Isaac Physics Skills

Linking concepts in pre-university physics

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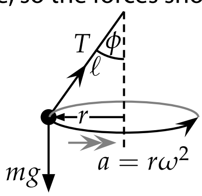
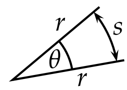
18 Conical pendulum

A particle of mass m at the end of a light string fixed to a point can be set in motion so that it moves in a horizontal circle centred below the point of suspension.

Example context: Fairground rides, mechanical speed controllers; examples are closely related to those on smooth banked tracks, as the normal reaction force of the track is replaced by tension in a string or rod. In the diagram, the tension in the string and the weight are not aligned, so the object is not in equilibrium. A resultant force of constant magnitude is directed horizontally towards the centre of the circle, so the forces should be resolved horizontally and vertically.





(the view from above)

$$s = r\theta \qquad \frac{s}{t} = r\frac{\theta}{t} = r\omega$$

Quantities:

T tension in the string (N)	a acceleration inwards $(\mathrm{m\,s^{-2}})$
r radius of orbit (m)	ω angular velocity (rad s^{-1})
ϕ angle to vertical (°)	f frequency (s^{-1}, Hz)
ℓ length of string (m)	t_p period (s)
v speed of object (m s^{-1})	θ angle of rotation (rad s^{-1})

Equations: $\quad F = ma \quad a_\text{centripetal} = r\omega^2 \quad v = r\omega \quad \omega = 2\pi f \quad t_\mathrm{p} = \dfrac{1}{f}$

18.1 A metal ball of mass m is attached to a light string of length ℓ and moves in a horizontal circular path at an angular velocity ω. Use diagrams to write down expressions for

a) the angular velocity ω of the ball in terms of ϕ, r and g,

b) the period of orbit, t_p, in terms of ϕ, r and g,

c) the [horizontal] acceleration of the ball, a in terms of ϕ and g,

d) the acceleration of the ball, a, in terms of m, T and ϕ,

e) the tension in the string, T, in terms of m, g, r and ω,

f) $\cos\phi$ in terms of ℓ and r,

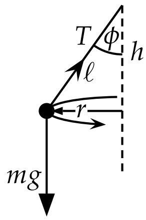
g) the angular velocity ω in terms of g, ℓ and r,

h) $\cos\phi$ in terms of g, r and ω,

i) v in terms of ϕ, r and g,

j) t_p in terms of v and a.

Example – *A small ball of mass* 0.60 *kg is suspended at the end of a light string of length* 0.80 *m attached to the ceiling. The ball travels in a horizontal circle about a vertical axis* 1.3 *times per second. How far below the ceiling is the ball?*
Resolving the forces on the sphere H and V, we obtain the two equations



$$T\sin\phi = mr\omega^2 \quad \text{and} \quad T\cos\phi = mg$$

$$\text{Dividing,}\quad \tan\phi = \frac{r\omega^2}{g} = \frac{r}{g}4\pi^2 f^2$$

$$\text{But also,}\quad \tan\phi = \frac{r}{h} = \frac{r}{g}4\pi^2 f^2$$

$$\text{Hence,}\quad h = \frac{9.81}{4\pi^2 \times 1.3^2} = 0.15\ \text{m}$$

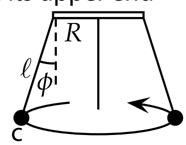
18.2 A small sphere of mass 2.0 kg, attached to the end of a light string of length 90 cm at 24° to the vertical, moves in a horizontal circle. Calculate

a) the tension T in the string, and

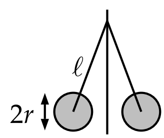
b) the height h by which the mass is raised above its position at rest.

18.3 A lead ball of mass 45 g is attached to the end of an 80 cm long light string and swung around in a horizontal circle at high speed. If the string snaps at a tension of 195 N, what is the maximum frequency of rotation f possible?

18.4 A fairground ride consists of several small carriages (**c**) each supported at its centre of mass by a light cable of length $\ell = 2.20$ m with its upper end attached to a supporting ring of radius $R = 3.40$ m from the axis of rotation. What is the period when the carriages are rotating so that the cables are inclined at $\phi = 30.0°$ to the vertical?



18.5 A mechanical governor consists of a narrow central axle to which are hinged to two light rods of length ℓ, each attached to the centres of spherical masses of radius r. At what angular velocity ω, in terms of g, ℓ and r, will the spheres lose contact with the axle?



18.6 A conical pendulum on Earth produces a period of 0.34 s for a 30° semi-angle of the cone. When the same pendulum is used on the Moon where $g = 1.6\ \text{m s}^{-2}$, what would be the period for double the semi-angle?

18.7 An aircraft travelling at 160 knots maintains its altitude during a circular banked "rate one turn", which is a $3.0°\ \text{s}^{-1}$ turning rate. At what angle to the horizontal are the wings of the plane? (1 knot = $0.514\ \text{m s}^{-1}$)

19 Vertical circles

It is helpful to calculate the forces on an object travelling in a vertical circle.

Example context: we can calculate the speed you would have to drive over a hump-back bridge in order to leave the ground, we can also calculate the minimum speed a roller coaster car requires in order to loop-the-loop.

Quantities:
- u speed at bottom $\left(\mathrm{m\,s^{-1}}\right)$
- v speed at top $\left(\mathrm{m\,s^{-1}}\right)$
- W weight (N)
- r radius of circle (m)
- m mass (kg)
- N normal reaction (N) + means ↑
- a centripetal acceleration $\left(\mathrm{m\,s^{-2}}\right)$
- F resultant force (N)

Equations: $F = ma \quad W = mg \quad a_{\text{top}} = \dfrac{v^2}{r} \quad a_{\text{bottom}} = \dfrac{u^2}{r}$

Gain in E_{GP} = Loss in E_{K}, so $mg \times 2r = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$

19.1 For an object travelling in a vertical circle (where upwards N are positive) write equations for

a) N for the mass at the bottom using W, m and a,

b) N for the mass at the bottom using m, r, g and u,

c) N for the mass at the top using W, m and a,

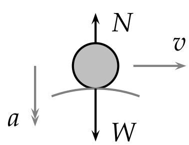
d) N for the mass at the top using m, r, g and v,

e) N for the mass at the top using m, r, g and u,

f) the speed v needed at the top if $N = 0$,

g) the speed u needed at the bottom if $N = 0$ at the top.

Example – *Calculate the normal reaction when a* 1200 *kg car is half way over a hump back bridge if it is travelling at* 13 *m* s^{-1}. *The radius of the bridge's arc is* 23 *m.*



Acceleration is downwards, so $W - N = ma$.

$$N = W - ma = mg - \frac{mv^2}{r} = m\left(g - \frac{v^2}{r}\right)$$

$$N = 1200 \times \left(9.81 - \frac{13^2}{23}\right) = 3000 \text{ N to 2sf.}$$

19.2 Calculate the normal reaction for the car in the Example at a speed of $8.0\ \mathrm{m\,s^{-1}}$.

19.3 For the car in the Example, calculate the speed at which the wheels would just leave the ground at the top of the bridge.

19.4 A 850 kg roller-coaster train goes over the top of a loop at 9.5 m s^{-1}. The loop has a radius of 4.5 m. Calculate the reaction force on the train. Use a negative number if the force is downwards.

19.5 Fill in the missing entries in the table below for a 70 kg person riding a loop-the-loop roller-coaster. Give N and a as negative if they point downwards.

Top or Bottom	r / m	Speed / m s^{-1}	a / m s^{-2}	N / N
Top	7.5	6.0	(a)	(b)
Bottom	7.5	6.0	(c)	(d)
Top	7.5	12.0	(e)	(f)
Bottom	(g)	15	30	(h)

19.6 A person feels weightless when $N = 0$. Calculate the speed a roller-coaster car would have to be travelling at the top of an $r = 4.5$ m loop in order for the riders to experience weightlessness at the top.

19.7 An 850 g radio-controlled car is driven in circles around the inside of a large (empty) pipe with a radius of 90 cm. It travels at a steady 4.0 m s^{-1}.

a) Is the car going quickly enough not to fall off the pipe's surface?

b) Calculate the normal reaction as the car passes the top.

c) Calculate the normal reaction as the car passes the bottom.

19.8 When roller-coaster riders describe their rides, they call the ratio N/mg the *g-force* (this is not a scientific term). In this formula, N is taken as positive if it is directed upwards through the rider's body towards their head. A roller-coaster is designed to give $N/mg = 2.5$ at both the top and the bottom of the ride. The loop is not circular. The rider sits in a train which runs around the inside of the loop. The top of the loop is curved with a 7.6 m radius.

a) State the value of N/mg for a rider sitting at rest in the train.

b) Calculate the speed of the train at the top of the loop.

c) If there is no friction, and the top of the loop is 21 m above the bottom, how fast will the train travel at the bottom of the loop?

d) Calculate the radius of the loop at the bottom of the track.

(d) Force diag. & Pythag.: $N = \sqrt{(mg)^2 + \left(\frac{mv^2}{r}\right)^2} = mg\sqrt{1 + \frac{v^4}{r^2 g^2}}$

(e) $a = \frac{v^2}{r}$ and $v = r\omega$, so that $a = \frac{v(r\omega)}{r} = v\omega$

(f) Resolving forces (H) and (V), $N \sin\theta = mr\omega^2$ and $N\cos\theta = mg$.

Dividing, $\tan\theta = \frac{r\omega^2}{g}$. Hence, $\omega = \sqrt{\frac{g}{r}\tan\theta}$

18 Conical pendulum

(a) Resolve (H): $T\sin\phi = mr\omega^2$ and (V): $T\cos\phi = mg$. Divide the equations,

$$\frac{T\sin\phi}{T\cos\phi} = \frac{mr\omega^2}{mg}. \quad \text{So,} \quad \tan\phi = \frac{r\omega^2}{g}. \quad \omega = \sqrt{\frac{g}{r}\tan\phi}$$

(b) $t_{\text{p}} = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$. So, $t_{\text{p}} = 2\pi\sqrt{\frac{r}{g\tan\phi}}$

(c) Resolving, $T\sin\phi = ma$ and $T\cos\phi = mg$. Then $a = g\tan\phi$

(d) From (c), the resolving horizontally equation, $a = \frac{T\sin\phi}{m}$

(e) From the resolving equations in (a), squaring and adding,

$(T\cos\phi)^2 + (T\sin\phi)^2 = T^2(\cos^2\phi + \sin^2\phi) = T^2 = (mg)^2 + (mr\omega)^2$.

Then, $T = mg\sqrt{1 + \frac{r^2\omega^4}{g^2}}$

(f) A sketch and using Pythagoras, $\cos\phi = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{\ell^2 - r^2}}{\ell} = \sqrt{1 - \frac{r^2}{\ell^2}}$

(g) $\tan\phi = \frac{r\omega^2}{g}$ and also $\tan\phi = \frac{r}{\sqrt{\ell^2 - r^2}}$. Equating, $\omega = \sqrt{\frac{g}{\sqrt{\ell^2 - r^2}}}$

(h) Same variables as (g), so $\tan\phi = \frac{\sin\phi}{\cos\phi} = \sqrt{\frac{1-\cos^2\phi}{\cos^2\phi}} = \sqrt{\frac{1}{\cos^2\phi - 1}}$

Thus, $\cos\phi = \frac{1}{1 + \tan^2\phi}$ and with $\tan\phi = \frac{r\omega^2}{g}$, then $\cos\phi = \frac{1}{\sqrt{1 + \frac{r^2\omega^4}{g^2}}}$

(i) Resolving (H) and (V), and dividing $\dfrac{T\sin\phi}{T\cos\phi} = \dfrac{v^2}{r}.\dfrac{1}{g} = \tan\phi = \dfrac{v^2}{rg}$

and so, $v = \sqrt{rg\tan\phi}$

(j) $a = \dfrac{v^2}{a}$, so $r = \dfrac{v^2}{a}$. Hence $t_{\text{p}} = \dfrac{2\pi r}{v} = \dfrac{2\pi}{v}\dfrac{v^2}{a} = \dfrac{2\pi v}{a}$

19 Vertical circles

(a) Acceleration is $\uparrow$ towards centre. $N - W = ma$, so $N = W + ma$

(b) $N = W + ma = mg + \dfrac{mu^2}{r} = m\left(g + \dfrac{u^2}{r}\right)$

(c) Acceleration is $\downarrow$ towards centre. $W - N = ma$, so $N = W - ma$

(d) $N = W - ma = mg - \dfrac{mv^2}{r} = m\left(g - \dfrac{v^2}{r}\right)$

(e) $N = mg - \dfrac{mv^2}{r}$, but $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - 2mgr$, so $mv^2 = mu^2 - 4gr$

$$N = mg - \frac{mu^2 - 4mgr}{r} = mg - \left(\frac{mu^2}{r} - 4mg\right) = 5mg - \frac{mu^2}{r}$$

(f) Using (d) with $N = 0$, $mg = \dfrac{mv^2}{r}$, so $v^2 = gr$ and $v = \sqrt{gr}$

(g) Using (e) with $N = 0$, $5mg = \dfrac{mu^2}{r}$, so $u^2 = 5gr$ and $u = \sqrt{5gr}$

20 Simple pendulum

(a) $x = l\theta$ From the definition of the radian.

(b) $60^\circ = 60 \times \dfrac{2\pi}{360} = 1.047$ rad So, $x = l\theta = 30$ cm $\times\, 1.047 = 31.4$ cm

(c) Component of weight in direction of $x = mg\sin\theta$

(d) $ma = -mg\sin\theta$ so $a = -g\sin\theta$

(e) $a = -g\sin\theta \approx -g\theta$

(f) $\theta = \dfrac{x}{l}$ so $a \approx -g\theta = -\dfrac{gx}{l}$

(g) $a = -\dfrac{g}{l}x$ so if $a = -\omega^2 x$ then $\omega^2 = \dfrac{g}{l}$

TABLE OF PHYSICAL CONSTANTS

Quantity & Symbol		**Magnitude**	**Unit**
Permittivity of free space	ϵ_0	8.85×10^{-12}	$\mathrm{F\,m^{-1}}$
Electrostatic force constant	$1/4\pi\epsilon_0$	8.99×10^{9}	$\mathrm{N\ m^{2}\ C^{-2}}$
Speed of light in vacuum	c	3.00×10^{8}	$\mathrm{m\,s^{-1}}$
Specific heat capacity of water	c_{water}	4180	$\mathrm{J\,kg^{-1}\,K^{-1}}$
Charge of proton	e	1.60×10^{-19}	C
Gravitational field strength on Earth	g	9.81	$\mathrm{N\,kg^{-1}}$
Universal gravitational constant	G	6.67×10^{-11}	$\mathrm{N\ m^{2}\ kg^{-2}}$
Planck constant	h	6.63×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	1.38×10^{-23}	$\mathrm{J\,K^{-1}}$
Mass of electron	m_{e}	9.11×10^{-31}	kg
Mass of neutron	m_{n}	1.67×10^{-27}	kg
Mass of proton	m_{p}	1.67×10^{-27}	kg
Mass of Earth	M_{Earth}	5.97×10^{24}	kg
Mass of Sun	M_{Sun}	2.00×10^{30}	kg
Avogadro constant	N_{A}	6.02×10^{23}	$\mathrm{mol^{-1}}$
Gas constant	R	8.31	$\mathrm{J\ mol^{-1}\ K^{-1}}$
Radius of Earth	R_{Earth}	6.37×10^{6}	m

OTHER INFORMATION YOU MAY FIND USEFUL

Electron volt	1 eV	=	1.60×10^{-19} J
Unified mass unit	1 u	=	1.66×10^{-27} kg
Absolute zero	0 K	=	-273 °C
Year	1 yr	=	3.16×10^{7} s
Light year	1 ly	=	9.46×10^{15} m
Parsec	1 pc	=	3.09×10^{16} m

PREFIXES

1 km = 1000 m	1 Mm = 10^{6} m	1 Gm = 10^{9} m	1 Tm = 10^{12} m
1 mm = 0.001 m	1 μm = 10^{-6} m	1 nm = 10^{-9} m	1 pm = 10^{-12} m