THE JOURNAL OF SYMBOLIC LOGIC
Volume 34, Number 3, Sept. 1969

# NONFINITIZABILITY OF CLASSES OF REPRESENTABLE CYLINDRIC ALGEBRAS[1]

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Cylindric algebras were introduced by Alfred Tarski about 1952 to provide an algebraic analysis of (first-order) predicate logic. With each cylindric algebra one can, in fact, associate a certain, in general infinitary, predicate logic; for locally finite cylindric algebras of infinite dimension the associated predicate logics are finitary. As with Boolean algebras and sentential logic, the algebraic counterpart of completeness is representability. Tarski proved the fundamental result that every locally finite cylindric algebra of infinite dimension is representable. This showed that the postulates selected to define cylindric algebras were adequate. However, it was also discovered at an early stage in the investigations that nonrepresentable cylindric algebras exist. The problem then arose to characterize the class of representable cylindric algebras in as simple a manner as possible. Tarski [9] showed that the class $\boldsymbol{RCA}_\alpha$ of representable cylindric algebras of dimension $\alpha$ is a variety, i.e., it can be characterized by a set of equations. A specific set of equations was not exhibited, although by Craig's method one could see that a primitive recursive set of equations characterizing $\boldsymbol{RCA}_\alpha$ exists. Since $\boldsymbol{RCA}_0 = \boldsymbol{CA}_0$ and $\boldsymbol{RCA}_1 = \boldsymbol{CA}_1$, the classes $\boldsymbol{RCA}_\alpha$ for $\alpha \leq 1$ have a simple characterization. Henkin showed that $\boldsymbol{RCA}_2$ can be characterized by adjoining two rather simple equations to the equations characterizing $\boldsymbol{CA}_2$. Monk [8] showed that $\boldsymbol{RCA}_3$ cannot be characterized by finitely many first-order axioms.

In §1 we show that for $3 \leq \alpha < \omega$, $\boldsymbol{RCA}_\alpha$ cannot be characterized by finitely many first-order axioms. In §2 we show that for $\omega \leq \alpha$ $\boldsymbol{RCA}_\alpha$ cannot be characterized by a finite schema of the type characterizing $\boldsymbol{CA}_\alpha$. In §3 we give explicitly a set of equations characterizing $\boldsymbol{RCA}_\alpha$, by an easy modification of an unpublished method of Ralph McKenzie. §4 gives some open problems. For extensions of these results to other algebraic versions of predicate logic see the immediately following paper of James Johnson.

I am grateful to Daniel Demaree, Leon Henkin, James Johnson and Ralph McKenzie for stimulating discussions about the problems treated in this paper.

**§0. Preliminaries.** We shall use the notation of the survey paper Henkin, Tarski [5]. A complete exposition of the algebraic theory of cylindric algebras will be found in Henkin, Monk, Tarski [4]. Indications of proofs of the results about cylindric algebras we use can be found in Henkin [3] and the first part of Monk [8].

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Received March 15, 1968.

[1] Research supported in part by NSF Grants GP 7387 and GP 6232-x.

In addition to results from [5] we shall use a construction of relativization, which we now describe. Let $\mathfrak{A} = \langle A, +, \cdot, -, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ be a $\boldsymbol{CA}_\alpha$ and $a \in A$. The *relativization of* $\mathfrak{A}$ *to* $a$ is the algebra $\mathfrak{Rl}_a\mathfrak{A} = \langle Rl_aA, +, \cdot, -', c'_\kappa, d'_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$, where $Rl_aA = \{x \in A : x \leq a\}$, $-'x = -x\cdot a$ for any $x \in Rl_aA$, $c'_\kappa x = c_\kappa x\cdot a$ for any $\kappa < \alpha$ and $x \in Rl_aA$, and $d'_{\kappa\lambda} = d_{\kappa\lambda}\cdot a$ for all $\kappa, \lambda < \alpha$. Two basic facts about relativization are as follows.

THEOREM 0.1. *$\mathfrak{Rl}_a\mathfrak{A}$ satisfies all of the postulates for $\boldsymbol{CA}_\alpha$'s given in* [5] *except possibly* P5 *and* P7.

The proof of 0.1 is routine.

THEOREM 0.2. *Let $\mathfrak{A}$ be a $\boldsymbol{CA}_\alpha$, $a \in A$, and suppose that $c_\kappa a\cdot c_\lambda a = a \leq c_\kappa(d_{\kappa\lambda}\cdot a)$ whenever $\kappa, \lambda < \alpha$ and $\kappa \neq \lambda$. Then $\mathfrak{Rl}_a\mathfrak{A}$ is a $\boldsymbol{CA}_\alpha$.*

PROOF. By 0.1, P5 and P7 of [5] must be checked. If $x \in Rl_aA$ and $\kappa < \lambda < \alpha$, then

$$\begin{aligned} c'_\kappa c'_\lambda x &= c_\kappa(c_\lambda x\cdot a)\cdot a = c_\kappa(c_\lambda x\cdot c_\kappa a\cdot c_\lambda a)\cdot a \\ &= c_\kappa(c_\lambda x\cdot c_\lambda a)\cdot c_\kappa a\cdot a = c_\kappa c_\lambda(x\cdot c_\lambda a)\cdot a \\ &= c_\kappa c_\lambda x\cdot a. \end{aligned}$$

Similarly $c'_\lambda c'_\kappa x = c_\kappa c_\lambda x\cdot a$, so P5 follows. To check P7, assume that $\kappa \neq \lambda, \mu$. Then

$$\begin{aligned} c'_\kappa(d'_{\lambda\kappa}\cdot d'_{\kappa\mu}) &= c_\kappa(d_{\lambda\kappa}\cdot d_{\kappa\mu}\cdot a)\cdot a = c_\kappa(d_{\kappa\lambda}\cdot d_{\lambda\mu}\cdot a)\cdot a \\ &= c_\kappa(d_{\kappa\lambda}\cdot a)\cdot d_{\lambda\mu}\cdot a = d_{\lambda\mu}\cdot a = d'_{\lambda\mu}. \end{aligned}$$

Thus $\mathfrak{Rl}_a\mathfrak{A}$ is a $\boldsymbol{CA}_\alpha$.

Another result concerning relativization which we shall use is deeper than the above two results and is a consequence of a more general theorem of Tarski. We shall state the more general theorem without proof, and give a complete proof for the consequence of it actually used below. Let $\mathfrak{A}$ be the $\boldsymbol{CS}_\alpha$ of all subsets of ${}^\alpha U$, and let $x \in {}^\alpha U$. Let $V = \{y \in {}^\alpha U : \{\kappa : \kappa < \alpha, y_\kappa \neq x_\kappa\}$ is finite$\}$. $V$ is called a *weak space*. Any subalgebra of $\mathfrak{Rl}_V\mathfrak{A}$ is called a *weak $\boldsymbol{CS}_\alpha$ with base $U$*. Henkin has shown in unpublished work that $\mathfrak{A} \in \boldsymbol{RCA}_\alpha$ iff $A$ is isomorphic to a subdirect product of weak $\boldsymbol{CS}_\alpha$'s. Now for any $\boldsymbol{CA}_\alpha$ $\mathfrak{A}$, let $\mathfrak{A}^* = \{x \in A : c_\kappa x\cdot c_\lambda x = x \leq c_\kappa(d_{\kappa\lambda}\cdot x)$ for any distinct $\kappa, \lambda < \alpha\}$. Tarski's theorem is as follows:

THEOREM. *If $\mathfrak{A} \subseteq \mathfrak{Rl}_b\mathfrak{B}$ with $\mathfrak{B} \in \boldsymbol{CS}_\alpha$, $b \in \mathfrak{B}^*$, and $\alpha \geq 2$, and if $a \in A$, then the following conditions are equivalent*:

(i) $a \in \mathfrak{A}^*$.

(ii) *$a$ is a union of pairwise disjoint weak spaces.*

A corollary of this theorem is that $\mathfrak{A} \in \boldsymbol{RCA}_\alpha$ if $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Rl}_b\mathfrak{B}$ for some $\mathfrak{B} \in \boldsymbol{CS}_\alpha$ and some $b \in \mathfrak{B}^*$.

The consequence we need is as follows; our proof is due to Don Pigozzi.

THEOREM 0.3. *Let $\mathfrak{A} \in \boldsymbol{RCA}_\alpha$, $a \in A$, and suppose that $a \in \mathfrak{A}^*$. Then $\mathfrak{Rl}_a\mathfrak{A} \in \boldsymbol{RCA}_\alpha$.*

PROOF. By Theorem 2.15 of Henkin, Tarski [5] and an easy induction it suffices to prove the following:

(1) If $\mathfrak{A}$ is neatly embedded in a $\boldsymbol{CA}_{\alpha+1}$ $\mathfrak{B}$, then there is a $b \in \mathfrak{B}^*$ such that $\mathfrak{Rl}_a\mathfrak{A}$ is isomorphic to a $\boldsymbol{CA}_\alpha$ neatly embedded in $\mathfrak{Rl}_b\mathfrak{B}$.

To prove (1), let $b = a\cdot c_0(d_{0\alpha}\cdot a)$. To establish the desired properties of $b$ we need the following results:

(2) if $\kappa, \lambda < \alpha$ and $\kappa \neq \lambda$, then $a = c_\kappa(d_{\kappa\lambda}\cdot a)\cdot c_\lambda(d_{\lambda\kappa}\cdot a)$.

Indeed, $a \leq c_\kappa(d_{\kappa\lambda}\cdot a)\cdot c_\lambda(d_{\lambda\kappa}\cdot a) \leq c_\kappa a\cdot c_\lambda a = a$.

(3) If $\kappa < \alpha$, $\kappa \neq 0$, then $a \cdot c_0(d_{0\alpha} \cdot a) = a \cdot c_\kappa(d_{\kappa\alpha} \cdot a)$.
In fact,

$$\begin{aligned} a \cdot c_0(d_{0\alpha} \cdot a) &= a \cdot c_0(d_{0\alpha} \cdot c_0(d_{0\kappa} \cdot a) \cdot c_\kappa(d_{0\kappa} \cdot a)) \\ &= a \cdot c_0(d_{0\kappa} \cdot a) \cdot c_0(d_{0\alpha} \cdot c_\kappa(d_{0\kappa} \cdot a)) \\ &= a \cdot c_0 c_\kappa(d_{0\kappa} \cdot a); \end{aligned}$$

(3) follows by symmetry. Now to establish the desired properties of $b$, suppose that $\kappa, \lambda < \alpha + 1$ and $\kappa \neq \lambda$. Clearly $c_\kappa b \cdot c_\lambda b \leq c_\kappa a \cdot c_\lambda a = a$. We have, say, $\kappa < \alpha$, and hence

$$\begin{aligned} c_\kappa b \cdot c_\lambda b &\leq a \cdot c_\kappa(a \cdot c_0(d_{0\alpha} \cdot a)) \\ &= a \cdot c_\kappa a \cdot c_0(d_{0\alpha} \cdot a) \quad \text{by (3)} \\ &= b. \end{aligned}$$

Furthermore,

$$\begin{aligned} c_\kappa(d_{\kappa\lambda} \cdot b) &= c_\kappa(d_{\kappa\lambda} \cdot a \cdot c_0(d_{0\alpha} \cdot a)) \\ &= c_\kappa(d_{\kappa\lambda} \cdot a) \cdot c_\kappa(d_{\kappa\alpha} \cdot a) \quad \text{by (3)} \\ &\geq b. \end{aligned}$$

Now for any $x \in Rl_\alpha \mathfrak{A}$ let $Fx = x \cdot c_0(d_{0\alpha} \cdot a)$. It is easily checked that $F$ is the desired isomorphism.

If $F$ is an isomorphism from a $\boldsymbol{CA}_\alpha$ $\mathfrak{A}$ onto a $\boldsymbol{CA}_\alpha$ neatly embedded in a $\boldsymbol{CA}_\beta$ $\mathfrak{B}$, we shall say that $F$ is a *neat embedding* of $\mathfrak{A}$ into $\mathfrak{B}$. If such an $F$ exists, we say that $\mathfrak{A}$ is *neatly embeddable* in $\mathfrak{B}$.

**§1. Nonfinitizability, finite-dimensional case.** Throughout this section assume that $3 \leq \alpha < \omega$. To establish the nonfinite axiomatizability of the class $\boldsymbol{RCA}_\alpha$ we shall construct an ultraproduct of nonrepresentable $\boldsymbol{CA}_\alpha$'s which is itself representable. To construct the nonrepresentable $\boldsymbol{CA}_\alpha$'s we shall modify the construction used in the second half of Monk [8] (but we do not assume any knowledge of that construction). Their nonrepresentability is established by using a simple graph-theoretical result of Greenwood and Gleason [1], while the representability of the ultraproduct is shown by applying the neat embedding theorem (Theorem 2.15 of Henkin, Tarski [5]).

With each natural number $\beta \geq \alpha - 1$ we shall now associate a $\boldsymbol{CA}_\alpha$ $\mathfrak{A}^\alpha_\beta$ (which is not yet, however, one of the nonrepresentable algebras we are after). If $R$ is an equivalence relation on a set $\Gamma$ of ordinals, we let $\binom{R}{2} = \{\{X, Y\}: X \text{ and } Y \text{ are distinct } R\text{-classes}\}$. If $f$ is a function with domain $\binom{R}{2}$, we write $f\kappa\lambda$ instead of $f\{\kappa/R, \lambda/R\}$, for $\kappa, \lambda \in \Gamma$ and $\kappa \not{R} \lambda$. Now we let

$$\begin{aligned} \beta'_\alpha = \{\langle R, f\rangle &: R \text{ is an equivalence relation on } \alpha, f \text{ maps} \\ &\tbinom{R}{2} \text{ into } \beta, \text{ and for all } \kappa, \lambda, \mu < \alpha, \text{ if } \kappa \not{R} \lambda \not{R} \mu \not{R} \kappa \\ &\text{then } |\{f\kappa\lambda, f\lambda\mu, f\mu\kappa\}| \neq 1\}. \end{aligned}$$

If $\langle R, f\rangle \in \beta'_\alpha$ and $\Gamma \subseteq \alpha$, by $\langle R, f\rangle \restriction \Gamma$ we mean $\langle S, g\rangle$, where $S = R \cap {}^2\Gamma$, $g: \binom{S}{2} \to \beta$, and $g\kappa\lambda = f\kappa\lambda$ whenever $\kappa, \lambda \in \Gamma$ and $\kappa \not{S} \lambda$. Next, for $\kappa, \lambda < \alpha$ we let

$$d^{\beta\alpha}_{\kappa\lambda} = \{\langle R, f\rangle \in \beta'_\alpha : \kappa R \lambda\};$$

for $\kappa < \alpha$ and $\langle R, f\rangle \in \beta'_\alpha$ we let

$$c_\kappa^{\beta\alpha}\{\langle R, f\rangle\} = \{\langle S, g\rangle \in \beta'_\alpha : \langle R, f\rangle \restriction \alpha \sim \{\kappa\} = \langle S, g\rangle \restriction \alpha \sim \{\kappa\}\};$$

finally, for $M \subseteq \beta'_\alpha$ and $\kappa < \alpha$ we let

$$c_\kappa^{\beta\alpha} M = \bigcup_{\langle R, f\rangle \in M} c_\kappa^{\beta\alpha}\{\langle R, f\rangle\}.$$

Here we have used the set-theoretical notation by which ${}^A B$ is the set of all functions mapping $A$ into $B$; each ordinal is identical with the set of all preceding ordinals, and in particular $2 = \{0, 1\}$. Further, letting $SA = \{X : X \subseteq A\}$ for all $A$, we define

$$\mathfrak{A}_\beta^\alpha = \langle S\beta'_\alpha, \cup, \cap, \sim, c_\kappa^{\beta\alpha}, d_{\kappa\lambda}^{\beta\alpha}\rangle_{\kappa,\lambda<\alpha}.$$

We will usually write $c_\kappa$, $d_{\kappa\lambda}$ in place of $c_\kappa^{\beta\alpha}$, $d_{\kappa\lambda}^{\beta\alpha}$, if no confusion is likely.

THEOREM 1.1. *If $\omega > \beta \geq \alpha - 1$, then $\mathfrak{A}_\beta^\alpha$ is a $CA_\alpha$.*

PROOF. We must check the postulates P1–P8 of Henkin, Tarski [5] (with $\omega$ replaced by $\alpha$). The postulates P1–P4, and P6, are easily checked. In view of the complete additivity of the $c_\kappa$'s and considerations of symmetry, to establish P5 it suffices to take arbitrary distinct $\kappa, \lambda < \alpha$ and $\langle R, f\rangle \in \beta'_\alpha$ and show that $c_\kappa c_\lambda\{\langle R, f\rangle\} \subseteq c_\lambda c_\kappa\{\langle R, f\rangle\}$. Suppose then that $\langle S, g\rangle \in c_\kappa c_\lambda\{\langle R, f\rangle\}$; say $\langle S, g\rangle \in c_\kappa\{\langle T, h\rangle\}$ with $\langle T, h\rangle \in c_\lambda\{\langle R, f\rangle\}$. Let $U = [R \cap {}^2(\alpha \sim \{\kappa\})] \cup [S \cap {}^2(\alpha \sim \{\lambda\})]$. We now distinguish three cases.

*Case* 1. $\exists\mu \neq \kappa, \lambda\ (\lambda R\mu S\kappa)$. Then let $V = U \cup \{\langle\kappa, \lambda\rangle, \langle\lambda, \kappa\rangle\}$. It is easily checked that $V$ is an equivalence relation on $\alpha$. Let $k\nu\rho = f\nu\rho$ for $\nu, \rho \neq \kappa$, $\nu \not{V} \rho$; this completely defines $k$ as a function mapping $\binom{V}{2}$ into $\beta$, and clearly $\langle V, k\rangle \in \beta'_\alpha$ and $\langle V, k\rangle \in c_\kappa\{\langle R, f\rangle\}$. If $\nu, \rho \neq \kappa, \lambda$ and $\nu \not{V} \rho$, then $k\nu\rho = f\nu\rho = h\nu\rho = g\nu\rho$. If $\nu \neq \kappa, \lambda$ and $\nu \not{V} \kappa$, then $k\nu\kappa = k\nu\lambda = f\nu\lambda = f\nu\mu = h\nu\mu = g\nu\mu = g\nu\kappa$. This establishes that $\langle S, g\rangle \in c_\lambda\{\langle V, k\rangle\}$.

*Case* 2. $\neg\exists\mu \neq \kappa, \lambda(\lambda R\mu S\kappa) \wedge \exists\mu \neq \kappa, \lambda(\lambda R\mu$ or $\mu S\kappa)$. Letting $V = U$, the details proceed as in Case 1.

*Case* 3. $\neg\exists\mu \neq \kappa, \lambda(\lambda R\mu$ or $\mu S\kappa)$. It is easily checked that $U$ is an equivalence relation on $\alpha$. For $\nu, \rho \neq \kappa$, $\nu \not{U} \rho$, let $k\nu\rho = f_{\nu\rho}$. For $\nu \neq \kappa, \lambda$, $\nu \not{U} \kappa$, let $k\nu\kappa = g\nu\kappa$. Finally, let $k\kappa\lambda$ be an element of $\beta \sim \{f\lambda\mu : \mu \neq \kappa, \lambda, f\lambda\mu = g\kappa\mu\}$ (using the assumption $\beta \geq \alpha - 1$). The desired conclusions are then easily checked.

Next, to establish P7, assume that $\kappa, \lambda \neq \mu$. Clearly $c_\mu(d_{\kappa\mu} \cap d_{\mu\lambda}) \subseteq d_{\kappa\lambda}$, so assume that $\langle R, f\rangle \in d_{\kappa\lambda}$. Let

$$S = [R \cap {}^2(\alpha \sim \{\mu\})] \cup \{\langle\mu, \nu\rangle, \langle\nu, \mu\rangle : \kappa R\nu\} \cup \{\langle\mu, \nu\rangle\}.$$

Clearly $S$ is an equivalence relation on $\alpha$. For $\nu, \rho \in \alpha \sim \{\mu\}$; $\nu \not{S} \rho$, let $g\nu\rho = f\nu\rho$. This defines $g$ completely as a mapping from $\binom{S}{2}$ into $\beta$, and it is obvious that $\langle R, f\rangle \in c_\mu\{\langle S, g\rangle\}$ and $\langle S, g\rangle \in d_{\kappa\mu} \cap d_{\mu\lambda}$.

Finally, by the complete additivity of $c_\kappa$, to verify P8 it is enough to note that a contradiction follows immediately from the following assumptions: $\langle R, f\rangle, \langle S, g\rangle, \langle T, h\rangle \in \beta'_\alpha$; $\langle R, f\rangle \neq \langle S, g\rangle$; $\kappa, \lambda \in \alpha$, $\kappa \neq \lambda$; $\langle T, h\rangle \in c_\kappa\{\langle R, f\rangle\} \cap c_\kappa\{\langle S, g\rangle\}$; $\langle R, f\rangle, \langle S, g\rangle \in d_{\kappa\lambda}$.

A fundamental property of the algebras $\mathfrak{A}_\beta^\alpha$ is expressed in the next theorem.

THEOREM 1.2. *If $3 \leq \alpha \leq \gamma < \omega$ and $\omega > \beta \geq \gamma - 1$, then $\mathfrak{A}^\alpha_\beta$ is neatly embeddable in $\mathfrak{A}^\gamma_\beta$.*

PROOF. For each $\langle R, f\rangle \in \beta'_\alpha$ let

$$F\{\langle R, f\rangle\} = \{\langle S, g\rangle \in \beta'_\gamma : \langle S, g\rangle \restriction \alpha = \langle R, f\rangle\};$$

for any $X \subseteq \beta'_\alpha$, let $FX = \bigcup_{\langle R,f\rangle \in \beta'_\alpha} F\{\langle R, f\rangle\}$. Clearly $F$ is a Boolean isomorphism from $\mathfrak{A}^\alpha_\beta$ into $\mathfrak{A}^\gamma_\beta$, $Fd_{\kappa\lambda} = d_{\kappa\lambda}$ for all $\kappa, \lambda < \alpha$, and $c_\kappa FX = FX$ whenever $\alpha \leq \kappa < \gamma$ and $X \subseteq \beta'_\alpha$. It remains to show that $F$ preserves $c_\kappa$ for $\kappa < \alpha$. First suppose that $\langle T, h\rangle \in c_\kappa FX$, say $\langle T, h\rangle \in c_\kappa\{\langle S, g\rangle\}$, $\langle S, g\rangle \in F\{\langle R, f\rangle\}$, $\langle R, f\rangle \in X$. Then with $\langle T, h\rangle \restriction \alpha = \langle U, k\rangle$ we clearly have $\langle T, h\rangle \in F\{\langle U, k\rangle\}$ and $\langle U, k\rangle \in c_\kappa\{\langle R, f\rangle\}$, so that $\langle T, h\rangle \in Fc_\kappa X$. Second, suppose that $\langle T, h\rangle \in Fc_\kappa X$. Say $\langle T, h\rangle \in F\{\langle S, g\rangle\}$, $\langle S, g\rangle \in c_\kappa\{\langle R, f\rangle\}$, and $\langle R, f\rangle \in X$. Let

$$U = [T \cap {}^2(\gamma \sim \{\kappa\})] \cup \{\langle \kappa, \kappa\rangle\} \cup \{\langle \kappa, \lambda\rangle, \langle \lambda, \kappa\rangle : \exists \mu \in \alpha \sim \{\kappa\}(\kappa R \mu T \lambda)\}.$$

It is easily checked that $U$ is an equivalence relation on $\gamma$. For $\lambda, \mu \in \gamma \sim \{\kappa\}$ and $\lambda \not{U} \mu$ let $k\lambda\mu = h\lambda\mu$. If $\kappa R \mu$ for some $\mu \in \alpha \sim \{\kappa\}$, then $k$ is completely defined, $k: \binom{U}{2} \to \beta$, $\langle U, k\rangle \in \beta'_\gamma$, $\langle T, h\rangle \in c_\kappa\{\langle U, k\rangle\}$, and $\langle U, k\rangle \in F\{\langle R, f\rangle\}$, as desired. Suppose now that $\kappa \not{R} \mu$ for all $\mu \in \alpha \sim \{\kappa\}$. For $\mu \in \alpha \sim \{\kappa\}$ let $k\kappa\mu = f\kappa\mu$. Let $k\kappa\mu$ be defined for $\mu \in \gamma \sim \alpha$ in such a way that $k\kappa\mu \notin \{k\kappa\nu : \nu \in \alpha \sim \{\kappa\}\}$ and $k\kappa\mu \neq k\kappa\mu'$ for $\mu, \mu' \in \gamma \sim \alpha$ and $\mu \not{U} \mu'$; this is possible since $\beta \geq \gamma - 1$. The desired conclusions again follow.

We shall now modify the algebras $\mathfrak{A}^\alpha_\beta$. Assuming that $\beta \geq \alpha - 1$, let $V^\alpha_\beta = \{\langle R, f\rangle \in \beta'_\alpha : R \cap {}^2(\alpha \sim 3) = \mathrm{Id} \cap {}^2(\alpha \sim 3)$ and for all $\kappa, \lambda$ such that $3 \leq \kappa < \lambda < \alpha, f\kappa\lambda = \kappa\}$. Here $\mathrm{Id} = \{(x, y) : x = y\}$. Let $\mathfrak{C}^\alpha_\beta$ be the 3-reduct of $\mathfrak{A}^\alpha_\beta$ (see p. 99 of Henkin, Tarski [5]) and let $\mathfrak{B}^\alpha_\beta = \mathfrak{Rl}_a \mathfrak{C}^\alpha_\beta$, where $a = V^\alpha_\beta$.

THEOREM 1.3. *For $\omega > \beta \geq \alpha - 1$, $\mathfrak{B}^\alpha_\beta$ is a homomorphic image of $\mathfrak{C}^\alpha_\beta$.*

PROOF. For any $X \in S\beta'_\alpha$ let $FX = X \cap V^\alpha_\beta$. As is well-known, $F$ is a Boolean homomorphism from $\mathfrak{C}^\alpha_\beta$ onto $\mathfrak{B}^\alpha_\beta$; it is also clear that $Fd_{\kappa\lambda} = d'_{\kappa\lambda}$ for all $\kappa, \lambda < 3$. To show that $F$ preserves $c_\kappa$ for $\kappa < 3$, let $X \in S\beta'_\alpha$; the verification reduces to checking that $c_\kappa X \cap V^\alpha_\beta = c_\kappa(X \cap V^\alpha_\beta) \cap V^\alpha_\beta$, and this is easy.

THEOREM 1.4. *For $\omega > \beta \geq \alpha - 1$, $\mathfrak{B}^\alpha_\beta$ is a simple $CA_3$.*

PROOF. By Theorem 1.3, $\mathfrak{B}^\alpha_\beta$ is a $CA_3$. To prove simplicity it suffices to take arbitrary $\langle R, f\rangle, \langle S, g\rangle \in V^\alpha_\beta$ and show that $\langle S, g\rangle \in c'_0 c'_1 c'_2\{\langle R, f\rangle\}$. Let

$$T = [R \cap {}^2(\alpha \sim \{2\})] \cup \{\langle 2, 2\rangle\} \cup \{\langle 2, \kappa\rangle, \langle \kappa, 2\rangle : \exists \lambda,\ 3 \leq \lambda < \alpha \text{ and } \kappa R \lambda S 2\};$$
$$U = [S \cap {}^2(\alpha \sim \{0\})] \cup \{\langle 0, 0\rangle\} \cup \{\langle 0, \kappa\rangle, \langle \kappa, 0\rangle : \exists \lambda,\ 3 \leq \lambda < \alpha \text{ and } \kappa S \lambda R 0\}.$$

It is easily checked that $T$ and $U$ are equivalence relations on $\alpha$, and $U \cap {}^2(\alpha \sim \{1\}) = T \cap {}^2(\alpha \sim \{1\})$. For $\kappa, \lambda \in \alpha \sim \{2\}$ and $\kappa \not{T} \lambda$, let $h\kappa\lambda = f\kappa\lambda$. If $\kappa T 2$ for some $\kappa \neq 2$, $h: \binom{T}{2} \to \beta$ is then completely defined. Otherwise, let $h\kappa 2 = g\kappa 2$ for $3 \leq \kappa < \alpha$. If $0 \not{T} 2$ and $0 T \kappa$ for some $\kappa \in \alpha \sim 3$, $h02$ is defined; if $0 \not{T} \kappa$ for all $\kappa \in \alpha \sim 2$, let $h02 \in \beta \sim \{h\kappa 2 : 3 \leq \kappa < \alpha\}$. If $1 \not{T} 2$ and $1 T \kappa$ for some $\kappa \in \alpha \sim \{1\}$, $h12$ is defined; if $1 \not{T} \kappa$ for all $\kappa \in \alpha \sim \{1\}$, let $h12 \in \beta \sim \{h\kappa 2 : \kappa \in \alpha \sim \{1\}, \kappa \not{T} 2\}$. Then it is easily checked that $\langle T, h\rangle \in c'_2\{\langle R, f\rangle\}$. Next, for $\kappa, \lambda \in \alpha \sim \{0\}$ and $\kappa \not{U} \lambda$, let $k\kappa\lambda = g\kappa\lambda$. If $\kappa U 0$ for some $\kappa \neq 0$, $k: \binom{U}{2} \to \beta$ is then completely defined.

Otherwise, let $k\kappa 0 = f\kappa 0$ for $3 \leq \kappa < \alpha$. If $2 \not U 0$ and $2 U \kappa$ for some $\kappa \in \alpha \sim 3$, $k02$ is defined; if $2 \not U \kappa$ for all $\kappa \in \alpha \sim \{1, 2\}$, let $k02 = h02$. If $1 \not U 0$ and $1 U \kappa$ for some $\kappa \in \alpha \sim \{1\}$, $k01$ is defined; if $1 \not U \kappa$ for all $\kappa \in \alpha \sim \{1\}$, let $k01 \in \beta \sim \{k0\kappa : 2 \leq \kappa < \alpha\}$. Then it is easily checked that $\langle U, k\rangle \in c_1'\{\langle T, h\rangle\}$ and $\langle S, g\rangle \in c_0'\{\langle U, k\rangle\}$. This completes the proof.

Our next object is to show that certain of the algebras $\mathfrak{B}^\alpha_\beta$ are nonrepresentable. By a *representation* of $\mathfrak{B}^\alpha_\beta$ we mean a pair $\langle F, U\rangle$ such that $F$ is an isomorphism from $\mathfrak{B}^\alpha_\beta$ into a $\boldsymbol{CS}_3$ with base $U$. By 1.4, $\mathfrak{B}^\alpha_\beta$ is representable iff it has a representation.

THEOREM 1.5. *Let $\omega > \beta \geq \alpha - 1$. Suppose $\langle F, U\rangle$ is a representation of $\mathfrak{B}^\alpha_\beta$, $\langle R, f\rangle \in V^\alpha_\beta$, $u$, $v$ are distinct elements of $U$, and $\langle u, v, v\rangle \in F\{\langle R, f\rangle\}$. Then $\langle v, u, u\rangle \in F\{\langle S, g\rangle\}$ for some $\langle S, g\rangle \in V^\alpha_\beta$ such that $0 \not S 1$ and $g01 = f01$.*

PROOF. First note that $\langle u, v, v\rangle \in \sim D_{01}$, so that $\langle R, f\rangle \notin d_{01}$ and hence $0 \not R 1$. We clearly have $\langle v, u, u\rangle \in F(d_{12} \cdot c_1(d_{01} \cdot c_0(d_{02} \cdot c_2\{\langle R, f\rangle\})))$; say $\langle v, u, u\rangle \in F\{\langle S, g\rangle\}$ with $\langle S, g\rangle \in d_{12} \cdot c_1(d_{01} \cdot c_0(d_{02} \cdot c_2\{\langle R, f\rangle\}))$. Clearly then $0 \not S 1$ and $g01 = f01$.

Now, still assuming that $\beta \geq \alpha - 1$, and $\langle F, U\rangle$ is a representation of $\mathfrak{B}^\alpha_\beta$, for each $\gamma < \beta$ let $T_\gamma = \{\{u, v\} : u, v \in U, u \neq v$ and $\langle u, v, v\rangle \in F\{\langle R, f\rangle\}$ for some $\langle R, f\rangle \in V^\alpha_\beta$ such that $0 \not R 1$ and $f01 = \gamma\}$. Let $\binom{U}{2}$ be the set of all $X \subseteq U$ with $|X| = 2$.

THEOREM 1.6. *Assume that $\omega > \beta \geq \alpha - 1$ and $\langle F, U\rangle$ is a representation of $\mathfrak{B}^\alpha_\beta$. Then $\langle T_\gamma : \gamma < \beta\rangle$ partitions $\binom{U}{2}$, and no $T_\gamma$ contains a triangle, i.e., for no $\gamma < \beta$ do there exist distinct elements $u, v, w \in U$ such that $\{u, v\}, \{v, w\}, \{w, u\} \in T_\gamma$.*

PROOF. Clearly $T_\gamma \cap T_\delta = 0$ for $\gamma < \delta < \beta$, using 1.5. If $u, v \in U$ and $u \neq v$, then $\langle u, v, v\rangle \in \sim D_{01} = F(-d_{01})$, so $\langle u, v, v\rangle \in F\{\langle R, f\rangle\}$ for some $\langle R, f\rangle \in -d_{01}$. Thus $0 \not R 1$, and $\{u, v\} \in T_\gamma$, where $\gamma = f01$. Thus $\langle T_\gamma : \gamma < \beta)$ partitions $\binom{U}{2}$. Now suppose $\gamma < \beta$ and $u, v, w$ are distinct elements of $U$ such that $\{u, v\}, \{v, w\}$, $\{w, u\} \in T_\gamma$. Using 1.5 we easily infer that $\langle u, v, v\rangle \in F\{\langle R, f\rangle\}$, $\langle v, w, w\rangle \in F\{\langle S, g\rangle\}$, and $\langle u, w, w\rangle \in F\{\langle T, h\rangle\}$ for certain $\langle R, f\rangle, \langle S, g\rangle, \langle T, h\rangle \in V^\alpha_\beta$ with $0 \not R 1$, $0 \not S 1$, $0 \not T 1$, $f01 = g01 = h01 = \gamma$. Note that since $\langle u, v, v\rangle \in D_{12} = Fd_{12}$, we have $\{\langle R, f\rangle\} \cap d_{12} \neq 0$, i.e., $\langle R, f\rangle \in d_{12}$ and so $1R2$. Similarly $1S2$, $1T2$. Now

$$\langle u, v, w\rangle \in F[c_2\{\langle R, f\rangle\} \cdot c_1\{\langle T, h\rangle\} \cdot c_0(d_{01} \cdot c_1\{\langle S, g\rangle\})];$$

say $\langle u, v, w\rangle \in F\{\langle N, t\rangle\}$ with $\langle N, t\rangle \in c_2\{\langle R, f\rangle\} \cdot c_1\{\langle T, h\rangle\} \cdot c_0(d_{01} \cdot c_1\{\langle S, g\rangle\})$; $\langle N, t\rangle \in c_0\{\langle M, s\rangle\}$ with $\langle M, s\rangle \in d_{01} \cdot c_1\{\langle S, g\rangle\}$. Then $0 \not N 1 \not N 2 \not N 0$, and $t01 = f01 = \gamma$; $t02 = h02 = h01 = \gamma$; $t12 = s12 = s02 = g02 = g01 = \gamma$. This contradicts $\langle N, t\rangle \in \beta'_\alpha$.

By a result of Greenwood and Gleason [1] we obtain from 1.6 the following corollary.

COROLLARY 1.7. *If $\omega > \beta \geq \alpha - 1$ and $\langle \mathscr{F}, \mathscr{U}\rangle$ is a representation of $\mathfrak{B}^\alpha_\beta$, then $|U| \leq 3 \cdot \beta!$, and so $\left|\binom{U}{2}\right| \leq 9 \cdot \beta!^2$.*

We now work in a different direction in order to get a lower bound for $\left|\binom{U}{2}\right|$.

THEOREM 1.8. *Assume that* $\alpha - 1 \leq \beta < \omega$, *and* $\langle F, U\rangle$ *is a representation of* $\mathfrak{B}^\alpha_\beta$. *Then*

$$\left|\binom{U}{2}\right| \geq (\beta - 2)^{2\alpha-6}.$$

PROOF. By 1.6 it suffices to get a lower bound for $|T_\gamma|$ for an arbitrary $\gamma < \beta$. Let $R = [\mathrm{Id} \cap {}^2(\alpha \sim \{1, 2\})] \cup \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$. Thus $R$ is an equivalence relation on $\alpha$. Let $M = \{\langle R, f\rangle : f\colon \binom{R}{2} \to \beta,\ f01 = \gamma,\ \{f0\kappa, f1\kappa\} \cap \{\kappa, \gamma\} = 0$ for $3 \leq \kappa < \alpha$, $f\kappa\lambda = \kappa$ for $3 \leq \kappa < \lambda < \alpha\}$. Clearly $M \subseteq V^\alpha_\beta$. For $\langle R, f\rangle \in M$ let $G\langle R, f\rangle = \{\{u, v\} \in \binom{U}{2} : \langle u, v, v\rangle \in F\{\langle R, f\rangle\}\}$. Clearly then $G$ maps $M$ into $ST_\gamma$. Furthermore, if $G\langle R, f\rangle \cap G\langle R, g\rangle \neq 0$, then $G\langle R, f\rangle = G\langle R, g\rangle$. To check this, it is enough to assume that $\langle u, v, v\rangle \in F\{\langle R, f\rangle\}$ and $\langle v, u, u\rangle \in F\{\langle R, g\rangle\}$ and prove that then $\langle s, t, t\rangle \in F\{\langle R, f\rangle\}$ iff $\langle t, s, s\rangle \in F\{\langle R, g\rangle\}$. The assumption gives easily

$$\langle u, v, v\rangle \in F\{\langle R, f\rangle\} \cap F[d_{12} \cap c_2(d_{02} \cap c_0(d_{01} \cap c_1\{\langle R, g\rangle\}))];$$

hence $\{\langle R, f\rangle\} = d_{12} \cap c_2(d_{02} \cap c_0(d_{01} \cap c_1\{\langle R, g\rangle\}))$, and the desired conclusion easily follows. Further, for any $\langle R, f\rangle \in M$ there are at most two $\langle S, g\rangle \in M$ with $G\langle R, f\rangle = G\langle S, g\rangle$. Hence

$$|T_\gamma| \geq \tfrac{1}{2}|M| \geq \tfrac{1}{2}(\beta - 2)^{\alpha-3}\cdot(\beta - 2)^{\alpha-3};$$

therefore

$$\left|\binom{U}{2}\right| \geq \beta\cdot\tfrac{1}{2}(\beta - 2)^{2\alpha-6} \geq (\beta - 2)^{2\alpha-6}.$$

The estimate of 1.8 is clearly very crude, but it suffices to give the following corollary which is basic for what follows.

COROLLARY 1.9. *For every* $\kappa \in \omega$ *there is an* $\alpha$, $3 \leq \alpha < \omega$, *such that* $\mathfrak{C}^\alpha_{\alpha+\kappa}$ *is a nonrepresentable* $CA_3$.

PROOF. By 1.3 it suffices to find $\alpha$, $3 \leq \alpha < \omega$, such that $\mathfrak{B}^\alpha_{\alpha+\kappa}$ is nonrepresentable. Let $\alpha = 10\cdot(\kappa + 4)!^2$. Then $9\cdot(\kappa + 4)!^2 \leq \alpha + \kappa - 2$, $(\kappa + 5)\cdot(\kappa + \alpha) \leq (\alpha + \kappa - 2)^2$, $(\kappa + 6)\cdot(\kappa + \alpha - 1) \leq (\alpha + \kappa - 2)^2$. If $\langle F, U\rangle$ is a representation of $\mathfrak{B}^\alpha_{\alpha+\kappa}$ we calculate as follows, using 1.7 and 1.8:

$$\begin{aligned}\left|\binom{U}{2}\right| &< 9\cdot(\alpha + \kappa)!^2 \\ &= 9\cdot(\kappa + 4)!^2\cdot[(\kappa + 5)(\kappa + 6)\cdots(\kappa + \alpha)]^2 \\ &\leq (\alpha + \kappa - 2)\cdot[(\alpha + \kappa - 2)^{\alpha+\kappa-\kappa-4}]^2 \\ &= (\alpha + \kappa - 2)^{2\alpha-7} \leq \left|\binom{U}{2}\right|,\end{aligned}$$

a contradiction.

Finally, we need a simple consequence of 1.2.

THEOREM 1.10. *If* $3 \leq \alpha < \omega$ *and* $\kappa < \omega$, *then* $\mathfrak{C}^\alpha_{\alpha+\kappa}$ *can be neatly embedded in a* $CA_{3+\kappa+1}$.

PROOF. By 1.2, there is a neat embedding $F$ of $\mathfrak{A}^\alpha_{\alpha+\kappa}$ into $\mathfrak{A}^{\alpha+\kappa+1}_{\alpha+\kappa}$. Now let $\delta\colon 3 + \kappa + 1 \to \alpha + \kappa + 1$ be defined by: $\delta\lambda = \lambda$ for $\lambda < 3$, $\delta\lambda = \alpha + \lambda - 3$

for $3 \leq \lambda < 3 + \kappa + 1$. Let $\mathfrak{B}$ be the $3 + \kappa + 1$, $\delta$-reduct of $\mathfrak{A}_{\alpha+\kappa}^{\alpha+\kappa+1}$. Clearly $F$ is a neat embedding of $\mathfrak{C}_{\alpha+\kappa}^{\alpha}$ into $\mathfrak{B}$.

Now we can give the first main result of this section.

THEOREM 1.11. *If $3 \leq \alpha < \omega$ and $\kappa < \omega$, then there is a nonrepresentable $\boldsymbol{CA}_\alpha$ which can be neatly embedded in a $\boldsymbol{CA}_{\alpha+\kappa}$.*

PROOF. We may assume that $\kappa \neq 0$. By 1.9 choose $\beta$ such that $3 \leq \beta < \omega$ and $\mathfrak{C}_{\beta+\alpha+\kappa-4}^{\beta}$ is a nonrepresentable $\boldsymbol{CA}_3$. By 1.10, $\mathfrak{C}_{\beta+\alpha+\kappa-4}^{\beta}$ can be neatly embedded in a $\boldsymbol{CA}_{\alpha+\kappa}$ $\mathfrak{B}$. We assume that $\mathfrak{C}_{\beta+\alpha+\kappa-4}^{\beta}$ is actually a subalgebra of the 3-reduct of $\mathfrak{B}$, with $c_\lambda x = x$ for $x \in C_{\beta+\alpha+\kappa-4}^{\beta}$ and $3 \leq \lambda < \alpha + \kappa$. Let $D = \{x \in B : c_\lambda x = x$ for all $\lambda$ with $\alpha \leq \lambda < \alpha + \kappa\}$, and let $\mathfrak{C}$ be the $\alpha$-reduct of $\mathfrak{B}$. $D$ is closed under all of the operations of $\mathfrak{C}$, and $d_{\lambda\mu} \in D$ for all $\lambda, \mu < \alpha$, so $D$ is the universe of some subalgebra $\mathfrak{D}$ of $\mathfrak{C}$. Clearly $\mathfrak{C}_{\beta+\alpha+\kappa-4}^{\beta}$ is neatly embedded in $\mathfrak{D}$, so by Theorem 2.12 of Henkin, Tarski [5], $\mathfrak{D}$ is nonrepresentable. $\mathfrak{D}$ is neatly embedded in $\mathfrak{B}$, so the proof is complete.

Theorem 1.11 furnishes a partial answer to the question raised on page 102 of Henkin, Tarski [5]. See §4 for some open problems in this connection.

The second main result of this section is the following theorem mentioned in the introduction.

THEOREM 1.12. *For $3 \leq \alpha < \omega$, $\boldsymbol{RCA}_\alpha$ is not finitely axiomatizable.*

PROOF. For each $\kappa < \omega$ let $\mathfrak{A}_\kappa$ be a nonrepresentable $\boldsymbol{CA}_\alpha$ neatly embedded in a $\boldsymbol{CA}_{\alpha+\kappa}$ $\mathfrak{B}_\kappa$; Theorem 1.11 assures the existence of such objects. Let $F$ be a nonprincipal ultrafilter over $\omega$. Then $\mathrm{P}_{\kappa<\omega}B_\kappa/F$ can easily be given the structure of a $\boldsymbol{CA}_{\alpha+\omega}$ in which $\mathrm{P}_{\kappa<\omega}\mathfrak{A}_\kappa/F$ is neatly embedded. By Theorem 2.15 of Henkin, Tarski [5], $\mathrm{P}_{\kappa<\omega}\mathfrak{A}_\kappa/F$ is representable. The basic result on ultraproducts now implies that $\boldsymbol{RCA}_\alpha$ is not finitely axiomatizable.

## §2. Nonfinitizability, infinite-dimensional case.

We begin with an infinite dimensional analog of 1.11:

THEOREM 2.1. *If $\omega \leq \alpha$ and $\kappa \in \omega$, then there is a nonrepresentable $\boldsymbol{CA}_\alpha$ which can be neatly embedded in a $\boldsymbol{CA}_{\alpha+\kappa}$.*

PROOF. By 1.9 there is a $\beta$, $3 \leq \beta < \omega$, such that $\mathfrak{A}_{\beta+\kappa-1}^{\beta}$ is nonrepresentable. Let

$$I = \{\Gamma : \beta \subseteq \Gamma \subseteq \alpha, |\Gamma| < \omega\}.$$

For each $\Gamma \in I$ let $\lambda_\Gamma < \omega$ and $t_\Gamma$ be such that $|\Gamma| = \beta + \lambda_\Gamma$ and $t_\Gamma$ maps $\beta + \lambda_\Gamma$ one-one onto $\Gamma$ with $t_\Gamma\mu = \mu$ for all $\mu < \beta$. Let $B_\Gamma$ be the following algebraic structure constructed from $\mathfrak{A}_{\beta+\lambda_\Gamma+\kappa-1}^{\beta+\lambda_\Gamma}$:

$$\mathfrak{B}_\Gamma = \langle \mathfrak{A}_{\beta+\lambda_\Gamma+\kappa-1}^{\beta+\lambda_\Gamma}, +, \cdot, -, c_{t_\Gamma^{-1}\mu}, d_{t_\Gamma^{-1}\mu, t_\Gamma^{-1}\nu} \rangle_{\mu,\nu\in\Gamma}.$$

$\mathfrak{B}_\Gamma$ is not a cylindric algebra, but clearly we may apply to $\mathfrak{B}_\Gamma$ all the results about $\boldsymbol{CA}_\alpha$'s with appropriate modifications. Thus we may think of $\mathfrak{B}_\Gamma$ as a "$\boldsymbol{CA}_\Gamma$". For each $\Gamma \in I$ let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$, and let $F$ be an ultrafilter on $I$ such that $M_\Gamma \in F$ for all $\Gamma \in I$. Let $C = \mathrm{P}_{\Gamma\in I}B_\Gamma/F$; we can give $C$ a natural structure as a $\boldsymbol{CA}_\alpha$, denoted by $\mathfrak{C}$. Clearly $\mathfrak{C}$ can be neatly embedded in a $\boldsymbol{CA}_{\alpha+\kappa}$, so it suffices to show that $\mathfrak{C}$ is nonrepresentable. Let $\mathfrak{D}$ be the $\beta$-reduct of $\mathfrak{C}$; it is enough to show that $\mathfrak{D}$ is nonrepresentable.

We now define $l \in \mathbf{P}_{\Gamma \in I} B_\Gamma$. For $\Gamma \in I$, let

$$l_\Gamma = \{\langle R, f\rangle \in (\beta + \lambda_\Gamma + \kappa - 1)'_{\beta+\lambda_\Gamma} : R = (R \cap {}^2\beta) \cup [\mathrm{Id} \cap {}^2(\beta + \lambda_\Gamma) \sim \beta)], \langle R, f\rangle \upharpoonright \beta \in (\beta + \kappa - 1)'_\beta, \text{ and for all } \mu < \lambda_\Gamma \text{ and } \nu < \beta + \mu, f(\nu, \beta + \mu) = \beta + \kappa + \mu - 1\}.$$

Let $x = l/F$, and let $\mathfrak{E} = \mathfrak{Rl}_x\mathfrak{D}$. Now

(1) $\quad c_\xi x \cdot c_\eta x = x \leq c_\xi(d_{\xi\eta} \cdot x)$ whenever $\xi, \eta < \beta$ and $\xi \neq \eta$.

Indeed, suppose $\Gamma \in I$ and $\langle R, f\rangle \in c_\xi l_\Gamma \cap c_\eta l_\Gamma$, say $\langle R, f\rangle \in c_\xi\{\langle S, g\rangle\} \cap c_\eta\{\langle T, h\rangle\}$ with $\langle S, g\rangle, \langle T, h\rangle \in l_\Gamma$. Clearly $R = (R \cap {}^2\beta) \cup [\mathrm{Id} \cap {}^2((\beta + \lambda_\Gamma) \sim \beta)]$. Let $\mu < \lambda_\Gamma$ and $\nu < \beta + \mu$. If $\nu \neq \xi$, then $f(\nu, \beta + \mu) = g(\nu, \beta + \mu) = \beta + \kappa + \mu - 1$. Further, if $\nu = \xi$ then $f(\nu, \beta + \mu) = h(\nu, \beta + \mu) = \beta + \kappa + \mu - 1$. To finish showing that $\langle R, f\rangle \in l_\Gamma$ it is enough to show that $f\xi\eta \in \beta + \kappa - 1$ in case $\xi R \eta$. Suppose this is not the case; say $f\xi\eta = \beta + \kappa + \mu - 1$, with $\mu < \lambda_\Gamma$. Then $f\xi\eta = f(\xi, \beta + \mu) = f(\eta, \beta + \mu)$, a contradiction. We have shown that $c_\xi l_\Gamma \cap c_\eta l_\Gamma \subseteq l_\Gamma$. It follows at once that $c_\xi x \cdot c_\eta x = x$. It is easily seen that $x \leq c_\xi(d_{\xi\eta} \cdot x)$.

From (1) and Theorem 0.3 we know that $\mathfrak{E}$ is a $CA_\beta$ and is representable if $\mathfrak{D}$ is. We shall now show that $\mathfrak{E}$ is nonrepresentable, and this will complete the proof. The nonrepresentability of $\mathfrak{E}$ will be established by defining an isomorphism from $\mathfrak{A}^\beta_{\beta+\kappa-1}$ into $\mathfrak{E}$. First we define $H: A^\beta_{\beta+\kappa-1} \to \mathbf{P}_{\Gamma \in I} B_\Gamma$. For $\langle R, f\rangle \in (\beta + \kappa - 1)'_\beta$ and $\Gamma \in I$ let $(H\{\langle R, f\rangle\})_\Gamma = \{\langle S, g\rangle\}$, where $S = R \cup [\mathrm{Id} \cap {}^2((\beta + \lambda_\Gamma) \sim \beta)]$, $f \subseteq g$, and for all $\mu < \lambda_\Gamma$ and $\nu < \beta + \mu$, $g(\nu, \beta + \mu) = \beta + \kappa + \mu - 1$. For $X \in A^\beta_{\beta+\kappa-1}$ let $(HX)_\Gamma = \bigcup_{\langle R, f\rangle \in X}(H\{\langle R, f\rangle\})_\Gamma$, and let $GX = HX/F$. Clearly $G0 = 0$, $G$ preserves $+$, and $G$ maps into $E$. Since $\mathfrak{C}$ is atomic (by the basic result on ultraproducts), $\mathfrak{E}$ is also atomic. Further,

(2) $\quad$ if $y$ is an atom of $\mathfrak{E}$, then $y = G\{\langle R, f\rangle\}$ for some $\langle R, f\rangle \in (\beta + \kappa - 1)'_\beta$.

Indeed, we may assume that $y = t/F$, where $t \leq l$ and $t_\Gamma$ is an atom of $\mathfrak{B}_\Gamma$ for each $\Gamma \in I$, say $t_\Gamma = \{\langle S_\Gamma, g_\Gamma\rangle\}$. For each $\langle R, f\rangle \in (\beta + \kappa - 1)'_\beta$ let

$$L_{Rf} = \{\Gamma \in I : \langle S_\Gamma, g_\Gamma\rangle \upharpoonright \beta = \langle R, f\rangle\}.$$

Then $I = \bigcup_{\langle R, f\rangle \in (\beta+\kappa-1)'_\beta} L_{Rf}$, so $L_{Rf} \in F$ for a certain $\langle R, f\rangle \in (\beta + \kappa - 1)'_\beta$. Clearly then $G\{\langle R, f\rangle\} = y$, so (2) holds.

It follows from (2) that $G$ maps onto $E$, and $G1 = 1$. If $\langle R, f\rangle, \langle S, g\rangle \in (\beta + \kappa - 1)'_\beta$ and $\langle R, f\rangle \neq \langle S, g\rangle$, clearly $G\{\langle R, f\rangle\} \neq G\{\langle S, g\rangle\}$. Hence we easily infer that $G$ preserves $-$. Since for $\langle R, f\rangle \in (\beta + \kappa - 1)'_\beta$ and $\Gamma \in I$ we have $(H\{\langle R, f\rangle\})_\Gamma \neq 0$, clearly $G$ is one-one. Using (2), one easily checks that $G$ preserves $c_\kappa$ and $d_{\kappa\lambda}$ for all $\kappa, \lambda < \beta$. This completes the proof.

For any $\alpha$, let $\mathscr{L}_\alpha$ be the elementary language for $CA_\alpha$'s. Symbols of $\mathscr{L}_\alpha$ will be indicated in bold-face type. For $\gamma$ a permutation of $\alpha$ we define $\gamma^+ : \mathit{Terms} \to \mathit{Terms}$ as follows: $\gamma^+\mathbf{v} = \mathbf{v}_\lambda$ for any variable $\mathbf{v}_\lambda$; for terms $\sigma$, $\tau$ and $\kappa, \lambda < \alpha$, $\gamma^+(\sigma \mathbin{\mathbf{+}} \tau) = \gamma^+\sigma \mathbin{\mathbf{+}} \gamma^+\tau$, $\gamma^+(\sigma \boldsymbol{\cdot} \tau) = \gamma^+\sigma \boldsymbol{\cdot} \gamma^+\tau$, $\gamma^+(\boldsymbol{-}\sigma) = \boldsymbol{-}\gamma^+\sigma$, $\gamma^+\mathbf{c}_\kappa\sigma = \mathbf{c}_{\gamma\kappa}\gamma^+\sigma$, $\gamma^+\mathbf{d}_{\kappa\lambda} = \mathbf{d}_{\gamma\kappa,\gamma\lambda}$. A class $\boldsymbol{K} \subseteq \boldsymbol{CA}_\alpha$ is *finite schema axiomatizable* if there exist finitely many equations $\sigma_0 \mathbin{\boldsymbol{=}} \tau_0, \cdots, \sigma_{\lambda-1} \mathbin{\boldsymbol{=}} \tau_{\lambda-1}$ such that $\boldsymbol{K}$ is characterized by $\{\gamma^+\sigma_\mu \mathbin{\boldsymbol{=}} \gamma^+\tau_\mu : \mu < \lambda, \gamma \text{ a permutation of } \alpha\}$.

It is easily seen that $\boldsymbol{CA}_\alpha$ and $\boldsymbol{RCA}_\alpha$ are not finitely axiomatizable for $\alpha \geq \omega$.

Clearly $\boldsymbol{CA}_\alpha$ is finite schema axiomatizable for any $\alpha$. This property does not extend to $\boldsymbol{RCA}_\alpha$:

THEOREM 2.2. *For $3 \leq \alpha$, $\boldsymbol{RCA}_\alpha$ is not finite schema axiomatizable.*

PROOF. The case $\alpha < \omega$ is given by Theorem 1.12. Now assume that $\omega \leq \alpha$. Suppose $\boldsymbol{RCA}_\alpha$ is finite schema axiomatizable. Say $\boldsymbol{RCA}_\alpha$ is characterized by $\{\gamma^+\sigma_\mu \boldsymbol{=} \gamma^+\tau_\mu : \mu < \lambda, \gamma \text{ a permutation of } \alpha\}$. By 2.1, for each $\kappa < \omega$ let $\mathfrak{A}_\kappa$ be a nonrepresentable $\boldsymbol{CA}_\alpha$ neatly embedded in a $\boldsymbol{CA}_{\alpha+\kappa}$ $\mathfrak{B}_\kappa$. For each $\kappa < \omega$ let $\gamma_\kappa$ be a permutation of $\alpha$ such that

$$\gamma_\kappa^+\sigma_0 \boldsymbol{=} \gamma_\kappa^+\tau_0 \wedge \cdots \wedge \gamma_\kappa^+\sigma_{\lambda-1} \boldsymbol{=} \gamma_\kappa^+\tau_{\lambda-1}$$

does not hold identically in $\mathfrak{A}_\kappa$. Let $\delta_\kappa = \gamma_\kappa \cup (\mathrm{Id} \cap {}^2[(\alpha + \kappa) \sim \alpha])$. Let $\mathfrak{A}'_\kappa$ be the $\alpha$, $\gamma_\kappa$-reduct of $\mathfrak{A}_\kappa$, and $\mathfrak{B}'_\kappa$ the $\alpha + \kappa$, $\delta_\kappa$-reduct of $\mathfrak{B}_\kappa$. Then the formula

$$\varphi = \sigma_0 \boldsymbol{=} \tau_0 \wedge \cdots \wedge \sigma_{\lambda-1} \boldsymbol{=} \tau_{\lambda-1}$$

does not hold identically in $\mathfrak{A}'_\kappa$. Let $F$ be a nonprincipal ultrafilter on $\omega$. Then $\varphi$ does not hold identically in $\mathbf{P}_{\kappa\in\omega}\mathfrak{A}_\kappa/F$, and $\mathbf{P}_{\kappa\in\omega}\mathfrak{A}_\kappa/F$ is neatly embedded in a $\boldsymbol{CA}_{\alpha+\omega}$ naturally built upon $\mathbf{P}_{\kappa\in\omega}\mathfrak{B}_\kappa/F$. Hence $\mathbf{P}_{\kappa\in\omega}\mathfrak{A}_\kappa/F$ is representable, and this is a contradiction.

**§3. A characterization of $\boldsymbol{RCA}_\alpha$.** In unpublished work, Ralph McKenzie has given a simple characterization of involuted semigroups $\langle A, \cdot, {}^\cup\rangle$ isomorphic to algebras $\langle A, |, {}^{-1}\rangle$ of binary relations. In this section we will describe explicitly equations characterizing $\boldsymbol{RCA}_\alpha$, by an easy modification of his method.

Suppose that $3 \leq \alpha$ and $\Gamma$ is a finite subset of $\alpha$, say with $|\Gamma| = \kappa$. We define a sequence $\lambda^\Gamma$ of natural numbers by recursion: $\lambda_0^\Gamma = 1$, and for $\mu < \omega$, $\lambda_{\mu+1}^\Gamma = \lambda_\mu^\Gamma + (\mu + 1)^2 \cdot \kappa \cdot (\lambda_\mu^\Gamma)^\kappa$. For $\mu < \omega$ and $\xi \in \Gamma$ let $C_\xi$ be the natural $\xi$-cylindrification on $\boldsymbol{S}({}^\alpha\lambda_\mu^\Gamma)$, and let $D_{\xi\eta}$ for $\xi, \eta \in \Gamma$ be the natural $\xi\eta$-diagonal on $\boldsymbol{S}({}^\alpha\lambda_\mu^\Gamma)$. $\boldsymbol{c}_{(\Gamma)}$ stands for $\boldsymbol{c}_{\nu_0} \cdots \boldsymbol{c}_{\nu_{\xi-1}}$, where $\Gamma = \{\nu_0, \cdots, \nu_{\xi-1}\}$ and $\nu_0 < \cdots < \nu_{\xi-1}$; $\sigma \oplus \tau$ is an abbreviation for $\sigma \cdot \boldsymbol{-}\tau \boldsymbol{+} \tau \cdot \boldsymbol{-}\sigma$. Now for $\mu < \omega$ and $\psi \in {}^{\mu+1}\boldsymbol{S}({}^\alpha\lambda_\mu^\Gamma)$, let $\sigma_\psi^\Gamma$ be the formal sum of the following terms:

$\mathbf{0}$;

$\boldsymbol{-c}_{(\Gamma)}[\boldsymbol{v}_\nu \oplus (\boldsymbol{-v}_\rho)]$ if $\nu < \rho \leq \mu$ and $\psi_\nu \neq {}^\alpha\lambda_\mu^\Gamma \sim \psi_\rho$;

$\boldsymbol{-c}_{(\Gamma)}[(\boldsymbol{v}_\nu \boldsymbol{+} \boldsymbol{v}_\rho) \oplus \boldsymbol{v}_\tau]$ if $\nu, \rho, \tau \leq \mu$ and $\psi_\tau \neq \psi_\nu \cup \psi_\rho$;

$\boldsymbol{-c}_{(\Gamma)}(\boldsymbol{c}_\xi \boldsymbol{v}_\nu \oplus \boldsymbol{v}_\rho)$ if $\xi \in \Gamma$, $\nu, \rho \leq \mu$, and there is a $\tau \leq \mu$ such that $\nu, \rho < \tau$ and $\psi_\rho \cap {}^\alpha\lambda_{\tau-1}^\Gamma \neq C_\xi(\psi_\nu \cap {}^\alpha\lambda_\tau^\Gamma) \cap {}^\alpha\lambda_{\tau-1}^\Gamma$;

$\boldsymbol{-c}_{(\Gamma)}(\boldsymbol{v}_\nu \oplus \boldsymbol{d}_{\xi\eta})$ if $\xi, \eta \in \Gamma$, $\nu \leq \mu$ and $D_{\xi\eta} \nsubseteq \psi_\nu$;

$\boldsymbol{-c}_{(\Gamma)}(\boldsymbol{v}_\nu \oplus \boldsymbol{d}_{01})$ if $0, 1 \in \Gamma$, $\nu \leq \mu$, and there are $\rho \leq \mu$, $x \in \psi_\rho$, $y \in \psi_\nu$, and $\xi \in \Gamma$ such that $y_0 = x_\xi$ and $(x \sim \{\langle \xi, x_\xi \rangle\}) \cup \{\langle \xi, y_1 \rangle\} \notin \psi_\rho$.

Next, for each $\mu < \omega$ let $\varphi_\mu^\Gamma$ be the following equation:

$$\prod_{\psi \in {}^{\mu+1}\boldsymbol{S}({}^\alpha\lambda_\mu^\Gamma)} \sigma_\psi^\Gamma = 0.$$

Note that $\varphi_\mu^\Gamma$ involves only the variables $v_0, \cdots, v_\mu$. For each $\alpha$, let $\Delta_\alpha$ be the set of axioms characterizing $CA_\alpha$.

THEOREM 3.1. *For $3 \leq \alpha < \omega$, $\boldsymbol{RCA}_\alpha$ is characterized by $\{\varphi_\mu^\alpha : \mu \in \omega\} \cup \Delta_\alpha$.*

PROOF. We omit superscripts $^\alpha$ in this proof. First we show that each $\varphi_\mu$ holds in each $RCA_\alpha$; it suffices to check that $\varphi_\mu$ holds in an arbitrary $\boldsymbol{CS}_\alpha\, \mathfrak{A}$, say with nonempty base $U$. Let $a_0, \cdots, a_\mu \in A$. First we construct a function $x \in {}^{\lambda_\mu}U$; $x$ is constructed from some partial functions $y_0, \cdots, y_\mu$. Let $y_0$ assign any element of $U$ to 0. Suppose $\nu < \mu$ and $y_\nu \in {}^{\lambda_\nu}U$ has been defined such that

(1) $$\begin{aligned}&\text{if } \rho, \gamma < \tau \leq \nu,\ \xi \in \alpha, \text{ and } C_\xi a_\rho = a_\gamma,\\ &\text{then } C_\xi(a_\rho \cap {}^\alpha y_\nu^* \lambda_\tau) \cap {}^\alpha y_\nu^* \lambda_{\tau-1} = a_\gamma \cap {}^\alpha y_\nu^* \lambda_{\tau-1}.\end{aligned}$$

Then we can pick a nonempty subset $S$ of $U$ with at most $(\nu + 1)^2 \cdot \alpha \cdot \lambda_\nu^\alpha$ elements such that if $\rho, \gamma \leq \nu$, $\xi \in \alpha$, $C_\xi a_\rho = a_\gamma$, and $x \in (a_\gamma \cap {}^\alpha y_\nu^* \lambda_\nu) \sim C_\xi(a_\rho \cap {}^\alpha y_\nu^* \lambda_\nu)$, then there is a $u \in S$ such that $(x \sim \{\langle \xi, x_\xi \rangle\}) \cup \{\langle \xi, u \rangle\} \in a_\rho$. Let $y_{\nu+1}$ be $y_\nu$ together with a map from $\lambda_{\nu+1} \sim \lambda_\nu$ onto $S$. Clearly then (1) holds with $\nu$ replaced by $\nu + 1$. Now let $x = y_\mu$, and

(2) $$\psi = \langle \{t \in {}^\alpha\lambda_\mu : x \circ t \in a_\nu\} : \nu \leq \mu \rangle.$$

Thus $\psi \in {}^{\mu+1}\boldsymbol{S}({}^\alpha\lambda_\mu)$. This direction in the proof will be finished when we show that $\sigma_\psi(a_0, \cdots, a_\mu) = 0$. To show this we look at the five kinds of terms in $\sigma_\psi$ that are not formally $\mathbf{0}$.

*Case* 1. Suppose $\nu < \rho \leq \mu$ and $\psi_\nu \neq {}^\alpha\lambda_\mu \sim \psi_\rho$. First suppose that $t \in \psi_\nu \cap \psi_\rho$. Then by (2), $x \circ t \in a_\nu \cap a_\rho$, so $a_\nu \neq -a_\rho$ and $-c_{(\alpha)}[a_\nu \oplus (-a_\rho)] = 0$. Second, suppose that $t \in {}^\alpha\lambda_\mu \sim (\psi_\nu \cup \psi_\rho)$. Then $x \circ t \in {}^\alpha U \sim (a_\nu \cup a_\rho)$, so again $a_\nu \neq -a_\rho$.

*Case* 2. $\nu, \rho, \tau \leq \mu$ and $\psi_\tau \neq \psi_\nu \cup \psi_\rho$. This case is treated similarly.

*Case* 3. $\xi \in \alpha$, $\nu, \rho < \tau \leq \mu$, and $\psi_\rho \cap {}^\alpha\lambda_{\tau-1} \neq C_\xi(\psi_\nu \cap {}^\alpha\lambda_\tau) \cap {}^\alpha\lambda_{\tau-1}$. Suppose that also $C_\xi a_\nu = a_\rho$. If $t \in \psi_\rho \cap {}^\alpha\lambda_{\tau-1}$, then $x \circ t \in a_\rho \cap {}^\alpha x^* \lambda_{\tau-1}$, and so by (1) $x \circ t \in C_\xi(a_\nu \cap {}^\alpha x^* \lambda_\tau)$, i.e., $t \in C_\xi(\psi_\nu \cap {}^\alpha\lambda_\tau)$. Conversely, $t \in C_\xi(\psi_\nu \cap {}^\alpha\lambda_\tau) \cap {}^\alpha\lambda_{\tau-1}$ easily implies that $t \in \psi_\rho \cap {}^\alpha\lambda_{\tau-1}$, a contradiction.

*Case* 4. $\xi, \eta < \alpha$, $\nu \leq \mu$, and $D_{\xi\eta} \cap {}^\alpha\lambda_\mu \nsubseteq \psi_\nu$. If $a_\nu = D_{\xi\eta}$ we easily arrive at a contradiction.

*Case* 5. $\rho, \nu \leq \mu$, $s \in \psi_\rho$, $t \in \psi_\nu$, $\xi < \alpha$, $t_0 = s_\xi$, and $w = (s \sim \{\langle \xi, s_\xi \rangle\}) \cup \{\langle \xi, t_1 \rangle\} \notin \psi_\rho$. Suppose $a_\nu = D_{01}$. Since $t \in \psi_\nu$, $x \circ t \in a_\nu = D_{01}$, so $xt_0 = xt_1$. Now $xw_\xi = xt_1 = xt_0 = xs_\xi$, so $x \circ w = x \circ s$. But $x \circ s \in a_\rho$ (since $s \in \psi_\rho$), and $x \circ w \notin a_\rho$ (since $w \notin \psi_\rho$), a contradiction.

Now we show that if $\mathfrak{A}$ is a $CA_\alpha$ satisfying $\{\varphi_\mu : \mu \in \omega\}$, then $\mathfrak{A}$ is representable. We may assume that $\mathfrak{A}$ is countable and simple, by Theorems 2.5 and 2.13 of Henkin, Tarski [5]. Let $\langle a_0, a_1, \cdots \rangle$ be an enumeration of elements of $A$ in which every element of $A$ occurs infinitely many times. Then

for every $\mu \in \omega$ there exists a $\psi \in {}^{\mu+1}\boldsymbol{S}({}^\alpha\lambda_\mu)$ such that

(a) if $\nu < \rho \leq \mu$ and $a_\nu = -a_\rho$, then $\psi_\nu = {}^\alpha\lambda_\mu \sim \psi_\rho$;

(b) if $\nu, \rho, \tau \leq \mu$ and $a_\tau = a_\nu + a_\rho$, then $\psi_\tau = \psi_\nu \cup \psi_\rho$;

(3) (c) if $\xi < \alpha, \nu, \rho < \tau \leq \mu$, and $c_\xi a_\nu = a_\rho$, then $\psi_\rho \cap {}^\alpha\lambda_{\tau-1} = C_\xi(\psi_\nu \cap {}^\alpha\lambda_\tau) \cap {}^\alpha\lambda_{\tau-1}$;

(d) if $\xi, \eta < \alpha$, $\nu \leq \mu$, and $a_\nu = d_{\xi\eta}$, then $D_{\xi\eta} \cap {}^\alpha\lambda_\mu \subseteq \psi_\nu$;

(e) if $\rho, \nu \leq \mu$, $x \in \psi_\rho$, $y \in \psi_\nu$, $\xi < \alpha$, and $y_0 = x_\xi$, then $(x \sim \{\langle \xi, x_\xi \rangle\}) \cup \{\langle \xi, y_1 \rangle\} \in \psi_\rho$.

This follows since $\mathfrak{A}$ satisfies $\varphi_\mu$, and hence $\sigma_\psi = 0$ in $\mathfrak{A}$ for some $\psi \in {}^{\mu+1}\boldsymbol{S}({}^\alpha\lambda_\mu)$. If $\mu' \leq \mu$ and $\psi$ satisfies (3), then $\langle \psi_\nu \cap {}^\alpha\lambda_{\mu'} : \nu \leq \mu' \rangle$ satisfies (3) with $\mu$ replaced by

$\mu'$. Hence by König's lemma there is a $\psi \in {}^{\omega}S({}^{\alpha}\omega)$ such that the following conditions hold:

(4) $\qquad$ if $\nu < \rho < \omega$ and $a_\nu = -a_\rho$, then $\psi_\nu = {}^{\alpha}\omega \sim \psi_\rho$;

(5) $\qquad$ if $\nu, \rho, \tau < \omega$ and $a_\tau = a_\nu + a_\rho$, then $\psi_\tau = \psi_\nu + \psi_\rho$;

(6) $\qquad$ if $\xi < \alpha$, $\nu, \rho < \omega$, and $c_\xi a_\nu = a_\rho$, then $\psi_\rho = C_\xi \psi_\nu$;

(7) $\qquad$ if $\xi, \eta < \alpha$, $\nu < \omega$, and $a_\nu = d_{\xi\eta}$, then $D_{\xi\eta} \subseteq \psi_\nu$;

(8) if $\nu, \rho < \omega$, $a_\nu = d_{01}$, $x \in \psi_\rho$, $y \in \psi_\nu$, $\xi < \alpha$, and $y_0 = x_\xi$, then $(x \sim \{\langle \xi, x_\xi \rangle\}) \cup \{\langle \xi, y_1 \rangle\} \in \psi_\rho$.

For any $a \in A$ let

(9) $\qquad \varphi a = \{x \in {}^{\alpha}\omega$: there is a $\mu < \omega$ such that $a_\mu = a$ and $x \in \psi_\mu\}$.

It is easily checked that $\varphi$ preserves the Boolean operations and $c_\xi$ for $\xi < \alpha$. Hence $\mathfrak{A}$ simple implies that $\varphi$ is one-one. Further, (7) and (8) yield

(10) $\qquad D_{\xi\eta} \subseteq \varphi d_{\xi\eta}$ for all $\xi, \eta < \alpha$;

(11) if $a \in A$, $x \in \varphi a$, $z \in \varphi d_{01}$, $\xi < \alpha$, and $z_0 = x_\xi$, then $(x \sim \{\langle \xi, x_\xi \rangle\}) \cup \{\langle \xi, z \rangle\} \in \varphi_a$.

From (10) and (11) it follows that $\mathfrak{A}$ is representable, by Halmos [2] and Johnson [6]. This completes the proof.

From Theorem 2.13 of Henkin, Tarski [5] we obtain

THEOREM 3.2. *For $\omega \leq \alpha$, $RCA_\alpha$ is characterized by* $\{\varphi_\mu^\Gamma : \mu \in \omega, \Gamma \subseteq \alpha, |\Gamma| < \omega\} \cup \Delta_\alpha$.

§4. **Open problems.** Several questions, some vague, some definite, remain open concerning the notions discussed in this paper. First of all, there is the question of giving "nicer" characterizations of $RCA_\alpha$ than the one in §3. In this connection, it would be interesting to have a better idea about when an equation holds in all $RCA_\alpha$'s but fails in some $CA_\alpha$. Several questions arise concerning neat embeddings. By 1.11 and 2.1, for all $\alpha \geq 3$ and $\kappa < \omega$ there is a $\lambda < \omega$ and a $CA_\alpha$ $\mathfrak{A}$ such that $\mathfrak{A}$ can be neatly embedded in a $CA_{\alpha+\kappa}$ but not in a $CA_{\alpha+\lambda}$; let us write $\langle \alpha, \kappa, \lambda \rangle \in R$ if there is such an algebra $\mathfrak{A}$. A characterization of $R$ in terms of $\alpha$ and $\omega$ is not known. It is reasonable to conjecture that $\langle \alpha, \kappa, \kappa + 1 \rangle \in R$ for every $\alpha \geq 3$ and $\kappa < \omega$, and this would characterize $R$. Again, for $\alpha \geq 3$ and $\kappa < \omega$ let $K_\kappa^\alpha$ be the class of all $CA_\alpha$'s which can be neatly embedded in $CA_{\alpha+\kappa}$'s. It is known (cf. Monk [7]) that each $K_\kappa^\alpha$ is an equational class, and $RCA_\alpha \subseteq \cdots \subseteq K_3^\alpha \subseteq K_2^\alpha \subseteq K_1^\alpha \subseteq K_0^\alpha = CA_\alpha$. It is not known whether $K_\kappa^\alpha = K_\lambda^\alpha$ for some $\alpha$ and some distinct $\kappa, \lambda < \alpha$. By 1.11 and 2.2, no class $K_\kappa^\alpha$ is equal to $RCA_\alpha$, and known counterexamples show that $K_1^\alpha \neq CA_\alpha$ for all $\alpha \geq 2$. It is also open whether $K_\kappa^\alpha$ is ever finitely axiomatizable. Finally, recalling that $\varphi_\mu$ involves variables $v_0, \cdots, v_\mu$, we may ask whether $RCA_\alpha$ can be characterized by equations involving (say) only three variables; alternatively, if every subalgebra of a $CA_\alpha$ $\mathfrak{A}$ generated by $\leq 3$ elements is representable, is $\mathfrak{A}$ representable? (This question was raised by William Craig.)

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