Isaac Mathematics

Pre-university mathematics for sciences

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Front cover image: An exponentially growing horn centred on a logarithmic spiral.
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Isaac Mathematics for Students and Teachers

Chapters 1–6 cover the Mathematics questions of levels 1–6 of the Isaac Sciences OPAL (Open Platform for Active Learning), corresponding to the last 3 years at school and to the foundation of university studies (1 – pre-A Level, 2–3 – AS, 2–5 – A level, 6 – further Maths and beyond). Chapter 7 addresses fundamental, perhaps unfamiliar, applications of mathematics that have been used in advanced Isaac events and for student extension.

The material of Chapters 1–6 is core to school mathematics courses, underpinning all A-level sciences and looking forward to university mathematics, sciences and engineering. It can be used by all students, even including those just doing mathematics without sciences.

All questions in this book can be answered on-line on Isaac at `isaacmaths.org` where there is immediate marking and feedback, with linked concept pages and several levels of hints. Many questions are straightforward, at least after practice(!), and are essential for the fluency and confidence required for creativity in higher maths and sciences. Other questions are challenging. In both cases, one learns by actively **doing**, hence the Isaac mantra "Put the mouse down and pick up the pencil". Working on paper is essential – answering on the OPAL is *not* the initial action.

Students should register (free) on Isaac since then progress is recorded (but remains private, except that set homework and shared with teachers). Indeed, problems done at any stage are declared as done, should they later be set as homework. Accomplishments on Isaac can be shared on CVs or for university applications.

Teachers registered on Isaac can apply for teacher status to set homework and have it marked, with the detailed results (question-by-question and student-by-student) instantly analysed for reporting.

JR & MW, Cambridge, 2018

Uncertainty and Significant Figures

In science, numbers represent values that have uncertainty and this is indicated by the number of significant figures in an answer.

Significant figures

When there is a decimal point (dp), all digits are significant, except leading (left-most) zeros: 2.00 (3 sf); 0.020 (2 sf); 200.1 (4 sf); 200.010 (6 sf).

Numbers without a dp can have an *absolute accuracy*: 4 people; 3 electrons.

Some numbers can be ambiguous: 200 could be 1, 2 or 3 sf (see below). Assume such numbers have the same number of sf as other numbers in the question.

Combining quantities

Multiplying or dividing numbers gives a result with a number of sf equal to that of the number with the smallest number of sf:

$x = 2.31, y = 4.921$ gives $xy = 11.4$ (3 sf, the same as x).

An absolutely accurate number multiplied in does not influence the above.

Standard form

On-line, and sometimes in texts, one uses a letter 'x' in place of a times sign and ^ denotes "to the power of":

1800000 could be 1.80x10^6 (3 sf) and 0.0000155 is 1.55x10^-5

(standardly, 1.80×10^6 and 1.55×10^{-5}).

The letter 'e' can denote "times 10 to the power of": 1.80e6 and 1.55e-5.

Significant figures in standard form

Standard form eliminates ambiguity: In $n.nnn \times 10^n$, the numbers before and after the decimal point are significant:

$191 = 1.91 \times 10^2$ (3 sf); 191 is $190 = 1.9 \times 10^2$ (2 sf); 191 is $200 = 2 \times 10^2$ (1 sf).

Answers to questions

Here, and on-line, give the appropriate number of sf: for example, when the least accurate data in a question is given to 3 significant figures, then the answer should be given to three significant figures; see above. Too many sf are meaningless; giving too few discards information. Exam boards also require consistency in sf.

Contents

Level 1

You might find it useful to look at the following on-line concept pages.
- Algebraic Manipulation - Rearranging Equations and Units - Level 1:
  isaacphysics.org/concepts/cm_algebra_manip

1. a) Rearrange the equation of motion $v = u + at$, where v is the speed of a particle which has been accelerating at a constant rate a for a time t from an initial speed u, to make t the subject.

   b) Rearrange the equation of motion $v = u + at$, this time to make a the subject.

   c) In the equation of motion for a uniformly accelerating body the distance s travelled in time t is given by $s = \frac{1}{2}(u + v)t$, where u and v are the initial and final speeds. Rearrange the equation to make u the subject.

   d) Looking again at $s = \frac{1}{2}(u + v)t$, rearrange the equation to make t the subject.

2. a) In the equation of motion for a body accelerating uniformly at a rate a, the distance s travelled in time t is given by $s = ut + \frac{1}{2}at^2$, where u is its initial speed. Rearrange the equation to find an expression for t assuming $u = 0$.

   b) Rearrange the equation $s = ut + \frac{1}{2}at^2$ again, this time without assuming $u = 0$, to make t the subject.

   c) In the equation of motion for a body accelerating uniformly at a rate a, the relationship between the distance travelled s and the initial and final speeds u and v is given by $v^2 = u^2 + 2as$. Rearrange the equation to find an expression for u.

   d) Rearrange $v^2 = u^2 + 2as$ again, to make s the subject.

3. a) Rearrange the equation $F = ma$, which relates the force F on a body to its mass m and acceleration a, to make a the subject of the equation.

   b) Rearrange the equation $W = mg$, which relates the weight W of a body to its mass m, to make m the subject of the equation.

4. a) Rearrange the equation $\rho = \dfrac{m}{V}$, relating the density ρ of a body to its mass m and volume V, to make m the subject of the equation.

   b) Rearrange $V = IR$, which relates the voltage V across a resistance R to the current I through it, to make I the subject of the equation.

   c) Considering again $V = IR$, make R the subject of the equation.

5. a) Rearrange $E_k = \frac{1}{2}mv^2$, which gives the kinetic energy E_k of a body of mass m travelling with speed v, to make v the subject of the equation.

   b) Rearrange $P = V^2/R$, which gives the power P dissipated in a resistance R when the voltage across it is V, to make V the subject.

6. Rearrange $F = GMm/r^2$, the expression for the gravitational force F between two masses M and m a distance r apart, to make r the subject.

7. a) The equation $F = ILB \sin\theta$ gives the force F on a length L of wire carrying a current I in a magnetic field B, when the magnetic field is at an angle of θ to the direction of current flow. Rearrange the equation to find an expression for θ.

   b) Rearrange $x = A\cos(2\pi ft + \phi)$, which gives the displacement x of an object oscillating at a frequency f with an amplitude A, to make t the subject.

8. Using $v = u + at$, find v if $u = 3.0 \text{ m s}^{-1}$, $a = 9.8 \text{ m s}^{-2}$ and $t = 2.0$ s.

9. Using $v = u + at$, find v if $u = 3.0 \text{ cm s}^{-1}$, $a = 9.8 \text{ m s}^{-2}$ and $t = 2.0$ ms.

10. Find the force, in newtons, on a body of mass 3.0 kg which is accelerating at 2.5 m s^{-2}.

1.2   Quadratic Equations – factorising, solving

> In addition to the concepts from previous sections, see:
> • Quadratic Equations - Level 1 and 2:
>   `isaacphysics.org/concepts/cm_algebra_quadr`

1. Consider the equation $3b^2 - 2b - 1 = 0$.

   a) Factorise the left hand side of the equation.

   b) Give the exact value of the root closest to zero.

2. Consider the equation $9q^2 + 9q + 2 = 0$.

   a) Factorise the left hand side of the equation.

   b) Give the exact value of the root closest to zero.

3. Consider the equation $-2s^2 - 5s + 25 = 0$.

   a) Factorise the left hand side of the equation.

   b) Give the exact value of the root closest to zero.

4. Consider the equation $8s^2 + 2s - 6 = 0$.

   a) Factorise the left hand side of the equation.

   b) Give the exact value of the root closest to zero.

5. Consider the equation $k + 3 = \dfrac{1 - k}{k + 2}$.

   a) Rearrange the equation to give a quadratic equation in which the right hand side is zero. (As a first step, eliminate the fraction by multiplying through by an appropriate expression.)

   b) Factorise the left hand side of the equation derived in part a).

   c) Find the roots of the equation.

6. Solve the equation $3p^2 - 6p - 4 = 0$. What is the solution closest to zero? Give your answer to 3 sf.

7. Solve the equation $m^2 + 3m + 1 = 0$. What is the solution closest to zero? Give your answer to 3 sf.

8. Solve the equation $-4z^2 + z + 1 = 0$. What is the solution closest to zero? Give your answer to 3 sf.

9. Find the value of v closest to zero if $\dfrac{3 - v}{1 - 3v} = \dfrac{2 + v}{1 + 2v}$. Give your answer to 3 sf.

10. Show that the solution to the equation $mp^2 + bp + k = 0$ can be written as $p = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$.

    a) Hence find an expression for γ in terms of one or more of the constants m, b and k in the original equation.

    b) Give also an expression for ω in terms of one or more of the constants m, b and k.

1.3   Simultaneous Equations – linear, quadratic

In addition to the concepts from previous sections, see:
- Simultaneous Equations:
  `isaacphysics.org/concepts/cm_algebra_simult`

1. a) Find p and q if $q = 4p + 8$ and $3q + 6p = 6$.

    b) Find r and s if $3s + 4r = 7$ and $2s = 3r - 4$.

2. Find a and b if $a + b = 3$ and $2(a - 1) = b - 1$.

3. Find, in terms of m_1, m_2 and r, expressions for a) l_1 and b) l_2 when $l_1 + l_2 = r$ and $m_1 l_1 = m_2 l_2$. (This gives the distances from their centre of mass of two masses a distance r apart.)

4. Starting with the equations of motion $v = u + at$ and $s = \frac{1}{2}(u + v)t$, eliminate v to give the expression for s in terms of u, a and t.

5. Find a pair of values of a and b which solve $b = 3a^2 + 3$ and $b = 2a + 4$.

6. Find a pair of values of r and s which solve $r = 2s^2 - 3s + 3$ and $r = 2s + 1$.

7. Find p and q if $p^2 + 2pq + 4q^2 = 7$ and $2p = q + 1$.

8. Starting with the equations $v = u + at$ and $s = ut + \frac{1}{2}at^2$, eliminate t to find an equation relating s, u, v and a. Give your answer as an equation with v^2 only on the left hand side.

9. A particle of mass $2M$ is travelling at speed u towards a stationary particle of mass M and collides head-on and elastically with it. After the collision both particles are moving – the mass $2M$ with speed v and the mass M with speed w. Using the laws of conservation of momentum and kinetic energy we can write down two simultaneous equations for the collision: $2Mu = 2Mv + Mw$ and $\frac{1}{2}(2M)u^2 = \frac{1}{2}(2M)v^2 + \frac{1}{2}Mw^2$.

    a) Find an expression for w, in terms of u.

    b) Find the corresponding expression for v, in terms of u.

10. A particle of mass M, travelling at speed u, collides head-on and elastically with a stationary particle of mass m. After the collision the particles of mass M and of mass m travel at (non-zero) speeds v and w respectively.

By applying the laws of conservation of momentum and kinetic energy we can write down two simultaneous equations for the collision: $Mu = Mv + mw$ and $\frac{1}{2}Mu^2 = \frac{1}{2}Mv^2 + \frac{1}{2}mw^2$

a) Find an expression for w, the speed of the particle of mass m after the collision, in terms of u, M and m.

b) Find the corresponding expression for v, the speed of the particle of mass M after the collision, in terms of u, M and m.

1.4   Trigonometry – angles, triangles

In addition to the concepts from previous sections, see
- Geometry - Angles and Triangles - Level 1:
  `isaacphysics.org/concepts/cm_geometry1`

1. Fig. 1.1a shows a triangle of side lengths a, b and c with angles A, B and C.

   a) Find the angle A if $B = 30°$ and $C = 70°$.

   b) Find the angle D if $A = 40°$ and $B = 60°$.

(a)   (b)
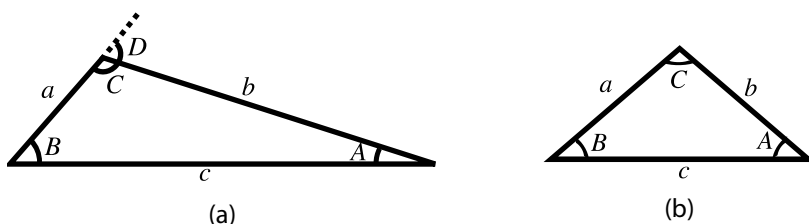
Figure 1.1

2. Fig. 1.1a shows a triangle of side lengths a, b and c with angles A, B and C.

   a) Find the angle A if $a = 10.0$ mm, $b = 14.0$ mm and $B = 65.0°$.

   b) Find the length c if $a = 10.0$ mm, $b = 6.00$ mm and $C = 40.0°$.

3. Fig. 1.1a shows a triangle of side lengths a, b and c with angles A, B and C.

   a) Find the length b if $a = 10.0$ mm, $A = 30.0°$ and $B = 70.0°$.

   b) Find the length c if $a = 10.0$ mm, $A = 60.0°$ and $B = 40.0°$.

4. Fig. 1.1a shows a triangle of side lengths a, b and c with angles A, B and C.
   Find the angle C if $a = 10.0$ mm, $b = 6.00$ mm and $c = 7.00$ mm.

5. Fig. 1.1a shows a triangle of side lengths a, b and c with angles A, B and C.

   a) Find the area of a triangle with $a = 10$ mm, $b = 4.0$ mm and $C = 70°$.

   b) Find the angle C if the area is 15.0 mm^2, $a = 10.0$ mm and $b = 4.00$ mm.

6. Fig. 1.1b shows an isosceles triangle of side lengths a, b and c, where $a = b = 10$ mm, and angles A, B and C, where $B = 50°$.

   a) Deduce the value of angle A.   b) Deduce the value of angle C.

   c) What is the side length c?

7. In Fig. 1.2a, OA and OB are radii of the circle centred at O, and the line EC is the tangent to the circle at B. Find the following angles.

   a) The angle OBA in terms of θ.   b) The angle ABD in terms of θ.

   c) The angle FBG in terms of θ.   d) The angle EBF in terms of θ.

(a)   (b)
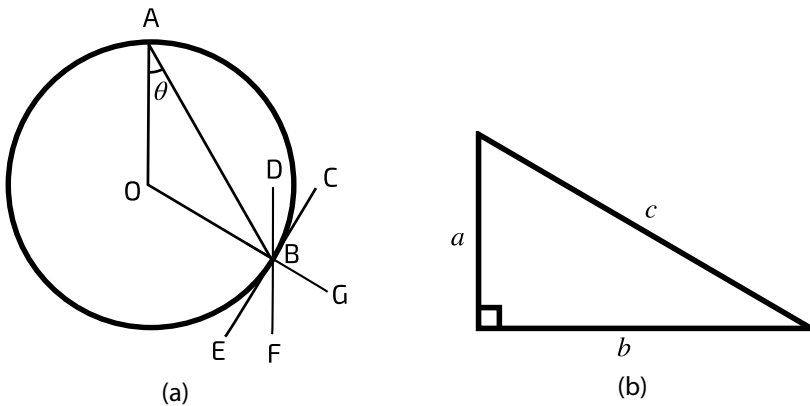
Figure 1.2

8. For a right-angled triangle, using the notation of Fig. 1.2b, find the length of the hypotenuse c in the following cases:

   a) when $a = 3$ cm, $b = 4$ cm,   b) when $a = 5.0$ cm, $b = 12$ cm,

   c) when $a = b = 3.00$ cm.

9. For a right-angled triangle, using the notation of Fig. 1.2b, find the length of the side a (to 2 sf) in the following cases:

   a) when $b = 6.0$ cm, $c = 10$ cm,   b) when $b = 10$ cm, $c = 26$ cm,

   c) when $b = 3.0$ cm, $c = 6.0$ cm.

10. An equilateral triangle has a perpendicular height of 2.00 cm.

   a) Find the length of the sides.   b) What is the area of the triangle?

1.5   Functions – evaluating, transforming, sketching

In addition to the concepts from previous sections, see:
- Functions - Simple Function Types - Level 1:
  `isaacphysics.org/concepts/cm_functions_simple`
- Functions - Transformations - Level 1:
  `isaacphysics.org/concepts/cm_functions_transformations`
- Trigonometric Relationships - Level 1:
  `isaacphysics.org/concepts/cm_trig1`

1. Consider the function $g(t) = (t-2)^3$. Find:

   a) $g(2)$,   b) $g(4)$,   c) $g(-1)$.

2. Consider the function $f(z) = 2z^2 - 4$. Find:

   a) $f(1)$,   b) $f(3)$,   c) $f(-3)$.

3. Consider the function $h(p) = \dfrac{2}{p} + 1$. Find:

   a) $h(1)$,   b) $h(-1)$,   c) $h(-2)$.

4. Consider the function $f(\theta) = 2\sin(2\theta)$. Find:

   a) $f(45°)$,   b) $f(-45°)$,   c) $f(90°)$.

5. Consider the function $k(\phi) = 3\tan(\phi + 90°)$. Find:

   a) $k(-90°)$,   b) $k(-45°)$,   c) $k(45°)$.

6. Sketch the graph of the function $g(x) = (x-2)^2 + 2$.

7. Sketch the graph of the function $f(x) = 3 - 2x$.

8. Sketch the graph of the function $f(x) = \dfrac{2}{x-1} + 1$.

9. Sketch the graph of the function $h(x) = \cos(2x) - 1$.

10. Sketch the graph of the function $k(x) = 2\sin(x - 45°) - 1$.

1.6   Graph Sketching – simple function types, trig functions

In addition to the concepts from previous sections, see
- Graph Interpreting - Level 1:
  `isaacphysics.org/concepts/cm_graph_interpreting`
- Graph Sketching - Level 1:
  `isaacphysics.org/concepts/cm_graph_sketching`

Toolkit #1. It is helpful to think about the following when interpreting a function $f(x)$.
1. What is the main shape of the function? (linear, quadratic, cubic, reciprocal, trigonometric, etc.)
2. Where are the zeros i.e. the values of x for which $f(x) = 0$?
3. What happens to the function when $x = 0$?
4. What happens to the function as x gets very large?
5. Does the function diverge anywhere within its range.

1. The graph of $y = f(x)$ is shown in Fig. 1.3a. Answer the **toolkit** questions about the function $f(x)$.

(a)   (b)

Figure 1.3

2. The graph of $y = f(x)$ is shown in Fig. 1.3b. Answer the **toolkit** questions about the function $f(x)$.

3. The graph of $y = f(x)$ is shown in Fig. 1.4a. Answer the **toolkit** questions about the function $f(x)$.

4. The graph of $y = f(x)$ is shown in Fig. 1.4b. Answer the **toolkit** questions about the function $f(x)$.
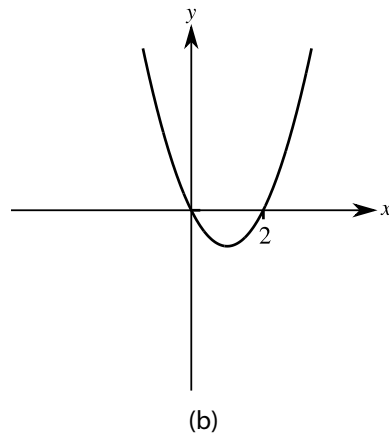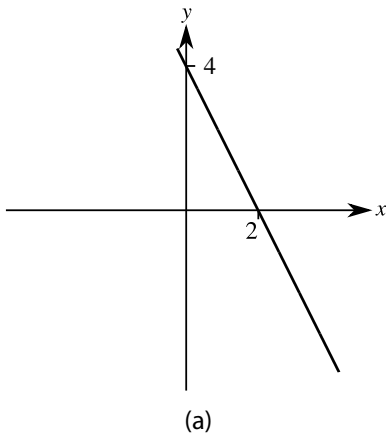

(a)   (b)
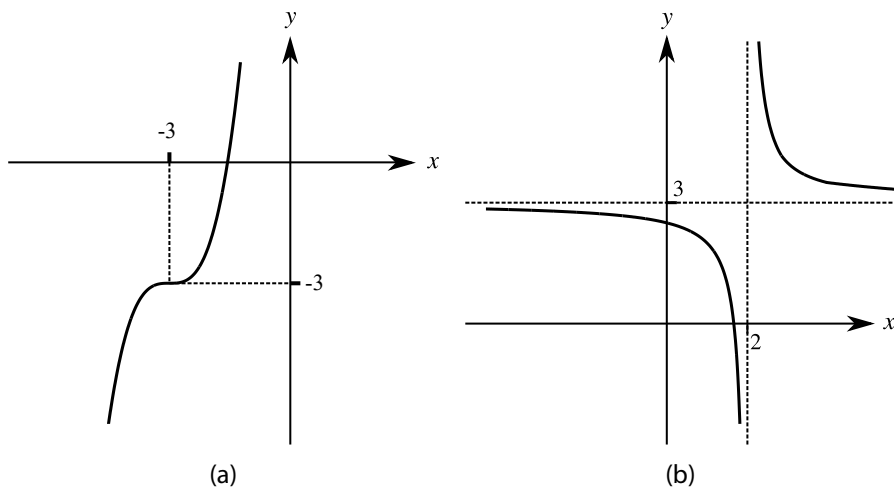
Figure 1.4

5. The graph of $y = f(x)$ is shown in Fig. 1.5. Answer the **toolkit** questions about the function $f(x)$.
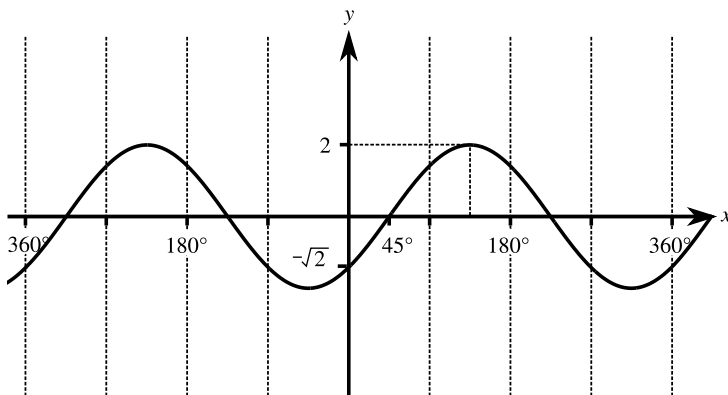
Figure 1.5

6. Answer the following questions about the linear functions $y = 2x + 3$ and $3y + 4x = 12$. Hence sketch graphs of the functions, labelling the key points on both axes.

   a) Consider $y = 2x + 3$. Deduce the value of x at which $y = 0$ (give your answer as an improper fraction) and the value of y when $x = 0$.

   b) Consider $3y + 4x = 12$. Deduce the value of x at which $y = 0$ and the value of y when $x = 0$.

7. Answer the following questions about the functions $f(x) = x^2 - 2x + 3$ and $g(x) = -2(x^3 + 1)$. Hence sketch the graphs of $y = f(x)$ and $y = g(x)$, labelling the key points on both axes.

   a) Deduce the value of $f(x)$ when $x = 0$.

   b) By completing the square deduce i. the value of x at which $f(x)$ has its minimum value and ii. the minimum value of $f(x)$.

   c) Deduce the value of $g(x)$ when $x = 0$.

   d) Deduce the value of x at which $g(x) = 0$.

8. Answer the following questions about the functions $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{1-x}$. Hence, using the same axes, sketch $y = f(x)$ and $y = g(x)$ between $x = -3$ and $x = 3$ and covering the range $y = -5$ to $y = 5$. Pay close attention to the shapes of the functions, label the key points on both axes and the point of intersection of the two functions.

   a) Answer the **toolkit** questions 2 to 5 about the function $f(x) = \dfrac{1}{x}$.

   b) Answer the **toolkit** questions 2 to 5 about the function $f(x) = \dfrac{1}{(1-x)}$.

   c) Find the x and y coordinates of the point of intersection of the functions $1/x$ and $1/(1-x)$. Label this point on your sketch graph.

9. A mass on the end of a spring is oscillating up and down so that at time t its height y above the ground is given by $y = p + q\cos(rt)$, where y is in metres and t is in seconds, and $p = 2.0$ m, $q = 0.50$ m and $r = 90°\,\text{s}^{-1}$. Find the height above the ground at $t = 0$ s, $t = 1$ s, $t = 2$ s and $t = 3$ s. Hence, sketch and label a graph of the height of the mass above the ground as a function of time from $t = 0$ to $t = 8$ s.

10. The mass m of a sphere, as a function of its radius x, is given by the formula

$$m(x) = \frac{4}{3}\pi\rho x^3$$

where ρ is the density of the sphere.

A balance has a scale which has not been zeroed properly, so that a reading of -95 g is displayed when nothing is placed on the balance. A number of spheres, made of the same plastic but with varying radius, are weighed individually using this balance. The apparent mass $M(x)$ of these spheres with respect to their radius x is given by

$$M(x) = ax^3 + b$$

If M is measured in g and x in cm find a and b to 2 sf given that the density of the plastic is roughly 1200 kg m^{-3}. Hence, sketch a graph of M as a function of x in the range $x = 0$ to $x = 5$ cm. Pay close attention to the shape of the function and label the key points on both axes.

Level 2

In addition to the concepts from previous sections, see:
- Algebraic Manipulation - Inequalities - Level 2:
  `isaacphysics.org/concepts/cm_algebra_manip_inequalities`
- Algebraic Manipulation - Index Notation - Level 2:
  `isaacphysics.org/concepts/cm_algebra_manip_index`

1. Rationalise the denominators of the following expressions.

   a) $\dfrac{3\sqrt{6}}{2\sqrt{18}}$.

   b) $\dfrac{4 - \sqrt{3}}{4 + 2\sqrt{3}}$.

2. Simplify the following expressions.

   a) $2\sqrt{20} + \sqrt{45} - 5\sqrt{5}$.

   b) $4(\sqrt{3} + 1)(\sqrt{3} - 1) - 2(2 + \sqrt{2})(1 + \sqrt{2})$.

3. Simplify the following expressions.

   a) $(4a^2b^3)^{\frac{1}{2}} \times (9ab^2)^{-\frac{3}{2}}$.

   b) $(8p^3q^2)^{\frac{2}{3}} \div (\dfrac{2p}{q^{\frac{1}{3}}})^5$.

   c) $(10^{-34})^{\frac{1}{2}}(10^{-10})^{\frac{1}{2}}(10^8)^{-\frac{5}{2}}$.

4. Solve the following inequalities.

   a) $3m + 8 \geq 2$.

   b) $2p + 5 < 4p - 7$.

5. Solve the following inequalities.

   a) $7 - 4a \leq -5$.

   b) $3 - 2(b + 1) \geq 6 + 3(2b + 1)$.

6. Solve the following inequalities.

   a) $3x^2 - 2x - 8 \leq 0$.

   b) $-2x^2 + 5 < 7x + 11$.

7. A mass m is suspended on a spring with spring constant k in a medium which damps its motion. The condition that it will oscillate after it has been displaced from equilibrium is $\dfrac{k}{m} > \dfrac{b^2}{4m^2}$ where b is called the damping constant. Find the range of masses over which it will oscillate.

8. A body of mass m and speed v can escape from a planet of mass M and radius R if the sum of its kinetic energy ($\frac{1}{2}mv^2$) and its gravitational potential energy ($-GMm/R$) is greater than or equal to zero, i.e. $\frac{1}{2}mv^2 - \dfrac{GMm}{R} \geq 0$ (G is the universal constant of gravitation).

   a) Find the range of speeds v over which it will escape. Give your answer as an inequality, with v on the left hand side.

   b) If the speed of the body has a fixed value, i.e. $v = v_0$, and the mass of the planet $M = \frac{4}{3}\pi R^3 \rho$, where ρ is its average density, find the range of radii R for which the body will escape.

> The dimensions of physical properties do not depend on specific units; here we use length L, time T and mass M as our fundamental dimensions. In any equation relating physical properties the dimensions must be the same on both sides. For example $force = mass \times acceleration$. Obviously mass has dimensions M. To deduce the dimensions of acceleration recall that acceleration = change in velocity over time; velocity (= change in displacement over time) has dimensions of LT^{-1} so acceleration has dimensions $(LT^{-1})(T^{-1}) = LT^{-2}$. Thus force has dimensions MLT^{-2}.

9. a) The kinetic energy of a body of mass m moving with speed v is equal to $\frac{1}{2}mv^2$. Find the dimensions of (kinetic) energy. Recall that the factor of $\frac{1}{2}$ in the expression is dimensionless.

   b) One type of "Planck unit" is defined as $h^{\frac{1}{2}}G^{\frac{1}{2}}c^{-\frac{5}{2}}$, where h is Planck's constant (dimensions ML^2T^{-1}), G is the universal constant of gravitation (dimensions $M^{-1}L^3T^{-2}$) and c is the speed of light (dimensions LT^{-1}). Find the dimensions of this "Planck unit".

2.2   Trigonometry – sin, cos, tan, triangles

In addition to the concepts from previous sections, see:
- Trigonometric Relationships - Level 3:
  `isaacphysics.org/concepts/cm_trig1`

1. For the range $-180° \leqslant \alpha \leqslant 180°$, consider all the values of α which satisfy $\sin \alpha = 0.2$.

   a) How many values of α, satisfying the equation, are in this range?

   b) What is the largest positive value of α satisfying the equation in this range? Give your answer to 3 sf.

2. For the range $-180° \leqslant \beta \leqslant 180°$, consider all the values of β which satisfy $\sin(2\beta) = -0.4$.

   a) How many values of β, satisfying the equation, are in this range?

   b) What is the smallest positive value of β in this range? Give your answer to 3 sf.

3. For the range $-180° \leqslant \alpha \leqslant 180°$, consider all the values of α which satisfy $\cos \alpha = -0.7$.

   a) How many values of α, satisfying the equation, are in this range?

   b) What is the largest value of α satisfying the equation in this range? Give your answer to 4 sf.

4. For the range $-180° \leqslant \alpha \leqslant 180°$, consider the values of α which satisfy the equation $\cos 4\alpha = 1$.

   a) How many values of α, satisfying the equation, are in this range?

   b) What is the smallest (in magnitude) negative value of α satisfying the equation in this range?

5. For the range $0 \leqslant \theta \leqslant 360°$, write down all the values of θ which have the following: a) $\sin \theta = \sqrt{3}/2$, b) $\sin \theta = -1/2$.

6. For the range $0 \leqslant \theta \leqslant 360°$, write down all the values of θ which have the following: a) $\cos \theta = -1/2$, b) $\cos \theta = 1/\sqrt{2}$.

7. Write down the exact values of the following: a) $\cos 150°$, b) $\cos 300°$.

8. Write down the exact values of the following: a) $\sin 120°$, b) $\sin 225°$.

9. Write down the exact values of the following: a) $\tan 135°$, b) $\tan 240°$.

10. Figure 2.1 shows a right-angled triangle with hypotenuse OP.

   a) Find OP if PQ $= 4.00$ cm and $A = 35.0°$.

   b) Find PQ if OQ $= 4.00$ cm and $A = 40.0°$.

   c) Finally, find OQ if OP $= 4.00$ cm and $A = 50.0°$.
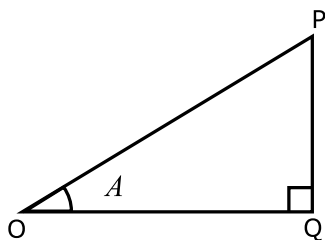
Figure 2.1

2.3   Simple Shapes – area, volume

In addition to the concepts from previous sections, see:
- Geometry - Shapes and their Properties - Level 2:
  `isaacphysics.org/concepts/cm_geometry2`

1. A cylinder of radius R contains some liquid; a solid metal cuboid with sides a, a and $2a$ is totally immersed in the liquid. Find an expression for the decrease in the height of the liquid in the cylinder when the cuboid is taken out of the liquid.

2. A cylinder of radius $2a$ contains water to a depth h. A cone with base radius a and perpendicular height a is lowered into the water. When the bottom of the cone is resting on the bottom of the beaker the water just covers the top of the cone. Find an expression for the initial depth of the water h in terms of a.

3. Four identical spherical water droplets coalesce to form one spherical drop. Find the factor (to 3 sf) by which the total surface area changes.

4. Find the volume of a sphere of radius 2.00 cm in:

   a) cm^3,   b) mm^3,   c) m^3.

5. Find the volume of a sphere which has a surface area of 0.45 m^2.

6. A sphere has volume V.

   a) Find an expression for the radius r of the sphere in terms of V.

   b) What is the surface area A in terms of V?

7. A cylinder with closed ends has a total surface area S; the radius of the base is a and the height is ka. Find an expression for a in terms of k and S.

8. A hollow steel cylinder of length 8.0 cm is to be constructed with an inner radius of 1.9 cm and an outer radius of 2.0 cm, using steel of density 7800 kg m^{-3}.

   a) Find the required volume of steel.

   b) Find the required mass of steel.

9. The pyramid in Fig. 2.2a has a square base of side a. The four identical triangles making up its sides each have a base of length a and perpendicular height b.

   a) Find in terms of a and b the total surface area S of the pyramid.

   b) Find in terms of a and b the total volume V of the pyramid.

   c) A cone has the same volume and height h as the pyramid; find, in terms of a, the radius r of its base.

Cylinder:

Box:
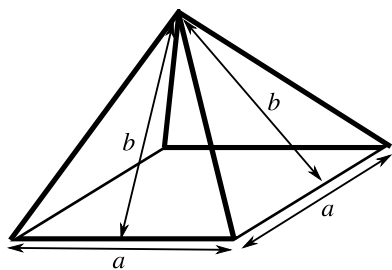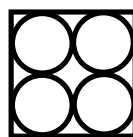
(a)   (b)

Figure 2.2

10. a) Four tennis balls are placed one above the other so that they just fit into a cylindrical tube as shown in Fig. 2.2b (i.e. each ball is in contact with the sides of tube and the top and the bottom balls are in contact with the lid and the base of the cylinder respectively). Find the fraction of the space inside the cylinder occupied by the balls. Give your answer to 3 sf.

   b) The same four balls are now placed in a box with a square cross-section so that they are touching the sides of the box and each other as shown in Fig. 2.2b and are in contact with the lid and the base of the box. Find the fraction of the space inside the box occupied by the balls. Give your answer to 3 sf.

2.4　　Vectors – notation; adding, resolving components

In addition to the concepts from previous sections, see:
- Vectors - Resolving Vectors - Level 2:
  `isaacphysics.org/concepts/cm_vectors2`
- Vectors - Describing and Adding Vectors - Level 2:
  `isaacphysics.org/concepts/cm_vectors`
- Potential Energy - Level 1: Constant g Field:
  `isaacphysics.org/concepts/cp_potential_energy`

1. Two vectors p and q are given by $p = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $q = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find the following to 3 sf.

   a) The magnitude of p.

   b) The magnitude of q.

   c) The angle that p makes with the x-axis.

   d) The vector $p + q$ in column vector form and give its magnitude.

   e) The column vector form of p - q and the angle this vector makes with the x-axis.

2. The vector $u = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$. Find an expression using unit vector notation for the unit vector $\hat{u}$ in the direction of u.

3. A vector a is of length 8.00 and makes an angle of $40°$ to the x-axis. What is the magnitude of its projection onto the x-axis and the y-axis? Now, express a using unit vector notation.

4. A boat sails 4.00 km at a bearing of $210°$.

   a) How far south of its starting point is its final position?

   b) How far west of its starting point is its final position?

5. A block of weight W is on a slope which makes an angle α to the horizontal as shown in Fig. 2.3a.

    a) What is the component of the block's weight W that acts perpendicular to the slope?

    b) The block slides a distance l along the slope. How far has the block moved horizontally?

    c) The block slides a distance l along the slope as in part b). How much gravitational potential energy has the block lost?

(a)   (b)
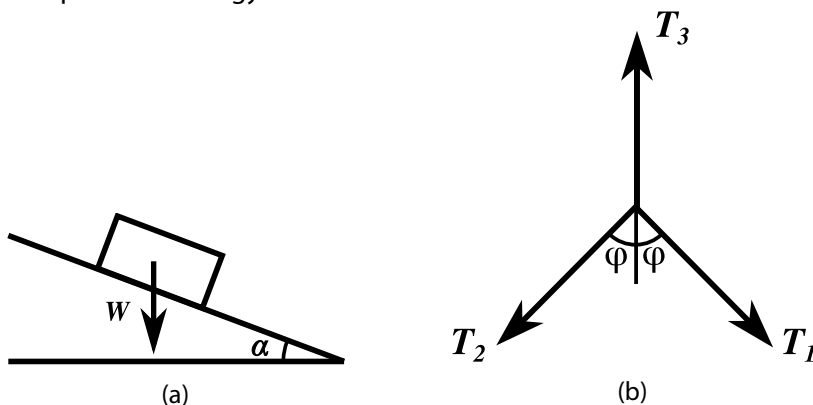
Figure 2.3

6. A body is acted on by three forces T_1, T_2 and T_3 in the (x, y)-plane as shown in Fig. 2.3b, where $T_3 = 20.0\hat{\jmath}$ N, $\phi = 20.0°$ and the body is equilibrium.

    a) Find the magnitude of T_1.

    b) Find the x component of T_2, taking left-to-right to be the positive direction.

7. The coordinates A, B and C of the vertices of a triangle are given by $(1, 1, -1)$, $(-1, 1, 1)$ and $(1, -1, 1)$ respectively. Find the following.

    a) The vector from A to B, expressed in unit vector notation.

    b) The vector from B to C, expressed in unit vector notation.

    c) The vector from C to A, expressed in unit vector notation.

    d) The side length (to 3 sf) of the equilateral triangle ABC.

8. Find the vector going from the point $(2,2,0)$ to $(1,4,3)$.

9. The coordinates of one vertex A of a triangle ABC are $(3,4,2)$ and the vectors from A to the other vertices B and C are $(1,1,4)$ and $(-3,2,-2)$ respectively.

   a) Find the coordinates of B.

   b) Find the coordinates of C.

   c) Find the length (to 3 sf) of the side a, i.e. the vector from B to C.

10. A vector $\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ has a length of 4.00 units. Answer the following questions.

    a) The vector lies in the (x, y)-plane, makes an angle of $30°$ with the x-direction and u_y is positive. What is u_x?

    b) The vector has $u_x = u_y = 2.00$ and u_z is negative. What is u_z?

    c) The vector is such that $u_z = 1.00$ and $u_y = 2u_x$. What is u_y?

2.5   Functions – polynomials, symmetry; transforming

In addition to the concepts from previous sections, see:
- Functions - Polynomials and Rational Functions - Level 2:
  `isaacphysics.org/concepts/cm_functions_polynomials`

1. The electrostatic potential $V(x,y)$ at a point in the (x,y)-plane with co-ordinates (x,y) due to a point charge lying in the (x,y)-plane at the origin, i.e. with coordinates $(0,0)$, is given by $V(x,y) = \dfrac{P}{\sqrt{x^2 + y^2}}$ where P is a constant proportional to the magnitude of the charge. Find expressions for:

   a) $V(a,0)$,   b) $V(0,b)$,   c) $V(a,b)$,   d) $V(a,a)$.

2. Investigate the transformations of the following functions.

   a) The functions $f(x) = x^2 + 2x + 1$ and $g(x) = f(x - a)$, where a is a constant. If $g(1) = 9$, find the value of a, given that it is positive.

   b) The functions $r(u) = \dfrac{2}{u - 2}$ and $s(u) = r(u) + b$, where b is a constant. If $s(0) = 1$, find the value of b.

   c) The functions $p(r) = \dfrac{1}{r}$ and $q(r) = p(r - c) + d$, where c and d are constants. If $q(0) = 1$ and $q(2) = 3$, find the values of c and d.

3. A particle of mass M moving at speed u collides head on with a stationary particle of mass m. No energy is lost in the collision. The speeds after the collision depend upon the relative magnitudes of M and m. Taking M as fixed we can express these speeds as a function of m. The speed $v(m)$ of mass M after the collision is given by $v(m) = \dfrac{M - m}{M + m}u$, and the speed $w(m)$ of mass m is given by $w(m) = \dfrac{2M}{M + m}u$. Find the expressions for v and w in terms of u when:

   a) $m = M$,   b) $m = 2M$,
   c) $m = M/2$,   d) $m = rM$ where r is any positive number.

4. Investigate the transformations of the following functions.

a) $f(x) = x^2 + 2x - 1$ is transformed into another function $g(x)$ by stretching it by a factor a in the x-direction, i.e. $g(x) = f(\frac{x}{a})$. If $g(2) = 2$, what is the value of a? (You may assume that a is positive).

b) $v(u) = \dfrac{3}{1 + 2u}$ is transformed into $w(u)$ by stretching it by a factor b in the vertical direction, i.e. $w(u) = bv(u)$. If $w(4) = \frac{1}{2}$, find b.

c) $f(x) = 2x + 3$ is transformed into $g(x)$ by stretching it by a factor p in the lateral direction and then by a factor of q in the vertical one, i.e. $g(x) = qf\left(\dfrac{x}{p}\right)$. If $g(1) = 2$ and $g(-2) = -3$, find the equation for $g(x)$.

The following question asks you to deduce the symmetry properties of a number of functions. There are three choices: 1) even - a function for which $f(x) = f(-x)$ which is also described as being symmetric about the vertical axis, 2) odd - a function for which $f(x) = -f(-x)$ which is also described as being antisymmetric about the vertical axis (or symmetric about zero), 3) neither even nor odd.

---

5. Decide the symmetry properties of the following functions. Where relevant you may assume that a and b are non-zero constants.

a) Decide which of the following functions are even:

ax^2, $ax^2 + b$, $ax^2 + bx^4$, $\dfrac{a}{x^2} + bx^2$, $(x - a)(x + a)$, $a \cos x$, $a(x + b)^2$, $\dfrac{a}{x^2} + b$, $(x - a)(x + b)\ (a \neq b)$, $a \sin x$, $x^2(a + bx)$.

b) Decide which of the following functions are odd:

ax, $\dfrac{a}{x}$, $\dfrac{a}{x} + bx^3$, $x(a + bx^2)$, $x^{1/3}$, $a \sin x$, $x^2(a + bx)$, $(x + a)^{1/3}$, $a \tan x$, $\dfrac{a}{x} + b$, $\dfrac{a}{x} + \dfrac{b}{x^3}$.

c) Decide which of the following functions are neither odd nor even:

$ax - b$, $x^2(ax + b)$, $(x - a)(x + a)^2$, $\dfrac{a}{(x - b)^2}$, $ax^{1/2}$, $a \tan(x + 45°)$, $a\left(\dfrac{1}{x^2} - \dfrac{1}{b^2}\right)$, $a(b - x)^{1/2}$, $\cos x + \sin x$, $x(ax^2 + b)$, $(x - a)(x + a)$.

6. A sinusoidal wave is travelling in the positive x-direction. Its displacement $\psi(x,t)$ at a point x at time t is given by $\psi(x,t) = \psi_0 \cos(\dfrac{2\pi x}{\lambda} - 2\pi f t)$ where ψ_0 is the amplitude, λ is the wavelength and f is the frequency of the wave; the speed of the wave $v = f\lambda$. Find expressions in terms of ψ_0, λ, f, x and t (as appropriate) for:

   a) $\psi(0,0)$,   b) $\psi(x + vt, t)$,   c) $\psi(x, t + \dfrac{x}{v})$.

7. Consider the polynomials $q(p) = p^4 + 2p^2 + 3p + 2$ and $r(p) = p^4 - 3p^3 - 2p^2 + 3$. Find:

   a) $q(p) + r(p)$,   b) $q(p) - r(p)$.

8. Consider $f(t) = t^3 - 2t^2 + 3t - 1$ and $g(t) = 4t^2 + 2t + 2$. Express each of the following as a single polynomial:

   a) $2f(t) + g(t)$,   b) $2f(t) - 3g(t)$.

9. Find the following polynomial products:

   a) $(s^2 - 1)(s^2 - 2s + 3)$,   b) $(u - 1)(3u - 2)(2u + 3)$.

10. Consider the polynomial product $(au^2 + bu + c)(du^3 + eu^2 + fu + g)$ where a, b, c, d, e, f and g are constants. Find expressions in terms of these constants for the coefficients of u^3 and u.

For more practice go to Section 3.2.

2.6   Differentiation – powers, stationary points

In addition to the concepts from previous sections, see:
- Calculus - Differentiation - Level 2:
  `isaacphysics.org/concepts/cm_differentiation`

1. a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $y = x^4$.

   b) Find the gradient of the curve $x = t^2$ at the points $t = 0$, $t = 3$ and $t = -3$.

2. a) Find $\dfrac{\mathrm{d}v}{\mathrm{d}u}$ if $v = \dfrac{1}{u}$.

   b) Find the equation of the tangent to the curve $v = \dfrac{1}{u}$ at $u = 2$.

   c) Find $\dfrac{\mathrm{d}F}{\mathrm{d}r}$ if $F = Ar^3$, where A is a constant.

3. a) Find the gradient of $w = z^{1/2} + z^{-1/2}$ at $z = 1/4$, $z = 1$ and $z = 4$.

   b) Find the roots of the curve $f(u) = u^3 - 4u$.

   c) Find the gradient at each of the roots of $f(u)$.

4. a) Find the gradient of the curve $t = 4s^{-3/4}$ at the point $s = 16$.

   b) Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ if $x = bt^{3/2}$.

   c) Find $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$ if $x = bt^{3/2}$.

5. a) Differentiate $ax^3 + (b/x) + c$ with respect to x, where a, b and c are constants.

   b) Differentiate $(2m + 3)(m - 1)$ with respect to m.

6. a) Find $\dfrac{\mathrm{d}v}{\mathrm{d}u}$ if $v = Bu^{-3}$.

   b) The electrostatic potential energy V of two equal charges q a distance r apart is given by $V = \dfrac{q^2}{(4\pi\epsilon_0 r)}$. The force between the two charges is given by $-\dfrac{\mathrm{d}V}{\mathrm{d}r}$; find an expression for this force. Interpret the sign.

7. Find the coordinates, nature and number of the stationary points of the following functions.

a) $y = 2x^3 - 24x - 5$.   b) $y = 2x^3 - 5x^2 + 4x + 6$.

8. Find the coordinates, nature and number of the stationary points of the following functions.

a) $p = q + (4/q)$.

b) $t = ay^4 - by^2 + c$, where a, b and c are all positive.

9. a) A particle is moving in one dimension. Its displacement s at time t is given by $s = ut + bt^2$. The velocity v of the particle at time t is given by the rate of change of displacement with time, i.e. $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$. Find an expression for the velocity.

b) The acceleration a of the particle at time t is given by the rate of change of velocity with time. Find an expression for the acceleration of the particle in part a).

c) The displacement of a body at time t is given by $x = \alpha t + \beta t^3$ where $\alpha = 4$ m s^{-1} and $\beta = 5$ m s^{-3}. Find the velocity of the body at $t = 2$ s.

d) Find the acceleration of the body in part c) at $t = 2$ s.

10. a) A particle is fired upwards into the air with a speed w and moves subsequently under the influence of gravity with an acceleration g downwards, such that its height h at time t is given by $h = wt - \frac{1}{2}gt^2$. Find an expression for its maximum height above its initial position.

b) The potential energy of two molecules separated by a distance r is given by $U = U_0 \left(\left(\dfrac{a}{r} \right)^{12} - 2 \left(\dfrac{a}{r} \right)^6 \right)$ where U_0 and a are positive constants. The equilibrium separation of the two molecules occurs when the potential energy is a minimum; find expressions for the equilibrium separation and the value of the potential energy at this separation.

For more practice go to Section 3.4.

2.7   Graph Sketching – powers of x, polynomials

In addition to the concepts from previous sections, see
- Graph Interpreting - Level 2:
  `isaacphysics.org/concepts/cm_graph_interpreting`
- Graph Sketching - Level 2:
  `isaacphysics.org/concepts/cm_graph_sketching`

Toolkit #2. It is helpful to think about the following when interpreting a function $f(x)$.
1. What is the main shape of the function? (linear, quadratic, cubic, reciprocal, trigonometric etc.)
2. Where are the zeros i.e. the values of x for which $f(x) = 0$?
3. What happens to the function when $x = 0$?
4. What happens to the function as x gets very large?
5. Does the function diverge anywhere within its range.
6. What are the symmetry properties of the function?

1. Fig. 2.4a shows a sketch of the variation of the gravitational field $g(r)$ with distance r from the centre of a spherical planet of radius R. The field varies with r in a different way inside and outside the planet. Identify the correct functional form of this field from the choices below; A is a constant.

$$g(r) = \begin{cases} Ar & \text{for} \quad r \le R \\ \dfrac{AR^3}{r^2} & \text{for} \quad r > R \end{cases} \qquad g(r) = \begin{cases} \dfrac{AR^3}{r^2} & \text{for} \quad r \le R \\ Ar & \text{for} \quad r > R \end{cases}$$

$$g(r) = \begin{cases} Ar & \text{for} \quad r \le R \\ \dfrac{AR^2}{r} & \text{for} \quad r > R \end{cases} \qquad g(r) = \begin{cases} Ar & \text{for} \quad r \le R \\ \dfrac{Ar^2}{R^3} & \text{for} \quad r > R \end{cases}$$

$$g(r) = \begin{cases} \dfrac{A}{r} & \text{for} \quad r \le R \\ Ar & \text{for} \quad r > R \end{cases} \qquad g(r) = \begin{cases} Ar & \text{for} \quad r \le R \\ \dfrac{A}{r^2} & \text{for} \quad r > R \end{cases}$$

If you are not familiar with the notation used in these answers,

$$g(r) = \begin{cases} Ar^n & \text{for} \quad r \le R \\ \dfrac{AR^m}{r^2} & \text{for} \quad r > R \end{cases}$$

means that $g(r) = Ar^n$ for $r \le R$ and $g(r) = \dfrac{AR^m}{r^2}$ for $r > R$, where n and m are particular powers.
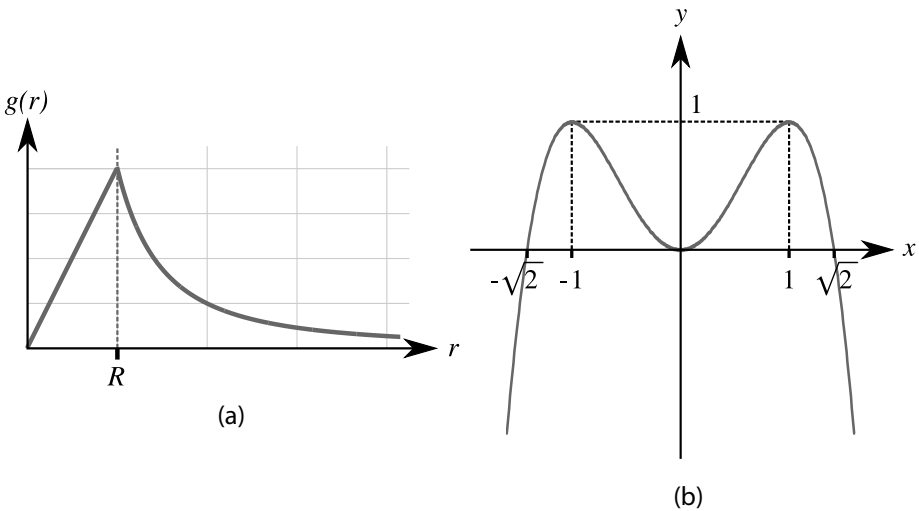

(a)

(b)

Figure 2.4

2. The graph shown in 2.4b is a polynomial of order 4, i.e. it has the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are constants. Use the information shown in the graph, including its symmetry, to deduce the values of the constants.

   a) From the symmetry of the graph deduce the value of b and d.

   b) Deduce the value of the constants e, a and c.

3. The magnetic field $B(r)$ at a perpendicular distance r from the centre of a current-carrying conductor of radius a is given by

$$B(r) = \begin{cases} \dfrac{Ar}{a^2} & \text{for} \quad r < a \\ \dfrac{A}{r} & \text{for} \quad r \geq a \end{cases}$$

Notice the two different functional forms inside and outside the wire. (If you are not familiar with the above notation it means that $B(r) = \dfrac{Ar}{a^2}$ for $r < a$ and $B(r) = \dfrac{A}{r}$ for $r \geq a$.) Which of the graphs in Fig. 2.5 represents this magnetic field?
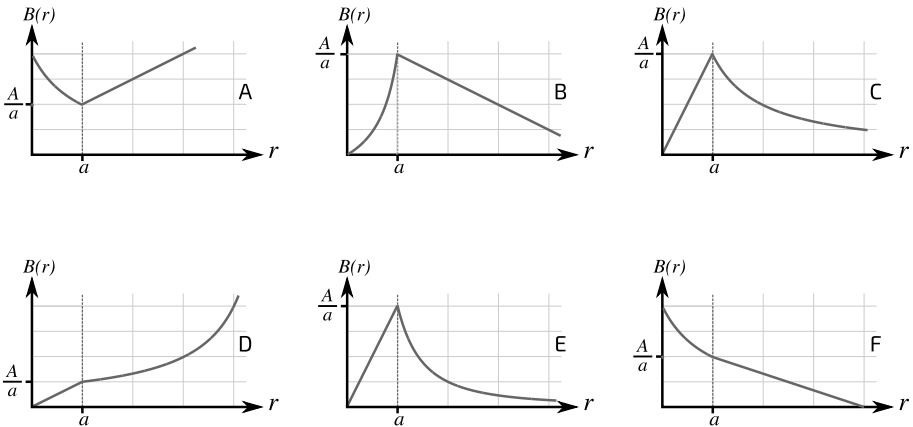

Figure 2.5

4. Answer the questions below about the cubic function $h(y) = y^3 - 8y^2 + 19y - 12 = (y-1)(y-3)(y-4)$. Hence, for a suitable range of values of y, sketch a graph of the function; consider its shape carefully and label the key points on both axes. (You are not expected to find the values of y and $h(y)$ at which the turning points of the function occur.)

a) Deduce the smallest value of y at which the function crosses the horizontal axis (i.e. where $h(y) = 0$).

b) Deduce the largest value of y for which $h(y) = 0$.

c) Deduce the value of h when $y = 0$.

d) What happens to the function when y gets very large?

5. Graphs of six functions are shown in 2.6. Deduce the symmetry properties of each function A to F, i.e. is it symmetric about the vertical axis (even), anti-symmetric about the vertical axis (odd, or symmetric about zero) or neither.
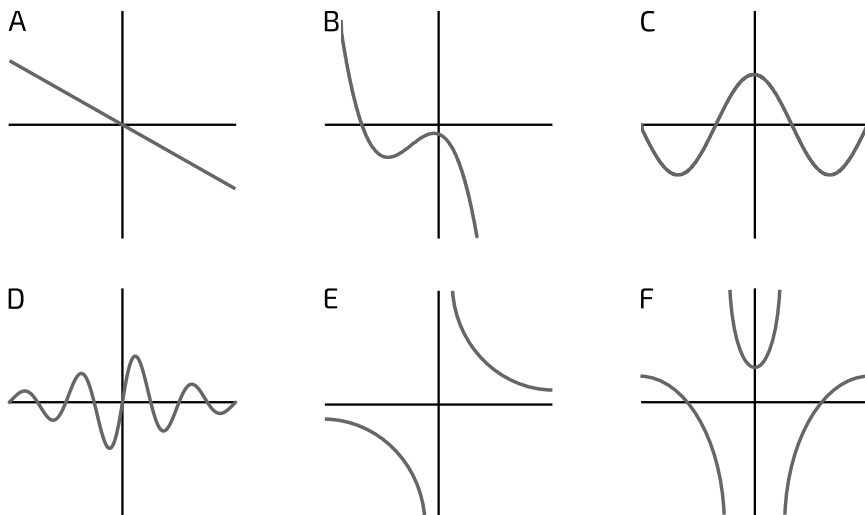

Figure 2.6

6. Answer the following questions about the functions $f(x) = x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{3}}$. Hence using the same axes, sketch graphs of $y = f(x)$ and $y = g(x)$ over the range $x = -9$ and $x = 9$. Pay close attention to the differences in the shapes of the two functions, and label the key points on both axes and the point of intersection of the two functions.

a) Deduce the value of $f(x)$ when $x = 0$.

b) What are the symmetry properties of the function $f(x)$?

c) Deduce the value of $g(x)$ when $x = 0$.

d) What are the symmetry properties of the function $g(x)$?

e) Find the x coordinate of the point of intersection of the functions $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$. Deduce the corresponding value of y. Label this point on your graph.

7. Consider the functions $f(x) = \dfrac{2}{x}$ and $g(x) = \dfrac{4}{x^2} - 2$.

a) Answer the **toolkit** questions about the function $f(x)$.

b) Answer the **toolkit** questions about the function $g(x)$.

c) Using the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ in the range $x = -4$ to $x = 4$. Pay close attention to the differences in the shapes of the two functions, and label the key points on both axes and the points of intersection of the two functions.

8. A polynomial of order 4 has the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e = a(x - p)(x - q)(x - r)(x - s)$ where a, b, c, d and e are constants, and $a < 0$. The constants p, q, r and s are such that $p < 0$ and $0 < q < r < s$. Answer the questions below about this function. Hence sketch a graph of the function; consider its shape carefully and label the key points on both axes. (You are not expected to find the values of x and $f(x)$ at which the turning points of the function occur.)

a) Give one of the values of x at which the function crosses the horizontal axis (i.e. where $f(x) = 0$)

b) Write down the relationship between e and a, p, q, r and s. From this you can deduce whether the value of f when $x = 0$ is positive or negative.

c) What happens to the function $f(x)$ when x gets very large?

9. A mass m is suspended on the end of a spring of spring constant k. The energy $U(x)$ stored in the system is made up of the elastic potential energy stored in the spring and the gravitational potential energy of the mass. It is measured relative to the point where the string is unstretched and is given by $U(x) = \frac{1}{2}kx^2 - mgx$, where x is the extension of the spring, measured downwards, and g is the acceleration due to gravity.

a) Find the values of x for which $U = 0$.

b) By considering the results from part a), or by completing the square, find the value of x at which the stored energy $U(x)$ is a minimum and deduce the value of U at that point.

c) Sketch the graph of $U(x)$ as a function of x; consider the shape of your function carefully and label the key points on both axes.

10. A pendulum consists of a small mass attached to one end of a light rod; the other end is attached to a pivot. The mass oscillates backwards and forwards and at time t its displacement from its equilibrium point vertically below the pivot is $x(t)$ and its speed is $v(t)$. At time t the displacement $x(t)$ of the mass from its equilibrium position is given by $x(t) = a\sin(\omega t + \phi)$, where x is in cm and t is in s, and $a = 3.0$ cm, $\omega = 90°\,\text{s}^{-1}$ and $\phi = 30°$. The speed $v(t)$ of the mass is given by $v(t) = u\cos(\omega t + \phi)$, where v is in cm s^{-1} and t is in s, and $u = 4.7$ cm s^{-1}, and $\phi = -30°$. Answer the questions below about $x(t)$ and $v(t)$. Hence sketch and label a graph of the displacement $x(t)$ and the speed $v(t)$ as a function of time from $t = -4$ s to $t = 4$ s.

a) Find the first time after $t = 0$ that the mass passes through its equilibrium position (i.e. $x(t) = 0$). (Give your answer to 2 sf.)

b) Find the second time after $t = 0$ that the mass passes through its equilibrium position. (Give your answer to 3 sf.)

c) Find the first time after $t = 0$ that the mass has its largest displacement from equilibrium (give your answer to 3 sf.) and deduce the maximum displacement of the mass.

d) Find the first time after $t = 0$ that the speed of the mass is zero (i.e. $v(t) = 0$). (Give your answer to 3 sf.)

e) Find the second time after $t = 0$ that the speed of the mass is zero. (Give your answer to 3 sf.)

f) Find the first time after $t = 0$ that the mass has its largest speed (give your answer to 2 sf.) and deduce the maximum speed of the mass.