Polynomials, Factors and Roots 4i

A Level
P P P

The polynomial $f(x)$ is given by $f(x) = 2x^3 + 9x^2 + 11x - 8$.

Part A Factors

Using the factor theorem decide whether $(2x - 1)$ is a factor of $f(x)$ or not.

- $(2x - 1)$ is not a factor of $f(x)$
- $(2x - 1)$ is a factor of $f(x)$

Part B Find quadractic factor

Express $f(x)$ as a product of a linear factor and a quadratic factor.

The following symbols may be useful: x

Part C Real roots

State the number of real roots to the equation $f(x) = 0$.

Gameboard:

Pure Maths Practice: Polynomials, Factors and Roots

Physics. You work it out.

Home Gameboard Maths Algebraic Division 5ii

Algebraic Division 5ii

A Level
P P P

Part A Quotient and Remainder 1

Find the quotient and remainder when $3x^4 - x^3 - 3x^2 - 14x - 8$ is divided by $x^2 + x + 2$.

Give the quotient.

The following symbols may be useful: x

Give the remainder.

The following symbols may be useful: x

Part B Quotient and Remainder 2

Find the quotient and remainder when $4x^3 + 8x^2 - 5x + 12$ is divided by $2x^2 + 1$.

Give the quotient.

The following symbols may be useful: x

Give the remainder.

The following symbols may be useful: x

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Gameboard:

Pure Maths Practice: Algebraic Division

Algebraic Division 5i

A Level

Part A Quotient and Remainder

Find the quotient and remainder when $x^4 + 1$ is divided by $x^2 + 1$.

State the quotient.

The following symbols may be useful: x

State the remainder.

Part B Find $f(x)$

When the polynomial $f(x)$ is divided by $x^2 + 1$, the quotient is $x^2 + 4x + 2$ and the remainder is $x - 1$. Find $f(x)$, simplifying your answer.

The following symbols may be useful: x

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Algebraic Division 3ii

A Level

The cubic polynomial $ax^3 - 4x^2 - 7ax + 12$ is denoted by $f(x)$.

Part A Value of a

Given that $(x - 3)$ is a factor of $f(x)$, find the value of the constant a.

The following symbols may be useful: a

Part B Remainder

Using this value of a, find the remainder when $f(x)$ is divided by $(x + 2)$.

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Gameboard:

Pure Maths Practice: Algebraic Division

Proof and Hollow Pyramids

A Level

A hollow pyramid shape can be made by stacking identical spheres.

Part A Square-based pyramids

The diagram below shows the first three pyramids in a sequence of square-based hollow pyramids.

Figure 1: Square-based hollow pyramids with sides made up of 4, 5 and 6 identical spheres.

Let the number of spheres that make up the k^{th} pyramid be S_k. From the list below, choose the correct expression for S_k.

- $8k + 21$
- $4k + 5$
- $8k + 13$
- $16k - 11$

Part B Triangle-based pyramids

The diagram below shows the first three pyramids in a sequence of triangle-based hollow pyramids.

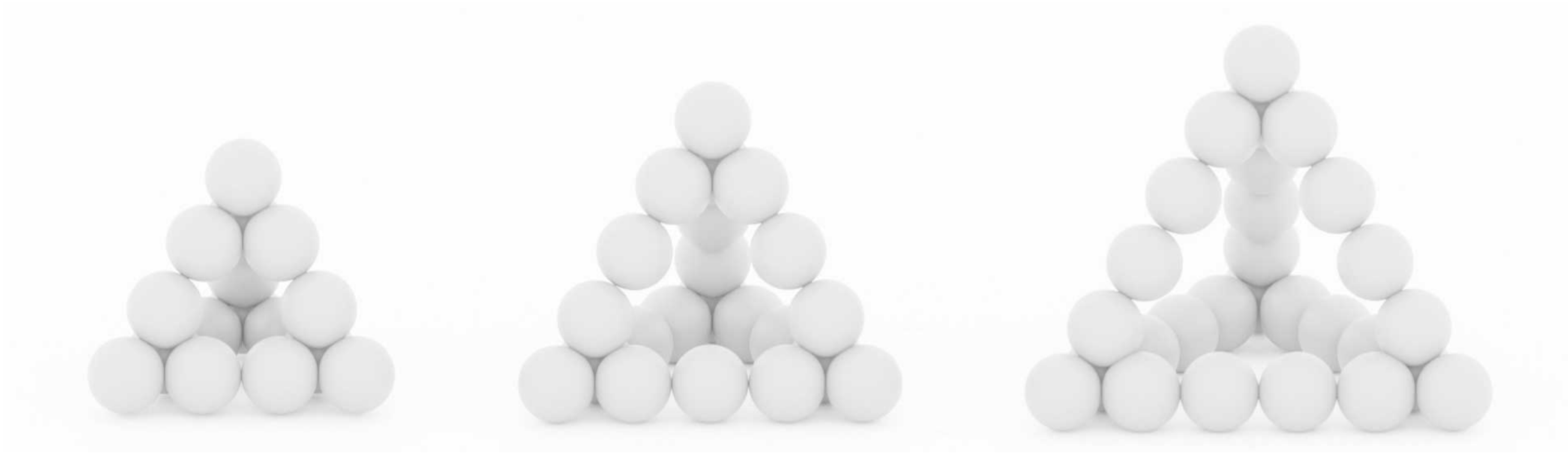

Figure 2: Triangle-based hollow pyramids with sides made up of 4, 5 and 6 identical spheres.

Find an expression for T_n, the number of spheres that make up the n^{th} pyramid in this sequence.

The following symbols may be useful: n

Part C Is rearrangement possible?

Prove that it is not possible to rearrange the spheres making up any square-based pyramid to produce a triangle-based pyramid (of any size) without either having spheres left over or needing extra spheres.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We will use proof by deduction.

Reasoning:

The number of spheres making up the k^{th} hollow square-based pyramid is given by $8k+13$. For any positive ______ value of k, $8k$ is ______. Hence, $8k+13$ is always ______.

The number of spheres making up the n^{th} hollow triangle-based pyramid is given by ______. For any positive ______ value of n, $6n$ is ______. Hence, ______ is always even.

Therefore, the number of spheres required to make a hollow square-based pyramid ______ the same as the number of spheres required to make a hollow triangle-based pyramid.

Conclusion:

Hence, it is not possible to rearrange the spheres making up any square-based pyramid to produce a triangle-based pyramid (of any size) without either having spheres left over or needing extra spheres.

Items:

integer | fractional | is always | even | odd | rational | $10n+6$ | can never be | $6n+10$

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Home Maths Number Arithmetic Proof Applied to Surface Areas

Proof Applied to Surface Areas

A Level

Consider a sphere with a radius r cm, where r is a rational number. Using proof by contradiction, show that the side length of a cube with the same surface area cannot also be a rational number of cm.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

Assumption:

Consider a sphere of radius r cm, where r is a rational number. Let the side length of a cube with the same surface area as the sphere be a cm. Assume that a is a rational number, in which case $a = \frac{b}{c}$, where b and c are integers with no common factor.

Reasoning:

The surface area of the sphere is ______. Because r is a rational number, $r = \frac{p}{q}$, where p and q are integers with no common factor. Hence, the surface area of the sphere may be written as ______.

The surface area of the cube is ______. Using $a = \frac{b}{c}$, the surface area may be written as ______.

The surface area of the sphere and the cube are equal. Hence, $4\pi\left(\frac{p}{q}\right)^2 = 6\left(\frac{b}{c}\right)^2$. Re-arranging this equation to give an expression for ______ gives ______.

As b, c, p and q are all integers, ______ must be ______ number. However, π is not ______ number.

Reasoning:

The assumption that a is rational has resulted in a contradiction. Hence, the assumption cannot be true. Therefore, the side length of a cube with the same surface area as a sphere of radius r cm, where r is a rational number, cannot be a rational number of cm.

Items:

a rational | $4\pi r^2$ | an irrational | $\pi = \frac{3b^2q^2}{2c^2p^2}$ | π | $6a^2$ | $\pi = \frac{3b^2p^2}{2c^2q^2}$ | a^3 | a real | $\frac{3b^2q^2}{2c^2p^2}$ | $4\pi\left(\frac{p}{q}\right)^2$ | $6\left(\frac{b}{c}\right)^2$

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Divisibility by Exhaustion

A Level Further A

C C C C C C

A sequence u_n is defined by $u_n = n^7 - n$, where $n \in \mathbb{N}$. The first four terms of this sequence are

$$0, 126, 2184, 16380, \ldots$$

What is the largest integer that will divide every term of this sequence?

Part A Factorise u_n

Factorise u_n completely.

The following symbols may be useful: n

Part B Divisibility by 2

Using your expression from part A, prove that every term in the sequence is divisible by 2.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof.

We know that $u_n = (n-1)n(n+1)(n^2+n+1)(n^2-n+1)$.

When n is even, it is divisible by 2 and we can see that ______ is a factor of u_n, so u_n is divisible by 2.

When n is odd, we can write $n =$ ______, where $k \in \mathbb{Z}$. Then ______ $=$ ______, so ______ is divisible by 2, and hence u_n is divisible by 2.

Therefore, u_n is divisible by 2 for any value of n. So every term in the sequence is divisible by 2.

Items:

n | $n+1$ | $2k$ | $n-1$ | n^2+n+1 | $2k+1$ | n^2-n+1

Part C Divisibility by 3

Using your expression from part A, prove that every term in the sequence is divisible by 3.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof.

We know that $u_n = (n-1)n(n+1)(n^2+n+1)(n^2-n+1)$.

When n is a multiple of 3, it is divisible by 3 and we can see that is a factor of u_n, so u_n is divisible by 3.

When $n = 3k+1$, where $k \in \mathbb{Z}$. Then $=$, so is divisible by 3, and hence u_n is divisible by 3.

When $n = 3k+2$, where $k \in \mathbb{Z}$. Then $=$, so is divisible by 3, and hence u_n is divisible by 3.

Therefore, u_n is divisible by 3 for any value of n. So every term in the sequence is divisible by 3.

Items:

n | $3k-3$ | $3k+1$ | $n-1$ | n^2-n+1 | $3k$ | $n+1$ | $3k+3$ | $3k+2$ | n^2+n+1

Part D Divisibility by 7

Using your expression from part A, prove that every term in the sequence is divisible by 7.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We know that $u_n = (n-1)n(n+1)(n^2+n+1)(n^2-n+1)$.

When n is a multiple of 7, it is divisible by 7 and we can see that is a factor of u_n, so u_n is divisible by 7.

When $n = 7k+1$, where $k \in \mathbb{Z}$, then $=$, so is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+2$, where $k \in \mathbb{Z}$, then $=$, so is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+3$, where $k \in \mathbb{Z}$, then $=$, so is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+4$, where $k \in \mathbb{Z}$, then $=$, so is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+5$, where $k \in \mathbb{Z}$, then $=$, so is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+6$, where $k \in \mathbb{Z}$, then $=$, so is divisible by 7, and hence u_n is divisible by 7.

Therefore, u_n is divisible by 7 for any value of n. So every term in the sequence is divisible by 7.

Items:

$7k+7$ | $n+1$ | $49k^2+63k+21$ | $7k$ | n^2-n+1 | n^2+n+1 | $n-1$ | n | $49k^2+35k+7$

Part E Largest Divisor

Prove that 42 is the largest integer that will divide every term of u_n.

We know that u_n is divisible by 2, 3 and 7. So we know that $2 \times 3 \times 7 =$ will divide u_n. Are there any larger integers that can do so?

Let's consider the first non-zero term, 126. We find that $126 \div 42 =$. This shows that the prime factorisation of 126 is . Hence, the only larger factors of 126 are and . Will these divide any other terms of u_n?

Looking at the next term, we find that $2184 \div$ $= \frac{104}{3}$, so does not divide 2184. Considering our other factor, we find that $2184 \div$ $= \frac{52}{3}$, so does not divide 2184 either.

Therefore, 42 is the largest integer that will divide every term of u_n.

Items:

45 42 18 $2^2 \times 3 \times 7$ 63 $2 \times 3^2 \times 7$ 2 126 3 5 $2 \times 3^2 \times 5$ 7

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Physics. You work it out.

Home Gameboard Maths Induction: Sequences 1i

Induction: Sequences 1i

Further A

P P P

The sequence u_1, u_2, $u_3 \ldots$ is defined by $u_1 = 2$ and $u_{n+1} = \frac{u_n}{1+u_n}$ for $n \geq 1$.

Part A u_2, u_3, and u_4

Find u_2.

The following symbols may be useful: u_2

Find u_3.

The following symbols may be useful: u_3

Find u_4.

The following symbols may be useful: u_4

Part B u_n in terms of n

Hence, suggest an expression for u_n in terms of n and use induction to prove your suggestion is correct.

The following symbols may be useful: n, u_n

Adapted with permission from UCLES, A Level, Jan 2013, Paper 4725, Question 10.

Gameboard:

Further Maths Practice: Induction - Sequences

Induction: Divisibility 1i

Further A

P P P

The sequence u_1, u_2, $u_3 \ldots$ is defined by $u_n = 5^n + 2^{n-1}$.

Part A u_1, u_2 and u_3

Find u_1.

The following symbols may be useful: u_1

Find u_2.

The following symbols may be useful: u_2

Find u_3.

The following symbols may be useful: u_3

Part B Divisibility

Hence, suggest a positive integer, other than 1, which divides exactly into every term of the sequence and prove it with induction by considering $u_{n+1} + u_n$.

Adapted with permission from UCLES, A Level, June 2014, Paper 4725, Question 10.

Gameboard:

Further Maths Practice: Induction - Divisibility

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Home Gameboard Maths Induction: Matrices 2i

Induction: Matrices 2i

Further A

P P P

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$.

$\mathbf{M}^n$ can be expressed in the form

$$\mathbf{M}^n = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Part A $\mathbf{M}^4$

Give an expression for $\alpha + \beta + \gamma + \delta$ when $n = 4$.

Part B $\mathbf{M}^n$

Hence, suggest a suitable form for $\mathbf{M}^n$ in terms of n and prove it with induction. Give an expression for $\alpha + \beta + \gamma + \delta$.

The following symbols may be useful: n

Adapted with permission from UCLES, A Level, Jan 2012, Paper 4725, Question 7.