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Homogeneous Boolean Algebras with Very Nonsymmetric Subalgebras

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We prove the following theorems.

Theorem 1 *For every Boolean algebra A there are extensions $C \supseteq B \supseteq A$ such that B and C are homogeneous, every endomorphism or automorphism of A extends to an endomorphism or automorphism of B, and no nontrivial one-one endomorphism of B extends to an endomorphism of C.*

Theorem 2 *Assume ($\diamond$). There is an ω_1-Souslin tree T such that the regular open algebra B of T is homogeneous and has a complete subalgebra A onto which no nontrivial automorphism of B restricts.*

These theorems were motivated by the following question raised by Štěpánek: Does there exist a complete homogeneous Boolean algebra B with a complete homogeneous subalgebra A such that no nontrivial automorphism of A extends to B? Here a Boolean algebra B is called homogeneous if every principal ideal $B \upharpoonright b = \{x \in B \mid x \leqslant b\}$ for $b \neq 0$ is isomorphic to B; because of $B \cong B \upharpoonright b \times B \upharpoonright -b$, $B \upharpoonright b$ is also called a factor of B. B is said to be rigid if it has no nontrivial automorphism.

Štěpánek's question arose from the following facts. Every Boolean algebra A can be embedded into a homogeneous complete algebra B such that every automorphism of A extends to B (see [4] and [5]). Every A can be embedded into a complete rigid B—of course, no nontrivial automorphism of A extends to B (see [7]). Every A can be embedded into a complete B without homogeneous or rigid factors such that either every or no nontrivial automorphism of A extends to B (see [8] and [9]).

We assume acquaintance with [6] for the proof of Theorem 1 and with [1] or [3] for Theorem 2.

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Proof of Theorem 1: Let A be given. Choose an ordinal α with $cf\,\alpha = \omega$ such that $|A| \leqslant \beth_\alpha$. Let $\kappa = \beth_\alpha$ and $\lambda = 2^\kappa$, so $\kappa^\omega = \lambda$.

Next, let A' be a Boolean algebra such that $|A'| = \kappa$ and each $A' \upharpoonright a$ where $a > 0$ contains a disjoint subset of power κ. By [4], there is a Boolean algebra A'' with $|A''| = \kappa$ such that the free product

$$B = A * A' * A''$$

is homogeneous; clearly $|B| = \kappa$ and each $B \upharpoonright b$ where $b > 0$ has a disjoint subset of power κ. B satisfies the conditions on A in the proof of Theorem 12 in [6]. Hence there is an atomless κ-complicated Boolean algebra B' such that

$$B \subseteq B' \subseteq (B * F)^{compl},$$

where F is the free Boolean algebra on λ free generators and D^{compl} denotes the completion of D. Note that the embedding from B to B' preserves all meets and joins existing in B. By [2], choose E such that

$$C = B' * E$$

is homogeneous.

Now let f be a nontrivial one-one endomorphism of B and assume that $\bar{f}$ is an endomorphism of C extending f. Choose $b \in B$ such that $b > 0$ and $b \cdot f(b) = 0$ and let $(a_\alpha)_{\alpha<\kappa}$ be a disjoint family in $B \upharpoonright b \backslash \{0\}$. By κ-complicatedness of B', there is an $S \subseteq \kappa$ satisfying:

(1) There is some $x \in B'$ such that $a_\alpha \leqslant x$ for $\alpha \in S$ and $a_\alpha \cdot x = 0$ for $\alpha \in \kappa \backslash S$

(2) There is no $y \in B'$ such that $f(a_\alpha) \leqslant y$ for $\alpha \in S$ and $f(a_\alpha) \cdot y = 0$ for $\alpha \in \kappa \backslash S$.

Write, since $\bar{f}(x) \in C = B' * E$,

$$\bar{f}(x) = \sum_{i<n} b_i \cdot e_i$$

where $b_i \in B'$, $e_i \in E$. But then $y = \sum_{i<n} b_i$ is an element of B' contradicting (2): for $\alpha \in S$, we have $a_\alpha \leqslant x$, $f(a_\alpha) \leqslant \bar{f}(x)$, so $f(a_\alpha) \leqslant \sum_{i<n} b_i$. For $\alpha \in \kappa \backslash S$, we have $a_\alpha \cdot x = 0$, $f(a_\alpha) \cdot \bar{f}(x) = 0$, so $f(a_\alpha) \cdot \sum_{i<n} b_i = 0$.

Proof of Theorem 2: Let $(S_\alpha)_{\alpha<\omega_1}$ be a sequence for ($\diamond$). It is sufficient to construct a normal Souslin tree T of length ω_1 with levels U_α and objects $g_{\alpha uv}$, $\tilde{\alpha}$ such that the following claims (1) to (4) are satisfied.

(1) (a) For $\beta < \alpha < \omega_1$ and $u, v \in U_\beta$, $g_{\alpha uv}$ is an automorphism of $T_\alpha = \bigcup_{\gamma<\alpha} U_\gamma$ such that $g_{\alpha uv}(u) = v$

(b) $g_{\alpha uv} \subseteq g_{\alpha' uv}$ for $\beta < \alpha \leqslant \alpha' < \omega_1$

(c) $g_{\lambda uv} = \bigcup \{g_{\alpha uv} | \beta < \alpha < \lambda\}$ if λ is a limit ordinal such that $\beta < \lambda < \omega_1$.

For $u, v \in U_\beta$, $\bigcup_{\beta<\alpha} g_{\alpha uv}$ is then an automorphism of T mapping u to v. The regular open algebra B of T will then be homogeneous.

(2) (a) For $\alpha < \omega_1$, $\underset{\alpha}{\sim}$ is an equivalence relation on U_α
(b) if $\beta < \alpha < \omega_1$, $y \underset{\alpha}{\sim} y'$ and $x, x' \in U_\beta$ are such that $x < y$, $x' < y'$, then $x \underset{\beta}{\sim} x'$
(c) if $\beta < \alpha < \omega_1$, $x \underset{\beta}{\sim} x'$, $y \in U_\alpha$ and $x < y$, then there are infinitely many $y' \in U_\alpha$ such that $x' < y'$ and $y \underset{\alpha}{\sim} y'$
(d) if $\beta < \alpha < \omega_1$, $x \in U_\beta$, then there are $y, y' \in U_\alpha$ such that $x < y, y'$ and $y \underset{\alpha}{\nsim} y'$.

The sequence $(\underset{\alpha}{\sim})_{\alpha<\omega_1}$ then gives rise to a complete subalgebra A of B (see [3]).

(3) $(S_\alpha)_{\alpha<\omega_1}$ diagonalizes each possible uncountable antichain of T and each possible nontrivial automorphism of B restricting to A.

The most complicated case to consider is: S_λ codes a maximal antichain a of T_λ plus a nontrivial automorphism ϕ of $T_\lambda \upharpoonright c$, where $c \subseteq \lambda$ is closed unbounded in λ. For two branches b, b' of length λ in T_λ, let $b \approx b'$ mean that for each $\alpha < \lambda$, $x_\alpha \underset{\alpha}{\sim} x'_\alpha$ where x_α (respectively, x'_α) is the unique element of $b \cap U_\alpha$ (respectively, $b' \cap U_\alpha$). Then choose U_λ such that the set Z of λ-branches in T_λ corresponding to points in U_λ satisfies:

(a) $\bigcup Z = T_\lambda$
(b) $b \cap a \neq \phi$ for $b \in Z$
(c) Z is closed under the obvious action of each $g_{\lambda uv}$ (where $u, v \in U_\beta$, $\beta < \lambda$) and of ϕ on the λ-branches of T_λ
(d) if $b \in Z$, $x \in b \cap U_\alpha$, $x \underset{\alpha}{\sim} x'$, then there is some $b' \in Z$ such that $x' \in b'$ and $b \approx b'$.

The existence of Z satisfying this countable list of requirements is most easily seen by a forcing style argument.

(4) If S_λ codes (a, ϕ) and U_λ is chosen as in (3), then there are $u, u', v, w \in U_\lambda$ such that $\phi(u) = v$, $\phi(u') = w$ under the obvious action of ϕ on U_λ and such that $u \underset{\lambda}{\sim} u'$ but $v \underset{\lambda}{\nsim} w$.

This guarantees that (the automorphism of B induced by) ϕ does not restrict to A.

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