Isaac Maths Skills

Using Essential GCSE Mathematics

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Contents

6 Percentages

Fractions, percentages and decimals are closely related. Percentage means "out of one hundred", so 20% is the fraction $\frac{20}{100}$, which can be cancelled down to $\frac{1}{5}$. $\frac{1}{5}$ can be written as the decimal 0.2.

Example 1 - Express 38% as (i) a fraction (ii) a decimal.

(i) $38\% = \frac{38}{100} = \frac{19}{50}$ (ii) $38\% = \frac{38}{100} = 38 \div 100 = 0.38$

Example 2 - Express the following as percentages: (i) $\frac{17}{20}$ (ii) 0.348.

(i) $\frac{20}{20}$ would be 100%, so $\frac{17}{20} = \frac{17}{20} \times 100\% = 85\%$

(ii) The number of digits 0.348 has after the decimal point is 3.

$$\therefore 0.348 = \frac{348}{1000} \qquad \frac{348}{1000} \times 100\% = \left(\frac{348}{10}\right)\% = 34.8\%$$

To calculate percentage changes, you must decide if you are starting with 100% of a quantity, or if you are given a different percentage and must calculate 100%.

Example 3 - There is a 25%-off sale. What is the sale price of a coat which was originally priced at £120?

In this example you are starting with 100%. If 25% is taken off, you will pay 75% of the starting cost, or $\frac{75}{100}$ of £120. $\frac{75}{100}$ is a scale factor, so multiply.

$$\frac{75}{100} \times £120 = £90$$

Example 4 – In the same 25%-off sale, there is a pair of shoes priced at £48. What was the original price?

This time you know what 75% of the original price is. Scale down to find 1%, then scale up to find 100%.

$$75\% \xrightarrow{\div 75} 1\% \xrightarrow{\times 100} 100$$

So the multiplier is $\frac{100}{75}$, or $\frac{4}{3}$. The original price was $\frac{4}{3} \times £48 = £64$.

6.1 Write these percentages as fractions.

(a) 30% (b) 5% (c) 0.5%

6.2 Write these percentages as decimals.

(a) 25% (b) 0.7% (c) 0.003%

6.3 Write these decimals as percentages.

(a) 0.10 (b) 0.01 (c) 0.005 (d) 2.00

6.4 Write these fractions as percentages.

(a) $\frac{3}{4}$ (b) $\frac{5}{8}$ (c) $\frac{7}{350}$

6.5 Which is larger in each case?

(a) 21.5% or $\frac{9}{20}$ (b) $\frac{7}{6}$ or 1.16 (c) 112% or $\frac{20}{18}$

6.6 Rank the following in order of size, starting with the smallest.

$$62\% \qquad \frac{5}{8} \qquad 0.629$$

6.7 Evaluate the following:

(a) 20% of £16 (b) 65% of 400 g (c) 160% of \$240

6.8 Calculate

(a) 27% of £24,000 (b) 15% of 75 (c) 7.5% of 6 kg

6.9 Write the first quantity as a percentage of the second.

(a) 3 minutes out of 2 hours

(b) £1.40 out of £40.00

(c) 366 g out of 3 kg

6.10 A family needs to buy a new washing machine and have it delivered.

Company A sells a machine at 15% off an original purchase price of £275, and charges £25 for delivery.

Company B sells the same machine. Their usual price is £300, but the machine is on sale at 20% off. Delivery is free.

From which company would it be cheaper to buy the machine?

6.11 Rohit saves 5% when purchasing a coat originally costing $£100$, and 8% on a sofa which originally costs $£200$. What is his overall saving as a percentage?

6.12 Rank the following in order of size, starting with the largest.

$$\frac{17}{20} \qquad 87\% \qquad \frac{7}{8} \qquad 0.889 \qquad \frac{349}{400}$$

6.13 A new process reduces the time to manufacture a product by 25%. If it takes 2 hours 21 minutes to make the product using the new process, how long did it take to make the item with the old process?

In a simple interest scheme, the interest added to an account each year is calculated as a percentage of the original amount deposited. The same amount of interest is added each year.

Example 5 - $£250$ is invested in a simple interest scheme at an interest rate of 4% per year. Calculate the total amount of money at the end of 5 years.

The amount of interest paid after each year is 4% of the starting $£250$. This is

$$4\% \times £250 = \frac{4}{100} \times £250 = \frac{4 \times £250}{100} = \frac{£1000}{100} = £10$$

After 5 years the amount of interest paid is $5 \times £10 = £50$.
The total amount of money in the account is therefore

$$£250 + £50 = £300$$

In a compound interest scheme, the interest added to an account each year is calculated as a percentage of the amount in the account during that year. Assuming that no money is taken out, the amount of interest added increases from year to year.

If the interest rate is $I\%$ per year, then after one year the amount in an account is $(100 + I)\%$ of the original amount. This is equivalent to multiplying the starting amount by the fraction $\frac{100+I}{100}$.

Example 6 - £250 is invested in a compound interest scheme at an interest rate of 4% per year. Calculate the total amount of money at the end of 5 years to the nearest penny.

For an interest rate of 4%, the multiplier is $\frac{100+I}{100} = \frac{104}{100} = 1.04$.

To find the amount after 5 years, apply the multiplier 5 times. To the nearest penny the amount in the account is £$250 \times 1.04^5 =$ £304.16.

6.14 Calculate, to the nearest penny, the amount that will be in an account if

(a) £$1,000$ is invested for one year in a simple interest scheme with a 5% interest rate.

(b) £$1,500$ is invested for two years in a simple interest scheme with a 3% interest rate.

(c) £$4,000$ is invested for one year in a compound interest scheme with a 4% interest rate.

(d) £$4,500$ is invested for three years in compound interest scheme with a 2.5% interest rate.

6.15 For each of the percentages below, calculate the amount in an account after 10 years if £$3,000$ is invested in a scheme with (i) simple interest (ii) compound interest. Give your answers to the nearest penny.

(a) 0.1% (b) 1% (c) 3% (d) 6%

6.16 Dan uses a balance to find the mass of some objects. The machine has an offset error, so it registers a mass of 9.0 g even when there is no mass on it.
Find the percentage error in his measurements when he weighs objects which have a true mass of

(a) 90 g (b) 720 g (c) 36.0 g

6.17 A carpenter is cutting a long, thin plank of wood into shorter lengths using a band saw. The saw blade is 2 mm wide, so each cut wastes 2 mm of wood as sawdust.

(a) How many 10 cm lengths can they cut from a plank with a total length of 2 m?

(b) What percentage of the original plank will not be used to make 10 cm lengths?

(c) What percentage of the plank will be turned into sawdust?

6.18 The average attendance at a sporting fixture went up by 40% every year. In year 2 the attendance was $35,000$. Find the attendance

(a) in year 3 (b) in year 1

6.19 Uranium has many isotopes. 99.274% of natural uranium is the isotope U_{238}. 720 out of $100,000$ atoms of natural uranium are the isotope U_{235}. What percentage of natural uranium is accounted for by the other isotopes?

6.20 For the formula $P = IV$, what is the percentage change in P if the current increases by 3.0% and the voltage falls by 10.0% at the same time?

11 Units

The table below shows a list of common unit prefixes.

n	μ	m	c	d	-	k	M	G
nano	micro	milli	centi	deci	-	kilo	mega	giga
$\times 10^{-9}$	$\times 10^{-6}$	$\times 10^{-3}$	$\times 10^{-2}$	$\times 10^{-1}$	-	$\times 10^{+3}$	$\times 10^{+6}$	$\times 10^{+9}$

For example, 1 cm $= 1 \times 10^{-2}$ m. This is equivalent to 1 cm $= 1 \times \frac{1}{100}$ m.

When converting from one set of units to another, write down a fact for each quantity you need to convert, turn that into a scale factor, and then apply this factor. For example, the fact 1 cm $= 1 \times \frac{1}{100}$ m tells us that to convert a quantity in centimetres to a quantity in metres, divide by 100. To go the other way, from a quantity in metres to a quantity in centimetres, multiply by 100. The scale factor is 100.

$$\begin{tikzcd} \text{m} \arrow[r, bend left, "\times 100"] & \text{cm} \arrow[l, bend left, "\div 100"] \end{tikzcd}$$

Example 1 – Convert 3 cm into metres.

Fact: $1\,\text{m} = 100\,\text{cm}$ Scale factor:

$$\begin{tikzcd} \text{m} \arrow[r, bend left, "\times 100"] & \text{cm} \arrow[l, bend left, "\div 100"] \end{tikzcd}$$

Answer: 3 cm $= 3 \div 100$ m $= 0.03$ m $= 3.0 \times 10^{-2}$ m

Sometimes you may need more than one scale factor. This often happens with time problems.

Example 2 – Change 2.5 hours into seconds.

Fact: 1 hour $= 60$ minutes 1 minute $= 60$ s

Scale factors:

$$\begin{tikzcd} \text{hours} \arrow[r, bend left, "\times 60"] & \text{min} \arrow[l, bend left, "\div 60"] \arrow[r, bend left, "\times 60"] & \text{s} \arrow[l, bend left, "\div 60"] \end{tikzcd}$$

Answer: 2.5 hours $= 2.5 \times 60 \times 60$ s $= 9000$ s

You may have a fact in which neither quantity is exactly 1 unit. In this case, scale down the fact to make one quantity exactly 1 unit in size, and then proceed as in the previous examples.

Example 3 – A cat weighs 8.0 lb (pounds). What is its weight in kilograms? 14.00 lb (one stone) is 6.3503 kg.

$$\begin{array}{cc} \text{lb} & \text{kg} \\ 14.00 & 6.3503 \\ \div 14 \downarrow & \downarrow \div 14 \\ 1.000 & 0.4536 \end{array}$$

Fact: 14.0 lb $= 6.3503$ kg

Divide by 14.00 to find how to convert 1.000 lb into kilograms.

Read off the scale factor.

$$\text{lb} \underset{\div 0.4536}{\overset{\times 0.4536}{\rightleftarrows}} \text{kg}$$

Answer: $8.0\ \text{lb} = (8.0 \times 0.4536)\ \text{kg} = 3.7\ \text{kg}$ to 2 s.f.

When dealing with units which involve powers other than one, or when handling compound units, convert each power of each unit in turn.

Example 4 – Convert $3\ \text{m}^3$ to $3\ \text{cm}^3$.

Fact:

$$1\ \text{m} = 100\text{cm}$$

$$\therefore 1\ \text{m}^3 = (100\ \text{cm})^3 = (100)^3\ \text{cm}^3$$

$$\Rightarrow 1\ \text{m}^3 = 100 \times 100 \times 100\ \text{cm}^3$$

$$\Rightarrow 1\ \text{m}^3 = 10^6\ \text{cm}^3$$

$$\text{m}^3 \underset{\div (10)^6}{\overset{\times (10)^6}{\rightleftarrows}} \text{cm}^3$$

Answer: $3\ \text{m}^3 = 3 \times 10^6\ \text{cm}^3$

Example 5 – Convert a density of $3\ \text{kg/m}^3$ to g/cm^3.

Fact: $1\ \text{kg} = 1000\ \text{g}$ $\qquad$ $1\ \text{m} = 100\ \text{cm}$

Scale factors: $\text{kg} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \text{g}$ $\qquad$ $\text{m}^3 \underset{\div (10)^6}{\overset{\times (10)^6}{\rightleftarrows}} \text{cm}^3$

Answer: $3\ \text{kg/m}^3 = 3\ \frac{\text{kg}}{\text{m}^3} = 3 \times \frac{1000}{10^6}\ \frac{\text{g}}{\text{cm}^3} = 3 \times \frac{10^3}{10^6}\ \frac{\text{g}}{\text{cm}^3}$

$$\therefore 3\ \text{kg/m}^3 = 3 \times 10^{-3}\ \text{g/cm}^3$$

Note: You may see units with a denominator part written in an alternative form with a negative index. For example, the units of speed, m/s, may also be written m s^{-1}, and the units of density, kg/m^3, may also be written kg m^{-3}.

11.1 Write:

(a) $2,300$ g in kilograms
(b) 0.002 g in milligrams
(c) 0.6 kg in grams
(d) 150 mg in grams

11.2 Write:

(a) 15 cm in metres
(b) 3.48 km in metres
(c) 67.2 km in centimetres
(d) 0.1 cm in kilometres

11.3 Write:

(a) 0.15 kg in grams
(b) 2.5 hours in seconds
(c) 365 days in seconds

11.4 Write:

(a) 3.31 mm in metres
(b) 15.2 μm in metres
(c) 60 km in millimetres
(d) 0.12 cm in nanometres

11.5 Perform the following unit conversions:

(a) 30 km/s to m/s
(b) 16 N/m^2 to kN/m^2
(c) $15,000$ Hz to MHz

11.6 Perform the following calculations, giving your answer in the units stated and to the given accuracy:

(a) 3.875×0.985 V, to 3 s.f., in units of V.

(b) 51.85 cm $\times 98.75$ cm, to 3 s.f., in units of cm^2.

(c) 106.75 m$^3 \div 43.1$, to 2 s.f., in units of m^3.

11.7 Concentrations are often written as values in moles per litre. An example is 4 moles/litre. 1 litre$=1$ dm^3, and 1 dm$=10$ cm. Convert 4 moles/litre into units of

(a) moles/m^3.

(b) moles/cm^3.

11.8 Perform the following unit conversions. In this question you will need to know that 1 mile $= 1609.34$ metres, "mph" stands for miles per hour, "kph" for kilometres per hour, and "m/s" for metres per second. Give your answers to 2 significant figures.

(a) 15 kph to m/s
(b) 33 m/s to kph
(c) 16 mph to m/s
(d) 64 kph to mph

11.9 A student claims that 15 cm is exactly the same as 6 inches. A conversion calculator states that 50 cm $= 1.64042$ feet. What is the percentage error in the student's claim? Give your answer to 2 significant figures.

11.10 Perform the following calculations, giving your answer in the units stated and to the given accuracy:

(a) 0.02511 cm$\times 78.34$ cm, to 3 s.f., in units of m^2.

(b) 91.25×0.00006751 V, to 2 s.f., in units of mV.

(c) Find, to 2 significant figures, the distance in metres covered by a car which travels for 45 minutes at 15 km per hour.

11.11 Put ticks in the table to show whether these expressions represent lengths, areas, volumes, or none of these. You are told that r and l have units of metres.

Expression	Length	Area	Volume	None of these
$\frac{4}{3}\pi r^3$				
$\pi r l$				
$\pi(r+l)$				
πr^2				
$r+l$				

11.12 Put ticks in the table to show whether these expressions represent lengths, areas, volumes, or none of these. You are told that π, p, q and s are unitless constants, and k, l and m have units of metres.

Expression	Length	Area	Volume	None of these
πm^2				
$plmk$				
$\frac{qm^2}{k}$				
$\frac{s\pi l}{mk}$				
$\frac{pqs}{2}$				

Algebra

12 Writing and Using Algebra

Algebra has its own terminology:

$4\pi x^2$	A term is made up of constants (such as 4 and π) and variables (such as x). Together the leading constants are known as the coefficient (here 4π). Like terms contain the same variables, to the same powers. They differ only in the values of their coefficients.
$4\pi x^2 - 3x$	An expression is made up of terms linked by operators ($+, -, \times, \div$).
$x - 5 = 4\pi x^2 - 3x$	An equation links two expressions with an equals sign.

If an equation is true for all possible values of the variable it is called an identity, and the $\equiv$ sign may be used. One side of an identity is effectively just a re-arrangement of the other. For example, $(2x + 3) - 4 \equiv 2x - 1$.

$7x^2 - 3x + 2$ and $5x^3 + 2x$ are polynomials. Polynomials have terms which contain only constants and positive, whole number powers of variables, linked together by addition or subtraction.

- If the highest power of the variable is 1, the polynomial is linear. For example, $2x - 5$ and $y + 3$.
- If the highest power of the variable is 2, the polynomial is quadratic. For example, $7x^2 + 4x - 2$ and $y^2 - 9$.
- If the highest power of the variable is 3, the polynomial is cubic. For example, $9x^3 + 2x^2 - 7x$ and $y^3 - 4$.

Writing algebra involves replacing words with variables, constants and operators. Brackets are often needed to ensure that applying the rules of BIDMAS when doing calculations will give correct answers.

Example 1 - Write the following as an equation:

"To find Q, add 6 to p, then divide by 5."

First add 6 to p: $p+6$ Next divide everything by 5: $\dfrac{p+6}{5}$

This is equal to Q: $Q = \dfrac{p+6}{5}$

12.1 (a) Write the following as an equation: "To find y, multiply x by four then subtract three."

(b) When $x = 5$ what is y?

12.2 (a) Write the following statement as an algebraic equation: "y is found by adding eight to six x."

(b) Find y if $x = 10$.

(c) Find y if $x = -5$.

12.3 A child says "Two p and three q make z."

(a) Write this statement as an equation.

(b) Find z if $p = 9$ and $q = -7$.

12.4 The costs of pieces of fruit are: apple 30 p, pear 35 p, banana 28 p and orange 25 p.

(a) Write an equation to find the total cost, C p, of d apples, e pears, f bananas and g oranges.

(b) What is the change from £10.00 if $d = 4$, $e = 4$, $f = 7$ and $g = 6$?

12.5 A gardener walks up and down his garden sowing seeds. The garden has length L, and he makes twelve round trips. In total he walks 336 m.

(a) Write an equation for this information.

(b) What is the length of the garden, L?

Superscripts and subscripts perform different roles. A superscript, such as the 2 in x^2, is used to indicate that a number or variable is raised to a power. Subscripts are used purely as labels. For example, the initial speed of a vehicle might be written as v_0, v_S or even v_{Start}. Numbers in subscripts are part of the label, and do not indicate that a mathematical operation is taking place.

Example 2 - the velocity of a car at time t is given by

$$v_t = v_0 + at$$

where v_0 is the initial velocity of the car and a is the acceleration. Find the value of v_t when $v_0 = 5$ m/s, $a = 2$ m/s^2 and $t = 8$ s.

$$v_t = 5 + 2 \times 8 = 5 + 16 = 21 \text{ m/s}$$

12.6 Using the equation $v_t = v_0 + at$, find v_t if

(a) $v_0 =0$ m/s, $a =3$ m/s^2 and $t =10$ s

(b) $v_0 =50$ mm/s, $a =2$ mm/s^2 and $t =4$ s.

(c) $v_0 =0.7$ km/s, $a =-0.04$ km/s^2 and $t =10$ s.

12.7 (a) If R is the number of rabbits now, and R_0 is the number of rabbits originally, write an equation for the statement "The number of rabbits now is twice the starting number of rabbits, minus 10 which have been sold."

(b) Find R if $R_0 = 210$.

Greek letters are commonly used in algebra in mathematics and the sciences. They can be manipulated in exactly the same way as Roman letters such as x and y. The table below shows those that are used most often and their names. On the left are lower case letters, and on the right are a smaller number of upper case letters.

	Name		**Name**		**Name**		**Name**
α	alpha	θ	theta	ρ	rho	Δ	delta
β	beta	λ	lambda	σ	sigma	Λ	lambda
γ	gamma	μ	mu	ϕ	phi	Σ	sigma
δ	delta	ν	nu	ω	omega	Φ	phi
ϵ, ε	epsilon	π	pi			Ω	omega

Example 3 - The resistance of a piece of wire, R, is equal to the resistivity of the wire ρ multiplied by the length of the wire l and divided by the wire's cross-sectional area A.

Multiply the wire's resistivity by its length. $\rho \times l$

Next divide by the cross-sectional area. $\dfrac{\rho \times l}{A}$

This is equal to the wire's resistance. $R = \dfrac{\rho \times l}{A}$

12.8 Λ is equal to ϕ minus ω.

(a) Write an equation for Λ.

(b) Find Λ for $\phi = 45^\circ$ and $\omega = 15^\circ$.

12.9 (a) Write this information as an equation: "To find γ start with 24 and subtract 4 times α, then divide the answer by 3."

(b) Find γ when $\alpha = 3$.

Simplifying "tidies up" algebra into a neater form. Simplifying includes collecting like terms together; using the rules of indices to combine different powers of a variable; and cancelling a common factor in the numerator and denominator of a fraction.

Example 4 - Simplify $\frac{1}{2} \times 4x^2 + 2p + 3\frac{p^2}{p}$.

The first term can be simplified by multiplying the $\frac{1}{2}$ and the 4 together. The third term can be simplified by cancelling a factor of p in the top and bottom of the fraction. Finally, combine like p terms.

$$\frac{1}{2} \times 4x^2 + 2p + 3\frac{p^2}{p} = 2x^2 + 2p + 3p = 2x^2 + 5p$$

In general it is good practice to simplify algebra whenever possible, even if not explicitly asked to do so.

12.10 Simplify:

(a) $3\alpha + 2\alpha$

(b) $5\lambda - \pi - 2\pi - \lambda$

(c) $M = M_0 + 3m + 5m - 6m + 4m$

12.11 Simplify:

(a) $3p - 6s + 2t - p + s$

(b) $\frac{3}{4}vw + \frac{1}{4}vw$

(c) $fg + gf + 2hj + jh$

12.12 Simplify:

(a) $2p \times 3q^2r + 4r \times 2pq^2$

(b) $\frac{1}{2} \times 2x^9 \div x^7 - 2x + x^2 + 20x$

12.13 A bar-tender is counting cans for stock-taking. He has x 4-packs, y 12-packs and z single cans.

(a) Write this information as an equation to find the total number of cans T.

(b) What is T if $x = 11$, $y = 10$ and $z = 7$?

12.14 A postman delivers mail to four houses. House 1 receives $3l$ letters and p parcels. House 2 receives $7l$ letters. House 3 receives $5l$ letters and $2p$ parcels. House 4 receives p parcels.

(a) Write an equation for the total number of items the four houses receive, T. Simplify your answer as far as possible.

(b) Assuming that the weight of a letter is 80 g and the weight of a parcel is 550 g, write an equation for W, the total weight in kilograms of the items delivered to the four houses.

12.15 A quantity called the discriminant is used in the calculation of solutions of quadratic equations.

(a) Using δ for the discriminant, write the following as an equation: "The discriminant is found by subtracting four times a times c from the square of b."

(b) Find δ if $b = 16$, $a = 1$ and $c = 4$.

(c) Find δ if $b = 100$, $a = 3$ and $c = 7$.

12.16 Write the following statements in algebra.

(a) α is twice β.

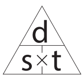
(b) α cubed is the same as γ squared.

$\beta = 2$ and γ is a positive integer.

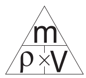
(c) Find the value of γ.

18 Formula Triangles

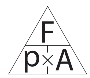
Formula triangles are used to save time re-arranging formulae which involve exactly three quantities. Here are some common formula triangles:



d Distance
s Speed
t Time



m Mass
ρ Density
V Volume

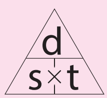


F Force
p Pressure
A Area

Note: Some books write formula triangles with a vertical line on the bottom instead of a multiplication symbol. The meaning is the same.

To find a formula from a formula triangle, cover up the quantity to be found. The pattern of the remaining letters shows how to calculate this quantity.

Example 1 - Use the formula triangle relating speed s, distance d and time t to write formulae for each of the three quantities in terms of the other two.



$s = \frac{d}{t}$

$d = s \times t$

$t = \frac{d}{s}$

Then performing a calculation using a formula triangle, it is necessary to make sure that the units you are using are consistent.

Example 2 - Find the time taken to travel 0.63 km at 4.2 m/s.

There is a mixture of units of length in the question (m and km). We will choose to do the calculation in metres. 0.63 km $= 630$ m.

$$t = \frac{d}{s} = \frac{630}{42} = 15 \text{ s}$$

18.1 Using the speed-distance-time triangle, or otherwise, find:

(a) s in m/s if $d = 200$ m and $t = 25$ s.

(b) d in m if $s = 10$ m/s and $t = 10$ s.

(c) t in s if $d = 540$ m and $s = 12$ m/s.

18.2 Using the speed-distance-time triangle, or otherwise, find:

(a) The speed in m/s if $d = 0.45$ km and $t = 2.5$ minutes.

(b) The distance travelled in m if $s = 0.3$ m/s and $t = 5$ minutes.

(c) The time a journey takes in s if $d = 720$ m, $s = 21.6$ km/hour.

18.3 Using a formula triangle or otherwise, find:

(a) The pressure in N/m^2 exerted by a force of 16.2 N on an area of 1.50 m^2.

(b) The force in Newtons (N) required to maintain a pressure of 15.0 N/m^2 on an area of 0.150 m^2.

(c) The volume of a lead block with a mass of 4.52 kg. Lead has a density of 11.3 g/cm^3

It is also possible to write your own formula triangles.

18.4 Write the following formulae as formula triangles.

(a) $V = IR$ (b) $a = F/m$ (c) $\frac{P}{V} = I$

18.5 Which of these formulae can be written as a formula triangle? Write a formula triangle where it is possible.

(a) $v = u + at$ (b) $\frac{Q}{C} = V$ (c) $L = a - b$

(d) Magnification, $M = \frac{\text{Image size, } i}{\text{Object size, } o}$

18.6 The concentration of salt in water, C g/cm^3, is found by dividing the mass of salt in grams, m, by the volume of water in cm^3, V.

(a) Create a formula triangle for concentration, mass and volume.

(b) Write a formula for volume in terms of mass and concentration.

(c) Find the volume of a solution with concentration 0.0020 g/cm^3 if the total mass of salt dissolved is 2.4 g.

(d) Write a formula for m in terms of C and V.

(e) Find m if $V = 1$ litre and $C = 0.004$ g/cm^3.

27 Graphs of Quadratic Functions

$y = x^2 + 1$, and $y = -x^2 + 2x + 5$ are examples of quadratic functions. A quadratic function can be plotted using a table of values.

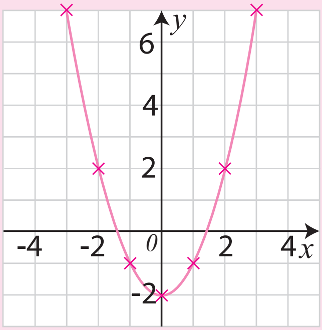
Example 1 – Plot a graph of $y = x^2 - 2$.

First, create a table.

x	-3	-2	-1	0	1	2	3
y	7	2	-1	-2	-1	2	7

Next, plot the points ($\times$).

Finally, draw a smooth curve through the points.



The shape of a quadratic function is called a parabola. A parabola has a minimum or maximum point. This is called the vertex or turning point of the graph. A parabola is symmetrical. The line of symmetry is parallel to the y-axis and passes through the vertex.

27.1 You are given the function $y = x^2 - 2x - 8$.

x	-3	-2	-1	0	1	2	3	4	5
y				-8				0	

(a) Complete the table of values.
(b) Use the table of values to draw the graph of the function.
(c) Where does the curve cross the x and y axes?
(d) Give the coordinates of the turning point.

27.2 For the function $y = x^2 - 6x + 8$:

x	-4	-2	0	2	4	6	8	10
y	48		8	0	0	8	24	

(a) Fill in the missing values.
(b) Predict the x-coordinate of the minimum.
(c) Find the y value of the minimum.
(d) Plot the curve, showing the position of the minimum.

27.3 For the function $y = (x-3)^2$,

(a) Construct a table of values for $-1 \leqslant x \leqslant 5$.

(b) Plot a graph of $y = (x-3)^2$

(c) What is the y-intercept?

(d) what is the equation of the line of symmetry?

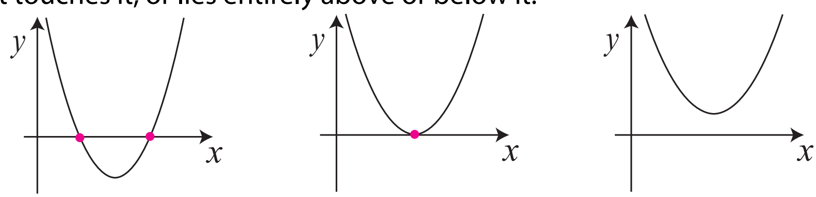
(e) Give the coordinates of the vertex.

The general form of a quadratic function is $y = ax^2 + bx + c$. When a quadratic function is written in this form, the sign of a determines the overall shape of the curve:

- If a is positive the curve has a minimum $\smile$ shape.
  For example, $y = 3x^2 + 4$ has a $\smile$ shape because 3 is positive.
- If a is negative the curve has a maximum $\frown$ shape.
  For example, $y = -2x^2 + 7$ has a $\frown$ shape because -2 is negative.

A quadratic function always crosses the y-axis once. The value of c is the y-intercept.

A quadratic function meets the x-axis in 2, 1 or 0 places. It crosses the x-axis, just touches it, or lies entirely above or below it.



On the x-axis the value of y is 0. To find where a quadratic function meets the x-axis, put $y = 0$ into the equation for the function and solve for x. The solutions are the roots of an equation with the form $0 = ax^2 + bx + c$.

Example 2 – Find where the curve $y = x^2 + 5x + 6$ crosses the x-axis.

On the x axis $y = 0$:

$0 = x^2 + 5x + 6 \qquad \Rightarrow 0 = (x+2)(x+3) \qquad \Rightarrow x = -2$ or $x = -3$

Therefore, the curve crosses the x-axis at $x = -2$ or $x = -3$.

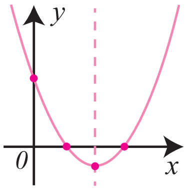
27.4 Without drawing graphs, find for each function:
(i) the y-intercept (ii) where the graph crosses the x-axis.
(a) $y = x^2 + x - 2$ (b) $y = x^2 + 6x + 5$ (c) $y = x^2 - 8x + 15$

27.5 (a) Plot a graph of the function $y = x^2 - 4x - 4$ for $-2 \leqslant x \leqslant 6$.
(b) Where does the graph cross the x-axis?

Sketching a graph is quicker than creating a table of values and putting them accurately onto graph paper. The goal is to identify the position of key points such as intercepts with the axes, and use these to draw a rough diagram showing the form of the graph.

To sketch a quadratic function, you need to find:

- the overall shape of the curve ($\smile$ or $\frown$)
- where the curve intercepts the y axis
- where the curve meets the x axis
- the position of the vertex



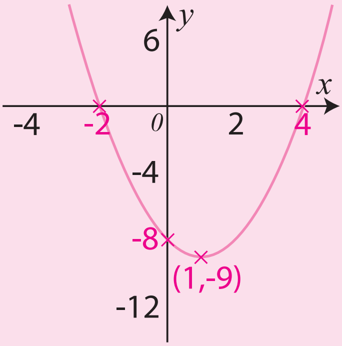
Example 3 – The curve $y = x^2 - 2x - 8$ crosses the x-axis at $x = -2$ and $x = 4$. Use this information to sketch $y = x^2 - 2x - 8$.

The curve is written in $y = ax^2 + bx + c$ form.

- $a = +1$, which is positive, so the basic shape is $\smile$.
- $c = -8$, so the y-intercept is at $y = -8$.

The curve is symmetrical. The curve crosses the x-axis at $x = -2$ and $x = 4$. The line of symmetry must be half-way between these x values at $x = 1$.

The vertex is on the line of symmetry, so it has $x = 1$. Putting this value into $y = x^2 - 2x - 8$ gives $y = -9$.



27.6 Find where the graphs of these quadratic equations meet or cross the x-axis, and use this information to sketch the graphs.

(a) $y = x^2 + 3x - 10$ (b) $y = -x^2 - 3x + 4$

27.7 (a) Sketch the graphs of $y = x^2$ and $y = -x^2$.

(b) Sketch the graph of $y = x^2 + 2$. Where does the curve cross the y-axis?

(c) Sketch the graph of $y = -x^2 - 2$. Where does the curve cross the y-axis?

27.8 Find where the graphs of these quadratic equations meet or cross the x-axis, and use this information to sketch the graphs.

(a) $y = 4x^2 - 24x + 27$ (b) $y = 3x^2 - 12$

27.9 The general form of a quadratic function is $y = ax^2 + bx + c$. You are told that a particular quadratic function intersects the x-axis at $x = 4$ and $x = 5$. For the given values of a, find: (i) the y-intercept (ii) the position of the vertex.

(a) $a = 1$ (b) $a = -1$

27.10 The graph of the equation $y = ax^2 + bx + 12$ intercepts the x-axis at $x = -3$ and $x = -2$. Find the values of the constants a and b.

27.11 The formula $s = ut + \frac{1}{2}at^2$ is used to calculate the height s of projectiles (such as balls) as a function of time.

Plot a graph of s against t for $0 \leqslant t \leqslant 7$, given that $u = 29.43$ m/s and $a = -9.81$ m/s^2.

(a) What is the maximum height reached? Give your answer to 3 s.f..

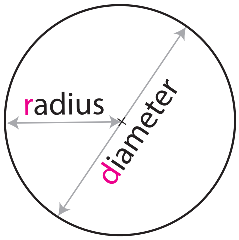
(b) How long does a projectile modelled by this graph take to return to its starting height? You may assume the projectile was launched at $t = 0$.

(c) At $t = 7$ s, what is the height of the projectile relative to its starting position? Give your answer to 3 s.f..

42 Circles and Circle Theorems

Circles are different from polygons because their boundaries are continuous curves rather than made up of straight lines.

The boundary of a circle is called the circumference. The distance between the centre and the circumference is the radius. A straight line from one side of a circle to the other through the centre is a diameter.



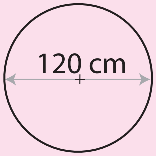
A diameter is twice the length of a radius: $d = 2r$. The length of the circumference, C, and the area of a circle, A, are given by the formulae:

$$C = 2\pi r \text{ or } C = \pi d \qquad A = \pi r^2 \text{ or } A = \frac{\pi d^2}{4}$$

π is an irrational number with the value $3.14159265\ldots$.

Example 1 – Find the circumference and area of a circle of diameter 120 cm to 3 s.f..

The radius is half the diameter.



$$r = \frac{d}{2} = \frac{120}{2} = 60 \text{ cm}$$

$\therefore C = 2\pi r$

$\Rightarrow C = 2 \times \pi \times 60$

$\Rightarrow C = 377$ cm to 3s.f.

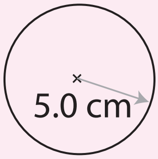
and $A = \pi r^2$

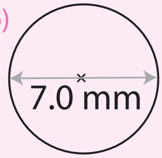
$\Rightarrow A = \pi \times 60^2$

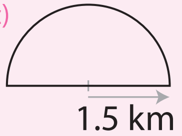
$\Rightarrow A = 11,300$ cm^2 to 3s.f.

In this exercise give your answers to 3 s.f. when rounding is required.

42.1 For each shape find (i) the length of the curve (ii) the area.

(a) 

(b) 

(c) 

42.2 (a) The circumference of a circle is 82.6 cm. Find the radius.

(b) The area of a circle is 156 cm^2. Find the radius.

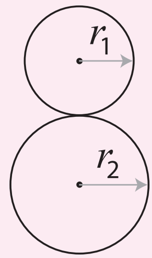
(c) The area of a semicircle is 16.4 cm^2. Find the diameter of the of the circle of which it is a part.

42.3 The diagram shows a "figure-8" consisting of two circles which touch at one point.

The radius of the upper circle, r_1, is 1.2 cm.
The radius of the lower circle, r_2, is 1.6 cm.

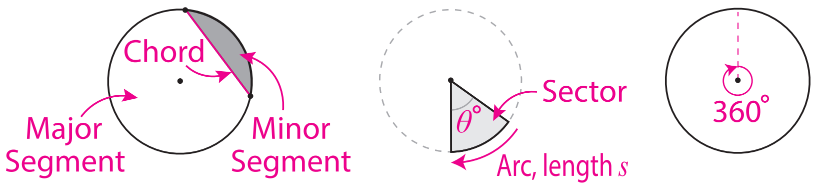
(a) What is the length of the edge of this figure?

(b) What is the area enclosed by the figure?



A chord is a straight line joining two points on the edge of a circle. A chord divides a circle into two segments. The major segment is the larger segment, and the minor segment is the smaller segment.

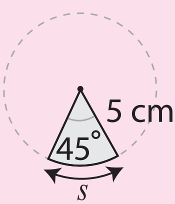
An arc is a part of the circumference of a circle. The area enclosed by an arc and two radii is a sector.



Drawing a full circle with compasses involves rotating $360°$ about the centre. Turning through $\theta°$ to draw an arc is therefore drawing $\frac{\theta°}{360°}$ of a full circle. Hence, the length of an arc and the area of the corresponding sector are $\frac{\theta°}{360°}$ of the circumference and area of a full circle.

$$\text{Arc length, } s = \frac{\theta°}{360°} \times 2\pi r \qquad \text{Sector area, } A = \frac{\theta°}{360°} \times \pi r^2$$

Example 2 – The diagrams hows a sector cut from a circle of radius 5 cm. Find the length of the arc, s, and the area of the sector, to 3 s.f..

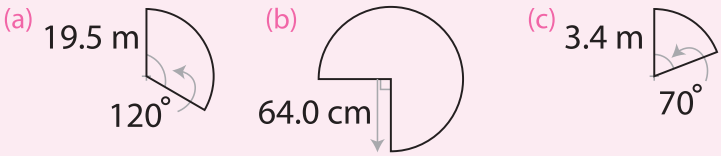


$$\text{Fraction of the circle} = \frac{45}{360} = \frac{1}{8}$$

$$s = \frac{1}{8} \times 2 \times \pi \times 5\text{ cm} = 3.93\text{ cm}$$

$$\text{Area} = \frac{1}{8} \times \pi \times 5^2\text{ cm} = 9.82\text{ cm}^2$$

42.4 For each sector, find to 3 s.f. (i) the length of the arc (ii) the area.



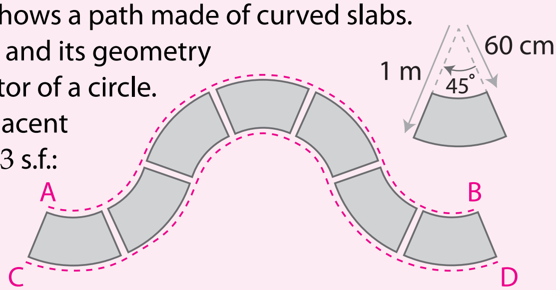
42.5 If A is the area of a circle, what angle would a sector of this circle have if the sector has area:

(a) $\frac{1}{4}A$ (b) $\frac{1}{12}A$ (c) $\frac{1}{72}A$

42.6 The diagram below shows a path made of curved slabs. Each slab is identical, and its geometry is based on a $45°$ sector of a circle. The gap between adjacent slabs is 2 cm. Find to 3 s.f.:

(a) The length AB.

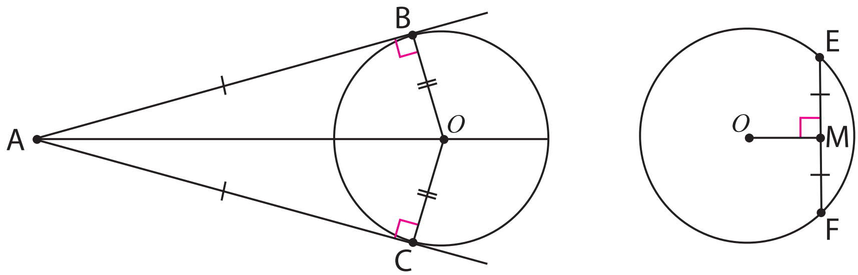
(b) The length CD.



§ There are a number of important circles theorems involving the geometry of tangents and chords.

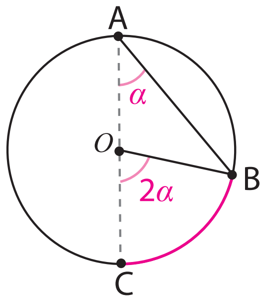
From any point outside a circle you can draw two tangents to the circle.

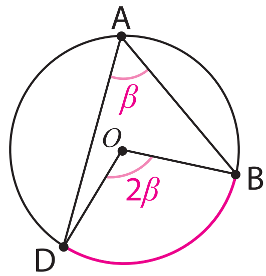
- The distance from the point to where it meets the circle is the same for the two tangents: AB $=$ AC.
- A tangent to a circle is perpendicular to the radius at the point of contact: A$\hat{\text{B}}O =$ A$\hat{\text{C}}O = 90°$.

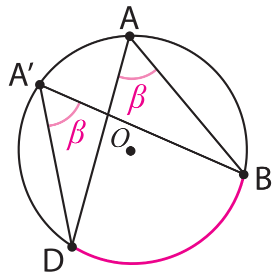


At the mid-point of a chord, the chord makes a right-angle with a line to the centre: E$\hat{\text{M}}O =$ F$\hat{\text{M}}O = 90°$.

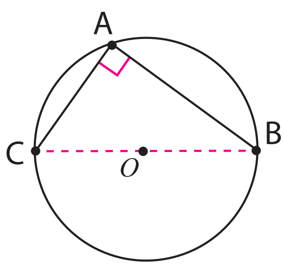
The diagram on the left below shows a useful property of circles. Triangle ABO is isosceles as AO and BO are both radii. If $B\hat{A}O = \alpha$, then $A\hat{B}O$ is also α. Hence $A\hat{O}B = 180° - 2\alpha$, and $B\hat{O}C = 2\alpha$. From this property a number of useful results can be derived.

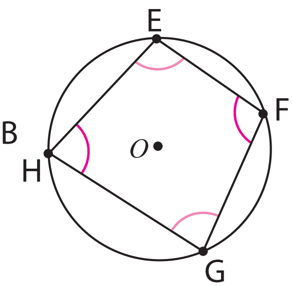


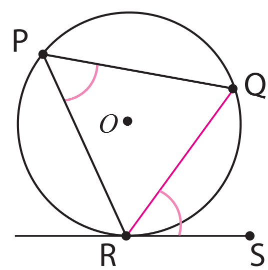




- The angle that an arc subtends at the centre is twice the angle that the arc subtends at a point on the circumference: $B\hat{O}D = 2 \times B\hat{A}D$.
- Angles subtended by an arc in the same segment are equal: $B\hat{A}D = B\hat{A'}D$.





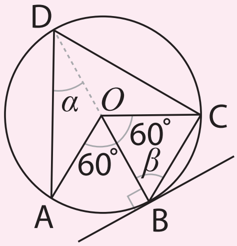


- At a point on the circumference, the angle subtended by a diameter is a right angle: $C\hat{A}B = 90°$.
- A quadrilateral is cyclic if its vertices all lie on a circle. For any cyclic quadrilateral, opposite angles sum to $180°$:
  $E\hat{F}G + E\hat{H}G = 180°$ and $F\hat{E}H + F\hat{G}H = 180°$.
- The angle between a tangent and a chord is equal to the angle subtended by the chord in the alternate (opposite) segment: $Q\hat{P}R = Q\hat{R}S$.

42.7 O is the centre of the circle. Find:

(a) α

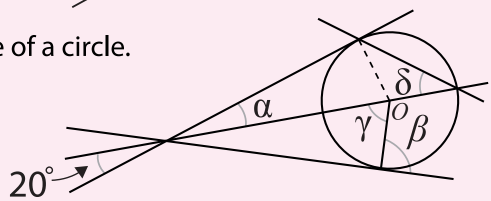
(b) β



42.8 Point O is the centre of a circle.

(a) α (c) γ

(b) β (d) δ



42.9 Point A has coordinates $(3, 1)$, and point B has coordinates $(17, 1)$.

(a) Work out the position of C, the mid-point of A and B.

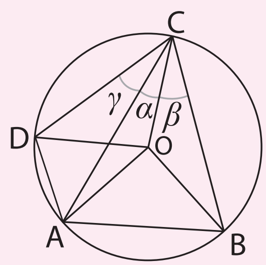
(b) Point D is 7 units from C, but not coincident with A or B. What is the value of the angle $A\hat{D}B$?

42.10 Point O is the centre of a circle. What are the values of the following angles in terms of α, β and γ?

(a) OÂC (d) AÔB

(b) OD̂C (e) AÔD

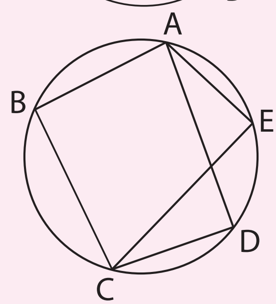
(c) BÔC



42.11 You are given the following diagram with the added information that AC is a diameter of the circle.
Find the value of the quantity:

$$\text{AB}^2 + \text{AD}^2 + \text{AE}^2 + \text{BC}^2 + \text{CD}^2 + \text{CE}^2$$

given that AC $= 10$ cm.



54 Sampling and Representations of Data

A data set that includes all individual values that could be gathered (the population) can be very large, so statistical analyses usually employ samples.

Samples should represent an entire population fairly, so if certain groups can be identified in the population (men, women, the elderly, the young, etc) then these groups should appear in the sample in the same ratio as they appear in the population. This is called stratified sampling. The selection of individuals within each group should be random to avoid bias in the results.

Example 1 – A transport survey collected data from 2000 people about how they get to work:

Walk	Bike or Car	Public Transport
375	1280	345

The surveyors ask follow-up questions from 400 people. How many must be randomly selected from each category to give a stratified sample?

The fraction of people who walk to work is $\frac{375}{2000} = \frac{3}{16}$. Therefore, the follow-up survey should include $\frac{3}{16} \times 400 = 75$ people who walk.

Doing similar calculations, the follow-up survey should include 256 who use a bike or car and 69 who use public transport.

Data collected about a quantity that can only take certain specific values is discrete. For example, the number of goats in a field can only be an integer. Data collected about quantities that can take an uninterrupted range of values, such as height or weight, is continuous.

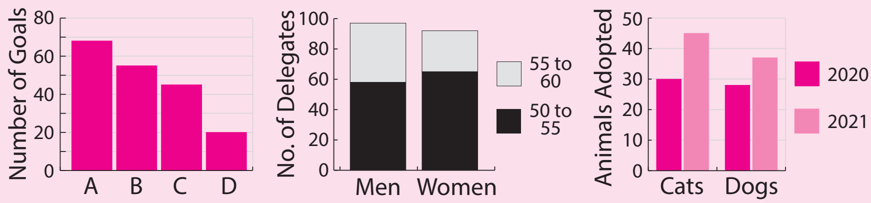
Sometimes a data set contains an outlier. This is a value very different from all the others.

The number of times a value appears in a data set is called the frequency of that value. For example, if a survey about food preference finds that 3 choose pizza, the frequency of pizza is 3. Frequency information is often recorded in a frequency table.

Food	Frequency
Curry	6
Roast	5
Chilli	5
Pizza	3

Bar charts are used to represent frequency data visually. There are several types which are shown in Example 2. Multiple bar charts and composite bar charts must include a key to indicate what the different bar types represent.

Example 2 – A bar chart, composite bar chart, and multiple bar chart.

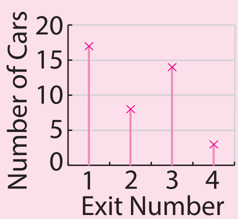


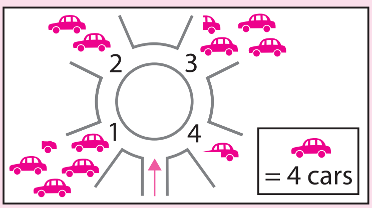
Tally charts are used to record frequency data quickly by hand. One stroke is made per piece of data, and every 5th stroke is drawn diagonally across the previous four to help with counting.

Vertical line charts are similar to bar charts, except vertical lines are used instead of bars. Pictograms use images for interesting visual presentations of data. They must be accompanied by a key to communicate what numerical value an image represents.

Example 3 – A study monitors where cars travel to after arriving at a roundabout from a motorway exit.

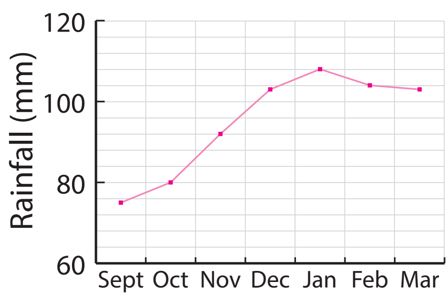
Road	Tally
1	卌 卌 卌 ||
2	卌 |||
3	卌 卌 ||||
4	|||





From left to a right: a tally chart, vertical line chart and pictogram.

Time series are used when data is collected at intervals over a period of time. For example, a weather station may record the number of millimetres of rain to fall each month. Data points are positioned in the centre of each time interval and connected by straight lines.

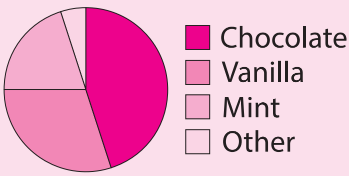


Pie charts are used to plot data that is in categories. Each category is shown as a sector of a circle. To find the angle of the sector, work out the fraction of the total that the category represents, then multiply this fraction by $360°$.

$$\text{Angle of sector} = \frac{\text{Number in category}}{\text{Total number}} \times 360°$$

Example 4 – A pie chart showing sales of ice cream on a summer day.

Flavour	**No. Sold**	**Degrees / °**
Chocolate	54	$162°$
Vanilla	36	$108°$
Mint	24	$72°$
Other	6	$18°$



120 ice creams were sold in total. The angle of the sector for chocolate is $\frac{54}{120} \times 360° = 162°$.

54.1 A hospital has a total of 9760 employees. They are divided into 4 categories:

Doctors	Nurses	Scientists	Administrators
1 240	5 020	1 080	?

(a) How many administrators work for the hospital?

A stratified sample is required for a survey of employees' opinions. 5% of the employees will be surveyed.

(b) How many employees will be included in the survey?

(c) Work out how many people must be randomly selected from each of the four employee categories to give a stratified sample.

54.2 A train company carried out a large survey on their passengers to learn about their reasons for travelling. The responses were:

Going on holiday 4%
Going shopping 27%
Visiting friends and family 7%
Travelling for work ? %
Other reasons 15%

(a) What percentage of the passengers were travelling for work?

(b) The number of people who answered the survey with "going shopping" was 540. How many people answered the survey with "visiting friends and family"?

(c) How many people were surveyed in total?

54.3 The birds visiting a garden were observed for 5 days. Draw a composite bar chart to display this data, with one column for each day of the week.

Bird	**Mon**	**Tue**	**Wed**	**Thur**	**Fri**
Robin	0	1	1	0	1
Pigeon	4	3	5	0	4
Wren	1	1	1	1	1
Magpie	3	3	2	0	3
Blackbird	2	0	2	0	2

54.4 The following sales of spring bulbs were recorded at a garden centre. 1 unit represents 10 packets.

Flower	**Year 1 / units**	**Year 2 / units**
Snowdrops	150	120
Crocuses	110	98
Daffodils	340	370
Hyacinths	75	54
Tulips	270	310

(a) Show the data on a labelled multiple bar chart.

(b) What is the difference between the two years in the total number of packets of bulbs sold?

(c) Which type of bulb showed the smallest percentage change in packets sold?

54.5 The tally chart shows the number of burgers sold in a restaurant on five weekdays.

(a) Display the results on a suitable pictogram using the image of a burger.

<table>
<tr><th>Day</th><th>Tally</th></tr>
<tr><td>Mon</td><td>𝍸 ||||</td></tr>
<tr><td>Tues</td><td>𝍸 ||</td></tr>
<tr><td>Wed</td><td>𝍸 𝍸 |</td></tr>
<tr><td>Thur</td><td>𝍸 |</td></tr>
<tr><td>Fri</td><td>𝍸 𝍸 𝍸 ||||</td></tr>
</table>

(b) If the price of each burger sold was £5.20, what was the restaurant's total sales income from burgers during these five days?

54.6 (a) The maximum and minimum daily temperatures reached on a patio were recorded for a week. Plot the data below as two time series on the same axes.

Day	Mon	Tue	Wed	Thur	Fri	Sat	Sun
Max. / °C	14	16	18	22	25	27	19
Min. / °C	3	2	5	8	9	7	6

(b) Which day had the smallest difference in maximum and minimum temperature?

(c) What was the greatest difference between maximum and minimum temperatures on one day?

54.7 (a) A survey is taken of the sales of meals at a food court in a shopping centre. Create a pie chart to display these results:

Stall	Pizza	Noodles	Curry	Salad	Burgers	Nachos
Meals	121	144	97	156	74	128

(b) What percentage of customers bought noodles?

(c) What fraction of purchases were pizzas?

54.8 A charity published the following figures for their sources of income:-

Government grants and fees 60% Donations 15%
Trading in shops 20% Other sources 5%

(a) Show these values on a pie chart.

(b) The sector for donations had an annual value of £$390\,000$. What is the total annual income for the charity?

54.9 An ecologist uses capture and recapture sampling to estimate the total number of terrapins, T, in a large pond.

(a) She captures 30 terrapins, marks their shells, then returns them to the pond. What fraction of the total did she capture?

(b) The terrapins disperse. Four days later, she captures 40 terrapins. 5 have her mark. What fraction of this sample are marked?

(c) The answer to (b) is an estimate of the fraction of turtles in the pond that are marked. Equate answers to (a) and (b) to estimate T.