F of X, an $\varepsilon \in {}^F2$, and a $\delta \in 2$ such that $\prod_{x \in F} x^{\varepsilon(x)} \cdot a^\delta = 0$. Take any $y \in X \backslash F$. Then $\prod_{x \in F}(x, 0)^{\varepsilon(x)} \cdot (y, 0) \cdot (a, b)^\delta = (0, 0)$. So Y is maximal independent. This shows that $\mathfrak{i}_{\mathrm{sp}}(A_0) \subseteq \mathfrak{i}_{\mathrm{sp}}(A_0 \times A_1)$. Similarly, $\mathfrak{i}_{\mathrm{sp}}(A_1) \subseteq \mathfrak{i}_{\mathrm{sp}}(A_0 \times A_1)$.

Now suppose that $\kappa \in \mathfrak{i}_{\mathrm{sp}}(A_0 \times A_1) \backslash (\mathfrak{i}_{\mathrm{sp}}(A_0) \cup \mathfrak{i}_{\mathrm{sp}}(A_1)$; we want to get a contradiction. Let X be a maximal independent subset of $A_0 \times A_1$ of size κ.

(1) $\kappa > \omega$.

In fact, suppose that $\kappa = \omega$. Then $\pi_0[X]$ is contained in a denumerable atomless subalgebra B of A_0. Let Y be a set of free generators of B. Since $\mathfrak{i}(A_0) > \omega$, there is a $y_0 \in A_0 \backslash Y$ such that $Y \cup \{y_0\}$ is still independent. It follows that if $x \in X$ and $\pi_0(x) \neq 0$, then $\pi_0(x) \cdot y_0 \neq 0 \neq \pi_0(x) \cdot -y_0$. Similarly we get an element $y_1 \in A_1$ such that if $x \in X$ and $\pi_1(x) \neq 0$, then $\pi_1(x) \cdot y_1 \neq 0 \neq \pi_1(x) \cdot -y_1$. Then $(y_0, y_1) \notin X$ and $X \cup \{(y_0, y_1)\}$ is still independent, contradiction. So (1) holds.

Temporarily fix $j \in 2$. We define

$$D_j = \{w : w \text{ is an } X\text{-monomial, and } \pi_j(v) \neq 0 \text{ for every } X\text{-monomial } v \leq w\}.$$

Thus if $w \in D_j$ and $v \leq w$ is an X-monomial, then $v \in D_j$.

(2) There is an X-monomial w such that $\pi_j(w) = 0$.

In fact, suppose not. Then $\langle \pi_j(a) : a \in X \rangle$ is an independent system in A_j. Since A_j does not have a maximal independent set of size κ, it follows that there is a $c \in A_j \backslash \{\pi_j(a) : a \in X\}$ such that $\langle \pi_j(a) : a \in X \rangle^\frown \langle c \rangle$ is independent. Take any $b \in A_0 \times A_1$ such that $\pi_j(b) = c$. Then $\langle a : a \in X \rangle^\frown \langle b \rangle$ is independent, contradicting the maximality of X. Thus (2) holds.

(3) If w is an X-monomial such that $\pi_j(w) = 0$, then $w \in D_{1-j}$.

For, suppose that v is an X-monomial such that $v \leq w$ and $\pi_{1-j}(v) = 0$. Then $v = 0$, contradiction. So (3) holds.

(4) $D_j \neq \emptyset$

This is true by (2) and (3), since j is arbitrary.

(5) If w is an X-monomial and $w \notin D_j$, then there is an X-monomial $v \leq w$ such that $v \in D_{1-j}$.

For, choose an X monomial $v \leq w$ such that $\pi_j(v) = 0$. Then $v \in D_{1-j}$ by (3).

Now let M_j be a maximal set of pairwise disjoint members of D_j. Thus M_j is countable. Let X_j be a denumerable subset of X such that $M_j \subseteq \langle X_j \rangle$, and let $Y_j = X \backslash X_j$.

(6) There exist an element b_j of $\langle X \rangle$ and a subalgebra C_j of $\langle X \rangle \restriction b_j$ such that C_j is free of size κ and the following conditions hold:

(a) For all $w \in D_j$ there is a nonzero $v \in C_j$ such that $v \leq w$.
(b) For all nonzero $v \in C_j$ there is a $w \in D_j$ such that $w \leq v$.

To prove this, we consider two cases.

Case 1. M_j is infinite. Let $b_j = 1$. Let J_j be the ideal of $\langle X_j \rangle$ generated by M_j, and let B_j be the subalgebra of $\langle X_j \rangle$ generated by J_j. Note that $B_j = J_j \cup \{a \in \langle X_j \rangle : -a \in J_j\}$. Let $C_j = \langle B_j \cup Y_j \rangle$. Since $B_j \leq \langle X_j \rangle$ and $Y_j = X \backslash X_j$, it follows that $b \cdot c \neq 0$ for all $b \in B_j^+$ and $c \in \langle Y_j \rangle^+$. So $C_j = B_j \oplus \langle Y_j \rangle$. By Lemma 10.17, C_j is free. Clearly $|C_j| = \kappa$.

To check (a), suppose that $w \in D_j$. By the maximality of M_j, there is a member r of M_j such that $w \cdot r \neq 0$. As a product of X-monomials, $w \cdot r$ is also an X-monomial. Write $w \cdot r = s_0 \cdot s_1$ with s_0 an X_j-monomial and s_1 a Y_j-monomial. So s_0 is the product of r with a part of w, and hence $s_0 \leq r$. It follows that $s_0 \in J_j \subseteq B_j$. This shows that $w \cdot r = s_0 \cdot s_1 \in C_j$. Thus $v = w \cdot r$ is as desired.

For (b), suppose that $v \in C_j^+$. Then there is an X_j-monomial d and a Y_j-monomial e such that $d \cdot e \leq v$. First suppose that $d \in J_j$. Then there exist $r_0, \ldots, r_{m-1} \in M_j$ such that $d \leq r_0 + \cdots + r_{m-1}$. Wlog $d \cdot r_0 \neq 0$. So $d \cdot r_0 \cdot e$ is an X-monomial and $d \cdot r_0 \cdot e \leq r_0 \in M_j \subseteq D_j$. Hence $d \cdot r_0 \cdot e \in D_j$. Since $d \cdot r_0 \cdot e \leq v$, this is as desired. Second, suppose that $d \notin J_j$. Then $-d \in J_j$, and so we get members $r_0, \ldots, r_{m-1}$ of M_j such that $-d \leq r_0 + \cdots + r_{m-1}$. Since M_j is infinite, there is a member s of M_j different from each r_i, and hence $s \leq d$. Clearly $s \in D_j$, so $s \cdot e \in D_j$, and $s \cdot e \leq v$, as desired.

Case 2. M_j is finite. Let $B_j = \sum M_j$ and $B_j = \langle X_j \rangle \restriction b_j$. Thus B_j is a denumerable atomless BA. Let C_j be the subalgebra of $\langle X \rangle \restriction b_j$ generated by $B_j \cup \{b_j \cdot y : y \in Y_j\}$. Now for each $c \in M_j$ let D_c be the subalgebra of $\langle X \rangle$ generated by $(\langle X_j \rangle \restriction c) \cup \{c \cdot y : y \in Y_j\}$. Then $C \cong \prod_{c \in M_j} D_c$. Clearly for each $c \in M_j$ we have $D_c = (\langle X_j \rangle \restriction c) \oplus \langle \{c \cdot y : y \in Y_j\} \rangle$. Moreover, $\langle X_j \rangle \restriction c$ is denumerable and $\langle \{c \cdot y : y \in Y_j\} \rangle$ is free, so again by Lemma 10.12, D_c is free of size $|X|$. Hence also C is free of size $|X|$ by the Handbook, Proposition 9.14.

To check (a), suppose that $w \in D_j$. By the maximality of M_j, there is a member r of M_j such that $w \cdot r \neq 0$. As a product of X-monomials, $w \cdot r$ is also an X-monomial. Write $w \cdot r = s_0 \cdot s_1$ with s_0 an X_j-monomial and s_1 a Y_j-monomial. So s_0 is the product of r with a part of w, and hence $s_0 \leq r$. It follows that $s_0 \in B_j$. This shows that $w \cdot r = s_0 \cdot s_1 \in C_j$. Thus $v = w \cdot r$ is as desired.

For (b), suppose that $v \in C_j^+$. Then there is an X_j-monomial $d \leq b_j$ and a Y_j-monomial e with $b_j \cdot e \neq 0$ such that $d \cdot e \leq v$. Since $d \leq b_j$, say wlog that $d \cdot r_0 \neq 0$. So $d \cdot r_0 \cdot e$ is an X-monomial and $d \cdot r_0 \cdot e \leq r_0 \in M_j \subseteq D_j$. Hence $d \cdot r_0 \cdot e \in D_j$. Since $d \cdot r_0 \cdot e \leq v$, this is as desired.

This finishes the proof of (6).

(7) π_j is injective on C_j.

For, suppose that $v \in C_j^+$. By (6)(b) choose $w \in D_j^+$ such that $w \leq v$. Then by the definition of D_j we have $\pi_j(w) \neq 0$, and so also $\pi_j(v) \not- 0$, proving (7).

Now since $\kappa \notin \mathfrak{i}_{\rm sp}(A_j)$, it follows from the easy direction in our theorem that $\kappa \notin \mathfrak{i}_{\rm sp}(A_j \restriction \pi_j(b_j))$. Now C_j is free of size κ, so also $\pi_j[C_j]$ is free of size κ. A set of free generators of $\pi_j[C_j]$ is not maximal, and so we can choose $w_j \in (A_j \restriction \pi_j(b_j))\backslash\pi_j[C_j]$ such that w_j is free over $\pi_j[C_j]$.

(8) $w_j \cdot \pi_j(d) \neq 0 \neq -w_j \cdot \pi_j(d)$ for all $d \in D_j$.

For, by (6)(a) choose $v \in C_j^+$ such that $v \leq d$. Then $0 \neq w_j \cdot \pi_j(v) \leq w_j \cdot \pi_j(d)$, so $w_j \cdot \pi_j(d) \neq 0$. Similarly, $-w_j \cdot \pi_j(d) \neq 0$. so (8) holds.

Now unfix j. Let $w = (w_o, w_1 0$. Suppose that v is any X-monomial. If $v \in D_0$, then $w \cdot v \neq 0 \neq -w \cdot v$ by (8). Suppose that $v \notin D_0$. By (5), let $s \leq v$ be an X-monomial such that $s \in D_1$. Then by (8) we get $w \cdot s \neq 0$, so $w \cdot v \neq 0$. Similarly $-w \cdot v \neq 0$.

This contradicts the maximality of X. $\square$

Corollary 10.19. *If A and B are atomless BAs, then* $\mathfrak{i}(A \times B) = \min\{\mathfrak{i}(A), \mathfrak{i}(B)\}$. $\square$

Proposition 10.20. *If I is an infinite set and $\langle A_i : i \in I\rangle$ is a system of atomless BAs, then*

$$\mathfrak{i}_{\rm sp}\left(\prod_{i\in I}^{\rm w} A_i\right) = \{\omega\} \cup \bigcup_{i\in I} \mathfrak{i}_{\rm sp}(A_i).$$

Proof. Let $B = \prod_{i\in I}^{\rm w} A_i$. Clearly $\mathfrak{i}_{\rm sp}(A_i) \subseteq \mathfrak{i}_{\rm sp}(B)$ for each $i \in I$.

Now to show that $\omega \in \mathfrak{i}_{\rm sp}(B)$, for convenience suppose that $I = \kappa$, an infinite cardinal. For each $\alpha < \kappa$ let $\langle x_{\alpha,i} : i \in \omega\rangle$ be a system of independent elements of A_α. We define $y_n \in B$ for each $n \in \omega$ by setting, for each $\alpha < \kappa$

$$y_n(\alpha) = \begin{cases} x_{\alpha,n-\alpha-1} & \text{if } \alpha < n, \\ 1 & \text{if } \alpha = n, \\ 0 & \text{if } n < \alpha. \end{cases}$$

We claim that $\langle y_n : n \in \omega\rangle$ is a maximal independent system of elements of B. To show that it is independent, it suffices to take any positive integer m and any $\varepsilon \in {}^m2$ and show that $z \stackrel{\rm def}{=} \prod_{n<m} y_n^{\varepsilon(n)} \neq 0$. If $\varepsilon(n) = 0$ for all $n < m$, then

$$z(m) = \prod_{n<m} -y_n(m) = 1.$$

So suppose that $\varepsilon(i)=1$ for some $i<m$, and take the least such i. Then

$$\begin{aligned} z(i) &= \prod_{n<m} y_n(i) \\ &= \prod_{n<i} -y_n(i)\cdot y_i(i)\cdot \prod_{i<n<m} y_n^{\varepsilon(n)}(i) \\ &= \prod_{i<n<m} x_{i,n-i-1}^{\varepsilon(n)} \neq 0. \end{aligned}$$

So $\langle y_n : n\in\omega\rangle$ is independent.

Now suppose that $w\in B^+\backslash\{y_n : n\in\omega\}$; we want to show that $\{y_n : n\in\omega\}\cup\{w\}$ is no longer independent. Wlog $\{\alpha<\kappa : w(\alpha)\neq 0\}$ is finite. If $w(\alpha)=0$ for all $\alpha<\omega$, then $y_0\cdot w=0$, as desired. So assume that there is an $\alpha<\omega$ such that $w(\alpha)\neq 0$, and take the greatest such. Let $v=w\cdot\prod_{n\leq\alpha} -y_n\cdot y_{n+1}$. We claim that $v=0$ (as desired). To show this, take any β,κ. If $\beta\leq\alpha$, then $v(\beta)\leq -y_\beta(\beta)=0$. If $\beta=\alpha+1$, then $v(\beta)\leq w(\beta)=0$. Finally, if $\alpha+1<\beta$, then $v(\beta)\leq y_{\alpha+1}(\beta)=0$.

This finishes the proof that $\omega\in \mathfrak{i}_{\rm sp}(B)$, and hence $\supseteq$ in our proposition holds.

For the other inclusion, suppose that

$$\kappa\in \mathfrak{i}_{\rm sp}(B)\backslash\left(\{\omega\}\cup\bigcup_{i\in I}\mathfrak{i}_{\rm sp}(A_i)\right);$$

we want to get a contradiction. Let X be a maximal independent subset of B of size κ. Wlog for all $x\in X$ the set $F_x \stackrel{\text{def}}{=} \{i\in I : x(i)\neq 0\}$ is finite. Let Y be an uncountable subset of X such that $\langle F_x : x\in Y\rangle$ forms a Δ-system, say with kernel G. Obviously $G\neq\emptyset$. Now

(1) $\langle x\restriction G : x\in X\rangle$ is independent in $\prod_{i\in G} A_i$.

In fact, suppose that K is a finite subset of X and $\varepsilon\in {}^K 2$. Choose $x,z\in Y\backslash K$. Then $x\cdot z\cdot\prod_{y\in K} y^{\varepsilon(y)}\neq 0$, so $\prod_{y\in K}(y\restriction G)^{\varepsilon(y)}\neq 0$, as desired for (1).

Now by Theorem 10.18 there is a $w\in\prod_{i\in G} A_i$ such that $\langle x\restriction G : x\in X\rangle^\frown\langle w\rangle$ is independent. Let v be the member of B whose restriction to G is w, with $v(i)=0$ for all $i\notin G$. For any finite subset K of X and any $\varepsilon\in {}^K 2$, choose $x,z\in Y\backslash K$; then for any $\delta\in 2$,

$$x\cdot z\cdot\prod_{y\in K} y^{\varepsilon(y)}\cdot v^\delta\neq 0.$$

But this contradicts the maximality of X. $\square$

A corollary of Proposition 10.20 is that $\mathfrak{i}(\prod_{i\in I}^w A_i)=\omega$. This has been generalized to moderate products by Kevin Selker.

Proposition 10.21. $\mathfrak{r}(A)\leq\mathfrak{i}(A)$ *for every atomless BA A.*

Proof. Let X be maximal independent with $|X| = \mathfrak{i}(A)$. Clearly the set

$$\left\{\prod_{x\in F} x^{\varepsilon(x)} : F \in [X]^{<\omega}, \varepsilon \in {}^F 2\right\}$$

is weakly dense. □

The results above leave the following questions open.

Problem 99. *Determine* $\mathfrak{i}\left(\prod_{i\in I} A_i\right)$ *in terms of* I *and* $\langle \mathfrak{i}(A_i) : i \in I\rangle$*, where* I *is infinite and each* A_i *is atomless.*

Problem 100. *Determine* $\mathfrak{i}\left(\prod_{i\in I}^B A_i\right)$ *in terms of* B *and* $\langle \mathfrak{i}(A_i) : i \in I\rangle$*, where* I *is infinite,* $\mathrm{Finco}(I) \le B \le \mathscr{P}(I)$*, each* A_i *is atomless* (*moderate products*)*.*

Problem 101. *Is* $\pi\chi_{\mathrm{inf}}(A) \le \mathfrak{i}(A)$ *for every atomless BA* A*?*

Concerning the spectrum function $\mathrm{Ind}_{\mathrm{Hs}}$, note that if $\mathrm{Ind}_{\mathrm{H}-}(A) \le \mu \le \mathrm{Ind}(A)$, then A has a homomorphic image B such that $\mu \le \mathrm{Ind}(B) \le \mu^\omega$. Moreover, if A has CSP, then this cannot be improved:

$$\mathrm{Ind}_{\mathrm{Hs}}(A) = \{\lambda : 2^\omega \le \lambda \le \mathrm{Ind}(A),\ \lambda^\omega = \lambda\} \text{ for CSP } A.$$

(These remarks are due to S. Koppelberg.)

The spectrum function $\mathrm{Ind}_{\mathrm{Ss}}$ is trivial: $\mathrm{Ind}_{\mathrm{Ss}}(A) = [\omega, \mathrm{Ind}A]$ for every infinite BA.

A BA A has *free caliber* κ if $\kappa \le |A|$ and $\forall X \in [A]^\kappa \exists Y \in [X]^\kappa$($Y$ is independent). $\mathrm{Freecal}(A)$ is the set of all $\kappa \le |A|$ such that A has free caliber κ.

Proposition 10.22. *For any BA* A*, if* $\mathrm{cf}(\kappa) = \omega$*, then* $\kappa \notin \mathrm{Freecal}(A)$*.*

Proof. If $\kappa = \omega$, take a disjoint $X \in [A]^\omega$; clearly this X shows that $\kappa \notin \mathrm{Freecal}(A)$.

Now suppose that $\kappa > \omega$ and $\mathrm{cf}(\kappa) = \omega$. We can assume that A has an independent subset of size κ. Let $\langle \mu_m : m \in \omega\rangle$ be a strictly increasing sequence of infinite cardinals with supremum κ, and suppose that $\langle x_\alpha : \alpha < \kappa\rangle$ is a system of independent elements of A. Then

$$\left\{x_\alpha \cdot x_{\mu_{m+1}} \cdot \prod_{n\le m} -x_{\mu_n} : \mu_m < \alpha < \mu_{m+1}\right\}$$

is a subset of A of size κ, but it does not have any independent subset of size κ. □

We give just a few results about free caliber; a more extensive treatment is in Monk [83]. The notion of free caliber is motivated by the central example of a free BA itself. By Theorem 9.16 of the Handbook, $\lambda \in \mathrm{Freecal}(\mathrm{Fr}(\kappa))$ for every uncountable regular cardinal $\lambda \le \kappa$. This result is generalized below.

Proposition 10.23. $\kappa \in \text{Freecal}(A \times B)$ iff *one of the following conditions holds:*

(i) $\kappa \in \text{Freecal}(A)$ *and* $|B| < \kappa$.
(ii) $\kappa \in \text{Freecal}(B)$ *and* $|A| < \kappa$.
(iii) $\kappa \in \text{Freecal}(A) \cap \text{Freecal}(B)$.

Proof. First suppose that $\kappa \in \text{Freecal}(A \times B)$. If $X \in [A]^\kappa$, then $\{(x,0) : x \in X\} \in [A \times B]^\kappa$, and so there is an independent $Z \in [\{(x,0) : x \in X\}]^\kappa$. Writing $Z = W \times \{0\}$, it is clear that W is independent and $W \in [X]^\kappa$. A similar argument works for B, and this shows that one of (i)–(iii) holds.

Second, suppose that (i) holds. Suppose that $X \in [A \times B]^\kappa$. Then $Y = \{a \in A : \exists b \in B[(a,b) \in X]\}$ has size κ, and hence there is an independent $Z \in [Y]^\kappa$. Clearly this gives rise to an independent $W \in [X]^\kappa$.

(ii) is similar. Now suppose that (iii) holds, and $X \in [A \times B]^\kappa$. For each $i < 2$ let

$$x \equiv_i y \quad \text{iff} \quad x, y \in X \text{ and } \pi_i(x) = \pi_i(y).$$

This gives two equivalence relations on X. Now for each $x \in X$ let $f(x) = (x/\equiv_0, x/\equiv_1)$. Clearly f is one-one. Hence there are κ many equivalence classes under $\equiv_0$ or under $\equiv_1$. This easily yields the desired conclusion. □

Proposition 10.24. *Let I be an infinite set, and $\langle A_i : i \in I\rangle$ a system of infinite BAs. Let κ be an infinite cardinal, and assume that $|I| < \text{cf}(\kappa)$. Then the following conditions are equivalent:*

(i) $\kappa \in \text{Freecal}\left(\prod^{\text{w}}_{i\in I} A_i\right)$.
(ii) $\exists i \in I[\kappa \le |A_i|]$, *and* $\kappa \in \bigcap\{\text{Freecal}(A_i) : \kappa \le |A_i|\}$.

Proof. Let $B = \prod^{\text{w}}_{i\in I} A_i$.

Assume (i). In particular, $\kappa \le |B|$. Suppose that $\forall i \in I[|A_i| < \kappa]$. Since $|I| < \text{cf}(\kappa)$, this is a contradiction. Clearly $\kappa \in \bigcap\{\text{Freecal}(A_i) : \kappa \le |A_i|\}$.

Now assume (ii), and suppose that $X \in \left[\prod^{\text{w}}_{i\in I} A_i\right]^\kappa$. For each $x \in X$ let $F_x = \{i \in I : x_i \ne 0\}$. We may assume that each set F_x is finite. Now

$$X = \bigcup_{G\in[I]^{<\omega}} \{x \in X : F_x = G\}.$$

Since $|[I]^{<\omega}| = |I| < \text{cf}(\kappa)$, there exist a $Y \in [X]^\kappa$ and a $G \in [I]^{<\omega}$ such that $F_x = G$ for all $x \in Y$. Let $Z = \{f \restriction G : f \in Y\}$. Then Z is a subset of $\prod_{i\in G} A_i$ of size κ. By Proposition 10.23 there is an independent $W \in [Z]^\kappa$. Clearly this gives an independent $U \in [Y]^\kappa$. □

Proposition 10.25. *Let I be an infinite set, and $\langle A_i : i \in I\rangle$ a system of infinite BAs. Let κ be an infinite cardinal, and assume that $\text{cf}(\kappa) \le |I|$. Then the following conditions are equivalent:*

(i) $\kappa \in \text{Freecal}\left(\prod^{\text{w}}_{i\in I} A_i\right)$.
(ii) *The following conditions hold:*

(a) $|I| < \kappa$.

(b) $\sup\{|A_i| : i \in I, |A_i| < \kappa\} < \kappa$.

(c) $|\{i \in I : \kappa \leq |A_i|\}| < \mathrm{cf}(\kappa)$.

(d) $\exists i \in I[\kappa \leq |A_i|]$.

(e) $\kappa \in \bigcap\{\mathrm{Freecal}(A_i) : i \in I, \kappa \leq A_i\}$.

Proof. Again let $B = \prod^{\mathrm{w}}_{i\in I} A_i$. Assume (i). In particular, $\kappa \leq |B|$. (a): If $\kappa \leq |I|$, let $\langle i_\alpha : \alpha < \kappa\rangle$ be a one-one sequence of elements of I. Let $X = \{\chi_{\{i_\alpha\}} : \alpha < \kappa$, where in general χ_F is the member of B such that $\chi_F(j) = 1$ for $j \in F$ and $\chi_F(j) = 0$ otherwise. Clearly X does not have an independent subset even of size 2, contradiction.

(b): Suppose that (b) fails. Then there is a sequence $\langle i(\alpha) : \alpha < \kappa\}$ of members of I such that $\langle |A_{i(\alpha)}| : \alpha < \kappa\rangle$ is strictly increasing with supremum κ. Define

$$X = \left\{\chi^a_{i(\alpha)} : \alpha < \kappa, a \in A_{i(\alpha)}\right\},$$

where $\chi^a_{i(\alpha)}$ is the member of B which is equal to a at $i(\alpha)$ and is 0 otherwise. Let $Y \in [X]^\kappa$ be independent. Then there are distinct $\alpha, \beta < \kappa$ and $a \in A_{i(\alpha)}$, $b \in A_{i(\beta)}$, such that $\chi^a_{i(\alpha)}, \chi^b_{i(\beta)} \in Y$. Since $\chi^a_{i(\alpha)} \cdot \chi^b_{i(\beta)} = 0$, this is a contradiction.

(c): Suppose that (c) fails. Then there is a one-one sequence $\langle i(\alpha) : \alpha < \mathrm{cf}(\kappa)\}$ of members of I such that $|A_{i(\alpha)}| = \kappa$ for all $\alpha < \mathrm{cf}(\kappa)$. Let $\langle X_\alpha : a < \mathrm{cf}(\kappa)$ be such that $\forall\alpha < \mathrm{cf}(\kappa)[X_\alpha \subseteq A_{i(\alpha)}]$ and $\langle |X_\alpha| : \alpha < \mathrm{cf}(\kappa)\rangle$ is strictly increasing with supremum κ. Define

$$Y = \left\{\chi^a_{i(\alpha)} : \alpha < \mathrm{cf}(\kappa), a \in X_\alpha\right\};$$

this gives a contradiction as in (b).

(d): Suppose that $\forall i \in I[|A_i| < \kappa]$. Then by (b) there is a cardinal $\lambda < \kappa$ such that $|A_i| \leq \lambda$ for all $i \in I$. Hence using (a),

$$|B| \leq \sum_{F\in[I]^{<\omega}} \left|\prod_{i\in F} A_i\right| \leq |I| \cdot \lambda < \kappa,$$

contradiction.

(e): Clear.

Now assume (ii), and suppose that $X \in [B]^\kappa$. Let $J = \{i \in I : |A_i| \geq \kappa\}$, $K = \{i \in I : |A_i| < \kappa\}$, and $C = \prod^{\mathrm{w}}_{i\in K} A_i$. Then

(1) $$|C| < \kappa.$$

In fact, by (b) we have $\lambda \stackrel{\text{def}}{=} \sup_{i\in K}|A_i| < \kappa$, and, using (a),

$$\begin{aligned}|C| &\le \left|\bigcup_{G\in[K]^{<\omega}}\{x\in C : \operatorname{supp}(x)=G\}\right|\\ &\le \sum_{G\in[K]^{<\omega}}\left|\prod_{i\in G}A_i\right|\\ &\le |I|\cdot\lambda<\kappa,\end{aligned}$$

proving (1).

Now again let $F_x = \{i\in I : x_i\neq 0\}$. Wlog each F_x is finite. Now

$$X=\bigcup\{\{x\in X : F_x\cap J=G\} : G\in[J]^{<\omega}\}.$$

Since $|[J]^{<\omega}| < \operatorname{cf}(\kappa)$ by (c), it follows that there exist $Y\in[X]^\kappa$ and $G\in[I]^{<\omega}$ such that $\forall x\in Y[F_x\cap J=G\}$. Let $D=\prod_{i\in G}A_i\times C$.

(2) $G\neq\emptyset$, and $\{x\restriction(G\cup K) : x\in Y\}\in[D]^\kappa$.

If $G=\emptyset$, then $F_x\subseteq K$ for each $x\in Y$. Then $\langle x\restriction K : x\in Y\rangle$ is a one-one function, so $|Y|=\kappa$ contradicts (1). For the second part of (2), suppose that $x,y\in Y$ with $x\neq y$. Choose $i\in I$ such that $x_i\neq y_i$. If $i\in K$, then $(x\restriction(G\cup K))_i\neq(y\restriction(G\cup K))_i$. If $i\in J$, then $i\in G$ since $i\in F_x$ or $i\in F_y$, so again $(x\restriction(G\cup K))_i\neq(y\restriction(G\cup K))_i$. This proves (2).

Now by (e), $\kappa\in\operatorname{Freecal}(A_i)$ for all $i\in G$ (since $G\subseteq J$). Hence by Proposition 10.24, $\kappa\in\operatorname{Freecal}\left(\prod_{i\in G}A_i\right)$. By (1) and Proposition 10.23, $\kappa\in\operatorname{Freecal}(D)$. It follows from (2) that there is a $W\in[Y]^\kappa$ such that $\{x\restriction(G\cup K) : x\in W\}$ is independent. Hence W itself is independent. □

Proposition 10.26. *Suppose that I is an infinite set, $\kappa\le\left|\prod_{i\in I}A_i\right|$, and for every system $\langle\lambda_i : i\in I\rangle$ of cardinals less that κ we have $\prod_{i\in I}\lambda_i<\kappa$. Then the following hold:*

(i) *There is an $i\in I$ such that $\kappa\le|A_i|$.*
(ii) $\kappa\in\operatorname{Freecal}\left(\prod_{i\in I}A_i\right)$ *iff* $\kappa\in\bigcap\{\operatorname{Freecal}(A_i) : \kappa\le|A_i|\}$.

Proof. Clearly the hypotheses imply (i). The direction $\Rightarrow$ in (ii) is clear. Now suppose that $\kappa\in\bigcap\{\operatorname{Freecal}(A_i) : \kappa\le|A_i|\}$, and $X\in\left[\prod_{i\in I}A_i\right]^\kappa$. Then there is an $i\in I$ such that $|\pi_i[X]|\ge\kappa$, as otherwise $|X|\le\prod_{i\in I}|\pi_i[X]|<\kappa$, contradiction. Hence the desired conclusion follows. □

Corollary 10.27. *If $\kappa\le\left|\prod_{i\in I}A_i\right|$, $\kappa=\lambda^+$, and $\lambda^{|I|}=\lambda$, then* (i) *and* (ii) *of Proposition* 10.26 *hold.* □

Of course Proposition 10.26 does not take care of all possibilities concerning free caliber and arbitrary products; the following problem is open.

Problem 102. *Describe* Freecal $\left(\prod_{i\in I} A_i\right)$ *in terms of* $\langle \mathrm{Freecal}(A_i) : i \in I\rangle$.

A particular infinite product to which Proposition 10.26 does not apply is $\prod_{n\in\omega} \mathrm{Fr}(\beth_n)$; it is natural to conjecture that this product has free caliber $\beth_\omega^+$. But Shelah [99], 6.11, shows that consistently the conjecture can be true or false. This answers Problem 35 of Monk [96].

We now turn to free products with respect to free caliber. The following general lemma will be useful.

Lemma 10.28. *If A and B are subalgebras of C and $A\cup B$ generates C, then every element $x \in C$ can be written in the form*

$$x = \sum_{i<m_x} (a_i^x \cdot b_i^x),$$

with each $a_i^x \in A$, $b_i^x \in B$. With x written in this form, let $\langle c_j^x : j < n_x\rangle$ be a one-one enumeration of the atoms of the finite subalgebra $\langle\{a_i^x : i < m_x\}\rangle$ of A, and let

$$d_j^x = \sum\{b_k^x : k < m_x, c_j^x \le a_k^x\}$$

for each $j < n_x$. Then

$$(*)\qquad x = \sum_{j<n_x} (c_j^x \cdot d_j^x);$$

$$-x = \sum_{j<n_x} (c_j^x \cdot -d_j^x).$$

Proof. We only give the derivation of $(*)$:

$$\begin{aligned}\sum_{i<m_x} (a_i^x \cdot b_i^x) &= \sum_{i<m_x} \sum\{c_j^x \cdot b_i^x : j < n_x, c_j^x \le a_i^x\}\\ &= \sum_{j<n_x} \left(c_j^x \cdot \sum\{b_i^x : i < m_x, c_j^x \le a_i^x\}\right)\\ &= \sum_{j<m_x} (c_j^x \cdot d_j^x).\end{aligned}$$ □

Note that by symmetry there is a similar result using atoms of $\langle\{b_j^x : j < m_x\}\rangle$.

Now we take first the case of a free product of two BAs, and κ regular.

Lemma 10.29. *Let κ be an infinite cardinal. Suppose that A and B are BAs, $X \in [A \oplus B]^\kappa$, $n \in \omega$, and for each $x \in X$ we have*

$$x = \sum_{i<n} (a_i^x \cdot b_i^x),$$

with each $a_i^x \in A$, $b_i^x \in B$, $\langle b_i^x : i < n\rangle$ *a partition of* B. *Suppose that* $i < n$, $\langle a_i^x : x \in X\rangle$ *is independent, and* $\{b_i^x : x \in X\}$ *has fip. Then* X *is independent.*

Proof. Suppose that $F, G \in [X]^{<\omega}$ with $F \cap G = \emptyset$. Then

$$\begin{aligned}\prod_{x\in F} x \cdot \prod_{x\in G} -x &= \prod_{x\in F}\sum_{k<n}(a_k^x \cdot b_k^x) \cdot \prod_{x\in G}\sum_{k<n}(-a_k^x \cdot b_k^x)\\ &\geq \prod_{x\in F}(a_i^x \cdot b_i^x) \cdot \prod_{x\in G}(-a_i^x \cdot b_i^x)\\ &\neq 0. \end{aligned}$$ □

Lemma 10.30. *Let* κ *be an infinite cardinal. Suppose that* A *and* B *are BAs, and* $\kappa \in \mathrm{Freecal}(A)$. *Suppose that* $X \in [A \oplus B]^\kappa$, $n \in \omega$, *and for each* $x \in X$ *we have*

$$x = \sum_{i<n}(a_i^x \cdot b_i^x),$$

with each $a_i^x \in A$, $b_i^x \in B$, $\langle b_i^x : i < n\rangle$ *a partition of* B. *Suppose that* $i < n$ *and* $|\{a_i^x : x \in X\}| = \kappa$ *while* $\{b_i^x : x \in X\}$ *has fip. Then there is a* $Y \in [X]^\kappa$ *such that* Y *is independent.*

Proof. There is a $Y \in [X]^\kappa$ such that $\langle a_i^x : x \in Y\rangle$ is independent. Y is independent by Lemma 10.29. □

Proposition 10.31. *Let* κ *be regular and uncountable. Then* $\kappa \in \mathrm{Freecal}(A_0 \oplus A_1)$ *iff* $\exists i < 2[\kappa \in \mathrm{Freecal}(A_i)]$ *and* $\kappa \in \bigcap\{\mathrm{Freecal}(A_i) : i < 2, \kappa \leq |A_i|\}$.

Proof. Clearly $\Rightarrow$ holds. Now assume the above conditions, and suppose that $X \in [A_0 \oplus A_1]^\kappa$. By Lemma 10.28, write each $x \in X$ in the form

$$x = \sum_{i<m_x}(a_i^x \cdot b_i^x),$$

where each $a_i^x \in A_0$, and $\langle b_i^x : i < m_x\rangle$ is a partition of A_1. Since $\mathrm{cf}(\kappa) > \omega$, there exist an $X_0 \in [X]^\kappa$ and an $n \in \omega$ such that $m_x = n$ for all $x \in X_0$.

Case 1. $\forall i < n[|\{a_i^x : x \in X_0\}| < \kappa]$. Then

$$X_0 = \bigcup\left\{\{x \in X_0 : \langle a_i^x : i < n\rangle = y\} : y \in \prod_{i<n}\{a_i^x : x \in X_0\}\right\},$$

so by the regularity of κ there exist an $X_1 \in [X_0]^\kappa$ and a system $\langle y_i : i < n\rangle$ of elements of A_0 such that $a_i^x = y_i$ for all $x \in X_1$ and $i < n$. By Lemma 10.28 again we can write each $x \in X_1$ as

$$x = \sum_{i<u}(c_i \cdot d_i^x)$$

with $\langle c_i : i < u\rangle$ a partition of A_0 and each $d_i^x \in A_1$. Now there is an $i < u$ such that $|\{d_i^x : x \in X_1\}| = \kappa$, as otherwise $|X_1| < \kappa$. By Lemma 10.30 there is an independent $Y \in [X_1]^\kappa$.

Case 2. $\exists i < n[|\{a_i^x : x \in X_0\}| = \kappa]$. Then there is an $X_1 \in [X_0]^\kappa$ such that $\langle a_i^x : x \in X_1\rangle$ is independent.

Subcase 2.1. $|\{b_i^x : x \in X_0\}| < \kappa$. Then there exist an $X_1 \in [X_0]^\kappa$ and an element c of B such that $b_i^x = c$ for all $x \in X_1$. By Lemma 10.29 X_1 is independent.

Subcase 2.2. $|\{b_i^x : x \in X_0\}| = \kappa$. Then there is an $X_1 \in [X_0]^\kappa$ such that $\langle a_i^x : x \in X_1\rangle$ and $\langle b_i^x : x \in X_1\rangle$ are independent. Then X_1 is independent by Lemma 10.29. □

Corollary 10.32. *Let κ be regular and uncountable. Suppose that I is finite with at least two elements. Then $\kappa \in \mathrm{Freecal}(\oplus_{i\in I} A_i)$ iff $\exists i \in I[\kappa \in \mathrm{Freecal}(A_i)]$ and $\kappa \in \bigcap\{\mathrm{Freecal}(A_i) : i \in I, \kappa \le |A_i|\}$.* □

To treat the case of singular κ, first recall Proposition 10.22, which implies that we only need to consider singular κ with $\mathrm{cf}(\kappa) > \omega$. We also need a new notion. A BA A has *precaliber* κ iff

$$\forall X \in [A]^\kappa \exists Y \in [X]^\kappa [Y \text{ has fip}].$$

We let $\mathrm{precal}(A) = \{\kappa : A \text{ has precaliber } \kappa\}$. Note that there is no assumption here that $\kappa \le |A|$; in fact, A vacuously has precaliber κ for every cardinal κ greater than $|A|$.

Proposition 10.33. *Suppose that κ is singular, $\mathrm{cf}(\kappa) \in \mathrm{precal}(A)$, $X \in [A]^\kappa$, and $a \in {}^X A$ with $|\mathrm{rng}(a)| < \kappa$. Then there is a $Y \in [X]^\kappa$ such that $\{a_x : x \in Y\}$ has fip.*

Proof. We claim

$$(*) \qquad \exists d \in {}^{\mathrm{cf}(\kappa)}\mathrm{rng}(a) \forall P \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}[|\{x \in X : \exists \alpha \in P(a_x = d_\alpha)\}| = \kappa].$$

For, define $x \equiv y$ iff $x, y \in X$ and $a_x = a_y$. Thus $\equiv$ is an equivalence relation on X with less than κ equivalence classes. If some class K has κ elements, let $d : \mathrm{cf}(\kappa) \to \mathrm{rng}(a)$ be constant with value a member of K. Then if $P \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ we have $\{x \in X : \exists \alpha \in P(a_x = d_\alpha)\} = K$, as desired. If no such class exists, then let $\langle Z_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ enumerate some of the classes so that $\langle |Z_\alpha| : \alpha < \mathrm{cf}(\kappa)\rangle$ is strictly increasing with supremum κ. Let $d_\alpha \in Z_\alpha$ for all $\alpha < \mathrm{cf}(\kappa)$. Clearly the desired conclusion follows again.

Now by the assumption $\mathrm{cf}(\kappa) \in \mathrm{precal}(A)$, choose $P \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ such that $\{d_\alpha : \alpha \in P\}$ has fip. Let $Y = \{x \in X : \exists \alpha \in P(a_x = d_\alpha)\}$. Clearly $\{a_x : x \in Y\}$ has fip. □

Proposition 10.34. *Suppose that κ is a singular cardinal, $\kappa \in \mathrm{precal}(B)$, $X \in [A \oplus B]^\kappa$, $n \in \omega$, and for every $x \in X$*

$$x = \sum_{i<n} a_i^x \cdot b_i^x,$$

with $a_i^x \in A$ *and* $b_i^x \in B$*. Suppose that* $i < n$, $|\{a_i^x : x \in X\}| = \kappa$, *and* $|\{b_i^x : x \in X\}| < \kappa$.

Then there is a $Y \in [X]^\kappa$ *such that* $|\{a_i^x : x \in Y\}| = \kappa$ *and* $\{b_i^x : x \in Y\}$ *has fip.*

Proof. Choose $Z \in [X]^\kappa$ such that $\langle a_i^x : x \in Z\rangle$ is one-one. By Proposition 10.33 choose $Y \in [Z]^\kappa$ such that $\{b_i^x : x \in Z\}$ has fip. □

Proposition 10.35. *If* κ *is regular and uncountable, then* $\kappa \in \mathrm{precal}(A \oplus B)$ iff $\kappa \in \mathrm{precal}(A) \cap \mathrm{precal}(B)$.

Proof. $\Rightarrow$ is clear. Now suppose that $\kappa \in \mathrm{precal}(A) \cap \mathrm{precal}(B)$, and $X \in [A \oplus B]^\kappa$. For each $x \in X$ write

$$x = \sum_{i<m_x} (a_i^x \cdot b_i^x),$$

where each $a_i^x \in A^+$ and $b_i^x \in B^+$. Then there exist n and $Y \in [X]^\kappa$ such that $m_x = n$ for all $x \in Y$. Choose $i < n$ such that $|\{a_i^x \cdot b_i^x : x \in Y\}| = \kappa$. Say by symmetry $|\{a_i^x : x \in Y\}| = \kappa$. Choose $Z \in [Y]^\kappa$ such that $\{a_i^x : x \in Z\}$ has fip.

Case 1. $|\{b_i^x : x \in Z\}| < \kappa$. Then there exist a $V \in [Z]^\kappa$ and a $c \in B^+$ such that $b_i^x = c$ for all $x \in V$. Clearly then V has fip.

Case 2. $|\{b_i^x : x \in Z\}| = \kappa$. Then there is a $V \in [Z]^\kappa$ such that $\{b_i^x : x \in V\}$ has fip. Hence V has fip. □

Lemma 10.36. *Let* κ *be a singular cardinal with* $\mathrm{cf}(\kappa) > \omega$*. Then* $\kappa \in \mathrm{Freecal}(A \oplus B)$ iff *one of the following conditions holds:*

(i) $\kappa \le |A|$, $|B| < \kappa$, $\kappa \in \mathrm{Freecal}(A)$, *and* $\mathrm{cf}(\kappa) \in \mathrm{precal}(B)$.
(ii) $\kappa \le |B|$, $|A| < \kappa$, $\kappa \in \mathrm{Freecal}(B)$, *and* $\mathrm{cf}(\kappa) \in \mathrm{precal}(A)$.
(iii) $\kappa \le |A|, |B|$, $\kappa \in \mathrm{Freecal}(A) \cap \mathrm{Freecal}(B)$, *and* $\mathrm{cf}(\kappa) \in \mathrm{precal}(A) \cap \mathrm{precal}(B)$.

Proof. Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of infinite cardinals with supremum κ.

For $\Rightarrow$, suppose that $\kappa \in \mathrm{Freecal}(A \oplus B)$. The assertions in (i)–(iii) concerning free caliber are clear. Suppose that $\kappa \le |A|$ and $|B| < \kappa$. To show that $\mathrm{cf}(\kappa) \in \mathrm{precal}(B)$, suppose that $Y \in [B]^{\mathrm{cf}(\kappa)}$. Let $\langle y_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a one-one enumeration of Y. For each $\alpha < \mathrm{cf}(\kappa)$, let $Z_\alpha \in [A]^{\lambda_\alpha}$. Let $X = \{a \cdot y_\alpha : \alpha < \mathrm{cf}(\kappa), a \in Z_\alpha\}$. Then $X \in [A \oplus B]^\kappa$, and so X has an independent subset W of size κ. Let $V = \{\alpha : (a \cdot y_\alpha) \in W$ for some $a \in Z_\alpha\}$. Then $|V| = \mathrm{cf}(\kappa)$ since $|X| = \kappa$. If F is a finite subset of V, for each $\alpha \in F$ choose $a(\alpha) \in Z_\alpha$ such that $(a(\alpha) \cdot y_\alpha) \in W$. Then $\prod_{\alpha \in F}(a(\alpha) \cdot y_\alpha) \neq 0$, and hence $\prod_{\alpha \in F} y_\alpha \neq 0$. So $\{y_\alpha : \alpha \in V\}$ has fip. It follows that B has precaliber $\mathrm{cf}(\kappa)$. The other precaliber assertions in (i)–(iii) are checked similarly.

Now for $\Leftarrow$, assume that one of (i)–(iii) holds, and suppose that $X \in [A \oplus B]^\kappa$. For each $x \in X$ write

$$x = \sum_{j<m_x} (a_j^x \cdot b_j^x),$$

with $a_j^x \in A$, $b_j^x \in B$, and $\langle b_j^x : j < m_x\rangle$ a partition of B. Then there exist $X_0 \in [X]^\kappa$ and $n \in \omega$ such that $m_x = n$ for all $x \in X_0$. We now consider several cases.

Case 1. $\exists X_1 \in [X_0]^\kappa \exists i < n[|\{b_i^x : x \in X_1\}| < \kappa$ and $|\{a_i^x : x \in X_1\}| = \kappa]$. By Proposition 10.34 choose $X_2 \in [X_1]^\kappa$ such that $|\{a_i^x : x \in X_2\}| = \kappa$ and $\{b_i^x : x \in X_2\}$ has fip. By Proposition 10.30 there is a $Z \in [X_2]^\kappa$ such that Z is independent.

Case 2. $\forall X_1 \in [X_0]^\kappa \forall i < n[|\{b_i^x : x \in X_1\}| < \kappa \to |\{a_i^x : x \in X_1\}| < \kappa]$, and there is an $i < n$ such that $|\{a_i^x : x \in X_1\}| = \kappa$. Choose $X_2 \in [X_1]^\kappa$ such that $\langle a_i^x : x \in X_2\rangle$ is independent. Applying our case condition to X_2 in place of X_1 we infer that $|\{b_i^x : x \in X_2\}| = \kappa$. Choose $X_3 \in [X_2]^\kappa$ such that $\langle b_i^x : x \in X_3\rangle$ is independent. Hence X_3 is independent by Lemma 10.29.

Case 3. $\forall X_1 \in [X_0]^\kappa \forall i < n[|\{b_i^x : x \in X_1\}| < \kappa \to |\{a_i^x : x \in X_1\}| < \kappa]$, and $\forall i < n[|\{a_i^x : x \in X_1\}| < \kappa$. For each $x \in X_1$ let $\langle c_j^x : j < p_x\rangle$ be a one-one enumeration of the atoms of $\langle\{a_i^x : i < n\}\rangle$. Then by Lemma 10.28 we can write

$$x = \sum_{j<p_x} (c_j^x \cdot d_j^x)$$

for each $x \in X_1$, where each d_j^x is in B. Choose $X_2 \in [X_1]^\kappa$ and q so that $p_x = q$ for all $x \in X_2$. Now $|\{c_j^x : x \in X_2\}| < \kappa$ for all $j < q$, so there is a $j < q$ such that $|\{d_j^x : x \in X_2\}| = \kappa$. Choose $X_3 \in [X_2]^\kappa$ so that $\langle d_j^x : x \in X_3\rangle$ is one-one. By Proposition 10.34, there is an $X_4 \in [X_3]^\kappa$ such that $\{c_j^x : x \in X_4\}$ has fip. Now the desired result follows from Lemma 10.30. $\square$

Proposition 10.37. *Let I be a finite set with at least two elements, and let κ be a singular cardinal with* $\mathrm{cf}(\kappa) > \omega$. *Then* $\kappa \in \mathrm{Freecal}(\oplus_{i\in I}A_i)$ iff *one of the following conditions hold:*

(i) *There is exactly one $i \in I$ such that $\kappa \leq |A_i|$; for this i we have* $\kappa \in \mathrm{Freecal}(A_i)$ *and* $\mathrm{cf}(\kappa) \in \bigcap_{j\in I\setminus\{i\}} \mathrm{precal}(A_j)$.

(ii) *There are at least two $i \in I$ such that $\kappa \leq |A_i|$,* $\kappa \in \bigcap\{\mathrm{Freecal}(A_i) : i \in I, \kappa \leq |A_i|\}$, *and* $\mathrm{cf}(\kappa) \in \bigcap_{i\in I} \mathrm{precal}(A_i)$.

Proof. We prove this by induction on $|I|$. The case $|I| = 2$ is given by Lemma 10.36. Now suppose that the proposition holds for I, with $|I| \geq 2$, and we are given $I \cup \{i\}$ with $i \notin I$. Let $B = \oplus_{j\in I}A_j$. Suppose that $\kappa \in \mathrm{Freecal}(B \oplus A_i)$. By Lemma 10.36 we have three cases.

Case 1. $\kappa \leq |\oplus_{j\in I}A_j|$, $|A_i| < \kappa$, $\kappa \in \mathrm{Freecal}(\oplus_{j\in I}A_j)$, and $\mathrm{cf}(\kappa) \in \mathrm{precal}(A_i)$. It is straightforward to check the desired conditions for $I \cup \{i\}$.

Case 2. $\kappa \leq |A_i|$, $|\oplus_{j\in I}A_j| < \kappa$, $\kappa \in \mathrm{Freecal}(A_i)$, and $\mathrm{cf}(\kappa) \in \mathrm{precal}(\oplus_{j\in I}A_j)$. This gives condition (i) for $I \cup \{j\}$, using Proposition 10.35.

Case 3. $\kappa \leq |\oplus_{j\in I}A_j|$, $\kappa \leq |A_i|$, $\kappa \in \mathrm{Freecal}(\oplus_{j\in I}A_j) \cap \mathrm{Freecal}(A_i)$, and $\mathrm{cf}(\kappa) \in \mathrm{precal}(\oplus_{j\in I}A_j) \cap \mathrm{precal}(A_i)$. By the inductive hypothesis, (ii) holds.

For the converse we have three possibilities.

Case 1. $\kappa \leq |A_i|$, $\forall j \in I[|A_j| < \kappa]$; $\kappa \in \text{Freecal}(A_i)$ and $\text{cf}(\kappa) \in \bigcap_{j\in I} \text{precal}(A_j)$. By Propositions 10.35 and 10.36 we get $\kappa \in \text{Freecal}(\oplus_{j\in I} A_i \oplus A_i)$.

Case 2. There is exactly one $j \in I$ such that $\kappa \leq |A_j|$; also $|A_i| < \kappa$; and $\kappa \in \text{Freecal}(A_j)$ and $\text{cf}(\kappa) \in \text{precal}(A_i) \cap \bigcap_{k\in I\setminus\{j\}} \text{precal}(A_j)$. By the inductive hypothesis we get $\kappa \in \text{Freecal}(\oplus_{k\in I} A_k)$, and hence $\kappa \in \text{Freecal}(\oplus_{j\in I} A_i \oplus A_i)$ by 10.36.

Case 3. (ii) holds for $I \cup \{i\}$. The desired conclusion is clear. □

Turning to arbitrary free products, the case of regular κ is simple:

Proposition 10.38. *Suppose that $|I| \geq 2$ and κ is uncountable and regular. Then $\kappa \in \text{Freecal}(\oplus_{i\in I} A_i)$ iff $\kappa \leq |\oplus_{i\in I} A_i|$ and $\forall i \in I[\kappa \leq |A_i| \to \kappa \in \text{Freecal}(A_i)]$.*

Proof. For I finite Corollary 10.32 gives the desired result. Now suppose that I is infinite. For brevity let $B = \oplus_{i\in I} A_i$. The implication $\Rightarrow$ is obvious. Now assume the two conditions, and suppose that $X \in [B]^\kappa$. For all $x \in X$ choose $F_x \in [I]^{<\omega}$ such that $x \in \oplus_{i\in F_x} A_i$. Then there exist an $X_0 \in [X]^\kappa$ and an integer m such that $\langle F_x : x \in X_0\rangle$ forms a Δ-system, say with kernel G, and $|F_x| = m$ for all $x \in X_0$. If $F_x = G$ for all $x \in X_0$, then the desired conclusion follows from Corollary 10.32. Hence assume that $F_x \neq G$ for all $x \in X_0$. For each $x \in X_0$ write

$$x = \sum_{i<n_x} (a_i^x \cdot b_i^x)$$

with each $a_i^x \in \oplus_{j\in G} A_j$, $b_i^x \in \oplus_{j\in F_x\setminus G} A_j$, $\langle b_i^x : i < n_x\rangle$ a partition of $\oplus_{j\in F_x\setminus G} A_j$. Then there exist $X_1 \in [X_0]^\kappa$ and p such that $n_x = p$ for all $x \in X_1$. Note by disjointness of supports that $\langle b_i^x : x \in X_1\rangle$ is independent for every $i < p$.

Case 1. $\exists i < p[|\{a_i^x : x \in X_1\}| = \kappa]$. Then there is an $X_2 \in [X_1]^\kappa$ such that $\langle a_i^x : x \in X_2\rangle$ is independent. Clearly also X_2 itself is independent.

Case 2. $\forall i < p[\{a_i^x : x \in X_1\}| < \kappa]$. For each $x \in X_1$ let $\langle c_j^x : j < q_x\rangle$ be the system of atoms of the algebra $\langle\{a_i^x : i < p\}\rangle$. By Lemma 10.28 write

$$x = \sum_{j<q_x} (c_j^x \cdot d_j^x)$$

with each $d_j^x \in \oplus_{k\in F_x\setminus G} A_k$. Take $X_2 \in [X_1]^\kappa$ and r such that $q_x = r$ for all $x \in X_2$. There is a $j < r$ such that $|\{d_j^x : x \in X_2\}| = \kappa$. Hence X_2 is independent. □

Corollary 10.39. *If κ is uncountable and regular, $|A_i| < \kappa$ for all $i \in I$, with I infinite, and $\kappa \leq |\oplus_{i\in I} A_i|$, then $\kappa \in \text{Freecal}(\oplus_{i\in I} A_i)$.* □

Taking each A_i of size 4, we obtain another proof of Theorem 9.16 in the Handbook.

Corollary 10.40. *If κ is uncountable and regular and I is a set with at least two elements, then $\kappa \in \text{precal}(\oplus_{i\in I} A_i)$ iff $\forall i \in I[\kappa \in \text{precal}(A_i)]$.*

Proof. Repeat the proof of Proposition 10.38, replacing "independent" by "fip". □

Now we take the case of singular κ. Here we need the well-known double delta system lemma. We give a proof of it which is due to Richard Laver First we have an elementary lemma about singular cardinals.

Lemma 10.41. *Let κ be a singular cardinal. Then there is a strictly increasing sequence $\langle\lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ of successor cardinals such that $\mathrm{cf}(\kappa) < \lambda_0$, $\sup_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha = \kappa$, and $\forall\alpha < \mathrm{cf}(\kappa)[\sup_{\beta<\alpha} \lambda_\beta < \lambda_\alpha]$.*

Proof. First let $\langle\mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be any sequence of cardinals less than κ but with supremum κ. Then set $\lambda_0 = (\mathrm{cf}(\kappa))^+$ and, for $0 < \alpha < \mathrm{cf}(\kappa)$,

$$\lambda_\alpha = \left(\max\left\{\mu_\alpha, \sup_{\beta<\alpha} \lambda_\beta\right\}\right)^+. \qquad \square$$

Lemma 10.42. (Double Δ-system lemma) *Suppose that κ is singular with $\mathrm{cf}(\kappa) > \omega$. Take $\langle\lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ as in Lemma 10.41. Suppose that $\langle A_\xi : \xi < \kappa\rangle$ is a system of finite sets. Then there exist $M \subseteq \mathrm{cf}(\kappa)$, G, and sequences $\langle N_\alpha : \alpha \in M\rangle$, $\langle F_\alpha : \alpha \in M\rangle$ such that the following conditions hold:*

(i) *$\langle N_\alpha : \alpha \in M\rangle$ is a system of pairwise disjoint subsets of κ, with $|N_\alpha| = \lambda_\alpha$ for all $\alpha \in M$.*

(ii) *$|M| = \mathrm{cf}(\kappa)$.*

(iii) *$\langle A_\xi : \xi \in N_\alpha\rangle$ is a Δ system with kernel F_α, for every $\alpha \in M$.*

(iv) *For distinct $\alpha, \beta \in M$ we have:*
   (a) *$F_\alpha \cap F_\beta = G$.*
   (b) *$A_\xi \cap A_\eta = G$ for all $\xi \in N_\alpha$ and $\eta \in N_\beta$.*

Proof. Write $\kappa = \bigcup_{\alpha<\mathrm{cf}(\kappa)} N'_\alpha$, where the N'_α's are pairwise disjoint and $|N'_\alpha| = \lambda_\alpha$ for all $\alpha < \mathrm{cf}(\kappa)$. For each $\alpha < \mathrm{cf}(\kappa)$ choose $N''_\alpha \in [N'_\alpha]^{\lambda_\alpha}$ so that $\langle A_\xi : \xi \in N''_\alpha\rangle$ is a Δ system, say with kernel F_α. Choose $M \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ so that $\langle F_\alpha : \alpha \in M\rangle$ is a Δ system, say with kernel G.

For each $\alpha \in M$ let

$$B_\alpha = \bigcup\left\{\bigcup_{\xi\in N''_\beta} A_\xi : \alpha \in M, \beta < \alpha\right\}.$$

Thus $\left|\bigcup_{\xi\in N''_\beta} A_\xi\right| = \lambda_\beta$ for each $\beta < \alpha$, and hence $|B_\alpha| < \lambda_\alpha$ by the choice of the sequence $\langle\lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$. Now

$$N''_\alpha = \bigcup\{\{\xi \in N''_\alpha : A_\xi \cap B_\alpha = D\} : D \in [B_\alpha]^{<\omega}\},$$

so there exist an $N'''_\alpha \in [N''_\alpha]^{\lambda_\alpha}$ and a finite C_α such that $A_\xi \cap B_\alpha = C_\alpha$ for all $\xi \in N'''_\alpha$. Note that $C_\alpha \subseteq F_\alpha$. In fact, take distinct $\xi, \eta \in N'''_\alpha$. Then $F_\alpha \cap B_\alpha = A_\xi \cap A_\eta \cap B_\alpha = A_\xi \cap B_\alpha \cap C_\alpha = C_\alpha$, so that $C_\alpha \subseteq F_\alpha$. Finally, let

$$(*) \qquad N_\alpha = N'''_\alpha \backslash \{\xi \in N'''_\alpha : (A_\xi \backslash F_\alpha) \cap F_\beta \neq \emptyset \text{ for some } \beta \in M\}.$$

Note that if $\alpha, \beta, \gamma \in M$ with $\beta \neq \gamma$, then $F_\beta \cap F_\gamma = G \subseteq F_\alpha$; so $(A_\xi \backslash F_\alpha) \cap F_\beta \cap (A_\xi \backslash F_\alpha) \cap F_\gamma = \emptyset$. From this fact and the fact that $|M| = \mathrm{cf}(\kappa) < \lambda_\alpha$ it follows that $|N_\alpha| = \lambda_\alpha$.

Now we check the conditions of the lemma. All of them are clear except for (iv)(b). For it, suppose by symmetry that $\beta < \alpha$, $\xi \in N_\alpha$, and $\eta \in N_\beta$. Then $A_\eta \subseteq B_\alpha$, and hence $A_\xi \cap A_\eta \subseteq A_\xi \cap B_\alpha = C_\alpha \subseteq F_\alpha$. Therefore

$$\begin{aligned} A_\xi \cap A_\eta &= A_\xi \cap A_\eta \cap F_\alpha \\ &\subseteq A_\eta \cap F_\alpha \\ &= (A_\eta \cap F_\alpha \cap F_\beta) \cup ((A_\eta \cap F_\alpha) \backslash F_\beta) \\ &= A_\eta \cap F_\alpha \cap F_\beta \\ &\subseteq G, \end{aligned}$$

using $(*)$. To show equality, pick $\xi' \in N_\alpha \backslash \{\xi\}$ and $\eta' \in N_\beta \backslash \{\eta\}$. Then

$$A_\xi \cap A_{\xi'} \cap A_\eta \cap A_{\eta'} = F_\alpha \cap F_\beta = G,$$

and so $G \subseteq A_\xi \cap A_\eta$. □

Theorem 10.43. *Let κ be singular with $\mathrm{cf}(\kappa) > \omega$, and suppose that $|I| \geq 2$ and $\kappa \leq |\oplus_{i \in I} A_i|$. Then $\kappa \in \mathrm{Freecal}(\oplus_{i \in I} A_i)$ iff* (i) *below holds, and exactly one of* (ii), (iii) *holds.*

(i) *If I is infinite and $G \in {}^{\mathrm{cf}(\kappa)}([I]^{<\omega})$ consists of pairwise disjoint sets, and if $\langle X_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ is such that*

$$\forall \alpha < \mathrm{cf}(\kappa)\, [X_\alpha \subseteq \oplus_{i \in G_\alpha} A_i] \text{ and } \sup_{\alpha < \mathrm{cf}(\kappa)} |X_\alpha| = \kappa,$$

then there is a system $\langle Y_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ such that

$$\forall \alpha < \mathrm{cf}(\kappa) [Y_\alpha \subseteq X_\alpha \text{ and } Y_\alpha \text{ is independent}] \text{ and } \sup_{\alpha < \mathrm{cf}(\kappa)} |Y_\alpha| = \kappa.$$

(ii) *There is an $i \in I$ such that*

$$\kappa \leq |A_i| \text{ and } \kappa \in \mathrm{Freecal}(A_i) \text{ and } |\oplus_{j \in I, j \neq i} A_j| < \kappa$$
$$\text{and } \mathrm{cf}(\kappa) \in \bigcap_{j \in I, j \neq i} \mathrm{precal} A_i.$$

(iii) *For all $i \in I$ we have*

$$|\oplus_{j \in I, j \neq i} A_j| \geq \kappa \text{ and } \mathrm{cf}(\kappa) \in \mathrm{precal}(A_i)$$
$$\text{and if } \kappa \leq |A_i| \text{ then } \kappa \in \mathrm{Freecal}(A_i).$$

Proof. The case I finite is given by Proposition 10.37. So suppose that I is infinite.

For $\Rightarrow$, suppose that $\kappa \in \mathrm{Freecal}\,(\oplus_{i\in I}A_i)$. Then (i) is clear. Clearly at most one of (ii), (iii) holds.

Case 1. $\forall i \in I[|\oplus_{j\in I,j\neq i}A_j| \geq \kappa]$. Then (iii) holds, using Proposition 10.36 for the precaliber assertion.

Case 2. $\exists i\in I[|\oplus_{j\in I,j\neq i}A_j|<\kappa]$. By Proposition 10.36 we have $\kappa\in\mathrm{Freecal}(A_i)$ and $\mathrm{cf}(\kappa) \in \mathrm{precal}\,(\oplus_{j\in I,j\neq i}A_j)$. Clearly then $\mathrm{cf}(\kappa) \in \bigcap_{j\in I,j\neq i}\mathrm{precal}(A_j)$.

Turning to $\Leftarrow$, we assume (i) and one of (ii), (iii). We now note that it suffices to treat these two cases:

(1) $\forall i \in I[\kappa \leq |A_i|,\ \kappa \in \mathrm{Freecal}(A_i)$, and $\mathrm{cf}(\kappa) \in \mathrm{precal}(A_i)]$.

(2) $\forall i \in I[|A_i| < \kappa$ and $\mathrm{cf}(\kappa) \in \mathrm{precal}(A_i)]$.

For, suppose that we have proved that $\kappa \in \mathrm{Freecal}\,(\oplus_{i\in I}A_i)$ in these two cases. If (ii) holds, then this conclusion follows immediately from Propositions 10.40 and 10.36(i), without using these cases. Assume that (iii) holds. Let $J = \{i \in I : |A_i| \geq \kappa\}$. If $J = \emptyset$, then (2) gives the desired result. So suppose that $J \neq \emptyset$. Then case (1) applies to J, and so by assumption $\kappa \in \mathrm{Freecal}\,(\oplus_{j\in J}A_j)$. Now Propositions 10.36 and 10.40 give the desired result.

We treat the cases (1) and (2) simultaneously. Take $\langle\lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ as in Lemma 10.41. Suppose that $X \in [\oplus_{i\in I}A_i]^\kappa$. For each $x \in X$ let $H_x \in [I]^{<\omega}$ be such that $x \in \oplus_{i\in H_x}A_i$. Let $f : \kappa \to X$ be a bijection, and define $K_\alpha = H_{f(\alpha)}$ for all $\alpha < \kappa$. Now we apply the double Δ-system lemma 10.42 to the system $\langle K_\alpha : \alpha < \kappa\rangle$ to obtain $M \subseteq \mathrm{cf}(\kappa)$, G, and sequences $\langle N_\alpha : \alpha \in M\rangle$, $\langle F_\alpha : \alpha \in M\rangle$ with the following properties:

(3) $\langle N_\alpha : \alpha \in M\rangle$ is a system of pairwise disjoint subsets of κ, with $|N_\alpha| = \lambda_\alpha$ for all $\alpha \in M$.

(4) $|M| = \mathrm{cf}(\kappa)$.

(5) $\langle K_\xi : \xi \in N_\alpha\rangle$ is a Δ system with kernel F_α, for every $\alpha \in M$.

(6) For distinct $\alpha, \beta \in M$ we have:

(a) $F_\alpha \cap F_\beta = G$.

(b) $K_\xi \cap K_\eta = G$ for all $\xi \in N_\alpha$ and $\eta \in N_\beta$.

Now

$$M = \bigcup\{\{\alpha \in M : |F_\alpha| = n\} : n \in \omega\},$$

so there exist an $M_0 \in [M]^{\mathrm{cf}(\kappa)}$ and $n \in \omega$ such that $|F_\alpha| = n$ for all $\alpha \in M_0$. Let $X_0 = f\left[\bigcup_{\alpha\in M_0} N_\alpha\right]$. So $X_0 \in [X]^\kappa$. For each $x \in X_0$ let $\alpha(x) \in M_0$ be such that $x \in f[N_{\alpha(x)}]$. Now

$$X_0 = \bigcup\{\{x \in X_0 : |H_x| = m\} : m \in \omega\},$$

so there exist an $X_1 \in [X_0]^\kappa$ and $m \in \omega$ such that $|H_x| = m$ for all $x \in X_1$.

Note that for any $x \in X_1$ we have $H_x = K_{f^{-1}(x)} \subseteq F_{\alpha(x)} \subseteq G$. Now there are several possibilities, and we first treat the most complicated one.

Case 1. $0 < |G| < n < m$. For each $x \in X_1$ write

$$x = \sum_{i<p_x} (a_i^x \cdot b_i^x \cdot c_i^x),$$

where each $a_i^x \in \oplus_{j\in G} A_j$, $\langle a_i^x : i < p_x\rangle$ a partition, $b_i^x \in \oplus_{j\in F_{\alpha(x)}\backslash G} A_j$ with $f^{-1}(x) \in N_{\alpha(x)}$, and $c_i^x \in \oplus_{j\in K_{f^{-1}(x)}\backslash F_{\alpha(x)}} A_j$. Then there exist $X_2 \in [X_1]^\kappa$ and q such that $p_x = q$ for all $x \in X_2$.

Now by Proposition 10.33 we have

(7) If $i < q$, $Y \in [X_2]^\kappa$, and $|\{a_i^x : x \in Y\}| < \kappa$, then there is a $Z \in [Y]^\kappa$ such that $\langle a_i^x : x \in Z\rangle$ has fip.

We now consider some subcases.

Subcase 1.1. There exist an $X_3 \in [X_2]^\kappa$ and an $i < q$ such that $\langle a_i^x : x \in X_3\rangle$ has fip and $|\{b_i^x \cdot c_i^x : x \in X_3\}| = \kappa$. Then there is an $X_4 \in [X_3]^\kappa$ such that $\langle b_i^x \cdot c_i^x : x \in X_4\rangle$ is one-one.

Now we consider two subsubcases.

Subsubcase 1.1.1. $|\{b_i^x : x \in X_4\}| < \kappa$. By Proposition 10.33 there is an $X_5 \in [X_4]^\kappa$ such that $\langle b_i^x : x \in X_5\rangle$ has fip. Since $\langle b_i^x \cdot c_i^x : x \in X_4\rangle$ is one-one, $\{b_i^x \cdot c_i^x : x \in X_5\}$ still has size κ, and so $|\{c_i^x : x \in X_5\}| = \kappa$. Hence there is an $X_6 \in [X_5]^\kappa$ such that $\langle c_i^x : x \in X_6\rangle$ is one-one. Now

(8) for distinct $x, y \in X_6$ we have $(K_{f^{-1}(x)}\backslash F_{\alpha(x)}) \cap (K_{f^{-1}(y)}\backslash F_{\alpha(y)}) = \emptyset$.

In fact, if $\alpha(x) = \alpha(y)$, then $K_{f^{-1}(x)} \cap K_{f^{-1}(y)} = F_{\alpha(x)}$ by (5), and the conclusion follows. If $\alpha(x) \neq \alpha(y)$, (6) gives the conclusion.

It follows from (8) that $\langle c_i^x : x \in X_6\rangle$ is independent. Then X_6 is independent. For, suppose that $Y, Z \in [X_6]^{<\omega}$ and $Y \cap Z = \emptyset$. Then

$$\begin{aligned}\prod_{x\in Y} x \cdot \prod_{x\in Z} -x &= \prod_{x\in Y}\sum_{j<q}(a_j^x \cdot b_j^x \cdot c_j^x) \cdot \prod_{x\in Z}\sum_{j<q}(a_j^x \cdot (-b_j^x + -c_j^x)\\ &\geq \prod_{x\in Y}(a_i^x \cdot b_i^x \cdot c_i^x) \cdot \prod_{x\in Z}(a_i^x \cdot (-b_i^x + -c_i^x)\\ &\geq \prod_{x\in Y}(a_i^x \cdot b_i^x \cdot c_i^x) \cdot \prod_{x\in Z}(a_i^x \cdot -c_i^x)\\ &\neq 0.\end{aligned}$$

Subsubcase 1.1.2. $|\{b_i^x : x \in X_4\}| = \kappa$. Then there is an $X_5 \in [X_4]^\kappa$ such that $\langle b_i^x : x \in X_5\rangle$ is one-one. Now for each $\alpha < \mathrm{cf}(\kappa)$ let $Z_\alpha = \{b_i^x : x \in X_5$ and $\alpha(x) = \alpha\}$, and

$$G'_\alpha = \begin{cases} F_{\alpha(x)}\backslash G & \text{if } \alpha = \alpha(x) \text{ for some } x \in X_5,\\ \emptyset & \text{otherwise.}\end{cases}$$

Now $\{b_i^x : x \in X_5\} = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Z_\alpha$, and $Z_\alpha \cap Z_\beta = \emptyset$ if $\alpha \neq \beta$. Hence

$$\sup_{\alpha<\mathrm{cf}(\kappa)} |Z_\alpha| = \kappa.$$

Thus the hypothesis of (i) holds with G' in place of G and $\langle Z_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ in place of $\langle X_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$. Hence there is a system $\langle Y_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ such that

$$\forall \xi < \mathrm{cf}(\kappa)[Y_\alpha \subseteq Z_\alpha \text{ and } Y_\alpha \text{ is independent] and } \sup_{\alpha<\mathrm{cf}(\kappa)} |Y_\alpha| = \kappa.$$

Now by (6)(a) we have $G'_\alpha \cap G'_\beta = \emptyset$ if $\alpha \neq \beta$. It follows that $\bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha$ is independent. Now there is an $X_6 \in [X_5]^\kappa$ such that $\{b_i^x : x \in X_6\} = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha$. Hence $\{b_i^x : x \in X_6\}$ is independent.

Now if $|\{c_i^x : x \in X_6\}| < \kappa$, then by Proposition 10.33 there is an $X_7 \in [X_6]^\kappa$ such that $\langle c_i^x : x \in X_7\rangle$ has fip. If $|\{c_i^x : x \in X_6\}| = \kappa$, then there is an $X_7 \in [X_6]^{<\omega}$ such that $\langle c_i^x : x \in X_7\rangle$ is one-one; by (8) we get that $\langle c_i^x : x \in X_7\rangle$ is independent, hence satisfies fip. It follows now that X_7 is independent. The argument here is very similar to that in Subsubcase 1.1.1.

Subcase 1.2. For all $i < q$ and $X_3 \in [X_2]^\kappa$, if $\langle a_i^x : x \in X_3\rangle$ has fip then $|\{b_i^x \cdot c_i^x : x \in X_3\}| < \kappa$. Now

(9) $\forall i < q \exists X_3 \in [X_2]^\kappa[\langle a_i^x : x \in X_3\rangle$ has fip].

In fact, take any $i < q$. If $|\{a_i^x : x \in X_2\}| = \kappa$, we can apply (1) to get the desired X_3. Otherwise apply (7).

By (9) there is an $X_3 \in [X_2]^\kappa$ such that for all $i < q$ the sequence $\langle a_i^x : x \in X_3\rangle$ has fip. By the subcase we are in, $\forall i < q[|\{b_i^x \cdot c_i^x : x \in X_4\}| < \kappa$. For each $x \in X_3$ let $\langle d_k^x : k < r_x\rangle$ be the system of atoms of $\langle\{b_j^x \cdot c_j^x : j < q\rangle$. Applying Lemma 10.28 we get $\langle e_k^x : k < r_x\rangle$ in $\oplus_{s\in G} A_s$ such that $x = \sum_{k<r_x}(e_k^x \cdot d_k^x)$ for all $x \in X_3$. Since $|\{d_k^x : k < r_x, x \in X_3\}| < \kappa$, there is a $k < r_x$ such that $|\{e_k^x : x \in X_3\}| = \kappa$. Then by (1) and Proposition 10.37 there is an $X_4 \in [X_3]^\kappa$ such that $\langle e_k^x : x \in X_4\rangle$ is independent. By Proposition 10.33 there is an $X_5 \in [X_4]^\kappa$ such that $\langle d_k^x : x \in X_4\rangle$ satisfies fip. Hence X_5 is independent.

Case 2. $G = \emptyset$, $n = 0$. Clearly then $m > 0$. Since $H_x \cap H_y = \emptyset$ for $x \neq y$, it follows that X_2 is independent.

Case 3. $G = \emptyset$, $0 < n < m$. For all $x \in X_1$ write

$$x = \sum_{i<p_x}(b_i^x \cdot c_i^x)$$

with each $b_i^x \in \oplus_{j\in F_{\alpha(x)}} A_j$, $f^{-1}(x) \in N_{\alpha(x)}$, $c_i^x \in \oplus_{j\in K_{f^{-1}(x)}\setminus F_{\alpha(x)}} A_j$, $\langle b_i^x : i < p_x\rangle$ a partition. Choose $X_2 \in [X_1]^\kappa$ and q so that $p_x = q$ for all $x \in X_2$.

(10) For all $i < q$ and $Y \in [X_2]^\kappa$ there is a $Z \in [Y]^\kappa$ such that $\langle b_i^x : x \in Z\rangle$ has fip.

To prove (10), let $i < q$ and $Y \in [X_2]^\kappa$. If $|\{b_i^x : x \in Y\}| < \kappa$, then by Proposition 10.33 there is a Z satisfying the desired condition. If $|\{b_i^x : x \in Y\}| = \kappa$, then by (1) there is an $Z \in [Y]^\kappa$ such that $\langle b_i^x : x \in Z\rangle$ is independent, hence has fip.

Now we apply (10) q times and obtain $X_3 \in [X_2]^\kappa$ such that for all $i < q$, $\langle b_i^x : x \in X_3\rangle$ has fip.

Subcase 3.1. There is an $i < q$ such that $|\{c_i^x : x \in X_3\}| = \kappa$. By (1), let $X_4 \in [X_3]^\kappa$ be such that $\langle c_i^x : x \in X_4\rangle$ is independent. By Proposition 10.29, X_4 is independent.

Subcase 3.2. $|\{c_i^x : x \in X_3\}| < \kappa$ for all $i < q$. Rewrite each $x \in X_3$ as

$$\sum_{i<r_x} d_i^x \cdot e_i^x$$

with $d_i^x \in \oplus_{j\in F_{\alpha(x)}} A_j$, $e_i^x \in \oplus_{j\in K_{f^{-1}(x)}\setminus F_{\alpha(x)}} A_j$, $\langle e_i^x : i < r_x\rangle$ a partition. Let $X_4 \in [X_3]^\kappa$ be such that $r_x = s$ for all $x \in X_4$. Now $|\{e_i^x : x \in X_4\}| < \kappa$ for all $i < s$, so there is an $i < s$ such that $|\{d_i^x : x \in X_4\}| = \kappa$. By Proposition 10.33 there is an $X_5 \in [X_4]^\kappa$ such that $\{e_i^x : x \in X_5\}$ has fip. Then by Lemma 10.30 there is an $X_6 \in [X_5]^\kappa$ such that X_6 is independent.

Case 4. $G = \emptyset$, $0 < n = m$. Then $x \in \oplus_{i\in F_\alpha} A_i$ for every $x \in X_1$ such that $f^{-1}(x) = \alpha$. Let $g : \mathrm{cf}(\alpha) \to M_0$ be a bijection, and for each $\alpha < \mathrm{cf}(\kappa)$ let $Y_\alpha = \{x \in X_1 : f^{-1}(x) = g(\alpha)\}$. Then (i) applies to give the desired result, since the F_α's are pairwise disjoint.

Case 5. $0 < |G| < n = m$. For each $x \in X_1$ write

$$x = \sum_{i<p_x} a_i^x \cdot b_i^x$$

where $a_i^x \in \oplus_{j\in G} A_j$, $\langle a_i^x : i < p_x\rangle$ is a partition, and $b_i^x \in \oplus_{j\in F_{\alpha(x)}\setminus G} A_j$. Take $X_2 \in [X_1]^\kappa$ such that $p_x = q$ for all $x \in X_2$.

Subcase 5.1. There is an $i < q$ such that $|\{b_i^x : x \in X_2\}| = \kappa$. Let $X_3 \in [X_2]^\kappa$ be such that $\langle b_i^x : x \in X_3\rangle$ is independent. Then take $X_4 \in [X_3]^\kappa$ such that $\{a_i^x : x \in X_4\}$ has fip. By Lemma 10.29 X_4 is independent.

Subcase 5.2. $|\{b_i^x : x \in X_2\}| < \kappa$ for all $i < q$. Rewrite each $x \in X_2$ in the form

$$x = \sum_{i<r_x} c_i^x \cdot d_i^x$$

with $c_i^x \in \oplus_{j\in G} A_j$, $d_i^x \in \oplus_{j\in F_{\alpha(x)}\setminus G} A_j$, with $\langle d_i^x : i < r_x\rangle$ a partition. Then proceed as in Subcase 5.1.

Case 6. $0 < |G| = n < m$. Then for $x, y \in X_1$ with $x \neq y$ we have $K_{f^{-1}(x)} \cap K_{f^{-1}(y)} = G$. This is very similar to Case 5.

Case 7. $0 < |G| = n = m$. The result follows from Proposition 10.37. □

Corollary 10.44. *For any cardinals κ, λ, if $\omega < \mathrm{cf}(\kappa)$ and $\kappa \le \lambda$, then $\kappa \in \mathrm{Freecal}(\mathrm{Fr}(\lambda))$.* □

The following problem concerns trying to characterize the classes Freecal(A). This is Problem 38 in Monk [96].

Problem 103. *For K any nonempty set of regular cardinals, are the following conditions equivalent?*

(i) *There is a BA A such that $K = \{\kappa : \kappa \in \mathrm{Freecal}(A)$ and κ is regular$\}$.*
(ii) *The following conditions hold, where $\mu = \min K$ and $\nu = \sup K$:*
   (a) *μ is uncountable.*
   (b) *For all $\lambda \in (\mu, \nu]$, if λ is regular and is not the successor of a singular cardinal, then $\lambda \in K$.*
   (c) *For all $\lambda \in (\mu, \nu]$, if $\lambda = \sigma^+$ for some singular σ with $\mu \le \mathrm{cf}\,\sigma$, then $\lambda \in K$.*

For background and evidence for this problem, see Monk [83].

The comparison of independence with the cardinal functions already introduced is simple: $\mathrm{Ind}(A) \le \mathrm{Irr}(A)$ for every infinite BA A, and the difference can be arbitrarily large, for example in an interval algebra; it is possible to have $\mathrm{Ind}(A)$ bigger than $\pi(A)$, for example in $\mathscr{P}(\kappa)$. $\mathrm{Depth}(A)$ can be much larger than $\mathrm{Ind}(A)$, for example in the interval algebra on κ. Note that there are some close relationships between independence and cellularity, though. The main fact here is Theorem 10.1 in the Handbook, a result due to Shelah. A simple application of that theorem is that if $(2^{\mathrm{c}(A)})^+ \le |A|$, then $(2^{\mathrm{c}(A)})^+ \le \mathrm{Ind}(A)$ by Corollary 10.9 of the Handbook. In particular, $|A| \le 2^{\max(\mathrm{c}(A),\mathrm{Ind}(A))}$. And if $|A|$ is strong limit, then $|A| = \max(\mathrm{c}(A), \mathrm{Ind}(A))$.

The following is Problem 32 in Monk [96].

Problem 104. *Assume that $\rho < \nu < \kappa \le 2^\rho < \lambda \le 2^\nu$ with κ and λ regular. Is there a κ-cc BA A of power λ with no independent subset of power λ?*

Shelah [99] has several partial results concerning this problem. A partial positive solution follows from 6.8 of Shelah [99], which says

If κ is weakly inaccessible and $\langle 2^\mu : \mu < \kappa\rangle$ is not eventually constant, then there is a κ-cc BA A of size $2^{<\kappa}$ with no independent subset of size κ^+.

This gives, consistently, several examples solving Problem 106 positively. For example, take a model in which $\kappa = \aleph_\alpha$ is weakly inaccessible and $2^{\aleph_\beta} = \aleph_{\alpha+\beta+1}$ for each regular $\aleph_\beta < \kappa$. Then $2^{<\kappa} = \aleph_{\alpha+\alpha}$. For Problem 104 one can take $\rho = \aleph_1$, $\nu = \aleph_3$, and $\lambda = \aleph_{\alpha+3}$.

The above result also solves Problem 34 of Monk [96] positively.

Another result from Shelah [99] is as follows:

(14.24 Conclusion) *If $\mu = \mu^{<\mu} < \theta = \theta^{<\theta}$, then for some μ-complete, μ^+-cc forcing P, in V^P: If B is a κ-cc BA of size $\geq \lambda$, $\mu^{<\kappa} = \mu$, and λ is a regular cardinal in $(\mu, \theta]$, then λ is a free caliber of B.*

This gives a negative solution of Problem 33 of Monk [96].

Problem 36 of Monk [96] asks whether $\beth_{\omega+1} \in \mathrm{Freecal}(\overline{\mathrm{Fr}(\beth_{\omega+1})})$. By 6.11 of Shelah [99] either answer to this question is consistent.

Concerning the possibility of a complete BA with empty free caliber set, 8.1 of Shelah goes as follows:

Assume GCH in the ground model, and let P be the partial order for adding $\aleph_{\omega_1}$ Cohen reals. Then in the generic extension we have $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} = \aleph_{\omega_1+1}$, and:

There is no complete BA A of size $2^{\aleph_1}$ such that $\mathrm{Freecal}(A) = \emptyset$; in fact, every complete BA of that size has free caliber $2^{\aleph_1}$.

This solves Problem 37 of Monk [96].

As mentioned in the introduction, there are bounded versions of independence. A set $X \subseteq A$ is *m-independent* (where m is a positive integer) if for every $Y \in [X]^m$ and every $\varepsilon \in {}^Y 2$ we have $\prod_{y \in Y} y^{\varepsilon y} \neq 0$. Then we set

$$\mathrm{Ind_n}(A) = \sup\{|X| : X \subseteq A \text{ and } X \text{ is } n\text{-independent}\}.$$

This notion is briefly studied in Monk [83], where the following problem is stated which is somewhat relevant to the notion; this is Problem 39 in Monk [96].

Problem 105. *Can one prove the following in ZFC? For every $m \in \omega$ with $m \geq 2$ there is an interval algebra having a subset P of size ω_1 such that for all $Q \in [P]^{\omega_1}$, Q has m pairwise comparable elements and also m independent elements.*

The condition in this problem is shown to be consistent in Monk [83].

Rosłanowski, Shelah [00] consider the finite version of independence more extensively, proving the following results (and more):

(1) If $n \geq 2$ and λ is an infinite cardinal, then there is a BA A such that $\mathrm{Ind}_n A = \lambda = |B|$ and $\mathrm{Ind}_{n+1} A = \omega$.
(2) If λ is an infinite cardinal and n is an even integer > 2, then there is a BA A such that $\mathrm{Ind}_n A = \lambda$ and $\mathrm{Ind}(A \times A) = \omega$.

11 π-Character

First of all, note that if F is a non-principal ultrafilter on a BA A, then $\pi\chi(F) \geq \omega$. To see this, suppose that X is a finite set of non-zero elements of A which is dense in F. Choose $y \in F$ such that $y < \prod(X \cap F)$. Then choose $x \in X$ such that $x \leq y \cdot \prod\{z \in F : -z \in X\}$. This clearly gives a contradiction, whether $x \in F$ or not.

The following equivalent definition of $\pi\chi(A)$ is frequently useful.

Proposition 11.1. *For any infinite BA A, $\pi\chi(A)$ is the smallest cardinal κ such that for every ultrafilter F on A there is an $X \in [A]^{\leq\kappa}$ such that X is dense in F.*

Proof. If F is an ultrafilter on A, then $\pi\chi(F) \leq \pi\chi(A)$, and there is a subset X of A of size $\pi\chi(F)$ which is dense in F. So $\pi\chi(A)$ is a cardinal κ of the type indicated. Now suppose that $\lambda < \kappa$. Then there is an ultrafilter F on A such that $\lambda < \pi\chi(F)$; so there is no $X \in [A]^{\leq\lambda}$ which is dense in F. This shows that $\pi\chi(A)$ is the least κ with the indicated property. $\square$

Note that if A is atomic, then the set of atoms of A is dense in any ultrafilter. Hence it is possible for $A \leq B$ with $\pi\chi(A) > \pi\chi(B)$. For example, let A be an uncountable free subalgebra of $\mathscr{P}(\omega)$.

Turning to our special kinds of subalgebras, first we have:

Proposition 11.2. *If A and B are infinite BAs such that $A \leq_\pi B$, then $\pi\chi(A) \leq \pi\chi(B)$.*

Proof. Let F be any ultrafilter on A, and extend it to an ultrafilter on B. Choose $X \in B^+$ which is dense in G, with $|X| = \pi\chi(G)$. For each $x \in X$ choose $y_x \in A^+$ such that $y_x \leq x$. Then $\{y_x : x \in X\}$ is dense in F. It follows that $\pi\chi(F) \leq \pi\chi(G)$. Since F is arbitrary, this shows that $\pi\chi(A) \leq \pi\chi(B)$. $\square$

An example of A, B with $A \leq_\pi B$ and $\pi\chi(A) < \pi\chi(B)$ is given by $A = \mathrm{Finco}(\omega_1)$ and $B = \mathscr{P}(\omega_1)$; see below, where we prove that $\pi\chi(\mathrm{Finco}(\kappa)) = \omega$ and that $\pi\chi(\mathscr{P}(\kappa)) \geq \kappa$ for any infinite cardinal κ.

Proposition 11.3. *If $A \leq_{\mathrm{rc}} B$ then $\pi\chi(A) \leq \pi\chi(B)$.*

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Proof. For each $b \in B$ let $C(b)$ be the smallest element of A which is above b. Let F be any ultrafilter on A, and extend it to an ultrafilter G on B. Let $X \subseteq B^+$ be dense in G with $|X| = \pi\chi(G)$. We claim that $\{C(x) : x \in X\}$ is dense in F. For, take any $a \in F$. Then also $a \in G$, so we can choose $x \in X$ such that $x \leq a$. Then $C(x) \leq a$, as desired. Thus $\pi\chi(F) \leq \pi\chi(G)$. □

Strict inequality is possible; for example, we show below that $\pi\chi(\mathrm{Fr}(\kappa)) = \kappa$ for every infinite cardinal κ. Hence Proposition 11.8 of the Handbook gives an example.

By the diagram at the end of Chapter 2, the conclusion $\pi\chi(A) \leq \pi\chi(B)$ also holds if $A \leq_{\mathrm{free}} B$ or $A \leq_{\mathrm{proj}} B$.

For the remaining special subalgebra notions we do not know what happens.

Problem 106. *Describe the relationship between $\pi\chi(A)$ and $\pi\chi(B)$ for the various special notions of A a subalgebra of B.*

Turning to homomorphic images, let $B = \mathscr{P}(\omega)$. Using the fact that B has an independent set of size ω_1, we obtain a homomorphism f from B onto an algebra A such that A is a subalgebra of the completion of the free algebra C on ω_1 free generators $\{x_\alpha : \alpha < \omega_1\}$, and C is a subalgebra of A. Then, we claim, $\pi\chi A = \omega_1$. For, suppose that F is an ultrafilter on A, and X is a countable subset of A. Then each element of X is a countable sum of monomials in the x_α's. If we take some α with x_α not in any of these monomials, then x_α (or $-x_\alpha$) is an element of F with no element of X below it. Thus A is a homomorphic image of B and $\pi\chi(B) = \omega < \omega_1 = \pi\chi(A)$.

We turn to products. Clearly $\pi\chi(A \times B) = \max(\pi\chi(A), \pi\chi(B))$ for any infinite BAs A and B. More generally, we have:

Theorem 11.4. $\pi\chi\left(\prod^{\mathrm{w}}_{i\in I} A_i\right) = \sup_{i\in I}\pi\chi(A_i)$ *for any system $\langle A_i : i \in I\rangle$ of BAs.*

Proof. We may assume that I is infinite. Since $\mathrm{Ult}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right)$ is the one-point compactification of the disjoint union of all of the spaces $\mathrm{Ult}(A_i)$, it suffices to prove the following:

(1) Let F be the ultrafilter on $\prod^{\mathrm{w}}_{i\in I} A_i$ consisting of all $x \in \prod^{\mathrm{w}}_{i\in I} A_i$ such that $\{i \in I : x_i \neq 1\}$ is finite. Then $\pi\chi(F) = \omega$.

To prove (1), let J be any denumerable subset of I. For each $j \in J$ we define an element x^j of $\prod^{\mathrm{w}}_{i\in I} A_i$ by setting, for each $i \in I$,

$$x^j_i = \begin{cases} 0 & \text{if } j \neq i, \\ 1 & \text{if } j = i. \end{cases}$$

We claim that $\{x^j : j \in J\}$ is dense in F. To see this, take any $y \in F$. Then there is a $j \in J$ such that $y_j = 1$. So $x^j \leq y$, as desired. □

Taking each A_i in Proposition 11.4 to be the two-element BA, we get:

Corollary 11.5. *$\pi\chi(\mathrm{Finco}(\kappa)) = \omega$ for any infinite cardinal κ.*

Note that the proof of Theorem 11.4 shows that π-character is attained in $\prod^{\mathrm{w}}_{i\in I} A_i$ iff there is an $i \in I$ such that $\pi\chi\left(\prod^{\mathrm{w}}_{i\in I} A_i\right) = \pi\chi(A_i)$ and $\pi\chi(A_i)$ is attained. Using this remark, we can describe the attainment property of π-character: for each uncountable limit cardinal κ there is a BA A with π-character κ not attained: we take the weak product of free algebras of the obvious sizes. On the other hand, if $\pi\chi A = \omega$, then it is attained, since any non-principal ultrafilter has infinite π-character by our initial remark.

Turning to arbitrary products, we have:

Theorem 11.6. *If $\langle A_i : i \in I\rangle$ is a system of non-trivial BAs with $\prod_{i\in I} A_i$ infinite, then $\pi\chi\left(\prod_{i\in I} A_i\right) \geq \max(|I|, \sup_{i\in I}\pi\chi(A_i))$.*

Proof. If $i \in I$ and G is an ultrafilter on A_i, then the set $F \stackrel{\text{def}}{=} \{y \in \prod_{i\in I} A_i : y_i \in G\}$ is an ultrafilter on $\prod_{i\in I} A_i$, and a subset of $\prod_{i\in I} A_i$ dense in F clearly gives rise to a subset of A_i with no more elements which is dense in G. Hence $\pi\chi(A_i) \leq \pi\chi\left(\prod_{i\in I} A_i\right)$.

Next, assume that I is infinite; we show that $|I| \leq \pi\chi\left(\prod_{i\in I} A_i\right)$. For each subset J of I let x_J be the characteristic function of J, considered as a member of $\prod_{i\in I} A_i$. Let F be any ultrafilter on $\prod_{i\in I} A_i$ containing all elements $x_{I\setminus J}$ such that $|J| < |I|$. Then, we claim, $\pi\chi(F) \geq |I|$. In fact, suppose that $X \subseteq A^+$, X is dense in F, and $|X| < |I|$. For each $y \in X$ choose $i(y) \in I$ such that $y_{i(y)} \neq 0$. Let $J = \{i(y) : y \in X\}$. Then the element $x_{I\setminus J}$ of F is not $\geq$ any element of X, contradiction. $\square$

Taking each A_i to be the two-element BA, we get:

Corollary 11.7. *$\pi\chi(\mathscr{P}(\kappa)) = \kappa$ for every infinite cardinal κ.*

Actually, $\pi\chi$ can jump tremendously in a product. This follows in an obvious way from the following theorem, which is an observation of Douglas Peterson based on Theorem 10.13 and its proof.

Theorem 11.8. *If A is an infinite atomless BA, then $\mathrm{c}(A) \leq \pi\chi\left(\prod_{i\in\omega\setminus 1} A^{*i}\right)$ (with notation as in Theorem 10.13).*

Proof. Wlog $\mathrm{c}(A) > \omega$. Let X and f be as in Theorem 10.13 and its proof, with X uncountable. Then by that proof, $\{-f_x : x \in X\}$ generates a proper filter in $\prod_{i\in\omega\setminus 1} A^{*i}$, and we extend it to an ultrafilter F. Since A is atomless, F is nonprincipal. We claim that $\pi\chi(F) \geq |X|$; this will prove the Theorem. Suppose that $Y \subseteq \left(\prod_{i\in\omega\setminus 1} A^{*i}\right)^+$, $|Y| < |X|$, and Y is dense in F; we want to get a contradiction. There exist a $y \in Y$ and an uncountable $Z \subseteq X$ such that $y \leq -f_z$ for all $z \in Z$. Say $y_i \neq 0$. Wlog y_i has the form $a^0 \cdot a^1 \cdot \ldots \cdot a^{i-1}$, where a^j is in the jth free factor of A^{*i}. Then $a^0 \cdot a^1 \cdot \ldots \cdot a^{i-1} \leq g_0(z) + \cdots + g_{i-1}(z)$ for all

$z \in Z$, so there is a $j < i$ such that $a^j \leq g_j(z)$. This being true for all $z \in Z$, and Z being infinite, it follows that there exist a $j < i$ and two distinct $z, w \in Z$ such that $a^j \leq g_j(z)$ and $a_j \leq g_j(w)$. Since $z \cdot w = 0$, it follows that $a^j = 0$, contradiction. □

The possibility of doing the above with interval algebras, which naturally arose in Chapter 10, is not so interesting here, since interval algebras can have high π-character (see the end of this chapter).

We turn to ultraproducts, giving some results of Douglas Peterson. Since $\pi\chi$ is a sup-min function, Theorems 6.5–6.7 hold. An additional result of the sort described in these theorems, with a proof using independent matrices, is the following theorem of Peterson: *If $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, F is a regular ultrafilter on I, and* $\text{ess.sup}^F_{i\in I}|A_i| \leq 2^{|I|}$, *then* $\pi\chi\left(\prod_{i\in I} A_i/F\right) \geq \text{cf}(2^{|I|})$. From 6.5–6.7 the following theorem follows, with a proof similar to that of Theorem 4.18:

Theorem 11.9. (GCH) *Suppose that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and F is a regular ultrafilter on I. Then* $\pi\chi\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \pi\chi A_i/F\right|$.

As usual, the result of Donder shows that it is consistent to always have $\geq$. Peterson has shown that it is consistent to have $<$ in Theorem 11.9 in the absence of GCH. See also Chapter 4 for an independent solution by Shelah. For $>$ we have the following extension of Theorem 11.8, which shows that $\pi\chi$ can jump very much in an ultraproduct.

Theorem 11.10. *If A is an infinite atomless BA, $X \subseteq A^+$ is disjoint, and F is a nonprincipal ultrafilter on ω, then, with notation as in Theorem* 10.13, $|X| \leq \pi\chi\left(\prod_{i\in\omega\setminus 1} A^{*i}/F\right)$.

Proof. By Theorem 10.13, if N is a finite subset of X then $\{n : \prod_{x\in N} -f_x(n) = 0\}$ is finite, and hence $\prod_{x\in N} -f_x/F \neq 0$. Thus $\{-f_x/F : x \in X\}$ has the finite intersection property, and we can let G be an ultrafilter on $\prod_{i\in\omega\setminus 1} A^{*i}/F$ containing this set. We claim that $\pi\chi(G) \geq |X|$, which will prove the theorem. To get a contradiction, suppose that $Y \subseteq (\prod_{i\in\omega\setminus 1} A^{*i}/F)^+$, $|Y| < |X|$, and Y is dense in G. Then there is a $y/F \in Y$ and an uncountable $X' \subseteq X$ such that $y/F \leq -f_x/F$ for all $x \in X'$. We may assume that $y_i \neq 0$ for all $i \in \omega$, and further that each y_i has the form $a_i^0 \cdot a_i^1 \cdot \ldots \cdot a_i^{i-1}$ with a_i^j from the jth factor. Now for any $x \in X'$ we have $y/F \leq -f_x/F$, and so there is an $i \in \omega$ such that $y_i \leq -f_x i$. Hence there is an $i \in \omega$ and an uncountable $X'' \subseteq X'$ such that $y_i \leq -f_x i$ for all $x \in X''$. Now we proceed to a contradiction as in the proof of Theorem 11.8. □

Next we describe π-character for free products:

Theorem 11.11. *If $\langle A_i : i \in I\rangle$ is a system of BAs each with at least* 4 *elements, then* $\pi\chi(\oplus_{i\in I} A_i) = \max(|I|, \sup_{i\in I}\pi\chi(A_i))$.

Proof. For brevity let $B = \oplus_{i\in I} A_i$. First take any $i \in I$; we show that $\pi\chi(A_i) \le \pi\chi(B)$. Let F be any ultrafilter on A_i, and extend F to an ultrafilter G on B. Suppose $X \subseteq B$ is dense in G. We may assume that each $x \in X$ has the form

$$x = \prod_{j\in M_x} y_j^x \tag{1}$$

for some finite subset M_x of I, where $y_j^x \in A_j$ for every $j \in M_x$. Now define $Y = \{y_i^x : x \in X, i \in M_x\}$. Then clearly Y is dense in F and $|Y| \le |X|$. This proves that $\pi\chi(A_i) \le \pi\chi(B)$.

Next, we show that $|I| \le \pi\chi(B)$, where we assume that I is infinite. For each $i \in I$ choose $a_i \in A_i$ such that $0 < a_i < 1$. Let F be an ultrafilter on B such that $a_i \in F$ for each $i \in I$; clearly such an ultrafilter exists. Suppose that $X \subseteq B$ is dense in F; we may assume that each $x \in X$ has the form (1) indicated above. Clearly then, by the free product property, we must have $|X| \ge |I|$.

Now let F be an ultrafilter on B. Then for each $i \in I$, $F \cap A_i$ is an ultrafilter on A_i, and so there is an $X_i \subseteq A_i$ of cardinality $\le \pi\chi(A_i)$ which is dense in $F \cap A_i$. Let

$$Y = \left\{ y : \text{there is a finite } J \subseteq I \text{ and a } b \text{ in } \textstyle\prod_{j\in J} X_j \text{ such that } y = \displaystyle\prod_{j\in J} b_j \right\}.$$

Clearly $|Y| \le \max(|I|, \sup_{i\in I}(A_i))$ and Y is dense in F, as desired. ☐

Corollary 11.12. *$\pi\chi(\mathrm{Fr}(\kappa)) = \kappa$ for every infinite cardinal κ.* ☐

Next we discuss the behaviour of $\pi\chi$ under unions.

Theorem 11.13. *Suppose that $\langle A_\alpha : \alpha < \kappa\rangle$ is a strictly increasing sequence of BAs with union B, where κ is regular. Let $\lambda = \sup_{\alpha<\kappa} \pi\chi(A_\alpha)$. Then $\pi\chi(B) \le \sum_{\alpha<\kappa} \pi\chi(A_\alpha) \le \max(\kappa, \lambda)$. Assume in addition that $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit $\alpha < \kappa$. Then $\pi\chi(B) \le \lambda^+$.*

Proof. Let F be an ultrafilter on B. Choose $X_\alpha \subseteq A_\alpha$ which is dense in $F \cap A_\alpha$, with $|X_\alpha| = \pi\chi(F \cap A_\alpha)$, for each $\alpha < \kappa$. Then $\bigcup_{\alpha<\kappa} X_\alpha$ is dense in F, and $|\bigcup_{\alpha<\kappa} X_\alpha| \le \sum_{\alpha<\kappa} \pi\chi(A_\alpha)$. So $\pi\chi(B) \le \sum_{\alpha<\kappa} \pi\chi(A_\alpha) \le \max(\kappa, \lambda)$.

Now we make the additional assumption indicated, and suppose that $\pi\chi(B) > \lambda^+$. Let F be an ultrafilter on B such that $\pi\chi(F) > \lambda^+$. Thus $\kappa > \lambda^+$ by the first part of this proof. Let $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda^+\}$. So, S is stationary in κ. For each $\alpha < \kappa$ let $X_\alpha \subseteq A_\alpha$ be dense in $F \cap A_\alpha$ with $|X_\alpha| \le \lambda$. For $\alpha \in S$ we then have $X_\alpha \subseteq A_{f(\alpha)}$ for some $f\alpha < \alpha$. Therefore f is constant, say equal to β, on some stationary subset of S. So X_β is dense in F, contradicting $\pi\chi(F) > \lambda^+$. ☐

In contrast to Theorem 6.4, we did not assert in 11.13 that $\kappa \le 2^\lambda$. In fact, for any infinite cardinal κ there is a strictly increasing continuous sequence $\langle A_\alpha : \alpha < \kappa\rangle$ of BAs such that $\pi\chi(A_\alpha) = \omega$ for all $\alpha < \kappa$. Namely, take a strictly increasing

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continuous sequence of subalgebras of Finco(κ) with union Finco(κ); recall that if $A \le$ Finco(κ), then A is isomorphic to Finco(λ) for some $\lambda \le \kappa$. (In Monk [90], $\kappa \le 2^\lambda$ was mistakenedly asserted.)

The upper bound λ^+ mentioned in Theorem 11.13 can be attained – take a sequence of free algebras.

Problem 107. *What is the relationship between $\pi\chi(A)$ and $\pi\chi(B)$ for A the one-point gluing of B?*

Proposition 11.14. $\pi\chi(\mathrm{Dup}(A)) \le \pi\chi(A)$ *for any infinite BA A.*

Proof. Suppose that F is a nonprincipal ultrafilter on Dup(A). By Proposition 1.19 there is an ultrafilter G on A such that $F = \{(a, X) \in \mathrm{Dup}(A) : a \in G\}$. Let M be dense in G with $|M| \le \pi\chi(A)$. We claim that $\{(c, \mathcal{S}(c)\backslash\{G\}) : c \in M\}$ is dense in F. For, let $(a, X) \in F$, with $a \in G$. Then $\mathcal{S}(a)\backslash(X \cup \{G\})$ is finite. For each $H \in \mathcal{S}(a)\backslash(X \cup \{G\})$ choose $b_H \in G\backslash H$. Let $d = a \cdot \prod\{b_H : H \in \mathcal{S}(a)\backslash(X \cup \{G\})\}$. Then $d \in G$. Choose $c \in M$ such that $c \le d$. Then $(c, \mathcal{S}(c)\backslash\{G\}) \le (a, X)$, as desired. □

Problem 108. *Characterize* $\pi\chi(\mathrm{Dup}(A))$ *in terms of A.*

Next, we consider moderate products.

Proposition 11.15. *Suppose that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, each A_i being a field of subsets of J_i. Assume that I is infinite and $J_i \cap J_j = \emptyset$ for all $i \ne j$. Let B be a field of subsets of I containing all the finite sets. Then*

$$\pi\chi\left(\prod_{i\in I}^{B} A_i\right) = \max\{\pi\chi(B), \sup_{i\in I} \pi\chi(A_i)\}.$$

Proof. We use some of the notation in the treatment of moderate products in Chapter 1. For brevity let $C = \prod_{i\in I}^B A_i$. Let H be a nonprincipal ultrafilter on B. Set $G = \langle\{\overline{b} : b \in H\}\rangle^{\mathrm{fi}}_C$.

(1) G is an ultrafilter on C.

For, suppose that $h(b, F, a)$ is any element of C, with $h(b, F, a)$ normal. If $b \in H$, then clearly $h(b, F, a) \in G$. Suppose that $I\backslash b \in H$. Now $K\backslash h(b, F, a) = h(I\backslash(b \cup F), F, a')$ for a certain a'. Since H is nonprincipal, we have $I\backslash F \in H$. So $I\backslash(b\cup F) \in H$ too, and consequently $K\backslash h(b, F, a) \in G$. This proves (1).

Let $X \subseteq C^+$ be dense in G, with $|X| = \pi\chi(G)$. For each $x \in X$ write $x = h(b_x, F_x, a^x)$. Let $Y = \{b_x \cup F_x : x \in X\}\backslash\{0\}$. We claim that Y is dense in H. For, suppose that $c \in H$. Choose $x \in X$ such that $x \subseteq \overline{c}$. Clearly then $b_x \cup F_x \subseteq c$. From this it follows that $\pi\chi(H) \le \pi\chi(G)$. Since H is arbitrary, we get $\pi\chi(B) \le \pi\chi(C)$.

Next, suppose that $i \in I$ and H is a nonprincipal ultrafilter on A_i. Let $G = \{c \in C : c \cap A_i \in H\}$. Clearly G is an ultrafilter on C. Let $X \subseteq C^+$ be dense in G, with $|X| = \pi\chi(G)$. Let $Y = \{x \cap A_i : x \in X\} \backslash \{0\}$. We claim that Y is dense in H. For, let $d \in H$. So also $d \in G$, so there is an $x \in X$ such that $x \leq d$. Now $d \subseteq A_i$, so $x = x \cap A_i$. So, indeed, Y is dense in H. It follows that $\pi\chi(H) \leq \pi\chi(G)$. Hence $\pi\chi(A_i) \leq \pi\chi(C)$.

This proves $\geq$ in the proposition. Suppose that actually $>$ holds. Let G be an ultrafilter on C such that $\pi\chi(G) > \max\{\pi\chi(B), \sup_{i \in I} \pi\chi(A_i)\}$.

Case 1. $\forall i \in I[J_i \notin G]$. Let $H = \{b \in B : \overline{b} \in G\}$. Thus H is a nonprincipal ultrafilter on B. Let $X \subseteq B^+$ be dense in H with $|X| = \pi\chi(H) < \pi\chi(G)$. We claim that $\{\overline{b} : b \in X\}$ is dense in G. For, suppose that $h(b, F, a) \in G$. Then $\overline{b} \in G$. Otherwise we get $h(b, F, a) \backslash \overline{b} \in G$, so that $\sum_{i \in F} a_i \in G$. This implies that $J_i \in G$ for some $i \in F$, contradiction. Now choose $c \in X$ such that $c \leq b$. Then $\overline{c} \leq \overline{b} \leq h(b, F, a)$. This proves the claim, which gives a contradiction.

Case 2. There is a (unique) $i \in I$ such that $J_i \in G$. Let $H = \{a \in A_i : a \in G\}$. Then H is a nonprincipal ultrafilter on A_i. Let $X \subseteq A_i^+$ be dense in H with $|X| = \pi\chi(H) < \pi\chi(G)$. We claim that X is also dense in G (contradiction). For, suppose that $h(b, F, a) \in G$. Then also $J_i \cap h(b, F, a) \in G$, and so $J_i \cap h(b, F, a) \in H$. Choose $x \in X$ with $x \subseteq J_i \cap h(b, F, a)$. So $x \subseteq h(b, F, a)$, as desired. □

For the exponential we have:

Proposition 11.16. $\pi\chi(A) \leq \pi\chi(\mathrm{Exp}(A))$ *for any infinite BA A.*

Proof. From Proposition 1.22(vi) and Sikorski's extension criterion it follows that there is a homomorphism f from $\mathrm{Exp}(A)$ onto A such that $f(\mathscr{V}(\mathcal{S}(a))) = a$ for all nonzero $a \in A$. Now suppose that D is an ultrafilter on A. Then $f^{-1}[D]$ is an ultrafilter on $\mathrm{Exp}(A)$, and we claim that $\pi\chi(D) \leq \pi\chi(f^{-1}[D])$. This will prove the proposition.

Let $X \subseteq \mathrm{Exp}(A)^+$ be dense in $f^{-1}[D]$, with $|X| = \pi\chi(f^{-1}[D])$. By Proposition 1.21 we may assume that each $x \in X$ has the form

$$x = \mathscr{V}(\mathcal{S}(a_0^x)) \cap \cdots \cap \mathscr{V}(\mathcal{S}(a_{m_x-1}^x)) \cap -\mathscr{V}(\mathcal{S}(b_0^x)) \cap \cdots \cap -\mathscr{V}(\mathcal{S}(b_{n_x-1}^x)),$$

with clear assumptions. Since each $x \in X$ is nonzero, from Proposition 1.22(iii) it follows that $a_0^x \cdot \ldots \cdot a_{m_x-1}^x$ is nonzero. Let

$$Y = \{a_0^x \cdot \ldots \cdot a_{m_x-1}^x : x \in X\}.$$

We claim that Y is dense in D; this will finish the proof.

Take any $c \in D$. So $\mathscr{V}(\mathcal{S}(c)) \in f^{-1}[D]$. Choose $x \in X$ such that $x \leq \mathscr{V}(\mathcal{S}(c))$. Thus

$$\mathscr{V}(\mathcal{S}(a_0^x)) \cap \cdots \cap \mathscr{V}(\mathcal{S}(a_{m_x-1}^x)) \cap -\mathscr{V}(\mathcal{S}(b_0^x)) \cap \cdots \cap -\mathscr{V}(\mathcal{S}(b_{n_x-1}^x)) \cdot -\mathscr{V}(\mathcal{S}(c)) = 0.$$

Since $x \neq 0$, it follows from Proposition 1.21(vi) that $a_0^x \cdot \ldots \cdot a_{m_x-1}^x \leq c$, as desired. □

An example with $<$ in Proposition 11.16 is given by the example worked out in Chapter 1, with κ an uncountable cardinal. We assume the notation there. By Corollary 11.5 we get $\pi\chi(A) = \omega$. We claim that $\pi\chi(\mathrm{Exp}(A)) \geq \kappa$. Let D be the ultrafilter on $\mathrm{Exp}(A)$ containing the set

$$\{a : -a \text{ is an atom}\} \cup \{x_\alpha : \alpha < \kappa\}.$$

Suppose that $X \subseteq \mathrm{Exp}(A)^+$ is dense in D and $|X| < \kappa$. Each $y \in X$ can be written in the form

$$y = F_y + m_y \cdot -G_y,$$

where m_y is a monomial in $\{x_\alpha : \alpha < \kappa\}$ and F_y and G_y are finite sums of atoms. Take $\beta < \kappa$ such that x_β is not in the support of any monomial m_y with $y \in X$, and also

$$\beta \notin \bigcup\{\Gamma : \{z_\Gamma\} \in F_y \text{ for some } y \in X\}.$$

Take $y \in X$ such that $y \leq x_\beta$. By the last condition on β we must have $F_y = \emptyset$. But then this implies that the monomial $m_y \cdot -x_\beta$ is finite, contradiction.

Concerning the derived functions of π-character, the first result is that $\mathrm{t}(A) = \pi\chi_{\mathrm{H}+}(A) = \pi\chi_{\mathrm{h}+}(A)$, where $\pi\chi_{\mathrm{h}+}(A) = \sup\{\pi\chi(F, Y) : F \in Y,\ Y \subseteq \mathrm{Ult}(A)\}$, and for where $\pi\chi_{\mathrm{h}+}(A) = \sup\{\pi\chi(F, Y) : F \in Y,\ Y \subseteq \mathrm{Ult}(A)\}$, and for any point x of any space X, $\pi\chi(x, X)$ is defined to be $\min\{|M| : M$ is a collection of non-empty open subsets of X and for every neighborhood U of x there is a $V \in M$ such that $V \subseteq U\}$. Such a set M is called a *local π-base* for x.

It is also convenient for this proof to have an algebraic version of free sequences. Let A be a BA. A *free sequence* in A is a sequence $\langle x_\xi : \xi < \alpha\rangle$ of elements of A such that if F and G are finite subsets of α, and $\forall\xi \in F\forall\eta \in G[\xi < \eta]$, then $\prod_{\eta\in F} x_\eta \cdot \prod_{\eta\in G} -x_\eta \neq 0$. Note that by taking $G = \emptyset$ it follows that $\langle x_\xi : \xi < \alpha\rangle$ has fip. Taking $F = \emptyset$ we see that $\langle -x_\xi : \xi < \alpha\rangle$ has fip.

Proposition 11.17. *For any infinite BA and any infinite ordinal α, the following are equivalent:*

(i) *A has a free sequence of length α.*
(ii) $\mathrm{Ult}(A)$ *has a free sequence of length α.*

Proof. (i)$\Rightarrow$(ii): Suppose that $\langle x_\xi : \xi < \alpha\rangle$ is a free sequence in A. For each $\xi < \alpha$ let F_ξ be an ultrafilter containing $\{x_\eta : \eta \leq \xi\} \cup \{-x_\eta : \xi < \eta < \alpha\}$. This is possible by the definition of free sequence. To show that $\langle F_\xi : \xi < \alpha\rangle$ is a free sequence in $\mathrm{Ult}(A)$, suppose to the contrary that $\xi < \alpha$ and $G \in \overline{\{F_\eta : \eta < \xi\}} \cap \overline{\{F_\eta : \xi \leq \eta < \alpha\}}$. Now $-x_\xi \in F_\eta$ for all $\eta < \xi$, so also $-x_\xi \in G$. Hence there is an η with $\xi \leq \eta < \alpha$ and $-x_\xi \in F_\eta$. This contradicts the definition of F_η.

Conversely, let $\langle F_\xi : \xi < \alpha\rangle$ be a free sequence in $\mathrm{Ult}(A)$. Then by the definition of free sequences in spaces, for each $\xi < \alpha$ there is a $y_\xi \in A$ such that $\{F_\eta : \eta < \xi\} \subseteq \mathcal{S}(-y_\xi)$ and $\{F_\eta : \xi \leq \eta\} \subseteq \mathcal{S}(y_\xi)$. Let $x_\xi = y_{1+\xi}$ for every $\xi < \alpha$.

Then $\langle x_\xi : \xi < \alpha\rangle$ is a free sequence in A. In fact, suppose that M and N are finite subsets of α and $\forall\xi \in M \forall\eta \in N[\xi < \eta]$.

Case 1. $M = \emptyset$. Now for any $\eta \in N$ we have $0 < 1 + \eta$, so $-x_\eta = -y_{1+\eta} \in F_0$. Hence $\prod_{\eta\in N} -x_\eta \in F_0$ and consequently $\prod_{\eta\in N} -x_\eta \neq 0$.

Case 2. $M \neq \emptyset$. Let ξ be the greatest member of M. For any $\rho \in M$ we have $\rho \leq \xi$, hence $1 + \rho \leq 1 + \xi$, and hence $x_\rho = y_{1+\rho} \in F_{1+\xi}$. For any $\eta \in N$ we have $\xi < \eta$, hence $1 + \xi < 1 + \eta$, and hence $-x_\eta = -y_{1+\eta} \in F_{1+\xi}$. Thus $\prod_{\rho\in M} x_\rho \cdot \prod_{\eta\in N} -x_\eta$ is a member of $F_{1+\xi}$, and hence is nonzero. □

This equivalence shows, in particular, that $\mathrm{Ind}(A) \leq \mathrm{t}(A)$. Note that tightness in these two free sequence senses has the same attainment properties: one is attained iff the other is.

Theorem 11.18. (Shapirovskiĭ) *For any infinite BA A we have* $\mathrm{t}(A) = \pi\chi_{\mathrm{H}+}(A) = \pi\chi_{\mathrm{h}+}(A)$.

Proof. First we show $\mathrm{t}(A) \leq \pi\chi_{\mathrm{h}+}(A)$. For brevity let $\kappa = \pi\chi_{\mathrm{h}+}(A)$. Suppose that $F \subseteq \bigcup Y$ with $Y \cup \{F\} \subseteq \mathrm{Ult}(A)$; we want to find $Z \subseteq Y$ with $|Z| \leq \kappa$ and $F \subseteq \bigcup Z$. By the definition of $\pi\chi$, let M be a collection of nonempty open subsets of $Y \cup \{F\}$ such that M is a local π-base for F in $Y \cup \{F\}$. We may assume that $F \notin Y$.

(1) $\forall a \in F[\mathcal{S}(a) \cap Y \neq \emptyset]$.

In fact, assume that $a \in F$. Now $F \subseteq \bigcup Y$, so there is a $G \in Y$ such that $a \in G$. Thus $G \in \mathcal{S}(a) \cap Y$, proving (1).

(2) $\forall V \in M[V \cap Y \neq \emptyset]$.

For, take any $V \in M$. Then V is a nonempty open subset of $Y \cup \{F\}$, so there is an $a \in A$ such that $\emptyset \neq \mathcal{S}(a) \cap (Y \cup \{F\}) \subseteq V$. By (1) it follows that $V \cap Y \neq \emptyset$.

Now let Z be a subset of Y of size at most κ such that $V \cap Z \neq \emptyset$ for all $V \in M$; this is possible by (2). We claim that $F \subseteq \bigcup Z$, showing that $\mathrm{t}(A) \leq \kappa$. For, take any $a \in F$. Choose $V \in M$ such that $V \subseteq \mathcal{S}(a)$. Then take $G \in Z \cap V$. Then $G \in \mathcal{S}(a)$, so $a \in G$, as desired.

Next we show that $\pi\chi_{\mathrm{h}+}(A) \leq \pi\chi_{\mathrm{H}+}(A)$. Given $Y \subseteq \mathrm{Ult}(A)$, let $\overline{Y}$ be the closure of Y, and recall from the duality theory that $\overline{Y}$ corresponds to a homomorphic image of A. So, we just need to show that $\pi\chi(Y) \leq \pi\chi(\overline{Y})$. Let $y \in Y$, and let M be a local π-base for y in $\overline{Y}$ with $|M| \leq \pi\chi(\overline{y})$. It suffices now to show that $\{U \cap Y : U \in M\}$ is a local π-base for y in Y. Suppose that W is a neighborhood of y in Y. Choose $a \in A$ so that $y \in \mathcal{S}(a) \cap Y \subseteq W$. Then $\mathcal{S}(a) \cap \overline{Y}$ is a neighborhood of y in $\overline{Y}$, so there is a $V \in M$ such that $V \subseteq \mathcal{S}(a) \cap \overline{Y}$. Then $V \cap Y \subseteq \mathcal{S}(a) \cap Y \subseteq W$, a desired.

Finally, we show that $\pi\chi_{\mathrm{H}+}(A) \leq \mathrm{t}(A)$. Note that if f is a homomorphism from A onto a BA B, $\langle b_\xi : \xi < \kappa\rangle$ is a free sequence in B, and $f(a_\xi) = b_\xi$ for all $\xi < \kappa$, then $\langle a_\xi : \xi < \kappa\rangle$ is a free sequence in A. Hence it suffices to show that if

B is a homomorphic image of A, F is an ultrafilter on B, and $\kappa = \pi\chi(F, B)$, then there is a free sequence of length κ in B. We may assume that κ is infinite, and hence F is not isolated in $\mathrm{Ult}(B)$. We construct $\langle b_\xi : \xi < \kappa\rangle$ by recursion. Suppose that b_η has been constructed for all $\eta < \xi$ so that the following conditions hold:

$(3)_\xi$ $b_\eta \in F$ for all $\eta < \xi$.

$(4)_\xi$ If $M, N \in [\xi]^{<\omega}$ and $\forall\eta \in M \forall\rho \in N[\eta < \rho]$, then $\prod_{\eta\in M} b_\eta \cdot \prod_{\rho\in N} -b_\rho \neq 0$.

Let $\mathscr{A}$ be the collection of all products described in $(4)_\xi$. Then $|\mathscr{A}| < \kappa$. It follows that $\{\mathcal{S}(u) : u \in \mathscr{A}\}$ is not a local π-base for F. Hence there is a neighborhood U of F such that $\forall u \in \mathscr{A}[\mathcal{S}(u) \not\subseteq U]$. Say $F \in \mathcal{S}(b_\xi) \subseteq U$. Then $\forall u \in \mathscr{A}[u \not\leq b_\xi]$. Now we verify $(3)_{\xi+1}$ and $(4)_{\xi+1}$. Actually $(3)_{\xi+1}$ is obvious. For $(4)_{\xi+1}$, suppose that $M, N \in [\xi + 1]^{<\omega}$ and $\forall\eta \in M\forall\rho \in N[\eta < \rho]$. If $\xi \notin M \cup N$, then $(4)_\xi$ gives the desired conclusion. Suppose that $\xi \in N$. Then the desired conclusion is clear. Finally, suppose that $\xi \in M$. It follows that $N = \emptyset$, and again the conclusion is clear. □

Note from the proof of Theorem 11.18 that one of $\pi\chi_{\mathrm{h}+}$ and $\pi\chi_{\mathrm{H}+}$ is attained iff the other is; and if $\pi\chi_{\mathrm{h}+}$ is attained, then so is t, in the free sequence sense.

It is possible to have $\pi\chi_{\mathrm{S}+}(A) > \pi\chi(A)$; this is true, for example, for $A = \mathscr{P}(\omega)$, using the fact that $\mathscr{P}(\omega)$ has a free subalgebra of size 2^ω.

Clearly $\pi\chi_{\mathrm{S}-}(A) = \pi\chi_{\mathrm{H}-}(A) = \omega$. On the other hand, $\pi\chi_{\mathrm{h}-}(A) = 1$ for any infinite BA A, since $\mathrm{Ult}(A)$ has a denumerable discrete subspace. If B is dense in A, then $\pi\chi(B) \leq \pi\chi(A)$. In fact, if F is an ultrafilter on A, let $X \subseteq A$ be dense in F with $|X| = \pi\chi(F)$. Wlog $X \subseteq B$. Hence X is dense in $F \cap B$, so $\pi\chi(F \cap B) \leq \pi\chi(F)$. This shows that, indeed, $\pi\chi(B) \leq \pi\chi(A)$. It is possible that $\pi\chi(B) < \pi\chi(A)$ when B is dense in A. For example, let A be the interval algebra on an uncountable cardinal κ and let B be $\mathrm{Finco}(\kappa)$; see the description of $\pi\chi$ for interval algebras below. These comments show that ${}_{\mathrm{d}}\pi\chi_{\mathrm{S}+}(A) = \pi\chi(A)$, but there is an example with ${}_{\mathrm{d}}\pi\chi_{\mathrm{S}-}(A) < \pi\chi(A)$ (contradicting a statement in Monk [90]).

The function $\pi\chi_{\mathrm{inf}}$ and related functions were discussed in Chapter 6.

We now consider the relationships of $\pi\chi$ with the other cardinal functions so far considered. Clearly $\pi\chi(A) \leq \pi(A)$ for any infinite BA A. The difference between $\pi\chi$ and π can be large, for example in a finite-cofinite algebra: see Corollary 11.5. $\pi\chi(A) > \mathrm{d}(A)$ for some free algebras A; a free algebra also shows that $\pi\chi(A)$ can be greater than Length$A$. It is easy to construct an example where $\pi\chi$ is much smaller than Ind. In fact, let A be a free BA on κ free generators. Then we construct a sequence $\langle B_n : n \in \omega\rangle$ of algebras by recursion. Let $B_0 = A$. Having constructed B_n, let B_{n+1} be an extension of B_n obtained by adding for each ultrafilter F on B_n an element $0 \neq y^n_F$ such that $y^n_F \leq b$ for all $b \in F$; it is easy to see that this is possible. Let $C = \bigcup_{n\in\omega} B_n$. Then $\mathrm{Ind}(C) \geq \kappa$, while $\pi\chi(C) = \omega$. For, let G be any ultrafilter on C. Then $\{y^n_{G\cap B_n} : n \in \omega\}$ is dense in G, showing that $\pi\chi(G) \leq \omega$.

If A is the interval algebra on an uncountable cardinal, then $\pi\chi(A) > \mathrm{Ind}(A)$, while $\mathrm{Depth}(A) > \pi\chi(A)$ for A the interval algebra on $1 + \omega^* \cdot (\kappa + 1)$; both of these results are clear on the basis of the description of $\pi\chi$ for interval algebras given at the end of this chapter.

There are two interesting positive results concerning the relationship of $\pi\chi$ with our earlier cardinal functions. The first of these is true for arbitrary non-discrete regular Hausdorff spaces, with no complications in the proof from the BA case. A *π-basis* for a point x in a space X is a collection $\mathscr{B}$ of nonempty open sets such that for every neighborhood U of x there is a $V \in \mathscr{B}$ such that $V \subseteq U$. $\pi\chi(x, X)$ is the least size of a π-basis for x, and $\pi\chi(X) = \sup_{x\in X} \pi\chi(x, X)$.

Theorem 11.19. $\mathrm{d}(X) \le \pi\chi(X)^{\mathrm{c}(X)}$ *for any non-discrete regular Hausdorff space X.*

Proof. For each $x \in X$ let $\mathcal{O}_x$ be a family of non-empty open subsets of X such that $|\mathcal{O}_x| \le \pi\chi(X)$ and for every neighborhood U of x there is a $V \in \mathcal{O}$ such that $V \subseteq U$. Now we define subsets $Y_\alpha \subseteq X$ and collections $\mathscr{P}_\alpha$ of open sets for $\alpha < (\mathrm{c}(X))^+$ by induction so that the following conditions hold:

(1) $|Y_\alpha| \le (\pi\chi(X))^{\mathrm{c}(X)}$;

(2) $|\mathscr{P}_\alpha| \le (\pi\chi(X))^{\mathrm{c}(X)}$.

Fix $x_0 \in X$. Set $Y_0 = \{x_0\}$ and $\mathscr{P}_0 = \mathcal{O}_{x_0}$. Suppose that Y_β and $\mathscr{P}_\beta$ have been defined for all $\beta < \alpha$. If α is a limit ordinal, set $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ and $\mathcal{P}_\alpha = \bigcup_{\beta<\alpha} \mathscr{P}_\beta$. Now suppose that α is a successor ordinal $\beta + 1$. Set

$$Q_\alpha = \{\mathcal{R} : \mathcal{R} \subseteq \mathscr{P}_\beta, |\mathcal{R}| \le \mathrm{c}(X), \overline{\bigcup \mathcal{R}} \ne X\}$$

Clearly $|Q_\alpha| \le \pi\chi(X)^{\mathrm{c}(X)}$. For every $\mathcal{R} \in Q_\alpha$ choose $\varphi_\mathcal{R} \in X \backslash \overline{\bigcup \mathcal{R}}$ and put

$$Y_\alpha = Y_\beta \cup \{\varphi_\mathcal{R} : \mathcal{R} \in Q_\alpha\}, \qquad \mathscr{P}_\alpha = \bigcup_{x\in Y_\alpha} \mathcal{O}_x.$$

This finishes the definition. Now we claim

(3) $L \stackrel{\text{def}}{=} \bigcup_{\alpha<(\mathrm{c}(X))^+} Y_\alpha$ is dense in X.

Since $|L| \le (\pi\chi(X))^{\mathrm{c}(X)}$, (3) finishes the proof. To prove (3), suppose that it is not true. Then by regularity, there is an open U such that $\overline{L} \subseteq U \subseteq \overline{U} \ne X$. Set $\mathcal{P}^* = \bigcup_{x\in L} \mathcal{O}_x$, and $\mathcal{T} = \{V \in \mathcal{P}^* : V \subseteq U\}$. Let $\mathcal{R}$ be a maximal disjoint subset of $\mathcal{T}$. Then $L \subseteq \overline{\bigcup \mathcal{R}}$; for, if $x \in L\backslash\overline{\bigcup \mathcal{R}}$, then $x \in U\backslash\overline{\bigcup \mathcal{R}}$, which is open, so there is a $V \in \mathcal{O}_x$ such that $V \subseteq U\backslash\overline{\bigcup \mathcal{R}}$, and $\mathcal{R} \cup \{V\}$ contradicts the maximality of $\mathcal{R}$. Also, $\overline{\bigcup \mathcal{R}} \subseteq \overline{\bigcup \mathcal{T}} \subseteq \overline{U} \ne X$. Since $\mathcal{R} \subseteq \mathscr{P}_\beta$ for some $\beta < (\mathrm{c}(X))^+$, it follows that $\mathcal{R} \in Q_\beta$ for some $\beta < (\mathrm{c}(X))^+$, and hence we get $\varphi_\mathcal{R} \in X\backslash\overline{\bigcup \mathcal{R}} \subseteq X\backslash L$, contradiction. □

Theorem 11.20. $\mathrm{d}(A) \cdot \pi\chi(A) = \pi(A)$ *for any infinite BA A.*

Proof. We already know that $\mathrm{d}(A) \le \pi(A)$ and $\pi\chi(A) \le \pi(A)$. Now let D be a dense subset of $\mathrm{Ult}(A)$ with $|D| = \mathrm{d}(A)$, and for each $F \in D$ let X_F be a local π-base for F of size $\le \pi\chi(A)$. Clearly $\bigcup_{F\in D} X_F$ is dense in A, as desired. □

There are many problems concerning the functions $\pi\chi_{\mathrm{Sr}}$ and $\pi\chi_{\mathrm{Hr}}$, so we restrict ourselves to the following vague questions.

Problem 109. *Give a purely cardinal number characterization of* $\pi\chi_{\mathrm{Sr}}$.

Problem 110. *Give a purely cardinal number characterization of* $\pi\chi_{\mathrm{Hr}}$.

Concerning $\pi\chi$ for special classes of algebras, we first give a description of what happens for interval algebras. Let L be a linearly ordered set with first element 0, and let A be the interval algebra on L. The ultrafilters on A are in one-one correspondence with the end segments of L different from L itself; corresponding to the ultrafilter F is the segment $\{a \in L : [0, a) \in F\}$. Given an end segment T of L, let κ be the type of a shortest cofinal sequence in $L\backslash T$ and λ the type of a shortest coinitial sequence in T. If both κ and λ are infinite, then $\pi\chi(F)$ is the minimum of κ and λ. If one is infinite and the other is 1, then $\pi\chi(F)$ is the infinite one. If both are 1, then $\pi\chi(F)$ is 1. From this description it is easy to construct a linear order L such that if A is the interval algebra on L then $\pi\chi(A) < \chi(A)$, with the difference arbitrarily large: for example, let κ be any infinite cardinal, and let L be $0 + \omega^* \cdot \kappa + \omega^*$. The above description implies that $\pi\chi(A) = \omega$, while if F is the ultrafilter corresponding to the end segment ω^*, then $\chi(F) = \kappa$. In this example we also have $\pi\chi(A) < \mathrm{Depth}(A)$. The description of $\pi\chi$ also shows that $\pi\chi(A) \le \mathrm{Depth}(A)$ for an interval algebra A.

If A is complete, then $\mathrm{c}(A) \le \pi\chi(A)$: in fact, suppose that $\pi\chi(A) < \mathrm{c}(A)$. Let X be disjoint in A with $\sum X = 1$ and $|X| = (\pi\chi(A))^+$. Let F be an ultrafilter on A such that $\sum(X\backslash Y) \in F$ for each $Y \subset X$ such that $|Y| < |X|$. Let Y be a π-base for F with $|Y| < |X|$. For each $y \in Y$ choose $x_y \in X$ such that $y \cdot x_y \ne 0$. Then $\{x_y : y \in Y\}$ is a π-base for $F \cap \langle X\rangle^{\mathrm{cm}}$ (where $\langle X\rangle^{\mathrm{cm}}$ is the complete subalgebra of A generated by X). But $-\sum_{y\in Y} x_y \in F \cap \langle X\rangle^{\mathrm{cm}}$, contradiction.

Also recall the result of K. Bozeman [91] given in Chapter 6 that under GCH we have $\pi(A) = \pi\chi(A)$ for A complete.

In Chapter 6 we gave an example of a complete algebra A with the property that $\mathrm{d}(A) < \pi(A)$; hence by Theorem 11.20 we have $\mathrm{d}(A) < \pi\chi(A)$ also.

$\pi\chi$ is characterized for tree algebras by the following theorem.

Theorem 11.21. *Let T be an infinite tree. Then* $\pi\chi(\mathrm{Treealg}(T)) = \sup\{\mathrm{cf}(C) : C$ *is an initial chain of T with finitely many immediate successors*$\}$.

Proof. We describe $\pi\chi(F)$ for each ultrafilter F on $\mathrm{Treealg}(T)$. Recall that the ultrafilters on $\mathrm{Treealg}(T)$ are in one-one correspondence with the initial chains of T, where if T has finitely many roots we exclude the empty chain (a correction

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of the description in the Handbook). Given an initial chain C, we let F_C be the filter generated by

$$\{T \uparrow t : t \in C\} \cup \{T\backslash(T \uparrow t) : t \in T\backslash C\}.$$

This is the ultrafilter associated with C. We now consider several cases.

Case 1. C has a maximal element t, and t has finitely many immediate successors. Then $\{t\} \in F_C$, which is thereby principal, so that $\pi\chi(F) = 1$.

Case 2. C has infinitely many immediate successors. Let M be a countable set of such immediate successors, and let $X = \{T \uparrow t : t \in M\}$. Then X is dense in F_C. So $\pi\chi(F_C) \leq \omega$ in this case.

Case 3. C has no maximal element, but has finitely many immediate successors. Let M be the set of all immediate successors of C, and let N be a cofinal subset of C of size $\mathrm{cf}(C)$. Then $\{(T \uparrow t)\backslash \bigcup_{s\in M}(T \uparrow s) : t \in N\}$ is dense in F_C. Suppose that X is dense in F_C but $|X| < \mathrm{cf}(C)$. Wlog each element $x \in X$ has the form $(T \uparrow t_x)\backslash \bigcup_{s\in P_x}(T \uparrow s)$. Choose $u \in N$ such that $t_x < u$ for all $x \in X$. Then $(T \uparrow u)\backslash \bigcup_{s\in M}(T \uparrow s) \in F_C$, and no element of X is below it, contradiction. Thus $\pi\chi(F_C) = \mathrm{cf}(C)$ in this case. □

For tree algebras we have $\pi\chi(A) \leq \mathrm{Depth}(A)$, since $\mathrm{Depth}(A) = \mathrm{t}(A)$ for them. The difference can be arbitrarily large; this is an observation of Douglas Peterson. Namely, given κ, consider the tree

$$T \stackrel{\mathrm{def}}{=} \{f : f : \alpha + 1 \to \omega \text{ for some } \alpha \leq \kappa\} \cup \{0\}$$

under $\subseteq$. Since every initial chain of T has countably many immediate successors, it follows that $\pi\chi(\mathrm{Treealg}(T)) = \omega$ by Theorem 11.21; but $\mathrm{Depth}(\mathrm{Treealg}(T)) = \kappa$.

In Dow, Monk [94] the relationship between depth and π-character for superatomic BAs is described. There is a superatomic BA A such that $\mathrm{Depth}(A) = \omega$ and $\pi\chi(A) = \omega_1$. If $\pi\chi(A) \geq \omega_2$, then $\pi\chi(A) = \mathrm{Depth}(A)$. See also Shelah, Spinas [99].

12 Tightness

Again we note first of all that if F is a non-principal ultrafilter in a BA A, then $\mathrm{t}(F) \geq \omega$. To see this, note that for each $x \in F$ there is a $y \notin F$ such that $0 < y < x$; hence there is an ultrafilter G_x such that $x \in G_x$ but $G_x \neq F$. Let $Y = \{G_x : x \in F\}$. Thus $F \subseteq \bigcup Y$. Suppose that Z is a finite subset of Y such that $F \subseteq \bigcup Z$. But it is a very elementary exercise to show that no ultrafilter is included in a finite union of other, different, ultrafilters. So, $\mathrm{t}(F) \geq \omega$, and hence $\mathrm{t}(A) \geq \omega$ for every infinite BA A.

From the algebraic form of the free sequence equivalent of tightness (see Chapter 11), it is clear that $\mathrm{Ind}(A) \leq \mathrm{t}(A)$.

Also, using the free sequence equivalent to the definition of tightness it is clear that if A is a subalgebra of B, then $\mathrm{t}(A) \leq \mathrm{t}(B)$. Clearly if $A \leq_{\mathrm{free}} B$, then tightness can increase arbitrarily from A to B; so the same applies for $\leq_{\mathrm{u}}$, $\leq_{\mathrm{proj}}$, $\leq_{\mathrm{rc}}$, $\leq_\sigma$, and $\leq_{\mathrm{reg}}$. If $A \leq_\pi B$ clearly tightness can increase in going from A to B. The increase can be arbitrary, as one sees by taking $A = \mathrm{Finco}(\kappa)$ and $B = \mathscr{P}(\kappa)$ for κ a cardinal. We have $\mathrm{t}(A) = \omega$ by the argument for weak products which follows next, while $\mathrm{t}(B) = 2^\kappa$ since $2^\kappa = \mathrm{Ind}(B) \leq \mathrm{t}(B)$. If $A \leq_{\mathrm{s}} B$, then $\mathrm{t}(A) = \mathrm{t}(B)$. This is true since clearly $\mathrm{t}(C) \leq \mathrm{t}(D)$ whenever C is a homomorphic image of D; then use Proposition 2.29. t can increase arbitrarily from A to B when $A \leq_{\mathrm{mg}} B$, as one sees by considering interval algebras.

From the definition of tightness it is clear that $\mathrm{t}(A \times B) = \max\{\mathrm{t}(A), \mathrm{t}(B)\}$. Furthermore, $\mathrm{t}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right) = \max(\omega, \sup_{i\in I}\mathrm{t}(A_i))$ for any system $\langle A_i : i \in I\rangle$ of non-trivial BAs with I infinite. By the topological description of weak products, to prove this it suffices to show that $\mathrm{t}(F) = \omega$ for the "new" ultrafilter $F \stackrel{\mathrm{def}}{=} \{x \in \prod^{\mathrm{w}}_{i\in I} A_i$: there is a finite subset G of I such that $x_i = 1$ for all $i \in I\backslash G\}$. To see this, first note that if $G \in \mathrm{Ult}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right)$ and $G \neq F$, then there is an $i_G \in I$ and an ultrafilter K_G on A_i such that $G = \left\{x \in \prod^{\mathrm{w}}_{i\in I} A_i : x_{i_G} \in K_G\right\}$. Next, for H a finite subset of I let

$$x_H(i) = \begin{cases} 1 & \text{if } i \in I\backslash H \\ 0 & \text{if } i \in H. \end{cases}$$

Now suppose that $F \subseteq \bigcup Y$ with $Y \subseteq \mathrm{Ult}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right)$. The case $F \in Y$ is easy, so suppose that $F \notin Y$. Now $H \stackrel{\mathrm{def}}{=} \{i_G : G \in Y\}$ is infinite; otherwise $x_H \in F$ gives a contradiction. (If $x_H \in G \in Y$, then $x_H(i_G) = 0$, contradiction.) Let Z be a

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countable subset of Y such that $\{i_G : G \in Z\}$ is infinite. Suppose that $x \in F$. Say $x_i = 1$ for all $i \in I \backslash L$, L finite. Choose $G \in Z$ such that $i_G \notin L$. Then $x \in G$, as desired.

Note that this argument again shows that tightness is attained in $\prod_{i\in I}^{\mathrm{w}} A_i$ iff there is an $i \in I$ such that $\mathrm{t}\left(\prod_{i\in I}^{\mathrm{w}} A_i\right) = \mathrm{t}(A_i)$ and tightness is attained in A_i (for infinite A_i's). From this, the attainment property of tightness follows: for each limit cardinal $\kappa > \omega$ there is a BA A with tightness κ not attained: take the weak product of $\langle A_\lambda : \omega < \lambda < \kappa, \lambda \text{ a cardinal}\rangle$, where A_λ is the free BA of size λ.

By Theorem 10.6, t can increase arbitrarily from algebras to their (full) product.

For the free sequence equivalents of tightness see Chapters 4 and 11. Concerning attainment in the free sequence sense, we first show

Theorem 12.1. *If κ is an infinite cardinal with $\mathrm{cf}(\kappa) > \omega$ and $\langle A_i : i \in I\rangle$ is a system of BAs none of which has a free sequence of type κ, then also $\prod_{i\in I}^{\mathrm{w}} A_i$ does not have a free sequence of type κ.*

Proof. Suppose that $\langle F_\alpha : \alpha < \kappa\rangle$ is a free sequence in $\mathrm{Ult}\left(\prod_{i\in I}^{\mathrm{w}} A_i\right)$. We think of $\mathrm{Ult}\left(\prod_{i\in I}^{\mathrm{w}} A_i\right)$ as the one-point compactification of the disjoint union of all of the spaces $\mathrm{Ult}(A_i)$. We may assume that the "new" ultrafilter G is not among the F_α's. For each $\alpha < \kappa$ let $i(F_\alpha)$ be the unique $i \in I$ such that $\{a \cdot \chi_{\{i\}} : a \in F\}$ is an ultrafilter on A_i, where

$$\chi_{\{i\}}(j) = \begin{cases} 1 & \text{if } \ i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Set $J = \{i(F_\alpha) : \alpha < \kappa\}$. Now for each $j \in J$ the set $\{\alpha < \kappa : i(F_\alpha) = j\}$ has size less than κ, since A_j does not have a free sequence of length κ. Hence $|J| \geq \mathrm{cf}(\kappa)$, since $\kappa = \bigcup_{j\in J}\{\eta < \kappa : i(F_\eta) = j\}$. It follows that there is a $\xi < \kappa$ such that $|\{i(F_\eta : \eta < \xi\}| = \omega$, and hence $|\{i(F_\eta) : \xi \leq \eta\}| = \mathrm{cf}(\kappa)$. Therefore

$$G \in \overline{\{F_\eta : \eta < \xi\}} \cap \overline{\{F_\eta : \xi \leq \eta < \kappa\}},$$

which contradicts the free sequence property. □

It follows from Theorem 12.1 that for every limit cardinal κ with $\mathrm{cf}(\kappa) > \omega$ there is a BA with tightness κ not attained in the free sequence sense.

Now we turn to the case of cofinality ω:

Theorem 12.2. *Let $\mathrm{t}(A) = \kappa$, where κ is a singular cardinal of cofinality ω. Then A has a free sequence of length κ.*

Proof. This will be a modification of the proof of Theorem 4.8; see also Theorem 7.3. An element $a \in A$ is called a *μ-element* if for some ideal I of $A \restriction a$, the algebra $(A \restriction a)/I$ has a strictly increasing sequence of type μ. Let $\langle \lambda_i : i < \omega\rangle$ be a strictly increasing sequence of infinite regular cardinals with supremum κ. We call an element $a \in A$ an *∞-element* if it is a λ_i-element for all $i < \omega$.

(1) If a is an ∞-element and $a = b + c$ with $b \cdot c = 0$, then b is an ∞-element or c is an ∞-element.

For, it is enough to show that for every $i < \omega$, either b is a λ_i-element or c is a λ_i-element. Suppose that for some $i < \omega$, neither b nor c is a λ_i-element. Let I be an ideal in $A \restriction a$ and $\langle [x_\alpha] : \alpha < \lambda_i \rangle$ a strictly increasing sequence of elements in $(A \restriction a)/I$. Now if $\alpha < \beta < \lambda_i$, then

$$x_\alpha \cdot b \cdot -(x_\beta \cdot b) = x_\alpha \cdot -x_\beta \cdot b \in I \cap (A \restriction b),$$

and hence in $A \restriction b$ we have $[x_\alpha \cdot b] \leq [x_\beta \cdot b]$. Hence there is an $\alpha < \lambda_i$ such that if $\alpha < \beta < \gamma < \lambda_i$ then $x_\gamma \cdot -x_\beta \cdot b \in I$. Similarly for c: there is an $\alpha' < \lambda_i$ such that if $\alpha' < \beta < \gamma < \lambda_i$, then $x_\gamma \cdot -x_\beta \cdot c \in I$. But then if $\max(\alpha, \alpha') < \beta < \gamma < \lambda_i$ we get $x_\gamma \cdot -x_\beta \in I$, contradiction. This proves (1).

Now we construct disjoint elements $a_0, a_1, \ldots$ such that a_i is a λ_i-element for all $i < \omega$. Note from Theorem 4.26 that 1 is an ∞ element. Suppose that a_i has been constructed for all $i < n$ so that $\prod_{i<n} -a_i$ is an ∞-element; so this holds for $n = 0$. Now there exists an ideal I in $A \restriction \prod_{i<n} -a_i$ with a sequence $\langle [x_\alpha] : \alpha < \lambda_{n+1} \rangle$ strictly increasing in $(A \restriction \prod_{i<n} -a_i)/I$. Then clearly

(2) x_{λ_n} is a λ_n-element.

Now by (1) and (2) there is a λ_n-element a_n such that $\prod_{i\leq n} -a_i$ is an ∞-element.

Now for each $i < \omega$ choose an ideal I_i in $A \restriction a_i$ such that $(A \restriction a_i)/I_i$ has a chain of type λ_i. Let $J = \langle \bigcup_{i<\omega} I_i \rangle^{Id}$. Then $J \cap (A \restriction a_i) = I_i$ for each $i < \omega$, and hence A/J has a chain of type λ_i for all $i < \omega$. Hence as in the proof of 4.8, A/J has a chain of type κ, as desired. □

We also recall from Theorem 11.18 that $\mathrm{t}(A) = \pi\chi_{\mathrm{H+}}(A) = \pi\chi_{\mathrm{h+}}(A)$. And, as mentioned after the proof of Theorem 11.18, $\pi\chi_{\mathrm{H+}}$ and $\pi\chi_{\mathrm{h+}}$ have the same attainment properties, while $\pi\chi_{\mathrm{H+}}$ attained implies that t is attained in the free sequence sense. Another of the attainment problems is answered by the following theorem.

Theorem 12.3. *Suppose that κ is a singular cardinal. Then tightness is not attained in* $\mathrm{Intalg}(\kappa)$.

Proof. Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ. By the Handbook, each ultrafilter on $\mathrm{Intalg}(\kappa)$ is determined by an end segment of κ not containing 0 (this last restriction is not found in the Handbook, but it is clearly necessary). If C is such an end segment of κ, then its associated ultrafilter F_C is generated by

$$\{[0, c) : c \in C\} \cup \{[c, \infty) : c \in \kappa \backslash C\}.$$

So, take any end segment C; we want to show that $\mathrm{t}(F_C) < \kappa$.

Case 1. $C = \emptyset$. In this case we claim that $\mathrm{t}(F_C) \leq \mathrm{cf}(\kappa)$. In fact, suppose that $F_C \subseteq \bigcup Y$, where $Y \subseteq \mathrm{Ult}(\mathrm{Intalg}(\kappa))$. For each $\alpha < \mathrm{cf}\kappa$ we have $[\lambda_\alpha, \infty) \in F_C$, so we can choose $G_\alpha \in Y$ such that $[\lambda_\alpha, \infty) \in G_\alpha$. We claim that $F_C \subseteq \bigcup\{G_\alpha : \alpha < \mathrm{cf}\kappa\}$ (as desired). In fact, let $x \in F_C$. Without loss of generality x has the form $[c, \infty)$ for some $c \in \kappa$. Choose $\alpha < \mathrm{cf}\kappa$ such that $c < \lambda_\alpha$. Then $[c, \infty) \supseteq [\lambda_\alpha, \infty) \in G_\alpha$, as desired.

Case 2. $C \neq \emptyset$. Let c be the least element of C. Then we claim that $\mathrm{t}(F_C) \leq \max(\omega, |c|)$. For, again suppose that $F_C \subseteq \bigcup Y$, where $Y \subseteq \mathrm{Ult}(\mathrm{Intalg}(\kappa))$. For each $d < c$ we have $[d, c) \in F_C$, and so we can choose $G_d \in Y$ such that $[d, c) \in G_d$. Now we claim that $F_C \in \bigcup\{G_d : d < c\}$, as desired. For, let $x \in F_C$. Wlog x has the form $[d, e)$ with $d \in \kappa\backslash C$ and $e \in C$. Then $x \in G_d$, as desired. $\square$

Corollary 12.4. *For every singular cardinal κ there is a BA A such that $\mathrm{t}(A) = \kappa$ not attained but A has a free sequence of type κ.* $\square$

This corollary answers Problem 29 of Monk [90].

The description of $\pi\chi$ for interval algebras given at the end of Chapter 11 shows that if κ is singular, then $\pi\chi(\mathrm{Intalg}(\kappa)) = \kappa$ not attained. Thus attainment in the free sequence sense does not imply attainment in the $\pi\chi_{\mathrm{H}+}$ sense, answering Problem 30 in Monk [90] negatively. But if $\mathrm{t}(A)$ is regular limit and it is attained in the free sequence sense then it is attained in the $\pi\chi_{\mathrm{h}+}$ sense. The argument here is a little lengthy, but will be useful in discussing character too. Let $\mathrm{t}(A) = \kappa$, κ regular limit, and suppose that $\langle F_\alpha : \alpha < \kappa\rangle$. is a free sequence in $\mathrm{Ult}(A)$. For each $\xi < \kappa$ choose $a_\xi \in A$ such that $\{F_\alpha : \alpha < \xi\} \subseteq \mathcal{S}(a_\xi)$ and $\mathcal{S}(a_\xi) \cap \{F_\alpha : \xi \leq \alpha < \kappa\} = \emptyset$. Then

$$\{-a_\xi : \xi < \kappa\} \cup \{x \in A : \{F_\alpha : \alpha < \kappa\} \subseteq \mathcal{S}(x)\}$$

has the finite intersection property. In fact, otherwise we would get $-a_{\xi_1} \cdot \ldots \cdot -a_{\xi_n} \cdot x = 0$, where $\{F_\alpha : \alpha < \kappa\} \subseteq \mathcal{S}(x)$. Choose $\alpha < \kappa$ with $\xi_i < \alpha$ for all $i = 1, \ldots, n$. Then $x \in F_\alpha$, so $a_{\xi_i} \in F_\alpha$ for some i, contradiction. So, let G be an ultrafilter containing the given set. Let $Y = \{F_\alpha : \alpha < \kappa\} \cup \{G\}$. We claim that $\pi\chi(G, Y) = \kappa$. For, suppose that $M \in [A]^{<\kappa}$ and $\{\mathcal{S}(x) \cap Y : x \in M\}$ is a π-base for G, where $\mathcal{S}(x) \cap Y \neq \emptyset$ for all $x \in M$. Then by the regularity of κ, there is an $x \in M$ and a $\Gamma \in [\kappa]^\kappa$ such that $\mathcal{S}(x) \cap Y \subseteq \mathcal{S}(-a_\xi) \cap Y$ for all $\xi \in \Gamma$. Then $\{F_\alpha : \alpha < \kappa\} \subseteq \mathcal{S}(-x)$. In fact, let $\alpha < \kappa$. Choose $\xi \in \Gamma$ such that $\alpha < \xi$. Then $F_\alpha \in \mathcal{S}(a_\xi)$, so $F_\alpha \notin \mathcal{S}(x)$, hence $F_\alpha \in \mathcal{S}(-x)$, proving that $\{F_\alpha : \alpha < \kappa\} \subseteq \mathcal{S}(-x)$. It follows that $-x \in G$ too. So $\mathcal{S}(x) \cap Y = 0$, contradiction.

We now give an example, due to J.C. Martínez [02], of a BA A in which for a certain singular cardinal κ, $t(A, F) = \kappa$ for some ultrafilter F, but A does not have a free sequence of order type κ. This answers Problems 41 and 42 of Monk [96]. In fact, let κ be any singular cardinal with $\mathrm{cf}(\kappa) > \omega$, and let $\langle\lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of infinite successor cardinals with supremum κ. For each $\alpha < \mathrm{cf}(\kappa)$ let G_α be the ultrafilter on $\mathrm{Intalg}(\lambda_\alpha)$ generated by the set

$\{[\xi, \lambda_\alpha) : \xi < \lambda_\alpha\}$. Define

$$A = \left\{ x \in \prod_{\alpha < \mathrm{cf}(\kappa)}^{\mathrm{w}} \mathrm{Intalg}(\lambda_\alpha) : \forall \alpha, \beta < \mathrm{cf}(\kappa)[x_\alpha \in G_\alpha \text{ iff } x_\beta \in G_\beta] \right\}.$$

Clearly A is a subalgebra of $\prod^{\mathrm{w}}_{\alpha<\mathrm{cf}(\kappa)} \mathrm{Intalg}(\lambda_\alpha)$. Then let

$$F = \{x \in A : \forall \alpha < \mathrm{cf}(\kappa)[x_\alpha \in G_\alpha]\}.$$

Clearly F is an ultrafilter on A. Clearly $|A| = \kappa$. We claim that $\mathrm{t}(F) = \kappa$. Obviously $\mathrm{t}(F) \le \kappa$. Suppose that $\mathrm{t}(F) < \kappa$. Choose $\alpha < \mathrm{cf}(\kappa)$ such that $\mathrm{t}(F) < \lambda_\alpha$. For each $\xi < \lambda_\alpha$ let H_ξ be an ultrafilter on A containing the set $\{x \in A : \xi \in x_\alpha\}$. Then $F \subseteq \bigcup_{\xi<\lambda_\alpha} H_\xi$. In fact, if $x \in F$, then $x_\alpha \in G_\alpha$, and so there is an $\eta < \lambda_\alpha$ such that $[\eta, \lambda_\alpha) \subseteq x_\alpha$; hence $\eta \in x_\alpha$ and so $x \in H_\eta$. Choose $M \in [\lambda_\alpha]^{\le \mathrm{t}(F)}$ such that $F \subseteq \bigcup_{\xi \in M} H_\xi$. Since λ_α is regular, choose $\eta \in \lambda_\alpha$ such that $\xi < \eta$ for all $\xi \in M$. Let $x \in A$ be such that $x_\alpha = [\eta, \lambda_\alpha)$. Then $x \in F \backslash \bigcup_{\xi \in M} H_\xi$, contradiction. This proves our claim that $\mathrm{t}(F) = \kappa$.

Now suppose that $\langle H_\xi : \xi < \kappa \rangle$ is a free sequence of ultrafilters on A; we want to get a contradiction. We may assume that $H_\xi \ne F$ for all $\xi < \kappa$. Now for each ultrafilter $K \ne F$ on A there is a unique $\alpha(K) < \mathrm{cf}(\kappa)$ such that $\{x_{\alpha(K)} : x \in K\}$ is an ultrafilter on $\mathrm{Intalg}(\lambda_{\alpha(K)})$.

$(*)$ If $M \subseteq \mathrm{Ult}(A) \backslash \{F\}$ and $\{\alpha(K) : K \in M\}$ is infinite, then $F \in \overline{M}$.

In fact, let $F \in \mathcal{S}(a)$ with $a \in A$. Then $\{\beta < \mathrm{cf}(\kappa) : a_\beta \ne 1\}$ is finite, and so there is a $K \in M$ such that $a_{\alpha(K)} = 1$. Thus $K \in \mathcal{S}(a)$, as desired in $(*)$.

Now we define a strictly increasing sequence $\langle \beta(n) : n \in \omega \rangle$ of ordinals less than $\mathrm{cf}(\kappa)$ by recursion. Let $\beta(0) = 0$. Now suppose that $\beta(n)$ has been defined. Then

$$|\{\xi < \kappa : \alpha(H_\xi) \le \alpha(H_{\beta(n)})\}| \le \sum_{\eta \le \alpha(H_{\beta(n)})} \lambda_\eta \le \lambda_\eta \cdot \lambda_\eta < \kappa;$$

hence there is a $\beta(n+1) < \kappa$ with $\beta(n) < \beta(n+1)$ and $\alpha(H_{\beta(n+1)}) > \alpha(H_{\beta(n)})$. Let $M = \{\beta(n) : n \in \omega\}$. Then $F \in \overline{M}$ by $(*)$. Let $\gamma = \sup_{n\in\omega} \beta(n)$. By the above argument we get a strictly increasing sequence $\langle \delta_n : n \in \omega \rangle$ of ordinals less than $\mathrm{cf}(\kappa)$ such that $\gamma = \delta_0$ and $\alpha(H_{\delta_n}) < \alpha(H_{\delta(m)})$ for $n < m$. Let $N = \{\delta_n : n \in \omega\}$. By $(*)$ again, $F \in \overline{N}$. Hence

$$F \in \overline{\{H_\xi : \xi < \gamma\}} \cap \overline{\{H_\xi : \gamma \le \xi < \kappa\}},$$

contradicting the fact that $\langle H_\xi : \xi < \kappa \rangle$ is a free sequence.

The following problem about attainment is still open; this is Problem 43 in Monk [96].

Problem 111. *Does attainment of tightness in the $\pi\chi_{\mathrm{H}+}$ sense imply attainment in the sense of the definition?*

We return to the discussion of products.

Theorem 12.5. *If $\langle A_i : i \in I\rangle$ is a system of non-trivial BAs, with I infinite, then* $\mathrm{t}(\prod_{i\in I} A_i) \geq \max(2^{|I|}, \sup_{i\in I}\mathrm{t}(A_i))$.

Proof. If $j \in I$, then A_j is isomorphic to a subalgebra of $\prod_{i\in I} A_i$; so $\mathrm{t}(A_i) \leq \mathrm{t}(\prod_{i\in I} A_i)$. Since independence is less than or equal to tightness, it also follows that $2^{|I|} \leq \mathrm{t}(\prod_{i\in I} A_i)$. □

Now we consider ultraproducts, giving some results of Douglas Peterson. Recall from the introduction that tightness is an order-independence function. For such functions we have the following theorem, which uses the notion of *depth* of a linear ordering, which is the supremum of cardinalities of well-ordered subsets of the ordering.

Theorem 12.6. *Suppose that k is an order-independence function, $\langle A_i : i \in I\rangle$ is a sequence of infinite BAs, with I infinite, F is an ultrafilter on I, and $\langle \kappa_i : i \in I\rangle$ is a sequence of cardinals such that $\kappa_i < k'(A_i)$ for all $i \in I$. Then $k\left(\prod_{i\in I} A_i/F\right) \geq$ Depth $\left(\prod_{i\in I} \kappa_i/F\right)$.*

Proof. For each $i \in I$ let $\langle a^i_\alpha : \alpha < \kappa_i\rangle$ be a sequence of elements of A_i such that for all finite $G, H \subseteq \kappa_i$, if $\langle \kappa_i, <, G, H\rangle \models \varphi$ then $\prod_{\alpha\in F} a^i_\alpha \cdot \prod_{\alpha\in H} -a^i_\alpha \neq 0$. Let $\lambda = \text{Depth}\left(\prod_{i\in I}\kappa_i/F\right)$. We consider two cases.

Case 1. λ is a successor cardinal. Let $\langle f_\alpha/F : \alpha < \lambda\rangle$ be a sequence of elements of $\prod_{i\in I}\kappa_i/F$ such that $f_\alpha/F < f_\beta/F$ if $\alpha < \beta$. Define $g_\alpha(i) = a^i_{f_\alpha(i)}$ for all $\alpha < \lambda$ and $i \in I$. Now suppose that G and H are finite subsets of λ such that $(\lambda, <, G, H) \models \varphi$. Let

$$K = \{i \in I : \forall \alpha, \beta \in G \cup H(\alpha < \beta \Rightarrow f_\alpha(i) < f_\beta(i))\}.$$

Then $K \in F$. By (2) in the definition of order-independence function we have $(\kappa_i, <, \{f_\alpha(i) : \alpha \in G\}, \{f_\alpha(i) : \alpha \in H\}) \models \varphi$ for each $i \in K$, and hence $\prod_{\alpha\in G} a^i_{f_\alpha(i)} \cdot \prod_{\alpha\in H} -a^i_{f_\alpha(i)} \neq 0$. Therefore $\prod_{\alpha\in G} f_\alpha/F \cdot \prod_{\alpha\in H} -g_\alpha/F \neq 0$, as desired.

Case 2. λ is a limit cardinal. Then $k\left(\prod_{i\in I} A_i/F\right) \geq \kappa$ for each successor $\kappa < \lambda$, by the above argument; hence $k\left(\prod_{i\in I} A_i/F\right) \geq \lambda$. □

Theorem 12.7. (GCH) *Suppose that $\langle A_i : i \in I\rangle$ is a system of infinite BAs, with I infinite, and F is a regular ultrafilter on I. Then* $\mathrm{t}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I}\mathrm{t}(A_i)/F\right|$.

Proof. Let $\kappa = \text{ess.sup}^F_{i\in I}\mathrm{t}(A_i)$.

Case 1. $\kappa \leq |I|$. The ultraproduct has an independent subset of size $2^{|I|}$, and the desired result follows.

Case 2. $\mathrm{cf}(\kappa) > |I|$. Then if κ is a successor cardinal, we may assume that $\mathrm{t}'(A_i) = \kappa^+$ for all $i \in I$, and hence by Theorem 12.6 we have $\mathrm{t}\left(\prod_{i\in I} A_i/F\right) \geq$ Depth $\left(\prod_{i\in I}\kappa/F\right) = \kappa$. The limit case clearly follows from this case.

Case 3. $\mathrm{cf}(\kappa) \le |I| < \kappa$. Using Lemma 3.17 or Lemma 3.18, we obtain a system $\langle\lambda_i : i \in I\rangle$ of infinite cardinals such that $\lambda_i < \mathrm{t}(A_i)$ for each $i \in I$, and $\mathrm{ess.sup}^F_{i\in I}\lambda_i = \kappa$. Hence by Theorem 12.6 again, and by the proof of Theorem 4.18, $\mathrm{t}\left(\prod_{i\in I} A_i/F\right) \ge \mathrm{Depth}\left(\prod_{i\in I}\lambda_i/F\right) \ge \kappa^+$. □

As usual, Donder's theorem then says that $\ge$ holds for any uniform ultrafilter, assuming $V = L$. The inequality can be strict, from the discussion of independence. A consistent example exists for the other direction by Rosłanowski, Shelah [98].

The tightness of free products is described by a theorem of Malyhim [72]; we give the result here. The proof we give is due to Todorčević. (private communication); he uses the idea of this proof to strengthen Malyhin's result.

Theorem 12.8. $\mathrm{t}(A \oplus B) = \max(\mathrm{t}(A), \mathrm{t}(B))$.

Proof. The inequality $\ge$ is clear. For the other inequality it suffices to show that if $\langle c_\alpha : \alpha < \theta\rangle$ is a free sequence in $A \oplus B$ with θ regular and uncountable, then either A or B has a free sequence of that length too. We use *free sequence* here in the algebraic sense described in Chapter 11. First we claim:

(1) We may assume that each c_α has the form $a_\alpha \cdot b_\alpha$ with $a_\alpha \in A$ and $b_\alpha \in B$.

To see this, first write $c_\alpha = \sum_{i<m_\alpha}(a_{\alpha i} \cdot b_{\alpha i})$ with each $a_{\alpha i} \in A$ and each $b_{\alpha i} \in B$. Since θ is regular and uncountable, we may assume that $m_\alpha = m$ does not depend on α. Now for each $\alpha < \theta$ let F_α be an ultrafilter on $A \oplus B$ such that $\{c_\xi : \xi \le \alpha\} \cup \{-c_\xi : \alpha < \xi < \theta\} \subseteq F_\alpha$. Then by the first part of the proof of Theorem 4.23 we get an ultrafilter G on $A \oplus B$ such that

(2) $|\{\alpha < \theta : a \in F_\alpha\}| = \theta$ for all $a \in G$.

Then $c_\alpha \in G$ for all $\alpha < \theta$; for if $-c_\alpha \in G$ we would get $-c_\alpha \in F_\beta$ for some $\beta \ge \alpha$ by (2), and this is impossible. It follows that for all $\alpha < \theta$ there is an $i < m$ such that $a_{\alpha i} \cdot b_{\alpha i} \in G$. Hence there exist an $i < m$ and a $\Gamma \in [\theta]^\theta$ such that $a_{\alpha i} \cdot b_{\alpha i} \in G$ for all $\alpha \in \Gamma$. Now let

$$K = \{\delta \in \Gamma : \forall H \in [\Gamma \cap \delta]^{<\omega}\exists\alpha \in (\max H, \delta)\forall\xi \in H(a_{\xi i} \cdot b_{\xi i} \in F_\alpha)\}.$$

We claim that K is unbounded in θ. For, let $\delta_0 < \theta$. For every finite $H \subseteq \Gamma \cap \delta_0$ we have $\prod_{\xi\in H}(a_{\xi i} \cdot b_{\xi i}) \in G$, and hence by (2) there is an $\alpha_H > \max H$ such that $\prod_{\xi\in H}(a_{\xi i} \cdot b_{\xi i}) \in F_{\alpha_H}$. Choose $\delta_1 \in \Gamma$ greater than δ_0 and all ordinals α_H for $H \in [\Gamma\cap\delta_0]^{<\omega}$. Then repeat the construction for δ_1, obtaining $\delta_2 \in \Gamma$, etc. Finally, let δ_ω be the least member of Γ greater than all δ_i, $i < \omega$. Clearly $\delta_\omega \in K$, proving the claim about K.

Let $\langle\delta_\xi : \xi < \theta\rangle$ enumerate K in increasing order. We claim, then, that $\langle a_{\delta_\xi i} \cdot b_{\delta_\xi i} : \xi < \theta\rangle$ is a free sequence in $A \oplus B$; this will prove the claim (1). To prove this, let M and N be finite subsets of θ such that each member of M is less than each member of N. We may assume that N is nonempty. Let ξ be the least member of N. We then apply the definition of K to its member δ_ξ to get an

$\alpha \in (\max\{\delta_\eta : \eta \in M\}, \delta_\xi)$ such that $a_{\delta_\eta i} \cdot b_{\delta_\eta i} \in F_\alpha$ for all $\eta \in M$. Note that we also have $-c_{\delta_\eta} \in F_\alpha$ for all $\eta \in N$. Now

$$\prod_{\eta\in M}(a_{\delta_\eta i} \cdot b_{\delta_\eta i}) \cdot \prod_{\eta\in N} -c_{\delta_\eta} \leq \prod_{\eta\in M}(a_{\delta_\eta i} \cdot b_{\delta_\eta i}) \cdot \prod_{\eta\in N} -(a_{\delta_\eta i} \cdot b_{\delta_\eta i}),$$

and the left side is in F_α and hence is nonzero, so the right side is nonzero too, and this proves that $\langle a_{\delta_\xi i} \cdot b_{\delta_\xi i} : \xi < \theta\rangle$ is a free sequence in $A \oplus B$.

So now we assume (1). We consider two cases.

Case 1. $\forall\alpha < \theta\exists\beta \geq \alpha\forall K \in [\alpha]^{<\omega}\forall L \in [\theta\backslash\beta]^{<\omega}(a_{KL} \stackrel{\text{def}}{=} \prod_{\xi\in K} a_\xi \cdot \prod_{\xi\in L} -a_\xi \neq 0)$. Define $\langle\alpha_\xi : \xi < \theta\rangle$ as follows. If α_η has been defined for all $\eta < \xi$, let $\beta_\xi = \sup_{\eta<\xi}\alpha_\eta$ and choose $\alpha_\xi > \beta_\xi$ such that $\forall K \in [\beta_\xi]^{<\omega}\forall L \in [\theta\backslash\alpha_\xi]^{<\omega}(a_{KL} \neq 0)$. Then $\langle a_{\alpha_\xi} : X < \theta\rangle$ is a free sequence in A. For, assume that M and N are finite subsets of θ, each member of M less than each member of N. Let $\xi = \sup_{\eta\in M}(\eta+1)$ ($\xi = 0$ if $M = 0$). Then $\{\alpha_\eta : \eta \in M\} \in [\beta_\xi]^{<\omega}$ and $\{\alpha_\eta : \eta \in N\} \in [\theta\backslash\alpha_\xi]^{<\omega}$, so $\prod_{\eta\in M} a_{\alpha_\eta} \cdot \prod_{\eta\in N} -a_{\alpha_\eta} \neq 0$, as desired.

Case 2. Case 1 fails: $\exists\alpha_0 < \theta\forall\beta \geq \alpha_0\exists K_\beta \in [\alpha_0]^{<\omega}\exists L \in [\theta\backslash\beta]^{<\omega}(a_{K_\beta L} = 0)$. So $\exists K \in [\alpha_0]^{<\omega}\exists\Gamma \in [\theta\backslash\alpha_0]^\theta\forall\beta \in \Gamma\exists L \in [\theta\backslash\beta]^{<\omega}(a_{KL} = 0)$. Hence we get $\langle L_\alpha : \alpha < \theta\rangle$ such that $\forall\alpha\forall\beta(\alpha < \beta < \theta \Rightarrow \forall\xi \in L_\alpha\forall\eta \in L_\beta(\xi < \eta))$ and $\forall\alpha < \theta[\forall\xi \in K\forall\eta \in L_\alpha(\xi < \eta))$ and $a_{KL_\alpha} = 0]$. Let $c_\alpha = \prod_{\xi\in L_\alpha} b_\xi$ for all $\alpha < \theta$. Then $\langle\overline{c}_\alpha : \alpha < \theta\rangle$ is a free sequence in B. For, suppose that M and N are finite subsets of θ, each member of M less than each member of N. Then, with $P = \bigcup_{\alpha\in N} L_\alpha$,

$$\begin{aligned} 0 &\neq \prod_{\xi\in K}(a_\xi \cdot b_\xi) \cdot \prod_{\alpha\in M,\xi\in L_\alpha}(a_\xi \cdot b_\xi) \cdot \prod_{\alpha\in N,\xi\in L_\alpha} -(a_\xi \cdot b_\xi) \\ &= \prod_{\xi\in K}(a_\xi \cdot b_\xi) \cdot \prod_{\alpha\in M,\xi\in L_\alpha}(a_\xi \cdot b_\xi) \cdot \sum_{\Gamma\subseteq P}\left(\prod_{\xi\in\Gamma} -a_\xi \cdot \prod_{\xi\in P\backslash\Gamma} -b_\xi\right), \end{aligned}$$

so choose $\Gamma \subseteq P$ so that

$$0 \neq \prod_{\xi\in K}(a_\xi \cdot b_\xi) \cdot \prod_{\alpha\in M,\xi\in L_\alpha}(a_\xi \cdot b_\xi) \cdot \prod_{\xi\in\Gamma} -a_\xi \cdot \prod_{\xi\in P\backslash\Gamma} -b_\xi.$$

Now if $L_\alpha \subseteq \Gamma$ for some $\alpha \in N$, then

$$\prod_{\xi\in K}(a_\xi \cdot b_\xi) \cdot \prod_{\alpha\in M,\xi\in L_\alpha}(a_\xi \cdot b_\xi) \cdot \prod_{\xi\in\Gamma} -a_\xi \cdot \prod_{\xi\in P\backslash\Gamma} -b_\xi \leq a_{KL_\alpha} = 0,$$

contradiction. So for all $\alpha \in N$ there is a $\xi \in L_\alpha\backslash\Gamma$. Thus

$$0 \neq \prod_{\xi\in K} b_\xi \cdot \prod_{\alpha\in M,\xi\in L_\alpha} b_\xi \cdot \prod_{\xi\in P\backslash\Gamma} -b_\xi \leq \prod_{\alpha\in M} c_\alpha \cdot \prod_{\alpha\in N}\sum_{\xi\in L_\alpha} -b_\xi = \prod_{\alpha\in M} c_\alpha \cdot \prod_{\alpha\in N} -c_\alpha,$$

as desired. $\square$

Theorem 12.9. *If $\langle A_i : i \in I\rangle$ is a system of BAs each with at least four elements, then* $\mathrm{t}(\oplus_{i\in I} A_i) = \max(|I|, \sup_{i\in I} \mathrm{t}(A_i))$.

Proof. Obviously $\mathrm{t}(A_j) \le \mathrm{t}(\oplus_{i\in I} A_i)$ for each $j \in I$; and $|I| \le \oplus_{i\in I} A_i$ since $\mathrm{Ind} \le \mathrm{t}$. Thus $\ge$ holds. To prove $\le$, let $\kappa = \max(|I|, \sup_{i\in I} \mathrm{t}(A_i))$, and suppose that $\langle c_\alpha : \alpha < \kappa^+\rangle$ is a free sequence in $\oplus_{i\in I} A_i$; we shall get a contradiction. For each $\alpha < \kappa^+$ there is a finite $S_\alpha \subseteq I$ such that $c_\alpha \in \oplus_{i\in S_\alpha} A_i$. We may assume that $S = S_\alpha$ does not depend on α. But then $\kappa^+ \le \sup_{i\in S} \mathrm{t}A_i$ by Theorem 12.8, contradiction. □

The behaviour of tightness in the free sequence sense under unions of chains of BAs is similar to the case of cellularity (Theorem 3.16). The definition of ordinary sup-function does not quite fit, but essentially the same proof can be used:

Theorem 12.10. *Let κ and λ be infinite cardinals, with λ regular. Then the following conditions are equivalent:*

(i) $\mathrm{cf}(\kappa) = \lambda$.

(ii) *There is a strictly increasing sequence $\langle A_\alpha : \alpha < \lambda\rangle$ of infinite Boolean algebras each with no free sequence of type κ such that $\bigcup_{\alpha<\lambda} A_\alpha$ has a free sequence of type κ.* □

In view of the equivalence of tightness with its free sequence variant, 12.10 also applies to tightness when κ is a successor cardinal. And actually 12.10 extends in the following form to tightness itself; this answers, negatively, Problem 31 in Monk [90].

Theorem 12.11. *Let κ and λ be infinite cardinals, with λ regular. Then the following conditions are equivalent:*

(i) $\mathrm{cf}(\kappa) = \lambda$.

(ii) *There is a strictly increasing sequence $\langle A_\alpha : \alpha < \lambda\rangle$ of Boolean algebras each with tightness less than κ such that $\bigcup_{\alpha<\lambda} A_\alpha$ has tightness κ.*

Proof. By the comment before the theorem, we assume that κ is a limit cardinal. (i)⇒(ii): Take a free BA of size κ and write it as an increasing union of smaller algebras in the obvious way.

(ii)⇒(i): Assume that (ii) holds and (i) fails. Let $\mu = \sup_{\alpha<\lambda} \mathrm{t}(A_\alpha)$; the first part of the proof will consist in showing that $\mu = \kappa$; to this end, suppose that $\mu < \kappa$. Fix ν such that $\mu < \nu < \kappa$. Let $\langle a_\alpha : \alpha < \nu\rangle$ be a free sequence in B. For each $\beta < \lambda$ let $S^\nu_\beta = \{\alpha < \nu : a_\alpha \in A_\beta\}$. Thus $S^\nu_\beta \subseteq S^\nu_\gamma$ for $\beta < \gamma < \lambda$, and $\nu = \bigcup_{\beta<\lambda} S^\nu_\beta$. If $\exists\beta < \lambda\forall\gamma \in (\beta,\lambda)[S^\nu_\beta = S^\nu_\gamma]$, then $\nu = S^\nu_\beta$ and so $\{a_\alpha : \alpha < \nu\} \subseteq A_\beta$, hence $\mathrm{t}(A_\beta) \ge \nu$, contradiction. Thus $\forall\beta < \lambda\exists\gamma \in (\beta,\lambda)[S^\nu_\beta \subset S^\nu_\gamma]$. Applying this to $\nu = \mu^+$ we get $\lambda \le \mu^+$; then applying it to $\nu = \mu^{++}$ we get $\mu^{++} = \bigcup_{\beta<\lambda} S^{\mu^{++}}_\beta$, so there is a $\beta < \lambda$ such that $|S^{\mu^{++}}_\beta| = \mu^{++}$, so A_β has a free sequence of type μ^{++}, which contradicts $\mathrm{t}(A_\beta) \le \mu$. This contradiction proves that $\mu = \kappa$.

Since each $\mathrm{t}(A_\alpha)$ is less than κ, from $\mu = \kappa$ it follows that $\mathrm{cf}(\kappa) \leq \lambda$; since (i) fails, we have in fact that $\mathrm{cf}(\kappa) < \lambda$. Now $\forall\alpha < \lambda\exists\beta \in (\alpha, \lambda)[\mathrm{t}(A_\alpha) < \mathrm{t}(A_\beta)]$, since otherwise we would have $\mu < \kappa$. Hence $\lambda \leq \sup_{\alpha<\lambda} \mathrm{t}(A_\alpha) = \kappa$. Since λ is regular, $\lambda < \kappa$. So κ is singular. Let $\langle \nu_\alpha : \alpha < \mathrm{cf}\kappa \rangle$ be a strictly increasing sequence of cardinals with supremum κ. For each $\alpha < \mathrm{cf}(\kappa)$ there is a $\beta_\alpha < \lambda$ such that $\mathrm{t}(A_{\beta_\alpha}) \geq \nu_\alpha$, since $\mu = \kappa$. Let $\gamma = \sup_{\alpha<\mathrm{cf}\kappa} \beta_\alpha$; then $\gamma < \lambda$ since $\mathrm{cf}\kappa < \lambda$. But then $\mathrm{t}(A_\gamma) = \kappa$, contradiction. $\square$

A more natural version of Theorem 12.10 for tightness itself would be the equivalence expressed in the following problem.

Problem 112. *Is the following true? Let κ and λ be infinite cardinals, with λ regular. Then the following conditions are equivalent:*

(i) $\mathrm{cf}(\kappa) = \lambda$.

(ii) *There is a strictly increasing sequence $\langle A_\alpha : \alpha < \lambda \rangle$ of Boolean algebras each having no ultrafilter with tightness κ such that $\bigcup_{\alpha<\lambda} A_\alpha$ has an ultrafilter with tightness κ.*

This was Problem 44 in Monk [96].

Theorem 12.12. $\mathrm{t}\left(\prod_{i\in I}^B A_i\right) = \max(\mathrm{t}(B), \sup_{i\in I} t(A_i))$.

Proof. For brevity let $C = \prod_{i\in I}^B A_i$. B and each A_i are isomorphic to subalgebras of C, so $\geq$ holds. For the other direction we use the description of the ultrafilters on C given in Theorem 1.6(x). Let U be an ultrafilter on C, and suppose that $\mathscr{A} \subseteq \mathrm{Ult}(C)$ with $U \subseteq \bigcup \mathscr{A}$.

Case 1. There exist $i \in I$ and an ultrafilter V on A_i such that $U = \{h(b, F, a) : (b, F, a)$ is normal, and $i \in b$ or $(i \notin b$ and $i \in F$ and $a_i \in V)\}$. Suppose that $x \in V$. Then $h(\emptyset, \{i\}, \{(i, x)\}) \in U$, and so there is a $W_x \in \mathscr{A}$ such that $h(\emptyset, \{i\}, \{(i, x)\}) \in W_x$. By Theorem 1.6(x) there is an ultrafilter Y_x on A_i such that

$$W_x = \{h(b, F, a) : (b, F, a) \text{ is normal, and } i \in b \text{ or } (i \notin b \text{ and } i \in F \text{ and } a_i \in Y_x)\}.$$

It follows that $x \in Y_x$. Thus $V \subseteq \bigcup_{x\in V} Y_x$. So there is a $M \in [V]^{\leq \mathrm{t}(V)}$ such that $V \subseteq \bigcup_{x\in M} Y_x$. We claim that $U \subseteq \bigcup_{x\in M} W_x$. In fact, take any normal $h(b, F, a) \in U$. If $i \in b$, then $h(b, f, a) \in W_x$ for all $x \in M$. Suppose that $i \notin b$. Then $a_i \in V$, so there is an $x \in M$ such that $a_i \in Y_x$. Then $h(b, f, a) \in W_x$. This proves the claim. Thus $\mathrm{t}(U) \leq \mathrm{t}(V)$.

Case 2. There is a nonprincipal ideal W on B such that $U = \{h(b, F, a) : h(b, F, a)$ is normal and $b \in W\}$. Suppose that $x \in W$. Then $h(x, \emptyset, \emptyset) \in U$, so there is a $V_x \in \mathscr{A}$ such that $h(x, \emptyset, \emptyset) \in V_x$. By Theorem 1.6(x) there is a nonprincipal ultrafilter Y_x on B such that $V_x = \{h(b, F, a) : h(b, F, a)$ is normal and $b \in Y_x\}$. Thus $x \in Y_x$. Hence $W \subseteq \bigcup_{x\in W} Y_x$. Choose $M \in [W]^{\leq \mathrm{t}(W)}$ such that $W \subseteq \bigcup_{x\in M} Y_x$. We claim that $U \subseteq \bigcup_{x\in M} V_x$. For, suppose that $h(b, F, a) \in U$, with

$h(b, F, a)$ normal. Thus $b \in W$, so there is an $x \in W$ such that $x \in Y_x$. Then $h(b, F, a) \in V_x$, as desired. $\square$

Problem 113. *Do there exist a system $\langle A_i : i \in I\rangle$ of BAs and a system $\langle F_i : i \in I\rangle$ of ultrafilters, each F_i an ultrafilter on A_i, such that, with B the one-point gluing using these inputs, $\mathrm{t}(B) < \mathrm{t}\left(\prod_{i\in I} A_i\right)$?*

If $\langle a_\alpha : \alpha < \kappa\rangle$ is a free sequence in A, then clearly $\langle (a_\alpha, \mathcal{S}(a_\alpha)) : \alpha < \kappa\rangle$ is a free sequence in $\mathrm{Dup}(A)$. Thus $\mathrm{t}(A) \le \mathrm{t}(\mathrm{Dup}(A))$.

Problem 114. *Is there a BA A such that $\mathrm{t}(A) < \mathrm{t}(\mathrm{Dup}(A))$?*

From Proposition 2.6 it follows that $\mathrm{t}(A) \le \mathrm{t}(\mathrm{Exp}(A))$ for any infinite BA A. The concrete example of the exponential described at the end of Chapter 1 shows that $<$ is possible here.

Lemma 12.13. *If X is a topological space and $X \in \overline{Z}$ in $\mathrm{Exp}(X)$, then $\bigcup Z$ is dense in X.*

Proof. Let $0 \ne U$ be open in X. Then $X \in \mathscr{V}(X, U)$, so there is an $F \in Z \cap \mathscr{V}(X, U)$. Thus $F \cap U \ne 0$, so $\bigcup Z \cap U \ne 0$. $\square$

Lemma 12.14. *For any Hausdorff space X we have $\mathrm{d}(X) \le \mathrm{t}(\mathrm{Exp}\,(X))$.*

Proof. Let Z be the collection of all finite non-empty subsets of X. Then Z is a subset of $\mathrm{Exp}\,(X)$. Moreover, $X \in \overline{Z}$, since if $\mathscr{V}(U_0, \ldots, U_{m-1})$ is any neighborhood of X, choose $a_i \in U_i$ for all $i < m$; then $\{a_i : i < m\} \in Z \cap \mathscr{V}(U_0, \ldots, U_{m-1})$. Now choose a subset Y of Z of size $\le \mathrm{t}(\mathrm{Exp}\,(X))$ such that $X \in \overline{Y}$. Then by Lemma 12.13, $\bigcup Y$ is dense in X. Clearly $|\bigcup Y| \le \mathrm{t}(\mathrm{Exp}\,(X))$. $\square$

We turn to derived functions for tightness. By Theorem 11.18 we have that $\mathrm{t}_{\mathrm{H}+}(A) = \mathrm{t}_{\mathrm{h}+}(A) = \mathrm{t}(A)$. Clearly $\mathrm{t}_{\mathrm{S}+}(A) = \mathrm{t}(A)$, $\mathrm{t}_{\mathrm{S}-}(A) = \omega$, and ${}_{\mathrm{d}}\mathrm{t}_{\mathrm{S}+}(A) = \mathrm{t}(A)$. In the algebra of Fedorchuk [75] we have $\mathrm{t}_{\mathrm{H}-}(A) \le \mathrm{t}(A) < \mathrm{Card}_{\mathrm{H}-}(A)$; so Problem 32 of Monk [90] was solved long ago. See Chapter 16 for Fedorchuk's algebra. Note that ${}_{\mathrm{d}}\mathrm{t}_{\mathrm{S}-}(A) \ne \mathrm{t}(A)$ in general; this can be seen by considering $\mathscr{P}(\omega)$ and its dense subalgebra consisting of the finite and cofinite subsets of ω.

Recall also our earlier results that $\mathrm{Depth}_{\mathrm{H}+}(A) = \mathrm{t}(A) = \pi\chi_{\mathrm{H}+}(A)$; see Theorems 4.26, 11.18. Next we mention more about the relationships between tightness and our previously introduced functions. By Theorem 4.26 we have $\mathrm{Depth}(A) \le \mathrm{t}(A)$ for any BA A; the difference can be big, for example in a free algebra. $\pi\chi(A) \le \mathrm{t}(A)$ by Theorem 11.18. We observed in Chapter 11 that one can have $\pi\chi(A) < \mathrm{Depth}(A)$ in an interval algebra with the difference arbitrarily large. This solves Problem 33 in Monk [90]. In particular, it is possible to have $\pi\chi(A) < \mathrm{t}(A)$ with the difference arbitrarily large.

It is obvious that $\mathrm{Ind}(A) \le \mathrm{t}(A)$; the difference is large in some interval algebras. Obviously $\mathrm{t}(A) \le |A|$. Note that $\mathrm{t}(A) \le \mathrm{s}(A) \le \mathrm{Irr}(A)$ by Theorem 3.30 and a later remark. Thus Problem 34 in Monk [90] has the obvious answer "no". $\mathrm{t}(A) > \pi(A)$ for $A = \mathscr{P}(\omega)$. $\mathrm{t}(A) > \mathrm{Length}(A)$ for A an uncountable free BA.

Length$(A) > \mathrm{t}(A)$ for A the interval algebra on the reals. $\mathrm{c}(A) > \mathrm{t}(A)$ for A an uncountable finite-cofinite algebra.

We also give the following result relating π with t; it is from Todorčević [90a].

Theorem 12.15. *For every infinite BA A there is a sequence $\langle a_\alpha : \alpha < \beta\rangle$ of nonzero elements of A such that $\{a_\alpha : \alpha < \beta\}$ is dense in A and for every subset Γ of β with no maximum element, the sequence $\langle a_\alpha : \alpha \in \Gamma\rangle$ is free iff $\{a_\alpha : \alpha \in \Gamma\}$ has the finite intersection property.* (Since Γ has a natural order from β, the meaning of "free" in this extended sense is clear.)

Proof. Let P be a maximal disjoint subset of A^+ such that $A \restriction b$ is π-homogeneous for every $b \in P$, that is, $\pi(A \restriction c) = \pi \restriction b)$ for every nonzero $c \leq b$. Temporarily fix $b \in P$. Let $\pi(A \restriction b) = \kappa_b$, and let $\langle c^b_\alpha : \alpha < \kappa_b\rangle$ enumerate a dense subset of $(A \restriction b)^+$ of size κ_b. Now we define $\langle a^b_\alpha : \alpha < \kappa_b\rangle$ by induction. Suppose that a^b_α has been defined for all $\alpha < \beta$. Let $\mathscr{F}$ be the collection of all nonzero elements of the form $c^b_\beta \cdot \prod_{\alpha\in F}(a^b_\alpha)^{\varepsilon(\alpha)}$ for F a finite subset of β and $\varepsilon \in {}^F2$. Then $\mathscr{F}$ is not dense in $A \restriction c^b_\beta$, so there is an $a^b_\beta \in (A \restriction c^b_\beta)^+$ such that $x \not\leq a^b_\beta$ for all $x \in \mathscr{F}$. This finishes the construction.

Concatenating the so obtained sequences $\langle a^b_\alpha : \alpha < \kappa_b\rangle$ in any order, we obtain a sequence $\langle a_\alpha : \alpha < \beta\rangle$ as desired in the theorem. In fact, first we check that $\{a_\alpha : \alpha < \beta\}$ is dense in A. Suppose that $a \in A^+$. Choose $b \in P$ such that $a\cdot b \neq 0$. There is a $\gamma < \kappa_b$ such that $c^b_\gamma \leq a\cdot b$. By construction, $a^b_\gamma \leq c^b_\gamma$, as desired. Next we check that for any subset Γ of β with no maximum element, $\langle a_\alpha : \alpha \in \Gamma\rangle$ is free iff $\{a_\alpha : \alpha \in \Gamma\}$ has the finite intersection property. $\Rightarrow$: obvious. $\Leftarrow$: Assume that $\{a_\alpha : \alpha \in \Gamma\}$ has the finite intersection property. Then there is a $b \in P$ such that each of the a_α's for $\alpha \in \Gamma$ is of the form a^b_γ. So without loss of generality we assume that $\{a^b_\alpha : \alpha \in \Gamma\}$ has the finite intersection property, and we want to show that $\langle a^b_\alpha : \alpha \in \Gamma\rangle$ is free. We prove

$(*)$ If F and G are finite subsets of γ and $F < G$, then $\prod_{\alpha\in F} a^b_\alpha \cdot \prod_{\alpha\in G} -a^b_\alpha \neq 0$.

This we do by induction on $|G|$. The case $G = 0$ is given. Assume that $(*)$ is true for G, and $G < \gamma \in \Gamma$. If $\prod_{\alpha\in F} a^b_\alpha \cdot \prod_{\alpha\in G} -a^b_\alpha \cdot c^b_\gamma = 0$, then also $\prod_{\alpha\in F} a^b_\alpha \cdot \prod_{\alpha\in G} -a^b_\alpha \cdot a^b_\gamma = 0$, and so $0 \neq \prod_{\alpha\in F} a^b_\alpha \cdot \prod_{\alpha\in G} -a^b_\alpha = \prod_{\alpha\in F} a^b_\alpha \cdot \prod_{\alpha\in G} -a^b_\alpha \cdot -a^b_\gamma$, as desired. If $\prod_{\alpha\in F} a^b_\alpha \cdot \prod_{\alpha\in G} -a^b_\alpha \cdot c^b_\gamma \neq 0$, then $\prod_{\alpha\in F} a^b_\alpha \cdot \prod_{\alpha\in G} -a^b_\alpha \cdot -a^b_\gamma \neq 0$ by construction. $\square$

A min-max version of tightness is defined as follows. A free sequence $\langle a_\xi : \xi < \alpha\rangle$ is *maximal* iff there is no $b \in A$ such that $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is a free sequence, where $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is the result of adjoining b at the end of the sequence $\langle a_\xi : \xi < \alpha\rangle$. Now we define

$$\mathfrak{f}_{\mathrm{sp}}(A) = \{|\alpha| : A \text{ has an infinite maximal free sequence of length } \alpha\};$$
$$\mathfrak{f}(A) = \min(\mathfrak{f}_{\mathrm{sp}}(A)).$$

These notions are studied in Monk [11].

Proposition 12.16. *If $\langle a_n : n \in \omega\rangle$ is a strictly decreasing sequence of elements of a BA A, with $1 > a_0$, then it is a free sequence.*

Proof. Suppose that $F, G \subseteq \omega$ with $F < G$. If $F = \emptyset \neq G$, then $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta = -a_\nu \neq 0$, where ν is the least member of G. If $F \neq \emptyset = G$, then $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta = a_\nu \neq 0$, where ν is the greatest member of F. If $F \neq \emptyset \neq G$, then $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta = a_\nu \cdot -a_\mu \neq 0$, where ν is the greatest member of F and μ is the least member of G. □

Proposition 12.17. *A free sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A is maximal iff for every $b \in A$ one of the following conditions holds:*

(i) *There is a finite $F \subseteq \alpha$ such that $\prod_{\xi\in F} a_\xi \cdot b = 0$.*

(ii) *There exist finite $F, G \subseteq \alpha$ such that $F < G$ and $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta \cdot -b = 0$.* □

Proposition 12.18. *Suppose that $\langle a_\xi : \xi < \kappa\rangle$ is a strictly decreasing sequence of elements of a BA A such that $\{a_\xi : \xi < \kappa\}$ generates an ultrafilter on A and $1 > a_0$. Then $\langle a_\xi : \xi < \kappa\rangle$ is a maximal free sequence.*

Proof. $\langle a_\xi : \xi < \kappa\rangle$ is a free sequence by Proposition 12.16. Clearly it is maximal. □

The following proposition gives a connection between maximal free sequences in a BA A and towers in homomorphic images of A.

Proposition 12.19. *Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is a maximal free sequence in an atomless BA A. For each $\xi \leq \alpha$ let F_ξ be an ultrafilter containing the set $\{a_\eta : \eta < \xi\} \cup \{-a_\eta : \xi \leq \eta < \alpha\}$. Let $I = \{x \in A : \forall \xi \leq \alpha[-x \in F_\xi]\}$. Then I is an ideal in A and $0 < [a_\eta]_I < [a_\xi]_I < 1$ if $\xi < \eta < \alpha$. Moreover, if α is a limit ordinal, then $\prod_{\xi<\alpha}[a_\xi]_I = 0$, while if $\alpha = \beta + 1$ then $[a_\beta]_I$ is an atom of A/I.*

Proof. Clearly I is an ideal on A. Now suppose that $\xi < \eta < \alpha$. If $\nu \leq \alpha$ and $a_\eta \cdot -a_\xi \in F_\nu$, then $\eta < \nu$, hence also $\xi < \nu$ and so $a_\xi \in F_\nu$, contradiction. Hence $\forall \nu \leq \alpha[-(a_\eta \cdot -a_\xi) \in F_\nu]$, and so $a_\eta \cdot -a_\xi \in I$ and consequently $[a_\eta]_I \leq [a_\xi]_I$. Suppose that $[a_\eta]_I = [a_\xi]_I$. Then $a_\xi \cdot -a_\eta \in I$, and so $-a_\xi + a_\eta \in F_{\xi+1}$. Also $a_\xi \in F_{\xi+1}$, so $a_\eta \in F_{\xi+1}$. Since $\xi + 1 \leq \eta$, this is a contradiction.

Thus we have shown that $[a_\eta]_I < [a_\xi]_I$ if $\xi < \eta < \alpha$. If $[a_\eta]_I = 0$, then $a_\eta \in I$. But $a_\eta \in F_\alpha$, contradiction. If $[a_\xi]_I = 1$, then $-a_\xi \in I$. But $-a_\xi \in F_0$, contradiction.

Now suppose that $0 < [b]_I < [a_\xi]_I$ for all $\xi < \alpha$. By the maximality of $\langle a_\xi : \xi < \alpha\rangle$ there are then two possibilities. If $\prod_{\xi\in F} a_\xi \cdot b = 0$ for some finite subset F of α, then $[b]_I = 0$, contradiction. Suppose that $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta \cdot -b = 0$, where $F < G$ are finite subsets of α. If ξ is the greatest member of F and η is the smallest member of G, then $[a_\xi]_I \cdot -[a_\eta]_I \leq [b]_I \leq [a_\eta]_I$, so that $[a_\xi]_I \cdot -[a_\eta]_I = 0$, contradiction. If ξ is the greatest member of F and $G = \emptyset$, then $[a_\xi]_I \cdot -[b]_I = 0$, hence $[a_\xi]_I \leq [b]_I < [a_\xi]_i$, contradiction. If η is the smallest element of G and

$F = \emptyset$, then $-[a_\eta]_I \cdot -[b]_I = 0$, so $-[a_\eta]_I \leq [b]_I \leq [a_\eta]_I$, so that $-[a_\eta]_I = 0$, contradiction. □

Proposition 12.20. *$\pi\chi_{\inf}(A) \leq \mathfrak{f}(A)$ for any atomless BA A.*

Proof. Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is a maximal free sequence. Suppose that $2 \leq m < \omega$. We claim that

$$\left\{\prod_{\xi\in F} a_\xi : F \in [\alpha]^{<\omega}\right\} \cup \left\{\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi : F, G \in [\alpha]^{<\omega}, F < G\right\}$$

is m-dense. To see this, let $\langle b_i : i < m\rangle$ be a weak partition of A. If there is an $i < m$ such that $\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi \cdot -b_i = 0$ for some finite $F < G$, this is as desired. If for every $i < m$ there is a finite F_i such that $\prod_{\xi\in F_i} a_\xi \cdot b_i = 0$, then with $G = \bigcup_{i<m} F_i$ we have

$$\prod_{\xi\in G} a_\xi = \left(\prod_{\xi\in G} a_\xi\right) \cdot (b_0 + \cdots + b_{m-1}) = 0,$$

contradiction. □

Theorem 12.21. *$\mathfrak{p}(A) \leq \mathfrak{f}(A)$ for any atomless BA A.*

Proof. Let $\langle a_\xi : \xi < \alpha\rangle$ be a maximal free sequence, with α an infinite ordinal. Clearly $\prod_{\xi\in F} a_\xi \neq 0$, for every finite $F \subseteq \alpha$. Suppose that $0 \neq b \leq a_\xi$ for every $\xi < \alpha$. Choose u with $0 < u < b$. First suppose that $\prod_{\xi\in G} a_\xi \cdot u = 0$ for some finite $G \subseteq \alpha$. Now $u < b \leq \prod_{\xi\in G} a_\xi$, so $u = 0$, contradiction. Suppose that $\prod_{\xi\in G} a_\xi \cdot \prod_{\eta\in H} -a_\eta \cdot -u = 0$ with finite $G < H$. If $H \neq \emptyset$, choose $\eta \in H$. Then $u < b \leq a_\eta$, so $-a_\eta < -u$, and it follows that $\prod_{\xi\in G} a_\xi \cdot \prod_{\eta\in H} -a_\eta = \prod_{\xi\in G} a_\xi \cdot \prod_{\eta\in H} -a_\eta \cdot -u = 0$, contradiction. Hence $H = \emptyset$. Hence $\prod_{\xi\in G} a_\xi \leq u < b \leq \prod_{\xi\in G} a_\xi$, contradiction. □

Example 12.22. There is an atomless BA A such that $\mathfrak{f}(A) < \mathfrak{i}(A)$. This is an algebra A of McKenzie, Monk [04]: with $\omega < \kappa < \lambda$ both regular, A has a strictly decreasing sequence of length κ which generates an ultrafilter, while $\mathfrak{i}(A) = \lambda$.

Kevin Selker has constructed under CH an atomless BA A such that $\mathfrak{f}(A) < \mathfrak{u}(A)$, where $\mathfrak{u}(A)$ is the smallest size of a set generating a nonprincipal ultrafilter.

Problem 115. *Describe the possibilities for* t_{Hs}.

Clearly $\mathrm{t}_{\mathrm{Ss}}(A) = [\omega, \mathrm{t}(A)]$ for any infinite BA A.

Problem 116. *Describe* $\mathrm{t}_{\mathrm{Sr}}(A)$ *in cardinal number terms.*

Even for small cardinals there are many specific problems related to Problem 117.

Problem 117. *Describe* $\mathrm{t}_{\mathrm{Hr}}(A)$ *in cardinal number terms.*

Again, for small cardinals there are many specific problems related to Problem 118.

There are several natural finite versions of tightness, using the free sequence equivalent. For $m, n \in \omega$, an *m, n-free sequence* is a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ such that if $\Gamma, \Delta \subseteq \alpha$ with $|\Gamma| = m$, $|\Delta| = n$, and $\Gamma < \Delta$, then $\prod_{\alpha\in\Gamma} a_\alpha \cdot \prod_{\beta\in\Delta} -a_\beta \neq 0$. Then we set

$$\mathrm{t}_{mn}(A) = \sup\{\kappa : \text{there is an } m, n\text{-free sequence of length } \kappa\}.$$

Similarly we get three more notions:

An *m-free sequence* is a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ such that if $\Gamma, \Delta \subseteq \alpha$ with $|\Gamma| = m$, Δ finite and $\Gamma < \Delta$, then $\prod_{\alpha\in\Gamma} a_\alpha \cdot \prod_{\beta\in\Delta} -a_\beta \neq 0$;

$$\mathrm{t}_m(A) = \sup\{\kappa : \text{there is an } m\text{-free sequence of length } \kappa\}.$$

$$\mathrm{ut}_{mn}(A) = \sup\{|X| : \forall Y \in [X]^m \text{ and } \forall Z \in [X]^n (Y \cap Z = 0 \Rightarrow \prod_{y\in Y} y \cdot \prod_{z\in Z} -z \neq 0)\};$$

$$\mathrm{ut}_m(A) = \sup\{|X| : \forall Y \in [X]^m \text{ and } \forall \text{ finite } Z (Y \cap Z = 0 \Rightarrow \prod_{y\in Y} y \cdot \prod_{z\in Z} -z \neq 0)\}.$$

These notions are studied in Rosłanowski, Shelah [94].

Concerning tightness for special classes of algebras, note first of all that $\mathrm{t}(A) = |A|$ whenever A is complete. The description of t for interval algebras is similar to that for $\pi\chi$. Since t coincides with $\mathrm{Depth}_{\mathrm{H}+}$, $\mathrm{t}(A) = \mathrm{Depth}(A)$ for A an interval algebra, by retractiveness. But it is of some interest to describe $\mathrm{t}(F)$ for each ultrafilter F on an interval algebra; this is a correction of the description in Monk [96]. Let A be the interval algebra on a linearly ordered set L with first element 0. Let C be a terminal segment of L not containing 0. The ultrafilter F_C determined by C is generated by

$$\{[a, \infty) : a \notin C\} \cup \{[0, c) : c \in C\}.$$

Case 1. $C = \emptyset$, and L has a greatest element a. Then F_C is the principal ultrafilter generated by $\{a\}$, and $\mathrm{t}(F_C) = 1$.

Case 2. $C = \emptyset$, and L does not have a greatest element. Let κ be the cofinality of L. We claim that $\mathrm{t}(F_C) = \kappa$. To prove this, let $\langle a_\xi : \xi < \kappa\rangle$ be a strictly increasing cofinal sequence of elements of L. For each $\xi < \kappa$ let G_ξ be an ultrafilter such that $[a_\xi, a_{\xi+1}) \in G_\xi$. Clearly $F_C \subseteq \bigcup_{\xi<\kappa} G_\xi$. If $M \in [\kappa]^{<\kappa}$, choose $\xi < \kappa$ such that $\eta < \xi$ for all $\eta \in M$. Then $[a_\xi, \infty) \in F_C \setminus \bigcup_{\eta\in M} G_\eta$. This shows that $\mathrm{t}(F_C) \geq \kappa$.

Now suppose that $F_C \subseteq \bigcup Y$, with $Y \subseteq \mathrm{Ult}(A)$. For each $\xi < \kappa$ choose $H_\xi \in Y$ such that $[a_\xi, \infty) \in H_\xi$. Clearly $F_C \subseteq \bigcup_{\xi<\kappa} H_\xi$. This shows that $\mathrm{t}(F_C) = \kappa$.

Case 3. C has a smallest element b and $L\setminus C$ has a greatest element a. Then F_C is the principal ultrafilter generated by $\{a\}$, and $\mathrm{t}(F_C) = 1$.

Case 4. C has a smallest element b, but $L\setminus C$ does not have a greatest element. Then $\mathrm{t}(F_C) = \kappa$, where κ is the cofinality of $L\setminus C$; see Case 2.

Case 5. C does not have a smallest element, but $L\backslash C$ has a greatest element a. Then $\mathrm{t}(F_C) = \kappa$, where κ is the coinitiality of C; see Case 2.

Case 6. C does not have a smallest element, and $L\backslash C$ does not have a greatest element. Let κ be the cofinality of $L\backslash C$ and λ the coinitiality of C. Then $\mathrm{t}(F_C) = \max(\kappa, \lambda)$. We prove this under the assumption that $\kappa \le \lambda$, by symmetry. Let $\langle a_\xi : \xi < \kappa\rangle$ be strictly increasing and cofinal in $L\backslash C$, and $\langle b_\xi : \xi < \lambda\rangle$ strictly decreasing and coinitial in C. For any $\xi < \kappa$ and $\eta < \lambda$ let $G_{\xi\eta}$ be an ultrafilter containing $[a_\xi, b_\eta)$. Clearly $F_C \subseteq \bigcup_{(\xi,\eta)\in\kappa\times\lambda} G_{\xi\eta}$. Suppose that $M \in [\kappa\times\lambda]^{<\lambda}$. Then there is a $\rho < \lambda$ such that $\eta < \rho$ whenever $(\xi, \eta) \in M$. Then $[a_0, b_\rho)$ is a member of $F_C\backslash\bigcup_{(\xi,\eta)\in M} G_{\xi\eta}$. This shows that $\mathrm{t}(F_C) \ge \lambda$.

Now suppose that $F_C \subseteq \bigcup Y$ with $Y \subseteq \mathrm{Ult}(A)$. For each $\xi, \eta) \in \kappa\times\lambda$ let $H_{\xi\eta} \in Y$ be such that $[a_\xi, b_\eta) \in H_{\xi\eta}$. Clearly $F \subseteq \bigcup_{(\xi,\eta)\in\kappa\times\lambda} H_{\xi\eta}$. Hence $\mathrm{t}(F_C) = \lambda$.

Concerning $\mathfrak{f}$ and interval algebras we have the following result.

Proposition 12.23. *Let L be a linear ordering.*

(i) *If $\langle a_\xi : \xi < \alpha\rangle$ is a strictly increasing sequence with lub b, with α a limit ordinal, then $\langle [a_\xi, b) : \xi < \alpha\rangle$ is a maximal free sequence in* $\mathrm{Intalg}(L)$.

(ii) *If $\langle a_\xi : \xi < \alpha\rangle$ is a strictly decreasing sequence with glb b, with α a limit ordinal, then $\langle [b, a_\xi) : \xi < \alpha\rangle$ is a maximal free sequence in* $\mathrm{Intalg}(L)$.

(iii) *Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is strictly increasing, $\langle b_\xi : \xi < \alpha\rangle$ is strictly decreasing, $\forall\xi < \alpha[a_\xi < b_\xi]$, and there is no element $c \in L$ such that $\forall\xi < \alpha[a_\xi < c < b_\xi]$. Then $\langle [a_\xi, b_\xi) : \xi < \alpha\rangle$ is a maximal free sequence in* $\mathrm{Intalg}(L)$.

(iv) *Suppose that κ and λ are distinct infinite cardinals, $\langle a_\xi : \xi < \kappa\rangle$ is strictly increasing, $\langle b_\xi : \xi < \lambda\rangle$ is strictly decreasing, $\forall\xi < \kappa\forall\eta < \lambda[a_\xi < b_\eta]$, and there is no element $c \in L$ such that $\forall\xi < \kappa\forall\eta < \lambda[a_\xi < c < b_\eta]$. Then there is a maximal free sequence of length* $\max(\kappa, \lambda)$ *in* $\mathrm{Intalg}(L)$.

Proof. (i): Let x be any nonzero element of $\mathrm{Intalg}(L)$. We consider two cases.

Case 1. For every component $[c, d)$ of x we have $b \le c$ or $d < b$. Clearly then there is a $\xi < \alpha$ such that $[a_\xi, b) \cap [c, d) = \emptyset$ for every component $[c, d)$ of x. Hence $[a_\xi, b) \cap x = \emptyset$, as desired in 12.17.

Case 2. There is a component $[c, d)$ of x such that $c < b \le d$. Then there is a $\xi < \alpha$ such that $[a_\xi, b) \subseteq [c, d) \subseteq x$. So $[a_\xi, b) \cdot -x = \emptyset$, again as desired in 12.17.

The proof of (ii) is similar, but (iii) is more complicated. Clearly $\langle [a_\xi, b_\xi) : \xi < \alpha\rangle$ is a free sequence in $\mathrm{Intalg}(L)$. Now suppose that x is a nonzero element of $\mathrm{Intalg}(L)$. If for every component $[c, d)$ of x there is a $\xi < \alpha$ such that $b_\xi < c$ or $d < a_\xi$, then there is a $\xi < \alpha$ such that $x \cap [a_\xi, b_\xi) = \emptyset$, as desired. So, suppose that there is a component $[c, d)$ of x such that for every $\xi < \alpha$ we have $c \le b_\xi$ and $a_\xi \le d$. Then by the hypothesis of (iii) there is a $\xi < \alpha$ such that $[a_\xi, b_\xi) \subseteq [c, d)$. Hence $[a_\xi, b_\xi)\backslash x = \emptyset$, as desired.

Now assume the hypothesis of (iv). By symmetry say $\kappa < \lambda$. For each $\varphi < \lambda$ write $\varphi = \kappa \cdot \rho + \xi$ with $\xi < \kappa$, and set $c_\varphi = [a_\xi, b_\varphi)$. Suppose that $F, G \in [\lambda]^{<\omega}$ and $F < G$. We may assume that $G \neq \emptyset$. First suppose also that $F \neq \emptyset$. Let φ be the greatest element of F and ψ the least element of G. Then $\bigcap F$ has the form $[a_\tau, b_\varphi)$ and $\cap_{e \in G} - e$ has the form $[0, a_\sigma) \cup [b_\psi, \infty)$. So $\bigcap F \cap \bigcap_{e \in G} -e$ contains $[b_\psi, b_\varphi)$ and hence is nonempty.

The case $F = \emptyset$ is also clear by this argument.

Thus we have a free sequence. To show that it is maximal, it suffices to show that $\{c_\varphi : \varphi < \lambda\}$ generates an ultrafilter. Let x be any member of A, with $0 < x < 1$. First suppose that x has a component $[u, v)$ with $u \notin L$ and $v \in L$. Choose $\xi < \kappa$ and $\eta < \lambda$ such that $u < a_\xi$ and $b_\eta < v$. Let $\varphi = \kappa \cdot \eta + \xi$. Then $\varphi \geq \eta$, and hence $b_\varphi \leq b_\eta$. So $c_\varphi \subseteq x$. Second, if x has no such component, then $-x$ does have such a component and so we get a $\varphi < \lambda$ such that $c_\varphi \subseteq -x$. □

Corollary 12.24. $\mathfrak{f}(A) \leq \mathrm{tow}(A)$ *for any atomless interval algebras A.*

Proof. By Theorem 4.82 and Proposition 12.23. □

For tree algebras we have $\mathrm{t}(\mathrm{Treealg}(T)) = \mathrm{Depth}(\mathrm{Treealg}(T))$ by retractiveness. Now take any ultrafilter F on $\mathrm{Treealg}(T)$. It corresponds to an initial chain C of T; see the Handbook. A description of $\mathrm{t}(F)$, due to Brenner [82], is as follows:

Theorem 12.25. *Let T be a tree with a single root and F an ultrafilter on* $\mathrm{Treealg}(T)$*. Let $C = \{t \in T : (T \uparrow t) \in F\}$. Then one of the following holds:*

(i) *C has a greatest element t, and t has only finitely many immediate successors. Then F is principal, and* $\mathrm{t}(F) = 1$*.*

(ii) *C has a greatest element t, and t has infinitely many immediate successors. Then* $\mathrm{t}(F) = \omega$*.*

(iii) *C has no greatest element. Then* $\mathrm{t}(F) = \mathrm{cf}(C)$*.*

Proof. (i) is obvious. For (ii), suppose that C has a greatest element t and t has infinitely many immediate successors. Suppose that $F \subseteq \bigcup Y$, where $Y \subseteq \mathrm{Ult}(\mathrm{Treealg}(T))$. Without loss of generality $F \notin Y$. We claim

(1) $S \overset{\mathrm{def}}{=} \{s : s$ is an immediate successor of t and $(T \uparrow s) \in G$ for some $G \in Y\}$ is infinite.

For, suppose that S is finite. Now $(T \uparrow t) \backslash \bigcup_{s \in S} (T \uparrow s) \in F$, so choose $G \in Y$ such that $(T \uparrow t) \backslash \bigcup_{s \in S} (T \uparrow s) \in G$. For every immediate successor s of t we have $(T \uparrow s) \notin G$. So $F = G$, contradiction. So (1) holds.

Let $U \in [S]^\omega$. For each $u \in U$ choose $G_u \in Y$ such that $(T \uparrow u) \in G_u$. Now suppose that $x \in F$. Without loss of generality x has the form $(T \uparrow t) \backslash \bigcup_{v \in V} (T \uparrow v)$ where V is a finite set of immediate successors of t. Choose $u \in U \backslash V$. Clearly $(T \uparrow t) \backslash \bigcup_{v \in V} (T \uparrow v) \in G_u$, as desired.

In the present case it is clear that F is nonprincipal, so $\mathrm{t}(F) = \omega$.

For (iii), suppose that C has no greatest element. Let $\langle s_\alpha : \alpha < \mathrm{cf}\, C\rangle$ be a strictly increasing cofinal sequence of elements of C.

First we show that $\mathrm{t}(F) \geq \mathrm{cf}(C)$. For each $\alpha < \mathrm{cf}(C)$ the set

$$\{T \uparrow s_\alpha\} \cup \{T \backslash (T \uparrow u) : u \text{ is an immediate successor of } s_\alpha\}$$
$$\cup \{T \backslash (T \uparrow v) : v \text{ and } s_\alpha \text{ are incomparable}\}$$

has the fip, as is easily seen; let G_α be an ultrafilter containing this set. We claim that $F \subseteq \bigcup_{\alpha < \mathrm{cf}(C)} G_\alpha$. For, suppose that $x \in F$. We may assume that

(2) $x = (T \uparrow r) \backslash \bigcup_{u \in U} (T \uparrow u)$ where $r \in C$, U is a set of immediate successors of r, and $U \cap C = 0$.

Choose $\alpha < \mathrm{cf}(C)$ such that $r \leq s_\alpha$. Clearly $x \in G_\alpha$, as desired.

Now suppose that $\Gamma \subseteq \mathrm{cf}(C)$ and $|\Gamma| < \mathrm{cf}(C)$. We claim that $F \not\subseteq \bigcup_{\alpha \in \Gamma} G_\alpha$. For, choose $\beta < \mathrm{cf}(C)$ such that $\Gamma < \beta$. Clearly $(T \uparrow s_\beta) \in F$ but $(T \uparrow s_\beta) \notin \bigcup_{\alpha \in \Gamma} G_\alpha$.

So, we have shown that $\mathrm{t}(F) \geq \mathrm{cf}(C)$.

Now suppose that $F \subseteq \bigcup Y$, $Y \subseteq \mathrm{Ult}(\mathrm{Treealg}\,(T))$. We want to find $Z \in [Y]^{\leq \mathrm{cf}(C)}$ such that $F \subseteq \bigcup Z$. We may assume that $F \notin Y$. For each $\alpha < \mathrm{cf}(C)$ let $y_\alpha = (T \uparrow s_\alpha)$ and let $G_\alpha \in Y$ be such that $y_\alpha \in G_\alpha$. We claim that $F \subseteq \bigcup_{\alpha < \mathrm{cf}(C)} G_\alpha$. Let $x \in F$. We may assume that x is as in (2). Choose $\alpha < \mathrm{cf}(C)$ such that $r < \alpha$. Then $y_\alpha \subseteq x$, and so $x \in G_\alpha$, as desired. $\square$

Concerning superatomic BAs, Shelah and Spinas [99] proved the following, improving a result of Dow, Monk [94]:

Suppose that $0^\sharp$ *exists. Let* B *be a superatomic Boolean algebra in the constructible universe* L*, and let* λ *be an uncountable cardinal in* V*. Then in* L *it is true that* $\mathrm{t}'(B) \geq \lambda^+$ *implies that* $\mathrm{Depth}'(B) \geq \lambda$.

This solves Problem 45 of Monk [96]. But the following is still open.

Problem 118. *Can one prove in ZFC that for every uncountable cardinal* λ*,* $\mathrm{t}'(B) \geq \lambda^+$ *implies that* $\mathrm{Depth}'(B) \geq \lambda$*?*

13 Spread

First note that any infinite BA A has an infinite disjoint subset D, which gives rise to an infinite discrete subspace of $\mathrm{Ult}(A)$. So $\mathrm{s}(A)$ is always infinite, for A infinite.

The following theorem gives some equivalent definitions of spread.

Theorem 13.1. *For any infinite BA A, $\mathrm{s}(A)$ is equal to each of the following cardinals:*

$\sup\{|X| : X$ *is a minimal set of generators of* $\langle X\rangle^{\mathrm{Id}}\}$*;*
$\sup\{|X| : X$ *is ideal-independent*$\}$*;*
$\sup\{|X| : X$ *is the set of all atoms in some homomorphic image of* $A\}$*;*
$\sup\{|\mathrm{At}(B)| : B$ *is an atomic homomorphic image of* $A\}$*;*
$\sup\{\mathrm{c}(B) : B$ *is a homomorphic image of* $A\}$*.*

Proof. Six cardinals are mentioned in this theorem; let them be denoted by $\kappa_0, \ldots, \kappa_5$ in the order that they are mentioned. In Theorem 3.29 we proved that $\kappa_0 = \kappa_2$, and in Theorem 3.30 that $\kappa_2 = \kappa_5$. It is obvious that $\kappa_1 = \kappa_2$. To show that $\kappa_3 \leq \kappa_4$, suppose that B is a homomorphic image of A with an infinite number of atoms. Let I be the ideal $\langle\{x : x \cdot a = 0 \text{ for every atom } a \text{ of } B\}\rangle^{\mathrm{Id}}$ of B. Clearly B/I is atomic with the same number of atoms as B. This shows that $\kappa_3 \leq \kappa_4$. Obviously $\kappa_4 \leq \kappa_5$ Finally, for $\kappa_5 \leq \kappa_3$, let B be a homomorphic image of A, and let D be an infinite disjoint subset of B. We show how to find an atomic homomorphic image C of B with exactly $|D|$ atoms. Let M be the subalgebra of B generated by D. Let f be an extension of the identity on D to a homomorphism of B into $\overline{D}$; the image of B under f is as desired. □

From these characterizations it follows that if A is a subalgebra or homomorphic image of B, then $\mathrm{s}(A) \leq \mathrm{s}(B)$. Clearly the difference can be arbitrarily large.

Clearly one can have $\mathrm{s}(B)$ much larger than $\mathrm{s}(A)$ when $A \leq_{\mathrm{free}} B$; so the same applies to $A \leq_{\mathrm{proj}} B$, $A \leq_{\mathrm{u}} B$, $A \leq_{\mathrm{rc}} B$, $A \leq_{\mathrm{reg}} B$, and $A \leq_{\sigma} B$. One can have $\mathrm{s}(A) < \mathrm{s}(B)$ also when $A \leq_{\pi} B$; for example, take $A = \mathrm{Fr}(\kappa)$ and $B = \overline{A}$. But when $A \leq_{\pi} B$, we have $\mathrm{s}(B) \leq |B| \leq 2^{|A|}$. If $A \leq_{\mathrm{s}} B$, in particular if $A \leq_{\mathrm{m}} B$, then $\mathrm{s}(A) = \mathrm{s}(B)$; this follows easily from Proposition 2.29. From Proposition 2.54 it is clear that one can have $A \leq_{\mathrm{mg}} B$ with $\mathrm{s}(B)$ much larger than $\mathrm{s}(A)$.

Note that all of the equivalents of spread given in Theorem 13.1 have the same attainment properties. The attainment question for spread has been thoroughly

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studied in topology, mainly by Juhász. We present these results, specialized to Boolean algebras.

For the first result we will use the general canonization lemma, 28.1 in Erdős, Hajnal, Máté, Rado [84]. We describe a special case of that lemma which we will use:

Define $\beth_0(\kappa) = \kappa$ *and* $\beth_{m+1}(\kappa) = \beth_m(2^\kappa)$ *for every* $m \in \omega$. *Suppose that* κ *is a strong limit singular cardinal, and* $\langle \lambda_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ *is a strictly increasing sequence of cardinals with supremum* κ *satisfying the following conditions:*

(i) $2^{\mathrm{cf}(k)} \le \lambda_0$.

(ii) $\beth_3(\lambda_\xi) < \lambda_\eta$ *for* $\xi < \eta < \mathrm{cf}(\kappa)$.

Suppose also that $\langle A_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ *is a system of pairwise disjoint sets, and* $|A_\xi| \ge (\beth_3(\lambda_\xi))^+$ *for all* $\xi < \mathrm{cf}(\kappa)$. *Let* $B = \bigcup_{\xi < \mathrm{cf}(\kappa)} A_\xi$. *Suppose that* $f : [B]^3 \to 2 \times 2$.

Then there is a system $\langle C_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ *satisfying the following conditions:*

(iii) $C_\xi \subseteq A_\xi$ *and* $|C_\xi| \ge \lambda_\xi^+$ *for each* $\xi < \mathrm{cf}(\kappa)$.

(iv) *Let* $D = \bigcup_{\xi < \mathrm{cf}(\kappa)} C_\xi$. *Then for any* $u, v \in [D]^3$, *if* $|u \cap C_\xi| = |v \cap C_\xi|$ *for all* $\xi < \mathrm{cf}(\kappa)$, *then* $f(u) = f(v)$.

Theorem 13.2. *If* κ *is strong limit singular and* $\kappa \le |\mathrm{Ult}(E)|$, *then* $\mathrm{Ult}(E)$ *has a discrete subset of size* κ.

Proof. Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be sequence of cardinals as in the above statement of the canonization lemma. Let $\prec$ be a well-order of $\mathrm{Ult}(E)$. $\langle A_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be a system of pairwise disjoint subsets of $\mathrm{Ult}(E)$, with $|A_\xi| = (\beth_3(\lambda_\xi))^+$ for all $\xi < \mathrm{cf}(\kappa)$, and with $A_\xi \prec A_\eta$ for $\xi < \eta < \mathrm{cf}(\kappa)$. Let $B = \bigcup_{\xi < \mathrm{cf}(\kappa)} A_\xi$. We will find a discrete subset of B of size λ. For distinct $F, G \in B$ choose $a_{FG} \in F \backslash G$. Let $\prec$ be a well-order of B. We now define a function $f : [B]^3 \to 2 \times 2$. For $F \prec G \prec H$, let $f(\{F, G, H\}) = (\varepsilon_1, \varepsilon_2)$, where

$$\varepsilon_1 = \begin{cases} 0 & \text{if } a_{GH} \in F, \\ 1 & \text{otherwise;} \end{cases} \qquad \varepsilon_2 = \begin{cases} 0 & \text{if } -a_{FG} \in H, \\ 1 & \text{otherwise.} \end{cases}$$

We now apply the above canonization lemma to get a system $\langle C_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ satisfying the following conditions:

(1) $C_\xi \subseteq A_\xi$ and $|C_\xi| \ge \lambda_\xi^+$ for each $\xi < \mathrm{cf}(\kappa)$.

(2) Let $D = \bigcup_{\xi < \mathrm{cf}(\kappa)} C_\xi$. Then for any $u, v \in [D]^3$, if $|u \cap C_\xi| = |v \cap C_\xi|$ for all $\xi < \mathrm{cf}(\kappa)$, then $f(u) = f(v)$.

(3) $C_\xi \prec C_\eta$ for $\xi < \eta < \mathrm{cf}(\kappa)$.

It suffices to find a discrete subset of D of size λ. In fact, let

$$\begin{aligned} Z = \{G \in D : G &\text{ has an immediate predecessor } F, \text{ an immediate successor } H, \\ &\text{and there is a } \xi < \mathrm{cf}(\kappa) \text{ such that} \{F, G, H\} \subseteq C_\xi\}. \end{aligned}$$

By conditions (1)–(3) it is clear that $|Z| = \lambda$. We claim that it is discrete. For, take any $G \in Z$; say that G has an immediate predecessor F an immediate successor H, $\xi < \mathrm{cf}(\kappa)$, and $\{F, G, H\} \subseteq C_\xi$. We want to show that G is isolated in Z. In fact, let $N = \mathcal{S}(-a_{FG}) \cap \mathcal{S}(a_{GH}) \cap Z$. Clearly $G \in N$ and $F, H \notin N$. Suppose that $K \in Z$ and $K \prec F$. Clearly $\forall \eta < \mathrm{cf}(\kappa)[|\{K, F, G\} \cap C_\eta\}| = |\{K, G, H\} \cap C_\eta|]$, so $f(\{K, F, G\}) = f(\{K, G, H\})$.

Case 1. $a_{FG} \in K$. Obviously then $K \notin N$.

Case 2. $-a_{FG} \in K$. Then $f(\{K, F, G\})$ has the form $(1, \varepsilon_2)$, so also

$$f(\{K, G, H\}) = (1, \varepsilon_2).$$

So $-a_{GH} \in K$ and hence $K \notin N$.

The case $H \prec K$ is similar: Clearly

$$\forall \eta < \mathrm{cf}(\kappa)[|\{K, F, G\} \cap C_\eta\}| = |\{K, G, H\} \cap C_\eta|],$$

so $f(\{K, F, G\}) = f(\{K, G, H\})$.

Case a. $-a_{GH} \in K$. Obviously then $K \notin N$.

Case b. $a_{GH} \in K$. Then $f(\{G, H, K\})$ has the form $(\varepsilon_1, 1)$, so also

$$f(\{F, G, K\}) = (\varepsilon_1, 1).$$

So $a_{FG} \in K$ and hence $K \notin N$. □

Corollary 13.3. *If* $\mathrm{s}(A)$ *is singular strong limit, then it is attained.* □

Our second attainment theorem for spread uses an argument (in Juhász [80]) easily adapted for hL and hd; so we treat all three simultaneously. To this end we need a definition, whose connection with hL will come later.

A sequence $\langle x_\xi : \xi < \kappa \rangle$ of distinct elements of a topological space X is *right-separated* provided that for every $\xi < \kappa$ the set $\{x_\eta : \eta \le \xi\}$ is open in $\{x_\xi : \xi < \kappa\}$;

Also recall from just before Theorem 6.12 the definition of left-separated.

Proposition 13.4. *Suppose that λ is a singular cardinal of cofinality ω, and suppose that A is an infinite BA. A subset X of* $\mathrm{Ult}(A)$ *is of type t, where $t < 3$, if $t = 0$ and X is discrete; or $t = 1$ and X is the range of a left-separated sequence; or $t = 2$ and X is the range of a right-separated sequence. Then for any $t < 3$, if λ is the supremum of sizes of type t subsets of* $\mathrm{Ult}(A)$, *then* $\mathrm{Ult}(A)$ *has a subset of type t and cardinality λ.*

Proof. Let $\langle \mu_n : n \in \omega \rangle$ be a strictly increasing sequence of uncountable regular cardinals with supremum λ. Let $t < 3$. Let $\langle D_n : n \in \omega \rangle$ be a system of subspaces of $\mathrm{Ult}(A)$ of type t such that $|D_n| = \mu_n$ for all $n \in \omega$, and let $Y = \bigcup_{n \in \omega} D_n$. It suffices to find a subspace of Y of type t and size λ. Let $Z = \bigcup\{U : U$ is an open subset of Y of size less than $\lambda\}$.

(1) If $W \subseteq Y$ has size κ with $\omega < \kappa < \lambda$ and κ is regular, then W has a subset of size κ and of type t.

In fact, $W = \bigcup_{n\in\omega}(W \cap D_n)$, so by the regularity of κ there is an $n \in \omega$ such that $|W \cap D_n| = \kappa$. So (1) holds.

Now to prove the Proposition we consider two cases.

Case 1. $|Z| < \lambda$. Thus $|Y\backslash Z| = \lambda$.

(2) Every nonempty open subspace of $Y\backslash Z$ has size λ.

In fact if U is a nonempty open subspace of $Y\backslash Z$, then there is an open subset V of Y such that $U = V \cap (Y\backslash Z)$. Then $|V| = \lambda$, and hence $|U| = \lambda$.

(3) There is a system $\langle a_n : n \in \omega\rangle$ of pairwise disjoint nonzero elements of A such that $\mathcal{S}(a_n) \cap (Y\backslash Z) \neq \emptyset$ for all $n \in \omega$.

In fact, suppose that we have constructed a_i for all $i < n$ so that they are nonzero, pairwise disjoint, $\mathcal{S}(a_i) \cap (Y\backslash Z) \neq \emptyset$ for all $i < n$, and $Y\backslash \left(Z \cup \bigcup_{i<n} \mathcal{S}(a_i)\right)$ is open in $Y\backslash Z$ and of size λ. Fix distinct $F, G \in Y\backslash \left(Z \cup \bigcup_{i<n} \mathcal{S}(a_i)\right)$, and let $a_n \in F$, $b_n \in G$, with $a_n \cdot b_n = 0$ and $\mathcal{S}(a_n) \cap (Y\backslash Z), \mathcal{S}(b_n) \cap (Y\backslash Z) \subseteq Y\backslash \left(Z \cup \bigcup_{i<n} \mathcal{S}(a_i)\right)$. This finishes the construction.

Now by (1) each set $\mathcal{S}(a_n) \cap (Y\backslash Z)$ contains a set E_n of size μ_n and type t, and then $\bigcup_{n\in\omega} E_n$ is a subset of $Y\backslash Z$ of size λ and type t.

Case 2. $|Z| = \lambda$. It suffices to find a subset of Z of size λ and type t. For each $\kappa < \lambda$ let

$$X_\kappa = \{F \in Z : \exists a_F \in F[|\mathcal{S}(a_F) \cap Y| < \kappa]\}.$$

Note that $Z = \bigcup_{\kappa<\lambda} X_\kappa$.

Subcase 2.1. $\forall\kappa < \lambda[|X_\kappa| < \lambda]$. We define $\langle F_m : m \in \omega\rangle$, $\langle n(i) : i < \omega\rangle$, and $\langle a_i : i \in \omega\rangle$ by recursion. Let $n(0) = 0$. Choose $F_0 \in Z\backslash X_{\mu_0}$. Choose $n(1) > n(0)$ so that $F_0 \in X_{\mu_{n(1)}}$. Let $a_0 \in F_0$ with $|\mathcal{S}(a_0) \cap Y| < \mu_{n(1)}$. Now $F_0 \notin X_{\mu_0}$, so $|\mathcal{S}(a_0) \cap Y| \geq \mu_0$. By (1), $\mathcal{S}(a_0) \cap Y$ has a subset of size μ_0 and type t.

Now suppose that F_i, a_i, $n(i)$, $n(i+1)$ have been defined for all $i \leq m$ so that for all $i \leq m$, $F_i \in Z\backslash X_{\mu_{n(i)}}$, $F_i \in X_{\mu_{n(i+1)}}$, $\mathcal{S}(a_i) \cap Y)$ has a subset of size $\mu_{n(i)}$ and type t, and $|\mathcal{S}(a_i) \cap Y| < \mu_{n(i+1)}$. Clearly these conditions hold for $i = 0$. Choose $F_{m+1} \in Z\backslash X_{\mu_{n(m+1)}}$. Say $F_{m+1} \in X_{\mu_{n(m+2)}}$ with $m+1 < m+2$. Choose $c \in F_{m+1}$ with $|\mathcal{S}(c) \cap Y| < \mu_{n(m+2)}$. Note that if $i \leq m$ then $a_i \notin F_{m+1}$. In fact, if $a_i \in F_{m+1}$, then, since $|\mathcal{S}(a_i) \cap Y| < \mu_{n(i+1)}$, we would get $F_{m+1} \in X_{n(i+1)} \subseteq X_{n(m+1)}$, contradiction. Let $a_{m+1} = c \cdot \prod_{i\leq m} -a_i$. Then $a_{m+1} \in F_{m+1}$; since $F_{m+1} \notin X_{\mu_{n(m+1)}}$, it follows that $\mathcal{S}(a_{m+1}) \cap Y$ has a subset of size $\mu_{n(m+1)}$ and type t.

This finishes the construction, and shows that Y has a subset of size λ and type t.

Subcase 2.2. There is a $\kappa < \lambda$ such that $|X_\kappa| = \lambda$. For each $F \in X_\kappa$ let $\mathscr{F}(F) = \mathcal{S}(a_F) \cap X_\kappa$. Thus $\mathscr{F} : X_\kappa \to [X_\kappa]^{<\kappa}$, and $\kappa < \lambda$. It follows from Hajnal's free set

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theorem that there is a set $S \subseteq X_\kappa$ of size λ which is free for $\mathscr{F}$, i.e., $\forall F, G \in S[F \neq G \to F \notin \mathscr{F}(G)]$. Thus $\{F\} = \mathcal{S}(a_F) \cap X_\kappa$ for all $F \in S$, so X_κ is discrete. Let $\langle F_\xi : \xi < \kappa\rangle$ enumerate S without repetitions. Take any $\xi < \kappa$. Then $\bigcup_{\eta \leq \xi} \mathcal{S}(x_{F_\eta})$ is an open set whose intersection with S is $\{F_\eta : \eta < \xi\}$; so $\langle F_\xi : \xi < \kappa\rangle$ is right-separated. Similarly, it is left-separated. $\square$

Assuming V=L, if κ is inaccessible but not weakly compact, then there is a BA A with spread κ not attained. Namely, one can take a κ-Suslin tree T in which every element has infinitely many immediate successors, and let $A = \mathrm{Treealg}(T)$. By the retractiveness of A, Theorem 13.1, and Theorem 3.59, A is as desired.

Theorem 13.5. *If κ is uncountable and weakly compact, and if* $\mathrm{s}(A) = \kappa$*, then* $\mathrm{s}(A)$ *is attained.*

Proof. We modify the proof of Theorem 13.2. Let $\langle \mu_\alpha : \alpha < \kappa\rangle$ be a strictly increasing sequence of uncountable regular cardinals with supremum κ. For each $\alpha < \kappa$ let D_α be a discrete subset of $\mathrm{Ult}(A)$ of size μ_α. Let $X = \bigcup_{\alpha < \mathrm{cf}(\kappa)} D_\alpha$. It suffices to find a discrete subset of X of size κ. For distinct $F, G \in X$ choose $a_{FG} \in F \backslash G$. Let $\prec$ be a well-order of X. We now define a function $f : [X]^3 \to 2 \times 2$. For $F \prec G \prec H$, let $f(\{F, G, H\}) = (\varepsilon_1, \varepsilon_2)$, where

$$\varepsilon_1 = \begin{cases} 0 & \text{if } a_{GH} \in F, \\ 1 & \text{otherwise;} \end{cases} \qquad \varepsilon_2 = \begin{cases} 0 & \text{if } -a_{FG} \in H, \\ 1 & \text{otherwise.} \end{cases}$$

Now $\kappa \to (\kappa)^3_4$; see Erdős, Hajnal, Máté, Rado [84], Theorem 29.6. Hence choose $Y \in [X]^\kappa$ and $(\varepsilon_0, \varepsilon_1) \in 2 \times 2$ such that $f \restriction [Y]^3$ takes on the constant value $(\varepsilon_0, \varepsilon_1)$. Let $Z = \{L \in Y : L$ has an immediate predecessor and an immediate successor$\}$. We claim that Z is discrete. To prove this, take any $G \in Z$. Let F be the immediate predecessor of G and H its immediate successor. Note that possibly $F \notin Z$ or $H \notin Z$. Let $b = a_{GH} \cdot -a_{FG}$. Thus $b \in G$, $b \notin F$, and $b \notin H$. Take any $K \in Z \backslash \{F, G, H\}$.

Case 1. $a_{GH} \in F$. Then $f(F, G, H) = 0$. If $K < F < G$, then $f(K, F, G) = 0$ and so $a_{FG} \in H$ and $b \notin H$. If $F < G < K$, then $f(G, H, K) = 0$ and so $a_{GH} \notin K$ and $b \notin K$.

Case 2. $a_{GH} \notin F$. Then $f(F, G, H) = 1$. If $K < G < H$, then $f(K, G, H) = 1$ and so $a_{GH} \notin K$ and $b \notin K$. If $F < G < K$, then $f(F, G, K) = 1$ and so $a_{FG} \in K$ and $b \notin K$. $\square$

The following two results are clear from topological duality.

Proposition 13.6. *If A and B are infinite BAs, then* $\mathrm{s}(A \times B) = \max(\mathrm{s}(A), \mathrm{s}(B))$. $\square$

Proposition 13.7. *Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs each with at least two elements. Then* $\mathrm{s}\left(\prod^{\mathrm{w}}_{i \in I} A_i\right) = \max(|I|, \sup_{i \in I} \mathrm{s}(A_i))$. $\square$

Clearly $\mathrm{s}\left(\prod_{i\in I}A_i\right)\geq\max(2^{|I|},\sup_{i\in I}\mathrm{s}(A_i))$. Shelah and Peterson independently observed that strict inequality is possible, thus answering Problem 35 of Monk [90]. Namely, let κ be the first limit cardinal bigger than 2^ω (thus κ has cofinality ω), let A be the finite-cofinite algebra on κ, and consider ${}^\omega A$. Then for any non-principal ultrafilter on ω we have

$$\kappa^\omega=|{}^\omega A|\geq\mathrm{s}({}^\omega A)\geq\mathrm{c}({}^\omega A/F)=\kappa^\omega$$

by the discussion of ultraproducts for cellularity. Thus ${}^\omega A$ gives a product where this inequality is strict.

Turning to ultraproducts, note that spread is an ultra-sup function, so Theorems 3.20–3.22 apply; Theorem 3.22 says that $\mathrm{s}\left(\prod_{i\in I}A_i/F\right)\geq\left|\prod_{i\in I}\mathrm{s}(A_i)/F\right|$ for F regular, and Donder's theorem says that under $V=L$ the regularity assumption can be removed. A consistent example with $>$ is described in Rosłanowski, Shelah [98]. Two further consistency results are as follows. Shelah [99] proved the following result, part of Conclusion 15.13:

If D is a uniform ultrafilter on κ, then there is a class of cardinals χ with the following properties:

(i) $\chi^\kappa=\chi$.

(ii) *There are BAs B_i for $i<\kappa$ such that:*

 (a) $\mathrm{s}(B_i)\leq\chi$ *for each $i<\kappa$, and hence* $|\prod_{i<\kappa}\mathrm{s}(B_i)/D|\leq\chi$.

 (b) $\mathrm{s}(\prod_{i<\kappa}B_i)=\chi^+$.

This solves Problem 46 of Monk [96].

The following is a result of Shelah, Spinas [00], part of Corollary 2.4:

There is a model in which there exist cardinals κ,μ, a system $\langle B_i:i<\kappa\rangle$ of BAs, and an ultrafilter D on κ such that $|\prod_{i<\kappa}\mathrm{s}(B_i)/D|=\mu^{++}$ *and* $\mathrm{s}(\prod_{i<\kappa}B_i)\leq\mu^+$.

This solves Problem 47 of Monk [96].

We turn to free products. .

Theorem 13.8. *If $\langle A_i:i\in I\rangle$ is a system of BAs each with at least 4 elements, then* $\mathrm{s}(\oplus_{i\in I}A_i)\geq\max(|I|,\sup_{i\in I}\mathrm{s}(A_i))$. □

Equality does not hold in Theorem 13.8, in general. For example, let A be the interval algebra on the reals. We observed in Corollary 3.48 that $\mathrm{s}(A)=\omega$. Here is a system of 2^ω ideal independent elements in $A\oplus A$: for each real number r, let $a_r=[r,\infty)\times[-\infty,r)$ (considered as an element of $A\oplus A$). Suppose that F is a finite subset of $\mathbb{R}$, $r\in\mathbb{R}\backslash F$, and $a_r\in\langle a_s:s\in F\rangle^{\mathrm{Id}}$. Thus

$$[r,\infty)\times[-\infty,r)\cdot\prod_{s\in F}([-\infty,s)+[s,\infty))=0.$$

But if $T \stackrel{\text{def}}{=} \{s \in F : r < s\}$ and $U \stackrel{\text{def}}{=} F \backslash T$, then

$$[r, \infty) \times [-\infty, r) \cdot \prod_{s \in F}([-\infty, s) + [s, \infty)) \geq$$

$$[r, \infty) \times [-\infty, r) \cdot \prod_{s \in T}[-\infty, s) \cdot \prod_{s \in U}[s, \infty) \neq 0,$$

contradiction.

We give now the proof that $|B| \leq 2^{\mathrm{s}(B)}$ for any BA B. It depends on several other results which are of interest. We will use the notion of a network introduced in Chapter 9.

Lemma 13.9. *If X is a Hausdorff space and $2^\kappa < |X|$, then there is a sequence $\langle F_\alpha : \alpha < \kappa^+ \rangle$ of closed subsets of X such that $\alpha < \beta$ implies $F_\beta \subset F_\alpha$.*

Proof. For each $f \in \bigcup_{\alpha < \kappa^+} {}^\alpha 2$ we define a closed subset X_f of X. Let $X_0 = X$. For $\mathrm{dmn}(f)$ limit, let $X_f = \bigcap_{\alpha < \mathrm{dmn}(f)} X_{f \restriction \alpha}$. Now suppose that X_f has been constructed. If $|X_f| \leq 1$, let $X_{f \frown \langle 0 \rangle} = X_{f \frown \langle 1 \rangle} = X_f$. Otherwise, let $X_{f \frown \langle 0 \rangle}$ and $X_{f \frown \langle 1 \rangle}$ be two proper closed subsets of X_f whose union is X_f. This finishes the construction. Clearly $\bigcup_{\mathrm{dmn}(f) = \alpha} X_f = X$ for all $\alpha < \kappa^+$. Now

(∗) there is an $f \in {}^{\kappa^+}2$ such that $|X_{f \restriction \alpha}| \geq 2$ for all $\alpha < \kappa^+$.

For, otherwise, for all $x \in X$ there is an $f \in \bigcup_{\alpha < \kappa^+} {}^\alpha 2$ such that $X_f = \{x\}$, and so

$$|X| \leq \left| \bigcup_{\alpha < \kappa^+} {}^\alpha 2 \right| = 2^\kappa,$$

contradiction. So (∗) holds, and it clearly gives the desired result. □

Theorem 13.10. $|A| \leq 2^{\mathrm{s}(A)}$ *for any BA A.*

Proof. To start with, we prove:

(1) $\mathrm{d}(A) \leq 2^{\mathrm{s}(A)}$.

In fact, suppose that (1) fails. Note that for every $Y \subseteq \mathrm{Ult}(A)$ of power $< \mathrm{d}(A)$ we have $\overline{Y} \neq \mathrm{Ult}(A)$. Hence one can construct two sequences $\langle F_\alpha : \alpha < (2^{\mathrm{s}(A)})^+ \rangle$ and $\langle a_\alpha : \alpha < (2^{\mathrm{s}(A)})^+ \rangle$ such that $a_\alpha \in F_\alpha \in \mathrm{Ult}(A)$ and $\mathcal{S}(a_\alpha) \cap \{F_\beta : \beta < \alpha\} = 0$ for all $\alpha < (2^{\mathrm{s}(A)})^+$. Let $X = \{F_\alpha : \alpha < (2^{\mathrm{s}(A)})^+\}$. Clearly F is one-one, so $|X| > 2^{\mathrm{s}(A)}$. By Lemma 13.9, let $\langle K_\alpha : \alpha < (\mathrm{s}A)^+ \rangle$ be a system of closed subsets of X such that $\alpha < \beta$ implies that $K_\beta \subset K_\alpha$. Say $F_{\beta_\alpha} \in K_\alpha \backslash K_{\alpha+1}$ for all $\alpha < (\mathrm{s}(A))^+$, and choose $b_\alpha \in A$ so that $F_{\beta_\alpha} \in \mathcal{S}(b_\alpha) \cap X \subseteq X \backslash K_{\alpha+1}$. Then

(2) $\mathcal{S}(b_\alpha) \cap \{F_{\beta_\gamma} : \gamma > \alpha\} = 0$.

For, suppose $\gamma > \alpha$ and $F_{\beta_\gamma} \in \mathcal{S}(b_\alpha)$. But $F_{\beta_\gamma} \in K_\gamma \subseteq K_{\alpha+1}$, contradiction.

Define $f : [(\mathrm{s}(A))^+]^2 \to 2$ as follows: $f\{\gamma,\delta\} = 0$ iff when $\gamma < \delta$ we have $\beta_\gamma > \beta_\delta$. We now use the partition relation $\mu^+ \to (\omega, \mu^+)$. Since there is no infinite decreasing sequence of ordinals, we get a subset Γ of $(\mathrm{s}(A))^+$ of size $(\mathrm{s}(A))^+$ such that if $\gamma, \delta \in \Gamma$ and $\gamma < \delta$, then $\beta_\gamma < \beta_\delta$. Hence for any $\alpha \in \Gamma$ we have

$$\mathcal{S}(a_{\beta_\alpha} \cdot b_\alpha) \cap \{F_{\beta_\gamma} : \gamma \in \Gamma\} = \{F_{\beta_\alpha}\},$$

and $\{F_{\beta_\gamma} : \gamma \in \Gamma\}$ is discrete, contradiction. So, we have finally proved (1).

Let Y be a subset of $\mathrm{Ult}(A)$ which is dense in $\mathrm{Ult}(A)$ and of cardinality $\mathrm{d}(A)$. Let

$$\mathcal{N} = \{\overline{Z} : Z \subseteq Y,\ |Z| \le \mathrm{t}(A)\}.$$

From (1), Theorem 5.18, and Lemma 13.9 we see that $|\mathcal{N}| \le 2^{\mathrm{s}(A)}$. So, we will be finished, by Theorem 13.9, after we show that $\mathcal{N}$ is a network for A. Let $F \in U$, with U open. Say $F \in \mathcal{S}(a) \subseteq U$. Now $F \in \mathrm{Ult}(A) = \overline{Y}$, so we can choose $Z \subseteq Y$ with $|Z| \le \mathrm{t}(A)$ such that $F \in \overline{Z}$. Let $Z' = V \cap Z$. Then $F \in \overline{Z'} \subseteq U$ and $\overline{Z'} \in \mathcal{N}$, as desired. □

By Theorem 13.1, spread can be considered to be an ordinary sup-function, and so its behaviour under unions is given by Theorem 3.16.

Concerning moderate products we have the following.

Proposition 13.11. *Suppose that I is infinite, κ is an infinite successor cardinal, $\mathrm{s}(B) < \kappa$, $|I| < \kappa$, and $s(A_i) < \kappa$ for every $i \in I$. Then $\mathrm{s}(\prod^B_{i\in I} A_i) < \kappa$.*

Proof. We write any element x of $\prod^B_{i\in I} A_i$ as $h(b_x, F_x, a_x)$, in normal form as described in Chapter 1. Now suppose that $\mathscr{A}$ is a discrete subset of $\mathrm{Ult}\left(\prod^B_{i\in I} A_i\right)$ such that $|\mathscr{A}| = \kappa$; we want to get a contradiction. For each $D \in \mathscr{A}$ choose $h(b_D, F_D, a_D)$ such that $\mathcal{S}(h(b_D, F_D, a_D)) \cap \mathscr{A} = \{D\}$.

Case A. $X \overset{\text{def}}{=} \{D \in \mathscr{A}$: Theorem 1.6(x)(b) holds for $D\}$ has size $> \mathrm{s}(B)$. For each $D \in X$ choose an ultrafilter E_D on B in accordance with Theorem 1.6(x)(b). Suppose that $D \ne D'$. Then $E_D \ne E_{D'}$. In fact, choose $x \in D\backslash D'$. Then $x = h(b_x, F_x, a_x)$. By Proposition 1.5(ii), $-x = h(-(b_x \cup F_x), F_x, a'_x)$ for some a'. Since $-x \in D'$, it follows that $b_x \in E_D \backslash E_{D'}$. Let $Y = \{E_D : D \in X\}$. Now we claim that $\mathcal{S}(b_D) \cap Y = \{E_D\}$ for each $D \in X$. Suppose that $H \in \mathcal{S}(b_D) \cap Y$. Say $H = E_K$ with $K \in X$. Now $b_D \in H$, so $h(b_D, F_D, a_D) \in K$. Hence $K = D$ and so $H = E_D$. Conversely, clearly $E_D \in \mathcal{S}(b_D) \cap Y$. This proves the claim, and contradicts $|Y| = |X| \ge \mathrm{s}(B)$.

Case B. Case A fails. It follows that $X \overset{\text{def}}{=} \{D \in \mathscr{A}$: Theorem 1.6(x)(a) holds for $D\}$ has size κ. Hence for each $D \in X$ there exist $i_D \in I$ and an ultrafilter H_D on A_i such that $D = \{h(b, F, a) : i_D \in b$ or $i_D \in F$ and $a(i_D) \in H_D\}$. Since $|X| > |I|$, there exist a $j \in I$ and a $Y \subseteq X$ such that $i_D = j$ for all $D \in Y$, with $|Y| = \kappa$. Clearly $H_D \ne H_{D'}$ for $D \ne D'$. Let $Z = \{H_D : D \in Y\}$. We claim that Z is discrete. For, take any $D \in Y$. $\mathcal{S}(a_D(j)) \cap Z = \{H_D\}$. In fact, if $H_E \in$

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$\mathcal{S}(a_D(j))$ with $E \in Y$, then $a_D(j) \in H_E$ and hence $h(b_D, F_D, a_D) \in E$. So $E \in \mathcal{S}(h(b_D, F_D, a_D)) \cap Y$ and so $E = D$. Conversely, clearly $H_D \in \mathcal{S}(a_D(j)) \cap Z$. □

Concerning one-point gluing, it is clear that $\mathrm{s}(C) = \max(\mathrm{s}(A), \mathrm{s}(B))$ if C is obtained from A and B by one-point gluing. Not having a good description for the spread of an infinite product, we also do not know what happens under one-point gluing in general:

Problem 119. *Completely describe the behaviour of spread under one-point gluing.*

If B is the Alexsandroff duplicate of A, clearly $\mathrm{s}(B) = |B| = 2^{|\mathrm{ult}(A)|}$.

By Proposition 2.6 we have $\mathrm{s}(\mathrm{Exp}(A)) \geq \mathrm{s}(A)$ for any infinite BA A.

Problem 120. *Is there a BA A such that* $\mathrm{s}(A) < \mathrm{s}(\mathrm{Exp}(A))$*?*

We turn to the derived functions for spread. The following facts are clear: $\mathrm{s}_{\mathrm{H}+}(A) = \mathrm{s}(A)$; $\mathrm{s}_{\mathrm{S}+}(A) = \mathrm{s}(A)$; $\mathrm{s}_{\mathrm{S}-}(A) = \omega$; $\mathrm{s}_{\mathrm{h}-}(A) = \omega$; ${}_{\mathrm{d}}\mathrm{s}_{\mathrm{S}+}(A) = \mathrm{s}(A)$. The algebra A of Fedorchuk [75] (constructed under $\diamondsuit$ and presented in Chapter 16) is such that $\mathrm{s}_{\mathrm{H}-}(A) \leq \mathrm{s}(A) < \mathrm{Card}_{\mathrm{H}-}(A)$. Thus Problem 36 of Monk [90] was solved long ago. It is also easy to see that $\mathrm{s}_{\mathrm{h}+}(A) = \mathrm{s}(A)$. The status of the derived function ${}_{\mathrm{d}}\mathrm{s}_{\mathrm{S}-}$ is not clear; note that ${}_{\mathrm{d}}\mathrm{s}_{\mathrm{S}-}(A) < \mathrm{s}(A)$ for $A = \mathscr{P}\kappa$.

Problem 121. *Completely describe ${}_{\mathrm{d}}\mathrm{s}_{\mathrm{S}-}$ in terms of the other cardinal functions.*

We turn to s_{mm} and $\mathrm{s}_{\mathrm{spect}}$, which were discussed in Monk [08]. They are defined as follows.

$$\mathrm{s}_{\mathrm{spect}}(A) = \{|X| : X \text{ is an infinite maximal ideal independent subset of } A\};$$
$$\mathrm{s}_{\mathrm{mm}}(A) = \min(\mathrm{s}_{\mathrm{spect}}(A)).$$

Note that $\{1\}$ is maximal ideal independent; this accounts for the restriction above that X is infinite.

A simple way of coming up with maximal ideal independent subsets is given in the following proposition.

Proposition 13.12. *If X is infinite, ideal independent, and generates a maximal ideal I, then X is maximal ideal independent.*

Proof. Let $y \in A \backslash X$. If $y \in I$, then $y \leq \sum F$ for some finite $F \subseteq I$, and so $X \cup \{y\}$ is ideal dependent. Suppose that $-y \in I$. Say $-y \leq \sum F$ with $F \in [X]^{<\omega}$. Choose $z \in X \backslash F$. Then $z \leq 1 = y + \sum F$, and so $X \cup \{y\}$ is ideal dependent. □

Proposition 13.13. $\mathrm{s}_{\mathrm{spect}}(A) \cup \mathrm{s}_{\mathrm{spect}}(B) \subseteq \mathrm{s}_{\mathrm{spect}}(A \times B)$.

Proof. Let X be maximal ideal-independent in A. Define $Y = \{(x, 1) : x \in X\}$. Clearly Y is ideal-independent in $A \times B$. To show that it is maximal, suppose that $(u, v) \in A \times B$. Then there are two possibilities.

Case 1. There is a finite subset F of X such that $u \leq \sum F$. We may assume that $F \neq \emptyset$. Then $(u, v) \leq \sum_{x \in F}(x, 1)$, as desired.

Case 2. There exist an $x \in X$ and a finite subset F of $X\backslash\{x\}$ such that $x \leq u + \sum F$. Again we may assume that $F \neq \emptyset$. Then $(x,1) \leq (u,v) + \sum_{y\in F}(y,1)$, as desired.

Hence the proposition follows by symmetry. $\square$

Problem 122. *Is* $\mathrm{s}_{\mathrm{spect}}(A \times B) = \mathrm{s}_{\mathrm{spect}}(A) \cup \mathrm{s}_{\mathrm{spect}}(B)$*?*

Problem 123. *Is* $\mathrm{s}_{\mathrm{mm}}(A \times B) = \mathrm{s}_{\mathrm{mm}}(A) \cup \mathrm{s}_{\mathrm{mm}}(B)$*?*

Theorem 13.14. *If K is a nonempty finite set of infinite cardinals, then*

$$\mathrm{s}_{\mathrm{spect}}\left(\prod_{\lambda\in K}\mathrm{Fr}(\lambda)\right) = K.$$

Proof. $\supseteq$ holds by Proposition 13.13. Suppose that $\kappa \in \mathrm{s}_{\mathrm{spect}}\left(\prod_{\lambda\in K}\mathrm{Fr}(\lambda)\right)\backslash K$. Let $L = \{\lambda \in K : \lambda < \kappa\}$ and $M = K\backslash L$. Assume that $L \neq \emptyset$; some obvious changes should be made in the following argument if $L = \emptyset$. Let X be a maximal independent subset of $\prod_{\lambda\in K}\mathrm{Fr}(\lambda)$ of size κ. For each $\lambda \in M$ let u_λ be a free generator of $\mathrm{Fr}(\lambda)$ not in the subalgebra generated by $\{x_\lambda : x \in X\}$. Now $|\prod_{\lambda\in L}\mathrm{Fr}(\lambda)| < \kappa$, so there is a $q \in \prod_{\lambda\in L}\mathrm{Fr}(\lambda)$ such that $X' \stackrel{\text{def}}{=} \{x \in X : x \restriction L = q\}$ has size greater than $\max(L)$. Let $f = q \cup \langle u_\lambda : \lambda \in M\rangle$. So $f \in \prod_{\lambda\in K}\mathrm{Fr}(\lambda)$ and $f \notin X$ (since clearly $M \neq \emptyset$). Hence $X \cup \{f\}$ is ideal-dependent. This gives two possibilities.

Case 1. There is a finite $F \subseteq X$ such that $f \leq \sum F$. It follows that $(\sum F)_\lambda = 1$ for all $\lambda \in M$. Choose $g \in X'\backslash F$. Then $g \leq \sum F$, contradiction.

Case 2. There exist a finite $F \subseteq X$ and a $g \in X\backslash F$ such that $g \leq f + \sum F$. For any $\lambda \in M$ we have $g_\lambda \cdot -u_\lambda \cdot -(\sum F)_\lambda = 0$, and hence $g_\lambda \cdot -(\sum F)_\lambda = 0$. Choose $h \in X'\backslash(F \cup \{g\})$. Then $g \leq h + \sum F$, contradiction. $\square$

For the next result, we define reg to be the class of all regular cardinals.

Theorem 13.15. *Suppose that K is an infinite set of infinite cardinals such that $|K| \leq \min(K)$. Then there is a BA A such that $K \subseteq \mathrm{s}_{\mathrm{spect}}(A)$ and $\mathrm{s}_{\mathrm{spect}}(A) \cap \mathrm{reg} \subseteq K$.*

Proof. Let $\mu = \min(K)$, let λ map μ onto K, and let $A = \prod^{\mathrm{w}}_{\alpha<\mu}\mathrm{Fr}(\lambda_\alpha)$. We claim that A is as desired.

The first inclusion in the proposition holds by Proposition 13.13. Now suppose that $\kappa \in (\mathrm{s}_{\mathrm{spect}}(A) \cap \mathrm{reg})\backslash K$. Let X be maximal ideal-independent of size κ. Let $L = \{\alpha < \mu : \kappa < \lambda_\alpha\}$, and let $M = \mu\backslash L$. For each $\alpha \in L$ let u_α be a free generator of $\mathrm{Fr}(\lambda_\alpha)$ not in $\langle\{x_\alpha : x \in X\}\rangle$.

(1) $M \neq \emptyset$.

For, suppose that $M = \emptyset$. Then $\kappa < \lambda_\alpha$ for each $\alpha < \mu$, and so $\kappa < \min(K) = \mu$.

(2) Some $x \in X$ has type II.

For, suppose not. Now $\bigcup_{x\in X}\operatorname{supp}(x)$ has size less than $\min(K)=\mu$, so we can choose $\alpha<\mu$ not in this union. Let y take the value u_α at α and 0 elsewhere. Clearly $y\notin X$ and $X\cup\{y\}$ is still ideal-independent, contradiction. So (2) holds.

We take x as in (2). Now let $y_\alpha=u_\alpha$ for all $i\in\operatorname{supp}(x)$, and $y_\alpha=0$ otherwise. Then $y\notin X$, so $X\cup\{y\}$ is ideal-dependent.

Case 1. $y\le\sum F$ for some finite $F\subseteq X$. We may assume that $x\in F$. Now for $\alpha\in\operatorname{supp}(x)$ we have $u_\alpha\le(\sum F)_\alpha$, and hence $(\sum F)_\alpha=1$. Since $x\in F$, it follows that $\sum F=1$, contradiction.

Case 2. There exist a finite $F\subseteq X$ and a $g\in X\backslash F$ such that $g\le y+\sum F$. It follows easily that $g\le\sum F$, contradiction.

This proves (1).

In particular, $\kappa>\mu$. Since κ is regular, it follows that there is a $G\in[\mu]^{<\omega}$ such that $X'\stackrel{\text{def}}{=}\{x\in X:\operatorname{supp}(x)=G\}$ has size κ. Now $|\prod_{\alpha\in G\cap M}\operatorname{Fr}(\lambda_\alpha)|<\kappa$, so there is a $q\in\prod_{\alpha\in G\cap M}\operatorname{Fr}(\lambda_\alpha)$ such that $Y\stackrel{\text{def}}{=}\{x\in X':x\restriction(G\cap M)=q\}$ has size κ. Note also that $G\cap L\ne\emptyset$, as otherwise $G=G\cap M$ and hence $|X'|<\kappa$, contradiction. Let $Y'=\{y\in Y:y\text{ has type I}\}$ and $Y''=Y\backslash Y'$. Now define

$$y_\alpha=\begin{cases}u_\alpha & \text{if } i\in G\cap L,\\ q_\alpha & \text{if } i\in G\cap M,\\ 0 & \text{otherwise.}\end{cases}$$

Since $G\cap L\ne\emptyset$, we have $y\notin X$. So $X\cup\{y\}$ is ideal-dependent. This gives two cases.

Case 1. There is a finite $F\subseteq X$ such that $y\le\sum F$. Then $(\sum F)\restriction(G\cap L)=1$. If $|Y'|=\kappa$, choose $g\in Y'$ such that $g\notin F$. Then $g\le\sum F$, contradiction. If $|Y''|=\kappa$, choose distinct $g,h\in Y''\backslash F$. Then $g\le\sum F+h$, contradiction.

Case 2. There exist a finite $F\subseteq X$ and a $g\in X\backslash F$ such that $g\le y+\sum F$. Then $g\restriction(G\cap L)\le(\sum F)\restriction(G\cap L)$ and also $g\restriction(\mu\backslash G)\le(\sum F)\restriction(\mu\backslash G)$. Choose $h\in Y\backslash(F\cup\{g\})$. Then $g\le h+\sum F$, contradiction. □

Corollary 13.16. *If K is an infinite set of regular cardinals and $|K|\le\min(K)$, then there is a BA A such that $\mathrm{s}_{\mathrm{spect}}(A)\cap\mathrm{reg}=K$.* □

Problem 124. *Is the assumption $|K|\le\min(K)$ in Proposition 13.16 necessary?*

Problem 125. *How can Corollary* 13.16 *be extended to singular cardinals in K?*

Proposition 13.17. $\mathfrak{r}(A)\le\mathrm{s}_{\mathrm{mm}}(A)$ *for any BA A.*

Proof. Suppose that X is maximal ideal-independent. Let

$$Y=X\cup\{-\sum F:F\in[X]^{<\omega}\}\cup\{b\cdot-\sum F:b\notin F,\ F\cup\{b\}\in[X]^{<\omega}\}.$$

Clearly the members of Y are nonzero. We claim that Y is weakly dense in A. For, suppose that $a \in A\backslash X$. Then $X \cup \{a\}$ is no longer ideal independent, so we have two cases.

Case 1. $a \leq \sum F$ for some $F \in [X]^{<\omega}$. Then $-\sum F \leq -a$, as desired.

Case 2. There exist a finite subset F of X and a $b \in X\backslash F$ such that $b \leq \sum F + a$. Then $b \cdot -\sum F \leq a$, as desired. □

Lemma 13.18. *Suppose that* $\mathrm{Fr}(\omega_1)$ *is a subalgebra of* A *such that* $I \stackrel{\mathrm{def}}{=} \langle\{x_\alpha : \xi < \omega_1\rangle_A^{\mathrm{id}}$ *is a maximal ideal of* A*, where* $\langle x_\alpha : \alpha < \omega_1\rangle$ *is a system of free generators of* $\mathrm{Fr}(\omega_1)$*. Also suppose that* $X \stackrel{\mathrm{def}}{=} \{x_\alpha : \alpha < \omega_1\}$ *is maximal ideal independent in* A*. Suppose that* Y *is an infinite partition of unity in* A*, with* $|Y| \leq \omega_1$*.*

Then A *has an extension* B *such that* X *is still maximal ideal independent in* B*,* $\langle\{x_\alpha : \xi < \omega_1\rangle_B^{\mathrm{id}}$ *is a maximal ideal of* B*, and* Y *is not a partition of unity in* B*.*

Proof. The main part of the proof is in establishing the following claim.

Claim. *There is a* $b \in X$ *such that* $b \not\leq \sum F$ *for all* $F \in [Y]^{<\omega}$*.*

We suppose that the claim does not hold. Thus

(1) For every $b \in X$ there is a finite $F_b \subseteq Y$ such that $b \leq \sum F_b$.

Then

(2) $y \in I$ for all $y \in Y$.

For, suppose that $y \in Y$ and $-y \in I$. Thus there is a finite $G \subseteq X$ such that $-y \leq \sum G$. Then $1 = y + \sum G = y + \sum_{b \in G} F_b$, contradiction. So (2) holds.

Thus for every $y \in I$ we can choose a finite $G_y \subseteq X$ such that $y \leq G_y$.

Now if $|Y| < \omega_1$, choose x_α not in the support of any element of $\bigcup_{y \in Y} G_y$. Now $x_\alpha \leq \sum F_{x_\alpha} \leq \sum_{y \in F_{x_\alpha}} G_y$, contradiction. Thus $|Y| = \omega_1$.

Let $\Gamma \in [\omega_1]^{\omega_1}$ be such that $\langle F_{x_\alpha} : \alpha \in \Gamma\rangle$ is a Δ-system, say with kernel H. Then if α and β are distinct elements of Γ we have

$$x_\alpha \cdot x_\beta \leq \left(\sum F_{x_\alpha}\right) \cdot \left(\sum F_{x_\beta}\right) \leq \sum_{y \in H} G_y. \tag{3}$$

Choose distinct $\alpha, \beta \in \Gamma$ so that $x_\alpha, x_\beta \notin \bigcup_{y \in G} G_y$. Then (3) gives a contradiction.

This proves the claim.

Choose $b \in X$ in accordance with the claim. Let $A(x)$ be a free extension of A, and let J be the ideal of $A(x)$ generated by

$$\{y \cdot x : y \in Y\} \cup \{x \cdot -b\}.$$

Clearly $A \cap J = \{0\}$. If $x \in J$, then we can write

$$x \leq y_1 \cdot x + \cdots + y_m \cdot x + x \cdot -b.$$

Mapping x to 1 yields $b \leq y_1 + \cdots + y_m$, contradicting the choice of b.

Thus $A(x)/J$ is as desired. □

Theorem 13.19. *There is a BA A such that* $\mathrm{s_{mm}}(A) = \omega_1$ *and* $\mathfrak{a}(A) = \omega_2$.

Proof. This is obtained by an obvious iteration from Lemma 13.18. □

For $A = \mathrm{Fr}(\kappa)$ with κ an uncountable cardinal, we have $\mathfrak{a}(A) = \mathrm{length_{mm}}(A) = \omega < \kappa = \mathrm{s_{mm}}(A)$. (This is easy to see.)

Let κ be uncountable, and let A be atomless an κ^+-saturated (in the model-theoretic sense). Let $B = {}^\kappa A^{\mathrm{w}}$, the weak κ-power of A. Then the following is easy to check:

$$\mathfrak{r}(B) = \mathfrak{i}(B) = \omega;\ \mathrm{s_{mm}}(B) \leq \kappa;\ \mathfrak{a}(B) = \mathfrak{p}(B) = \kappa;\ \mathrm{tow}(B) \geq \kappa^+.$$

Proposition 13.20 (C. Bruns)**.** $\mathfrak{p}(A) \leq \mathrm{s_{mm}}(A)$ *for any infinite BA A.*

Proof. Let X be maximal ideal independent of size $\mathrm{s_{mm}}(A)$. Then $\sum X = 1$, since if $a \in A^+$ and $a \cdot x = 0$ for all $x \in X$ it is clear that $X \cup \{a\}$ is still ideal independent. Also, if $F \in [X]^{<\omega}$ we have $\sum F < 1$; for if $\sum F = 1$ and $x \in X \backslash F$ then $x \leq \sum F$, contradiction. Hence the inequality of the proposition follows. □

Problem 126. *Is there an atomless BA A such that* $\mathrm{s_{mm}}(A) < \mathfrak{i}(A)$*?*

Now we show that it is consistent to have $\mathrm{s_{mm}}(\mathscr{P}(\omega)/\mathrm{fin})$ less than 2^ω. The argument is a modification of exercises (A12), (A13) in chapter VIII of Kunen [80]; the essential argument is given in the following lemma.

Lemma 13.21. *Let M be a c.t.m. of ZFC. Suppose that κ is an infinite cardinal and $\langle a_i : i < \kappa\rangle$ is a system of infinite subsets of ω such that $\langle [a_i] : i < \kappa\rangle$ is ideal independent, where $[x]$ denotes the equivalence class of x modulo the ideal fin of $\mathscr{P}(\omega)$. Then there is a generic extension $M[G]$ of M using a ccc partial order such that in $M[G]$ there is a $d \subseteq \omega$ with the following two properties:*

(i) $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d]\rangle$ *is ideal independent.*

(ii) *If $x \in (\mathscr{P}(\omega) \cap M)\backslash(\{a_i : i < \kappa\} \cup \{\omega\backslash d\})$, then $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d], [x]\rangle$ is not ideal independent.*

Proof. Let I be the ideal of $\mathscr{P}(\omega)/\mathrm{fin}$ generated by $\{[a_i] \cdot [a_j] : i < j < \kappa\}$, and let A be the quotient algebra $(\mathscr{P}(\omega)/\mathrm{fin})/I$. Let f be the natural homomorphism of $\mathscr{P}(\omega)$ onto A. Note that $f(a_i) \neq 0$ for all $i < \kappa$, by ideal independence. Let B be the subalgebra of A generated by $\{f(a_i) : i < \kappa\}$. Thus B is an atomic BA with $\{f(a_i) : i < \kappa\}$ its set of atoms. By Sikorski's extension theorem, let $h : \mathscr{P}(\omega) \to \overline{B}$ extend f, where $\overline{B}$ is the completion of B.

Let $P = \{(b, y) : b \in \ker(h) \text{ and } y \in [\omega]^{<\omega}\}$. We define $(b, y) \le (b', y')$ iff $b \supseteq b'$, $y \supseteq y'$, and $y \cap b' \subseteq y'$. Clearly this gives a ccc partial order of P. Let G be any P-generic filter over M, and let $d = \bigcup_{(b,y)\in G} y$.

(1) If R is a finite subset of κ and $i \in \kappa\backslash R$, then $a_i \cap \bigcap_{j\in R}(\omega\backslash a_j) \cap d$ is infinite.

In fact, let R and i be as in the hypothesis of (1). For any natural number n let

$$E_n = \left\{(b, y) \in P : \exists m > n \left[m \in a_i \cap \bigcap\nolimits_{j\in R}(\omega\backslash a_j) \cap y\right]\right\}.$$

Clearly it suffices to show that each such set E_n is dense in P. Suppose that $(b, y) \in P$. Then $(a_i \cap \bigcap_{j\in R}(\omega\backslash a_j))\backslash b$ is infinite. For, if it is a finite set c, then

$$a_i \subseteq \bigcup\nolimits_{j\in R} a_j \cup b \cup c,$$

and upon applying h we would get $h(a_i) \le \sum_{j\in R} h(a_j)$, which is clearly impossible. Thus the indicated set is infinite. We can hence choose m in it with $m > n$. Clearly $(b, y \cup \{m\}) \in E_n$ and $(b, y \cup \{m\}) \le (b, y)$, proving (1).

(2) If R is a finite subset of κ, then $\omega\backslash \left(d \cup \bigcup_{i\in R} a_i\right)$ is infinite.

In fact, let R be a finite subset of κ. For any natural number n let

$$F_n = \left\{(b, y) \in P : \exists m > n \left[m \in b\backslash \left(y \cup \bigcup\nolimits_{i\in R} a_i\right)\right]\right\}.$$

We claim that F_n is dense in P. For, suppose that $(b, y) \in P$. Then the set $\omega\backslash \left(y \cup \bigcup_{i\in R} a_i\right)$ is infinite. For, if it is a finite set c, then we get

$$\omega = c \cup y \cup \bigcup\nolimits_{i\in R} a_i,$$

and applying h we get $1 = \sum_{i\in R} h(a_i)$, which is clearly impossible. Choose $(b, y) \in F_n \cap G$, and then choose $m > n$ such that $m \in b\backslash \left(y \cup \bigcup_{i\in R} a_i\right)$. We claim that $m \notin d$; by the arbitrariness of n, this will prove (2). Suppose that $m \in d$. Choose $(c, z) \in G$ with $m \in z$. Then choose $(d, w) \in G$ with $(d, w) \le (b, y), (c, z)$. Thus $m \in w$ since $m \in z$. Also, $m \in b\backslash y$. This contradicts $(d, w) \le (b, y)$. Hence (2) holds.

(3) $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d]\rangle$ is ideal independent.

For, suppose not. There are two possibilities.

Case 1. There are a finite subset R of κ and an $i \in \kappa\backslash R$ such that $[a_i] \le [\omega\backslash d] + \sum_{j\in R}[a_j]$. This contradicts (1).

Case 2. There is a finite subset R of κ such that $[\omega\backslash d] \le \sum_{i\in R}[a_i]$. This contradicts (2).

Thus (3) holds.

(4) If $b \in \ker(h)$, then $b \cap d$ is finite.

In fact, clearly $\{(c, y) \in P : b \subseteq c\}$ is dense in P, so we can choose $(c, y) \in G$ such that $b \subseteq c$. Then $b \cap d \subseteq y$ (as desired). For, suppose that $m \in b \cap d$. Choose $(e, z) \in G$ such that $m \in z$. Then choose $(r, w) \in G$ such that $(r, w) \leq (e, z), (c, y)$. Then $m \in w \cap c \subseteq y$.

(5) If $x \in (\mathscr{P}(\omega) \cap M) \backslash (\{a_i : i < \kappa\} \cup \{\omega \backslash d\})$, then $\langle [a_i] : i < \kappa \rangle^\frown \langle [\omega \backslash d], [x] \rangle$ is not ideal independent.

To prove this, we consider two cases. First, if $x \in \ker(h)$, then $[x] \leq [\omega \backslash d]$ by (4), as desired. Second, if $x \notin \ker(h)$, choose $i < \kappa$ such that $h(a_i) \leq h(x)$. So $a_i \backslash x \in \ker(h)$, and so by (4) we get $[a_i] \leq [x] + [\omega \backslash d]$, as desired. $\square$

Theorem 13.22. *It is consistent with* $2^\omega > \omega_1$ *that* $\mathrm{s_{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) = \omega_1$.

Proof. Start with a c.t.m. M of ZFC$+2^\omega > \omega_1$. Iterate the construction of Lemma 13.21 ω_1 times, obtaining a generic filter G over M. Then $M[G]$ is as desired, using Lemma 5.14 of Chapter VIII of Kunen [80]. $\square$

Problem 127. *What is the exact place of* $\mathrm{s_{mm}}(\mathscr{P}(\omega)/\mathrm{fin})$ *among the other continuum cardinals?*

Turning to the relationships of spread to our other functions, we first list out the things already proved: $\mathrm{c_{H+}}(A) = \mathrm{s}(A)$ by Theorem 3.30; $\mathrm{Depth_{h+}}(A) = \mathrm{s}(A)$ in Theorem 4.26; $\mathrm{t}(A) \leq \mathrm{s}(A)$ in Theorem 5.18; and $|A| \leq 2^{\mathrm{s}(A)}$ in Theorem 13.10. Now we prove the important fact that $\pi(A) \leq \mathrm{s}(A) \cdot (\mathrm{t}(A))^+$ for any infinite BA A, following Todorčević [90a]. The result he proves is somewhat stronger, and to state it we need two definitions. First, $\mathrm{dd}(A)$ is the least cardinality of a collection of discrete subsets of $\mathrm{Ult}(A)$ whose union is dense in $\mathrm{Ult}(A)$. Second, $\mathrm{f}'(A)$ is the smallest cardinal such that A does not have a free sequence of length $\mathrm{f}'(A)$. Thus if $\mathrm{t}(A)$ is attained in the free sequence sense, then $\mathrm{f}'(A) = (\mathrm{t}(A))^+$, while $\mathrm{t}(A) = \mathrm{f}'(A)$ otherwise.

Theorem 13.23. $\mathrm{dd}(A) \leq \mathrm{f}'(A)$, *and* $\pi(A) \leq \mathrm{s}(A) \cdot \mathrm{f}'(A)$ *for any infinite BA A.*

Proof. Choose $\langle a_\alpha : \alpha < \beta \rangle$ in accordance with Theorem 12.15. Let $E = \{a_\alpha : \alpha < \beta\}$. Now we define $D_\gamma \subseteq \mathrm{Ult}(A)$ and $S_\gamma \subseteq E$ for $\gamma < \mathrm{f}'(A)$ by induction. Suppose that they have been defined for all $\gamma < \delta$. Let

$$S_\delta = \{x \in E : \mathcal{S}(x) \cap D_\gamma = \emptyset \text{ for all } \gamma < \delta\}.$$

Then we let D_δ be a maximal subset of $\bigcup_{x \in S_\delta} \mathcal{S}(x)$ having at most one element in common with each $\mathcal{S}(x)$ for $x \in S_\delta$. This finishes the construction. Note that D_δ is discrete: if $F \in D_\delta$, choose $x \in S_\delta$ such that $F \in \mathcal{S}(x)$. Then $D_\delta \cap \mathcal{S}(x) = \{F\}$ by the defining property of D_δ.

For each $F \in D_\delta$ choose $g(F) \in S_\delta$ such that $F \in \mathcal{S}(g(F))$. Then g is a one-one function, and its range is $S'_\gamma \stackrel{\text{def}}{=} \{x \in S_\delta : D_\delta \cap \mathcal{S}(x) \neq \emptyset\}$. Now

(1) $\mathcal{S}(x) \cap \bigcup_{\delta < \mathrm{f}'(A)} D_\delta \neq \emptyset$ for all $x \in E$.

For, suppose that (1) fails for a certain $x \in E$. Then

(2) $\mathcal{S}(x) \subseteq \bigcup_{y \in S'_\delta} \mathcal{S}(y)$ for all $\delta < \mathrm{f}'(A)$.

For, suppose that (2) fails for a certain $\delta < \mathrm{f}'(A)$. Choose $F \in \mathcal{S}(x) \setminus \bigcup_{y \in S'_\delta} \mathcal{S}(y)$. Now if $G \in D_\delta \cap \mathcal{S}(y)$ with $y \in S_\delta$, then $y \in S'_\delta$ and so $F \notin \mathcal{S}(y)$. Also, $D_\delta \cap \mathcal{S}(x) = \emptyset$ by (1) failing. So $D_\delta \cup \{F\}$ has at most one element in common with each $\mathcal{S}(y)$ for $y \in S_\delta$, and $F \notin D_\delta$, contradicting the maximality of D_δ. Thus (2) holds.

Now if $\gamma < \delta < \mathrm{f}'(A)$, then $S'_\gamma \cap S'_\delta = \emptyset$, since if $z \in S'_\gamma \cap S'_\delta$, then $\mathcal{S}(z) \cap D_\gamma = \emptyset$ because $z \in S'_\delta \subseteq S_\delta$, but $\mathcal{S}(z) \cap D_\gamma \neq \emptyset$ by the definition of S'_γ, contradiction. It follows now that for any $F \in \mathcal{S}(x)$ we have $F \in \mathcal{S}(y)$ for a collection of $\mathrm{f}'(A)$ y's, and this contradicts the condition of Theorem 12.15. So we have proved (1).

By (1) we have $\mathrm{dd}(A) \leq \mathrm{f}'(A)$, since E is dense in A. Moreover, $E \subseteq \bigcup_{\delta < \mathrm{f}'(A)} S'_\delta$, since by (1), if $x \in E$ then there is a $\delta < \mathrm{f}'(A)$ such that $\mathcal{S}(x) \cap D_\delta \neq \emptyset$, and hence $x \in S'_\delta$. Hence

(3) $\bigcup_{\delta < \mathrm{f}'(A)} S'_\delta$ is dense in A.

Now $|D_\delta| = |S'_\delta|$ for all $\delta < \mathrm{f}'(A)$. Hence

$$\pi(A) \leq \left| \bigcup_{\delta < \mathrm{f}'(A)} S'_\delta \right| \leq \mathrm{s}(A) \cdot \mathrm{f}'(A),$$

as desired. □

Note that $\mathrm{t}(A)$ can be much smaller than $\mathrm{s}(A)$, for example in the finite-cofinite algebra on an infinite cardinal κ. Also note that, obviously, $\mathrm{c}(A) \leq \mathrm{s}(A)$; and the difference is big in, e.g., free algebras. We have $\mathrm{s}(A) > \mathrm{Length}(A)$ for A a free algebra; $\mathrm{s}(A) < \mathrm{Length}(A)$ for A the interval algebra on the reals. Also, $\mathrm{s}(A) > \pi(A)$ for $A = \mathscr{P}(\kappa)$. The interval algebra of a Suslin line provides an example of a BA A with $\mathrm{s}(A) = \omega$ and $\mathrm{d}(A) > \omega$. In fact, clearly $\mathrm{s}(A) = \mathrm{c}(A)$ for A an interval algebra, by the retractiveness of interval algebras.

An example with $\mathrm{s}(A) = \omega < \mathrm{d}(A)$ cannot be given in ZFC; this follows from the following rather deep results. Juhász [71] showed that under the assumption of MA+$\neg$CH, for every compact Hausdorff space X, if $\mathrm{s}(X) = \omega$ then $\mathrm{hL}(X) = \omega$. Todorčević [83] showed that it is consistent with MA+$\neg$CH that for every regular space X, if $\mathrm{s}(X) = \omega$ then $\mathrm{hL}(X) = \omega$. Hence it is consistent that for every BA A, if $\mathrm{s}(A) = \omega$ then $\mathrm{hL}(A) = \omega = \mathrm{hd}(A)$.

Recall also Theorem 8.11, which says that $\mathrm{Irr}(A) \leq \mathrm{s}(A \oplus A)$.

Bounded versions of spread can be defined as follows. For m a positive integer, a subset X of A is called *m-ideal-independent* if for all distinct $x_0, \ldots, x_m \in X$ we have $x_0 \not\leq x_1 + \cdots + x_m$. Then we let $\mathrm{s}_m(A) = \sup\{|X| : X \subseteq A$ and X is m-ideal-independent$\}$. For these functions see Rosłanowski, Shelah [98].

14 Character

First note that we can define $\chi(A)$ as a sup; namely, for any ultrafilter F on A let $\chi(F) = \min\{|X| : X$ is a set of generators of $F\}$ – then $\chi(A) = \sup\{\chi(F) : F$ is an ultrafilter on $A\}$.

Character can increase in going from an algebra to a subalgebra. To construct an example of this sort, first notice that if A is the finite-cofinite algebra on an infinite cardinal κ, then $\chi(A) = \kappa$, as is easily seen. The algebra that we want is the Alexsandroff duplicate of the free algebra A on κ free generators, where κ is any infinite cardinal. By Theorem 14.6 below, $\chi(\mathrm{Dup}(A)) = \chi(A) = \kappa$. The finite-cofinite algebra on $\mathrm{Ult}(A)$ is isomorphic to a subalgebra of $\mathrm{Dup}(A)$, and by the previous remark it has character 2^κ.

An atomless example of this sort can be obtained as follows. Take A and $\mathrm{Dup}(A)$ as above, and let M be the set of atoms of $\mathrm{Dup}(A)$. Then there is an isomorphism from $\mathrm{Dup}(A)$ onto an algebra C of subsets of M, containing all singletons $\{x\}$ with $x \in M$. For each $x \in M$ let $A_x = \mathrm{Fr}(\omega)$. Then $D \stackrel{\mathrm{def}}{=} \prod^{C}_{x\in M} A_x$ has character κ by the above and Proposition 14.4 below. Now $\prod^{\mathrm{w}}_{x\in M} A_x$ is an atomless subalgebra of D, and its character is 2^κ.

Below we show that $\chi(A \oplus B) = \max(\chi(A), \chi(B))$. Hence if $A \leq_{\mathrm{free}} B$, then $\chi(A) \leq \chi(B)$, and the difference can be arbitrarily large. Thus there are BAs A, B with $\chi(B)$ arbitrarily greater than $\chi(A)$, and with $A \leq_{\mathrm{u}} B$, $A \leq_{\mathrm{proj}} B$, $A \leq_{\mathrm{rc}} B$, $A \leq_\sigma B$, and $A \leq_{\mathrm{reg}} B$.

The two examples given in detail above show that it is possible to have $C \leq_\pi D$ with $\chi(C) > \chi(D)$; so this applies to $\leq_{\mathrm{reg}}$ also. On the other hand, we have $\chi(A) < \chi(B)$ for $A = \mathrm{Fr}(\omega)$ and $B = \overline{A}$; see the discussion of $\mathfrak{u}$ below.

It is clear by topological duality that $\chi(A \times B) = \sup(\chi(A), \chi(B))$. Then by Propositions 2.29 and 14.1 it follows that $\chi(B) \leq \chi(A)$ whenever $A \leq_{\mathrm{s}} B$. On the other hand, if F is an ultrafilter on B, with $B = A(x)$, and $x^\delta \in F$ with $\delta \in 2$, and if X generates $F \cap A$ with $|X| = \chi(F \cap A)$, then $X \cup \{x^\delta\}$ generates F; it follows that $\chi(A) \leq \chi(B)$. So $\chi(A) = \chi(B)$ if $A \leq_{\mathrm{s}} B$. Hence also $\chi(A) = \chi(B)$ if $A \leq_{\mathrm{m}} B$.

It is clear from Propositions 2.52 and 14.18 that there exist A, B with $A \leq_{\mathrm{mg}} B$ and $\chi(B)$ arbitrarily larger than $\chi(A)$.

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The facts mentioned answer many natural questions concerning the special subalgebra notions; but other questions remain.

Problem 128. *Completely describe the possibilities for character with respect to the various subalgebra notions.*

Proposition 14.1. *If A is a homomorphic image of B, then $\chi(A) \leq \chi(B)$.*

Proof. Let f be a homomorphism from B onto A; if $F \in \mathrm{Ult}(A)$, then $f^{-1}[F] \in \mathrm{Ult}(B)$, and if we choose $X \subseteq f^{-1}[F]$ with $|X| \leq \chi(B)$ such that X generates $f^{-1}[F]$, then $f[X]$ generates F. □

For a weak product we have $\chi(\prod^{\mathrm{w}}_{i\in I} A_i) = \max(|I|, \sup_{i\in I}\chi(A_i))$. To show this, it suffices to show that $\chi(F) = |I|$ for the "new" ultrafilter F. This ultrafilter is defined as follows. For each subset M of I, let x_M be the element of $\prod_{i\in I} A_i$ such that $x_M(i) = 1$ if $i \in M$ and $x_M(i) = 0$ for $i \notin M$. Then F is the set of all $y \in \prod^{\mathrm{w}}_{i\in I} A_i$ such that $x_M \leq y$ for some cofinite subset M of I. So, it is clear that $\chi(F) \leq |I|$. If X is a set of generators for F with $|X| < |I|$, we may assume that X is closed under $\cdot$, and then there is a $y \in X$ such that $y \subseteq x_M$ for infinitely many cofinite subsets M of I; this is clearly impossible.

As usual, weak products enable us to discuss the attainment problem. Any infinite BA has a non-principal ultrafilter, and hence if A has character ω, then it is attained. Next, if κ is a singular cardinal, then we can construct a BA A with $\chi(A) = \kappa$ not attained. Namely, let $\langle \mu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence of infinite cardinals with sup κ. For each $\xi < \mathrm{cf}(\kappa)$ let A_ξ be the free BA on μ_ξ free generators; thus $\chi(A_\xi) = \mu_\xi$. By the above remarks on weak products, $\prod^{\mathrm{w}}_{\xi<\mathrm{cf}\kappa} A_\xi$ has character κ not attained.

For regular limit cardinals, there is an old theorem of Parovičenko [67] which characterizes attainment; this solves Problem 37 of Monk [90]. We give this theorem here; it requires a definition. If X is a topological space and $x \in X$, the *character* of x is $\min\{|\mathscr{O}| : \mathscr{O}$ is a neighborhood base of $x\}$. Clearly for any BA A and any $F \in \mathrm{Ult}(A)$, the character of F as a point in $\mathrm{Ult}(A)$ coincides with $\chi(F)$ as defined above.

Theorem 14.2. *Let κ be a limit cardinal. Then the following conditions are equivalent:*

(i) *κ is weakly compact.*
(ii) *For every compact Hausdorff space X of size at least κ, X has a point of character at least κ.*
(iii) *For every BA A, if $\chi(A) = \kappa$, then A has an ultrafilter with character κ.*

Proof. (i)⇒(ii): Assume (i), and suppose that X is a compact Hausdorff space with $|X| \geq \kappa$ such that X has no point of character $\geq \kappa$.

(1) We may assume that $|X| = \kappa$.

For, let $Y \in [X]^\kappa$. We claim that $|\overline{Y}| = \kappa$. Since the character of a point of $\overline{Y}$ is clearly still $< \kappa$, (1) follows from the claim. By Theorem 9.49, it suffices to show that if $y \in \overline{Y}$, then there is a $Z \in [Y]^{<\kappa}$ such that $y \in \overline{Z}$, since then $\overline{Y} = \bigcup_{Z\in[Y]^{<\kappa}} \overline{Z}$, and by κ being strongly inaccessible $|\overline{Y}| = \kappa$ follows. Let $\mathscr{U}$ be an open neighborhood base for y of size $< \kappa$. For every $U \in \mathscr{U}$ choose $z_U \in U \cap Y$. Clearly $Z \stackrel{\text{def}}{=} \{z_U : u \in \mathscr{U}\}$ is as desired.

So we now assume that $|X| = \kappa$. Let $X = \{x_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, let $\mathscr{U}_\alpha$ be an open neighborhood base for x_α of size $< \kappa$. Set $\mathscr{F}_\alpha = \{F : X\backslash F \in \mathscr{U}_\alpha\}$. So $\mathscr{F}_\alpha$ is a collection of closed sets, $\bigcup \mathscr{F}_\alpha = X\backslash\{x_\alpha\}$, and $|\mathscr{F}_\alpha| < \kappa$. Let T be the collection of all functions f such that there is an $\alpha < \kappa$ such that $\operatorname{dmn} f = \alpha$, $\forall\beta < \alpha(f_\beta \in \mathscr{F}_\beta)$, and $\bigcap_{\beta<\alpha} f_\beta \neq 0$. Thus T is a tree under $\subseteq$.

(2) $\forall\alpha < \delta\exists f \in T(\operatorname{dmn} f = \alpha)$.

For,

$$\begin{aligned}0 \neq X\backslash\{x_\beta : \beta < \alpha\} &= \bigcap_{\beta<\alpha} \bigcup \mathscr{F}_\alpha \\ &= \bigcup_{f\in\prod\limits_{\beta<\alpha}\mathscr{F}_\beta} \bigcap_{\beta<\alpha} f_\beta,\end{aligned}$$

so there is an $f \in \prod_{\beta<\alpha} \mathscr{F}_\beta$ such that $\bigcap_{\beta<\alpha} f_\beta \neq 0$. Thus f is as desired in (2).

(3) Every level of T has size $< \kappa$.

This is true since κ is strongly inaccessible, so that $\left|\prod_{\beta<\alpha} \mathscr{F}_\beta\right| < \kappa$ for every $\alpha < \kappa$.

Now by the weak compactness of κ, let f with domain κ be a branch through T. By compactness, $\bigcap_{\alpha<\kappa} f_\alpha \neq 0$. But if $y \in \bigcap_{\alpha<\kappa} f_\alpha$, then $y \neq x_\alpha$ for all $\alpha < \kappa$, contradiction.

(ii)⇒(iii): obvious.

(iii)⇒(i): Assume that κ is not weakly compact; we want to find a BA A with character κ not attained. By the comments on attainment above, we may assume that κ is regular. Let L be a linear order of size κ such that neither κ nor κ^* embeds in L. By replacing points of L by ordinals less than κ we may assume that each ordinal less than κ embeds in L. More precisely, write $L = \{a_\alpha : \alpha < \kappa\}$ with no repetitions. let $M = \{(\beta, a_\alpha) : \beta \leq \alpha,\ \alpha < \kappa\}$, ordered anti-lexicographically. We show that M has no increasing chain of type κ. For, suppose that $\langle(\beta_\xi, x_\xi) : \xi < \kappa\rangle$ is such a chain. Since κ is regular, the set $\{x_\xi : \xi < \kappa\}$ has κ elements, and hence determines a chain in L of type κ, contradiction. Similarly, M has no decreasing chain of type κ. Let $A = \operatorname{Intalg}(M)$. Then by the description of character for interval algebras below, A has the desired properties. □

To treat arbitrary direct products, note that obviously $\operatorname{t}(A) \leq \chi(A)$; hence $\operatorname{Ind}(A) \leq \chi(A)$, and so clearly $\chi(\prod_{i\in I} A_i) \geq \max(2^{|I|}, \sup_{i\in I}\chi(A_i))$. Shelah and

Peterson independently observed that strict inequality is possible. This solves Problem 38 in Monk [90]. The same example used for spread works here: let κ be the first limit cardinal $> 2^\omega$, let A be the finite-cofinite algebra on κ, and consider ${}^\omega A$. Recall that character does not increase when going to a homomorphic image.

We now discuss ultraproducts, giving some results of Douglas Peterson. Character is a sup-min function, and so Theorems 6.5–6.9 apply. Then a proof similar to that of Theorem 4.18 shows that if GCH holds then $\chi\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I}\chi A_i/F\right|$ for F regular, and Donder's theorem says that under $V = L$ the regularity assumption can be removed. The following result of Shelah, Spinas [00], part of Corollary 2.7, is relevant here, solving Problem 48 of Monk [96]:

There is a model in which there exist cardinals κ, μ, a system $\langle B_i : i < \kappa\rangle$ of BAs, and an ultrafilter D on κ such that

$$\left|\prod_{i<\kappa}\chi(B_i)/D\right| = \mu^{++} \quad \textit{and} \quad \chi\left(\prod_{i<\kappa} B_i/D\right) \leq \mu^+.$$

On the other hand, it is easy to give an example in which $>$ holds. Let κ be any infinite cardinal such that $\kappa^\omega = \kappa$. As we saw above, the Alexsandroff duplicate A of a free BA on κ generators has character κ and cellularity 2^κ. By Theorem 11.8 this implies that $\chi(A^{*i}) = \kappa$ for all $i \in \omega\backslash 1$ while $\chi\left(\prod_{i\in\omega\backslash 1} A^{*i}/F\right) = 2^\kappa$ for any nonprincipal ultrafilter F on $\omega\backslash 1$. (See below for the character of free products.) The assumption $\kappa^\omega = \kappa$ implies that $\left|\prod_{i\in\omega\backslash 1}\chi(A^{*i})/F\right| = \kappa$.

Proposition 14.3. *If $\langle A_i : i \in I\rangle$ is a system of BAs each with at least four elements, then $\chi(\oplus_{i\in I}A_i) = \max(|I|, \sup_{i\in I}\chi(A_i))$.*

Proof. For $\geq$, first let $j \in I$ and let $F \in \mathrm{Ult}(A_j)$. Let G be any ultrafilter on $\oplus_{i\in I}A_i$ which includes F. Suppose that $X \subseteq G$ generates G. Without loss of generality, each member of X is a product of elements from distinct A_i's. Then it is clear that $X \cap A_j \subseteq F$ and $X \cap A_j$ generates F. So $\chi(F) \leq |X|$. It follows that $\chi(F) \leq \chi(G) \leq \chi(\oplus_{i\in I}A_i)$. Hence $\chi(A_j) \leq \chi(\oplus_{i\in I}A_i)$. It is clear that $\chi H \geq |I|$ for any ultrafilter H on $\oplus_{i\in I}A_i$. Altogether, this proves $\geq$. For $\leq$, for any ultrafilter G on $\oplus_{i\in I}A_i$, and for each $i \in I$ let $X_i \subseteq G \cap A_i$ generate $G \cap A_i$, with $|X_i| = \chi(G\cap A_i)$. Clearly the set of all finite products of elements of $\bigcup_{i\in I} X_i$ generates G, as desired. □

Problem 129. *Describe the behaviour of character under unions.*

Proposition 14.4. *Assume the framework for moderate products, where B and each A_i are infinite BAs. Then $\chi(\prod^B_{i\in I} A_i) = \max(\chi(B), \sup_{i\in I}\chi(A_i))$.*

Proof. It suffices to look at the characterization of ultrafilters on $\prod^B_{i\in I} A_i$ given in Proposition 1.6(x) and show that the character of an ultrafilter U on $\prod^B_{i\in I} A_i$ is the same as the character of the associated ultrafilter on B or on A_i.

Case 1. There exist an $i \in I$ and an ultrafilter V on A_i such that $U = \{h(b, F, a) : (b, F, a)$ is normal, and $i \in b$ or $(i \notin b$ and $i \in F$ and $a_i \in V)\}$. Let $X \subseteq V$ generate V, with $|X| = \chi(V)$. Wlog X is closed under $\cap$. We claim that

$$\{h(\{i\}, \emptyset, \emptyset)\} \cup \{h(\emptyset, \{i\}, \{(i, x)\}) : x \in X\}$$

is a subset of U which generates it. Clearly it is a subset of U. Now suppose that (b, F, a) is normal and $h(b, F, a) \in U$. If $i \in b$, then $h(\{i\}, \emptyset, \emptyset) \subseteq h(b, F, a)$, as desired. Suppose that $i \notin b$, $i \in F$, and $a_i \in V$. Choose $x \in X$ such that $x \subseteq a_i$. Then $\{h(\emptyset, \{i\}, \{(i, x)\}) \subseteq h(b, F, a)$, as desired. This proves our claim. It follows that $\chi(U) \leq \chi(V)$.

Now suppose that $Y \subseteq U$ generates U and $|Y| = \chi(U)$. Wlog Y is closed under $\cap$. We claim that

$$Z \stackrel{\text{def}}{=} \{x : h(\emptyset, \{i\}, \{(i, x)\}) \in Y\}$$

is a subset of V which generates it. Clearly it is a subset of V. Now suppose that $y \in V$. Then $h(\emptyset, \{i\}, \{(i, y)\}) \in U$, so we can choose $h(b, F, a) \in Y$ such that $h(b, F, a) \subseteq h(\emptyset, \{i\}, \{(i, y)\})$. It follows that $b = \emptyset$, $F = \{i\}$, and $a_i \subseteq y$. Thus $a_i \in Z$, as desired. This proves our claim. Hence $\chi(U) = \chi(V)$.

Case 2. There is a nonprincipal ultrafilter W on B such that $U = \{h(b, F, a) : (b, F, a)$ is normal and $b \in W\}$. Let $X \subseteq W$ generate W, with $|X| = \chi(W)$. Wlog X is closed under $\cap$. Then we claim that

$$\{h(b, \emptyset, \emptyset) : b \in X\}$$

generates U. For, let (b, F, a) be normal with $h(b, F, a) \in U$. Choose $v \in X$ such that $v \leq b$. Then $h(v, \emptyset, \emptyset) \subseteq h(b, F, a)$, as desired. This shows that $\chi(U) \leq \chi(W)$.

Now let $Y \subseteq U$ generate U, with $|Y| = \chi(U)$. Wlog Y is closed under $\cap$. We claim that $Z \stackrel{\text{def}}{=} \{b :$ there is a normal (b, F, a) with $h(b, F, a) \in Y\}$ generates W. Clearly it is a subset of W. Now suppose that $c \in W$. Then $h(c, \emptyset, \emptyset) \in U$, so there is a normal (b, F, a) such that $h(b, F, a) \in Y$ and $h(b, F, a) \subseteq h(c, \emptyset, \emptyset)$. Then $b \in Z$ and $b \subseteq c$, as desired.

This proves that $\chi(U) = \chi(W)$. $\square$

Proposition 14.5. *Let C be the one-point gluing of BAs A, B with respect to ultrafilters F, G on A, B respectively. Then $\chi(C) = \max(\chi(A), \chi(B))$.*

Proof. Let H be an ultrafilter on $A \times B$.

(1) $\chi(H \cap C) \leq \max(\chi(H), \chi(F), \chi(G))$.

In fact, wlog $(1, 0) \in H$. Let $X \subseteq H$ generate H with $|X| = \chi(H)$. Let $Z \subseteq G$ generate G with $|Z| = \chi(G)$. Let

$$W = \{(a, w) : a \in F, w \in Z, \exists b[(a, b) \in X]\} \cup \{(a, 0) : a \notin F \text{ and } \exists b[(a, b) \in X]\}.$$

Suppose that $(u,v) \in H \cap C$. Choose $(a,b) \in X$ such that $(a,b) \le (u,v)$.

Case 1. $a \in F$. Then $u \in F$ and $v \in G$. Choose $w \in Z$ such that $w \le v$. Then $(a,w) \in W$ and $(a,w) \le (u,v)$. Obviously $(a,w) \in C$. Since $(a,b) \in H$ and $(1,0) \in H$, we have $(a,0) \in H$ and hence $(a,w) \in H$.

Case 2. $a \notin F$. Clearly $(a,0) \in H \cap C$ and $(a,0) \le (u,v)$.

From (1) it follows that $\chi(C) \le \chi(A \times B)$.

Still with H an ultrafilter on $A \times B$ and $(1,0) \in H$, choose $X \subseteq H \cap C$ generating $H \cap C$, with $|X| = \chi(H \cap C)$. Let

$$W = X \cup \{(a,0) : a \in F \text{ and } \exists c[(a,c) \in X]\}.$$

Suppose that $(a,b) \in H$.

Case 1. $a \in F$ and $b \in G$. Then there exists $(c,d) \in X$ such that $(c,d) \le (a,b)$.

Case 2. $a \in F$ and $b \notin G$. Then $(a,0) \in W$ and $(a,0) \le (a,b)$.

Case 3. $a \notin F$ and $b \in G$. Then $(a,0) \in W$ and $(a,0) \le (a,b)$.

Case 4. $a \notin F$ and $b \notin G$. Then there exists $(c,d) \in X$ such that $(c,d) \le (a,b)$.

Clearly $W \subseteq H$.

So we have $\chi(H) \le \chi(H \cap C)$. Hence $\chi(A \times B) \le \chi(C)$. □

Proposition 14.6. $\chi(A) = \chi(\mathrm{Dup}(A))$ *for any infinite BA A.*

Proof. Let $\kappa = \chi(A)$. Take any nonprincipal ultrafilter F on $\mathrm{Dup}(A)$. By Proposition 1.19 there is an ultrafilter G on A such that $F = \{(a,X) : a \in G$ and $\mathcal{S}(A) \triangle X$ is finite$\}$. Let M generate G, M closed under $\cdot$, $|M| = \chi(G)$. We claim that $\{(a, \mathcal{S}(a) \backslash \{G\} : a \in M\}$ generates F. For, suppose that (a,Y) is any element of F; thus $\mathcal{S}(a) \backslash Y$ is finite. For each $H \in \mathcal{S}(a) \backslash (Y \cup \{G\})$ choose $a_H \in F \backslash H$. Then let $b = a \cdot \prod_{H \in \mathcal{S}(a) \backslash (Y \cup \{G\})} a_H$. Thus $b \in F$. Choose $c \in M$ such that $c \le b$. Then $(c, \mathcal{S}(c) \backslash \{G\}) \le (a,Y)$, as desired.

Conversely, suppose that G is a nonprincipal ultrafilter on A. Let $F = \{(a,X) \in \mathrm{Dup}(A) : a \in G\}$. So F is a nonprincipal ultrafilter on $\mathrm{Dup}(A)$. Let M be a subset of F of size at most $\pi\chi(\mathrm{Dup}(A))$ which is dense in F. For each $a \in G$ choose $(b_a, X_a) \in M$ such that $(b_a, X_a) \le (a, \mathcal{S}(a))$. We claim that $\{b_a : a \in G\}$ is dense in G. Suppose that $a \in G$. Then $b_a \le a$, as desired. □

Proposition 14.7. $\chi(A) \le \chi(\mathrm{Exp}(A))$ *for any infinite BA A.*

Proof. Let G be an ultrafilter on A. Then the set $\{\mathcal{V}(\mathcal{S}(a)) : a \in G\}$ has fip by Lemma 1.22(iii). Let F be the filter on $\mathrm{Exp}(A)$ generated by this set. Then F is an ultrafilter. To prove this, note that if $a \notin G$ then $-\mathcal{V}(\mathcal{S}(a)) \in F$. In fact, $-a \in G$, so $\mathcal{S}(-a) \in F$. By Lemma 1.22(v) we then get $-\mathcal{V}(\mathcal{S}(a)) \in F$. Now by Proposition 1.21 it follows that F is an ultrafilter.

Now suppose that $X \subseteq G$ is dense in G, with X closed under $\cdot$ and $|X| = \chi(G)$. Then clearly $\{\mathcal{V}(\mathcal{S}(a)) : a \in G\}$ is dense in F. So $\chi(F) \le \chi(G)$.

Conversely, suppose that $Y \subseteq F$ is dense in F with $|Y| = \chi(F)$. We may assume that each member of Y has the form $\mathcal{V}(\mathcal{S}(a))$. Then $\{a : \mathcal{V}(\mathcal{S}(a)) \in Y\}$ is dense in G. So $\chi(G) \le \chi(F)$. $\square$

Note that the example at the end of Chapter 1 gives an infinite BA A such that $\chi(A) < \chi(\mathrm{Exp}(A))$.

Lemma 14.8. $\chi(A) \le \mathrm{t}(\mathrm{Exp}\,(A))$*; in fact, every closed set in* $\mathrm{Ult}(A)$ *has a neighborhood basis with at most* $\mathrm{t}(\mathrm{Exp}\,(A))$ *elements.*

Proof. Let F be a closed subset of $\mathrm{Ult}(A)$. Then

$$F \in \overline{\{U : U \text{ clopen}, F \subseteq U\}}.$$

For, suppose that $F \in \mathscr{V}(U_0, \ldots, U_{m-1})$ with each U_i clopen. Then $U_0 \cup \cdots \cup U_{m-1} \in \mathscr{V}(U_0, \ldots, U_{m-1})$, as desired.

It follows that there is a subset $\mathscr{O}$ of $\{U : U \text{ clopen}, F \subseteq U\}$ such that $|\mathscr{O}| \le \mathrm{t}(\mathrm{Exp}\,(A)$ and $F \in \overline{\mathscr{O}}$. Then $\mathscr{O}$ is the desired neighborhood base for F. For, suppose $F \subseteq W$ with W clopen. Then $F \in \mathscr{V}(W)$, so there is a $U \in \mathscr{O}$ such that $U \in \mathscr{V}(W)$. So $U \subseteq W$, as desired. $\square$

Problem 130. *Describe* $\chi(\mathrm{Exp}(A))$.

We turn to the derived functions for character. By a remark above, we have $\chi_{\mathrm{H}+}(A) = \chi(A)$ for any infinite BA A. Koszmider [99] has shown that it is consistent to have $\mathrm{Card}_{\mathrm{H}-}(A) = \omega_2 = 2^\omega$ while $\chi_{\mathrm{H}-}(A) = \omega_1$. This solves Problem 39 in Monk [90].

We also do not know the status of $\chi_{\mathrm{S}+}(A)$; we observed above that it can happen that $\chi_{\mathrm{S}+}(A) > \chi(A)$. Clearly $\chi_{\mathrm{S}-}(A) = \omega$ for any infinite BA A. The topological version of character is this: for any space X and any $x \in X$, $\chi(x, X)$ is the minimum of the cardinalities of neighborhood bases for x in X, and $\chi X = \sup\{\chi(x, X) : x \in X\}$. Clearly then $\chi_{\mathrm{h}+}(A) = \chi(A)$, and $\chi_{\mathrm{h}-}(A) = 1$ for any infinite BA A, since A has an infinite discrete subspace. The function χ_{inf} is of some interest; recall from the introduction that $\chi_{\mathrm{inf}}(A) = \inf\{\chi(F) : F \in \mathrm{Ult}A\}$ for any infinite BA A. This function has value 1 if A has an atom. Hence it is natural to modify it slightly; and in view of the special case $\mathscr{P}(\omega)/\mathrm{fin}$, we use $\mathfrak{u}(A)$ instead of $\chi_{\mathrm{inf}}(A)$. Thus by definition,

$$\mathfrak{u}(A) = \min\{\chi(F) : F \text{ is a nonprincipal ultrafilter on } A\}.$$

There is a classical result of concerning $\mathfrak{u}$:

Theorem 14.9. $2^{\mathfrak{u}(A)} \le |\mathrm{Ult}A|$ *for any atomless BA* A.

Proof. For brevity set $\kappa = \mathfrak{u}(A)$. It clearly suffices to construct a function f mapping ${}^{<\kappa}2$ into A such that

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(1) For each $s \in {}^{<\kappa}2$, the set $\{f(s \restriction \alpha) : \alpha \leq \operatorname{dom} s\}$ has the finite intersection property;

(2) $f(s^\frown 0) \cdot f(s^\frown 1) = 0$ for each $s \in^{<\kappa} 2$.

Suppose $s \in {}^{<\kappa}2$ and $f(s \restriction \alpha)$ has been defined for all $\alpha \in \operatorname{dom} s$. By the induction hypothesis, $\{f(s \restriction \alpha) : \alpha \in \operatorname{dom} s\}$ has the finite intersection property; since this set has $< \kappa$ elements, it does not generate an ultrafilter, and hence there is a $a \in A$ such that both a and $-a$ fail to be in the filter generated by it. Hence if we set $f(s^\frown 0) = a$ and $f(s^\frown 1) = -a$ we extend our function f so that (1) and (2) will hold. This completes the proof. □

Kevin Selker has obtained the following results concerning $\mathfrak{u}$ and moderate products:

(1) *Let $\langle A_i : i \in I\rangle$ be a system of infinite BAs, and B an infinite BA, satisfying the hypotheses for the formation of a moderate product, and let $C = \prod_{i\in I}^B A_i$. Then $\mathfrak{u}_{\mathrm{spect}}(C) = \mathfrak{u}_{\mathrm{spect}}(B) \cup \bigcup_{i\in I} \mathfrak{u}_{\mathrm{spect}}(A_i)$.*

(2) *If K is an infinite set of infinite cardinals, and either $|K| < \sup(K)$ or $|K| = \sup(K)$ is singular, then there is an atomless BA A such that $\mathfrak{u}_{\mathrm{spect}}(A) = K$.*

It is easy to see that $\mathfrak{p} \leq \mathfrak{u}$, and $\pi\chi_{\mathrm{inf}} \leq \mathfrak{u}$ is obvious. There is a BA A with $\mathfrak{u} < \mathfrak{i}$; see McKenzie, Monk[04]; and there is a BA A with $\mathfrak{u} < \mathrm{s_{mm}}$; see Monk [08].

If A is complete, then $\omega < \mathfrak{u}(A)$. In fact, suppose that F is a nonprincipal ultrafilter on A and X is a countable set which generates F. Let $X = \{x_i : i < \omega\}$. Without loss of generality, $1 = x_0 > x_1 > \cdots$. Then

(1) $\prod X = 0$.

In fact, suppose that $\prod X \neq 0$. If $\prod X \in F$, then there is an $i \in \omega$ such that $x_i \leq \prod F < x_{i+1}$, contradiction. If $-\prod X \in F$, then there is an $i \in \omega$ such that $x_i \leq= -\prod X$, and then $\prod X = x_i \cdot \prod X = 0$, contradiction. So (1) holds.

Let $a_i = x_i \cdot -x_{i+1}$ for all $i \in \omega$. Now take any nonzero $b \in A$. Let i be maximum such that $b \leq x_i$. Then $b \cdot a_i = b \cdot x_i \cdot -x_{i+1} = b \cdot -x_{i+1} \neq 0$. This shows that $\sum_{i\in\omega} a_i = 1$. If $\sum_{i\in\omega} a_{2i} \in F$, choose $j \in \omega$ such that $x_j \leq \sum_{i\in\omega} a_{2i}$. We may assume that $j = 2k+1$ for some k. Then $a_{2k+1} = x_{2k+1} \cdot -x_{2k+2} \leq \sum_{i\in\omega} a_{2i} \in F$, contradiction. A similar contradiction is reached if $\sum_{i\in\omega} a_{2i+1} \in F$.

There is an atomic BA A such that $\mathfrak{f}(A) < \mathfrak{u}(A)$. Let $A = \mathscr{P}(\omega)$. Thus by the above, $\mathfrak{u}(A) > \omega$. We claim that $\mathfrak{A} = \omega$. For each $n \in \omega$ let $a_n = \omega\backslash(n+1)$. Thus $a_0 \neq \omega$ and $\langle a_n : n \in \omega\rangle$ is strictly decreasing; hence by Proposition 12.16 it is a free sequence. To show that it is maximal we apply Proposition 12.17. Suppose that $x \subseteq \omega$. If x is finite, choose $n \in \omega$ with $x \subseteq n+1$. Thus $a_n \cap x = 0$, as desired in Proposition 12.18(i). Suppose that x is infinite. Take any $m \in x$ with $m > 0$. Then $a_{m-1} \cap (\omega\backslash a_m) \cap (\omega\backslash x) = 0$, as desired in Proposition 12.17(ii).

Kevin Selker has shown under CH that there is an atomless BA A such that $\mathfrak{f}(A) < \mathfrak{u}(A)$.

Problem 131. *Can one prove in ZFC that there is an atomless BA A such that $\mathfrak{f}(A) < \mathfrak{u}(A)$?*

In Kunen [80], Exercise (A10) in Chapter VIII, a model is constructed with $\mathfrak{u}(\mathscr{P}(\omega)/\mathrm{fin}) < 2^\omega$; that model also has $\mathfrak{f}(\mathscr{P}(\omega)/\mathrm{fin}) = \mathfrak{u}(\mathscr{P}(\omega)/\mathrm{fin})$.

We also have $\mathrm{s}_{\mathrm{mm}}(A) = \omega$ for $A = \mathscr{P}(\omega)$. In fact, the set of all singletons is clearly maximal ideal independent. Kevin Selker has shown under CH that there is an atomless BA A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{u}(A)$. (Theorem 2.10 in Monk [08], asserting that there is a BA satisfying this inequality, has a faulty proof.)

Problem 132. *Can one prove in ZFC that there is an atomless BA A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{u}(A)$?*

For the cardinal function alt, see Chapter 9.

Theorem 14.10. *Let B be a homomorphic image of A, and G a nonprincipal ultrafilter on B. Then* $\mathrm{alt}(A) \le \chi(G)$.

Proof. Let f be a homomorphism from A onto B, and let $\langle b_\alpha : \alpha < \chi(G)\rangle$ be an enumeration of a set of generators of G. For each $\alpha < \chi(G)$ choose $a_\alpha \in A$ such that $f(a_\alpha) = b_\alpha$. Now clearly for each $\alpha < \chi(G)$ the set $\{a_\xi : \xi < \alpha\} \cup \{x \in A : f(x) = 1\}$ has fip, so we can let F_α be the filter generated by this set. Clearly

(1) $f^{-1}[G]$ is an ultrafilter on A.

(2) $F_\alpha \subseteq f^{-1}[G]$ for each $\alpha < \chi(G)$.

In fact, if M is a finite subset of α, $x \in A$, $f(x) = 1$, and $x \cdot \prod_{\xi\in M} a_\xi \le c$, then $\prod_{\xi\in M} b_\xi \le f(c)$, hence $f(c) \in G$ and $c \in f^{-1}[G]$. So (2) holds.

Clearly

(3) $\alpha < \beta$ implies that $F_\alpha \subseteq F_\beta$.

(4) If $\alpha < \chi(G)$, then $F_\alpha \neq f^{-1}[G]$.

In fact, choose $c \in G$ such that c is not in the filter generated by $\{b_\xi : \xi < \alpha\}$, and choose $d \in A$ such that $f(d) = c$. Then $d \in f^{-1}[G]$ but clearly $d \notin F_\alpha$. Next,

(5) $\bigcup_{\alpha<\chi(G)} F_\alpha = f^{-1}[G]$.

In fact, take any $c \in f^{-1}[G]$, and choose a finite $M \subseteq \chi(G)$ such that $\prod_{\xi\in M} b_\xi \le f(c)$. Then $f(-\prod_{\xi\in M} a_\xi + c) = 1$, so that $-\prod_{\xi\in M} a_\xi + c \in F_\alpha$, where $M \subseteq \alpha < \chi(G)$. Now $\prod_{\xi\in M} a_\xi \cdot (-\prod_{\xi\in M} a_\xi + c) \le c$, so $c \in F_\alpha$, proving (5).

From (1)–(5) we see that a subsequence of $\langle F_\alpha : \alpha < \chi(G)\rangle$ is strictly increasing with union $f^{-1}[G]$. □

Corollary 14.11. $\mathrm{alt}(A) \le \chi_{\mathrm{H}-}A \le \mathrm{card}_{\mathrm{H}-}(A)$ *for any infinite BA A.* □

Corollary 14.12. $\mathrm{alt}(A) = \omega$ *iff* $\text{p-alt}(A) = \omega$ iff $\mathrm{Card}_{\mathrm{H}-}(A) = \omega$ *iff* $\chi_{\mathrm{H}-}(A) = \omega$, *for any infinite BA A.*

Proof. See Proposition 9.4. □

We turn to the relationships of character with our previously treated functions. Obviously $\mathrm{t}(A) \leq \chi(A)$ for any infinite BA A; the difference can be big – for example for the finite-cofinite algebra on an infinite cardinal κ. Now consider the possibility that $\chi(A) > \mathrm{s}(A)$. By the comment at the end of the last Chapter, plus the fact that $\chi(A) \leq \mathrm{hL}(A)$ (easy, and proved in Chapter 15), it is consistent that $\mathrm{s}(A) = \omega$ implies $\chi(A) = \omega$. The Kunen line, constructed in Chapter 8, has uncountable character but countable spread; it was constructed using CH. To show that it has uncountable character, assume the notation in the construction.

(∗) If $U \in \tau_{\omega_1}$ is open and $\{\xi < \omega_1 : x_\xi \in U\}$ is uncountable, then U is not compact in τ_{ω_1}.

To see this, note that for each $\xi \in U$ we have $\xi \in U \cap \mathscr{P}(\xi) \in \tau_{\omega_1}$. Thus $\{U \cap \mathscr{P}(\xi) : \xi \in U\}$ is an open cover of U, and it clearly has no finite subcover, proving (∗).

Now suppose that the new point y of Y has a countable base $\{\{y\} \cup U_m : m \in \omega\}$; we may assume that these are clopen. Hence $Y \backslash (\{y\} \cup U_m) = X \backslash U_m$ is open in Y and hence in X; and it is also compact. It follows from (∗) that $X \backslash U_m$ is countable. Choose $\alpha \in \bigcap_{m \in \omega} U_m$. Then $Y \backslash \{\alpha\}$ is an open neighborhood of y not containing any of the basis sets $\{y\} \cup U_m$, contradiction.

Under $\diamondsuit$ there is a BA A such that $\mathrm{s}(A) < \chi(A)$; see Chapter 17. The following problem is open, however.

Problem 133. *Can one construct in ZFC a BA A such that* $\mathrm{s}(A) < \chi(A)$*?*

This was Problem 49 in Monk [96]. It is equivalent to the problem whether one can construct in ZFC a BA A such that $\mathrm{s}(A) < \mathrm{hL}(A)$; see the end of Chapter 15.

An example of a BA A with $\mathrm{c}(A) > \chi(A)$ is provided by the Alexsandroff duplicate of the free algebra on κ free generators, as described at the beginning of this chapter. The interval algebra on $\mathbb{R}$ gives an example of an algebra A with $\mathrm{Length}(A) > \chi(A)$.

Now we turn to Arhangelskiĭ's theorem that $|\mathrm{Ult}(A)| \leq 2^{\chi(A)}$ for any infinite BA A. We need some lemmas.

Lemma 14.13. *If $Y \subseteq \mathrm{Ult}A$ and $F \subseteq \bigcup Y$ has the finite intersection property, then there is an ultrafilter G such that $F \subseteq G \subseteq \bigcup Y$.*

Proof. Let G be maximal among the filters H such that $F \subseteq H \subseteq \bigcup Y$. Suppose that G is not an ultrafilter; say $a, -a \notin G$. Then $\langle G \cup \{a\}\rangle^{\mathrm{fi}} \not\subseteq \bigcup Y$. Say $b \in G$ and $b \cdot a \notin \bigcup Y$. Similarly obtain $c \in G$ such that $c \cdot -a \notin \bigcup Y$. Choose $H \in Y$ such that $b \cdot c \in H$. Then $b \cdot c \cdot a \notin H$ and $b \cdot c \cdot -a \notin H$, contradiction. □

Note that for any subset Y of $\mathrm{Ult}(A)$, the closure of Y is $\{F \in \mathrm{Ult}(A) : F \subseteq \bigcup Y\}$.

Lemma 14.14. *If $Z \subseteq \mathrm{Ult}(A)$ is closed, then $\mathrm{Ult}(A) \backslash Z$ is the union of at most*

$$\max\{\omega, |Z|, \sup_{G \in Z} \chi(G)\}$$

clopen sets.

Proof. For every $G \in Z$ let $\{a^G_\alpha : \alpha < \chi(G)\}$ be a set of generators of G, closed under multiplication. Let

$$B = \{a^{G_0}_{\alpha_0} + \cdots + a^{G_{n-1}}_{\alpha_{n-1}} : n \in \omega,\ G_0, \ldots, G_{n-1} \in Z,\ \alpha_i < \chi(G_i) \text{ for all } i < n\},$$

and let $C = \{y : -y \in B \cap \bigcap Z\}$. We claim that

$$\mathrm{Ult}(A)\backslash Z = \bigcup_{y \in C} \mathcal{S}(y),$$

which gives the desired result. $\supseteq$ is clear. Now suppose that $F \in \mathrm{Ult}(A)\backslash Z$. For every $G \in Z$ choose $b_G \in F\backslash G$; say $a^G_{\alpha(G)} \leq -b_G$. Then

$(*)$ There exist an integer $n \in \omega$ and elements $G_0, \ldots, G_{n-1} \in Z$ with the property that $a^{G_0}_{\alpha(G_0)} + \cdots + a^{G_{n-1}}_{\alpha(G_{n-1})} \in H$ for all $H \in Z$.

Otherwise, $L \stackrel{\text{def}}{=} \{-a^{G_0}_{\alpha(G_0)} \cdot \ldots \cdot -a^{G_{n-1}}_{\alpha(G_{n-1})} : n \in \omega,\ G_0, \ldots, G_{n-1} \in Z\}$ has the finite intersection property and is contained in $\bigcup Z$. Hence by Lemma 14.13, there is an ultrafilter K such that $L \subseteq K \subseteq \bigcup Z$. Hence $K \in Z$ and $-a^K_{\alpha(K)} \in K$, contradiction.

We choose $n \in \omega$ and $G_0, \ldots, G_{n-1} \in Z$ as in $(*)$. Let y be the element $-a^{G_0}_{\alpha(G_0)} \cdot \ldots \cdot -a^{G_{n-1}}_{\alpha(G_{n-1})}$. Then $y \in C$ and $F \in \mathcal{S}(y)$, as desired. □

Lemma 14.15. *If* $Y \subseteq \mathrm{Ult}(A)$ *and* $|Y| \leq \chi(A)$, *then* $|\overline{Y}| \leq 2^{\chi(A)}$.

Proof. For every ultrafilter G on A let $\{a^G_\alpha : \alpha < \chi(A)\}$ be a set of generators of G, and set $f(G) = \{\{F \in Y : a^G_\alpha \in F\} : \alpha < \chi(A)\}$. Thus $f(G) \in [\mathscr{P}(Y)]^{\leq\chi(A)}$. Hence it is enough to show that $f \restriction \overline{Y}$ is one-one. Suppose that G and H are distinct ultrafilters on A such that $G, H \in \overline{Y}$. Say $a^G_\alpha \in G\backslash H$, and choose $a^H_\beta \leq -a^G_\alpha$. Suppose that $f(G) = f(H)$; then there is a $\gamma < \chi(A)$ such that $\{F \in Y : a^G_\gamma \in F\} = \{F \in Y : a^H_\beta \in F\}$. Then $a^G_\alpha \cdot a^G_\gamma \in G$; say then $a^G_\alpha \cdot a^G_\gamma \in F \in Y$. Then $a^H_\beta \in F$, $a^G_\alpha \in F$, and $-a^G_\alpha \in F$, contradiction. □

Now we are ready for the proof of Arhangelskiĭ's theorem.

Theorem 14.16. $|\mathrm{Ult}(A)| \leq 2^{\chi(A)}$ *for any infinite BA* A.

Proof. Suppose that $2^{\chi(A)} < |\mathrm{Ult}(A)|$. Fix an ultrafilter F on A. For each $f \in \bigcup_{\alpha < (\chi(A))^+} {}^\alpha(2^{\chi(A)})$ we define a closed set $X_f \subseteq \mathrm{Ult}(A)$ and a $G_f \in \mathrm{Ult}(A)$. Let $X_0 = \mathrm{Ult}(A)$ and $G_0 = F$. For $\mathrm{dmn}(f)$ limit let $X_f = \bigcap_{\alpha < \mathrm{dmn} f} X_{f\restriction\alpha}$, and if $X_f \neq 0$ choose $G_f \in X_f$, and otherwise let $G_f = F$. Now suppose that $\mathrm{dmn}(f)$ is a successor ordinal $\alpha + 1$. Let $g = f \restriction \alpha$, and set $Y_g = \{G_{g\restriction\beta} : \beta \leq \alpha\}$. Thus $|Y_g| \leq \chi(A)$, so by Lemma 14.15, $|\overline{Y}_g| \leq 2^{\chi(A)}$, and so by Lemma 14.14 we can let $\langle a^g_\beta : \beta < 2^{\chi(A)}\rangle$ be such that $\mathrm{Ult}(A)\backslash\overline{Y}_g = \bigcup_{\beta < 2^{\chi(A)}} \mathcal{S}(a^g_\beta)$ and set $X_f = X_g \cap \mathcal{S}\left(a^g_{f(\alpha)}\right)$. Again let $G_f \in X_f$ if $X_f \neq 0$, and $G_f = F$ otherwise. This finishes the construction.

Now choose

$$H \in \mathrm{Ult}(A)\backslash \bigcup \left\{ \overline{\{G_{f\restriction\beta} : \beta \le \mathrm{dmn}(f)\}} : f \in \bigcup_{\alpha<(\chi(A))^+} {}^{\alpha}(2^{\chi(A)}) \right\}.$$

Now we define f mapping $(\chi(A))^+$ into $2^{\chi(A)}$ by induction. Suppose that $f(\beta)$ has been defined for all $\beta < \alpha$. Now $H \notin \overline{\{G_{f\restriction\beta} : \beta \le \alpha\}}$, so there is a $\gamma < 2^{\chi(A)}$ such that $H \in X_{(f\restriction\alpha)\cup\{(\alpha,\gamma)\}}$; set $f(\alpha) = \gamma$. Thus $H \in X_{f\restriction\alpha}$ for all $\alpha < (\chi(A))^+$. We claim that $\langle G_{f\restriction(\beta+1)} : \beta < (\chi(A))^+\rangle$ is a free sequence, which contradicts $\mathrm{t}(A) \le \chi(A)$. Let $\alpha < (\chi(A))^+$ and suppose that

$$K \in \overline{\{G_{f\restriction(\beta+1)} : \beta < \alpha\}} \cap \overline{\{G_{f\restriction(\beta+1)} : \alpha \le \beta < (\chi(A))^+\}}.$$

Then $K \in \overline{\{G_{f\restriction\beta} : \beta \le \alpha\}}$, so $K \in \mathrm{Ult}(A)\backslash X_{f\restriction(\alpha+1)}$, and this set is open, so there is a $\beta \ge \alpha$ such that $G_{f\restriction(\beta+1)} \in \mathrm{Ult}(A)\backslash X_{f\restriction(\alpha+1)} \subseteq \mathrm{Ult}(A)\backslash X_{f\restriction(\beta+1)}$, contradiction. □

Problem 134. *Characterize* $\chi_{\mathrm{Hs}}(A)$.

Problem 135. *Characterize* $\chi_{\mathrm{Ss}}(A)$.

Problem 136. *Characterize* $\chi_{\mathrm{Sr}}(A)$.

Problem 137. *Characterize* $\chi_{\mathrm{Hr}}(A)$.

We describe character for interval algebras; this corrects the description given in Monk [96].

Proposition 14.17. *Let L be a linear order with first element 0, let F be an ultrafilter on $A \stackrel{\mathrm{def}}{=} \mathrm{Intalg}(L)$, and let $T = \{a \in L : [0, a) \in F\}$ be the end segment determined by F. Note that $0 \notin T$. Then:*

(i) *If $T = [b, \infty)$ and b has an immediate predecessor c, then F is the principal ultrafilter determined by $\{c\}$.*
(ii) *If $T = [b, \infty)$ and b does not have an immediate predecessor, then the character of F is the cofinality of $[0, b)$.*
(iii) *If $T = (b, \infty) \ne \emptyset$ and b does not have an immediate successor, then the character of F is the coinitiality of $[b, \infty)$.*
(iv) *If T is empty and L has a greatest element b, then F is the principal ultrafilter generated by $\{b\}$*
(v) *If T is empty and L does not have a greatest element, then the character of F is the cofinality of L.*
(vi) *If T is nonempty and there is no glb for T in L, then the character of F is the maximum of the left and right characters of the gap $(L\backslash T, T)$.* □

Theorem 14.18. $\mathfrak{p}(A) = \mathfrak{u}(A)$ *for every interval algebra A.*

Proof. As remarked above, $\mathfrak{p}(A) \leq \mathfrak{u}(A)$ for any atomless BA A. Hence it suffices to show that $\mathfrak{u}(A) \leq \mathfrak{p}(A)$ for any atomless interval algebra. So, let L be a dense linear order with first element 0, and let $A = \text{Intalg}(L)$. Suppose that $X \subseteq A$, $\sum X = 1$, $\sum F \neq 1$ for every finite subset F of X, and $|X| = \mathfrak{p}(A)$. Wlog each member of X has the form $[u, v)$ with $u < v \leq \infty$. Clearly $\{-x : x \in X\}$ has fip, so this set is included in an ultrafilter U. It suffices now to show that the character of U is $\leq |X|$. Let

$$M = \{w \in L : \exists v([v, w) \in X \text{ and } [w, \infty) \in U)\};$$
$$N = \{v \in L : \exists w([v, w) \in X \text{ and } [0, v) \in U\}.$$

Let $T = \{a \in L : [0, a) \in U\}$ be the end segment determined by U. We may assume that U is nonprincipal, and hence (ii), (iii), (v), or (vi) of Proposition 14.17 holds.

Case 1. (ii) holds; so $T = [b, \infty)$, where b does not have an immediate predecessor. Thus the character of U is the cofinality of $[0, b)$. It suffices now to show that M is cofinal in $[0, b)$. Suppose not; take $c < b$ such that $M \cap (c, b) = \emptyset$, and also choose d with $c < d < b$. Choose $[x, y) \in X$ such that $[c, d) \cap [x, y) \neq \emptyset$. Then $\max(c, x) < \min(d, y)$, so $x < b$ and hence $[x, \infty) \in U$. Hence $-[x, y) \cap [x, \infty) = [y, \infty) \in U$. So $y \in M$ and $c < y < b$, contradiction.

Case 2. (iii) holds; so $T = (b, \infty)$, where b does not have an immediate successor. The character of U is the coinitiality of (b, ∞). Then N is coinitial in (b, ∞). For, suppose not. Take $b < d$ with $N \cap (b, d) = \emptyset$, and take c with $b < c < d$. Choose $[x, y) \in X$ such that $[c, d) \cap [x, y) \neq \emptyset$. Then $\max(c, x) < \min(d, y)$, so $b < c < y$, hence $[0, y) \in U$. So $-[x, y) \cap [0, y) = [0, x) \in U$, and so $b < x < d$ and $x \in N$, contradiction.

Case 3. (v) holds. So T is empty and L does not have a greatest element. Hence the character of U is the cofinality of L. We claim that M is cofinal in L. Suppose not. Then there is a $c \in L$ such that $M \cap [c, \infty) = \emptyset$. Take d with $c < d$. Choose $[x, y) \in X$ such that $[x, y) \cap [c, d) \neq \emptyset$. Then $y \in M \cap (c, \infty)$, contradiction.

Case 4. (vi) holds, so that T is nonempty and there is no glb for T in L; we also assume that the left character of the gap $(L\backslash T, T)$ is $\geq$ the right character. Then we claim that M is cofinal in $L\backslash T$. Suppose not; say $c \in L\backslash T$ and $M \cap [c, \infty) \cap (L\backslash T)) = \emptyset$. Choose $d > c$ with $d \in L\backslash T$. Let $[x, y) \in X$ be such that $[c, d) \cap [x, y) \neq \emptyset$. Then $x < d$, so $[x, \infty) \in U$; hence $[y, \infty) = -[x, y) \cap [x, \infty) \in U$. So $y \in L\backslash T$, $c < y$, and $y \in M$, contradiction.

Case 5. (vi) holds, so that T is nonempty and there is no glb for T in L; we also assume that the right character of the gap $(L\backslash T, T)$ is $\geq$ the left character. Then we claim that N is coinitial in T. Suppose not. Say $d \in L$ and $N \cap [0, d) \cap L = \emptyset$. Choose $c \in L$ such that $c < d$. Then choose $[x, y) \in X$ such that $[x, y) \cap [c, d) \neq \emptyset$. Then $c < y$, so $y \in L$ and $[0, y) \in U$. Hence $[0, x) = -[x, y) \cap [0, y) \in U$. Then $x \in d$ and $x \in N$, contradiction. □

For tree algebras we have the following theorem (Brenner [82]):

Theorem 14.19. *Let T be a tree, and set $A = \mathrm{Treealg}(T)$. Then $\chi(A) = \sup\{|\{x : x$ is an immediate successor of $C\}|, \mathrm{cf}(C) : C$ an initial chain of $T\}$.*

Proof. Let $\kappa = \sup\{|\{x : x$ is an immediate successor of $C\}|, \mathrm{cf}(C) : C$ an initial chain of $T\}$. Let F be an ultrafilter of A and let C be the associated initial chain. We shall show that F has a set of generators of size at most κ; this will prove $\leq$.

If C has a maximal element x, then $\{(T\uparrow x)\backslash\bigcup_{y\in F}(T\uparrow y) : F$ a finite set of immediate successors of $x\}$ generates F, as desired.

Suppose that C has no maximal element. Let $\langle x_\alpha : \alpha < \mathrm{cf}(C)\rangle$ be an increasing cofinal sequence in C. Then the set

$$\{(T\uparrow x_\alpha)\backslash\bigcup_{y\in F}(T\uparrow y) : \alpha < \mathrm{cf}(C),\ F \text{ a finite set of immediate successors of } C\}$$

generates F, as desired.

Conversely, Let C be an initial chain of T and F the associated ultrafilter. Suppose X generates F and $|X| < \max(|\{x : x$ is an immediate successor of $C\}|, \mathrm{cf}(C))$. Without loss of generality each element $x \in X$ has the form $(T\uparrow t_x)\backslash\bigcup_{y\in F_x}(T\uparrow y)$, where F_x is a finite set of successors of t_x. If C has a maximal element z, then $|X| < \{u : u$ is an immediate successor of $z\}$, so there is an immediate successor u of z such that $z \notin \bigcup_{x\in X} F_x$. Then $(T\uparrow z)\backslash(T\uparrow u) \in F$, but no element of X is $\leq$ it, contradiction. Suppose that C has no maximal element. If $|X| < \mathrm{cf}(C)$, choose $z \in C$ such that $t_x < z$ for all $x \in X$; then $T\uparrow z$ is in F but has no element of X less than it. Finally, if $\mathrm{cf}\,C \leq |X|$, then $|X| < \{u : u$ is an immediate successor of $C\}$, and we get a contradiction as above. □

$\chi(A)$ for pseudo-tree algebras has been characterized by Jennifer Brown.

Problem 138. *Characterize $\chi(A)$ for A a complete BA.*

Concerning superatomic algebras we have the following result (due to Monk):

Theorem 14.20. *$\chi(A) = |A|$ for A superatomic.*

Proof. Let λ be a regular cardinal $\leq |A|$; we want to show that $\chi(A) \geq \lambda$. Let R be a complete system of representatives of atoms (of all levels) of A. We may assume that the top atoms of R form a finite partition of unity. Recall the notion of *rank* of an element of A: this is the least α such that $a \in I_{\alpha+1}$. An element $a \in A$ is *big* if $|\{x \in R : x \leq^* a\}| \geq \lambda$. ($u \leq^* v$ means that if β is the rank of u then $u/I_\beta \leq v/I_\beta$). Note that at least one of the top atoms of R is big. Let α be minimum such that there is a big $a \in R$ of rank α, and fix such an a, and the ultrafilter F associated with it. Note that if x is any element of rank less than α, then x is small. In fact, otherwise suppose that x is of smallest rank $\beta < \alpha$ such that x is big. Then $x/I_\beta = c_1/I_\beta + \cdots + c_m/I_\beta$ for certain $c_1, \ldots, c_m \in R$ such

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that each c_i/I_β is an atom. Thus $x \cdot -c_1 \cdot \ldots \cdot -c_m \in I_\beta$, and hence it is small. If $y \in R$ and $y \leq^* x$, say y has rank γ. Then

$$\begin{aligned} y/I_\gamma \leq x/I_\gamma &= (x \cdot c_1)/I_\gamma + \cdots + (x \cdot c_m)/I_\gamma + (x \cdot -c_1 \cdot \ldots \cdot -c_m)/I_\gamma \\ &\leq c_1/I_\gamma + \cdots + c_m/I_\gamma + (x \cdot -c_1 \cdot \ldots \cdot -c_m)/I_\gamma, \end{aligned}$$

and so y/I_γ is $\leq$ one of these last summands. This means that one of $c_1, \ldots, c_m, x \cdot -c_1 \cdot \ldots \cdot -c_m$ is big, contradiction.

We claim that $\chi(F) \geq \lambda$ (as desired). In fact, suppose that $X \subseteq F$ generates F, where $|X| < \lambda$. If $b \in R$, $b \leq^* a$, and b has rank less than α, then $-b \in F$, and hence there is an $x_b \in X$ such that $x_b \leq -b$. Since λ is regular there is an $S \subseteq R$ with $|S| \geq \lambda$ and an $x \in X$ such that each member of S has rank less than α and $x \leq -b$ for each $b \in S$. Thus $b \leq -x$, and $b \leq^* a \cdot -x$. So, $a \cdot -x$ is big.

Case 1. $a/I_\alpha \leq x/I_\alpha$. Then $a \cdot -x \in I_\alpha$, so $a \cdot -x$ is small, contradiction.

Case 2. $a/I_\alpha \cdot x/I_\alpha = 0$. Then $a \cdot x \in I_\alpha$, and hence $-a + -x \in F$ and $-x \in F$, contradiction. □

15 Hereditary Lindelöf Degree

Recall from just before Proposition 13.4 the definition of a right-separated sequence in a topological space. Let α be an ordinal. A sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of a BA A is *right separated* provided that the following two conditions hold:

(i) If $F \in [\alpha]^{<\omega}$, then $\prod_{\xi\in F} -a_\xi \neq 0$.
(ii) If $F \cup \{\eta\} \in [\alpha]^{<\omega}$ with $\forall\xi \in F[\xi < \eta]$, then $a_\eta \cdot \prod_{\xi\in F} -a_\xi \neq 0$.

Note that if α is a limit ordinal, then (i) follows from (ii).

Theorem 15.1. *For any infinite BA A,* $\mathrm{hL}(A)$ *is equal to each of the following cardinals:*

sup$\{\kappa$*:there is an ideal not generated by less than κ elements*$\}$*;*
sup$\{\kappa$*:there is a strictly increasing sequence of ideals of length κ*$\}$*;*
sup$\{\kappa$*:there is a strictly increasing sequence of filters of length κ*$\}$*;*
sup$\{\kappa$*:there is a strictly increasing sequence of open sets of length κ*$\}$*;*
sup$\{\kappa$*:there is a strictly decreasing sequence of closed sets of length κ*$\}$*;*
sup$\{\kappa$*:there is a right-separated sequence in* $\mathrm{Ult}(A)$ *of length κ*$\}$*;*
sup$\{\kappa$*:there is a right-separated sequence in A of length κ;*
min$\{\kappa$*:every open cover of a subspace of* $\mathrm{Ult}(A)$ *has a subcover of size $\leq \kappa$*$\}$.

Proof. Nine cardinals are mentioned; let them be denoted by $\kappa_0, \ldots, \kappa_8$ in their order of mention (starting with hL itself). First we take care of easy relations: $\kappa_2 = \kappa_3$ since, if I is an ideal then $I^f \stackrel{\text{def}}{=} \{a \in A : -a \in I\}$ is a filter, and $I \subset J$ iff $I^f \subset J^f$; similarly, going from filters to ideals. So $\kappa_2 = \kappa_3$ follows. Next, $\kappa_2 \leq \kappa_4$. For, if I is an ideal, let $I^u = \bigcup_{a\in I} \mathcal{S}(a)$. Then I^u is open, and $I \subset J$ implies $I^u \subset J^u$. (If $a \in J\backslash I$, then $\mathcal{S}(a) \subseteq J^u$, of course, but $\mathcal{S}(a) \not\subseteq I^u$, since otherwise compactness of $\mathcal{S}(a)$ would easily yield $a \in I$.) This shows $\kappa_2 \leq \kappa_4$. It is clear that $\kappa_4 = \kappa_5$, by taking complements. $\kappa_4 \leq \kappa_6$: If $\langle U_\alpha : \alpha < \kappa\rangle$ is a strictly increasing sequence of open sets, for every $\alpha < \kappa$ choose $x_\alpha \in U_{\alpha+1}\backslash U_\alpha$. Clearly $\langle x_\alpha : \alpha < \kappa\rangle$ is right-separated. $\kappa_6 = \kappa_7$: First suppose that $\langle F_\alpha : \alpha < \kappa\rangle$ is right-separated in $\mathrm{Ult}(A)$. For all $\alpha < \kappa$ choose $a_\alpha \in A$ such that $F_\alpha \in \mathcal{S}(a_\alpha) \cap \{F_\beta : \beta < \kappa\} \subseteq \{F_\beta : \beta \leq \alpha\}$. We claim that $\langle a_\alpha : \alpha < \kappa\rangle$ is right-separated in A. To see this, suppose that Γ is a finite subset of κ, $\alpha < \kappa$, and $\beta < \alpha$ for all $\beta \in \Gamma$. Thus $a_\alpha \in F_\alpha$.

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If $\beta \in \Gamma$, then $a_\beta \notin F_\alpha$ by the choice of a_β, and hence $-a_\beta \in F_\alpha$. Therefore the element $a_\alpha \cdot \prod_{\beta\in\Gamma} -a_\beta$ is in F_α, and hence it must be non-zero, as desired. Second, suppose that $\langle a_\alpha : \alpha < \kappa\rangle$ is right-separated in A. Then for each $\alpha < \kappa$ the set $\{a_\alpha\} \cup \{-a_\beta : \beta < \alpha\}$ has the finite intersection property, and hence is included in an ultrafilter F_α. It is easy to check that $\langle F_\alpha : \alpha < \kappa\rangle$ is right-separated in $\mathrm{Ult}(A)$, as desired. $\kappa_8 \leq \kappa_0$: for any subspace X of $\mathrm{Ult}(A)$, any cover of X has a subcover of power $\leq \mathrm{L}(X) \leq \kappa_0$, so $\kappa_8 \leq \kappa_0$.

It remains only to prove that $\kappa_0 \leq \kappa_1$, $\kappa_1 \leq \kappa_2$, and $\kappa_7 \leq \kappa_8$. For the first one, suppose that $X \subseteq \mathrm{Ult}(A)$ and $\mathcal{O}$ is an open cover of X with no subcover of power λ; we construct an ideal not generated by λ or fewer elements. Let

$$I = \langle\{a \in A : \exists U \in \mathcal{O}(\mathcal{S}(a) \cap X \subseteq U)\}\rangle^{\mathrm{Id}}.$$

Suppose that I is generated by J, where $|J| \leq \lambda$. For every $a \in J$ there is a finite subset $\mathscr{P}_a$ of $\mathcal{O}$ such that $\mathcal{S}(a) \cap X \subseteq \bigcup \mathscr{P}_a$. Let $\mathcal{O}' = \bigcup_{a\in J} \mathscr{P}_a$. We claim that $\mathcal{O}'$ covers X, which is the desired contradiction. Indeed, let $x \in X$. Say $x \in U \in \mathcal{O}$. Say $x \in \mathcal{S}(a) \cap X \subseteq U$. Choose a finite subset F of J such that $a \leq \sum F$. Then $x \in \mathcal{S}(a) \cap X \subseteq \bigcup_{b\in F} \bigcup \mathscr{P}b$, as desired.

Next, $\kappa_1 \leq \kappa_2$: suppose that I is an ideal not generated by fewer than λ elements. Then it is easy to construct a sequence $\langle a_\alpha : \alpha < \lambda\rangle$ of elements of I such that $a_\alpha \notin \langle\{a_\beta : \beta < \alpha\}\rangle^{\mathrm{Id}}$ for all $\alpha < \lambda$. Thus $\langle\langle\{a_\beta : \beta < \alpha\}\rangle^{\mathrm{Id}} : \alpha < \lambda\rangle$ is a strictly increasing sequence of ideals, as desired.

For $\kappa_7 \leq \kappa_8$, suppose that λ is a regular cardinal $\leq \kappa_7$ and $\langle x_\alpha : \alpha < \lambda\rangle$ is right separated. Thus for each $\alpha < \lambda$ we can choose an open set U_α such that $U_\alpha \cap \{x_\xi : \xi < \lambda\} = \{x_\xi : \xi < \alpha + 1\}$. Then $\{U_\alpha : \alpha < \lambda\}$ is a cover of $\{x_\xi : \xi < \lambda\}$ with no subcover of size $< \lambda$. Hence $\lambda < \kappa_8$, and this shows that $\kappa_7 \leq \kappa_8$. □

In Theorem 15.1, eight of the nine equivalents involve sups, and thus give rise to attainment problems. The proof of the theorem shows the following: attainment is the same for κ_2 and κ_3, for κ_4 and κ_5, and for κ_6 and κ_7; moreover, attainment in the sense κ_2 implies attainment in the sense κ_4, attainment in the sense κ_4 implies attainment in the sense κ_6, and attainment in the sense κ_1 implies attainment in the sense κ_2. It is also easy to see that attainment in the sense κ_4 implies attainment in the sense κ_2. In fact, if $\langle U_\alpha : \alpha < \kappa_3\rangle$ is an increasing sequence of open sets, for each $\alpha < \kappa_3$ let $I_\alpha = \{a : \mathcal{S}a \subseteq U_\alpha\}$. Clearly I_α is an ideal. To show properness, pick $F \in U_{\alpha+1}\backslash U_\alpha$. Say $F \in \mathcal{S}a \subseteq U_{\alpha+1}$. Thus $a \in I_{\alpha+1}\backslash I_\alpha$. And attainment in the sense κ_6 implies attainment in the sense κ_4. In fact, suppose that $\langle F_\alpha : \alpha < \kappa\rangle$ is right separated. For each $\alpha < \kappa$ choose $a_\alpha \in F_\alpha$ such that $\mathcal{S}a_\alpha \cap \{F_\beta : \beta > \alpha\} = 0$, and let $U_\alpha = \bigcup_{\beta<\alpha} \mathcal{S}a_\beta$. Note that $F_\alpha \in U_{\alpha+1}\backslash U_\alpha$, as desired. Also note that if $\mathrm{hL}A$ is regular, then attainment in the right-separated sense implies attainment in the defined sense.

Thus we have seen that there are only three versions of the definition of hL that might lead to different attainment properties: hL as defined, in the ideal-generated sense, and in the right-separated sense, where we know only that attainment in the ideal-generated sense implies attainment in the right-separated sense. In Rosłanowski Shelah [01a] further results are obtained. By Hypothesis 1.1 and Theorem 1.4 of this paper we have:

Suppose that $\langle \chi_i : i < \mathrm{cf}(\lambda)\rangle$ *is a strictly increasing sequence of infinite cardinals such that* $(2^{\mathrm{cf}(\lambda)})^+ < \chi_0$ *and* $\lambda = \sup_{i<\mathrm{cf}(\lambda)} \chi_i$. *Then for any BA A such that* $\mathrm{hL}(A) = \lambda$, *all (three) versions of* hL *are attained.*

On the other hand, in 3.7 they show that it is consistent to have a BA with hL attained in the right-separated sense but not in the ideal-generation sense.

These results leave the following problem open; this is a version of Problem 50 on Monk [96].

Problem 139. *Completely describe the relations between these attainment possibilities for* hL.

There are two important positive results concerning attainment.

Theorem 15.2. *If* $\mathrm{hL}(A)$ *is singular strong limit, then it is attained in the ideal-generated sense.*

Proof. Since $\mathrm{hL}(A) \le |A| \le |\mathrm{Ult}(A)|$, we can apply Theorem 13.2 to get a discrete subset X of $\mathrm{Ult}(A)$ of size $\mathrm{hL}(A)$. Let $I = \{a \in A : \exists F \in [X]^{<\omega}[\mathcal{S}(a) \cap X = F]\}$. Clearly I is an ideal in A; we claim that it is not generated by fewer than $\mathrm{hL}(A)$ elements. In fact, suppose that $Y \in [I]^{<\mathrm{hl}(A)}$ generates I. For each $y \in Y$ choose $F_y \in]X]^{<\omega}$ such that $\mathcal{S}(y) \cap X = F_y$. Take any $x \in X \setminus \bigcup_{y\in Y} F_y$, and then choose $a \in A$ such that $\mathcal{S}(a) \cap X = \{x\}$. Thus $a \in I$. Let Z be a finite subset of Y such that $a \le \sum Z$. Then

$$x \in X \cap \bigcup_{z\in Z} \mathcal{S}(z) = \bigcup_{z\in Z} (X \cap \mathcal{S}(z)) = \bigcup_{z\in Z} F_z,$$

contradiction. □

The second attainment result is given in Proposition 13.4: $\mathrm{hL}(A)$ is attained for $\mathrm{hL}(A)$ singular of countable cofinality.

We turn to algebraic operations.

If A is a subalgebra or homomorphic image of B, then $\mathrm{hL}(A) \le \mathrm{hL}(B)$. For special subalgebras, note that one can have $A \le_{\mathrm{free}} B$ with $\mathrm{hL}(B)$ arbitrarily greater than $\mathrm{hL}(A)$; hence the same applies to $\le_{\mathrm{u}}$, $\le_{\mathrm{proj}}$, $\le_{\mathrm{rc}}$, $\le_\sigma$, and $\le_{\mathrm{reg}}$. If $A \le_\pi B$, then $|B| \le 2^{|A|}$; so hL cannot jump arbitrarily. But for $A = \mathrm{intalg}(\mathbb{R})$ we have $\mathrm{hL}(A) = \omega$ (see the remarks on interval algebras at the end of this chapter), $\mathrm{hL}(\overline{A}) = 2^\omega$ by the Balcar, Franěk theorem since obviously $\mathrm{Ind}(B) \le \mathrm{hL}(B)$ for any BA B, and $A \le_\pi \overline{A}$. For $A \le_{\mathrm{s}} B$ we clearly have $\mathrm{hL}(A) = \mathrm{hL}(B)$; one can use Proposition 2.29 to see this. Clearly one can have $A \le_{\mathrm{mg}} B$ with $\mathrm{hL}(A)$ arbitrarily smaller than $\mathrm{hL}(B)$.

Proposition 15.3. $\mathrm{hL}(A \times B) = \max(\mathrm{hL}(A), \mathrm{hL}(B))$.

Proof. We use the ideal-generation equivalent of the definition of hL. If I is an ideal in A not generated by less than κ elements, then $\{(a,b) \in A \times B : a \in I\}$ is an ideal in $A \times B$ not generated by less than κ elements. Hence by symmetry $\geq$ holds. Conversely, let I be an ideal in $A \times B$ not generated by less than κ elements. Define

$$J = \{a \in A : (a,0) \in I\};$$
$$K = \{b \in B : (0,b) \in I\}.$$

Then J and K are ideals in A, B respectively. If J is generated by X with $|X| < \kappa$ and K is generated by Y with $|Y| < \kappa$, let $Z = \{(a, 0 :)a \in X\} \cup \{(0,b) : b \in Y\}$. Then Z generates I, since if $(a,b) \in I$, then $(a,0),(0,b) \in I$, hence $a \in J$ and $b \in I$; choosing $U \in [X]^{<\omega}$ and $V \in [Y]^{<\omega}$ such that $a \leq \sum U$ and $b \leq \sum V$, let $W = \{(a,0) : a \in U\} \cup \{(0,b) : b \in V\}$; then $W \subseteq Z$ and $(a,b) \leq \sum W$. From this $\leq$ follows. □

Proposition 15.4. *For I infinite,* $\mathrm{hL}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right) = \max\left(|I|, \sup_{i\in I} \mathrm{hL}(A_i)\right)$.

Proof. For brevity let $B = \prod^{\mathrm{w}}_{i\in I} A_i$. By Proposition 15.3 we have $\mathrm{hL}(A_j) \leq \mathrm{hL}(B)$ for each $j \in I$. Now let $K = \{a \in B : \{i \in I : a_i \neq 0\}$ is finite$\}$; we claim that K is not generated by less than $|I|$ elements. For, suppose that $X \in [K]^{<\omega}$. Choose $i \in I \setminus \bigcup_{a\in X}\{j \in I : a_j \neq 0\}$. Then the element of B which is 1 at i and 0 elsewhere is not in the ideal generated by X. This proves $\geq$ in the proposition.

For $\leq$, let κ be the right side of our equation, and let K be an ideal on B; we want to show that K is generated by a set with at most κ elements. For each $i \in I$ let $L_i = \{a \in A_i : \exists b \in K[b_i = a]\}$. Then L_i is an ideal in A_i, so it can be generated by a set X_i with at most κ elements. Let $Y = \bigcup_{i\in I} L_i$. Thus $|Y| \leq \kappa$. Now we have two cases.

Case 1. Every element of K is of type I. Then we claim that Y generates K. For, suppose that $b \in K$. For each $i \in I$ there is a finite $Y_i \subseteq X_i$ such that $b_i \leq \sum Y_i$. Let $J = \{i \in I : b_i \neq 0\}$. Then $b \leq \sum_{i\in J} Y_i$, as desired.

Case 2. K has an element c of type II. Let $J \stackrel{\mathrm{def}}{=} \{i \in I : c_i \neq 1\}$. Then for any element $b \in K$, $b \leq c + \sum_{i\in I\setminus J} Y_i$ with Y_i as in Case 1, as desired. □

Note that $\mathrm{Ind}(A) \leq \mathrm{hL}(A)$, using the equivalent concerning algebraic right-separated sequences. Hence it is clear that $\mathrm{hL}\left(\prod_{i\in I} A_i\right) \geq \max(2^{|I|}, \sup_{i\in I}\mathrm{hL}(A_i))$. Strict inequality is possible, as was noticed by Shelah and Peterson independently, solving Problem 44 of Monk [90]. Again the example used for spread applies here.

Concerning ultraproducts, note that hL is an order-independence function, and hence Theorem 12.6 holds. By the proof of Theorem 12.7 it follows that under GCH we have $\mathrm{hL}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{hL}(A_i)/F\right|$ for F regular, and Donder's theorem says that under $V = L$ the regularity assumption can be removed. The example of Laver for depth works for hL also: it is consistent to have a situation where $\mathrm{hL}\left(\prod_{i\in I}A_i/F\right) > \left|\prod_{i\in I}\mathrm{hL}(A_i)/F\right|$ for F regular; see also Rosłanowski. Shelah [98] for another consistent example.

Also note the following results. In Shelah [99], part of 15.13, the following is proved:

If D is a uniform ultrafilter on κ then there is a class of cardinals χ with $\chi^\kappa = \chi$ such that there are BAs B_i for $i < \kappa$ such that:

(i) $\mathrm{hL}(B_i) \le \chi$, *hence* $|\prod_{i<\kappa}\mathrm{hL}(B_i)/D| \le \chi$;
(ii) $\mathrm{hL}(\prod_{i<\kappa}B_i/D) = \chi^+$.

This solves Problem 51 of Monk [96]. In Shelah, Spinas [00], part of Corollary 2.4 says the following:

There is a model in which there exist cardinals κ, μ, a system $\langle B_i : i < \kappa\rangle$ of BAs, and an ultrafilter D on κ such that

$$\left|\prod_{i<\kappa}\mathrm{hL}(B_i)/D\right| = \mu^{++} \quad \textit{and} \quad \mathrm{hL}\left(\prod_{i<\kappa}B_i/D\right) \le \mu^+.$$

This solves Problem 52 of Monk [96].

Next come free products:

Theorem 15.5. *If $\langle A_i : i \in I\rangle$ is a system of BAs each with at least 4 elements, then*

$$\max(|I|, \sup_{i\in I}\mathrm{hL}(A_i)) \le \mathrm{hL}\left(\oplus_{i\in I}A_i\right) \le \max\left(|I|, 2^{\sup_{i\in I}\mathrm{s}(A_i)}\right).$$

Proof. The first inequality is easy; use the right-separated equivalent to see that $\mathrm{hL}(A_j) \le \mathrm{hL}\left(\oplus_{i\in I}A_i\right)$ for each $j \in I$, and use the fact that $\mathrm{Ind}\left(\oplus_{i\in I}A_i\right) \le \mathrm{hL}\left(\oplus_{i\in I}A_i\right)$ to see that $|I| \le \mathrm{hL}\left(\oplus_{i\in I}A_i\right)$.

For the second,

$$\begin{aligned}\mathrm{hL}\left(\oplus_{i\in I}A_i\right) \le |\oplus_{i\in I}A_i| &= |I|\cdot\sup_{i\in I}|A_i| \\ &\le |I|\cdot\sup_{i\in I}2^{\mathrm{s}}(A_i) \\ &\le \max\left(|I|, 2^{\sup_{i\in I}\mathrm{s}(A_i)}\right).\end{aligned}$$ □

The inequalities in Theorem 15.5 are sharp, in the sense that all possibilities can occur. Thus both are equalities if $I = \omega_1$ and each A_i is a four-element algebra. The first is an equality and the second not for $I = \omega_1$ and each A_i the free BA on ω_1 free generators. The first is a strict inequality and the second an equality for $A \oplus A$, where $A = \mathrm{Intalg}\,\mathbb{R}$. Finally, both inequalities are strict for $A \oplus A$, where A is the tree algebra on a Suslin tree in which each element has infinitely many immediate successors, and $\neg$CH holds. It is clear that $\neg$CH is needed to get an example where both inequalities are strict.

Proposition 15.6. $\mathrm{hL}\left(\prod^B_{i\in I}A_i\right) = \max(\mathrm{hL}(B), |I|, \sup_{i\in I}\mathrm{hL}(A_i))$.

Proof. Let $C = \prod_{i \in I}^{B} A_i$. Then $\mathrm{hL}(B) \le \mathrm{hL}(C)$ since B is isomorphic to a subalgebra of C. Also, $\mathrm{hL}(A_i) \le \mathrm{hL}(C)$ by Proposition 15.3. By Proposition 15.4 we have $\mathrm{hL}\left(\prod_{i \in I}^{\mathrm{w}} A_i\right) \ge |I|$; since $\prod_{i \in I}^{\mathrm{w}} A_i$ is a subalgebra of C, it follows that $|I| \le \mathrm{hL}(C)$. So we have proved $\ge$.

Now let κ be the indicated maximum. We show that an arbitrary ideal K on C can be generated by $\le \kappa$ elements; this will prove $\le$. Let $L = \{b \in B : h(b, 0, 0) \in K\}$. Then L is an ideal on B, and hence can be generated by a set X of size $\le \kappa$. For each $i \in I$ let

$$M_i = \{a \in A_i : h(\emptyset, \{i\}, \{(i, a)\}) \in K\}.$$

Then M_i is an ideal of A_i. Say M_i is generated by a set Y_i of size at most κ. Now let

$$N \stackrel{\text{def}}{=} \{h(b, 0, 0) : b \in X\} \cup \{h(0, \{i\}, \{(i, a) : a \in Y_i\})\}.$$

Clearly $|N| \le \kappa$. We claim that it generates K. Let $h(b, F, a)$ be any element of K. Then also $h(b, 0, 0) \in K$, and so there is a finite $P \subseteq X$ such that $b \le \sum P$. Also, for each $i \in F$ we have $h(0, \{i\}, \{(i, a_i)\}) \in K$, hence $a_i \in M_i$, so there is a finite $Q_i \subseteq Y_i$ such that $a_i \le \sum Q_i$. Then

$$h(b, F, a) \le \sum\nolimits_{c \in P} h(c, 0, 0) + \sum\nolimits_{i \in F} \sum_{c \in Q_i} H(0, \{i\}, \{(i, c)\}),$$

as desired. □

Proposition 15.7. *If C is the subalgebra of $A \times B$ obtained by one-point gluing with respect to ultrafilters F, G on A, B respectively, then* $\mathrm{hL}(C) = \max(\mathrm{hL}(A), \mathrm{hL}(B))$.

Proof. Clearly $\mathrm{hL}(C) \le \max(\mathrm{hL}(A), \mathrm{hL}(B))$, by Proposition 15.3. Now if I is an ideal on A not generated by less than κ elements, then $\{(a, b) \in C : a \in I\}$ is an ideal on C not generated by less than κ elements. By symmetry, the equation in the proposition follows. □

Since clearly $\mathrm{c}(A) \le \mathrm{hL}(A)$ for any BA A, we must have $\mathrm{hL}(B) = |B|$ if B is the Alexsandrov duplicate of an algebra A.

By Proposition 2.6 we have $\mathrm{hL}(A) \le \mathrm{hL}(\mathrm{Exp}(A))$. It is open whether $\le$ can be replaced by $=$ here:

Problem 140. *Is there a BA A such that* $\mathrm{hL}(A) < \mathrm{hL}(\mathrm{Exp}(A))$*?*

Theorem 15.8. $\mathrm{hL}(A) \le \mathrm{s}(\mathrm{Exp}\,(A))$.

Proof. Let $\langle F_\alpha : \alpha < \kappa \rangle$ be a strictly decreasing sequence of closed subsets of $\mathrm{Ult}(A)$. We claim that $D \stackrel{\text{def}}{=} \{F_{\alpha+1} : \alpha < \kappa\}$ is a discrete set of points of $\mathrm{Exp}\,(A)$. For, let $\alpha < \kappa$. Choose $x \in F_\alpha \backslash F_{\alpha+1}$ and $y \in F_{\alpha+1} \backslash F_{\alpha+2}$. Then there is a clopen subset S of $\mathrm{Ult}(A)$ such that $y \in S$, $F_{\alpha+2} \cap S = \emptyset$, and $x \notin S$. And there is a clopen U such that $F_{\alpha+1} \subseteq U$ and $x \notin U$. Now $\mathscr{V}(U, S) \cap D = \{F_{\alpha+1}\}$. For, obviously $F_{\alpha+1} \in \mathscr{V}(U, S)$. Suppose that $\alpha < \beta$ and $F_{\beta+1} \in \mathscr{V}(U, S)$. Then $F_{\beta+1} \subseteq F_{\alpha+2}$ and $F_{\beta+1} \cap S \neq \emptyset$, contradicting $F_{\alpha+2} \cap S = \emptyset$. Suppose that $\beta < \alpha$

and $F_{\beta+1} \in \mathscr{V}(U,S)$. Then $F_\alpha \subseteq F_{\beta+1}$, so $x \in F_{\beta+1}$. But $x \notin U$ and $x \notin S$, contradiction. $\square$

Concerning derived functions of hL, we mention these obvious facts:

$$\mathrm{hL}(A) = \mathrm{hL}_{\mathrm{H}+}(A) = \mathrm{hL}_{\mathrm{S}+}(A) = \mathrm{hL}_{\mathrm{h}+}(A) = {}_{\mathrm{d}}\mathrm{hL}_{\mathrm{S}+}(A);$$

and $\mathrm{hL}_{\mathrm{S}-}(A) = \mathrm{hL}_{\mathrm{h}-}(A) = \omega$. The following theorem is a corollary of 14.12, using the fact given below that $\chi(A) \leq \mathrm{hL}(A)$ for any infinite BA.

Theorem 15.9. *If* $\mathrm{hL}(A) = \omega$*, then* $\mathrm{Card}_{\mathrm{H}-}(A) = \omega$*.* $\square$

Now we consider the derived function $\mathrm{hL}_{\mathrm{mm}}$. Here we have several choices as to how to define this function, depending on which equivalent definition from Theorem 15.1 we take. We consider three possibilities: right-separated in the algebraic and in the topological senses, and increasing sequences of ideals.

A right-separated sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A is *maximal* iff there does not exist a $b \in A$ such that $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is right-separated. right-separated sequences exist by Zorn's lemma.

Proposition 15.10. *Let* $\langle a_\xi : \xi < \alpha\rangle$ *be a right-separated sequence in a BA A. Then it is maximal* iff $\{a_\xi : \xi < \alpha\}$ *generates a maximal ideal.*

Proof. Let I be the ideal generated by $\{a_\xi : \xi < \alpha\}$. Clearly I is a proper ideal.

$\Rightarrow$: take any $b \in A$. Then $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is not right-separated, and this gives two possibilities.

Case 1. There is a finite $F \subseteq \alpha$ such that $-b \cdot \prod_{\xi\in F} -a_\xi = 0$. Then $-b \in I$.

Case 2. There is a finite $F \subseteq \alpha$ such that $b \cdot \prod_{\xi\in F} -a_\xi = 0$. Then $b \in I$.

$\Leftarrow$: Given $b \in A$, if $b \in I$, then $b \cdot \prod_{\xi\in F} -a_\xi = 0$ for some finite $F \subseteq \alpha$. If $-b \in I$, then $-b \cdot \prod_{\xi\in F} -a_\xi = 0$ for some finite $F \subseteq \alpha$. $\square$

Proposition 15.11. $\mathfrak{u}(A)$ *is equal to* $\min\{|\alpha| :$ *there is a maximal right-separated sequence of length* $\alpha\}$*, for any BA A.*

Proof. Let $\lambda = \min\{|\alpha| :$ there is a maximal right-separated sequence of length $\alpha\}$. From Proposition 15.10 it follows that $\mathfrak{u}(A) \leq \lambda$. Now let $\kappa = \mathfrak{u}(A)$, and let X be a set of size κ which generates a nonprincipal maximal ideal I. Write $X = \{x_\alpha : \alpha < \kappa\}$. We now define a sequence $\langle y_a : \alpha < \kappa\rangle$ by recursion. If $y_\beta \in X$ has been defined for all $\beta < \alpha$, then $\{y_\beta : \beta < \alpha\}$ does not generate I, so we can let y_α be x_γ with γ minimum such that $x_\gamma \notin \langle\{y_\beta : \beta < \alpha\rangle^{\mathrm{id}}$. This finishes the definition. Clearly $\langle\{y_\alpha : \alpha < \kappa\}\rangle^{\mathrm{id}} = I$. We claim that $\langle\{y_\alpha : \alpha < \kappa\}\rangle$ is right separated. To see this, first suppose that $F \in [\kappa]^{<\omega}$. Then $\sum_{\alpha\in F} y_\alpha \neq 1$ since I is nonprincipal. Second, suppose additionally that $F < \beta$. Then $y_\beta \not\leq \sum_{\alpha\in F} y_\alpha$, as desired. Thus we have produced a maximal right-separated sequence of length κ. So $\lambda \leq \mathfrak{u}(A)$. $\square$

By Proposition 15.11, the function $\mathrm{hL}_{\mathrm{mm}}$ defined in terms of right-separated sequences of elements of a BA is not a new function.

Looking at the equivalent in Theorem 15.1 involving a sequence of ideals, from the definition of the function alt in Chapter 9 we see that alt is the $\mathrm{hL_{mm}}$ function in this case.

We turn to a variant of minimum-maximum functions associated with hL, involving a right-separated sequence of ultrafilters. We modify this notion so that it can be applied to sequences of non-limit length. We call a sequence $\langle F_\xi : \xi \le \alpha\rangle$ of ultrafilters *right-separated* iff the sequence does not have any repetitions, and for every $\xi < \alpha$ the set $\{F_\eta : \eta \le \xi\}$ is open in $\{F_\eta : \eta \le \alpha\}$. Note that we insist that every such sequence have length a successor ordinal.

Proposition 15.12. *Suppose that α is a limit ordinal, and $\langle F_\xi : \xi < \alpha\rangle$ is a sequence of distinct ultrafilters on A. Then the following conditions are equivalent:*

(i) *For every $\xi < \alpha$, the set $\{F_\eta : \eta \le \xi\}$ is open in $\{F_\eta : \eta < \alpha\}$.*
(ii) *$\{F_\xi : \xi < \alpha\} \ne \mathrm{Ult}(A)$, and for every $G \notin \{F_\xi : \xi < \alpha\}$, the sequence $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G\rangle$ is right-separated.*
(iii) *For some $G \notin \{F_\eta : \eta < \alpha\}$, the sequence $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G\rangle$ is right-separated.*

Proof. (i)⇒(ii): First we show that $\{F_\xi : \xi < \alpha\} \ne \mathrm{Ult}(A)$. For each $\xi < \alpha$ we have $F_\xi \in \{F_\eta : \eta \le \xi\}$, which is open in $\{F_\eta : \eta < \alpha\}$. Hence there is an $a_\xi \in F_\xi$ such that

$$(*) \qquad \mathcal{S}(a_\xi) \cap \{F_\eta : \eta < \alpha\} \subseteq \{F_\eta : \eta \le \xi\}.$$

We claim that $\{-a_\xi : \xi < \alpha\}$ has f.i.p. For, suppose that $\xi(0) < \cdots < \xi(m) < \alpha$ and $-a_{\xi(0)} \cdot \ldots \cdot -a_{\xi(m)} = 0$. Choose $\xi < \alpha$ with $\xi(m) < \xi$; this is possible since α is a limit ordinal. Then $a_{\xi(0)} + \cdots + a_{\xi(m)} = 1 \in F_\xi$, so $a_{\xi(i)} \in F_\xi$ for some i. This contradicts $(*)$. Clearly any ultrafilter containing $\{-a_\xi : \xi < \alpha\}$ is different from each F_ξ.

Next, suppose that $G \notin \{F_\eta : \eta < \alpha\}$. Take any $\xi < \alpha$. Let U be an open set in $\mathrm{Ult}(A)$ such that $U \cap \{F_\eta : \eta < \alpha\} = \{F_\eta : \eta \le \xi\}$. Then $(U\backslash\{G\}) \cap (\{F_\eta : \eta < \alpha\} \cup \{G\}) = \{F_\eta : \eta \le \xi\}$.

Finally, clearly $\{F_\xi : \xi < \alpha\}$ is open in $\{F_\xi : \xi < \alpha\} \cup \{G\}$. This checks that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G\rangle$ is right-separated.

(ii)⇒(iii): Trivial.

(iii)⇒(i): Assume (iii), and suppose that $\xi < \alpha$. Let U be an open set such that $U \cap (\{F_\eta : \eta < \alpha\} \cup \{G\}) = \{F_\eta : \eta \le \xi\}$. Then $U \cap \{F_\eta : \eta < \alpha\} = \{F_\eta : \eta \le \xi\}$, as desired. □

Proposition 15.13. *Let $\langle F_\xi : \xi < \alpha\rangle$ be a sequence of distinct ultrafilters on A, and let G be an ultrafilter on A. Then the following conditions are equivalent:*

(i) *$\langle F_\xi : \xi < \alpha\rangle^\frown\langle G\rangle$ is a maximal right-separated sequence.*
(ii) *G is the only ultrafilter on A such that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G\rangle$ is a right-separated sequence.*

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Proof. (i)⇒(ii): Assume (i), and suppose that $H \neq G$ is an ultrafilter such that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle H\rangle$ is right-separated. We claim then that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G, H\rangle$ is right-separated. To prove this, first take any $\xi < \alpha$. Then there are open sets U, V in $\mathrm{Ult}(A)$ such that

$$U \cap (\{F_\eta : \eta < \alpha\} \cup \{G\}) = \{F_\eta : \eta \leq \xi\} \quad \text{and}$$
$$V \cap (\{F_\eta : \eta < \alpha\} \cup \{H\}) = \{F_\eta : \eta \leq \xi\}.$$

Then $U \cap V \cap (\{F_\eta : \eta < \alpha\} \cup \{G, H\}) = \{F_\eta : \eta \leq \xi\}$, as desired. We also need to show that $\{F_\eta : \eta < \alpha\} \cup \{G\}$ is open in $\{F_\eta : \eta < \alpha\} \cup \{G, H\}$; but this is obvious.

(ii)⇒(i): Assume (ii), and suppose that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G, H\rangle$ is right-separated. In particular, $G \neq H$. So we will get a contradiction by showing that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle H\rangle$ is right-separated. Take any $\xi < \alpha$. Let U be open such that $U \cap (\{F_\eta : \eta < \alpha\} \cup \{G, H\}) = \{F_\eta : \eta \leq \xi\}$. Then $U \cap (\{F_\eta : \eta < \alpha\} \cup \{H\}) = \{F_\eta : \eta \leq \xi\}$, as desired. □

Proposition 15.14. *Let $\langle F_\xi : \xi \leq \alpha\rangle$ be a maximal right-separated sequence of ultrafilters. For each $\xi < \alpha$ let a_ξ be such that $F_\xi \in \mathcal{S}(a_\xi) \cap \{F_\eta : \eta \leq \alpha\} \subseteq \{F_\eta : \eta \leq \xi\}$. Then $\langle a_\xi : \xi < \alpha\rangle$ is a maximal right-separated sequence of elements of A.*

Proof. Since $-a_\xi \in F_\alpha$ for all $\xi < \alpha$, condition (i) in the definition of right-separated holds. For condition (ii), suppose that $\xi < \alpha$ and Γ is a finite set such that $\Gamma < \xi$. Then $a_\xi \cdot \prod_{\eta\in\Gamma} -a_\eta \in F_\xi$, so (ii) holds. For maximality, suppose that $\{a_\xi : \xi < \alpha\}$ does not generate a maximal ideal. Then $\{-a_\xi : \xi < \alpha\}$ does not filter-generate an ultrafilter, but it is a subset of F_α, so there is an ultrafilter $G \neq F_\alpha$ such that $\{-a_\xi : \xi < \alpha\} \subseteq G$. Clearly $\langle F_\xi : x < \alpha\rangle^\frown\langle G\rangle$ is right-separated, contradiction. □

Proposition 15.15. $\mathrm{Intalg}(\kappa)$ *has a maximal right-separated sequence of ultrafilters, of length $\kappa + 1$.*

Proof. For each $\xi < \kappa$, the set

$$\{[\eta, \kappa) : \eta < \xi\} \cup \{[0, \eta) : \xi \leq \eta\}$$

clearly has f.i.p., and we let F_ξ be an ultrafilter containing this set. Furthermore, the set

$$\{[\eta, \kappa) : \eta < \kappa\}$$

has f.i.p., and we let F_κ be an ultrafilter containing this set.

Clearly all of these ultrafilters are distinct. To show that the sequence $\langle F_\xi : \xi \leq \kappa\rangle$ is right-separated, suppose that $\eta < \kappa$. Then clearly $\mathcal{S}([0, \eta)) \cap \{F_\xi : \xi \leq \kappa\} = \{F_\xi : \xi \leq \eta\}$, as desired.

Finally, we claim that $\{F_\xi : \xi \leq \kappa\} = \mathrm{Ult}(\mathrm{Intalg}(\kappa))$; hence $\langle F_\xi : \xi \leq \kappa\rangle$ is maximal. For, take any ultrafilter G. If $[\eta, \kappa) \in G$ for all $\eta < \kappa$, then $G = F_\kappa$. Otherwise, take the least ξ such that $[\xi, \kappa) \notin G$. Then $G = F_\xi$. □

Problem 141. *Does every BA have a maximal right-separated sequence of ultrafilters?*

Problem 142. *Describe* $\mathrm{hL_{Hs}}(A)$ *in cardinal number terms.*

Problem 143. *Describe* $\mathrm{hL_{Sr}}(A)$ *in cardinal number terms.*

Problem 144. *Describe* $\mathrm{hL_{Hr}}(A)$ *in cardinal number terms.*

On the relationship of hL with the previously defined functions: obviously $\mathrm{s}(A) \leq \mathrm{hL}(A)$ for any infinite BA A. Next, $\chi(A) \leq \mathrm{hL}(A)$. In fact, suppose that F is any ultrafilter on A; we want to find a subset X of F which generates F and has at most $\mathrm{hL}(A)$ elements. The set $\{\mathcal{S}(a) : -a \in F\}$ covers $\mathrm{Ult}(A)\backslash\{F\}$. Hence there is a subset X of F such that $\{\mathcal{S}(a) : -a \in X\}$ also covers $\mathrm{Ult}(A)\backslash\{F\}$, and $|X| \leq \mathrm{hL}(A)$. We claim that X generates F. For suppose that $a \in F$. Then $X \cup \{-a\}$ does not have the finite intersection property; otherwise, there would exist an ultrafilter G containing this set – then $G \neq F$, so $b \in G$ for some b such that $-b \in X$, contradiction. But $X \cup \{-a\}$ not having the finite intersection property means that a is in the filter generated by X, as desired.

The following theorem is due to Todorčević [90]:

Theorem 15.16. $|A| \leq \mathrm{Irr}(A) \cdot (\mathrm{hL}(A))^+$ *for any infinite BA* A.

Proof. Let $\theta = \mathrm{Irr}(A) \cdot (\mathrm{hL}(A))^+$ and $\kappa = (\mathrm{hL}(A))^+$. Assume that $|A| > \theta$, in order to work for a contradiction. Wlog $|A| = \theta^+$. Write A as a strictly increasing sequence $\langle A_\xi : \xi < \theta^+\rangle$ of subalgebras of size $\leq \theta$. Let $S_0 = \{\delta < \theta^+ : \mathrm{cf}(\delta) = \kappa\}$. So S_0 is stationary in θ^+. For each $\delta \in S_0$ choose $a_\delta \in A_{\delta+1}\backslash A_\delta$. Now fix $\delta \in S_0$. Define

$$I_\delta = \{b \in A_\delta : b \cdot a_\delta = 0\};\ J_\delta = \{b \in A_\delta : b \cdot -a_\delta = 0\}.$$

Note that I_δ and J_δ are ideals in A_δ. Let I'_δ be the ideal of A generated by I_δ. Thus $I'_\delta = \{a \in A : a \leq b \text{ for some } b \in I_\delta\}$. Now I'_δ has a generating set of size $\leq \mathrm{hL}(A)$; so I_δ itself has such a generating set. Since κ is a regular cardinal $> \mathrm{hL}(A)$, there is an $f(\delta) < \delta$ such that a generating set for I_δ is a subset of $A_{f(\delta)}$. So by Fodor's theorem there is a $\xi_0 < \theta^+$ and a stationary subset S_1 of S_0 such that $f(\delta) = \xi_0$ for all $\delta \in S_1$. Similarly, we can get a stationary subset S_2 of S_1 and a $\xi_1 < \theta^+$ such that every ideal J_δ for $\delta \in S_2$ has a generating set in A_{ξ_1}. Let ξ_2 be the maximum of ξ_0, ξ_1. We now claim that $\langle a_\delta : \delta \in S_2\rangle$ is irredundant, which, of course, is a contradiction. To prove this claim we first show

($*$) Suppose $\xi_2 < \delta < \delta_0 < \cdots < \delta_n$ are elements of S_2 and $\varepsilon \in {}^{n+1}2$. Suppose that $c \in A_\delta$ and

$$c \cdot \prod_{i \leq n} a_{\delta_i}^{\varepsilon i} \leq a_\delta.$$

Then there exist $b_0, \ldots, b_n \in A_{\xi_2}$ such that

$$c \cdot \prod_{i \leq n} a_{\delta_i}^{\varepsilon(i)} \leq c \cdot \prod_{i \leq n} b_i \leq a_\delta.$$

We prove $(*)$ by induction on n; the following argument will work when $n = 0$ and also for the inductive step. Assume the hypothesis of $(*)$. Let $d = \left(c \cdot \prod_{i<n} a_{\delta_i}^{\varepsilon i}\right) \cdot -a_\delta$. Then $d \in A_{\delta_n}$ and $d \leq -a_{\delta_n}^{\varepsilon(n)}$.

Case 1. $\varepsilon(n) = 1$. Then $d \in I_{\delta_n}$. Hence there is an $x \in A_{\xi_0} \cap I_{\delta_n}$ such that $d \leq x$. So $d \leq x \leq -a_{\delta_n}^{\varepsilon(n)}$.

Case 2. $\varepsilon(n) = 0$. Then $d \in J_{\delta_n}$, so there is an $x \in A_{\xi_1} \cap J_{\delta_n}$ such that $d \leq x$. So again $d \leq x \leq -a_{\delta_n}^{\varepsilon(n)}$.

So, in either case we get an $x \in A_{\xi_2}$ such that $d \leq x \leq -a_{\delta_n}^{\varepsilon(n)}$. It follows that

$$c \cdot \prod_{i \leq n} a_{\delta_i}^{\varepsilon(i)} \leq c \cdot -x \cdot \prod_{i<n} a_{\delta_i}^{\varepsilon(i)} \leq a_\delta,$$

so we have started the induction if $n = 0$, and continued the induction otherwise.

Now suppose that $\langle a_\delta : \xi_2 < \delta \in S_2 \rangle$ is redundant. So we can find $\delta < \delta_0 < \cdots < \delta_n$ with $\xi_2 < \delta$ such that a_δ is generated by $A_\delta \cup \{a_{\delta_0}, \ldots, a_{\delta_n}\}$. Therefore a_δ is a finite sum of elements of the form

$$c \cdot \prod_{i \leq n} a_{\delta_i}^{\varepsilon i},$$

where $c \in A_\delta$. By $(*)$, every such intersection can be replaced by one of the form

$$c \cdot \prod_{i \leq n} b_i$$

for some $b_0, \ldots, b_n \in A_{\xi_2}$. It follows that $a_\delta \in A_\delta$, contradiction. $\square$

The BA of the Kunen line constructed in Chapter 8 (assuming CH) has size and character ω_1 (see Chapter 14), hence hereditary Lindelöf degree ω_1, and countable spread.

If one can construct in ZFC a BA A such that $\mathrm{s}(A) < \mathrm{hL}(A)$, then one can also construct in ZFC a BA B such that $\mathrm{s}(B) < \chi(B)$. For, let I be an ideal of A such that I is not generated by fewer than $(\mathrm{s}(A))^+$ elements, and let $B = I \cup -I$.

An example where $\chi(A) < \mathrm{hL}(A)$ is provided by the Alexsandroff duplicate of a free algebra; see Chapter 14. An example with $\mathrm{hL}(A) < \mathrm{d}(A)$ is provided by the interval algebra on a complete Suslin line, using the argument of Lemma 3.43. On the other hand, we have the following result of Juhász [75]:

Theorem 15.17. (MA $+ \neg$*CH*) *If* $\mathrm{hL}(A) = \omega$, *then* $\mathrm{hd}(A) = \omega$.

Proof. Suppose that $\mathrm{hL}(A) = \omega$ but $\mathrm{hd}(A) > \omega$. Note that A satisfies ccc, since $\mathrm{c}(A) \leq \mathrm{hL}(A)$. By Theorem 6.15, there is a homomorphism f from A onto a BA B such that $\pi(B) > \omega$. We define $\langle b_\xi : \xi < \omega_1 \rangle$ by recursion, as follows. If b_ξ has been defined for all $\xi < \eta$, then $|\langle \{b_\xi : \xi < \eta\} \rangle| < \pi(B)$, and hence there is an element $b_\eta \in B^+$ such that no nonzero element of $\{\prod_{\xi\in F} b_\xi : F \in [\eta]^{<\omega}\}$ is below b_η. Clearly this constructs a sequence of distinct elements. For each $\xi < \omega_1$ let a_ξ be such that $f(a_\xi) = b_\xi$. Now by Lemma 3.5 there is an $S \in [\omega_1]^{\omega_1}$ such that $\{-a_\xi : \xi \in S\}$ has fip. Clearly then a strictly increasing enumeration of $\{-a_\xi : \xi \in S\}$ gives a right-separate sequence, contradiction. $\square$

These observations leave the following question open; this is Problem 53 in Monk [96]:

Problem 145. *Is there an example in ZFC of a BA A such that* $\mathrm{hL}(A) < \mathrm{d}(A)$*?*

This problem is equivalent to the problem of constructing in ZFC a BA A such that $\mathrm{hL}(A) < \mathrm{hd}(A)$; see the end of Chapter 16.

Bounded versions of hL can be defined as follows. For m a positive integer, a sequence $\langle x_\alpha : \alpha < \kappa\rangle$ of elements of A is said to be *m-right-separated* provided that if $\Gamma \in [\kappa]^m$, $\alpha < \kappa$, and $\beta < \alpha$ for all $\beta \in \Gamma$, then $a_\alpha \cdot \prod_{\beta\in\Gamma} -a_\beta \neq 0$. Then we define

$$\mathrm{hL}_m(A) = \sup\{\kappa : \text{there is an } m\text{-right-separated sequence in } A\}.$$

For this notion see Rosłanowski Shelah [98].

For an interval algebra A we have $\mathrm{hL}(A) = \mathrm{c}(A)$. In fact, suppose that A is the interval algebra on L and I is an ideal of A. Define $a \equiv b$ iff $a, b \in L$ and either $a = b$ or else if, say, $a < b$, then $[a, b) \in I$. Then $\equiv$ is a convex equivalence relation on L. For each $\equiv$-class k having more than one element, let $\langle a^k_\alpha : \alpha < \lambda_k\rangle$ be a strictly decreasing coinitial sequence in k (with $\lambda_k = 1$ if k has a first element), and let $\langle b^k_\alpha : \alpha < \mu_k\rangle$ be a strictly increasing cofinal sequence in k (with $\mu_k = 1$ if k has a greatest element), and with $a^k_0 < b^k_0$. Note that there are at most $\mathrm{c}(A)$ $\equiv$-classes with more than one element, and always $\lambda_k, \mu_k < (\mathrm{c}(A))^+$. Hence

$$\{[a^k_\alpha, b^k_\beta) : k \text{ an } \equiv\text{-class with more than one element, } \alpha < \lambda_k,\ \beta < \mu_k\}$$

is a collection of at most $\mathrm{c}(A)$ elements which generates I; so $\mathrm{hL}(A) \leq \mathrm{c}(A)$ by Theorem 15.1.

For any tree algebra B on an infinite pseudo-tree T we also have $\mathrm{c}(B) = \mathrm{hL}(B)$. For, Treealg$(T)$ embeds in an interval algebra A, and we may assume that Treealg(T) is dense in A (extend the identity from Treealg(T) onto itself to a homomorphism from A into the completion of Treealg(T), and then take the image of A). Hence

$$\mathrm{hL}(A) \geq \mathrm{hL}(\mathrm{Treealg}(T)) \geq \mathrm{c}(\mathrm{Treealg}()T) = \mathrm{c}(A) = \mathrm{hL}(A).$$

16 Hereditary Density

We begin again with some equivalent definitions, which are similar to the case of hereditary Lindelöf degree. Recall the definition of left-separated sequence from Chapter 6, before Theorem 6.12.

Theorem 16.1. *For any infinite BA A,* $\mathrm{hd}(A)$ *is equal to each of these cardinals:*

$\sup\{\kappa$*:there is a strictly decreasing sequence of ideals of length* $\kappa\}$*;*
$\sup\{\kappa$*:there is a strictly decreasing sequence of filters of length* $\kappa\}$*;*
$\sup\{\kappa$*:there is a strictly decreasing sequence of open sets of length* $\kappa\}$*;*
$\sup\{\kappa$*:there is a strictly increasing sequence of closed sets of length* $\kappa\}$*;*
$\sup\{\kappa$*:there is a left-separated sequence of length* $\kappa\}$*;*
$\min\{\kappa$*:every subspace* S *of* $\mathrm{Ult}(A)$ *has a dense subset of power* $\leq \kappa\}$*;*
$\sup\{\pi(B) : B$ *is a homomorphic image of* $A\}$*;*
$\sup\{\mathrm{d}(B) : B$ *is a homomorphic image of* $A\}$*.*

(Note that *left-separated* can be taken in the topological or algebraic sense.)

Proof. This time there are nine cardinals, named $\kappa_0, \ldots, \kappa_8$ in their order of mention, starting with hd itself. The following relationships are easy, following the pattern of the proof of Theorem 15.1: $\kappa_1 = \kappa_2$; $\kappa_1 \leq \kappa_3$; $\kappa_3 = \kappa_4$; $\kappa_3 \leq \kappa_5$; and $\kappa_0 = \kappa_6$. Furthermore, $\kappa_8 = \kappa_0$ by Theorem 5.20, $\kappa_0 = \kappa_5$ by Theorem 6.12, and $\kappa_0 = \kappa_7$ by Theorem 6.15. Hence only two inequalities remain.

$\kappa_6 \leq \kappa_2$: Suppose that X is a subspace of $\mathrm{Ult}(A)$, and $\mathrm{d}(X) = \kappa$; we construct a strictly decreasing sequence of filters of type κ. By induction let

$$F_\alpha \in X \backslash \overline{\{F_\beta : \beta < \alpha\}}$$

for each $\alpha < \kappa$. Then set $C_\alpha = \bigcap_{\beta \leq \alpha} F_\beta$. Thus $\langle C_\alpha : \alpha < \kappa\rangle$ is a decreasing sequence of filters. It is strictly decreasing, since if $\alpha < \kappa$ we can choose $a \in F_{\alpha+1}$ such that $\mathcal{S}(a) \cap \{F_\beta : \beta \leq \alpha\} = 0$, so that $-a \in C_\alpha \backslash C_{\alpha+1}$.

$\kappa_5 \leq \kappa_6$: Suppose $\langle x_\alpha : \alpha < \kappa\rangle$ is left separated, where κ is regular. Clearly then $\{x_\alpha : \alpha < \kappa\}$ has no dense subset of power $< \kappa$. □

The equivalents in Theorem 16.1 give rise to eight possible attainment problems, on the face of it. However, proofs of previous results set some limits:

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Proof of Theorem 5.20:

Attainment in the κ_8 sense implies attainment in the κ_0 sense;

Proof of Theorem 6.12:

Attainment in the κ_0 sense implies attainment in the κ_5 sense;

For $\mathrm{hd}(A)$ regular, attainment in the κ_5 sense implies attainment in the κ_0 sense;

Proof of Theorem 6.15:

attainment in the κ_7 sense implies attainment in the κ_5 sense;

attainment in the κ_0 sense implies attainment in the κ_7 sense;

Proof of Theorem 16.1:

attainment for $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are equivalent;

attainment in the κ_0 sense implies attainment in the κ_2 sense.

Now we note two other implications.

κ_1 *attained implies* κ_5 *attained.* Suppose that $\langle I_\alpha : \alpha < \kappa_1\rangle$ is a strictly decreasing sequence of ideals. For each $\alpha < \kappa_1$ choose $a_\alpha \in I_\alpha \backslash I_{\alpha+1}$. Then $\langle a_\alpha : \alpha < \kappa_1\rangle$ is left-separated.

κ_5 *attained implies* κ_1 *attained.* Similarly.

In Rosłanowski, Shelah [01a] there are two results about attainment for hd.

(1) Theorem 1.5 says:

If $2^{\mathrm{cf}(\lambda)} < \lambda$*, then in any BA with* hd *equal to* λ*, all equivalent versions are attained.*

(2) In Section 4, it is shown that it is consistent to have a singular cardinal λ and a BA A with $\mathrm{hd}(A) = \lambda$, with attainment in the left-separated sense but not in the sense κ_7.

Thus the following problem is open; this is Problem 54 in Monk [96].

Problem 146. *Completely describe the relationships between the attainment problems for* hd.

Corollary 16.2. hd *is attained in the sense of left-separation for strong limit singular cardinals.*

Proof. Suppose that λ is strong limit singular and $\mathrm{hd}(A) = \lambda$. By Theorem 13.2, $\mathrm{Ult}(A)$ has a discrete subset of size λ. By a remark before Theorem 13.2, A has an atomic homomorphic image B with λ atoms; hence $\pi(B) = \lambda$, and the remarks on attainment above give the desired result. □

Also recall from Proposition 13.4 that hd is attained in the left-separated sense for singular cardinals of confinality ω.

If A is a subalgebra or homomorphic image of B, then $\mathrm{hd}(A) \le \mathrm{hd}(B)$. We now consider special subalgebras. One can have $A \le_{\mathrm{free}} B$ with $\mathrm{hL}(B)$ arbitrarily larger than $\mathrm{hL}(A)$. Hence the same applies to $\le_{\mathrm{proj}}$, $\le_{\mathrm{u}}$, $\le_{\mathrm{rc}}$, $\le_{\mathrm{reg}}$, and $\le_\sigma$. The situation for $\le_\pi$ is similar to that for hL. Also, the arguments for hL and $\le_{\mathrm{s}}$ and $\le_{\mathrm{mg}}$ work for hd too.

Proposition 16.3. $\mathrm{hd}(A \times B) = \max(\mathrm{hd}(A), \mathrm{hd}(B))$.

Proof. Suppose that $\langle I_\alpha : \alpha < \kappa\rangle$ is a strictly decreasing sequence of ideals in A. For each $\alpha < \kappa$ let $J_\alpha = \{(a,0) : a \in I_\alpha\}$. Then $\langle J_\alpha : \alpha < \kappa\rangle$ is a strictly decreasing sequence of ideals in $A \times B$. A similar argument holds for B, so $\ge$ holds.

Now let $\kappa = \max(\mathrm{hd}(A), \mathrm{hd}(B))$, and suppose that $\langle K_\alpha : \alpha < \kappa^+\rangle$ is a strictly decreasing sequence of ideals in $A \times B$; we want to get a contradiction. Let $L_\alpha = \{a \in A : (a,0) \in K_\alpha\}$ and $M_\alpha = \{b \in B : (0,b) \in K_\alpha\}$, for each $\alpha < \kappa^+$. Then L_α and M_α are ideals in A and B respectively. Moreover, if $\alpha < \beta$, then $L_\beta \subseteq L_\alpha$ and $M_\beta \subseteq M_\alpha$, with $L_\beta \ne L_\alpha$ or $M_\beta \ne M_\alpha$. Since $\mathrm{hd}(A) \le \kappa$, the set $\{L_\alpha : \alpha < \kappa^+\}$ has size at most κ, and so there is a $\Gamma \in [\kappa^+]^{\kappa^+}$ such that $L_\alpha = L_\beta$ for all $\alpha, \beta \in \Gamma$. Then $\langle M_\alpha : \alpha \in \Gamma\rangle$ is one-one, contradicting $\mathrm{hd}(B) \le \kappa$. $\square$

Proposition 16.4. $\mathrm{hd}\left(\prod^{\mathrm{w}}_{i\in I} A_i\right) = \max(|I|, \sup_{i\in I} \mathrm{hd}(A_i))$.

Proof. It is convenient to assume that I is a cardinal κ, and by Proposition 16.3 we may assume that κ is infinite. For brevity let $B = \prod^{\mathrm{w}}_{\alpha<\kappa} A_\alpha$. By Proposition 16.3 we have $\mathrm{hd}(A_\beta) \le \mathrm{hd}(B)$ for all $\beta < \kappa$. Now for each $\alpha < \kappa$ let J_α consist of all $x \in \prod^{\mathrm{w}}_{\alpha\in\kappa} A_\alpha : x$ is of type 1, with support contained in $\kappa\backslash\alpha\}$. Clearly this gives a strictly decreasing sequence of ideals. Thus we have proved $\ge$.

Now let $\lambda = \max(|I|, \sup_{i\in I} \mathrm{hd}(A_i))$. Suppose that $\langle K_\alpha : \alpha < \lambda^+\rangle$ is a strictly decreasing sequence of ideals in B; we want to get a contradiction.

Case 1. There is an $\alpha < \lambda^+$ such that some $a \in K_\alpha$ has type 2. Then with M the support of a, let $L_\beta = \{a \restriction M : a \in K_\beta\}$ for each $\beta \in [\alpha, \lambda^+)$. Then

(1) $\langle L_\beta : \beta \in [\alpha, \lambda^+)\rangle$ is a strictly decreasing sequence of ideals in $\prod_{\gamma\in M} A_\gamma$.

In fact, suppose that $\alpha \le \beta < \gamma < \lambda^+$. Take any $b \in K_\beta\backslash K_\gamma$. Then $b \restriction M \in L_\beta$. Suppose that $b \restriction M \in L_\gamma$. Say $b \restriction M = c \restriction M$ with $c \in K_\gamma$. Then also $a + c \in K_\gamma$, and $b \le a + c$, so $b \in K_\gamma$, contradiction. Thus $b \restriction M \notin L_\gamma$. This proves (1).

But (1) contradicts Proposition 16.3.

Case 2. For each $\alpha < \lambda^+$, every element of K_α has type 1. For each $\alpha < \lambda^+$ and $\beta < \kappa$ let $N^\beta_\alpha = \{a \in A_\beta : \chi^\beta_a \in K_\alpha\}$, where we define $\chi^\beta_a \in B$ by

$$\chi^\beta_a(\gamma) = \begin{cases} a & \text{if } \gamma = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\langle N^\beta_\alpha : \alpha < \lambda^+\rangle$ is a decreasing sequence of ideals in A_β. Hence there is a $\gamma_\alpha < \lambda^+$ such that $N^\beta_\delta = N_{\gamma_\alpha}$ for all $\delta \in [\gamma_\alpha, \lambda^+)$. Let $\varepsilon < \lambda^+$ be such that

$\gamma_\alpha < \varepsilon$ for all $\alpha < \kappa$. Choose $a \in K_\varepsilon \backslash K_{\varepsilon+1}$. Let a have support P. For each $\beta \in P$ we have $a_\beta \in N^\beta_\varepsilon = N^\beta_{\varepsilon+1}$, and so $\chi^\beta_{a_\beta} \in K_{\varepsilon+1}$. Now $a = \sum_{\beta\in P}\chi^\beta_{a_\beta} \in K_{\varepsilon+1}$, so $a \in K_{\varepsilon+1}$, contradiction. $\square$

Obviously $\mathrm{s}(A) \leq \mathrm{hd}(A)$, and hence $\mathrm{Ind}(A) \leq \mathrm{hd}(A)$. It follows that for arbitrary products we have, as usual, $\mathrm{hd}\big(\prod_{i\in I} A_i\big) \geq \max(2^{|I|}, \sup_{i\in I}\mathrm{hd}(A_i))$. Shelah and Peterson independently noticed that strict inequality is possible; this answers Problem 49 of Monk [90]. The example for spread can be used here.

The situation for ultraproducts is similar to that for hL. Problems 55 and 56 of Monk [96] were solved by Shelah [99] and Spinas [00] as follows:

Shelah [99], part of 15.13: *Suppose that D is a uniform ultrafilter on κ. Then there is a class of cardinals χ such that $\chi^\kappa = \chi$ and there are BAs B_i for $i < \kappa$ such that $\mathrm{hd}(B_i) \leq \chi$ for each $i < \kappa$, hence $\prod_{i<\kappa}\mathrm{hd}(B_i)/D \leq \chi$, while $\mathrm{hd}(\prod_{i<\kappa} B_i/D) = \chi^+$.*

Shelah, Spinas [00], part of 2.4:

It is consistent that there exist cardinals κ, μ, BAs B_i for $i < \kappa$, and an ultrafilter D on κ such that $\prod_{i<\kappa}\mathrm{hd}(B_i)/D = \mu^{++}$ while $\mathrm{hd}(\prod_{i<\kappa} B_i/D) = \mu^+$.

For free products, the analog of Theorem 15.5 holds, with essentially the same proof:

Theorem 16.5. *If $\langle A_i : i \in I\rangle$ is a system of non-trivial BAs, for brevity let $\lambda = \sup_{i\in I}\mathrm{hd}(A_i)$; then*

$$\max(|I|, \sup_{i\in I}\mathrm{hd}(A_i)) \leq \mathrm{hd}(\oplus_{i\in I}A_i) \leq \max\left(|I|, 2^{\sup_{i\in I}\mathrm{s}(A_i)}\right). \qquad \square$$

The inequalities in Theorem 16.5 are sharp. This is seen as for Theorem 15.5, except for both $<$; for this case one can take S such that $\mathbb{Q} \subseteq S \subseteq \mathbb{R}$, $|S| = \aleph_1$, let $A = \mathrm{Intalg}(S)$, and consider $A \oplus A$, assuming $\neg$CH.

Another important fact about free products is given in the following theorem.

Theorem 16.6. *For infinite BAs A and B we have $\mathrm{s}(A\oplus B) \geq \min(\mathrm{hL}(A), \mathrm{hd}(B))$.*

Proof. Let $\kappa = \min(\mathrm{hL}(A), \mathrm{hd}(B))$, and let $\lambda^+ \leq \kappa$. Let $\langle a_\alpha : \alpha < \lambda^+\rangle$ be right-separated in A, and let $\langle b_\alpha : \alpha < \lambda^+\rangle$ be left-separated in B. We claim that $\langle a_\alpha \cdot b_\alpha : \alpha < \lambda^+\rangle$ is ideal independent. For, suppose that $\Gamma \in [\lambda^+]^{<\omega}$ and $\alpha \in \lambda^+\backslash\Gamma$. Let $\Delta = \{\beta \in \Gamma : \beta < \alpha\}$. then

$$a_\alpha \cdot b_\alpha \cdot \prod_{\beta\in\Gamma} -(a_\beta \cdot b_\beta) \geq \left(a_\alpha \cdot \prod_{\beta\in\Delta} -a_\beta\right) \cdot \left(b_\alpha \cdot \prod_{\beta\in\Gamma\backslash\Delta} -b_\beta\right) \neq 0,$$

as desired. $\square$

Proposition 16.7. *In the setup for moderate products,*

$$\mathrm{hd}\left(\prod\nolimits_{i\in I}^{B} A_i\right) = \max(|I|, \mathrm{hd}(B), \sup_{i\in I} \mathrm{hd}(A_i)).$$

Proof. For brevity let $C = \prod_{i\in I}^{B} A_i$.

Clearly $\geq$ holds; see the proof of Proposition 15.6. Now let κ be the right side of the above equation, and suppose that $\langle K_\alpha : \alpha < \kappa^+\rangle$ is a strictly decreasing sequence of ideals of C. Let $L_\alpha = \{b \in B : h(b, \emptyset, \emptyset) \in K_\alpha\}$. Clearly L_α is an ideal in B, and $L_\beta \subseteq L_\alpha$ whenever $\alpha < \beta < \kappa^+$. Hence there is an ordinal α such that $L_\beta = L_\alpha$ for all $\beta \in (\alpha, \kappa^+)$. For each $i \in I$ and $\alpha < \kappa^+$ let $M_{i\alpha} = \{a \in A_i : h(\emptyset, \{i\}, \{(i, a)\}) \in K_\alpha\}$. Then $M_{i\alpha}$ is an ideal in A_i, and $M_{i\beta} \subseteq M_{i\alpha}$ whenever $\alpha < \beta < \kappa^+$. Hence there is an ordinal $\beta_i < \kappa^+$ such that $M_{i\gamma} = M_{i\beta_i}$ for all$\gamma \in (\beta_i, \kappa^+$. Let $\gamma = \sup(\alpha, \sup_{i\in I} \beta_i)$. Then $\gamma < \kappa$, and $K_\delta = K_\gamma$ for all $\delta \in (\gamma, \kappa^+)$ (contradiction). In fact, if $h(b, F, a) \in K_\delta$, then $h(b, \emptyset, \emptyset) \in K_\delta$, hence $b \in L_\delta$, hence $b \in L_\gamma$, hence $h(b, \emptyset, \emptyset) \in K_\gamma$. Also, if $i \in F$, then $h(\emptyset, \{i\}, \{(i, a_i)\}) \in K_\delta$, hence $a_i \in M_{i\delta}$, hence $a_i \in M_{i\gamma}$, hence $h(\emptyset, \{i\}, \{(i, a_i)\}) \in K_\gamma$. So

$$h(b, F, a) = h(b, \emptyset, \emptyset) \cup \bigcup_{i\in F} h(\emptyset, \{i\}, \{(i, a_i)\}) \in K_\gamma. \qquad \square$$

Proposition 16.8. *Let C be the one-point gluing of A and B with respect to ultrafilters F on A and G on B. Then* $\max(\mathrm{hd}(A), \mathrm{hd}(B)) \leq \mathrm{hd}(C)$.

Proof. Suppose that $\langle I_\alpha : \alpha < \kappa\rangle$ is a strictly decreasing sequence of ideals in A. Define $J_\alpha = \{(a, b) \in C : a \in I_\alpha\}$. Then $\langle J_\alpha : \alpha < \kappa\rangle$ is a decreasing sequence of ideals in C. If $a \in I_\alpha \backslash I_{\alpha+1}$, then $(a, 1) \in J_\alpha \backslash J_{\alpha+1}$ if $a \in F$, and $(a, 0) \in J_\alpha \backslash J_{\alpha+1}$ if $a \notin F$. Thus $\langle J_\alpha : \alpha < \kappa\rangle$ is strictly decreasing. A similar argument works for B. $\square$

Problem 147. *Do there exist BAs A, B and ultrafilters F, G on A, B respectively such that if C is the associated one-point gluing, then* $\max(\mathrm{hd}(A), \mathrm{hd}(B)) < \mathrm{hd}(C)$*?*

Similarly to hL, it is clear that $\mathrm{hd}(A) = |A|$ for A the Alexsandroff duplicate of a BA.

Theorem 16.9. $\mathrm{hd}(A) \leq \mathrm{t}(\mathrm{Exp}\,(A))$.

Proof. We use the fact that $\mathrm{hd}(A) = \sup\{d(B) : B \text{ a homomorphic image of } A\}$, given in Theorem 16.1. Let B be any homomorphic image of A. Then by Proposition 2.7, $\mathrm{Exp}\,(B)$ is a homomorpic image of $\mathrm{Exp}\,(A)$, so $\mathrm{d}(B) \leq \mathrm{t}(\mathrm{Exp}\,(B)) \leq \mathrm{t}(\mathrm{Exp}\,(A))$, using Lemma 12.14. $\square$

Lemma 16.10. *For any infinite BA A and any $X \subseteq A$ the following are equivalent:*

(i) *X generates A.*

(ii) *$\{\mathcal{S}(x) : x \in X\}$ separates points in* $\mathrm{Ult}(A)$.

Proof. (i)⇒(ii): Suppose that $F, G \in \mathrm{Ult}(A)$, $F \neq G$. If $\forall x \in X(x \in F$ iff $x \in G)$, then $\forall x \in \langle X\rangle(x \in F$ iff $x \in G)$, by an easy argument.

(ii)⇒(i): Suppose that $a \in A\backslash\langle X\rangle$. Let $\mathscr{A} = \{x_0^{\varepsilon_0} \cdot \ldots \cdot x_{m-1}^{\varepsilon_{m-1}}$: each $x_i \in X$ and $x_0^{\varepsilon_0} \cdot \ldots \cdot x_{m-1}^{\varepsilon_{m-1}} \leq a\}$. Then $\bigcup_{b\in\mathscr{A}} \mathcal{S}(b) \subset \mathcal{S}(a)$ by compactness, since $a \notin \langle X\rangle$, so choose $F \in \mathcal{S}(a)\backslash \bigcup_{b\in\mathscr{A}} \mathcal{S}(b)$. Now $(F \cap \langle X\rangle) \cup \{-a\}$ has fip, and so is contained in an ultrafilter G. But then F and G are distinct ultrafilters which cannot be separated by $\{\mathcal{S}(x) : x \in X\}$. □

Concerning derived functions, we have the following obvious facts:

$$\mathrm{hd}(A) = \mathrm{hd}_{\mathrm{H}+}(A) = \mathrm{hd}_{\mathrm{S}+}(A) = \mathrm{hd}_{\mathrm{h}+}(A) = {}_{\mathrm{d}}\mathrm{hd}_{\mathrm{S}+}(A);$$

and $\mathrm{hd}_{\mathrm{S}-}(A) = \mathrm{hd}_{\mathrm{h}-}(A) = \omega$.

Now we consider min-max versions of hd. We extend the definition of left separated sequences, algebraic or topological, to sequences indexed by any infinite ordinal; see Chapter 6, discussion around Theorem 6.12. We take the definition of $\mathrm{hd}_{\mathrm{spect}}$ and $\mathrm{hd}_{\mathrm{mm}}$ in the algebraic sense. Thus

$$\begin{aligned}\mathrm{hd}_{\mathrm{spect}}(A) = \{|\alpha| : \alpha \geq \omega &\text{ and there is a maximal left-separated sequence}\\ &\text{of elements of } A \text{ of length } \alpha\};\\ \mathrm{hd}_{\mathrm{mm}}(A) = \min(\mathrm{hd}_{\mathrm{spect}}(A)).&\end{aligned}$$

Proposition 16.11. *A sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A is a maximal left-separated sequence* iff *it is a left-separated sequence and*

$$\left\{a_\xi \cdot \prod\nolimits_{\eta\in F} -a_\eta : \xi \in \alpha,\ F \in [\alpha]^{<\omega},\ \textit{and } \xi < F\right\}$$

is dense in A. □

Proposition 16.12. $\mathrm{hd}_{\mathrm{mm}}(A) = \pi(A)$ *for any infinite BA A.*

Proof. This is immediate from Proposition 16.11 and Lemma 6.14. □

There are BAs A with $|\mathrm{hd}_{\mathrm{spect}}(A)| > 1$. For example, $\mathrm{hd}_{\mathrm{spect}}(\mathscr{P}(\omega))$ has at least the two members $\omega, 2^\omega$, since $\pi(\mathscr{P}(\omega)) = \omega$ and $\mathscr{P}(\omega)$ has an independent subset of size 2^ω. The existence of a BA A with $|\mathrm{hd}_{\mathrm{spect}}(A)| > 2$ is necessarily a consistency problem, since $|A| \leq 2^{\pi(A)}$.

Problem 148. *Is it consistent to have an infinite BA A such that* $|\mathrm{hd}_{\mathrm{spect}}(A)| > 2$*?*

If we generalize the notion of a left-separated sequence of ultrafilters to arbitrary ordinal lengths, we see that such a sequence is maximal iff it contains all ultrafilters. Thus the min-max function is trivial for hd in this sense.

We consider one more variant of hd and its min-max function. Clearly a strictly decreasing sequence of ideals in a BA A is maximal iff its intersection is the trivial ideal $\{0\}$. Let $\mathrm{hd}^{\mathrm{id}}_{\mathrm{mm}}(A)$ be the least size of the index set of a maximal strictly decreasing sequence of ideals in A.

Proposition 16.13. $\mathrm{hd}^{\mathrm{id}}_{\mathrm{mm}}(A) \le \mathrm{cf}(\mathrm{d}(A))$.

Proof. By Theorem 5.1 we may assume that A is a subalgebra of $\mathscr{P}(\kappa)$, where $\kappa = \mathrm{d}(A)$. For each $\alpha < \kappa$ let $I_\alpha = \{a \in A : a \subseteq [\alpha, \kappa)\}$. Clearly I_α is an ideal in A, and $I_\beta \subseteq I_\alpha$ if $\alpha < \beta$. Suppose that $I_\alpha = \{0\}$. Then $\forall a \in A^+[a \cap \alpha \neq 0]$. Then the mapping $a \mapsto (a \cap \alpha)$ is an isomorphism from A into $\mathscr{P}(\alpha)$, contradicting the choice of κ. $\square$

Problem 149. *What is the exact place of* $\mathrm{hd}^{\mathrm{id}}_{\mathrm{mm}}$ *among the other cardinal functions?*

Now we want to go into a result of Fedorchuk [75], which provides an example for several of the questions in Monk [90]: assuming $\diamondsuit$, there is a BA A with $\mathrm{hd}(A) = \omega$ and $\mathrm{Card}_{\mathrm{H-}}A = \omega_1$. This is a weakened form of his main theorem. We give a construction of such a BA due to Kunen [75] (quite different from that of Fedorchuk but done upon looking at that article and noticing some problems with the construction).

Special $\diamondsuit$-sequences. This material is from Kunen [75] (except for the name *special* and the proof of the lemma). If $f \in {}^{\omega_1}({}^{\omega_1}\omega_1)$, let $f\restriction\restriction\alpha = \langle f_\xi \restriction \alpha : \xi < \alpha\rangle$. A *special* $\diamondsuit$*-sequence* is a sequence $\langle f^\alpha : \alpha < \omega_1\rangle$ such that each $f^\alpha \in {}^\alpha({}^\alpha\omega_1)$ and for all $f \in {}^{\omega_1}({}^{\omega_1}\omega_1)$ the set $\{\alpha < \omega_1 : f\restriction\restriction\alpha = f^\alpha\}$ is stationary.

Lemma 16.14. $\diamondsuit$ *implies that there is a special* $\diamondsuit$*-sequence.*

Proof. Let H be a one-one function from ω_1 onto $\omega_1 \times \omega_1$. If $x \in \omega_1 \times \omega_1$, we write $x = (x_0, x_1)$. For $f \in {}^{\omega_1}({}^{\omega_1}\omega_1)$ we define $\tilde{f} \in {}^{\omega_1}\omega_1$ by $\tilde{f}(\alpha) = f_{(H(\alpha))_0}((H(\alpha))_1)$. Define $G : {}^{\omega_1}\omega_1 \to {}^{\omega_1}2$ by

$$G_h(\beta) = \begin{cases} 1 & \text{if } h\big((H(\beta))_0\big) = (H(\beta))_1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $F : {}^{\omega_1}2 \to {}^{\omega_1}\omega_1$ by

$$(F(k))(\beta) = \begin{cases} \gamma & \text{if } \ k(H^{-1}(\beta,\gamma)) = 1 \text{ and } k(H^{-1}(\beta,\delta)) = 0 \text{ for all } \delta \neq \gamma, \\ 0 & \text{if there is no such } \gamma. \end{cases}$$

Let $\chi : \mathscr{P}(\omega_1) \to {}^{\omega_1}2$ be the natural bijection. Let $C_0 = \{\alpha < \omega_1 : H[\alpha] = \alpha \times \alpha\}$. So, C_0 is club. In fact, suppose $\alpha < \omega_1$ is limit and $C_0 \cap \alpha$ is unbounded in α. If $\beta < \alpha$, choose $\gamma \in C_0 \cap \alpha$ such that $\beta < \gamma$. Then $H(\beta) \in \gamma \times \gamma \subseteq \alpha \times \alpha$. On the other hand, if $\beta, \gamma < \alpha$, choose $\delta \in C_0 \cap \alpha$ such that $\beta, \gamma < \delta$. Then $(\beta, \gamma) \in \delta \times \delta$, so there is an $\varepsilon \in \delta$ such that $H(\varepsilon) = (\beta, \gamma)$. Since $d < \alpha$, this shows that $(\beta, \gamma) \in H[\alpha]$. Thus C_0 is closed. To show that C_0 is unbounded, let $\beta < \omega_1$. Let $\gamma_0 = \beta$. Choose $\gamma_1 > \gamma_0$ such that $H[\gamma_0] \subseteq \gamma_1 \times \gamma_1$, and choose $\gamma_2 > \gamma_1$ such that $H^{-1}[\gamma_1 \times \gamma_1] \subseteq \gamma_2$. Continuing this way, the supremum of all γ_i is in C_0.

Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. For $\alpha \in C_0$ let $f^\alpha \in {}^\alpha({}^\alpha\omega_1)$ be defined by

$$f^\alpha_\beta(\gamma) = (F(\chi_{A_\alpha}))(H^{-1}(\beta,\gamma)).$$

Let $f^\alpha \in {}^\alpha({}^\alpha\omega_1)$ be arbitrary if $\alpha \notin C_0$.

Now suppose that $f \in {}^{\omega_1}({}^{\omega_1}\omega_1)$. Let $B = \chi^{-1}(G_{\tilde{f}})$. Now

$$C_1 \stackrel{\text{def}}{=} \{\alpha : \forall \xi < \alpha (f_\xi \restriction \alpha \in {}^\alpha\alpha)\}$$

is club. In fact, to show that C_1 is closed, suppose that $\alpha < \omega_1$ is a limit ordinal and $C_1 \cap \alpha$ is unbounded in α. Take any $\xi < \alpha$. For any $\beta \in C_1$ with $\xi < \beta < \alpha$ we have $f_\xi \restriction \beta \in {}^\beta\beta$, and $C_1 \cap \alpha$ is unbounded in α, so $f_\xi \restriction \alpha \in {}^\alpha\alpha$. Thus $\alpha \in C_1$. To show that C_1 is unbounded in ω_1, let $\beta < \omega_1$. Choose γ_1 such that for all $\xi < \beta$ we have $f_\xi \restriction \beta \in {}^\beta\gamma_1$. Then choose γ_2 such that for all $\xi < \gamma_1$ we have $f_\xi \restriction \gamma_1 \in {}^{\gamma_1}\gamma_2$. Continuing this way, we let δ be the supremum of all γ_i. We claim that $\delta \in C_1$. For, suppose that $\xi < \delta$. Then for all $i \in \omega$ such that $\xi < \gamma_i$ we have $f_\xi \restriction \gamma_i \in {}^{\gamma_i}\gamma_{i+1}$. Hence $f_\xi \restriction \delta \in {}^\delta\delta$. In fact, let $\varepsilon < \delta$. Say $\varepsilon < \varphi_i$. Then $f_\xi(\varepsilon) \in \gamma_{i+1} \subseteq \delta$, as desired.

Hence $C_0 \cap C_1 \cap \{\alpha < \omega_1 : \alpha \cap B = A_\alpha\}$ is stationary. We claim that if α is in this set, then $f \restriction\restriction \alpha = f^\alpha$; this will finish the proof. Suppose $\xi < \alpha$. We want to show that $f_\xi \restriction \alpha = f^\alpha_\xi$. Let $\gamma < \alpha$. Then

$$\begin{aligned}
\tilde{f}(H^{-1}(\xi,\gamma)) &= f_\xi(\gamma);\\
G_{\tilde{f}}(H^{-1}(H^{-1}(\xi,\gamma), f_\xi(\gamma))) &= 1;\\
H^{-1}(H^{-1}(\xi,\gamma), f_\xi(\gamma)) &\in \alpha \cap B;\\
H^{-1}(H^{-1}(\xi,\gamma), f_\xi(\gamma)) &\in A_\alpha;\\
\chi_{A_\alpha}(H^{-1}(H^{-1}(\xi,\gamma), f_\xi(\gamma))) &= 1.
\end{aligned}$$

It is easily checked that if $\delta \neq f_\xi(\gamma)$ then $\chi_{A_\alpha}(H^{-1}(H^{-1}(\xi,\gamma),\delta)) = 0$. In fact, we need to show that $H^{-1}(H^{-1}(\xi,\gamma),\delta) \notin A_\alpha$; equivalently, that $H^{-1}(H^{-1}(\xi,\gamma),\delta) \notin B$, i.e., $\chi_B(H^{-1}(H^{-1}(\xi,\gamma),\delta)) = 0$, which means that $G_{\tilde{f}}(H^{-1}(H^{-1}(\xi,\gamma),\delta)) = 0$, i.e., $\tilde{f}(H^{-1}(\xi,\gamma) \neq \delta$, i.e., $f_\xi(\gamma) \neq \delta$, as desired. Therefore $(F\chi_{A_\alpha})H^{-1}(\xi,\gamma) = f_\xi(\gamma)$. It follows that $f^\alpha_\xi(\gamma) = f_\xi(\gamma)$. □

Kunen's construction. We assume $\diamondsuit$; so CH is available also. Fix a special $\diamondsuit$-sequence $\langle f^\alpha : \alpha < \omega_1\rangle$. For any space Y, a point $y \in Y$ is a *strong limit point* of $\mathscr{H} \subseteq \mathscr{P}(Y)$ if for all neighborhoods V of y there is an $H \in \mathscr{H}$ such that $y \notin H$ and $H \subseteq V$. For $\alpha \leq \beta \leq \omega_1$ define $\pi^\beta_\alpha : {}^\beta 2 \to {}^\alpha 2$ by $\pi^\beta_\alpha(g) = g \restriction \alpha$; thus π^β_α is continuous. We claim

(1) there is an enumeration $\langle q_\alpha : \alpha < \omega_1\rangle$ of $\bigcup_{\sigma<\omega_1} {}^\sigma 2$ such that every element is repeated ω_1 times and $q_\alpha \in {}^{\sigma_\alpha}2$ with $\sigma_\alpha \leq \alpha$.

To prove (1), for each $\sigma < \omega_1$ let ${}^\sigma 2 = \{h_{\sigma\xi} : \xi < \omega_1\}$ with each element repeated ω_1 times. Let H enumerate $\omega_1 \times \omega_1$ under its natural order ($(\alpha,\beta) < (\gamma,\delta)$ iff $[\max(\alpha,\beta) < \max(\gamma,\delta)$ or $(\max(\alpha,\beta) = \max(\gamma,\delta)$ and $\alpha < \gamma)$ or $(\max(\alpha,\beta) = \max(\gamma,\delta)$ and $\alpha = \gamma$ and $\beta < \delta)]$). As is well known, H has domain ω_1. By induction on α one can show that $H(\alpha)_0, H(\alpha)_1 \leq \alpha$ for all $\alpha < \omega_1$. In fact, clearly $H(0) = (0,0)$, so it holds for $\alpha = 0$, Suppose that it holds for α. Let $\beta = H(\alpha)_0$ and $\gamma = H(\alpha)_1$. Then $H(\alpha+1)$ is determined as in the following table.

Case:	$\beta+1<\gamma$	$\beta+1=\gamma$	$\gamma<\beta$	$\gamma=\beta$
$H(\alpha+1)$:	$(\beta+1,\gamma)$	$(\gamma,0)$	$(\beta,\gamma+1)$	$(0,\gamma+1)$

Thus $H(\alpha+1)_0, H(\alpha+1)_1 \leq \alpha+1$. Now suppose that α is limit and $H(\beta)_0, H(\beta)_1 \leq \beta$ for all $\beta < \alpha$. For each $\beta < \alpha$ let $K(\beta) = \sup(H(\beta)_0, H(\beta)_1)$.

Case 1. For all $\beta < \alpha$ there is a $\gamma \in (\beta,\alpha)$ such that $K(\beta) < K(\gamma)$. Then $H(\alpha) = (0, \sup_{\beta<\alpha} K(\beta))$, and the inductive step works.

Case 2. There is a $\beta < \alpha$ such that $K(\beta) = K(\gamma)$ for all $\gamma \in (\beta,\alpha)$.
Subcase 2.1. $H(\beta)_0 < H(\beta)_1$. Thus $H(\beta)_1 = K(\beta)$.

Subsubcase 2.1.1. $\delta \stackrel{\text{def}}{=} \sup_{\beta<\gamma<\alpha} H(\gamma)_0 < K(\gamma)$. Then $H(\alpha) = (\delta, K(\beta))$, as desired.

Subsubcase 2.1.2. $\delta = K(\gamma)$. Then $H(\alpha) = (0, \delta+1)$, as desired.
Subcase 2.2. $H(\beta)_0 > H(\beta)_1$. So $H(\beta)_0 = K(\beta)$. Let $\theta = \sup_{\beta<\gamma<\alpha} H(\gamma)_1$.
Subsubcase 2.2.1. $\theta < K(\beta)$. Then $H(\alpha) = (K(\beta), \theta)$, as desired.
Subsubcase 2.2.2. $\theta = K(\beta)$. Then $H(\alpha) = (0, K(\beta)+1)$, as desired.

This finishes our inductive proof. Now let $q_\alpha = h_{H(\alpha)_0 H(\alpha)_1}$ for all $\alpha < \omega_1$. Clearly (1) holds.

Our space will be a closed subspace of ${}^{\omega_1}2$. By induction on $\alpha \leq \omega_1$ we will define $X_\alpha \subseteq {}^\alpha 2$ and $p_\alpha \in X_\alpha$ so that:

(2) X_α is closed and nonempty.
(3) If $\alpha \leq \beta$ then $\pi^\beta_\alpha[X_\beta] = X_\alpha$.
(4) If $q_\alpha \in X_{\sigma_\alpha}$, then $p_\alpha \restriction \sigma_\alpha = q_\alpha$.
(5) $p_\alpha{}^\frown 0, p_\alpha{}^\frown 1 \in X_{\alpha+1}$.
(6) If $\alpha \leq \beta$, $\{f^\alpha_\xi : \xi < \alpha\} \subseteq X_\alpha$, and $h \in X_\alpha$ is an accumulation point of $\{f^\alpha_\xi : \xi < \alpha\}$, then every point k in $X_\beta \cap (\pi^\beta_\alpha)^{-1}[\{h\}]$ is a strong limit point of $\{X_\beta \cap (\pi^\beta_\alpha)^{-1}[\{f^\alpha_\xi\}] : \xi < \alpha\}$.

Before actually making this construction we check that (2)–(6) yield the desired properties of $X \stackrel{\text{def}}{=} X_{\omega_1}$.

X is hereditarily separable: If not, let $f = \langle f_\xi : \xi < \omega_1\rangle$ be a left-separated sequence in X. Let

$$\begin{aligned}C = \{\alpha < \omega_1 : &\text{ for all clopen } N \subseteq X_\alpha \\ &(N \cap \{f_\xi \restriction \alpha : \xi < \alpha\} = 0 \text{ iff } (\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\xi : \xi < \omega_1\} = 0) \text{ and} \\ &(|N \cap \{f_\xi \restriction \alpha : \xi < \alpha\}| = 1 \text{ iff } |(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\xi : \xi < \omega_1\}| = 1)\}.\end{aligned}$$

Then C is club. To prove this, first note, obviously:

(7) $N \cap \{f_\beta \restriction \alpha : \beta < \alpha\} \neq 0$ implies that $(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\beta : \beta < \omega_1\} \neq 0$, if $\alpha < \omega_1$ and N is a clopen subset of X_α.

(8) $|N \cap \{f_\beta \restriction \alpha : \beta < \alpha\}| \geq 2$ implies that $|(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\beta : \beta < \omega_1\}| \geq 2$, if $\alpha < \omega_1$ and N is a clopen subset of X_α.

Now to prove that C is closed, suppose that $\alpha < \omega_1$ is a limit ordinal and $\alpha \cap C$ is unbounded in α. Suppose that N is clopen in X_α and $(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\xi : \xi < \omega_1\} \neq 0$. Say $\xi < \omega_1$ and $f_\xi \restriction \alpha \in N$. Write $N = U^\alpha_g \stackrel{\text{def}}{=} \{h \in X_\alpha : g \subseteq h\}$, where $g \in {}^F2$ for some finite $F \subseteq \alpha$. Say $F \subseteq \beta < \alpha$, $\beta \in C$. Then $f_\xi \restriction \beta \in U^\beta_g$, i.e., $(\pi^{\omega_1}_\beta)^{-1}[U^\beta_g] \cap \{f_\eta : \eta < \omega_1\} \neq 0$ so, since $\beta \in C$, we get $U^\beta_g \cap \{f_\eta \restriction \beta : \eta < \beta\} \neq 0$. Hence choose $\eta < \beta$ with $f_\eta \restriction \beta \in U^\beta_g$. Hence $f_\eta \restriction \alpha \in U^\alpha_g = N$, and $N \cap \{f_\eta \restriction \alpha : \eta < \alpha\} \neq 0$, as desired. For the other part of C, suppose that N is clopen in X_α and $(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\xi : \xi < \omega_1\} \geq 2$. Say $\xi, \eta < \omega_1$, $\xi \neq \eta$, and $f_\xi \restriction \alpha, f_\eta \restriction \alpha \in N$. Write $N = U^\alpha_g \stackrel{\text{def}}{=} \{h \in X_\alpha : g \subseteq h\}$, where $g \in {}^F2$ for some finite $F \subseteq \alpha$. Say $F \subseteq \beta < \alpha$, $\beta \in C$. Then $f_\xi \restriction \beta, f_\eta \restriction \beta \in U^\beta_g$, i.e., $|(\pi^{\omega_1}_\beta)^{-1}[U^\beta_g] \cap \{f_\eta : \eta < \omega_1\}| \geq 2$ so, since $\beta \in C$, we get $|U^\beta_g \cap \{f_\eta \restriction \beta : \eta < \beta\}| \geq 2$. Hence choose distinct $\rho, \sigma < \beta$ with $f_\rho \restriction \beta, f_\sigma \restriction \beta \in U^\beta_g$. Hence $f_\rho \restriction \alpha, f_\sigma \restriction \alpha \in U^\alpha_g = N$, and $|N \cap \{f_\eta \restriction \alpha : \eta < \alpha\}| \geq 2$, as desired. This proves that C is closed.

To prove that C is unbounded, suppose that $\alpha_0 < \omega_1$. Choose α_1 such that $\alpha_0 < \alpha_1$ and for all clopen $N \subseteq X_{\alpha_0}$ we have

$$(\pi^{\omega_1}_{\alpha_0})^{-1}[N] \cap \{f_\xi : \xi < \omega_1\} \neq 0 \Rightarrow \exists \xi < \alpha_1 (f_\xi \restriction \alpha_0 \in N) \text{ and}$$
$$|(\pi^{\omega_1}_{\alpha_0})^{-1}[N] \cap \{f_\xi : \xi < \omega_1\}| \geq 2 \Rightarrow |N \cap \{f_\xi \restriction \alpha_0 : \xi < \alpha_1\}| \geq 2.$$

This is possible since there are only countably many clopen sets $N \subseteq X_{\alpha_0}$. Continuing in this fashion with $\alpha_2, \alpha_3, \ldots$, we see that $\alpha_\omega = \sup_{n<\omega} \alpha_n$ is the desired member of C.

By the definition of special $\diamondsuit$-sequences, fix $\alpha \in C$ such that $f \restriction\restriction \alpha = f^\alpha$.

(9) $f_\eta \restriction \alpha \in \overline{\{f^\alpha_\beta : \beta < \alpha\}}$ for all $\eta < \omega_1$.

For, if $f_\eta \restriction \alpha \in N$ with N clopen, then $(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\nu : \nu < \omega_1\} \neq 0$ so, since $\alpha \in C$, $N \cap \{f_\beta \restriction \alpha : \beta < \alpha\} \neq 0$, as desired.

(10) For all $\eta < \omega_1$, if $f_\eta \restriction \alpha$ is an isolated point of $\{f_\beta \restriction \alpha : \beta < \alpha\}$, then $\eta < \alpha$.

For, let N be clopen such that $N \cap \{f_\beta \restriction \alpha : \beta < \alpha\} = \{f_\eta \restriction \alpha\}$. Say $f_\eta \restriction \alpha = f_\gamma \restriction \alpha$ with $\gamma < \alpha$. Now since $\alpha \in C$ we get $|(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\beta : \beta < \omega_1\}| = 1$. Since f_η and f_γ are both in $(\pi^{\omega_1}_\alpha)^{-1}[N] \cap \{f_\beta : \beta < \omega_1\}$, it follows that $\eta = \gamma$, as desired.

From (9) and (10) it follows that $f_\alpha \restriction \alpha$ is an accumulation point of $\{f_\beta \restriction \alpha : \beta < \alpha\} = \{f^\alpha_\beta : \beta < \alpha\}$. Hence by (6) f_α is a strong limit point of $\{X_{\omega_1} \cap (\pi^{\omega_1}_\alpha)^{-1}[\{f^\alpha_\xi\}] : \xi < \alpha\}$. So f_α is a limit point of $\{f_\xi : \xi < \alpha\}$, which contradicts the left-separatedness.

Now we show that $\chi_{\mathrm{H-}}A \geq \omega_1$, where $A = \mathrm{Clop}X_{\omega_1}$. By Corollary 14.9 it suffices to show that X_{ω_1} has no one-one convergent sequences. So, assume that $\lim_{n\to\infty} g_n = h$ with all g_n distinct and different from h. Choose $f \in {}^{\omega_1}({}^{\omega_1}2)$ so that $\{f_\alpha : \alpha < \omega_1\} = \{g_n : n \in \omega\}$. Choose α so that $f\lceil\lceil\alpha = f^\alpha$, all of the functions $g_n \restriction \alpha$, $h \restriction \alpha$ are distinct, and $\{f_\beta : \beta < \alpha\} = \{g_n : n \in \omega\}$. Then $h \restriction \alpha$ is an accumulation point of $\{f^\alpha_\xi : \xi < \alpha\}$. Say $h \restriction \alpha = q_\beta$ with $\alpha \leq \beta$. Then $p_\beta{}^\frown 0$ and $p_\beta{}^\frown 1$ are both in $X_{\beta+1}$ and extend $h \restriction \alpha$. This gives by (6) two distinct points h, l which are strong limit points of $X_{\omega_1} \cap \{(\pi^{\omega_1}_\alpha)^{-1}[\{f^\alpha_\xi\}] : \xi < \alpha\}$. Thus both are limit points of $\{g_n : n \in \omega\}$, contradiction.

Next we do the construction to yield (2)–(6). As soon as a space X_α is constructed, fix $p_\alpha \in X_\alpha$ such that (4) holds. Let X_0 be the one-point space. For δ limit, let $X_\delta = \{g \in {}^\delta 2 : \forall \alpha < \delta(g \restriction \alpha \in X_\alpha)\}$. It is straightforward to check (2)–(6) then. Now we do the crucial step from X_δ to $X_{\delta+1}$. We now define a nested clopen basis $\langle K_n : n \in \omega\rangle$ of p_δ. First let $\langle K'_n : n \in \omega\rangle$ be any such basis, with $K'_0 = X_\delta$. If there is no $\alpha \leq \delta$ such that $\{f^\alpha_\xi : \xi < \alpha\} \subseteq X_\alpha$ and $p_\delta \restriction \alpha$ is an accumulation point of $\{f^\alpha_\xi : \xi < \alpha\}$, let $K_n = K'_n$ for all $n \in \omega$. Otherwise, let $\{\alpha_n : n \in \omega\}$ enumerate all $\alpha \leq \delta$ such that $\{f^\alpha_\xi : \xi < \alpha\} \subseteq X_\alpha$ and $p_\delta \restriction \alpha$ is an accumulation point of $\{f^\alpha_\xi : \xi < \alpha\}$, each one enumerated infinitely many times by both even and odd integers. Now define K_n by induction as follows. $K_0 = X_\delta$. If K_n has been defined, by (6) for $\beta = \delta$, p_δ is a strong limit point of $\{X_\delta \cap (\pi^\delta_{\alpha_n})^{-1}[\{f^{\alpha_n}_\xi\}] : \xi < \alpha_n\}$, so there is a $\xi < \alpha_n$ such that $p_\delta \notin X_\delta \cap (\pi^\delta_{\alpha_n})^{-1}[\{f^{\alpha_n}_\xi\}] \subseteq K_n$. We let $p_\delta \in K_{n+1} \subseteq K'_n \cap (K_n \backslash (\pi^\delta_{\alpha_n})^{-1}[\{f^{\alpha_n}_\xi\}])$, K_{n+1} clopen. Thus the following condition holds:

(11) $\{K_n : n \in \omega\}$ is a nested clopen base for p_δ, and if $\alpha \leq \delta$ is such that $\{f^\alpha_\xi : \xi < \alpha\} \subseteq X_\alpha$ and $p_\delta \restriction \alpha$ is an accumulation point of $\{f^\alpha_\xi : \xi < \alpha\}$, then there are infinitely many even and infinitely many odd n such that $\exists\xi(p_\delta \notin X_\delta \cap (\pi^\delta_\alpha)^{-1}[\{f^\alpha_\xi\}] \subseteq K_n \backslash K_{n+1})$.

Finally, we set

$$\begin{aligned} X_{\delta+1} = &\{g \in {}^{\delta+1}2 : g \restriction \delta = p_\delta\} \cup \\ &\bigcup_{n \text{ even}} \{g \in {}^{\delta+1}2 : g \restriction \delta \in K_n \backslash K_{n+1},\ g\delta = 0\} \cup \\ &\bigcup_{n \text{ odd}} \{g \in {}^{\delta+1}2 : g \restriction \delta \in K_n \backslash K_{n+1},\ g\delta = 1\}. \end{aligned}$$

It remains only to check (2)–(6) for $\delta + 1$. All except (6) are easy, and (6) is obvious if $\alpha = \delta + 1$. So assume that $\alpha \leq \delta$, $h \in X_\alpha$ is an accumulation point of $\{f^\alpha_\xi : \xi < \alpha\}$, $k \in X_{\delta+1} \cap (\pi^{\delta+1}_\alpha)^{-1}[\{h\}]$. To show that k is a strong limit point of $\{X_{\delta+1} \cap (\pi^{\delta+1}_\alpha)^{-1}[\{f^\alpha_\xi\}] : \xi < \alpha\}$, let $k \in U^{\delta+1}_g$ with $g \in {}^F 2$, $F \subseteq \delta + 1$, F finite. Without loss of generality, $\delta \in F$.

Case 1. $k \restriction \delta \neq p_\delta$. Say $k \restriction \delta \in K_n \backslash K_{n+1}$ with n even. Thus $k(\delta) = 0 = g(\delta)$. Thus $k \restriction \delta \in U^\delta_{g\restriction\delta} \cap (K_n \backslash K_{n+1})$, so by (6) for δ choose $\xi < \alpha$ such that

$$k \restriction \delta \notin (\pi^\delta_\alpha)^{-1}[\{f^\alpha_\xi\}] \subseteq U^\delta_{g\restriction\delta} \cap (K_n \backslash K_{n+1}).$$

It follows that $k \notin (\pi_\alpha^{\delta+1})^{-1}[\{f_\xi^\alpha\}] \subseteq U_g^{\delta+1}$, since if $l \in (\pi_\alpha^{\delta+1})^{-1}[\{f_\xi^\alpha\}]$ then $l \restriction \delta \in K_n \backslash K_{n+1}$ and n is even, so $l(\delta) = 0 = g(\delta)$.

Case 2. $k \restriction \delta = p_\delta$. Say $g(\delta) = 0$. Choose m even such that $K_m \subseteq U^\delta_{g\restriction\delta}$ and $p_\delta \notin X_\delta \cap (\pi_\alpha^\delta)^{-1}[\{f_\xi^\alpha\}] \subseteq K_m \backslash K_{m+1}$ for some $\xi < \alpha$. Then $k \notin X_{\delta+1} \cap (\pi_\alpha^{\delta+1})^{-1}[\{f_\xi^\alpha\}] \subseteq U_g^{\delta+1}$, as desired.

This completes Kunen's construction.

Another generalization of Fedorchuk's example is found in Koszmider [99]; see Chapter 3, (20) in the discussion of $\mathrm{c_{Hr}}$.

On the relationships of hd with the other functions, note also that by Theorem 16.1 we have $\pi(A) \leq \mathrm{hd}(A)$. $\pi(A)$ is strictly less than $\mathrm{hd}(A)$ in $\mathscr{P}(\kappa)$, for example. And we have $\mathrm{s}(A) < \mathrm{hd}(A)$ for A the interval algebra on a Suslin line, and $\mathrm{hd}(A) < \chi(A)$ for a Kunen line (Chapter 8).

There is a model of ZFC with a BA A such that $\mathrm{s}(A), \mathrm{d}(A) < \mathrm{hd}(A)$ (a remark of I. Juhász (email message in February, 1995). Namely, take a model with $\mathrm{MA}(\sigma\text{-centered}) + \exists$ a 0-dimensional Suslin line S, let K be a compactification of ω such that $K\backslash\omega = S$, and let $A = \mathrm{clop}(K)$. (See Weiss, van Mill [84].)

From the result that $\pi(A) \leq \mathrm{s}(A) \cdot (\mathrm{t}(A))^+$ it follows that $\mathrm{hd}(A) \leq \mathrm{s}(A) \cdot (\mathrm{t}(A))^+$. It is also true that $\mathrm{hd}(A) \leq \mathrm{Irr}(A)$. In fact, we have $\mathrm{hd}(A) = \pi_{\mathrm{H}+}A$, and for any homomorphic image B of A we have $\pi(B) \leq \mathrm{Irr}(B) \leq \mathrm{Irr}(A)$.

If one can construct in ZFC a BA A such that $\mathrm{hL}(A) < \mathrm{hd}(A)$, then one can also construct in ZFC a BA B such that $\mathrm{hL}(B) < \mathrm{d}(B)$ (see problem 145). In fact, A has a homomorphic image such that $\mathrm{hL}(A) < \mathrm{d}(B)$, and $\mathrm{hL}(B) \leq \mathrm{hL}(A)$. Similarly for $\mathrm{hL} < \pi$.

The following problems are open; these are Problems 57 and 58 in Monk [96].

Problem 150. *Can one construct in ZFC a BA A such that* $\mathrm{s}(A) < \mathrm{hd}(A)$*?*

Problem 151. *Can one construct in ZFC a BA A such that* $\mathrm{hd}(A) < \chi(A)$*?*

Bounded versions of hd can be defined as follows. For m a positive integer, a sequence $\langle x_\alpha : \alpha < \kappa\rangle$ of elements of A is said to be *m-left-separated* provided that if $\Gamma \in [\kappa]^m$, $\alpha < \kappa$, and $\alpha < \beta$ for all $\beta \in \Gamma$, then $a_\alpha \cdot \prod_{\beta\in\Gamma} -a_\beta \neq 0$. Then we define

$$\mathrm{hd}_m(A) = \sup\{\kappa : \text{there is an } m\text{-left-separated sequence in } A\}.$$

For this notion, see Rosłanowski, Shelah [98].

17 Incomparability

We begin with one important equivalent definition:

Theorem 17.1. *For any infinite BA A we have* $\mathrm{Inc}(A) = \sup\{|T| : T$ *is a tree included in* $A\}$.

(Note that when we say that T is a tree included in A, we mean merely that T is a subset of A which is a tree under the induced ordering; there is no assumption that incomparable elements (in T) are disjoint (in the dual of A).)

Proof. Since any incomparable set is a tree having only roots, the inequality $\leq$ is clear. To show equality, suppose that κ is regular and A has no incomparable set of size κ; we show that A has no tree of size κ. Suppose T is a tree of size κ. By Theorem 4.25 of Part I of the BA handbook, A has a dense subset D of size $< \kappa$. Now each level of T is an incomparable set, and hence has fewer than κ elements. Hence T has at least κ levels. Let T' be a subset of T of power κ consisting exclusively of elements of successor levels. For each $d \in D$ let

$$M_d = \{t \in T' : \text{if } s \text{ is the immediate predecessor of } t, \text{ then } d \leq t \cdot -s\}.$$

Thus $T' = \bigcup_{d \in D} M_d$, so there is a $d \in D$ such that $|M_d| = \kappa$. But then M_d is incomparable, contradiction: if $y, z \in M_d$ and $y < z$, then $y \leq u$ where u is the immediate predecessor of z, and $d \leq z \cdot -u$, hence $d \cdot y = 0$, contradicting $d \leq y$. $\square$

Note that if $\mathrm{Inc}(A)$ is attained, then it is obviously attained in the tree sense. The converse also holds, as Todorčević pointed out in a letter to the author several years before Monk [90] appeared; this solves Problem 52 in Monk [90]. We give this result here, following the proof in an email message from Shelah of December 1990.

Theorem 17.2. *If A is an infinite BA and there is a tree $T \subseteq A$ with $|T| = \mathrm{Inc}(A)$, then A has an incomparable subset of power* $\mathrm{Inc}(A)$.

Proof. By the proof of Theorem 17.1 we may assume that $\lambda \overset{\text{def}}{=} \mathrm{Inc}(A)$ is singular. Let $\langle \kappa_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of cardinals with supremum λ

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and with $\mathrm{cf}(\lambda) < \kappa_0$. Without loss of generality, T has no level of size λ. Now we consider two cases.

Case 1. For every $\alpha < \mathrm{cf}(\lambda)$ there is a β such that T has at least κ_α elements of level β. For any ordinal β let $\mathrm{lev}_\beta(T)$ be the set of elements of T of level β. By an easy construction we obtain a strictly increasing sequence $\langle \beta_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$ of ordinals such that $|\mathrm{lev}_{\beta_{\alpha+1}} T| > \max(\kappa_\alpha, |\mathrm{lev}_{\beta_\alpha}(T)|)$ for all $\alpha < \mathrm{cf}(\lambda)$. For every $\alpha < \mathrm{cf}(\lambda)$ let S_α be a subset of $\mathrm{lev}_{\beta_{\alpha+1}} T$ of power $(\max(\kappa_\alpha, |\mathrm{lev}_{\beta_\alpha}|))^+$ such that all elements of S_α have the same predecessors at level β_α. Note that if $\alpha < \mathrm{cf}\lambda$ then

$$R_\alpha \stackrel{\mathrm{def}}{=} \{t \in S_\alpha : t \le s \text{ for some } s \in S_\gamma \text{ with } \alpha < \gamma < \mathrm{cf}(\lambda)\}$$

has power at most $\mathrm{cf}(\lambda)$. Now the set $\bigcup_{\alpha<\mathrm{cf}\lambda}(S_\alpha \backslash R_\alpha)$ is incomparable of size λ, as desired.

Case 2. Case 1 fails to hold. Then clearly T must have at least λ levels. Hence $\mathrm{Depth}(A) = \lambda$ and $\mathrm{c}(A) = \lambda$, so by the Erdős–Tarski theorem A has a disjoint subset of power λ; it is also an incomparable subset. $\square$

Concerning attainment of Inc, several things are known. Milner and Pouzet [86] proved a general result, of which a special case is that if $\mathrm{Inc}(A) = \lambda$ with $\mathrm{cf}(\lambda) = \omega$, then $\mathrm{Inc}(A)$ is attained. has shown that if 2^ω is weakly inaccessible, then there is a BA of size 2^ω with incomparability 2^ω not attained. The statement in Monk [90] about attainment due to Shelah has been withdrawn by him. Instead, in Shelah [92] he proves that $\mathrm{Inc}(A)$ is always attained for singular cardinals. We give this interesting proof here.

Theorem 17.3. Inc *is attained for singular cardinals.*

Proof. Suppose that $\mathrm{Inc}(B) = \lambda$, with λ singular. Now we choose $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ and $\langle A_i : i < \mathrm{cf}(\lambda)\rangle$ so that the following conditions hold:

(1) $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ is a strictly increasing sequence of regular cardinals all greater than $\mathrm{cf}(\lambda)$ and with supremum λ.
(2) $\langle A_i : i < \mathrm{cf}(\lambda)\rangle$ is a system of incomparable sets in B with $|A_i| = \lambda_i^+$ for all $i < \mathrm{cf}(\lambda)$.

It is clear that this can be done. In addition, if possible we choose these things so that the following condition holds:

(3) If $i < j < \mathrm{cf}(\lambda)$, $x \in A_i$, and $y \in A_j$, then $y \not\le x$.

Now let $A = \bigcup_{i<\mathrm{cf}(\lambda)} A_i$. The following notation will also be useful. Let $C \subseteq B$. For any $x \in B$, $C \uparrow x = \{y \in C : y \ge x\}$, $C \downarrow x = \{y \in C : y \le x\}$, and for any cardinal μ, $C \uparrow_\mu = \{x \in C : |C \uparrow x| < \mu\}$ and $C \downarrow_\mu = \{x \in C : |C \downarrow x| < \mu\}$.

Case 1. There is a $\mu < \lambda$ such that $|A \uparrow_\mu| = \lambda$. For each $x \in (A \uparrow_\mu)$ let $f(x) = (A \uparrow_\mu) \uparrow x$. Thus $f : (A \uparrow_\mu) \to \mathscr{P}(A \uparrow_\mu)$ and $|f(x)| < \mu < \lambda$ for all $x \in (A \uparrow_\mu)$. Hence by Hajnal's free set Theorem (see the Handbook, Part III, p.

1231) we get $E \subseteq (A \uparrow_\mu)$ of size λ such that $x \notin f(y)$ for all distinct $x, y \in E$. So E is incomparable, as desired.

Case 2. There is a $\mu < \lambda$ such that $|A \downarrow_\mu| = \lambda$. Similarly.

Case 3. For every $i < \mathrm{cf}(\lambda)$ there is an $x_i \in A$ such that $\lambda_i < |A \uparrow x_i| < \lambda$. We now define a function $\mu : \mathrm{cf}(\lambda) \to \mathrm{cf}(\lambda)$ by induction. Having defined μ_j for all $j < i$, choose $\mu_i < \mathrm{cf}\lambda$ so that $\lambda_{\mu_i} > \sum_{j<i}(|A \uparrow x_{\mu_j}| + \lambda_{\mu_j})$. Now let $A'_i = A_{\mu_i}$, $\lambda'_i = \lambda_{\mu_i}$, $x'_i = x_{\mu_i}$ for all $i < \mathrm{cf}(\lambda)$. Then (1)–(2) hold for A'_i and λ'_i, (3) holds for the A'_i's if it held for the A_i's, and

(4) If $j < i < \mathrm{cf}\,\lambda$, then $\sum_{j<i} |A \uparrow x'_j| < \lambda'_i$.

Now for $i < \mathrm{cf}(\lambda)$ we have $\left|(A \uparrow x'_i) \backslash \bigcup_{j<i}(A \uparrow x'_j)\right| > \lambda'_i$, $\mathrm{cf}\,\lambda < \lambda_i$, and $A = \bigcup_{j<\mathrm{cf}\,\lambda} A_j$, so there is an $\alpha(i) < \mathrm{cf}\,\lambda$ such that

$$\left| A_{\alpha(i)} \cap (A \uparrow x'_i) \backslash \bigcup_{j<i} (A \uparrow x'_j) \right| > \lambda'_i.$$

Clearly $\alpha(i) \geq \mu_i$. We define $\beta : \mathrm{cf}(\lambda) \to \mathrm{cf}(\lambda)$ by induction. If $\beta(j)$ has been defined for all $j < i$, choose $\beta(i) < \mathrm{cf}(\lambda)$ such that $\sup_{j<i} \max\{\alpha(\beta(j)), \mu_{\beta(j)}\} < \mu_{\beta(i)}$. Now choose

$$A^*_i \subseteq A_{\alpha(\beta(i))} \cap (A \uparrow x'_{\beta(i)}) \backslash \bigcup_{j<i} (A \uparrow x'_{\beta(j)})$$

of size $\lambda'_{\beta(i)}$. Note that $\alpha(\beta(j)) < \mu_{\beta(i)} \leq \alpha(\beta(i))$ for $j < i < \mathrm{cf}\,\lambda$. Let $\lambda^*_i = \lambda'_{\beta(i)}$. Then (1)–(2) hold for A^*_i and λ^*_i, (3) holds for the A^*_i's if it held for the A_i's (since $A^*_i \subseteq A_{\alpha(\beta(i))}$), and

(5) if $i < j < \mathrm{cf}(\lambda)$, $x \in A^*_i$, and $y \in A^*_j$, then $x \not\leq y$.

For, otherwise $x'_{\beta(i)} \leq x \leq y \notin (A \uparrow x'_{\beta(i)})$, contradiction. Now let $A^\star_i = \{x : -x \in A^*_i\}$. Then (1)–(3) hold for $A^\star_i$ and $\lambda^\star_i$. Therefore by the initial choices, (3) itself holds. So, (3) and (5) hold for A^*_i and λ^*_i. It follows that $\bigcup_{i<\mathrm{cf}(\lambda)} A^*_i$ is an incomparable set of size λ, as desired.

Case 4. For every $i < \mathrm{cf}(\lambda)$ there is an $x \in A$ such that $\lambda_i < |A \downarrow x| < \lambda$. Similar to Case 3.

Case 5. None of the previous cases. By $\neg$Case 3, there is an $i(*) < \mathrm{cf}(\lambda)$ such that for all $x \in A$ we have $\neg(\lambda_{i(*)} < |A \uparrow x| < \lambda)$. By $\neg$Case 2, $|A \downarrow_{\lambda^+_{i(*)}}| < \lambda$, and by $\neg$Case 1, $|A \uparrow_{\lambda^+_{i(*)}}| < \lambda$. Choose $x^* \in A \backslash ((A \downarrow_{\lambda^+_{i(*)}}) \cup (A \uparrow_{\lambda^+_{i(*)}}))$. Thus $|A \uparrow x^*| \geq \lambda^+_{i(*)}$, so by the choice of $i(*)$ we have $|A \uparrow x^*| = \lambda$. Also, $|A \downarrow x^*| \geq \lambda^+_{i(*)} > \mathrm{cf}(\lambda)$, so there is a $j(*) < \mathrm{cf}(\lambda)$ such that $|(A \downarrow x^*) \cap A_{j(*)}| \geq \mathrm{cf}(\lambda)$. Choose distinct $y_i \in (A \downarrow x^*) \cap A_{j(*)}$ for $i < \mathrm{cf}(\lambda)$.

For each $i < \mathrm{cf}(\lambda)$ let $A'_i = A_i \cap (A \uparrow x^*)$. Thus A'_i is an incomparable set and

$$\left|\bigcup_{i<\mathrm{cf}(\lambda)} A'_i\right| = \left|(A \uparrow x^*) \cap \bigcup_{i<\mathrm{cf}(\lambda)} A_i\right| = |(A \uparrow x^*) \cap A| = \lambda.$$

Finally, $\{y_i + x \cdot -x^* : i < \mathrm{cf}(\lambda),\ x \neq x^*,\ x \in A'_i\}$ is an incomparable set of size λ, as desired. □

These results leave the following problem open.

Problem 152. *Is it true that for every regular limit cardinal κ there is a BA with* $\mathrm{Inc}(A) = \kappa$ *not attained?*

Now we turn to algebraic operations, as usual. If A is a subalgebra or homomorphic image of B, then $\mathrm{Inc}(A) \leq \mathrm{Inc}(B)$. The difference here can clearly be large. For our special subalgebras, the difference can be large when $A \leq_{\mathrm{free}} B$, and hence also when $A \leq_{\mathrm{proj}} B$, $A \leq_u B$, $A \leq_{\mathrm{rc}} B$, $A \leq_{\mathrm{reg}} B$, and $A \leq_\sigma B$. Clearly this is also true for $A \leq_\pi B$ and for $A \leq_{\mathrm{mg}} B$. But we have the following problem.

Problem 153. *Completely describe the behaviour of* Inc *for special subalgebras.*

If A is a subalgebra of B, then, easily, $\mathrm{Inc}(A \times B) \geq |A|$; in fact, $\{(a, -a) : a \in A\}$ is an incomparable set in $A \times B$. Hence if A is cardinality-homogeneous and has no incomparable set of size $|A|$, then A is rigid (this follows from some elementary facts concerning automorphisms; see the article in the BA handbook about automorphisms). Thus the incomparability of a product can jump from that in a factor – for example, if A is such that $\mathrm{Inc}(A) < |A|$, we have $\mathrm{Inc}(A \times A) = |A|$. Finally, $\mathrm{Inc}(A \oplus B) = \max(|A|, |B|)$ if $|A|, |B| \geq 4$, since $A \oplus C \cong A \times A$ if $|C| = 4$.

Ultraproducts: Inc is an ultra-sup function, so Theorems 3.20–3.22 hold, Theorem 3.22 saying that $\mathrm{Inc}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Inc}A_i/F\right|$ for F regular, and Donder's theorem says that under $V = L$ the regularity assumption can be removed. In Shelah [97a] it is shown that $>$ is consistent, but we do not know whether this can be done in ZFC; this is problem 59 in Monk [96],

Problem 154. *Do there exist in ZFC a system $\langle A_i : i \in I\rangle$ of infinite BAs, I infinite, and a regular ultrafilter F on I such that* $\mathrm{Inc}\left(\prod_{i\in I} A_i/F\right) > \left|\prod_{i\in I} \mathrm{Inc}A_i/F\right|$*?*

In the other direction, we have the following result of Shelah and Spinas [00], part of 1.7:

It is consistent that there exist cardinals κ, μ, an ultrafilter D on κ, and a system $\langle B_i : i < \kappa\rangle$ of interval algebras, such that $\left|\prod_{i<\kappa} \mathrm{Inc}(B_i)/D\right| = \mu^{++}$ *while* $\mathrm{Inc}(\prod_{i<\kappa} B_i/D) \leq \mu^+$.

This answers problem 60 of Monk [96].

Concerning derived functions of incomparability, we mention only a result of Shelah (email message of December 1990), solving Problem 53 in Monk [90]:

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Theorem 17.4. *If* $\mathrm{Inc}(A) = \omega$, *then* $\mathrm{Card}_{\mathrm{H-}}(A) = \omega$.

Proof. Suppose that $\mathrm{Inc}(A) = \omega < \mathrm{Card}_{\mathrm{H-}}(A)$. Without loss of generality, assume that A is a subalgebra of $\mathscr{P}(\omega)$ containing all of the finite subsets of ω. Hence there is an $a \in A$ such that both a and $\omega \backslash a$ are infinite. Let F be a nonprincipal ultrafilter on $A \restriction a$. Then we can construct $\langle a_\alpha : \alpha < \chi(F) \rangle$, each $a_\alpha \subseteq a$, such that for each $\alpha < \chi(F)$, the element a_α is not in the filter generated by $\{a_\beta : \beta < \alpha\}$; in particular, $\beta < \alpha \Rightarrow a_\beta \not\leq a_\alpha$. Similarly we get a nonprincipal ultrafilter G on $A \restriction -a$ and a sequence $\langle b_\alpha : \alpha < \chi(G) \rangle$ of subelements of $-a$ such that $\beta < \alpha \Rightarrow b_\beta \not\leq b_\alpha$. Say $\chi(F) \leq \chi(G)$. Then $\langle a_\alpha + -b_\alpha \cdot -a : \alpha < \chi(F) \rangle$ is a system of incomparable elements. By 14.10, $\mathrm{alt}(A) \leq \chi(F)$, so $\mathrm{a}(A) = \omega$. Hence 14.12 gives a contradiction. □

Next we consider "small" versions of Inc. Thus

$$\mathrm{Inc}_{\mathrm{spec}}(A) = \{|X| : X \subseteq A \text{ is maximal incomparable}\};$$
$$\mathrm{Inc}_{\mathrm{mm}}(A) = \min(\mathrm{Inc}_{\mathrm{spec}}(A)).$$

Charles Scherer has studied these notions and proved the following:

(1) For any infinite cardinal κ, $\mathrm{Inc}_{\mathrm{mm}}(\mathrm{finco}(\kappa)) = \omega$.

(2) $\mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\kappa)) = \omega$.

(3) $\mathrm{Inc}_{\mathrm{mm}}(A) = \omega$ if A has $\mathrm{Fr}(\omega)$ as a dense subalgebra.

The following is a slight generalization of (2);

Proposition 17.5. $[\omega, \kappa] \subseteq \mathrm{Inc}_{\mathrm{spect}}(\mathscr{P}(\kappa))$. *Moreover, if* $\kappa < 2^\mu \leq 2^\kappa$, *then* $2^\mu \in \mathrm{Inc}_{\mathrm{spect}}(\mathscr{P}(\kappa))$.

Proof. Suppose that $\omega \leq \nu \leq \kappa$. Let $M \subseteq \kappa$ with $|M| = \nu$. Then $\{\{\alpha\} : \alpha \in M\} \cup \{\kappa \backslash M\}$ is maximal incomparable of size ν. Now suppose that $\kappa < 2^\mu \leq 2^\kappa$. Then we may assume that $\mu \leq \kappa$. Let N be a subset of κ of size μ, and let A be maximal incomparable in $\mathscr{P}(N)$ of size 2^μ. (For example, extend an independent subset of $\mathscr{P}(N)$ of size 2^μ to a maximal incomparable set.) Then $\{\{\alpha\} : \alpha \in \kappa \backslash N\} \cup A$ is maximal incomparable of size 2^μ. □

Problem 155. *If* κ *is an infinite cardinal and* $\kappa < \mu < 2^\kappa$, μ *not a power of* 2, *is there a maximal incomparable set in* $\mathscr{P}(\kappa)$ *of size* μ?

Proposition 17.6. $\mathrm{Inc}_{\mathrm{spect}}(\mathrm{Fr}(\kappa)) = \{\kappa\}$. □

Proposition 17.7. *Suppose that* X *is incomparable in* A. *Then* X *is maximal incomparable iff* $\forall a \in A \backslash X \exists x \in X (a \leq x$ *or* $x \leq a)$. □

Proposition 17.8. *Suppose that* X *is maximal incomparable in* A. *Then*

(i) $\sum X = 1$.
(ii) $\prod X = 0$.
(iii) $\{a \in A : -a \in X\}$ *is maximal incomparable.*

Proof. (i): if $\sum X \neq 1$, then there is an $a \in A$ such that $a \cdot x = 0$ for all $x \in X$, and $a \neq 0$. Then $X \cup \{a\}$ is still incomparable, contradiction.

(iii): clear; and (ii) follows from (i) and (iii). □

The equivalent version of Inc given in Theorem 17.1 gives rise to a another "small" function. If A is a BA and T_1, T_2 are trees contained in A (i.e., trees in the induced ordering from A), then we say that T_2 *extends* T_1 iff $T_1 \subseteq T_2$ and for every $x \in T_2$ and $y \in T_1$, if $x \leq y$ then $x \in T_1$; so T_2 is an end-extension of T_1, but there may be new roots and elements above them. We can apply Zorn's lemma to get maximal trees in A under this partial ordering. We define

$$\begin{aligned}\mathrm{Inc}^{\mathrm{tree}}_{\mathrm{spect}}(A) = \{|T| : {} & T \text{ is an infinite maximal tree} \\ & \text{contained in } A \text{ and } 0 \in T, 1 \notin T\}; \\ \mathrm{Inc}^{\mathrm{tree}}_{\mathrm{mm}}(A) = {} & \min(\mathrm{Inc}^{\mathrm{tree}}_{\mathrm{spect}}(A)).\end{aligned}$$

We require that $1 \notin T$ in order to make the notion nontrivial. Since we take end extensions, if we allowed 1 as an element we could take a well-ordered chain of any length that is present in the algebra and adjoin 1 to it to obtain a maximal tree.

Proposition 17.9. $\{\kappa, 2^\kappa\} \subseteq \mathrm{Inc}^{\mathrm{tree}}_{\mathrm{spect}}(\mathscr{P}(\kappa))$ *for any infinite cardinal* κ.

Proof. $\{\emptyset\} \cup \{\{\alpha\} : \alpha < \kappa\}$ is a maximal tree. If $X \subseteq \mathscr{P}(\kappa)$ is independent with $|X| = 2^\kappa$, then we can extend X to a maximal tree. □

Problem 156. *Is* $\mathrm{Inc}^{\mathrm{tree}}_{\mathrm{spect}}(\mathscr{P}(\kappa)) = \{\kappa, 2^\kappa\}$*?*

Proposition 17.10. $\mathrm{Inc}^{\mathrm{tree}}_{\mathrm{spect}}(\mathrm{Fr}(\kappa)) = \{\kappa\}$ *for any infinite cardinal* κ.

Proof. Suppose that T is a tree in $\mathrm{Fr}(\kappa)$ and $|T| < \kappa$; we want to show that T is not maximal. Let $\langle x_\alpha : \alpha < \kappa\rangle$ be a system of free generators for $\mathrm{Fr}(\kappa)$. Choose α such that x_α is not in the support of any element of T. Clearly $T \cup \{x_\alpha\}$ is a tree extending T. □

Proposition 17.11. *Let* T *be a tree in* A*. Then the following are equivalent:*

(i) T *is maximal.*

(ii) *For every* $x \in A\backslash T$ *one of the following conditions holds:*

  (a) *There are incomparable* $u, v \in T$ *such that* $u + v \leq x$*.*

  (b) *There is a* $u \in T$ *such that* $x < u$*.*

Proof. (i)⇒(ii): Assume (i), and suppose that $x \in A\backslash T$. Then either $x = 1$, hence (a) holds, $T \cup \{x\}$ is not a tree, or it is a tree but is not an end extension of T. If $T \cup \{x\}$ is not a tree, then there exist $u, v \in T$ such that either u and v are incomparable and both are less than x, giving (a), or v and x are incomparable and both are less than u, and this gives (b). If $T \cup \{x\}$ is a tree but not an end extension of T, this gives (b).

(ii)→(i): obvious. □

Some more remarks:

1. For $A = \mathrm{Fr}(\kappa)$ with $\kappa > \omega$ we have: $\mathrm{Inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) = \kappa$, $\mathfrak{a}(A) = \omega$, $\mathrm{Length}_{\mathrm{mm}}(A) = \omega$, and $\mathfrak{s}(A) = \omega$.
2. For $A = \mathscr{P}(\omega)$ we have: $\mathrm{Inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) = \omega$ and $\mathfrak{i}(A) > \omega$.

We finish our discussion of small incomparability with some facts about $\mathscr{P}(\omega)/\mathrm{fin}$.

Theorem 17.12. *If T is a nonempty countable collection of infinite, co-infinite subsets of ω, then there is an infinite subset X of ω such that $X\backslash Y$ and $Y\backslash X$ are infinite for all $Y \in T$.*

Proof. Let $T = \{Y_i : i < \omega\}$ possibly with repetitions (necessary if T is finite). We define increasing finite subsets $F_0, F_1, \ldots$ and $G_0, G_1, \ldots$ of ω by recursion. Suppose that F_j and G_j have been defined for all $j < i$. Let $H_i = \bigcup_{j<i} F_j$ and $K_i = \bigcup_{j<i} G_j$. Now for all $j < i$ choose

$$\begin{aligned} m_{ij} &\in \omega\backslash(H_i \cup K_i \cup Y_j \cup \{n_{ik} : k < j\}) \text{ and} \\ n_{ij} &\in Y_j\backslash(H_i \cup K_i \cup \{m_{ik} : k \le i\}). \end{aligned}$$

Let $F_i = H_i \cup \{m_{ij} : j < i\}$ and $G_i = K_i \cup \{n_{ij} : j < i\}$.

(1) $F_i \cap G_i = \emptyset$ for all i.

We prove this by induction on i. Assume that it is true for all $j < i$. Then $H_i \cap K_i = \emptyset$. We have $m_{ij} \notin K_i$ for all $j < i$, so $F_i \cap K_i = \emptyset$. Also, $n_{ij} \notin H_i$ for all $j < i$, so $G_i \cap H_i = \emptyset$. And $n_{ij} \notin \{m_{ik} : k < i\}$, so $F_i \cap G_i = \emptyset$.

Now let $X = \bigcup_{i<\omega} F_i$ and $Z = \bigcup_{i<\omega} G_i$.

(2) $X\backslash Y_j$ is infinite for all $j < \omega$.

For, if $i > j$, then $m_{ij} \in X\backslash Y_j$, and if $i > k > j$, then $m_{kj} \in F_k \subseteq H_i$, and so $m_{ij} \neq m_{kj}$. So (2) holds.

(3) $Y_j\backslash X$ is infinite for all $j < \omega$.

For, if $i > j$, then $n_{ij} \in Z \cap Y_j$, and if $i > k > j$, then $n_{kj} \in G_k \subseteq K_i$, and so $n_{ij} \neq n_{kj}$. Hence $Y_j \cap Z$ is infinite. By (1), (3) follows. □

Corollary 17.13. $\omega < \mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}), \mathrm{Inc}_{\mathrm{mm}}^{\mathrm{tree}}(\mathscr{P}(\omega)/\mathrm{fin})$. □

Theorem 17.14. (MA) *If T is a collection of infinite, co-infinite subsets of ω and $|T| < 2^\omega$, then there is an infinite subset X of ω such that $X\backslash Y$ and $Y\backslash X$ are infinite for all $Y \in T$.*

Proof. Again we apply MA to the partially ordered set $\mathrm{Fn}(\omega, 2)$. For each $a \in T$ and $i \in \omega$ let

$$D_{ai} = \{f \in \mathrm{Fn}(\omega, 2) : \exists j > i[j \in \mathrm{dmn}(f) \cap a \text{ and } f(j) = 0]\}.$$

Then each set D_{ai} is dense. For, suppose that $g \in \mathrm{Fn}(\omega, 2)$. Choose $j > i$ with $j \in a$ and $j \notin \mathrm{dmn}(g)$, and let $f = g \cup \{(j, 0)\}$. Clearly $g \subseteq f \in D_{ai}$.

For each $a \in T$ and $i \in \omega$ let

$$E_{ai} = \{f \in \mathrm{Fn}(\omega, 2) : \exists j > i[j \in \mathrm{dmn}(f)\backslash a \text{ and } f(j) = 1]\}.$$

Then each set E_{ai} is dense, with proof as above.

Now if G is a generic filter intersecting all of these dense sets, then $X \stackrel{\text{def}}{=} \{j \in \omega : j \in \mathrm{dmn}(\bigcup G) \text{ and } (\bigcup G)(j) = 1\}$ is the set desired in the theorem. $\square$

Corollary 17.15. (MA) $\mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) = \mathrm{Inc}^{\mathrm{tree}}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) = 2^\omega$. $\square$

Problem 157. *Is it consistent that* $\mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) < 2^\omega$ *or* $\mathrm{Inc}^{\mathrm{tree}}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) < 2^\omega$*?*

Problem 158. *Is it consistent that* $\mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) \neq \mathrm{Inc}^{\mathrm{tree}}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin})$*?*

Now we turn to connections with our other functions. From the Handbook Theorem 4.25 it follows that $\pi(A) \leq \mathrm{Inc}(A)$ for any infinite BA A; hence $\mathrm{hd}(A) \leq \mathrm{Inc}(A)$, by an easy argument. An example in which they are different is the interval algebra A on the reals. In fact, $\mathrm{hd}(A) = \omega$ by Theorem 16.1, and an incomparable set of size 2^ω is provided by

$$\{[0, r) \cup [1 + r, 2) : r \in (0, 1)\}.$$

Much effort has been put into constructing BAs A in which $\mathrm{Inc}(A) < |A|$. An example with $|A|$ a successor cardinal is found in Shelah [97a]. An example of such an algebra is an algebra of Bonnet, Shelah [85]; their algebra is an interval algebra, and has power $\mathrm{cf}(2^\omega)$, so that if $\mathrm{cf}(2^\omega)$ is not limit one gets such an algebra. In any case, every incomparable set in their algebra has size less than $\mathrm{cf}(2^\omega)$. We give this construction here.

Throughout the next results, let $\lambda = 2^\omega$ and $\kappa = \mathrm{cf}(\lambda)$.

A subset A of ${}^nS$ is *good* iff each member of A is one-one.

If $(L, \leq)$ is a linear order, $n \geq 2$, and $\varepsilon \in {}^n2$, then we define $x \leq_\varepsilon y$ iff $x, y \in {}^nL$ and for all $k < n$, $\varepsilon(k) = 1 \rightarrow x_k \leq y_k$, while $\varepsilon(k) = 0 \rightarrow y_k \leq x_k$. Suppose that $|L| = \lambda$. We say that L is *hyper-rigid* iff $\forall n \geq 2 \forall \varepsilon \in {}^n2$[every good antichain of $({}^nL, \leq_\varepsilon)$ has size less than λ].

We say that a subset P of $\mathbb{R}$ is κ*-dense* iff every nonempty open interval in $\mathbb{R}$ contains κ elements of P.

Lemma 17.16. *If P is a κ-dense hyper-rigid subset of $\mathbb{R}$, then* $\mathrm{Inc}(\mathrm{Intalg}(P)) < \kappa$.

Proof. Suppose that A is an antichain in $\mathrm{Intalg}(P)$ of size κ. For each $a \in A$ write

$$a = [x^a_0, x^a_1) \cup \cdots \cup [x^a_{2m(a)}, x^a_{2m(a)+1})$$

with $-\infty \leq x^a_0 < x^a_1 < \cdots < x^a_{2m(a)} < x^a_{2m(a)+1} \leq \infty$. Then there exist a subset A' of A of size κ and an integer n such that $m(a) = n$ for all $a \in A'$. Let B be the set of all sequences $\langle(r_i, r'_i) : i \le 2n\rangle$ of pairs of rational numbers such that $r_0 < r'_0 < \cdots < r_{2n} < r'_{2n}$. Then

$$\begin{aligned} A' = \bigcup\{\{&a \in A' : x_0^a < r_0 < r'_0 < x_1^a < \cdots < x_{2n}^a < r_{2n} < r'_{2n} < x_{2n+1}^a\} \\ &: \langle(r_i, r'_i) : i \le 2n\rangle \in B\}. \end{aligned}$$

It follows that there exist a $A'' \in [A]^\kappa$ and a $\langle(r_i, r'_i) : i \le 2n\rangle \in B$ such that for all $a \in A''$,

$$x_0^a < r_0 < r'_0 < x_1^a < \cdots < x_{2n}^a < r_{2n} < r'_{2n} < x_{2n+1}^a.$$

Now take any $k < 2n+2$. We define $a \equiv a'$ iff $a, a' \in A''$ and $x_k^a = x_k^{a'}$. This gives an equivalence relation, and hence there is a $A''' \in [A'']^\kappa$ such that one of the following conditions holds:

(1) $x_k^a = x_k^{a'}$ for all $a, a' \in A'''$.

(2) $x_k^a \ne x_k^{a'}$ for all distinct $a, a' \in A'''$.

Repeating this argument finitely many times, we arrive at a set $A_0 \in [A''']^\kappa$ such that one of the following conditions hold for all $k \le 2n+1$:

(3) $x_k^a = x_k^{a'}$ for all $a, a' \in A_0$.

(4) $x_k^a \ne x_k^{a'}$ for all distinct $a, a' \in A_0$.

Let $R = \{k \le 2n+1 : \forall a, a' \in A_0[a \ne a' \to x_k^a \ne x_k^{a'}]\}$. Clearly $R \ne \emptyset$. Write $R = \{k_0, \ldots, k_{p-1}\}$ with $k_0 < \cdots < k_{p-1}$. Define $\varepsilon \in {}^p2$ by setting

$$\varepsilon(i) = \begin{cases} 1 & \text{if } k_i \text{ is odd,} \\ 0 & \text{if } k_i \text{ is even.} \end{cases}$$

Now define $c^a \in {}^pP$ for any $a \in \mathrm{Intalg}(P)$ by setting

$$c^a(i) = x_{k_i}^a.$$

Then we claim

(5) For any $a, a' \in A_0$, $a \le a'$ iff $c^a \le_\varepsilon c^{a'}$.

In fact, suppose that $a, a' \in A_0$. Then

$$\begin{aligned} a \le a' \quad \text{iff} \quad & \forall i \le n[[i \text{ even } \to x_i^{a'} \le x_i^a] \\ & \quad \text{and } [i \text{ odd } \to x_i^a \le x_i^{a'}]] \\ \text{iff} \quad & c^a \le_\varepsilon c^{a'}. \end{aligned}$$

Since A_0 is an antichain, it follows that $\{c^a : a \in A_0\}$ is a good antichain of $({}^pP, \le_\varepsilon)$. Since it is of size κ, this is a contradiction. $\square$

Let $P \subseteq \mathbb{R}$, $n \geq 2$, and $\varepsilon \in {}^nP$.

A subset A of ${}^nP$ is *separate* iff the following two conditions hold:

(1) There is a sequence $\langle r_i : i < 2n\rangle$ of rational numbers such that for all $a \in A$,

$$r_0 < a_0 < r_1 < r_2 < a_1 < \cdots < a_{n-2} < r_{2n-3} < r_{2n-2} < a_{n-1} < r_{2n-1}.$$

(2) If $a, b \in A$ are distinct and $k < n$, then $a_k \neq b_k$.

Proposition 17.17. *Suppose that A is separate, with notation as above. For each $k < n$ let $I_k = (r_{2k}, r_{2k+1})$. Then the I_k's are pairwise disjoint, and $a_k \in I_k$ for all $k < n$.* $\square$

Suppose that A is separate, with notation as above, and suppose that $0 \leq l < n$. Then we define

$$\begin{aligned}
a[l] &= \langle a_0, a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_n\rangle \quad \text{for any } a \in A;\\
A[l] &= \{a[l] : a \in A\};\\
A_l &= \{a_l : a \in A\};\\
\psi_l(a) &= a[l];\\
\pi_l(a[l]) &= a_l.
\end{aligned}$$

Note here that ψ_l is one-one and $\pi_l(a[l]) = \pi_l(\psi_l^{-1}(a[l]))$.

Now we define $a \leq^l_\varepsilon b$ iff $a, b \in {}^{n-1}\mathbb{R}$ and the following two conditions hold:

(1) If $k < l$ and $\varepsilon(k) = 1$, then $a_k \leq b_k$;
If $k < l$ and $\varepsilon(k) = 0$, then $b_k \leq a_k$.

(2) If $l \leq k < n-1$ and $\varepsilon(k+1) = 1$, then $a_k \leq b_k$;
If $l \leq k < n-1$ and $\varepsilon(k+1) = 0$, then $b_k \leq a_k$.

Also we define $u \leq_1 v$ iff $u, v \in \mathbb{R}$ and $u \leq v$, and $u \leq_0 v$ iff $u, v \in \mathbb{R}$ and $v \leq u$.

Proposition 17.18. *For $a, b \in A$ we have*

$$a \leq_\varepsilon b \quad \text{iff} \quad a[l] \leq^l_\varepsilon b[l] \text{ and } a_l \leq_{\varepsilon(l)} b_l.$$

Proof.

$$\begin{aligned}
&a[l] \leq^l_\varepsilon b[l] \text{ and } a_l \leq_{\varepsilon(l)} b_l \quad \text{iff}\\
&\qquad \forall k < l[[\varepsilon(k) = 1 \to a_k \leq b_k] \text{ and } [\varepsilon(k) = 0 \to b_k \leq a_k]]\\
&\qquad \text{and } \forall k[l \leq k < n-1 \text{ and } \varepsilon(k+1) = 1 \to (a[l])_k \leq (b[l])_k]\\
&\qquad \text{and } \forall k[l \leq k < n-1 \text{ and } \varepsilon(k+1) = 0 \to (b[l])_k \leq (a[l])_k]\\
&\qquad \text{and } a_l \leq_{\varepsilon(l)} b_l
\end{aligned}$$

iff $\forall k < l[[\varepsilon(k) = 1 \to a_k \le b_k]$ and $[\varepsilon(k) = 0 \to b_k \le a_k]]$
and $\forall k[l \le k < n-1$ and $\varepsilon(k+1) = 1 \to a_{k+1} \le b_{k+1}]$
and $\forall k[l \le k < n-1$ and $\varepsilon(k+1) = 0 \to b_{k+1} \le a_{k+1}]$
and $a_l \le_{\varepsilon(l)} b_l$
iff $a \le_\varepsilon b$. $\square$

Proposition 17.19. *Let A be a separate subset of ${}^n\mathbb{R}$. Then the following are equivalent:*

(i) *A is an antichain of $({}^n\mathbb{R}, \le_\varepsilon)$.*
(ii) *There is an $l < n$ such that π_l is a decreasing function from $(A[l], \le^l_\varepsilon)$ to $(A_l, \le_{\varepsilon(l)})$.*
(iii) *For every $l < n$, π_l is a decreasing function from $(A[l], \le^l_\varepsilon)$ to $(A_l, \le_{\varepsilon(l)})$.*

Proof. (i)⇒(iii): Assume (i), and suppose that $l < n$ and $a[l] \le^l_\varepsilon b[l]$. Then $a \not\le_\varepsilon b$, and so by Proposition 17.18, $a_l \not\le_{\varepsilon(l)} b_l$. Hence $b_l <_{\varepsilon(l)} a_l$.

(iii)⇒(ii): obvious.

(ii)⇒(i): Assume that (i) is false. Say $a \ne b$ and $a \le_\varepsilon b$. Then by Proposition 17.18, $a_l \le_{\varepsilon(l)} b_l$. $\square$

A subset A of ${}^n\mathbb{R}$ is a *nice antichain* iff the following conditions hold:

(1) A is good and separate.

(2) A is an antichain of size κ.

(3) For every $l < n$, the set $A[l]$ does not have an antichain of size κ in $({}^{n-1}\mathbb{R}, \le^l_\varepsilon)$.

Proposition 17.20. *Let P be a κ-dense subset of $\mathbb{R}$. Then the following conditions are equivalent:*

(i) *P is hyper-rigid.*
(ii) *For each $n \ge 2$ and $\varepsilon \in {}^n2$ there is no nice antichain in $({}^nP, \le_\varepsilon)$.*

Proof. Obviously (i) implies (ii). Now assume that P is not hyper-rigid. Let $n \ge 2$ be smallest such that for some $\varepsilon \in {}^n2$ the set $({}^nP, \le_\varepsilon)$ has a good antichain A of size κ.

Since A is good, every member of A is one-one, and so for every $a \in A$ there is a permutation σ_a of n such that $a_{\sigma_a(0)} < \cdots < a_{\sigma_a(n-1)}$. Hence there exist an $A' \in [A]^\kappa$ and a permutation τ of n such that $\forall a \in A'[\sigma_a = \tau]$. let $A'' = \{a \circ \tau : a \in A'$.

Now for each $a \in A''$ choose a strictly increasing sequence $\langle r^a_i : i < 2n\rangle$ of rational numbers such that $r^a_{2k} < a_k < r^a_{2k+1}$ for all $k < n$. Then there exist a single strictly increasing sequence $\langle s_i : i < 2n\rangle$ of rational numbers and an $A'' \in [A]]^\kappa$ such that $s_{2k} < a_k < s_{2k+1}$ for all $k < n$.

Now we claim

(1) $S_0 \stackrel{\text{def}}{=} \{a_0 : a \in A''\}$ has size κ.

In fact, suppose not. Then there exist $A''' \in [A'']^\kappa$ and t such that $a_0 = t$ for all $a \in A'''$. By Proposition 17.20, $(A''', \leq_\varepsilon)$ is isomorphic to $(A'''[0], \leq^0_\varepsilon)$. Clearly $A'''[0]$ is a good antichain in ${}^{n-1}P$, contradicting the minimality of n.

By (1), there is a subset A''' of A'' of size κ such that for any two distinct $a, b \in A'''$ we have $a_0 \neq b_0$.

Now we apply this argument to $1, \ldots, n-1$; we end up with a set $A_0 \in [A'']^\kappa$ such that for all distinct $a, b \in A_0$ and all $k < n$, $a_k \neq b_k$. Thus A_0 is nice. $\square$

Theorem 17.21. (Corominas, Shelah) *Suppose that λ is an infinite cardinal, and $(E, \leq)$ is a poset of size λ. Suppose that D is a subset of E of size $< \text{cf}(\lambda)$ such that the following conditions hold:*

(i) *If $x < y$ in E, then there is a $d \in D$ such that $x \leq d \leq y$.*
(ii) *For any $x \in E$ there are $d_1, d_2 \in D$ such that $d_1 \leq x \leq d_2$.*

Then every subset F of E of size λ either contains an antichain of size λ or a chain isomorphic to $\mathbb{Q}$.

Proof. Assume the hypothesis, with F a subset of E of size λ. Let G be a subset of F of size λ, and let $G_* = G \cup D$. For each $x \in G_*$ we define $\varepsilon_x = (\varepsilon_x^-, \varepsilon_x^+) \in 2 \times 2$ as follows.

Let $G_x^- = \{t \in G : t \leq x\}$ and $G_x^+ = \{t \in G : x \leq t\}$. Then we set

$$\begin{aligned}
\varepsilon_x^- &= \begin{cases} 0 & \text{if } |G_x^-| < \lambda, \\ 1 & \text{if } |G_x^-| = \lambda; \end{cases} \\
\varepsilon_x^+ &= \begin{cases} 0 & \text{if } |G_x^+| < \lambda, \\ 1 & \text{if } |G_x^+| = \lambda; \end{cases} \\
G_*^- &= \{x \in G_* : \exists d \in D[x \leq d \text{ and } \varepsilon_d^- = 0]\}; \\
G_*^+ &= \{x \in G_* : \exists d \in D[d \leq x \text{ and } \varepsilon_d^+ = 0]\};
\end{aligned}$$

Since $|D| < \text{cf}(\lambda)$ and $|G_d^-| < \lambda$ for each $d \in D$, it follows that $|G_*^-| < \lambda$. Similarly, $|G_*^+| < \lambda$. Hence $N(G) \stackrel{\text{def}}{=} G \backslash (G_*^- \cup G_*^+)$ has size λ.

(1) $\varepsilon_x^-(G) = \varepsilon_x^-(N(G))$.

In fact,

$$\{t \in G : t \leq x\} = \{t \in G_*^- : t \leq x\} \cup \{t \in G_*^+ : t \leq x\} \cup \{t \in N(G) : t \leq x\}.$$

Now $|G_*^-| < \lambda$ and $|G_*^+| < \lambda$; so $|\{t \in G : t \leq x\}| = \lambda$ iff $|\{t \in N(G) : t \leq x\}| = \lambda$. So (1) holds.

Similarly,

(2) $\varepsilon_x^+(G) = \varepsilon_x^+(N(G))$.

Case 1. For every subset G of F of size λ there is an $x \in G$ such that $\varepsilon_x = (1,1)$. Choose $a(1/2) \in F$ such that $\varepsilon_{a(1/2)} = (1,1)$. Then the sets $F_{1/4} \stackrel{\text{def}}{=} \{x \in F : x \le a_{1/2}\}$ and $F_{3/4} \stackrel{\text{def}}{=} \{x \in F : a_{1/2} \le x\}$ have size λ. Choose $a(1/4) \in F_{1/4}$ and $a(3/4) \in F_{3/4}$ so that $\varepsilon_{a(1/4)} = \varepsilon_{a(3/4)} = (1,1)$. Then the following sets have size λ:

$$\begin{aligned}
&F_{1/8} \stackrel{\text{def}}{=} \{x \in F_{1/4} : x \le a(1/4)\},\\
&F_{3/8} \stackrel{\text{def}}{=} \{x \in F_{1/4} : a(1/4) \le x\} = \{x \in F_{1/4} : a(1/4) \le x \le a(1/2)\},\\
&F_{5/8} \stackrel{\text{def}}{=} \{x \in F_{3/4} : x \le a(3/4)\} = \{x \in F_{3/4} : a(1/2) \le x \le a(3/4)\},\\
&F_{7/8} \stackrel{\text{def}}{=} \{x \in F_{3/4} : a(3/4) \le x\}.
\end{aligned}$$

Now suppose that $k \ge 3$ and we have defined $a_{(2s-1)/2^{k-1}}$ for all s such that $1 \le s \le 2^{k-2}$ so that the following sets have size λ:

$$\begin{aligned}
F_{1/2^k} &\subseteq \{x \in F : x \le a(1/2^{k-1})\};\\
F_{(2s-1)/2^k} &\subseteq \{x \in F : a((s-1)/2^{k-1}) \le x \le a(s/2^{k-1})\} \quad \text{for } 1 < 2s-1 < 2^k - 1\\
F_{(2^k-1)/2^k} &\subseteq \{x \in F : a((2^{k-1}-1)/2^{k-1}) \le x\}.
\end{aligned}$$

Note that these conditions hold for $k = 3$. We now choose $a((2s-1)/2^k)$ for all s such that $1 \le s \le 2^{k-1}$ so that $\varepsilon_{a((2s-1)/2^k)}(F_{(2s-1)/2^k}) = (1,1)$. It follows that the following sets have size λ:

$$\begin{aligned}
F_{1/2^{k+1}} &= \{x \in F_{1/2^k} : x \le a(1/2^k)\};\\
F_{3/2^{k+1}} &= \{x \in F_{1/2^k} : a(1/2^k) \le x\};\\
F_{(2s-1)/2^{k+1}} &= \{x \in F_{(2t-1)/2^k} : x \le a((2t-1)/2^k)\}\\
&\qquad\qquad \text{if } s = 2t-1 \text{ and } 5 \le 2s-1 < 2^{k+1}-2;\\
F_{(2s-1)/2^{k+1}} &= \{x \in F_{(2t-1)/2^k} : a((2t-1)/2^k) \le x\}\\
&\qquad\qquad \text{if } s = 2t \text{ and } 5 \le 2s-1 < 2^{k+1}-2;\\
F_{(2^{k+1}-2)/2^{k+1}} &= \{x \in F_{(2^k-1)/2^k} : a((2^k-1)/2^k) \le x\};\\
F_{(2^{k+1}-1)/2^{k+1}} &= \{x \in F_{(2^k-1)/2^k} : x \le a((2^k-1)/2^k)\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
F_{1/2^{k+1}} &\subseteq \{x \in F : x \le a(1/2^k)\};\\
F_{(2s-1)/2^{k+1}} &\subseteq \{x \in F : a((s-1)/2^k) \le x \le a(s/2^k)\} \quad \text{for } 1 < 2s-1 < 2^{k+1}-1\\
F_{(2^{k+1}-1)/2^{k+1}} &\subseteq \{x \in F : a((2^k-1)/2^k) \le x\}.
\end{aligned}$$

This finishes the inductive construction, and it shows that $\mathbb{Q}$ is embedded in F via the mapping $s/2^k \mapsto a(s/2^k)$ for $s < 2^k$.

Case 2. There is a subset G of F of size λ such that $\varepsilon_x \neq (1,1)$ for every $x \in G$.

(3) $\forall x, y \in N(G)[x < y \to \varepsilon_x = (0,1)$ and $\varepsilon_y = (1,0)]$.

In fact, choose $d \in D$ such that $x \le d \le y$. Now $x \notin G_*^-$, so $\varepsilon_d^- = 1$, i.e., $|G_d^-| = \lambda$. Now $G_d^- = \{t \in G : t \le d\}$, and $\{t \in G : t \le d\} \subseteq \{t \in G : t \le y\}$. Hence $|\{t \in G : t \le y\}| = \lambda$, so $\varepsilon_y^- = 1$. Similarly, $y \notin G_*^+$. so $\varepsilon_d^+ = 1$, i.e., $|G_d^+| = \lambda$. Now $G_d^+ = \{t \in G : d \le t\}$, and t $\in G : d \le t\} \subseteq \{t \in G : x \le t\}$. Hence $|\{t \in G : x \le t\}| = \lambda$, so $\varepsilon_x^+ = 1$. By the case we are in, $\varepsilon_x \neq (1,1) \neq \varepsilon_y$, so $\varepsilon_x = (0,1)$ and $\varepsilon_y = (1,0)$, as desired in (3).

(4) There do not exist $a, b, c \in N(G)$ such that $a < b < c$.

In fact, otherwise by (3) we get $\varepsilon_b = (1,0) = (0,1)$, contradiction.

Now it follows that $N(G)$ has an antichain of size λ. In fact, let X be a maximal antichain in $N(G)$. We may assume that $|X| < \lambda$. Now for each $u \in N(G)\backslash X$ there is an $x \in X$ such that u and x are comparable. Let $Y = \{u \in N(G)\backslash X : \exists x \in X[x \le u]\}$ and $Z = \{u \in N(G)\backslash X : \exists x \in X[u \le x]\}$. Then $N(G)\backslash X = Y \cup Z$, so $|Y| = \lambda$ or $|Z| = \lambda$. Now Y is an antichain. For, suppose that $u, v \in Y$ and $u < v$. Choose $x \in X$ such that $x \le u$. Then $x < u < v$, contradicting (4). Similarly Z is an antichain. □

Let $p \ge 2$, $B \subseteq {}^p\mathbb{R}$, and let f be an increasing function from $(B, \le)$ into $(\mathbb{R}, \le)$. Let $b \in {}^p\mathbb{R}$. We define

$$f(b^-) = \sup\{f(t) : t \in B \text{ and } t \le b\},$$
$$f(b^+) = \inf\{f(t) : t \in B \text{ and } b \le t\}.$$

Note that $f(b^-) = -\infty$ if there is no $t \in B$ such that $t \le b$, and $f(b^+) = \infty$ if there is no $t \in B$ such that $b \le t$.

Now let D be a countable subset of ${}^p\mathbb{R}$, and let f be an increasing function from $(D, \le)$ into $(\mathbb{R}, \le)$. Then an element $b \in {}^p\mathbb{R}$ is a *good point* for f iff either $b \in D$ or $f(b^-) = f(b^+)$. Let $D^* = G(D)$ be the set of good points for f. We define a function $D(f) = f^*$ as follows. If $b \in D$, then $f^*(b) = f(b)$. If $b \in D^*\backslash D$, then $f^*(b) = f(b^-)$ $(= f(b^+))$.

Proposition 17.22. *f^* is an increasing function.*

Proof. Suppose that $b, c \in D^*$ and $b < c$. If $b, c \in D$, then $f^*(b) = f(b) \le f(c) = f^*(c)$. If $b \in D$ and $c \in D^*\backslash D$, then $f^*(b) = f(b) \le f(c^-) = f^*(c)$. If $b \in D^*\backslash D$, then $f^*(b) = f(b^+) \le f(c) = f^*(c)$. Finally, if $b, c \in D^*\backslash D$, then $f^*(b) = f(b^-) \le f(c^-) = f^*(c)$. □

The function f^* is called the *entire extension* of f.

The same definitions work for a decreasing function f.

Proposition 17.23. *Let $n \geq 2$ and $\varepsilon \in {}^2n$. Let A be a nice antichain in $({}^n\mathbb{R}, \leq_\varepsilon)$. Let $l < n$. Let D_l be a countable topologically dense subset of $A[l]$. Let π_l be the decreasing function from $(A[l], \leq^l_\varepsilon)$ onto $(A_l, \leq_{\varepsilon(l)})$ according to Proposition* 17.19*, and let φ_l be its restriction to D_l. Let D^*_l be the domain of the entire extension φ^*_l of φ_l.*

Then $A[l]\backslash D^*_l$ has size less than $\mathrm{cf}(2^\omega)$.

Proof. Let $\kappa = \mathrm{cf}(2^\omega)$. Wlog $\varepsilon(k) = 1$ for all k. Let $S = A[l]\backslash D^*_l$; so we want to show that $|S| < \kappa$. Let $c \in A[l]$. Then $c = a[l]$ for a unique $a \in A$, and $\pi_l(c) = a_l$. Now suppose that $c \in S$. In particular, $c \notin D_l$. Now we have

$$\begin{aligned}
\varphi_l(c^-) &= \inf\{\varphi_l(v) : v \in D_l \text{ and } v < c\};\\
\varphi_l(c^+) &= \sup\{\varphi_l(v) : v \in D_l \text{ and } c < v\};\\
\pi_l(c^-) &= \inf\{\pi_l(v) : v \in A[l] \text{ and } v < c\};\\
\pi_l(c^+) &= \sup\{\pi_l(v) : v \in A[l] \text{ and } c < v\}.
\end{aligned}$$

Then

(1) $\varphi_l(c^+) < \varphi_l(c^-)$.

(2) $\varphi_l(c^-) \geq \pi_l(c^-) \geq \pi_l(c^+) \geq \varphi_l(c^+)$.

Now let $S_- = \{c \in S : \varphi_l(c^-) > \pi_l(c^-)\}$, $S_0 = \{c \in S : \pi_l(c^-) > \pi_l(c^+)\}$, $S_+ = \{c \in S : \pi_l(c^+) > \varphi_l(c^+)$. Clearly $S = S_- \cup S_0 \cup S_+$. So it suffices to show that each of S_-, S_0, S_+ has size less than $\mathrm{cf}(\lambda)$.

S_0 has size less than κ. Suppose not. For each $c \in S_0$ let $r(c)$ be a rational such that $\pi_l(c^-) > r(c) > \pi_l(c^+)$. Then there exist a rational s and a subset S' of S_0 of size κ such that $r(c) = s$ for all $c \in S'$. Then S' does not have a chain of length 3 or more. For, suppose that $c < d < e$ in S'. Then

$$s > \pi_l(c^+) \geq \pi_l(d) \geq \pi_l(e^-) > s,$$

contradiction. It follows that S' has an antichain of size κ, contradiction.

S_- has size less than κ. In fact, take any $c \in S_-$. Thus $\varphi_l(c^-) > \pi_l(c^-)$, so there is a $d_c \in A[l]$ with $d_c < c$ and $\varphi_l(c^-) > \pi_l(d_c) \geq \pi_l(c^-)$. Let $U(d_c, c) = \prod_{k<n-1}(d_k, c_k)$. Then $U(d_c, c)$ is a nonempty subset of ${}^{n-1}\mathbb{R}$ and $U(d_c, c) \cap D_l = \emptyset$. In fact, suppose that $w \in U(d_c, c) \cap D_l$. So $d_c < w < c$, hence

$$\pi_l(w) \leq \pi_l(d_c) < \varphi_l(c^-) \leq \varphi_l(w) = \pi_l(w),$$

contradiction. It follows that also $U(d_c, c) \cap A[l] = \emptyset$. Now for each $c \in S_-$ choose $r(c) \in {}^{n-1}\mathbb{Q}$ such that $d_c < r(c) < c$. Suppose that $|S_-| \geq \kappa$. Then there exist $s \in {}^{n-1}\mathbb{Q}$ and $T \subseteq S_-$ with $|T| = \kappa$ such that $r(c) = s$ for all $c \in T$. Then T is an antichain. For, suppose that $c, c' \in T$ with $c < c'$. Then $d_{c'} < r(c') = r(c) < c < c'$, hence $c \in (d_{c'}, c') \cap A[l]$, contradiction.

S_+ has size less than κ: this is similar to S_-. $\square$

Suppose that $p \geq 1$, $\eta \in {}^p2$, $\theta \in 2$, and D is a countable subset of ${}^p\mathbb{R}$. Let $\leq_1$ be the usual order on $\mathbb{R}$ and $\leq_0$ its converse. Let $M(p,\eta,\theta,D)$ be the set of all decreasing functions from $(D,\leq_\eta)$ into $(\mathbb{R},\leq_\theta)$. Let $M^*(p,\eta,\theta,D)$ be the set of all entire extensions of members of $M(p,\eta,\theta,D)$. Clearly $M(p,\eta,\theta,D)$ is of size 2^ω, and hence so is $M^*(p,\eta,\tau,D)$. Let $M^*(p,\eta,\theta) = \bigcup\{M^*(p,\eta,\theta,D) : D$ a countable subset of ${}^p\mathbb{R}\}$, $M^*(p) = \bigcup\{M^*(p,\eta,\theta) : \eta \in {}^p2, \theta \in 2\}$, and $M^* = \bigcup\{M^*(p) : p \geq 1\}$. Clearly all of these sets have size 2^ω. For each $f \in M^*$ let $n(f)$ be the unique positive integer such that $f \in M^*(n(f))$.

Lemma 17.24. *There is a κ-dense hyperrigid subset of $\mathbb{R}$.*

Proof. Let $M^* = \bigcup_{\alpha<\kappa} M^*_\alpha$ with each M^*_α of size less than 2^ω, and $\langle M^*_\alpha : \alpha < \kappa\rangle$ increasing. Let $\langle I_\alpha : \alpha < \kappa\rangle$ enumerate all intervals (r,r') with r,r' rational, each interval repeated κ times.

Now we define $\langle x_\alpha : \alpha < \kappa\rangle$ by recursion. Let x_0 be any element of $\mathbb{R}$. Suppose that x_α has been constructed for all $\alpha < \beta$. Let $P_\beta = \{x_\alpha : \alpha < \beta\}$. Let T_β be the set of all $f(a)$ for $f \in M^*_\beta$ and $a \in {}^{n(f)}P_\beta$. Clearly $|T_\beta| < 2^\omega$ and so also $|P_\beta \cup T_\beta| < 2^\omega$. Choose $x_\beta \in I_\beta \backslash (P_\beta \cup T_\beta)$.

Let $P = \{x_\alpha : \alpha < \kappa\}$. Then P is κ-dense, since each interval (r,r') is repeated κ many times. To show that P is hyperrigid, suppose not. Then by Proposition 17.22, there exist $n \geq 2$, $\varepsilon \in {}^n2$, and a nice antichain A for $({}^nP,\leq_\varepsilon)$. Let $a \in A$. Then a is one-one. For each $i < n$ choose $\alpha(i) < \kappa$ such that $a_i = x_{\alpha(i)}$. Let $q(a)$ be the k such that $\alpha(k)$ is the largest element of $\{\alpha(0),\ldots,\alpha(n-1)\}$. Let A' be a subset of A of size κ and $l < n$ such that $q(a) = l$ for all $a \in A'$. By Proposition 17.21 let π_l be the decreasing function from $(A'[l],\leq^l_\varepsilon)$ to $(A_l,\leq_{\varepsilon(l)})$. Let $\theta = \varepsilon(l)$ and let $\leq^l_\varepsilon$ be ε_η, where $\eta(k) = \varepsilon(k)$ for $k < l$ and $\eta(k) = \varepsilon(k+1)$ for $l \leq k < n-1$. Let $f = \varphi^*_l \in M^*$. Say $f \in M^*_\alpha$. Let $D = \text{dmn}(f)$. Then $A'[l]\backslash D$ has size less than κ, so $A'[l] \cap D$ has size κ. Choose a so that $a[l] \in (A'[l] \cap D)\backslash P^{n-1}_\alpha$. Say $a_i = x_{\gamma(i)}$ for all $i < n$. Then if $i < n$ and $i \neq l$ we have $\varphi(i) < \gamma(l)$. Let $\beta = \gamma(l)$. Since $a \notin P^{-1}_\alpha$, it follows that $\alpha < \gamma(k)$ for some $k < n$. So $\alpha < \beta$. Let $b = a[l]$. Then $f \in M^*_\beta$ and $b \in P^{n-1}_\beta$, so $x_\beta = f(b) \in T_\beta$, contradicting the definition of x_β. □

Theorem 17.25. *There is a subset P of $\mathbb{R}$ of size* $\text{cf}(2^\omega)$ *such that* $\text{intalg}(P)$ *does not have an incomparable subset of size* $\text{cf}(2^\omega)$. □

Another example of a BA with incomparability less than cardinality is Rubin's algebra [83] (constructed assuming $\diamondsuit$). Baumgartner [80] showed that it is consistent to have MA, $2^\omega = \omega_2$, and every uncountable BA has an uncountable incomparable subset.

We give a construction, using $\diamondsuit$, of a BA A with $\text{Inc}A < |A|$; the example is due to Baumgartner, Komjath [81], and settles another question which is of interest. It depends on some lemmas. For all the lemmas let A be a denumerable atomless subalgebra of $\mathscr{P}(\omega)$, and let I be a maximal ideal in A. We consider a partial

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ordering $P = \{(a,b) : a \in I,\ b \in A\backslash I,\ a \subseteq b\}$, ordered by: $(a,b) \preceq (c,d)$ iff $a \supseteq c$ and $b \subseteq d$.

Lemma 17.26. *The following sets are dense in P:*

(i) *For each $m \subseteq I$, the set $D_1(m) \stackrel{\text{def}}{=} \{(a,b) \in P :$ either $\forall c \in m(c \not\subseteq b)$ or $\exists c \in m(c \subseteq a)\}$.*

(ii) *For each $c \in A$, the set $D_2(c) \stackrel{\text{def}}{=} \{(a,b) \in P : \neg(a \subseteq c \subseteq b\})$.*

(iii) *For each $c \in I$, the set $D_3(c) \stackrel{\text{def}}{=} \{(a,b) \in P : c \subseteq a \cup (\omega\backslash b)\}$.*

(iv) *The set $D_4 \stackrel{\text{def}}{=} \{(a,b) \in P : a \neq 0\}$.*

Proof. Suppose $(a,b) \in P$. (i) If $\forall c \in m(c \not\subseteq b)$, then $(a,b) \in D_1(m)$, as desired. Otherwise there is a $c \in m$ such that $c \subseteq b$, and then $(a \cup c, b) \in P$, $(a \cup c, b) \preceq (a,b)$, and $(a \cup c, b) \in D_1(m)$, as desired.

(ii) If $a \subseteq c \subseteq b$, then there are two cases:

Case 1. $c \in I$. Thus $c \subset b$. Choose disjoint non-empty d, e such that $b\backslash c = d \cup e$. Since $1 \notin I$, one of d, e is in I; say $d \in I$. Then $(a \cup d, b) \in P$, $(a \cup d, b) \preceq (a,b)$, and $(a \cup d, b) \in D_2(c)$.

Case 2. $c \notin I$. We can similarly find $d \subset (c\backslash a)$ such that $d \notin I$; then $(a, a \cup d)$ is the desired element showing that $D_2(c)$ is dense.

(iii) The element $(a \cup (c \cap b), b)$ shows that $D_3(c)$ is dense.

(iv) If $a \neq 0$, we are through. Otherwise choose non-empty disjoint c, d such that $c \cup d = b$. One of c, d, say c, is in I; then (c, b) is as desired. $\square$

Lemma 17.27. *Suppose that D is dense in P. Then so are the following sets:*

(i) *$S(D) \stackrel{\text{def}}{=} \{(a,b) \in P : (\omega\backslash b, \omega\backslash a) \in D\}$.*

(ii) *For $e, f \in I$, $T(D,e,f) \stackrel{\text{def}}{=} \{(a,b) \in P : ((a\backslash e) \cup f, (b\backslash e) \cup f) \in D\}$.*

Proof. Suppose $(a,b) \in P$. (i) We can choose $(c,d) \preceq (\omega\backslash b, \omega\backslash a)$ so that $(c,d) \in D$. Thus $\omega\backslash b \subseteq c$ and $d \subseteq \omega\backslash a$, so $\omega\backslash c \subseteq b$ and $a \subseteq \omega\backslash d$, which shows that $(\omega\backslash d, \omega\backslash c)$ is a desired element.

(ii) By Lemma 17.26(iii), choose $(a',b') \preceq (a,b)$ such that $e \cup f \subseteq a' \cup (\omega\backslash b')$. Let $e' = a' \cap e$ and $f' = a' \cap f$. By density of D, choose $(x,y) \in D$ such that $(x,y) \preceq ((a'\backslash e) \cup f, (b'\backslash e) \cup f)$. Now let

$$a'' = (x\backslash(e \cup f)) \cup e' \cup f' \text{ and } b'' = (y\backslash(e \cup f)) \cup e' \cup f'.$$

It is easy to check that $(a'',b'') \in T(D,e,f)$ and $(a'',b'') \preceq (a,b)$. $\square$

Now suppose that M is a countable collection of subsets of I; then we let $\mathcal{D}(M)$ be the smallest collection of dense sets in P such that

(1) every set $D_1(m)$, $D_2(c)$, $D_3(e)$, D_4 is in $\mathcal{D}(M)$ for $m \in M$, $c \in A$, $e \in I$.
(2) if $D \in \mathcal{D}(M)$ and $e, f \in I$, then $S(D)$,$T(D,e,f) \in \mathcal{D}9M)$.

A subset $x \subseteq \omega$ is *M-generic* if for all $D \in \mathcal{D}(M)$ there is an $(a,b) \in D$ such that $a \subseteq x \subseteq b$. Note that because $D_2(c) \in \mathcal{D}(M)$ for all $c \in A$ we have $x \notin A$ in such a case.

Lemma 17.28. *For every M as above, there is a subset $x \subseteq \omega$ which is M-generic.*

Proof. Let $\langle D_0, D_1, \ldots\rangle$ enumerate all members of $\mathcal{D}(M)$. Now we define (a_0, b_0), $(a_1, b_1), \ldots$ by induction: $a_0 = 0$ and $b_0 = \omega$. Having defined (a_i, b_i), choose (a_{i+1}, b_{i+1}) so that $(a_{i+1}, b_{i+1}) \preceq (a_i, b_i)$ and $(a_{i+1}, b_{i+1}) \in D_i$. Let $x = \bigcup_{i<\omega} a_i$. Clearly x is as desired. $\square$

Lemma 17.29. *Let x be M-generic and set $B = \langle A \cup \{x\}\rangle$. Then every element of $B\backslash A$ is M-generic. B is atomless, and $B \supset A$. Moreover, for any $a \in I$ we have $x \cap a \in I$ and $a\backslash x \in I$.*

Proof. First we prove the final statement. In fact, choose $(c,d) \in D_3(a)$ so that $c \subseteq x \subseteq d$. Thus $a \subseteq c \cup (\omega\backslash d)$. It follows that $a \cap x \subseteq c \cap a \subseteq a \cap x$, as desired. And $a\backslash x \subseteq a\backslash d \subseteq a\backslash x$, as desired.

Now we claim:

(1) Every element of $B\backslash A$ has one of the two forms $(x\backslash e) \cup f$ or $((\omega\backslash x)\backslash e) \cup f$ for some $e, f \in I$.

In fact, take any element y of $B\backslash A$; we can write it in the form $y = (e \cap x) \cup (f\backslash x)$, where $e, f \in A$. By the above we cannot have $e, f \in I$. Now $-y = ((\omega\backslash e) \cap x) \cup ((\omega\backslash f)\backslash x)$, so by the above we also cannot have $-e, -f \in I$. So we have two cases:

Case 1. $e \in I$ and $f \notin I$. Then $y = ((\omega\backslash x)\backslash(\omega\backslash f)) \cup (e \cap x)$, which is in one of the desired forms.

Case 2. $e \notin I$ and $f \in I$. Then $y = (x\backslash(\omega\backslash e)) \cup (f\backslash x)$, which again is in one of the desired forms. So (1) holds.

Next

(2) If $e, f \in I$, then $(x\backslash e) \cup f$ is M-generic.

For, given $D \in \mathcal{D}(M)$, we also have $T(D, e, f) \in \mathcal{D}(M)$, and hence there is an $(a,b) \in T(D,e,f)$ such that $a \subseteq x \subseteq b$. Then $(a\backslash e) \cup f \subseteq (x\backslash e) \cup f \subseteq (b\backslash e) \cup f$, and $((a\backslash e) \cup f, (b\backslash e) \cup f) \in D$, as desired.

Finally, since $\mathcal{D}(M)$ is closed under the operation S, it follows easily that $\omega\backslash x$ is M-generic. From (1) and (2) the first conclusion of the lemma now follows.

That B is atomless follows from what has already been shown, plus the fact that $D_4 \in \mathcal{D}(M)$. And B is a proper extension of A since $D_2(c) \in \mathcal{D}(M)$ for every $c \in A$. $\square$

Lemma 17.30. *Under the hypotheses of Lemma* 17.29*,*

(i) *for all $a \in I$ and all $b \in B$, if $b \subseteq a$ then $b \in A$.*

(ii) *$I' \stackrel{\text{def}}{=} \langle I \cup \{x\}\rangle^{\mathrm{Id}}$ is a maximal ideal in B.*

Proof. (i): By Lemma 17.29, if $b \notin A$ then b is M-generic, so by the last comment of Lemma 17.29, $a \cap b \in A$; so, of course, $b \not\subseteq a$.

(ii): First suppose that I' is not proper; write $\omega = x \cup a$ with $a \in I$. Thus $\omega \backslash a \subseteq x$ so, since $a \backslash x \in A$ by Lemma 17.29, its complement is also in A, and $x = (\omega \backslash a) \cup x \in A$, contradiction. So, I' is a proper ideal.

Since $u \in I'$ or $-u \in I'$ for every $u \in A \cup \{x\}$, and $A \cup \{x\}$ generates B, it follows that I' is maximal. □

Example 17.31. (The Baumgartner–Komjath algebra.) We construct a BA A such that $\mathrm{Inc}(A) = \omega = \mathrm{Length}(A)$, while $\chi(A) = \omega_1$. $\diamondsuit$ is assumed.

Let $\langle S_\alpha : \alpha \in \omega_1 \rangle$ be a $\diamondsuit$-sequence, and let $\langle a_\alpha : \alpha \in \omega_1 \rangle$ be a one-one enumeration of $\mathcal{P}(\omega)$. For each $\beta < \omega_1$ let $m_\beta = \{a_\alpha : \alpha \in S_\beta\}$.

We define sequences $\langle A_\alpha : \alpha < \omega_1 \rangle$, $\langle I_\alpha : \alpha < \omega_1 \rangle$, $\langle M_\alpha : \alpha < \omega_1 \rangle$ by induction, as follows. Let A_0 be a denumerable atomless subalgebra of $\mathscr{P}(\omega)$, and let I_0 be a maximal ideal in A_0. If we have defined a denumerable atomless subalgebra A_α of $\mathscr{P}(\omega)$ and a maximal ideal I_α of A_α, we let $M_\alpha = \{m_\beta : \beta \le \alpha,$ and $m_\beta \subseteq I_\alpha\}$. Let x_α be M_α-generic (with respect to A_α and I_α), and let $A_{\alpha+1} = \langle A_\alpha \cup \{x_\alpha\} \rangle$ and $I_{\alpha+1} = \langle I_\alpha \cup \{x_\alpha\} \rangle^{\mathrm{Id}}$. For α a limit ordinal let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Finally, let $A = \bigcup_{\alpha<\omega_1} A_\alpha$ and $I = \bigcup_{\alpha<\omega_1} I_\alpha$.

From the above lemmas it is clear that each A_α is atomless, and hence A is atomless. Furthermore, I is a maximal ideal of A, and $|A \restriction a| = \omega$ for every $a \in I$. Moreover, $|A| = \omega_1$. The filter dual to I has character ω_1, so $\chi(A) = \omega_1$. In fact, let $F = \{a \in A : -a \in I\}$. Thus F is an ultrafilter on A. Assume that $\chi(F) = \omega$; say N is a countable generating set for F. Say $N \subseteq A_\alpha$, $\alpha < \omega_1$. Choose $b \in A_{\alpha+1} \backslash A$. Wlog $b \in F$. Then there exist $a_0, \ldots, a_{m-1} \in N$ such that $a_0 \cdot \ldots \cdot a_{m-1} \le b$. So $-b \le -a_0 + \cdots + -a_{m-1} \in A_\alpha \cap I$, so by Lemma 17.32, $-b \in A_\alpha$, contradiction.

Suppose that m is an uncountable incomparable set. Now trivially $a \subseteq b$ iff $\omega \backslash b \subseteq \omega \backslash a$, so we may assume that $m \subseteq I$. And wlog $m \cap A_0 = 0$. Let $S = \{\alpha : a_\alpha \in m\}$, and let Z be the set of all α satisfying the following two conditions:

(1) $\{a_\beta : \beta \in S \cap \alpha\} = m \cap A_\alpha$.
(2) For all $b \in A_\alpha \backslash I_\alpha$, if there is a $c \in m$ such that $c \subseteq b$, then there is a $\beta \in S \cap \alpha$ such that $a_\beta \subseteq b$.

Clearly Z is club. Hence by the $\diamondsuit$ property, choose $\alpha \in Z$ such that $S_\alpha = S \cap \alpha$.

(3) $m_\alpha \subseteq A_\alpha \cap m$.

For, let $x \in m_\alpha$. Say $x = a_\beta$ with $\beta \in S_\alpha = S \cap \alpha$. Since $\alpha \in Z$, we get $x \in m \cap A_\alpha$.

For each $c \in A \backslash A_0$ let $\rho(c)$ be the least β such that $c \in A_{\beta+1} \backslash A_\beta$. Now pick $c \in m \backslash m_\alpha$. Thus $c \notin A_0$. Write $\rho(c) = \beta$. Now $\beta \ge \alpha$: if $\beta < \alpha$, then $c \in A_\alpha \cap m$, so, since $\alpha \in Z$, we have $c = a_\gamma$ for some $\gamma \in S \cap \alpha = S_\alpha$, so $c \in m_\alpha$, contradiction.

Since c is M_β-generic (with respect to A_β and I_β) and $m_\alpha \in M_\beta$, there is a $(a, b) \in D_1 m_\alpha$ such that $a \subseteq c \subseteq b$. By the definition of $D_1 m_\alpha$ and since

c is not comparable with any element of m_α, we must have $\forall c' \in m_\alpha(c' \not\subseteq b)$. Choose b' with $b' \in A_0$ or ($b' \notin A_0$ and $\rho b'$ minimum) such that $c \subseteq b' \notin I_\alpha$ and $\forall c' \in m_\alpha(c' \not\subseteq b')$.

Now $b' \notin A_0$ and $\rho(b') \geq \alpha$: suppose not. Then $b' \in A_\alpha \backslash I_\alpha$. Now for any γ, if $\gamma \in S \cap \alpha$ then $\gamma \in S_\alpha$, $a_\gamma \in m_\alpha$, and $a_\gamma \not\subseteq b'$. This contradicts (2) for α.

Say $\rho(b') = \gamma \geq \alpha$. Now b' is M_γ-generic (with respect to A_γ and I_γ) and $m_\alpha \in M_\gamma$, so there is a $(a'', b'') \in D_1 m_\alpha$ such that $a'' \subseteq b' \subseteq b''$; note that $a'', b'' \in A_\gamma$. For any $c' \in m_\alpha$ we have $c' \not\subseteq a''$; hence by the definition of $D_1 m_\alpha$ we have $\forall c' \in m_\alpha(c' \not\subseteq b'')$. Since $c \subseteq b'' \notin I_\alpha$ and $b'' \in A_0$ or ($b'' \notin A_0$ and $\rho b'' < \gamma$), this contradicts the minimality of $\rho b'$. Thus we have shown that A has no uncountable incomparable set.

If C is an uncountable chain in A, we may assume that $C \subseteq I$. We define $\langle c_\alpha : \alpha < \omega_1 \rangle$. Suppose $c_\beta \in C$ has been constructed for all $\beta < \alpha$. Say $\{c_\beta : \beta < \alpha\} \subseteq A_\gamma$. Then $\{c : c \in C \text{ and } c \leq c_\beta \text{ for some } \beta < \alpha\} \subseteq A_\gamma$ by Lemma 17.30(i). So, we can choose $c_\alpha \in C$ such that $c_\beta < c_\alpha$ for all $\beta < \alpha$. The sequence so constructed shows that Depth$A = \omega_1$; hence Inc$A = \omega_1$, contradiction. □

Problem 159. *Can one construct in ZFC a BA A such that* $\mathrm{Inc}(A) < \chi(A)$*?*

This is Problem 61 in Monk [96]. This problem is equivalent to constructing in ZFC a BA A such that $\mathrm{Inc}(A) < \mathrm{hL}(A)$; see the argument at the end of Chapter 15.

We should mention that Shelah [80], and independently van Wesep, showed that it is consistent to have 2^ω arbitrarily large and to have a BA of size 2^ω whose length and incomparability are countable.

We conclude this chapter with some remarks about incomparability in subalgebras of interval algebras. By Theorem 15.22 of Part I of the BA handbook, if κ is uncountable and regular, and B is a subalgebra of an interval algebra and $|B| = \kappa$, then B has a chain or incomparable subset of size κ. M. Bekkali has shown that it is consistent that this no longer holds for singular cardinals.

An important combinatorial equivalent for incomparability in interval algebras has been established by Shelah. Let μ be an infinite cardinal and let L be a linear order. We say that L is *μ-entangled* if for every $n \in \omega$, every system $\langle t^i_\zeta : i < n, \zeta < \mu \rangle$ of pairwise distinct elements of L with $t^0_\zeta < t^1_\zeta < \cdots < t^{n-1}_\zeta$, and every $w \subseteq n$ there exist $\zeta < \xi < \mu$ such that $\forall i < n(i \in w \text{ iff } t^i_\zeta < t^i_\xi)$. Shelah [90] showed that the following conditions are equivalent, for μ regular and uncountable:

(1) L is μ-entangled;
(2) If $\langle a_\alpha : \alpha < \mu \rangle$ is a sequence of elements of Intalg(L) then there exist $\alpha < \beta < \mu$ such that $a_\alpha \leq a_\beta$;
(3) There is no incomparable subset of Intalg(L) with μ elements.

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18 Hereditary Cofinality

Theorem 18.1. *For any infinite BA A,* h-cof$A$ *is equal to each of:*

sup$\{|T| : T \subseteq A,\ T$ *well-founded*$\}$*;*

sup$\{\kappa :$ *there is an* $a \in {}^{\kappa}A$ *such that for all* $\alpha, \beta < \kappa$*, if* $\alpha < \beta$ *then* $a_\alpha \not\geq a_\beta\}$.

Proof. Call these three cardinals κ_0, κ_1. κ_2 respectively. Suppose that $\kappa_1 < \kappa_0$. Let X be a subset of A having no cofinal subset of power $\leq \kappa_1$. We construct elements $\langle x_\alpha : \alpha < \kappa_1^+\rangle$ by induction: if x_α has been defined for all $\alpha < \beta$, with $\beta < \kappa_1^+$, then $\{x_\alpha : \alpha < \beta\}$ is not cofinal in X, so there is an $x_\beta \in X$ such that $x_\beta \not\leq x_\alpha$ for all $\alpha < \beta$. This finishes the construction. Now $\{x_\alpha : \alpha < \kappa_1^+\}$ is not well founded, so there exist $\alpha 0, \alpha 1, \ldots, < \kappa_1^+$ such that $x_{\alpha 0} > x_{\alpha 1} > \cdots$. Choose $i < j$ such that $\alpha i < \alpha j$. Then $x_{\alpha j} < x_{\alpha i}$ is a contradiction.

Suppose $\kappa_0 < \kappa_1$. Let T be a well-founded subset of A of power κ_0^+. If T has κ_0^+ incomparable elements, this is a contradiction. So T has $\geq \kappa_0^+$ levels. Let T' consist of all elements of T of level $< \kappa_0^+$. Let $X \subseteq T'$ be a cofinal subset of T' of cardinality $\leq \kappa_0$. Then choose $a \in T'$ of level greater than the levels of all members of X; clearly this is impossible.

Thus we have shown that $\kappa_0 = \kappa_1$. Next, we show that $\kappa_2 \leq \kappa_1$. Suppose that κ and a are as in the definition of κ_2; we show that $\{a_\alpha : \alpha < \kappa\}$ is well founded. Suppose not: say $a_{\alpha_0} > a_{\alpha_1} > \cdots$. Then there exist $m < n$ such that $\alpha_m < \alpha_n$. Since $a_{\alpha_m} > a_{\alpha_n}$, this contradicts the defining property of a.

Finally, suppose that $\kappa_2 < \kappa_1$; we shall get a contradiction. Let T be well founded, with $|T| > \kappa_2$. Write $T = \{a_\alpha : \alpha < \kappa\}$, with $\kappa > \kappa_2$ and a one-one. Now because $\kappa > \kappa_2$, it follows that for each $\Gamma \in [\kappa]^\kappa$ there are $\alpha, \beta \in \Gamma$ such that $\alpha < \beta$ and $a_\alpha > a_\beta$. Let $\Delta = \{\{\alpha, \beta\} : \alpha < \beta < \kappa$ and not$(a_\alpha > a_\beta)\}$. Then from the partition relation $\kappa \to (\kappa, \omega)_2$ we obtain $\alpha_0 < \alpha_1 < \cdots$ in κ such that $a_{\alpha_0} > a_{\alpha_1} > \cdots$, contradicting T well founded. □

Concerning ultraproducts, h-cof is a sup-min function, so Theorems 6.5–6.9 hold. Moreover, h-cof$\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \text{h-cof}A_i/F\right|$ under GCH for F regular, by a proof similar to that of Theorem 4.18, and Donder's theorem says that under $V = L$ the regularity assumption can be dropped. The inequality $>$ is consistently possible by Rosłanowski, Shelah [98]. $<$ is consistently possible by Shelah [03]; see the comment at the end of that paper. This solves problem 62 of Monk [96].

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The equivalences in Theorem 18.1 give rise to two "small" versions of h-cof.

First, a *general* h-cof *sequence* in a BA A is a one-one sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A such that $\{a_\xi : \xi < \alpha\}$ is well founded under the order $<$ of the Boolean algebra, and for all $\xi, \eta < \alpha$, if $\xi < \eta$ then $rk(a_\xi) \le rk(a_\eta)$.

Proposition 18.2. *Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is a general* h-cof *sequence and $a_\xi \neq 1$ for all $\xi < \alpha$. Then $\langle a_\xi : \xi < \alpha\rangle^\frown\langle 1\rangle$ is a maximal general* h-cof *sequence.* □

Thus this notion of maximal general h-cof sequence is rather trivial. So we define an h-cof *sequence* to be a general h-cof sequence with all entries different from 1. Zorn's lemma can be applied to see that maximal h-cof sequences exist. Now we define

$$\begin{aligned}\text{h-cof}_{\text{spect}}(A) &= \{|\alpha| : A \text{ has a maximal h-cof sequence of length } \alpha\},\\ \text{h-cof}_{\text{mm}}(A) &= \min(\text{h-cof}_{\text{spect}}(A)).\end{aligned}$$

Now consider $A \stackrel{\text{def}}{=} \text{Finco}(\kappa)$ with κ an infinite cardinal. Then the following is an h-cof sequence: $a_i = \{1, \ldots, i+1\}$ for each $i < \omega$, and $a_\omega = \kappa\backslash\{0\}$. For atomless algebras we have:

Proposition 18.3. *If A is atomless and $\langle a_\xi : \xi < \alpha\rangle$ is a maximal* h-cof *sequence, then α is a limit ordinal.*

Proof. If $\alpha = \beta + 1$, let a_α be such that $a_\beta < a_\alpha < 1$. □

Proposition 18.4. *If $\langle a_\xi : \xi < \alpha\rangle$ is an* h-cof *sequence, then $\sum_{\xi<\alpha} a_\xi = 1$.*

Proof. Suppose that $\sum_{\xi<\alpha} a_\xi \neq 1$. Let $b \in A^+$ be disjoint from each a_ξ. Then $\langle a_\xi : \xi < \alpha\rangle^\frown\langle -b\rangle$ is an h-cof-sequence. □

Proposition 18.5. $\text{h-cof}_{\text{mm}}(A) \le \text{tow}(A)$ *for any infinite BA A.* □

Problem 160. *Is* $\text{h-cof}_{\text{mm}} = \mathfrak{p}$*?*

The second version of a small h-cof function is as follows. A sequence $\langle a_\xi : \xi < \alpha\rangle$ is an h-cof-*sequence of the second kind* provided that $a_\xi \neq 1$ for all $\xi < \alpha$, and for all $\xi < \eta < \alpha$ we have $a_\xi \not\geq a_\eta$. Note that the condition $a_\xi \neq 1$ is implied by the second condition except if $\alpha = \xi + 1$. So for limit α this condition can be omitted. Now we set

$$\begin{aligned}\text{h-cof}^2_{\text{spect}}(A) = \{|\alpha| : \text{ there is a maximal h-cof-sequence of}\\ \text{the second kind } \langle a_\xi : \xi < \alpha\rangle\};\\ \text{h-cof}^2_{\text{mm}}(A) = \min(\text{h-cof}^2_{\text{spect}}(A)).\end{aligned}$$

Proposition 18.6. *An* h-cof-*sequence of the second kind $\langle a_\xi : \xi < \alpha\rangle$ is maximal iff $\{-a_\xi : \xi < \alpha\}$ is dense in A.*

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Proof. $\Rightarrow$: Suppose that $\{-a_\xi : \xi < \alpha\}$ is not dense. Choose $0 \neq b \in A$ such that $-a_\xi \not\leq b$ for all $\xi < \alpha$. Then $-b \neq 1$ and $-b \not\leq a_\xi$ for all $\xi < \alpha$, and so $\langle a_\xi : \xi < \alpha\rangle^\frown\langle -b\rangle$ is an h-cof-sequence and it follows that $\langle a_\xi : \xi < \alpha\rangle$ is not maximal.

$\Leftarrow$: Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is not maximal. Say that $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is still an h-cof-sequence. Choose $\xi < \alpha$ such that $-a_\xi \leq -b$. Then $b \leq a_\xi$, contradiction. □

Corollary 18.7. $\pi(A) = \text{h-cof}^2_{\text{mm}}(A)$.

Proof. $\leq$ holds by Proposition 18.6. For the other direction, let $\langle a_\xi : \xi < \pi(A)\rangle$ enumerate a system of elements of A^+ which is dense in A. Now we define a system $\langle b_\xi : \xi < \pi(A)\rangle$ of elements of A^+. Let $b_0 = a_0$. Suppose that b_η has been defined for each $\eta < \xi$ so that

$$(*) \qquad \forall\rho < \xi\exists\eta < \xi[b_\eta \leq a_\rho].$$

Now $\{b_\eta : \eta < \xi\}$ is not dense in A, so there is a $\rho < \pi(A)$ such that $b_\eta \not\leq a_\rho$ for all $\eta < \xi$. Let b_ξ be a_ρ, ρ minimum among such ordinals. By $(*)$, $\xi \leq \rho$, so $(*)$ holds for $\xi+1$. Thus $b_\eta \not\leq b_\xi$ whenever $\eta < \xi$. So $-b_\xi \not\leq -b_\eta$ whenever $\eta < \xi$. So it suffices to show that $\{b_\xi : \xi < \pi(A)\}$ is dense in A. In fact, we claim that for every $\rho < \pi(A)$ there is an $\eta < \pi(A)$ such that $b_\eta \leq a_\rho$. This follows from $(*)$. □

Thus $\{\kappa, 2^\kappa\} \subseteq \text{h-cof}^2_{\text{spect}}(\mathscr{P}(\kappa)) \subseteq [\kappa, 2^\kappa]_{\text{card}}$.

Problem 161. *What are the possibilities for* $\text{h-cof}^2_{\text{spect}}(A)$*?*

Concerning relationships to our other functions, the main facts that $\text{Inc}(A) \leq \text{h-cof}(A) \leq |A|$ and $\text{hL}(A) \leq \text{h-cof}(A)$. To see that $\text{hL}(A) \leq \text{h-cof}(A)$, suppose that $\langle x_\alpha : \alpha < \kappa\rangle$ is a right-separated sequence of elements of A. Then it is also well founded. For, suppose that $x_{\alpha_0} > x_{\alpha_1} > \cdots$. Choose $i < j$ with $\alpha_i < \alpha_j$. Then $x_{\alpha_i} > x_{\alpha_j}$, so $x_{\alpha_j} \cdot -x_{\alpha_i}$, contradiction. It is obvious that $\text{Inc}(A) \leq \text{h-cof}(A) \leq |A|$. An example in which $\text{hL}A < \text{h-cof}A$ is provided by the interval algebra A on the reals. In fact, in Lemma 3.45 we showed that $\text{hL}(A) = \omega$, and clearly $\text{Inc}(A) = 2^\omega$, and hence $\text{h-cof}(A) = 2^\omega$. Since $\chi(A) \leq \text{hL}(A) \leq \text{h-cof}(A)$, the Baumgartner, Komjath algebra of Chapter 17 provides an example where $\text{Inc}(A) < \text{h-cof}(A)$.

Problem 162. *Can one construct in ZFC a BA A with the property that* $\text{Inc}(A) < \text{h-cof}(A)$*?*

This is problem 63 in Monk [96].

For interval algebras the equality $\text{h-cof}(A) = \text{Inc}(A)$ holds. This was proved by Shelah [90]. In fact, let $A = \text{Intalg}(I)$, and suppose that $\text{Inc}(A) < \text{h-cof}(A)$. Let $\mu = (\text{Inc}(A))^+$, and by Theorem 18.1 let T be a well-founded subset of A of power μ. Now each level of T is an incomparable set, so there are at least μ levels. Let $\langle a_\alpha : \alpha < \mu\rangle$ be a sequence of elements of T such that a_α has level α for each

$\alpha < \mu$. Then by the result at then end of Chapter 17 there exist $\alpha < \beta < \mu$ such that $-a_\alpha \leq -a_\beta$. So $a_\beta < a_\alpha$, which is impossible.

To complete the picture, it remains to provide an example in which h-cof is less than cardinality. We are going to describe a construction due to Rubin [83]. It requires $\diamondsuit$, but it will be used later too. It is relevant to many of our functions and problems.

Example 18.8. Rubin's construction is not direct, but goes by way of more general considerations. Let A be a BA. A *configuration* for A is, for some $n \in \omega$, an $(n+3)$-tuple $\langle a, c_1, c_2, b_1, \ldots, b_n\rangle$ such that $a, b_1, \ldots, b_n$ are pairwise disjoint, each $b_i \neq 0$, $c_1 \subseteq a + \sum_{i=1}^n b_i$, $a + c_1 \leq c_2$, and $(c_2 - c_1) \cdot b_i \neq 0$ for all $i = 1, \ldots, n$. (See Figure 18.9.)

Now we call a subset P of A *nowhere dense for configurations* in A, for brevity nwdc in A, if for every $n \in \omega\backslash 1$ and all disjoint $a, b_1, \ldots, b_n$ with each $b_i \neq 0$, there exist c_1, c_2 such that $\langle a, c_1, c_2, b_1, \ldots, b_n\rangle$ is a configuration and $P \cap (c_1, c_2) = 0$. Rubin's theorem that we are aiming for says that, assuming $\diamondsuit$, there is an atomless BA A of power ω_1 such that every set which is nwdc in A is countable. Before proceeding to the proof of this theorem, let us check that for such an algebra we have h-cof$(A) = \omega$. Suppose that P is an uncountable subset of A. Thus P is not nwdc, so we get $n \in \omega\backslash 1$ and disjoint $a, b_1, \ldots, b_n$ with each $b_i \neq 0$ such that

(1) For all c_1, c_2, if $\langle a, c_1, c_2, b_1, \ldots, b_n\rangle$ is a configuration then $P \cap (c_1, c_2) \neq 0$.

Let $c_0 = a + b_1 + \cdots + b_n$. Choose d_0 such that $a \leq d_0 \leq c_0$ and $d_0 \cdot b_i \neq 0 \neq -d_0 \cdot b_i$ for all i; this is possible since A is atomless. Thus $\langle a, d_0, c_0, b_1, \ldots, b_n\rangle$ is a configuration, hence choose $c_1 \in P$ with $d_0 < c_1 < c_0$. Then $c_1 \cdot b_i \geq d_0 \cdot b_i \neq 0$ for all i, and $a \leq c_1$. Choose d_1 so that $a \leq d_1 \leq c_1$ and $d_1 \cdot b_i \neq 0 \neq c_1 \cdot -d_1 \cdot b_i$ for all i. then $\langle a, d_1, c_1, b_1, \ldots, b_n\rangle$ is a configuration, so choose $c_2 \in P$ with $d_1 < c_2 < c_1$. Continuing in this fashion, we get elements $c_1, c_2, \ldots$ of P such that $c_1 > c_2 > \cdots$, which means that P is not well founded. This shows that h-cof$A = \omega$.

To do the actual construction leading to Rubin's theorem, we need another definition and two lemmas. Let A be a BA and assume that $P \subseteq B \subseteq A$. We say that *$P$ is B-nowhere dense for configurations in A*, for brevity P is B-nwdc in A, if for every $n \in \omega\backslash 1$ and all disjoint $a, b_1, \ldots, b_n \in A$ with each $b_i \neq 0$, there exist $c_1, c_2 \in B$ such that $\langle a, c_1, c_2, b_1, \ldots, b_n\rangle$ is a configuration and $P \cap (c_1, c_2) = 0$. Thus to say that P is nwdc in A is the same as saying that P is A-nwdc in A.

An important tool in the construction is the general notion of the free extension $A(x)$ of a BA A obtained by adjoining an element x (and other elements necessary when it is adjoined); this is the free product of A with a BA with four elements $0, x, -x, 1$. We need only this fact about this procedure:

Lemma 18.9. *Let A be a BA and $A(x)$ the free extension of A by an element x. Suppose that $\langle a_i : i \in I\rangle$ is a system of disjoint elements of A, $\langle b_i : i \in I\rangle$ is another system of elements of A, and $b_i \leq a_i$ for all $i \in I$. Let $I = \langle\{(a_i \cdot x)\triangle b_i : i \in I\rangle^{\mathrm{Id}}$,*

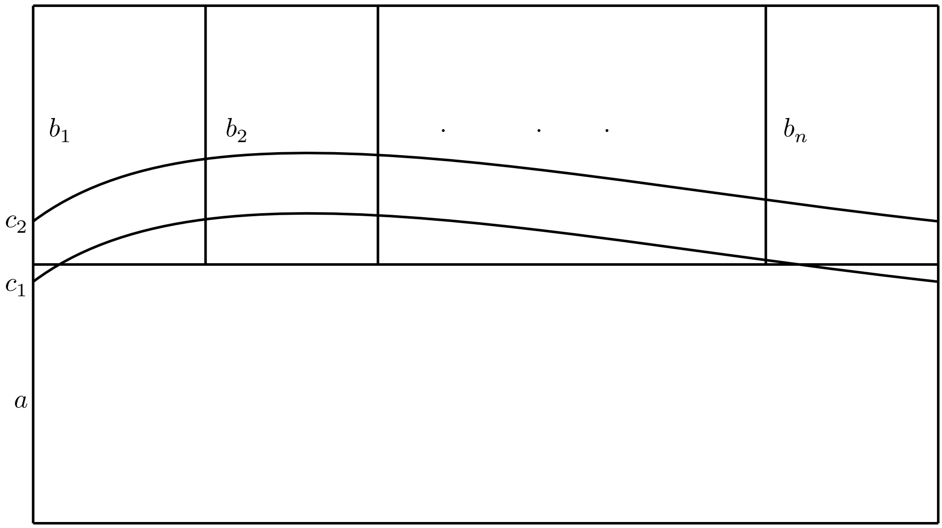

Figure 18.9

and let k be the natural homomorphism from $A(x)$ onto $A(x)/I$. Then $k \restriction A$ is one-one.

Proof. Suppose $kc = 0$, with $c \in A$. Then there exist $i(0), \ldots, i(m) \in I$ such that

$$c \leq (a_{i(0)} \cdot x) \triangle b_{i(0)} + \cdots + (a_{i(m)} \cdot x) \triangle b_{i(m)};$$

letting f be a homomorphism of $A(x)$ into A such that f is the identity on A and $fx = b_{i(0)} + \cdots + b_{i(m)}$, we infer that $c = 0$, as desired. $\square$

Note that the effect of the ideal I in Lemma 18.10 is to subject x to the condition that $x \cdot a_i = b_i$ for all $i \in \omega$. Now we prove the main lemma:

Lemma 18.10. *Let A be a denumerable atomless BA and for each $i < \omega$ we have $P_i \subseteq B_i \subseteq A$ with P_i is B_i-nwdc for A. Then there is a countable proper extension A' of A such that A is dense in A' and P_i is B_i-nwdc for A' for all $i < \omega$.*

Proof. Let $A(x)$ be a free extension of A by an element x; we shall obtain the desired algebra A' by the procedure of Lemma 18.10; thus we will let $A' = A(x)/I$, with I specified implicitly by defining a_j's and b_j's. Let $\langle s_n : n < \omega \rangle$ be an enumeration of the following set:

$$\{\langle 0, a \rangle : a \in A\} \cup \{\langle 1, a, b, c \rangle : a, b, c \text{ are disjoint elements of } A\} \cup$$
$$\{\langle 2, a, b, c, b_1, \ldots, b_k, i \rangle : a, b, c \text{ are disjoint elements of } A,\ k \in \omega \backslash 1$$
$$b_1, \ldots, b_k \text{ are disjoint non-zero elements of } A, \text{ and } i < \omega\}.$$

As we shall see, $\langle s_n : n < \omega\rangle$ is a list of things to be done in coming up with the ideal I. We will take care of the objects s_i by induction on i. Suppose that we have already taken care of s_i for $i < n$, having constructed a_j and b_j for this purpose, $j \in J$, so that J is a finite set, $b_j \le a_j$ for all $j \in J$, the a_j's are pairwise disjoint, and $\sum_{j\in J} a_j < 1$. Let $u = \sum_{j\in J} a_j$ and $v = \sum_{j\in J} b_j$. We want to take care of s_n so that these conditions (called the "list conditions") will still be satisfied. Note that under I, $x \cdot u$ will be equivalent to v, and $-x \cdot u$ will be equivalent to $u \cdot -v$. Now we consider three cases, depending upon the value of the first term of s_n.

Case 1. The first term of s_n is 0; say $s_n = \langle 0, a\rangle$, where $a \in A$. We want to add new elements a_k and b_k to our lists in order to insure that $[x] \ne [a]$ in $A(x)/I$, where in general $[z]$ denotes the equivalence class of $z \in A(x)$ with respect to I. Thus the fact that this case is taken care of for all s_n of this type in our list will insure merely that $A(x)/I$ is a proper extension of A. If $a+u \ne 1$, choose e so that $0 < e < -(a+u)$, and set $a_k = b_k = e$. Then in the end we will have $(e\cdot x)\triangle e \in I$, hence $0 < [e] \le [x]$, and $[e]\cdot[a] = 0$, so $[x] \ne [a]$. Clearly the list conditions still hold. Now suppose that $a + u = 1$. Thus $-u \le a$, and $-u \ne 0$. Choose e with $0 < e < -u$. Let $a_k = e$ and $b_k = 0$. Then in the end we will have $e \cdot x \in I$, hence $[e] \cdot [x] = 0$, and $0 < [e] \le [a]$, so $[a] \ne [x]$. And again the list conditions hold.

Case 2. The first term of s_n is 1; say $s_n = \langle 1, a, b, c\rangle$, where a, b, c are disjoint elements of A. We consider the element $t \stackrel{\text{def}}{=} a + b \cdot x + c \cdot -x$; we want to fix things so that if $[t]$ is non-zero then there will be some element $w \in A$ such that $0 < [w] \le [t]$. This will insure that A will be dense in $A(x)/I$. Now

$$t = a + b \cdot x \cdot u + b \cdot x \cdot -u + c \cdot -x \cdot u + c \cdot -x \cdot -u,$$

and under I this is equivalent to

$$a + b \cdot v + u \cdot -v \cdot c + b \cdot -u \cdot x + c \cdot -u \cdot -x.$$

Let $a' = a + b \cdot v + u \cdot -v \cdot c$, $b' = b \cdot -u$, $c' = c \cdot -u$; thus a', b', c' are disjoint. If $a' \ne 0$, we don't need to add anything to our lists. Suppose that $b' \ne 0$. Then choose e with $0 < e < b'$, and add a_k, b_k to our lists, where $a_k = b_k = e$; this assures that $[e] \le [x]$, hence $0 < [e] \le [t]$; clearly the list conditions hold. If $c' \ne 0$ a similar procedure works. Finally, if $a' = b' = c' = 0$, then $[t] = 0$, and again we do not need to add anything.

Case 3. The first term of s_n is 2; say $s_n = \langle 2, a, b, c, b_1, \ldots, b_k, i\rangle$, where a, b, c are disjoint elements of A, $k \in \omega\backslash 1$, $b_1, \ldots, b_k$ are disjoint non-zero elements of A, and $i < \omega$. Let t be as in case 2. Case 3 is the crucial case, and here we will do one of three things: (1) make t equivalent to an element of A; (2) make sure that $[t] \cdot [b_j] \ne 0$ for some $j = 1, \ldots, k$; (3) find $c_1, c_2 \in B_i$ with $P_i \cap (c_1, c_2) = 0$ so that $\langle [t], [c_1], [c_2], [b_1], \ldots, [b_k]\rangle$ is a configuration. Thus this step will assure in the end that P_i is B_i-nwdc for A'. In fact, assume that the construction is completed. To show that P_i is B_i-nwdc for A', suppose that $k \in \omega$, $a', b'_1, \ldots, b'_k \in A'$ are disjoint with each $b'_i \ne 0$. Since A is dense in A', choose $b_i \in A$ with $0 < b_i \le b'_i$ for each i.

Write $a' = a + b \cdot [x] + c \cdot [-x]$ with a, b, c pairwise disjoint elements of A. Say $\langle 2, a, b, c, b_1, \ldots, b_k \rangle = s_n$. If (1) was done, the desired conclusion follows since P_i is B_i-nwdc for A. Since a' is disjoint from each b_i, (2) could not have been done. If (3) was done, the desired conclusion is clear.

Let a', b', c' be as in Case 2. If $\sum_{j=1}^k b_i \cdot a' \neq 0$, then (2) will automatically hold, and we do not need to add anything to our lists. If there is a j, $1 \leq j \leq k$, such that $b_j \cdot b' \neq 0$, let e be such that $0 < e < b_j \cdot b'$, and adjoin a_l, b_l to our lists, where $a_l = b_l = e$; then we will have $[e] \leq [x]$, and $[b_j] \cdot [t] \neq 0$, which means that (2) holds – and the list conditions are ok. Similarly if $b_j \cdot c' \neq 0$ for some j. If $b' + c' = 0$, then $[t] = [a']$, i.e., (1) holds. Thus we are left with the essential situation: $b' + c' \neq 0$, and $(a' + b' + c') \cdot b_j = 0$ for all $j = 1, \ldots, k$. First of all we use the fact that P_i is B_i-nwdc for A, applied to $a', b' + c', b_1, \ldots, b_k$, to get $c_1, c_2 \in B_i$ such that $P_i \cap (c_1, c_2) = 0$ and $\langle a', c_1, c_2, b' + c', b_1, \ldots, b_k \rangle$ is a configuration. This time we add elements a_l, a_m, b_l, b_m to our lists, where $a_l = c_1 \cdot (b' + c')$, $b_l = c_1 \cdot b'$, $a_m = (b' + c') \cdot -c_2$, and $b_m = c' \cdot -c_2$. Clearly $a_l \cdot a_m = 0$ and both elements are disjoint from previous a_j's. Obviously $b_l \leq a_l$ and $b_m \leq a_m$. Next, since $\langle a', c_1, c_2, b' + c', b_1, \ldots, b + k \rangle$ is a configuration, $c_2 \cdot -c_1 \cdot (b' + c') \neq 0$, and since this element is disjoint from all previous a_j's as well as from a_l and a_m it follows that $u + a_l + a_m < 1$. Thus the list conditions hold. It remains only to show that in the end $\langle [t], [c_1], [c_2], [b_1], \ldots, [b_k] \rangle$ is a configuration. The only things not obvious are that $[c_1] \leq [t + \sum_{j=1}^k b_j]$ and $[t] \leq [c_2]$. Since $\langle a', c_1, c_2, b' + c', b_1, \ldots, b_k \rangle$ is a configuration, we have $c_1 \leq a' + b' + c' + b_1 + \cdots + b_k$. Hence to show that $[c_1] \leq [t + \sum_{j=1}^k b_j]$, it suffices to prove that $[c_1 \cdot (b' + c')] \leq [t]$, which is done as follows. First note that our added elements a_l, a_m, b_l, b_m assure that $[x \cdot c_1 \cdot (b' + c')] = [c_1 \cdot b']$ and $[x \cdot (b' + c') \cdot -c_2] = [c' \cdot -c_2]$, hence $[c_1 \cdot b'] \leq [x]$ and $[c' \cdot -c_2] \leq [x]$. Now,

$$\begin{aligned}
[t] &\geq [b' \cdot x + c' \cdot -x] \\
&\geq [b' \cdot c_1 \cdot x + c' \cdot -x \cdot c_1] \\
&= [c_1 \cdot b' + c' \cdot c_1 \cdot -x] \\
&= [c_1 \cdot b' + (c' \cdot c_1) \cdot -(c' \cdot c_1 \cdot x)] \\
&\geq [c_1 \cdot b' + (c' \cdot c_1) \cdot -((b' + c') \cdot c_1 \cdot x)] \\
&= [c_1 \cdot b' + (c' \cdot c_1) \cdot -(c_1 \cdot b')] \\
&= [c_1 \cdot b' + c_1 \cdot c'] \\
&= [c_1 \cdot (b' + c')].
\end{aligned}$$

To show that $[t] \leq [c_2]$, it suffices to show that $[t \cdot (b' + c')] \leq [c_2]$, and that is done like this:

$$\begin{aligned}
[t \cdot (b' + c')] &= [t \cdot (b' + c') \cdot c_2 + t \cdot (b' + c') \cdot -c_2] \leq [c_2 + t \cdot (b' + c') \cdot -c_2] \\
&= [c_2 + b' \cdot x \cdot (b' + c') \cdot -c_2 + c' \cdot -x \cdot (b' + c') \cdot -c_2] \\
&= [c_2 + b' \cdot c' \cdot -c_2 + c' \cdot -c_2 \cdot -x] = [c_2].
\end{aligned}$$

This completes the construction and the proof. $\square$

Example 18.8 (Conclusion). Recall that we are trying to construct, using $\diamondsuit$, an atomless BA A of power ω_1 such that every nwdc subset of A is countable. We shall define by induction an increasing sequence $\langle A_\alpha : \alpha < \omega_1,\ \alpha \text{ limit}\rangle$ of countable BAs, and a sequence $\langle P_\alpha : \alpha < \omega_1,\ \alpha \text{ limit}\rangle$, such that: the universe of A_α is α; A_ω is atomless and is dense in A_α for all limit $\alpha < \omega_1$; $P_\alpha \subseteq A_\alpha$ for all limit $\alpha < \omega_1$, and P_α is A_α-nwdc for A_β whenever α, β are limit ordinals $< \omega_1$ with $\alpha \le \beta$.

Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. Let A_ω be a denumerable atomless BA. If λ is a limit of limit ordinals, $\lambda < \omega_1$, let $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$. If S_λ is nwdc for A_λ, let $P_\lambda = S_\lambda$, and let $P_\lambda = 0$ otherwise. Now suppose that α is a limit ordinal $< \omega_1$, and A_β and P_β have been defined for all limit ordinals $\beta \le \alpha$. By Lemma 18.13 let $A_{\alpha+\omega}$ be a BA with universe $\alpha + \omega$ such that A_α is dense in $A_{\alpha+\omega}$ and P_β is A_β-nwdc for $A_{\alpha+\omega}$ for all limit $\beta \le \alpha$. And again choose $P_{\alpha+\omega} = S_{\alpha+\omega}$ if $S_{\alpha+\omega}$ is nwdc for $A_{\alpha+\omega}$, and let it be 0 otherwise. This completes the inductive definition. Let $A = \bigcup\{A_\alpha : \alpha \text{ limit},\ \alpha < \omega_1\}$.

Clearly A is atomless and of power ω_1. Now suppose, in order to get a contradiction, that P is an uncountable nwdc subset of A. Let

$$F = \{\alpha : \alpha < \omega_1, \alpha \text{ limit, and } (A_\alpha, P \cap \alpha) \preceq_{\text{ee}} (A, P)\}.$$

Here $\preceq_{\text{ee}}$ means "elementary substructure". Clearly F is club in ω_1. Now by the $\diamondsuit$-property, the set $S \stackrel{\text{def}}{=} \{\alpha < \omega_1 : \alpha \cap P = S_\alpha\}$ is stationary, so we can choose $\alpha \in F \cap S$. Clearly nwdc can be expressed by a set of first-order formulas; so $P \cap \alpha$ is nwdc in A_α. Since $P \cap \alpha = S_\alpha$, the construction then says that $P_\alpha = S_\alpha$. Since P is uncountable, choose $a \in P \backslash P_\alpha$, and then choose $c_1, c_2 \in A_\alpha$ so that $\langle a, c_1, c_2\rangle$ is a configuration (this means just so that $c_1 \le a \le c_2$) and $P_\alpha \cap (c_1, c_2) = 0$; this is possible, since if $a \in A_\beta$ with $\alpha \le \beta$, then P_α is A_α-nwdc for A_β by the construction. But then we have

$$(A_\alpha, P_\alpha) \models \forall x[P(x) \to x \notin (c_1, c_2)];$$
$$(A, P) \models P(a) \wedge x \in (c_1, c_2).$$

This contradicts the fact that $(A_\alpha, P_\alpha) \preceq_{\text{ee}} (A, P)$. $\square$

19 Number of Ultrafilters

This cardinal function is rather easy to describe, at least if we do not try to go into the detail that we did for cellularity, for example. If A is a subalgebra or homomorphic image of B, then $|\mathrm{Ult}(A)| \leq |\mathrm{Ult}(B)|$. For weak products we have $|\prod^{\mathrm{w}}_{i\in I} A_i| = \max(\omega, \sup_{i\in I}|\mathrm{Ult}(A_i)|)$. The situation for full products is more complicated:

$$\left|\mathrm{Ult}\left(\prod_{i\in I} A_i\right)\right| \leq 2^{2^\kappa},$$

where $\kappa = \sum_{i\in I} \mathrm{d}(A_i)$. This follows from the following two facts:

$$\prod_{i\in I} A_i \rightarrowtail \prod_{i\in I} \mathscr{P}(\mathrm{d}(A_i)) \cong \mathscr{P}\left(\dot{\bigcup}_{i\in I} \mathrm{d}(A_i)\right),$$

where "$\rightarrowtail$" means "is isomorphically embeddable in", and "$\dot{\bigcup}$" means "disjoint union". Next, clearly $|\mathrm{Ult}(\oplus_{i\in I} A_i)| = \prod_{i\in I} |\mathrm{Ult} A_i|$. We give some observations due to Douglas Peterson concerning ultraproducts and the number of ultrafilters. Clearly $\mathrm{Ult}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Ult} A_i/F\right|$. And if $\mathrm{ess.sup}^F_{i\in I}|A_i| \leq |I|$ and F is regular, then $|\mathrm{Ult}\left(\prod_{i\in I} A_i/F\right)| = 2^{2^{|I|}}$. This follows from one of the results stated for independence, for example.

Concerning relationships to our other functions, we mention only that $|A| \leq |\mathrm{Ult}(A)|$; and $2^{\mathrm{Ind}(A)} \leq |\mathrm{Ult}(A)|$ if $\mathrm{Ind}(A)$ is attained. This last assumption is needed. For example, if κ is an uncountable strong limit cardinal and A_α is the free BA of size $|\alpha + \omega|$ for each $\alpha < \kappa$, then $\prod^{\mathrm{w}}_{\alpha<\kappa} A_\alpha$ has independence κ and only κ ultrafilters. (These remarks are due to L. Heindorf, and correct a mistake in Monk [90].)

About $|\mathrm{Ult}(A)|$ for A in special classes of BAs: first recall from Theorem 17.10 of Part I of the BA handbook that $|\mathrm{Ult}(A)| = |A|$ for A superatomic. If A is not superatomic, then $|\mathrm{Ult}(A)| \geq 2^\omega$, since A has a denumerable atomless subalgebra B, and obviously $|\mathrm{Ult}(B)| = 2^\omega$.

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20 Number of Automorphisms

This cardinal function is not related very much to the preceding ones. To start with, we state some general facts about the size of automorphism groups in BAs; for proofs or references, see the chapter on automorphism groups in the BA handbook.

1. If A is denumerable, then $|\mathrm{Aut}(A)| = 2^\omega$.
2. If $0 \neq m \in \omega$ and $\kappa > \omega$, then there is a BA A with $|A| = \kappa$ such that $|\mathrm{Aut}(A)| = m!$.
3. If $|\mathrm{Aut}(A)| < \omega$, then $|\mathrm{Aut}(A)| = m!$ for some $m \in \omega$.
4. If MA and $|\mathrm{Aut}(A)| = \omega$, then $|A| \geq 2^\omega$.
5. If $2^\omega \leq \kappa$, then there is a BA A such that $|\mathrm{Aut}(A)| = \omega$ and $|A| = \kappa$.
6. If $\omega < \kappa \leq \lambda$, then there is a BA A with $|A| = \lambda$ and $|\mathrm{Aut}(A)| = \kappa$.
7. If $\omega \leq \kappa$, then there is a BA A with $|A| = \kappa$ and $|\mathrm{Aut}(A)| = 2^\kappa$.
8. Any BA can be embedded in a rigid BA.
9. Any BA can be embedded in a homogeneous BA.

Now we discuss algebraic operations on BAs vis-à-vis automorphism groups. If A is a subalgebra or homomorphic image of B, then $|\mathrm{Aut}(A)|$ can vary in either direction from $|\mathrm{Aut}(B)|$: embedding a rigid BA A into a homogeneous BA B, we get $|\mathrm{Aut}(A)| < |\mathrm{Aut}(B)|$, while if we embed a free BA A in a rigid BA B we get $|\mathrm{Aut}(A)| > |\mathrm{Aut}(B)|$; any rigid BA A is the homomorphic image of a free BA B, and then $|\mathrm{Aut}(A)| < |\mathrm{Aut}(B)|$; and finally, embed $A \stackrel{\text{def}}{=} \mathscr{P}(\omega)$ into a rigid BA B, and then extend the identity on A to a homomorphism from B onto A – this gives $|\mathrm{Aut}(A)| > |\mathrm{Aut}(B)|$.

Now we consider products. There are two fundamental, elementary facts here. First, $|A| \leq |\mathrm{Aut}(A \times A)|$ for any BA A. This is easily seen by the following chain of isomorphisms, starting from any element $a \in A$ to produce an automorphism f_a of $A \times A$:

$$\begin{aligned} A \times A &\stackrel{g}{\cong} (A \restriction a) \times (A \restriction -a) \times (A \restriction a) \times (A \restriction -a) \\ &\stackrel{h}{\cong} (A \restriction a) \times (A \restriction -a) \times (A \restriction a) \times (A \restriction -a) \quad \stackrel{g^{-1}}{\cong} A \times A, \end{aligned}$$

where g is the natural mapping and h interchanges the first and third factors, leaving the second and fourth fixed. If $a \neq b$, then $f_a \neq f_b$; in fact, say $a \not\leq b$;

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then $f_a(a,0) = (0,a)$ while $f_b(a,0) = (a \cdot -b, a \cdot b) \neq (0,a)$. This proves that $|A| \leq |\mathrm{Aut}(A \times A)|$. The second fact is that the group $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ embeds isomorphically into $\mathrm{Aut}(A \times B)$; an isomorphism F is defined like this, for any $f \in \mathrm{Aut}(A)$, $g \in \mathrm{Aut}(B)$, $a \in A$, $b \in B$: $(F(f,g))(a,b) = (fa, gb)$. Putting these two elementary facts together, we have $|A|$, $|\mathrm{Aut}(A)|$ both $\leq |\mathrm{Aut}(A \times A)|$. Shelah in an email message of December 1990 showed that actually equality holds (this solves Problem 56 of Monk [90]):

Theorem 20.1. *If A is an infinite BA, then* $|\mathrm{Aut}(A \times A)| = \max(|A|, |\mathrm{Aut}(A)|)$.

Proof. First note that $A' \stackrel{\text{def}}{=} (A \times A) \restriction (1,0)$ and $A'' \stackrel{\text{def}}{=} (A \times A) \restriction (0,1)$ are both isomorphic to A. Now for any $b \in A \times A$ let $G_b = \{g \in \mathrm{Aut}(A \times A) : g(1,0) = b\}$. Then

$(*)$ For any $b \in A \times A$, $|G_b| \leq |\mathrm{Aut}(A)|^2$.

For, take any $b \in A \times A$ and fix $f \in G_b$ (if $G_b \neq 0$). Note that for any $g \in G_b$, $f^{-1}g(1,0) = (1,0)$; so $(f^{-1} \circ g) \restriction A' \in \mathrm{Aut}(A)'$, and similarly $(f^{-1} \circ g) \restriction A'' \in \mathrm{Aut}(A)''$. Now the map

$$g \mapsto ((f^{-1} \circ g) \restriction A', (f^{-1} \circ g) \restriction A'')$$

is clearly one-one, so $(*)$ follows.

By $(*)$,

$$|\mathrm{Aut}(A \times A)| = \sum\nolimits_{b \in A \times A} |G_b| \leq |A \times A| \cdot |\mathrm{Aut} A|^2,$$

and the theorem follows. □

For weak products, we have $\sup_{i \in I} |\mathrm{Aut}(A_i)| \leq \left|\mathrm{Aut}\left(\prod^{\mathrm{w}}_{i \in I} A_i\right)\right|$ by the above remarks; a similar statement holds for full products – in fact, the full direct product of groups $\prod_{i \in I} \mathrm{Aut}(A)_i$ is isomorphically embeddable in $\mathrm{Aut}\left(\prod_{i \in I} A_i\right)$.

The situation for free products is much like that for products. By Proposition 11.11 of the BA handbook, Part I, every automorphism of A extends to one of $A \oplus B$; so $|\mathrm{Aut}(A \oplus B)| \geq \max(|\mathrm{Aut}(A)|, |\mathrm{Aut}(B)|)$. And $|A| \leq |\mathrm{Aut}(A \oplus A)|$. In fact, choose $a \in A$ with $0 < a < 1$. Then $|(A \oplus A) \restriction (a \times -a)| = |A|$, $(A \oplus A) \restriction (a \times -a) \cong (A \oplus A) \restriction (-a \times a)$, and

$$A \oplus A \cong [(A \oplus A) \restriction (a \times -a)] \times [(A \oplus A) \restriction (-a \times a)] \times [(A \oplus A) \restriction c]$$

for some c, so our statement follows from the above considerations on products. Actually, Shelah showed that there is an infinite BA A such that $|A|$ and $|\mathrm{Aut}(A)|$ are both smaller than $|\mathrm{Aut}(A \oplus A)|$ (in an email message of December 1990; S. Koppelberg supplied some details and simplifications in January 1992). This solves Problem 57 in Monk [90]. Namely, we start with an uncountable cardinal κ and a system $\langle B_\alpha : \alpha < \kappa \rangle$ of rigid BAs of size κ such that $B_\alpha \restriction b \not\cong B_\beta \restriction c$ if $\alpha < \beta < \kappa$ and $b \in B^+_\alpha$, $c \in B^+_\beta$; for the existence of such a system see Shelah [83]. Let $A = \prod^{\mathrm{w}}_{\alpha < \kappa} B_\alpha$. Then A is also rigid, as is easy to check. We claim that

$A \oplus A$ has 2^κ automorphisms. To see this we use duality. Recall that $\mathrm{Ult}(A)$ is homeomorphic to the one-point compactification of $\bigcup_{\alpha<\kappa} \mathrm{Ult}(B_\alpha)$. For each $\Gamma \subseteq \kappa$ we define $f_\Gamma : \mathrm{Ult}(A) \times \mathrm{Ult}(A) \to \mathrm{Ult}(A) \times \mathrm{Ult}(A)$ by

$$f_\Gamma(F,G) = \begin{cases} (G,F) & \text{if } F,G \in \mathrm{Ult} B_\alpha \text{ for some } \alpha \in \Gamma, \\ (F,G) & \text{otherwise.} \end{cases}$$

Obviously f_γ is one-one and onto, and it is easy to check that it is continuous. Since $f_\Gamma \neq f_\Delta$ for $\Gamma \neq \Delta$, this exhibits 2^κ autohomeomorphisms, as desired.

Concerning the relationship between ultraproducts and automorphisms, it is easy to check that the inequality $\mathrm{Aut}\left(\prod_{i\in I} A_i/F\right) \geq \left|\prod_{i\in I} \mathrm{Aut}(A)_i/F\right|$ holds. If CH holds and A is a rigid BA of power $\aleph_1$, then ${}^\omega A/F$ has at least $\aleph_1$ automorphisms; this is true because ${}^\omega A/F$ is ω_1-saturated and of power $\aleph_1$.

As mentioned at the beginning of this section, $|\mathrm{Aut}(A)|$ is not strongly related to our previous cardinal functions. An example with the property that $|\mathrm{Aut}(A)| < \mathrm{Depth}(A)$ is provided by embedding the interval algebra on κ into a rigid BA A. A similar procedure can be applied for independence and π-character, and these three examples show similar things for all of our preceding functions. And recall from the chapter on incomparability that if A is cardinality-homogeneous and has no incomparable subset of size $|A|$, then A is rigid.

Concerning automorphisms of special kinds of BAs, first note that $|\mathrm{Aut}(A)| = 2^\kappa$ for A the interval algebra on κ. In fact, every automorphism of A is induced by a permutation of κ; so we just need to describe 2^κ permutations of κ that give rise to automorphisms of A. For each $\alpha < \kappa$ we can consider the transposition $(\omega\cdot\alpha+1, \omega\cdot\alpha+2)$. For each $\varepsilon \in {}^\kappa 2$ let f_ε be the permutation of κ which, on the interval $[\omega\cdot\alpha, \omega\cdot\alpha+\omega)$, is this transposition if $\varepsilon\alpha = 1$, and is the identity there otherwise. It is easy to see that the function on A induced by f_ε maps into A, and hence is an automorphism, as desired.

If A is infinite and superatomic, then $|\mathrm{Aut}(A)| \geq 2^\omega$. In fact, we may assume that A is a subalgebra of some power-set algebra $\mathscr{P}(\kappa)$, and $\{\alpha\} \in A$ for all $\alpha < \kappa$. Let a be a representative of an atom of A at level 1. Suppose that f is a permutation of a such that f^2 is the identity. Extend f to all of κ by letting $f\alpha = \alpha$ if $\alpha \in \kappa\backslash a$. Now we claim that $x \in A$ implies that $f[x] \in A$; this will show that f induces an automorphism of A, hence proving the theorem.

Case 1. $a/I_1 \leq x/I_1$, where I_1 is the ideal of A generated by its atoms. Then $a\backslash x$ is finite. Hence

$$\begin{aligned} f[x] &= f[x\cap a] \cup f[x\backslash a] = (f[a]\backslash f[a\backslash x]) \cup f[x\backslash a] \\ &= (a\backslash f[a\backslash x]) \cup (x\backslash a) \in A, \end{aligned}$$

since $f[a\backslash x]$ is finite.

Case 2. $a/I_1 \cdot x/I_1 = 0$. Thus $a \cap x$ is finite. Hence

$$f[x] = f[a\cap x] \cup f[x\backslash a] = f[a\cap x] \cup (x\backslash a) \in A,$$

since $f[a\cap x]$ is finite.

21 Number of Endomorphisms

The main relationships of $|\text{End}(A)|$ to our previous functions are the following two easily established facts: $|\text{Ult}(A)| \leq |\text{End}(A)|$ and $|\text{Aut}(A)| \leq |\text{End}(A)|$. If A is the BA of finite and cofinite subsets of an infinite cardinal κ, then $|\text{Ult}(A)| = \kappa$ while $|\text{Aut}(A)| = |\text{End}(A)| = 2^\kappa$. For an infinite rigid BA A we have $|\text{Aut}(A)| < |\text{End}(A)|$. Furthermore, we have:

Theorem 21.1. $|\text{End}(A)| \leq |\text{Ult}(A)|^{\text{d}(A)}$ *for any infinite BA A.*

Proof. Let D be a dense subset of $\text{Ult}(A)$ of cardinality $\text{d}(A)$. Then any continuous function from $\text{Ult}(A)$ into $\text{Ult}(A)$ is determined by its restriction to D. Hence the theorem follows by duality. □

It is more interesting to construct a BA A such that $|A| = |\text{Ult}(A)| = |\text{End}(A)|$, and we will spend the rest of this chapter discussing this. An easy example of this sort is the interval algebra of the reals, and we first want to generalize the argument for this. (Here we are repeating part of Monk [89].)

Theorem 21.2. *Suppose that L is a complete dense linear ordering of power $\lambda \geq \omega$, and D is a dense subset of L of power κ, where $\lambda^\kappa = \lambda$. Let A be the interval algebra on L. Then $|A| = |\text{End}(A)| = \lambda$.*

Proof. Recalling the duality for interval algebras from Part I of the BA handbook, we see that $\text{Ult}(A)$ is a linearly ordered space of size λ with a dense subset (in the topological sense) of power κ. Now apply Theorem 21.1. □

Corollary 21.3. *If A is the interval algebra on $\mathbb{R}$, then $|A| = |\text{End}(A)| = 2^\omega$.* □

Recalling a construction of more general linear orders of the type described in Theorem 21.2 (see Monk [89]), we get

Corollary 21.4. *If μ is an infinite cardinal and $\forall \nu < \mu(\mu^\nu = \mu)$, then there is a BA A such that $|A| = |\text{End}(A)| = 2^\mu$.* □

Corollary 21.5. (GCH) *If κ is infinite and regular, then there is a BA A such that $|A| = |\text{End}(A)| = \kappa^+$.* □

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Corollary 21.6. *Let λ be strong limit, let L consist of all members of ${}^{\lambda}2$ which are not eventually 1, and let A be the interval algebra on L (which is lexicographically ordered). Then $|A| = |\mathrm{End}(A)| = 2^{\lambda}$.*

Proof. Let D consist of all members $f \in {}^{\lambda}2$ such that there is an α with $f\alpha = 0$ and $f\beta = 1$ for all $\beta > \alpha$. Then D is dense in L and Theorem 21.2 applies. $\square$

Corollary 21.6 was pointed out by Shelah (answering Problems 58 and 59 in Monk [90].) We mention one more result connecting $|A|$ and $|\mathrm{End}(A)|$:

Theorem 21.7. *If A is infinite, then $|\mathrm{End}(A)| \geq 2^{\omega}$.*

Proof. If A has an atomless subalgebra, then $|\mathrm{End}(A)| \geq |\mathrm{Ult}(A)| \geq 2^{\omega}$. So suppose that A is superatomic. Then there is a homomorphism f from A onto B, the finite-cofinite algebra on ω: if $[a]$ is an atom of $A/\langle \mathrm{At}(A)\rangle^{\mathrm{Id}}$, then f can be taken to be the composition of the natural onto mappings

$$A \to A \restriction a \to C \to B,$$

where C is a finite-cofinite algebra and B is the finite-cofinite algebra on ω. There is an isomorphism g of B into A. If X is any subset of ω with $\omega \backslash X$ infinite, then $B/\langle \{i\} : i \in X\rangle^{\mathrm{Id}}$ is isomorphic to B, and so there is an endomorphism k_X of B with kernel $\langle \{i\} : i \in X\rangle^{\mathrm{Id}}$. Clearly the endomorphisms $g \circ k_X \circ f$ of A are distinct for distinct X's. $\square$

Corollary 21.8. ($\omega_1 < 2^{\omega}$). *There is no BA A with the property that $|A| = |\mathrm{End}(A)| = \omega_1$.* $\square$

22 Number of Ideals

The main relationships with our earlier functions are: $|\mathrm{Ult}(A)| \leq |\mathrm{Id}(A)|$ and $2^{\mathrm{s}(A)} \leq |\mathrm{Id}(A)|$; both of these facts are obvious. Also recall the deep Theorem 10.10 from Part I of the BA handbook: if A is an infinite BA, then $|\mathrm{Id}(A)|^{\omega} = |\mathrm{Id}(A)|$. This result is due to Shelah [86b]; there he also proves that if κ is a strong limit cardinal of size at most $|A|$, then $|\mathrm{Id}(A)|^{<\kappa} = |\mathrm{Id}(A)|$. Note that $|\mathrm{Ult}(A)| < |\mathrm{Id}(A)|$ for A the finite-cofinite algebra on an infinite cardinal κ.

Next, we show that $|\mathrm{Id}(A)| = 2^{\omega}$ for the interval algebra A on the reals; thus A has the property that $|A| = |\mathrm{Ult}(A)| = |\mathrm{Id}(A)|$. For each ideal I on A, let $\equiv_I$ be defined as follows: $a \equiv_I b$ iff $a = b$ or else if, say $a < b$, then $[a, b) \in I$. Thus $\equiv_I$ is a convex equivalence relation on $\mathbb{R}$. Now define the function f by setting, for any ideal I,

$$\begin{aligned} f(I) = \{(r,s,\varepsilon) : {} & \text{there is an equivalence class } a \text{ under } \equiv_I \text{ such that } |a| > 1 \\ & \text{and } a \text{ has left endpoint } r, \text{ right endpoint } s, \text{ and} \\ & \varepsilon = 0,1,2,3 \text{ according as } a \text{ is } [r,s], [r,s), (r,s], \text{ or } (r,s)\}. \end{aligned}$$

Clearly f is a one-one function; since $f(I) \in (\mathbb{R}\times\mathbb{R}\times 4)^{\leq\omega}$, it follows that $|\mathrm{Id}(A)| = 2^{\omega}$, as desired.

A rigid BA A shows that $|\mathrm{Aut}(A)| < |\mathrm{Id}(A)|$ is possible. Koppelberg, Shelah [95] show that if μ is a strong limit cardinal satisfying $\mathrm{cf}(\mu) = \omega$ and $2^{\mu} = \mu^+$, then there is a Boolean algebra B such that $|B| = |\mathrm{End}(B)| = \mu^+$ and $|\mathrm{Id}(B)| = 2^{\mu^+}$. This answers Problem 60 of Monk [90]. Also, in an email message of December 1990 Shelah showed that under suitable set-theoretic hypotheses there is a BA A such that $|\mathrm{Id}(A)| < |\mathrm{Aut}A|$, answering problem 61 from Monk [90]. This result is easy to see from known facts. Namely, let T be a Suslin tree in which each element has infinitely many immediate successors, and with more than ω_1 automorphisms. Assume CH. Then $A \stackrel{\text{def}}{=} \mathrm{Treealg}(T)$ has more than ω_1 automorphisms. By the characterization of the cellularity of tree algebras given in Chapter 3, $\mathrm{c}(A) = \omega$, and since $\mathrm{hL}(A) = \mathrm{c}(A)$ (see the end of Chapter 15), by the equivalents at the beginning of Chapter 15 every ideal in A is countably generated, and hence A has only ω_1 ideals. The following problem is open.

Problem 163. *Can one construct in ZFC a BA A such that $|\mathrm{Id}(A)| < |\mathrm{Aut}(A)|$?*

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23 Number of Subalgebras

First we note the following simple result:

Proposition 23.1. *If B is a homomorphic image of A, then* $|\mathrm{Sub}(B)| \le |\mathrm{Sub}(A)|$.

Proof. Let f be a homomorphism from A onto B. With each subalgebra C of B associate the subalgebra $f^{-1}[C]$ of A. $\square$

It is also obvious that if B is a subalgebra of A, then $|\mathrm{Sub}(B)| \le |\mathrm{Sub}(A)|$.

Now we give some results from Shelah [92]; the main fact is that $|\mathrm{End}(A)| \le |\mathrm{Sub}(A)|$. This answers Problem 63 from Monk [90]. Let $\mathrm{Psub}(A)$ be the collection of all subsets of A closed under $+$, $\cdot$, and $-$ (as a binary operation – namely, $a - b = a \cdot -b$). The part $|\mathrm{Id}(A)| \le |\mathrm{Sub}(A)|$ in the next theorem is due to James Loats, and can be proved more easily.

Theorem 23.2. $|\mathrm{Psub}(A)| = |\mathrm{Sub}(A)|$ *for any infinite BA A. In particular,* $|\mathrm{Id}(A)| \le |\mathrm{Sub}(A)|$.

Proof. First, it is clear that $|A| \le |\mathrm{Sub}(A)|$, since $a \mapsto \{0, 1, a, -a\}$ ($a \in A$) is a 2-to-1 mapping from A into $\mathrm{Sub}A$. Hence for the theorem it suffices to show that the set of infinite members of $\mathrm{Psub}(A)$ has cardinality at most $|\mathrm{Sub}(A)|$. To this end, choose for every infinite $X \in \mathrm{Psub}(A)$ an element a_X of X such that there are elements $u, v \in X$ with $0 < u < a_X < v < 1$; then let $Y[X]$ be the subalgebra generated by $X \restriction a_X$ and let $Z[X]$ be the subalgebra generated by $X \restriction -a_X$. Note that $Y[X]$ consists of all elements of the form $y + -z$ with $y \in X \restriction a_X$, and either $z \in X \restriction a_X$ or $z = 1$, and similarly for $Z[X]$. Now we claim that for $X \ne X'$ we have $Y[X] \ne Y[X']$ or $Z[X] \ne Z[X']$, from which the desired conclusion clearly follows. Suppose that this claim fails; say $X \backslash X' \ne 0$. Let $a = a_X$ and $b = a_{X'}$. We claim next that $-a \le b$ or $a \le -b$. For, take any $x \in X \backslash X'$. Then we can write

$$\begin{array}{llll} x \cdot a = y + -z, \ y \in X' \restriction b, & \text{and} & z \in X' \restriction b & \text{or} \quad z = 1, \\ x \cdot -a = u + -v, \ u \in X' \restriction -b, & \text{and} & v \in X' \restriction -b & \text{or} \quad v = 1. \end{array}$$

Since $x \notin X'$, we have $z \ne 1$ or $v \ne 1$; this gives in the first case $-x + -a = z \cdot -y \le b$, hence $-a \le b$, and $a \le -b$ in the second case, proving our latest claim.

Say without loss of generality that $-a \le b$. Choose $b' \in X'$ such that $b < b' < 1$. So $0 < -b' < -b$. Thus $-b' \in X \restriction -b$, so $-b' = s + -t$, where $s \in X \restriction -a$, and

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$t \in X \upharpoonright a$ or $t = 1$. If $t = 1$, then $-b' = s \leq -a \leq b$ and $-b' \leq -b$, contradiction. If $t \neq 1$, then $b' = t \cdot -s \leq -a \leq b$, so $-b \leq -b'$, contradiction. □

Lemma 23.3. $|\mathrm{Aut}(A)| \leq |\mathrm{Sub}(A)|$ *for A infinite.*

Proof. The idea is to associate with each automorphism of A a sequence of 8 members of $\mathrm{Psub}(A)$, in a one-one fashion; clearly this will prove the theorem. Let f be any automorphism. Define

$$J^f = \{x \in A : f(y) = y \text{ for all } y \leq x\};$$
$$I^f = \{x \in A : x \cdot f(x) = 0\}.$$

Clearly we have:

(1) If $x \in I^f$ then $f(x) \in I^f$.
(2) $J^f \cup I^f$ is dense in A.

To prove (2), let $a \in A^+$, and suppose that $a \notin J^f$. Then there is a $b \leq a$ such that $b \neq f(b)$. If $b \not\leq f(b)$, then $0 \neq b \cdot -f(b) \leq a$ and $b \cdot -f(b) \in I^f$. If $f(b) \not\leq b$, then $0 \neq b \cdot -f^{-1}(b) \leq a$ and $b \cdot -f^{-1}(b) \in I^f$.

Next, let X be a maximal subset of I^f such that $x \cdot f(y) = 0$ for all $x, y \in X$. Let $I_1^f = \langle X \rangle^{\mathrm{Id}}$ and $I_2^f = \langle \{f(x) : x \in X\rangle^{\mathrm{Id}}\}$. Then let

$$I_0^f = \{y \in I^f : f(y) \in I_1^f \text{ and } y \cdot x = 0 \text{ for all } x \in I_1^f\}.$$

Let $I_3^f = \langle \{f(x) : x \in I_2^f\}\rangle^{\mathrm{Id}}$. Thus

(3) I_i^f is an ideal $\subseteq I^f$ for $i < 4$.
(4) $I_i^f \cap I_{i+1}^f = \{0\}$ for $i < 3$.
(5) $I_0^f \cup I_1^f \cup I_2^f$ is dense in I^f.

To prove (5), suppose that $x \in (I^f)^+$ but there is no non-zero member of the indicated union which is below it. Since $x \in I^f$, we have $x \cdot f(x) = 0$. Now since there is no nonzero element of I_1^f below x, we have $x \notin X$, and this gives two cases.

Case 1. There is a $z \in X$ such that $x \cdot f(z) \neq 0$. But $x \cdot f(z) \in I_2^f$, contradiction.

Case 2. There is a $z \in X$ such that $z \cdot f(x) \neq 0$. Choose w such that $f(w) = z \cdot f(x)$. Since $w \leq x$, we have $w \in (I^f)^+$, and since $f(w) \leq z$ we have $f(w) \in I_1^f$. For any $t \in I_1^f$ we have $t \cdot x = 0$ (otherwise $0 \neq t \cdot x \leq x$ and $t \cdot x \in I_1^f$), hence $t \cdot w = 0$. Thus $w \in I_0^f$, contradiction, proving (5).

For $i < 3$ we now define a member C_i^f of $\mathrm{Psub}A$: $C_i^f = \{x + f(x) : x \in I_i^f\}$. Clearly each C_i^f is closed under $+$. For any $x, y \in I_i^f$ we have $x \cdot f(y) = 0$, and from this it follows that C_i^f is closed under $\cdot$ and $-$.

We have now defined our sequence

$$\langle J^f, I_0^f, I_1^f, I_2^f, I_3^f, C_0^f, C_1^f, C_2^f\rangle$$

of 8 members of $\mathrm{Psub}(A)$. It remains just to show that if two automorphisms f and g give rise to the same sequence $\langle J, I_0, I_1, I_2, I_3, C_0, C_1, C_2\rangle$, then $f = g$. By (2) and (5) it is enough to show that they agree on $J \cup I_0 \cup I_1 \cup I_2$. Suppose that $y \in I_0$. Thus $y + f(y) \in C_0$, so there is a $z \in I_0$ such that $y + f(y) = z + g(z)$. Now $0 = y \cdot g(z)$ since $g(z) \in I_1$, so $y \leq z$. Similarly $z \leq y$, so $y = z$. So $y + f(y) = y + g(y)$. But $y \in I^f$, so $y \cdot f(y) = 0$. Similarly $y \cdot g(y) = 0$, so $f(y) = g(y)$. The cases of I_1 and I_2 are similar. □

Theorem 23.4. $|\mathrm{End}(A)| \leq |\mathrm{Sub}(A)|$ *for any infinite BA A.*

Proof. Let $\mu = |\mathrm{Sub}(A)|$, and suppose that $\mu < |\mathrm{End}(A)$. With each $f \in \mathrm{End}(A)$ associate the pair $(\mathrm{Kernel}(f), \mathrm{Range}(f))$. The number of such pairs is at most $|\mathrm{Id}(A)| \times |\mathrm{Sub}(A)|$, which by 23.2 is μ. Thus there is a set E of μ^+ endomorphisms with the same kernel I and range R. For each $f \in E$ we define a mapping g_f from A/I onto R by $g_f(x/I) = f(x)$; clearly g_f is well defined and is an isomorphism from A/I onto R. Fix $h \in E$. Then $\{g_f \circ g_h^{-1} : f \in E\}$ is a set of μ^+ different automorphisms of R. Thus by 23.3,

$$\mu < |\mathrm{Aut}(R)| \leq |\mathrm{Sub}(R)| \leq |\mathrm{Sub}A|,$$

contradiction. □

Another result in Shelah [92] is that $|\mathrm{Aut}(A)|^\omega \leq |\mathrm{Sub}(A)|$; we shall not give the proof.

For any BA A, let $\mathrm{Pend}(A) = \{f : f$ is a homomorphism from a subalgebra of A onto another subalgebra of $A\}$. Such homomorphisms are called *partial endomorphisms* of A.

Theorem 23.5. $|\mathrm{Pend}(A)| = |\mathrm{Sub}(A)|$ *for any infinite BA A.*

Proof. $\geq$ is clear, since with each subalgebra B one can associate the identity mapping on B, a partial endomorphism of A. For $\leq$ we proceed as in the proof of Theorem 23.4: this time we associate with each partial endomorphism a triple consisting of its domain, kernel, and range; otherwise the details are similar. □

Theorem 23.6. *If $A \times B$ is infinite, then* $|\mathrm{Sub}(A \times B)| = \max(|\mathrm{Sub}(A)|, |\mathrm{Sub}(B)|)$.

Proof. We prove the equivalent statement that if A is infinite and $a \in A$, then $|\mathrm{Sub}(A)| = \max(|\mathrm{Sub}(A \restriction a)|, |\mathrm{Sub}(A \restriction -a)|)$. The inequality $\geq$ is obvious. Let $\mu = \max(|\mathrm{Sub}(A \restriction a)|, |\mathrm{Sub}(A \restriction -a)|)$, and suppose that $\mu < |\mathrm{Sub}(A)|$. Now we associate with each subalgebra B of A five objects:

$$C_0^B = \{x \cdot a : x \in B\}, \text{ a subalgebra of } A \restriction a;$$
$$C_1^B = \{x \cdot -a : x \in B\}, \text{ a subalgebra of } A \restriction -a;$$

$$I_0^B = \{x \in B : x \le a\}, \text{ an ideal of } C_0^B;$$
$$I_1^B = \{x \in B : x \le -a\}, \text{ an ideal of } C_1^B;$$

and g_B, the isomorphism from C_0^B/I_0^B onto C_1^B/I_1^B such that $g_B((x\cdot a)/I_0^B) = (x\cdot -a)/I_1^B$ for all $x \in B$. Now B can be reconstructed from these five objects:

$$B = \langle I_0^B \cup I_1^B \cup \{x+y : x \in C_0^B, y \in C_1^B, \text{ and } g_B(x/I_0^B) = y/I_1^B\}\rangle$$

In fact, the direction $\subseteq$ is clear. For $\supseteq$ it suffices to show that if $x \in C_0^B$, $y \in C_1^B$, and $g_B(x/I_0^B) = y/I_1^B$ then $x+y \in B$. Say $x = x'\cdot a$ and $y = y'\cdot -a$, with $x', y' \in B$. Now $(x'\cdot -a)/I_1^B = (y'\cdot -a)/I_1^B$, so $(x'\cdot -a)\triangle(y'\cdot -a) \in B$. Now

$$\begin{aligned}(x+y)\triangle(x'\cdot -a)\triangle(y'\cdot -a) &= (x'\cdot a)\triangle(y'\cdot -a)\triangle(x'\cdot -a)\triangle(y'\cdot -a)\\ &= (x'\cdot a)\triangle(x'\cdot -a) = x' \in B,\end{aligned}$$

so $x+y \in B$.

Now there is a set X of μ^+ subalgebras of A such that $C_i^B = C_i^D$ and $I_i^B = I_i^D$ for all $B, D \in X$ and all $i < 2$. Fix $B \in X$. Now $\{g_D^{-1}\circ g_B : D \in X\}$ is a set of automorphisms of C_0^B/I_0^B, and by 23.1 and 23.3,

$$|\mathrm{Aut}(C_0^B/I_0)| \le |\mathrm{Sub}(C_0^B/I_0)| \le |\mathrm{Sub}C_0^B| \le \mu,$$

so there are distinct $D, E \in X$ for which $g_D = g_E$. This contradicts the noted fact about reconstruction. $\square$

Note that $2^{\mathrm{Irr}(A)} \le |\mathrm{Sub}(A)|$ if $\mathrm{Irr}(A)$ is attained. An example A for which $|\mathrm{Id}(A)| < |\mathrm{Sub}(A)|$ is provided by the interval algebra A on the reals. We noted in the last chapter that $|\mathrm{Id}(A)| = 2^\omega$. Since $\mathrm{Irr}(A) = 2^\omega$ attained, we have $|\mathrm{Sub}A| = 2^{2^\omega}$.

The above theorems imply that $|\mathrm{Sub}(A)|$ is our biggest cardinal function. Its size is, of course, always at most $2^{|A|}$. For most algebras, this value is actually attained. It is also quite interesting to construct a BA A in which $|\mathrm{Sub}(A)|$ is as small as possible. The only algebra we know of where this is the case is Rubin's algebra A from Chapter 18. We now go through the proof that $|\mathrm{Sub}(A)| = \omega_1$; thus $|A| = |\mathrm{Ult}(A)| = |\mathrm{Id}(A)| = |\mathrm{Sub}(A)|$. We call an element $a \in A$ *countable* provided that $A \upharpoonright a$ is countable.

Lemma 23.7. *A has only countably many countable elements.*

Proof. Let P be the set of all countable elements of A, and suppose that P is uncountable. Then it is easy to construct $\langle a_\alpha : \alpha < \omega_1\rangle \in {}^{\omega_1}P$ such that if $\alpha < \beta < \omega_1$ then $a_\beta \not\le a_\alpha$. Since $\text{h-cof}(A) = \omega$, the set $P' \stackrel{\text{def}}{=} \{a_\alpha : \alpha < \omega_1\}$ is not well founded; say $a_{\alpha(0)} > a_{\alpha(1)} > \cdots$. Choose $i, j \in \omega$ such that $i < j$ and $\alpha(i) < \alpha(j)$. Then $a_{\alpha(i)} > a_{\alpha(j)}$ contradicts the choice of the a_β's. $\square$

Note that the collection of countable elements of A forms an ideal, which we denote by $C(A)$. To proceed further, we have to go back to the main property of Rubin's algebra. For this purpose we introduce the following notation. A *preconfiguration* in a BA A is a sequence $\langle a, b_1, \ldots, b_n\rangle$ of pairwise disjoint elements of A with each $b_i \neq 0$, $n > 0$. Given such a preconfiguration, a subset P of A is *dense at* $\langle a, b_1, \ldots, b_n\rangle$ provided that for all c_1, c_2, if $\langle a, c_1, c_2, b_1, \ldots, b_n\rangle$ is a configuration then $P \cap (c_1, c_2) \neq 0$. Thus the main property of Rubin's BA A is that if P is an uncountable subset of A then there is a preconfiguration $\langle a, b_1, \ldots, b_n\rangle$ of A such that P is dense at $\langle a, b_1, \ldots, b_n\rangle$. For both of the next two lemmas we advise the reader to draw a diagram along the lines of the one in Chapter 18 to see what is going on.

Lemma 23.8. *Assume that $P \subseteq A$, P is uncountable, $\langle a, b_1, \ldots, b_n\rangle$ is a preconfiguration of A, P is dense at $\langle a, b_1, \ldots, b_n\rangle$, $a \leq b \leq a + \sum_{i=1}^n b_i$, $b \cdot b_i \neq 0$ for $i = 1, \ldots, n$, and $b \cdot -a \notin C(A)$. Then $P \cap [a, b]$ is uncountable.*

Proof. Suppose that $P \cap [a, b]$ is countable, and let Q be the closure of $P \cap [a, b]$ under $+$; so Q is countable also. Say $b \cdot b_i \notin C(A)$. Pick any $c \leq b \cdot b_i$, $c \neq 0$, such that $c' \stackrel{\text{def}}{=} a + c + \sum_{j \neq i} b_j \notin Q$. Then pick c_1, c_2 so that $a + c_i < c'$ and $\langle a, c_i, c', b_1, \ldots, b_n\rangle$ is a configuration for $i = 1, 2$, and $c_1 + c_2 = c'$. Then pick $d_1, d_2 \in P$ so that $c_i < d_i < c'$ for $i = 1, 2$. But $d_1, d_2 \in [a, b]$ and $d_1 + d_2 = c'$, so $c' \in Q$, contradiction. □

Lemma 23.9. *Every subalgebra of A is the union of countably many closed intervals.*

Proof. Suppose that B is a subalgebra of A which is not the union of countably many closed intervals. Let $\langle [x_\alpha, y_\alpha] : \alpha < \omega_1\rangle$ enumerate all of the closed intervals contained in B. Now B contains ω_1 elements pairwise inequivalent with respect to $C(A)$; hence it is easy to construct a sequence $\langle z_\alpha : \alpha < \omega_1\rangle \in {}^{\omega_1}B$ with the following two properties:

(1) $z_\alpha \notin \bigcup_{\beta<\alpha} [x_\beta, y_\beta]$ for each $\alpha < \omega_1$;
(2) $z_\alpha \triangle z_\beta \notin C(A)$ for distinct $\alpha, \beta < \omega_1$.

Let $D = \{z_\alpha : \alpha < \omega_1\}$. Since D is somewhere dense, let $\langle a, b_1, \ldots, b_n\rangle$ be a preconfiguration of A such that D is dense at $\langle a, b_1, \ldots, b_n\rangle$. Choose any b such that $a \leq b \leq a + \sum_{i=1}^n b_i$ and $b \cdot b_i \neq 0 \neq b_i \cdot -b$ for all $i = 1, \ldots, n$. We show that $[a, b] \subseteq B$. Take any $d \in [a, b]$. Choose e_1, e_2 with the following properties: $d = e_1 \cdot e_2$; $a \leq e_i \leq a + \sum_{j=1}^n b_j$; $e_i \cdot b_j \neq 0$ for $i = 1, 2$, $j = 1, \ldots, n$. Thus $\langle a, d, e_i, b_1, \ldots, b_n\rangle$ is a configuration, so we can choose $f_i \in D \cap (d, e_i)$ for $i = 1, 2$. Then $f_1 \cdot f_2 = d$, and so $d \in B$ (since $D \subseteq B$). Thus, indeed, $[a, b] \subseteq B$. By an easy argument, $[a, b] \cap D$ has at least two elements. Since distinct elements of D are inequivalent mod $C(A)$, it follows that $b \cdot -a \notin C(A)$. Hence by Lemma 23.8, $D \cap [a, b]$ is uncountable. But this clearly contradicts the construction of D. □

With these lemmas available we can now prove that $|\mathrm{Sub}(A)| = \omega_1$. We claim that each subalgebra B of A is generated by an ideal along with a countable set. In fact, write $B = \bigcup_{i<\omega}[a_i, b_i]$. Let I be the ideal generated by $\{b_i \cdot -a_i : i < \omega\}$. Then clearly B is generated by $I \cup \{b_i : i < \omega\}$, as required. Now every ideal is countably generated. This follows from the fact that $\mathrm{hL}(A) \leq \text{h-cof}(A) = \omega$, proved in Chapter 18, and one of the equivalents of hL given in Chapter 15. This being the case, it follows that there are exactly ω_1 ideals in A, since $\omega_1^\omega = \omega_1$ by virtue of CH (which follows from $\diamondsuit$, which we are assuming). Now $|\mathrm{Sub}A| = \omega_1$ is clear, again using CH.

In Cummings, Shelah [95] it is shown that it is consistent (relative to a large cardinal assumption) that every infinite BA A has $2^{|A|}$ subalgebras. This answers Problem 62 in Monk [90]. We give part of this consistency proof.

Lemma 23.10. *If κ is a strong limit cardinal $\leq |A|$, then A has an irredundant subset of size κ.*

Proof. Let B be a subalgebra of A of size κ. Then B has an irredundant subset of size κ; this subset is irredundant in A too. □

Let $(*)$ be the conjunction of the following statements:

(1) For every infinite cardinal κ, 2^κ is weakly inaccessible.
(2) For all infinite cardinals λ, κ, if $\kappa \leq \lambda < 2^\kappa$, then $2^\lambda = 2^\kappa$.
(3) For every infinite cardinal κ and every BA A of size 2^κ, A has an irredundant subset of size 2^κ.

Cummings and Shelah show that there is a model in which $(*)$ holds, assuming GCH + six supercompact cardinals.

Theorem 23.11. *Assuming GCH + six supercompact cardinals, there is a model in which every infinite BA A has $2^{|A|}$ subalgebras.*

Proof. We assume $(*)$. Let

$$C = \{\mu : \mu \text{ is strong limit or } \exists\theta \geq \omega[\mu = 2^\theta]\}.$$

Clearly C is a closed unbounded class of cardinals. Let A be any infinite BA, and let $\nu \in C$ be minimum such that $|A| < \nu$. Note that $\omega \in C$. Let $\mu = \sup\{\nu \in C : \nu < |A|\}$. Thus $\mu \in C$ and $\mu \leq |A|$. Now $2^\mu \in C$, so $|A| < 2^\mu$. Let B be a subalgebra of A of size μ.

Case 1. $\mu = 2^\theta$ for some θ. Then by $(*)(3)$, B has an irredundant subset of size μ.

Case 2. μ is strong limit. Then by Lemma 23.10, B has an irredundant subset of size μ.

Now $\mu \leq |A| < 2^\mu$, so by $(*)(2)$, $2^\mu = 2^{|A|}$, and the desired conclusion follows. □

It is possible to have $|\mathrm{Aut}(A)| < |\mathrm{Sub}(A)|$: take a rigid BA. One can even have $|\mathrm{End}(A)| < |\mathrm{Sub}(A)|$, for example in the interval algebra on the reals.

24 Other Cardinal Functions

There are many other cardinal functions besides the 21 that we have discussed in the preceding chapters. In this chapter we give a list of some natural ones; some of these have been explicitly mentioned earlier. We also mention some facts and problems about them, without trying to be exhaustive.

Functions mentioned in the previous text

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Some additional natural functions

81. **The tree algebra number.** For any BA A, the *tree algebra number* of A, denoted by $\mathrm{ta}(A)$, is the supremum of cardinalities of subalgebras of A isomorphic to tree algebras. This number is clearly greater or equal to cellularity. It is dominated by d, since if A is isomorphic to a tree algebra and A is a subalgebra of B, then $|A| = \mathrm{d}(A) \leq \mathrm{d}(B)$. Note that an infinite free algebra always has tree algebra number $\aleph_0$. So $\mathrm{ta}(B)$ can be much smaller than $\mathrm{d}(B)$.

82. **The pseudo-tree algebra number**. For any BA A, the *pseudo-tree algebra number* of A, denoted by $\mathrm{pta}(A)$, is the supremum of cardinalities of subalgebras of A isomorphic to pseudo-tree algebras. Clearly $\mathrm{Length}(A) \leq \mathrm{pta}(A)$, $\mathrm{ta}(A) \leq \mathrm{pta}(A)$, and $\mathrm{pta}(A) \leq \mathrm{Irr}(A)$. Any infinite free algebra has pseudo-tree number $\aleph_0$. In a large free algebra we thus have $\mathrm{pta}(A) = \aleph_0$ while $\mathrm{d}(A)$ is large.

83. **The semigroup algebra number**. For any BA A, the *semigroup algebra number* of A, denoted by $\mathrm{sa}(A)$, is the supremum of cardinalities of subalgebras of A which are semigroup algebras. We have $\mathrm{Ind}(A) \leq \mathrm{sa}(A)$ and $\mathrm{pta}(A) \leq \mathrm{sa}(A)$. It is not clear whether $\mathrm{sa}(A) \leq \mathrm{Irr}(A)$.

84. **The tail algebra number**. For any BA A, the *tail algebra number* of A, denoted by $\mathrm{tla}(A)$, is the supremum of cardinalities of subalgebras of A isomorphic to tail algebras. Clearly $\mathrm{sa}(A) \leq \mathrm{tla}(A)$. It may be that $\mathrm{tla}(A) = |A|$ for every infinite BA A.

85. **Disjunctiveness.** The *disjunctiveness* of A, $\mathrm{dj}(A)$, is the supremum of cardinalities of disjunctive subsets of A. Clearly $\mathrm{tla}(A) \leq \mathrm{dj}(A)$ and $\mathrm{s}(A) \leq \mathrm{dj}(A)$. It may be that $\mathrm{dj}(A) = |A|$ for every infinite BA A.

86. **Minimality**. The *minimality* of A, $\mathrm{m}(A)$, is the supremum of cardinalities of minimally generated subalgebras of A. Clearly $\mathrm{pta}(A) \leq \mathrm{m}(A)$. For every infinite free BA A we have $\mathrm{m}(A) = \aleph_0$.

87. **Initial chain algebra number**. This number, denoted by $\mathrm{ic}(A)$, is the supremum of cardinalities of subalgebras of A isomorphic to the initial chain algebra on some tree. If A is free, then $\mathrm{ic}(A) = \omega$.

88. **Initial chain algebra number for pseudo trees**. Similarly, for pseudo-trees; denoted by $\mathrm{icp}(A)$ Thus $\mathrm{ic}(A) \leq \mathrm{icp}(A)$. If A is free, then $\mathrm{icp}(A) = \omega$.

89. **Superatomic number**. This is the supremum of cardinalities of superatomic subalgebras of a BA; denoted by $\mathrm{spa}(A)$. If A is free, then $\mathrm{spa}(A) = \omega$. We have $\mathrm{ic}(A) \leq \mathrm{spa}(A)$.

90. **The order-ideal number**. This is the supremum of the cardinality of a system of ideals ordered by inclusion; we denote it by $\mathrm{oi}(A)$ Clearly $\mathrm{hL}(A) \leq \mathrm{oi}(A)$ and $\mathrm{hd}(A) \leq \mathrm{oi}(A)$. It is possible to have $|A| < \mathrm{oi}(A)$; this is true for $A = \mathrm{Intalg}(\mathbb{Q})$, for example. It is not clear whether $\mathrm{oi}(A) < |A|$ is possible.

Dimensions of Boolean algebras

Heindorf [91] introduced an interesting notion of *dimension* of Boolean algebras, which gives rise to several cardinal functions. Let $\mathscr{A}$ be a non-empty class of nontrivial BAs. Since every BA is embeddable in a product of two-element BAs, every BA is embeddable in a product of members of $\mathscr{A}$. Hence the following definition makes sense: the *$\mathscr{A}$-dimension* of a BA A is the smallest cardinal number κ such that A can be embedded in a product of κ members of $\mathscr{A}$ (not necessarily distinct members). This $\mathscr{A}$-dimension is denoted by $\mathscr{A}$-dim$A$. It is natural to consider this notion for natural classes $\mathscr{A}$. Various one-element classes $\mathscr{A}$ are natural, of course. At the opposite extreme we can consider the notion for our natural proper classes – the class of all free algebras, of all superatomic algebras, etc. Heindorf investigated the three cases (1) all free algebras, (2) all superatomic algebras, (3) all interval algebras. Some cases are trivial; for example, with $\mathscr{A}$ the class of all complete BAs, the $\mathscr{A}$-dimension of any BA is 1, since any BA can be embedded in a complete BA. We list some dimension functions which may be interesting.

91. int-dim, where $\mathscr{A}$ is the class of all interval algebras.

92. sa-dim, where $\mathscr{A}$ is the class of all superatomic algebras.

93. free-dim, where $\mathscr{A}$ is the class of all free algebras.

94. tree-dim, where $\mathscr{A}$ is the class of all tree algebras.

95. ptree-dim, where $\mathscr{A}$ is the class of all pseudo-tree algebras.
96. sg-dim, where $\mathscr{A}$ is the class of all semigroup algebras.
97. mg-dim, where $\mathscr{A}$ is the class of all minimally generated algebras.
98. ic-dim, where $\mathscr{A}$ is the class of all initial chain algebras.
99. dj-dim, where $\mathscr{A}$ is the class of all disjunctively generated algebras.
100. finco-dim, where $\mathscr{A}$ is the class of all finite-cofinite algebras on some infinite set.
101. $\{\text{Finco}(\omega)\}$-dim.
102. $\{\text{Intalg}(\mathbb{R})\}$-dim.

25 Diagrams

In this chapter we give several diagrams for the relationships between the main 21 functions that we have considered. Thus this chapter summarizes the main text. But it also turns out that we have some new things to say upon considering these relationships thoroughly.

For each of the diagrams, we need to do the following:

(1) For each edge, indicate where the relation is proved, and give an example where the functions involved are different. Also, if the difference is indicated as "small", indicate where that is stated in the text, while if the difference is "large", indicate an example. Recall that a difference is "small" if there is some limitation on the difference. It is "large" if for every infinite cardinal κ, there is an example where the difference is at least κ.

(2) Show that there are not any relations except those indicated in the diagrams. It suffices to do this just for crucial places in the diagram. For example, if in the diagram for the general case we give an algebra A in which $\mathrm{Length}A < \pi\chi A$, this will also be an example for $\mathrm{Length}A < \text{h-cof}A$ and $\mathrm{Depth}A < \pi\chi A$. As will be seen, we have not been completely successful in either of these two tasks; there are several open problems left.

The main diagram, edges and "large" and "small" indications

See the diagram on top of the next page.

25.1. Depth $\leq$ Length. This relation is obvious from the definitions. The difference is small by the Erdős–Rado theorem.

25.2. Depth $\leq$ c. Again, this is obvious from the definitions. The difference is large in the finite-cofinite algebra on an infinite cardinal κ.

25.3. Depth $\leq$ t. This is proved in Chapter 4; see 4.26. The difference is big in a free algebra.

25.4. $\pi\chi \leq$ t. See Chapter 11, Theorem 11.18. The difference can be large in an interval algebra. Namely, given an uncountable cardinal κ, one can take a complete linear order L in which the elements have character (ω, λ) or (λ, ω) for each $\lambda \leq \kappa$,

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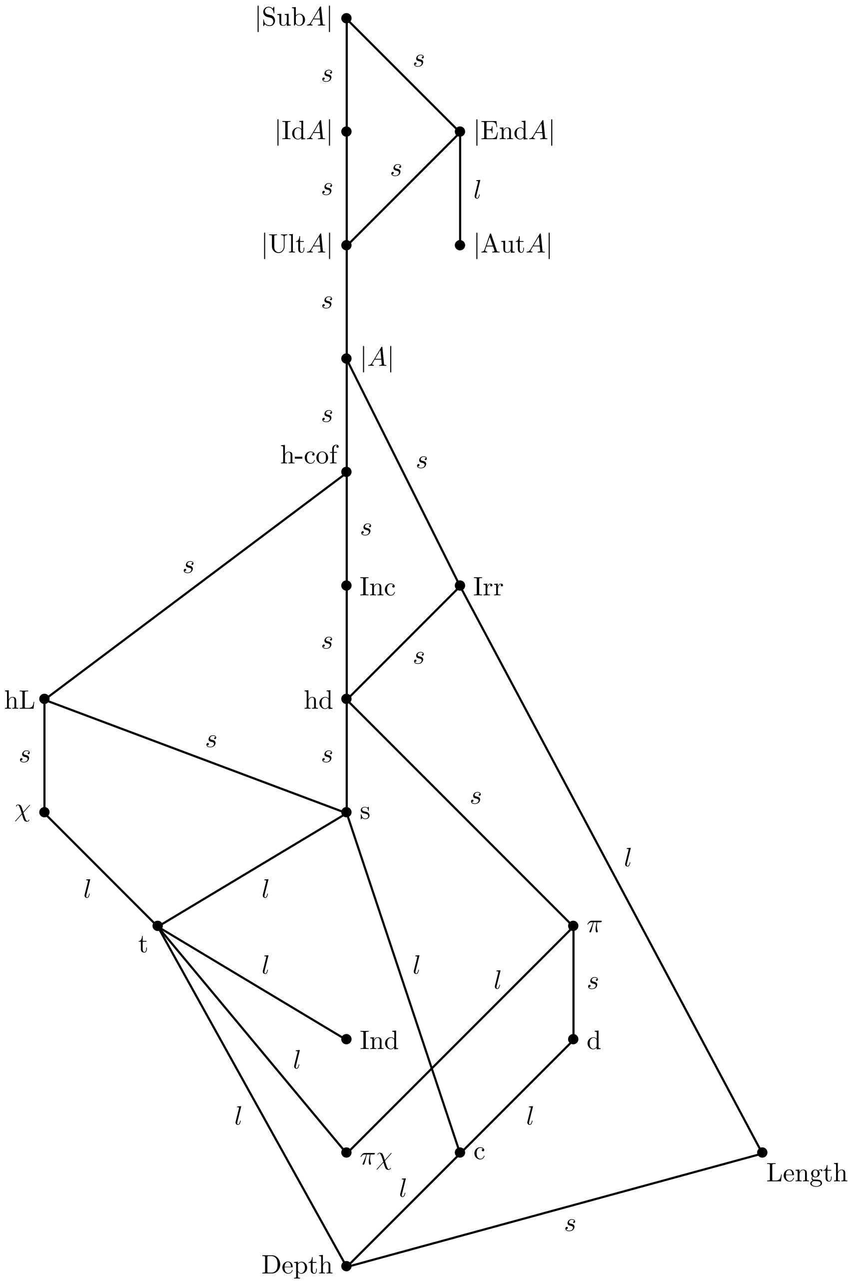
The general case

$l =$ difference can be large; $s =$"small" difference



with the cofinality and coinitiality of L being ω; see Hausdorff [1908]. See the ends of Chapters 11 and 12.

25.5. $\pi\chi \leq \pi$. Obvious from the definitions. The difference is large in a finite-cofinite algebra; see Chapter 11.

25.6. $\mathrm{c} \leq \mathrm{s}$. This is an easy consequence of Theorem 3.30. The difference is large in free algebras.

25.7. $\mathrm{c} \leq \mathrm{d}$. See Corollary 5.2. The difference is large in free algebras.

25.8. Length $\leq$ Irr. Obvious from the definitions. The difference is large in free algebras.

25.9. Ind $\leq$ t. This is clear by the free sequence characterization of tightness. The difference can be large in some interval algebras, since Depth $\leq$ t.

25.10. $\mathrm{d} \leq \pi$. Obvious from the topological versions of these functions. The difference is small. See the end of Chapter 6 for an example where they differ.

25.11. $\mathrm{t} \leq \chi$. Obvious from the definitions. The difference is big in a finite-cofinite algebra on a large cardinal κ.

25.12. $\mathrm{t} \leq \mathrm{s}$. See Theorem 5.18. The difference is big in a finite-cofinite algebra on a cardinal κ.

25.13. $\pi \leq \mathrm{hd}$. See Theorem 6.15. The difference is small, since $\pi(A) \leq \mathrm{hd}(A) \leq |A| \leq 2^{\pi(A)}$. The functions differ in $\mathscr{P}(\kappa)$, for example.

25.14. hd $\leq$ Irr. See the end of Chapter 16. They differ in the interval algebra on the reals.

25.15. $\chi \leq \mathrm{hL}$. See after Problem 144. The difference is small, since $\mathrm{hL}(A) \leq |A| \leq |\mathrm{Ult} A| \leq 2^{\chi(A)}$. They differ on the Alexsandroff duplicate of a free algebra; see Proposition 14.7.

25.16. $\mathrm{s} \leq \mathrm{hL}$. Obvious from the definitions. The difference is small, since $|A| \leq 2^{\mathrm{s}(A)}$ for any BA A by 13.10. They differ in a Kunen line (constructed under CH; see page 378). Whether there is an example in ZFC is an open question

Problem 164. *Can one construct in ZFC a BA A such that* $\mathrm{s}(A) < \mathrm{hL}(A)$*?*

25.17. $\mathrm{s} \leq \mathrm{hd}$. Obvious from the definitions. The difference is small (see above). They differ on the interval algebra of a Suslin line. Whether there is an example in ZFC is open (Problem 153).

25.18. Irr $\leq$ Card. Obvious from the definitions. The difference is small; from Theorem 4.23 of Part I of the Boolean algebra handbook it follows that $|A| \leq 2^{\mathrm{Irr}(A)}$. A compact Kunen line (constructed under CH) gives a BA in which they are different (see Chapter 8). It is open to give an example in ZFC. (Problem 91).

25.19. $\mathrm{hL} \leq$ h-cof. See Chapter 18. The difference is small, since $\chi(A) \leq \mathrm{hL}(A)$, and so $\mathrm{hL}(A) \leq \text{h-cof}(A) \leq |A| \leq 2^{\chi(A) \leq 2^{\mathrm{hL}}(A)}$, using Chapter 14. They differ on the interval algebra on the reals; see Chapter 18.

25.20. hd $\leq$ Inc. This is an easy consequence of Theorem 4.25 of the BA handbook, part I. The difference is small, since $\mathrm{s} \leq \mathrm{hd}$, Inc $\leq$ Card, and $|A| \leq 2^{\mathrm{s}(A)}$ for any BA. They differ on Intalg($\mathbb{R}$).

25.21. Inc $\le$ h-cof. Obvious from Theorem 18.1. The difference is small since by the above $|A| \le 2^{\mathrm{Inc}(A)}$. They differ on the Baumgartner, Komjath algebra (see Chapter 18); this was constructed using $\diamondsuit$, and it remains a problem to get an example with weaker assumptions. (Problem 162)

25.22. h-cof $\le$ Card. Obvious from the definitions. The difference is small (see above). Since $\mathrm{Inc}(A) = \text{h-cof}(A)$ for an interval algebra (see Chapter 18), we get an example with $\text{h-cof}(A) < |A|$.

25.23. Card $\le$ |Ult|. This is well known; see the Handbook Part I, Theorem 5.31. The difference is, of course, small. They differ in an infinite free algebra.

25.24. |Ult| $\le$ |End|. This is obvious. The difference is small. They differ for the finite-cofinite algebra on ω; see Theorem 21.7.

25.25. |Aut| $\le$ |End|. Also obvious. The difference is large, as shown by a rigid BA.

25.26. |Ult| $\le$ |Id|. Again obvious. The difference is small. They differ on the finite-cofinite algebra on an infinite cardinal.

25.27. |Id| $\le$ |Sub|. See Chapter 23. The difference is small. They differ for the interval algebra on the reals.

25.28. |End| $\le$ |Sub|. See Chapter 23. The difference is small. They differ for the interval algebra on the reals.

The main diagram: no other relationships

Keep in mind that we only treat "crucial" relations; other possibilities are supposed to follow from these.

25.29. Length $< \pi\chi$: an uncountable free algebra: see Chapter 11.

25.30. Length $<$ c: the finite-cofinite algebra on an uncountable cardinal.

25.31. Length $<$ Ind: an uncountable free algebra.

25.32. $\pi\chi <$ Depth: see the example in Chapter 11.

25.33. d $< \pi\chi$: some free algebras.

25.34. Ind $<$ Depth: the interval algebra on an uncountable cardinal.

25.35. Ind $< \pi\chi$: true in the interval algebra on an uncountable cardinal; see Chapter 11.

25.36. $\pi <$ Ind: $\mathcal{P}(\kappa)$. The difference is small.

25.37. $\chi <$ c: the Alexsandroff duplicate of an infinite free algebra; Chapter 14.

25.38. hL $<$ d: The interval algebra of a complete Suslin line. It is not known if this is possible in ZFC. (Problem 145)

25.39. hL $<$ Inc: The interval algebra on the reals; see Chapter 17.

25.40. Inc $< \chi$: The Baumgartner-Komjath algebra constructed under $\diamondsuit$; see Chapter 17. It is not known if this is possible under weaker hypotheses. See Problem 159.

25.41. Inc $<$ Length: Constructed by Shelah in ZFC using entangled linear orders.

25.42. Irr $<\chi$: The compact Kunen line, constructed using CH. No example is known in ZFC; this is problem 65 in Monk [96].

Problem 165. *Can one construct in ZFC a BA A such that* $\mathrm{Irr}A < \chi A$*?*

This problem is equivalent to the problem of constructing in ZFC a BA A such that $\mathrm{Irr}(A) < \mathrm{hL}(A)$; see the argument at the end of Chapter 15.

25.43. h-cof $<$ Length: Constructed by Shelah in ZFC using entangled linear orders; see Shelah [91].
25.44. $|\mathrm{Aut}| <$ Depth: embed a large interval algebra in a rigid algebra.
25.45. $|\mathrm{Aut}| < \pi\chi$: a rigid complete BA of large cellularity gives an example.
25.46. $|\mathrm{Aut}| <$ Ind: embed a large free algebra in a rigid algebra.
25.47. $|\mathrm{Id}| < |\mathrm{Aut}|$: see Chapter 22; possible under some set-theoretic assumptions. No example is known in ZFC (Problem 163)
25.48. $|\mathrm{Ult}| < |\mathrm{Aut}|$: the finite-cofinite algebra on ω; see Chapter 20.
25.49. $|\mathrm{End}| < |\mathrm{Id}|$: See Chapter 22.

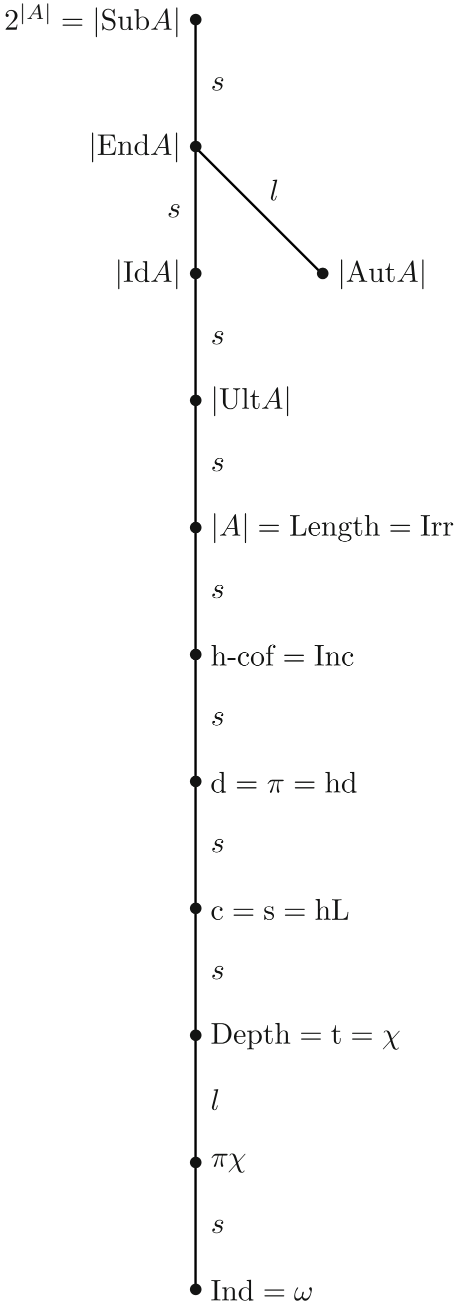
The interval algebra diagram: the edges, indicated equalities, and the "large" and "small" indications

(See the next page for the diagram.)

25.50. Ind $=\omega$: this is one of the main results about interval algebras; see Part I of the BA handbook.
25.51. $\omega \leq \pi\chi$, difference possibly large. See the description of $\pi\chi$ for interval algebras in Chapter 11.
25.52. Depth $=$ t $=\chi$: see Chapter 14.
25.53. $\pi\chi \leq$ Depth, difference possibly large. See Chapter 11.
25.54. c=s=hL: see Chapter 15.
25.55. Depth $\leq$ c, with the difference small. The difference is small since $|A| \leq 2^{\mathrm{Depth}(A)}$ for an interval algebra A; *this implies smallness for the next few that we consider also.* For an example where they differ, see Chapter 4.
25.56. d $=\pi=$ hd: obvious from the retractiveness of interval algebras.
25.57. c $\leq$ d. They differ in the interval algebra of a Suslin line; see Chapter 5. In fact, for any infinite cardinal κ the following two conditions are equivalent:
(1) there is an interval algebra A such that $\kappa = \mathrm{c}A < \mathrm{d}A$;
(2) there is a κ^+-Suslin line (or tree).
Thus the problem of getting an example in ZFC of a BA A such that $\mathrm{c}(A) < \mathrm{d}(A)$ is equivalent to the set-theoretical question of proving in ZFC that there is for some infinite κ a κ^+-Suslin tree.
25.58. d $\leq$ Inc. They differ in the interval algebra on the reals.
25.59. Inc $=$ h-cof. See Chapter 18.
25.60. h-cof $\leq$ Card. Shelah [91] constructed an example where they differ in ZFC.

Interval algebras

$l =$ difference can be large; $s =$"small" difference



25.61. Card $\le$ |Ult|. They differ for the interval algebra on the rationals.

25.62. |Ult| $\le$ |Id|. They differ on the interval algebra on κ.

25.63. |Aut| $\le$ |End|, the difference large: take an infinite rigid interval algebra.

25.64. |Id| $\le$ |End|: follows from retractiveness. A Suslin line with more than ω_1 automorphisms gives an example where they are different, assuming CH. And if an example of an interval algebra A such that $|\mathrm{Id}(A)| < |\mathrm{End}(A)|$ can be given in ZFC, then assuming GCH + (there is no uncountable inaccessible) one can show that for some infinite κ there is a κ^+-Suslin tree. For, suppose that A is an interval algebra such that $|\mathrm{Id}(A)| < |\mathrm{End}(A)|$, GCH holds, and there are no uncountable inaccessibles. Thus $|A| = |\mathrm{Ult}(A)| = |\mathrm{Id}(A)|$ and $|\mathrm{End}(A)| = |A|^+$. If $\mathrm{c}(A) = |A|$, then $\mathrm{c}(A)$ is attained (since there are no uncountable inaccessibles), and hence $|A| < |\mathrm{Id}(A)|$, contradiction. Thus $\mathrm{c}(A) < |A|$, and $|A| = |\mathrm{c}(A)|^+$. If $\mathrm{d}(A) < |A|$, then $|\mathrm{End}(A)| \le |\mathrm{Ult}(A)|^{\mathrm{d}(A)} = |A|^{\mathrm{d}(A)} = |A|$, contradiction. So $\mathrm{c}(A) < \mathrm{d}(A)$, and A must be the interval algebra on a $(\mathrm{c}(A))^+$-Suslin tree. So, the problem of existence of such interval algebras A in ZFC is stronger than the above set-theoretical question; this is problem 69 in Monk [96]:

Problem 166. *Can one construct in ZFC an interval algebra A such that* $|\mathrm{Id}(A)| < |\mathrm{End}(A)|$*?*

25.65. |End| $\le$ |Sub|. They differ on the interval algebra on the reals.

25.66. |Aut| < Ind: an infinite rigid interval algebra.

25.67. |Id| < |Aut|: as in the case of Suslin trees (see Chapter 22), one can construct a Suslin line with more than ω_1 automorphisms. Again, the question of existence of such algebras in ZFC is a strong set-theoretical hypothesis; this is problem 70 in Monk [96]:

Problem 167. *Can one construct in ZFC an interval algebra A such that* $|\mathrm{Id}A| < |\mathrm{Aut}A|$*?*

The tree algebra diagram: the indicated equalities and inequalities, and the "large" and "small" indications

See the diagram on the next page.

Let T be a tree, and let $A = \mathrm{Treealg}(T)$.

25.68. Ind$A = \omega$, since A can be embedded in an interval algebra.

25.69. The difference between Ind and $\pi\chi$ can be arbitrarily large by the description of $\pi\chi$ for tree algebras.

25.70. $\pi\chi \le \mathrm{t}$ in general.

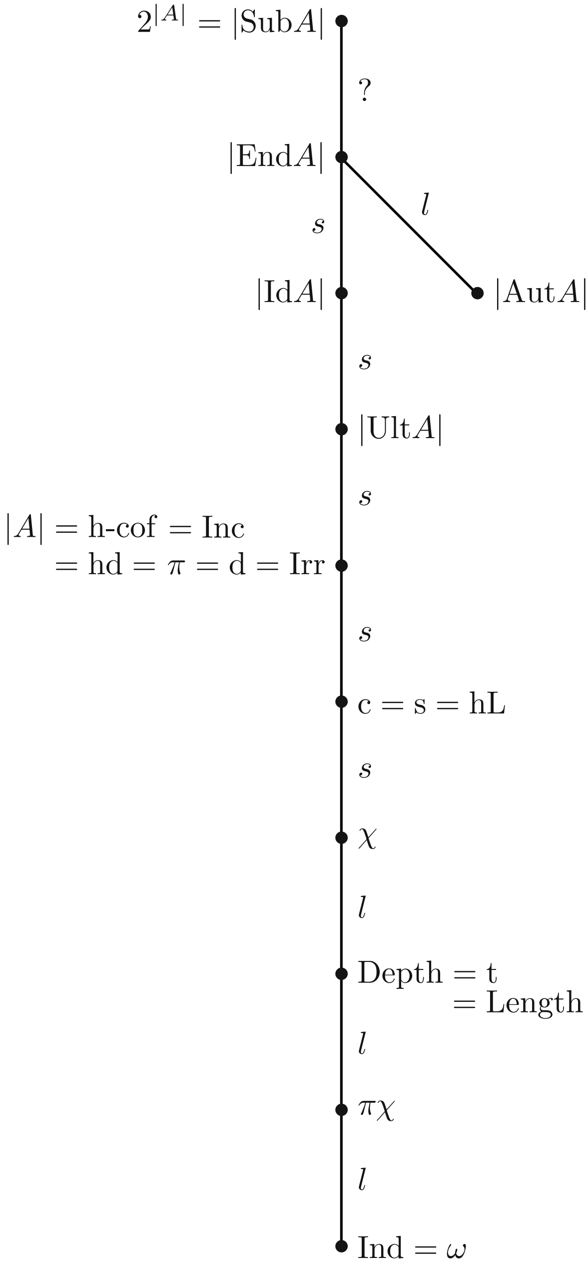
25.71. Depth = t, since tree algebras are retractive; see Chapter 4.

25.72. Length = Depth by the Brenner, Monk theorem (Handbook, p. 269).

25.73. The difference between $\pi\chi$ and Depth can be arbitrarily large; Chapter 11.

Tree algebras

$l =$ difference can be large; $s =$"small" difference



25.74. $\chi A = \sup\{|\text{set of immed. succ. of } C|, cf(C) : C \text{ an initial chain}\}$; see Chapter 14. Finco κ is an example where χ is high and depth low.

25.75. In Chapter 3 we showed that

$$\mathrm{c}(A) = \max\{|\{t \in T : t \text{ has finitely many immed. succ.}\}|, \mathrm{Inc}(T)\}.$$

From this it is easy to see that $\chi \le \mathrm{c}$. In fact, suppose that C is an initial chain of T whose order type is an infinite regular cardinal. Obviously |set of immediate succ. of $C| \le \mathrm{c}$. If $|\{x \in C : x \text{ has finitely many immediate successors}\}| = |C|$, clearly $|C| \le c$. If this set has power less than $|C|$, one can choose an element to the side of x for $|C|$ many elements $x \in C$, and this gives an incomparable subset of T of size $|C|$, and so again $|C| \le c$. So this shows that $\chi \le \mathrm{c}$. An Aronszajn non-Suslin tree with every element having infinitely many immediate successors is an example in which $\chi < c$. The difference has to be small by the general diagram.

25.76. s=c by Chapter 3 plus the fact that tree algebras are retractive.

25.77. To see that s=hL, take any infinite tree T. Now Treealg(T) embeds in an interval algebra A, and we may assume that Treealg(T) is dense in A (extend the identity from Treealg(T) onto itself to a homomorphism from A into the completion of Treealg(T), and then take the image of A). Hence

$$\mathrm{hL}(A) \ge \mathrm{hL}(\mathrm{Treealg}(T)) \ge \mathrm{c}(\mathrm{Treealg}(T)) = \mathrm{c}(A) = \mathrm{hL}(A).$$

25.78. Suppose that $\mathrm{c}(A) < |A|$. Then clearly T is a tree such that $\mathrm{Inc}(T) < |T|$ and $|T|$ has height $|T|$ but no chains of length $|T|$. Moreover, since $\mathrm{c}(A) < |A|$, the cardinal $|A|$ must be a successor. Thus T is a generalized Suslin tree on a successor cardinal.

25.79. d=hd, since tree algebras are retractive, $\mathrm{d}(B) \le \mathrm{d}(A)$ for B a subalgebra of A, and by Theorem 16.1.

25.80. Recall from Chapter 5 that $\mathrm{d}(A) = |A|$ for A a tree algebra; hence by the general diagram,

$$|A| = \text{h-cof}(A) = \mathrm{Inc}(A) = \mathrm{hd}(A) = \pi(A) = \mathrm{d}(A) = \mathrm{Irr}(A).$$

25.81. For T a chain with order type a regular cardinal we have $|A| = |\mathrm{Ult}(A)|$, since A is superatomic. For T the full binary tree of height ω we have $|A| = \omega$ and $|\mathrm{Ult}(A)| = 2^\omega$.

25.82. For $A = \mathrm{Finco}(\kappa)$ we have $|\mathrm{Ult}(A)| = \kappa$ and $|\mathrm{Id}(A)| = 2^\kappa = |\mathrm{Aut}(A)|$.

25.83. There are rigid tree algebras.

25.84. $|\mathrm{Id}(A)| \le |\mathrm{End}(A)|$ by retractiveness. See Chapter 22 for an example where they differ (consistently). If $\mathrm{c}(A) = |A|$ and cellularity is attained, then $|\mathrm{Id}(A)| = 2^{|A|}$, so that such an example is impossible. On the other hand, if $\mathrm{c}(A) < |A|$, then see 25.78. So no such example is possible in ZFC alone.

25.85. $2^{|A|} = |\mathrm{Sub}(A)|$ since $\{T \uparrow t : t \in T\}$ is an irredundant set.

25.86. $|\mathrm{End}(A)| \leq |\mathrm{Sub}(A)|$ in general; no example of a tree algebra is known where they differ. By 25.85, the problem reduces to the following question; this is problem 71 of Monk [96].

Problem 168. *Is there a tree algebra A such that* $|\mathrm{End}(A)| < 2^{|A|}$*?*

The tree algebra diagram: no other relationships

25.87. $|\mathrm{Aut}A| <$ others: there are rigid tree algebras.

25.88. See Chapter 22 for an example of a tree algebra where $|\mathrm{Id}(A)| < |\mathrm{Aut}(A)|$ (consistently). It is open to construct such an example in ZFC (Problem 166).

25.89. $|\mathrm{Ult}(A)| < |\mathrm{Aut}(A)|$ for the interval algebra on an infinite cardinal.

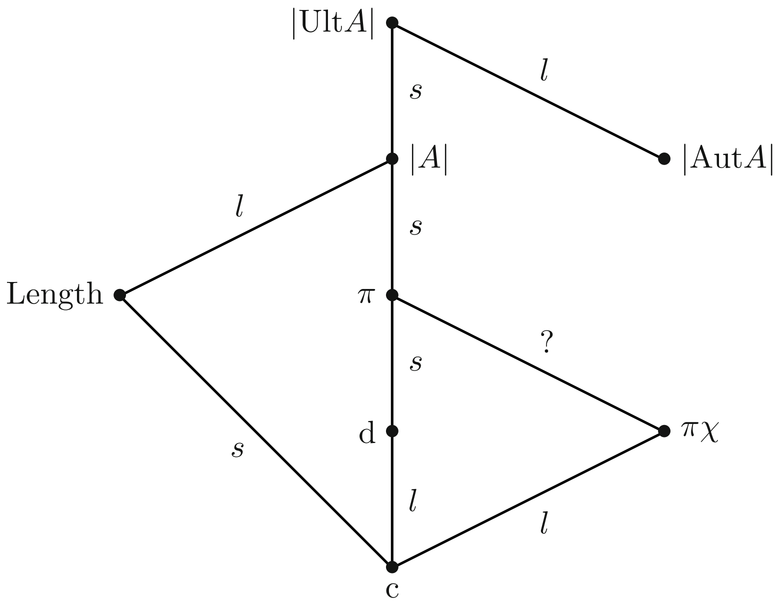
The complete BA diagram: the indicated equalities and inequalities and the "large" and "small" indications

In fact, the "small" indications are clear.

25.90. $\mathrm{c} = \mathrm{Depth}$: obvious.
25.91. $\mathrm{c} < \mathrm{Length}$: $\mathcal{P}(\omega)$.

Complete BAs

$l =$ difference can be small; $s =$"small" difference



25.92. c < d: completions of free algebras.

25.93. c < $\pi\chi$: see 25.94.

25.94. d < π: completion of the free algebra on ω_1 free generators.

25.95. $\pi\chi < \pi$: under GCH these are equal, by an argument of Bozeman. It is not known if this is true in ZFC. See Problem 74.

25.96. Length < Card: a large ccc algebra.

25.97. π < Card: $\mathcal{P}(\kappa)$.

25.98. card = Ind = t = s = χ = hL = Irr = s = hd = Inc = h-cof: these equalities all follow from |Ind| = Card (attained), which is a consequence of the Balcar, Franěk theorem.

25.99. $|\mathrm{Ult}(A)| = |\mathrm{Id}(A)| = |\mathrm{Sub}(A)| = |\mathrm{End}(A)| = 2^{|A|}$: again true since any infinite complete BA A has an independent subset of size $|A|$.

25.100. |Aut| < |Ult|: a rigid complete algebra.

The complete BA diagram: no other relations

25.101. π < Length: $\mathscr{P}(\omega)$.

25.102. Length < $\pi\chi$: the completion of a free algebra.

25.103. Length < d: the completion of a free algebra.

25.104. Under GCH we have d $\leq \pi\chi$, by the result of Bozeman and Chapter 11; it is not known whether this holds in ZFC.

25.105. |Aut| < c: embed $\mathcal{P}(\kappa)$ in a rigid BA.

25.106. Card < |Aut|: the completion of the free BA of size 2^ω.

Diagram for superatomic BAs: the indicated relations, and the "large" and "small" indications

See below.

25.107. Ind = ω since every superatomic BA has countable independence.

25.108. Ind < Depth: Any interval algebra on an uncountable cardinal provides an example; the difference can be arbitrarily large.

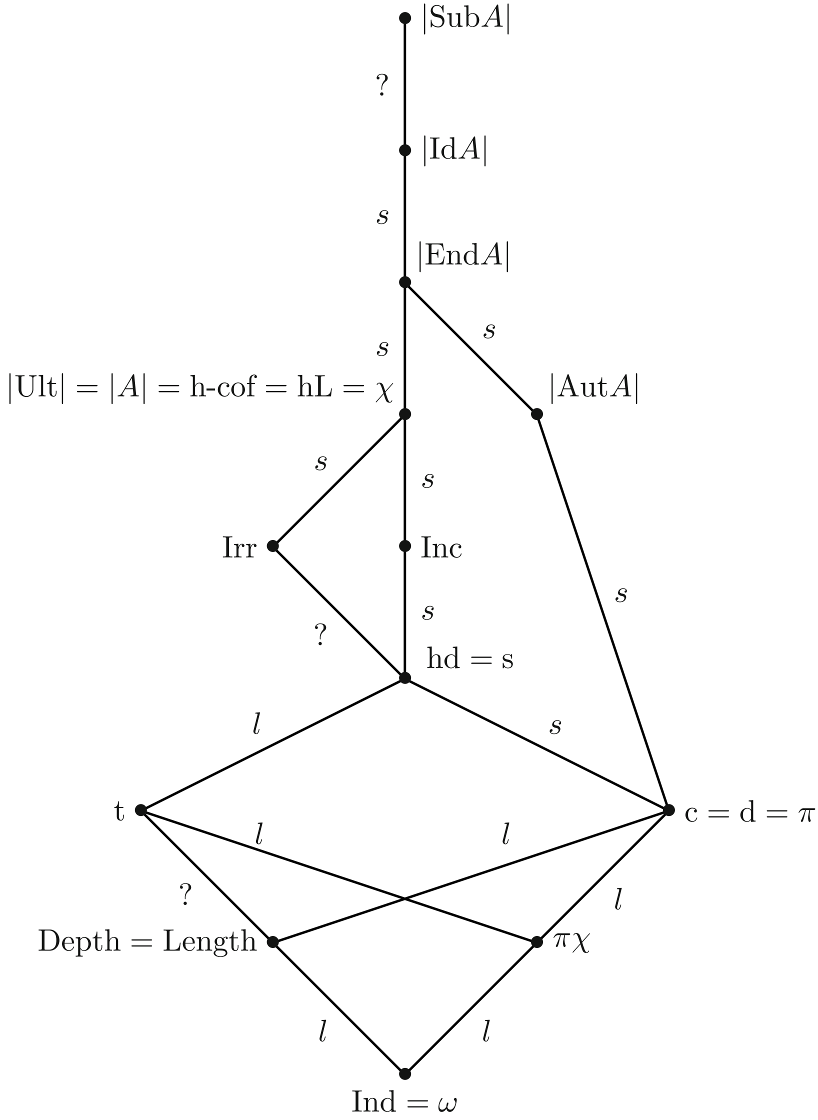
25.109. Depth = Length by Rosenstein [82] Corollary 5.29, p. 88.

25.110. Ind < $\pi\chi$: the interval algebra on a cardinal provides an example with the difference arbitrarily large.

25.111. Depth < t. There is an even stronger example, with c < t. Let $\langle a_\alpha : \alpha < \omega_1\rangle$ be a system of subsets of ω such that for $\alpha < \beta < \omega_1$ we have $a_\alpha \backslash a_\beta$ finite and $a_\beta \backslash a_\alpha$ infinite. For the existence of such a sequence see Blass [10], 6.4. Let A be the subalgebra of $\mathscr{P}(\omega)$ generated by the singletons together with the a_α's. Clearly A is as desired. On the other hand, in Dow, Monk [94] it is shown that if $\kappa \to (\kappa)^{<\omega}_2$, then any superatomic BA with tightness κ^+ also has depth at least κ. This shows that the difference between tightness and depth cannot be arbitrarily large.

Superatomic BAs

$l =$ difference can be small; $s =$"small" difference



25.112. $\pi\chi < \mathrm{t}$: see Chapter 11; the difference can be large.

25.113. Obviously $\mathrm{c} = \mathrm{d} = \pi =$ number of atoms.

25.114. Depth $<$ c: they differ in a finite-cofinite algebra, where the difference can be large.

25.115. $\pi\chi <$ c: see Chapter 11; the difference can be large.
25.116. hd $=$ s: see the characterizations of hd and s.
25.117. t $<$ s, and the difference can be arbitrarily large: a finite-cofinite algebra.
25.118. c $<$ s: Take a family $\mathscr{A}$ of 2^ω almost disjoint subsets of ω, and consider the BA generated by $\mathscr{A} \cup \{\{i\} : i \in \omega\}$.
25.119. s $<$ Inc. Under $\diamondsuit$, Shelah constructed a thin-tall BA A with countable spread. Then $A \times A$ has countable spread too, while its incomparability is ω_1. The example of Bonnet, Rubin [11], also constructed using $\diamondsuit$, can also be used for this purpose. We do not know whether there is an example in ZFC; this is Problem 72 in Monk [96]:

Problem 169. *Is there an example in ZFC of a superatomic BA A such that* $\mathrm{s}A < \mathrm{Inc}A$*?*

25.120. Rosłanowski, Shelah [00] showed that it is consistent to have a superatomic BA in which s is less than Irr. We do not know whether this can be done in ZFC; this is a version of Problem 73 of Monk [96]:

Problem 170. *Can one construct in ZFC a superatomic BA A such that* $\mathrm{s}(A) < \mathrm{Irr}(A)$*?*

25.121. card $=$ h-cof $=$ hL $= \chi$: $\chi =$ Card by Chapter 14, and the other equalities follow.
25.122. In Bonnet, Rubin [11] a superatomic algebra of power ω_1 is constructed using $\diamondsuit$ in which Inc is countable.

The following is Problem 74 in Monk [96]:

Problem 171. *Can one construct in ZFC a superatomic algebra A with the property that* $\mathrm{Inc}(A) < |A|$*?*

25.123. In Rosłanowski, Shelah [00] a superatomic BA is forced in which $\mathrm{Irr}(A) < |A|$.

Problem 75 in Monk [96] is:

Problem 172. *Can one construct in ZFC a superatomic algebra A with the property that* $\mathrm{Irr}(A) < |A|$*?*

25.124. $|A| = |\mathrm{Ult}A|$: see the Handbook.
25.125. $|A| < |\mathrm{End}(A)|$ in a finite-cofinite algebra.
25.126. M. Rubin constructed under $\diamondsuit$ a BA A such that $|\mathrm{Aut}(A)| < |A|$ (unpublished, December 1992) in particular, $|\mathrm{Aut}(A)| < |\mathrm{End}(A)|$. In Shelah [01] an algebra A with $|\mathrm{Aut}(A)| < |\mathrm{End}(A)|$ was constructed in ZFC. This solves Problem 76 of Monk [96].

25.127. c $< |\mathrm{Aut}|$. The BA of finite and cofinite subsets of κ gives an example where they differ.

25.128. $|\mathrm{End}(A)| \le |\mathrm{Id}(A)|$ for A superatomic. For, let κ be the number of atoms of A. Then $|\mathrm{Ult}(A)| = |A| \le 2^\kappa \le |\mathrm{Id}(A)|$, and hence by Chapter 21, $|\mathrm{End}(A)| \le |\mathrm{Ult}(A)|^{\mathrm{d}(A)} \le 2^\kappa \le |\mathrm{Id}(A)|$. In the example of 25.118 we have $|\mathrm{End}(A)| = 2^\omega$ and $|\mathrm{Id}(A|) = 2^{2^\omega}$.

25.129. Rosłanowski, Shelah [00] showed that it is consistent to have a superatomic BA A where $|\mathrm{Id}(A)| < |\mathrm{Sub}(A)|$. We do not know whether this can be done in ZFC; this is a version of Problem 77 of Monk [96]:

Problem 173. *In ZFC, can one have* $|\mathrm{Id}(A)| < |\mathrm{Sub}(A)|$ *in a superatomic BA?*

Superatomic BAs, no additional relationships

25.130. $\pi\chi <$ Depth: see Chapter 11.

25.131. Depth $< \pi\chi$: Dow, Monk [94] constructed an example.

25.132. $\mathrm{t} < \mathrm{c}$: the finite-cofinite algebra on κ.

25.133. Rosłanowski, Shelah [00] showed that it is consistent to have a superatomic BA A with the property that $\mathrm{Inc}(A) < \mathrm{Irr}(A)$. It is open whether this can be done in ZFC; this is a version of Problem 78 of Monk [96]:

Problem 174. *Can one construct in ZFC a superatomic BA A such that* $\mathrm{Inc}(A) < \mathrm{Irr}(A)$*?*

25.134. Also Rosłanowski, Shelah [00] showed that it is consistent to have a superatomic BA A with the property that $\mathrm{Irr}(A) < \mathrm{Inc}(A)$. It is open whether this can be done in ZFC; this is a version of Problem 79 of Monk [96]:

Problem 175. *Can one construct in ZFC a superatomic BA A such that* $\mathrm{Irr}(A) < \mathrm{Inc}(A)$*?*

25.135. Card $< |\mathrm{Aut}|$: a finite-cofinite algebra.

25.136. $|\mathrm{Aut}|$ small relative to "lower" functions. Recall 25.127. Shelah [01] constructs a superatomic BA A such that $|\mathrm{Aut}(A)| < |A|$. This solves problem 80 of Monk [96].

Rosłanowski, Shelah [00] showed that it is consistent to have a superatomic BA A such that $|\mathrm{Aut}(A)| < \mathrm{t}(A)$. It is open to do this in ZFC; this is a version of Problem 81 of Monk [96]:

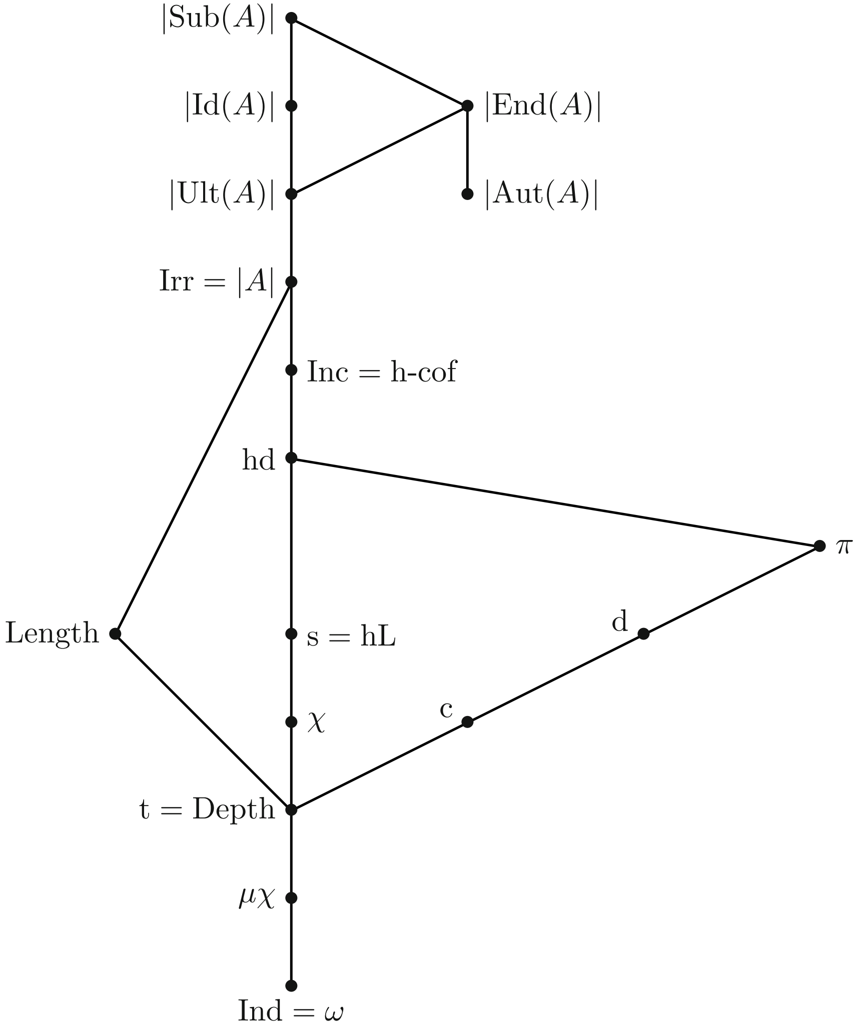
Problem 176. *In ZFC, is there a superatomic BA A such that* $|\mathrm{Aut}(A)| < \mathrm{t}(A)$*?*

25.137. The relationship between Card and $|\mathrm{Aut}|$ is not completely clear, but we indicate some other facts; see 25.136. Recall from Chapter 20 that an infinite superatomic BA has at least 2^ω automorphisms. The initial chain algebra A on ${}^{\le\omega}2$ is such that $|A| = |\mathrm{Aut}(A)| = 2^\omega$.

Now assume that $2^\omega = \omega_2$, $2^{\omega_1} = \omega_3$, and $2^{\omega_2} = \omega_4$. Let T be the tree ${}^{\le\omega}\omega_1$, and let $A = \mathrm{Init}(T)$, the initial chain algebra on T. Note that $|A| = \omega_2$. For each permutation φ of ω_1 there is an automorphism φ' of A such that $\varphi'(T \downarrow t) =$

$T \downarrow (\varphi \circ t)$ for every $t \in T$. If $\varphi \neq \psi$, then $\varphi' \neq \psi'$. Thus this gives $2^{\omega_1} = \omega_3$ automorphisms; since A has only ω_1 atoms, it follows that $|\mathrm{Aut}A| = \omega_3$. Note that $2^{|A|} = \omega_4$. Thus $|A| < |\mathrm{Aut}(A)| < 2^{|A|}$.

Initial chain algebras over a pseudo-tree

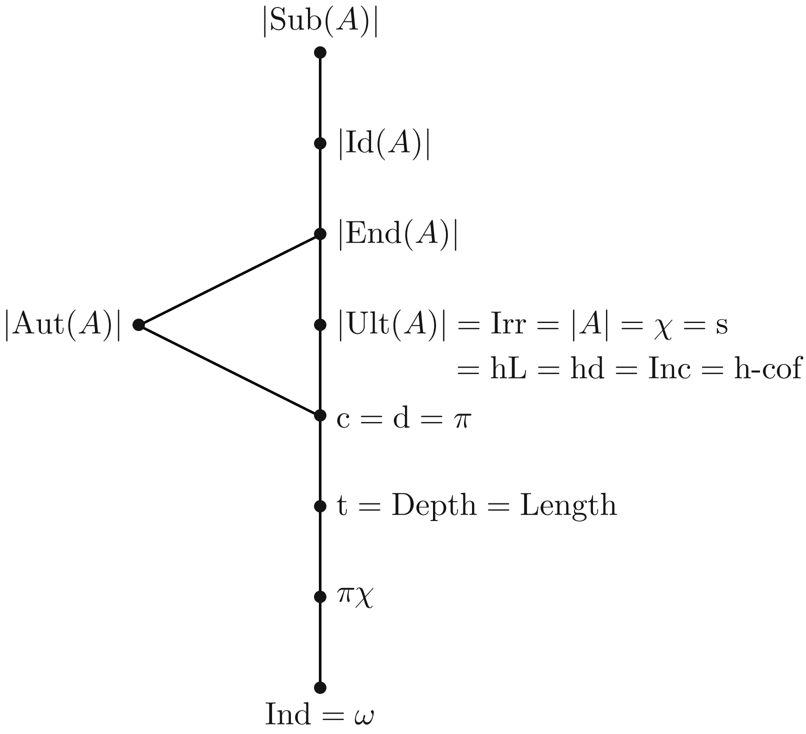


For this diagram, see Baur [00]. Examples in interval algebras exist for $\omega < \pi\chi$, $\pi\chi <$ t, t $<$ c, $\chi <$ s, s $<$ hd, hd $<$ Inc, Depth $<$ Length, c $<$ d. An example with $d <$ hd is given by the full binary tree ${}^{<\omega}2$; this also works for t $< \chi$ and Length $<$ *card*. The Bonnet, Shelah interval algebra gives an example with Inc $<$ *card*, assuming that $\mathrm{cf}(2^{\omega})$ is a successor cardinal.

Problem 177. *Is there an example in ZFC with* Inc $<$*card for initial chain algebras?*

A rigid interval algebra shows that one can have $|\mathrm{Aut}(A)| < |\mathrm{End}(A)|$ for initial chain algebras.

Initial chain algebras over a tree



See Baur [00]. $\mathrm{t} < \mathrm{c}$ for $\mathrm{init}(T)$, T having only roots, uncountably many of them. $\mathrm{Ind} < \pi\chi$ for $\mathrm{init}(\omega_1)$.

Small functions, atomless BAs

See the diagram on top of the next page.

25.138. $\mathfrak{r} \le \mathfrak{i}$: see Proposition 10.21.
25.139. $\mathfrak{r} \le \pi\chi_{\mathrm{inf}}$: see Theorem 6.28.
25.140. $\mathfrak{p} \le \mathfrak{f}$: Theorem 12.21.
25.141. $\mathfrak{p} \le \mathfrak{u}$: this is easy to check.
25.142. $\mathfrak{p} \le \mathrm{s}_{\mathrm{mm}}$: Proposition 13.20.
25.143. $\mathfrak{p} \le \mathfrak{a}$: Proposition 4.47(iii).

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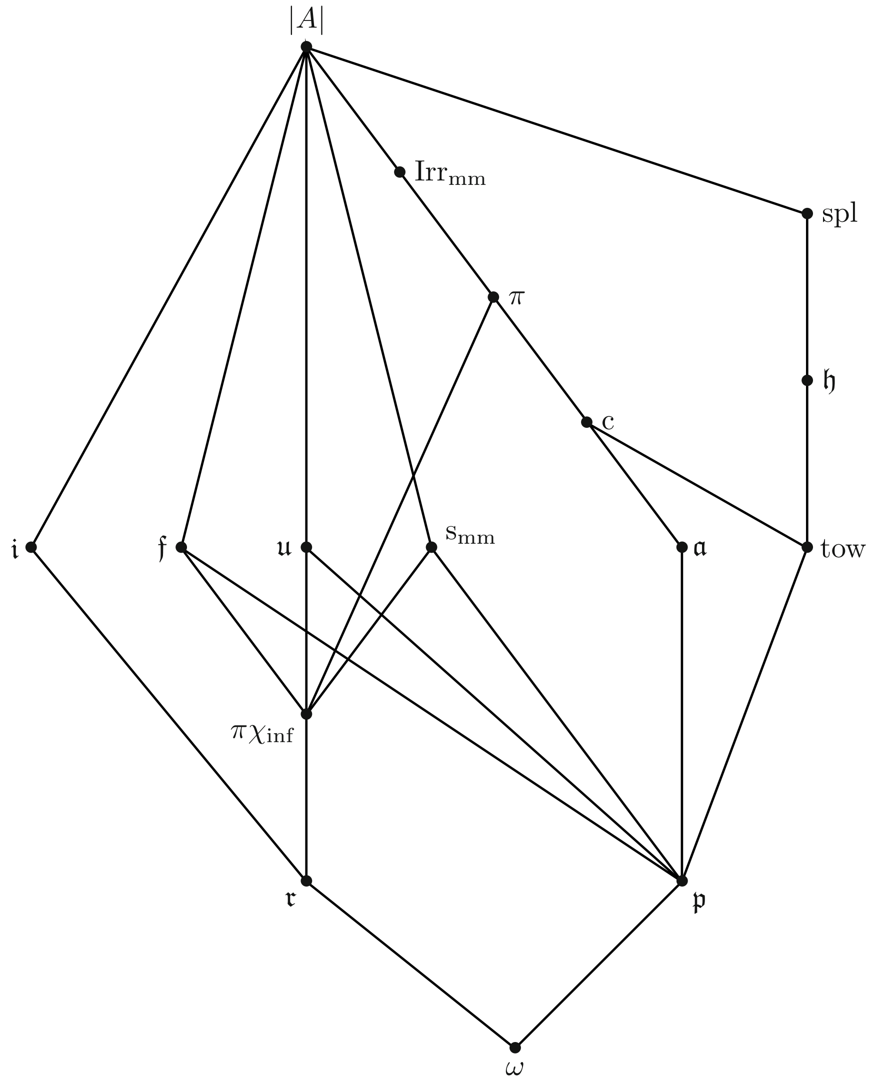
Small functions, atomless Boolean algebras



25.144. $\mathfrak{p} \le \mathrm{tow}$: Proposition 4.47(iii).
25.145. $\pi\chi_{\mathrm{inf}} \le \mathfrak{f}$: Proposition 12.20.
25.146. $\pi\chi_{\mathrm{inf}} \le \mathfrak{u}$: obvious.
25.147. $\pi\chi_{\mathrm{inf}} \le \mathrm{s}_{\mathrm{mm}}$: We will apply Theorem 6.28. Let X be maximal ideal independent with $|X| = \mathrm{s}_{\mathrm{mm}}(A)$. Suppose that $2 \le m < \omega$. We claim that

$$\left\{y \cdot \prod\nolimits_{x\in F} -x : y \in X, F \in [X\backslash\{y\}]^{<\omega}\right\}$$

is m-dense. For, suppose that $\langle b_i : i < m\rangle$ is a weak partition of A. For any $i < m$ we have two possibilities:

(1) There is a finite $F \in [X]^{<\omega}$ such that $b_i \leq \sum F$.
(2) There exist $x \in X$ and a finite $F \subseteq X \setminus \{x\}$ such that $x \leq b_i + \sum F$.

If (2) holds for some i, clearly the desired conclusion follows. If (1) holds for all i, ideal independence of X is contradicted.

25.148. $\pi\chi_{\mathrm{inf}} \leq \pi$: obvious.
25.149. $\mathfrak{a} \leq \mathrm{c}$: obvious.
25.150. $\mathrm{tow} \leq \mathrm{c}$: obvious.
25.151. $\mathrm{tow} \leq \mathfrak{h}$: Proposition 4.62.
25.152. $\mathfrak{h} \leq \mathrm{spl}$: Proposition 4.63.
25.153. $\pi \leq \mathrm{Irr}_{\mathrm{mm}}$: Handbook, Proposition 4.23.

Concerning counterexamples, we have the following results and problems.

25.154. It is a problem to find a BA A such that $\mathfrak{i}(A) < \pi\chi(A)$. (Problem 101).
25.155. An example with $\mathfrak{i}(A) < \mathfrak{p}(A)$ is given in Monk [01a].
25.156. For an example with $\mathfrak{f}(A) < \mathfrak{i}(A)$ see example 12.22.
25.157. Under CH there is a BA A with $\mathfrak{f}(A) < \mathfrak{u}(A)$. It is a problem to do this in ZFC. (Problem 131)
25.158. A BA A with $\mathfrak{f}(A) < \mathrm{s}_{\mathrm{mm}}(A)$ is described in Monk [11].
25.159. Concerning $\mathfrak{f}$ and $\mathfrak{a}$ we have the following problem:

Problem 178. *Is there an atomless BA A such that* $\mathfrak{f}(A) < \mathfrak{a}(A)$*?*

25.160. A BA A with $\mathfrak{f}(A) < \mathrm{tow}(A)$ is given in Monk [11].
25.161. For a BA A such that $\mathfrak{u}(A) < \mathfrak{i}(A)$ see McKenzie, Monk [04].
25.162. It is a problem to get a BA with $\mathfrak{u} < \mathfrak{f}$:

Problem 179. *Is there an atomless BA A such that* $\mathfrak{u}(A) < \mathfrak{f}(A)$*?*

25.163. An example with $\mathfrak{u} < \mathrm{s}_{\mathrm{mm}}$ is given in Monk [08].
25.164. An example with $\mathfrak{u} < \mathfrak{a}$ is given in Monk [01a].
25.165. An example with $\mathfrak{u} < \mathrm{tow}$ is given in Monk [01a].
25.166. It is a problem to get a BA with $\mathrm{s}_{\mathrm{mm}} < \mathfrak{i}$. (Problem 126)
25.167. It is a problem to get a BA with $\mathrm{s}_{\mathrm{mm}} < \mathfrak{f}$:

Problem 180. *Is there an atomless BA A such that* $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{f}(A)$*?*

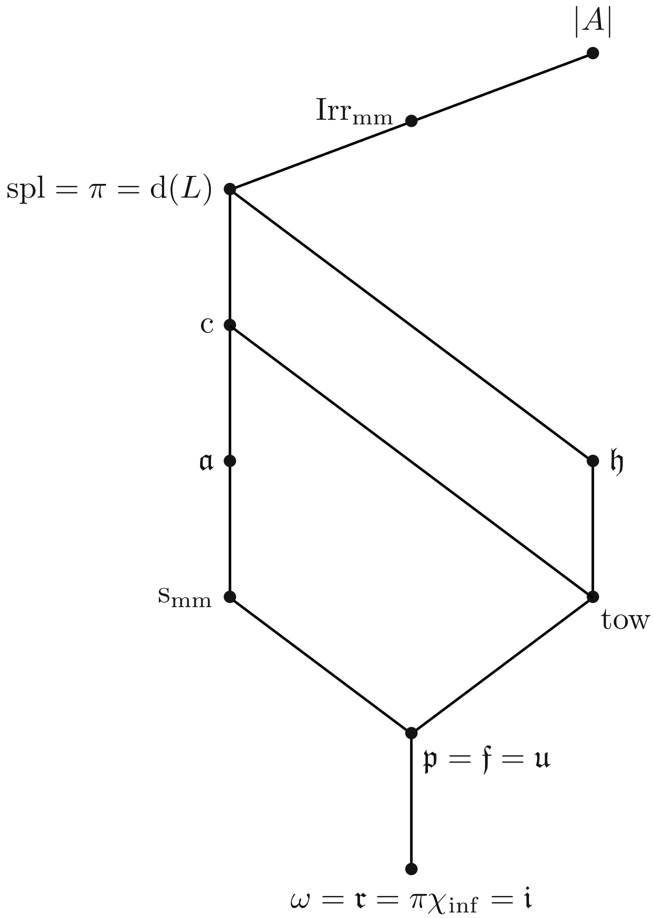
25.168. Under CH there is a BA A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{u}(A)$, but it is a problem to get one in ZFC. (Problem 134)
25.169. A BA A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{a}(A)$ is given in Theorem 13.19.
25.170. See Chapter 13 for a BA A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathrm{tow}(A)$.
25.171. The exact place of $\mathrm{Irr}_{\mathrm{mm}}$ in the diagram is not known:

Problem 181. *Are there examples with* $\mathrm{Irr}_{\mathrm{mm}}$ *less than the other functions?*

25.172. An example with $\mathrm{spl}(A) < \mathfrak{r}(A)$ is given in Monk [01a].

25.173. An example with $\mathrm{spl}(A) < \mathfrak{a}(A)$ is given in McKenzie, Monk [04].

Small functions, atomless interval algebras



25.174. $\mathfrak{i} = \mathfrak{r} = \omega$ in interval algebras, since an interval algebra does not have an uncountable independent subset.

25.175. $\pi\chi_{\mathrm{inf}}(A) = \omega$ for A an interval algebra, since there is a point or gap with left or right character ω in any linear order; see the description of $\pi\chi_{\mathrm{inf}}$ for interval algebras at the end of Chapter 11.

25.176. For $\mathfrak{p} = \mathfrak{f} = \mathfrak{u}$ for interval algebras, see Monk [12].

25.177. $\mathfrak{p} \leq \mathrm{s}_{\mathrm{mm}}$: true in general.

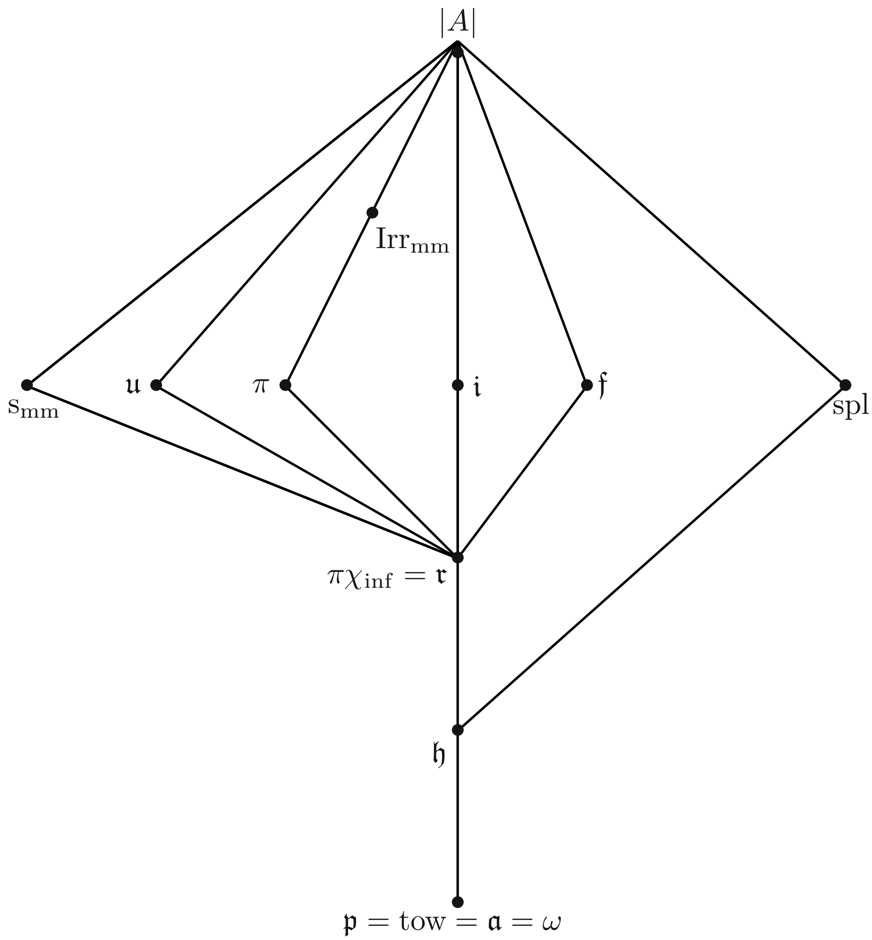
25.178. $\mathfrak{f} \leq \mathrm{tow}$ for interval algebras: see Corollary 12.24.

25.179. For $\mathrm{s_{mm}} \le \mathfrak{a}$ for interval algebras, see Monk [12].
25.180. For $\mathrm{spl} = \pi = \mathrm{d}(L)$ see Monk [01a].
25.181. For an example of an interval algebra with $\omega < \mathfrak{p}$ see Monk [01a].
25.182. For an example of an interval algebra with $\mathfrak{a} < \mathrm{tow}$ see Monk [02].
25.183. It is a problem to find an interval algebra with $\mathfrak{h} < \mathrm{s_{mm}}$:

Problem 182. *Is there an atomless interval algebra A such that $\mathfrak{h}(A) < \mathrm{s_{mm}}(A)$?*

25.184. For an example of an atomless interval algebra A such that $\mathfrak{h}(A) < \mathrm{spl}(A)$, see Monk [01a].
25.185. It is a problem whether there is an atomless interval algebra A with $\pi(A) < \mathrm{Irr_{mm}}(A)$:

Small functions, atomless complete BAs

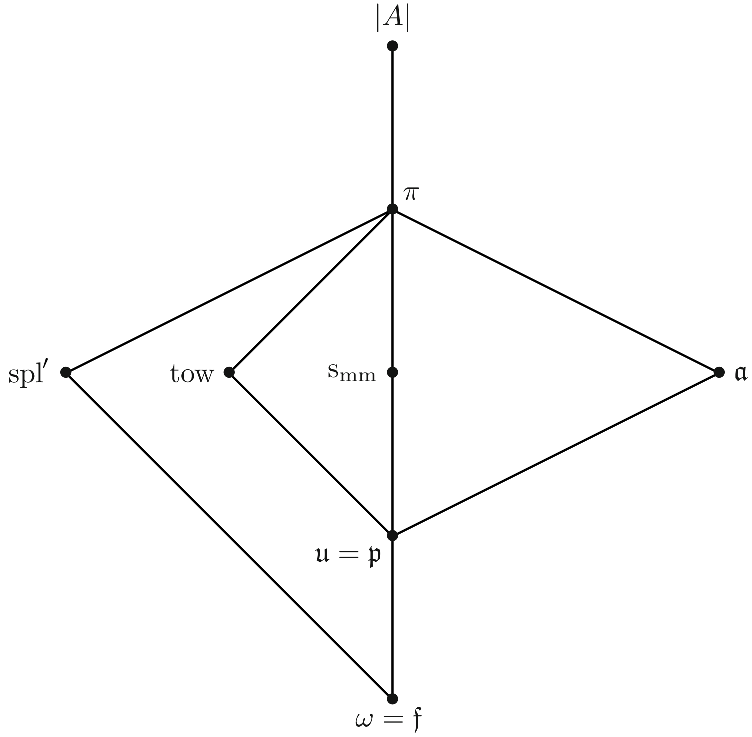


Problem 183. *Is there is an atomless interval algebra A with $\pi(A) < \mathrm{Irr_{mm}}(A)$?*

Small functions, atomless complete BAs

25.186. Clearly $\mathfrak{p} = \mathrm{tow} = \mathfrak{a} = \omega$ in any complete BA.
25.187. It is shown in Monk [01a] that $\mathfrak{h} \leq \mathfrak{r}$ for complete BAs.
25.188. See Theorem 6.46 for $\pi\chi_{\mathrm{inf}} = \mathfrak{r}$ for complete BAs.
25.189. In general $\mathfrak{r} \leq \mathfrak{i}, \mathfrak{f}, \mathfrak{u}, \mathrm{s_{mm}}, \pi$, and $\pi \leq \mathrm{Irr_{mm}}$.

Small functions, superatomic Boolean algebras



25.190. There are several questions concerning whether the diagram is exactly as indicated. We mention only examples that are known, described in Monk [01a]: $\omega < \mathfrak{h}$, $\mathfrak{i} < \mathfrak{u}$, $\mathfrak{u} < \mathrm{spl}$, $\mathfrak{i} < \mathrm{spl}$, $\mathrm{spl} < \mathfrak{r}, \mathfrak{i}, \mathfrak{u}$.

Here $\mathrm{spl}'(A) = \min\{|X| : \forall a \in A[|A \upharpoonright a| \geq \omega \to \exists x \in X[x \cdot a \neq 0 \neq -x \cdot a]]\}$.

25.191. $\mathfrak{f} = \omega$ is shown in Monk [11].
25.192. $\mathfrak{u} = \mathfrak{p}$ is shown in Monk [01a].
25.193. There are examples in Monk [12] with $\mathrm{s_{mm}} < \mathrm{spl}$, $\mathrm{tow} < \mathrm{s_{mm}}$, and $\mathrm{s_{mm}} <$ tow. Other possible improvements of the diagram are unknown.

26 Examples

We determine our cardinal functions on the following examples, as much as possible; see also the following table:

1. The finite-cofinite algebra on κ.
2. The free algebra on κ free generators.
3. The interval algebra on the reals.
4. $\mathscr{P}(\kappa)$.
5. The interval algebra on κ.
6. $\mathscr{P}(\omega)/\text{fin}$.
7. The Alexsandroff duplicate of a free algebra.
8. The completion of a free algebra.
9. The countable-cocountable algebra on ω_1.
10. A compact Kunen line.
11. The Baumgartner-Komjath algebra.
12. The Rubin algebra.

We do not have to consider all of our 21 functions for each of them, since usually the determination of some key functions says what the rest are; see the diagrams.

1. The finite-cofinite algebra on κ

1. $\mathrm{c}(A) = \kappa$.
2. $\mathrm{t}(A) = \omega$. See the beginning of Chapter 12.
3. $|\mathrm{Ult}(A)| = \kappa$.
4. $|\mathrm{Aut}(A)| = 2^\kappa$.

2. The free BA on κ free generators

1. $\mathrm{Length}(A) = \mathrm{c}(A) = \omega$. See Handbook, Part I, Corollaries 9.17 and 9.18.
2. $\mathrm{d}(A) =$ the smallest cardinal λ such that $\kappa \leq 2^\lambda$; see Corollary 5.6.
3. $\pi\chi(A) = \kappa$: see Corollary 11.12.
4. $\mathrm{Ind}(A) = \kappa$.
5. $|\mathrm{Ult}(A)| = 2^\kappa$.
6. $|\mathrm{Aut}(A)| = 2^\kappa$.

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3. The interval algebra on the reals

1. $\pi(A) = \omega$.
2. $\mathrm{Inc}(A) = 2^\omega$. For example, $\{[r, r+1) : r \in \mathbb{R}\}$ is incomparable.
3. $|\mathrm{End}(A)| = 2^\omega$; Corollary 21.3.
4. $|\mathrm{Aut}(A)| = 2^\omega$; this is clear by the above, since it is easy to exhibit 2^ω automorphisms.

4. $\mathscr{P}(\kappa)$

1. $\mathrm{c}(A) = \kappa$.
2. $\pi(A) = \kappa$.
3. $\pi\chi(A) = \kappa$. An easy argument gives this.
4. $\mathrm{Length}(A) = \mathrm{Ded}(\kappa)$. See Chapter 7.
5. $|\mathrm{Ult}(A)| = 2^{2^\kappa}$.
6. $|\mathrm{Aut}(A)| = 2^\kappa$.

5. The interval algebra on κ

1. $\pi\chi(A) = \kappa$. See the end of Chapter 11. $\pi\chi(A)$ is attained if κ is regular, otherwise not.
2. $|\mathrm{Ult}(A)| = \kappa$. See Theorem 17.10 of Part I of the BA handbook.
3. $|\mathrm{Aut}(A)| = 2^\kappa$. See the end of Chapter 20.

6. $\mathscr{P}(\omega)/\mathrm{fin}$

1. $\mathrm{Depth}(A) \geq \omega_1$. It is consistent that it is ω_1 and $\neg$CH holds. Under MA, it is 2^ω. See the end of Chapter 4.
2. $\mathrm{Length}(A) = 2^\omega$.
3. $\mathrm{c}(A) = 2^\omega$.
4. $\pi\chi(A) \geq \mathrm{cf}2^\omega$. $\mathrm{Con}(2^\omega = \aleph_{\omega_1} + \pi\chi(A) = \omega_1)$; see van Mill [84], p. 558.
5. $\mathrm{Ind}(A) = 2^\omega$.
6. $|\mathrm{Ult}(A)| = 2^{2^\omega}$.
7. $|\mathrm{Aut}(A)|$ can consistently be 2^ω or 2^{2^ω}; see van Mill [84], p. 537.

7. The Alexsandroff duplicate of a free BA

We use notation as in Chapter 14. Thus B is a free BA of size κ and $\mathrm{Dup}(B)$ is its Alexsandroff duplicate.

1. $\mathrm{c}(\mathrm{Dup}(B)) = 2^\kappa$.
2. $\chi\mathrm{Dup}(B) = \kappa$. See Chapter 14.
3. $\mathrm{Length}(\mathrm{Dup}(B)) = \omega$. In fact, suppose that $Y \subseteq \mathrm{Dup}(B)$, Y a chain, $|Y| = \omega_1$. Define $(a, X) \equiv (b, Z)$ iff $(a, X), (b, Z) \in Y$ and $a = b$. Then since B has no uncountable chains, there are only countably many $\equiv$-classes. So there is a class, say K, which has ω_1 elements; say that a is the first member of each ordered pair in K. Then $M \stackrel{\text{def}}{=} \{X : (a, X) \in K\}$ is of size ω_1, is a chain

under inclusion, and if $X, Z \in M$ with $X \subseteq Z$, then $Z \backslash X$ is finite. Clearly this is impossible.

4. $\mathrm{Ind}(\mathrm{Dup}(B)) = \kappa$.
5. $\pi\chi(\mathrm{Dup}(B)) = \omega$. (This corrects a mistake in Monk [90].) For, let H be a non-principal ultrafilter on $\mathrm{UltDup}(B)$. By the description of ultrafilters in Proposition 1.19, there is an ultrafilter G on B such that

$$H = \{(a, X) : a \in G,\ X \subseteq \mathrm{Ult}(B),\ \mathcal{S}(a) \triangle X \text{ is finite}\}.$$

Let $\langle x_\alpha : \alpha < \kappa \rangle$ be the system of free generators of B (without repetitions). Choose $\varepsilon \in {}^\kappa 2$ such that $x_\alpha^{\varepsilon\alpha} \in G$ for all $\alpha < \kappa$. For each $n < \omega$ let F_n be an ultrafilter on B such that $x_\alpha^{\varepsilon\alpha} \in F_n$ for all $\alpha \neq n$ and $x_n^{1-\varepsilon n} \in F_n$. We claim that $\{(0, \{F_n\}) : n < \omega\}$ is dense in H. For, let $y \in H$. Without loss of generality y has the form

$$(x_{\alpha_1}^{\varepsilon\alpha_1} \cdot \ldots \cdot x_{\alpha_m}^{\varepsilon\alpha_m}, X), \quad \mathcal{S}(x_{\alpha_1}^{\varepsilon\alpha_1} \cdot \ldots \cdot x_{\alpha_m}^{\varepsilon\alpha_m}) \triangle X \text{ finite}.$$

Choose $n < \omega$ so that $n \neq \alpha_1, \ldots, \alpha_m$ and $F_n \notin \mathcal{S}(x_{\alpha_1}^{\varepsilon\alpha_1} \cdot \ldots \cdot x_{\alpha_m}^{\varepsilon\alpha_m}) \triangle X)$. Since $F_n \in \mathcal{S}(x_{\alpha_1}^{\varepsilon\alpha_1} \cdot \ldots \cdot x_{\alpha_m}^{\varepsilon\alpha_m})$, it follows that $F_n \in X$, as desired.
6. $|\mathrm{Ult}(\mathrm{Dup}(B))| = 2^\kappa$. See the description of ultrafilters in Chapter 14.
7. $|\mathrm{Id}(\mathrm{Dup}(B))| = 2^{2^\kappa}$.
8. In an email message of January 1992, Sabine Koppelberg shows that $\mathrm{Aut}(\mathrm{Dup}(A))$ has exactly 2^κ elements, answering Problem 64 in Monk [90]. She also showed that $|\mathrm{End}(\mathrm{Dup}(A))| = 2^{2^\kappa}$. We give her proofs here.
   (a) For any BA A, let

$$\mathrm{Dup}'(A) = \langle \mathcal{S}[A] \cup \{\{F\} : F \in \mathrm{Ult}(A)\rangle^{\mathscr{P}\mathrm{Ult}(A)}.$$

   For A atomless, $\mathrm{Dup}(A)$ is isomorphic to $\mathrm{Dup}'(A)$; an isomorphism is given by $f(a, X) = X$ for all $(a, X) \in \mathrm{Dup}(A)$, as is easily checked.
   (b) Any automorphism of A induces an automorphism of $\mathrm{Dup}(A)$. Namely, if f is an automorphism of A, define $f^+(a, X) = (f(a), \{f[F] : F \in X\})$; it is easy to check that f^+ is an automorphism of $\mathrm{Dup}(A)$. Clearly $f^+ \neq g^+$ for distinct f, g.
   (c) In $\mathrm{Dup}'(A)$, $\{F\} = \bigcap_{a \in F} \mathcal{S}(a)$. Hence if f and g are automorphisms of $\mathrm{Dup}'(A)$ and $f \restriction \mathcal{S}[A] = g \restriction \mathcal{S}[A]$, then $f = g$.
   (d) From (a)–(c) it follows that $|\mathrm{Aut}(\mathrm{Dup}(A))| = 2^\kappa$ for A free on κ free generators.
   (e) Suppose that A is atomless. Let f be a homomorphism from $\mathcal{S}[A]$ into $\mathrm{Dup}'(A)$ and g a homomorphism from $\mathrm{Finco}(\mathrm{Ult}(A))$ into $\mathrm{Dup}'(A)$. Then $f \cup g$ extends to an endomorphism of $\mathrm{Dup}'(A)$ iff the following condition holds:

   (∗) If $a \in A$, M is a finite subset of $\mathrm{Ult}(A)$, and $M \subseteq \mathcal{S}(a)$, then $g(M) \subseteq f(\mathcal{S}(a))$.

In fact, $\Rightarrow$ is obvious. For $\Leftarrow$, to apply Sikorski's criterion suppose that $\mathcal{S}(a) \cap N = 0$, where $a \in A$ and $N \in \text{Finco}(\text{Ult}(A))$. If N is cofinite, then $\mathcal{S}(a)$ is finite, hence $a = 0$, so $f(\mathcal{S}(a)) \cap g(N) = 0$. If N is finite, then $f(\mathcal{S}(a)) \cap g(N) = 0$ by $(*)$.

(f) Suppose that π is a one-one mapping of $\text{Ult}(A)$ into $\text{Ult}(A)$. Then we can define an endomorphism π^+ of $\text{Finco}(\text{Ult}(A))$ by setting $\pi^+(\mathscr{F}) = \{\pi(F) : F \in \mathscr{F}\}$ for $\mathscr{F}$ a finite subset of $\text{Ult}(A)$, and $\pi^+(\mathscr{F}) = \text{Ult}(A) \backslash \pi^+(\text{Ult}(A) \backslash \mathscr{F})$ for $\mathscr{F}$ a cofinite subset of $\text{Ult}(A)$. Then for f a homomorphism from $\mathcal{S}[A]$ into $\text{Dup}'(A)$ the criterion $(*)$ for $f \cup \pi^+$ to extend to an endomorphism of $\text{Dup}'(A)$ becomes

$(**)$ If $a \in A$ and $F \in \mathcal{S}(a)$, then $\pi(F) \in f(\mathcal{S}(a))$.

(g) Now suppose that A is free on κ free generators. We show that $\text{Dup}(A)$ has 2^{2^κ} endomorphisms. Now A is isomorphic to $A \oplus A$; let h be an isomorphism of $A \oplus A$ onto A. Let k be the embedding of A onto the first factor of $A \oplus A$, and l the embedding onto the second factor. Set $f = h \circ k$. Thus f is a one-one endomorphism of A, so the dual mapping f^{-1} from $\text{Ult}(A)$ to $\text{Ult}(A)$ is onto. Now

$(\star)$ For each $G \in \text{Ult}(A)$, the set $\{F : f^{-1}[F] = G\}$ has at least two elements.

In fact, we claim that for any non-zero element b of A the set $f[G] \cup \{h(l(b))\}$ has the finite intersection property. If not, there is an element $a \in G$ such that $f(a) \cdot h(l(b)) = 0$; since $f = h \circ k$, it follows that $k(a) \cdot l(b) = 0$, contradiction. This claim being true, if we take $b \in A \backslash \{0, 1\}$, we can get ultrafilters F, F' in A such that $f[G] \cup \{h(l(b))\} \subseteq F$ and $f[G] \cup \{h(l(-b))\} \subseteq F'$. Then $F \neq F'$ and $f^{-1}[F] = G = f^{-1}[F']$, as desired in $(\star)$.

Now take a function π from $\text{Ult}(A)$ into $\text{Ult}(A)$ such that $\pi G \in \{F : f^{-1}[F] = G\}$ for all $G \in \text{Ult} A$. Let $f'[\mathcal{S}(a)] = \mathcal{S}(f(a))$ for all $a \in A$. Then f' is a homomorphism from $\mathcal{S}[A]$ into $\text{Dup}'(A)$. If $G \in \mathcal{S}(a)$, then $\pi(G) \in \{F : f^{-1}[F] = G\}$, so $f^{-1}[\pi(G)] = G$, $a \in f^{-1}[\pi(G)]$, $f(a) \in \pi(G)$, and $\pi(G) \in \mathcal{S}(f(a)) = f'[\mathcal{S}(a)]$. This means that $(**)$ holds for π and f', and so $f' \cup \pi^+$ extends to an endomorphism of $\text{Dup}'(A)$. Clearly $\pi^+ \neq \sigma^+$ for distinct π, σ, so we have exhibited 2^{2^κ} endomorphisms of $\text{Dup}'(A)$, as desired.

8. The completion of a free algebra

Let B be a free algebra of size κ, A its completion.

1. $\text{c}(A) = \omega$.
2. $\text{Length}(A) = 2^\omega$. In fact, $\geq$ is clear. Suppose that L is a chain of size $(2^\omega)^+$. Using a well-ordering of L and the partition relation $(2^\omega)^+ \to (\omega_1)^2_\omega$, we get an uncountable well-ordered chain in A, contradiction.

3. $\mathrm{d}(A)$ is the least cardinal λ such that $\kappa \leq 2^\lambda$. For, this is true for B itself by Chapter 5, and an application of Sikorski's extension theorem shows that it is true of A.
4. $\pi\chi(A) = \kappa$. For, $\leq$ is clear. Suppose that F is an ultrafilter on A and D is a π-base for F. Without loss of generality $D \subseteq B$. Then D is dense in $F \cap B$, so $|D| \geq \kappa$.
5. $\pi(A) = \kappa$ by the same argument.
6. $|A| = \kappa^\omega$.
7. $|\mathrm{Ult}(A)| = 2^{\kappa^\omega}$.
8. $|\mathrm{Aut}(A)| = 2^\kappa$. In fact, any automorphism of A is uniquely determined by its restriction to B, and there are only 2^κ mappings of B into A. On the other hand, there are at least 2^κ automorphisms of A.

9. The countable-cocountable algebra on ω_1

1. $\mathrm{Depth}(A) = \omega_1$, by an easy argument.
2. $\mathrm{Length}(A) = 2^\omega$.
3. $\pi(A) = \omega_1$.
4. $\mathrm{Ind}(A) = 2^\omega$.
5. $\pi\chi(A) = \omega_1$: let F be the ultrafilter of cocountable sets. Suppose that D is dense in F, with $|D| \leq \omega$. Without loss of generality the members of D are singletons. But then $\omega_1 \backslash \bigcup D \in F$, contradiction.
6. $|A| = 2^\omega$.
7. $|\mathrm{Ult}(A)| = 2^{2^\omega}$.
8. $|\mathrm{Aut}(A)| = 2^{\omega_1}$.

10. A compact Kunen line

Recall that this Boolean algebra was constructed using CH.

1. $\mathrm{Irr}(A) = \omega$. See Chapter 8.
2. $\chi(A) = \omega_1$. This was proved in the discussion following Corollary 14.12.
3. $|A| = |\mathrm{Ult}(A)| = \omega_1$. Clear from the construction.
4. $|\mathrm{End}(A)| = \omega_1$ by 3 and Theorem 21.1.
5. Although we have not been able to determine $\mathrm{Inc}(A)$, the algebra $A \times A$ has incomparability ω_1, and still has the other important properties of A: $|A \times A| = |\mathrm{Ult}(A \times A)| = \omega_1$, $\chi(A \times A) = \omega_1$, and $\mathrm{Irr}(A \times A) = \omega$. The set $\{(a, -a) : a \in A\}$ is incomparable, showing that $\mathrm{Inc}(A \times A) = \omega_1$. Obviously $|A \times A| = |\mathrm{Ult}(A \times A)| = \omega_1$ and $\chi(A \times A) = \omega_1$. To see that $\mathrm{Irr}(A \times A) = \omega$, note from the Handbook volume 1, example 11.6, that $(A \times A) \oplus (A \times A) \cong (A \oplus A)^4$, and then apply Heindorf's theorem in Chapter 8.

Problem 184. *Determine* $\mathrm{Inc}(A)$, $|\mathrm{Aut}(A)|$, $|\mathrm{Id}(A)|$, *and* $|\mathrm{Sub}(A)|$ *for the compact Kunen line of Chapter* 8.

This is problem 96 in the Monk [96].

Table of examples

Example	Depth	$\pi\chi$	c	Length	Ind	d	t	π	χ	s	Irr
Fincoκ	ω	ω	κ	ω	ω	κ	ω	κ	κ	κ	κ
Frκ	ω	κ	ω	ω	κ	(1)	κ	κ	κ	κ	κ
Intalg$\mathbb{R}$	ω	ω	ω	2^ω	ω	ω	ω	ω	ω	ω	2^ω
$\mathscr{P}\kappa$	κ	κ	κ	Dedκ	2^κ	κ	2^κ	κ	2^κ	2^κ	2^κ
Intalgκ	κ	κ	κ	κ	ω	κ	κ	κ	κ	κ	κ
$\mathscr{P}\omega/\mathrm{fin}$	(2)	(3)	2^ω	2^ω	2^ω	2^ω	2^ω	2^ω	2^ω	2^ω	2^ω
Dup	ω	ω	2^κ	ω	κ	2^κ	κ	2^κ	κ	2^κ	2^κ
$\overline{\mathrm{Fr}}(\kappa)$	ω	κ	ω	2^ω	κ^ω	(1)	κ^ω	κ	κ^ω	κ^ω	κ^ω
Cblcoω_1	ω_1	ω_1	ω_1	2^ω	2^ω	ω_1	2^ω	ω_1	2^ω	2^ω	2^ω
CKL	ω	ω	ω	ω	ω	ω	ω	ω	ω_1	ω	ω
BK	ω	ω	ω	ω	ω	ω	ω	ω	ω_1	ω	?
Rubin	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω

Notes:
Dup is the Alexsandroff duplicate of the free BA of size κ.
CKL is the compact Kunen line constructed in chapter 8.
BK is the Baumgartner, Komjath algebra constructed in chapter 17.
Rubin is the algebra constructed in chapter 18.

(1) The least λ such that $\kappa \leq 2^\lambda$.
(2) The depth is $\geq \omega_1$; various possibilities are consistent.
(3) $\geq \mathrm{cf}2^\omega$.

(*Table continued on the next page*)

11. The Baumgartner, Komjath algebra

Recall that this BA was constructed using $\diamondsuit$. See Chapter 17.

1. $\mathrm{Inc}(A) = \omega$.
2. $\mathrm{Length}(A) = \omega$.
3. $\chi(A) = \omega_1$.
4. $|\mathrm{Ult}(A)| = \omega_1$. To see this, first note that each ultrafilter on A is determined by the membership of the elements x_α or their complements. Hence this equality follows from the following fact:

(∗) If F and G are ultrafilters on A, $x_\alpha \in F\cap G$, and $F\cap A_{\alpha+1} = G\cap A_{\alpha+1}$, then $F = G$.

In fact, suppose that $\beta \in (\alpha, \omega_1)$. If $x_\beta \in F$, then $x_\alpha \cap x_\beta \in F$; but by construction $x_\alpha \cap x_\beta \in A_{\alpha+1}$, so $x_\alpha \cap x_\beta \in G$ and so $x_\beta \in G$. The same argument works if $\omega\backslash x_\beta \in F$, so $F \subseteq G$ and hence $F = G$.

5. $|\mathrm{End}(A)| = \omega_1$ by 4 and Theorem 21.1.
6. Since A has a nonzero element a such that $A \restriction a$ is countable, $A \restriction a$ has ω_1 automorphisms, and so the same is true of A itself.

Problem 185. *Determine* Irr(A), $|\mathrm{Id}(A)|$, *and* $|\mathrm{Sub}(A)|$ *for the Baumgartner, Komjath algebra.*

This is problem 97 in Monk [96].

Example	hL	hd	Inc	h-cof	Card	$\|$Ult$\|$	$\|$Aut$\|$	$\|$Id$\|$	$\|$End$\|$	$\|$Sub$\|$
Fincoκ	κ	κ	κ	κ	κ	κ	2^κ	2^κ	2^κ	2^κ
Frκ	κ	κ	κ	κ	κ	2^κ	2^κ	2^κ	2^κ	2^κ
Intalg$\mathbb{R}$	ω	ω	2^ω	2^ω	2^ω	2^ω	2^ω	2^ω	2^ω	2^{2^ω}
$\mathscr{P}\kappa$	2^κ	2^κ	2^κ	2^κ	2^κ	2^{2^κ}	2^κ	2^{2^κ}	2^{2^κ}	2^{2^κ}
Intalgκ	κ	κ	κ	κ	κ	κ	2^κ	2^κ	2^κ	2^κ
$\mathscr{P}\omega/\mathrm{fin}$	2^ω	2^ω	2^ω	2^ω	2^ω	2^{2^ω}	(1)	2^{2^ω}	2^{2^ω}	2^{2^ω}
Dup	2^κ	2^κ	2^κ	2^κ	2^κ	2^κ	2^κ	2^{2^κ}	2^{2^κ}	2^{2^κ}
$\overline{\mathrm{Fr}}(\kappa)$	κ^ω	κ^ω	κ^ω	κ^ω	κ^ω	2^{κ^ω}	2^κ	2^{κ^ω}	2^{κ^ω}	2^{κ^ω}
Cblcoω_1	2^ω	2^ω	2^ω	2^ω	2^ω	2^{2^ω}	2^{ω_1}	2^{2^ω}	2^{2^ω}	2^{2^ω}
CKL	ω_1	ω	?	ω_1	ω_1	ω_1	?	?	ω_1	?
BK	ω_1	ω	ω	ω_1	ω_1	ω_1	ω_1	?	ω_1	?
Rubin	ω	ω	ω	ω	ω_1	ω_1	?	ω_1	ω_1	ω_1

Notes:
Dup is the Alexsandroff duplicate of the free BA of size κ.
CKL is the compact Kunen line constructed in chapter 8.
BK is the Baumgartner, Komjath algebra constructed in chapter 17.
Rubin is the algebra constructed in chapter 18.

(1) Consistently 2^ω or 2^{2^ω}.

12. The Rubin algebra

This algebra was also constructed using $\diamondsuit$.

1. h-cof$(A) = \omega$; see Chapter 18.
2. $\mathrm{Irr}(A) = \omega$; see Rubin [83].
3. $|A| = \omega_1$. This is clear from the construction in Chapter 18.
4. $|\mathrm{Sub}(A)| = \omega_1$. See Chapter 23.

We do not know about $|\mathrm{Aut}(A)|$, although the construction can be changed to make A rigid; see Shelah [91] for more details.

27 Problems

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Problem 2. *Is every disjunctively generated BA isomorphic to a tail algebra?* Page 70

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Problem 6. *Describe the possibilities for chain conditions in ultraproducts with respect to countably complete ultrafilters.* Page 109

Problem 7. *Does $A \leq_{\mathrm{s}} B$ imply that $\mathfrak{a}(B) \leq \mathfrak{a}(A)$?* Page 123

Problem 8. *Is it true that for all infinite BAs A, B one has*

$$\mathfrak{a}(A \oplus B) = \min(\mathfrak{a}(A), \mathfrak{a}(B))?$$

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Problem 9. *Suppose that K is a set of infinite cardinals without greatest element, with $\sup(K)$ inaccessible. Is there a BA A such that $\mathfrak{a}_{\mathrm{spect}}(A) = K$?* Page 127

Problem 10. *Describe in terms not involving BAs the sets K, L such that there exist BAs A, B with $A \leq_{\mathrm{reg}} B$, $\mathfrak{a}_{\mathrm{spect}}(A) = K$, and $\mathfrak{a}_{\mathrm{spect}}(B) = L$.* Page 127

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Problem 13. *Give a purely cardinal number characterization of c_{Sr}.* Page 137

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Problem 14. *Can one prove in ZFC that there is a BA A with*

$$c_{\mathrm{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),\ (\omega_1,\omega_1),(\omega_1,\omega_2)\}?$$ Page 140

Problem 15. *Can one prove in ZFC that there is a BA A with*

$$c_{\mathrm{Sr}}(A) = \{(\omega,\omega),(\omega,\omega_1),\ (\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}?$$ Page 141

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Problem 18. *Can one prove in ZFC that there is a BA A such that*

$$c_{\mathrm{Hr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2)\}?$$ Page 148

Problem 19. *Is it consistent that* $c_{\mathrm{Hr}}(A) = \{(\omega,\omega_1),(\omega,\omega_2),(\omega_1,\omega_1),(\omega_2,\omega_2)\}$ *for some BA A?* Page 149

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$$c_{\mathrm{Hr}}(A) = \{(\omega,\omega),(\omega,\omega_1),(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_2)\}?$$ Page 150

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Problem 22. *Are the following conditions equivalent, for any linearly ordered set L?*

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(ii) *L is* κ*-saturated.* Page 153

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Problem 24. *If K is a nonempty set of infinite cardinals, is there a tree algebra A such that* $\mathfrak{a}_{\mathrm{spect}}(A) = K$*?* Page 153

Problem 25. *For each infinite cardinal* κ*, give a criterion, purely in terms of the tree T, for there to exist a partition of size* κ *in* $\mathrm{Treealg}(T)$*.* Page 153

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Problem 42. *Given nonempty sets $K \subseteq L$ of cardinals with $\omega \notin K$, are there BAs $A \le_{\sigma} B$ such that* $\mathfrak{p}_{\mathrm{spect}}(A) = K$ *and* $\mathfrak{p}_{\mathrm{spect}}(B) = L$*?* Page 189

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(i) *There is a BA A such that $K = \{\kappa : \kappa \in \mathrm{Freecal}(A)$ and κ is regular$\}$.*

(ii) *The following conditions hold, where $\mu = \min K$ and $\nu = \sup K$:*

  (a) *μ is uncountable.*

  (b) *For all $\lambda \in (\mu, \nu]$, if λ is regular and is not the successor of a singular cardinal, then $\lambda \in K$.*

  (c) *For all $\lambda \in (\mu, \nu]$, if $\lambda = \sigma^+$ for some singular σ with $\mu \leq \mathrm{cf}\,\sigma$, then $\lambda \in K$.* Page 371

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