Solving Equations & Logs 3ii

A Level

Part A Express log

Express $\log_3(4x+7) - \log_3 x$ as a single logarithm.

The following symbols may be useful: `ln()`, `log()`, `x`

Part B Solve equation

Hence solve the equation $\log_3(4x+7) - \log_3 x = 2$. Give your answer in decimal form.

Part C Use logs

Use logarithms to solve the equation $7^x = 2^{x+1}$, giving the value of x correct to 3 significant figures.

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Solving Equations & Logs 2ii

A Level

P P P

Part A Solve equation

Use logarithms to solve the equation $5^{3w-1} = 4^{250}$, giving the value of w correct to 3 significant figures.

Part B Find expression

Given that $\log_x(5y + 1) - \log_x 3 = 4$, express y in terms of x.

The following symbols may be useful: x, y

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Solving Equations & Logs 3i

A Level

P P P

Part A Solve equation

Solve the equation $2^{4x-1} = 3^{5-2x}$, giving your answer in the form $x = \frac{\log_{10} a}{\log_{10} b}$.

The following symbols may be useful: `log()`, `x`

Part B Find integer

Find the smallest integer n which satisfies the inequality $7^{2n} > e^{600}$.

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Integrating x^{-1}

A Level

The usual rule for integrating polynomials $\int x^n \, \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ breaks down for x^{-1}. In this question we explore the properties of this integral. This will give us insight into common $(\ln x)$ logarithms and also the exponential function (e^x).

The fundamental property of a logarithm is that $\log ab = \log a + \log b$ regardless of your choice of base. Here we will define

$$L(y) = \int_1^y \frac{1}{x} \mathrm{d}x$$

and show that our function $L(y)$ does indeed have the property of a logarithm.

To work out $L(ab)$ we will break the integral into two sections

$$L(ab) = \int_1^{ab} \frac{1}{x} \mathrm{d}x = \int_1^{a} \frac{1}{x} \mathrm{d}x + \int_a^{ab} \frac{1}{x} \mathrm{d}x$$

so

$$L(ab) = L(a) + \int_a^{ab} \frac{1}{x} \mathrm{d}x.$$

Which substitution will be most suitable to express $\int_a^{ab} \frac{1}{x} \mathrm{d}x$ in terms of our function L?

- ◯ $z = bx$
- ◯ $z = \frac{x}{a}$
- ◯ $z = ax$
- ◯ $z = \frac{x}{b}$

Once the appropriate substitution has been made, we find that $\int_a^{ab} \frac{1}{x} \mathrm{d}x$ is equal to

- ◯ $\int_1^{b} \frac{a}{z} \mathrm{d}z = a\,L(b)$
- ◯ $\int_1^{a} \frac{1}{z} \mathrm{d}z = L(a)$
- ◯ $\int_1^{b} \frac{1}{z} \mathrm{d}z = L(b)$
- ◯ $\int_1^{ab} \frac{1}{z} \mathrm{d}z = L(ab)$

You have shown that $L(ab) = L(a) + L(b)$ and therefore that our function $L(z) = \int_1^{z} \frac{1}{z} \mathrm{d}z$ is some kind of logarithm.

Part B Logarithm base

As in the previous section, we write $L(a) = \int_1^a \frac{1}{x} \mathrm{d}x$. By definition, this means that $\frac{\mathrm{d}L}{\mathrm{d}x} = \frac{1}{x}$.

We already know that $L(x)$ has the properties of a logarithm. This means that if $y = L(x)$, then $x = g^y$ for some unknown constant g, which will be the base of the logarithms.

Remembering that $\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$, we can combine the information above to show that $\frac{\mathrm{d}\,g^y}{\mathrm{d}y}$ is equal to

- $\frac{1}{y}$
- g^y
- y
- $\frac{1}{g^y}$

One of the defining features of the exponential function e^x is that $\frac{\mathrm{d}\,\mathrm{e}^x}{\mathrm{d}x} = \mathrm{e}^x$. The number e is also the base of the natural logarithms $\ln(x)$.

It follows that $g^x = \mathrm{e}^x$ and that accordingly $L(x) = \log_e x = \ln x$.

We therefore know (at least for positive x) that $\int \frac{1}{x} \mathrm{d}x = \ln x + C$.

Part C Expansion first term

In this section, we investigate the exponential function and how it might be evaluated.

We make an assumption that the function $y = \mathrm{e}^x$ can be written as polynomial with fixed co-efficients a_n:

$$\mathrm{e}^x = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots + a_n\,x^n + \cdots$$

What must be the value of a_0?

Part D Exponential expansion co-efficient

In this section, we continue to investigate the exponential function and how it might be evaluated.

We assume that the function $y = e^x$ can be written as polynomial with fixed co-efficients a_n:

$$e^x = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots + a_n\,x^n + \cdots$$

One property of the exponential function e^x is that $\dfrac{\mathrm{d}\,\mathrm{e}^x}{\mathrm{d}x} = \mathrm{e}^x$.

Using this information, write an expression for $\dfrac{a_n}{a_{n-1}}$.

The following symbols may be useful: `n`

Part E Exponential expansion first terms

In this section, we continue to investigate the exponential function and how it might be evaluated.

We assume that the function $y = \mathrm{e}^x$ can be written as polynomial with fixed co-efficients a_n:

$$\mathrm{e}^x = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots + a_n\,x^n + \cdots$$

Use the answers to the previous questions to write the expansion of e^x up to and including the x^4 term. Do **not** use factorial notation in your answer (write 24 rather than $4!$).

The following symbols may be useful: `e`, `x`

Part F Value of e

Use your expansion up to and including the x^4 term from the last question to calculate the value of e to three significant figures.

Exponential equation 3

A Level P P P Further A P P P

Solve the following for m: $\dfrac{1}{9^m} = 27^{1-m}$.

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Exponential equation 2

A Level P P P Further A P P P

Solve the following for x: $3^x = \dfrac{1}{9^{x-\frac{9}{4}}}$.

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Energy decay

A Level Further A

A steel bar is tapped on one end and the resulting pulse of energy travels backwards and forwards along the bar. A very small fraction α of its energy is lost on each reflection so that after n reflections the fraction of its initial energy left is $(1-\alpha)^n$. It takes a time τ to travel from one end of the bar to the other.

Part A Time for energy to halve

Find an expression for the time it takes for the energy in the pulse to halve.

The following symbols may be useful: `alpha`, `ln()`, `log(number , base)`, `tau`

Part B Time for energy to fall by factor of 100

Find an expression for the time it takes for the energy in the pulse to fall by a factor of 100.

The following symbols may be useful: `alpha`, `ln()`, `log(number , base)`, `tau`

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Apparent magnitudes

A Level Further A

The apparent magnitude m of an astronomical object describes on a logarithmic scale how bright an object appears to an observer. It is related to its actual brightness or energy flux F (i.e. the energy arriving at the Earth per unit area per second) in the following way. Consider two objects with magnitudes m_1 and m_2 and brightnesses F_1 and F_2; the relationship between these quantities is

$$\frac{F_1}{F_2} = 100^{(m_2 - m_1)/5}.$$

Part A Sun and Moon

The magnitude of the Sun is -26.8 and it is a factor of 4.8×10^5 brighter than the full Moon. Find the magnitude of the full Moon.

Part B Supernova 1987A

Supernova 1987A was discovered in the nearby dwarf galaxy the Large Magellanic Cloud and, with a magnitude of $+2.9$, it was visible with the naked eye. It was subsequently discovered that its progenitor was a blue supergiant with a magnitude of $+12.2$. Find the ratio of the brightness of Supernova 1987A to that of its progenitor (give your answer to 2 sig figs).

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Logarithmic equations 3

A Level P P P Further A P P P

Solve the following logarithmic equations.

Part A $\log_3 \sqrt{b} = 2.$

Find b if $\log_3 \sqrt{b} = 2.$

Part B $\log_2(x^2) - \log_2 3 = \log_2 48.$

Solve the following for x: $\log_2(x^2) - \log_2 3 = \log_2 48.$

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