Isaac Physics Skills

Linking concepts in pre-university physics

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TABLE OF PHYSICAL CONSTANTS

Quantity & Symbol		**Magnitude**	**Unit**
Permittivity of free space	ϵ_0	8.85×10^{-12}	$\mathrm{F\,m^{-1}}$
Electrostatic force constant	$1/4\pi\epsilon_0$	8.99×10^{9}	$\mathrm{N\ m^2\ C^{-2}}$
Speed of light in vacuum	c	3.00×10^{8}	$\mathrm{m\,s^{-1}}$
Specific heat capacity of water	c_{water}	4180	$\mathrm{J\,kg^{-1}\,K^{-1}}$
Charge of proton	e	1.60×10^{-19}	C
Gravitational field strength on Earth	g	9.81	$\mathrm{N\,kg^{-1}}$
Universal gravitational constant	G	6.67×10^{-11}	$\mathrm{N\ m^2\ kg^{-2}}$
Planck constant	h	6.63×10^{-34}	J s
Boltzmann constant	k_{B}	1.38×10^{-23}	$\mathrm{J\,K^{-1}}$
Mass of electron	m_{e}	9.11×10^{-31}	kg
Mass of neutron	m_{n}	1.67×10^{-27}	kg
Mass of proton	m_{p}	1.67×10^{-27}	kg
Mass of Earth	M_{Earth}	5.97×10^{24}	kg
Mass of Sun	M_{Sun}	2.00×10^{30}	kg
Avogadro constant	N_{A}	6.02×10^{23}	$\mathrm{mol^{-1}}$
Gas constant	R	8.31	$\mathrm{J\ mol^{-1}\ K^{-1}}$
Radius of Earth	R_{Earth}	6.37×10^{6}	m

OTHER INFORMATION YOU MAY FIND USEFUL

Electron volt	1 eV	=	1.60×10^{-19} J
Unified mass unit	1 u	=	1.66×10^{-27} kg
Absolute zero	0 K	=	$-273\ ^{\circ}\mathrm{C}$
Year	1 yr	=	3.16×10^{7} s
Light year	1 ly	=	9.46×10^{15} m
Parsec	1 pc	=	3.09×10^{16} m

PREFIXES

1 km = 1000 m	1 Mm = 10^{6} m	1 Gm = 10^{9} m	1 Tm = 10^{12} m
1 mm = 0.001 m	1 μm = 10^{-6} m	1 nm = 10^{-9} m	1 pm = 10^{-12} m

31 Deriving kinetic theory

We create a mathematical model using Newton's laws for the particles in a gas. When we have done this, we find it predicts many aspects of bulk gas behaviour correctly. To do this, we assume that the gas is an **ideal gas**.

Example context: explaining how the volume, pressure and temperature of a gas change by considering the collisions of the particles in the gas with each other and the walls of the container. This allows you to predict the thermodynamic behaviour of a gas without having to do an experiment.

Quantities:

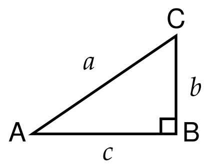
- Δ "A change in"
- F force (N)
- t time taken (s)
- s distance travelled (m)
- A area of face (m^2)
- N number of molecules
- m mass of a particle (kg)
- p momentum of a particle $(\mathrm{kg\,m\,s^{-1}})$
- c speed of a particle $(\mathrm{m\,s^{-1}})$
- P pressure of a gas $(\mathrm{N\,m^{-2}}$ or Pa$)$
- V volume of gas (m^3)
- $\overline{c^2}$ mean-square speed $(\mathrm{m^2\,s^{-2}})$
- u, v, w components of velocity in the x, y, z directions $(\mathrm{m\,s^{-1}})$

Equations:

$$F = \frac{\Delta m v}{\Delta t} \qquad P = \frac{F}{A}$$

$$v = \frac{s}{t} \qquad p = m \times \text{velocity}$$

$$\Delta p = p_{\text{after}} - p_{\text{before}}$$

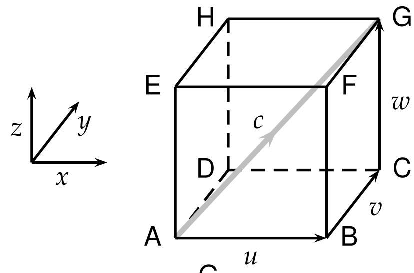


$a^2 = b^2 + c^2$ (Pythagoras)

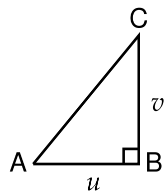
Assumptions about ideal gases:

1. The volume of a particle is so small compared to the volume of the gas, we can ignore it.
2. There are no attractive forces between particles, only collision forces.
3. Particle movement is continuous and random.
4. Particle collisions are perfectly elastic, so there is no loss of kinetic energy.
5. Collision time is very short in comparison with the time between impacts.
6. There are enough molecules for statistics to be applied.

Example – *Consider a gas particle of mass m with speed c. We can write c in terms of 3 velocity components, u, v and w in the x, y and z directions respectively. Prove that $c^2 = u^2 + v^2 + w^2$ using Pythagoras' Theorem.*

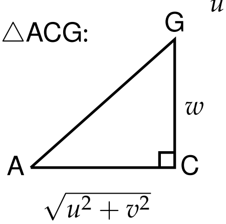


In $\triangle$ABC:



$$\text{AC}^2 = u^2 + v^2$$

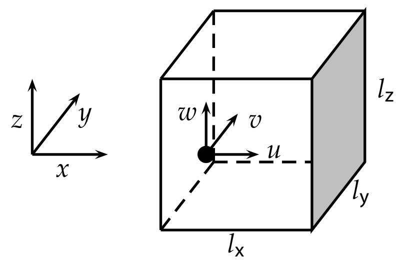
In $\triangle$ACG:



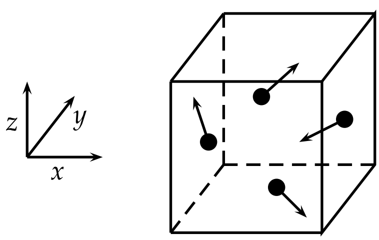
$$\text{AG}^2 = \left(\sqrt{u^2+v^2}\right)^2 + w^2$$
$$= u^2 + v^2 + w^2$$

So $c^2 = u^2 + v^2 + w^2$

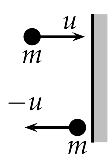
31.A The particle is in a box of dimensions l_{x}, l_{y}, l_{z}. The box represents the volume of the gas. The shaded faces represent the collisions. Write down the formula for the volume of the box V in terms of l_{x}, l_{y} and l_{z}.



We can think of the gas as a group of N particles moving around randomly, hitting the sides of the container. As the motion is random, we expect the average speeds in different directions to be the same.



Let's consider one particle moving in the positive x direction. The particle collides with the container wall.

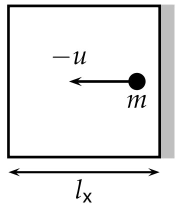


31.B Write an expression for the change in momentum of the particle, Δp, in terms of m and u. Pay attention to which direction is positive.

31.C Write an expression for the average force F_{particle} on the particle (from the wall), to cause the change in momentum of the particle. The time between collisions with the wall is Δt.

31.D Use Newton's Third Law of Motion to write down an expression for the average force, F_{wall}, of the particle on the wall over time Δt. Pay attention to the sign.

31.E Between collisions the particle will travel to the other side of the container and back again. Find an expression for Δt in terms of u and l_{x}.



31.F Now substitute your expression for Δt from 31.E into your equation in 31.D and simplify it. This will give you a new expression for the force of the particle on the wall, F_{wall}, in terms of m, u, and l_{x}.

31.G The average pressure exerted by the particle on the wall may be written as $F_{\mathrm{wall}}\,/\,A$, where A is the area of the wall. Use your answer to 31.F to find an expression for the average pressure P_1 due to this one molecule in terms of u, m and:

a) l_{x}, l_{y}, and l_{z}

b) the volume, V, of the container. Use your answer from 31.A.

We now have an expression for the pressure P_1 on the container due to the collision of one particle. From here on we refer to this particle as 'particle 1' and label its speed as c_1 and its velocity components as u_1, v_1 and w_1. There are actually N particles in the gas. They each have the same mass m, but will have different velocities. For example, 'particle 2' has velocity components u_2, v_2 and w_2, has speed c_2, and will cause a pressure P_2.

31.H Up until now, we have assumed that our particle was only moving in the x direction. Does the expression for P_1 derived in question 31.G change if v_1 and w_1 are not necessarily zero?

31.I By looking at your reasoning for particle 1, write down an expression for the pressure P_2 on the same wall in terms of m, u_2, v_2, w_2 and V.

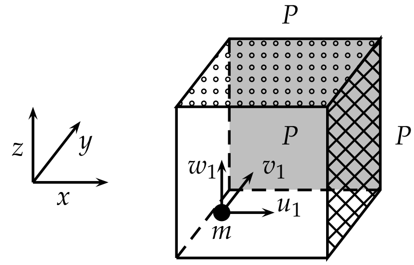
The total pressure on this wall will be the sum of the pressures due to all of the individual particles: $P = P_1 + P_2 + \ldots$.

31.J Use your expression from 31.G (b) to write the equation for total pressure P in terms of m, V, u_1, u_2 and the other x components of velocity. Assume that all the particles have the same mass, m.

31.K Find an expression for the average squared x component of velocity $\overline{u^2}$ if there are N molecules whose squared velocity components are u_1^2, u_2^2 and so on.

31.L Use your answer to 31.K to re-write the pressure from 31.J in terms of m, V, N and $\overline{u^2}$.

We now have an expression for the pressure of the particles on the right hand wall. As the particles are moving randomly, they exert the same pressure on the other walls as well.



We now take into account the fact that the molecules are not just moving in the x direction.

31.M The y components of each molecules' velocity are written v_1, v_2, v_3 and so on. Use you answer to 31.K to write expressions (when there are N particles) for:

a) the average squared y velocity component $\overline{v^2}$ and

b) the average squared z velocity component $\overline{w^2}$.

31.N In question 31.L, you wrote an equation linking P and $\overline{u^2}$. By thinking of collisions with the back wall causing an equal pressure, write a similar equation linking P and $\overline{v^2}$. Then, by thinking of collisions with the top wall, write a similar equation linking P and $\overline{w^2}$.

31.O In the example, we saw that $c^2 = u^2 + v^2 + w^2$, where c^2 is the square speed of one molecule. Applied to particle 1 this means that $c_1^2 = u_1^2 + v_1^2 + w_1^2$. Use this information to write an equation relating $\overline{c^2}$ (the mean square speed), to $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ (the mean square velocity components).

31.P Use your answers to questions 31.N and 31.O to write an equation for the pressure P in terms of the mean square speed $\overline{c^2}$.

This equation beautifully links the macroscopic behaviour of a gas (PV) with the average (square) speed of the N microscopic gas particles.

> **Exercise** – *Copy the diagram of the box and see how far you can progress with the proof without looking at all the steps. Remember:*
>
> 1. 1 *particle*
> 2. N *particles*
> 3. 3 *dimensions.*

The parts of this section are lettered A, B, C…to match with the implementation of this question online.

32 Gas laws, density and kinetic energy

We can combine the Gas Law with the molar mass equation to calculate the density of a gas if we know the molar or molecular mass. We can also relate the temperature of a gas to the average kinetic energy of its molecules.

Gas density is important as it will determine whether the gas will rise or fall relative to the medium it is in, for example in a hot air balloon: the hot air in the balloon is less dense than the air surrounding the balloon. Our questions here also show the link between temperature and molecular energy for a gas.

It is useful to re-write the kinetic theory equation derived in section 31 in terms of gas density.

Quantities:		
	P pressure $\left(\mathrm{N\,m^{-2}}\right)$	n number of moles of gas (mol)
	V volume of a gas $\left(\mathrm{m^3}\right)$	M total mass of gas (kg)
	T temperature (K)	N number of particles in a gas
	m mass of particle (kg)	$\overline{c^2}$ mean-square speed $\left(\mathrm{m^2\,s^{-2}}\right)$
	ρ density of a gas $\left(\mathrm{kg\ m^{-3}}\right)$	m_u molecular mass (u)
	M_M molar mass $\left(\mathrm{kg\,mol^{-1}}\right)$	$\overline{K}$ mean molecule kinetic energy (J)

Equations:

$$PV = nRT \qquad n = \frac{M}{M_\mathrm{M}} \qquad PV = \frac{Nm\overline{c^2}}{3} \qquad n = \frac{N}{N_A} \qquad \rho = \frac{M}{V}$$

$$PV = Nk_\mathrm{B}T \qquad \overline{K} = \tfrac{1}{2}m\overline{c^2} \qquad m_u = \frac{m}{1.66\times10^{-27}\ \mathrm{kg}}$$

32.1 Use the equations above to derive expressions for

a) P in terms of M, M_M, V, R and T,

b) ρ in terms of M_M, P, R and T,

c) ρ in terms of m, P, k_B and T,

d) ρ in terms of P and $\overline{c^2}$,

e) $\overline{K}$ in terms of k_B and T.

Example 1 – *What is the density of carbon dioxide gas (CO_2) at a pressure of 110 kPa and a temperature of $30\,^\circ$C? The molar mass of C is 12 g mol^{-1} and the molar mass of O is 16 g mol^{-1}. Remember to convert g to kg.*
The molar mass of the molecule is $(12 + 2\times16)$ g $= 0.044$ kg
$T = 273 + 30 = 303$ K

$$\rho = \frac{M_M P}{RT} = \frac{0.044 \times 110 \times 10^3}{8.31 \times 303} = 1.9222 \text{ kg m}^{-3} = 1.9 \text{ kg m}^{-3} \text{ (2 s.f.)}$$

32.2 What is the density of a sulfuric acid gas cloud on Venus if the temperature is 467 °C and the pressure is 9308 kPa? The chemical formula for sulfuric acid is H_2SO_4.

Element	Molar mass / g mol^{-1}
H	1
S	32
O	16

32.3 Use your answer to 32.1 to complete the following table containing information on different gases: (give your answers to 2 s.f.)

Chemical formula	Molecular mass / u	Temperature / K	Pressure / kPa	Density / kg m^{-3}
NO_2	46	500	115	(a)
HCl	36.5	(b)	120	277
NH_3	17	723	(c)	57.3

Example 2 – *What is the density of a gas at a pressure of* 101 *kPa if the root mean square velocity* $c_{rms} = \sqrt{\overline{c^2}}$ *of the particles is* 500 *m s*$^{-1}$*?*

$$\rho = \frac{3P}{\overline{c^2}} = \frac{3 \times 101 \times 10^3}{500^2} = 1.212 \text{ kg m}^{-3} = 1.21 \text{ kg m}^{-3} \text{ (3 s.f.)}$$

This is a typical value for air at room temperature.

32.4 What is the density of a gas at a pressure of 150 kPa if the mean square speed of the particles is 9.0×10^4 m^{2}s^{-2}?

32.5 What is the pressure needed for a gas of density 1.2 kg m^{-3} to have a root mean square speed of 330 m s^{-1}?

32.6 Calculate the mean kinetic energy of molecules in a gas at 15 °C.

32.7 Calculate the temperature at which the mean molecular kinetic energy is 1.60×10^{-21} J.

32.8 Within a gas mixture at equilibrium, the mean kinetic energy of each type of molecule is the same. This is because the temperature is uniform. In a mixture of helium ($m = 4.00$ u) and nitrogen ($m = 28.0$ u),

a) state which molecules typically move faster, and

b) calculate the ratio $\overline{c^2_{\text{helium}}} \;/\; \overline{c^2_{\text{nitrogen}}}$.

(b) From a): $r = \dfrac{mv\sin\theta}{qB}$ and $v_\perp = \dfrac{2\pi r}{T} = v\sin\theta$.

$$\text{So } r = \frac{mv\sin\theta}{qB} = \frac{T}{2\pi}v\sin\theta. \text{ Therefore } T = \frac{2\pi m}{qB}$$

(c) $v_\perp^2 + v_\parallel^2 = v^2\sin^2\theta + v^2\cos^2\theta = v^2\left(\sin^2\theta + \cos^2\theta\right) = v^2$.

Thus $v = \sqrt{v_\perp^2 + v_\parallel^2}$.

(d) From c): $T = \dfrac{2\pi m}{qB}$ and $s_\mathrm{p} = v_\parallel T = v\cos\theta\dfrac{2\pi m}{qB}$.

Re-arranging gives $q/m = \dfrac{2\pi}{B}v\cos\theta s_\mathrm{p}$.

30 Vectors and fields – mass spectrometer

(a) $F_\mathrm{B} = ma$ so $Bqv = \dfrac{mv^2}{r}$. Rearranging gives $r = \dfrac{mv}{Bq}$

(b) $qV_\mathrm{a} = \frac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2qV_\mathrm{a}}{m}}$. Now using our result for r from (a),

$$r = \frac{mv}{Bq} = \frac{m}{Bq}\sqrt{\frac{2qV_\mathrm{a}}{m}} = \sqrt{\frac{2mV_\mathrm{a}}{B^2q}}$$

(c) From (a): $r = \dfrac{mv}{Bq}$ so $\dfrac{q}{m} = \dfrac{v}{Br}$

(d) From (b): $r^2 = \dfrac{2mV_\mathrm{a}}{B^2q}$ so $\dfrac{q}{m} = \dfrac{2V_\mathrm{a}}{B^2r^2}$

(e) $F_\mathrm{E} = F_\mathrm{B}$ so $qE = qvB$ and $E = vB$. So $V_\mathrm{s} = Ed = vBd$

31 Deriving kinetic theory

(A) $V = l_x\, l_y\, l_z$

(B) $\Delta p = -mu - mu = -2mu$

(C) $F_\mathrm{particle} = \dfrac{\Delta p}{\Delta t} = -\dfrac{2mu}{\Delta t}$

(D) $F_{\text{wall}} = -F_{\text{particle}} = \dfrac{2mu}{\Delta t}$

(E) Using velocity $= \dfrac{\text{displacement}}{\text{time}}$, time $= \dfrac{\text{displacement}}{\text{velocity}}$, $\Delta t = \dfrac{2l_x}{u}$

(F) $F_{\text{wall}} = \dfrac{2mu}{2l_{\mathsf{x}}/u} = \dfrac{mu^2}{l_x}$

(G) a) $P_1 = \dfrac{F_{\text{wall}}}{l_y l_z} = \dfrac{mu^2}{l_x l_y l_z}$ b) $V = l_x l_y l_z$ so $P_1 = \dfrac{mu^2}{V}$

(H) No, as v and w do not change in the collision

(I) $P_2 = \dfrac{mu_2^2}{V}$

(J) $P = P_1 + P_2 + \ldots = \dfrac{mu_1^2}{V} + \dfrac{mu_2^2}{V} + \ldots = \dfrac{m}{V}\left(u_1^2 + u_2^2 + \ldots\right)$

(K) $\overline{u^2} = \dfrac{u_1^2 + u_2^2 + u_3^2 + ... + u_N^2}{N}$

(L) $\dfrac{PV}{m} = u_1^2 + u_2^2 + \ldots = N\overline{u^2}$ so $P = \dfrac{Nm\overline{u^2}}{V}$

(M) a) $\overline{v^2} = \dfrac{v_1^2 + v_2^2 + v_3^2 + \ldots}{N}$ b) $\overline{w^2} = \dfrac{w_1^2 + w_2^2 + w_3^2 + \ldots}{N}$

(N) $P = \dfrac{Nm}{V}\overline{v^2}$ using y components of velocity on back wall

$P = \dfrac{Nm}{V}\overline{w^2}$ using z components of velocity on top wall

(O)
$$\begin{aligned}\overline{c^2} &= \frac{c_1^2 + c_2^2 + \ldots}{N} = \frac{\left(u_1^2 + v_1^2 + w_1^2\right) + \left(u_2^2 + v_2^2 + w_2^2\right) + \ldots}{N} \\ &= \frac{u_1^2 + u_2^2 + \ldots}{N} + \frac{v_1^2 + v_2^2 + \ldots}{N} + \frac{w_1^2 + w_2^2 + \ldots}{N} \\ &= \overline{u^2} + \overline{v^2} + \overline{w^2}\end{aligned}$$

(P) $\overline{u^2} = \dfrac{PV}{Nm} = \overline{v^2} = \overline{w^2}$ so $\overline{c^2} = \overline{u^2} + \overline{v^2} + \overline{w^2} = \dfrac{3PV}{Nm}$ and so

$PV = \dfrac{Nm\overline{c^2}}{3}$

32 Gas laws, density and kinetic energy

(a) $PV = nRT$ and $n = \dfrac{M}{M_\mathrm{M}}$ so $PV = \dfrac{MRT}{M_\mathrm{M}}$ and $P = \dfrac{MRT}{M_\mathrm{M}V}$

(b) From (a) $V = \dfrac{MRT}{M_\mathrm{M}P}$ so $\rho = \dfrac{M}{V} = \dfrac{M}{MRT/M_\mathrm{M}P} = \dfrac{M_\mathrm{M}P}{RT}$

(c) $\rho = \dfrac{M}{V} = \dfrac{Nm}{V} = \dfrac{Nm}{Nk_\mathrm{B}T/P} = \dfrac{mP}{k_\mathrm{B}T}$

(d) $PV = \dfrac{Nm\overline{c^2}}{3}$ so $P = \dfrac{Nm}{V} \cdot \dfrac{\overline{c^2}}{3} = \dfrac{\rho\overline{c^2}}{3}$ and $\rho = \dfrac{3P}{\overline{c^2}}$

(e) $PV = Nk_\mathrm{B}T = \frac{1}{3}Nm\overline{c^2}$ so $m\overline{c^2} = 3k_\mathrm{B}T$ and $\bar{K} = \dfrac{m\overline{c^2}}{2} = \dfrac{3k_\mathrm{B}T}{2}$

33 Capacitors and resistors

(a) $Q = CV_0 e^{-t/RC}$ (or $Q_0 e^{-t/RC}$)

(b) $Q = CV_0(1 - e^{-t/RC})$ or $Q_0(1 - e^{-t/RC})$

(c) $Q_0/V_0 = C$ so $Q_0 = CV_0$

(d) $V_\mathrm{R} = V_\mathrm{C} = V_0 e^{-t/RC}$

(e) $V_\mathrm{R} = V_0 - V_\mathrm{C} = V_0 e^{-t/RC}$ i.e. the same as when discharging

(f) $Q = CV_\mathrm{C} = CV_\mathrm{R} = C(IR) = I \times RC$

(g) $Q = -RC\dfrac{\mathrm{d}Q}{\mathrm{d}t}$ so the differential equation is $\dfrac{\mathrm{d}Q}{\mathrm{d}t} = \dfrac{-Q}{RC}$

(h) $I_0 = \dfrac{V_0}{R}$

(i) $I_0 = \dfrac{(Q_0/C)}{R} = \dfrac{Q_0}{RC}$

(j) Time to discharge at constant current $= \dfrac{Q_0}{I_0} = RC$

(k) $t = RC$, so $\dfrac{Q}{Q_0} = e^{-1} = 0.37$