Teachers’ Manual

for

Using Essential GCSE Mathematics

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Brief list of contents

Introduction

The Isaac Physics book Using Essential GCSE Mathematics is designed to be:

- a resource for students to use alongside other GCSE maths materials during their GCSE years
- a guide for studying GCSE maths as an entrant from other educational systems
- a reference document for any students using maths in other numerate subjects at GCSE
- an additional question bank for maths teachers and teachers from other subjects who need materials for supplementing their own subject

Beyond GCSE the book will be useful for:

- revising and extending GCSE performance for those students in transition to studying maths at A level, and in the early months of the A level course
- a continuing skills resource for students who use maths in other A level subjects without the benefit of A level maths
- a means for more mature learners to revise and extend their GCSE maths skills for the workplace, or for continuing education

We have written the book after many years of experience trying to ensure that maths skills and knowledge transfer easily between maths and user-of-maths classrooms, and also between different stages of learning. In practical terms some major issues we have borne in mind are:

- the gradation of mathematical ability and experience in classrooms where maths is used rather than taught, at GCSE and beyond
- the necessity to consider how learners with special needs (including those of very high ability) can approach materials
- the encouragement of independent learning

This book is written with the GCSE Mathematics 9-1 syllabus in mind, and presented by topics so that teacher-users and students can locate suitable practice material easily.

In the book you will find:

- explanations in each chapter as a basis for independent study, with straightforward examples
- questions which move from straightforward practice of the topic through to problems. Exercises begin with Foundation material where this exists. The level of demand tends towards grade 9 through each exercise, and in some cases beyond grade 9 to provide something to for all students.
- indication of material which is usually studied for grades 4-9 marked with § symbols. More subtle breaks are found in some text and question sets to indicate to the teacher using maths where Foundation students are likely to begin to be challenged by questions.

The connectivity of the topics means that questions in one chapter will at times also be suitable practice for another chapter, and we indicate that in this handbook and on-line.

In particular we have selected questions to illustrate ideas which occur in subjects which use maths, and a brief listing of questions by related subject will also be available on-line.

In this manual to accompany the book you will find the following information for each chapter:

- The study aims for the chapter
- Methods and procedures that are required as prior-learning
- Common mistakes and mis-conceptions
- Comments for individual questions
- Suggestions for presenting and linking maths topics where there are obvious transferences between subjects

At the end of the manual there are also some suggestions for using selections of questions as bridging materials between GCSE maths and A level maths and STEM courses.

We are well aware that there are also pedagogical areas of preference or even dissention about what constitutes a good method to offer students for certain mathematical issues. Where it is possible, we have attempted to provide a range of materials to cater for all tastes, and teachers can then decide for themselves if relevant sections of the book suit their purposes or not. For example, Formula Triangles, which cause frequent discussions, appear in a short chapter of their own.

This handbook is written to be useful to a wide range of teachers, from maths specialists to STEM teachers who have learned maths appropriate to their own university courses, but who are inexperienced in teaching mathematics as a subject. Consequently, we have attempted to offer ways of handling some basic issues which arise when maths is put to use, as well as dealing with more rigorous issues. We hope that, in the manual, all teachers will find items which are helpful.

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General notes for this manual

This manual is provided on the Isaac Physics website alongside three spreadsheets which supply the following information:

1. **GCSE maths syllabus mapping** A syllabus guide showing a matching of the book chapters, and the questions within them, to the syllabus requirements for GCSE mathematics published by the DfE, and to the specifications published by a range of examination boards.
2. **Maths skills for GCSE science subjects** A guide for science teachers showing the maths skills listed for GCSE science by the DfE, and the locations within the book of material associated with each skill.
3. **STEM question finder** A question finder for STEM teachers and user-of-maths teachers, where problems set in context are listed under a selection of headings.

Syllabus requirements

Syllabus references (see Spreadsheet 1 GCSE maths syllabus mapping) have been prepared for this book for the 7 most common examination boards that offer GCSE-level maths in the UK: Edexcel/Pearson, AQA, WJEC, OCR, IGCSE, CCEA (Northern Ireland) and SQA (Scotland). With the exception of IGCSE, CCEA and SQA the contents of the syllabus are governed by the DfE document GCSE mathematics: subject content and assessment objectives .

Closest matchings to this document are given for the remaining boards, and limited references are provided where material occurs in qualifications other than mathematics. For example, GCSE Further Mathematics for CCEA. Where material is not specified at a comparable stage of learning no references are provided. The authors draw attention to the fact the matchings cannot be exhaustive because of the number of qualifications involved, and teachers should regard this only as a general guide. KS3 learning, or its equivalent, is assumed.

For teachers unfamiliar with the different examination boards it may be useful to know the following information. Edexcel/Pearson, AQA and WJEC boards all present their syllabus in the same form as the DfE document, with the 97 specification descriptors divided into 6 sections which will be referenced in this handbook as N=Number (16 descriptors), A=Algebra (25), R=Ratio (16), G=Geometry (25), P=Probability (9) and S=Statistics (6). OCR and IGCSE use different groupings of topics. OCR uses 12 numerical categories and IGCSE 9 numerical divisions. CCEA divides the syllabus into maths modules (M). M1, M2, M5 and M6 are Foundation and M3, M4, M7 and M8 are Higher material. Certain topics which match the DfE document occur in GCSE Further Mathematics, so those topics are recorded in the syllabus table. SQA follows National qualifications which are studied sequentially. This teachers' guide refers to National 4 and National 5 in Mathematics. Each National Level also offers a qualification in Applications of Mathematics, but these qualifications are not referenced here because the essential learning is covered in National Mathematics.

Questions in Using Essential GCSE Mathematics are grouped into topics with a theme for each chapter. Within a chapter, questions generally share the same major syllabus descriptor. Occasionally, to reinforce specific connections of reasoning between topics, a chapter will contain a question where the primary designation is a different syllabus descriptor. Other cross-connections are reflected in the subsidiary connections columns in the spreadsheet **GCSE maths syllabus mapping** provided on-line and they are summarised in this manual. The cross-connections in the manual are shown relative to the DfE syllabus document, but further breakdown for individual examination boards is given in the on-line spreadsheet. Teachers can therefore use these notes or search on-line to accumulate additional connected questions for their own use. The list of connections is not exhaustive - it is intended to be a helpful place to start. It should also be noted that, by virtue of question-grouping, not all specification descriptors can be the target of a chapter of their own.

Inevitably, in the early chapters of the book, it has been necessary to presume some familiarity with ideas that are revised and studied at greater depth in subsequent chapters. For example, Chapter 3 on BIDMAS assumes some ability to handle fractions which are studied in detail in Chapter 4. We have attempted to keep to a minimum the places where these assumptions occur, and we have limited assumptions to employing material which should be secure from KS3. Relevant references to other related chapters, both earlier and later in the book, are listed in the commentaries for each individual chapter.

In choosing questions for Using Essential GCSE Mathematics we have tried to include examples so that the whole maths course is covered in outline. However, the book is primarily a resource for users of maths, so coverage of some parts of the maths syllabus has to be limited in order to include applications which we know students will find beneficial if they proceed to a variety of studies beyond GCSE. Thus, the book is a useful resource for augmenting and revising GCSE maths as a subject, but it is not a stand-alone resource for teaching the subject from scratch.

Teachers of GCSE science may well wish to locate questions which are directly related to maths skills specified for their subjects at GCSE, so links between the required maths skills and chapters of the book are shown in Spreadsheet 2 **Maths skills for GCSE science subjects**. To help teachers in selected subjects which use maths, Spreadsheet 3 **STEM question finder**, contains a listing of problems in contexts which could be useful for Biology, Chemistry, Physics, Engineering (to represent technical subjects), Economics and Geography at GCSE and early A level.

There are also a few items in this book which are not specified by the DfE but which the authors have found from their own experience to be extremely useful in STEM settings. For example, Chapter 12 Writing and Using Algebra includes a table of Greek letters, and some questions that use some of those letters, because a lot of students (particularly those with low reading scores) falter when asked to substitute values for unfamiliar letters, even when the calculation is straightforward.

Qualification levels

The DfE syllabus defines material which is common to Foundation and Higher teaching, and then additional material for Higher level only. In the text of the book the material for Higher level is identified by the § symbol, but the questions which follow can be attempted by any student on the basis of the explanations provided ahead of the questions.

What is less obvious for user-of-maths teachers is the general effect on processing which is seen by splitting students into Foundation and Higher classes. In addition to the difference in subject knowledge specified by the DfE the following observations may be helpful:

Foundation students:

- see fewer multi-variable substitutions
- generally perform calculations with fewer stages of working
- have less experience with rearrangements involving powers and roots
- have less practice spotting for themselves where units must be changed within a calculation
- can become very adept at performing familiar calculations, but will suffer from lack of confidence when asked to “work backwards from the answer”. They will be able to do so, but the process may take a while and they will need encouragement.

- have less success when asked to extend their methods/mathematical reasoning. For example, expanding factor brackets to give a quadratic expression may be done by rote-learning an acronym (e.g. FOIL shown in Chapter 14), instead of multiplying one bracket systematically by the terms in the second bracket. The systematic method can be extended to any number of factor brackets, but a rote-learned acronym leads the student into confusion if they try to use it for more factor brackets.

In some of the exercises there is a break in the coloured background of the questions. This indicates for users-of-maths where Foundation students may begin to struggle. The questions will not necessarily be impossible for them, but processing will take longer. In these teachers' notes we have tried to offer a more detailed analysis for chapters where a significant number of questions relate to Foundation material, and we have tried to highlight common misconceptions and methods that teachers can use to promote better reasoning.

Style of questions

Learning and testing of GCSE (and advanced) maths is structured so that command words such as "find" and "calculate" provide clarity of instruction. Unfortunately, general questions posed in the real world rarely use these selected command words, and students can be baffled if they are faced with a problem which is inherently very simple, but which does not tell them in familiar command language what to do. Higher ability students naturally analyse any problem and produce their own command structure. But other students who depend upon scaffolding often search real-world problems in vain for a phrase which links them to their bank of learned responses. The problem then becomes one of linguistic connections as much as mathematical connections.

This book is intended to give a broad experience for students handling GCSE-level maths, so must address phraseologies typical of real-world questions. Hence, all questions only require GCSE knowledge, but not all questions are necessarily of the exact form required for an examination question in terms of the command language employed. For example, some questions ask "What is….?" Students can be encouraged to replace this language for themselves with phrases such as "Find the value of the variable and give its units". We hope that this may help some students to become more adept at analysing what is required within a problem.

Provision of answers

It is a policy at Isaac Physics that authors do not make available any pdf of answers or solutions, so that students are not able to discover them by searching the book and/or website. Thus, students must engage with the questions they are set, work to produce answers and learn from their mistakes. The same is true of this teachers' manual, in that we discuss common errors but do not give the answers, so should a student come across the handbook, they are still not able to extract answers without doing work themselves.

Significant figures

In maths exams the guidance is often to calculate answers to 3 figures. In physics, students are usually expected to work out the number of significant figures by looking at the number of figures for the variables: the number of significant figures in the answer matches that for the variable with fewest significant figures (the answer cannot be more accurate than the information given). In most questions we given specific guidance so that students do not waste time worrying about significance. Where no guidance is given, then answers should have the significance which fits the context i.e. match the least number of significant figures.

On-line question boards

If the website version of this book is accessed, students and teachers will see default question boards for each chapter. The defaults are chosen with the assumptions that the book is being used for revision purposes close to examination, and that some challenge is required. Some chapters have one board which is suitable for both Foundation and Higher, and some chapters have separate boards for Foundation and Higher. For a few chapters where practice is particularly useful, there are 3 boards: one basic practice board which is initial practice for all students, and then a choice of either a follow-on Foundation board or a follow-on Higher board. The follow-on boards contain extra material and a raised level of difficulty.

We draw attention to the fact that the questions in the book are not necessarily intended simply for examination purposes: many of them are intended to explore use of maths in STEM and other contexts. This means that the questions may take students longer to process than straightforward exam practice questions, and Foundation students will find questions demanding even when the knowledge-base is Foundation material. Consequently, teachers should be wary of setting a whole question board, at either Foundation or Higher, without assessing the time that their students will need to complete it. Teachers may also wish to reconfigure Higher boards to include more straightforward questions if their class contains only a small proportion of students aiming for grades 8 and 9. It is also possible to set Foundation boards followed by Higher boards to give the full range of practice. The website will record questions which have already been attempted so that students are not required to repeat their work.

These boards will obviously not suit every purpose, and teachers are encouraged to use the question-setting tools on Isaac physics to create boards for their own use. In particular, creating a customized board will allow combining questions from different chapters.

Students with special needs

Between them the authors have accumulated experience in teaching students with a significant range of abilities, needs and starting points, and we have tried to provide materials that can be employed to address some of the issues that we have come across.

There is not space to discuss many learning support techniques in the text of this guide, so we are presuming that general reading and processing support is available for those students with special needs. In addition, for teachers of other subjects using maths, we offer the following suggestions to minimize errors, and to make life easier for students with reading troubles such as dyslexia:

- Encourage underlining or highlighting of variables and their values.
- Extract the variables and values to a list beside the formula where they will be substituted. If it is helpful, then the rest of the text can then be obscured while the substitution is achieved.
- Encourage students to create their own look-up tables of general information. This includes the usual formulae lists, but it also includes such things as multiplication tables if they are a difficulty: multiplication tables can be prepared by students using addition if necessary (including in an examination), and having tables to hand helps students to focus on the content of questions without distraction to attend to processing troubles.
- Insist that all lines of working are shown. For example, when rearranging an equation write the inverse operation explicitly on both sides of the equation and demonstrate the cancelling. Similarly, inequalities are best handled using simple inverse operations one at a time, and not employing a strategy of multiplying by -1 at the same time as reversing the inequality (see Chapter 31).

- Be aware that the precision of mathematical communication may not be appreciated/used by those students who struggle to read text, particularly those who have visual disturbance: the written and typed page appears untidy to them anyway, so either they do not read details, or they ignore details because they do not expect them to be significant. For example, these students may not notice the difference between I and I' for object and image, they may place decimal repeat dots over the wrong digits (if they see the markers at all), and they may need constant reminding to under-line vectors in written work. It helps these students enormously if teachers set them some work which focusses on picking out and using these details.

List of mathematics acronyms found in chapter-by-chapter commentary

BIDMAS/BODMAS Brackets, Indices (Order), Division, Multiplication, Addition, Subtraction

FOIL Firsts, Outers, Inners, Lasts

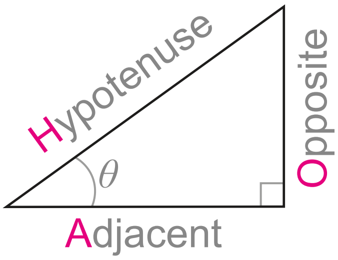
HCF Highest Common Factor

IQR Inter Quartile Range ($Q_3 – Q_1$)

LCM Lowest Common Multiple

Q_1, Q_2, Q_3 First, second, third Quartile

SOHCAHTOA For a right-angled triangle with a known angle θ, and sides labelled as shown:



$$\sin(\theta) = \frac{opposite}{hypotenuse} \quad \cos(\theta) = \frac{adjacent}{hypoenuse} \quad \tan(\theta) = \frac{opposite}{adjacent}$$

$$\mathrm{S} = \frac{\mathrm{O}}{\mathrm{H}} \qquad \mathrm{C} = \frac{\mathrm{A}}{\mathrm{H}} \qquad \mathrm{T} = \frac{\mathrm{O}}{\mathrm{A}}$$

suvat The suvat formulae are used for modelling motion with **constant acceleration**.

s: displacement

u: initial velocity

v: final velocity

a: acceleration

t: time

Chapter 1 Solving a Maths Problem

The aim of this chapter is to provide a framework for solving problems that look horrible at GCSE, but are actually relatively simple if analysed correctly at the beginning. Students often regard this sort of chapter as optional rather than informative, so teachers may like to encourage engagement with the material by setting the example as a problem to the students and then revealing the answer later.

This example has been chosen because many students see a mechanics diagram and simply decide that it will be impossible for them without even reading the question. But the solution to the given problem employs methods which should be very familiar to students studying at Foundation level.

The key step is the first step of realizing that the minimum length of rope allows the box to touch the ground but with no slack. So this is the diagram to draw. Of course, there are many assumptions, and these could be considered as part of a class discussion. In particular, students should realise that the rope must be attached by some means at both ends. They can consider the case where the rope is tied to a ring or a hook. Then more rope is needed for the knot, but the height of the ring or hook must be subtracted. Also, attention can be drawn to the number of significant figures: the lengths are given in whole metres, so even if the actual length of rope varies by a few cm, that value will not be significant. Other assumptions include: the ground is flat and horizontal, the rope does not stretch (extend), etc.

Possible source of error: missing the conversion of 50cm to 0.5m so that units are consistent.

Point of challenge: marking right angles where horizontal and vertical lines intersect. For some students it is a “eureka” scenario to realise that marking horizontal and vertical directions essentially fixes their diagram to a cartesian grid which they can handle.

In some situations, it can also be useful for a class to discuss in general why they perceive this question to be a difficult one. Of course, for some individuals there are issues with interpreting information on diagrams, and recording what they see. This may be especially true for students requiring learning support, and those students may find it helpful to verbalise what they see and write that down (or use their TA) subsequently. But in general, students who opt out of a question on sight may do so simply because of anecdotal influence: if the peer group decides that what appears to be physics is difficult, then a lot of students are immediately disinclined to attempt the materials. Science teachers may well want to pursue this topic to allay fears in science as well as maths.

DfE connections: A4 general algebra, A21 writing an equation, G20 Pythagoras’ Theorem (2-D)

Related chapters of this book: Chapter 12 Writing and Using Algebra,

Chapter 38 Pythagoras’ Theorem

Chapter 2 Factors (numerical)

Aims: * identify prime numbers

* write numbers as the product of prime factors

* find HCF and LCM

DfE syllabus main objective: N4 concepts of prime numbers and factors

Skills assumed: correct use of four basic operations (+, −, ×, ÷), familiarity with tables of multiples

Associated skills and knowledge: set notation, notation for indices

DfE connections: N6 squares (Q4), N7 calculating roots (Q8), A4 indices (Q9), N6 powers(Q10)

Related chapters of this book: Chapter 13 Indices and Roots

General comments:

- A common source of error is to identify 1 as a prime number.
- The first prime number is 2 (factors 2 × 1), and by definition this is the only even prime: if *E* is any Even number except 2, then *E* has at least 3 factors: E, 2 and 1.
- Systematic decomposition into prime factors begins with dividing the starting value by the smallest prime number which is a factor, and applying that factor as many times as possible, then working similarly through the other prime numbers in ascending order, until the remaining value is also prime. But this is not the only way in which to create a factor tree. Branches of the tree can be established using any factors and each branch decomposed until primes are reached. For example, a systematic search for prime factors of 24 will yield 2 × 2 × 2 × 3. The same result will be achieved by writing 24 = 4 × 6 and then 24 = (2 × 2) × (2 × 3).

Notes for specific questions:

2.1 and 2.4 Set notation should be familiar to the students. The elements listed in the set are the only values which can be used in the question. (Note: It is not necessary for members of a set to be related mathematically, although it is often the case that there is a connection.)

2.4b Possible wrong answer 22 (1 is treated as prime)

2.5a Possible wrong answer 1 and 2 (1 is treated as prime)

2.5b Possible wrong answer 1 and 7 (1 is treated as prime)

2.5c Possible wrong answer 1 and 29 (1 is treated as prime)

2.6a Possible wrong answer 45 (multiplying together without considering multiples of 3)

2.6b Possible wrong answer 80 (multiplying together without considering multiples of 8 and 10)

2.9 Students frequently define a power loosely as "the number of times a factor is multiplied by itself". This statement leads them into confusion because for *n* appearances of the factor there are (n-1) multiplication operations, so the student is unsure whether to write *n* or (n-1) for the index. For these students reinforce learning that the power is the count of how many times the factor itself is used. Practice of handling indices is provided in Chapter 13 Indices and Roots.

2.10c Encourage the systematic decomposition of 5460 and 3248700: the decomposition seems to be a large task, but it is still an efficient method. If students try to work from 5460 upwards, then it is possible to work out that 5460 must be multiplied by 5 to generate 00 in the two right hand digits, but finding the other factors is hard work.

Chapter 3 BIDMAS/BODMAS and Substituting Values Into Formulae

Aims: * apply BIDMAS reliably

* substitute and use numerical values, including combining operators and signs of real values

* use numerical answers to work out a BIDMAS-based procedure to generate that result

DfE syllabus main objective: N3 prioritising and performing operations

Skills assumed: handling and combining signed real values, including combining signs with operators

squares, cubes, and square roots of integers,

processing simple fractions

Associated skills and knowledge: recognising inequalities,

using exponential functions (interest calculations)

DfE connections: N1 ordering values (Q3), N2 working with operators and directed numbers

N6 using powers and roots, N8 calculating exactly with fractions

A2 substituting numerical values, R16 compound interest (Q11)

Related chapters of this book: Chapter 4 Fractions, Chapter 12 Writing and Using Algebra

General comments:

- We mention combining operators and signs of real values as assumed knowledge. However, teachers may well find it helpful to offer a quick summary of the rules where they are required:

"+" with "+" gives "+", "+" with "−" gives "−", "−" with "+" gives "−", "−" with "−" gives "+"

Notes for specific questions:

3.1c Possible wrong answer 3.5 – division not done first

3.2, 3.5 Higher level students will be able to deduce likely positions for brackets by looking at the size of the answer value. Foundation level students may need to proceed by experimentation. Remind them that brackets always occur in pairs, and instruct them to simply try pairs of brackets in different places until the correct answer is found. To prevent students giving up easily, warn them that they may need to try several different positions for the brackets.

3.2b Some Foundation students can have great difficulty when dealing with zero, so may arrive at 0 + 11 = 11 as an addition idea to aim for, but may need a hint about how to achieve zero from the remaining integers.

3.4 Encourage working on paper to show the fractions at each stage.

3.5 This question will be a challenge for Foundation students, but clues for all parts may be found by looking at the size of the answer. Also, for part c) the sign of the answer is a clue.

3.6 Possible wrong answer 22 (mistake combining operator and sign)

3.7c Possible wrong answer 22 (mistake combining operator and sign)

3.8a Possible wrong answer 4 (mistake combining operator and sign)

3.9c There is potential for proceeding towards a wrong answer by treating -*u* as -8, but most students are likely to back-check quickly when they see they have produced an awkward fraction with a denominator of 7, and find their error.

3.11 This question is a standard compound interest question, and has been placed in the BIDMAS section because the values can be put into a calculator as a substitution exercise.

The question can also be attempted by explicit repeated application of a multiplier. Evaluate the bracket as required by BIDMAS, and apply this n times to the capital amount C. This second idea relates to multiplicative (geometrical) sequences (Chapter 19) and is also a practical example of an exponential function (Chapter 33). Teachers may like to reserve this question for follow-up in those chapters.

Other questions involving interest are found in Chapter 6 Percentages, questions 6.14 and 6.15.

3.12 This question requires straightforward substitution, but students may be very wary of it because of the subscript notation. Subscripts are covered in Chapter 12 Writing and Using Algebra.

Chapter 4 Fractions

Aims: * apply the four operations with fractions

* understand that cancelling down and creating equivalent fractions are inverse procedures

* interpret fractions and their individual numerators and denominators as operators

* solve problems, including those with mixed fractions

DfE syllabus main objective: N8 calculate exactly with fractions

DfE subsidiary objective: N12 interpret fractions as operators

Skills assumed: conversion of simple text statements into equations, fluency with multiples

Associated skills and knowledge: basic understanding of fractions to KS3

DfE connections: N1 ordering values (Q9), N13 units (Q14), R3 expressing fractions (Q15)

Related chapters of this book: Chapter 2 Factors,

Chapter 5 Decimals and Rational and Irrational Numbers, Chapter 6 Percentages,

Chapter 12 Writing and Using Algebra

General comments:

- This is a chapter where work to KS3 and GCSE material is consolidated into one topic and therefore more than one DfE objective is addressed.

- By the time they reach GCSE many students feel confused about fractions. They have generated fractions by practical approaches, involving sorting of objects and division of groups of objects, they have calculated numerically and they have seen abstract algebraic approaches. But they have struggled to relate these ideas. In our experience, these students breathe a sigh of relief when instructed to think of a fraction as simply giving two instructions: multiply by the top number and divide by the bottom number. The step to combining these as one operator may, or may not, help further, but simply reducing the issue to one of using multiplication tables can be a revelation.

- Not all multiplication tables come to mind easily, so encourage students, even those of higher ability, to create useful look-up tables of multiples down the side of their working sheet, before they try to tackle problems. This is equally applicable to the general process of long division (see Chapter 5 for decimal long division), where students often try to work out multiples several times during the awkward division instead of creating a list of multiples first.

- Students sometimes ask why they use the operator “×” for the word “of” when writing algebra for “a fraction of a quantity”. A fraction is a scaling operator, and scaling is achieved by applying a multiplier. An example of this is question 4.13.

- Foundation students are not accustomed to seeing fractions with a negative sign in questions such as converting to mixed fractions. But negative values turn up in all sorts of problems (for example intercepts for straight lines), so we have included some negative fractions in the early practice questions.

Notes for specific questions:

4.8 Foundation students may not have seen simplifying with more than two terms. Encourage the cancelling in stages, and if it is necessary, demonstrate that cancelling can occur anywhere along the products of numerator and denominator. To do this present the products in a different order so that the values which cancel appear as numerator and denominator of a visible fraction.

For example:

$$\frac{2}{5}\times\frac{10}{3}\times\frac{9}{8} = \frac{2}{8}\times\frac{10}{5}\times\frac{9}{3} = \frac{1}{4}\times\frac{2}{1}\times 3 = \frac{2}{4}\times\frac{1}{1}\times 3 = \frac{1}{2}\times\frac{3}{1} = \frac{3}{2}\times\frac{1}{1} = \frac{3}{2}$$

4.8b Rewrite the division operation first to give two consecutive multiplications, and then cancel in stages.

4.8c Rewriting the division, and dealing with the pair of minus signs, can be done in either order, but treating the signs first simplifies the writing task.

4.10b Possible wrong answer 121/216 – not following BIDMAS.

4.11a Possible wrong answer 19/5 – not seeing the word “extra”.

4.14 Not all students will have met feet and inches, so teachers may need to supply details of Imperial units and conversion ratios. Students sometimes ask why they need to do conversions to and from metric now, so there is an opportunity to discuss where this might be relevant, such as plumbing, or general engineering.

4.15 Possible wrong answer 0.43m – multiplying 24/50 by 8/9. Encourage students to write an equation, with the original length represented by a variable, for example L. They will then see that 8/9 is the scaling factor for L, rather than the scaling factor for 24/50. This question could be used to supplement material in Chapter 12 Writing and using Algebra.

Chapter 5 Decimals and Rational and Irrational Numbers

Aims: * work interchangeably between decimals and fractions

* recognise irrational numbers

* work with recurring decimals and fractions

DfE syllabus main objective: N10 work with decimals and fractions

Skills assumed: use of basic fractions, equivalent fractions

Associated skills and knowledge: place value

DfE connections: N2 place value, N8 calculating exactly with fractions,

R8 ratios and fractions (Q7, Q11)

Related chapters of this book: Chapter 4 Fractions, Chapter 6 Percentages

General comments:

- Handling decimals should be very familiar by GCSE. This chapter serves to revise the decimal system, and to provide practice in converting recurring decimals to and from fractions.
- This is one chapter where students with reading issues may struggle because they have difficulty identifying the decimal-repeat notation above the digits. Encourage students to read the question twice, and to highlight the notation. Remind them that, if the decimal repeat has more than one digit, then pattern-repeat dots must occur in pairs.

Notes for specific questions:

7 and 11 Some students may not have worked with the idea that a known fraction/decimal conversion can be used to evaluate or convert other fractions which are multiples in either the numerator or the denominator (or both).

7 This question is accessible for Foundation students, but they may need a hint to write parts c) and d) in the form of a multiple of 1/9 (respectively multiples ×3 and ×(1/2) i.e. divide by 2).

8 Encourage a line of working to evaluate the expressions, where that is possible.

11d There is a possibility that some students do not appreciate the difference between the recurring decimal pattern and -0.90. Ask them to write out the first 4 decimal places for both numbers and subtract the values so that they see there is a numerical difference.

Chapter 6 Percentages

Aims: * convert systematically between percentages, fractions and decimals

* find percentages of a quantity, and one quantity as a percentage of another

* evaluate an original quantity, given the value of a percentage change

DfE syllabus main objective: R9 percentages

Skills assumed: handling decimals, simple fractions

Associated skills and knowledge: ordering values by size

DfE connections: N1 ordering values, N12 fractions and percentages as operators,

N13 using standard units (Q 9,13), R16 compound interest and exponentials (Q 14,15,18)

Related chapters of this book: Chapter 4 Fractions,

Chapter 5 Decimals and Rational and Irrational numbers

General comments:

- Finding percentages and fractions of a quantity, converting between systems of units, and using ratio (for example to find lengths for similar geometric figures) are all problems requiring scaling. Hence relevant operations are multiplying (scale up) and dividing (scale down). If more than one operation is required, combining all of the relevant scaling operations will result in one multiplier that can be applied to find the desired final quantity.
  In examinations at GCSE, percentage calculations are often set on the non-calculator papers. These questions can be solved using a multiplier, but they can also be approached by finding easy percentages or fractions of the quantity and adding those results together. For example, with a calculator, 17.5% is found by applying the multiplier 0.175. Proceeding without a calculator, the problem can be solved by finding 10%, dividing this by 2 to get 5%, dividing that again by 2 to get 2.5% and adding all three results together. This is a perfectly adequate strategy for solving that particular real problem, and Foundation level students find this method very accessible. However, some students think this means that they have to learn "different" chunking methods for every possible percentage calculation. They then overload with learning all their examples and trying to match one to the question in hand, without ever really understanding how to find a general percentage.
  Consequently, we suggest always finding the required multiplier first. Then proceed to work out how this can be broken down if the problem is to be processed without a calculator.

Notes for specific questions:

6.6, 6.12 Remind the students to choose a common system for writing the values before they compare them. Either decimal or percentage form gives clear comparison.

6.11 Possible wrong answer 6.5% - simply finding the value mid-way between the discount percentages.

6.13 Possible wrong answer 2 hours 56.25mins – finding 25% of the new time and adding it on. Remind the students to write down the percentage of the old time that is still used, and refer them to Example 4 in the book.

6.14 and 6.15 Another compound interest calculation is found in Chapter 3, question 3.11.

6.16 Percentage errors are found generally using the equation:

$$Percentage\ error = \frac{Measured\ value - True\ value}{True\ value} \times 100\%$$

For this question the difference in values is always 9g and errors are then calculated with reference to true mass. Students may attempt to find the balance display value and use this instead of the true mass. This is most obvious in c) which will incorrectly give an answer of 20%.

6.16b Students may ask about significant figures. The mass is given to 2sf, so 2sf should be used in the answer, but teachers may decide to accept the exact answer which has 3sf.

6.17a Possible wrong answer 20 – not allowing for the wastage with each cut.

6.18b Possible wrong answer 21,000 – finding 60% of year 2.

6.20 A calculation with more than one percentage change is rarely seen at GCSE but can be tackled by Higher Tier students. Suggest starting with

$$I_{new} = I_{old} \times 1.03$$

$$P_{new} = P_{old} \times 0.90$$

Then substitute these into

$$P_{new} = I_{new} \times V_{new}$$

to produce

$$P_{new} = (I_{old} \times 1.03) \times (V_{old} \times 0.90) = I_{old} \times V_{old} \times (1.03 \times 0.90)$$

The result of the calculation will be of the form $P_{new} = k\, P_{old}$, where k is a decimal multiplier. If P_{old} is the original 100%, the percentage change is $(k–1)\times 100\%$.

If $k = 1$, then there is no percentage change.

If $k > 1$, the percentage has increased and $(k – 1)\times 100\%$ is positive.

If $k < 1$, then the overall percentage change is a decrease, and $(k – 1)\times 100\%$ is negative.

Remind the students that they must interpret their result, and their answer must communicate whether P increases or decreases.

Chapter 7 Ratio

Aims: * write and process expressions of ratio

* relate ratio to fractions and to linear scaling

* employ systematic scaling methods for conversion problems

DfE syllabus main objective: R5 Expressing and applying ratio

Skills assumed: handling simple fractions, calculating percentages

Associated skills and knowledge: symbols used for currencies

DfE connections:

R3/R6/R8/N11 ratio with fractions and linear functions, R4 notation, R9 percentages,

R10 proportionality (Q7,8)

Related chapters of this book: Chapter 4 Fractions, Chapter 11 Units, Chapter 34 Proportionality,

Chapter 36 Real World Graphs and Kinematics, Chapter 45 Scale Drawings and Bearings.

General comments:

- Students are frequently nervous about ratio because they struggle with fractions. For these students ensure that they begin with an integer form for the ratio statement. If they imagine distributing objects in that ratio, then they should be able to see that the number of objects needed to perform the distribution once is the sum (S) of the integers. Dividing a large quantity of objects is then reduced to how many times a group of size S can be distributed. This requires the multiplication table for S. As with fractions (see Chapter 4), some students will benefit greatly from writing out the multiplication table at this point: they can then concentrate on processing the ratio statements without worrying about multiplication or division.
  Although they may have established the group size for themselves, students can still need reminding that this also means that S is the denominator when the distribution is shown as fractions.

- Although some students are capable of working out scaling factors in their heads without error, it is preferable that all students know, and write down, a logical method for working out conversion factors, especially if more than one operation is required.
  We show one method for working out scaling factors in Example 3. Begin with stating a known fact for two quantities on the top line. Then write the scale factor which applies from left to right and show this with an arrow. If required, show the reciprocal scale factor operating form right to left with an arrow. Work vertically down for scaling each individual quantity, and work across to scale between quantities.

- When converting between quantities, students should appreciate that the ratio symbol could be replaced by an equality, so any operation performed on one column must also be performed on the other column. Scaling one quantity to unity provides the multiplier to convert across to the second quantity (and hence the inverse operation to convert back to the first quantity). If the conversion is two-step, involving one vertical and one horizontal scaling operation, the order of operations does not matter (multiplication is commutative).

The same method is used in other contexts such as converting units (see Chapter 11 Units).

- We recommend to teachers who are users of maths that they require their students to show conversion routes explicitly. We find that students often guess whether they should multiply or divide by values they are given. If a question is marked as wrong, they try the inverse operation without gaining understanding of the appropriate reasoning. The usual explanation for the guess-work is that they do not have a regular procedure for doing conversions and they will benefit greatly from being required to work using a logical and reproducible method.

- More scaling and conversion questions can be found in Chapter 6 Percentages, Chapter 11 Units, Chapter 34 Proportionality, Chapter 36 Real World Graphs and Kinematics and Chapter 45 Scale Drawings and Bearings.

Notes for specific questions:

This list does not include all possible errors of writing ratios in the wrong order or using the inverse operation for conversions.

7.3b Teachers may decide to set this question with or without calculators. Without a calculator students may become stuck when they find that, as fractions, cherries are: Recipe 1 50/500 or 5/50, Recipe 2 40/380 or 4/38. Suggest writing Recipe 1 as 4/40. This facilitates comparison without requiring a division.

7.8a Possible wrong answer £14.59 – ratios the wrong way round. Students should see that their answer ought to be less than the input value for Euros, but they do not always think about this.

7.8b may show a similar error for the opposite conversion.

7.10 Students may find it helpful to present all of the parts to this question in two columns with A:B as the “fact” at the top, and different multipliers applied down the columns for the question parts. For part b) this makes it clear that they multiply by x on both sides of the ratio statement. We have deliberately placed the algebraic scale factor first in order to encourage writing algebraically, but many students will process the numerical parts first, and then do part b) to describe what happened with numbers, rather than create the algebra and substitute.

7.11 For ESL students it may be necessary to explain that the word “pixel” is a shortened form of “picture element”.

7.12e Common wrong answer 5.7% - students often forget that, for a comparison, they must give the magnitude of the percentage **and** whether it is an increase or decrease.

Chapter 8 Rounding, Limits of Accuracy and Bounds

Aims: * round values: accuracy to given decimal places or significant figures

* state error intervals using inequality notation

* Higher only: use upper and lower bounds

DfE syllabus main objective: N15 rounding and accuracy

DfE subsidiary objective: N16 upper and lower bounds (Higher only)

Skills assumed:

Associated skills and knowledge: place value, inequality notation

DfE connections: N8 calculating exactly with fractions, R9 percentages,

formulae for area and volume – G16 cuboid (Q13), G17 sphere (Q7)

Related chapters of this book:

General comments:

- To introduce the idea of "closest" or "nearest" value, teachers can ask for the smallest value which would round up, and the largest value which would round down, to give a target number. For example, if D is 5.1 cm to the nearest mm, this is the same as writing $5.05 \le D < 5.15$ cm. Some students benefit from a magnified display showing this in action: if the target is specified to the nearest 1mm, then values are accounted for by taking a 1mm window and placing it centrally on a given target of 5.1 mm on a ruler. Then half of the window falls below 5.1, to 5.05, and the other half of the window is above 5.1 to 5.15 cm.

- The book does not use the notation of 5.1 ± 0.05 cm, but teachers can introduce this if appropriate.

Notes for specific questions:

8.7 Students may need reminding that percentage errors are found using the equation:

$$Percentage\ error = \frac{Measured\ value - True\ value}{True\ value} \times 100\%$$

8.11, 8.12 When combining bounds for variables, encourage statements of which bounds to use before substitution of values. For example, if upper bounds for variables a and b are a_{max} and b_{max}, then the upper bound for the sum of a and b is $a_{max} + b_{max}$.

The usual sources of error are incorrect combinations of upper and lower values. Hints are given to this effect when students tackle the questions on the website.

Chapter 9 Approximation

Aims: * perform quick calculations using values rounded to numbers which are easy to process

* be aware that an estimate suggests the size to expect for an answer to a problem

DfE syllabus main objective: N14 estimation and approximation

Skills assumed: rounding of values, basic fractions, percentages

Associated skills and knowledge:

DfE connections: N2 working with operators and directed numbers,

N3 prioritising and performing operations, N12 fractions and percentages as operators (Q1)

Related chapters of this book: Chapter 3 BIDMAS and Substitution of Values,

Chapter 10 Standard Form

General comments:

- A common difficulty that students experience is deciding what constitutes sensible rounding. Teachers will put in place their own guidelines for this, so the website allows a for range of answers to these questions.

- Generally useful questions are
    1. "Is it nearly a whole number?"
    2. "Is it nearly 5 or 10?"
    3. "How many powers of 10 do you have?"
    4. "Do you need BIDMAS?"
    5. "Do you think the full answer will be bigger than this or smaller than this?"

- Students need constant encouragement to use approximation to check if their answer to a question appears to be sensible. This is especially important with problems involving ratio and conversion of units, where there is a lot of scope for producing answers which are wrong by several orders of magnitude. An example might be finding the value of 3.01×2.15m in cm, giving the answer in Standard Form. A suitable approximation would be 6×10^2cm. Using the inverse conversion for units would return 6×10^{-2}cm. If they think about it, students should realise that this second answer is far too small, because they know that there are 100cm in 1m, so they should expect an answer involving 10^2.

- Questions where teachers could ask for an estimate are found in various places in the book, for example Chapter 11 on units, questions 11.6 and 11.10.

Notes for specific questions:

Chapter 10 Standard Form

Aims: * convert to and from Standard Form

* perform operations with numbers in Standard Form

* use Standard Form in problems with very large and very small numbers

DfE syllabus main objective: N9 Standard Form

Skills assumed: using powers of 10, applying rules of indices, converting standard units

Associated skills and knowledge: place value, using operators with signed integers

DfE connections: A4 rules of indices

Related chapters of this book: Chapter 11 Units, Chapter 13 Indices and Taking Roots

General comments:

- Students may need a reminder about how to write powers using indices. Reference can be made to the introductory summaries and tables in Chapter 11 Units (for powers of 10) and Chapter 13 Indices and Taking Roots (using algebra). To demonstrate how to write powers of 10, begin with a familiar positive integer power, say 1000, written as $10 \times 10 \times 10$ and then as 10^3. Then divide by 10 explicitly. Repeat until $10 \div 10 = 1$, which, continuing the pattern, is 10^0. (Note: anything else to the power 0 is also 1.) Subsequent division of 1 by 10 then leads to 10^{-1} and so on, providing decimal places, and negative integer powers when written in index form.

- Rules of indices are revised in Chapter 13. For handling Standard Form students need to recall that when multiplying numbers, the indices are added, and when dividing numbers, the indices are subtracted. If necessary, this can be demonstrated explicitly. In this case, we recommend that the calculation is arranged so that values which cancel in the numerator and denominator are vertically aligned, so that it is easy to see what will remain:

$$(4 \times 10^5) \div (2 \times 10^2) = \frac{4}{2} \times \frac{10}{10} \times \frac{10}{10} \times 10 \times 10 \times 10 = 2 \times 10^3$$

- Remind students that the power of 10 in Standard Form may be positive or negative. Therefore, when adding or subtracting indices, they must take the power with its sign. For example:

$$(4.5 \times 10^6) \div (1.5 \times 10^{-2}) = 3 \times 10^{6-(-2)} = 3 \times 10^8$$

- Other skills which may need revising before attempting problems with Standard Form are substituting values and generally applying BIDMAS (see Chapter 3). In order to ensure that both parts of Standard Form numbers remain together when applying BIDMAS, we recommend that each number is presented in brackets, as shown in the two examples in the commentary above.
- When asked to add or subtract Standard Form numbers with a different power of 10, students who rely on procedure tend to turn both values into ordinary numbers, even if this gives them

many zeros (and much more scope for error). Encourage these students that they only need to decide to use the power of 10 shown for one of the numbers, and convert the other value to a number in the same power of 10. Refer them also to the examples shown in the text for Chapter 10.

Notes for specific questions:

10.5 Unit conversions and prefixes are summarized in Chapter 11 Units. Encourage the systematic conversion of units by deducing required multipliers from a known fact (instead of inspired guessing). One suitable method is shown in Example 3 in the text for Chapter 7 Ratio, and the method is discussed in detail in this teachers' guide under general comments for that chapter.

10.6 – 10.10 Most errors arise because students do not handle the powers of 10 correctly, especially where processing must be done by converting one value into a form that shares a common power with a second quantity.

10.7d Common error 2.7×10^2 – difficulty dividing by a negative power.

10.14, 10.15 These questions were written as pre-cursors to discussions of orders of magnitude for STEM topics. Teachers may like to follow up this idea, and possibly link with Chapter 11 Units.

10.16 Students may ask why the expression for energy is negative. An electron is considered to have an energy level of zero when it is stationary and not bound into an atom. If an electron is bound into an atom, then energy would have to be put in to free the electron from the atom. To remove an electron from the innermost energy level of a hydrogen atom ($n = 1$ in the question) requires an input of 13.6eV. So, the innermost energy level within the atom is negative, with a value of -13.6eV.

Chapter 11 Units

Aims: * ensure familiarity with common unit prefixes

* systematically deduce and apply conversion factors between units

* correctly handle units within problems

DfE syllabus main objective: N13 use standard units

DfE subsidiary objective: G14 use standard units of measure and related quantities

Skills assumed: handling powers of 10, applying general scaling factors

Associated skills and knowledge: index notation

DfE connections: R1/R11 compound units, R5 ratio, R9 percentages (Q9)

Related chapters of this book: Chapter 7 Ratio, Chapter 10 Standard Form

General comments:

- As with other conversion-based topics, there is a strong tendency for students to indulge in inspired guesswork for multipliers when converting units, rather than apply themselves to writing out a method. We strongly recommend that teachers set some of these questions as an on-paper exercise so that they can insist that method is shown, and then check that the method is reasoned correctly.

- The method to deduce and apply multipliers shown in this chapter of the book is the same as that used in other chapters, for example Chapter 6 Percentages and Chapter 7 Ratio. These other chapters contain questions which could be added to a practice board for conversions if teachers require such a board. More examples are also found in Chapter 36 Real-World Graphs and Kinematics.

- The obvious errors for questions are using inverse operations for conversion factors, and mis-handling indices. Suggest that the students ask themselves, "Did I expect my answer to be bigger than, or smaller than, what I started with?" If students have produced answers which tend in the opposite direction from their expectations, then they should check their starting conversion fact and their scaling factors. Checking of this form ties in with DfE objective N14 checking calculations using estimation.
The possible wrong answers are far too numerous to list, so only selected wrong responses are shown in the question notes below.

Notes for specific questions:

11.7 Students may struggle with this question for several reasons. 1) A starter fact is given in the text for dm and cm, but students may be wary of it simply because they do not often use dm. 2) Students often learn that 1litre = 1000cm^3, which is useful as a quick route to part b), but they do not practise doing the conversion using the dm to cm relationship, so it is unfamiliar. Hence, we set part a) first so that they do not use a remembered result. 3) The conversion occurs in the denominator. There is an example in the book to show how this works – Example 5. In question 11.7a converting to m^3 results in dividing by 10^{-3}, which, using the rules of indices, gives multiplication by 10^{+3} (positive integer shown explicitly to make the answer clear).

Although general methods are usually preferred, there is an alternative approach to problems of this sort. In spoken English, 4 moles/litre is read as "four moles per litre". Students should see that "per" therefore specifies "in 1" and the statement becomes "four moles in one litre". Ask students to rephrase Part a) to specify moles in 1 cubic metre, and then the problem reduces to finding how many dm^3 (litres) are in $1m^3$ and thereby multiplying by 10^3.

Whichever method is used, 11.7b should follow with less difficulty.

11.8 Students may make mistakes because they do not read the question carefully and mix up m/s and mph. They may also try to do more than one part of the conversion in one line of working – insist that time and distance conversions are done in separate steps.

11.9 As in other exercises, students may need reminding that percentage errors are found using the equation:

$$Percentage\ error = \frac{Measured\ value - True\ value}{True\ value} \times 100\%$$

11.10a Likely incorrect answer $1.97 \times 10^{+4} m^2$ – inverse conversion factor applied.

11.10b Likely incorrect answer 0.0000062mV – inverse conversion factor applied.

11.11, 11.12 These questions are no longer required at GCSE, although they were used in previous specifications. They are very useful for STEM students, so we have included them as transition material (marked t on the syllabus spreadsheet) suitable for those students progressing to additional studies in maths and physics. The principle is that the units of a quantity can be deduced from a formula or expression for the quantity, and hence the type of quantity can be inferred.

By replacing each variable of length with metres, and ignoring constants, expressions will be reduced to units of m, m^2, m^3, or none of these, so it is possible to select those quantities which are lengths, areas and volumes.

In the general method of dimensional analysis, length, mass and time are all combined to identify the type of a quantity and find its units. The method is often used in physics to provide a check on the validity of equations: the measured units of the target variable must be the same as the units which are derived algebraically from the formula given to calculate the variable. For example, a formula for a variable measured in m^3 must contain only terms with **overall** units of volume. If a formula is proposed with separate terms in area and volume then there is not equality of dimensions and the formula is incorrect.

Chapter 12 Writing and Using Algebra

Aims: * translate text statements into algebraic expressions and formulae

* understand and use superscript and subscript notations

* work with variables represented by letters from different alphabets

* simplify algebra that has been created and solve equations where appropriate

DfE syllabus main objective: A21 translate into algebra and solve equations

Skills assumed: substitution of values, ordering of operations (BIDMAS), solving equations, combining indices

Associated skills and knowledge: superscript and subscript notations, Greek characters

DfE connections: A2 substituting numerical values, A4 simplify and manipulate algebra,

A17 solving linear equations

Related chapters of this book: Chapter 3 BIDMAS/BODMAS and Substituting Values Into Formulae,

Chapter 13 Indices and Taking Roots, Chapter 23 Solving Linear Equations

General comments:

- On the whole, the main difficulty with these questions is confidence. Matters that worry students include 1) identifying the correct variables and operators to write algebra themselves, 2) taking the initiative to allocate variables for themselves ("let *x* represent the value of"), and 3) dealing with unfamiliar notation. Allocation of variables tends to happen more easily as algebra becomes more familiar, so this chapter concentrates on concerns 1 and 3.

- At the beginning, students may benefit from a practice of highlighting or underlining important parts of the text and then writing above those parts the corresponding variable name or the operation. By the time they reach the end of the text, much of the algebraic expression or equation is already visible, and can be transcribed. Gaps can be filled in by referring back to the text, and the algebra can be rewritten into a more familiar form if required. For example, writing question 12.3 in the order of the text results in allocation to variable *z* as the last statement, and so *z* appears on the right-hand side of the equation, rather than on the left.

Several problems in other chapters of the book include the skill of writing equations from text. Some are listed below:

Finding the original length of a panel of wood: Chapter 4, question 4.15

General writing of algebraic equations with indices Chapter 13 questions 13.9 and 13.10.

Calculating cooking times Chapter 17 question 9

Converting between Centigrade and Fahrenheit Chapter 17 question 10

Notes for specific questions:

12.5 Possible error: statements involve $12L$ – ignore return journey.

12.9 Possible error in equation so that using BIDMAS only applies dividing by 3 to second term – brackets or common denominator required.

12.11c May not be fully simplified – some students do not recall that multiplication is commutative, so *gf* and *fg* can be combined etc.

12.14b Likely wrong answers: 1) mixing pence and pounds because 80 pence cost per letter is not converted to £, 2) giving answer entirely in pence not £ – not reading question correctly.

Chapter 13 Indices and Taking Roots

Aims: * understanding and application of rules of indices

* calculations with familiar powers and roots of small integers

* Higher only: use of fractional indices

DfE syllabus main objective: A4 rules of indices

Skills assumed: familiarity with squares of numbers 1-10, higher powers of 2, 3 and 4,

writing simple algebra

Associated skills and knowledge:

DfE connections: N6 using powers and roots, N7 calculate with roots and indices,

A21 translate into algebra and solve equations (Q9, 10)

Related chapters of this book: Chapter 12 Writing and Using Algebra, Chapter 21 Surds

General comments:

Questions in this chapter are mainly straightforward practice.

Common sources of error are:

1) the index is not applied correctly to the quantity on the immediate left-hand side of the index. In particular, in taking a power of a bracket, the index is not applied to everything in the bracket. For example $(ab)^2$ evaluated as ab^2 when the correct answer is a^2b^2.

2) dividing by a negative power not returning a positive index. For example $a^{14} \div a^{-1}$ evaluated as a^{13} when the correct answer is $a^{(14 - (-1))} = a^{15}$.

Notes for specific questions:

13.5c Possible wrong answer $(3a^3)/2$ – squaring 3 as well as a^2.

13.8c Possible wrong answer $9a^7$ – squaring 3 as well as a^4.

13.23b Possible wrong answer a^{13} – incorrect handling of negative indices.

Chapter 14 Expanding

Aims: * multiply a single term with a bracket

* find the product of two binomial brackets

* Higher only: more complicated expansions of brackets

DfE syllabus main objective: A4 simplify and manipulate algebra

Skills assumed: using indices

Associated skills and knowledge:

DfE connections:

Related chapters of this book: Chapter 15 Factorising I: Common factors,

Chapter 16 Factorising II: Quadratic Expressions

General comments:

- This chapter is essentially straightforward and stand-alone.
- Watch out for errors handling "− × −".
- The skill of using index notation to simplify terms is tested to a greater extent in the later questions, most of which apply only to Higher studies.

Notes for specific questions:

14.3b Possible wrong answer is $- x^2 – x$. "− × − = +" not applied.

14.5b Possible wrong answer is $12p – 15r$. "− × − = +" not applied.

14.6c Possible error: constant term of -14. "− × − = +" not applied.

14.11c Students may note this is a difference of two squares problem. They may only offer the two terms of the answer, rather than show full working with cancelling out of the terms in s.

14.11d This may be the first time students have used powers of 2 in a problem of this type. Suggest that working is written in full, which for the first term gives $2^n \times 2^n$, and then apply the rules for indices. Some students will see that the product takes a difference of two squares form, so they already recognise that two of the terms cancel out each other, and those students may offer only two terms for working, even if asked to give full working.

14.14 This question is included to give practice expanding binomials when the brackets contain other variables. Students are unlikely to be familiar with notation for writing powers of trigonometric functions, so expect to see $(\sin x)^2$ and not $\sin^2 x$, and similarly $(\cos x)^2$ and not $\cos^2 x$.

In particular, teachers should watch for incorrect presentations such as $\sin x^2$ and $\cos x^2$. $(\sin x)^2$ may be written as $\sin^2 x$, but this is not the same as $\sin (x^2)$. If students are confused, then we suggest that $\sin x$ is replaced by S and $\cos x$ is replaced by C to work through the question, and then the substitution is removed in the answer line.

Note that this question is extension material.

Chapter 15 Factorising I: Common Factors

Aims: * identify factors which are common to all terms in an expression

* extract these factors as a term by which to multiply a bracket

DfE syllabus main objective: A4 taking out common factors

Skills assumed: using indices

Associated skills and knowledge:

DfE connections: A1 using and interpreting algebraic notation

Related chapters of this book: Chapter 14 Expanding, Chapter 16 Factorising II: Quadratic Expressions

General comments:

- Extracting a common term is the opposite procedure from multiplying a term with a bracket. If students struggle with factorising, and they have not studied Chapter 14 Expanding, it may help them to do Chapter 14 before the current chapter. That way, they will see how a common term is used as a multiplier. This may make it clear to them why they identify constants and variables which are common, and can therefore be taken outside a bracket.

- Students often forget that coefficients can be expressed as products of numerical factors, so they miss taking out common numerical factors. Encourage the writing of coefficients as products of prime factors if there may be a common numerical factor.

- Most importantly, students often forget that all coefficients are the product of the coefficient itself and 1. And in particular, if no coefficient is shown then the coefficient has a value of 1. This becomes an issue if, at any position in the expression, all visible constants and variables would be removed as a common term. Without showing the coefficient as 1, students tend to leave the position in the expression as a blank space or "0", not realising that when they re-expand their answer they will have a term missing.
  Consequently, we recommend that:
  1. At least in early practice, students write in a coefficient of 1 explicitly before they begin factorising.
  2. Teachers insist that the resulting common term and bracket are re-expanded to back-check that the original algebra is returned. If students see this check as part of the algebraic process, they save themselves much frustration later on.

- We also note that many students (especially those with reading issues) benefit from circling all the common factors that they will be taking outside the bracket. This makes it very clear what to write outside the bracket, and anything left behind, including operators, is copied into the bracket.

Notes for specific questions:

15.5b Likely wrong answer $9(x + 1)$ – incorrect expansion of $-3(x - 1)$. Ask students to check their integer term.

15.6b Likely wrong answer $-4x$. Incorrect expansion of -4 with bracket (check integer term).

15.9a Likely wrong answer -24. Subtraction of $(-2)^3$ incorrect.

15.9b Likely wrong answer $-n^3(n - 1)$. Extraction of $-n^3$ not completed correctly. Suggest that students re-expand their result to try to find their error.

15.10c Likely wrong answer $2\pi r^2(h - (2/3)l)$. Squaring of r in bracket, but no squaring of 2.

Chapter 16 Factorising II: Quadratic Expressions

Aims: * understand the principles for finding binomial factors

* find two binomial factors for $x^2 + bx + c$

* recognise the difference of two squares form for quadratic expressions

* Higher only: find two binomial factors for $ax^2 + bx + c$ with $a \neq 1$

DfE syllabus main objective: A4 factorising quadratic expressions

Skills assumed: general factorising by extracting common terms

Associated skills and knowledge:

DfE connections: N8 working with surds (Q12)

Related chapters of this book: Chapter 14 Expanding, Chapter 15 Factorising I: Common Factors,

Chapter 21 Surds and Rationalising a Denominator

General comments:

- However hard teachers work to encourage methodical approaches, students often continue to find factors of quadratic expressions by trial and error. If students are continuing with maths beyond GCSE, then developing a systematic way of handling quadratics will be very beneficial for them. In the notes for this chapter of the book we show one method of deriving values for the integers to use in the brackets. We have found that this particular method appeals to students because it draws on skills which are familiar, there is an obvious logic to the method, and it is relatively simple where the coefficient of the x^2 term is 1. The method extends logically to cater for other values of that coefficient, but in that case more patience is required to match the pairings of integers which contribute to the correct coefficient for the term in x.

- The documented method of handling factors is very helpful for students with dyslexia. If they set out the columns of factors neatly, then they can circle the operators at the top, and the correct row of factors when they discover it, and then copy what they see in the circles into their factor brackets. This turns what could be a very confused page of working into a straightforward procedure.

- On the whole, this method also reduces errors which result when attempting to apply the rule "$- \times - = +$" (see Chapters 14 Expanding and 15 Factorising I: Common Factors).

- Recognising the difference of two squares is useful for general algebra at GCSE, but it is also worth encouraging students to look for it because it becomes a tool to employ by choice in maths beyond GCSE. In Chapter 21 Surds and Rationalising a Denominator, Higher level students use the difference of two squares to eliminate surds from denominators. Teachers may like to set questions from Chapter 21 at this point to reinforce applications of the concept of the difference of two squares. A numerical use of the difference of two squares is also found in question 16.9.

Notes for specific questions:

16.6 Difference of two squares - for the first two parts students will probably write down the product of factors without working.

16.6c Encourage two steps for this factorisation.

16.9 If students are puzzled by this, then suggest they write $x^2 - 17^2$ in factor form first, then ask what they think they could do next.

16.11c Students may need a hint to remove a common numerical factor before factorising into two brackets.

16.12c Encourage an extra working step to remove the largest possible numerical factor.

Chapter 17 Rearranging and Changing the Subject

Aims: * understand that rearranging involves "undoing" algebra, so requires inverse operations

* appreciate that rearranging essentially uses "reverse BIDMAS"

* rearrange familiar and unfamiliar formulae, and solve problems requiring rearrangements

DfE syllabus main objective: A5 rearranging formulae to change the subject

Skills assumed: general understanding of order of operations and inverse operations

Associated skills and knowledge:

DfE connections: N3 using operations, including inverse operations,

A2 substituting numerical values, A17-19, 21, 22 solving equations

Related chapters of this book: Chapter 20 Functions, Chapter 23 Solving Linear Equations

Also Chapter 18 Formula Triangles if teachers use this technique

General comments:

- This is a key chapter for STEM students, and we recommend that teachers set as many as possible of these questions. We have deliberately included several questions using formulae from physics specifications, as well as some wider applications such as population density calculations.
  Other questions which involve rearranging algebra are found throughout the book.

- As with solving equations, we recommend that students adopt an organised approach to dealing with each phase of rearrangement:

  1. Decide which operation must be undone. This will be the operation which is least significant in BIDMAS i.e. undo subtractions and additions first.
  2. Write down the inverse of your selected operation on the side of the equation where you want to undo the algebra. Call this side 1 of the equation.
     Write exactly the same procedure on the opposite side (side 2) of the equation as well.
     For rearranging purposes remember that multiplicative operations apply to **all** terms to scale up the whole equation, so you may need to place terms on each side of the equation in brackets if you are going to multiply or divide.
  3. Do what you have written down.
     Use cancelling to show the elimination, and therefore make clear what remains on the first side of the equation.
  4. Underneath your working write a new equation, copying down everything that is not cancelled out. The inverse procedure term now appears as part of the new equation on side 2.

Example:

Rearrange the following equation to make r the subject:

$$4k + 5 = r^2 + k - 2$$

- The coefficient of r^2 is +1, so terms in r remain on the right-hand side. Call this side 1 because all other terms on that side will be transferred. For all practical purposes the other two terms on side 1 could be tackled in either order, but given the guidelines of BIDMAS, -2 should be treated first, so +2 to both sides

- Side 2 Side 1

$$4k + 5 + 2 = r^2 + k \cancel{- 2 + 2}$$

  Show cancelling and rewrite

$$4k + 7 = r^2 + k$$

- Subtract k from both sides as the inverse of $+k$

$$4k - k + 7 = r^2 \cancel{+ k - k}$$

$$3k + 7 = r^2$$

- Take the square root of both sides as the inverse of r^2. At GCSE it is often acceptable to assume only the positive square root for this rearrangement , but it is good practice for later maths studies and STEM to show the possible existence of both positive and negative roots (see Chapter 13 Indices and Taking Roots). Certainly, in practical situations two results would be considered for numerical square roots of real numbers unless the constraint $r \geq 0$ was given to exclude negative values for r. So,

$$\pm\sqrt{3k + 7} = r$$

  Therefore

$$r = (\pm)\sqrt{3k + 7}$$

  Teachers may also note that in this example a real square root can only be found if $k \geq -7/3$. At GCSE students will be given any information that they need about the numbers that they are handling, and they will be guided if they have to specify a range of values which can be considered. Consequently, teachers should not expect students to state a condition such as $k \geq -7/3$. However, we have mentioned this in case questions are asked about situations where the value under a square root sign could become negative.

Teachers will obviously also use discretion to decide when, if at all, students can leave out any of the steps in presentation. Guidance concerning the possibility of two roots can be provided as is most appropriate for individual students.

- Although it is a natural outcome of the process, Foundation students may be disconcerted when their rearrangement places the target variable on the right-hand side of the equation as an outcome. With practice they should naturally rewrite the equation with the target variable in the standard position on the left-hand side as the last line of their answer.

- Rearrangements which lead to negative coefficients for the target variable at any stage tend to cause anxiety and errors. We think it is helpful for students to be guided towards rearrangements that give positive coefficients for the target variable as soon as possible. If the target only occurs in one term, and the coefficient is negative, then a good first step is to perform the additive inverse for this term, so that the variable appears on the opposite side with a positive coefficient. Then other steps are more straightforward.

- We suggest always guiding students towards using term-by-term procedures, and away from procedures that look like short-cuts.
  One example that occurs is where students "swap all the plus and minus signs". This is a procedure which can be adopted without any understanding, and errors result. Here is an example of a common error which is introduced when signs are swapped before brackets are expanded:

  Compare the correct answer:

Take the statement	$-s = p - 3(q - r)$
Expand first	$-s = p - 3q + 3r$
Apply inverse operations	$+s = -p + 3q - 3r$

  With a common student's version:

Take the statement	$-s = p - 3(q - r)$
Swap all + and – signs	$+s = -p + 3(q + r)$
And expand	$+s = -p + 3q + 3r$

  In the first version, with the expansion done initially, $(-3) \times (-r)$ results in a term $+3r$, and applying the inverse operations term by term leaves $-3r$ in the final answer. In their version, the student has accidentally applied an extra sign change to the term in r.

- The same procedures which are used in rearrangements are also adopted when i) finding the inverse of a function (Chapter 20) and ii) solving equations (Chapter 23), so we have noted these topics as links.

Notes for specific questions:

17.1 We have deliberately asked for E to be the subject in Part a) so that, if students follow on with that working, rearrangement for I in Part b) leads to a positive coefficient for I. Questions set in physics may ask immediately for I or r to become the subject, so teachers may wish to emphasise that creating a positive coefficient for the target variable is a good first step even if it is not requested in the question.

17.2-17.4 All of these questions are of the form $y = mx + c$ and teachers may wish to mention how this relates to straight lines and finding the x-intercept.

17.6b Foundation students do know that "square root" is the inverse of "square", but they may need encouragement to write all of the variables under a single root sign.

17.9, 17.10 If students struggle with writing the algebra, then more practice converting text into algebra can be found in Chapter 12 Writing and Using Algebra.

17.9c Foundation students often guess values for m until they find the required time. Encourage students to work backwards from the cooking time.

17.10 Possible errors: 1) missing brackets, 2) inverted fractions (9/5 for 5/9 and vice versa).

17.14 Possible source of error – not reading that Population is given in Millions. Ask students to calculate a value for C and notice that the size of C as then calculated is different from other values in the Population column by several orders of magnitude. Do they think that Scotland is so densely populated compared with England? Can those figures give the correct value for the UK as a whole in line 1? Can the students explain what has happened?

17.15 Students may not have seen this notation before. Ask them to suggest what they think it means.

Chapter 18 Formula Triangles

Aims: * use familiar formula triangles in calculations

* write formula triangles from suitable algebra and information

DfE syllabus main objective: A5 rearranging formulae

Skills assumed: substitution of numerical values, conversion of standard units

Associated skills and knowledge:

DfE connections: A2 substituting values, A21 translate into algebra and solve equations,

R1 compound units, R11 using compound units

Related chapters of this book: Chapter 17 Rearranging and Changing the Subject

General comments:

- The authors are well aware that the use of formula triangles is a hotly-debated issue in the mathematics community. What is observable is that some students who rely on memory (often those who will not progress through the more demanding STEM A levels) find the visual representation easier to remember than algebra. For special needs students in particular, memorizing the triangles means that they can copy down the relationships that they need without confusion. We therefore include a short chapter on the triangles and teachers can use it, or not, as they see fit.

- One recommendation we do make is that teachers and students use the versions of triangles which display the multiplication operator in the bottom line of the triangle, and not an alternative version where there is a vertical line and no operator: Without an operator the triangles do not lead directly to formulae.

- It is often true that those students who rely on the triangles do not appreciate that formula triangles only work for exactly 3 variables with a relationship $a=b\times c$. Teachers may wish to follow-up the issue and include questions 18.4-18.6. However, if students are unlikely to progress with maths and/or STEM, then limiting attention to questions 18.1–18.3 will provide practice with more familiar ideas.

- If teachers would like another question, then Chapter 17 Question 14 on population density can be tackled using a formula triangle.

Notes for specific questions:

18.2, 18.3c Note the mixtures of units.

Chapter 19 Sequences

Aims: * recognise familiar integer sequences (squares, cubes, Fibonnaci)

* recognise arithmetic and geometric sequences by common difference and common ratio respectively

* use and derive simple position-to-term and term-to-term rules

* Higher only: simple treatment of quadratic sequences

DfE syllabus main objective: A24 recognise and use sequences

Subsidiary DfE objectives: A23 generate terms of sequences,

A25 deduce expressions to calculate n^{th} terms

Skills assumed: substitution of values, writing and using algebra

Associated skills and knowledge:

DfE connections: A4 using indices, R12 ratio and scale factors, G19 similar figures (Q15,16)

Related chapters of this book: Chapter 40 Symmetry and Similarity

General comments:

- Some mathematics teachers may be disappointed that we have not shown subscript notation, u_n, in the text. Not all students meet this at Foundation level, and we wished to make the text accessible for all students, so we leave it to teachers to introduce subscript notation where it is suitable for their own students. We have deliberately presented questions 19.6-19.14 with no notation in them at all, so that these questions can be used with any notation. We presume that students will be using subscript notation by the time they are attempting questions 19.15 and 19.16, and these questions are presented accordingly.
  Subscripts are mentioned briefly in Chapter 12 Writing and Using Algebra.

- Some students may recognise the phrase "find an expression for the n^{th} term" but not the more formal equivalent "find the position-to-term rule".

- Algebraic topics provide good opportunities to work towards material found in qualifications beyond GCSE. The questions at the end of the exercise are designed with this in mind, and they should offer a degree of challenge to the high-performing students.

- While not marked as a specific connection, teachers may like to draw attention to repeated use of a multiplier in other contexts, such as calculating compound interest. Examples of interest calculations are found in Chapter 3, question 3.11 and Chapter 6 questions 6.14 and 6.15.

- Students frequently avoid questions on sequences because they struggle with algebraic notation. User-of-maths teachers may find it helpful to use a column format of some sort and use a separate step to introduce notation. One possible table for finding term-to-term rules would be:

n	Notation	Value of term	Term-to-term
1	T(1)	7	
2	T(2)	11	Add 4 to previous term T(2)= T(1) + 4
3	T(3)	15	Add 4 to previous term T(3)= T(2) + 4
4	T(4)	19	Add 4 to previous term T(4)= T(3) + 4

Generalise: T(n) = T(n – 1) + 4

If the students work with columns 1 and 3 when the topic is new to them, they can focus on the terms of the sequence. From their values they can write a general word-description in column 4, such as "add 4" or "+4". This stage is shown in black. Once this is completed without anxiety over algebra, the notation column can be filled in, and then the term descriptors from column 2 can be used to formalize the term-to-term movement in column 4 (procedures shown in blue). The generalization step, which may present a challenge to Foundation students, is shown in red. Once the notation becomes more familiar, students can be asked to fill the columns in numerical order.

- Students generally have less trouble with the algebra for finding the position-to-term rule or "n^{th}" term. The book shows one possible layout for this using a table in Example 3. However, some students benefit from a more visual method. For example:

Write down sequence and find differences between terms

7		11		15		19		23
	+4		+4		+4		+4	

Write down values of n

n	1		2		3		4		5
	7		11		15		19		23
		+4		+4		+4		+4	

Note the differences between terms are four times the differences between values of n

		+1		+1		+1		+1		
n	1		2		3		4		5	×4
	7		11		15		19		23	
		+4		+4		+4		+4		

Hence create a row for $4n$

n	1		2		3		4		5	
	7		11		15		19		23	
		+4		+4		+4		+4		Both rows are +4
$4n$	4		8		12		16		20	
		+4		+4		+4		+4		

Note terms of sequence are always greater than terms of $4n$ by +3. Hence formula for T(n) is $4n + 3$.

n	1		2		3		4		5	
	7		11		15		19		23	
		+4		+4		+4		+4		Differ by 3
$4n$	4		8		12		16		20	
$4n + 3$	7		11		15		19		23	

$$T(n) = 4n + 3$$

The key stages are 1) seeing that the gaps for the values are 4 times the gaps for changes in n, so the sequence is based on the multiplication table for 4 (which will give a contribution of $4n$ towards the n^{th} term), and 2) a correction is then applied to shift the whole multiplication table to the correct start point by adding or subtracting a constant. A useful analogy is making a set of wooden stairs which must meet the landing at the top of the stairs to avoid a trip-step, so the bottom step of the stairs is adjusted slightly to make the staircase fit.

- Geometric sequences: students who struggle with indices do not like writing down their multiplier with a power, even if they have correctly found, and applied, the multiplier. Ensure that every use of the multiplier between terms is shown in their working, and remind them that the power required is the number of times their multiplier is used. If the start value is S and the multiplier is written as m, then the n^{th} term is Sm^{n-1} because the multiplier has been applied zero times to the start (first) term. There are two common sources of confusion with this:
  - If the value of S is the same as the value of m, then the n^{th} term will have the value m^n. Students should understand that this is a special case for the index and is true only when $S = m$.
  - Students notice that the expression Sm^{n-1} is very similar to the general expression for calculating exponentials (see Chapter 33 Graphs of Standard Functions), but then have trouble applying the power of (n-1) in contextual problems written as exponentials. A typical expression for exponential terms is ab^t, where a is the starting value, b is the multiplier which is applied to a, and the index t is the number of times the multiplier is used. t often represents time periods in the contexts which are familiar at GCSE. For example, this could be years of an investment, the number of doubling times for bacterial growth, or the number of half-life decay periods for radioactive materials. Remind students that the **first term** in their sequence of values in these cases is the starting value before time has elapsed: when $\boldsymbol{t = 0}$, the value is a ($b^0 = 1$). For example, the quantity £a is invested. In contrast, in this chapter the sequence starts with the first term $\boldsymbol{n = 1}$. Thus t, the number of times the multiplier b is used when studying exponentials, always lags 1 unit behind n, the position when studying numerical sequences. So, t could be replaced by (n-1).

Notes for specific questions:

19.1,19.2 Students should be able to spot most of these missing terms. If a procedure is required for finding missing terms, it is useful to create some form of table format (see comments above) and fill in what is known. Ask the students to try writing down differences between neighbouring terms. Is there a pattern? If that doesn't produce a pattern, then the next thing to try is using a multiplier. Can they make a pattern now? If neither of these approaches works, do they have a special sequence that they have been shown?

19.3-19.5 Students may find the notation intimidating, and fail to engage because they have no confidence that they know what to do. Remind students that T(n) is read as "term n", and encourage them to read each statement in the question first. For evaluating terms, encourage the students to make a table of values of n they should use (see comments above), and then for each value of n, substitute the value into the right-hand side of the equation given for the n^{th} term.

19.6,19.7 As above with finding missing terms, use a table format (see comments above) and fill in what is known. Ask the students to try writing down differences between neighbouring terms. Is there a pattern? If that doesn't produce a pattern, then the next thing to try is using a multiplier. Can they make a pattern now?

19.8-19.10 These questions require some use of notation. Teachers who are in user-of-maths subjects may like to confer with their maths department about notation that is preferred in their setting.

19.9b This part of the question involves more demanding simplification. Most students arrive at the form $a \times 2^{n-1}$, although they may hesitate over writing down the power of 2. Once this form has been written, ask them if their value of a happens to be a power of 2 also.

19.10b Note that all terms have a negative sign. This can be extracted for working purposes, but must appear correctly in the final expression.

19.11c Has changes required in numerator and denominator, so the multiplier must be of the form (a/b). The web-site hint suggests that expressing all terms in fractional form is therefore a good idea, so that the patterns of changes in numerator and denominator are as clear as possible. For example, 3 should be written as (3/1).

19.14 Students may need a reminder that the coefficient of n^2 is half of the second difference.

19.15b A hint may be required to express the answer as a single power of 2.

19.16c There are many ways to express the formula, so teachers may like to draw attention to the linear scale factor (5) and suggest that powers of this factor would be a sensible presentation.

Chapter 20 Functions

Aims: * know that a function takes an input value and performs specified instructions to return an output value

* recognise different notations for writing functions

* evaluate simple functions

* find original input values given an output

* Higher only: handle simple inverse functions and composite functions

DfE syllabus main objective: A7 functions

Skills assumed: substitution of numerical values, understanding of inverse operations, simple fractions

Associated skills and knowledge:

DfE connections: N3 prioritising and performing operations, A2 substituting numerical values,

G21 exact values of trigonometric functions (Q9)

Related chapters of this book: Chapter 3 BIDMAS/BODMAS and Substituting Values Into Formulae,

Chapter 12 Writing and Using Algebra

General comments:

- A big issue surrounding functions is notation: there are several possible forms that students may see, and students who are challenged by reading tasks may spend a significant time working out what is being communicated. We have attempted to offer a selection of questions so that teachers can choose what is most suitable for their students.

- For working out the answers to functions, students may find it helpful to replace all the function notation with “output = ” and then show stages of substitution and tidying up as if using equations.

- The questions for this chapter can be completed without a calculator, but teachers may suggest checking of some of the later answers using a calculator. Foundation students can be instructed to use a calculator for question 20.9 if they struggle with exact values of sin(t), although they should recognise the values as a requirement of DfE objective G21 by the end of their course (se Chapter 41 Trigonometry).

- The procedures used to find the input that results in a certain output, are the same ideas that are used in rearranging formulae, and some students may refer to that as “reverse BIDMAS” – see also general comments for Chapter 17 Rearranging and Changing the Subject. The principle of using inverse operations also applies to solving equations (Chapter 23 Solving Linear Equations). Solving equations as a method for finding original input values is shown in the note below for question 20.10.

Notes for specific questions:

20.3a There are various possible number machines that could be drawn. The most basic begins with *x*, multiplies by 2, adds 3 etc. However, some students who are happy with algebra may expand the bracket and simplify. We leave it to individual teachers to decide which route to adopt.

20.4-20.8 To avoid errors, encourage students to show all relevant lines of working, with substitution, simplifying of individual terms, and then combination of those results.

20.6b Foundation students sometimes feel very confused when handling zero, so may ask the value of 0^2. They may also think the function returns 1 because they attribute "no effect" to the zero in the numerator.

20.9 All students can do this question with a calculator. If students have studied exact values in trigonometry, refer them to Chapter 41 Trigonometry to revise values of $\sin(\theta)$ for $\theta = 0°$, $30°$, and $90°$. They may need to know that the value for $180°$ is the same as that for $0°$.

20.10-20.12 These questions all require the student to work backwards from the answer, without using a formal request to find an inverse function. Some students may try to do this by writing inverse operations. Many students will realise that because they know the answer, they can write an equation, and then solve that equation. For example, 20.10 could be written as

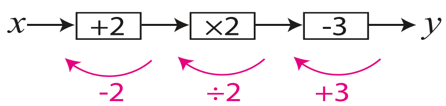
$$10 = 2x + 6.$$

If students find that working backwards is genuinely too demanding, the values given for outputs have all been chosen so that sensible trial and error with possible original inputs should find the correct input and output pairs.

20.13 Presents a familiar number machine. The aim of the question is to introduce the idea of working towards an inverse function algebraically instead of working numerically.

20.13a Ask the students to copy the diagram. Write each given value of input, *x*, underneath *x* on their diagram. Then follow the arrows and operators to the right-hand side, writing down intermediate values when they do a calculation, and placing the answer under *y* on the right-hand side

20.13b Students may need reminding that they are going to find out how to "undo" the forward procedure. The following method shows how to use the given machine in reverse, but it does not complete the full derivation of the inverse function. (A complete derivation of an inverse function is shown in the book in Example 4.)



To move from output back to input, students need to write a line of working to record how to go from right to left using inverse operations. Begin by writing the forward number machine. Underneath each forward operation, write down the inverse operation.

To use the reverse machine, write the value given for *y* underneath *y* on the reverse line of working, and follow the inverse operations from right to left, recording intermediate values, until a final value, identifying the original input, is written below *x*.

Now ask the students to use that value of *x* as the test input for the forward machine, to check that the starting *y* value is returned.

2.14-2.16 These questions draw together algebraic skills. They are difficult GCSE material, and provide a useful introduction to A level maths.

2.16b,c Students may need a reminder to use a bracket for the output of the first function when it becomes the input for the second function.

Chapter 21 Surds and Rationalising a Denominator – Higher only

Aims: * become proficient with handling surds

* rationalise a denominator which is a multiple of a surd

* rationalise a denominator which is a binomial expression with one or both terms surds

DfE syllabus main objective: N8 calculating with surds

Skills assumed: calculating with squares and square roots, multiplication of two binomial brackets

Associated skills and knowledge:

DfE connections: A4 general algebra

Related chapters of this book: Chapter 14 Expanding,

Chapter 16 Factorising II: Quadratic Expressions (difference of two squares)

General comments:

- The more mathematically able students should understand the principles of this chapter, but will still be prone to making errors because they rush through the questions, and often try to do too much processing without writing down any working. Explicit working to show the following procedures is useful:
    1. Writing each square root as the product of roots of square numbers and roots of primes. This makes it more obvious where it is possible to simplify by a) extracting an integer as the root of a square number, and b) combing pairs of roots of the same prime to give the prime number itself. Finally, simplify the entire answer. For example:

$\sqrt{10} \times \sqrt{45} = \sqrt{2} \times \sqrt{5} \times \sqrt{9} \times \sqrt{5} = \sqrt{2} \times \sqrt{5} \times 3 \times \sqrt{5} = \sqrt{2} \times 3 \times (\sqrt{5})^2 = \sqrt{2} \times 3 \times 5 = 15\sqrt{2}$

    2. When rationalising the denominator, show the multiplication term(s) in both numerator and denominator, and then multiply out. It is particularly important to emphasise that, wherever binomials (two terms) are involved, these should be placed in a bracket. This includes adding brackets when the original numerator and/or denominator are also binomials, but have not been placed in brackets. This is demonstrated in Example 6 of the text.

- Most of the errors in the questions will probably arise from either missing brackets, or from mis-handling/forgetting minus signs, especially if the negative sign is in the denominator.

- The text for this chapter notes that a quadratic which has the form of the difference of two squares can be written as the product of conjugate factors (the central operators are different, but the brackets are otherwise the same). This idea is used in reverse to create a rational value for the denominator. The general principle is used widely in A level maths.

Notes for specific questions:

21.11a Possible wrong answer $4 - 4\sqrt{2}$: losing minus sign in denominator.

21.11c Possible wrong answer from losing leading minus sign.

Chapter 22 Algebraic Fractions – Higher only

Aims: * simplify algebraic fractions

* combine algebraic fractions, applying common denominators where necessary

DfE syllabus main objective: A4 algebraic fractions

Skills assumed: handling numerical fractions, general algebra, factors of quadratic expressions

Associated skills and knowledge:

DfE connections:

Related chapters of this book: Chapter 4 Fractions, Chapter 14 Expanding,

Chapter 15 Factorising I: Common Factors, Chapter 16 Factorising II: Quadratic Expressions

General comments:

- Competency with algebraic fractions is highly desirable for students who intend to progress to A level maths and A level further maths.

- One key principle is to check that brackets are present, and to add brackets if necessary, so that (where intended) operations are applied to all terms in a numerator and all terms in a denominator.

- We recommend that teachers insist on a line of working to explicitly factorise numerators and denominators, before attempting to cancel any factors common to numerator and denominator. Cancelling can only occur if a factor occurs in every term in numerator and denominator, or for binomial brackets, exactly the same bracket must be a factor of numerator and denominator. Unnecessary errors arise because students 1) attempt to cancel factors quickly to avoid writing, and do not look to see that their chosen factors appear in every term, or 2) they guess a binomial factor incorrectly.

- A coefficient is a numerical multiplier in front of a term in an algebraic expression. The integer 1 is always present as a factor of a coefficient (this is also discussed in Chapter 15 Factorising I: Common Factors). If no coefficient is shown, then the coefficient is 1. If another numerical coefficient is shown, then 1 is still a factor. Students often find it helpful to consider this "hidden 1" as the position where they will multiply top and bottom by a chosen factor to create a common denominator.

- Working out a common denominator may be a matter of inspection. For example, one denominator may be a numerical multiple of another. In other cases, the common denominator will be a product of the individual denominators.

- When a common denominator has been decided, encourage students to proceed term by term though the expression. For each term, work out what the multiplying factor should be for the denominator of that term to become the common denominator. Then, multiply by this factor in both numerator and denominator. This is the opposite of cancelling and is shown in Example 5. When dealing with algebraic fractions, the factor concerned could be a binomial expression, or even a more complicated expression, so as noted above, use brackets to enclose the multiplying factors everywhere where they appear. Move on to the next term until all terms have been treated in this way. Then multiply out each term and check that all share the same denominator.

- The most common errors occur in handling minus signs. A particular place to check working is the subtraction of a fraction where the numerator is a bracket also containing a minus sign. The subtraction effectively multiplies everything in the numerator that follows by (−1), but students often forget to apply the multiplication by (−1) to every term.

- The following example illustrates several of the comments above:

  Express as a single fraction $\frac{1}{a-1} - \frac{1}{a+2}$

  Neither denominator shares any common factors with the other. Therefore the common denominator will be the product of the individual values: (a - 1)(a + 2).

  The multiplications of the fractions are $\frac{1}{(a-1)} \times \frac{(a+2)}{(a+2)} - \frac{1}{(a+2)} \times \frac{(a-1)}{(a-1)}$

  Note that the binomials must be placed in brackets.

  Expanding gives $\frac{a+2-(a-1)}{(a-1)(a+2)}$

  And applying the multiplication by -1 to all of the remaining bracket in the numerator, and simplifying

$$\frac{a+2-a+1}{(a-1)(a+2)} = \frac{3}{(a-1)(a+2)}$$

Notes for specific questions:

22.7a The numerator does not factorise into two binomials. Examination questions often factorise neatly because that is a skill that is being tested. We have deliberately included a question that remains relatively awkward because students should not expect that real problems always produce neat algebra.

Chapter 23 Solving Linear Equations

Aims: * solve linear equations in one unknown algebraically

* manipulate equations where the unknown appears on both sides

* construct, solve and interpret linear equations in context

DfE syllabus main objective: A17 solve linear equations algebraically (one unknown)

Subsidiary DfE objective: A21 translate into algebra and solve equations

Skills assumed: handling inverse operations

Associated skills and knowledge:

DfE connections: A5 rearranging formulae to change the subject, N13 units (Q7)

A4 algebraic fractions (Q14 Higher only)

Related chapters of this book: Chapter 3 BIDMAS/BODMAS and Substituting Values Into Formulae,

Chapter 12 Writing and Using Algebra, Chapter 17 Rearranging and Changing the Subject,

Chapter 20 Functions

General comments:

- Linear equations contain one or more terms of the unknown to the first power, and no higher powers. The only other constituents of the equation are operators and constants (multiplicative or additive).

- Equations are created, and evaluated using substitution of values, by the convention BIDMAS (see Chapter 3). If a linear equation is to be solved, then this is "undoing" the creation procedures, so this is sometimes referred to as "reverse BIDMAS".
  The priority of procedures for solving (or "undoing") equations is therefore the same as changing the subject of a formula (Chapter 17) and applying an inverse function (Chapter 20).

- A general equation can involve any number of variables, so after rearrangement the new equation for the unknown, say x, will involve the other variables also. In the case of a linear equation in one variable, all other quantities are constants, so the equation can be solved to find a value for x.

- In a complicated problem, preparing the solution may involve several phases of algebra, including expanding brackets and gathering together several terms in the unknown. It is good policy for students to present each algebraic objective with complete working. As with solving equations, we recommend that students adopt an organised approach to dealing with each phase of rearrangement:
    1. Decide which operation must be undone. This will be the operation which is least significant in BIDMAS i.e. undo subtractions and additions first.

2. Write down the inverse of your selected operation on the side of the equation where you want to undo the algebra. Call this side 1 of the equation.
   Write exactly the same procedure on the opposite side (side 2) of the equation as well. For rearranging purposes remember that multiplicative operations apply to **all** terms to scale up the whole equation, so you may need to place terms on each side of the equation in brackets if you are going to multiply or divide.
3. Do what you have written down.
   Use cancelling to show the elimination, and therefore make clear what remains on the first side of the equation.
4. Underneath your working write a new equation, copying down everything that is not cancelled out. The inverse procedure term now appears as part of the new equation on side 2.

An example of this approach is given in the notes for Chapter 17 Rearranging and Changing the Subject.

Teachers will obviously use discretion to decide when, if at all, students can leave out any steps and just write down their new equation.

- When there are terms in the unknown on both sides of the equation, encourage students to gather terms on the side where the coefficient of the unknown will be positive. Inexperienced students worry when this places their variable on the right-hand side of the equation. Reassure them that they can rewrite their answer on the last line, with the variable on the left-hand side and its value on the right-hand side as usual.

Notes for specific questions:

23.5a Expand first or divide both sides by 2.

23.5c Remove the denominator first.

23.7 Look for consistency with units.

23.8 Given only a glance, this appears to be a question involving an equation in more than one unknown. However, the first equation has 1Y on both sides, so reduces immediately to an equation in X only, and therefore each equation can be solved independently.

23.9 This question appears to be complicated at first sight, so may present a challenge to less confident Foundation students. Again, it appears that students will need to handle equations in more than one unknown, but techniques to solve multi-variable equations simultaneously are not required. Encourage the students to translate the text into algebra, and then see if it is possible to solve one of their equations to give a value which can be substituted into the other equations.

Students should begin with assigning a letter to each of the three variables: blocks, pens and packets. We have deliberately used two variables beginning with "p" so that the students really think about what their algebra means: always using the first letter of a word can obscure the general idea that any letter can be used. What is important is that the assignment is written down. So, for example, students could begin with "Let f = the number of blocks, g = the number of pens, and h = the number of packets". If students really struggle with associating general letters, then try using b, p and q.

This question is typical of "wordy" questions where students with special needs require assistance. We recommend having hard copies of this question available, and encourage the students to write their algebra above the text first. Remind them that the purpose of the question is to write and solve equations, and encourage them to propose that "balances" can be replaced by "=". This may be familiar to some students from KS2 and KS3 where pictorial approaches to solving equations often draw objects on a set of scales. Similarly, "weigh" can be seen as 1) using a set of scales, or 2) the weight is the answer to an investigation, or 3) the value is the output of a calculation, so "weigh" is also a place where they can replace text with "=".

23.11 Likely error is failing to divide the whole algebraic expression by 5. If students have studied number machines (see Chapter 20 Functions), this question can be tackled very effectively using a number machine presentation.

23.12a Possible wrong answer 9/4 – not multiplying both terms in left-hand bracket by (-3).

23.12b Remove denominators one at a time, or realise this will be the same as multiplying through once by 6.

23.13 Remind students to begin with assigning a variable name/letter. This is a good place to consider the nature of the variable (real, integer etc). Part b) illustrates that consideration of the context should be used when offering interpretation of answers. Is the answer sensible?

23.14 This question is marked for Higher students, but any students who understand the principles of inverse operations could try the challenge of scaling up whole equations using each denominator in turn.

Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines

Aims: * know that graphs of linear functions are straight lines, general form $y = mx + c$

* understand and calculate gradient

* understand and find intercepts with x- and y- axes

* find properties by comparison with the general form

* identify parallel lines and lines of forms $y = c$, $x = k$

* find equation of a straight line from 2 points or 1 point with gradient

* Higher only: identify perpendicular lines

DfE syllabus main objective: A9 straight line graphs

Subsidiary DfE objective: A10 gradients and intercepts of linear functions

Skills assumed: substitution of values, graph drawing and graph reading: axes and co-ordinates

Associated skills and knowledge: (x,y) co-ordinate system

DfE connections: A12 graphs of linear functions, A17 solving linear equations (x-intercept) (Q11),

G2 perpendicular distance (Q5)

Related chapters of this book: Chapter 23 Solving Linear Equations

General comments:

- Students should be familiar with many concepts from KS3, but they do not necessarily pay attention to details, particularly when asked to read information from a graph. Ensure that they:
    1. Decide on sensible scales for their axes and label positions at intervals on the axes.
    2. Read the scales on the axes if given a graph, and turn this into an understanding of the worth of one square or grid marking.
    3. Label any graphs with the function that they have drawn.
    4. Mark any intercepts and known points, and draw straight lines with a ruler.

  The practices of recording and reading all available details on a graph are vital tools for handling and understanding graphs of real-world situations, when there are titles and units to read as well as the representation of the function itself.

- Calculations of gradients between points often go awry because the co-ordinates are not substituted in a consistent order. Ensure that there is a line of working showing substitutions with appropriate signs and operators. For example, the gradient of the line between (5,-2) and (-3,4):

$$Gradient = \frac{Change\ in\ y\ co\text{-}ordinate}{Change\ in\ x\ co\text{-}ordinate}$$

$$= \frac{(-2)-(4)}{(5)-(-3)} = \frac{(-6)}{(8)} = \frac{-3}{4}$$

A common error here would be for students to take positive start values for both x and y, even when that choice does not employ the coordinates in correct pairings, resulting in a value of (+)3/4 for the gradient.

Some students find it helpful to write one pair of co-ordinates above the other, and then show working as if using an ordinary column subtraction for the x values and the y values.

- Other common errors, especially for those students with reading difficulties, involve loss of minus signs for gradient and intercept when comparing an equation with $y = mx + \mathrm{c}$. One suggestion is that the equation to be evaluated is written directly below a general equation, and then the two vertical columns which result for m and c can be circled with the relevant signs.

- As a way of encouraging students to work from first principles, in the notes in the book we have not listed the following formulae relating to gradients and finding equations of lines:

For a line of gradient m passing through the point (x_1, y_1): $m = \frac{(y - y_1)}{(x - x_1)}$

From this, the equation of the line can be found using: $y - y_1 = m(x - x_1)$

The equation of the line through points (x_1, y_1) and (x_2, y_2) can be found using the formula:

$$\frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Quoting these formulae is often a suitable starting point for tackling questions, particularly for Higher Tier students, but we suggest that students should be required to show their substitutions explicitly, and not simply write down their answers.

- Most students use the comparison of their equation of a straight line, with the general equation $y = mx + \mathrm{c}$, as a convenient way of finding m and c simply because they are told that this works. For those students who are more mathematically-minded there is an opportunity to discuss the use of general equations, and also comparison of coefficients, in more depth. A familiar example, which is found in Chapter 28, is the general quadratic expression:

$$ax^2 + bx + c$$

and the use of the coefficients a, b, and c in the formula for solving a quadratic equation.

- Organizing the equation of a line into the form $y = mx + \mathrm{c}$ requires understanding of applying inverse operations. Finding the x-intercept requires rearrangement of the straight line equation to make x the subject, which also requires inverse operations. Guidelines and good practices for these tasks are found in Chapter 17 Rearranging and Changing the Subject, and Chapter 23 Solving Linear Equations.

- To avoid confusion since the case is avoided at GCSE, the notes for this chapter do not include discussion of the situation where perpendicular lines are parallel to the cartesian axes. In this case one gradient will be zero, but the other will be undefined (the result of attempting to divide by zero is undefined). Hence, the product of the gradients cannot be -1. For instances where the tangent is parallel to the x-axis, the equation of the normal is $x = k$, and where the tangent is parallel to the y-axis the equation of the normal is $y = k$.

Notes for specific questions:

24.5 If students struggle with knowing where to sketch lines parallel to the axes, suggest making a list of a few points first. For example, for the line $x = 2$, list and join up (2,-1), (2,1), (2,3) and (2,6).

24.7, 24.8 Common errors are losses of minus signs (see notes above).

24.9, 24.10 Students may need a reminder that, for comparison with the general equation, the working equation must be rearranged so that the coefficient of y is 1.

24.12 These questions are increasingly complicated and some Foundation students will be challenged by the number of working steps required.

24.14, 24.15 Some high-achieving students may be able to write down the equations of the lines by inspection, but they still use observation and reasoning to do that. We recommend that all students are encouraged to write down any points that they use, with their reasoning: it is our experience that the students who try to skip these steps of recording often find it difficult to express their answers when they proceed to A level maths, because they have not practised writing down what they see and how to use that information.

Chapter 25 Simultaneous Equations 1 – Two Linear Equations

Aims: * understand when a simultaneous solution exists for 2 linear equations

* solve 2 linear equations algebraically

* estimate the simultaneous solution for 2 linear equations graphically

DfE syllabus main objective: A19 solve 2 simultaneous equations in 2 variables

Skills assumed: scaling of entire equations, general algebraic manipulations, graphs of linear functions

Associated skills and knowledge:

DfE connections: A9 straight line graphs, A10 gradients and intercepts of linear functions,

A12 sketching and interpreting graphs, G16 calculating area (Q10)

Related chapters of this book: Chapter 23 Solving Linear Equations,

Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines,

Chapter 30 Simultaneous Equations 2 – Where One is Quadratic

General comments:

- There are three places in the algebraic method where errors are common:
    1. Individual equations are not scaled throughout
    2. The wrong operation is selected to eliminate a variable
    3. The addition or subtraction of equations is not performed consistently

  Many of the troubles arise because students do not want to write out their new equations, so we suggest that teachers insist that the working equations are written as a new pair, even if one of the original equations is unchanged. Also insist that the equations are aligned underneath each other, and that the operation of elimination is shown explicitly, so that addition or subtraction is performed on a column-by-column basis in the usual way.

- If students struggle with solving equations which have been reduced to one variable, then practice questions for this topic can be found in Chapter 23 Solving Linear Equations. Practice in constructing straight line graphs is found in Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines.

- In the chapter notes we have not commented on the case where the two linear equations are scaled versions of each other. Graphically, this corresponds to two lines drawn in the same place. The number of solutions (points where the lines intersect) is infinite.

- For algebraic solutions most students learn to scale up one or both equations to match the numerical coefficients for one of the two variables, and then either add or subtract the resulting equations to eliminate their target variable. Unfortunately, they often have limited understanding of why this works. If time permits, many students will find it helpful to explore the algebra. Suggestions are:

1) Create a common parameter. Rearrange both equations to place either the multiple of x on the left-hand side, with other terms on the right, or the multiple of y on the left-hand side and other terms on the right. Scale the new equations until the multiples on the left-hand sides match. This gives a common parameter which connects the two equations. Then equate the right-hand sides to each other to give an equation in a single variable, and solve that equation,

2) Use substitution of one equation into the other. Rearrange one equation to make either x or y the subject. The substitute the expression derived for x or y into the second equation. Solve the resulting equation. The rearranging step often produces fractions for the substitution step; students may struggle to handle the fractional coefficients, so this method can be prone to error.

In both cases students should be asked to spot where working steps coincide with steps from their rote-learned method.

- Higher Tier only. This is a place where the connectivity of maths can be explored with students who are interested. Higher Tier students benefit in particular from working with the algebraic rearrangement mentioned as suggestion 1 in the previous comment above. A connection could possibly be made with using a common intermediate quantity to prove the Sine Rule (see the commentary for Chapter 50 Applications of Trigonometry to Geometry). In this case the common intermediate is the vertical height of a larger triangle. Question 41.9 of Chapter 41 Trigonometry employs the same idea of finding the height of a structure in two different ways to work out an unknown.

  The idea of using a common quantity to pass information and/or link algebraic expressions is a general principle, and the technique may be met as "passing a parameter": If a relationship cannot be established directly between two variables, but it is known that they are both related to a third variable, then that third variable is used as a connecting device. Beyond GCSE, the idea is expressed in theory in A level mathematics in the topic parametric equations.

  There is also a connection to correlation studies, where two very different variables show correlation of some description on a bivariate plot. The original variables may be individually linked to a third variable, and it is the links with this third variable that lead to the observed relationship on the bivariate plot (see commentary for Chapter 57 Correlation). In experimental sciences, observation of correlation may lead to searches to see if a third connecting variable can be discovered.

Notes for specific questions:

25.1 This may be a new question for some Foundation students who are accustomed only to being asked to solve pairs of equations which have a simultaneous solution.

25.4 This is a suitable question to suggest that students try a substitution method as an option.

25.9 This can be solved algebraically or graphically. If the graphical route is chosen, students may need encouraging to use the portion of the y-axis from £25.00, with a suitable axis break shown, so that the values are plotted on a practical scale. Teachers of STEM may find this a useful example of choosing axes on the basis of comparison with $y = mx + c$, and selecting start-points and scales for the best representation of experimental data.

25.10c This question requires clear planning, although the execution is straightforward. There are many expressions which can be used to find the area, but we suggest that teachers insist on full working so that they can assess the geometrical thinking that was applied.

Chapter 26 Solving Quadratic Equations 1 By Factorising

Aims: * understand that an individual equation can be solved when it contains only one variable

* solve quadratic equations algebraically by factorising

* Higher only: apply knowledge of solutions to recreate an original equation

DfE syllabus main objective: A18 solve quadratic equations algebraically by factorising

Skills assumed: factorising expressions into simple binomial brackets

Associated skills and knowledge:

DfE connections: A4 factorising quadratic expressions

Related chapters of this book: Chapter 16 Factorising II: Quadratic Expressions

Chapter 23 Solving Linear Equations, Chapter 27 Graphs of Quadratic Functions

Chapter 28 Solving Quadratic Equations 2 – The Quadratic Formula,

Chapter 29 Solving Quadratic Equations 3 – Completing the Square

General comments:

- Students may need reminding that before they can solve a quadratic equation they must rearrange and simplify the algebra to give an equation in the form

$$ax^2 + bx + c = 0$$

  Some students who struggle with algebra find it easier to see why this is so when they begin to look at graphs of quadratic functions. The graphical interpretation of this equation is discussed in the notes for Chapter 27 Graphs of Quadratic Functions.

- Success in answering questions in this chapter requires competence with factorising quadratic expressions into binomial brackets. If students are insecure with factorising, then we recommend some practice using questions from Chapter 16 before tackling this present chapter. Early questions in this chapter only use x^2 terms with a coefficient of 1, so are suitable for Foundation students. Questions after the visual break in the exercise offer an increased degree of difficulty. Foundation students may like to try some of them as a challenge.

- Students often struggle with solving quadratic equations which lack a constant term. Instead of factorizing, they fall into the temptation to divide through by x, as if it is a constant. But x is a variable, not a constant: dividing by the variable reduces the order of the equation (the highest power present is reduced by 1), and so removes a solution.
  Teachers may wish to suggest that two binomial brackets are always shown when factorizing quadratic expressions. Then, when there is no constant term, one of the brackets is $(x + 0)$.
  For example:

$$x^2 - 5x = (x + 0)(x - 5)$$

This makes it more evident that there are still 2 solutions to be found. When the first bracket is equated to zero then $x = 0$, and the other solution is $x = 5$.
If the students are asked to re-expand the brackets shown above, that should also help them to see why there is no constant in the resulting polynomial.

- Quadratic equations which occur in real-world situations, and in algebra which results from calculations, often need major manipulation to achieve the correct form for solution. One example would be finding the intersection of a straight line $y = 2 + x$ with the curve $y = \frac{1}{x}$ (where $x > 0$). If the right-hand sides of these two equations are equated, then the variable appears in the numerator on one side and in the denominator on the other side. In such cases, rather than be presented with a set of rules, it is really helpful for developing mathematicians to learn to ask questions of the type "What is awkward here?" and "What would I like this to be?" They should then expect to propose a way to approach the problem based on their own control of the algebra. In the example above, the decision would be to multiply both sides of the equation by x to eliminate the denominator.
  The exercise for this chapter contains several examples where manipulations of this sort are required.

Notes for specific questions:

26.3 Loss of a solution may occur in all parts if students attempt to divide through by x, so encourage the use of two binomial brackets (see third note above). In part c) division through by 5 is permitted because division by a constant merely scales down the equation but does not remove a solution.

26.4c Encourage division throughout by a constant to scale down the whole equation.

26.5 Remind students that they need to rearrange their equation into a form where the right-hand side is 0.

26.6-26.11 These questions have been chosen to illustrate useful manipulations.

26.6d Multiplying through by x on both sides of an equation does not change the number of solutions: it merely displays the equation in a standard polynomial form.

26.7 Foundation students could be offered this question. They should be able to do part a), but may need a hint for part b) to take the square root of everything when handling the term in x^2.

26.8 Encourage division or multiplication by a constant throughout to scale the equations.

26.9 Students should realise that the first requirement is algebraic removal of denominators.

26.10 Students may not have tried working backwards from solutions to generate a quadratic equation which is satisfied by the solutions. They should begin by writing down factor brackets which lead to the solutions, and then expand the brackets to produce an equation. What should be emphasized is that this equation is one possible answer that fits the values for p: the result could be multiplied throughout by any non-zero constant, so without other conditions there is an infinite number of possible original equations. Hence, the question defines the coefficients to be the smallest possible integers.

26.11 For parts a) and c), if students need a hint, then suggest that they consider possible factors of the constants. In part b) where the value of p is the constant, they should note that the question specifies that solutions are integers. Are solutions always integers when the different values of p are substituted?

Chapter 27 Graphs of Quadratic Functions

Aims: * plot graphs of quadratic functions using tabulated values

* know the key features of shape, intercepts, symmetry, and position and nature of the vertex

* use key features to sketch a graph

DfE syllabus main objective: A11 quadratic functions – graphical

Skills assumed: plotting and reading values on cartesian axes

Associated skills and knowledge:

DfE connections: A12 sketching functions, A18 solving quadratic equations – graphical,

A13 sketch translations and reflections (Q7), A5 use standard formulae (Q11)

A14 plotting real-world graphs (Q11)

Related chapters of this book: Chapter 26 Solving Quadratic Equations 1 – By Factorising,

Chapter 28 Solving Quadratic Equations 2 – The Quadratic Formula,

Chapter 29 Solving Quadratic Equations 3 –Completing the Square

General comments:

- Students often comment that it is only when they begin drawing graphs of quadratic functions that they really understand the significance of the form required to solve a quadratic equation:

  $$ax^2 + bx + c = 0$$

  Consider the two functions: $y = ax^2 + bx + c$ and $y = 0$

  Where the graphs of the functions intersect, the y value is the same, so using y to connect the individual equations,

  $$ax^2 + bx + c = 0$$

  This mathematical statement can be read in words as "the quadratic curve $y = ax^2 + bx + c$ meets the line y equals zero". But the line $y = 0$ is the x-axis. So the word statement for finding the intersections becomes "the quadratic curve meets the x-axis", which can be a "eureka" moment, especially for students who work particularly well with visual representations.

- Students usually engage quickly with the three general positions that the graph of a quadratic can take: crossing the x-axis twice, touching the x-axis once, and not touching or crossing the x-axis at all. They also see that this means respectively 2, 1 and 0 numerical solutions to $ax^2 + bx + c = 0$. However, there can be confusion relating these to the appropriate binomial factors.
  1. Writing down binomial factors is the opposite of solving using factors. For a solution at $x = a$, the binomial factor is $(x - a)$. If students do not consider how a solution is calculated, they make the error of copying the sign of the solution into the binomial bracket. Show them that $x = a$ can rearranged to $x - a = 0$. Therefore

$(x - a)$ is one of the binomial brackets i.e. the sign in the bracket is the opposite sign to that of the solution.

2. Students understand that the case where there are 2 different numerical solutions is coupled with 2 different factor brackets. So when there is 1 numerical solution some students write down just one bracket. They may realise that this cannot be correct, because that provides only x^1 i.e. a linear function, but they cannot recall what else to do. Rather than telling them that the answer is a repeated bracket, we suggest a short investigation to follow the movement of the two roots when a curve, which starts with 2 solutions, is raised parallel to the y-axis until it touches the x-axis. Students should be encouraged to draw in the 2 roots each time and describe their relative separation. Guide them to the conclusion that "the roots get closer and closer to each other until **they are on top of each other**". From this they should arrive at the idea that there are two coincident solutions, so there are still two brackets, but the brackets are the same (hence only one numerical answer). In the simplest case of x^2, the constants in the factor brackets are both zero.

- Sketching may be a new skill for some students. Encourage them to list the intercepts and the vertex first, and transfer the information to a set of axes, drawing in the line of reflection if they wish. Then they should connect their features with a carefully hand-drawn curve. Some students will need a lot of reassurance that recording the features on their sketch is what is required, not great accuracy in drawing. Students sometimes make very large sketches because they think that they have to reproduce what they would have drawn on graph paper. Again, reassurance will be required, and encouragement that it is adequate to do sketches that are to smaller scales than accurate plots on graph paper.

Notes for specific questions:

27.2b Requires understanding that the parabola is symmetrical about a line parallel to the y-axis and passing through the vertex. Thus the line of symmetry cuts the x-axis at the mid-point between the two roots.

27.5 Students are accustomed to seeing a lot of quadratic equations with integer solutions. Encourage them that they must read off exactly what they see having drawn a curve carefully through the points. They should not try to alter their curve to produce integer solutions just because other curves have produced integer answers.

27.6, 27.7 Foundation students will find these questions difficult, but the questions could be set as a challenge.

27.6b The curve has a maximum, which is less familiar to Foundation students. Foundation students will also dislike handling a negative coefficient for x^2. Maths teachers will have a preferred route for their students to handle this.

When finding the x-intercepts, one route is to start by setting y equal to 0, and then applying inverse operations explicitly to all terms, to give an equation with a positive coefficient for x^2. (This is equivalent to multiplying throughout by -1, but teachers may not wish to introduce the idea at this stage.) The resulting equation can be solved to locate the roots. At this stage, Foundation students will probably need a reminder that the x coordinate of the maximum lies mid-way between the roots. Once found, this value can be substituted into the given equation to find the y coordinate of the maximum.

27.9 This question may be unfamiliar, so some students may need a hint to use the solutions to write down binomial factor brackets (see second general comment above).

27.10 This question is demanding even for Higher Tier students. Students may need a hint about the x-intercepts to work towards constants in the binomial factors. In this question they must also make use of the value of the y-intercept. They may require a hint to consider the relationship between the y-intercept and the product of their binomial brackets when $x = 0$.

27.11 If students have not seen this sort of question before in maths or physics, they may need a hint to look at the signs of the velocity and acceleration values. Which way is this projectile travelling? Why is there a negative value for the acceleration due to gravity? How does that help with the interpretation of the answer in part c)?

If further practice with projectiles is required, another question can be found in Chapter 28 question 28.7.

Chapter 28 Solving Quadratic Equations 2 – The Quadratic Formula – Higher only

Aims: * predict the number of (real) solutions based on the value of $b^2 - 4ac$

* calculate roots for a given quadratic equation

* apply the formula in problems, which may include construction of the quadratic equation

DfE syllabus main objective: A18 solve quadratic equations using the quadratic formula

Skills assumed:

Associated skills and knowledge:

DfE connections: A21 translate into algebra and solve equations (Q6), G16 area (Q6),

A5 use standard formulae (Q7)

Related chapters of this book: Chapter 26 Solving Quadratic Equations 1 – By Factorising,

Chapter 27 Graphs of Quadratic Functions

Chapter 29 Solving Quadratic Equations 3 – Completing the Square

General comments:

- Students may, or may not, know the term discriminant for $b^2 - 4ac$, so we have not used this term in the notes for the chapter. Teachers may obviously introduce the term if they wish.

- Students can see in the notes for Chapter 27 Graphs of Quadratic Functions that there are three graphical cases to consider for intersection of a quadratic curve with the line $y = 0$: the curve crosses the line in two places (2 real roots), just touches the line at one place (1 real root), and a third case where the curve and line do not meet at all (no real roots).

Notes for specific questions:

28.4c We do not ask students to interpret their answer for this question, but they may benefit from considering why there is more than one answer for b.

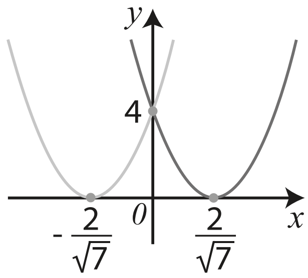
This is most easily explored graphically.

The graph of $y = 7x^2 + bx + 4$ has the following features:

i) it is convex (" ∪ " shaped) because the coefficient of x^2 is positive,
ii) it has a y-intercept of +4.

Also, if there is only one solution to $7x^2 + bx + 4 = 0$, the curve touches the x-axis once.

There are two curves that satisfy all three of these constraints:



The curves are mirror images of one another in the *y*-axis.

28.5c Reassure students that algebraic coefficients can be substituted into the quadratic formula in the same way as numerical coefficients.

28.6 Encourage the use of a diagram, and look for the written allocation of a variable for the width of the strip.

A likely error is finding the maximum length not the maximum width.

28.7 This question deals only with vertical travel of the ball. If students are not familiar with studying projectiles, they may ask how to include the horizontal component. The two components can be treated independently: gravity only affects the vertical component. There is no component of the gravitational force in the perpendicular (horizontal) direction, so horizontal motion is at a constant velocity.

Students may also need to think about how many times the ball will be at a given height.

If more practice with projectiles is required, another question can be found in Chapter 27 Graphs of Quadratic Functions, question 27.11.

Chapter 29 Solving Quadratic Equations 3 – Completing the Square – Higher only

Aims: * rewrite quadratic expressions in completed square form

* use completed square form to solve quadratic equations

* use completing the square to locate vertices of quadratic curves and sketch the curves

DfE syllabus main objective: A11 properties of quadratic functions and completing the square

Subsidiary DfE objective: A18 solving quadratic equations by completing the square

Skills assumed: competency with algebra

Associated skills and knowledge:

DfE connections: A8 working in all four quadrants (Q9), A12 sketching functions (Q9, 10)

Related chapters of this book: Chapter 26 Solving Quadratic Equations 1 – By Factorising,

Chapter 27 Graphs of Quadratic Functions

Chapter 28 Solving Quadratic Equations 2 – The Quadratic Formula

Chapter 35 Function and Graph Transformations

General comments:

- Errors often arise when extracting the coefficient of x^2 as a factor of the first two terms. If this coefficient, a, is an integer factor of b, the coefficient of x, then students generally proceed correctly. Otherwise, it is necessary to envisage the term in x as multiplied by a fraction $^a/_a$, so that a factor a can be extracted from both the x^2 and x terms. This is another instance where students are sometimes helped by the idea of a "hidden 1" to create the fraction they require (for more details see the comments for Chapter 22 Algebraic Fractions). An example of extracting a factor in this way is:

$$5x^2 - 2x - 1 = 5x^2 - 2 \times 1x - 1 = 5x^2 - 2 \times \frac{5}{5}x - 1 = 5\left(x^2 - {}^{2x}/_{5}\right) - 1$$

- To complete the square it is necessary to produce two factor brackets which are exactly the same, so the constant in the bracket must be half the target coefficient of x: this is not an inspired guess – it is because in multiplying out the brackets, the constant produces a multiple of x twice – once from each bracket.
  Note that the constant may be a fraction.

- When the completed square form is finished, students should always expand their result to check that they recover the correct polynomial. Squaring brackets which contain fractions is particularly prone to error, so students should be advised to check their working especially carefully in these cases.

- It is possible to complete the square and locate the vertex of a quadratic curve without formally considering transformations of a curve. However, students benefit greatly from being able to tie together the algebra of completing the square and the understanding that they can describe the results in terms of transformations of the curve $y = x^2$.

Consider creating the graph for the following equation in completed square form:

$$y = a(x + b)^2 + c$$

The graph of $y = x^2$ undergoes three transformations: 1) an x-translation: if the bracket to be squared is $(x + b)$, the graph is translated in the x direction by $-b$, 2) a stretch with scale factor a parallel to the y-axis and away from the x-axis (if a is 1 there will be no stretch seen), and 3) a translation by $+c$ units parallel to the y-axis. If the y-translation is negative (c is negative) then there will be 2 real roots where the quadratic curve crosses the x-axis; if the translation is positive (c is positive) then there are no real roots; if $c = 0$ there is one root. For more details see Chapter 35 Function and Graph Transformations.

Notes for specific questions:

29.10 This question looks deceptively straightforward, with the vertex even shown clearly. Students may find a completed square quickly, but then struggle when the correct intercept does not result from the simplest possible application of their repeated factor. Ask them if they have allowed for the coefficient of x^2. A further hint would be asking them if the required intercept is bigger than, or smaller than, their solution.

Chapter 30 Simultaneous Equations 2 – Where One is Quadratic – Higher only

Aims: * simultaneous equations, with one quadratic - rearrange and solve algebraically

* interpret simultaneous solutions graphically

* consider extension of the technique to other pairs of curves

DfE syllabus main objective: A19 solve linear/quadratic equations simultaneously

Skills assumed: algebraic substitution

Associated skills and knowledge:

DfE connections: A18 solving quadratic equations, A12 sketching and interpreting graphs (Q5),

A16 equations of circles (Q8, 9, 10)

Related chapters of this book: Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines,

Chapter 25 Simultaneous Equations 1 – Two Linear Equations,

Chapter 26 Solving Quadratic Equations 1 – By Factorising,

Chapter 27 Graphs of Quadratic Functions

Chapter 28 Solving Quadratic Equations 2 – The Quadratic Formula,

Chapter 29 Solving Quadratic Equations 3 – Completing the Square,

Chapter 33 Graphs of Standard Functions

General comments:

- This is one of the chapters where we have taken the opportunity to provide a raised level of challenge in the later questions of the exercise (questions 7-10), where students are asked to apply their knowledge to more general intersection of curves.

- Teachers may wish to follow up the ideas in the notes concerning tangency:
    i. A straight line is a tangent to any curve if it just touches the curve in exactly one place (the point of tangency).
    ii. If a quadratic meets the x-axis in only one place, the x-axis (y = 0) is a tangent to the curve at this location.
    iii. If solving a quadratic equation and a linear equation simultaneously yields only one solution, the graph of the linear equation is a tangent to the graph of the quadratic equation at the coordinates given by the solution.
    iv. The straight line perpendicular to a tangent at the point of tangency is known as the normal. The relationship between gradients of a tangent and its normal is the same as for any other pair of perpendicular lines: the product of the gradients is -1, provided that the lines are not parallel to the axes. For cases where the tangent is parallel to the x-axis, the equation of the normal has the form x = constant (for example x = 3); and where the tangent is parallel to the y-axis the equation of the normal is y = constant.

Notes on gradients of perpendicular lines are found in Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines, and three questions are available in that chapter: questions 24.18 - 24.20.

v. In many physical applications both the tangent and its normal are of interest. For example, consider the motion of an object constrained to move in a circle at the end of a rope. At any instant the velocity of the object is directed along a tangent to the circle, and the force on the object from the rope is directed along a normal. The topic of circular motion is considered further in the comments for Chapter 42 Circles and Circle Theorems.

Notes for specific questions:

30.6 Students may need a hint about how to deal with *a* when rearranging into a quadratic that can be solved. *a* will become part of the constant in that solvable equation. Asking if brackets would be useful for the constant would provide a second hint if this is required. When a straight line is a tangent to a quadratic there is only one point at which they meet i.e. the quadratic which must be solved has one solution. So, students could set the value of the discriminant ($b^2 - 4ac$ in the quadratic formula) to zero, or they could compare their results with a completed square.

30.7 Simultaneous solution of two quadratic functions is not listed as a requirement at GCSE, but it is a natural extension to the syllabus.

30.8 Students should realise that they need to use the linear equation to substitute for one of the variables in the equation for the circle. From this they can find values of either x or y at the intersections. Corresponding values of the second variable can be established by substituting their answers into the linear relationship. It is always wise to back-check that the coordinate pairs also work in the circle equation.

30.9 GCSE students are usually only introduced to circles which have the origin for their centre. In this case we require two circles, so we have used a centre of (5,0) as well. The algebra for the question is not dependent upon familiarity with circle geometry beyond GCSE level. It would be instructive, if time permits, for the students to plot both circles and label the points of intersection.

30.10 Substitution for y^2 is, by far, the more efficient method of tackling this question. The most usual place for errors to occur is in the scaling up of the equation which has a denominator. We suggest that students are encouraged to show their multiplication of every term explicitly.

Again, we have used a circle with a centre other than (0,0).

We strongly recommend that students plot the graphs for this question so that they can see the locations of the solutions. Students could be asked to propose the maximum number of real solutions that could occur in a general case with these types of curves. Why would this be so, both algebraically and graphically?

What do they see if they intersect the same circle with $x = \frac{y^2}{2} - 4$?

Chapter 31 Inequalities

Aims: * understand and use algebraic notations and graphical representations

* solve linear inequalities

* solve problems by employing linear inequalities

DfE syllabus main objective: A22 linear inequalities

Skills assumed: setting up and solving linear equations

Associated skills and knowledge:

DfE connections: N1 order positive and negative numbers, N5 systematic listing strategies (Q8),

A21 translate into algebra and solve equations, R10 proportion (Q10),

R11 use compound units (Q10)

Related chapters of this book: Chapter 12 Writing and Using Algebra,

Chapter 23 Solving Linear Equations,

Chapter 32 Plotting Inequalities and Solving Quadratic Inequalities

General comments:

- Many errors arise where the coefficient of the unknown is negative: students often learn to "multiply through by -1 and reverse the inequality", but then forget about reversing the inequality when they attempt questions. This may be because they fail to appreciate why the technique works. The notes for the chapter consider the inequality $-x < 0$. For $-x$ to have a value less than 0, x itself must be positive. So $x > 0$. Comparing these two inequalities, both sides have been multiplied by -1, and the inequality sign has been reversed.

- To avoid troubles with sign changes and reversing inequalities, we strongly recommend that students be encouraged to rearrange/group terms in x by addition and subtraction so that the coefficient of x is always positive in their working inequality statement. Then no change is required for the inequality sign.

- If students are struggling with rearranging algebra into a suitable form to solve their inequality, then they may benefit from practice first with general rearranging of algebra (see Chapter 17) and solving linear equations (see Chapter 23).

Notes for specific questions:

31.1, 31.2 These are straightforward number questions. Possible sources of error: mis-read of inequalities, carelessness with open or closed circles for end-points.

31.3c Likely wrong answer $P < -2$. The question introduces a negative coefficient for the variable, and therefore the need to also reverse the inequality if multiplying through by (-1). As explained in the comments, we recommend that the first step should be the addition of $5P$ to both sides of the inequality, followed by subtraction of 10, and then division by a positive number to find the solution.

31.4 This question introduces more algebraic manipulations. Possible sources of error: incorrect identification of inverse operations, or the order in which to do them. Watch out that the same operation is applied to both sides of the inequality

31.5 To translate words into algebra, it may be necessary to focus the students to begin with the variable and build the statement.

Common error: No bracket in part b) for (a + 3), so a typical wrong answer is $3+ 6a -5 \leq 19$, leading to $a \leq 7/2$.

31.6 This question will be challenging for Foundation students, but they may like to try it. Parts a) – c) all combine max,max or min,min, so these are straightforward. Part d) a common wrong answer is -2. Suggest that students look at the sign of y, and the size of values for x. What should they use to get a more negative answer? Part e) requires max – min. Common errors are (max – max) = 2 and –(min + min) = -1.

31.7 Each part of this question can be approached by 1) using both inequalities together and applying the chosen inverse operation to all parts at the same time, or 2) by writing and dealing with two separate inequalities.

Errors may arise with the first method where the inverse is not applied to all parts.

31.8 Students may find it helpful to lay this out as a table in x and y. The pattern of the additional constraint can be seen easily in a table.

This question could be used as an additional example for graphing linear inequalities (Chapter 32).

31.9 This question involves a change of variable from P to T where T=P/9. T must also be a whole number.

31.10 Understanding what is happening in this problem is conceptually demanding, although easy to process once formulated. The first problem is for students to realise that the runners run for the same time but go different distances. Runner A travels twice the distance of B in the same time. So the whole distance is split 2/3 to A and 1/3 to B. Students should recall that speed = distance/time. Teachers may need to emphasise that the time was the longest they could take, and as time taken for the same distance goes down, speed increases. Therefore speed ≥ distance/150s.

The most common errors are 1) A travels 1/3 of the distance, so has half the speed of B. Ask students to back-check their answers against the data in the question, where they see that A should be twice the speed, not half the speed, 2) speed ≤ d/150 because the inverse dependency of speed and time was not formulated correctly.

This question could be used as additional material when studying proportionality (see Chapter 34).

Chapter 32 Plotting Inequalities and Solving Quadratic Inequalities – Higher only

Aims: * construct graphs using standard conventions for inequalities

* understand how constraints can be turned into inequalities

* solve quadratic inequalities

* find and interpret regions of graphs subject to multiple constraints

DfE syllabus main objective: A22 represent inequalities on a graph

Skills assumed: graph drawing, factorising of quadratic expressions

Associated skills and knowledge:

DfE connections: A9 straight line graphs, A18 solving quadratic equations,

A21 translate into algebra and solve equations, A16 equations of circles (Q8)

Related chapters of this book: Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines,

Chapter 26 Solving Quadratic Equations 1 – By Factorising,

Chapter 27 Graphs of Quadratic Functions, Chapter 33 Graphs of Standard Functions

General comments:

- If students need to revise graphs of straight lines, then notes and practice questions can be found in Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines.

- In order to focus on the inequalities, in this chapter solutions to quadratic equations can all be found by factorising (see Chapter 26).

- If students have difficulty sorting out which parts of a graph satisfy their inequality, they may find it helpful to translate the algebra back into words. For example:

  Find the values of x which obey the following: $ax^2 + bx + c > 0$

  Students could envisage, or sketch graphs of $y = ax^2 + bx + c$ and $y = 0$ (the x-axis). The question would then be read in words as "for what values of x does the quadratic curve $y = ax^2 + bx + c$ lie above, and not on, the x-axis?"

- Many of the errors encountered will concern values which lie on the boundary defined by an inequality: mis-reading or mis-writing inequalities will cause inclusion of values that are not required and vice versa.

- Question 31.8 could be used as an additional example for graphing linear inequalities.

Notes for specific questions:

32.4 Students will probably find this question difficult. It is always a good idea to decide if there are minimum and maximum values for each variable. x is an area of a flowerbed, so cannot be a negative quantity. How can this be turned into an inequality for x?

Students may ask if they could write $x \leq 10$. They could do so, but as they will see when they plot their graph, this does not add information because it is implied by the other conditions in the question ($x + y \leq 20$ and $10 \leq y \leq 20$). They could also write $0 \leq x \leq 10$, although only $x \geq 0$ is necessary.

32.5-32.7 Students should be encouraged to adopt the good policy of making a quick sketch to help them identify which values are required. In all cases saying in words what is required can be very helpful (see comments above).

32.8 This question uses a circle (see Chapter 33) where the centre is not the origin, but all the necessary information is given for a sketch to be made. Students should realise that they need to work with the circle

$$(x-4)^2 + (y-3)^2 = 5^2$$

Chapter 33 Graphs of Standard Functions

Aims: * identify, plot and sketch $y = \frac{1}{x}$, $y = x^3$

* identify and use asymptotes

* Higher only: simple exponential functions $y = ak^x$ (k > 0), circles with centre (0,0)

DfE syllabus main objective: A12 graphs of linear, quadratic, cubic, reciprocal, exponential functions

Skills assumed: plotting and sketching graphs

Associated skills and knowledge:

DfE connections: A19 solve simultaneous equations graphically (Q2,5),

N8 calculating with surds (Q8,9,11), A18 factorise quadratic equations (Q11)

Related chapters of this book: Chapter 19 Sequences,

Chapter 24 Graphs and Co-ordinate geometry of Straight Lines,

Chapter 27 Graphs of Quadratic Functions, Chapter 35 Function and Graph Transformations,

Chapter 42 Circles and Circle Theorems

General comments:

- This chapter concentrates on functions and graphs which are not studied in detail elsewhere in the book, so questions here are mainly (although not exclusively) concerned with $y = \frac{1}{x}, y = x^3$, simple exponentials and circles. Graphs of straight lines and quadratic functions are found respectively in Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines and Chapter 27 Graphs of Quadratic Functions.

- Students usually plot the curves $y = x^3$ and $y = \frac{1}{x}$ in class-work, and with the help of the graphs shown in the book notes, they should be able to recognise these basic functions. So, in order to maximise the teaching value of questions involving these functions, questions 33.1 and 33.2-33.5 all involve an additional constant, which translates the whole curve parallel to the y-axis. Question 33.5 also involves a reflection in the x-axis. Teachers have the option of asking students to simply work out values of the function, and plot their results (Foundation), but the questions could also be considered by Higher students as examples of transformations (see Chapter 35 Function and Graph Transformations).

- The first six questions can all be done without a calculator. If students say they cannot calculate 1.5^3, they should consider if they could write the value in another form to make their calculation easier: this is a good example of turning the value into a fraction, because they can certainly calculate 3^3 and 2^3. However, teachers may prefer to allow calculators so that the students concentrate on drawing the functions.

- When asked to state the result of attempting to divide by zero, GCSE students offer a variety of answers. Common wrong suggestions are 0,1 and infinity. **The correct answer is that the result of attempting to divide by zero is "undefined",** which many students find extremely surprising because they are under the impression that maths "always gives right answers". The benefits of discussion are very limited for Foundation students who are unlikely to progress with mathematics of number and abstract concepts, but some Higher students will be interested in some of the ideas below.

- What is important is that all students must be able to sketch reciprocal functions, so the graphical implications of what happens on the lines $x = 0$ and $y = 0$ (the axes) cannot be ignored. If students have not undertaken the task before, then ask them to find the results of dividing 1 by a few values of x, some of which become very much less than 1. For example, x values of 2, 1, ½, 1/10,1/100,$1/10^4$ could be used. Students soon realise that very small values of x produce very large values for the reciprocal. They should also investigate the use of very small negative values for x, and deduce that division produces very large negative results for $\frac{1}{x}$.

  This provides a rationale for why dividing by zero cannot simply be "∞": depending upon whether 0 is approached from the positive side or the negative side, it appears that attempting to calculate $\frac{1}{0}$ could produce $+\infty$ or $-\infty$!

  The important outcome for graph work is that, for both x and y axes, the reciprocal function approaches the straight line of the axis but never touches it or crosses it. The term for a straight line which is approached but never touched or crossed is "asymptote", and beyond GCSE students learn that asymptotes are shown as dotted lines (if the asymptote is an axis, then the axis may be extended with a dotted line). This is the same sort of convention as showing inequalities graphically, where a dotted line is a boundary of a region, but the values on the boundary do not satisfy the inequality.
  At GCSE, teachers may decide to accept asymptotes drawn as solid lines, but the reciprocal curves must not touch or cross the lines.

- Infinity is a concept describing increase beyond bounds. The concept of infinity is employed in STEM applications such as testing of electrical circuits. An open circuit is said to have "infinite resistance". The resistance in the circuit is so high that no current can flow, which may be because of deliberate disconnection, or because a component has broken and there is no conductive path through the circuit.

- Exponential functions: A typical expression for exponential terms is ab^t, where a is the starting value, b is the multiplier which is applied to a, and the index t is the number of times the multiplier is used. t often represents time periods in the contexts which are familiar at GCSE. For example, this could be years of an investment involving compound interest, the number of doubling times for bacterial growth, or the number of half-life decay periods for radioactive materials.

  Remind students that the **first term** in their sequence of values in these cases is the starting value before time has elapsed: when $\boldsymbol{t = 0}$, $b^0 = 1$ and $ab^0 = a$. For the contexts mentioned above, the starting values are the capital invested, the initial size of the bacterial colony, and the starting mass of radioactive material.

- A sequence of values where a common ratio relates each term to the previous term is described as a geometric sequence (repeated "enlargement" by the same multiplier, which can be greater then, or less than, 1). Students often see the connection with exponentials, but become confused because treatment of sequences refers to the position, n, of the term to derive the power for the multiplier, starting from $n = 1$ (whereas exponentials are started from $t = 0$). Thus, if a is the starting value of a sequence, and the common ratio (multiplier) is r, then the second term is ar, the third is ar^2, and so on, giving the n^{th} term ar^{n-1}. Further discussion of sequences can be found in the notes for Chapter 19.

- Applications of exponential growth to a colony of bacteria can be found in Chapter 34 Proportionality, question 34.12, and Chapter 36 Real World Graphs, question 36.13.

- Calculations of compound interest are found in Chapter 3, question 3.11 and Chapter 6 questions 6.14 and 6.15.

- Other questions which involve the algebra of circles are 30.10 and 32.8, both of which include a significant degree of challenge.

- Circles are relevant to a lot of physical applications such as considering the motion of an object constrained to move in a circle at the end of a rope. Comments for these applications can be found in the teachers' notes for Chapter 42 Circles and Circle Theorems, where the geometrical properties of circles are considered.

Notes for specific questions:

33.1 Students should be able to match the general shapes of the graphs with their equations without worrying about the details of any transformations. If Foundation students are worried because they have not seen a y-translation (+2) for a cubic, or a stretch (multiplier of 2) applied to ($1/x$), reassure them that they need to look only at the general shapes of the graphs and compare them with the graphs in the notes section of the book. Higher students could be asked to sketch their own copies of the graphs in this question and label them, including marking all of the values of the intercepts with the axes.

33.2b It is not possible to produce a numerical answer for an attempt to divide by zero: the result is undefined. Do not accept responses that refer to infinity as the result of dividing by zero (see comment above).

33.2c Note that there are 2 answers.

33.3 Question 33.2 considers why x cannot take the value 0 when calculating y coordinates for the curve $y = \frac{1}{x}$, so this current question merely shows the exclusion of zero from the domain of x. If teachers have not set question 33.2, they may like to ask students to explain why $x \neq 0$ is included with the statement of the function in the current question.

33.6 This question asks only for tabulated values and plots of, respectively, a growth curve and a decay curve. Teachers may like to ask for real-world situations where each type of curve may be seen. Some possible answers, all with respect to time, are: growth – investments with compound interest, number of bacteria in a colony; decay – radioactive decay.

33.10, 33.11 These questions are intended to stretch Higher candidates.

33.10 Students are accustomed to seeing exponential equations in terms of x and y, so they may need reassuring that t can be used in exactly the same way as x. Part a) requires the use of $t = 0$. In part b) encourage substitution of values and then rearranging into the form of a fraction as a step to finding the value of *b*.

33.11 For part b) students should be able to solve simultaneously the cubic $y = x^3 - x$ with $y = 0$ to create an equation for the intersection of the cubic with the *x*-axis. They may need a reminder to create three binomial factor brackets so that they retain three solutions to their equation. The cubic equation has no constant term, so they should know from studies of factorizing quadratic expressions that this will lead to at least one factor bracket of the form $(x + 0)$ – see the third comment for Chapter 26 Solving Quadratic Equations 1 – By Factorising. Some students may show this as simply extracting a common factor *x*, instead of a binomial bracket, but they must follow this by showing how there is an associated solution for *x*. They must not divide their whole equation by *x* because they will lose a solution.

Note that this function is used in Chapter 34 question 6, where a sketch is shown. If teachers wish to set both questions we suggest that the students are guided to do 33.11 first.

Chapter 34 Proportionality

Aims: * understand and apply direct proportion

* understand and use inverse proportion

* Higher only: use proportionality with a range of other relationships

DfE syllabus main objective: R13 work with equations that describe direct and inverse proportion

Subsidiary DfE objective: R10 solve proportion problems using algebra and graphs

Skills assumed: rearranging equations

Associated skills and knowledge:

DfE connections: A2 substituting values, A5 rearranging formulae, A10 linear functions,

A12 sketch and interpret graphs of functions, R9 percentages (Q6), R11 compound units (Q7),

A8 simplifying surds (Q13)

Related chapters of this book: Chapter 7 Ratio, Chapter 12 Writing and Using Algebra,

Chapter 17 Rearranging and Changing the Subject, Chapter 33 Graphs of Standard Functions,

Chapter 36 Real-World Graphs and Kinematics, Chapter 40 Symmetry and Similarity

General comments:

- One common misconception is to think that two variables that have *any* positive linear relationship, or correlation, are also proportional to each other "because they go up or down together". It is important to ensure that students understand that for two variables to be in proportion, then they must always be in exactly the same ratio. The only straight lines which describe this situation are straight lines **passing through the origin**.
  A practical example is the sharing of a bag of sweets between two people (A and B) in the ratio 2:1. If 3 sweets are taken out at a time, then the first four totals of sweets given out to the two people, written down as (A,B) are (2,1), (4,2), (6,3), and then (8,4). The first person in the list will always have twice as many as the second person, **provided that they both begin with no sweets**, i.e. a starting position of (0,0). Now compare the results if the second person were to begin with 3 sweets, and the distribution occurred as before. From a starting position of (0,3) the same allocations of sweets would result in the two people holding the following: (2,4), (4,5), (6,6) and (8,7). The ratio is different in each case.

  Practice in handling direct proportion questions, and particularly conversions such as exchanging currency, can be found in various chapter of the book. Some useful examples by question number (in the form chapter.question) are: currency 7.7, 7.8, 36.5, dilution of liquid plant food 36.2, similarity and enlargement 40.4 and 40.6, scale drawing 45.2.

- Students may write their equations for inverse proportion as $y = k\left(\frac{1}{x}\right)$ instead of $y = \frac{k}{x}$.

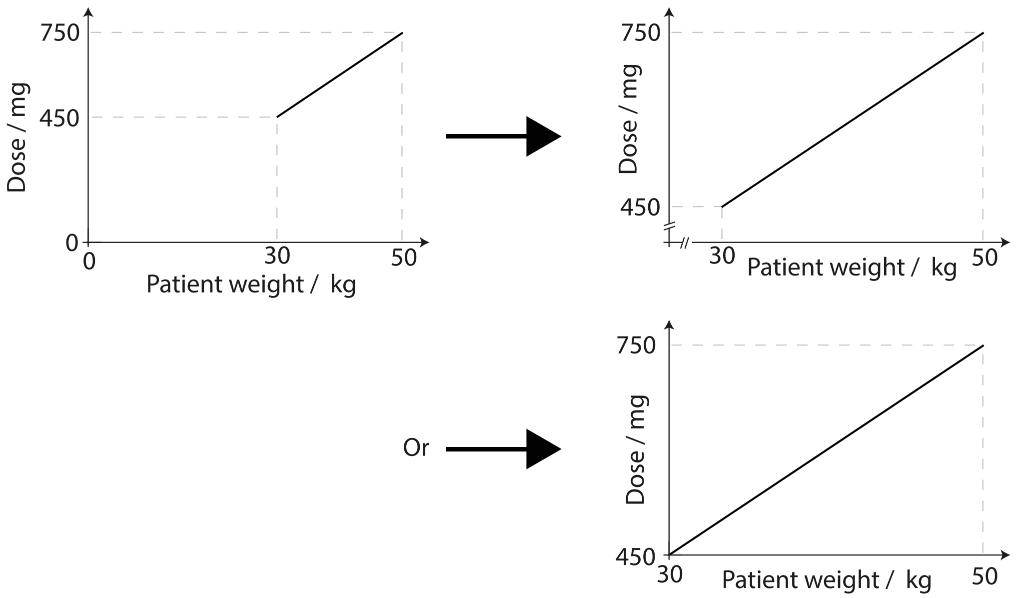
- Question 10 of Chapter 31 (runners meeting) could be used as additional material for inverse proportion if required.

- It is likely that the later questions in the chapter will be new to many students. The questions have been included to provide a higher level of challenge, and at the same time a direct link to STEM applications which could be encountered in post-GCSE courses.

Notes for specific questions:

34.3 The most likely error is that students compute *G* as a function of *F*, instead of *F* as a function of *G*, because the value of *G* is given first.

34.5b Note that the line on the graph starts at 30kg. The most useful graphs for practical purpose will show only this part: then the scales can be as large as possible, thus giving the greatest accuracy in plotting points and reading values from the graph. The axes must be labelled correctly, and break marks should be shown in the axes where this is appropriate.



34.5c This can be read from an accurate graph, or it can be calculated. If calculation is used, then teachers should insist that full working is shown.

34.5d Answers may be in mg or g.

34.7 For some Foundation students this question will be a challenge because they are accustomed to calculating from equations without considering that they can make predictions based on the algebra rather than numbers. If students are stuck on part c), then asking them what they can calculate using the first sentence is usually a good hint. Teachers may wish to ask Higher students to produce an equation of the form $v_1 t_1 = v_2 t_2$ instead of calculating the distance as an intermediate parameter.

34.8 This question looks daunting, but can be tackled by straightforward substitution and rearrangement.

34.12 Students may think this question looks difficult. Encourage them to use standard procedures, but with *A* in place of *k* for their constant of proportionality. Given that they have the correct equation in part a), part b) requires substitution. For part c) ask them to write down the appropriate value of *t*, and substitute that in their original equation from part a). What is the numerical value of the power of 2? For part d) students may list the number of bacteria at the end of each doubling period, or they may find the end:start ratio of bacteria and then express that as a power of 2. The number of doubling periods can then be converted into a total time.

34.13 This question also looks daunting, especially to students who do not do physics. However, it is highly scaffolded, and what is required is confidence to proceed with instructions at every step, and substitute values where this can be done. For part c) it may help students to rewrite the square root as $\frac{\sqrt{l}}{\sqrt{g}}$.

Chapter 35 Function and Graph Transformations – Higher only

Aims: * understand and use transformations parallel to the *x*- and *y*-axes

* understand and use reflection in the *x*- and *y*-axes

* extend understanding to describing transformations

DfE syllabus main objective: A13 sketch translations and reflections of a given function

Skills assumed: sketching of graphs showing intercepts and turning points where appropriate

Associated skills and knowledge:

DfE connections: A11 properties of quadratic functions, A12 sketch and interpret graphs of functions,

G7 transformations of shapes, G8 combining transformations of shapes

Related chapters of this book: Chapter 14 Expanding, Chapter 33 Graphs of Standard Functions,

Chapter 48 Shape Transformations, Chapter 49 Graphs of Trigonometric Functions

General comments:

- This is a chapter where we have included challenging material which is not specified at GCSE but which can be tackled using only GCSE knowledge. For example, it is unusual for students to write down the equation of a function which results from a transformation (question 6).

- Questions on graphs from other chapters of the book can also be used as illustrations of transformations, such as Chapter 33 Graphs of Standard Functions, questions 1, 3, 5, and 6, and various questions from Chapter 49 Graphs of Trigonometric Functions.

- Stretches parallel to the axes were included in previous GCSE specifications, but these are not required in the current specification.

Notes for specific questions:

35.3b The key feature in this question is location of the vertical asymptote. For the function $y = \frac{1}{x}$ the vertical asymptote has the equation $x = 0$ (along the *y*-axis). The denominator of the equation $y = \frac{1}{x-1}$ will become zero if $x = 1$, so the asymptote is the line $x = 1$. Thus the asymptote is translated 1 unit in the positive *x* direction, and the two parts to the curve of $y = \frac{1}{x}$ are translated also 1 unit in the positive *x* direction. Note that this means there is now an intercept with the *y*-axis at -1. Students can find this value and include it on their sketch.

35.3c Students can find the intercept with the *x*-axis and include it on their sketch.

35.4 Students may need prompting to label the intercepts on their sketches.

35.5 Both parts of this question involve two transformations of $y = \frac{1}{x}$. We suggest that students should be encouraged to write down the transformations as a list, and then sketch each stage to build up the correct graph. This promotes logical development, and is also excellent preparation for looking at the behaviour of functions in maths studies beyond GCSE.

Guiding students to a suitable order for successive transformations is helpful to them. For example, for 35.5a it is appropriate to perform a reflection in the x-axis (or y-axis) and then consider the y translation.

35.6 Students are required to locate the roots and sketch this curve in question 33.11 (Chapter 33 Graphs of Standard Functions). If question 33.11 has not been set previously, teachers may like to suggest that students do that question before 35.6.

In part a) knowing the position of the (one) x intercept should provide a clue to the location of the new cubic with respect to the axes.

For part b) the notes in the book for this chapter show algebraically that a translation of $y = f(x)$ by $+a$ units parallel to the x-axis leads to a new curve $g(x) = f(x - a)$. Some students find it helpful to describe this in words. A typical statement would be "I am drawing this curve 3 units further on, so I have to look back 3 units to find out what to draw".

There are various ways in which the expression for $h(x)$ can be presented, depending on whether, or not, teachers wish to see expansion to give the polynomial.

35.7 The statement of intention at the beginning of this question makes it appear daunting at GCSE, but each individual step should be possible for most Higher students. If the expansion in part a) causes trouble, then refer students to Examples 4 and 5 in Chapter 14 Expanding, where multiplying three brackets is shown, and then they could complete questions 14.12 and 14.13.

35.8 Any method of finding the roots is acceptable, but some methods take longer than others. The quadratic equation factorises, so it is probably quickest to factorise and note that the x coordinate of the vertex is mid-way between the roots. Students may also complete the square to find the vertex and the roots. If students need to revise solving quadratics, then practice is found in Chapters 26, 28 and 29. Chapter 27 covers Graphs of Quadratics. Students should be able to identify the invariant points by observation or by deduction from the algebra.

Chapter 36 Real-World Graphs and Kinematics

Aims: * read and interpret graphs from real-world contexts

* use distance-time and speed-time graphs

* understand the difference between distance and displacement, and between speed and velocity

* understand and use the “suvat” equations for problems in kinematics

* Higher only: find instantaneous and average rate of change values

* Higher only: recognise that the area under a graph may represent a physical quantity. If so, identify the quantity, estimate its size and give the units

DfE syllabus main objective: A14 graphs of real contexts including kinematics

DfE subsidiary objectives: A15 gradients of graphs and areas under graphs, R15 rates of change

Skills assumed: graph plotting

Associated skills and knowledge:

DfE connections: A2 substitution of values, A5 use of standard formulae,

A12 sketch and interpret graphs of functions, R5 ratio,

R10 solve proportion problems using algebra and graphs, R11 compound units,

R14 graphs, rate of change and proportion, R16 general iterative processes (Q13)

Related chapters of this book: Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines,

Chapter 33 Graphs of Standard Functions, Chapter 34 Proportionality, Chapter 57 Correlation

General comments:

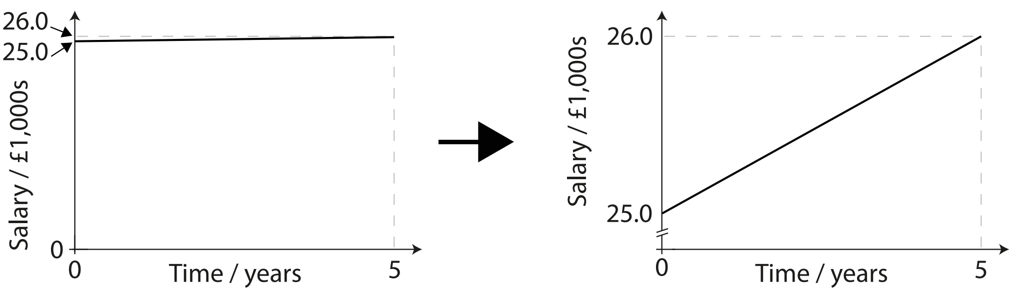
- Analyses of real-world data and graphs usually require a combination of skills and knowledge. It is very difficult to reflect all the connectivity and methods required for every question, so the following comments are designed to reflect the major connections and issues.

- Students sometimes see real-world graphs and decide they look too complicated to handle. It is important for these students to gain sufficient confidence to read the axes and decide what sort of information they are given. Then they should say what they see for the graph, with details about important features and where they occur. Encourage accurate observation expressed in clear English. There may, or may not, be obvious conclusions to be drawn.

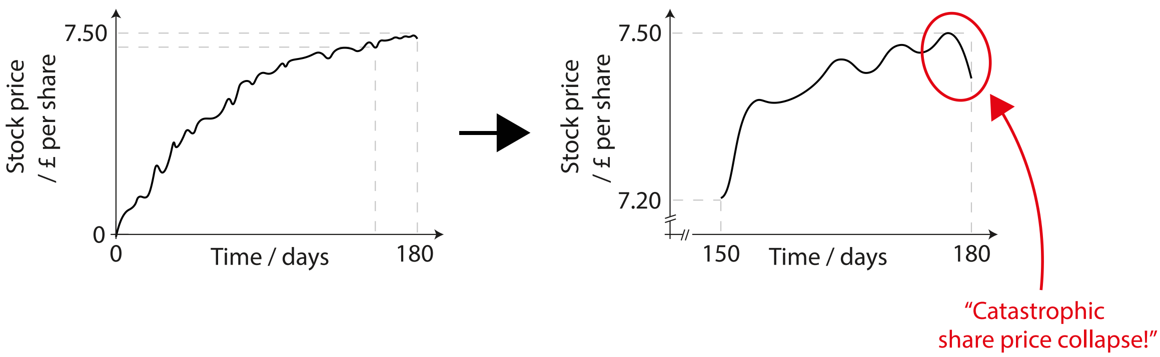
  On a practical note, if students practise this pattern of “observation first”, they will often find that they have answered most of the questions before they have even read what they are required to do.

- In order to interpret observations made about a graph, it is essential that students first read the title (if given) and all the information supplied on the axes. For each axis they should know what quantity is plotted, what range of values is covered, and the units. We suggest that students (particularly those doing Foundation level) make a statement for both axes of what one grid square is worth, giving numerical size and the units. This focusses attention and makes it much easier to read off values correctly.

- It is also important to note if there are axis-breaks. These may be employed usefully so that accurate working is possible with data from a small part of a larger graph. However, in real-world situations axis-breaks may also be used to give a deliberately misleading effect: the scale of part of a graph is enlarged, or the range on one of the axes is changed, so the extracted part of the graph is not comparable with its appearance if the entire graph is plotted.
  An example would be showing an annual wage growth if £200 is added each calendar year to a starting salary of £25,000. If the whole graph is sketched then it is squeezed at the top of the page and the £200 increment is barely visible: £200 is an annual rise for the first year of only 0.8%. However, if the wage axis (*y*) shows values beginning at £25,000 and is marked in increments of £200 up to £26,000, the graph looks steep and this suggests a very generous wage rise plan.



  In a second example the value of a new share is tracked, showing a generally rising trend which tends to level off in the last 30 days. The second graph of data for the last 30 days, replotted using a magnified *y*-axis scale, has been annotated to suggest a huge fall in share price at the end of the period, although the drop is only about 10pence in £7.50. The "meteoric" rise of about 20pence in the same time at the beginning of the graph is ignored.



- When the shape of a graph is discussed, reference should be made to the context, including units. If there is additional information in the question it is often advisable for students to add this to their graph, or a copy of the graph. For example, if a cyclist riding in km per hour starts from place A and stops twice on the journey at places B and C, before arriving at D, then encourage students to mark information about A, B, C and D on their graph. Is the vertical axis marked as Distance from A in km, rather than just Distance in km? A must be at 0km on this axis. Can the other places be marked on the axis also? And so on.

- For kinematics, distinction is not always made at GCSE between distance and displacement, and between speed and velocity. We have deliberately included this section because students studying STEM subjects, even at GCSE, may need to understand these differences, and may already be working with them in physics. In order for the text to remain accessible, we have not referred specifically to vectors, but teachers may wish to discuss with students that they are handling vectors when considering displacement and velocity (and acceleration).

- Students must also be clear that "suvat" only applies in situations of **constant acceleration**. It is our experience that many students do not appreciate this until well into their A level courses.

- The majority of students should know that they construct a tangent to a curve by drawing a straight line that touches the curve at one point only, with the angles between the curve and the line on each side of the point of tangency as equal as possible. Where the curvature is changing rapidly this can be difficult to do accurately, so the gradient of a tangent found in this way can only be considered to be an approximation.

- At GCSE only <u>estimates</u> of area under a graph are required. Students are also only expected to identify already-familiar quantities with the area under graphs (for example distance as the area under a time and speed graph). However, it will benefit students intending to do STEM subjects, if they learn how to identify other quantities. Area is found by multiplying width and height. So the units on the axes can be multiplied together to find the units for the area. Are these the units of a known physical quantity? Using a time and speed graph as an example:

  Area = a speed × a time   Units are $\frac{\mathrm{m}}{\mathrm{s}} \times \mathrm{s} = \mathrm{m}$

  Distances (lengths) are measured in m. The area represents the distance covered in the journey shown.

Notes for specific questions:

36.1 This graph is a complicated curve, which could be daunting if students have no method for trying to understand it. Encourage observations of the axes and details of the curve. Encourage them to talk about their experience of busy times and low times for traffic flow.

Likely wrong answer for part c): students do not notice the time period for comment is 3-4pm.

36.2, 36.5 Both questions could be used as supplementary material for practice of ratio, conversions and units.

36.3 This is an example where locations can be marked on the graph, including showing Ben's journey from Marhampton to Scorville. If letters are used for places (M and S) then the students should write a key: M = Marhampton, S = Scorville.

36.4 This question does not ask for a commentary about the phases of this journey, but teachers may like to ask questions about acceleration for each stage. Students should note that the final portion of the graph is a variable deceleration. It is also instructive to ask students to sketch a possible terrain, along the lines of steeply down, more steeply down, flat, going up.

Possible error – misunderstanding when the cyclist starts to climb.

36.5b The trap here is that students simply try to replace dollar values on the left-hand graph with euros, and end up with a very strange scale. Suggest that pairs of values (£,€) are recorded, and a new graph constructed on a sensible scale. Some students may wish to use conversion multipliers to find the rate of exchange i.e. gradient of the graph.

36.6e Likely error – students do not consider the direction of motion: they are asked for velocity not speed.

36.8c Students may think that 6 km/s^2 and 24 km/s are large values. This is true, but they are not mistakes. Suggest that students compare the velocity with the speed of light in a vacuum: 3×10^8 m/s or 3×10^5 km/s.

36.9 This question requires calculation of the gradient, for which the units are kWh÷h i.e. kW. While this is a practical example of using rates, students who do not do physics can be very puzzled because the resulting unit appears not to mention time, although rates are generally per second, per minute, per hour, or some other time period. The Watt is a unit of power, and 1 Watt is expenditure of 1 Joule per second, so the Watt does measure the rate of change of energy and includes time. The energy exchange for many real-world physical processes involves many thousands of Joules, so power is measured most conveniently in kW. The kWh (3.600×10^6 J: the total energy used by taking 1 kW of power (1×10^3 W) for 1 hour (3600 seconds)) is a convenient derived unit that avoids using large numbers or standard form. This is the unit that appears, for example, on domestic electricity bills. An average household may use about 9kWh of electricity per day running appliances (not including heating).

36.11c A possible error is using the mean of the speed values at $t = 0$ and $t = 10$ minutes. Calculation should involve distance travelled.

36.12 Part b) requires a tangent. Part c) uses values on the curve at the end-points of the time period. There are many ways of estimating the area for part d). Encourage either counting squares, or the use of large regular shapes and simple forms of adjustment at the edges: this is an estimate not a calculation.

36.13 Part b) employs values on the curve at the end-points of the time period. The key word in part c) is "instantaneous", so this part requires construction of a tangent. Part d) can be answered correctly by eliminating one of the possibilities, but if time permits teachers could encourage students to take up a very challenging concept for GCSE, and derive the correct growth model for themselves without looking at the formulae. Ask students to write down readings from the graph as ordered pairs of (t, N) for values of t at 0, 20, 40 and 60 minutes. What happens to N by the end of each time period? What sort of curve describes this behaviour? What is the start value? What is the multiplier to move from one term in N to the next? What counts how many times the multiplier is used? How can you make this counting number from the time measurements you have made? This last question is very difficult at GCSE, but students may be able to express a view such as "Each step takes 20 minutes, so I need to apply multiples of 20, which means I find values of $t/20$".

This question can be used as supplementary material for Chapter 33 Graphs of Standard Functions, or for looking at sequences (Chapter 19).

Chapter 37 Numerical Methods – Higher only

Aims: * find approximate solutions to equations using decimal search

* find approximate solutions to equations using interval bisection

* find approximate solutions to equations using iterative sequences

DfE syllabus main objective: A20 numerical solution using iteration

Skills assumed: rearranging equations

Associated skills and knowledge: function notation, subscript notation

DfE connections: R16 work with general iterative processes, N6 estimate powers and roots (Q7)

Related chapters of this book:

General comments:

- The national syllabus requires some use of iterative processes, but the extent to which numerical methods are studied varies greatly depending on the examination board chosen.

- We have included three methods of finding approximate solutions because all of them are useful if students progress with maths and STEM subjects. Teachers can choose to use whatever fits their course. Even if the material is not a requirement, teachers may consider this to be suitable extension work for more able Higher students.

- Observing how iterative calculations approach the roots of equations often helps students to understand more about graphs that they have drawn. For example, the sign change when evaluating a function on both sides of a non-repeated root can be interpreted as "the function lies above the *x*-axis when the value is positive, and the function lies below the *x*-axis when the value is negative".

Notes for specific questions:

37.7 Students should be encouraged to find their own starting values for this question by writing down relevant squares and cubes of integers. For example, if $\sqrt{41}$ was required, the first step would be to note that $6^2 = 36$ and $7^2 = 49$. Therefore

$$36 < 41 < 49 \quad \text{so} \quad \sqrt{36} < \sqrt{41} < \sqrt{49} \quad \text{i.e.} \quad 6 < \sqrt{41} < 7$$

Chapter 38 Pythagoras' Theorem

Aims: * find the length of any side in a right-angled triangle given the other two sides

* know the first few integer solutions to the theorem (Pythagorean triples)

* apply Pythagoras' Theorem to solving problems

DfE syllabus main objective: G20 Pythagoras' Theorem

Subsidiary DfE objective: G6 applying Pythagoras' Theorem

Skills assumed: rearranging equations, handling squares and roots

Associated skills and knowledge:

DfE connections: N6 integer powers and roots (Q1), N8 calculate exactly with surds (Q11, 12),

A5 using standard formulae and rearranging, R2 scale factors, G19 apply similarity(Q6)

Related chapters of this book: Chapter 7 Ratio, Chapter 13 Indices and Taking Roots,

Chapter 17 Rearranging and Changing the Subject,

Chapter 21Surds and Rationalising a Denominator, Chapter 40 Symmetry and Similarity

General comments:

- Pythagoras' Theorem should be a familiar topic, but it is still a good idea for students to do a sketch of their chosen triangle and label the hypotenuse (or the longest side if they are testing for a right angle). Where no diagram is given, then it is essential that students draw their own diagram and label it fully. The problems in the exercise are good examples of questions which require a line of working to show analysis of the steps to the answer, followed by selection of Pythagoras' Theorem as the tool to find a missing side of a triangle where that is required.

- The most common errors occur with rearranging the formula. Students should write down the formula as a first line of working and then show their inverse operations explicitly as they usually do for rearranging and changing the subject (Chapter 17), and for solving linear equations (Chapter 23).

- Question 38.1 can be achieved without a calculator, and teachers may prefer to specify no calculators to practise handling integer squares and square roots. Questions 38.11 and 38.12 have been designed to give Higher students practice with manipulating surds, so these questions can be done without a calculator. All other questions require a calculator.

Notes for specific questions:

38.3b Students may see fractions, and attempt to solve the whole problem with fractions. Finding the squares as fractions is good practice. However, teachers may wish to encourage students to use a calculator thereafter.

38.5 Encourage a statement of the form: Distance = AB + BH + HA. Then choice of Pythagoras for the unknown (see comments above).

38.6-38.8 These questions may be a challenge for Foundation students because all of the questions link several ideas and concepts. However, nothing beyond Foundation studies is required and students should be encouraged to persist, starting out with a clear sketch.

38.6 Students may be tempted to label the sides of the rectangle as 3 cm and 4 cm. Ask them to re-read the question. Suggest that they extract the triangle they will use from the rectangle, mark the right angle, and label the measurement that they do have with its units. They could then draw a separate right-angled triangle with the two shorter sides labelled as 3 and 4, but with no units. What is the length of the longest side of this triangle? What can they say about a relationship between the two triangles they have drawn? Expect the answer "similar" or "enlargement". So how can they move from one triangle to the other? Reinforce the concept that relationships of this kind are always multiplicative not additive, and encourage the use of arrows between the triangles, writing the scale factor above the arrows.

Other questions of this type are found in Chapter 40 Symmetry and Similarity, particularly questions 40.4 and 40.6.

If more general practice with scaling values is required, then Example 3 in Chapter 7 Ratio shows how to calculate scale factors and use them to perform monetary conversions. Some useful examples of scaling and ratio, listed by question number (in the form chapter.question) are: currency 7.7, 7.8, 36.5, dilution of liquid plant food 36.2, scale drawing 45.2.

38.7 This question has been designed to appear to be a 3-D geometry question, but a 2-D figure is clearly marked. If students struggle with it, suggest that they obscure the left-hand side of the diagram. Can they see a quadrilateral? One of the main learning objectives of the question is transcribing information from the text to a diagram, so ensure that students draw the correct trapezium and put on all the measurements. Can they see how to cut this up to make a triangle and another shape they can use? They may need hints such as "There is a slant height marked. Do you think that would be the longest or the shortest side of a triangle?" This is a challenge for Foundation students, but if they label every line on the two shapes they should be able to shade in the correct triangle to finish the question.

38.8 Did the students draw a diagram? This question becomes a standard question once the inequality has been used to mark the hypotenuse of the triangle.

38.9-38.12 Teachers should expect a separate diagram, or suitable clear annotation of a 3-D sketch, for each triangle considered in the 3-D questions (see Example 4 in the chapter).

38.11, 38.12 If practice with handling surds is required, suitable questions are found in Chapter 21 Surds and Rationalising a Denominator, questions 21.1 – 21.7.

Chapter 39 Angles and Shapes

Aims: * know and use angle properties of intersecting lines

* know and use angle properties of triangles and quadrilaterals

* find internal and external angles of regular polygons

* apply knowledge of angle properties to problems

DfE syllabus main objective: G3 properties of angles

Skills assumed:

Associated skills and knowledge: notation for geometric figures

DfE connections: G1 conventions in geometry, G4 properties of quadrilaterals,

G6 applying angle facts, A21 translate into algebra and solve equations (Q 3, 10),

G20 Pythagoras' Theorem (Q5, 9)

Related chapters of this book: Chapter 12 Writing and Using Algebra, Chapter 38 Pythagoras' Theorem

General comments:

- Much of this material should be very familiar from KS3. GCSE increases the emphasis on more formal use of basic facts in conjecture, derivation and proof. Consequently, the presentation of answers for many of these questions is a key issue: each deduction should be accompanied by a justification statement, and the last line of working is the equivalent of a conclusion. For example, if asked to identify a quadrilateral, students should observe that, say, all four sides are equal and all angles are 90°, and conclude that therefore this shape is a square.

- Students should be able to observe information given on drawings using standard notation, and should be able to construct diagrams using this notation to communicate properties involving equal lengths, parallel lines and equal angles. This can be a good place to emphasise that sketches do not have to be accurate representations – shape should be a good guide, but sketches work because the annotations communicate everything that is needed.

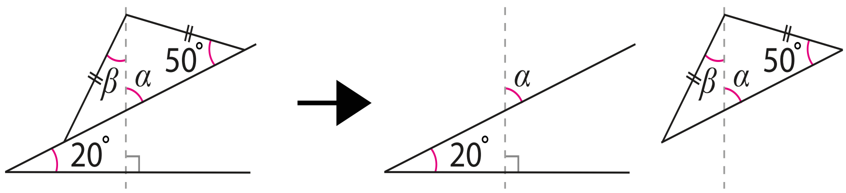
- Students often become confused and make mistakes because they try to cope with small, crowded drawings, if they have a drawing at all. Encourage students to copy the drawing from the question at the outset, or make a drawing if there is no figure given. They should label everything that they can label, or work out, before reading on to discover what they are asked to do next. If figures are complicated, then it can be useful to make subsidiary sketches of the relevant sections of the main drawing for each question part. Some students find it helpful to mask parts of the drawing to reduce their field of view, so that they can focus on features and

Notes for specific questions:

39.2a Students may find it helpful to extend the top horizontal line towards the left-hand side, to reveal α as the sum of two smaller angles which can be deduced.

This is a plane figure, not 3-D axes: no direction vectors are shown to indicate axes perpendicular to the set of parallel lines, and only one right angle is marked.

39.2b This figure resembles a triangle on an inclined plane. It may be helpful to re-draw this figure in two parts, with a set of vertically opposite angles for finding α, and a triangle for finding β:



39.3b Guide students to writing their equation in full as the sum of three individual angles. Simplifying could follow in this part of the question or in part c).

39.4 This is a complicated figure for Foundation students to handle. Suggest that they draw and name shapes ABC and ADBE individually, then consider the links with the lines AB and EF.

39.5 Encourage labelling in full and the addition of the line AC. For part b) students may need to be asked which formula links the lengths of the sides of a right-angled triangle.

For part c) expect Foundation students to use symmetry for the length of half of BD, and then Pythagoras twice. Higher students may offer the Cosine Rule (see Chapter 50 Applications of Trigonometry to Geometry).

39.6 The key to this question is working out the implications of the statement that EFC is a straight line. One approach might be to suggest that students begin by marking an angle between EFC and one of the parallel lines (e.g. angle BCF), assign to it a variable such as α, and ask the students to mark all the other angles that have the same value. They will need to think about bisecting the angles of the quadrilaterals (ABCF and AFDE). For example, bisecting angle AFD into angle AFE and angle EFD. Which other angles can they find, and how do they justify their working? How can they mark the lengths of EF and BF?

For part c) students may find it helpful to shade in triangles of area equal to the area of triangle AFE.

39.7 Parts a) and b) are standard. For part c) there are various ways of setting up a triangle, but the key step is finding all the angles for the triangle in which α is marked.

39.8 The way in which this question is presented may be a challenge for Foundation students. Encourage them to step through the question, and especially for part b) to draw a careful, larger-scale, labelled diagram of the triangle formed by extending the sides a and c. What do they know about the angles?

39.9 This question is more appropriate for Higher students because of the number of connected skills that are required. The key step is to label all possible lengths and then come to a conclusion about the nature of one of the triangles that can be seen.

39.10 The most likely wrong answer to part a) is $y + 10^\circ = x$. Suggest that students review how they used the information about the sizes of the angles involved.

Chapter 40 Symmetry and Similarity

Aims: * identify symmetry elements in shapes

* understand congruence and similarity and apply the principles to triangles and other shapes

* calculate and apply linear scale factors for similar shapes

* Higher only – scale areas and volumes given a linear scale factor

DfE syllabus main objective: G19 concepts of congruence and similarity

Skills assumed: basic understanding and application of ratio and proportion

Associated skills and knowledge:

DfE connections: G1 conventions in geometry, G3 – G7 properties of shape and angle,

N13 standard units (Q7), R6 scaling, ratio and fractions (Q4), R11 compound units (Q9),

R12 scale factors and similarity, G20 Pythagoras' Theorem and Trigonometry (Q6)

Related chapters of this book: Chapter 7 Ratio, Chapter 11 Units, Chapter 34 Proportionality,

Chapter 38 Pythagoras' Theorem, Chapter 39 Angles and Shapes, Chapter 41 Trigonometry,

Chapter 44 Surface Area and Volume

General comments:

- GCSE introduces more quantified treatments of similarity. Consequently, several of the questions in this exercise are chosen to link with STEM contexts.

- The most important issue in numerical treatments of similarity is ensuring that calculation and use of the scale factor is multiplicative and not additive. We refer teachers to comments for Chapter 7 Ratio and Chapter 34 Proportionality where finding scale factors is discussed in more depth, and applied in various situations such as currency conversion. Example 3 in Chapter 7 shows a suitable method for finding conversion factors.
  With similar figures, pairs of corresponding sides in the two figures are always in the same ratio. If, for example, one pair of corresponding sides has lengths of 4cm and 7cm, the ratio 4:7 also applies to all other corresponding pairs. Writing this ratio as $1:{}^{7}/_{4}$ means that moving from the smaller to the larger shape means scaling up by a factor ${}^{7}/_{4}$. In the reverse direction the multiplier is ${}^{4}/_{7}$ (which is equivalent to dividing by the multiplier used in the forward direction).

  If students would benefit from practice with ratio and conversion problems, some suitable questions (in the form chapter.question) are: currency 7.7, 7.8, 36.5, dilution of liquid plant food 36.2, scale drawing 45.2. Question 38.6 is also a suitable problem which can be done by considering similar triangles, but it requires use of Pythagoras' Theorem in addition to the scaling task.

- Students know in principle that corresponding angles in similar shapes are equal, and that the linear scale factor between the shapes applies to calculating lengths of corresponding sides.

Even so, some students may attempt to apply the linear scale factor to angle size as well as side length.

Notes for specific questions:

40.3 Students may well try to do this question without any diagrams. We recommend that teachers ask to see sketches of every triangle with all of the angles labelled. The question could be set to illustrate Pythagoras' Theorem and trigonometry, in which case the lengths of the unmarked sides of all of the triangles could be found also.

40.4 These are standard scenarios, so they provide a good opportunity to create sketches of pairs of triangles with conversion arrows forwards and backwards between them to indicate scaling factors. Students should write down all the ratios that they have for pairs of sides, and find the conversion factor from the pair which has no unknown. Note that part b) has a pair of sides in the ratio x:(x + 4.5). Encourage students to write the expression for the larger side in a bracket and then apply their scale factor to the whole bracket.

40.6 This question requires connection of several principles, so Foundation students may take a considerable time to finish it. There several ways of answering parts b) and c). If students struggle, remind them to mark all the angles on both triangles. If they have studied trigonometry of special angles (see Chapter 41 Trigonometry), they may recognise the ratio AB:AC when expressed in its simplest form, and so write down an expression for BC. BC can also be found using Pythagoras' Theorem. In triangle DEF the sides could be found by trigonometry, as an alternative to applying the scale factor found in part a) to the length of BC. However, note that the exact value of EF is required, so whichever route is taken, the answer will involve a surd (see the notes for Chapter 41), and students may need to toggle the S↔D button on their calculators to see the correct form.

This question can be used as an additional question for Chapter 41 Trigonometry.

40.7 Foundation students will find this question a challenge. Example 3 in Chapter 7 of the book shows that the first step when using scale factors should be to write down a fact. This can be in words such as "The image we see is the metal grain multiplied up by ×300." Then turn this statement into an equation. Suggest that the calculation is done in mm first. Then perform a conversion from mm to µm. This may require practice questions to revise conversion of units (see Chapter 11 Units). Part b) reverses the procedure found in part a). Part c) must be expressed clearly to show the ratio which has been calculated.

Likely wrong answer Al:Ni = 2:1. This is the ratio of the grain sizes in the images, and does not take into account the different magnification factors.

40.9 Teachers may prefer to ask students to calculate the volume scale factor separately. Alternatively, the cubing of the linear scale factor may occur within the calculation of the larger volume.

Note the change to kg in part b).

Students may need a prompt about the formula for calculation of density. Questions 18.1 and 18.2 offer practice handling density, mass and volume. Although Chapter 18 is called Formula Triangles, the questions can be solved with or without the aid of formula triangles, and teachers can indicate which method they would like students to use.

Chapter 41Trigonometry

Aims: * know and apply the trigonometric ratios for sine, cosine and tangent

* use trigonometric functions to find missing sides or angles in right-angled triangles

* be familiar with exact values of all three functions for 0º, 30º, 45º, 60º, sin and cos for 90º

* apply this knowledge to solving problems

DfE syllabus main objective: G20 trigonometric ratios (and Pythagoras' Theorem)

Subsidiary DfE objective: G21 exact values of trigonometric ratios for selected angles

Skills assumed:

Associated skills and knowledge:

DfE connections: G6 applying angle facts, A5 rearranging formulae,

A21 set up and solve simultaneous equations (Q9)

Related chapters of this book: Chapter 13 Indices and Taking Roots,

Chapter 17 Rearranging and Changing the Subject, Chapter 18 Formula Triangles,

Chapter 38 Pythagoras' Theorem, Chapter 40 Symmetry and Similarity

General comments:

- Students should be encouraged to label the sides of each triangle that they use, identifying the angle of interest and the sides opposite, adjacent and hypotenuse. If they use a new angle, it is a good idea to use a new sketch to avoid confusion from crossed-out labels.

- Most students eventually find some way to memorise the trigonometric ratios. Formula triangles are one optional method which teachers may, or may not, find helpful. We have included the triangles for the benefit of those students who have learned this method.

- The notation for the inverse functions often causes anxiety because of confusion with reciprocals. A simple way for students to tell these apart is looking at the precise position of the instruction "$^{-1}$". A function such as sin (x) can be read as "sine of x", and specifies operations that will be done to the variable x. The inverse function $\sin^{-1}(x)$ can be read as "inverse sine of x". The instruction to "undo" the sine function (written as " $^{-1}$") goes after the sin, not the x. In contrast, a reciprocal indicates a power of x, for example x^{-1} is $^1/_x$, so the power occurs after the variable to which it applies.

- When rearranging a sin, cos or tan equation to find an unknown angle, we recommend that the inverse operation is written on both sides of the equation, just as we encourage students to write down inverse operations when solving equations. The inverse should be applied to the whole numerical side of the equation, so brackets should be used:

$$\sin\ (\alpha) = {}^{\sqrt{3}}/_{2}$$

$$\sin^{-1}\ (sin(\alpha)) = \sin^{-1}\ ({}^{\sqrt{3}}/_{2})$$

$$\alpha = \sin^{-1}\ ({}^{\sqrt{3}}/_{2})$$

$$\alpha = 60°$$

- User-of-maths teachers should not depend on many students having easy recall of the trigonometric ratios for special angles, especially if students study Foundation maths. We suggest that, if these angles will be used repeatedly, and not supplied in the subject text, some preparatory practice is set. A suitable task would be to derive from first principles the results for special angles summarized in the notes for this chapter. Questions 41.6 and 41.7 offer straightforward application of these results.

- Even if students have good recall of the results for special angles, they do not always look for these types of triangles, so reminders may be necessary. Attention can also be drawn to any instruction to give an "exact" answer, which is a clue that surds may be needed in this context. ("Exact" is often a clue to expect fractions in other contexts.)

- Some suitable additional practice questions can be found in Chapter 45 Scale Drawings and Bearings: questions 45.5, 45.10 and 45.11.

Notes for specific questions:

41.1-41.3 These are basic practice questions. Encourage students to do their own sketch for each part and label it fully.

41.4c Foundation students may need a prompt that they know about applying trigonometry to right-angled triangles, so how can they create a right angle? What is the best position for the line they will draw? If necessary, remind them that their experience with triangles suggests that symmetry is usually useful. They may need a further reminder at the end of the question to find *z* itself.

Likely wrong answer 39.0° to 3sf – answer was rounded part-way through the calculation and doubled later. A good place for students to appreciate the problem of not retaining plenty of working figures.

41.5 Students should be encouraged to produce a labelled sketch.

41.6 This question is intended to illustrate the usefulness of knowing results for standard angles, and as such does not require a calculator. Ratios should be written down and cancelled to their simplest form. Angles can then be found using the table of values in the book if teachers do not wish to derive the special angle results. Alternatively, teachers can suggest that calculators can be employed.

41.7 "Exact values" implies that surds will appear in the answer. One way of tackling these questions is to write down a pair of similar triangles each time: the fully labelled target triangle and the similar standard special triangle. Then identify the scale factor to move from the standard to the target triangle (see Chapter 40 Symmetry and Similarity).

41.9, 41.10 These questions require manipulations which are more suitable for Higher Tier students.

41.9 This question is demanding because it involves finding a horizontal unknown before the value of *y* can be determined. For many students this will not be obvious. Encourage the construction of a fully labelled diagram, including allocating a variable to BD. For example, call this distance *x*. Then suggest that students draw separate diagrams for triangles ADC and BDC. Can they find two equations involving both *x* and *y*, and rearrange them into the form *y* = ….? If so, then the right-hand sides of the equations can be equated with each other, eliminating the common variable, *y*, and leaving an equation which can be solved for *x*. Students may be initially reluctant to eliminate *y* when that is the variable they ultimately want to find, but they should realise that their original equations for *y* require a value for *x*. Some students may need a reminder that, having found *x*, they are actually required to find *y*.

Students may also be worried that they are being guided to find simultaneous equations which involve sine functions. Encourage them to step through the working methodically.

The procedure of passing information from a first triangle to a second triangle which shares a common side, is the basis of proving the Sine Rule. Teachers may wish to pursue this connection: the proof is shown in the commentary for Chapter 50 Applications of Trigonometry to Geometry. Question 41.9 can be used as a supplementary question for Chapter 50.

41.10 This question is much shorter than 41.9 but it has been placed last because part b) requires explicit use of the inverse function. A suitable form of working is shown above in the comment concerning rearrangement of trigonometric equations.

When finding *r* it is important to note that

$$\sin^{-1}\left(\frac{A}{B}\right) \neq \frac{\sin^{-1}(A)}{\sin^{-1}(B)}$$

so

$$\sin^{-1}\left(\frac{\sin(p)}{q}\right) \neq \frac{p}{\sin^{-1}(q)}$$

Expressions of the form $\sin^{-1}\left(\frac{A}{B}\right)$ cannot be simplified further and should be left as they are.

Chapter 42 Circles and Circle Theorems

Aims: * revise the vocabulary associated with circles

* calculate circumference and area of entire circles

* find fractions of the area and circumference given a sector of a circle (and vice versa)

* Higher only – know and apply circle theorems

DfE syllabus main objectives: G9 circles

Subsidiary DfE objectives: G18 arc lengths, angles and areas, Higher G10 circle theorems

Skills assumed: writing and handling fractions, using properties of triangles

Associated skills and knowledge:

DfE connections: G17 circumference and area of circles, G2 loci (Q9),

A21 translate into algebra and solve equations (Q6), G20 Pythagoras' Theorem (Q11)

Related chapters of this book: Chapter 39 Angles and Shapes, Chapter 43 Perimeter and Area,

Chapter 46 Constructions and Loci

General comments:

- Calculations of the circumferences and areas of circles are included in this chapter to provide a full treatment of circles. Further practice of these calculations, and applications to problems, may be found in Chapter 43 Perimeter and Area.

- In many physical applications the tangent to a circle and the normal at the point of contact are of interest. For example, when considering the motion of an object constrained to move in a circle at the end of a rope. The rope acts as the radius of the circular motion, and the tangential velocity of the object at any instant is perpendicular to the line of the rope (i.e. along a tangent). If the rope is released, the object will move along the tangent, unless acted upon by another force.

- The algebraic relationship between the gradient of a tangent and a normal is the same as for any other pair of perpendicular lines: the product of the gradients is -1 provided that the lines are not parallel to the cartesian axes. For cases where the tangent is parallel to the x-axis, the equation of the normal is x = constant, and where the tangent is parallel to the y-axis then equation of the normal is y = constant.
  Notes on gradients of perpendicular lines are found in Chapter 24 Linear Functions and three questions are available in that chapter: questions 24.18 - 24.20.

Notes for specific questions:

42.3 Students may see immediately what they need to do, but encourage statements of their plans to sum the circumferences and the areas before they embark on any calculations.

42.5 Encourage students to see this as a scaling problem. The first statement is the fact to be used. For this problem they require a statement of the form: "Area *A* is drawn using a full turn of 360°". Then underneath scale both sides of the statement appropriately.

42.6 The most likely errors are forgetting to allow for the gaps between the slabs. Students should count the slabs, and gaps between the slabs, and write down an equation for AB and an equation for CD to show how to calculate the total lengths required. This question is relatively complicated for Foundation students because they have to calculate two different arc lengths as well as construct the initial equations. Their equations may be limited to word equations, whereas Higher students can be encouraged to allocate variables to the different arcs.

42.7 – 42.11 Encourage students to copy the diagrams and mark on all the radii to help them to identify isosceles triangles. Also, they should mark right angles, and any other groups of lines of equal length. Where they can they should then start to work out other angles. They will need to draw on their wider knowledge of geometry. Extending radii and adding diameters are often helpful things to do. Some students do not like to take the initiative to add to their diagrams, so they will benefit from encouragement to experiment in pencil. They can always rub out their efforts if the result does not produce a useful result such as showing an angle in a semicircle.

42.9b If they are stuck, students could try to plot a few possible places for D. However, suggest that they ask themselves "Where are all the points that are 7 units from C?" They may associate this with the idea of a locus (see Chapter 46 Constructions and Loci), or they may make the link to the appropriate definition.

42.10 Encourage a large sketch so that there is space for writing in the angles.

Chapter 43 Perimeter and Area

Aims: * find the perimeter and area of basic shapes

* find the perimeter and area of composite shapes

* solve problems involving perimeter and area

DfE syllabus main objective: G17 perimeter and area of plane shapes

Subsidiary DfE objective: G16 areas of plane shapes

Skills assumed: analysis of composite shapes

Associated skills and knowledge:

DfE connections: A21 translate into algebra and solve equations, G20 Pythagoras' Theorem,

R9 percentages (Q10)

Related chapters of this book: Chapter 38 Pythagoras' Theorem, Chapter 39 Angles and Shapes,

Chapter 42 Circles and Circle Theorems, Chapter 44 Surface Area and Volume

General comments:

- Much of this work is familiar from KS3. A key issue is ensuring that presentations and arguments are of a form suitable for GCSE level.

- Other practice questions involving perimeters and areas of circles can be found in Chapter 42 Circles and Circle Theorems.

Notes for specific questions:

43.2b Students may work out, or quote, the required Pythagorean triple.

43.4a Students may work out, or quote, the required Pythagorean triple.

43.8b Encourage rearrangement before substitution of values.

43.9 Encourage students to offer a statement of what they will calculate for the individual shapes before working out values. They may calculate, or quote, the required Pythagorean triple for part a).

43.10 Foundation students may need guidance with this question.

Students may need a hint to leave their expressions for area in terms of π and a.

They may also need reassurance that percentages are comparative, and may exceed 100%.

43.11 Part a) Encourage labelling of a diagram, from which it should be clear that expressions for the areas of the triangle and the circle will be in terms of the radius, r. For part b) students should expect to write an equation for their answer in words, and then substitute appropriate formulae.

They may carry forward the expression for the area of the triangle from part a), or they may attempt to find the length of a side of the square shown in the diagram.

Chapter 44 Surface Area and Volume

Aims: * find surface area and volume of cuboids, cylinders, prisms, spheres, cones and pyramids

* find surface area and volume of composite solids

DfE syllabus main objective: G17 surface area and volume

Skills assumed: identification of surfaces of a solid using nets or listing

Associated skills and knowledge:

DfE connections: G16 areas of plane shapes, G14 units of measure (Q4,8), N9 standard form (Q6),
R9 percentages (Q7), G20 Pythagoras' Theorem (Q9,10)

Related chapters of this book: Chapter 11 Units, Chapter 38 Pythagoras' Theorem,
Chapter 39 Angles and Shapes, Chapter 42 Circles and Circle Theorems,
Chapter 43 Perimeter and Area

General comments:

- In principle, the first 8 questions in this exercise can be completed by Foundation students, but some students may need support to write down an explicit plan for their answer. Questions 44.9 and 44.10 involve Pythagoras' Theorem with 3-D considerations. Foundation students are not required to analyse 3-D figures, but could try these questions as a challenge.

- In all questions it is a good principle to list surfaces and volumes that must be calculated (preferably with a diagram), and create at least a word equation for how to combine them. In particular, students often do not identify correctly those surfaces which occur in pairs (say the faces of a cuboid) and those surfaces which may or may not be in pairs (for example the ends of a cylinder, which could be closed, or open at one end).

- Where values are summed, students may need reminding that they should keep extra working figures until they have completed the summation. Otherwise, they may accumulate rounding errors.

Notes for specific questions:

44.4 Encourage students to write down the equation for the volume of a cylinder and then rearrange the equation before substituting values. Some students may need a reminder that they have the diameter and not the radius.

44.7 Some students may be able to write down immediately the percentage of S which is occupied by the beam.

44.8 This question is modelled on a laboratory experiment that can be demonstrated to students at GCSE, or undertaken by students with sufficient laboratory experience. (It is also often used as a practical for A level chemistry.) Details for an experiment are given on the Royal Society of Chemistry website at https://edu.rsc.org/resources/making-nylon-the-nylon-rope-trick/755.article

Given that this is based on an experiment, teachers may also wish to discuss modelling assumptions that could be made when using the question in the exercise. For example, i) the "rope" will contain a volume of liquid as well as the polymer, but in presenting the answer the rope is considered to contain only polymer, ii) the profile of the rope is considered to be a perfect cylinder, etc.

44.9 Students should draw a fully-labelled diagram and indicate the triangle they will use to find the other measurement which they will need.

44.10 If students need a hint, they can be referred in the first instance to Example 2 in the book.

Chapter 45 Scale Drawings and Bearings

Aims: * create and interpret scale drawings

* understand and use bearings (3-figure format with clockwise rotation from North)

* solve problems involving bearings and separation distances of objects

DfE syllabus main objective: G15 maps, scale drawings and bearings

DfE subsidiary objectives: R2 using scale factors, R12 comparing lengths and scale factors

Skills assumed: use of protractor, basic understanding of ratio and scale factors

Associated skills and knowledge:

DfE connections: N13 units, N14 estimating (Q1,7), R4 ratio notation (Q4), G14 standard units (Q7),

G19 concept of similarity (Q2,9), G20 trigonometry (Q5,8,10,11),

G21 trigonometry exact values (Q8,11)

Related chapters of this book: Chapter 7 Ratio, Chapter 11 Units, Chapter 38 Pythagoras' Theorem,

Chapter 40 Symmetry and Similarity, Chapter 41 Trigonometry, Chapter 43 Perimeter and Area

General comments:

- This chapter draws heavily on material covered in other parts of the syllabus. We have attempted to show the major connections with other topics. If teachers need linked practice questions, then suitable materials are available in the related chapters listed above.

- Many of the errors which occur are related to incorrectly calculating and applying scale factors. Suitable questions to practice calculation of scale factors are (in the form chapter.question): currency 7.7, 7.8, 36.5, dilution of liquid plant food 36.2, similarity and enlargement 40.4 and 40.6.

- One method of presenting scale factor calculations is shown in Example 3 of Chapter 7. All scaling procedures require observation or knowledge of the ratio between the two quantities involved. This information may be found by direct measurements from scale drawings, using a key provided with data, reading of lengths recorded on sketches, knowledge of conversion factors for standard units, finding out currency exchange rates, and many other different sources. In Example 3 of Chapter 7, and in the commentary notes for this teachers' guide, we refer to the ratio information as the working "fact".

- Two of the questions (8 and 10) in this exercise could also be approached using vectors. Teachers of mathematics may not prefer this route at GCSE, but physicists may well find it very appropriate. For reference, the DfE mathematics objectives would be G24 translations as vectors, and G25 applying vectors.

- Students sometimes struggle with the idea of specifying 3 figures for bearings when the first 1 or 2 figures may be zeros. The usual reason is that they attribute no significance to those digits, having learned that in handling general number, leading zeros for values > 1.0 are not shown. The bearings convention gives information for spatial orientation, and the leading zeros convey important data.

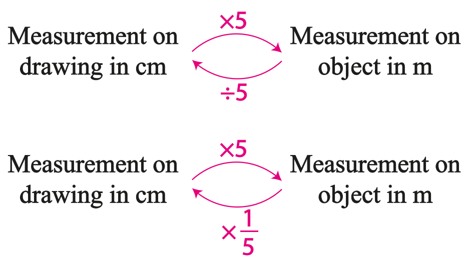
  If students imagine being the pilot of an aircraft with a crackly radio signal, and the pilot hears a turning command of just "two", they cannot be sure if this is 2°, 20°, or 200°. If they hear three figures, for example "zero, two, zero" they can be sure that they haven't missed any information on the bad radio signal, and they know that they definitely want a heading of 020°.

- Several of these questions can be approached by scale drawing as an alternative to using trigonometry. If scale drawing is used, it is important for students to appreciate that they need to use a large scale so that i) their drawing is as accurate as possible, and ii) they do not need to extend lines in order to measure angles. Less mathematically-inclined students are tempted to try to measure angles with lines shorter than the radius of their protractor, and they may not take the initiative to extend the existing lines unless reminded to do so.

Notes for specific questions:

45.2 Remind students that the "fact" written in part a) allows them to show a multiplier from left to right. The inverse operation, dividing by the multiplier that they found, shows how to perform the conversion in the opposite direction i.e. from right to left. Some, but not all, students will know that dividing by the original multiplier can also be written as a new multiplier which is the reciprocal of the old one. For example:

1 cm on drawing : 5 m on object



45.3 The most common error is omission of leading zeros from the bearings.

45.4 Remind students to write the "fact" used to find the scale factor in suitable units (they are asked for answers in cm). If they wish, they can use columns to tabulate all their answers tidily.

45.5 This question can be done by scale drawing or by trigonometry. Encourage separate labelled drawings or sketches for the two parts, each with an arrow showing the correct direction between A and B. Students should mark the angle they wish to find, and then look at their diagram to find other angles of the same value, or angles which sum to the total they require. They may need reminding about alternate angles and parallel lines (see Chapter 39 Angles and Shapes).

Look for missing leading zeros for part a).

45.6a Encourage students to assess what they need to do for this question, and note that some measurements occur more than once. It is efficient to state the conversion "fact" and multiplier, and then tabulate the required measurements and conversions before beginning the drawing.

45.6b Common wrong answer: 13.6cm – overlap of one radius-length between circle and cone not accounted for correctly.

45.7 Part a) likely error is missing the leading zero for the bearing.

Part c) should be measured with a ruler and then the measured length turned into km by comparison with the printed scale.

Part d) is intended to encourage practical solutions to problems. String is notoriously difficult to use (thick string does not follow curves exactly, some string extends when pulled etc) so students should appreciate that this is a device to produce a good estimate for the distance and not an exact answer. Values in km and miles should be found using the printed scales.

Part e) should be tackled by counting squares. There are various methods to account for partly-filled squares around the edge, for example pairing boxes that sum to one square, or counting only squares that are at least half-covered. One method of assessing area in this way is shown in Example 7 in Chapter 36 Real-World Graphs and Kinematics. (The section in which this appears is marked for Higher Tier students, but the text and method are accessible for Foundation students.) We leave it to teachers to give guidance as appropriate for their students.

45.8 This question can be started by scale drawing or by trigonometry. If scale drawing is used, students will still need to use a trigonometry procedure for part b) because a distance measured from a small scale drawing will not be accurate to 3 sf. The values have been chosen so that students can use the 1, 1, $\sqrt{2}$ (45°) Pythagorean triangle and scale up, or they can use sin or cos expressions (for trigonometry see Chapter 41).

45.9 This question was designed to give a clear conversion from cm to m. In part b), Foundation students can be guided to converting measurements from scale to actual values, and then checking their answer for the correct units. Higher students should know that a linear scale factor s scales area by a factor s^2 (see Chapter 40 Symmetry and Similarity), so those students have two options for their approach to part b).

A likely wrong answer is 0.53m^2 – incorrect use of scale factor.

45.10 Foundation students will find this question a challenge. Encourage students to draw a new diagram for each part of the question, with an arrow pointing from B to A on the line BA. They should mark the distances they know, and the angle they wish to find, and write down a statement of how to build up their chosen angle.

Parts a) and c) can be answered by scale drawing, or the unknown angle can be found using trigonometry. Part a) can be answered using a special angle triangle but part c) requires general trigonometry. Part b) does not require trigonometry.

45.11 This question presents a challenge, but can be attempted by Foundation students. If the sketches for parts a) and b) are correct, then two different right-angled triangles can be drawn where one side is ST. For one triangle the third vertex is R, and for the other triangle the third vertex is Q. The distances travelling North can be found using special triangles, or using general trigonometry, and the distance QR will follow.

Chapter 46 Constructions and Loci

Aims: * use ruler and compasses to bisect angles and lines, and construct a perpendicular

* understand the geometry which applies to constructions

* understand the geometry of loci relating to points, lines and intersecting lines

* know the conventions for presenting loci

* solve problems involving loci

DfE syllabus main objective: G2 constructions and loci

Skills assumed:

Associated skills and knowledge:

DfE connections: G4 properties of plane figures, G15 measuring lines and angles,

G20 Pythagoras' Theorem (Q7)

Related chapters of this book: Chapter 38 Pythagoras' Theorem, Chapter 39 Angles and Shapes,

Chapter 42 Circles and Circle Theorems, Chapter 45 Scale Drawings and Bearings

General comments:

- It is usual procedure in constructing bisectors of angles and lines to show just the arcs which are necessary to find intersection positions. Some students find it much easier to see why those arcs are used if they are shown the entire circles from which those arcs are derived. In the same way, with constructions some students benefit from drawing in the isosceles triangles, equilateral triangles, rhombuses and kites which connect the intersections of arcs. This helps them to see the symmetry involved.

- Inclusion, or otherwise, of the boundaries of loci follows the same basic rules as showing inequalities, where non-inclusion is a broken line and inclusion is a solid line. This connection with graphs of inequalities has not been marked as a link to follow, but teachers may wish to mention it to reinforce a cohesive approach to communicating mathematics.

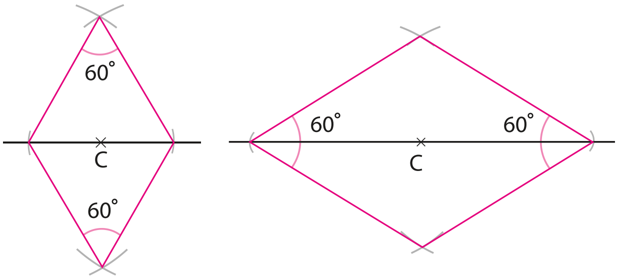
- The website implementations of the questions for this chapter have been designed to test understanding of the underlying principles of the constructions, but, by the very practical nature of these problems, the physical skills of the students can only be assessed by marking an on-paper exercise.

- There are situations where there is more than one way to construct solutions to problems. Where possible these have been considered on the website (see, for example, question 46.6 below), but teachers may need to discuss the various methods within the classroom.

Notes for specific questions:

46.3 Students will find it easier to form a plan for this construction if they first make a rough sketch of a kite and mark on the equal sides and the perpendicular diagonals. Sketching the equal sides should help them to envisage the arcs that will intersect to create the short diagonal in their construction.

46.6 Two constructions to solve this problem are shown on the website when students work through the question on-line. One method is to construct two equilateral triangles, with a common side along the line containing C becoming the short diagonal. The perpendicular bisector through C becomes the long diagonal. In the second method the line containing C becomes the long diagonal, and the short diagonal lies along the perpendicular bisector.



46.7 Many students will need a hint for this question. Given that the required length is in surd form, it is sensible to think about calculations of lengths which often involve square roots. What rule do they know that involves squares of sides of a geometrical figure, and taking square roots to find the lengths of sides? Can the students find two squares which sum to 40? So what integer values will they give to the sides of the triangle? The problem then reduces to constructing a right-angled triangle based on a perpendicular to a line, and then measuring off their chosen values on the two perpendicular sides. They should measure the length of the third side, and then find the value of $\sqrt{40}$ on a calculator to check that their measured value is correct. Teachers can give appropriate guidance for the accuracy expected in construction depending upon the age and experience of the students.

46.8 The most likely error in this question is incorrect marking of the boundary arcs of the region for points more than (but not including) 30cm from vertices. The straight lines which form parts of the edges of the triangle are included in the locus.

46.9 Sources of error may be:

1) conversion of the distances. Some students may find it easier to express the scaling in the form 0.5cm represents 100m, and then convert multiples of 100m.
2) showing the boundaries of the area with rocks: boats may be on the boundaries but not inside the area, so the boundaries are not included in the forbidden zone.

Chapter 47 Vectors

Aims: * know that vectors have both size and direction

* use column notation for 2D vectors

* recognise vectors in printed form and know how to show vectors in handwritten form

* add and subtract vectors and find scalar multiples of vectors

* use vectors to solve problems

* Higher only – use vectors to construct geometric arguments and proofs

DfE syllabus main objective: G25 combine, scale and apply vectors

DfE subsidiary objective: G24 describe translations as 2D vectors

Skills assumed: plotting on coordinate axes

Associated skills and knowledge:

DfE connections: A21 translate into algebra and solve equations (Q10,15),

A19 solving simultaneous equations (Q15), G15 bearings (Q11),

G20 Pythagoras' Theorem (Q11,14)

Related chapters of this book: Chapter 23 Solving Linear Equations,

Chapter 25 Simultaneous Equations 1 – Two Linear Equations,

Chapter 38 Pythagoras' Theorem, Chapter 41 Trigonometry,

Chapter 45 Scale Drawings and Bearings

General comments:

- GCSE treatments of vectors generally focus on algebraic manipulations of vectors more than physical applications. If physics teachers wish to set questions which rely on displacements from an origin, general calculation of components of a given vector, or calculation of the magnitude of a resultant, then it will be necessary to scaffold questions, or at least guide the students, because these applications are not specified for GCSE mathematics. Components parallel to the coordinate axes can be found by trigonometry, and magnitudes can be found using Pythagoras' Theorem, given that the terminology is explained.

- Many GCSE students do not appreciate the importance of mathematical notation as a means of communicating very specific information. Consequently, for both algebra and labelling diagrams, they often miss out the underlining or top arrow for a handwritten vector, and copy the alphabetic letter as an ordinary variable. While it can be very tiresome to maintain accuracy in presenting vectors, we recommend that teachers persist with this from the outset because poor notation will be penalized, sometimes at GCSE, but certainly if carried beyond GCSE.

- Students sometimes regard writing vectors as a lot of work, and teachers may be happy to receive just the answers for simple questions. However, we recommend that a line of working should be written if there is an intermediate step when finding a scalar multiple of a vector, especially if the scalar is a fraction.

- Even with clear working, errors are common when attempting to subtract a vector for which one or both components are negative.

- It is relatively uncommon at GCSE to find unknowns by forming equations for the x components and the y components, but we have included some questions of this type as a good foundation for those going on to more advanced maths (questions 47.10 and 47.15).

Notes for specific questions:

47.4d, 47.5 Encourage a separate line of working for the scalar multiples.

47.6d Some students will try to write ***p*** as a scalar multiple of ***q*** because they are not confident about fractional scale factors.

47.7 Encourage students to write down a simple vector which appears to be common to several of the vectors given. In this case $\binom{1}{2}$ would be a suitable suggestion. The other vectors should be written as a scalar multiple of the chosen vector where a multiple exists.

47.8 Encourage a separate line of working for the scalar multiples.

47.9c Encourage students to show the vector sum explicitly and not just the answer.

47.9d Check that the vectors ***p*** and ***a*** written in part c) are drawn nose-to-tail, and that all the vectors on the diagram are labelled, and have the correct direction.

47.10 Begin by writing separate equations for the x component and the y component. The y component equation has only one unknown (the scale factor p), so solve this for p. Then substitute into the x component equation to find x.

47.11c Encourage students to draw a diagram with the measurements that they know, and mark on the angle they wish to find. What technique do they know for finding an angle in a right-angled triangle? They may need to revise the notes given in Chapter 41 Trigonometry. They may also need a reminder about 3-figure bearings (see Chapter 45 Scale Drawings and Bearings).

The most likely error is that students will miss the leading zero for the bearing.

47.11d The shortest distance between two points is the straight line distance between them. Students should have a diagram from part c) showing a right-angled triangle, comprising a walk due East, then a walk due North, and the hypotenuse indicating the resultant vector from the start point to the end point. Which side of the triangle corresponds to the shortest distance? How would they normally calculate such a distance in a right-angled triangle? This question has been chosen to use a Pythagorean triple, which many of the students may remember or look up in the notes. In this case they can quote the result. If not, then they can perform the full calculation.

47.12 Encourage students to copy and label their own diagram, and then add the vectors they are asked to calculate. Encourage them to do a new diagram if the old one becomes crowded.

47.13 A hint may be required to label each set of parallel vectors for ***s*** and ***t***. The vectors ***s*** and ***t*** describe translations only – they are not tied to any one starting point. For example, the vector from E to D is also ***s***. Students should remember that the hexagon is described as regular, so they know that the sides are of equal length and there are three sets of parallel sides. They can also label radii that connect the vertices to the centre (shown as dotted lines to provide another hint).

M and *N* should be added to the diagram when required, and a new diagram drawn if the old one is too crowded.

47.14 This question is challenging even for Higher students because the scale factor is unknown until the end of the question. Some students may be tempted to read on and use the given scale factor from the beginning. If they do so, this shows initiative, but they could then go back and re-work the question with the scale factor as an unknown.

Note that the same scale factor applies throughout the question, so the road-map is an enlargement of a triangle with two sides described by the vectors $\binom{2}{2}$ and $\binom{-2}{1}$. To go from York to Driffield one can take the direct route, or go via Market Weighton. Encourage students to express the route via Market Weighton as the sum of the two known vectors, remembering that one leg is into Market Weighton, so the vector to use for that leg is the opposite of travelling out of Market Weighton on that road. The sum, including the scale factor k, starts in York and ends up in Driffield, so this resultant describes the direct route. Simplify the result to one vector multiplied by the scale factor *k*.

For part d) find the magnitude, using Pythagoras' Theorem with the *x* and *y* components, and then multiply by the scale factor to find the actual distance.

47.15 This is a more complicated example of simultaneous equations for the *x* and *y* components.

Chapter 48 Shape Transformations

Aims: * identify and use translation, reflection, rotation and enlargement (positive scale factors)

* describe transformations accurately

* identify points that are invariant under given transformations

* Higher only – apply enlargements with negative scale factors

* Higher only – work with combinations of transformations

DfE syllabus main objective: G7 relating shapes using transformations

Subsidiary DfE objective: G8 work with combinations of transformations

Skills assumed: use of coordinates in all four quadrants

Associated skills and knowledge:

DfE connections: A8 work with coordinates in all four quadrants, A9 using straight lines,

G16 area of plane shapes (Q8), G19 similarity and area (Q8,10), G24 translations as vectors

Related chapters of this book: Chapter 39 Angles and Shapes, Chapter 40 Symmetry and Similarity,

Chapter 47 Vectors

General comments:

- Much of the content for this chapter is familiar from KS3. Fractional scale factors are introduced for all students. Higher students also meet negative scale factors and combinations of transformations. Given the degree of familiarity with the material, the questions for the chapter are designed to increase connectivity and provide a level of challenge.

- A common source of error is lack of attention to labelling.

Notes for specific questions:

48.1 The equation for the line of reflection is $y = 0$. All points with a y-coordinate of zero lie on the line of reflection and are therefore invariant under the reflection.

48.2a The labelling of an image of A as A' is a common convention in physics and may be used in maths. Students may not have met this convention, or they may not see the relevance, so their image may be labelled as ABC.

48.3c It may be useful for students to draw in the line of reflection and label it on their diagram.

48.4 This question introduces the idea of successive transformations, but Foundation students (who usually only perform an individual transformation) can approach the question by considering each new part as a separate procedure.

48.5 This question uses successive transformations, but it can be approached by considering transformations one at a time. For part a) students should be prompted to be as specific as possible with their answer, and describe the symmetry of their named quadrilateral by including the word isosceles. In order to answer part e), students may find it helpful to obscure shape B to avoid confusion, or draw a separate diagram with just A and C. The notes for the chapter list the details which are required to describe different transformations correctly.

48.6 This question does not ask for scale factors as well as centres of enlargement, but teachers could ask for the scale factors in a written exercise.

48.8a The straightforward way to find the area of the square is to note that the x-axis bisects the square, and each part is half of a rectangle. Students may try to use Pythagoras' Theorem, which is valid, but it will take several lines of working with fractions, so teachers may prefer to divert these efforts to the geometrical considerations.

48.cd Students may work out the new area on their grid, but many of the students will have studied areas for similar figures (see Chapter 40 Symmetry and Similarity) and will know that, for a linear scale factor s, the area is scaled by s^2.

48.9 Students should expect to show working/draw diagrams to justify their answers.

48.10 Parts b) and d) each create a similar figure. As mentioned above for 48.8c, for a linear scale factor s, the area is scaled by s^2.

Chapter 49 Graphs of Trigonometric Functions – Higher only

Aims: * draw and interpret graphs of sin x°, cos x°, tan x°

* solve simultaneous equations involving trigonometric functions

* sketch curves which are transformations of trigonometric functions

DfE syllabus main objective: A12 recognise, sketch and interpret graphs of sin x°, cos x°, tan x°

Subsidiary DfE objective A13 sketch translations and reflections of a given function

Skills assumed: sketching of graphs, understanding of the principles for solving simultaneous equations

Associated skills and knowledge: understanding of degrees as a unit

DfE connections: G8 transformations of shapes, A19 solving simultaneous equations, G20 trigonometric ratios and their application, G21 exact values of trigonometric ratios

Related chapters of this book: Chapter 25 Simultaneous Equations 1 – Two Linear Equations, Chapter 30 Simultaneous Equations 2 – Where One is Quadratic

Chapter 35 Function and Graph Transformations, Chapter 41 Trigonometry

General comments:

- Many of these questions extend the students beyond the standard GCSE syllabus for handling trigonometric functions. Questions 49.1 – 49.3 cover specified material, and listing of the required values of x can be done either directly from memory or from well-drawn graphs. The remaining questions can be approached with GCSE knowledge, with additional application of a y-stretch scale factor for some functions. A y-stretch scale factor has already been met with quadratic functions, so the principle should be understood.
  The questions with a raised level of challenge have been chosen because they will be particularly helpful to students expecting to study maths and physics beyond GCSE.

- We have chosen to work with special angles for the questions in this exercise. Students may need to revise trigonometry of special angles (see Chapter 41 Trigonometry) before attempting the exercise if teachers are setting the challenge of completing the questions without using a calculator. Some teachers may prefer to retain calculators to focus on the procedures of solving equations. Evaluation of inverse trigonometric functions is considered below.

- Students are not asked to identify transformations which have been applied to a trigonometric function, but the questions do require sketching of some transformed curves. Calculations can be approached by tabulating values of x. If teachers wish to discuss the transformations, and relate them to transformations of functions met elsewhere in the syllabus, then general comments can be found in the notes for Chapter 35 Function and Graph Transformations.

- Solution of the simultaneous equations involves understanding the principle that solutions are found where two graphs meet. In the case of trigonometric functions, the cyclic behaviour means that there can be many solutions, the total being related to the number of cycles selected (see comment below).

- In the case of trigonometric equations, rearrangement of the algebra for the simultaneous expressions will require the use of an inverse trigonometric function at some stage. When rearranging a sin, cos or tan equation to find an unknown angle, the inverse should be applied to the whole numerical side of the equation, so brackets should be used. For example:

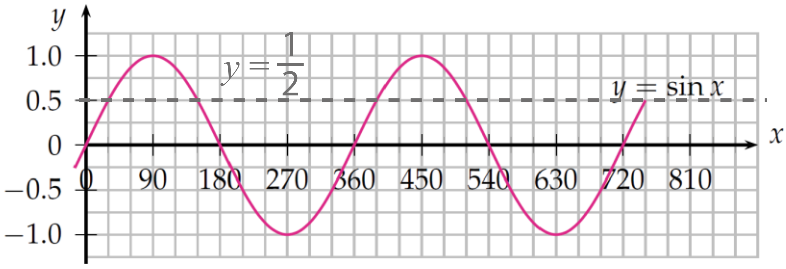
$$\sin\ (\alpha) = (\frac{\sqrt{3}}{2} - \frac{1}{2})$$

$$\sin^{-1}\ (\sin(\alpha)) = \ \sin^{-1}\ (\frac{\sqrt{3}}{2} - \frac{1}{2})$$

$$\alpha = \ \sin^{-1}\ (\frac{\sqrt{3}}{2} - \frac{1}{2})$$

$$\alpha = 21.5° \text{ to 3sf}$$

- Where students are asked to solve equations for $x°$, the value they find will be the smallest value where intersection of the appropriate graphs occurs. The number of intersections of the graphs must then be considered, and all other solutions of the equations must be found using the symmetry and the number of cycles to deduce the other positions where there is also a solution. For example, if $\sin x° = 0.5$, then the smallest value of $x°$ where this is true is $x = 30°$.



By symmetry, it will also be true when is $x = 150°$. If $0 \leq x \leq 360°$ then these are the only solutions. But if $0 \leq x \leq 720°$ then there is a second cycle and so there are two more solutions found by adding 360° to the answers for the first cycle. The additional solutions are $x = 390°$ and $x = 510°$.
Students are not required to solve any equations involving transformations parallel to the x-axis where the inverse trigonometric step leads to an expression for x.

Notes for specific questions:

49.2 Students may need a prompt to sketch their graph for the specified range of values of x.

49.3 This question introduces the concept of intersecting of graphs with multiple solutions.

49.4b A y-stretch with scale factor 2 is applied to the cosine function, so the maximum values are +2 and the minimum values are -2. Equating right-hand sides and rearranging produces a first value for x which should be recognized from studies of special triangles. The other values are found using symmetry.

49.5 The question does not ask students to identify the two transformations which are applied to sin $x°$, and the order in which they occur. However, teachers may like to add identification of the transformations to class-work. If this is required, then it is important that descriptions include the direction in which each transformation is applied. In this question both transformations operate parallel to the y-axis.

49.6 Some students may reach for their calculator and find $\sin^{-1}(\frac{1}{\sqrt{2}})$. Others may recognise the acute angle (angle less than 90°) which produces a sin of $(\frac{1}{\sqrt{2}})$. In both cases attention should be drawn to the request in the question for two graphs. The procedure in this question "runs in reverse" the procedure for solving a pair of equations graphically, and one of the outcomes from the question is two equations for y. So what should the two graphs (or two lines on the same graph) be?

49.7 Stretch (scaling) parallel to the x-axis is not specified at GCSE in the list of required transformations: this question should be attempted by tabulation of values. Students could be asked what they observe after their values have been plotted. After how many degrees does the cycle begin again for the transformed functions?

49.8 This question can be approached by tabulating values. In part a) some students will quickly realise that the graph of $y = \cos(x° - 90°)$ is a translation of $y = \cos x°$ by 90° in the positive x direction, and so the result is the same as $y = \sin x°$. In part b) students could be asked to identify the two transformations applied to $y = \sin x°$. The transformation parallel to the y-axis may be described as a reflection in the x-axis, or as a reflection in the line $y = 0$, or as a scaling operation parallel to the y-axis with scale factor -1.

49.9 This question is written as a graphical exercise, but teachers may wish to challenge students to use simultaneous equations and solve for $x°$. Encourage the students to rearrange the trigonometry to give ($\sin x°/\cos x°$), and then substitute ratios from SOHCAHTOA. What ratio results from that? Can they draw the graph for this function? Where are values +1? Can they use symmetry to find where values are -1? Are these the same values that they found for the intersections of the two curves given in the question?

49.10 This question is intended to challenge even the better students. Some students will notice the factor $\frac{1}{\sqrt{2}}$. Suggest to the students that a good question to ask themselves in maths is "What would I like this equation to be?" The simplest answer in this case is to choose values of x which reduce the equation to $y = \frac{1}{\sqrt{2}}$ (or multiples of $\frac{1}{\sqrt{2}}$). What values of $x°$ would yield an integer for ($x°/45°$)? Which values will give $y = \frac{1}{\sqrt{2}}$ for the overall equation? (Hint: Did you remember that the original equation uses $(x°/45°)^2$?) Do any other integers produce solutions? (Hint: What is the maximum value of $\cos x°$?) Students could illustrate this on their sketch.

Chapter 50 Applications of Trigonometry to Geometry

Aims: * understand and apply the Sine Rule

* understand and apply the Cosine Rule

* find the area of a triangle using the formula $Area = \frac{1}{2}ab\sin C$

DfE syllabus main objectives: G22 Sine Rule and Cosine Rule, G23 use of $Area = \frac{1}{2}ab\sin C$

Skills assumed: handling of trigonometric ratios

Associated skills and knowledge:

DfE connections: G20 trigonometric ratios, G15 bearings (Q8), G17 area of a circle (Q7), R9 using percentages

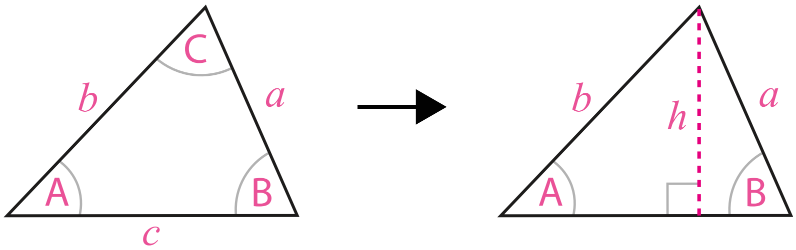
Related chapters of this book: Chapter 41 Trigonometry, Chapter 43 Perimeter and Area, Chapter 45 Scale Drawings and Bearings

General comments:

- The key to answering questions well is a fully labelled diagram.

- Students often need a reminder that there are two possible numerical results for an angle found using the Sine Rule: one angle is acute and one is obtuse. Reference to a diagram may indicate which answer is required. Both solutions should be offered if there is no clear geometrical basis for rejecting one of them.

- The notes for this chapter in the book do not include derivation of the Sine Rule. For those teachers who are users of maths, the following shows a standard derivation in which a pair of simultaneous equations is created:
  A perpendicular is used to divide a triangle into two smaller triangles with a common side, the height, *h*:



Two expressions are found for the height using trigonometry.

Left-hand triangle:	$\frac{h}{b} = \sin(A)$	therefore	$h = b\sin(A)$
Right-hand triangle:	$\frac{h}{a} = sin(B)$	therefore	$h = a\sin(B)$

Equate using h and rearrange: $b\sin(A) = a\sin(B)$, therefore $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$

A similar expression can be found involving $\frac{c}{\sin(C)}$, leading to $\frac{a}{sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$

Teachers may wish to emphasise the connectivity with simultaneous equations, in which case question 41.9 (from Chapter 41 Trigonometry) is a suitable example because it requires expression of a measurement in two different ways.

Notes for specific questions:

50.3 Note that in part a) the given angle is obtuse, so the remaining angles are both acute, and there will only be one solution from the Sine Rule. In part b) a sketch will indicate that two solutions could apply. For part c) note that $a < b$, so students should be able to draw a conclusion about the sizes of angles opposite these sides. How many solutions do they expect this time?

50.4 Part a) is symmetrical, so it could also be solved using simple trigonometry. Part c) requires a two-step process to find the unknown.

50.5b The triangle is symmetrical, so simple trigonometry could be applied.

50.5c The Cosine Rule can be applied here, but if students examine the values given, they may see a familiar ratio. Suggest first that they scale down by a common integer factor. Does this give them a clue? Then, what is still awkward about two of the sides? Why not scale down again throughout to leave integers on those sides? What ratio is left now? So what is angle γ? Students may need to refer to the notes for Chapter 41 Trigonometry, to find the table of trigonometric ratios for special angles.

50.8 Students may need a reminder that bearings are always given with 3 figures, and rotation clockwise from North (see Chapter 45 Scale Drawings and Bearings).

50.9 The question states that planting is based on the area calculation. Students are not required to discuss assumptions about how much of the area can be planted usefully, or any layouts for planting etc – this would be far too complicated.

Chapter 51 Probability Laws and Outcomes

Aims: * understand and use terminology associated with probability

* know that the probabilities of an exhaustive set of outcomes of an event sum to 1

* create a sample space which lists all possible outcomes for an event and work out the probability of a selected outcome

* know that events may be dependent or independent

* know how to combine probabilities

* work with experimental probability

DfE syllabus main objective: P7 probability theory and probability spaces

Skills assumed: manipulating fractions

Associated skills and knowledge: set notation

DfE connections: N5 listing strategies, P1 recording probabilities, P2 bias and prediction,

P3 relative frequency and theory, P4 probabilities of an exhaustive set of outcomes sum to one,

P5 effect of sample size, P6 systematic enumeration, P8 combining events,

A21 writing and solving equations

Related chapters of this book: Chapter 12 Writing and Using Algebra,

Chapter 23 Solving Linear Equations, Chapter 52 Tree Diagrams and Venn Diagrams,

Chapter 53 Conditional Probability

General comments:

- Simple probability is familiar from KS3 and most students will have accrued experience with basic questions with clear guidance. Much of the challenge to solving probability questions lies in working out what the given information means, and especially whether the outcome of one event changes the probabilities for other events thereafter (dependent events). The questions in this exercise are designed to test skills of analysis and planning in various contexts, and students should write down clear reasoning and working at every stage.

- Question 51.10 is more accessible for Foundation students than questions 51.8 and 51.9, so Foundation candidates could be encouraged to try question 51.10 after 51.6 or after 51.7. Question 51.10 has been offered at the end because the first nine questions all develop calculation of theoretical probabilities, but question 10 requires comparison of experimental and theoretical results, and has no scaffold for creating the sample space.

- Foundation students may find question 51.7 difficult, and they will find questions 51.8 and 51.9 particularly challenging, although that should not prevent them from trying to work out an answer, so they may need several hints to make progress in a sensible time. Questions 51.7 and 51.9 involve dependency, so a good question for them to ask themselves is: “After I do the first

pick, what do I have left?" If the response is recorded clearly, then the next part of the question is a new event using the recorded answer.
Question 51.8 may need a hint to allocate a variable, say *x*, to the probability of getting a 1.

- Questions 51.7 and 51.9 could be used as additional practice for Chapter 52 Tree Diagrams and Venn Diagrams.

Notes for specific questions:

51.1 Students may tabulate their answer or they may list combinations. If listing is used, encourage students to be systematic in their approach, taking the description "plain" and moving through the list of icing from left to right, and then taking "chocolate" and proceeding in the same way. Only Higher students are required to demonstrate the product rule for counting the possible outcomes (for each value of a first quantity m, they will have a combination with every value of a second quantity n, so they should expect $m \times n$ combinations). Foundation students usually understand the principle, but may not generate their results systematically.

51.2, 51.3 Encourage working which states, with reasons, what outcomes will be chosen. For example, in question 51.2b, students should write a statement that if outcome ≥ 4, then they are selecting 4, 5 and 6. Some students will be able to write down the answers for these questions without showing working, but recording reasoning is a skill to be practised as much as working out numerical values, so teachers may need to employ discretion over how much working to expect depending upon the experience of their students.

51.4 Part c) of the question applies a procedure to the table from part a), so encourage production of a new table of the values of the sums. The outcomes chosen can then be circled in the new table. The new table should be labelled in some way to show what has been calculated: some students will show the summation operation in the top left-hand corner, while others may provide a title.

Students may start to work out products of the two numbers for part d). They may need a hint that working out the values is not required. In order for the product to be a square, what must be true about the values on the cards that the children place on the table?

51.5 Students should realise what they are trying to calculate, but those with less experience may have difficulty formulating their answer. How many possibilities are there? What do they add up to? Can you write a word equation for this adding up? For Foundation students it is probably easier to combine the fractions before rearranging. So they could be guided to i) put in the values that you know, but do not rearrange yet, ii) simplify the fractions, and iii) now rearrange your equation to find the unknown value. For Higher students it is good practice to rearrange the algebra first and substitute at the end. If they are required, then practice questions for fractions can be found in Chapter 4 Fractions.

51.7 Encourage students to begin each part of this question with a statement of the form "toffees = …, mint creams = ….., total = ……" Then it will be easier for them to write down a fraction. Fractions should be simplified if possible.

51.8 Students are going to generate an equation in one unknown, so the first step is to assign a variable to the unknown. The probabilities of producing a 3 and a 4 are multiples of other probabilities, so use a variable that describes P(1), P(2), P(5) and P(6). Write the other two probabilities in terms of this variable. Are there any other possible values from the number generator? If not, then the set of outcomes is exhaustive (accounts for all possibilities), so what is the sum of the probabilities? This should lead to an equation that can be solved to give the value of the variable. Note that the last line of working must list the probabilities for all of the outcomes – some students may forget to work out P(3) and P(4).

51.9a Part a) is straightforward, but if students have not used data tables for a while they may need reminding that it is useful to sum the rows and write the results on the right-hand side, and to sum the columns and write those results below the table. The total found by adding all rows is the same as the total found by adding all columns. This total should be recorded also.

Some students are not methodical about how they select results that obey given conditions. One suggestion could be that they highlight cells in the table that fulfil the first condition (which in this question will give a highlighted column) and then the next condition (which will give them a row). The number of marshmallows that obey both conditions will be highlighted twice. The number in this cell must then be expressed as a fraction of the total.

51.9b Encourage students to identify if the second event is with replacement or without replacement. If a marshmallow is put into a drink, can it be replaced in the bag?

The events are only concerned with colour, which requires the totals for the rows, so the table can be replaced with a statement of "Pink = …, White = ….., Total = ……." for event 1. What is the probability of taking a pink marshmallow?

Given that a pink marshmallow was taken in Event 1, the numbers of Pink and White marshmallows can be written down for Event 2, and the new probability for picking a pink marshmallow can be found.

The key to understanding how to combine the sequential probabilities is to realise that only a **fraction** of the outcomes with the desired result from the first event (Pink) will also lead to the desired result for the second event (Pink). Fractions of a quantity are found by scaling, which is multiplicative. The probability of the chosen result from event 1 followed by the chosen result from event 2 is the product of the individual probabilities:

$$P(\text{Pink then Pink}) = P(\text{Pink event 1}) \times P(\text{Pink event 2})$$

51.9c What is P(Not Pink then Not Pink)? For this question students can think specifically of White and Pink, so the probability to be found can be written as P(White then White).

51.10 Part a) is a fraction, which should be simplified.

Part b) is unstructured, and students are expected to form their own sample space, and identify the outcomes as the sum of the numbers on the dice. Students have probably constructed a suitable table before, so a sufficient hint, if one is required, could be asking students how they could show all the possible results in a table, and where they would record the operation that they used to work out their answers (at the top left of the table, or as a title). All results of 7 must be counted and this value expressed as a fraction of all outcomes.

Part c) requires a scaling-up of the result from part b): probabilities are expressed out of 1, which represents a single trial. Students should be able to identify the correct multiplier.

For part d) the result of the comparison should be clear. For unbiased dice the experimental probability would be expected to approach the theoretical probability. In this question is that true? The conclusion is justified because the number of trials is large.

Chapter 52 Tree Diagrams and Venn Diagrams

Aims: * understand, construct and interpret tree diagrams for independent and dependent events

* be familiar with set notation and its use with Venn diagrams

* understand, construct and interpret Venn diagrams

DfE syllabus main objective: P8 tree diagrams and other diagrams for displaying probabilities of combined events

Skills assumed: analysis of dependency of events

Associated skills and knowledge: set notation

DfE connections: P2 bias and prediction, P4 probabilities of an exhaustive set of outcomes sum to one,

P6 systematic enumeration, P7 probability theory and probability spaces,

A21 writing and solving equations

Related chapters of this book: Chapter 23 Solving Linear Equations,

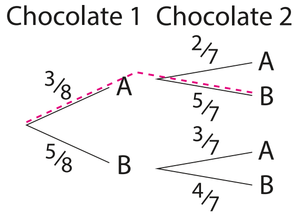
Chapter 51 Probability Laws and Outcomes, Chapter 53 Conditional Probability

General comments:

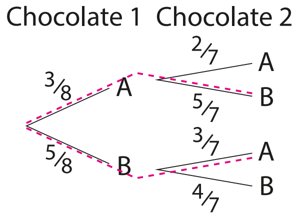
- This chapter presumes that students have a basic understanding of probability, and that they understand how to determine whether two events are dependent or independent. Practice questions for these topics can be found in Chapter 51 Probability Laws and Outcomes.

- With tree diagrams many errors can be avoided by full labelling of the diagram during its construction.
  Each event should be named with at least a title of Event 1 etc, but often a more helpful title can be written, such as "coin". Students may also find it useful to write a note to themselves about important features of each event. For example, where the number of objects changes if events are dependent, then comments such as "first event is out of 6", and then "second event is out of 5" act as reminders of what the denominators of fractions will be when the students express probabilities.
  Ensure that students draw the branches for each event under its title, and that they label each part of a branch with the outcome it represents, and either its probability or an unknown (if the probability is not known). Some students find it helpful to create a column listing the combined outcomes at the very ends of the branches, to the right-hand side of the tree, and use this to select the outcomes they require.

- It is important that students remember that finding the **probability of any one branch** occurring is **multiplicative** for the probabilities of the consecutive events, but **choosing several branches is additive** because they are choosing "this branch **and** that branch" to fulfil given conditions. The commentary for question 51.9 (Chapter 51 Probability Laws and Outcomes) considers how to explain that probabilities must be multiplied for combined events along a single branch. The suggestion is to state that only a **fraction** of the outcomes with a desired result from the first event will also go on to have the desired outcome from the second event. Fractions of a quantity are found by scaling, which is multiplicative. The following example starts with eight chocolates, 3 of type A and 5 of type B. Two chocolates are chosen without replacement of the first chocolate. Some selected probabilities are calculated to illustrate the procedures.



P(A then B) = $\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$



P(A then B) = $\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$

P(B then A) = $\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$

P(Two different chocolates) = P(A then B) + P(B then A)

P(Two different chocolates) = $\frac{15}{56} + \frac{15}{56} = \frac{30}{56} = \frac{15}{28}$

- Attention to full and correct labelling is also necessary with Venn Diagrams. It may be worth emphasizing that in the real world, busy people who commission reports often only look at the diagrams and read the summary. So, everything that can be on the diagrams should be there. For Venn diagrams the labelling should communicate if i) the diagram shows individual members of sets, or ii) the display gives total counts of particular outcomes.
  Students often need a reminder to label the universal set, $\mathscr{E}$.

- Questions 51.7 and 51.9 could be used as additional practice for the current chapter if teachers specify that tree diagrams should be presented.

- The notes in the book do not include the formula
  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
  Teachers may wish to discuss this.

Notes for specific questions:

52.1 Students should be able to identify that the pattern is the same every day. For part c) ensure that correct probabilities are calculated, or at least indicated, individually for ST and TS, and then the values are summed afterwards.

52.2 Students should identify the "with replacement" situation. For part c) check that both required branches of the tree are selected. Part d) can be calculated using probabilities for green followed by green, or by summing the answers from parts b) and c) and subtracting that result from 1.

In the second case, students should show clearly somewhere in their working that they use summation to 1.

52.3b Some students can be confused when given only one set to consider, and write down (9/124) because they do not understand that the diagram shows 9 people visited “the library only”. In conversation this might be described as “visited only the library” or “just used the library”. It is often helpful for these students to copy down the circle for the Library set, or highlight the whole of the Library circle, and then obscure everything else on the original Venn diagram. Then they can see that to represent all of the visitors to the library they must add two values (9 + 6), so that they include people who visited both the Library and the Toilets, as well as people who only used the Library.

52.4a The key piece of information is the number of people who both drive to work and work in an office, so this should be entered onto the diagram first. Then students can work out the number of people who drive to work but do not work in an office, and the number of people who work in an office but do not drive to work. In order to find out how many people are accounted for in the two sets put together, the students should take one whole set, including those people who do both activities, and then add those members of the second set who only do the second activity. Obscuring the second set to start with, and then revealing those members not already counted is a useful visual way of dealing with the problem.

A common error with Venn diagrams is that students forget to check how many people in the survey have not been accounted for because they do not do any of the listed activities.

52.5 Part e) is most likely to cause trouble. Students may like to think of this as a job of copying from the diagram the values which are inside the circles for the sets, but missing out the values in the common overlap.

52.6 Students should be able to decide if this is “with replacement”, or “without replacement”. It will probably be easier to follow the choice of socks, if students write probabilities using fractions in terms of numbers of socks, without cancelling, until the tree branches are labelled. When the labelling is finished, many of the fractions will cancel so that numbers are easier to handle.
Part d) may be calculated using the result from part c), or it may be calculated explicitly.

52.7 Foundation students may be over-awed by this question because it involves three tosses of the coin, and most questions at that level are restricted to two events. The question is a straightforward extension of familiar questions, so we recommend that Foundation students should be encouraged to try the 3-stage tree, including a column to the right of the tree to label every branch with the sequence of choices along it, **written in the order they appear** moving from left to right. That way students will retain all possible permutations of Heads and Tails: not all students appreciate that i) each different arrangement is an outcome in the sample space, and ii) all outcomes must be identified on the tree so that the correct branches can be chosen when conditions for selection are given.
For parts c) – e) students should record their interpretation of the conditions given. For example, if the condition were two tails or more, then given three coins, this means 2 out of 3 coins, or 3 out of 3 coins, show tails. The desired branches would be TTH, THT, HTT and TTT.
Some students may need a hint for part e) so that they see a useful link with part d), and do not waste a lot of time calculating a list of probabilities.

52.8 Students could copy and complete the diagram as it is. However, some students (particularly those with dyslexia) may find the large number of digits confusing. These students may find the question easier to approach if rolling 5 or 6 is defined as outcome A, and rolling any of 1-4 is defined as outcome B. These definitions should appear as a key alongside the much-simplified tree.

52.9 For part c) a possible wrong answer is 66/70 or 33/35 if $P(B \cap L)$ (the overlap region) gets counted twice. (This usually happens when totals for the individual sets are simply summed: a procedure which only works if sets do not intersect at all.) Students who offer this answer should be suspicious that they have made a mistake because 12/70 children are not in either set B or set L (that is do not have hot Breakfast or hot Lunch). So $P(B \cup L)$ students who have at least one hot meal (only breakfast + only Lunch + both) should account for 58 children.

Part d) can be confusing. Suggest that students find out how many students are in set L altogether. Out of 70, how many students are therefore in L'? Ask for a diagram showing only set L and ask the students to shade in L'. Now add that part of set B which represents children having hot breakfast only and not hot lunch. Did they draw that in the shaded region? So what is $P(B \cap L')$?

52.10 This question looks very difficult, but essentially relies on one principle: whenever an event occurs, the sum of the probabilities is always 1. Students can see that they have two events and two unknowns, so they may well expect that they will produce simultaneous equations, with one from each event. Can they write an equation in x and y for Event A? Remind them that, having done so, they can scale up their equation so that they remove the denominator.

Can they write an equation for Event B? They may need a hint to consider only the branches of Event B happening after one of a, or b, or c. What is the sum of the probabilities in each of these three separate cases? Can they write an equation in x and y now? From here, students should be able to solve their equations to find x and y.

Chapter 53 Conditional Probability – Higher only

Aims: * understand that applying a condition results in a new sample space

* be familiar with the formula $P(B|A) = \frac{P(A \cap B)}{P(A)}$

* solve problems involving conditional probability

DfE syllabus main objective: P9 conditional probability

Skills assumed: competency with basic probability and understanding of dependency

Associated skills and knowledge: set notation

DfE connections: P4 probabilities of an exhaustive set of outcomes sum to one,

P6 systematic enumeration, P7 probability theory and probability spaces,

P8 tree and Venn diagrams, N5 systematic listing and the product rule (Q3)

Related chapters of this book: Chapter 51 Probability Laws and Outcomes,

Chapter 52 Tree Diagrams and Venn Diagrams

General comments:

- The aim of these questions is to challenge understanding, so we have limited the exercise to three questions. Maths teachers will recognise that there is a significant overlap in level of demand with A level mathematics.

Notes for specific questions

53.1 Creating the Venn diagram is a standard procedure, with a key step of finding out how many people predict either Year 1 only or Year 2 only. Parts b) – d) are also straightforward, given that the Venn diagram is correct. Part e) is less common. Students understand that the expected occurrence of an outcome can be described as "never", "likely" (with a given probability), or "certain", but questions that deal with the extremities of probability can still be a surprise.

53.2 Parts a) – c) are straightforward. Parts b) and c) provide a foundation for part d).

53.3 Students can work out the number of outcomes in the sample space using the product rule. (For each value of a first quantity *m*, they will have a combination with every value of a second quantity *n*, so they should expect $m \times n$ combinations). Although they do not need to work out all the summed values, it is worth laying out the sample space as a table and filling in the answers, rather than relying on listing the outcomes for each part of the question: the table is only done once, and it is easier to spot any errors by looking at patterns in a table.

Part c) should be presented with clear working to show the outcomes that make up the new sample space i.e. the outcomes which satisfy the condition that the total score ≤ 5, and which of those outcomes obey the condition that at least one of the dice scored a 2.

Chapter 54 Sampling and Representations of Data

Aims: * know and use statistical terminology

* understand the basic principles behind sampling a population, and avoiding bias

* know that data can be discrete or continuous

* display data appropriately, including using tables, charts, pictograms, pie charts and time series

DfE syllabus main objective: S2 construct and interpret tables, charts and diagrams to display data

Skills assumed: basic calculations with percentages, fractions and scale factors, use of protractor

Associated skills and knowledge: designing and carrying out methods of data collection

DfE connections: S1 sampling and populations, S4 comparing distributions, R9 percentages,

N11 fractions and ratio, N12 fractions and percentages as operators, R2 scale factors,

R3 writing fractions (Q9), R7 equality of ratios (Q9), A21 writing and solving equations (Q9)

Related chapters of this book: Chapter 4 Fractions, Chapter 6 Percentages, Chapter 7 Ratio,

Chapter 23 Solving Linear Equations

General comments:

- Much of the material in this chapter is familiar from earlier Key Stages. It is important to look for errors carried forward, and especially to check that diagrams are fully labelled: two common errors are forgetting to give keys, and units missing from the axes.

- As well as doing the questions provided in the exercise, students may use the notes in the book to plot data that they planned and gathered themselves. This chapter assumes that data collection in general has been covered in earlier years, and that students are able to design data collection sheets. However, errors often arise. Some important things to check for questions on data collection sheets are: i) Can all possible responses be recorded in one, and only one, of the categories offered, and ii) Does the number of people questioned match the number of data points? Students should attempt to explain any discrepancies and teachers may need to use discretion concerning the usefulness of the data.

- Tally charts should be familiar, but teachers may like to revise how tally charts are created before starting the exercise. Even at GCSE students sometimes mis-record a tally. One error in practical situations may be an individual misplacement of one count, when a tally is done in a hurry, and students should be encouraged to check record sheets for errors of this type when they have finished a survey. A more serious common error is consistently grouping 5 tallies vertically and then using a sixth stroke to cross the first five. Does each group mean a count of 5 or a count of 6? Students should be clear that every stroke is a count of 1, and that the standard grouping is 5 strokes, shown as 4 vertical strokes plus 1 inclined stroke.

- Time series are introduced at GCSE and students do not always appreciate that: i) data collected over a time period should be shown at the middle of the period because there is no information to show how the data amassed within the period, and ii) there is no indication of behaviour between the data points, so straight lines should be used to connect the midpoint values.

- Handling data often draws upon skills with fractions and percentages, and treating these as operators to scale other values. If questions to practise individual skills are required, these are available in Chapter 4 Fractions, Chapter 6 Percentages, and Chapter 7 Ratio.

- For teachers who are users of maths, and who may be accessing applied maths questions without the maths skills section, we add a note on calculating percentages. If at all possible, it is recommended that percentages are calculated by use of scale factors. For example, 10% (or a fraction 10/100) means applying a multiplier of 0.1. This technique can be applied in one line of working whatever the percentage that is required, and the method uses principles which apply throughout the GCSE course for other topics such as ratio, conversions and similarity. Students may offer methods based on chunking and sub-dividing, finding 10%, then from that 5% and so on. This method works, and may offer a one-off approach to a problem, but for more complicated questions a lot of work is involved, and calculations can become very cumbersome and prone to error. It is also very difficult for students to appreciate the overall effect of scaling all their values by the same factor.

Notes for specific questions:

54.2 For part a) students need to identify the total percentage reached when all passengers in the survey have been included. Parts b) and c) require a method for general scaling using percentages (user-of-maths teachers please see note above). Students may be able to write down the appropriate multiplier in one step. Alternatively, they may show working in two steps, by scaling down to find 1% and then scaling that result up to find any other percentage. A suitable method of scaling beginning from a known "fact" is shown in Example 3 of Chapter 7 Ratio. In the current question a suitable starting fact is supplied for the category "going shopping".

54.4 The most likely error is that students will miss the information given at the start that 1 box = 10 packets.

For part a) ensure that the chart is fully labelled, with a key. The data can be shown as boxes or packets (with the correct key).

Part b) common wrong answer 7 - students should note which quantity is required (boxes or packets).

For part c) students may find it useful to add columns to the chart to show the change in the number of each bulb sold, and then an expression of the change as a percentage of the original value for that type of bulb. Converting fractions to percentages is covered in Chapter 6 Percentages, with a typical calculation shown in Example 2. Students should note that the percentages are for types of bulb and not a percentage of total sales.

54.7 For part a) students should sum the number of meals bought. They can then find the number of degrees corresponding to one meal, and hence scale up to find the sector angle for each type of meal. Alternatively, they may write each sector as a fraction of 360°, as shown in Example 4. Teachers should check for correct labelling of the chart and the presence of a key.

The answer to part b) should be cancelled to its simplest form.

54.8 Students may be worried by this question because they do not have monetary values in part a). But if asked, they know that the first step in creating a pie chart is to find sector angles. Can they find 60% of 360°?

Part b) reduces to using a known fact to calculate a scale factor – see commentary above for question 54.2. Then scale up to find the total annual income. The question does not ask for the value in Standard Form, but students may use it for their answer.

54.9 This question illustrates the method of capture-recapture for estimating populations, which is used particularly in the biological sciences. It is sometimes assumed that the user knows how to apply the method, but this is not always true, so question 54.9 is designed to illustrate the procedure. In particular, students often ask what their second sample represents. So part c) of this question includes explanation and explicit instruction to equate the results from parts a) and b).

Chapter 55 Summary Statistics

Aims: * understand mean, median and mode as averages, and calculate and apply appropriately

* understand range as a measure of spread

* Higher only – find Quartiles, and use Inter-Quartile Range for assessing the spread of data

* Higher only – box plots and their use

DfE syllabus main objective: S4 interpret and analyse distributions of data

Skills assumed: ordering and sorting of values, basic use of percentages, fractions and ratio

Associated skills and knowledge: Sigma notation

DfE connections: S2 tables, charts and diagrams for displaying data, S5 statistics and populations

Related chapters of this book: Chapter 54 Sampling and Representations of Data,

Chapter 56 Grouped Data and Diagrams

General comments:

- Statistics is one of the topics that students are likely to use in the work-place, so it is important that they practise presenting data in a way that is directly useful to an employer. Busy executives commission reports, which must be complete in their analysis, but when the report is received the executive may initially just look through the graphs, and then read the (executive) summary to find out what conclusions were reached. Each result given in the summary must therefore be a complete statement of what data was recorded, what quantity was calculated, the value of the quantity with its accuracy and units, and (if appropriate) an interpretation in the context of the report. Students sometimes find it irksome to write each statement, feeling that they are merely repeating themselves. It may help them to persist if they are reminded that what they are practising is a good skill for the jobs market.

- The first section of this exercise provides basic practice for examining data and assessing location and spread. No questions requiring comparison of samples or populations are provided for Foundation students, but teachers could use data from question 55.10 leaving out the IQR in part a) and the box plots in part b) if students are not studying for Higher Tier.

- Students often try to hurry through sorting data into numerical order, and make mistakes. Encourage them to write a new ordered list, and to lightly cross out each value from the original list once that value is transferred. If there are a lot of values to be stored in only a few categories, then a tally chart is a good way of sorting the data. The ordered data stream can be written out from the tally chart afterwards.

- Outliers are mentioned briefly in Chapter 54 Sampling and Representations of Data, but no guideline is given for deciding if a value is an outlier or not. As a general rule for GCSE, students will only be asked to identify an outlier because it obviously does not fit an otherwise well-defined pattern. Should students ask about a more numerical guideline, one commonly-used "rule of thumb" is to employ the quartiles (Q_1 and Q_3) and the IQR ($Q_3 – Q_1$), and a value is described as an outlier if it is more than 1.5× the IQR below the lower quartile or above the upper quartile. i.e.

$$\text{value} < Q_1 - 1.5(Q_3 - Q1) \qquad \text{or} \qquad \text{value} > Q_3 + 1.5(Q_3 - Q_1)$$

This rule will only be meaningful if students are already familiar with quartiles.

Notes for specific questions:

55.1 Students may worry that the mean is not an integer (as prey is captured one animal at a time) and attempt to round their answer because they do not appreciate what it is that they learn from a non-integer answer. To produce a mean, all of the capture results are added together and then it is assumed that the total creatures are shared equally over all of the days involved. The answer for this average is simply a number, which may, or may not, be an integer. The value must then be interpreted in terms of the capture of creatures. For this question, the correct value for the mean tells them that, using numbers smoothed out over a month, the cat captures more than one whole creature per day, but well below two creatures per day. The mean does not say which creatures are captured on any day, or exactly how many creatures are captured on any one day.

55.2 The most likely problem will be errors in handling the negative values in part b).

55.4 This is a question where a tally chart could be useful for sorting values.

55.5 This should be a familiar style of problem, where one or two individuals have very much larger salaries than the other employees.

55.6 Part a) is a "working backwards" problem, so Foundation students may need a hint to allocate a variable to the unknown, for example "call it x". When they include this value, how many data items do they have altogether? Then they should write out the equation for the mean, showing addition of all of the contributing values, including x, explicitly in the numerator. They know the number of data items in the denominator, and they know the answer to the mean on the right-hand side. Can they rearrange the equation and solve for x?

Ordering the data numerically is required to answer parts b) and d), and the most likely problem is that students forget to include the value that they calculated in part a) to finish the data list.

For part c) one way of assessing which value(s), if any, could be an outlier, is to tabulate the departure from the mean for each data value.

55.7 Students should show working for this question either by copying the table and adding a column to calculate the actual number of wickets represented in each row (see Example 2), or by writing an explicit calculation showing Wickets × Frequency for each of the four rows of the table. Working should also show the formula and substitution for calculation of the mean. The answer may be presented as an exact value, or as a decimal (1dp would be appropriate).

55.8 To find quartiles students must remember that, when they are finding the middle value with an even number of data points, the answer they use is mid-way between the two most central values in the list they are using (see Example 5). For part b) the data must be ordered first. Students should expect to show the ordered values as a line of working.

55.10 If teachers wish to set this question as a data comparison without IQR and box plot, then students can use part a) items i) and ii) (mean and range), and then make their comparison of spread using the range instead of the IQR.

For part c) students should bear in mind the need to write a full statement of the information that they extract and the conclusion they draw from it, so that a reader has evidence in context. In the question as set, answers should include the IQR and/or range for men, the IQR and/or range for women, and the evaluation.

55.11 This question is intended to illustrate a situation that could occur in a real setting. Before experimental results are published, calculations should always be checked, and preferably by another person who starts from the beginning so that they are not influenced by something they have seen.

Part b) is a slightly more difficult version of finding an unknown.

Chapter 56 Grouped Data and Diagrams

Aims: * know how classes are defined to cover all possible data values

* find median and modal classes

* know how to calculate an estimate of the mean, and understand why this is an estimate

* Higher only – create and interpret histograms

* Higher only – create and interpret cumulative frequency diagrams

DfE syllabus main objectives: S4 interpret and analyse distributions of data (grouped data),

S3 histograms and cumulative frequency graphs,

Skills assumed: general presentation and analysis of statistical diagrams,

understanding of mean, median and mode

Associated skills and knowledge:

DfE connections:

Related chapters of this book: Chapter 8 Rounding, Limits of Accuracy and Bounds,

Chapter 54 Sampling and Representations of Data, Chapter 55 Summary Statistics

General comments:

- One issue with grouped data can be identifying the correct class boundaries. For continuous data there must be no gaps between classes. Definitions for classes must be such that any possible measurement can be assigned to a class, and there should be only one class where any specific measurement belongs.

- In mathematics, Foundation students are accustomed to handling grouped data with the same width for all classes. Teachers of subjects using maths should be wary of expecting Foundation students to handle data in their own subjects where there are different class widths. Taking the mid-point of a class as the "best representation" for all members of the class is a familiar concept, so in principle Foundation students could find an estimated mean for data presented with unequal class widths, as well as using equal class widths. But, Foundation students are unlikely to appreciate that heights of the bars on a frequency graph of the same data are not comparable in the same way as they are on familiar bar graphs.

Notes for specific questions:

56.1b Students should know that a disadvantage of grouping data is that exact information about individual values is lost. Even so, in part b) they sometimes try to find a value for the median. The median can be estimated by linear interpolation, but this is beyond the scope of GCSE level. For this question remind students that they are only asked for the **class** in which the median occurs.

56.2 Part a): some teachers may prefer to discuss class boundaries first, but the midpoint is symmetrically disposed with respect to class boundaries, so it can be determined by bisecting the gap shown for the rounded integer values. The error students usually make is to bisect the gap, but think they must give an integer and round back up again. So their answer is 55cm, even though they know that the gap is 9cm and the middle of the class is at 54.5cm.

Part b): Writing down class boundaries will not be familiar for all students. The notes for the chapter explain that students should consider what happens with rounding values. Encourage students to think about finding the smallest value that would be written as 50cm, and the largest value that would still be written as 59cm. If revision of rounding is required, teachers can refer to Chapter 8 Rounding, Limits of Accuracy and Bounds.

56.3 Part b) As in 56.1b, students should be dissuaded from expecting to know the median value: they can only find the class in which it occurs.

Part d) encourage addition of columns to the data table to show the price of 1 tree, and then the total value of all the trees for that height class.

56.4 Students should add a column for frequency density to their data table, with units (per kg). Check the histogram for correct labels, class boundaries and frequency density values.

56.5 The frequency density has no units in this question.

Teachers should expect to see additional columns created for calculating the mean, and for the cumulative frequency.

56.6 Constructing the cumulative frequency curves and box plots covers familiar material. Most issues are with part c) and concern how to work out and present the argument. First of all, students should examine the question and decide what it is they are looking for, **before** they jump to their plots. They are asked for the harder paper. If they take an examination, do they expect to get more marks or fewer marks if the paper is harder? The answer gives them their first statement: “On a harder paper the students expect to get…..marks.” The turn to the plots and record comparisons for the positions of the two curves in general (more marks and fewer marks), and read off Q_1, median, and Q_3 for the two curves. Make statements about the relative values of each of the measured quantities. Then base a conclusion on those statements.

56.7 This question may be unfamiliar to some students because they have not constructed tables of data from a histogram. They will need to rearrange the equation

$$frequency\ density, f.d. = \frac{frequency}{class\ width}$$

Working should be shown, and a table is a good way to present the working. Column headings for the final frequency table should include “Pay in £1 000s, *p*”, and Frequency.

For part c) students must remember that they are dealing with pay in thousands of £.

56.8 This question illustrates handling of real-life data. For part c) students can comment on IQR and range, and reach conclusions based on both comparisons. In this case the conclusions agree, and the result of the trial is clear. In the real world, statistics are often much more complicated to interpret, but the first stage is always to write down the individual comparisons and conclusions. In many situations this will lead to recommendations for a second set of trials, followed by evaluation, and there may be more cycles of investigation beyond that. Teachers may wish to discuss data cycles with their classes.

Chapter 57 Correlation

Aims: * identify correlation on a bivariate plot

* know that correlation does not imply causation

* construct a straight line of best fit for linear correlation

* use a line of best fit to find values, and predict pairs of values using interpolation and extrapolation, with understanding of the limitations

DfE syllabus main objective: S6 scatter graphs and correlation

Skills assumed: use of coordinates and straight lines

Associated skills and knowledge:

DfE connections: A10 graphs of linear functions, R11 using compound units (Q2,3),

A9 equations of straight lines (Q4), A12 recognising and interpreting graphs (Q4),

R4 using ratio (Q5)

Related chapters of this book: Chapter 7 Ratio,

Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines

General comments:

- A major issue with correlated variables is the assumption of causation with no evidence for the claim. For two variables a "rule of thumb" numerical relationship may often be observed for part, or all, of a bivariate plot of data. Within the confines of the data, it is a reasonable working hypothesis that other pairs of values can be estimated using the behaviour seen in the surrounding part of the graph. This says nothing about the reasons for the behaviour, and no causation can be implied. There may be a causal connection between the variables, but often there is no connection at all.
  If correlation exists, it may be that a third variable is involved, and it is links with this third variable that lead to the observed relationship. Often the third variable would be time. For example, the price of a specified rail ticket can be expected to rise over time. A child grows taller over time. It may be that a plot of the price of a rail ticket against the height of a child shows some correlation. But the relationship is not causal: growth of the child does not influence the price of the ticket, and the ticket price does not change the child's height.

- If correlation exists, students must be careful to identify correlation as linear where an approximate straight-line relationship exists: simply stating that the variables are correlated does not communicate the form of the relationship.

- It is a good policy for students to create a habit of commenting on the reliability of predictions that they make from lines of best fit, even if they are not explicitly asked to do so. If interpolation is used, then this should give a good estimate of values of the variables (reliable). If extrapolation is used, students should explain their answer was derived by assuming that the relationship between variables extends beyond the range of the data, and their answer may not be reliable (or could be misleading). Reference to the context of the problem may also help with assessment of reliability (see part d) of question 57.2 and part c) of question 57.3).

- Some lines of best fit pass through the origin, but some students will always try to draw a straight line which includes the origin, even when the data clearly suggest otherwise. Remind them that the line of best fit is given that name because it is drawn to represent the recorded data, and it cannot be assumed that the line will pass through the origin.

- If students wish to revise construction and properties of straight lines, then practice questions can be found in Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines.

Notes for specific questions:

57.1 For those sketches which students identify as linearly correlated, they could also be asked to specify strong or weak, and positive or negative.

57.2 Finding the gradient of a straight line should be familiar, but students can be referred to Example 3 in the notes for Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines, if they wish to revise the concept. The most likely error is swapping the order of **either** the two *x*-coordinates, **or** the two *y*-coordinates (but not both). Students are more likely to spot their own mistakes if they show full working with clear substitution.

For parts c) and d) extrapolation is required, and in the context of the problem the answer would be qualified as unreliable. In this question extending the correlation suggests that a child of 6 months would participate in craft for the given time. Relating this to the context, a child of 6 months is unlikely to be capable of undertaking any craft activities, and extending the correlation cannot be justified.

57.3 The best straight line is close to all the points with one exception, so correlation is strong and interpolation would be reliable. For part d) extrapolation is required, but only for a short time period. While reliability falls for predictions outside the range of the data, the combination of a strong relationship, and a short extrapolation in time, suggest that, as long as there is no financial upheaval, the predicted answer is fairly reliable.

57.4 Suggest that students write out the equation of a straight line;

$$y = mx + c$$

By comparing each equation with this general form, they can find the gradient, *m*, and the intercept, *c*, for each of the given equations, and hence identify the correct equation.

A common source of error is taking the magnitude for *m* and *c*, but omitting the signs that go with them.

For part c) the most likely wrong answer is ii) – the label on the *x*-axis has not been read, or has been mis-read.

57.5 This real-world question uses lines of best fit to illustrate the change in neutron:proton ratio in nuclei as the atomic number of elements increases. The data have been chosen from two regions of the periodic table where the n:p ratios approximate to different integers. Not all elements in a region are chosen, and if there is more than one isotope of an element, then the choice is an abundant stable isotope that provides strong correlation. The question does not require an explanation for the changing n : p ratio, and students who do not study physics can complete the question as an observational exercise. Physics and chemistry specialists may wish to discuss nucleons in more detail.

For part d) the most likely error is that students plot $(p + n)$, which is given in the table, instead of n.

Preparation for Sixth Form and other courses

When entering a new educational phase or environment, students thrive best if they have:

- Reliable recall of knowledge and methods learned previously
- Fluidity in handling algebra, and other rearrangements and conversions
- A high degree of connectivity of mathematical ideas so that they can use analogy

From a teaching perspective, classes make more progress if there is:

- An extensive common knowledge base, and preferably common methodology

Entry into Sixth Form courses, particularly those involving mathematics, has always posed a difficulty in this respect because i) students cease to reinforce memory of what they have learned once they have taken their GCSEs, and ii) most Sixth Forms are a merger of students from a variety of GCSE settings.

One way of tackling these issues is to set incoming students some revision questions, so that they begin from a common position. A subsidiary, but frequently useful, outcome of using such bridging material is that some students who would struggle part-way through the A level course may decide not to start the course, and transfer to a different subject which will be a greater success for them. Assessing students at the outset is especially helpful for maths teachers, who have an extra issue because students can opt for just A level maths, or they can take both maths and further maths A levels: setting bridging work may identify students who have done well at GCSE, but will find further maths courses a challenge.

For teachers who wish to direct students to Using Essential GCSE Mathematics for refreshing maths between GCSE and A level courses, or as preparation for entering the UK system post-GCSE, we suggest below some questions that would provide a useful background.

The first three sections of questions are suggestions to cover general revision of some selected topics for: 1) important basic skills, 2) augmenting basic skills for students doing STEM subjects at A level (or the equivalent), but not including A level maths, 3) material preparatory for studying A level maths (with or without further maths). The STEM questions relate to skills for handling experimental data, and for mathematicians the choices have been made to concentrate on developing algebraic competency.

Students will have time limitations, so in each section we indicate questions from a few selected chapters to give a good starting point for each category of study. If teachers require students, in particular STEM students, to look at more than one section, then guidance will be required to avoid creating a large workload. Hence, we have not suggested using whole chapters for bridging work, but if teachers prefer to use chapters, the on-line chapter-by-chapter question boards can be used in the usual way.

Teachers may well want to choose material from extra chapters to broaden the revision experience, so in section 4 we suggest some other questions which might be useful. These questions include both additional topics, and more synoptic applications of basic principles.

In section 5 we have listed some more challenging questions which could be used for testing understanding or raising the level of challenge for students who are considering A level further maths.

Please note that these lists are not exhaustive: they are intended to provide a basic resource which teachers can adapt for their current students.

Suggested questions for study:

Section 1 Basic skills for students who are:

- joining or rejoining the UK system post-GCSE
- users of maths in non-STEM subjects at A level or the equivalent
- preparing to start STEM A levels, but without studying A level maths
- wanting to check their skill level before studying A level maths or further maths

Chapter 4 Fractions (questions 4, 5, 10)
Chapter 6 Percentages (8, 9, 11)
Chapter 7 Ratio (6, 8, 11)
Chapter 11 Units (1, 5, 10)
Chapter 12 Writing and Using Algebra (2, 7, 11, 12)
Chapter 13 Indices and Taking Roots (1, 5, 9)
Chapter 17 Re-arranging and Changing the Subject (3, 4, 7, 9)
Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines (5, 8, 12, 14)
Chapter 54 Sampling and Representations of Data (2, 4, 6, 9)

Section 2 Supporting STEM questions suitable for:

- augmenting basic skills for STEM students not studying A level maths

Chapter 10 Standard Form (questions 5, 10, 13)
Chapter 23 Solving Linear Equations (4, 5, 7, 12)
Chapter 34 Proportionality (3, 5, 11)
Chapter 36 Real-World Graphs and Kinematics (1, 3, 5)
Chapter 38 Pythagoras' Theorem (3, 5, 7)
Chapter 55 Summary Statistics (3, 5, 6)
Chapter 57 Correlation (2, 3, 4)

Section 3 Pre-A level maths topics suitable for:

- students intending to study A level maths (with or without further maths)

Chapter 14 Expanding (4, 7, 10)
Chapter 15 Factorising I: Common Factors (7, 8)
Chapter 16 Factorising II: Quadratic Expressions (questions 1, 5, 11)
Chapter 20 Functions (6, 9, 13)
Chapter 21 Surds and Rationalising a Denominator (3, 7, 8, 10)
Chapter 24 Graphs and Co-ordinate Geometry of Straight Lines (15, 17, 19)
Chapter 27 Graphs of Quadratic Functions (4, 6, 8)
Chapter 38 Pythagoras' Theorem (8, 10, 12)
Chapter 41 Trigonometry (2, 3, 7, 9, 10)

Section 4 Useful general questions suitable for:

- increasing the number of topics that are revisited
- increasing the connectivity of ideas

Chapter 4 Fractions (question 15)
Chapter 6 Percentages (16, 19)
Chapter 8 Rounding, Limits of Accuracy and Bounds (7)
Chapter 10 Standard Form (16)
Chapter 11 Units (11)
Chapter 25 Simultaneous Equations 1 – Two Linear Equations (9)
Chapter 31 Inequalities (5, 9)
Chapter 34 Proportionality (10)
Chapter 36 Real-World Graphs and Kinematics (11, 12)
Chapter 40 Symmetry and Similarity (9)
Chapter 42 Circles and Circle Theorems (6)
Chapter 43 Perimeter and Area (10)
Chapter 44 Surface Area and Volume (4)
Chapter 51 Probability Laws and Outcomes (4, 5)
Chapter 52 Tree Diagrams and Venn Diagrams (1)
Chapter 55 Summary Statistics (7)
Chapter 56 Grouped Data and Diagrams (1)

Section 5 Additional questions for:

- testing higher level skills
- raising the level of challenge

Chapter 6 Percentages (question 20)
Chapter 11 Units (12)
Chapter 12 Writing and Using Algebra (16)
Chapter 14 Expanding (11, 13)
Chapter 16 Factorising II: Quadratic Expressions (12, 13)
Chapter 20 Functions (16)
Chapter 21 Surds and Rationalising a Denominator (13)
Chapter 22 Algebraic Fractions (3, 9)
Chapter 23 Solving Linear Equations (13, 14)
Chapter 26 Solving Quadratic Equations 1 – By Factorising (9, 11)
Chapter 28 Solving Quadratic equations 2 – The Quadratic Formula (6)
Chapter 29 Solving Quadratic Equations 3 – Completing the Square (8)
Chapter 30 Simultaneous Equations 2 – Where One is Quadratic (9, 10)
Chapter 33 Graphs of Standard Functions (11)
Chapter 34 Proportionality (12, 13)
Chapter 36 Real-World Graphs and Kinematics (10, 13)
Chapter 39 Angles and Shapes (10)
Chapter 51 Probability Laws and Outcomes (8)
Chapter 52 Tree Diagrams and Venn Diagrams (10)