Isaac Maths Skills

Using Essential GCSE Mathematics

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Using Essential GCSE Mathematics

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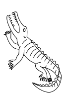 used with kind permission of M. J. Rutter.

About this book

This book is designed to provide practice for GCSE-level mathematics. It can be used by those taking GCSE mathematics courses, and also by students in other subjects who need to learn or brush up on their knowledge of particular topics. The goals of the book are to help students master the skills they learn at GCSE level, and act as a resource for students who need to use these skills in their courses at A-level.

The book covers all major areas of mathematics in the various GCSE-level courses that are available. Where you see the § symbol, this indicates material that would usually be taught only in a Higher level (grades 4-9) GCSE course. However, don't let this put you off! The intent of the book is to help you learn mathematics, so if you would like to read the text or try the questions in a section that is outside the confines of your course then we would strongly encourage you to challenge yourself and have a go!

The focus of this book is on helping students master the skills they learn at GCSE. Being a short book, it is not intended as a complete stand-alone resource for learning GCSE mathematics from scratch. Likewise, you can use it as part of general maths revision for GCSE exams, but it is not intended to be a complete revision guide.

Teachers are encouraged to use whichever sections of the book would be helpful for their classes.

JNW & SAW
Aylesbury, 2021

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Aylesbury, 2021

Using Isaac Physics with this book

An online version of this book is implemented on the Isaac Physics website at `https://isaacphysics.org/books/maths_book_gcse`. In line with other subjects on the website, students may enter their answers to the questions in the book for immediate marking and feedback, and teachers are able to create bespoke worksheets using their own selections of questions from the book for use in class or as homework.

Contents

Solving Maths Problems

1 Solving Maths Problems

In each chapter of this book there are questions to help you practise your mathematics skills. While the style of question varies, there are a number of general problem-solving strategies that apply to all topics.

- Whenever possible, draw a labelled diagram showing all the information in the question.
- Try to write algebra for information which you are given in words.
- When a question includes a graph, read the labels on the axes to find the quantities that are plotted and their units.
- Write down all your working as you go along. This can be time-consuming, but is worth it as it will help you get correct answers.
- If you get stuck, read the question again to check what the question is asking for and whether there is any information you have missed.
- Make sure you give your answers in the form the question requires. Use a suitable number of significant figures and state the units.

When tackling a multi-step problem, a useful approach is to start by writing out in words a quick plan for finding a solution. This doesn't have to be a long paragraph - bullet points will do. Don't worry about whether you know how to carry out every part of the plan. Just get the plan down on paper.

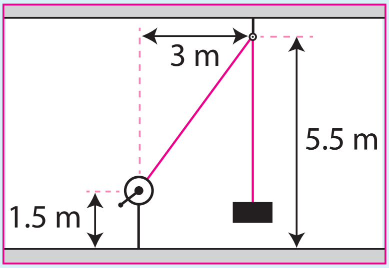
Next, break your plan down into a series of individual steps and work out how to do each one. List the quantities you wish to calculate, and write down the formulae you intend to use. If you find there is a step you don't know how to do, try looking up the topic where that step is covered elsewhere in the book (or in your class notes). Consider if you have done something similar before which is appropriate to use as an analogy.

Finally, carry out your plan.

To see this in action, Example 1 shows a rather complicated-looking problem. Do not worry if you do not know how to solve the problem yourself at the moment. The techniques that you need are covered later in the book.

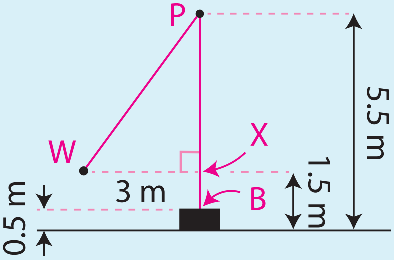
Focus instead on the approach that is taken - drawing a diagram, writing a plan, and then carrying out the plan.

Example 1 – A winch and a small pulley are used to lower a heavy box . The arrangement is shown in the diagram. If the height of the box is 50 cm, what is the minimum length of rope needed to lower the box to the ground?



The rope goes from the winch to the pulley and down to the box. The rope has to be long enough for the box to reach the ground. The diagram shows the box in the air, so the first task is to draw a new diagram.

We keep the diagram simple, label lengths that we know, and use our experience with this type of problem to mark on extra lines and right-angles that might be useful. The height of the box is converted into metres.



Next, plan how to solve the problem. We need to find the length of the rope. The rope has two sections: WP and PB. We will find the length of each section separately, then add them together.

$$\text{Minimum length of rope needed } = WP + PB$$

To find WP: WXP is a right-angled triangle, so we can use Pythagoras' Theorem: $WP^2 = WX^2 + XP^2$. From the diagram, $WX = 3$ m and $XP = 5.5 - 1.5 = 4$ m.

$$\therefore WP^2 = 3^2 + 4^2 = 25 \qquad \Rightarrow WP = 5\text{ m}$$

PB is equal to the height of the pulley above the ground minus the height of the box. So, $PB = 5.5 - 0.5 = 5.0$ m.

Therefore,

$$\text{Minimum length of rope needed } = 5.0 + 5.0 = 10.0\text{ m}$$

Skills

2 Factors

An integer is a whole number, for example 12.

A sum is the result of adding numbers together. A difference is the result of subtracting one number from another. A product is the result of multiplying numbers together. A quotient is the result of dividing one number by another.

If a number is multiplied by itself, a power (index) records how many times that number is used. For example, $2 \times 2 \times 2 = 2^3$.

The first 5 powers of the integers 2 to 5 are listed in the table below.

$a^{(1)}$	a^2	a^3	a^4	a^5
2	4	8	16	32
3	9	27	81	243
4	16	64	256	$1\,024$
5	25	125	625	$3\,125$

A factor (divisor) is an integer that divides into a number exactly. For example, 2 is a factor of 6 because $6 \div 2 = 3$, and 3 is a whole number.

A prime number is an integer which has only two factors: 1 and itself. An example is 13, which only has factors 1 and 13. 1 is not considered to be a prime number.

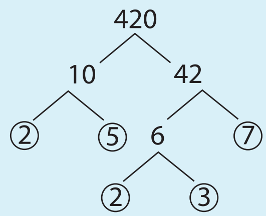
A prime factor is a factor that is also a prime number. For example, 13 is a prime factor of 26.

A factor tree can be used both to find the prime factors of a number and to write the number as a product of prime factors. Writing a number in terms of its prime factors is called factor decomposition.

Example 1 – Write 420 as a product of prime factors.

At each branching of the tree, take out another factor. In the first branching, 10 is a factor of 420. $420 \div 10$ is 42.

If a factor is a prime factor, circle it. Keep going until every branch ends in prime factors.



Written as a product of prime factors, $420 = 2 \times 2 \times 3 \times 5 \times 7$.

2.1 You are given a set of integers: $\{-5, -2, -1, 3, 4, 7\}$. Write down two numbers which give:

(a) A sum of 3. (b) A difference of 6. (c) A product of 2.

2.2 Beginning with 2, Adam starts making a list of prime numbers in order of increasing size. He doesn't skip any numbers.

(a) What is the sum of the first 9 numbers on the list?

(b) What is the 10^{th} number on the list?

2.3 Write each number as a product of prime factors:

(a) 66 (b) 210 (c) 182

2.4 You are given the following set of positive integers: $\{1, 3, 4, 5, 6, 9, 12, 13\}$. For this set, find:

(a) The sum of the square numbers.

(b) The sum of the prime numbers.

(c) The sum of the numbers which are multiples of 2.

2.5 Write down or find:

(a) The two prime numbers separated by 1.

(b) The smallest pair of prime numbers that differ by 6.

(c) Three pairs of prime numbers that each have a sum of 30.

(d) The product of the numbers in the answer to (a), added to the sum of all eight numbers in the answers to (b) and (c).

A product of prime factors can be written using index notation. For example, $6300 = 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 = 2^2 \times 3^2 \times 5^2 \times 7$.

The highest common factor (HCF) of two numbers is the largest number that is a factor of both numbers. For example, the HCF of 28 and 42 is 14.

1 is not a prime number, so is not listed in a prime factor decomposition. However, 1 is a factor of every integer and can be a highest common factor. For example, the HCF of 3 and 5 is 1.

The lowest common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers. For example, the LCM of 8 and 12 is 24.

Example 2 shows how to use factor decomposition to find an HCF or LCM.

Example 2 – What are the HCF and LCM of 84 and 140?

$$84 = 2 \times 2 \times 3 \times 7 \qquad 140 = 2 \times 2 \times 5 \times 7$$

(i) The factors that are common to 84 and 140 are 2, 2 and 7. Therefore the HCF is $2 \times 2 \times 7 = 28$.

(ii) The LCM contains all the prime factors that make up the two numbers. The LCM is $2 \times 2 \times 3 \times 5 \times 7 = 420$.

2.6 By listing the multiples of each number, find the lowest common multiples of

(a) 3 and 15 (b) 8 and 10 (c) 18 and 24

2.7 By writing each number in terms of prime factors, find the HCF and LCM of

(a) 6 and 40 (b) 15 and 70 (c) 24 and 25

2.8 (a) Write $44\,100$ as a product of prime factors.

(b) Use your answer to part (a) to find the square root of $44,100$.

2.9 Find the prime factors of the following numbers, writing your answers in the form $a^p \times b^q \times c^r$, where a, b and c are prime numbers.

(a) 28 (b) 90 (c) 96

2.10 (a) A number is written as $2^2 \times 3 \times 7$. What is the number?

(b) A number is written as $2 \times 3^3 \times 5^2 \times 7$. What is the number?

(c) You are told that $5\,460$ is a factor of $3\,248\,700$. Using factor decomposition, find the value of $3\,248\,700 \div 5\,460$.

3 BIDMAS / BODMAS and Substituting Values Into Formulae

The acronym BIDMAS or BODMAS is used as a way to help remember the order in which in which operations are done in a mathematical calculation. By following BIDMAS rules, all users of mathematics do the same operations in the same order, and so always arrive at the same answers when evaluating expressions or formulae.

- B Brackets (do calculations inside brackets first)
- I/O Indices or Order (calculate powers next)
- DM Division and Multiplication
- AS Addition and Subtraction

Division and multiplication are done together, working through a calculation from left to right. The same is true for addition and subtraction.

Example 1 – Evaluate $3 - \frac{(9-5)^2}{2}$.

First evaluate the brackets, then evaluate the indices:

$$3 - \frac{(9-5)^2}{2} = 3 - \frac{4^2}{2} = 3 - \frac{16}{2}$$

Next, do the division. Then, finally, the subtraction.

$$3 - \frac{(9-5)^2}{2} = 3 - 8 = -5$$

When dealing with fractions, treat the numbers above the dividing line and the numbers below the dividing line as though they were in separate brackets. Evaluate them before performing the division.

Roots (square roots, cube roots etc.) are calculated along with I/O because taking a root is the inverse operation to raising to a power. Calculate everything included under a root sign as if it was in a bracket before taking the root.

Example 2 – Evaluate $2\sqrt{\frac{70-30}{5\times 2}} + 1$.

First, evaluate the top and bottom of the fraction as though they were in separate brackets. Then do the division under the square root sign.

$$2\sqrt{\frac{70-30}{5\times 2}}+1=2\sqrt{\frac{40}{10}}+1=2\sqrt{4}+1$$

Next, take the square root. Then do the multiplication, followed by the addition.

$$\Rightarrow 2\sqrt{\frac{70-30}{5\times 2}}+1=2\times\sqrt{4}+1=2\times 2+1=4+1=5$$

3.1 Evaluate the following using BIDMAS rules:

(a) $3-5$ (b) $8\div 16+5$ (c) $1+6\div 2$

3.2 Insert brackets in these calculations so that they are correct:

(a) $2\times 9+4=26$ (c) $5\times 5+2\times 2>60$

(b) $6-6\times 7+11=11$ (d) $6\times 8+3\times 7=174$

3.3 Insert $>$, $=$ or $<$ to make the following statements correct:

(a) $5^2\ ?\ 2^2$ (b) $(2+1)^3\ ?\ 4^2$ (c) $\sqrt{36}\ ?\ \sqrt{25}+\sqrt{81}$

3.4 Evaluate the following using BIDMAS rules:

(a) $1+8\div 3$ (b) $\frac{1\times 7}{4-2}$ (c) $1\frac{1}{2}-\frac{7}{42\div 3}$

3.5 Insert brackets in these calculations so that they are correct:

(a) $1+7+1\div 18=\frac{1}{2}$ (c) $\frac{-5}{16-6\times 5-1}=\frac{1}{3}$

(b) $4+5\div 2+2=\frac{13}{2}$ (d) $\sqrt{3^2+1\times 2-4}=\sqrt[3]{64}$

Substituting values into a formula means putting numerical values (including minus signs) in place of algebraic variables and using the BIDMAS rules to calculate a value.

Example 3 – Substitute $q=7$, $r=2$ and $s=8$ into the formula $p=\sqrt{q^2+2rs}$ to find the value of p.

Insert the values into the appropriate places in the formula, then perform the calculation in BIDMAS order.

$$p=\sqrt{7^2+2\times 2\times 8}=\sqrt{49+2\times 2\times 8}=\sqrt{49+32}=\sqrt{81}=9$$

3.6 Given $a = -4$, $b = -6$ and $c = 2$, find the value of $3(c - b) - \frac{a}{2}$.

3.7 If $a = 2$, $b = -3$ and $c = 5$, find the value of p when

(a) $p = 2a + 5$ (b) $p = 3b + 2c$ (c) $p = 4ac - 6b$

3.8 $x = 3$, $y = -5$ and $z = 2$. Find the following:

(a) $(x - y)^2$ (b) $y + z(x + z)^2$ (c) $yz + \frac{xz}{y}$

3.9 Find the values of p, q and r if $s = 6$, $t = 3$ and $u = -8$.

(a) $p = \frac{1}{2}u - st$ (b) $3q = ut + s$ (c) $2r = \frac{2t-u}{7} + 4$

3.10 Evaluate

(a) $2 + (3^2 - 5)^2$ (b) $5 + \sqrt{7^2 - 13}$ (c) $1 + \frac{4}{3} + \frac{1+4}{3}$

The following questions use formulae from real-world applications.

3.11 Interest calculations:

$$A = C\left(1 + \frac{I}{100}\right)^n$$

A is the amount of money in an account after n years if an amount of capital C is invested with a compound interest rate of $I\%$. Calculate the value of A to the nearest penny if:

(a) $C = £100$, $I = 4$ and $n = 2$

(b) $C = £750$, $I = 3.2$ and $n = 6$

(c) £100 is invested at a rate of 12% for 4 years.

3.12 Gas laws in physics:

$$V = V_0\left(1 + \frac{T}{273}\right)$$

V_0 is the volume of a gas at a temperature of $0°$C. V is the volume the same amount of gas would occupy at a temperature $T°$C if the gas has the same pressure.

(a) Find V if $V_0 = 22.4$ m^3 and $T = 273°$C.

(b) Find V if $V_0 = 0.14$ m^3 and $T = 136.5°$C.

4 Fractions

The number on the top of a fraction is the numerator and the number on the bottom is the denominator. The line between the numerator and the denominator is equivalent to "divided by", i.e. $\frac{3}{4}$ is equivalent to $3 \div 4$.

Dividing by 0 is not possible, and all divisions by zero, including fractions with 0 in the denominator, are undefined and cannot be worked out.

A numerical fraction with the same number in both the numerator and denominator has a value of 1 because anything divided by itself is 1.

$$\frac{3}{3} = \frac{1}{1} = 1$$

In proper fractions such as $\frac{3}{4}$, the numerator is smaller than the denominator, while in improper fractions the numerator is larger than the denominator. Mixed fractions have both an integer and a fraction part.

Example 1 – Write (i) $5\frac{1}{4}$ as an improper fraction (ii) $\frac{9}{4}$ as a mixed number.

(i) In words, $5\frac{1}{4}$ is "five and a quarter". 1 is the same as four quarters, so 5 is the same as twenty quarters. Therefore,

$$5\frac{1}{4} = 5 + \frac{1}{4} = \frac{20}{4} + \frac{1}{4} = \frac{21}{4}$$

(ii) First, rewrite the numerator as a multiple of the denominator plus a remainder: $9 = 2 \times 4 + 1$

Then, $\frac{9}{4} = \frac{2\times4+1}{4} = \frac{2\times4}{4} + \frac{1}{4}$, so $\frac{9}{4} = 2 + \frac{1}{4} = 2\frac{1}{4}$

When multiplying or dividing fractions, always start by converting any mixed numbers to improper fractions. To do the division or multiplication it can be helpful to think of the second fraction as a pair of scale factors.

Example 2 – Calculate (i) $\frac{3}{7} \times \frac{2}{5}$ (ii) $\frac{3}{7} \div \frac{2}{5}$.

(i) In words, $\frac{2}{5}$ is "two divided by five". So, starting with $\frac{3}{7}$, scale up by 2 ($\times 2$) and down by 5 ($\div 5$).

$$\frac{3}{7} \times \frac{2}{5} = \frac{3 \times 2}{7 \times 5} = \frac{6}{35}$$

(ii) Multiply and divide are inverse operations. So if multiply scales $\frac{3}{7}$ up by 2 and down by 5, then divide scales $\frac{3}{7}$ down by 2 and up by 5. This is equivalent to multiplying by $\frac{5}{2}$.

$$\frac{3}{7} \div \frac{2}{5} = \frac{3}{7} \times \frac{5}{2} = \frac{3 \times 5}{7 \times 2} = \frac{15}{14}$$

If the numerator and denominator of a fraction have the same factor, this factor can be cancelled to leave a simpler equivalent fraction. There are several ways this can be written:

$$\frac{6}{8} = \frac{3 \times 2}{4 \times 2} = \frac{3}{4} \times \frac{2}{2} = \frac{3}{4} \times 1 = \frac{3}{4} \quad \text{or} \quad \frac{6}{8} = \frac{3 \times \cancel{2}}{4 \times \cancel{2}} = \frac{3}{4}$$

Note: Cancelling down is only possible when every term in both the numerator and denominator has the same factor.

Creating an equivalent fraction with a denominator that is a multiple of the starting denominator is the opposite of cancelling down. For example, to find a fraction equivalent to $\frac{1}{6}$ with a denominator of 18, start by noting that $18 = 6 \times 3$. Write $\frac{1}{6}$ as $\frac{1}{6} \times 1$, then rewrite the 1 as $\frac{3}{3}$:

$$\frac{1}{6} = \frac{1}{6} \times 1 = \frac{1}{6} \times \frac{3}{3} = \frac{1 \times 3}{6 \times 3} = \frac{3}{18}$$

To add or subtract fractions, first re-write them as equivalent fractions with a common denominator. Next, add or subtract the numerators and simplify the result.

Example 3 – Calculate (i) $\frac{2}{3} + \frac{3}{4}$ (ii) $\frac{2}{3} - \frac{3}{4}$.

The lowest common multiple (LCM) of the denominators of $\frac{2}{3}$ and $\frac{3}{4}$ is 12. Rewriting both fractions as equivalent fractions with a denominator of 12,

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \qquad \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$$

(i) $\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12} = 1\frac{5}{12}$.

(ii) $\frac{2}{3} - \frac{3}{4} = \frac{8}{12} - \frac{9}{12} = \frac{-1}{12} = -\frac{1}{12}$.

4.1 Convert the following into mixed fractions:

(a) $\frac{29}{3}$ (b) $\frac{17}{13}$ (c) $-\frac{105}{4}$

4.2 Convert the following into improper fractions:

(a) $3\frac{1}{5}$ (b) $9\frac{2}{7}$ (c) $-4\frac{24}{25}$

4.3 Write as proper fractions in their simplest form:

(a) $\frac{16}{18}$ (b) $\frac{56}{74}$ (c) $-\frac{96}{120}$

4.4 Multiply the following fractions:

(a) $\frac{6}{7} \times \frac{2}{3}$ (b) $1\frac{1}{3} \times \frac{3}{8}$ (c) $2\frac{1}{6} \times 2\frac{5}{26}$

4.5 Divide the following fractions:

(a) $\frac{2}{3} \div \frac{8}{9}$ (b) $1\frac{5}{6} \div \frac{5}{12}$ (c) $5\frac{1}{7} \div 4\frac{1}{8}$

4.6 Add the following fractions:

(a) $\frac{1}{3} + \frac{1}{9}$ (b) $\frac{2}{3} + \frac{3}{4}$ (c) $\frac{11}{12} + \frac{1}{18} + 1\frac{1}{3}$

4.7 Subtract the following fractions:

(a) $\frac{5}{12} - \frac{1}{4}$ (b) $\frac{6}{7} - \frac{1}{6}$ (c) $2\frac{2}{3} - 1\frac{7}{11}$

4.8 Cancel these fractions down to their simplest form:

(a) $\frac{3}{14} \times \frac{49}{8} \times \frac{5}{21}$ (b) $\frac{5}{9} \times \frac{22}{15} \div \frac{11}{27}$ (c) $-\frac{3}{5} \div -\frac{3}{5}$

4.9 Rank the following fractions by size, starting with the smallest.

$$\frac{19}{24} \qquad \frac{7}{12} \qquad \frac{2}{3} \qquad \frac{3}{4}$$

4.10 Calculate the following:

(a) $1\frac{1}{8} - 2\frac{5}{6} + 3\frac{1}{4}$ (b) $5\frac{1}{2} \div 1\frac{4}{11} \times 7\frac{1}{5}$

4.11 A baker works 6 days a week. Every day she uses $\frac{3}{5}$ of a sack of bread flour for regular loaves of bread. On Saturday she bakes special loaves in addition, and this needs an extra $\frac{4}{5}$ of a sack of flour.

(a) What is the total amount of flour used each week, expressed as a mixed fraction?

(b) If there is 20 kg of flour in a sack, how many kilograms of flour does the baker use each week?

4.12 A large group of friends ordered some pizzas to share. 1 person ate half a pizza, 3 people ate $\frac{3}{5}$ of a pizza each, and 7 people ate $\frac{2}{3}$ of a pizza each. What is the minimum number of pizzas the friends ordered?

4.13 Two children are given pocket money for helping with the gardening. Child 1 receives $\frac{2}{5}$ths of the money and immediately spends $\frac{3}{4}$ of his share on a magazine.

(a) What fraction of the original amount was spent on the magazine?

(b) If the total amount of money that was split between the children was $£12.50$, what was the price of the magazine?

4.14 A man is putting up some metal supports for a fence. A support is 4 ft in length and $\frac{3}{16}$ths of this is driven into the ground. At the top of the support the man screws in a decorative knob which is $3\frac{1}{2}$ inches tall. How high above ground level is the top of the knob?

4.15 A carpenter is making decorative panels. She cuts a panel out of a piece of wood and measures its length. The panel is $\frac{24}{50}$ m long, which is $\frac{8}{9}$ths of the length of the original piece of wood. How long was the original piece?

5 Decimals and Rational and Irrational Numbers

Decimals, based on powers of 10, show whole numbers to the left of the point and fractions of numbers to the right. Each step to the left increases the power of 10 by one. Each step to the right divides by 10.

Example 1 – Write two thousand and three hundredths in figures.

1000s	100s	10s	1s	•	$\frac{1}{10}$ths	$\frac{1}{100}$ths
2	0	0	0	•	0	3

2000.03

To convert a fraction into a decimal, use long division.

Example 2 – Convert $\frac{7}{8}$ to a decimal.

$\frac{7}{8}$ is equivalent to "7 divided by 8". The calculation is shown on the right.

$$\therefore \frac{7}{8} = 0.875$$

$$\begin{array}{rl} 0.875 & \\ 8\overline{)7.000} & \\ -\underline{6.4} & \leftarrow 8\times8=64 \\ 0.60 & \\ -\underline{0.56} & \leftarrow 8\times7=56 \\ 0.040 & \\ -\underline{0.040} & \leftarrow 8\times5=40 \\ 0.000 & \end{array}$$

Decimal numbers are either terminating or non-terminating. A terminating decimal, such as 26.18304, has a finite number of non-zero values after the decimal point, while a non-terminating decimal goes on forever.

Rational numbers are numbers that can be written as a fraction with integers in both the numerator and denominator. Terminating decimals are always rational. Example 3 shows how to convert a terminating decimal into an equivalent fraction.

Example 3 – Convert the number 3.04 into a fraction. Write your answer as a fraction in its simplest form.

3.04 is the same as $3 + 0.04$. The number 0.04 has two places after the decimal point. Therefore, 0.04 is equivalent to $4 \div 100$.

$$\therefore 3.04 = 3 + \tfrac{4}{100} = 3 + \tfrac{1}{25} = 3\tfrac{1}{25}$$

Recurring decimals have a pattern of digits after the decimal point that repeats forever. They can also always be written as fractions. For example, $0.33333... = \frac{1}{3}$.

The repeating part of a recurring decimal can be indicated using dots. Dots are placed over the first and last digits of the repeating unit. For example, $14.769769769... = 14.\dot{7}6\dot{9}$. If the repeating unit contains only one number, the dot is placed over a single digit. For example $0.66666... = 0.\dot{6}$.

Irrational numbers are numbers that cannot be written as fractions. The square roots of prime numbers are all irrational numbers, as is π. Integer multiples of irrational numbers, such as 2π, are always irrational.

5.1 Write as decimals:

(a) Sixty-two point four. (b) $5\frac{33}{100}$

(c) Five hundred and six and four thousandths.

5.2 Convert these fractions to decimals.

(a) $\frac{3}{4}$ (b) $\frac{34}{50}$ (c) $\frac{5}{2}$ (d) $\frac{27}{6}$

5.3 Convert the following decimals to fractions.

(a) 0.30 (b) 0.75 (c) 0.18

5.4 Convert the following decimals to mixed fractions.

(a) 1.16 (b) 2.07 (c) 5.72

5.5 Convert these fractions to recurring decimals.

(a) $\frac{2}{9}$ (b) $\frac{16}{11}$ (c) $\frac{3}{7}$

5.6 Here is a list of numbers: $-1, 7, \sqrt{2}, 3, \frac{5}{2}, \pi$.

(a) Identify the irrational numbers in the list.

(b) Find the product of the remaining numbers, giving your answer as a decimal.

5.7 Using $\frac{1}{9} = 0.\dot{1}$, deduce without using a calculator the recurring decimals for

(a) $\frac{2}{9}$ (b) $\frac{7}{9}$ (c) $\frac{1}{3}$ (d) $\frac{1}{18}$

5.8 Expressed in their simplest form, which of the following are irrational?

$$\frac{22}{7} \qquad \frac{\pi}{\pi} \qquad \sqrt{2}\times\sqrt{2} \qquad 6\pi \qquad \sqrt{25}$$

§ A method for turning a recurring decimal into an equivalent fraction is shown in example 4.

Example 4 – Convert the number $14.\dot{7}6\dot{9}$ into a fraction.

Let $x = 14.769769769...$

The repeating unit is three digits long. Therefore, multiply x by 10 three times; this is equivalent to multiplying by $10^3 = 1000$.

$$1000x = 1000 \times 14.769769769...$$
$$\Rightarrow 1000x = 14769.769769769...$$

Subtract x from both sides to eliminate the digits after the decimal point.

$$1000x - x = 14769.769769769... - 14.769769769...$$
$$\Rightarrow 999x = 14769 - 14 = 14755$$

Divide both sides by 999.

$$x = \frac{14755}{999} \quad \therefore 14.\dot{7}6\dot{9} = \frac{14755}{999} = 14\frac{769}{999}$$

5.9 Convert these recurring decimals to fractions.

(a) $0.\dot{4}\dot{5}$ (b) $0.\dot{1}463\dot{4}$ (c) $3.\dot{2}$

5.10 Convert these fractions to recurring decimals.

(a) $\frac{17}{9}$ (b) $\frac{5}{66}$ (c) $-\frac{5}{13}$

5.11 Use the information below to write each fraction as a decimal.

$$\frac{1}{11} = 0.\dot{0}\dot{9} \quad \frac{1}{7} = 0.\dot{1}4285\dot{7} \quad \frac{1}{13} = 0.\dot{0}7692\dot{3}$$

(a) $\frac{1}{33}$ (b) $-\frac{2}{33}$ (c) $\frac{1}{21}$

Write these decimals as fractions.

(d) $-0.\dot{9}\dot{0}$ (e) $0.\dot{4}2857\dot{1}$ (f) $0.\dot{3}0769\dot{2}$

6 Percentages

Fractions, percentages and decimals are closely related. Percentage means "out of one hundred", so 20% is the fraction $\frac{20}{100}$, which can be cancelled down to $\frac{1}{5}$. $\frac{1}{5}$ can be written as the decimal 0.2.

Example 1 – Express 38% as (i) a fraction (ii) a decimal.

(i) $38\% = \frac{38}{100} = \frac{19}{50}$ (ii) $38\% = \frac{38}{100} = 38 \div 100 = 0.38$

Example 2 – Express the following as percentages: (i) $\frac{17}{20}$ (ii) 0.348.

(i) $\frac{20}{20}$ would be 100%, so $\frac{17}{20} = \frac{17}{20} \times 100\% = 85\%$

(ii) The number of digits 0.348 has after the decimal point is 3.

$$\therefore 0.348 = \frac{348}{1000} \qquad \frac{348}{1000} \times 100\% = \left(\frac{348}{10}\right)\% = 34.8\%$$

To calculate percentage changes, you must decide if you are starting with 100% of a quantity, or if you are given a different percentage and must calculate 100%.

Example 3 – There is a 25%-off sale. What is the sale price of a coat which was originally priced at £120?

In this example you are starting with 100%. If 25% is taken off, you will pay 75% of the starting cost, or $\frac{75}{100}$ of £120. $\frac{75}{100}$ is a scale factor, so multiply.

$$\frac{75}{100} \times £120 = £90$$

Example 4 – In the same 25%-off sale, there is a pair of shoes priced at £48. What was the original price?

This time you know what 75% of the original price is. Scale down to find 1%, then scale up to find 100%.

$$75\% \xrightarrow{\div 75} 1\% \xrightarrow{\times 100} 100$$

So the multiplier is $\frac{100}{75}$, or $\frac{4}{3}$. The original price was $\frac{4}{3} \times £48 = £64$.

6.1 Write these percentages as fractions.

(a) 30% (b) 5% (c) 0.5%

6.2 Write these percentages as decimals.

(a) 25% (b) 0.7% (c) 0.003%

6.3 Write these decimals as percentages.

(a) 0.10 (b) 0.01 (c) 0.005 (d) 2.00

6.4 Write these fractions as percentages.

(a) $\frac{3}{4}$ (b) $\frac{5}{8}$ (c) $\frac{7}{350}$

6.5 Which is larger in each case?

(a) 21.5% or $\frac{9}{20}$ (b) $\frac{7}{6}$ or 1.16 (c) 112% or $\frac{20}{18}$

6.6 Rank the following in order of size, starting with the smallest.

$$62\% \qquad \frac{5}{8} \qquad 0.629$$

6.7 Evaluate the following:

(a) 20% of £16 (b) 65% of 400 g (c) 160% of \$240

6.8 Calculate

(a) 27% of £24 000 (b) 15% of 75 (c) 7.5% of 6 kg

6.9 Write the first quantity as a percentage of the second.

(a) 3 minutes out of 2 hours

(b) £1.40 out of £40.00

(c) 366 g out of 3 kg

6.10 A family needs to buy a new washing machine and have it delivered.

Company A sells a machine at 15% off an original purchase price of £275, and charges £25 for delivery.

Company B sells the same machine. Their usual price is £300, but the machine is on sale at 20% off. Delivery is free.

From which company would it be cheaper to buy the machine?

6.11 Rohit saves 5% when purchasing a coat originally costing $£100$, and 8% on a sofa which originally costs $£200$. What is his overall saving as a percentage?

6.12 Rank the following in order of size, starting with the largest.

$$\frac{17}{20} \qquad 87\% \qquad \frac{7}{8} \qquad 0.889 \qquad \frac{349}{400}$$

6.13 A new process reduces the time to manufacture a product by 25%. If it takes 2 hours 21 minutes to make the product using the new process, how long did it take to make the item with the old process?

In a simple interest scheme, the interest added to an account each year is calculated as a percentage of the original amount deposited. The same amount of interest is added each year.

Example 5 – $£250$ is invested in a simple interest scheme at an interest rate of 4% per year. Calculate the total amount of money at the end of 5 years.

The amount of interest paid after each year is 4% of the starting $£250$. This is

$$4\% \times £250 = \frac{4}{100} \times £250 = \frac{4 \times £250}{100} = \frac{£1000}{100} = £10$$

After 5 years the amount of interest paid is $5 \times £10 = £50$.
The total amount of money in the account is therefore

$$£250 + £50 = £300$$

In a compound interest scheme, the interest added to an account each year is calculated as a percentage of the amount in the account during that year. Assuming that no money is taken out, the amount of interest added increases from year to year.

If the interest rate is $I\%$ per year, then after one year the amount in an account is $(100 + I)\%$ of the original amount. This is equivalent to multiplying the starting amount by the fraction $\frac{100+I}{100}$.

Example 6 – £250 is invested in a compound interest scheme at an interest rate of 4% per year. Calculate the total amount of money at the end of 5 years to the nearest penny.

For an interest rate of 4%, the multiplier is $\frac{100+I}{100} = \frac{104}{100} = 1.04$.

To find the amount after 5 years, apply the multiplier 5 times. To the nearest penny the amount in the account is £250 $\times 1.04^5 =$ £304.16.

6.14 Calculate, to the nearest penny, the amount that will be in an account if

(a) £1 000 is invested for one year in a simple interest scheme with a 5% interest rate.

(b) £1 500 is invested for two years in a simple interest scheme with a 3% interest rate.

(c) £4 000 is invested for one year in a compound interest scheme with a 4% interest rate.

(d) £4 500 is invested for three years in compound interest scheme with a 2.5% interest rate.

6.15 For each of the percentages below, calculate the amount in an account after 10 years if £3 000 is invested in a scheme with (i) simple interest (ii) compound interest. Give your answers to the nearest penny.

(a) 0.1% (b) 1% (c) 3% (d) 6%

6.16 Dan uses a balance to find the mass of some objects. The machine has an offset error, so it registers a mass of 9.0 g even when there is no mass on it.
Find the percentage error in his measurements when he weighs objects which have a true mass of

(a) 90 g (b) 720 g (c) 36.0 g

6.17 A carpenter is cutting a long, thin plank of wood into shorter lengths using a band saw. The saw blade is 2 mm wide, so each cut wastes 2 mm of wood as sawdust.

(a) How many 10 cm lengths can they cut from a plank with a total length of 2 m?

(b) What percentage of the original plank will not be used to make 10 cm lengths?

(c) What percentage of the plank will be turned into sawdust?

6.18 The average attendance at a sporting fixture went up by 40% every year. In year 2 the attendance was $35\,000$. Find the attendance

(a) in year 3

(b) in year 1

6.19 Uranium has many isotopes. 99.274% of natural uranium is the isotope U_{238}. 720 out of $100\,000$ atoms of natural uranium are the isotope U_{235}. What percentage of natural uranium is accounted for by the other isotopes?

6.20 For the formula $P = IV$, what is the percentage change in P if the current I increases by 3.0% and the voltage V falls by 10.0% at the same time?

7 Ratio

A ratio compares the sizes of two or more quantities with each other.
A proportion is the amount of one quantity as a fraction of the total amount.

Example 1 – In a playground there are 10 boys, 12 girls and 8 adults.

(i) What is the ratio of girls to boys in its simplest form?
The ratio of girls to boys is $12 : 10$. This cancels down to $6 : 5$.

(ii) What proportion of the people in the playground are adults?
The total number of people in the playground is $10 + 12 + 8 = 30$. The proportion of of the people that are adults is $\frac{8}{30} = \frac{4}{15}$.

Example 2 shows how to share out an amount according to a given ratio.

Example 2 – Two friends, Peter and Alice, share out 220 flower bulbs in the ratio $2 : 3$. Work out how many bulbs they each receive.

Think of the ratio as dividing the bulbs into portions. If Peter receives 2 portions, and Alice 3 portions, there are $2 + 3 = 5$ portions overall.

The size of one portion is therefore $220 \div 5 = 44$ bulbs. Peter receives $2 \times 44 = 88$ bulbs, and Alice receives $3 \times 44 = 132$ bulbs.

7.1 Calculate the ratio of what each person has, giving your answer as the simplest possible ratio of whole numbers in the form A : B:

(a) A has 10 potatoes, B has 20 potatoes.

(b) A has 80 kg of coal, B has 60 kg of coal.

(c) A is paid £12.60 per hour, B is paid £14.40 per hour.

(d) A has 51 models, B has 68 models.

7.2 £120 is divided between two people, A and B. Calculate how much each person would get if the ratio of what they receive (£A : £B) is

(a) $1 : 1$ (b) $2 : 1$ (c) $3 : 2$ (d) $1.5 : 1$ (e) $11 : 13$

7.3 (a) A fizzy orange drink is made by mixing 1 part orange juice with 3 parts lemonade. What proportion of the drink is lemonade?

(b) The table shows how many grams of each ingredient are needed in two cherry cake recipes. Which contains the higher proportion of cherries?

	Flour	Sugar	Butter	Cherries	Milk
Recipe 1	200	100	100	50	50
Recipe 2	160	80	70	40	30

7.4 When it is painted onto new plaster, a certain type of emulsion paint has to be watered down in a ratio:

Volume of paint : Volume of water 10 : 1

Work out in litres how much water is needed for the following volumes of undiluted paint:

(a) 1 litre (b) 10 litres (c) 3.5 litres

(d) After one job, 4.4 litres of watered-down paint was left over. How much undiluted paint was wasted?

7.5 Blocks of printer paper are to be ordered for two offices as the stock has run out. Office 1 has 6 desks, and office 2 has 8 desks.

(a) What is the ratio of the number of desks in the two offices in its simplest form? Give your answer in the form Office 1 : Office 2.

(b) Each desk needs to be supplied with two blocks of paper. Blocks of paper come in multiples of 5. How many blocks of paper will be ordered?

(c) After putting 2 blocks of paper onto each desk, the left over paper is put into the store room. What fraction of the paper that is ordered ends up in the storeroom?

7.6 450 sweets are divided between three people, A, B and C. Calculate how many each person would get if the ratio of what they receive (A : B : C) is

(a) 1 : 1 : 1 (b) 3 : 1 : 1 (c) 2 : 5 : 2 (d) 22 : 0 : 23 (e) 1 : 3 : 14

Given the ratio between two quantities, and the value of one of them, it is possible to find the value of the other by scaling up or down. This is the basis of all conversion calculations such as converting units or currencies.

Example 3 – A man exchanges £120 for \$150 at a currency exchange.

(i) What is the exchange rate for turning £1.00 into dollars?

Fact: £120 = \$150

Scale factors:

$$\begin{array}{ccc} \text{£} & & \text{\$} \\ 120.00 & : & 150.00 \\ \div 120 \downarrow & & \downarrow \div 120 \\ 1.00 & : & 1.25 \end{array}$$

Answer: £1.00 = \$1.25

(ii) To the nearest penny, how many pounds would be equivalent to \$278.00?

Fact: £1.00 = \$1.25

Scale factors:

$$\begin{tikzcd} \text{£} \arrow[r, bend left, "\times 1.25"] & \text{\$} \arrow[l, bend left, "\div 1.25"] \end{tikzcd}$$

Answer: $\$278 = £(278 \div 1.25) = £222.40$

7.7 A currency converter shows that the ratio of £1 in the UK to the Japanese Yen on one particular day is £1 : ¥150.

(a) Convert £33 to Yen.

(b) Converting ¥2355 will give how many Pounds?

7.8 On a certain day the exchange rate between euros and pounds is €1.10 : £0.95. Giving your answers to the nearest penny or €0.01,

(a) Convert €12.60 to pounds.

(b) Convert £278.00 to euros.

7.9 A recipe for cheese biscuits requires 150 g of flour, 75 g of butter, and 50 g of grated cheese.

(a) Write the quantities as a ratio in its simplest form.

(b) How many 12.5 g biscuits will this recipe make?

(c) What fraction of each biscuit is flour? What fraction of each biscuit is butter? What fraction of each biscuit is cheese?

7.10 Two children are picking strawberries. In the time that child A picks 200 g, child B picks 300 g.

(a) Express this as a ratio A : B in the form $1 : n$.

(b) If child A picks x g, how many grams are picked by child B?

(c) If child A picks 1.2 kg, what does child B pick?

(d) If child B picks a total of 2.25 kg, what does child A pick?

7.11 The resolution of a digital display is limited by the number of pixels. Assuming that pixels are square, find in its simplest form the ratio $($screen width$)$: $($screen height$)$ for the following screens.

(a) A full HD screen has $1\,920$ pixels in the horizontal direction and $1\,080$ pixels in the vertical direction.

(b) A 4k screen has $3\,840$ pixels in the horizontal direction and $2\,160$ pixels in the vertical direction.

7.12 A mortar used for binding bricks together in a wall uses 4 parts sand to 1 part cement, as well as water and a liquid plasticiser. For the two solid ingredients:

(a) Write down the ratio of cement to sand.

(b) Give the quantities of cement and sand as (i) fractions and (ii) percentages of the overall amount of solid ingredients.

Concrete used in load-bearing situations contains an aggregate (such as gravel) as a solid ingredient in addition to sand and cement. A typical mix for concrete would be 2 parts sand, to 4 parts gravel, to 1 part cement.

(c) Write down the ratio cement : sand : gravel.

(d) What fraction of the solid ingredients is cement?

(e) Find, to 2 s.f., the difference between the percentage of cement in the concrete mix, and the percentage of cement in the mortar mix.

8 Rounding, Limits of Accuracy and Bounds

When the answer to a calculation is a decimal with a lot of digits, a decision has to be made about the accuracy to which the answer is stated, i.e. how many figures to write down.

Truncation, which is sometimes used by digital instruments such as calculators, is where all the decimal places after a cut-off point are simply discarded. For example, if 1.418263748 is truncated after the fourth decimal place, the digits $63748...$ are discarded to leave 1.4182.

Rounding is where a figure is approximated by the nearest value with the desired accuracy. Rounding may be to a set number of significant figures (s.f.) or decimal places (d.p.). The key principle is to look at one more digit than the desired accuracy requires. If the value of this digit is 5 or more, then when the number is rounded the final digit is rounded up by 1.

Example 1 - Round 2.5647 to (i) 3 significant figures (ii) 1 decimal place.

(i) The third significant figure is the 6. 2.5647 is between 2.56 and 2.57.

The fourth significant figure is a 4. It is not necessary to round up the third significant figure, because 2.5647 is closer to 2.56 than to 2.57.

$$\therefore 2.5647 = 2.56 \text{ to } 3 \text{ s.f..}$$

(ii) The first decimal place is the 5. 2.5647 is between 2.5 and 2.6.

The second decimal place is a 6. It is necessary to round up the first decimal place, because 2.5647 is closer to 2.6 than to 2.5.

$$\therefore 2.5647 = 2.6 \text{ to } 1 \text{ d.p..}$$

When rounding decimals where all the digits before the decimal point are zero, the first significant figure is the first non-zero digit after the decimal point.

Example 2 - Round 0.009984 to 3 significant figures.

The first significant figure is the first 9. The fourth significant figure is a 4, so it will not be necessary to round up. $0.009989 = 0.00999$ to 3 s.f..

When calculating a value from an expression or formula, it is important to round only the final answer. Rounding too early leads to a loss of accuracy, because subsequent steps in the calculation are using approximated values. Always state the number of significant figures to which you give an answer.

In real-world applications the precision of measurements is limited. The final answer to a calculation cannot be given to more significant figures than the values from which it is calculated. In general it is appropriate to find the value which is stated to the smallest number of significant figures, and give your answer to the same number of significant figures.

Example 3 - Use the formula $V = IR$ to find the value of V when $I = 0.5943$ A and $R = 1.2\ \Omega$.

Using the formula gives $V = 0.5943 \times 1.2 = 0.71316$ V. The value for I is given to 4 s.f., and the value of R is given to 2 s.f.. The answer should therefore be stated to 2 s.f..

$$V = 0.71\ \Omega \text{ to 2 s.f.}$$

8.1 Round the following to the stated accuracy:

(a) 81.63 to 2 s.f. (b) 0.0027356 to 3 s.f. (c) 0.49999 to 3 s.f.

8.2 Round the following to the stated accuracy:

(a) 6432 to 2 s.f. (b) 58.743 to 3 s.f. (c) 1.84338 to 2 s.f.

8.3 Round the following to the stated accuracy:

(a) 10.6845 (2 d.p.) (b) 10.6845 (1 d.p.) (c) 0.0347859 (2 d.p.)

8.4 Round the following to the stated accuracy:

(a) 0.0347859 (3 d.p.) (b) 0.00382 (2 d.p.) (c) -1.57864 (3 d.p.)

8.5 Find:

(a) 155.68 to the nearest integer

(b) -227.1848 to the nearest 10

(c) $43\,608$ to the nearest $1\,000$

(d) 0.0054545 to the nearest 1000^{th}

8.6 Here is a list of values

$$\frac{3}{4} \qquad \sqrt{3} \qquad \frac{5}{8} \qquad 6 \qquad 2\pi$$

(a) Which of these values cannot be written as terminating decimals?

(b) Which values would need to be rounded if you were asked to write each value as a decimal number correct to 2 significant figures?

(c) How many values would need to be rounded if you were asked to write each value as a decimal number correct to 3 significant figures?

8.7 Two students try the following question: "Calculate the volume of a sphere of diameter 2.30 cm using the formula $V = \frac{4}{3}\pi r^3$. Give your answer correct to 2 significant figures."

(a) Student A sees that this is a 2 significant figures question and rounds the value of the radius before using the formula. He then also rounds the answer he gets from the formula. What value does he get for the volume?

(b) Student B uses all the figures for the radius, and only rounds the final answer. What value do they get for the volume?

(c) Which student calculates the volume correctly?

(d) What is the percentage error of the answer of the student who calculates incorrectly?

Stating a quantity to a set number of significant figures or decimal places means that the quantity lies within a range of values which would all round to the same answer. This range is called an error interval and can be expressed as an inequality between upper and lower bounds (limits).

In Example 4 the lower bound is 2.475 mm, and the upper bound is 2.485 mm. The inequality signs for the two bounds are different; p can be equal to the lower bound, but cannot be equal to the upper bound.

Example 4 - The length of an object, l, is 2.48 mm to 3 significant figures. Write down the error interval for l as an inequality.

When rounding to 2 significant figures, a value of l below 2.475 mm would round down to 2.47 mm, and a value of 2.485 mm would round up to 2.49 m.

Therefore, 2.475 mm $\leqslant p < 2.485$ mm.

§ When dealing with a quantity that has to be an integer, there are two ways of writing the upper limit of the inequality.

Example 5 - The number of people in a stadium, p, is $85\,000$, to 2 significant figures. What are the minimum and maximum values of p? Write down the range of values for p as an inequality.

The minimum value of p is $84\,500$. If there were any fewer people, p would round down to $84\,000$ when rounding to 2 significant figures.

The maximum value of p is $85\,499$. If there were any more people, p would round up to $86\,000$ when rounding to 2 significant figures.

When expressed as an inequality, $84\,500 \leqslant p \leqslant 85\,499$.

The alternative inequality is $84\,500 \leqslant p < 85\,500$.

8.8 Write error intervals for the following rounded values.

(a) $l = 12$ m to 2 s.f.
(b) $v = 31.4$ m/s to 3 s.f.
(c) $V = 4.78$ litres to 2 d.p.
(d) $f = 6.178$ Hz to 3 d.p.
(e) $n = 60$ to 1 s.f.
(f) $m = 22.3$ g to 1 d.p.

8.9 (a) Give the maximum and minimum numbers of people that these rounded values could represent:

(i) 420 to 2 s.f. (ii) $2\,200$ to 2 s.f.

(b) State, as an inequality, the number of people in a crowd with a size of $29\,500$ to 3 s.f..

8.10 Find the upper and lower bounds of the following quantities:

(a) 120 km to 2 s.f. (b) 2.0 minutes to 1 d.p. (c) 51.2 g to 3 s.f.

§ When more than one variable is used in a calculation, upper and lower bounds for the final answer are found by considering the upper and lower bounds of the individual variables from which the answer is calculated.

Example 6 - To one decimal place, $a = 11.1$ and $b = 14.5$. Find
(i) Find the upper bound of $a + b$. (ii) The lower bound of $a - b$.

The error intervals for a and b are $11.05 \leqslant a < 11.15$ and $14.45 \leqslant b < 14.55$.

(i) The upper bound of $a + b$ is found by adding the upper bounds of a and b. The upper bound of $a + b$ is $11.15 + 14.55 = 26.7$.

(ii) The lower bound of $a - b$ is found by subtracting the upper bound of b from the lower bound of a. The lower bound of $a - b$ is $11.05 - 14.55 = -4.5$.

8.11 To two significant figures, $a = 150$ and $b = 720$. Find:
(a) The upper bound of $a + b$.
(b) The lower bound of $a + b$.
(c) The upper bound of $a - b$.
(d) The lower bound of $a - b$.

8.12 To one decimal place, $a = 5.3$ and $b = 2.9$. Find, giving your answer to 3 s.f.:
(a) The upper bound of $a \times b$.
(b) The lower bound of $a \div b$.

8.13 A cuboid has sides of length 1.58 m, 2.39 m and 6.44 m. Giving your answers to 4 significant figures, what are
(a) its maximum volume?
(b) its minimum surface area?

8.14 Find the upper and lower bounds of G, where $G = 2p + q$ and p and q have the values $p = 1.6$ cm (1 d.p.) and $q = 7.4$ cm (1 d.p.).

9 Approximation

To estimate the answer to a calculation, proceed through the calculation in BIDMAS order, at each stage using numbers that are close to the true values but easier to work with. When estimating the signs $\approx$ or $\simeq$ are used, which mean "approximately equal to."

Example 1 – Estimate the value of the calculation $\frac{102\times13.5}{8.2+1.6}$.

Replace 102 by 100, 13.5 by 14, 8.2 by 8 and 1.6 by 2.

$$\frac{102\times13.5}{8.2+1.6}\approx\frac{100\times14}{8+2}\quad\frac{100\times14}{10}=\frac{100}{10}\times14=10\times14=140$$

When doing a difficult calculation, it is a good idea to use estimation to find a rough value for the answer to compare with your accurate solution. This will help you spot mistakes, such as mis-placed decimal points. This is particularly important if you are doing calculations on a calculator.

9.1 Estimate

(a) $\frac{3}{5}$ of 24 (b) $\frac{5}{8}$ of 38 (c) 53% of £25.50

9.2 Estimate the values of the following:

(a) $10(4.3+7.1)$ (b) $\frac{2.1\times5.1}{12.3}$ (c) $\frac{\sqrt{4.1^2+3.1^2}}{2.1}$

9.3 Estimate the values of the following:

(a) $\frac{8.9\times6.1}{5.1\div2.2}$ (b) $\frac{7.28+5.01}{9.23-5.09}$ (c) 75% of £179

9.4 By estimating an answer, work out which value is correct for each of the following calculations:

(a) 62×51	(i) 2902	(ii) 3162	(iii) 3692
(b) $1+1.21\div5.1$	(i) 1.01	(ii) 1.24	(iii) 1.502
(c) $\frac{62\,500-1700}{3.1}$	(i) 1846	(ii) 9723	(iii) 19 613
(d) $32+(\frac{202}{51})^3$	(i) 94.1	(ii) 150.7	(iii) 278.1

9.5 Estimate the following:

(a) The time needed to put 12 screws into a piece of wood if each screw takes an average of 5.2 seconds.

(b) The mass in grams of one tin of beans if a 4-pack has a mass of 1.63 kg.

(c) The distance travelled in 22 minutes at 50 miles per hour.

9.6 Suzie is buying food for a barbecue. She needs 60 rolls and 60 sausages. Rolls come in packs of 6, each pack costing 82 p. Sausages come in packs of 8, costing £2.95.

(a) Estimate how much Suzie will have to spend.

(b) How many £10 notes should Suzie withdraw at a cashpoint to cover the cost?

9.7 Estimate the income of a theatre that sells the following tickets for one of its performances:

Price per seat / £	Number sold
15.50	41
27.50	122
58.00	285
72.00	53

9.8 A quantity surveyor estimates the cost of materials for a building project. 102 houses are to be built. There will be a few Large properties. The rest will be a mixture of Medium and Small properties. There will be more of the Medium size than the Small size.

(a) Which of these options best fits this description?

(i) 25 Large houses, 45 Medium and 32 Small
(ii) 6 Large houses, 48 Medium and 48 Small
(iii) 5 Large houses, 57 Medium and 50 Small

(b) If the cost of materials is £$110\,000$ for a Large house, £$73\,000$ for a Medium house and £$58\,000$ for a Small house, estimate the total cost of materials for building all the houses.

10 Standard Form

Standard form is useful for writing very large or small numbers compactly. To write a number in standard form, express it as $a \times 10^b$, where $1 \leqslant a < 10$ and b is an integer.

Example 1 - Write (i) 253.84 (ii) -0.003578 in standard form.

(i) $253.84 = 2.5384 \times 10^2$ (ii) $-0.003578 = -3.578 \times 10^{-3}$

2 1
253.84

1 2 3
-0.003578

Numbers such as 31×10^3 and 0.87×10^{-3} contain powers of 10 but are not in standard form because 31 and 0.87 are not in the range $1 \leqslant a < 10$. To write them in standard form, first write 31 and 0.87 in standard form. Then combine the powers of 10 by adding them together.

Example 2 - Write (i) 31×10^3 (ii) 0.87×10^{-3} in standard form.

(i) In standard form 31 is 3.1×10^1.

$$\therefore 31 \times 10^3 = 3.1 \times 10^1 \times 10^3 = 3.1 \times 10^4$$

(ii) In standard form 0.87 is 8.7×10^{-1}.

$$\therefore 0.87 \times 10^{-3} = 8.7 \times 10^{-1} \times 10^{-3} = 8.7 \times 10^{-4}$$

10.1 Put the following numbers into standard form:
(a) 11.7 (b) 0.5 (c) 123 (d) 0.0021

10.2 Put the following numbers into standard form:
(a) $1\,103\,457$ (b) 0.000456 (c) 7310.08 (d) 0.00000315

10.3 Write these values as ordinary numbers:
(a) 2.6×10^4 (b) 4.9×10^{-2} (c) 5.17×10^3 (d) 8.8×10^{-6}

10.4 Put the following numbers into standard form:
(a) 47×10^2 (b) 12.34×10^6 (c) 0.56×10^{-5} (d) 101×10^{-3}

10.5 Write the following quantities in units of metres in standard form:
(a) 900 km (b) 0.01 mm (c) 20×10^{-5} m (d) $1\ \mu$m

When adding or subtracting numbers in standard form, start by re-writing the numbers in terms of the same power of ten. Next, do the addition or subtraction. Finally, re-write the numbers in standard form if necessary.

Example 3 - Evaluate, giving your answers in standard form:
(i) $3.2 \times 10^3 + 2.9 \times 10^4$ (ii) $1.1 \times 10^{-2} - 9.8 \times 10^{-3}$

(i) 2.9×10^4 is the same as 29×10^3.

$$\therefore 3.2 \times 10^3 + 2.9 \times 10^4 = 3.2 \times 10^3 + 29 \times 10^3 = 32.2 \times 10^3$$

In standard form, 32.2×10^3 is 3.22×10^4.

(ii) 1.1×10^{-2} is the same as 11×10^{-3}.

$$\therefore 1.1 \times 10^{-2} - 9.8 \times 10^{-3} = 11 \times 10^{-3} - 9.8 \times 10^{-3} = 1.2 \times 10^{-3}$$

1.2×10^{-3} is already written in standard form.

When multiplying numbers in standard form, multiply the numbers in front of the powers of 10, and combine the powers of the powers of 10 by adding the powers. Likewise, when dividing numbers in standard form, divide the numbers in front of the powers of 10, and subtract the powers of 10. Finally, check that the answer is in standard form.

Example 4 - Evaluate, giving your answers in standard form:
(i) $(7.5 \times 10^2) \times (2.0 \times 10^3)$ (ii) $(8.2 \times 10^4) \div (2.0 \times 10^2)$

(i)
$$\begin{aligned}(7.5 \times 10^2) \times (2.0 \times 10^3) &= (7.5 \times 2.0) \times (10^2 \times 10^3)\\ &= 15 \times 10^5 = 1.5 \times 10^6\end{aligned}$$

(ii)
$$\begin{aligned}(8.2 \times 10^4) \div (2.0 \times 10^2) &= (8.2 \div 2.0) \times (10^4 \div 10^2)\\ &= 4.1 \times 10^2\end{aligned}$$

Evaluate the following without using a calculator, giving your answers in standard form:

10.6 (a) $7.3 \times 10^2 + 4.8 \times 10^3$ (c) $4.2 \times 10^{-2} - 1.1 \times 10^{-3}$
(b) $9.7 \times 10^4 + 9.7 \times 10^3$ (d) $5.27 \times 10^5 - 2.8 \times 10^4$

10.7 (a) $(5 \times 10^2) \times (6 \times 10^3)$ (c) $(6.2 \times 10^2) \div (3.1 \times 10^3)$
(b) $(1.8 \times 10^6) \times (3 \times 10^3)$ (d) $(5.4 \times 10^4) \div (2 \times 10^{-2})$

10.8 (a) $1.6 \times 10^3 + 4.0 \times 10^4$ (c) $(2.5 \times 10^2) \times (5.0 \times 10^6)$
(b) $2.6 \times 10^{-4} - 1.7 \times 10^{-5}$ (d) $(4.2 \times 10^3) \div (1.4 \times 10^{-2})$

10.9 Evaluate the following with a calculator. Give your answers in standard form, using 3 s.f. where rounding is necessary.

(a) $2500 + 4.0 \times 10^3$ (c) $(2.81 \times 10^3) \div 9.81$
(b) $3.3 \times (5.0 \times 10^6)$ (d) $\sqrt{\frac{8.0 \times 10^6}{4.1}}$

10.10 Evaluate the following without using a calculator, giving your answers in standard form:
(a) $3.3 \times 10^{-4} + 1.9 \times 10^{-2}$ (c) $(2.6 \times 10^{-2}) \times (4.3 \times 10^{-7})$
(b) $2.9 \times 10^{-6} - 5.7 \times 10^{-5}$ (d) $(8.0 \times 10^{-3}) \div (6.4 \times 10^{-9})$

10.11 A ream of paper contains 500 sheets. Fabi measured the height of a ream of A4 printer paper stacked vertically to be 4.9 cm.

(a) In millimetres, what is the thickness of one sheet of paper?

(b) Write the value from part (a) in metres in standard form.

10.12 The number of seconds in a year is approximately $\pi \times 10^7$. Using this approximation, find the number of seconds in 250 years, giving your answer in standard form to 2 s.f..

10.13 Avogadro's number, 6.02×10^{23}, is the number of molecules in 1 mole of a substance. How many molecules are in 0.05 moles? Give your answer in standard form.

10.14 (a) Write $(\frac{1}{2} \times 10^9) \times (5 \times 10^8)$ in standard form.

(b) You are told that 6×10^8 is the same as $(3 \times 10^n) \times (2 \times 10^5)$. What is the value of n?

10.15 Round the following to the nearest power of 10. State the power.
(a) $0.6 \times 10^7 \times 4.5 \times 10^8$ (b) $(2 \times 10^6) \div (4.5 \times 10^{-3})$

10.16 On the atomic scale energies are measured in a unit called the electron-Volt (eV). $1 \text{ eV} = 1.602 \times 10^{-19}$ J. The energy levels of the electron in a Hydrogen atom are given by the formula below, where n is an integer. Find the energies E_1, E_2, E_3 and E_4 in Joules. Give your answers in standard form to 2 s.f..

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

11 Units

The table below shows a list of common unit prefixes.

n	μ	m	c	d	-	k	M	G
nano	micro	milli	centi	deci	-	kilo	mega	giga
$\times 10^{-9}$	$\times 10^{-6}$	$\times 10^{-3}$	$\times 10^{-2}$	$\times 10^{-1}$	-	$\times 10^{+3}$	$\times 10^{+6}$	$\times 10^{+9}$

For example, 1 cm $= 1 \times 10^{-2}$ m. This is equivalent to 1 cm $= 1 \times \frac{1}{100}$ m.

When converting from one set of units to another, write down a fact for each quantity you need to convert, turn that into a scale factor, and then apply this factor. For example, the fact 1 cm $= 1 \times \frac{1}{100}$ m tells us that to convert a quantity in centimetres to a quantity in metres, divide by 100. To go the other way, from a quantity in metres to a quantity in centimetres, multiply by 100. The scale factor is 100.

$$\text{m} \underset{\div 100}{\overset{\times 100}{\rightleftarrows}} \text{cm}$$

Example 1 – Convert 3 cm into metres.

Fact: $1\,\text{m} = 100\,\text{cm}$ Scale factor: $\text{m} \underset{\div 100}{\overset{\times 100}{\rightleftarrows}} \text{cm}$

Answer: $3\,\text{cm} = 3 \div 100\,\text{m} = 0.03\,\text{m} = 3.0 \times 10^{-2}\,\text{m}$

Sometimes you may need more than one scale factor. This often happens with time problems.

Example 2 – Change 2.5 hours into seconds.

Fact: 1 hour $= 60$ minutes 1 minute $= 60$ s

Scale factors: $\text{hours} \underset{\div 60}{\overset{\times 60}{\rightleftarrows}} \text{min} \underset{\div 60}{\overset{\times 60}{\rightleftarrows}} \text{s}$

Answer: 2.5 hours $= 2.5 \times 60 \times 60$ s $= 9000$ s

You may have a fact in which neither quantity is exactly 1 unit. In this case, scale down the fact to make one quantity exactly 1 unit in size, and then proceed as in the previous examples.

Example 3 – Some old scales read 8.0 lb (pounds) when a cat is put on them. What is this in kilograms? 14.00 lb (one stone) is 6.3503 kg.

Fact: $14.0\ \text{lb} = 6.3503\ \text{kg}$

Divide by 14.00 to find how to convert 1.000 lb into kilograms.

Read off the scale factor.

$$\begin{array}{cc} \text{lb} & \text{kg} \\ 14.00 & 6.3503 \\ \downarrow \div 14 & \downarrow \div 14 \\ 1.000 & 0.4536 \end{array}$$

$$\begin{tikzcd} \text{lb} \arrow[r, bend left, "\times 0.4536"] & \text{kg} \arrow[l, bend left, "\div 0.4536"] \end{tikzcd}$$

Answer: $8.0\ \text{lb} = (8.0 \times 0.4536)\ \text{kg} = 3.7\ \text{kg}$ to 2 s.f.

When dealing with units which involve powers other than one, or when handling compound units, convert each power of each unit in turn.

Example 4 – Convert $3\ \text{m}^3$ to cm^3.

Fact:

$$\begin{aligned} 1\ \text{m} &= 100\ \text{cm} \\ \therefore 1\ \text{m}^3 &= (100\ \text{cm})^3 = (100)^3\ \text{cm}^3 \\ \Rightarrow 1\ \text{m}^3 &= 100 \times 100 \times 100\ \text{cm}^3 \\ \Rightarrow 1\ \text{m}^3 &= 10^6\ \text{cm}^3 \end{aligned}$$

$$\begin{tikzcd} \text{m}^3 \arrow[r, bend left, "\times(10)^6"] & \text{cm}^3 \arrow[l, bend left, "\div(10)^6"] \end{tikzcd}$$

Answer: $3\ \text{m}^3 = 3 \times 10^6\ \text{cm}^3$

Example 5 – Convert a density of $3\ \text{kg/m}^3$ to g/cm^3.

Fact: $1\ \text{kg} = 1000\ \text{g}$ $\quad 1\ \text{m} = 100\ \text{cm}$

Scale factors:

$$\begin{tikzcd} \text{kg} \arrow[r, bend left, "\times 1000"] & \text{g} \arrow[l, bend left, "\div 1000"] \end{tikzcd} \qquad \begin{tikzcd} \text{m}^3 \arrow[r, bend left, "\times(10)^6"] & \text{cm}^3 \arrow[l, bend left, "\div(10)^6"] \end{tikzcd}$$

Answer: $3\ \text{kg/m}^3 = 3\ \frac{kg}{m^3} = 3 \times \frac{1000}{10^6}\ \frac{g}{cm^3} = 3 \times \frac{10^3}{10^6}\ \frac{g}{cm^3}$

$\therefore 3\ \text{kg/m}^3 = 3 \times 10^{-3}\ \text{g/cm}^3$

Note: You may see units with a denominator part written in an alternative form with a negative index. For example, the units of speed, m/s, may also be written m s^{-1}, and the units of density, kg/m^3, may also be written kg m^{-3}.

11.1 Write:
(a) 2 300 g in kilograms
(b) 0.002 g in milligrams
(c) 0.6 kg in grams
(d) 150 mg in grams

11.2 Write:
(a) 15 cm in metres
(b) 3.48 km in metres
(c) 67.2 km in centimetres
(d) 0.1 cm in kilometres

11.3 Write:
(a) 0.15 kg in grams
(b) 2.5 hours in seconds
(c) 365 days in seconds

11.4 Write:
(a) 3.31 mm in metres
(b) 15.2 μm in metres
(c) 60 km in millimetres
(d) 0.12 cm in nanometres

11.5 Perform the following unit conversions:
(a) 30 km/s to m/s (b) 16 N/m^2 to kN/m^2 (c) 15 000 Hz to MHz

11.6 Perform the following calculations, giving your answer in the units stated and to the given accuracy:
(a) 3.875×0.985 V, to 3 s.f., in units of V.
(b) 51.85 cm $\times$ 98.75 cm, to 3 s.f., in units of cm^2.
(c) 106.75 m^3 $\div$ 43.1, to 2 s.f., in units of m^3.

11.7 Concentrations are often written as values in moles per litre. An example is 4 moles/litre. 1 litre $=$ 1 dm^3, and 1 dm $=$ 10 cm. Convert 4 moles/litre into units of
(a) moles/m^3.
(b) moles/cm^3.

11.8 Perform the following unit conversions. In this question you will need to know that 1 mile $=$ 1 609.34 metres, "mph" stands for miles per hour, "kph" for kilometres per hour, and "m/s" for metres per second. Give your answers to 2 significant figures.
(a) 15 kph to m/s
(b) 33 m/s to kph
(c) 16 mph to m/s
(d) 64 kph to mph

11.9 A student claims that 15 cm is exactly the same as 6 inches. A conversion calculator states that 50 cm $=$ 1.64042 feet. What is the percentage error in the student's claim? Give your answer to 2 significant figures.

11.10 Perform the following calculations, giving your answer in the units stated and to the given accuracy:

(a) 0.02511 cm $\times$ 78.34 cm, to 3 s.f., in units of m^2.

(b) 91.25×0.00006751 V, to 2 s.f., in units of mV.

(c) Find, to 2 significant figures, the distance in metres covered by a car which travels for 45 minutes at 15 km per hour.

11.11 Put ticks in the table to show whether these expressions represent lengths, areas, volumes, or none of these. You are told that r and l have units of metres.

Expression	Length	Area	Volume	None of these
$\frac{4}{3}\pi r^3$				
$\pi r l$				
$\pi(r+l)$				
πr^2				
$r+l$				

11.12 Put ticks in the table to show whether these expressions represent lengths, areas, volumes, or none of these. You are told that π, p, q and s are unitless constants, and k, l and m have units of metres.

Expression	Length	Area	Volume	None of these
πm^2				
$plmk$				
$\frac{qm^2}{k}$				
$\frac{s\pi l}{mk}$				
$\frac{pqs}{2}$				

Algebra

12 Writing and Using Algebra

Algebra has its own terminology:

$4\pi x^2$	A term is made up of constants (such as 4 and π) and variables (such as x). Together the leading constants are known as the coefficient (here 4π). Like terms contain the same variables, to the same powers. They differ only in the values of their coefficients.
$4\pi x^2 - 3x$	An expression is made up of terms linked by operators ($+, -, \times, \div$).
$x - 5 = 4\pi x^2 - 3x$	An equation links two expressions with an equals sign.

If an equation is true for all possible values of the variable it is called an identity, and the $\equiv$ sign may be used. One side of an identity is effectively just a re-arrangement of the other. For example, $(2x+3) - 4 \equiv 2x - 1$.

$7x^2 - 3x + 2$ and $5x^3 + 2x$ are polynomials. Polynomials have terms which contain only constants and positive, whole number powers of variables, linked together by addition or subtraction.

- If the highest power of the variable is 1, the polynomial is linear. For example, $2x - 5$ and $y + 3$.
- If the highest power of the variable is 2, the polynomial is quadratic. For example, $7x^2 + 4x - 2$ and $y^2 - 9$.
- If the highest power of the variable is 3, the polynomial is cubic. For example, $9x^3 + 2x^2 - 7x$ and $y^3 - 4$.

Writing algebra involves replacing words with variables, constants and operators. Brackets are often needed to ensure that applying the rules of BIDMAS when doing calculations will give correct answers.

Example 1 - Write the following as an equation:

"To find Q, add 6 to p, then divide by 5."

First add 6 to p: $p+6$ Next divide everything by 5: $\dfrac{p+6}{5}$

This is equal to Q: $Q = \dfrac{p+6}{5}$

12.1 (a) Write the following as an equation: "To find y, multiply x by four then subtract three."

(b) When $x = 5$ what is y?

12.2 (a) Write the following statement as an algebraic equation: "y is found by adding eight to six x."

(b) Find y if $x = 10$.

(c) Find y if $x = -5$.

12.3 A child says "Two p and three q make z."

(a) Write this statement as an equation.

(b) Find z if $p = 9$ and $q = -7$.

12.4 The costs of pieces of fruit are: apple 30 p, pear 35 p, banana 28 p and orange 25 p.

(a) Write an equation to find the total cost, C p, of d apples, e pears, f bananas and g oranges.

(b) What is the change from £10.00 if $d = 4$, $e = 4$, $f = 7$ and $g = 6$?

12.5 A gardener walks up and down his garden sowing seeds. The garden has length L, and he makes twelve round trips. In total he walks 336 m.

(a) Write an equation for this information.

(b) What is the length of the garden, L?

Superscripts and subscripts perform different roles. A superscript, such as the 2 in x^2, is used to indicate that a number or variable is raised to a power. Subscripts are used purely as labels. For example, the initial speed of a vehicle might be written as v_0, v_S or even v_{Start}. Numbers in subscripts are part of the label, and do not indicate that a mathematical operation is taking place.

Example 2 - the velocity of a car at time t is given by

$$v_t = v_0 + at$$

where v_0 is the initial velocity of the car and a is the acceleration. Find the value of v_t when $v_0 = 5$ m/s, $a = 2$ m/s^2 and $t = 8$ s.

$$v_t = 5 + 2 \times 8 = 5 + 16 = 21 \text{ m/s}$$

12.6 Using the equation $v_t = v_0 + at$, find v_t if

(a) $v_0 = 0$ m/s, $a = 3$ m/s^2 and $t = 10$ s.

(b) $v_0 = 50$ mm/s, $a = 2$ mm/s^2 and $t = 4$ s.

(c) $v_0 = 0.7$ km/s, $a = -0.04$ km/s^2 and $t = 10$ s.

12.7 (a) If R is the number of rabbits now, and R_0 is the number of rabbits originally, write an equation for the statement "The number of rabbits now is twice the starting number of rabbits, minus 10 which have been sold."

(b) Find R if $R_0 = 210$.

Greek letters are commonly used in algebra in mathematics and the sciences. They can be manipulated in exactly the same way as Roman letters such as x and y. The table below shows those that are used most often and their names. On the left are lower case letters, and on the right are a smaller number of upper case letters.

	Name		Name		Name		Name
α	alpha	θ	theta	ρ	rho	Δ	delta
β	beta	λ	lambda	σ	sigma	Λ	lambda
γ	gamma	μ	mu	ϕ	phi	Σ	sigma
δ	delta	ν	nu	ω	omega	Φ	phi
ϵ, ε	epsilon	π	pi			Ω	omega

Example 3 - The resistance of a piece of wire, R, is equal to the resistivity of the wire ρ multiplied by the length of the wire l and divided by the wire's cross-sectional area A.

Multiply the wire's resistivity by its length. $\rho \times l$

Next divide by the cross-sectional area. $\dfrac{\rho \times l}{A}$

This is equal to the wire's resistance. $R = \dfrac{\rho \times l}{A}$

12.8 Λ is equal to ϕ minus ω.

(a) Write an equation for Λ.

(b) Find Λ for $\phi = 45^\circ$ and $\omega = 15^\circ$.

12.9 (a) Write this information as an equation: "To find γ start with 24 and subtract 4 times α, then divide the answer by 3."

(b) Find γ when $\alpha = 3$.

Simplifying "tidies up" algebra into a neater form. Simplifying includes collecting like terms together; using the rules of indices to combine different powers of a variable; and cancelling a common factor in the numerator and denominator of a fraction.

Example 4 - Simplify $\frac{1}{2} \times 4x^2 + 2p + 3\frac{p^2}{p}$.

The first term can be simplified by multiplying the $\frac{1}{2}$ and the 4 together. The third term can be simplified by cancelling a factor of p in the top and bottom of the fraction. Finally, combine like p terms.

$$\frac{1}{2} \times 4x^2 + 2p + 3\frac{p^2}{p} = 2x^2 + 2p + 3p = 2x^2 + 5p$$

In general it is good practice to simplify algebra whenever possible, even if not explicitly asked to do so.

12.10 Simplify:

(a) $3\alpha + 2\alpha$

(b) $5\lambda - \pi - 2\pi - \lambda$

(c) $M = M_0 + 3m + 5m - 6m + 4m$

12.11 Simplify:

(a) $3p - 6s + 2t - p + s$

(b) $\frac{3}{4}vw + \frac{1}{4}vw$

(c) $fg + gf + 2hj + jh$

12.12 Simplify:

(a) $2p \times 3q^2r + 4r \times 2pq^2$

(b) $\frac{1}{2} \times 2x^9 \div x^7 - 2x + x^2 + 20x$

12.13 A bar-tender is counting cans for stock-taking. He has x 4-packs, y 12-packs and z single cans.

(a) Write this information as an equation to find the total number of cans T.

(b) What is T if $x = 11$, $y = 10$ and $z = 7$?

12.14 A postman delivers mail to four houses. House 1 receives $3s$ letters and t parcels. House 2 receives $7s$ letters. House 3 receives $5s$ letters and $2t$ parcels. House 4 receives t parcels.

(a) Write an equation for the total number of items the four houses receive, N. Simplify your answer as far as possible.

(b) Assuming that the cost to send a letter is 80 pence and the cost to send a parcel is £5.50, write an equation for C, the total cost in pounds to send all the items that were delivered.

12.15 A quantity called the discriminant is used in the calculation of solutions of quadratic equations.

(a) Using δ for the discriminant, write the following as an equation: "The discriminant is found by subtracting four times a times c from the square of b."

(b) Find δ if $b = 16$, $a = 1$ and $c = 4$.

(c) Find δ if $b = 100$, $a = 3$ and $c = 7$.

12.16 Write the following statements in algebra.

(a) α is twice β.

(b) α cubed is the same as γ squared.

$\beta = 2$ and γ is a positive integer.

(c) Find the value of γ.

13 Indices and Taking Roots

The table below illustrates the meaning of positive and negative indices (powers).

a^{-3}	a^{-2}	a^{-1}	a^0	a^{+1}	a^{+2}	a^{+3}
$\frac{1}{a\times a\times a}$	$\frac{1}{a\times a}$	$\frac{1}{a}$	1	a	$a \times a$	$a \times a \times a$

- Increasing the power of a by 1 is equivalent to multiplying by a, and decreasing the power of a by 1 is equivalent to dividing by a.
- Anything raised to the power 0 has the value 1, i.e. $a^0 = 1$. The exception is 0^0, the value of which is undefined.
- Negative indices indicate reciprocals. A number can be written in the numerator with a negative power or in the denominator with a positive power. For example, $3^{-2} = \frac{1}{3^2}$.

The rules for indices show how to combine powers of a single number:

- When multiplying powers, indices add:

$$2^3 \times 2^5 = 2^{3+5} = 2^8 \qquad a^m \times a^n = a^{m+n}$$

- When dividing powers, indices subtract:

$$3^4 \div 3^3 = 3^{4-3} = 3^1 \qquad a^m \div a^n = a^{m-n}$$

- When raising a power to a power (”power on power”), indices multiply:

$$(2^3)^4 = 2^{3\times4} = 2^{12} \qquad (a^m)^n = a^{m\times n}$$

13.1 (a) Express the following in index form with a single power:
(i) $2^2 \times 2^3$ (ii) $3^9 \div 3^7$ (iii) $5^{-2} \times 5^4$

(b) Evaluate the following, writing your answers without indices:
(i) $2^2 \times 2^3$ (ii) $3^9 \div 3^7$ (iii) $5^{-2} \times 5^4$

Simplify the following, giving your answers in index form:

13.2 (a) $2^4 \times 2^{-6}$ (b) $3^3 \div 3^4$ (c) $5^{-3} \times 5$

13.3 (a) $10^{-1} \times 10^{-1}$ (b) $10^{-1} \div 10^1$ (c) $3^{-1} \times 9$

13.4 (a) $a^2 \times a^3$ (b) $a^7 \div a^8$ (c) $\frac{a^1}{a^5}$

13.5 (a) $a^2 \times a^{-5}$ (b) $(a^3)^2 \div a$ (c) $3(a^2)^2 \div 6a$

13.6 (a) $3^0 \div 3$ (b) $10^2 \times 10^3 \times 10^{-1}$ (c) $2^2 \times 2^{-1} \times 2^4$

13.7 (a) $(10^2)^2$ (b) $(10^2)^3$ (c) $(10^{-2})^3$ (d) $\frac{10^3}{10^6}$ (e) $\frac{10^3}{(10^2)^3}$

13.8 (a) $a^6 \times a^{-4} \times a^3$ (b) $a^2 \times a^3 \div a$ (c) $\frac{3(a^4)^2}{a}$ (d) $((a^2)^2)^3$

13.9 A quantity p is found by multiplying the value of x^2 by the value of x^3 and then dividing by 2.

(a) Write this information algebraically, and simplify as far as possible.

(b) Evaluate p when $x = 3$.

13.10 (a) A quantity F is found by calculating a squared, multiplying by b cubed, and finally dividing by c. Write this formula algebraically using indices.

(b) You are told additionally that $c = a$. Rewrite your formula in terms of a and b only, simplifying the indices as far as possible.

(c) Find F for $a = 7$ and $b = 2$.

Taking a root is the inverse operation to raising a number to a power. Squaring a positive or negative number always produces a positive result. Hence, taking the square root of a positive number can produce two answers. Since the squares of both positive and negative numbers are positive, there are no real numbers which square to give a negative number.

The number included in a root sign indicates the type of root. $\sqrt[2]{}$ is a square root, $\sqrt[3]{}$ is a cube root, and so on. A $\sqrt{}$ symbol without a number is always assumed to be a square root. By convention, the $\sqrt{}$ sign indicates that you are to find the positive value of the square root.

$$3^3 = 9, \quad \sqrt[2]{9} = 3 \qquad\qquad 4^3 = 64, \quad \sqrt[3]{64} = 4$$

There are instances where both the positive and negative solutions to a square root are required, such as solving quadratic equations. In these cases, you can indicate the need to find both solutions by putting the $\pm$ symbol in front of the square root. For example, $\pm\sqrt{9}$ has solutions $+3$ and -3.

13.11 Write down the values of

(a) $\sqrt{25}$ (b) $\sqrt{49}$ (c) $\sqrt{8^2+17}$

13.12 Calculate the following, simplifying as far as possible.

(a) $\sqrt[2]{4}+\sqrt[2]{16}$ (b) $\sqrt[3]{27}+\sqrt[3]{8}$ (c) $\sqrt[2]{9+16}$

13.13 Find

(a) $\sqrt{5^2}$ (b) $\sqrt{4^3}$ (c) $\sqrt{9^2+12^2}$

13.14 Write down the values of

(a) $\sqrt{4}$ (b) $(\sqrt{11})^2$ (c) $9-(\sqrt{3})^2$

13.15 Calculate the following, simplifying as far as possible.

(a) $\sqrt[2]{9}+\sqrt[2]{64}$ (b) $\sqrt[3]{27}-\sqrt[2]{16}$ (c) $\sqrt{4}\times\sqrt{9}$

13.16 Calculate the following, simplifying as far as possible.

(a) $-\sqrt[2]{9}+\sqrt[4]{81}$ (b) $2\sqrt[2]{4}+5\sqrt[3]{8}$ (c) $1-7\sqrt[3]{8}+\sqrt[2]{81}$

13.17 Calculate the following, simplifying as far as possible.

(a) $\frac{\sqrt[2]{16}}{2\sqrt[2]{4}}-\frac{\sqrt[3]{27}}{\sqrt[3]{8}}$ (b) $\frac{\sqrt[2]{16}-\sqrt[3]{27}}{2\sqrt[2]{4}-\sqrt[3]{8}}$ (c) $2\sqrt{9}-4\sqrt[5]{243}$

13.18 Write down the values of

(a) $\sqrt[3]{64}$ (b) $\sqrt[3]{-64}$ (c) $\sqrt[3]{8}$ (d) $\sqrt[3]{-27}$

§ Raising a number to the power n, then taking the n^{th} root, gives back the original number. From the "power on power" rule, we know that raising a number to the power n, then raising the result to the power $\frac{1}{n}$, also has this property.

$$\sqrt[n]{a^n}=a \qquad (a^n)^{\frac{1}{n}}=a^{n\times\frac{1}{n}}=a^1=a$$

This result suggests that taking an n^{th} root is equivalent to raising a number to the power $\frac{1}{n}$, and this is indeed the case. Fractional indices are used for roots, and the general rule is:

$$\sqrt[n]{m}=a^{\frac{m}{n}}$$

e.g. $\sqrt{9}=9^{\frac{1}{2}}$ $\qquad\sqrt[3]{8}=8^{\frac{1}{3}}$ $\qquad\sqrt[3]{8^2}=(8^2)^{\frac{1}{3}}=a^{\frac{2}{3}}$

13.19 Evaluate:

(a) $27^{\frac{1}{3}}$ (b) $32^{\frac{1}{5}}$ (c) $\sqrt[2]{4^3}$

13.20 Write the following in the form $\sqrt[n]{a^m}$.

(a) $p^{\frac{5}{7}}$ (b) $q^{\frac{3}{4}}$ (c) $r^{\frac{5}{3}}$

13.21 Write the following with fractional indices.

(a) $\sqrt[3]{a^4}$ (b) $\sqrt[7]{b^9}$ (c) $(\sqrt[5]{z})^2$

13.22 Evaluate:

(a) $9^{\frac{3}{2}}$ (b) $16^{\frac{5}{4}}$ (c) $(-8)^{\frac{2}{3}}$

13.23 Simplify the following to a single power of a in the numerator:

(a) $\frac{a^3}{(a^1)^7}$ (b) $\frac{(a^7)^2}{a^{-1}}$ (c) $(a^7)^{\frac{1}{2}}$ (d) $\sqrt[4]{\frac{a^7 \times a^4}{a^6}}$

14 Expanding

Expanding is the process of multiplying out one or more brackets. Example 1 shows how to expand a single bracket with a term in front of it.

> Example 1 - Expand the expression $3x(5x-1)$.
>
> Multiply the $3x$ by the two terms in the bracket, and add the results together.
>
> $$3x \times 5x = 15x^2 \qquad\qquad 3x \times -1 = -3x$$
>
> $$\therefore 3x(5x-1) = 15x^2 - 3x$$

Expand and simplify:

14.1 (a) $3(y+2)$ (b) $p(3-r)$ (c) $2(x+3-y)$

14.2 (a) $4(r+1)+3$ (b) $2(1+s)+6(s-1)$ (c) $z(z-1)+2z-1$

14.3 (a) $-2(3+x)$ (b) $-x(x-1)$ (c) $2xy(x-2)$

14.4 (a) $2x(x+7)$ (b) $a(1+a)+a^2+3$ (c) $\frac{1}{2}y(2y+4)$

14.5 (a) $3(2m-1)+2(m+3)-5(3+m)$

(b) $3p(4+2r)-3r(2p-5)$

Expanding a pair of brackets means multiplying the whole of the second bracket by each of the terms in the first bracket taken one after the other (including any minus signs).

$$(x-5)(2x-3) \qquad (x-5)(2x-3)$$

$$(x-5)(2x-3) = x(2x-3)\ -5(2x-3)$$

$$(x-5)(2x-3) = 2x^2 - 3x\ -10x + 15 = 2x^2 - 13x + 15$$

You may have seen the expansion of a pair of binomial brackets (brackets which contain exactly two terms) summarised by the acronym FOIL:

Firsts Outers

$(x-5)(2x-3)$

Inners Lasts

Example 2 - Expand and simplify the expression $(x+2)(x-3)$.

To multiply out the brackets, go through the FOIL acronym in order, multiplying pairs of terms together, and at the end add up the result.

$(\times \quad x)$	Firsts:	$x \times x = x^2$	Outers:	$x \times -3 = -3x$
$(\times +2)$	Inners:	$+2 \times x = +2x$	Lasts:	$2 \times -3 = -6$

$$\therefore (x+2)(x-3) = x^2 - 3x + 2x - 6 = x^2 - x - 6$$

Example 3 - Expand and simplify $(x-12)^2$.

Squaring a bracket means multiplying it by itself, so $(x-12)^2$ can be rewritten as two brackets:

$$(x-12)^2 = (x-12)(x-12)$$

This can be multiplied out in the same way as Example 2, to give:

$$(x-12)^2 = x^2 - 24x + 144$$

Expand and simplify the following:

14.6	(a)	$3t(t-5)+2(t-5)$	(b)	$(a+2)(a+3)$	(c)	$(a-2)(a-7)$
14.7	(a)	$(m+5)(m-2)$	(b)	$(n-7)(n+3)$	(c)	$(p-4)(p-9)$
14.8	(a)	$(x+4)^2$	(b)	$(n-2)^2$	(c)	$(2p-5)^2$
14.9	(a)	$(x+2)(x+0)$	(b)	$(x-13)(x+13)$	(c)	$(a+b)(a-b)$
14.10	(a)	$(h+3)(h-3)$	(b)	$(2x-1)(x+3)$	(c)	$(3x+2)(5x-1)$
14.11	(a)	$(2x+0)(5-3x)$	(c)	$(\frac{1}{2}s+1)(\frac{1}{2}s-1)$		
	(b)	$(pq+p^2)(pq+q^2)$	(d)	$(2^n+2)(2^n-2)$		

§ The same principle that lies behind the FOIL acronym also applies when multiplying out brackets containing more than two terms. First, multiply every term in the first bracket by every term in the second bracket separately, including the signs. Then collect like terms together and simplify the result.

Example 4 - Expand and simplify the expression $(x^2 + x - 5)(x - 3)$.

To multiply out the brackets, multiply terms together separately, and then add up the results.

$(\times \quad x)$	$x^2 \times x = x^3$	$x \times x = x^2$	$-5 \times x = -5x$
$(\times -3)$	$x^2 \times -3 = -3x^2$	$x \times -3 = -3x$	$-5 \times -3 = 15$

$$(x^2 + x - 5)(x - 3) = x^3 + x^2 - 5x - 3x^2 - 3x + 15$$
$$(x^2 + x - 5)(x - 3) = x^3 - 2x^2 - 8x + 15$$

When it is necessary to expand more than two brackets, tackle the problem in stages. Multiply out and simplify one pair of brackets at a time.

Example 5 - Expand and simplify the expression $(2x - 1)(x + 2)(x - 1)$.

Start by expanding and simplifying the $(x + 2)$ and $(x - 1)$ brackets.

$$(2x - 1)(x + 2)(x - 1) = (2x - 1)(x^2 - x + 2x - 2)$$
$$(2x - 1)(x + 2)(x - 1) = (2x - 1)(x^2 + x - 2)$$

Now multiply out the $(2x - 1)$ and $(x^2 + x - 2)$ brackets.

$$(2x - 1)(x + 2)(x - 1) = 2x^3 + 2x^2 - 4x - x^2 - x + 2$$
$$(2x - 1)(x + 2)(x - 1) = 2x^3 + x^2 - 5x + 2$$

Expand and simplify the following:

14.12 (a) $(x + 1)(x + 2)(x + 3)$ (c) $(x - 3)(x - 7)^2$

(b) $(x - 5)(2x - a + 4)$ (d) $(\frac{1}{x} + 5)(3x^2 - 9)$

14.13 (a) $(2x - 1)(x + 2)(\frac{1}{2} - x)$ (c) $(\frac{1}{4}x + x^2)(4x + \frac{1}{x^2})$

(b) $5(2x + 7)(3x - \frac{1}{2})(x + 1)$ (d) $2(2x + 1)(3x - 7)^2$

14.14 (a) $(\sin x + 7)(\cos x + \sin x)$

(b) $(\sin x - \cos x)(\sin x + \cos x)$

15 Factorising I: Common Factors

Factorising is the opposite of expanding brackets.

To factorise an expression such as $4ax + 2a$, start by identifying the factors that are common to every term. Take these factors outside a bracket, and leave everything else inside the bracket. Remember that 1 is a factor of any term, so if all the other factors are removed you are left with a 1. This is shown in Example 1(ii).

Example 1 – Factorise (i) $28bq - 21b$ (ii) $4ax + 2a$.

(i) 7 and b are factors of both terms. The common factor to take out is therefore $7b$. Hence, $28bq - 21b$ factorises to

$$28bq - 21b = 7 \times 4 \times b \times q - 7 \times 3 \times b = 7b(4q - 3)$$

(ii) 2 and a are factors of both terms. The common factor to take out is therefore $2a$. Hence, $4ax + 6a$ factorises to

$$4ax + 2a = 2a \times 2x + 2a \times 1 = 2a(2x + 1)$$

In some problems, it is necessary to expand brackets and collect terms before factorising:

Example 2 – Factorise $3(2 - x) + 5x + 4$.

First expand the bracket: $3(2 - x) + 5x + 4 = 6 - 3x + 5x + 4$

Next collect like terms: $3(2 - x) + 5x + 4 = 2x + 10$

Then take out common factors: $3(2 - x) + 5x + 4 = 2(x + 5)$

When factorising expressions which contain π, treat π in the same way as algebraic quantities like x or a.

Example 3 – Factorise $28\pi^2x^2 + 12\pi x$.

4, π and x are factors of both terms. The common factor to take out is therefore $4\pi x$. Hence, $28\pi^2x^2 + 12\pi x$ factorises to

$$28\pi^2x^2 + 12\pi x = 4\pi x \times 7\pi x + 4\pi x \times 3 = 4\pi x(7\pi x + 3)$$

Factorise and simplify the following as fully as possible.

15.1 (a) $6x - 3$ (b) $2x^2 + 3x$ (c) $2(x+1) + 3(x+1)$

15.2 (a) $5a - 10b$ (b) $2x^2 - 4x$ (c) $3p + 6pq$

15.3 (a) $4a + 6b$ (b) $3a^3 + 6ab$ (c) $10ab^2c^2 + 15a^2bc$

15.4 (a) $2a^3b + 4cd - 10e$ (c) $6ST - 3S^2 + 21ST$
(b) $pqr - 2qr + 3pr$

15.5 (a) $2ab^2 + 4ab$ (c) $5pqr^2 + 10r \times \frac{pqr}{2} + 2pr$
(b) $6(2x+1) - 3(x-1) + 6$

15.6 Simplify as fully as possible, then factorise:
(a) $2(x+y) + 3(x-y) + 6y$ (b) $\frac{8-4x-4(x-2)}{2}$

15.7 Simplify then factorise:
(a) $\frac{\pi r^2}{r} + 2\pi ra$ (b) $\frac{\lambda}{\theta}(\alpha\theta^2 + \beta\theta^3)$

15.8 Simplify the following, factorising if possible:
(a) $3x^2 \times 2a \times ax^3$ (c) $3c \times (\frac{1}{2}x)^2 \times 8c^2x + 4c^3$
(b) $7p \times \frac{1}{2}x^2 \div \frac{p}{4} - 7x$

15.9 (a) Evaluate $-n^4 - n^3$ for $n = -2$.

(b) Factorise $-n^4 - n^3$ as fully as possible.

(c) Use your answer to part (b) to evaluate the expression $-n^4 - n^3$ for $n = 5$.

15.10 Factorise as fully as possible:

(a) $x \times 5^2 + x \times y \times 10^2$

(b) $2x \cos\theta - 2x$

(c) $\pi(2r)^2h - \frac{4}{3}\pi r^2 l$

16 Factorising II: Quadratic Expressions

A quadratic polynomial has the general form $ax^2 + bx + c$, where a, b and c are constants. Examples include:

$$x^2 + 2x + 1 \qquad -2p^2 + 3p \qquad 5y^2 - 1 \qquad \frac{3}{2}s^2 - \frac{1}{3}s + 17$$

To factorise a quadratic expression which has no constant term, such as $12x^2 + 9x$, start by identifying the common factor of both terms. This will include an x.

> Example 1 – Factorise $12x^2 + 9x$.
>
> The common factor of both terms is $3x$. $12x^2 + 9x$ factorises to
> $$12x^2 + 9x = 3x \times 4x + 3x \times 3 = 3x(4x + 3)$$

Factorising a quadratic expression which includes a constant is the opposite of multiplying out a pair of brackets which each contain two terms (binomial brackets). Consider expanding the brackets $(x + 2)(x + 3)$:

$$(x + 2)(x + 3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6$$

- The coefficient of x in the answer is equal to the sum of the constants in the brackets, $(+5) = (+2) + (+3)$.
- The constant in the answer is equal to the product of the constants in the brackets, $(+6) = (+2) \times (+3)$.

These observations allow us to work backwards and turn a quadratic expression into a pair of brackets.

> Example 2 – Factorise $x^2 - 4x - 12$.
>
> We wish to turn this expression into a pair of brackets $(x + p)(x + q)$.
>
> p and q multiply to give -12:
> The magnitude (size) of this number tells us that the magnitudes of p and q are factors of 12. List the factors of 12 in ordered columns.
>
> | | |
> |---|---|
> | 1 | 12 |
> | 2 | 6 |
> | 3 | 4 |
>
> The sign of -12 is negative. In order for two numbers to multiply to give -12, one of them must be positive and one negative.

p and q add to give -4:
This is a negative number, which tells us that the minus sign must go with the larger factor of 12. Put the "-" over the column of larger numbers in the factor table.

+	−
1	12
2	6
3	4

Finally, find the row of the factor table which add up to give -4. This is the second row. This tells us that $p = 2$, and $q = -6$.

$$(+1) + (-12) = -11$$
$$(+2) + (-6) = -4$$
$$(+3) + (-4) = -1$$

The quadratic factorises as $x^2 - 4x - 12 = (x + 2)(x - 6)$

A quadratic expression such as $x^2 - 25,$ where there is no linear term and the constant is subtracted, factorises to a pair of brackets $(x + p)(x - p)$. The reason for this is that terms in x automatically cancel when multiplying out brackets which differ only in central sign:

$$(x + p)(x - p) = x^2 - px + px - p^2 = x^2 - p^2$$

Due to the $x^2 - p^2$ here, expressions such as $x^2 - 25$ are said to have difference of two squares form.

Example 3 – Factorise $x^2 - 25$.

$$x^2 - 25 = x^2 - p^2 \qquad \Rightarrow p^2 = 25 \qquad \Rightarrow p = 5$$
$$\therefore x^2 - 25 = (x + 5)(x - 5)$$

Factorise the following expressions.

16.1 (a) $x^2 + x$ (b) $x^2 - 2x$ (c) $6x^2 - 21x$

16.2 (a) $3x^2 - 15x$ (b) $2x + 4x^2$ (c) $10x - 15x^2$

16.3 (a) $a^2 + 7a + 6$ (b) $a^2 + 3a - 40$ (c) $a^2 - 11a + 24$

16.4 (a) $c^2 + 9c + 20$ (b) $b^2 + 3b - 18$ (c) $c^2 + 15c + 36$

16.5 (a) $p^2 - 10p + 21$ (b) $r^2 + 2r - 48$ (c) $s^2 + 11s - 80$

16.6 (a) $p^2 - 4$ (b) $r^2 - 36$ (c) $2m^2 - 32$

16.7 (a) $x^2 - 9x - 22$ (b) $B^2 - 8B + 16$ (c) $25 - y^2$

16.8 (a) $x^2 + 2x + 1$ (b) $x^2 + 10x + 25$ (c) $x^2 - 18x + 81$

16.9 Without calculating either 23^2 or 17^2, find the value of $23^2 - 17^2$.

§ Factorising a quadratic expression with both constant and linear terms is more difficult when the co-efficient of x^2 is not equal to 1.

Example 4 – Factorise $6x^2 - 11x - 10$.

We wish to turn this expression into a pair of brackets $(rx+p)(sx+q)$.

First, consider the $6x^2$ term. The only contribution to this is $rx \times sx$. Therefore, $r \times s = 6$. Hence r and s are either 1 & 6, or 2 & 3.

Next, consider the constant -10. The only contribution to this is $p \times q$. Therefore, $p \times q = -10$. The "$-$" tells us p and q have opposite signs. p and q are -1 & 10, 1 & -10, -2 & 5 or 2 & -5.

Finally, consider the $-11x$ term. There are two contributions to this, $rx \times q$ and $p \times sx$. Therefore, $rq + sp = -11$. Combine pairings of r and s with pairings of p and q to find one which satisfies this equation. The combination $r = 2$ & $s = 3$, $p = -5$ & $q = 2$ works.

$$\therefore 6x^2 - 11x - 10 = (2x-5)(3x+2)$$

Finally, check the answer by multiplying out:

$$(2x-5)(3x+2) = 6x^2 + 4x - 15x - 10 = 6x^2 - 11x - 10 \quad \checkmark$$

Quadratic expressions in difference of two squares form can factorise to a pair of brackets where the constants in the brackets are surds.

Example 5 – Factorise $x^2 - 6$.

$$x^2 - 6 = x^2 - p^2 \qquad \Rightarrow p^2 = 6 \qquad \Rightarrow p = \sqrt{6}$$
$$\therefore x^2 - 6 = (x+\sqrt{6})(x-\sqrt{6})$$

Take out a common factor and then factorise fully.

16.10 (a) $3a^2 + 18a + 15$ (b) $2a^2 + 12a + 18$ (c) $2a^2 + 4a - 30$

Factorise the following expressions.

16.11 (a) $2a^2 + 5a - 12$ (b) $6a^2 - 19a + 10$ (c) $20a^2 + 6a - 8$

16.12 (a) $4p^2 - 9$ (b) $p^2 - 5$ (c) $16p^2 - 200$ (d) $7p^2 - 3$

16.13 (a) $\frac{x^2}{3} + \frac{2x}{3} + \frac{1}{3}$ (c) $3\pi x^2 + 14\pi x - 5\pi$

(b) $x^2 - \frac{1}{2}x - 3$ (d) $(\sin\theta)^2 - 4\sin\theta + 4$

17 Re-arranging and Changing the Subject

Changing the subject of an equation or formula means re-arranging until the variable chosen to be the subject is on its own. This gives an equation or formula that can be used to find the value of the subject.

The order of operations when rearranging is is equivalent to applying BIDMAS in reverse order: start with "undoing" addition and subtraction, and work through BIDMAS backwards, keeping brackets until last. Finally, simplify the result if possible.

Example 1 - Re-arrange the formula $v = u + at$ to make t the subject.

Calculating v from t involves multiplying t by a, then adding u. To make t the subject, these operations are undone in reverse order. First u is subtracted from both sides, then both sides are divided by a.

$$v = u + at$$

$\downarrow$ Subtract u from both sides.

$$v - u = u + at - u$$

$$v - u = at$$

$\downarrow$ Divide both sides by a.

$$\frac{v-u}{a} = \frac{at}{a}$$

$$\frac{v-u}{a} = t$$

Finally, write the formula the other way round so that it is in the form $t = \ldots$.

$$\Rightarrow t = \frac{v-u}{a}$$

17.1 Rearrange $V = E - Ir$:

(a) To make E the subject. (b) To make I the subject.

17.2 (a) Rearrange $m = 3 + 2n$ to make n the subject.

(b) Expand and simplify $p = 4(2s - r) - 5r$.

(c) Rearrange your answer to part (b) to make s the subject.

17.3 (a) Rearrange $y = mx + c$ to make x the subject.

(b) Simplify the following: $Q = (5p + 2) - 2(p - 3) - 4$.

(c) Rearrange your answer to part (b) to make p the subject.

17.4 It is given that $y = 2x - 13$.

(a) Re-arrange this equation to make x the subject.

(b) If $x = 4$, find y.

(c) Find the value of x if $y = -43$.

17.5 For the formula $F = BIl$, F is measured in Newtons (N), B is measured in Teslas (T), I is measured in Amps (A) and l is measured in metres.

(a) Find F if $B = 0.19$ T, $I = 0.5$ A and $l = 1.2$ m.

(b) Find B if $F = 0.36$ N, $I = 3.0$ A and $l = 2.0$ m.

17.6 Re-arrange $F = \frac{GMm}{r^2}$ to give an expression for

(a) m (b) r

Problems which involve circles or spheres often produce formulae which include π. When re-arranging these formulae, π can be treated like any other algebraic letter.

Example 2 - Re-arrange the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$ to make r the subject.

$$V = \frac{4}{3}\pi r^3 \qquad \downarrow \text{Multiply both sides by 3.}$$

$$3V = 4\pi r^3 \qquad \downarrow \text{Divide both sides by 4.}$$

$$\frac{3V}{4} = \pi r^3 \qquad \downarrow \text{Divide both sides by } \pi.$$

$$\frac{3V}{4\pi} = r^3$$

$$\therefore r^3 = \frac{3V}{4\pi} \qquad \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$$

Exchange sides so r^3 is on the left, then take the cube root of both sides.

17.7 The surface area A of a sphere is given by $A = 4\pi r^2$, where r is the radius.

(a) Rearrange the formula to make r the subject.

(b) If $A = 4$ cm^2 find r to 2 decimal places, stating the units.

17.8 Re-arrange $s = ut + \frac{1}{2}at^2$ to give an expression for

(a) u (b) a

17.9 The cooking time for a type of pudding is "40 minutes per kilo and then 30 minutes extra." Let T be the cooking time in minutes.

(a) Write an equation for T in terms of the mass m in kilograms.

(b) If $m = 500$ g what is the cooking time?

(c) A pudding was cooked for 1 hour 20 minutes. What was its mass in kg?

17.10 "Fahrenheit" and "Centigrade" are two scales for measuring temperature.

(a) Using C and F for temperatures in Centigrade and Fahrenheit, write a conversion formula according to the following instructions: To find a Centigrade reading, take the measurement in Fahrenheit, subtract 32, and then take $\frac{5}{9}$ths of that value.

(b) Use your formula to convert a Fahrenheit reading of 98.4 to Centigrade, giving your answer to 3 s.f..

(c) Re-arrange your formula to convert Centigrade readings into Fahrenheit.

(d) Convert 50.0 Centigrade into Fahrenheit.

When changing the subject of a formula, if the variable that is to become the new subject appears more than once, a factorisation step may be required.

Example 3 - Re-arrange $A = 3r + 2\pi r$ to make r the subject.

$$A = 3r + 2\pi r$$
$$A = r(3 + 2\pi)$$

r appears twice on the right hand side. In order to make r the subject, we need to factorise so that r only appears once.

$$\Rightarrow r = \frac{A}{3 + 2\pi}$$

Then, divide both sides by $(3 + 2\pi)$, and write the formula in the form $r = \ldots$.

17.11 It is given that $y = (x-3)^3 + 4$.

(a) Re-arrange the original equation to make x the subject.

(b) Find the value(s) of x corresponding to a y-value of 31.

17.12 (a) Re-arrange the formula $P = \sigma A T^4$ to make T the subject.

(b) Temperature measured in Kelvin, T, is related to temperature measured in Centigrade, C, by the formula $T = C + 273$. A beaker of liquid is heated by $35\ ^\circ$C from $20\ ^\circ$C. What is the final temperature of the liquid in Kelvin?

17.13 This is a formula for calculating the area of a complicated shape:

$$A = \frac{5}{2}\pi r^2 + 2bh + \frac{3}{2}b(h+2)$$

(a) Find A when $r = 20$ cm, $b = 10$ cm and $h = 15$ cm.

(b) Find r in terms of π if $A = 60$ cm^2, $b = 2$ cm and $h = 3$ cm.

17.14 The table below shows population density figures for the UK in 2019.

Nation / Country	Population (Millions)	Land Area (km^2)	Population Density (people km^{-2})
The UK	67	244 000	A
England	56.3	B	432
Scotland	C	77 000	70
Wales	3.1	D	152
Northern Ireland	1.9	E	137

$$\text{Population density} = \frac{\text{Population}}{\text{Land Area}}$$

Use the formula and the information in the table to find the values of the five missing figures A to E. Give your answers to 2 s.f..

17.15 Simplify and re-arrange the following to make the stated variable the subject.

(a) $c^2 = 4 - 2t; t$

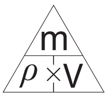
(b) $6 - t^3 = 4p - 3; t$

(c) $3x + y = 6xy^2 - (2+x); x$

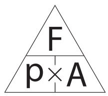
(d) $4y = 3x^3 + y + 4; x$

18 Formula Triangles

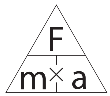
Formula triangles can be used to save time re-arranging formulae which involve exactly three quantities in a relationship of the form $a = b \times c$. These formulae can always be rearranged algebraically using the method in the previous chapter. Here are some common formula triangles:



m Mass
ρ Density
V Volume



F Force
p Pressure
A Area

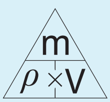


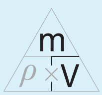
F Force
m Mass
a Acceleration

Note: Some books write formula triangles with a vertical line on the bottom instead of a multiplication symbol. The meaning is the same.

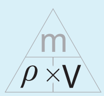
To find a formula from a formula triangle, cover up the quantity to be found. The pattern of the remaining letters shows how to calculate this quantity.

Example 1 - Use the formula triangle relating mass m, density ρ and volume V to write formulae for each of the three quantities in terms of the other two.

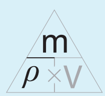




$\rho = \dfrac{m}{V}$



$m = \rho \times V$



$V = \dfrac{m}{\rho}$

When performing a calculation using a formula triangle, it is necessary to make sure that the units you are using are consistent.

Example 2 - Find the volume of a block of steel which has a mass of 0.156 kg and a density of 7.8 g/cm^3.

There is a mixture of units of mass in the question (g and kg). We will choose to do the calculation in grams. 0.156 kg = 156 g.

$$V = \frac{m}{\rho} = \frac{156}{7.8} = 20 \text{ cm}^3$$

18.1 Using the mass-density-volume triangle, or otherwise, find:

(a) ρ in g/cm^3 if $m = 200$ g and $V = 25$ cm^3.

(b) m in g if $\rho = 10$ g/cm^3 and $V = 10$ cm^3.

(c) V in cm^3 if $m = 0.54$ g and $\rho = 6$ g/cm^3.

18.2 Using the mass-density-volume triangle, or otherwise, find:

(a) The density in g/cm^3 if $m = 0.045$ kg and $V = 2.5$ cm^3.

(b) The volume in cm^3 if $m = 50$ g, $\rho = 4\,000$ kg/m^3.

(c) The mass in kg if $\rho = 7\,500$ kg/m^3 and $V = 1\,000$ cm^3.

18.3 Using a formula triangle, or otherwise, find:

(a) The pressure in N/m^2 exerted by a force of 16.2 N on an area of 1.50 m^2.

(b) The force in Newtons (N) required to maintain a pressure of 15.0 N/m^2 on an area of 0.150 m^2.

(c) The area in cm^2 of a surface which experiences a pressure of 11.3 N/cm^2 from a uniformly applied force of 4.52 kN.

It is also possible to write your own formula triangles.

18.4 Write the following formulae as formula triangles.

(a) $V = IR$ (b) $v = s/t$ (c) $\frac{P}{V} = I$

18.5 Which of these formulae can be written as a formula triangle? Write a formula triangle where it is possible.

(a) $v = u + at$ (b) $\frac{Q}{C} = V$ (c) $L = a - b$

(d) Magnification, $M = \frac{\text{Image size, } i}{\text{Object size, } o}$

18.6 The concentration of salt in water, C g/cm^3, is found by dividing the mass of salt in grams, m, by the volume of water in cm^3, V.

(a) Create a formula triangle for concentration, mass and volume.

(b) Write a formula for volume in terms of mass and concentration.

(c) Find the volume of a solution with concentration 0.0020 g/cm^3 if the total mass of salt dissolved is 2.4 g.

(d) Write a formula for m in terms of C and V.

(e) Find m if $V = 1$ litre and $C = 0.004$ g/cm^3.

19 Sequences

A sequence is an ordered list of values related by a rule. In sequence notation, $\mathsf{T}(1)$ is the value of the first term in the sequence, $\mathsf{T}(2)$ the value of the second term, and so on. Using n for the position of a term within a sequence, $\mathsf{T}(n)$ is the value of the n^{th} term.

A term-to-term description of a sequence explains how to calculate the value of a term from the value of the previous term (or terms).

> Example 1 – For the sequence $1, 4, 7, 10, 13, ...$, the term-to-term description is:
> $$\mathsf{T}(1) = 1, \quad \mathsf{T}(n) = \mathsf{T}(n-1) + 3$$
> The value of the first term is 1: $\mathsf{T}(1) = 1$.
>
> The value of each term is greater than the previous term by $+3$.
> $\mathsf{T}(2) = \mathsf{T}(1) + 3$, $\mathsf{T}(3) = \mathsf{T}(2) + 3$, and so on. The value of the n^{th} term is 3 greater than the $(n-1)^{\text{th}}$ term: $\mathsf{T}(n) = \mathsf{T}(n-1) + 3$.

A position-to-term description of a sequence explains how to calculate the value of a term from its position in the list, n.

> Example 2 – For the sequence $1, 4, 7, 10, 13, ...$, the position-to-term description is:
> $$\mathsf{T}(n) = 3n - 2$$
> The value of the first term is: $\mathsf{T}(1) = 3 \times 1 - 2 = 1$.
> The value of the second term is: $\mathsf{T}(2) = 3 \times 2 - 2 = 4$. And so on.

There are a number of standard sequences to know. These include:

The squared numbers: $1, 4, 9, 16, 25, 36, 49, 64, 81...$
Each term is the square of its position in the sequence, $\mathsf{T}(n) = n^2$.

The cubed numbers: $1, 8, 27, 64, 125, 216, ...$
Each term is the cube of its position in the sequence, $\mathsf{T}(n) = n^3$.

The Fibonnaci Sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$
Each term is the sum of the two previous terms,
$$\mathsf{T}(1) = 1, \ \mathsf{T}(2) = 1, \quad \mathsf{T}(n) = \mathsf{T}(n-1) + \mathsf{T}(n-2)$$

The triangular numbers: $1, 3, 6, 10, 15, 21, 28, \ldots$

To get from one term in the sequence to the next add n (the position of the new term) to the value of the previous term.

$$\mathsf{T}(1) = 1, \quad \mathsf{T}(n) = \mathsf{T}(n-1) + n \qquad\qquad \mathsf{T}(n) = \frac{1}{2}n(n+1)$$

19.1 Find the missing terms in these sequences:

(a) $4, 6, 8, ?, 12, ?$ (b) $1, 4, ?, ?, 25, 36$ (c) $1, ?, 2, 3, 5, ?, 13$

19.2 Find the missing terms in these sequences:

(a) $1, 8, ?, 64, ?, 216$ (b) $4, 7, 10, 13, ?, ?, 22$ (c) $2, ?, 8, ?, 32, 64$

19.3 Find the first 5 terms of these sequences from the rules for the n^{th} terms:

(a) $\mathsf{T}(n) = n$ (b) $\mathsf{T}(n) = 2n + 3$ (c) $\mathsf{T}(n) = 5 - 3n$

19.4 Find terms two to six of the sequences given by these term-to-term rules:

(a) $\mathsf{T}(1) = 6, \quad \mathsf{T}(n) = \mathsf{T}(n-1) + 3$

(b) $\mathsf{T}(1) = 3, \quad \mathsf{T}(n) = \mathsf{T}(n-1) - 2$

(c) $\mathsf{T}(1) = 2, \quad \mathsf{T}(n) = 2\mathsf{T}(n-1)$

19.5 Generate the first four terms of the sequences described by the following rules for the n^{th} term:

(a) $\mathsf{T}(n) = 2n^3$ (b) $\mathsf{T}(n) = (n+1)^2$ (c) $\mathsf{T}(n) = \frac{1}{2}n(n+1)$

In an arithmetic sequence (also called an arithmetic progression), to go from one term to the next the same numerical constant is added on each time. This constant is called the common difference.

Example 3 – The table shows the start of an arithmetic sequence. $\mathsf{T}(n)$ is the value of the term at position n in the sequence. Find an expression for the n^{th} term, $\mathsf{T}(n)$.

n	1	2	3	4	5	6	7	8
$\mathsf{T}(n)$	3	5	7	9	11	13	15	17

Each term in the sequence is greater than the previous term by $+2$. Therefore make a table of multiples of 2.

n	1	2	3	4	5	6	7	8
$2n$	2	4	6	8	10	12	14	16

Now compare the multiples of 2 with the terms of $\mathsf{T}(n)$. Each term in $\mathsf{T}(n)$ is greater than the corresponding term in $2n$ by $+1$.

n	1	2	3	4	5	6	7	8
$2n$	2	4	6	8	10	12	14	16
$2n+1$	3	5	7	9	11	13	15	17

Therefore the n^{th} term of $\mathsf{T}(n)$ is equal to $2n+1$, $\quad \mathsf{T}(n) = 2n+1$.

In a geometric sequence (also called a geometric progression), to go from one term to the next involves multiplication by the same numerical constant each time. This constant is called the common ratio.

Example 4 – A geometric sequence begins $5, 15, 45, 135, 405, \ldots$.
Find term-to-term and position-to-term expressions for the value of the n^{th} term in the sequence.

(i) Each term in the sequence is equal to $3\times$ the previous term.
$\therefore \mathsf{T}(n) = 3 \times \mathsf{T}(n-1)$, with the first term $\mathsf{T}(1) = 5$.

(ii) The value of the first term is 5. To get to the n^{th} term from the first term involves multiplication by 3, $(n-1)$ times.
$\therefore \mathsf{T}(n) = 5 \times 3^{n-1}$.

19.6 Find the missing terms in these sequences:
(a) $3, 7, 11, ?, ?, 23$ (b) $7, 5, 3, ?, -1, ?$ (c) $6, 2, -2, ?, -10, ?$

19.7 Find the missing terms in these sequences:
(a) $6, 12, 24, ?, 96, ?$ (b) $?, 6, 18, 54, ?$ (c) $\frac{1}{2}, ?, 2, ?, 8, 16$

19.8 For each sequence, find
(i) the term-to-term rule (ii) the position-to-term rule.
(a) $5, 7, 9, 11, 13$ (b) $9, 5, 1, -3, -7$

19.9 For each sequence, find
(i) the term-to-term rule (ii) the position-to-term rule.
(a) $1, 3, 9, 27, 81$ (b) $4, 8, 16, 32, 64$

19.10 For each sequence, find
(i) the next two terms in the sequence
(ii) the term-to-term rule
(iii) the position-to-term rule.
(a) $6, 12, 18, 24, 30$ (b) $-1, -4, -16, -64, -256$

§ The differences between the terms of a sequence are called first differences, and the differences between first differences are called second differences. In a quadratic sequence such as $7, 19, 39, 67, 103, \ldots$, the first differences follow an arithmetic progression and the second differences all have the same value.

A general formula for the n^{th} term of a quadratic sequence is $\mathrm{T}(n) = an^2 + bn + c$, where a is equal to half the second difference value.

Example 5 – Find the n^{th} term of the sequence $7, 19, 39, 67, 103, \ldots$.

Find the first and second differences:

n	1	2	3	4	5
$\mathrm{T}(n)$	7	19	39	67	103

+12 +20 +28 +36 First differences

+8 +8 +8 Second differences

The value of the second differences is $+8$. a in the formula for the n^{th} term is equal to half of this value. $\therefore a = 4$, and $\mathrm{T}(n) = 4n^2 + bn + c$.

To find b and c, write the first two terms of the sequence in terms of b and c and solve these equations simultaneously.

$$\mathrm{T}(1) = 4(1)^2 + b(1) + c = 7 \quad \Rightarrow 4 + b + c = 7 \quad \Rightarrow b + c = 3$$

$$\mathrm{T}(2) = 4(2)^2 + b(2) + c = 19 \quad \Rightarrow 16 + 2b + c = 19 \quad \Rightarrow 2b + c = 3$$

Subtracting $b + c = 3$ from $2b + c = 3$ gives $b = 0$, and substituting this into $b + c = 3$ gives $c = 3$. Therefore, $\mathrm{T}(n) = 4n^2 + 3$.

19.11 Find the missing terms in these sequences:

(a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ?, ?$ (b) $4, 6, 9, \frac{27}{2}, ?, ?$ (c) $\frac{27}{4}, ?, 3, 2, ?, \frac{8}{9}$

19.12 Find the missing terms in these sequences:

(a) $8, ?, 14, 17, ?$ (b) $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, ?, \frac{1}{48}, ?$ (c) $\frac{1}{3}, -\frac{1}{9}, ?, ?, \frac{1}{243}, -\frac{1}{729}$

19.13 Find the missing terms in these sequences:

(a) $1, \sqrt{2}, ?, 2\sqrt{2}, 4, ?$ (c) $12, -4\sqrt{3}, ?, \frac{-4}{\sqrt{3}}, \frac{4}{3}, ?$

(b) $3, \frac{3}{\sqrt{2}}, ?, \frac{3}{2\sqrt{2}}, ?, \frac{3}{4\sqrt{2}}$

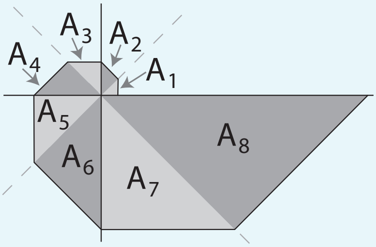
19.14 For each of these quadratic sequences, find -

(i) the next two terms in the sequence

(ii) the position-to-term rule.

(a) $3, 6, 11, 18, 27$ (b) $4, 21, 48, 85, 132$ (c) $66, 45, 28, 15, 6$

19.15 The diagram shows a spiral construction. Each triangle has two $45°$ angles. The areas of the triangles form the start of a sequence, and the area of the first triangle is $A_1 = \frac{1}{2}$.

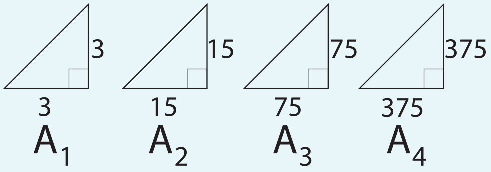


(a) What are the values of the second and third terms in the sequence, A_2 and A_3?

(b) Write down a formula for the area A_n in terms of n.

(c) What is the total area of the first 8 terms in the sequence?

19.16 The diagram below shows the first four triangles in a sequence. A_n is the area of the n^{th} triangle in the sequence.



(a) Find the areas of the first four triangles in the sequence.

(b) Find a formula for the area of the n^{th} term in the sequence.

20 Functions

A function is a set of instructions for turning one number (the input) into another (the output). Functions have only one possible output for each input. Functions can be illustrated using number (function) machines.

Example 1 – The diagram below shows a number machine which turns an input x into an output y. The input is first multiplied by 2, then 5 is subtracted.

$$x \longrightarrow \boxed{\times 2} \longrightarrow \boxed{\text{-5}} \longrightarrow y$$

(i) Find the output when the input is 6
The output is $(6 \times 2) - 5 = 12 - 5 = 7$.

(ii) Find an equation for y in terms of x.
The output of the function is $y = 2x - 5$.

There are several ways to write a function, each of which means something slightly different:

$x \longrightarrow 2x + 4$	• When the input is x, the output is $2x + 4$.
$y = 2x + 4$	• When the input is x, the output is $2x + 4$. y is equal to the output.
$f(x) = 2x + 4$	• When the input is x, the output of the function named "f" is $2x + 4$.

The trigonometric function $\sin(\theta)$ is an example of a function written in $f(x)$ form. $\sin$ is the function name, and θ is the value of the input.

In the example below, the function has one input, represented by the variable x. The function uses this input several times in producing the output (the terms $3x^2$ and $-x$).

Example 2 – Evaluate the function $y = 3x^2 - x - 3$ when $x = 2$.

$$y = 3(2)^2 - 2 - 3 = 3 \times 4 - 2 - 3 = 12 - 2 - 3 = 7$$

Example 3 – For $s = 5$, evaluate (i) $s \longrightarrow 3 - s^2$ and (ii) $g(s) = \frac{20}{5+s}$.

(i) When $s = 5$, the value of the function is $3 - (5)^2 = 3 - 25 = -22$.

(ii) $g(5) = \frac{20}{5+5} = \frac{20}{10} = 2$

20.1 Find the value of the function $y = x + 3$ when:
(a) $x = 2$ (b) $x = -3$ (c) $x = \frac{1}{3}$

20.2 (a) Draw a number machine to illustrate the function $y = 3x - 4$.
Find the value of the function $y = 3x - 4$ when:
(b) $x = 5$ (c) $x = 1$ (d) $x = -5$

20.3 (a) Construct a number machine to calculate y given that $y = 5(2x + 3) + 1$.
Find the values of this function when:
(b) $x = 3$ (c) $x = 0$ (d) $x = -1$

20.4 Evaluate the function $y = \frac{1}{4}x^2 + \frac{1}{4}$ for
(a) $x = 1$ (b) $x = 2$ (c) $x = -2$

20.5 Evaluate $s \longrightarrow \frac{1}{s} + s$ for
(a) $s = 1$ (b) $s = 5$ (c) $s = 2$

20.6 Evaluate $t \longrightarrow \frac{t^2}{1+t^2}$ for:
(a) $t = 1$ (b) $t = 0$ (c) $t = -2$

20.7 Find the values of these functions when $x = 2$.
(a) $f(x) = 5x + 1$ (c) $h(x) = x^3 + x^2 + x + 1$
(b) $g(x) = x^2 - 3x + 7$

20.8 Evaluate $Q(R) = \frac{4}{R^2} - \frac{1}{R} + 2$ for:
(a) $R = 2$ (b) $R = 4$ (c) $R = \frac{1}{2}$

20.9 Find the values of $u(t) = 3 \sin t$ for:
(a) $t = 90°$ (b) $t = 180°$ (c) $t = 30°$

20.10 Find the values of x such that $x \longrightarrow 2x + 6$ has a value of
(a) 10 (b) -10

20.11 Find the values of y, such that $y \longrightarrow \frac{y}{2} + \frac{3}{2}$ has a value of

(a) 2 (b) 6

20.12 Find the values of r, such that $V(r) = \frac{18}{r}$ has a value of

(a) 6 (b) $\frac{1}{2}$

20.13 $x \rightarrow \boxed{+2} \rightarrow \boxed{\times 2} \rightarrow \boxed{-3} \rightarrow y$

(a) Use the number machine to find y when (i) $x = 2$ (ii) $x = -4$.

(b) Find the value of x that produces a y value of 25.

§ An inverse function is essentially using a function machine running in reverse. The inverse operations are carried out, in the reverse order.

Example 4 – For the function machine below, write an equation to represent (i) the function (ii) the inverse function.

$x \rightarrow \boxed{\times 3} \rightarrow \boxed{+2} \rightarrow \boxed{\div 5} \rightarrow y$

The diagram tells us that in this problem the input to the machine is called x, and the output is called y.

$x \xrightarrow{\times 3} \xrightarrow{+2} \xrightarrow{\div 5} y$

(i) The input is on the left. The (forwards) function works from left to right. First the input is multiplied by 3, then 2 is added, then the result is divided by 5. This gives $y = \frac{3x+2}{5}$.

$y \xleftarrow{\div 3} \xleftarrow{-2} \xleftarrow{\times 5} x$

(ii) In the inverse function the input x is put into the machine from the right hand side. The machine is run in reverse from right to left. First the input is multiplied by 5, then 2 is subtracted, then the result is divided by 3. This gives for the output $y = \frac{5x-2}{3}$.

When a function is written in $f(x)$ form, the inverse is indicated by a superscript $^{-1}$ between the name of the function and the bracket containing the variable. For example, the inverse of $\sin(x)$ is written $\sin^{-1}(x)$.

A composite function is where the output of one function is used as the input to a second function.

Example 5 – The diagram shows a composite function where the output of function A is used as the input of function B.

$$x \longrightarrow \overset{A}{\boxed{+4}} \longrightarrow (x+4) \longrightarrow \overbrace{\boxed{\div 3} \longrightarrow \boxed{+5}}^{B} \longrightarrow \frac{(x+4)}{3}+5$$

Brackets are used here to indicate that the whole of $(x+4)$ is used as the input to function B. The output of function B could be tidied up at the end to give $\frac{(x+4)}{3}+5=\frac{1}{3}x+\frac{4}{3}+5=\frac{1}{3}x+6\frac{1}{3}$.

A composite function made up of a function and its inverse has the property that the output is always equal to the input. The operations in the inverse function effectively "undo" the operations in the original function.

20.14 $$x \longrightarrow \boxed{+1} \longrightarrow \boxed{\times 4} \longrightarrow \boxed{-5} \longrightarrow y$$

(a) Write an equation for the function represented by this number machine.

(b) Write an equation for the inverse function.

(c) Evaluate the inverse function to find y when $x=11$.

20.15 $$x \longrightarrow \boxed{-1} \longrightarrow \boxed{\times 7} \longrightarrow \boxed{+3} \longrightarrow y$$

(a) For this number machine, write an equation for the function. Simplify your equation as far as possible.

(b) Using the number machine, find an equation for the inverse function. Simplify this equation as far as possible.

(c) For the inverse function, what value of x gives a y value of 24?

20.16 The composite function $h(x)$ uses the output of function A as the input to function B. Find an expression for $h(x)$ when

(a) Function A is "divide the input by four"; function B is "add 2 to the input, then multiply by 5".

(b) Function A is "multiply by 2, then subtract 3"; function B is "divide by 5, then subtract 1".

(c) Function A is "multiply by 2, then subtract 3"; function B is "add 3, then divide by 2".

21 § Surds and Rationalising a Denominator

Square numbers such as 4, 16 and 25 have integer square roots. The square roots of the other positive integers are irrational.

Surds are expressions that include an instruction to take a root which has an irrational answer. Examples are $\sqrt{2}$, $5\sqrt{3}$ and $\sqrt[3]{11}$. Keeping $\sqrt{\ }$ signs in an answer keeps the answer exact.

Square roots can be simplified when the number under the root sign has a factor which is a square number.

Example 1 – Simplify $\sqrt{18}$.

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{3^2 \times 2} = \sqrt{3^2} \times \sqrt{2} = 3\sqrt{2}$$

When two surds are multiplied together, the numbers in front of the square root signs multiply, and the numbers under the square root signs multiply.

Example 2 – Simplify $2\sqrt{3} \times 5\sqrt{7}$.

$$2\sqrt{3} \times 5\sqrt{7} = 10\sqrt{21}$$

When two surds are divided, the numbers in front of the square root signs are divided, and the numbers under the square root signs are divided.

Example 3 – Simplify $5\sqrt{3} \div 10\sqrt{7}$.

There are two equivalent ways of writing this,

$$5\sqrt{3} \div 10\sqrt{7} = \frac{5}{10}\sqrt{\frac{3}{7}} = \frac{1}{2}\sqrt{\frac{3}{7}} \quad \text{or} \quad 5\sqrt{3} \div 10\sqrt{7} = \frac{5\sqrt{3}}{10\sqrt{7}} = \frac{1}{2}\sqrt{\frac{3}{7}}$$

Addition and subtraction of surds is only possible when the number under the square root is the same. An initial simplification step may be needed.

Example 4 – Calculate $5\sqrt{32} + 8\sqrt{2}$

$$5\sqrt{32} = 5\sqrt{16 \times 2} = 5\sqrt{16}\sqrt{2} = 5 \times 4\sqrt{2} = 20\sqrt{2}$$

$$\therefore 5\sqrt{32} + 8\sqrt{2} = 20\sqrt{2} + 8\sqrt{2} = 28\sqrt{2}$$

These rules apply in a similar way when combining cube roots. When multiplying or dividing roots of a different type, such as a square root and a cube root, the numbers in front of the roots can be combined but the roots cannot. For example, $2\sqrt[3]{7} \times 5\sqrt[4]{6}$ cannot be simplified further than $10\sqrt[3]{7}\sqrt[4]{6}$.

Simplify as far as possible.

21.1	(a) $\sqrt{2} \times \sqrt{2}$	(b) $(2\sqrt{3})^2$	(c) $5\sqrt{5} \times 2\sqrt{2}$
21.2	(a) $\frac{\sqrt{5}}{\sqrt{20}}$	(b) $\frac{7\sqrt{7}}{\sqrt{56}}$	(c) $\frac{-2\sqrt{11}}{\sqrt{44}}$
21.3	(a) $\sqrt{6} \times \sqrt{12}$	(b) $\sqrt{2} \times \sqrt{6} \times \sqrt{15}$	(c) $\frac{\sqrt{51}\sqrt{34}}{\sqrt{6}}$
21.4	(a) $\sqrt{\frac{27}{4}}$	(b) $\sqrt{\frac{7}{36}}$	(c) $3\sqrt{\frac{54}{243}}$
21.5	(a) $\sqrt{8} \times \sqrt{20}$	(b) $\sqrt{27} \times \sqrt{30}$	(c) $\frac{9\sqrt{75}}{\sqrt{15}}$
21.6	(a) $2\sqrt{3} \times \sqrt{21}$	(b) $(5\sqrt{7}) \times 2\sqrt{14}$	(c) $4\sqrt{33} \div 2\sqrt{27}$
21.7	(a) $\frac{3\sqrt{5}\times2\sqrt{2}}{4\sqrt{30}}$	(b) $\frac{5\sqrt{10}\times2\sqrt{20}}{4\sqrt{2}}$	(c) $\frac{2\sqrt{6}\times4\sqrt{12}}{2\sqrt{2}}$

Expand and simplify.

21.8 (a) $(\sqrt{2}+\sqrt{7})(\sqrt{3}-\sqrt{7})$ (c) $(\sqrt{2}+\sqrt{7})(\sqrt{2}-\sqrt{7})$
(b) $(\sqrt{5}-2\sqrt{3})(\sqrt{5}-3\sqrt{3})$ (d) $(\sqrt{5}-2\sqrt{3})^2$

21.9 (a) $(3\sqrt{2}+5\sqrt{2})(3\sqrt{2}-2\sqrt{5})$ (c) $(\sqrt{5}-2\sqrt{3})(\sqrt{5}-2\sqrt{3})$
(b) $(2\sqrt{5}+5\sqrt{2})^2$ (d) $8(\sqrt{6}+2\sqrt{3})(\sqrt{6}-3\sqrt{3})$

When a fraction has a surd in the denominator, the value of the denominator is irrational. Rationalising the denominator means finding an equivalent fraction without a surd in the denominator. There are two cases.

The first case is fractions with denominators of the form $a\sqrt{b}$, such as $\frac{10}{3\sqrt{5}}$. These fractions can be rationalised by multiplying by $\frac{\sqrt{b}}{\sqrt{b}}$.

Example 5 - Rationalise the denominator of $\frac{10}{3\sqrt{3}}$.

Multiply the fraction by $\frac{\sqrt{3}}{\sqrt{3}}$.

$$\frac{10}{3\sqrt{3}} = \frac{10}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{10\times\sqrt{3}}{3\sqrt{3}\times\sqrt{3}} = \frac{10\times\sqrt{3}}{3\times3} = \frac{10\times\sqrt{3}}{9} = \frac{10\sqrt{3}}{9}$$

The second case is fractions with denominators of the form $a + b\sqrt{c}$, such as $\frac{3+2\sqrt{7}}{2+5\sqrt{7}}$. To understand how to rationalise denominators of this form, first consider multiplying out a pair of binomial brackets which differ only in the sign of one of the terms:

$$(x + y)(x - y) = x^2 - y^2$$

The result is the difference of two squares. This is rational even if one or both of x and y are surds. For example,

$$\begin{aligned}(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) &= (\sqrt{5})^2 - \sqrt{5}\sqrt{3} + \sqrt{5}\sqrt{3} - (\sqrt{3})^2 \\ \Rightarrow (\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) &= 5 - 3 = 2\end{aligned}$$

Therefore, to rationalise a fraction with a denominator of the form $a + b\sqrt{c}$, multiply by $\frac{a-b\sqrt{c}}{a-b\sqrt{c}}$. This approach works even if a is itself a surd.

Example 6 - Rationalise the denominator of $\frac{3+2\sqrt{7}}{2+5\sqrt{7}}$.

Multiply the fraction by $\frac{2+5\sqrt{7}}{2+5\sqrt{7}}$.

$$\frac{3 + 2\sqrt{7}}{2 + 5\sqrt{7}} = \frac{3 + 2\sqrt{7}}{2 + 5\sqrt{7}} \times \frac{2 - 5\sqrt{7}}{2 - 5\sqrt{7}} = \frac{(3 + 2\sqrt{7}) \times (2 - 5\sqrt{7})}{(2 + 5\sqrt{7}) \times (2 - 5\sqrt{7})}$$

$$\therefore \frac{3 + 2\sqrt{7}}{2 + 5\sqrt{7}} = \frac{6 - 15\sqrt{7} + 4\sqrt{7} - 70}{4 - 10\sqrt{7} + 10\sqrt{7} - 175} = \frac{-64 - 11\sqrt{7}}{-171} = \frac{64 + 11\sqrt{7}}{171}$$

Expressions such as $(\sqrt{5} + \sqrt{3})$ and $(\sqrt{5} - \sqrt{3})$, which differ only in the sign of one of the terms, are called conjugate surds.

Rationalise the denominators of the following. Leave the numerators fully expanded and simplified as far as possible.

21.10 (a) $\frac{1}{\sqrt{7}}$	(b) $\frac{2}{\sqrt{11}}$	(c) $\frac{7}{3\sqrt{3}}$
21.11 (a) $\frac{4}{1+\sqrt{2}}$	(b) $\frac{3}{5-\sqrt{3}}$	(c) $-\frac{6}{7+\sqrt{5}}$
21.12 (a) $\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$	(b) $\frac{1+\sqrt{5}}{\sqrt{5}-\sqrt{3}}$	(c) $\frac{\sqrt{2}-\sqrt{11}}{\sqrt{7}-\sqrt{11}}$
21.13 (a) $\frac{2\sqrt{5}+3\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$	(b) $\frac{1-3\sqrt{5}}{7\sqrt{3}-\sqrt{2}}$	(c) $\frac{2(\sqrt{5}-\sqrt{7})}{13(\sqrt{5}+\sqrt{2})}$

22 § Algebraic Fractions

Algebraic fractions such as $\frac{x}{x+1}$ can be manipulated and combined in an analogous way to numerical fractions such as $\frac{2}{5}$.

A fraction with the same expression in both the numerator and denominator has a value of 1.

$$\frac{3}{3} = \frac{1}{1} = 1 \qquad \text{Rule: } \frac{a}{a} = \frac{1}{1} = 1 \qquad \frac{3x^2}{3x^2} = \frac{1}{1} = 1$$

When fractions are multiplied, the numerators are multiplied and the denominators are multiplied. In the example below, note the use of brackets around each numerator and denominator.

$$\frac{3}{7} \times \frac{2}{5} = \frac{3 \times 2}{7 \times 5} \qquad \text{Rule: } \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Example 1 – $\dfrac{x}{x+3} \times \dfrac{x+1}{x+2} = \dfrac{x \times (x+1)}{(x+3) \times (x+2)}$

Dividing one fraction by another is equivalent to multiplying the first fraction by the second fraction inverted.

$$\frac{2}{9} \div \frac{5}{11} = \frac{2}{9} \times \frac{11}{5} = \frac{2 \times 11}{9 \times 5} = \frac{22}{45} \qquad \text{Rule: } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Example 2 – $\dfrac{x+5}{x+7} \div \dfrac{x+3}{x+4} = \dfrac{x+5}{x+7} \times \dfrac{x+4}{x+3} = \dfrac{(x+5)(x+4)}{(x+7)(x+3)}$

If both the numerator and denominator in a fraction have the same factor, this factor can be cancelled to leave a simpler equivalent fraction.

Example 3 – $\dfrac{2(x+2)}{3x(x+2)} = \dfrac{2}{3x} \times \dfrac{x+2}{x+2} = \dfrac{2}{3x} \times \dfrac{1}{1} = \dfrac{2}{3x}$

Note: A factor can only be cancelled when it multiplies every term in the numerator and every term in the denominator. Factorisation of the numerator and/or denominator is often needed before cancellation is possible.

Example 4 – Simplify the algebraic fraction $\frac{x(x+1)+2x}{(x+1)(x+3)}$ as far as possible:

$(x+1)$ multiplies every term in the denominator but not every term in the numerator, so it cannot be cancelled. The numerator needs to be properly factorised before cancellation is possible.

$$\frac{x(x+1)+2x}{(x+1)(x+3)} = \frac{x^2+x+2x}{(x+1)(x+3)} = \frac{x^2+3x}{(x+1)(x+3)} = \frac{x(x+3)}{(x+1)(x+3)}$$

$(x+3)$ multiplies every term in the denominator and every term in the numerator, so it can be cancelled. There are no further common factors. Hence, the answer is

$$\frac{x(x+1)+2x}{(x+1)(x+3)} = \frac{x}{x+1}$$

Addition or subtraction of algebraic fractions is analogous to addition of numerical fractions. First find a common denominator, then add or subtract the numerators, and finally simplify the result.

$$\frac{2}{5} + \frac{3}{7} = \frac{2\times 7}{5\times 7} + \frac{5\times 3}{5\times 7} = \frac{14+15}{35} = \frac{29}{35}$$

$$\text{Rule:}\quad \frac{a}{b} + \frac{c}{d} = \frac{a\times d}{b\times d} + \frac{b\times c}{b\times d} = \frac{ad+bc}{bd}$$

Example 5 – Sum $\frac{2}{x}$ and $\frac{4}{x+1}$ and simplify the result.

$$\frac{2}{x} + \frac{4}{x+1} = \frac{2(x+1)}{x(x+1)} + \frac{4x}{(x+1)x} = \frac{2(x+1)+4x}{(x+1)x}$$

$$\Rightarrow \frac{2}{x} + \frac{4}{x+1} = \frac{2x+2+4x}{(x+1)x} = \frac{6x+2}{(x+1)x} = \frac{2(3x+1)}{x(x+1)}$$

In this exercise leave your answers as fully factorised as possible.

22.1 Simplify:

(a) $\frac{2x}{x(x+1)}$ (b) $\frac{(x+1)(x+3)}{(x+3)(x+5)}$ (c) $\frac{x+1}{x^2+2x+1}$

22.2 Simplify:

(a) $4x^2 \div 2x^3$

(b) $3(x+1) \div 6(x+1)(x+2)$

(c) $6x \times 2x \div (x(x+2))$

(d) $a^2 \times (\frac{x^{\frac{1}{2}}a^{-\frac{1}{2}}}{ax}) \times (ax^{-\frac{5}{2}})^{-1}$

22.3 Simplify:

(a) $\frac{x}{x+5} \times \frac{x+5}{x+7}$ (b) $\frac{x^2}{x+3} \times \frac{2x+6}{x^3}$ (c) $\frac{x}{(x+3)(x+4)} \times \frac{x+4}{x(x+3)}$

22.4 (a) $\frac{x^2}{5x^2+15} \div \frac{x}{4x^3+6}$ (b) $\frac{x}{x+2} \div \frac{5x^2-7x}{2x+4}$ (c) $\frac{x^3}{4x^4} \div -\frac{x+1}{x^2}$

Express the following as a single fraction in as simple a form as possible.

22.5 (a) $\frac{1}{x} + \frac{1}{x}$ (b) $\frac{1}{x} + \frac{1}{2x}$ (c) $\frac{2}{x} - \frac{3}{4x}$

22.6 (a) $\frac{1}{x} + \frac{1}{x+1}$ (b) $\frac{1}{x-2} + \frac{2}{x+2}$ (c) $\frac{2x}{x-1} - \frac{15}{x+1}$

22.7 (a) $\frac{x+1}{x+2} + \frac{2x}{x+3}$ (b) $\frac{x+2}{x+3} - \frac{x-4}{x-5}$ (c) $\frac{x}{x-7} - \frac{2x}{3x+5}$

22.8 Simplify the following:

(a) $\frac{x^2+x}{x^2-1}$ (b) $\frac{x(x+2)}{x^2+5x+6}$ (c) $\frac{x^2+3x-10}{x^2+12x+35}$

22.9 Write as a single fraction, simplifying as far as possible:

(a) $\frac{6}{x^2-25} + \frac{3}{x-5}$ (b) $\frac{1}{x+3} + \frac{2}{x-3} + \frac{3}{x^2-9}$ (c) $\left(\frac{(x+a)^2}{x^2-a^2}\right)^{-1}$

22.10 Simplify:

(a) $\frac{x^2-1}{x+1}$ (b) $\frac{2x^4-32}{x^2-4}$ (c) $(x^3+9x^2+20x) \div (x^3-25x)$