Integrating Factors 1

Further A

P P P

Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + (a+x)y = \mathrm{e}^{-x}.$$

Find the general solution for y as a function of x.

The following symbols may be useful: , `a`, `c`, `e`, `k`, `x`, `y`

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RC Circuit (Integrating Factors)

Further A

P P P

A circuit consists of a capacitor C, a resistor R and a switch in series with a battery of emf V_0. The switch is initially open and the capacitor is uncharged. At $t = 0$ the switch is closed. The equation for the charge q on the capacitor as a function of time t after the switch is closed is

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = V_0.$$

Find how the charge on the capacitor varies with time t given that $q = 0$ at $t = 0$.

Find the equation for the charge q on the capacitor as a function of time t.

The following symbols may be useful: `C, R, V_0, e, q, t`

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Undamped Pendulum (2nd Order)

Further A University

The equation describing the small-angle oscillations of a simple pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\theta$$

where θ is its angular displacement from the vertical at time t, l is the length of the pendulum and g is the acceleration due to gravity. Find an expression for θ as a function of t given that $\theta = \alpha$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \beta$ at $t = 0$.

Find the equation for θ as a function of t given that $\theta = \alpha$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \beta$ at $t = 0$.

The following symbols may be useful: `alpha`, `beta`, `g`, `l`, `t`, `theta`

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Mass on Spring (2nd Order)

Further A University
C C C P P P

A mass m on a spring is subjected to a damping force. The equation describing its displacement x from its equilibrium position as a function of time t is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - b\frac{\mathrm{d}x}{\mathrm{d}t},$$

where $-kx$ is the force from the spring and $-b\frac{\mathrm{d}x}{\mathrm{d}t}$ is the force due to damping. The damping coefficient b is related to the spring constant k by $k = \frac{4b^2}{25m}$. Find an expression for the subsequent motion of the mass given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = V$ at $t = 0$.

Find the equation describing the subsequent motion of the mass given that $x = 0$ and $\frac{dx}{dt} = V$ at $t = 0$.

The following symbols may be useful: `V, b, e, m, t, x`

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Damped Pendulum (2nd Order)

Further A

The equation describing the displacement x of the bob of a damped pendulum from its equilibrium position is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega_0^2 x - 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t}$$

where ω_0 is the angular frequency of undamped oscillations of the pendulum and γ is related to the damping. Assuming $\omega_0 > \gamma$ find an equation for x at time t given that $x = X$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ at $t = 0$. (You will find it helpful to define a new constant ω_1 such that $\omega_1^2 = \omega_0^2 - \gamma^2$.)

Find an equation for x at time t given that $x = X$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ at $t = 0$.

The following symbols may be useful: `X, e, gamma, omega_1, t, x`

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Inhomogeneous Equation (2nd Order)

Further A

Find the solution of the equation

$$\frac{\mathrm{d}^2p}{\mathrm{d}q^2} - 4\frac{\mathrm{d}p}{\mathrm{d}q} + 3p = 3q - 1$$

given that $p = 2$ and $\frac{\mathrm{d}p}{\mathrm{d}q} = -1$ when $q = 0$.

Find the solution of the equation.

The following symbols may be useful: `e, p, q`

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Forced Oscillator (2nd Order)

Further AUniversity

C C C P P P

The equation of motion of a forced oscillator is given by

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \omega_0^2 z = Z_0 \sin(\omega_1 t)$$

Given that $\omega_0 \neq \omega_1$ find the solution for z given that $z = 0$ and $\frac{\mathrm{d}z}{\mathrm{d}t} = 0$ at $t = 0$.

Fnd the solution for z given that $z = 0$ and $\frac{\mathrm{d}z}{\mathrm{d}t} = 0$ at $t = 0$.

The following symbols may be useful: `Z_0, omega_0, omega_1, t, z`

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Differential Equations: General Applications 2i

During an industrial process substance X is converted into substance Z. Some of the substance X goes through an intermediate phase, and is converted into substance Y, before being converted into substance Z. The situation is modelled by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.3x - 0.2y \quad \text{and} \quad \frac{\mathrm{d}z}{\mathrm{d}t} = 0.2y + 0.1x$$

where x, y and z are the amounts in kg of X, Y and Z at time t hours after the process starts.

Initially there is $10\,\mathrm{kg}$ of substance X and nothing of substances Y and Z. The amount of substance X decreases exponentially. The initial rate of decrease is $4\,\mathrm{kgh^{-1}}$.

Part A Expression for x

Find an expression for x.

The following symbols may be useful: `e`, `t`

Part B $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$

Show that $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = k$ where k is a constant.

State the value of k.

Comment on this result in the context of the industrial process.

- The total amount of all three substances increases throughout the process.
- The total amount of all three substances decreases throughout the process.
- The total amount of all three substances is zero throughout the process.
- The total amount of all three substances is constant throughout the process.

Part C Expression for y

Find an expression for y in terms of t.

The following symbols may be useful: `e, t`

Part D Maximum amount of Y

Determine the maximum amount of substance Y present during the process.

Part E Time to produce $9\,\mathrm{kg}$ of substance Z

How long does it take to produce $9\,\mathrm{kg}$ of substance Z? Give your answer to 3 significant figures.