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CHAINS IN BOOLEAN ALGEBRAS

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0. Introduction

We consider chains (linearly ordered subsets) in arbitrary Boolean algebras (BA's). Specifically, we study mainly the cardinal function *depth*, where depth A is the supremum of $|X|$, X a subset of A well-ordered under the Boolean ordering (the name comes from Juhász [14], and is more intuitive for inverse well-orderings). Our main results are: Theorem 1.1.1. If depth $A=\kappa$ and $\mathrm{cf}\kappa=\omega$, then depth is attained (for κ singular and $\mathrm{cf}\kappa>\omega$, by Corollary 1.2.6 this no longer holds); Theorem 1.3.1. If $\mathrm{cf}\kappa>\omega$, A has no chain of type $\mathrm{cf}\kappa$, and B has no chain of type κ, then $A * B$ has no chain of type κ; Theorem 1.7.11 (GCH). If $|A|=\lambda^+$, A has a chain of type κ, and $\aleph_0\leq\kappa\leq\mu\leq\lambda$, then A has a subalgebra of power μ and depth κ (there are counterexamples for $\mu=\lambda^+$). We also consider more briefly two other cardinal functions. The *ordinal depth* of A is the supremum of the order types of well-ordered subsets of A. It is never attained, and has the form $\omega^\alpha\cdot n$ with either $n>1$ or else $n=1$ and α a singular ordinal with $\mathrm{cf}\alpha>\omega$. The *length* of A is the supremum of $|X|$, X a chain in A. Our main results about this are Theorem 3.3. If length $A=\kappa$ with $\mathrm{cf}\kappa=\omega$, then length is attained (with counterexamples for singular κ having $\mathrm{cf}\kappa>\omega$); and Theorem 1.3.1 with 'depth' replaced by 'length'. In the course of our investigations we have run across several problems which we cannot solve; we list five of them.

Several papers are related to this one. Boolean algebras generated by chains have been extensively studied: see [19, 16, 20, 21, 12]. Chains in Boolean algebras in general have not been studied so much. Jakubik [13], Day [6], and Grätzer [9, 10] have considered maximal chains in Boolean algebras.

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We use standard set-theoretical and algebraic notation. MA stands for Martin's axiom. For any infinite cardinal λ we let $\beth_0\lambda=\lambda$, $\beth_{\alpha+1}\lambda=2^{\beth_\alpha\lambda}$, and $\beth_\mu\lambda=\bigcup_{\alpha<\mu}\beth_\alpha\lambda$ for μ limit. Further, $\beth_\alpha=\beth_\alpha\omega$. A Boolean algebra is identified with its

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universe. A is κ-complete if any subset of A of power $<\kappa$ has a supremum. $\mathscr{P}\kappa$ denotes the BA of all subsets of κ, $\mathscr{P}_\lambda\kappa$ the set of all subsets of κ of power λ, while $\mathscr{P}_{<\lambda}\kappa$ denotes the set of all subsets of κ of power $<\lambda$. A satisfies the κ-chain condition, κ-c.c., if every collection of pairwise disjoint elements of A has power $<\kappa$. The *cellularity* of A, cell A, is the supremum of cardinalities of sets of pairwise disjoint elements of A. If A is any BA and $a \in A$, then $A \upharpoonright a$ is $\{x: x \leq a\}$ considered as a BA. $f: A \twoheadrightarrow B$ indicates that f is a homomorphism from A onto B. Given a linearly ordered set L, the *interval algebra* on L is the algebra of subsets of L generated by the half-open intervals $[a, b)$ and $[a)$. A BA A is *cardinality-homogeneous* if $|A \upharpoonright a| = |A|$ whenever $0 \neq a \in A$. For A a BA and $X \subseteq A$, Sg X is the subalgebra generated by X and Ig X the ideal generated by X. $\mathrm{Fr}\langle X_\alpha : \alpha < \kappa\rangle$ is the free algebra with free generators $\langle X_\alpha : \alpha < \kappa\rangle$. For any X, $X^1 = X$ and $X^0 = -X$. Further notation is introduced as needed.

The following basic lemma is frequently used below.

Lemma 0. *Let X be a chain in $A \times B$ of infinite cardinality κ. Then the projections of X are chains, and one of them has size κ. If X has order type κ, then X has a subset of order type κ on which one of the two projections is one-to-one.*

To prove the last part of this lemma, define for $i = 0, 1$ $x \equiv_i y$ iff $x, y \in X$ and $x_i = y_i$, where $\forall Z \in X (Z = (Z_0, Z_1))$. If there are κ $\equiv_i$-classes, then a subset Y of X with exactly one member in common with each $\equiv_i$-class is as desired. One of $\equiv_0$, $\equiv_1$ has κ classes.

1. Depth

For any BA A we let

$$\text{depth } A = \omega \cup \sup\{|X| : X \subseteq A,\ X \text{ well-ordered by the Boolean ordering}\}.$$

We consider several aspects of this notion: in order of our exposition—attainment; behavior under products; free products; amalgamated free products; reduced products; homomorphisms; subalgebras; unions; automorphisms; special algebras.

1.1. Attainment

We say that depth is *attained* in A if there is a well-ordered $X \subseteq A$ with $|X| = \text{depth } A$. Our basic result here is:

Theorem 1.1.1. *If* depth $A = \kappa$ *and* cf$\kappa = \omega$, *then depth is attained.*

Proof. We may assume that $\kappa > \omega$. Let $\langle \lambda_i : i < \omega\rangle$ be a strictly increasing sequence of cardinals with supremum κ, where $\lambda_0 = 0$. Now we call $a \in A$ an ∞-element if λ_i is embeddable in $A \restriction a$ for all $i < \omega$. Thus 1 is an ∞-element. We claim:

$$\text{If } a \text{ is an } \infty\text{-element and } a = b + c \text{ with } b \cdot c = 0, \text{ then } b \text{ is an } \infty\text{-element or } c \text{ is an } \infty\text{-element.} \tag{1}$$

For, by Lemma 0 it is clear that for every $i < \omega$, λ_i is embeddable in b or in c. So (1) holds.

Using (1) we construct a sequence $\langle a_i : i < \omega \rangle$ of elements of A by induction; suppose a_j has been constructed for all $j < i$ so that $\prod_{j<i} -a_j = b$ is an ∞-element. Let $\langle c_\alpha : \alpha < \lambda_{i+1} \rangle$ be an isomorphic embedding of λ_{i+1} into b. By (1), one of c_{λ_i} and $b - c_{\lambda_i}$ is an ∞-element, while clearly λ_i is embeddable in both of these elements. So we can choose $a_i \leq b$ so that λ_i is embeddable in a_i and $\prod_{j \leq i} -a_j$ is an ∞-element. This finishes the construction.

For each $i < \omega$ let $\langle b_{i\alpha} : \alpha < \lambda_i \rangle$ be an embedding of λ_i in a_i. Note that $a_i \cdot a_j = 0$ if $i < j < \omega$. Hence the following sequence $\langle d_\alpha : \alpha < \kappa \rangle$ is clearly the desired embedding of κ into A. Given $\alpha < \kappa$, there is a unique i such that $\lambda_i \leq \alpha < \lambda_{i+1}$. We let $d_\alpha = a_0 + \cdots + a_i + b_{i+1,\alpha}$.

We show in Corollary 1.2.6 below that Theorem 1.1.1 is best possible.

1.2. Products

The behaviour of depth in products is quite clear, so we state the facts without detailed proofs.

Theorem 1.2.1. depth $\prod_{i \in I} A_i = |I| \cup \sup_{i \in I} \text{depth } A_i$.

Theorem 1.2.2. *Let* $\kappa = \sup_{i \in I} \text{depth } A_i$, *and suppose that* κ *is regular. Then the following conditions are equivalent*:

(i) depth $\prod_{i \in I} A_i$ *is not attained*;

(ii) $|I| < \kappa$, *and* $\forall\, i \in I$ (A_i *has no chain of type* κ).

Theorem 1.2.3. *Let* $\kappa = \sup_{i \in I} \text{depth } A_i$, *and suppose that* κ *is singular. Then the following conditions are equivalent*:

(i) depth $\prod_{i \in I} A_i$ *is not attained*;

(ii) $|I| < \kappa$, $\forall\, i \in I$ (A_i *has no chain of type* κ), $|\{i \in I : \text{depth } A_i = \kappa\}| < \text{cf}\,\kappa$, *and* $\sup\{\text{depth } A_i : i \in I, \text{depth } A_i < \kappa\} < \kappa$.

Proof. (i)$\Rightarrow$(ii) is clear. Assume (ii), and suppose that $\langle x_\alpha : \alpha < \kappa \rangle$ is strictly increasing in $\prod_{i \in I} A_i$. Define

$$J_i = \{\alpha < \kappa : x_\alpha i < x_{\alpha+1} i\} \quad \text{for } i \in I;$$

$$K = \{i \in I : \text{depth } A_i = \kappa\};$$

$$\lambda = \sup\{\text{depth } A_i : i \in I, \text{depth } A_i < \kappa\}.$$

Then $\lambda<\kappa, \forall i \in I \setminus K$ ($|J_i| \leq \lambda$), $|K|<\mathrm{cf}\kappa$, and $\forall i \in K(|J_i|<\kappa)$. So $|\bigcup_{i\in I} J_i|<\kappa$. Hence for $\alpha \in \kappa \setminus \bigcup_{i\in I} J_i$ we have $x_\alpha = x_{\alpha+1}$, which is impossible.

We also discuss depth for the *weak product* of a system $\langle A_i : i \in I\rangle$ of BA's, defined as

$$\prod_{i\in I}^{\mathrm{w}} A_i = \left\{ f \in \prod_{i\in I} A_i : \{i : f_i \neq 0\} \quad \text{or} \quad \{i : f_i \neq 1\} \text{ is finite} \right\}.$$

Theorem 1.2.4. *Let* $\sup_{i\in I}\text{depth } A_i = \kappa$ *with* $\mathrm{cf}\kappa > \aleph_0$. *Then the following conditions are equivalent*:

(i) $\prod_{i\in I}^{\mathrm{w}} A_i$ *has no chain of type* κ;

(ii) $\forall i \in I$ (A_i *has no chain of type* κ).

For the proof of this theorem (easy), see the more general Theorem 1.10.2.

Corollary 1.2.5. depth $\prod_{i\in I}^{\mathrm{w}} A_i = \sup_{i\in I}$ depth A_i.

Corollary 1.2.6. *If* κ *is a limit cardinal with* $\mathrm{cf}\kappa > \aleph_0$, *then there is a* BA *of depth* κ *which is not attained.*

1.3. Free products

If $A = \langle A_i : i \in I\rangle$ is a system of BA's, a *free product* of A is a pair (B, f) such that B is a BA, $f = \langle f_i : i \in I\rangle$ is a system of isomorphic embeddings $f_i : A_i \to B$, $\bigcup_{i\in I} f_i^* A_i$ generates B, and if $i \in {}^nI$ is one-to-one and $a \in \prod_{j<n}(A_{i(j)} \setminus \{0\})$, then $\prod_{j<n} f_{i(j)} a_j \neq 0$. Usually we assume that each f_i is an inclusion map, speak of *the* free product of A, and write $B = *_{i\in I} A_i$. For two BA's A_0 and A_1 we use the notation $A_0 * A_1$.

Our main result on free products and depth is as follows. Its proof is so written that the analogous theorem for length (Theorem 3.6) follows with practically no change. (The proof is much easier for regular κ.)

Theorem 1.3.1. *Suppose that* $\mathrm{cf}\kappa > \omega$, A *has no chain of type* $\mathrm{cf}\kappa$, *and* B *has no chain of type* κ. *Then* $A * B$ *has no chain of type* κ.

Proof. An element x of $A * B$ is of *length* n if we can write

$$x = \sum_{i<n} a_i \cdot b_i,$$

where $\forall i < n$ $(0 \neq a_i \in A, 0 \neq b_i \in B)$ and for all distinct $i, j < n$, $b_i \cdot b_j = 0$, and x cannot be so written for any $m < n$. Note that then $a_i \neq a_j$ for all distinct $i, j < n$. Clearly it suffices to show that for all n, $A * B$ has no chain of type κ all terms of which have length n. Assume that n is minimum such that this is not true. Say $X \subseteq A * B$, X a chain of type κ, each member x of X of length n,

$$x = \sum_{i<n} a_i^x \cdot b_i^x$$

where $\forall\, i<n$ $(0\neq a_i^x\in A, 0\neq b_i^x\in B)$ and $b_i^x\cdot b_j^x=0$ whenever $i<j<n$. If $n=1$, then $a_0^x\leq a_0^y$ and $b_0^x\leq b_0^y$ whenever $x, y\in X$ and $x<y$. Since A has no chain of type $\mathrm{cf}\kappa$ it follows that there is a $Y\subseteq X$ of size κ with $a_0^x=a_0^y$ for all $x, y\in Y$. But then $b_0^x<b_0^y$ whenever $x, y\in Y$ and $x<y$, contradicting our assumption on B. Thus $n>1$.

For $x, y\in X$ with $x<y$, and $i<n$ we have $a_i^x\cdot b_i^x\leq y\leq\sum_{j<n} b_j^y$, and hence

$$\text{if } x, y\in X \text{ with } x<y, \text{ and } i<n, \text{ then } b_i^x\leq\sum_{j<n} b_j^y. \tag{1}$$

Furthermore,

$$\text{if } x, y\in X;\ x<y;\ i, j<n;\ \text{and } b_i^x\cdot b_j^y\neq 0, \text{ then } a_i^x\leq a_j^y. \tag{2}$$

For, the hypotheses imply that $a_i^x\cdot b_i^x\cdot\prod_{k<n}(-a_k^y+-b_k^y)=0$; multiplying by b_j^y gives the conclusion of (2).

Now we shall use (1) and (2) to prove

$$\text{there is a } g\in{}^Xn \text{ such that } a_{gx}^x\leq a_{gy}^y \text{ whenever } x, y\in X \text{ and } x<y. \tag{3}$$

To prove (3), note that if $x, y\in X$ and $x<y$ then $\{g\in{}^Xn: a_{gx}^x\leq a_{gy}^y\}$ is closed in the space ${}^Xn$ (with discrete topology on n and product topology). Thus by compactness of ${}^Xn$ it suffices to show that if $x_1<\cdots<x_m$ in X, then there is a $g\in{}^Xn$ such that $a_{gx_1}^{x_1}\leq\cdots\leq a_{gx_m}^{x_m}$. However, this is clear from (1) and (2) by a simple inductive argument.

Now define $x\equiv y$ iff $x, y\in X$ and $a_{gx}^x=a_{gy}^y$. There are $<\mathrm{cf}\kappa$ equivalence classes, so some equivalence class has κ elements. We may take this equivalence class in place of X; relabeling, we simply assume that each a_0^x is a constant a_0. Now for any $x\in X$ the element $x\cdot -a_0$ has length $<n$, so $|\{x\cdot -a_0: x\in X\}|<\kappa$. Hence by Lemma 0 the set $\{x\cdot a_0: x\in X\}$ has κ elements. Thus cutting down again we may assume

$$\text{for all } x\in X \text{ and } i<n \text{ we have } a_i^x\leq a_0 \tag{4}$$

Now if $x, y\in X, x<y, 0<i<n$, and $b_0^x\cdot b_i^y\neq 0$, then by (2) and (4) $a_0\leq a_i^y\leq a_0$, contradiction (see the initial remarks to this proof). Thus by (1),

$$\text{if } x, y\in X \text{ and } x<y, \text{ then } b_0^x\leq b_0^y. \tag{5}$$

Next, for any $x\in X$ and $0\leq i<n$ let

$$\Delta_i^x=\{y\in X: x<y \quad\text{and}\quad b_i^x\leq b_0^y\}.$$

We claim now that there is a function $f: X\to\{1,\cdots, n-1\}$ such that

$$\text{if } x, y\in X, x<y, \text{ and } y\notin\Delta_{fx}^x, \text{ then } a_{fx}^x\leq a_{fy}^y \text{ and } \Delta_{fy}^y\subseteq\Delta_{fx}^x; \tag{6}$$

$$\text{if } x\in X, a_{fx}^x\leq a_j^x \text{ with } 0<j<n \text{ and } j\neq fx, \text{ then } \Delta_{fx}^x\subset\Delta_j^x. \tag{7}$$

To prove this claim we again use the compactness of ${}^X\{1,\ldots, n-1\}$; it is sufficient to take any finite sequence $x_0<\cdots<x_{m-1}$ of elements of X and

construct a function $f:\{x_0,\dots,x_{m-1}\}\to\{1,\dots,n-1\}$ satisfying (6) and (7) with X replaced by $\{x_0,\dots,x_{m-1}\}$. We construct f by induction. Let $a^{x_0}_{fx_0}$ be maximal among all $a^{x_0}_j$ for $j>0$; then both (6) and (7) vacuously hold (recall again that $a^x_k\neq a^x_l$ for $k\neq l$). Now suppose that $fx_0,\dots,fx_i$ have been constructed so that (6) and (7) hold.

Case 1. $x_{i+1}\in\Delta^{x_i}_{fx_i}$. Again let $a^{x_{i+1}}_{fx_{i+1}}$ be maximal among all $a^{x_{i+1}}_j$ for $j>0$. So (7) holds vacuously, and (6) also holds vacuously, since $x_{i+1}\in\Delta^{x_j}_{fx_j}$ for any $j\leqslant 1$, as is easily seen, using (5).

Case 2. $x_{i+1}\notin\Delta^{x_i}_{fx_i}$. Thus $b^{x_i}_{fx_i}\nleqslant b^{x_{i+1}}_0$. Now

$$\text{there is a } j>0 \text{ such that } b^{x_i}_{fx_i}\cdot b^{x_{i+1}}_j\neq 0 \quad\text{and}\quad \Delta^{x_{i+1}}_j\subseteq\Delta^{x_i}_{fx_i}. \tag{8}$$

For, otherwise for all $j>0$ with $b^{x_i}_{fx_i}\cdot b^{x_{i+1}}_j\neq 0$ choose $z_j\in\Delta^{x_{i+1}}_j\setminus\Delta^{x_i}_{fx_i}$; note that there exist such j by (1), and that then $b^{x_{i+1}}_j\leqslant b^{z_j}_0$ and $b^{x_i}_{fx_i}\nleqslant b^{z_j}_0$. Let z_{j_0} be maximal among such z_j. Then

$$b^{x_i}_{fx_i}\leqslant b^{x_{i+1}}_0+\sum\{b^{x_{i+1}}_j : b^{x_i}_{fx_i}\cdot b^{x_{i+1}}_j\neq 0,\ j>0\}\leqslant b^{z_{j_0}}_0$$

a contradiction. Thus (8) holds, and from it we see by (2) that there is a $j>0$ such that $a^{x_i}_{fx_i}\leqslant a^{x_{i+1}}_j$ and $\Delta^{x_{i+1}}_j\subseteq\Delta^{x_i}_{fx_i}$. We let fx_{i+1} be such a j with $\Delta^{x_{i+1}}_j$ minimal, and among all k with $\Delta^{x_{i+1}}_j=\Delta^{x_{i+1}}_k$ one with a^{x+1}_j maximal. So (6) holds for i and $i+1$. Now suppose $j<i$ and $x_{i+1}\notin\Delta^{x_j}_{fx_j}$. If $x_i\in\Delta^{x_j}_{fx_j}$, then $b^{x_j}_{fx_j}\leqslant b^{x_i}_0\leqslant b^{x_{i+1}}_0$, contradiction. So $x_i\notin\Delta^{x_j}_{fx_j}$, and (6) for j and i yields (6) for j and $i+1$. For (7), suppose $a^{x_{i+1}}_{fx_{i+1}}\leqslant a^{x_{i+1}}_j$, $j>0$, and $j\neq fx_{i+1}$. Thus by the minimality and maximality choices, $\Delta^{x_{i+1}}_j\nsubseteq\Delta^{x_{i+1}}_{fx_{i+1}}$, i.e., $\Delta^{x_{i+1}}_j\setminus\Delta^{x_{i+1}}_{fx_{i+1}}\neq 0$. It remains to show that $\Delta^{x_{i+1}}_{fx_{i+1}}\subseteq\Delta^{x_{i+1}}_j$. Choose $v\in\Delta^{x_{i+1}}_j\setminus\Delta^{x_{i+1}}_{fx_{i+1}}$ and let $u\in\Delta^{x_{i+1}}_{fx_{i+1}}$. Thus $b^{x_{i+1}}_{fx_{i+1}}\leqslant b^u_0$, $b^{x_{i+1}}_j\leqslant b^v_0$, and $b^{x_{i+1}}_{fx_{i+1}}\nleqslant b^v_0$. So $v<u$ and $b^{x_{i+1}}_j\leqslant b^v_0\leqslant b^u_0$, as desired. We have now proved that f exists satisfying (6) and (7).

From (6) we easily obtain

$$\text{if } x, y\in X \text{ and } x<y, \text{ then } \Delta^y_{fy}\subseteq\Delta^x_{fx}. \tag{9}$$

Now define $x\sim y$ iff $x, y\in X$ and $x=y$, or say $x<y$ and whenever $x\leqslant r\leqslant u\leqslant y$, then $a^x_{fx}\leqslant a^r_{fr}\leqslant a^u_{fu}\leqslant a^y_{fy}$. So $\sim$ is an equivalence relation and the $\sim$-classes are convex. Also

$$\text{if } x, y\in X,\ x<y, \text{ and } x\nsim y, \text{ then } b^x_{fx}\leqslant b^y_0, \text{ and if } i<n \text{ and } a^x_{fx}\leqslant a^x_i, \text{ then } b^x_i\leqslant b^y_0. \tag{10}$$

For, suppose $x, y\in X$, $x<y$, and $b^x_{fx}\nleqslant b^y_0$; we show that $x\sim y$. So, assume that $x\leqslant r\leqslant u\leqslant y$. By (9), $\Delta^u_{fu}\subseteq\Delta^x_{fx}$, so $y\notin\Delta^u_{fu}$ and hence by (6) $a^u_{fu}\leqslant a^y_{fy}$. If $u\in\Delta^x_{fx}$, then $b^x_{fx}\leqslant b^u_0\leqslant b^y_0$, a contradiction. Thus $u\notin\Delta^x_{fx}$ hence $u\notin\Delta^r_{fr}$ by (9) and so $a^r_{fr}\leqslant a^u_{fu}$ by (6). If $r\in\Delta^x_{fx}$, then $b^x_{fx}\leqslant b^r_0\leqslant b^y_0$, a contradiction. So $r\notin\Delta^x_{fx}$ hence $a^x_{fx}\leqslant a^r_{fr}$ by (6). Thus $x\sim y$, as desired. For the second part of (10), assume its hypothesis: by the first part, assume $0\neq i\neq fx$. By (7), $\Delta^x_{fx}\subset\Delta^x_i$. Hence $y\in\Delta^x_i$, i.e., $b^x_i\leqslant b^y_0$. We have established (10).

Now if S is a selector for $\sim$-classes, then $\{b_0^x : x \in S\}$ is a chain in B of size $|S|$ by (5) and (10); so, $|S| < \kappa$; i.e., $|X/\sim| < \kappa$. Now suppose that $x_0/\sim$ has κ elements for some $x_0 \in X$. Since $\{a_{fx}^x : x \in x_0/\sim\}$ has $<\mathrm{cf}\kappa$ elements, there is a $Y \subseteq x_0/\sim$ of power κ on which a_{fx}^x has some constant value a'. Relabeling, we may assume that $fx = 1$ for all $x \in Y$, and hence each $x \in Y$ has the form

$$x = a_0 \cdot b_0^x + a' \cdot b_1^x + \sum_{1<i<n} a_i^x \cdot b_i^x.$$

Then Lemma 0 gives a contradiction, since both $\{x \cdot a' : x \in Y\}$ and $\{x \cdot -a : x \in Y\}$ consist of elements of length $<n$. We have shown

$$|X/\sim| < \kappa, \text{ and each } \sim\text{-class has size } <\kappa. \tag{11}$$

Now we relabel in order to assume that $fx = 1$ for all $x \in X$.

Note by (11) that κ is singular. For any $\sim$-class k, $\{a_1^x : x \in \kappa\}$ is a chain; hence if $|k| \geq \lambda^+$ with $\lambda < \kappa$, there is a subset k' of k of power λ^+ such that $a_1^x = a_1^y$ for all $x, y \in k'$. Since there are $\sim$-classes of arbitrarily big size $<\kappa$, clearly there is a subset X' of X of power κ such that, if $x, y \in X'$ and $x < y$ and $x \sim y$, then $a_1^x = a_1^y$. Hence

$$n > 2. \tag{12}$$

For assume that $n = 2$. Note that if E and E' are equivalence relations on X' each with $<\kappa$ classes, then $E \cap E'$ also has $<\kappa$ classes. We apply this remark to $\sim$, $\sim'$, and $\sim''$, where $x \sim' y$ iff $b_0^x + b_1^x = b_0^y + b_1^y$ and $x \sim'' y$ iff $b_0^x = b_0^y$. $\sim''$ has $<\kappa$ classes by (5). $\sim'$ has $<\kappa$ classes by (1), since $b_0^x + b_1^x \leq b_0^y + b_1^y$ for $x \leq y$. So $E = \sim \cap \sim' \cap \sim''$ has $<\kappa$ classes. But if xEy, then $x = y$, a contradiction.

Now for any $x \in X'$ let

$$u_x = \sum \{a_i^x \cdot b_i^x : a_i^x > a_1^x\},$$
$$v_x = \sum \{a_i^x \cdot b_i^x : a_i^x \geq a_1^x\},$$
$$w_x = \sum \{a_i^x \cdot b_i^x : a_i^x \not\geq a_1^x\}.$$

Note that $v_x = u_x + a_1^x \cdot b_1^x$ by a remark at the beginning of this proof. Then

$$\text{if } x, y \in X',\ x < y, \text{ and } x \sim y, \text{ then } u_x \leq u_y. \tag{13}$$

To prove this, assume its hypotheses; we are to show $\sum\{a_i^x \cdot b_i^x : a_i^x > a_1^x\} \leq \sum\{a_i^y \cdot b_i^y : a_i^y > a_1^y\}$.

To this end, take any i with $a_i^x > a_1^x$; we want to show that $a_i^x \cdot b_i^x \cdot -\sum\{a_i^y \cdot b_i^y : a_i^y > a_1^y\} = 0$. By de Morgan's law and the distributive law this means that if $\Gamma = \{i : a_i^y > a_1^y\}$ and $\Theta \subseteq \Gamma$ we want to show that

$$a_i^x \cdot b_i^x \prod_{j \in \Theta} -a_j^y \cdot \prod_{j \in \Gamma \setminus \Theta} -b_j^y = 0. \tag{14}$$

So suppose that $b_i^x \cdot \prod_{j \in \Gamma \setminus \Theta} -b_j^y \neq 0$. Then by (1) there is a $k \in n \setminus (\Gamma \setminus \Theta)$ such that

$b_i^x \cdot b_k^y \neq 0$, and thus by (2) $a_i^x \leq a_k^y$. Hence $a_1^y = a_1^x < a_i^x \leq a_k^y$, so $k \in \Gamma$ and hence $k \in \Theta$. So

$$a_i^x \cdot \prod_{j \in \Theta} -a_j^y \leq a_i^x \cdot -a_k^y = 0,$$

as needed to prove (14) and hence (13). We also have

$$\text{if } x, y \in X' \text{ and } x < y, \text{ then } v_x \leq v_y. \tag{15}$$

For, if $x \sim y$ the proof is as above. Suppose $x \not\sim y$. Let $\Gamma = \{i : a_i^y \geq a_1^y\}$ and suppose that $\theta \subseteq \Gamma$ and $a_i^x \geq a_1^x$; we want to prove (14) again. Now $0 \in \Gamma$, and if $0 \in \theta$ we are through. Suppose $0 \in \Gamma \setminus \theta$. Then by (10), $b_i^x \leq b_0^y$, so (14) holds, and (15) follows.

Now we consider two cases

Case 1. $|\{v_x : x \in X'\}| < \kappa$. Then there is an $X'' \subseteq X'$ of power κ such that, if $x, y \in X''$, $x < y$, and $x \sim y$, then $v_x = v_y$. For any $x \in X''$ let

$$z_x = a_0 \cdot \sum \{b_i^x : a_i^x \geq a_1^x\} + w_x.$$

From (15) we have

$$\text{if } x, y \in X'' \text{ and } x < y, \text{ then } \sum \{b_i^x : a_i^x \geq a_1^x\} \leq \sum \{b_i^y : a_i^y \geq a_1^y\}; \text{ if moreover } x \sim y, \text{ then } = \text{ holds.} \tag{16}$$

Hence we can show

$$\text{if } x, y \in X'', x < y, \text{ and } x \sim y, \text{ then } z_x < z_y. \tag{17}$$

For, $x < y$ and $v_x = v_y$, so $w_x < w_y$ and then (16) yields (17).

$$\text{If } x, y \in X'', x < y, \text{ and } x \not\sim y, \text{ then } z_x \leq z_y. \tag{18}$$

For, in view of (16) it is enough to suppose that $a_i^x \not\geq a_1^x$ and $\Gamma \subseteq \{j < n : a_j^y \not\geq a_1^y\}$ and show that

$$a_i^x \cdot b_i^x \cdot \prod_{j \in \Gamma} -a_j^y \cdot \prod_{j \in n \setminus \Gamma} -b_j^y = 0.$$

Suppose $b_i^x \cdot \prod_{j \in n \setminus \Gamma} -b_j^y \neq 0$. Then there is a $k \in \Gamma$ such that $b_i^x \cdot b_k^y \neq 0$, so $a_i^x \leq a_k^y$, as desired.

From (17) and (18), by eliminating the last element, if there is one, from each $\sim$-class for X'' we get $X''' \subseteq X''$ of power κ on which z is strictly increasing. Since each z_x is of length $< n$, this is a contradiction.

Case 2. $|\{v_x : x \in X'\}| = \kappa$. We may assume that $v_x = x$ for all $x \in X'$. Then

$$\text{if } x, y \in X' \quad \text{and} \quad x < y, \quad \text{then} \quad u_x \leq u_y. \tag{19}$$

For, if $x \sim y$ this is given in (13), so suppose $x \not\sim y$. Take i with $a_i^x > a_1^x$. Then by (10) $b_i^x \leq b_0^y$, and $a_i^x b_i^x \leq u_y$ will follow from $a_0 = a_0^y > a_1^y$ (which holds because y has length n).

Since each u_x has length $<n$, by (19) we see that $|\{u_x: x\in X'\}|<\kappa$. Hence there is an $X''\subseteq X'$ of power κ such that $x, y\in X''$ and $x\sim y$ imply $u_x=u_y$.

For all $x\in X''$ let

$$z_x=a_0\cdot\sum\{b_i^x: a_i^x>a_1^x\}+a_1^x\cdot b_1^x.$$

Then

$$\text{if } x, y\in X'', x<y, \quad\text{and}\quad x\sim y, \quad\text{then}\quad z_x<z_y. \tag{20}$$

This is proved just like (17).

$$\text{If } x, y\in X'' \text{ and } x<y, \text{ then } z_x\leqslant z_y. \tag{21}$$

For, we want to show, when $x\not\sim y$,

$$a_1^x\cdot b_1^x\cdot(-a_1^y+-b_1^y)\cdot\prod\{-b_i^y: a_1^y<a_i^y\}=0.$$

Since $b_1^x\leqslant b_0^x$ by (10), this is clear.

Now from (20) and (21), as in Case 1, we find a subset X''' of X'' of power κ on which z is strictly increasing. But this contradicts (12).

Using Theorem 1.3.1 it is rather easy to describe depth for free products in general, and attainment for them. That is the purpose of the rest of our results of this section.

Theorem 1.3.2. *Let κ be regular and uncountable. Suppose that for every $i\in I$ a BA A_i has no chain of type κ. Then $*_{i\in I}A_i$ has no chain of type κ.*

Proof. Suppose $\langle x_\alpha:\alpha<\kappa\rangle$ is a strictly increasing sequence in $*_{i\in I}A_i$. For each $\alpha<\kappa$ let $F_\alpha\subseteq I$ be a finite support of x_α, i.e., $x_\alpha\in *_{i\in F_\alpha}A_i$. We may assume that $\langle F_\alpha:\alpha<\kappa\rangle$ forms a Δ-system, say with kernel G, that is, $F_\alpha\cap F_\beta=G$ for all distinct $\alpha,\beta\in\kappa$. We may also assume that $|F_\alpha|$ is constant and that this constant value is minimal. By Theorem 1.3.1, we may assume that $F_\alpha\neq G$ for all $\alpha<\kappa$; fix $s_\alpha\in F_\alpha\setminus G$ for each $\alpha<\kappa$. Write

$$x_\alpha=\sum_{i<n_\alpha}a_i^\alpha\cdot b_i^\alpha$$

where $0<a_i^\alpha<1$, $a_i^\alpha\in A_{s_\alpha}$, $b_i^\alpha\neq 0$ is in the subalgebra generated by $\bigcup_{k\in F_\alpha\setminus\{s_\alpha\}}A_k$, and $i<j\Rightarrow a_i^\alpha\cdot a_j^\alpha=0$, for all $\alpha<\kappa$ and $i,j<n_\alpha$. We may assume that $n_\alpha=m$ is constant. If $m=1$, then for $\alpha<\beta$ we have $a_0^\alpha\cdot b_0^\alpha\cdot -a_0^\beta=0$, which is impossible by freeness. Thus $m>1$. If $\alpha<\beta$, then

$$\sum_{i<m}a_i^\alpha\cdot b_i^\alpha\cdot\prod_{j<m}(-a_j^\beta+-b_j^\beta)=0,$$

so for every $i<m$ and $\Gamma\subseteq m$ we have

$$a_i^\alpha\cdot b_i^\alpha\cdot\prod_{j\in\Gamma}-a_j^\beta\cdot\prod_{j\in m\setminus\Gamma}-b_j^\beta=0.$$

Taking $\Gamma = m$ we get by freeness $\sum_{i<m} a_i^\beta = 1$. Taking $i, j < m$ and $\Gamma = m\setminus\{j\}$ we then get $b_i^\alpha \leq b_i^\beta$. Since this is true whenever $\alpha < \beta < \kappa$ and $i, j < m$, and since $|F_\alpha|$ was minimal, there is an $\alpha < \kappa$ such that $b_i^\beta = b_i^\beta$ for all $\beta > \alpha$ and all $i, j < m$. So we may assume that x_α has the form $(\sum_{i<m} a_i^\alpha) \cdot c$ all $a < \kappa$, which is clearly impossible.

Corollary 1.3.3. depth $*_{i\in I}A_i = \sup_{i\in I}$depth A_i.

Theorem 1.3.4. *Let depth $B = \kappa$. If A has a chain of type* cfκ, *then $A * B$ has a chain of type κ.*

Proof. We may assume that κ is singular. let $\langle\lambda_\xi : \xi < \mathrm{cf}\kappa\rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum κ. Let $\langle a_\xi : \xi < \mathrm{cf}\kappa\rangle$ be strictly increasing in A, and for each $\xi < \mathrm{cf}\kappa$ let $\langle b_{\xi\eta} : \eta < \lambda_\xi\rangle$ be strictly increasing in B. Now for $\lambda_0 \leq \alpha < \kappa$, choose $\xi < \mathrm{cf}\kappa$ with $\lambda_\xi \leq \alpha < \lambda_{\xi+1}$ and let

$$c_\alpha = a_\xi + b_{\xi+1,\alpha} \cdot a_{\xi+1} \cdot -a_\xi.$$

It is easily checked that $\langle c_\alpha : \lambda_0 \leq \alpha < \kappa\rangle$ is strictly increasing in $A * B$.

The following two corollaries summarize our results on depth in free products.

Theorem 1.3.5. *Let* $B = *_{i\in I} A_i$, depth $B = \kappa$, cf$\kappa > \aleph_0$, *κ singular. Then the following are equivalent*:

(i) *B has no chain of type κ*;

(ii) *there is a unique $i \in I$ such that*: depth $A_i = \kappa$, *A has no chain of type κ, and for $j \neq i$, A_j has no chain of type* cfκ.

Theorem 1.3.6. *Let* $*_{i\in I} A_i = B$, depth $B = \kappa > \aleph_0$, *κ regular. Then the following are equivalent*:

(i) *B has no chain of type κ*;

(ii) *for all $i \in I$, A_i has no chain of type κ.*

1.4. Amalgamated free products

Let (A, f, B) be such that A is a BA, $B = \langle B_i : i \in I\rangle$ is a system of BA's, and $f = \langle f_i : i \in I\rangle$ is a system such that f_i is an isomorphic embedding of A into B_i for each $i \in I$. An *A-amalgamated free product of* (A, f, B) is a pair (C, g) such that C is a BA, $g = \langle g_i : i \in I\rangle$ is a system of isomorphic embeddings $g_i : B_i \to C$, $g_i \circ f_i = g_j \circ f_j$ for all $i, j \in I$, for distinct $i, j \in I$ we have $g_i^* B_i \cap g_j^* B_j = g_i^* f_i^* A$, $\bigcup_{i\in E} g_i^* B_i$ generates C, and if $i \in {}^nI$ is one-one, $b \in \prod_{j<n} B_{ij}$, and $\prod_{j<n} g_{ij} b_j = 0$, then there exists a $a \in {}^nA$ such that $\forall\, j < n$ $(b_j \leq f_{ij} a_j)$ and $\prod_{j<n} a_j = 0$. Usually we assume that all f_i, g_i are inclusion maps, speak of *the* A-amalgamated free product, and write $C = *_{i\in I}^A B_i$. For two BA's B_0, B_1 we use the notation $B_0 *_A B_1$. Note that then

$b_0 \cdot b_1 = 0$ (with $b_0 \in B_0$, $b_1 \in B_1$) implies the existence of $a \in A$ with $b_0 \leq a$, $b_1 \leq -a$. We also note the following facts about this notion.

$$\left(\mathop{*}_{i\in I}^{A} B_i\right) \upharpoonright a \cong \mathop{*}_{i\in I}^{A\upharpoonright a} (B_i \upharpoonright a) \quad \text{for } a \in A,$$

$$\mathop{*}_{i\in I}^{A} B_i \cong \prod_{x\in \mathrm{At}A} \mathop{*}_{i\in I} (B_i \upharpoonright x) \quad \text{for } A \text{ finite}.$$

Using these facts one can easily show.

Theorem 1.4.1. *For a finite, Theorems* 1.3.2–1.3.3, 1.3.6 *hold for A-amalgamated free products.*

Theorem 1.3.4 does not carry over to A-amalgamated free products. For example, let κ be a singular cardinal with $\mathrm{cf}\kappa > \aleph_0$, let A be a four-element BA, with atoms a, $-a$, and choose $B, C \supseteq A$ so that $B \upharpoonright a$ and $C \upharpoonright a$ have depth $\mathrm{cf}\kappa$ attained, $B \upharpoonright -a$ is denumerable, while $C \upharpoonright -a$ has depth not attained (see Corollary 1.2.6). Thus B has a chain of type $\mathrm{cf}\kappa$ and depth $C = \kappa$, but $B *_A C$ has no chain of type κ. An amalgamated version of Theorem 1.3.5 for finite A is:

Theorem 1.4.2. *Let* $\kappa = \mathrm{depth} *_{i\in I}^{A} B_i$, *where* A *is finite, and suppose that* κ *is singular with* $\mathrm{cf}\kappa > \aleph_0$. *Then the following are equivalent*:

(i) $*_{i\in I}^{A} B_i$ *has no chain of type* κ;

(ii) *for every atom* a *of* A, $*_{i\in I} (B_i \upharpoonright a)$ *has no chain of type* κ.

Now we go to the general case, where A can be infinite. Here the situation is much different from the case of free products, as is shown by the following result.

Theorem 1.4.3. *Let* $\kappa, \lambda, \mu, \nu$ *be infinite cardinals such that* $\kappa \leq \lambda, \mu$ *and* $\lambda, \mu \leq \nu$. *Then there exist* BA's A, B, C *with* $A \subseteq B, C$, depth $A = \kappa$, depth $B = \lambda$, depth $C = \mu$, and $\mathrm{depth}(B *_A C) = \nu$.

Proof. First we show the following, which is of some independent interest.

There exist ccc BA's A, B, C such that $|A| = |B| = |C| = \nu$ and $B *_A C$ has a chain of type ν. (1)

For, let $B = \mathrm{Fr}\langle x_\alpha : \alpha < \nu\rangle$, and let $A = Sg(\{x_0 \cdot x_\alpha \cdot -x_\beta : 0 < \alpha < \beta < \nu\} \cup \{-x_0 \cdot x_\alpha \cdot -x_\beta : 0 < \alpha < \beta < \nu\})$. Let C be free on $\langle y_\alpha : \alpha < \nu\rangle$, with $B \cap C = A$ and for all $0 < \alpha < \beta < \nu$, $x_0 \cdot x_\alpha \cdot -x_\beta = y_0 \cdot y_\alpha \cdot -y_\beta$ and $-x_0 \cdot x_\alpha \cdot -x_\beta = -y_0 \cdot y_\alpha \cdot -y_\beta$. Now for $0 < \alpha < \nu$ let $z_\alpha = x_0 \cdot x_\alpha \cdot -y_0 \cdot y_\alpha$ (in $B *_A C$). Then $z_\alpha < z_\beta$ if $0 < \alpha < \beta < \nu$. In fact, $z_\alpha \cdot -z_\beta = x_0 \cdot x_\alpha \cdot -y_0 \cdot y_\alpha \cdot -x_\beta + x_0 \cdot x_\alpha \cdot -y_0 \cdot y_\alpha \cdot -y_\beta$, and $a = x_0 \cdot x_\alpha \cdot -x_\beta \in A$ shows that $x_0 \cdot x_\alpha \cdot -y_0 \cdot y_\alpha \cdot -x_\beta = 0$; similarly, $x_0 \cdot x_\alpha \cdot -y_0 \cdot y_\alpha \cdot -y_\beta = 0$. On the other hand, $z_\beta \cdot -z_\alpha \geq x_0 \cdot x_\beta \cdot -x_\alpha \cdot -y_0 \cdot y_\beta$, and there is no $a \in A$ such that $x_0 \cdot x_\beta \cdot -x_0 \leq a$ and $a \cdot -y_0 \cdot y_\beta = 0$, since then also $a \cdot -x_0 \cdot x_\beta = 0$; letting D be the interval algebra on ν and taking a homomorphism f from B into D such that $fx_0 = (1, 0)$ and $fx_\xi = (\xi, \xi)$ for all $\xi \neq 0$ gives a contradiction. Thus (1) holds.

Now let A', B', C' be, respectively, interval algebras on κ, λ, μ. Now

$$(A' * B') *_A (A' * C') \cong A' * B' * C'. \tag{2}$$

In fact, let $f(a \cdot b) = a \cdot b$ for $a \in A', b \in B'$ and also for $a \in A', b \in C'$. To show that f extends to a homomorphism, suppose $(u \cdot b) \cdot (v \cdot c) = 0$, where $u, v \in A'$ and $b \in B', c \in C'$ (considered as an element of the left side of (2)). Thus there is an $a \in A$ such that $u \cdot b \leq a$ and $a \cdot v \cdot c = 0$. Hence $u \leq a$ and $v \cdot a = 0$, so $u \cdot v = 0$, and $u \cdot b \cdot v \cdot c = 0$ as an element of the right side. Clearly f is one-one and onto, so (2) holds.

By Corollary 1.3.3 we have depth $(A' * B' * C') = \lambda \cup \mu$. Set $A'' = A \times A'$, $B'' = B \times (A' * B')$, $C'' = C \times (A' * C')$. Then, using a fact stated before Theorem 1.4.1, and (2),

$$B'' *_{A''} C'' \cong (B *_A C) \times [(A' * B') *_{A'} (A' * C')] \cong (B *_A C) \times (A' * B' * C'),$$

and hence A'', B'', C'' are as desired.

A natural question arises concerning how the depth of $B *_A C$ changes when A is kept fixed. We present some results about this question, and then state two open questions in a more precise form. Our first result depends on the following two set-theoretical lemmas, whose proofs use standard techniques.

Lemma 1.4.4. *Let $\langle a_\alpha : \alpha < \kappa \rangle$ be a system of subsets of X, where $|X| = \kappa$ and $|a_\alpha| = \kappa$ for each $\alpha < \kappa$. Then there is a $b \subseteq X$ such that $|b \cap a_\alpha| = |a_\alpha \setminus b| = \kappa$ for all $\alpha < \kappa$.*

Proof. Let f be a one-one function from κ onto $\kappa \times \kappa$; for any $\alpha < \kappa$ we write $f\alpha = ((f\alpha)0, (f\alpha)1)$. By recursion we pick for all $\alpha < \kappa$ distinct $\gamma_\alpha, \delta_\alpha \in a_{(f\alpha)0} \setminus \{\gamma_\beta, \delta_\beta : \beta < \alpha\}$. Set $b = \{\gamma_\alpha : \alpha < \kappa\}$; note that for any $\varepsilon < \kappa$ we have $b \cap a_\varepsilon \supseteq \{\gamma_\alpha : (f\alpha)0 = \varepsilon\}$ and $a_\varepsilon \setminus b \supseteq \{b_\alpha : (f_\alpha)0 = \varepsilon\}$ so the desired conditions hold.

Lemma 1.4.5. *Let κ be an infinite regular cardinal, $\langle a_\alpha : \alpha < \kappa \rangle$ a system of subsets of X each of power κ. Then there is a system $\langle b_\alpha : \alpha < \kappa^+ \rangle$ of subsets of X such that $|b_\alpha \cap a_\beta| = \kappa$ whenever $\alpha < \kappa^+$, $\beta < \kappa$, and $|b_\alpha \setminus b_\beta| < \kappa = |b_\beta \setminus b_\alpha|$ whenever $\alpha < \beta < \kappa^+$.*

Proof. A system $\langle b_\alpha : \alpha < \kappa \rangle$ is *good* if

$$\forall \alpha, \gamma < \kappa \ (|b_\alpha \cap a_\gamma| = \kappa); \tag{1}$$

$$\forall \Gamma \in P_{<\kappa}\kappa \ \forall \gamma \in \kappa \ \left| a_\gamma \Big/ \bigcup_{\alpha \in \Gamma} b_\alpha \right| = \kappa. \tag{2}$$

There is a good system: choose b by Lemma 1.4.4 and let $b_\alpha = b$ for all $\alpha < \kappa$. Clearly now it suffices to show that if $\langle b_\alpha : \alpha < \kappa\rangle$ is good then there is a $c \subseteq \kappa$ such that

$$\forall\, \alpha < \kappa\, (|b_\alpha \setminus c| < \kappa = |c \setminus b_\alpha| \quad\text{and}\quad |a_\alpha \cap c| = \kappa); \tag{3}$$

$$\forall\, \Gamma \in P_{<\kappa}\kappa\; \forall\, \gamma \in \kappa \left(\left| a_\gamma \Big/ \left(\bigcup_{\alpha\in\Gamma} b_\alpha \cup c\right)\right| = \kappa\right). \tag{4}$$

Let f be a one-one function from κ onto $\kappa \times \kappa$. By recursion we pick for all $\alpha < \kappa$ distinct $\gamma_\alpha, \delta_\alpha \in a_{(f\alpha)0} \setminus (\bigcup_{\beta<\alpha} b_\beta \cup \{\gamma_\beta, \delta_\beta : \beta < \alpha\})$. Then set $c = \kappa \setminus \{\gamma_\alpha : \alpha < \kappa\}$. Then for any $\alpha < \kappa$ we have $b_\alpha \setminus c \subseteq \{\gamma_\beta : \beta \leqslant \alpha\}$, $c \setminus b_\alpha \supseteq \{\delta_\beta : \beta < \alpha\}$, $a_\alpha \cap c \supseteq \{\delta_\beta : (f\beta)0 = \alpha\}$. Moreover, if $\Gamma \in P_{<\kappa}\kappa$, then by the regularity of κ there is a $\beta \in \kappa$ with $\Gamma \subseteq \beta$, and for any $\alpha < \kappa$ we have $a_\alpha \setminus (\bigcup_{\gamma\in\Gamma} b_\gamma \cup c) \supseteq \{\gamma_\varepsilon : \varepsilon > \beta, (f\varepsilon)0 = \alpha\}$. Thus (3) and (4) hold.

Theorem 1.4.6. *Let κ be an infinite regular cardinal, A a BA of cardinality κ. Suppose there is a homomorphism $f : A \twoheadrightarrow D$, where D is a subalgebra of $\mathcal{P}\kappa$ containing all subsets of power $<\kappa$. Then there exist BA's $B, C \supseteq A$ satisfying the κ^+-c.c. such that $\beta *_A C$ has a chain of type κ^+.*

Proof. Define $g : A \to A \times \mathcal{P}\kappa$ by: $ga = (a, fa)$ for all $a \in A$. Let $B = C = A \times \mathcal{P}\kappa$; we claim that $B *_A C$ is as desired. To see this, choose by Lemma 1.4.4 $d \subseteq \kappa$ such that for all $a \in A$ with $|fa| = \kappa$ we have $|fa \cap d| = |fa \setminus d| = \kappa$. By Lemma 1.4.5 let $\langle a_\alpha : \alpha < \kappa^+\rangle$ be a system of subsets of d such that $|a_\alpha \cap fa| = \kappa$ whenever $\alpha < \kappa^+$, $a \in A$, and $|fa| = \kappa$, and such that $|a_\alpha \setminus a_\beta| < \kappa = |a_\beta \setminus a_\alpha|$ for $a < \beta < \kappa^+$. Let $\langle b_\alpha : \alpha < \kappa^+\rangle$ be subsets of $\kappa \setminus d$ with analogous properties. Then, we claim, $\langle (0, a_\alpha) \cdot (0, b_\alpha) : \alpha < \kappa^+\rangle$ is a κ^+-chain in $B *_A C$ (where $(0, a_\alpha)$, $(0, b_\alpha)$ are considered as elements of B, C respectively). If $\alpha < \beta < \kappa$, since $|a_\alpha \setminus a_\beta| < \kappa$ choose $x \in A$ such that $fx = a_\alpha \setminus a_\beta$. Thus $(0, a_\alpha) \cdot -(0, a_\beta) \leqslant gx$, and $gx \cdot (0, b_\alpha) = 0$ since $fx \leqslant d$ and $b_\alpha \leqslant \kappa \setminus d$. This shows that $(0, a_\alpha) \cdot (0, b_\alpha) \cdot -(0, a_\beta) = 0$. Similarly $(0, a_\alpha) \cdot (0, b_\alpha) \cdot -(0, b_\beta) = 0$, so $(0, a_\alpha) \cdot (0, b_\alpha) \leqslant (0, a_\beta) \cdot (0, b_\beta)$. Suppose equality holds. Choose $a \in A$ such that $(0, a_\beta \setminus a_\alpha) \leqslant (a, fa)$ and $(0, b_\beta) \cdot (a, fa) = 0$. Then $|fa| = \kappa$ while $b_\beta \cap fa = 0$, a contradiction.

As a corollary to the proof of Theorem 1.4.6 we get:

Corollary 1.4.7. *Let κ be an infinite regular cardinal. Suppose A is a BA with no chain of type κ^+. Suppose there is a homomorphism $f : A \twoheadrightarrow D$, where D is the BA of subsets of κ of power $<\kappa$ and their complements. Then there exist $B, C \supseteq A$ such that $B *_A C$ has a chain of type κ^+, while neither B nor C have such chains.*

Proof. Let g, B, C be as in the proof of Theorem 1.4.6. Choose $d \subseteq \kappa$ with $|d| = |\kappa \setminus d| = \kappa$. By Lemma 1.4.5 let $\langle a_\alpha : \alpha < \kappa^+\rangle$ be a system of subsets of d such that $|a_\alpha \setminus a_\beta| < \kappa = |a_\beta \setminus a_\alpha|$ for $\alpha < \beta < \kappa^+$; let $\langle b_\alpha : \alpha < \kappa^+\rangle$ be a system of subsets of $\kappa \setminus c$ with analogous properties; then proceed as in the proof of Theorem 1.4.6. (Note that $|fa| = \kappa$ implies $|\kappa \setminus fa| < \kappa$.)

Note that the hypothesis of Theorem 1.4.6 holds for every denumerable BA A. Assuming $\kappa^{\xi} = \kappa$, it holds for every BA A of power κ which has a free subalgebra of power κ, in particular for every complete BA of power κ (by [1]). The hypothesis fails, however, for $\kappa = \omega_1$ and A the BA of finite and cofinite subsets of ω_1, and also for $\kappa = 2^{\aleph_0}$ and A the interval algebra on $\mathbb{R}$ (even assuming that κ is regular). For these two choices of A the hypothesis of Corollary 1.4.7 holds with $\kappa = \aleph_0$. But for A the interval algebra on ${}^{\kappa}2$ (with lexicographic order), $\kappa \geq \omega_1$, it is not clear how to use Theorem 1.4.6 or Corollary 1.4.7 to find $B, C \supseteq A$ with $\mathrm{Depth}(B *_A C) > \max(\mathrm{depth}\, B, \mathrm{depth}\, C)$. By modifying an argument in [11] one sees that under Martin's axiom there is a chain of type 2^{ω} in $P\omega *_A P\omega$, where A is the BA of finite and cofinite subsets of ω.

For A incomplete we have the following consistency result of Saharon Shelah.

Theorem 1.4.8. *In V assume $\lambda = \lambda^{\lambda} < \mu$, A is an incomplete BA, and $|A| < \lambda$ (hence $\omega < \lambda$).*

Then there is a forcing extension V^G preserving cardinals in which there are BA's B, C such that $A \subsetneq B$, $A \subseteq C$, B and C have depth $\leq \lambda$, $B *_A C$ has a chain of length μ, and $\mu \leq 2^{\lambda}$.

Proof. Say $X \subseteq A$ and $\sum^A X$ does not exist. Let $I = \mathrm{Ig}\, X$ and $J = \{a \in A : \forall x \in X(x \cdot a = 0)\}$. Then $I \cap J = \{0\}$, $I \cup J$ is dense in A, and there is no $a \in A$ such that $x \leq a$ for all $x \in I$ and $x \cdot a = 0$ for all $x \in J$. Let

$$P = \{(T, f, g) : T \in P_{<\lambda}\mu, f : T \times T \to I \quad \text{and} \quad g : T \times T \to J\}.$$

We define $(T, f, g) \leq (T', f', g')$ iff $T \supseteq T'$, $f \supseteq f'$, $g \supseteq g'$ ((T, f, g) is stronger). Clearly P is $(<\lambda)$-closed and satisfies the λ^+-c.c. We force with it to get V^G; thus V^G preserves cardinals. Clearly $\{(T, f, g) : \alpha \in T\}$ is dense for each $\alpha < \mu$, so $\bigcup_{(T,f,g) \in G} T = \mu$.

Let $f = \bigcup_{(T',f',g') \in G} f'$ and $g = \bigcup_{(T',f',g') \in G} g'$.

Let B be a free extension of A by $\{b_\alpha : \alpha < \mu\}$ subject to $b_\alpha \cdot -b_\beta \leq f(\alpha, \beta)$ whenever $\alpha < \beta < \mu$ and $b_\alpha \cdot a = 0$ whenever $\alpha < \mu$ and $a \in J$. Let C be similarly defined with c_α's, g, and I in place of b_α's, f, and J.

Clearly $b_\alpha \cdot c_\alpha \leq b_\beta \cdot c_\beta$ for all $\alpha < \beta < \mu$, in $B *_A C$. Now suppose $\alpha < \beta < \mu$ but $b_\beta \cdot -b_\alpha \cdot c_\beta = 0$; we shall get a contradiction, which hence will show that $b_\alpha \cdot c_\alpha < b_\beta \cdot c_\beta$. Let $\bar{A}$ be the completion of A, D the interval algebra on μ, $E = [\bar{A} \restriction \sum I) * D] \times (\bar{A} \restriction -\sum I)$, and h the natural embedding of A into E. For each $\gamma < \mu$ let $d_\gamma = ([0, \gamma), 0)$. Then $d_\gamma \cdot -d_\delta = 0 \leq hf(\gamma, \delta)$ whenever $\gamma < \delta < \mu$, and $d_\gamma \cdot ha = 0$ whenever $\gamma < \mu$ and $a \in J$. Hence there is a homomorphism k from B into E such that $k \restriction A = h$ and $kb_\gamma = d_\gamma$ for all $\gamma < \mu$. Now from $b_\beta \cdot -b_\alpha \cdot c_\beta = 0$

we get an $a \in A$ such that $b_\beta \cdot -b_\alpha \leq a$ and $a \cdot c_\beta = 0$. Thus, applying k,

$$([\alpha, \beta), 0) \leq (a \cdot \sum I, a \cdot -\sum I).$$

Hence by $*$-freeness, $\sum I \leq a$. Working similarly with C, we get $a = \sum I$, contradiction.

Thus $B *_A C$ has a chain of type μ. Next, $\mu \leq 2^\lambda$. Indeed, let k be a one-one map of $\mu \times \lambda$ onto $\mu \times \mu$ in M. Fix $0 \neq a \in I$, and for each $\alpha < \mu$ let $D_\alpha = \{\beta < \lambda : fk(\alpha, \beta) = \alpha\}$. If $\alpha, \beta < \mu$ and $\alpha \neq \beta$, then

$$\{(T', f', g') \in P : \exists\, \gamma < \lambda\ (k(\alpha, \gamma), k(\beta, \gamma) \in T' \times T' \quad \text{and} \quad f'k(\alpha, \gamma) = a \quad \text{while} \quad f'k(\beta, \gamma) = 0)\}$$

is dense. Thus $D_\alpha \neq D_\beta$. So, $\mu \leq 2^\lambda$.

It remains only to show that B and C have depth $\leq \lambda$; by symmetry we work only with B. First two preliminary statements. For any $\Gamma, \Delta \in P_{<\omega}\mu$ let

$$a_{\Gamma\Delta} = \prod \{f(\alpha, \beta) : \alpha \in \Gamma, \beta \in \Delta, \alpha < \beta\}.$$

Then

For $x = \prod_{\alpha \in \Gamma} b_\alpha \cdot \prod_{\alpha \in \Delta} -b_\alpha$, if $x \neq 0$, then $\forall\, d \in A$ $(d \cdot x = 0$ iff $d \cdot a_{\Gamma\Delta} = 0)$. (1)

For, $x \leq a_{\Gamma\Delta}$, so $\Leftarrow$ is clear. For $\Rightarrow$, let $B' = \mathrm{Sg}\{b_\gamma : \gamma \in \Gamma \cup \Delta\}$. Let $y_\alpha = a_{\Gamma\Delta}$ for all $\alpha \in \Gamma$ and $y_\alpha = 0$ for all $\alpha \in \Delta$. Then there is a homomorphism h from B' onto A such that h is the identity on A and $hb_\alpha = d_\alpha$ for all $\alpha \in \Gamma \cup \Delta$. It easily follows that $d \cdot a_{\Gamma\Delta} = 0$.

For every non-zero $y \in B$ there is an $a \in A$ such that $a \neq 0$ and for every non-zero $a' \leq a$ with $a' \in A$ we have $a' \cdot y \neq 0$. (2)

In fact, we may assume that y has the form $d \cdot x$ with $d \in A$ and $x = \prod_{\alpha \in \Gamma} b_\alpha \cdot \prod_{\alpha \in \Delta} -b_\alpha$. Then $a = d \cdot a_{\Gamma\Delta}$ is easily seen to work, using (1).

Now we begin the proof that B has depth $\leq \lambda$. Suppose $\langle z_\alpha : \alpha < \lambda^+ \rangle$ is strictly increasing in B. By a Δ-system argument and other cutting down we may assume that

$$\begin{gathered} z_\alpha = d \cdot d' \cdot e^\alpha, \quad d \in A, \quad z_\alpha \in \mathrm{Sg}(A \cup \{b_\gamma : \gamma \in \Gamma_\alpha\}), \quad \Gamma_\alpha \in P_{<\omega}\mu, \\ \langle \Gamma_\alpha : \alpha < \lambda^+ \rangle \text{ a } \Delta\text{-system with kernel } K, \quad d' = \prod_{\gamma \in E} b_\gamma \cdot \prod_{\gamma \in F} -b_\gamma, \\ e^\alpha = \sum_{\Omega \in \Theta_\alpha} \prod_{\gamma \in \Omega} b_\gamma \cdot \prod_{\gamma \in \Gamma'_\alpha \setminus \Omega} -b_\gamma, \quad \Gamma'_\alpha = \Gamma_\alpha \setminus K, \quad \Theta_\alpha \subseteq P\Gamma'_\alpha. \end{gathered} \tag{3}$$

Now by (2), using the density of $I \cup J$ in A, for each $\alpha < \lambda^+$ choose $a_\alpha \neq 0$, $a_\alpha \in I \cup J$ so that $a' \cdot z_{\alpha+1} \cdot -z_\alpha \neq 0$ whenever $0 \neq a' \leq a_\alpha$, $a' \in A$. Cutting down, we may assume $a_\alpha = a$ for all $\alpha < \lambda^+$. Then $\langle z_\alpha \cdot a : \alpha < \lambda^+ \rangle$ is strictly increasing, so we may assume that $d \leq a$. In fact, clearly $d = a$. If $d \in J$, then each $z_\alpha = d$ since

$b_\gamma \cdot d = 0$ for all γ, contradiction. Thus

$$d \in I, \text{ and } a' \cdot z_{\alpha+1} \cdot -z_\alpha \neq 0 \text{ whenever } 0 \neq a' \leq d,\ a' \in A. \tag{4}$$

Now from (1) and (3) we get

$$\begin{aligned} &\forall\, \alpha, \beta < \lambda^+ \,(z_\alpha \leq z_\beta \text{ iff} \\ &\quad \forall\, \Omega \in \Theta_\alpha \forall\, \Xi \in P\Gamma'_\beta \setminus \Theta_\beta (d \cdot a_{E\cup\Omega\cup\Xi,\, F\cup(\Gamma_\alpha\setminus\Omega)\cup(\Gamma_\beta\setminus\Xi)} = 0)). \end{aligned} \tag{5}$$

Note that z, Γ', Θ, a, and f are all in V^G but not in V. Choose $p \in G$ so that

$$\begin{aligned} &p \Vdash \langle z_\alpha : \alpha < \lambda^+ \rangle \text{ is strictly increasing and (5) holds and } \forall\, \alpha < \lambda^+ \\ &(\Gamma'_\alpha \in P_{<\omega}\mu \text{ and } \Theta_\alpha \subseteq P\Gamma'_\alpha). \end{aligned} \tag{6}$$

Now we work in V. For each $\alpha < \lambda^+$ we find $p_\alpha \in P$, $\Gamma''_\alpha \in P_{<\omega}\mu$, and $\Theta'_\alpha \subseteq P\Gamma''_\alpha$ such that $p_\alpha \leq p$, $p_\alpha \Vdash \Gamma'_\alpha = \Gamma''_\alpha$ and $\Theta_\alpha = \Theta'_\alpha$, and $\Gamma''_\alpha \subseteq T^\alpha$, where $p_\alpha = (T^\alpha, f^\alpha, g^\alpha)$. By a Δ-system argument and further cutting down we may assume that $\langle T^\alpha : \alpha < \lambda^+ \rangle$ forms a Δ-system with kernel L, $f^\alpha \restriction L \times L = f^\beta \restriction L \times L$ for all $\alpha, \beta < \lambda^+$, similarly for g's, $\langle T^\alpha, < \rangle \cong \langle T^\beta, < \rangle$ under a unique isomorphism $\pi_{\alpha\beta}$ which is the identity on L, $\pi_{\alpha\beta}[\Gamma''_\alpha] = \Gamma''_\beta$, $\pi_{\alpha\beta}$ takes Θ'_α to Θ'_β, and $f^\alpha(\gamma, \delta) = f^\beta(\pi_{\alpha\beta}\gamma, \pi_{\alpha\beta}\delta)$ for all $\gamma < \delta$ both in T^α. Now fix $\alpha < \beta < \lambda^+$, and extend $\pi_{\alpha\beta}$ to a permutation π of μ which also acts as an automorphism of P and of names, with π^2 the identity. Let

$$a' = \sum_{\Omega \in \Theta'_\beta} a_{E\cup\Omega,\, F\cup(\Gamma''_\beta\setminus\Omega)} \cdot d,$$

where a is calculated using f^β rather than f, and define $q = (T', f', g')$ as follows: $T' = T^\alpha \cup T^\beta$, and for all $\gamma, \delta \in T$,

$$f'(\gamma,\delta) = \begin{cases} f^\alpha(\gamma,\delta), & \text{if } \gamma, \delta \in T^\alpha, \\ f^\beta(\gamma,\delta), & \text{if } \gamma, \delta \in T^\beta, \\ a', & \text{otherwise,} \end{cases} \qquad f'(\gamma,\delta) = \begin{cases} g^\alpha(\gamma,\delta), & \text{if } \gamma, \delta \in T^\alpha, \\ g^\beta(\gamma,\delta), & \text{if } \gamma, \delta \in T^\beta, \\ 0, & \text{otherwise.} \end{cases}$$

Then $q \leq p_\alpha$ and $q \leq p_\beta$. Since $\alpha < \beta$, by $p \Vdash$ '(5) holds' we get $\Omega \in \Theta'_\beta$ and $\Xi \in P\Gamma''_\alpha \setminus \Theta_\alpha$ such that $q \Vdash d \cdot a'' \neq 0$, where $a'' = a_{E\cup\Omega\cup\Xi,\, F\cup(\Gamma''_\beta\setminus\Omega)\cup(\Gamma''_\alpha\setminus\Xi)}$. But by the definition of q and a' we then get

$$q \Vdash d \cdot a'' = d \cdot a_{E\cup\Omega,\, F\cup(\Gamma''_\beta\setminus\Omega)} \cdot a_{E\cup\Xi,\, F\cup(\Gamma''_\alpha\setminus\Xi)}.$$

Applying π, we get

$$\pi q \Vdash d \cdot a_{E\cup\pi[\Omega],\, F\cup\pi[\Gamma''_\alpha\setminus\Omega]} \cdot a_{E\cup\pi[\Xi],\, F\cup\pi[\Gamma''_\alpha\setminus\Xi]} \neq 0.$$

But $\pi[\Omega] \in \Theta'_\alpha$ and $\pi[\Xi] \in P\Gamma''_\beta \setminus \Theta_\beta$; since $\pi p_\alpha = p_\beta$ we still have $\pi q \leq p$, so this contradicts $p \Vdash$ '(5) holds'.

Shelah remarks that instead of forcing in Theorem 1.4.8 one can do the same thing in L, with $\mu = \lambda^+$, using a known theorem.

We now give an upper bound for the types of chains in $B *_A C$, for fixed A.

Theorem 1.4.9. *Let* $\aleph_0 \leq |A| \leq \lambda$ *and let* $B, C \supseteq A$ *have no chains of type* λ^+. *Then* $B *_A C$ *has no chain of type* $(\beth_\omega\lambda)^+$.

Proof. We use the notation *length* n introduced in the proof of Theorem 1.3.1. Now let $m_1 = 2$, and for $n > 1$ let $m_n = m_{n-1} + n^2 + n$. To establish the theorem it suffices to prove

$$\text{for every } n \in \omega \setminus 1,\ B *_A C \text{ does not have a chain of type } (\beth_{m_n}\lambda)^+, \text{ all of whose members have length } n. \tag{1}$$

We prove (1) by induction on n, assuming in each case, by contradiction, that $\langle x_\alpha : \alpha < (\beth_{m_n}\lambda^+) \rangle$ is a chain in $B *_A C$, each x_α of length n, say

$$x_\alpha = \sum_{i<n} b_i^\alpha \cdot c_i^\alpha$$

with $b_i^\alpha \in B$, $c_i^\alpha \in C$ for $i < n$, $c_i^\alpha \cdot c_j^\alpha = 0$ for $i < j < n$.

First suppose $n = 1$. Then for $\alpha < \beta < (\beth_2\lambda)^+$ we have $b_0^\alpha \cdot c_0^\alpha \cdot -b_0^\beta = 0$, so there is an $a_{\alpha\beta} \in A$ such that $b_0^\alpha \cdot -b_0^\beta \leq a_{\alpha\beta}$ and $c_0^\alpha \cdot a_{\alpha\beta} = 0$. By the Erdös–Rado theorem there is an $a' \in A$ and a $\Gamma \subseteq (\beth_2\lambda)^+$ of order type $(\beth_1\lambda)^+$ such that for all $\alpha, \beta \in \Gamma$ with $\alpha < \beta$ we have $a_{\alpha\beta} = a'$. Since $b_0^\alpha \cdot -a' \leq b_0^\beta \cdot -a'$ for $\alpha, \beta \in \Gamma$ and $\alpha < \beta$, there is a terminal segment Δ of Γ such that $b_0^\alpha \cdot -a' = b_0^\beta \cdot -a'$ for all $\alpha, \beta \in \Delta$; let this common value be d. Note that for $\alpha \in \Delta$ we have $x_\alpha \leq c_0^\alpha \leq -a'$, so $x_\alpha = d \cdot c_0^\alpha$. For $\alpha, \beta \in \Delta$ and $\alpha < \beta$ we have $d \cdot c_0^\alpha \cdot -c_0^\beta = 0$, so there is an $a''_{\alpha\beta} \in A$ such that $d \leq a''_{\alpha\beta}$ and $c_0^\alpha \cdot -c_0^\beta \cdot a''_{\alpha\beta} = 0$. Thus by the Erdös–Rado theorem again, there is a $\theta \subseteq \Delta$ of order type λ^+ and there is an $a''' \in A$ such that for all $\alpha, \beta \in \theta$ with $\alpha < \beta$ we have $\alpha''_{\alpha\beta} = a'''$. Then $x_\alpha \leq d \leq a'''$ for each $\alpha \in \theta$, and $c_0^\alpha \cdot a''' \leq c_0^\beta \cdot a'''$ whenever $\alpha, \beta \in \theta$ and $\alpha < \beta$. This is clearly impossible.

Now assume inductively that $n > 1$. Then if $\alpha, \beta < (\beth_{m_n}\lambda)^+$ and $\alpha < \beta$, we have

$$0 = b_0^\alpha \cdot c_0^\alpha \cdot \prod_{j<n} (-b_j^\beta + -c_j^\beta) \geq b_0^\alpha \cdot c_0^\alpha \cdot \prod_{j>n} -c_j^\beta,$$

so there is an $a_{\alpha\beta} \in A$ such that $b_0^\alpha \leq a_{\alpha\beta}$ while $c_0^\alpha \cdot \prod_{j<n} -c_j^\beta \cdot a_{\alpha\beta} = 0$. Hence by the Erdös–Rado theorem again, there is an $a' \in A$ and a $\Gamma_0 \subseteq (\beth_{m_n}\lambda)^+$ of order type $(\beth_{m_n-1}\lambda)^+$ such that for all $\alpha, \beta \in \Gamma_0$ with $\alpha < \beta$ we have $a_{\alpha\beta} = a'$. Thus for all $\alpha \in \Gamma_0$ we have $b_0^\alpha \leq a'$ while for $\alpha, \beta \in \Gamma_0$ and $\alpha < \beta$ we have $c_0^\alpha \cdot \prod_{j<n} -c_j^\beta \cdot a' = 0$. Now $\langle x_\alpha \cdot -a' : \alpha \in \Gamma_0 \rangle$ has all terms of length $<n$, so by the induction hypothesis it is eventually constant. Hence we may assume (passing to $\langle x_\alpha \cdot a' : \alpha \in \Gamma_0 \rangle$, and deleting an initial segment) that $b_i^\alpha \leq a'$, $c_i^\alpha \leq a'$ for all $\alpha \in \Gamma_0$ and $i < n$. Thus

$$\langle x_\alpha : \alpha \in \Gamma_0 \rangle \text{ is strictly increasing,} \tag{2}$$

$$\text{for all } \alpha, \beta \in \Gamma_0 \text{ with } \alpha < \beta \text{ we have } c_0^\alpha \leq \sum_{j<n} c_j^\beta. \tag{3}$$

Repeating this argument n times we arrive at $\Gamma_{n-1} \subseteq (\beth_{m_n}\lambda)^+$ such that

$$\text{the order type of } \Gamma_{n-1} \text{ is } (\beth_{m_n-n}\lambda)^+, \tag{4}$$

$$\langle x_\alpha : \alpha \in \Gamma_{n-1} \rangle \text{ is strictly increasing,} \tag{5}$$

$$\text{for all } \alpha, \beta \in \Gamma_{n-1} \text{ with } \alpha < \beta, \text{ and for all } i < n, \ c_i^\alpha \leq \sum_{j<n} c_j^\beta. \tag{6}$$

Now let $\langle (k_i, l_i) : i < n^2 \rangle$ be an enumeration of $n \times n$. We construct $d_0, \ldots, d_{n^2-1}$ and $\Gamma_n, \ldots, \Gamma_{n^2+n-1}$ by induction. Suppose Γ_{i-1} has been constructed so that $\Gamma_{i-1} \subseteq \Gamma_{n-1}$, $n \leq i < n^2+n-1$, and Γ_{i-1} has order type $(\beth_{m_n-1})^+$. Let $(k_{i-n}, l_{i-n}) = (k, l)$. For all $\alpha, \beta \in \Gamma_i$ with $\alpha < \beta$ we have $b_k^\alpha \cdot c_k^\alpha \cdot \prod_{j<n}(-b_j^\beta + -c_j^\beta) = 0$; meeting with c_l^β we get $b_k^\alpha \cdot c_k^\alpha \cdot -b_l^\beta \cdot c_l^\beta = 0$, so there is an $a_{\alpha\beta} \in A$ such that $b_k^\alpha \cdot -b_l^\beta \leq a_{\alpha\beta}$ and $c_\kappa^\alpha \cdot c_l^\beta \cdot a_{\alpha\beta} = 0$. Thus by the Erdös–Rado theorem there is a $d_{i-n} \in A$ and a $\Gamma_i \subseteq \Gamma_{i-1}$ such that Γ_i has order type $(\beth_{m_n-i-1}\lambda)^+$ and for all $\alpha, \beta \in \Gamma_i$ with $\alpha < \beta$, $a_{\alpha\beta} = d_{i-n}$. This finishes the induction. For any $s, t < n$ let $d'_{st} = d_i$, where $(k_i, l_i) = (s, t)$. Thus with $p = n^2 + n - 1$ we have

$$\Gamma_p \subseteq (\beth_{m_n}\lambda)^+, \text{ and it has order type } (\beth_{m_n-1}\lambda)^+, \tag{7}$$

$$\langle x_\alpha : \alpha \in \Gamma_p \rangle \text{ is strictly increasing,} \tag{8}$$

$$\text{for all } \alpha, \beta \in \Gamma_p \text{ with } \alpha < \beta, \text{ and for all } i < n, \ c_i^\alpha \leq \sum_{j<n} c_j^\beta, \tag{9}$$

$$\text{for all } \alpha, \beta \in \Gamma_p \text{ with } \alpha < \beta, \text{ and for all } i, j < n, \ b_i^\alpha - b_j^\beta \leq d'_{ij} \text{ and } c_i^\alpha \cdot c_j^\beta \cdot d'_{ij} = 0. \tag{10}$$

Now for each $\varepsilon \in {}^{n \times n}2$ let $e_\varepsilon = \prod_{i,j<n} d_{ij}^{\prime\,\varepsilon ij}$. It is enough now to prove that for any such ε, the sequence $\langle x_\alpha \cdot e_\varepsilon : \alpha \in \Gamma_p \rangle$ is eventually constant. So we may assume that $b_i^\alpha, c_i^\alpha \leq e_\varepsilon$ for all $\alpha \in \Gamma_p$ and all $i < n$. Hence

$$\text{for all } \alpha, \beta \in \Gamma_p \text{ with } \alpha < \beta \text{ and for all } i, j < n, \ c_i^\alpha \cdot c_j^\beta \neq 0 \text{ implies that } b_i^\alpha \leq b_j^\beta, \tag{11}$$

$$\text{for all } \alpha \in \Gamma_p \text{ and all distinct } i, j < n, \ b_i^\alpha \neq b_j^\alpha. \tag{12}$$

Using (9), (11), (12) we can now proceed as in the first part of the proof of Theorem 1.3.1 to obtain the desired conclusion. (Or one can easily check that $\langle x_\alpha : \alpha \in \Gamma_p \rangle$ is a chain in $B * C$, contradicting Theorem 1.3.1 itself.)

As the referee pointed out, essentially the same proof as for Theorem 1.4.9. shows that if $\aleph_0 \leq |A| \leq \lambda < \kappa$, κ is weakly compact, and $B, C \supset A$ have no chains of type κ, then also $B *_A C$ has no chain of type κ.

Theorems 1.4.6–1.4.9 leave the open the following specific questions.

Problem 1. (In ZFC) For every infinite BA A, do there exist BA's $B, C \supseteq A$ and an infinite cardinal κ such that B and C have no chains of type κ but $B *_A C$ does?

Problem 2. For every infinite BA A, is there a cardinal κ such that if $B, C \supseteq A$ and $|B|, |C| \geq \kappa$, then $\mathrm{depth}(B *_A C) = \max(\mathrm{depth}\, B, \mathrm{depth}\, C)$?

1.5. Ultraproducts

We consider how the depths of factors A_i in an ultraproduct $\prod_{i \in I} A_i/F$ are related to the depth of the ultraproduct itself; to a lesser extent we deal with reduced products. The few results we state here are easy consequences of various known results. We consider two types of results: those most interesting when $|I|$ is big, i.e., $|I| \geq |A_i|$ for all $i \in I$, and those with small index set I.

Clearly we have:

Theorem 1.5.1. *Let A be the BA of finite and cofinite subsets of a cardinal κ, and let F be a countably complete ultrafilter on a set I Then ${}^I A/F$ has no chain of type $\omega + \omega$.*

Now recall that a filter F on a set I is called κ-regular if there is an $X \subseteq F$ with $|X| = \kappa$ such that every infinite subset of X has empty intersection. Thus κ-regular ultrafilters are in a sense the opposite of countably complete ultrafilters. (For motivation of this concept and the others of this section — uniform and good ultrafilters — see [4, 5].) The next result is in a sense opposite to Theorem 1.5.1; it is an obvious consequence of the following (easy) theorem of W. Hodges (see [4, Exercise 4.3.28]). If F is κ-regular on I, then ${}^I\langle \omega, < \rangle/F$ has a chain of type κ^+:

Theorem 1.5.2. *If F is κ-regular on I and A_i is an infinite BA for each $i \in I$, then $\prod_{i \in I} A_i/F$ has a chain of type κ^+.*

In contrast to Theorems 1.5.1 and 1.5.2, Laver has shown using a model of Woodin that it is consistent to have a uniform ultrafilter on ω_1 such that $|{}^{\omega_1}\omega/F| = \omega_1$.

Now let us consider the possibility of improving κ^+ in Theorem 1.5.2 to 2^κ.

Lemma 1.5.3. *If $\kappa \geq \aleph_0$, then there is a non-principal filter F on κ such that for every non-principal $G \supseteq F$, ${}^\kappa\langle \omega, < \rangle/G$ has a chain of type 2^κ.*

Proof. Let $\mathscr{S} \subseteq {}^\kappa\omega$ be a family of large ω-oscillation, with $|\mathscr{S}| = 2^\kappa$ (see [5, p. 77]). Let $\langle f_\alpha : \alpha < 2^\kappa \rangle$ enumerate $\mathscr{S}$ without repetitions. For $\alpha < \beta < 2^\kappa$ let $J_{\alpha\beta} = \{\gamma < \kappa : f_\alpha\gamma < f_\beta\gamma\}$. Clearly the intersection of any finite number of $J_{\alpha\beta}$'s is infinite, so $\{J_{\alpha\beta} : \alpha < \beta < 2^\kappa\} \cup \{\Gamma \subseteq \kappa : |\kappa \setminus \Gamma| < \omega\}$ generates a proper filter F. Clearly F is as desired.

Corollary 1.5.4. *Under the hypothesis of Lemma 1.5.3, if A_α is an infinite BA for each $\alpha<\kappa$, then $\prod_{\alpha<\kappa} A_\alpha/G$ has a chain of type 2^κ.*

Along the same lines we have

Theorem 1.5.5 (MA). *If F is any proper filter on ω containing all cofinite sets, and if A_i is an infinite BA for all $i\in\omega$, then $\prod_{i\in\omega} A_i/F$ has a chain of type 2^ω.*

Proof. The theorem is immediate from the following consequence of MA:

$$\text{if } \mathscr{F}\subseteq{}^\omega\omega \text{ and } |\mathscr{F}|<2^\omega, \text{ then there is a } g\in{}^\omega\omega \text{ such that for all } f\in\mathscr{F} \text{ there is an } m\in\omega \text{ with } fn<gn \text{ for all } n\geqslant m. \tag{$*$}$$

On the other hand we have the following result; its proof is analogous to the similar statement about non-embeddability of ω_2 into ${}^\omega\omega$ under eventual dominance; [15].

Theorem 1.5.6. *It is relatively consistent to have $2^{\aleph_0}>\aleph_1$ and* depth$({}^\omega A/F)=\aleph_1$, *where A is the BA of finite and cofinite subsets of ω and F is the filter of cofinite subsets of ω. In fact, one can start with a model M of CH and let G be a generic set for introducing κ Cohen reals, where κ is regular $>\aleph_1$; $M[G]$ is the desired model.*

The last result we mention about big index sets concerns ultraproducts of finite BA's.

Theorem 1.5.7. *Assume that A_i is a finite BA for all $i\in I$, that F is a κ^+-good ultrafilter on I, and that $\prod_{i\in I} A_i/F$ is infinite. Then $\prod_{i\in I} A_i/F$ has a chain of type κ^+.*

Proof. For each $i\in I$ let $\langle B_i,<\rangle$ be a maximal chain in A_i. Let $C=\prod_{i\in I} B_i/F$. Since $\prod_{i\in I} A_i/F$ is infinite, F is countably incomplete. By [4, 6.1.8], C is a κ^+-saturated. Now the ordered set $D=1+(\omega^*+\omega)\cdot\kappa^++\omega^*$ is an elementary substructure of C (see [4, Exercise 5.5.10]), so $\prod_{i\in I} A_i/F$ has a chain of type κ^+.

The following consistency result of Shelah shows that Theorem 1.5.7 is best possible when $I=\omega$.

Theorem 1.5.8. *Assume $V\models CH$, let κ be any uncountable cardinal in V, and let P be the partial order for adding κ Sacks reals side-by-side. Then in V^P there is a non-principal ultrafilter F on ω such that* depth $\prod_{i\in\omega} A_i/F=\omega_1$, *where A_i is the BA of all subsets of i, for each $i\in\omega$.*

Proof. Recall that

$$\begin{aligned}P=\{p\colon\ & p \text{ is a function, Dmn } p\in P_{<\omega_1}\kappa,\\ & \text{and } p_\alpha \text{ is a perfect subset of } [0,1]\\ & \text{for every } \alpha\in \text{Dmn } p\}.\end{aligned}$$

with $p \leqslant q$ iff $\mathrm{Dmn}\, p \supseteq \mathrm{Dmn}\, q$ and $p_\alpha \subseteq q_\alpha$ for all $\alpha \in \mathrm{Dmn}\, q$ (p is stronger). Recall that P satisfies the ω_2-c.c. and V^P preserves cardinals; $2^\omega \geqslant \kappa$ in V^P. By a result of Laver, let F' be a Ramsey ultrafilter in V which generates a Ramsey ultrafilter F in V^P. By Theorem 1.5.7, we only need to show that $\prod_{i \in \omega} A_i/F$ has no chain of type ω_2. So, arguing by contradiction, suppose $p \in P$ and

$$p \Vdash \langle \dot{f}_\alpha/F \colon \alpha < \omega_2 \rangle \text{ is strictly increasing,}$$

where $\dot{f}$ is a name for an α-sequence of elements of $\prod_{i \in \omega} A_i$. Now by applying the fusion lemma one can show

$$\begin{aligned} &\forall \alpha < \omega_2\, \exists\, q_\alpha \leqslant p\, \forall\, r \leqslant q_\alpha\, \forall\, i \in \omega\, \forall\, j \in i\, (r \Vdash j \in \dot{f}_\alpha i \text{ iff} \\ &r \restriction \mathrm{Dmn}\, q_\alpha \Vdash j \in \dot{f}_\alpha i) \text{ and } (\Vdash j \notin \dot{f}_\alpha i \text{ iff } r \restriction \mathrm{Dmn}\, q_\alpha \Vdash j \notin \dot{f}_\alpha i). \end{aligned} \tag{1}$$

Now we may assume that $\langle \mathrm{Dmn}\, q_\alpha \colon \alpha < \omega_2 \rangle$ forms a Δ-system with kernel Δ, and for all $\alpha, \beta < \omega_2$ we have $q_\alpha \restriction \Delta = q_\beta \restriction \Delta$ and there is a unique order-isomorphism $\Pi_{\alpha\beta}$ of $\mathrm{Dmn}\, q_\alpha \setminus \Delta$ onto $\mathrm{Dmn}\, q_\beta \setminus \Delta$ such that $\forall\, \gamma \in \mathrm{Dmn}\, q_\alpha \setminus \Delta$ $(q_\alpha \gamma = q_\beta \Pi_{\alpha\beta} \gamma)$. Thus $\Pi_{\alpha\beta} q_\alpha = q_\beta$. Extend $\Pi_{\alpha\beta}$ to a permutation of κ, still denoted by $\Pi_{\alpha\beta}$. Now we write $\alpha \sim \beta$ iff for all $i \in \omega$, all $j \in i$, and all $r \leqslant q_\alpha$, $r \Vdash i \in \dot{f}_\alpha j$ iff $\Pi_{\alpha\beta} r \Vdash i \in \dot{f}_\beta j$, and $r \Vdash i \notin \dot{f}_\alpha j$ iff $\Pi_{\alpha\beta} r \Vdash i \notin \dot{f}_\beta j$. Clearly there are only ω_1 equivalence classes, so fix equivalent $\alpha \neq \beta$. Then $r = q_\alpha \cup q_\beta \in P$. Now

$$r \Vdash \exists\, X \in F'\, \forall\, i \in X\, \forall\, j \in i\, (j \in \dot{f}_\alpha i \to j \in \dot{f}_\beta i).$$

We claim

$$\Pi_{\alpha\beta} r \Vdash \exists\, X \in F'\, \forall\, i \in X\, \forall\, j \in i\, (j \in \dot{f}_\beta i \to j \in \dot{f}_\alpha i); \tag{3}$$

since $\Pi_{\alpha\beta} r \leqslant q_\beta \leqslant p$, this will be a contradiction.

To prove (3), take any $s \leqslant \Pi_{\alpha\beta} r$. Then $\Pi_{\beta\alpha} s \leqslant r$, so by (2) there is a $t \leqslant \Pi_{\beta\alpha} s$ and an $X \in F'$ such that

$$t \Vdash \forall\, i \in X\, \forall\, j \in i\, (j \in \dot{f}_\alpha i \to j \in \dot{f}_\beta i). \tag{4}$$

Since $\Pi_{\alpha\beta} t \leqslant s$, it suffices now to show

$$\Pi_{\alpha\beta} t \Vdash \forall\, i \in X\, \forall\, j \in i\, (j \in \dot{f}_\beta i \to j \in \dot{f}_\alpha i). \tag{5}$$

So, let $u \leqslant \Pi_{\alpha\beta} t$, $i \in X$, $j \in i$, and assume that $u \Vdash j \in \dot{f}_\beta i$. Since $u \leqslant \Pi_{\alpha\beta} r \leqslant q_\beta$, from $\alpha \sim \beta$ we get $\Pi_{\alpha\beta} u \Vdash j \in \dot{f}_\alpha i$. Now $\Pi_{\alpha\beta} u \leqslant t$, so by (4), $\Pi_{\beta\alpha} u \Vdash j \in \dot{f}_\beta i$. Since $\Pi_{\beta\alpha} u \leqslant r \leqslant q_\beta$, from $\alpha \sim \beta$ we get $u \Vdash j \in \dot{f}_\alpha i$, as desired.

Now we briefly discuss ultraproducts with $|I|$ small, i.e., with $|I| \leqslant |A_i|$ for all $i \in I$. The following is an obvious consequence of the Erdös–Rado theorem.

Theorem 1.5.9. *Let $\kappa \geqslant \aleph_0$, let A be a partial ordering with no chains of type λ^+, where $\kappa \leqslant \lambda$, and let F be a filter on κ. Then ${}^\kappa A/F$ has no chain of type $(2^\lambda)^+$.*

Since our earlier results on big index sets show that it is possible for ${}^\kappa A/F$ to have a chain of type 2^κ even for A denumerable (see Corollary 1.5.4), the most plausible upper bound for the depth of ${}^{\kappa}A/F$ is $\max(2^{\kappa}, \text{depth } A)$, a number in general smaller than that provided by Theorem 1.5.8. For example, if F is an ultrafilter on ω, then ${}^{\omega}\mathcal{P}(2^{\omega})/F$ has no chain of type $(2^{\omega})^{+}$. In fact, suppose $\langle [f_\alpha]: \alpha < (2^{\omega})^{+}\rangle$ is strictly increasing in ${}^{\omega}P(2^{\omega})/F$. Define $g: (2^{\omega})^{+} \to {}^{\omega}(2^{\omega})$ by letting $g_\alpha i$ be $\min(f_{\alpha+1} i \setminus f_\alpha i)$ when this is non-empty, 0 otherwise. There is an infinite $\Gamma \subseteq (2^{\omega})^{+}$ on which g is constant. If $\alpha, \beta \in \Gamma$ with $\alpha + 1 < \beta$, a contradiction is easily reached. On the other hand, a result of Laver shows that it is consistent with CH that there is a BA A satisfying the $\aleph_2$-c.c. such that for a certain filter F on ω, ${}^{\omega}A/F$ has a chain of type ω_2, so that this plausible upper bound fails. (Laver's result is that it is consistent with CH to have an $\aleph_2$-Souslin tree T such that for each $\alpha < \omega_2$ there is an enumeration $\langle x_{\alpha n}: n \in \omega\rangle$ of some of the elements of T of level α such that if $\alpha < \beta < \omega_2$, then there is an $m \in \omega$ such that for all $n \geqslant m$, $x_{\alpha n} < x_{\beta n}$. By standard procedures, we may assume that every element of T has infinitely many immediate successors. For each $t \in T$, let $R_t = \{s \in T: t \leqslant s\}$, and let A be the subalgebra of PT generated by $\{R_t: t \in T\}$. Then $\{R_t: t \in T\}$ is dense in A, and so A satisfies ω_2-c.c. Let F be the filter of cofinite subsets of ω. For all $\alpha < \omega_2$ and $n \in \omega$ let $f_\alpha n = x_{\alpha n}$. Then $\langle [f_\alpha]: \alpha < \omega_2\rangle$ is a chain in ${}^{\omega}A/F$ of type ω_2.) Shelah has shown that it is consistent to have $\kappa < \lambda$ arbitrary, λ regular, GCH below λ, 2^{λ} arbitrarily large, and there is a BA B with depth $B \leqslant \lambda$ but for some filter F on κ, ${}^{\kappa}B/F$ has a chain of length 2^{λ}. Thus we have the following problem.

Problem 3. Is it consistent with ZFC that for all $\kappa, \lambda \geqslant \aleph_0$, if A is a BA with no chains of type λ and F is a filter on κ, then ${}^{\kappa}A/F$ has no chain of type $\max((2^{\kappa})^{+}, \lambda)$?

1.6. Homomorphisms

Without further restrictions homomorphisms have no effect on depth, as is shown by the following obvious results.

Theorem 1.6.1. *For each infinite κ there is a BA A of power κ such that if $f: A \twoheadrightarrow B$, then B has a chain of type $|B|$.*

Theorem 1.6.2. *If $\lambda \geqslant \aleph_0$, then there is a BA A of power λ such that if $\lambda \geqslant \kappa \geqslant \mu \geqslant \aleph_0$, then A has a homomorphic image of power κ with depth μ.*

Theorem 1.6.3. *If A is a BA and $\aleph_0 \leqslant \kappa \leqslant \lambda \geqslant |A|$, then there is a BA $B \twoheadrightarrow A$ with $|B| = \lambda$ and* depth $B = \kappa$.

Briefly, an algebra A as in Theorem 1.6.1 is the interval algebra on κ; for Theorem 1.6.2 take $A = B \times C$, where B is a free algebra on λ generators and C is the interval algebra on λ. Finally, for Theorem 1.6.3 take $B = D \times E$ with D a free algebra on λ generators and E the interval algebra on κ.

1.7. Subalgebras

The behaviour of depth under subalgebra formation is more involved. First we give a simple result about embedding in algebras with given depth.

Theorem 1.7.1. *Suppose* depth $A \leqslant \kappa$ *and* $\kappa, |A| \leqslant \lambda$. *Then there is a BA* $B \supseteq A$ *of power* λ *and depth* κ. *We may assume that* depth B *is attained. If* κ *is a limit cardinal and* depth $A < \kappa$ *with* $\mathrm{cf}\kappa > \aleph_0$, *or if* depth $A = \kappa$ *is not attained, we may assume that* depth B *is not attained.*

Proof. For the first part (depth B attained), let $B = A \times C \times D$ with C of power κ and depth κ attained, D of power λ and depth $\aleph_0$. (Recall that for any BA's A and B with $|A| > 1$ we have $A \rightarrowtail A \times B$.) For the second part take for C instead a BA of power κ and depth κ not attained. (See Theorem 1.2.4 and Corollaries 1.2.5–1.2.6).

The converse question is more difficult: given a BA A, what are the depths of its subalgebras? Some natural limitations follow from the following theorem of Rubin [21].

> Let $(J, <)$ be a linear order, and let B be a subset of the interval algebra on J closed under $\cap$, with $|B| = \kappa$ regular. Assume either that there is no system of κ pairwise disjoint intervals in J, or that in B there is no subset of power κ consisting of pairwise incomparable elements. Then there is in B a chain of cardinality κ. (*)

We combine this with a well-known result about linear orders (see [2]):

> Suppose $2 \leqslant \lambda \leqslant \kappa \geqslant \aleph_0$ and μ is minimal such that $\kappa < \lambda^\mu$ (thus $\lambda^\mu \leqslant 2^\kappa$). Then there is a linear order L of power λ^μ with a dense subset of power κ. (**)

Thus we have:

Theorem 1.7.2. *Suppose* $2 \leqslant \lambda \leqslant \kappa \leqslant \aleph_0$ *and* μ *is minimal such that* $\kappa < \lambda^\mu$. *Let* L *be as in* (**). *Let* A *be the interval algebra on* L. *Then if* B *is a subalgebra of* A, $\kappa < \nu \leqslant |B|$, *and* ν *is regular, then* B *has a chain of cardinality* ν.

(By using results about families of pairwise incomparable elements in BA's — see [7] — one can obtain similar conclusions.)

Corollary 1.7.3 (GCH). *For every* $\kappa \geqslant \aleph_0$ *there is a BA* A *of power* κ^+ *such that every subalgebra* $B \subseteq A$ *of power* κ^+ *has a chain of cardinality* κ^+, *and hence* depth $B \geqslant \kappa$ (*by Theorem* 3.1).

Note that if a BA A has a chain of type κ^+, then it has a subalgebra of power κ^+ with no uncountable chains, namely if $\langle a_\alpha : \alpha < \kappa^+ \rangle$ is strictly increasing in A, then $\mathrm{Sg}\{a_{\alpha+1} \cdot -a_\alpha : \alpha < \kappa^+\}$ is as indicated.

Our main results about subalgebras show that, assuming GCH, this corollary gives the only restriction on depth for A of successor power (see Corollary 1.7.11). We state our results independently of GCH, and first consider situations with depth attained. Thus we assume

$$\aleph_0 \leqslant \kappa \begin{Bmatrix} \leqslant \text{depth } A \leqslant \\ \leqslant \lambda < \end{Bmatrix} |A|$$

and try to find a subalgebra B of power λ and depth κ attained. If $\lambda = \kappa$ and A has a chain of type κ, there is no problem. Next we suppose $\kappa < \lambda$ and A has large families of pairwise disjoint elements.

Lemma 1.7.4. *Suppose A has a chain of type $\kappa \geqslant \aleph_0$ and has a set of λ pairwise disjoint elements, $\lambda > \kappa$. Then A has a subalgebra B of power λ and depth κ attained.*

Proof. Let $\langle a_\alpha : \alpha < \kappa \rangle$ be strictly increasing, $\langle b_\alpha : \alpha < \lambda \rangle$ pairwise disjoint, $C = \mathrm{Sg}\{a_\alpha : \alpha > \kappa\}$, $D = \mathrm{Sg}\{b_\alpha : \alpha < \lambda\}$, $B = \mathrm{Sg}(C \cup D)$. Suppose $\langle x_\alpha : \alpha < \kappa^+ \rangle$ is strictly increasing in B. Say

$$x_\alpha = \sum_{i < m_\alpha} c_i^\alpha \cdot d_i^\alpha ,$$

with $c_i^\alpha \in C$, $c_i^\alpha \cdot c_j^\alpha = 0$ for $i \neq j$, $d_i^\alpha \in D$. We may assume that m_α is constant $= m$ and c_i^α is constant $= c_i$; since $c_i \cdot c_j = 0$ for $i \neq j$, we may assume that $m = 1$. A contradiction is now easily reached.

Lemma 1.7.5. *Let $\kappa \geqslant \aleph_0$. If $\langle c_\alpha : \alpha < \kappa \rangle$ is strictly increasing in A and $B \subseteq A \restriction c_0$ is free, then $C = \mathrm{Sg}(B \cup \{c_\alpha : \alpha < \kappa\})$ has depth κ attained.*

Proof. Let $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow (A \restriction c_0) \times (A \restriction -c_0)$ be the natural isomorphism. Then f maps C into $B \times \mathrm{Sg}\{c_\alpha \cdot -c_0 : \alpha < \kappa\}$, so the desired conclusion follows from Section 1.2.

For the next few theorems we shall use the following result of Shelah [22]:

> If A satisfies the κ-c.c., κ is regular, γ is regular, $\forall \mu < \lambda$ $(\mu^{<\kappa} < \lambda)$, and $\lambda \leqslant |A|$, then A has a free subalgebra of power λ. $(***)$

Theorem 1.7.6. *Suppose $|A| = (2^\lambda)^+$, A has a chain of type κ, $\aleph_0 \leqslant \kappa \leqslant \mu \leqslant \lambda^+$. Then A has a subalgebra of power μ and depth κ attained.*

Proof. We may assume that $\kappa < \mu = \lambda^+$; and by Lemma 1.7.4 we may assume that A satisfies the λ^+-c.c. let $\langle c_\alpha : \alpha < \kappa \rangle$ be strictly increasing. Let D be a maximal system of pairwise disjoint elements such that for all $d \in D$, either $d \leq c_\alpha$ for some $\alpha < \kappa$ or $d \cdot c_\alpha = 0$ for all $\alpha < \kappa$. Thus $\sum D = 1$. Note that

$$\forall \mu < (2^\lambda)^+ \, (\mu^{<\lambda^+} < (2^\lambda)^+).$$

Hence by $(***)$ and Lemma 1.7.5 we may assume that $|A \upharpoonright d| \leq 2^\lambda$ for all $d \in D$. But since $\sum D = 1$, the natural homomorphism

$$A \to \prod_{d \in D} A \upharpoonright d$$

is one–one, while by the λ^+-c.c., $|D| \leq \lambda$. Hence $|A| \leq (2^\lambda)^\lambda = 2^\lambda$, contradiction.

Corollary 1.7.7. *Suppose $|A| = \lambda$, strong limit, $\aleph_0 \leq \kappa \leq \mu < \lambda$, A has a chain of type κ. Then A has a subalgebra B of power μ with depth κ attained.*

Lemma 1.7.8. *Suppose $|A| = \lambda^+$, λ strong limit. Assume either*
(i) *λ is regular; or*
(ii) *λ is singular and A satisfies the $\mathrm{cf}\lambda$-c.c.*
Suppose A has a chain of type κ, $\aleph_0 \leq \kappa \leq \mu \leq \lambda$. Then A has a subalgebra B of power μ with depth κ attained.

Proof. We may assume that $\kappa < \mu = \lambda$. Let $\langle c_\alpha : \alpha < \kappa \rangle$ by strictly increasing. By Lemma 1.7.4 we may assume that A satisfies the λ-c.c. By $(***)$ we have: if $x \in A$, $|A \upharpoonright x| = \lambda^+$, then $A \upharpoonright x$ has a free subalgebra of power λ^+. Thus if we form D as in the proof of Theorem 1.7.6, we may assume that $|A \upharpoonright d| \leq \lambda$ for all $d \in D$. So we get a contradiction as in that proof.

Lemma 1.7.9. *Suppose $|A| = \lambda^+$, λ is strong limit singular, and A is cardinality-homogeneous. Then A has a subalgebra B of cardinality λ and depth $\aleph_0$.*

Proof. We may assume, by Lemma 1.7.4, that A satisfies the λ-c.c. By Erdös and Tarski [8] there is a $\mu < \lambda$ such that A satisfies the μ-c.c. Hence by $(***)$ we get

$$\text{If } \mu < \nu < \lambda \text{ and } 0 \neq x \in A, \text{ then there is a free subalgebra } C \subseteq A \upharpoonright x \text{ with } |C| \geq \nu. \tag{1}$$

Now by Lemma 1.7.8(ii) we may assume that there is a system $\langle c_\alpha : \alpha < \mathrm{cf}\lambda \rangle$ of pairwise disjoint elements of A. By (1) we can choose for each $\alpha < \mathrm{cf}\lambda$ a free subalgebra $C_\alpha \subseteq A \upharpoonright c_\alpha$ such that $\sup_{\alpha<\lambda} |C_\alpha| = \lambda$. We now let

$$B = \mathrm{Sg}\Big(\bigcup_{\alpha < \mathrm{cf}\lambda} C_\alpha \Big).$$

To see that B is as desired, note that each element of B has the form

$$\sum_{\alpha \in \Gamma} d_\alpha \quad \text{or} \quad \prod_{\alpha \in \Gamma} -d_\alpha,$$

where Γ is a finite subset of cfλ and $d \in \prod_{\alpha\in\Gamma} C_\alpha$. Hence if we let $f: A \twoheadrightarrow \prod_{\alpha<\mathrm{cf}\lambda} (A \upharpoonright c_\alpha)$ be the natural homomorphism it follows that $f \upharpoonright B$ is one–one into $\prod^{\mathrm{w}}_{\alpha<\mathrm{cf}\lambda} C_\alpha$. Hence B has depth $\aleph_0$ by Section 1.2.

Theorem 1.7.10. *Suppose $|A| = \lambda^+$, λ strong limit, A has a chain of type κ, $\aleph_0 \leq \kappa \leq \mu \leq \lambda$. Then A has a subalgebra of power μ and depth κ attained.*

Proof. Again we may assume $\kappa < \mu = \lambda$. By Lemma 1.7.8 we may assume that λ is singular and that A does not satisfy the cfλ-c.c. Also by Lemma 1.7.4 we may assume that A satisfies the λ-c.c. Let $\langle a_\alpha : \alpha < \kappa\rangle$ be a chain of type κ. Let D be a collection of pairwise disjoint elements satisfying the following conditions:

D is maximal such that for all $d \in D$ either (a) $\exists\, \alpha < \kappa$ $(d \leq a_{\alpha+1} \cdot -a_\alpha)$ or (b) $\exists$ limit $\alpha < \kappa$ $(d \leq a_\alpha$ and $\forall\, \beta < \alpha$ $(d \cdot a_\beta = 0))$ or (c) $\forall\, \alpha < \kappa$ $(a_\alpha \cdot d) = 0$; (1)

$$|D| \geq \mathrm{cf}\lambda; \tag{2}$$

$$\forall\, d \in D\, (A \upharpoonright d \text{ is cardinality-homogeneous}). \tag{3}$$

We now consider two cases.

Case 1. $\exists\, d \in D$ $(|A \upharpoonright d| = \lambda^+)$. By Lemma 1.7.9, $A \upharpoonright d$ has a subalgebra B of power λ and depth $\aleph_0$. Then the desired subalgebra of A is

$$\mathrm{Sg}(B \cup \{a_\beta \cdot -a_{\alpha+1} : \alpha+1 < \beta < \kappa\}) \quad \text{if} \quad d \leq a_{\alpha+1} \cdot -a_\alpha$$

in case (1)(a) holds, and a similar construction works for (1)(b) and (1)(c).

Case 2. $\forall\, d \in D$ $(|A \upharpoonright d| \leq \lambda)$. Then

There is a system $\langle d_\alpha : \alpha < \mathrm{cf}\lambda\rangle$ of distinct elements of D and a system $\langle C_\alpha : \alpha < \mathrm{cf}\lambda\rangle$ such that each C_α is a subalgebra of $A \upharpoonright d_\alpha$, $\sup_{\alpha<\mathrm{cf}\lambda} |C_\alpha| = \lambda$, each C_α of depth $\aleph_0$. (4)

For this is true by Corollary 1.7.7 if $|\{d \in D : |A \upharpoonright d| = \lambda\}| \geq \mathrm{cf}\lambda$. Assume that $|\{d \in D : |A \upharpoonright d| = \lambda\}| < \mathrm{cf}\lambda$. Then

$$\sup\{|A \upharpoonright d| : |A \upharpoonright d| < \lambda\} = \lambda. \tag{5}$$

For, otherwise by the maximality of D

$$|A| \leq \lambda^{|\{d\in D : |A\upharpoonright d| = \lambda\}|} \cdot \prod \{|A \upharpoonright d| : |A \upharpoonright d| < \lambda\} \leq \lambda,$$

a contradiction. So (5) holds, and we easily obtain (4) again by using Theorem 1.7.6. Now we let

$$B = \mathrm{Sg}\Big(\{a_\alpha : \alpha < \kappa\} \cup \bigcup_{\alpha<\mathrm{cf}\lambda} C_\alpha\Big).$$

We need to show that B has no chain of type κ^+. Suppose that $\langle x_\alpha : \alpha < \kappa^+\rangle$ is such a chain. By the procedure in the proof of Lemma 1.7.4 we may assume that for each $\beta < \kappa^+$ we have $x_\beta = y \cdot z_\beta$, where $y \in \mathrm{Sg}\{a_\alpha : \alpha < \kappa\}$ and $z_\beta \in \mathrm{Sg} \bigcup_{\alpha < \mathrm{cf}\lambda} C_\alpha$. Now for every $\alpha < \kappa$ and $d \in D$ we have $a_\alpha \cdot d = 0$ or $d \leq a_\alpha$, so by the idea of the proof of Lemma 1.7.9 we can assume that $z_\beta \in \mathrm{Sg} \bigcup \{C_\alpha : \alpha < \mathrm{cf}\lambda, d_\alpha \leq \lambda\}$. Under the natural homomorphism $f : A \to \prod \{A \restriction d_\alpha : d_\alpha \leq y\}$, the subalgebra $\mathrm{Sg}^{A \restriction y} \bigcup \{C_\alpha : \alpha < \mathrm{cf}\lambda, d_\alpha \leq \lambda\}$ goes isomorphically into $\prod^w \{C_\alpha : d_\alpha \leq y\}$, and $f(y \cdot z_\alpha) < f(y \cdot z_\beta)$ for $\alpha < \beta < \kappa^+$. This contradicts Corollary 1.2.5.

Corollary 1.7.11 (GCH). Suppose $|A| = \lambda^+$, *A has a chain of type* κ, $\aleph_0 \leq \kappa \leq \mu \leq \lambda$. *Then A has a subalgebra of power* μ *and depth* κ *is attained.*

Now we consider algebras A of limit cardinality. By Corollary 1.7.7 we still have to consider: subalgebras B with $|B| = |A|$, where we do not have any negative result like Corollary 1.7.3.

First we consider $|A|$ strongly inaccessible. here two results give a fairly complete picture. We u: the following result of Shelah [22]:

> If κ is weakly compact, $\kappa \geq |A|$, and A satisfies the κ-c.c., then A has a free subalgebra of power κ. $(****)$

Hence by the method of proof of Theorem 1.7.6 we obtain:

Theorem 1.7.12. *Suppose* $|A| = \lambda$, *weakly compact*, $\aleph_0 \leq \kappa \leq \mu \leq \lambda$, *and A has a chain of type* κ. *Then A has a subalgebra B of power* μ *with depth* κ *attained.*

On the other hand, assuming $V = L$, if κ is strongly inaccessible but not weakly compact then there is a κ-Souslin tree, and hence a linear order of power κ without a κ-powered system of pairwise disjoint intervals; hence by $(*)$ we get:

Theorem 1.7.13 ($V = L$). *If* κ *is strongly inaccessible but not weakly compact, then there is a BA A of power* κ *such that every subalgebra of A of power* κ *has a chain of power* κ.

Now we turn to the case λ singular; Theorem 1.7.15, here, is due to Shelah. We need the following lemma, which we state without proof.

Lemma 1.7.14. *Suppose* $|A| = \lambda$, λ *a singular strong limit cardinal, and A is cardinality-homogeneous. Then A has a subalgebra B of power* λ *and depth* $\aleph_0$.

Theorem 1.7.15. *Suppose* $|A| = \lambda$, λ *a singular strong limit cardinal, A has a chain of type* κ, *and* $\aleph_0 \leq \kappa \leq \mu \leq \lambda$. *Then A has a subalgebra of power* μ *and depth* κ *attained.*

Proof. Without loss of generality we may assume that $\kappa < \mu = \lambda$, and A satisfies the θ-c.c. for some $\theta < \lambda$. Let $\langle a_\alpha : \alpha < \kappa \rangle$ be a chain of type κ, and let D be as in the proof of Lemma 1.7.10 except for (2); the further details are similar to that proof, using Lemma 1.7.14 rather than Lemma 1.7.9.

Our main results above — Theorems 1.7.6, 1.7.12, 1.7.15, Corollaries 1.7.7, 1.7.11 and Lemma 1.7.10 — also extend to the case depth κ not attained, when κ is a limit cardinal and $\aleph_0 < \mathrm{cf}\,\kappa$.

Various assumptions in our results are necessary. For example, assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_3$. In the notation of Comfort and Negrepontis [5] let $L = \Lambda(\aleph_1) \setminus \mathscr{L}(\aleph_1)$, and let A be the interval algebra on L. Then $|A| = \aleph_3$, A satisfies ω_2-c.c., and every subalgebra of A of power $\aleph_2$ has depth $> \aleph_0$. Thus GCH is needed in Corollary 1.7.11. As another example, assume $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, and $2^{\aleph_2} = \aleph_{\omega_3}$. Let $L = \Lambda(\aleph_2) \setminus \mathscr{L}(\aleph_2)$, and let A be the interval algebra on L. Then $|A| = \aleph_{\omega_3}$; depth $A = \aleph_2$, A satisfies the ω_3-c.c., and A has no subalgebras of power $\aleph_3$ and depth $\aleph_0$; this blocks an obvious generalization of Theorem 1.7.15.

1.8. Unions

The behaviour of depth under unions is clear; it is given in the following theorem.

Theorem 1.8.1. *For infinite cardinals κ and λ the following two conditions are equivalent.*

(i) $\mathrm{cf}\,\kappa = \mathrm{cf}\,\lambda$;

(ii) *there is a strictly increasing sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of BA's with union B such that for all $\alpha < \kappa$, A_α has no chain of type λ, but B does.*

Proof. (i)$\Rightarrow$(ii) is clear. Assume (ii). Then it is easy to check that $\mathrm{cf}\,\lambda \leq \mathrm{cf}\,\kappa \leq \lambda$. If λ is regular, (i) follows. Hence assume that λ is singular; say $\langle \nu_\alpha : \alpha < \mathrm{cf}\,\lambda \rangle$ is a strictly increasing sequence of regular cardinals with supremum λ, with $\mathrm{cf}\,\kappa < \nu_0$. Let $\langle \mu_\alpha : \alpha < \mathrm{cf}\,\kappa \rangle$ be a strictly increasing sequence of ordinals with supremum κ. Then $\forall\, \alpha < \mathrm{cf}\,\lambda\ \exists\, \beta < \mathrm{cf}\,\kappa\ (|\{x_\gamma : \gamma < \nu_\alpha\} \cap A_{\mu_\beta}| = \nu_\alpha)$, where we assume that $\langle x_\gamma : \gamma < \lambda \rangle$ is strictly increasing in B. Then $\mathrm{cf}\,\lambda < \mathrm{cf}\,\kappa$ gives a contradiction.

1.9. Automorphisms

The following theorem can be proved by modifying Monk and Rassbach [18].

Theorem 1.9.1. *For each κ, λ with $\aleph_0 \leq \lambda \leq \kappa \leq \aleph_0$ there are 2^κ isomorphism types of rigid BA's of power κ and depth λ; for κ regular each algebra is cardinality-homogeneous and depth is attained.*

The modifications are as follows. For κ regular, instead of B_Γ in [18] take $A * B_\Gamma$, A the interval algebra on λ. The singular case is based on the regular case as in [18].

Corollary 1.9.2. *Suppose $\aleph_0 \leq \kappa \leq \mu$, $\aleph_0 \leq \lambda \leq \mu$, μ regular. Then there is a BA A of power μ with depth λ attained and $|\mathrm{Aut}\, A| = \kappa$.*

Proof. Let B be rigid of power μ, cardinality-homogeneous, and depth λ attained, and let C be rigid of power κ and depth $\aleph_0$. If $\kappa = \mu$, let $A = B \times B$. If $\kappa < \mu$, let $A = B \times C \times C$. (See [17]).

1.10. Classes of BA's

It is natural to ask about depth for various special kinds of BA's. If A is complete, then depth A = cell A, with one attained iff the other is; by [8], non-attainment is only possible when cell A is weakly inaccessible.

Now consider the class of κ-complete BA's, where κ is infinite and regular. If A is κ-complete but not complete, then cell $A \geq \kappa$ and hence depth $A \geq \kappa$; if cell $A = \kappa$ then it is attained, and so A has a chain of type κ. Now we shall make use of the following modification of Lemma 0:

Lemma 0′. *If X is a chain in $\prod_{i\in I} A_i$ of type κ, κ an infinite regular cardinal, and if $|I| < \kappa$, then there is a subset Y of X of power κ and an $i \in I$ such that the ith projection is one–one on Y.*

Using Lemma 0′ we can generalize Theorem 1.1.1 as follows.

Theorem 1.10.1. *Suppose $\kappa \geq \aleph_0$, κ regular, A is κ-complete, but not complete,* depth $A = \lambda$, *and* $\mathrm{cf}\lambda \leq \kappa$. *Then the depth of A is attained.*

Proof. By the remarks preceding Lemma 0′, we may assume that $\lambda > \kappa$, so that λ is singular. Now let $\langle \mu_\alpha : \alpha < \mathrm{cf}\lambda \rangle$ be a strictly increasing sequence of regular cardinals with sup λ. Set

$$J = \{a : \text{there is an } \alpha < \mathrm{cf}\lambda \text{ such that } \mu_\alpha \text{ is not embeddable in } a\}.$$

By Lemmas 0 and 0′ we have

$$J \text{ is a } (\mathrm{cf}\lambda)\text{-complete ideal.} \tag{1}$$

Case 1. For all $\alpha < \mathrm{cf}\lambda$, μ_α is embeddable in J. Then

$$\text{For all } x \in J \text{ and all } \alpha < \mathrm{cf}\lambda,\ \mu_\alpha \text{ is embeddable in } J \cap (A \restriction -x). \tag{2}$$

In fact, let $x \in J$ and $\alpha < \mathrm{cf}\lambda$. We may assume that μ_α is not embeddable in x. If $\langle C_\beta : \beta < \mu_\alpha \rangle$ is strictly increasing in J, then by Lemma 0, $\langle C_\beta \cdot -x : \beta < \mu_\alpha \rangle$ has a subsequence which is strictly increasing, as desired in (2).

Now we define $\langle x_\alpha : \alpha < \mathrm{cf}\lambda \rangle$ by induction. Suppose $x_\beta \in J$ has been defined for all $\beta < \alpha$, where $\alpha < \mathrm{cf}\lambda$. Thus $\sum_{\beta<\alpha} x_\beta \in J$ by (1). By (2), choose $x_\alpha \in J \cap (A \restriction \prod_{\beta<\alpha} -x_\beta)$ so that μ_α is embeddable in x_α. Now we can finish as at the end of the proof of Theorem 1.1.1.

Case 2. There is an $\alpha < \mathrm{cf}\lambda$ such that μ_α is not embeddable in J. Choose $\beta < \alpha$ such that $\mathrm{cf}\lambda < \mu_\beta$. Let $\langle a_\gamma : \gamma < \mu_{\beta+1} \rangle$ be strictly increasing in A. For each $\delta < \mathrm{cf}\lambda$ let

$$\tau\delta = \mu_\beta + \mu_\alpha \cdot \delta,$$

and set $b_\delta = a_{\tau(\delta+1)} \cdot -a_{\tau\delta}$. Then the b_δs are pairwise disjoint, and μ_α is embeddable in each b_δ, consequently $b_\delta \notin J$. So the conclusion follows again.

To show that Theorem 1.10.1 is 'best possible', we need the following generalization of Theorem 1.2.4. For κ infinite let

$$^{\kappa}\prod_{i \in I}^{w} A_i = \left\{ f \in \prod_{i \in I} A_i : \{i : f_i \neq 0\} \text{ or } \{i : f_i \neq 1\} \text{ has power } < \kappa \right\}.$$

Theorem 1.10.2. *Assume κ is infinite and regular, and $\sup_{i \in I} \mathrm{depth}\, A_i = \lambda$, where $\kappa < \mathrm{cf}\lambda$. Then the following conditions are equivalent:*

(i) *$\mathrm{depth}\, ^{\kappa}\prod_{i \in I}^{w} A_i$ is not attained;*

(ii) *$\forall i \in I$ (A_i has no chain of type λ).*

Proof. Obviously (i)$\Rightarrow$(ii). Now assume (ii), but suppose that $\langle x_\alpha : \alpha < \lambda \rangle$ is a chain of type λ in $^{\kappa}\prod_{i \in I}^{w} A_i$. For each $y \in {}^{\kappa}\prod_{i \in I}^{w} A_i$ let $Sy = \{i \in I : y_i \neq 0\}$. Thus $|Sy| < \kappa$ or $|I \setminus Sy| < \kappa$. Clearly $\alpha < \beta < \lambda$ implies $Sx_\alpha \subseteq Sx_\beta$.

Case 1. $|Sx_\alpha| < \kappa$ for all $\alpha < \lambda$. Since $\kappa < \mathrm{cf}\lambda$ it follows that there is an $\alpha < \lambda$ such that $Sx_\alpha = Sx_\beta$ whenever $\alpha \leqslant \beta < \lambda$. So we may assume that $Sx_\alpha = T$ for all $\alpha < \lambda$. Thus $\langle x_\alpha \restriction T : \alpha < \lambda \rangle$ is a chain of type λ in $\prod_{i \in T} A_i$, contradicting Theorem 1.2.2 or 1.2.3.

Case 2. For some $\alpha < \lambda$ we have $|Sx_\alpha| = \kappa$. So we may assume that $|\{i \in I : x_\alpha i \neq 1\}| < \kappa$ for all $\alpha < \lambda$, and the argument goes just as in Case 1.

Using Theorem 1.10.2, if λ is singular and $\aleph_0 \leqslant \kappa < \mathrm{cf}\lambda$ with κ regular it is easy to construct a κ-complete BA B of depth λ not attained. Namely, let $\langle \mu_\alpha : \alpha < \mathrm{cf}\lambda \rangle$ be a strictly increasing sequence of infinite cardinals with supremum λ, and take $^{\kappa}\prod_{\alpha < \mathrm{cf}\lambda}^{w} \mathcal{P}\mu_\alpha$.

Now consider the algebras $\mathcal{P}\kappa/\mathcal{P}_{<\lambda}\kappa$. The most studied of these algebras is $\mathcal{P}\omega/\mathcal{P}_{<\omega}\omega = A$. It is well known, and easy to see, that A has depth $\geqslant \aleph_1$. Hechler [1] has shown under MA that depth $A = 2^{\aleph_0}$. On the other hand, adding Cohen reals shows that it is consistent to have depth $A = \aleph_1$ and $2^{\aleph_0} > \aleph_1$; see [15]. In the case of the algebras $\mathcal{P}\kappa/\mathcal{P}_{<\omega}\kappa$ we can restrict attention to those with cellularity $\geqslant \kappa^+$, i.e., by [5, 12.2], to those with $\kappa^\omega > \kappa$. Thus the following theorem due to the referee settles this case.

Theorem 1.10.3. *For any infinite cardinal κ we have $\mathrm{depth}(P\kappa/P_{<\omega}\kappa) = \max(\kappa, \mathrm{depth}(P\omega/P_{<\omega}\omega))$. Hence for $\kappa \geqslant 2^\omega$ we get $\mathrm{depth}(P\kappa/P_{<\omega}\kappa) = \kappa$.*

Proof. Suppose the theorem is false, and let κ be minimum for which there is a counterexample. Clearly $\kappa > \omega$. Let $\nu = \max(\text{depth } P\omega/P_{<\omega}\omega, \kappa)$, and let $\langle A_\alpha : \alpha < \nu^+\rangle$ be a strictly decreasing (mod finite) sequence of subsets of κ. Then

$$\forall \xi < \kappa\ \exists \alpha_\xi < \nu^+\ \forall \beta, \gamma \geq \alpha_\xi\ (A_\beta \cap \xi = A_\gamma \cap \xi \text{ (mod finite)}). \tag{1}$$

In fact, if $\xi < \kappa$, then $\langle A_\beta \cap \xi : \beta < \nu^+\rangle$ is a decreasing (mod finite) sequence of subsets of ξ, so by the choice of κ such an α_ξ exists.

Letting $\beta = \sup\{\alpha_\xi : \xi < \kappa\}$, we see that $A_\gamma \cap \xi = A_\beta \cap \xi$ (mod finite) for all $\gamma \geq \beta$ and all ξ. We may assume that $\beta = 0$.

For each $\gamma > 0$ let $B_\gamma = A_0 \setminus A_\gamma$. Thus $\langle B_\gamma : 0 < \gamma < \nu^+\rangle$ is strictly increasing (mod finite). Moreover, for each $0 < \gamma < \nu^+$,

$$B_\gamma \text{ has order type } \omega \text{ and is cofinal in } \kappa \text{ (thus cf } \kappa = \omega). \tag{2}$$

In fact, B_γ is infinite since $\langle A_\delta : \delta < \nu^+\rangle$ is *strictly* decreasing. If $\delta_0 < \cdots < \delta_\omega$ are in B_γ, then $A_0 \cap (\delta_\omega + 1) = A_\gamma \cap (\delta_\omega + 1)$ (mod finite) is contradicted. If $B_\gamma \subseteq \xi$, with $\xi < \kappa$, then $A_0 \cap \xi = A_\gamma \cap \xi$ (mod finite) is contradicted. So (2) holds.

Let $\langle \lambda_n : n \in \omega\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Now for each $\alpha < \nu^+$ let $C_\alpha = \{(m, n) \in \omega \times \omega : |B_\alpha \cap \kappa_m| \leq n\}$. Then the following statement contradicts $\nu^+ > \text{depth}(P\omega/P_{<\omega}\omega)$:

$$\langle C_\alpha : \alpha < \nu^+\rangle \text{ is a strictly decreasing (mod finite).} \tag{3}$$

To prove (3), suppose $\alpha < \beta < \nu^+$. Choose m' so that $B_\alpha \setminus B_\beta \subseteq \kappa_{m'}$. Let $s = |B_\alpha \cap \kappa_{m'}|$, and choose $m \geq m'$ so that $|(B_\beta \setminus B_\alpha) \cap \kappa_m| > s$. Then for any $p \geq m$ we have

$$\begin{aligned} |B_\beta \cap \kappa_p| &= |B_\beta \cap B_\alpha \cap \kappa_p| + |(B_\beta \setminus B_\alpha) \cap \kappa_p| \\ &> |B_\beta \cap B_\alpha \cap \kappa_p \setminus \kappa_{m'}| + s \\ &= |B_\alpha \cap \kappa_p \setminus \kappa_{m'}| + s = |B_\alpha \cap \kappa_p|. \end{aligned}$$

Thus $C_\beta \setminus (m \times 1) \subseteq C_\alpha$, while $(p, |B_\alpha \cap \kappa_p|) \in C_\alpha \setminus C_\beta$ for each $p \geq m$, as desired.

It is also well known and easy to see that $\text{depth}(P\kappa/P_{<\kappa}\kappa) > \kappa$ for all $\kappa \geq \aleph_0$. If we take a ground model M satisfying GCH and any cardinal $\kappa \geq \omega$ in M, then using the usual conditions to make $2^\kappa = \kappa^{++}$ we get a model in which $\text{depth}(P_\kappa/P_{<\kappa}\kappa) = \kappa^+$.

Concerning the depth of interval algebras we make the following remarks.

(1) If κ is regular but not weakly compact, then there is a linear order of power κ such that the interval algebra on L has no chain of type κ. For, we just take L so that κ and κ^* are not embeddable in L. If $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing in the interval algebra on L, by κ regular $>\omega$ we may write

$$a_\alpha = [b_{\alpha 1}, b_{\alpha 2}) \cup \cdots \cup [b_{\alpha n}, b_{\alpha, n+1})$$

for each $\alpha<\kappa$. Then $\alpha<\beta$ implies $b_{\beta 1}\leqslant b_{\alpha 1}$, so we may assume all $b_{\alpha 1}$ are equal; then we continue with $b_{\alpha 2},\ldots$, getting a contradiction.

(2) If κ is weakly compact, then every linear order L of power κ embeds κ or κ^*, and hence the interval algebra on L has a chain of type κ.

(3) If κ is singular, then there is a linear order L of power κ such that L does not embed κ or κ^* but the interval algebra on L has a chain of type κ. To prove (3), suppose that $\langle\lambda_\alpha:\alpha<\mathrm{cf}\kappa\rangle$ is a strictly increasing continuous sequence of cardinals with sup κ, and with $\lambda_0=0$. We define $\gamma<'\delta$ iff $\gamma,\delta<\kappa$ and there is an $\alpha<\mathrm{cf}\kappa$ such that $\gamma<\lambda_{\alpha+1}\leqslant\delta$ or else $\lambda_\alpha\leqslant\delta<\gamma<\lambda_{\alpha+1}$. Then under $<'$, κ does not embed κ or κ^* (see [5, p. 166]). For each $\gamma<\kappa$ let $a_\gamma=[0,\lambda_\alpha)\cup[\gamma,\lambda_{\alpha+1})$, where $\lambda_\alpha\leqslant\gamma<\lambda_{\alpha+1}$. Then $\langle a_\gamma:\gamma<\kappa\rangle$ is strictly increasing, as desired.

(4) It is consistent with ZFC for there to exist a singular cardinal κ and a linear order L of power κ such that for some $\lambda<\kappa$, neither λ nor λ^* are embeddable in L; for such an L, λ (if regular) is not embeddable in the interval algebra on L. For, we can take a model in which $\aleph_\omega<2^\omega$ and choose $L\subseteq\mathbb{R}$ of power $\aleph_\omega$, taking $\lambda=\aleph_1$.

(5) (Remark due to R. Laver.) If κ is singular, $\mathrm{cf}\kappa$ is weakly compact, L is a linear ordering of power κ, and for every $\lambda<\kappa$ either λ or λ^* is embeddable in L, then κ is embeddable in the interval algebra on L. (ω is counted as weakly compact.) For, we may assume that each $\lambda<\kappa$ is embeddable in L (otherwise take L^*). Let $\langle\lambda_\alpha:\alpha<\mathrm{cf}\kappa\rangle$ be a strictly increasing sequence of regular cardinals with sup κ, all $>\mathrm{cf}\kappa$. For each $\alpha<\mathrm{cf}\kappa$ let $\langle a_{\alpha\xi}:\xi<\lambda_\alpha\rangle$ be strictly increasing in L. Set

$$P=\{\{\alpha,\beta\}:\alpha<\beta<\mathrm{cf}\kappa\quad\text{and}\quad\exists\,\xi<\lambda_\beta\;\forall\,\eta<\lambda_\alpha\;(a_{\alpha\eta}<a_{\beta\xi})\}.$$

By the weak compactness of $\mathrm{cf}k$ we then have: there is a $\Gamma\in\mathscr{P}_{\mathrm{cf}\kappa}\mathrm{cf}\kappa$ such that $\mathscr{P}_2\Gamma\subseteq P$ or $\mathscr{P}_2\Gamma\subseteq\mathscr{P}_2\mathrm{cf}\kappa\setminus P$. We can assume $\Gamma=\mathrm{cf}\kappa$.

Case 1. $\mathscr{P}_2\Gamma\subseteq P$. Fix $\beta<\mathrm{cf}\kappa$. For all $\alpha<\beta$ choose $\xi_\alpha<\lambda_\beta$ such that $\forall\,\eta<\lambda_\alpha$ $(a_{\alpha\eta}<a_{\beta\xi_\alpha})$. Let $\nu_\beta=\bigcup_{\alpha<\beta}\xi_\alpha$. Since λ_β is regular $>\mathrm{cf}\kappa$, we have $\nu_\beta<\lambda_\beta$. Thus if $\alpha<\beta$ and $\eta<\lambda_\alpha$, then $a_{\alpha\eta}<a_{\beta\nu_\beta}$. Hence

$$\langle a_{\beta\eta}:\beta<\mathrm{cf}\kappa,\,\nu_\beta\leqslant\eta\leqslant\lambda_\beta\rangle$$

is a chain of type κ in L.

Case 2. $\mathscr{P}_2\Gamma\subseteq\mathscr{P}_2\mathrm{cf}\kappa\setminus\mathscr{P}$. Fix $\alpha<\beta<\mathrm{cf}\kappa$. For each $\xi<\lambda_\beta$ choose $\eta_\xi<\lambda_\alpha$ so that $a_{\alpha\eta_\xi}>a_{\beta\xi}$. Since $\lambda_\alpha<\lambda_\beta$ and λ_β is regular, there is a $\nu_\beta<\lambda_\alpha$ such that $\eta_\xi=\nu_\beta$ for λ_β many $\xi<\lambda_\beta$. Now with α still fixed, let $\rho_\alpha=\bigcup_{\alpha<\beta<\mathrm{cf}\kappa}\nu_\beta$. Since λ_α is regular $>\mathrm{cf}\kappa$, we have $\rho_\alpha<\lambda_\alpha$. Thus if $\alpha<\beta$ and $\rho_\alpha\leqslant\eta<\lambda_\alpha$, $\xi<\lambda_\beta$, then $a_{\alpha\eta}>a_{\beta\xi}$. Hence

$$\langle[a_{\alpha+1,\rho_{\alpha+1}},a_{\alpha+1,\eta})\cup[a_{\alpha\rho_\alpha}):\alpha<\mathrm{cf}\kappa,\,\rho_{\alpha+1}\leqslant\eta<\lambda_{\alpha+1}\rangle$$

is strictly increasing in the interval algebra on L.

(6) If κ is singular and $\mathrm{cf}\kappa$ is not weakly compact, then there is a linear order L of power κ such that for every $\lambda<\kappa$, L embeds λ, but the interval algebra on L

does not embed κ. For, let L' be a linear order of power $\mathrm{cf}\kappa$ which does not embed $\mathrm{cf}\kappa$ or $(\mathrm{cf}\kappa)^*$. Let $\langle\lambda_\alpha \cdot \alpha < \mathrm{cf}\kappa\rangle$ be a strictly increasing sequence of cardinals with $\sup\kappa$. Let f be a one–one function mapping L' onto $\mathrm{cf}\kappa$. Set $L = \{(a, \alpha): a \in L', \alpha < \lambda_{fa}\}$, and define

$$(a, \alpha) < (b, \beta) \quad \text{iff} \quad a < b, \quad \text{or} \quad a = b \quad \text{and} \quad \alpha < \beta.$$

Clearly if $\langle x_\xi : \xi < \kappa\rangle$ is increasing or decreasing, then $\exists\, \xi < \kappa\ \forall\, \eta \geqslant \xi\ (x_\xi = x_\eta)$. Hence the interval algebra on L does not embed κ (see the proof for (1) above).

2. Ordinal depth

For any BA A, the *ordinal depth* of A is the supremum of the order types of subsets of A well-ordered under the Boolean ordering. Our main result, Theorem 2.6, specifies the form of ordinal depth.

The following lemma is well known.

Lemma 2.1. *If $\Gamma \subseteq \omega^\alpha$, then the order type of Γ or of $\omega^\alpha \setminus \Gamma$ is ω^α.*

Proof. We proceed by induction on α. For $\alpha \leqslant 1$ the lemma is clear. Assume it for α. If for infinitely many n the order type of $\Gamma \cap [\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1))$ is ω^α, then the order type of Γ is $\omega^{\alpha+1}$ (assuming $\Gamma \subseteq \omega^{\alpha+1}$). Otherwise the order type of $\omega^{\alpha+1} \setminus \Gamma$ is $\omega^{\alpha+1}$. Assume inductively α limit, $\Gamma \subseteq \omega^\alpha$. If the order type of $\Gamma \cap \omega^\beta$ is ω^β for cofinally many $\beta < \alpha$, then Γ has order type ω^α; otherwise $\omega^\alpha \setminus \Gamma$ does. This completes the proof.

Hence we obtain:

Lemma 2.2. *If $\Gamma \subseteq \omega^\alpha \cdot 2n$, $n \in \omega$, then the order type of Γ or of $(\omega^\alpha \cdot 2n) \setminus \Gamma$ is $\geqslant \omega^\alpha \cdot n$.*

Lemma 2.3. *Suppose that A has ordinal depth $\geqslant \omega^\alpha$, where α is a successor ordinal. Then A has a chain of type ω^α.*

Proof. Say $\alpha = \beta + 1$. We call $a \in A$ an ∞-*element* if $\omega^\beta \cdot i$ is embeddable in a for all $i \in \omega$. The procedure of the proof of Theorem 1.1.1 can be applied if we show

if a is an ∞-element and $a = b + c$ with $b \cdot c = 0$, then b is an ∞-element or c is an ∞-element. (1)

To prove (1), it is enough to show that for all $n \in \omega$, $\omega^\beta \cdot n$ is embeddable in b or in c. Let $\langle d_\gamma : \gamma < \omega^\beta \cdot 2n\rangle$ be an embedding into a. Set $\Gamma = \{\gamma < \omega^\beta \cdot 2n : d_\gamma \cdot \beta < d_{\gamma+1} \cdot b\}$. Applying Lemma 2.2 gives the desired result.

Similarly we obtain:

Lemma 2.4. *If A has ordinal depth $\geq\omega^\alpha$, where α is a limit ordinal with $\mathrm{cf}\alpha=\omega$, then A has a chain of type ω^α.*

Using Lemma 2.1 we easily obtain:

Lemma 2.5. *If A and B have no chains of type ω^α, then neither does $A\times B$.*

Theorem 2.6. *Ordinal depth is never attained. If A has ordinal depth δ, then δ has the form $\omega^\alpha\cdot n$, where if $n=1$, then α is a limit ordinal and $\mathrm{cf}\alpha>\omega$.*

Proof. Let A have ordinal depth δ. If it is attained, say $\langle a_\xi:\xi<\delta\rangle$ be strictly increasing. Note that $\delta\geq\omega$. Thus $a_1\neq 0$, and the sequence $\langle a_\xi\cdot -a_1: 1\leq\xi<\delta\rangle$ with 1 adjoined at the end is strictly increasing of type $\delta+1$, contradiction. So, ordinal depth is not attained. In particular δ is a limit ordinal. Write

$$\delta=\omega^\alpha\cdot n+\gamma,$$

where $0\neq n\in\omega$ and $\gamma<\omega^\alpha$. Suppose $\gamma\neq 0$. Then let $\langle b_\xi:\xi\leq\omega^\alpha\cdot n\rangle$ be strictly increasing. Then $\langle b_\xi\cdot -b_\gamma:\gamma\leq\xi<\omega^\alpha\cdot n\rangle$ followed by $\langle b_\nu\cdot -b_\gamma+b_\xi:\xi<\gamma\rangle$, where $\nu=\omega^\alpha\cdot n$, is strictly increasing of type δ, contradiction. So, $\gamma=0$. If $n=1$, then α is a limit ordinal and $\mathrm{cf}\alpha>\omega$, by Lemmas 2.3 and 2.4.

Finally, we construct examples showing that the depths described in Theorem 2.6 can actually occur. To this end we introduce a standard sequence of ideals in a BA A:

$$I_0^A=\{0\};\qquad I_\lambda^A=\bigcup_{\alpha<\lambda}I_\alpha^A \text{ for } \lambda \text{ a limit ordinal;}$$

$$I_{\alpha+1}^A=\{a\in A: a/I_\alpha^A \text{ is a finite sum of atoms of } A/I_\alpha^A\}.$$

The following lemma is well known; it can be easily proved by induction on β.

Lemma 2.7. *Let A have an ordered basis $\langle a_\xi:\xi\leq\omega^\alpha\cdot n\rangle$, where $n\in\omega$, and $\beta\leq\alpha$. Then A/I_β^A has an ordered basis $\langle[a_\xi]:\xi\leq\omega^\alpha\cdot n, \xi=\omega^\beta\cdot\gamma\rangle$ for some $\gamma\geq 0$.*

Theorem 2.8. *Let A have an ordered basis of type $\omega^\alpha\cdot n+1$, where $n>0$. Then the ordinal depth of A is $\omega^\alpha\cdot(n+1)$.*

Proof. Let $\langle a_\xi:\xi\leq\omega^\alpha\cdot n\rangle$ be the ordered basis, where $a_0=0$ and $a_\gamma=1$, $\gamma=\omega^\alpha\cdot n$. For each $\beta<\omega^\alpha$ the sequence $\langle a_\xi\cdot -a_\beta:\beta\leq\xi\leq\omega^\alpha\cdot n\rangle$ followed by $\langle a_\xi+-a_\beta:\xi<\beta\rangle$ is of type $\omega^\alpha\cdot n+\beta$. Thus the ordinal depth of A is at least $\omega^\alpha\cdot(n+1)$. Suppose $\langle b_\xi:\xi\leq\omega^\alpha\cdot(n+1)\rangle$ is strictly increasing in A. It is easy to prove by induction that $\langle[b_\xi]:\xi=\omega^\beta\cdot\gamma$ for some $\gamma\geq 0\rangle$ is strictly increasing in A/I_β^A for each $\beta\leq\alpha$. Thus for $\beta=\alpha$, A/I_α^A has a chain of length $n+2$. This contradicts the fact that by Lemma 2.7, A/I_α^A is a finite algebra with n atoms.

Theorem 2.9. *Suppose α is a limit ordinal with $\mathrm{cf}\alpha > \omega$. Then there is a BA with ordinal depth ω^α.*

Proof. For each $\beta < \alpha$ let B_β be a BA with ordered basis ω^β, and let $A = \prod_{\beta<\alpha}^{w} B_\beta$. Using Lemma 2.5 it is easy to check that A has ordinal depth ω^α.

3. Length

For any BA A we define the *length* of A to be the supremum of $|X|$ such that X is a linearly ordered subset of A (under the Boolean ordering). An obvious application of the Erdös–Rado theorem yields:

Theorem 3.1. length $A \leq 2^{\text{depth } A}$.

Theorem 3.2. *If $\aleph_0 \leq \kappa \leq \lambda \leq 2^\kappa$, then there is a BA with length λ and depth κ.*

Proof. We may assume that $\lambda = 2^\kappa$. In the notation of [5] we can then take the interval algebra on $\Lambda(\kappa)\setminus\mathscr{L}(\kappa)$.

The proof of Theorem 1.1.1 can be easily modified to yield:

Theorem 3.3. *If A has length κ with $\mathrm{cf}\kappa = \omega$, then length is attained.*

The analog of Theorem 1.2.1 for length fails. For example, if A is any denumerable BA then ${}^{\omega}A$ has length 2^ω. This is because ${}^{\omega}A$ embeds $\mathscr{P}\mathbb{Q}$, and $\langle\{q \in \mathbb{Q}: q < r\}: r \in \mathbb{R}\rangle$ is a chain in $\mathscr{P}\mathbb{Q}$ of size 2^ω. We do not have a good description of the length of direct products of BA's. This question is related to some problems treated in the literature. If L is a linearly ordered set and $K \subseteq L$, we call K *dense* in L if for all $a, b \in L$, if $a < b$, then there is a $c \in K$ such that $a < c < b$. For each infinite cardinal κ let Ded κ be the sup of all λ such that there is a linearly ordered set of power λ with a dense subset of power κ. In [2] one can find a lot of information on this function Ded. In particular, the following facts proved there are relevant to our length problem.

For every $\kappa \geq \omega$, $\kappa < \text{Ded } \kappa$. (1)

There is a model of ZFC in which $2^\omega = \aleph_{\omega_1}$, $2^{\omega_1} = \aleph_{\omega_1+1}$, and Ded $\omega_1 < 2^{\omega_1}$. (2)

The reason for the relevance of this notion to length of products is that $\mathscr{P}I$ can be isomorphically embedded in $\prod_{i\in I} A_i$ (if all A_i have more than one element), and we have the following simple lemma (see [2, Theorem 2.1]).

Lemma 3.4. *For κ and λ infinite cardinals with $\kappa \leq \lambda$ the following conditions are* *equivalent*:

(i) *There is a linear ordering* L *of power* λ *with a dense subset* K *of power* κ.

(ii) $\mathcal{P}\kappa$ *has a chain of size* λ.

From Lemma 3.4 we get the following lower and upper bounds for length of products.

Theorem 3.5. *If all* A_i *have more than one element, then* $\max(\mathrm{Ded}\,|I|, \sup_{i\in I} \mathrm{length}\, A_i) \leqslant \mathrm{length} \prod_{i\in I} A_i \leqslant \prod_{i\in I} \mathrm{length}\, A_i$.

The following two examples shed some light on these estimates.

Example 3.6.1. For each $i<\omega$ let A_i be the interval algebra on $\beth_0 \times \cdots \times \beth_i$ (lexicographically ordered). Thus $\mathrm{Ded}\,\omega = 2^\omega = \beth_1$, while $\sup_{i\in I} \mathrm{length}\, A_i = \beth_\omega$. But the length of $\prod_{i\in\omega} A_i$ is $\prod_{i\in\omega} \mathrm{length}\, A_i = \beth_{\omega+1}$. In fact, for any $x \in \prod_{i<\omega} \beth_i$ define $g_x \in \prod_{i\in\omega} A_i$ by $g_x i = x \restriction (i+1)$ for all $i \in \omega$. Then if $x<y$ in the lexicographic order we also have $g_x < g_y$; so g is the desired order-isomorphism.

Example 3.6.2. We take the model mentioned in (2) above, and consider ${}^{\omega_1}\mathcal{P}\omega$. We have $\mathrm{Ded}\,\omega = \mathrm{length}\, \mathcal{P}\omega = \aleph_{\omega_1}$, $\mathrm{length}\, {}^{\omega_1}\mathcal{P}\omega = \mathrm{length}\, \mathcal{P}\omega_1 = \aleph_{\omega_1}$, but ${}^{\omega_1}\mathrm{length}\, \mathcal{P}\omega = {}^{\omega_1}\aleph_{\omega_1} \geqslant \aleph_{\omega_1+1}$.

Problem 4. Describe exactly length $\prod_{i\in I} A_i$ in terms of $|I|$ and length A_i, $i \in I$.

In connection with the above results it is perhaps appropriate at this point to make some remarks on maximal chains in power set algebras; although a careful study of maximal chains is not within the scope of this paper, we can clarify some results and problems in the literature. First we note that (1) and (2) above, along with Lemma 3.4, show that [10, Exercise II.4.28] (= [9, Exercise 2.10.28]) should have the assumption GCH, there being counterexamples otherwise. Second [10, Problem II.11] is clarified somewhat by the following result:

$$\text{if } X \text{ is a maximal chain in } \mathcal{P}\omega, \text{ then } |X| = \omega \quad \text{or} \quad |X| = 2^\omega. \tag{3}$$

In fact, X can be embedded in $\mathbb{R}$ (see [3]), and any uncountable set of reals contains an $\aleph_1$-dense set (i.e., a set with the property that between any two of its members are exactly $\aleph_1$ others). In fact, we may assume that $|X| = \aleph_1$. Define $a \equiv b$ iff $a, b \in X$ and, with $a<b$, $|(a,b)| \leqslant \omega$. Then there are only countably many equivalence classes with more than one element, and each is countable, so the set of elements in one-element classes is $\aleph_1$-dense.

Thus if X is uncountable then it contains a dense-in-itself subset Y, and by maximality must contain all cuts of Y; hence $|X| = 2^\omega$.

Now we return to our discussion of length. By Lemma 0 it is clear that $\mathrm{length}(A \times B) = \max(\mathrm{length}\, A, \mathrm{length}\, B)$, with attainment for $A \times B$ iff attainment for a factor with length that of $A \times B$. Hence it is easy to establish Theorem 1.2.4 and Corollaries 1.2.5, 1.2.6 with depth replaced by length.

Concerning free products and length, first we mention:

Theorem 3.7. *Suppose that* $\mathrm{cf}\kappa > \omega$, *A has no chain of power* $\mathrm{cf}\kappa$, *and B has no chain of power* κ. *Then* $A * B$ *has no chain of power* κ.

As already mentioned, the proof of Theorem 3.7 is almost identical to that of Theorem 1.3.1. Now a modification of Theorem 1.3.2 with type replaced by power follows, and Corollary 1.3.3 holds with depth replaced by length. It is perhaps surprising that Theorem 1.3.4 does *not* hold with depth replaced by length. In fact, suppose $\lambda < \mathrm{cf}\kappa < \kappa$, and $\mathscr{P}\lambda$ has a chain C of size μ, where $\mathrm{cf}\kappa \leq \mu < \kappa$; see our discussion above concerning when this can take place. Let A be the interval algebra on C, and let B be any BA of length κ not attained (this is possible by the version of Corollary 1.2.6 for length). Then $A * B$ has no chain of size κ. For, suppose that X is such a chain. For each $x \in X$ write

$$x = \sum_{i < m_x} a_i^x \cdot b_i^x,$$

where $a_i^x \cdot a_j^x = 0$ for $i < j < m_x$, $a_i^x \in A$, $b_i^x \in B$. For each $\alpha \in \lambda$ let

$$T_\alpha = \{b_i^x \colon x \in X, i < m_x, \text{and} \quad \alpha \in a_i^x\}.$$

Then T_α is a chain in B (cf. (2) in the proof of Theorem 1.3.1), so $|T_\alpha| < \kappa$. Thus $T = \bigcup_{\alpha \in \lambda} T_\alpha$ has size $< \kappa$. Let $B' = \mathrm{Sg}\, T$. Then $X \subseteq A * B'$ and so $|X| < \kappa$, contradiction.

However, by the method of proof of Theorem 1.3.4 we have:

Theorem 3.8. *Let* length $B = \kappa > \mathrm{cf}\kappa$. *If A has a chain of type* $\mathrm{cf}\kappa$, *then* $A * B$ *has a chain of size* κ.

For κ singular we thus know the following. Using Lemma 3.4, Theorem 3.7, (1) and the above argument,

> if $\mathrm{cf}\kappa$ is a successor cardinal, κ singular, then there is an A of length $\mathrm{cf}\kappa$ and a B of length κ such that $A * B$ has no chain of size κ. (4)

By the argument for Theorems 3.1 and 3.8, we have

> if $\forall \lambda < \mathrm{cf}\kappa\ (2^\lambda \leq \mathrm{cf}\kappa)$, $\mathrm{cf}\kappa <$ length $A \leq \kappa$, and length $B = \kappa$, then $A * B$ has a chain of size κ. (5)

> If $\mathrm{cf}\kappa =$ length $A < \kappa$ and $\mathrm{cf}\kappa$ is weakly compact, then if length A is attained and length $B = \kappa$ it follows that $A * B$ has a chain of size κ. (6)

The following was noticed by the referee, generalizing Theorem 3.8.

> If $\mathrm{cf}\kappa < \kappa$, A has a chain of size $\mathrm{cf}\kappa$ having $\mathrm{cf}\kappa$ pairwise disjoint intervals each with at least two elements, and length $B = \kappa$, then $A * B$ has a chain of size κ. (7)

In fact, let $\langle a_\alpha : \alpha < \mathrm{cf}\kappa\rangle$ and $\langle a'_\alpha : \alpha < \mathrm{cf}\kappa\rangle$ be such that $\forall\, \alpha < \mathrm{cf}\kappa\ (a'_\alpha < a_\alpha)$ and for all distinct $\alpha, \beta < \mathrm{cf}\kappa$, $a_\alpha < a'_\beta$ or $a_\beta < a'_\alpha$, with all $a_\alpha, a'_\alpha \in A$. For each $\alpha < \mathrm{cf}\kappa$ let C_α be a chain in B of size λ_α, where $\langle \lambda_\alpha : \alpha < \mathrm{cf}\kappa\rangle$ is strictly increasing with sup κ. Then for each $\alpha < \mathrm{cf}\kappa$ and $b \in C_\alpha$ let

$$x_{\alpha b} = a_\alpha \cdot b + a'_\alpha \cdot -b.$$

It is easily checked that $\langle x_{\alpha b} : \alpha < \mathrm{cf}\kappa, b \in C_\alpha\rangle$ forms a chain of size κ.

The above facts leave several questions open; the following is a simple one.

Problem 5. Let $\omega < \mathrm{cf}\kappa < \kappa$, and let L be a dense linear ordering of power $\mathrm{cf}\kappa$ with no dense subset of power $< \mathrm{cf}\kappa$ and with no family of $\mathrm{cf}\kappa$ pairwise disjoint intervals. Let A be the interval algebra on L, and suppose that length $B = \kappa$. Then does $A * B$ have a chain of size κ?

Recall that the ordering arising from a $\mathrm{cf}\kappa$-Souslin tree satisfies the conditions in Problem 5.

We have not investigated the behaviour of length under amalgamated free products or ultraproducts.

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