Isaac Physics Skills

Linking concepts in pre-university physics

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TABLE OF PHYSICAL CONSTANTS

Quantity & Symbol		**Magnitude**	**Unit**
Permittivity of free space	ϵ_0	8.85×10^{-12}	$\mathrm{F\,m^{-1}}$
Electrostatic force constant	$1/4\pi\epsilon_0$	8.99×10^{9}	$\mathrm{N\ m^2\ C^{-2}}$
Speed of light in vacuum	c	3.00×10^{8}	$\mathrm{m\,s^{-1}}$
Specific heat capacity of water	c_{water}	4180	$\mathrm{J\,kg^{-1}\,K^{-1}}$
Charge of proton	e	1.60×10^{-19}	C
Gravitational field strength on Earth	g	9.81	$\mathrm{N\,kg^{-1}}$
Universal gravitational constant	G	6.67×10^{-11}	$\mathrm{N\ m^2\ kg^{-2}}$
Planck constant	h	6.63×10^{-34}	J s
Boltzmann constant	k_{B}	1.38×10^{-23}	$\mathrm{J\,K^{-1}}$
Mass of electron	m_{e}	9.11×10^{-31}	kg
Mass of neutron	m_{n}	1.67×10^{-27}	kg
Mass of proton	m_{p}	1.67×10^{-27}	kg
Mass of Earth	M_{Earth}	5.97×10^{24}	kg
Mass of Sun	M_{Sun}	2.00×10^{30}	kg
Avogadro constant	N_{A}	6.02×10^{23}	$\mathrm{mol^{-1}}$
Gas constant	R	8.31	$\mathrm{J\ mol^{-1}\ K^{-1}}$
Radius of Earth	R_{Earth}	6.37×10^{6}	m

OTHER INFORMATION YOU MAY FIND USEFUL

Electron volt	1 eV	=	1.60×10^{-19} J
Unified mass unit	1 u	=	1.66×10^{-27} kg
Absolute zero	0 K	=	$-273\ ^\circ$C
Year	1 yr	=	3.16×10^{7} s
Light year	1 ly	=	9.46×10^{15} m
Parsec	1 pc	=	3.09×10^{16} m

PREFIXES

1 km = 1000 m	1 Mm = 10^{6} m	1 Gm = 10^{9} m	1 Tm = 10^{12} m
1 mm = 0.001 m	1 μm = 10^{-6} m	1 nm = 10^{-9} m	1 pm = 10^{-12} m

25 Energy and fields – closest approach

It is useful to work out how close a charged particle can get to an atomic nucleus from its speed, its kinetic energy or the temperature of the material.

Example context: In Geiger and Marsden's gold foil experiment, alpha particles were fired at gold nuclei. Depending on how close they came to the nucleus, they were deflected by different angles. In nuclear fusion, small nuclei will only fuse if they come close enough for the strong nuclear force to act.

Quantities:

- V electric potential (V)
- U energy (J)
- v initial speed $\left(\mathrm{m\,s^{-1}}\right)$
- m mass (kg)
- Q charge of nucleus (C)
- q charge of projectile (C)
- r minimum separation (m)
- T absolute temperature (K)

Equations: $V = \dfrac{Q}{4\pi\epsilon_0 r}$ $\quad U = qV \quad U = \dfrac{mv^2}{2} \quad U = \dfrac{3k_\mathrm{B}T}{2}$

25.1 Assume that the large nucleus is held stationary in the material by interatomic forces. Use the equations to derive expressions for

a) the distance of closest approach r in terms of energy U,

b) the distance of closest approach r given the initial speed v,

c) the distance of closest approach r given the temperature T.

Example 1 – *Calculate the distance of closest approach of a* 10 *MeV alpha particle to a gold nucleus (with 79 protons) in a direct collision where the nucleus remains stationary.*
We work out the energy of the alpha particle, remembering that it has two protons $U = qV = 2 \times 1.6 \times 10^{-19} \times 1.0 \times 10^{7} = 3.2 \times 10^{-12}$ J. Now we calculate the distance of closest approach

$$r = \frac{Qq}{4\pi\epsilon_0 U} = \frac{\left(2 \times 1.6 \times 10^{-19}\right)\left(79 \times 1.6 \times 10^{-19}\right)}{4\pi\epsilon_0 \times 3.2 \times 10^{-12}} = 1.14 \times 10^{-14}\ \mathrm{m}$$

25.2 Calculate the distance of closest approach of a proton travelling at $2.0 \times 10^{6}\ \mathrm{m\,s^{-1}}$ to a copper nucleus with 29 protons. Assume that the copper nucleus remains stationary.

25.3 Repeat question 25.2 for a proton with half the speed.

25.4 How much kinetic energy would a proton need to have to approach a gold nucleus (79 protons) and to come within 2 fm of it?

25.5 Fill in the missing entries in the table below for collisions with a stationary gold nucleus.

Projectile	Energy	Closest approach distance
Proton	3.0×10^{-12} J	(a)
Proton	(b)	7.2×10^{-15} m
Alpha particle	(c)	18 fm
Positron	50 keV	(d)

25.6 For collisions of an alpha particle with a gold nucleus held stationary, calculate

a) the distance of closest approach for $v = 1.5 \times 10^6$ m s^{-1},

b) the distance of closest approach for $v = 3.0 \times 10^6$ m s^{-1},

c) the initial speed needed for $r = 7.0$ fm,

d) the initial speed needed for $r = 3.5$ fm.

Example 2 – *What temperature would be needed in order for two deuterium* $\left({}^2_1\mathrm{H}\right)$ *nuclei of typical speeds to come within* 1.0 *fm of each other?*

We need to remember that half of the energy for the approach comes from each nucleus. The total kinetic energy will be $U = 2 \times \frac{3}{2} k_\mathrm{B} T = 3 k_\mathrm{B} T$.

$$3k_\mathrm{B}T = U = \frac{Qq}{4\pi\epsilon_0 r}, \text{ so } T = \frac{qQ}{4\pi\epsilon_0 r}\frac{1}{3k_\mathrm{B}} = \frac{Qq}{12\pi\epsilon_0 k_\mathrm{B} r}$$

$$T = \frac{\left(1.6 \times 10^{-19}\right)^2}{12\pi\epsilon_0 k_\mathrm{B} \times 1.0 \times 10^{-15}} = 5.6 \times 10^9 \text{ K}$$

25.7 What is the distance of closest approach of two gold nuclei if they have kinetic energies typical of material at 1.4×10^7 K?

25.8 What would be the wavelength of a proton which could come within 3.0 fm of a uranium nucleus with 92 protons which was held stationary? Hint: you may wish to revise the link on page 5.

26 Orbits

An orbit is the path that an object follows in a gravitational or electromagnetic field. This includes the paths of the planets around the Sun.

Example context: The planets of the solar system orbit around the Sun due to the gravitational force of attraction between the planets and the Sun. To accelerate a particle in orbit in a particle accelerator the magnetic field strength must be increased so that the radius of the particles orbit remains the same and the particles do not collide with the walls of the accelerator.

Quantities:
- G Newton's gravitational constant $\left(\mathrm{N\,m^2\,kg^{-2}}\right)$
- g gravitational field strength $\left(\mathrm{N\,kg^{-1}}\right)$
- E electric field strength $\left(\mathrm{N\,C^{-1}}\right)$
- B magnetic flux density (T)
- a centripetal acceleration $\left(\mathrm{m\,s^{-2}}\right)$
- ϵ_0 permittivity of free space $\left(\mathrm{F\,m^{-1}}\right)$
- q, Q charge (C)
- r radius of orbit (m)
- F centripetal force (N)
- T orbital period (s)
- m, M mass (kg)
- v velocity $\left(\mathrm{m\,s^{-1}}\right)$

Equations:

$$g = \frac{GM}{r^2} \qquad E = \frac{Q}{4\pi\epsilon_0 r^2} \qquad a = \frac{v^2}{r} \qquad F = ma \qquad v = \frac{2\pi r}{T}$$

$$F = mg \qquad F = qE \qquad F = qvB \qquad r^3 \propto T^2$$

26.1 A moon of mass m moves at speed v in a circular orbit around a planet of mass M

a) Use the equations above to obtain v in terms of G, M and r.

b) Use the equations above to derive Kepler's Third Law: $r^3 \propto T^2$,

c) What is the constant of proportionality r^3/T^2 in terms of G and M?

26.2 A positron of charge $+q$ and mass m enters a magnetic field B travelling at a speed v perpendicular to the direction of the magnetic field.

a) Derive an expression for r in terms of q, B, m and v.

b) If we now change the particle from a positron to a proton, keeping the magnetic field and the velocity of the particle the same, what would happen?

26.3 Calculate the radius of the Moon's orbit around the Earth given that Moon takes approximately 27 days to orbit the Earth and the mass of the Earth is 6.0×10^{24} kg.

26.4 Astronauts on the International Space Station appear weightless because both they and the space station have the same centripetal acceleration and therefore there is no contact force between the astronauts and the floor of the space station. They are in free-fall. What is the centripetal acceleration of the international space station in orbit at a height $h = 400$ km above the surface of the Earth?

Example – *Venus takes* 225 *Earth days to orbit the Sun at an average distance of* 1.08×10^8 *km. What is the mass of the Sun according to this data?*

$$r^3 = \frac{GM}{4\pi^2}T^2 \text{ therefore } M = \frac{4\pi^2 r^3}{GT^2}$$

$$M = \frac{4\pi^2(1.08 \times 10^{11})^3}{6.67 \times 10^{-11}\ (225 \times 24 \times 3600)^2} \approx 1.97 \times 10^{30} \text{ kg}$$

26.5 Calculate the orbital period of Jupiter in units of Earth years given that the mass of the Sun, $M = 2.0 \times 10^{30}$ kg, the mass of Jupiter, $m = 1.9 \times 10^{27}$ kg and the average radius of Jupiter's orbit around the sun is $R = 7.8 \times 10^8$ km.

26.6 Calculate the ratio of the radii of the orbits of Phobos and Deimos, which are the moons of Mars. The mass of Mars is $M = 6.4 \times 10^{23}$ kg, the mass of Phobos $m_1 = 11 \times 10^{15}$ kg and the mass of Deimos $m_2 = 1.5 \times 10^{15}$ kg. The period of Phobos's orbit is $T_1 = 7.7$ hours and of Deimos's orbit is $T_2 = 30.4$ hours.

26.7 61 Cygni is a wide binary star system. It contains two stars of nearly equal mass which orbit once around their mid point every 659 years. They are 1.26×10^{13} m apart. Assuming that the two stars have equal mass, calculate

a) the speed of the stars,

b) the total mass of the system.

26.8 Find an expression for the the ratio of the gravitational field to the electric field, g/E, for an electron that is in orbit at a radius r around the central proton of a hydrogen atom.

26.9 In a particle accelerator protons are accelerated in the $+x$-direction until they have a velocity of $v = 6.5 \times 10^6$ m s^{-1}. They then pass into a magnetic field of strength 0.1 T that is oriented in the $+y$-direction.

a) In which direction do the protons accelerate when they first enter the magnetic field?

b) What is the radius of the orbital path that the protons take?

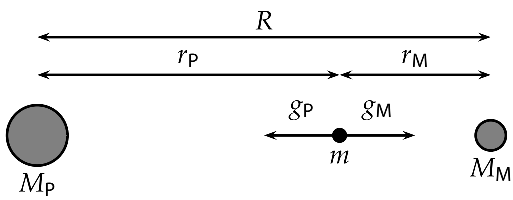
27 Vectors and fields – between a planet and a moon

It is helpful to be able to calculate the gravitational field at points between a planet and its moon.

Example context: Satellites can be placed in orbits in complicated systems if we understand the overall gravitational field. The motion of stars in our galaxy can be used to measure the total mass of the galaxy, and thereby estimate the amount of dark matter in the galaxy.

Quantities:
- m mass of object (kg)
- F force on object (N)
- M_P mass of planet (kg)
- r_P point – planet distance (m)
- M_M mass of moon (kg)
- r_M point – moon distance (m)
- g field at point $\left(\mathrm{N\,kg^{-1}}\right)$
- R planet – moon distance (m)
- g_P field at point due to planet $\left(\mathrm{N\,kg^{-1}}\right)$
- g_M field at point due to moon $\left(\mathrm{N\,kg^{-1}}\right)$

All distances are measured to the centres of planets and moons.



Equations: $F = mg \quad g_\mathrm{P} = \dfrac{GM_\mathrm{P}}{r_\mathrm{P}^2} \quad g_\mathrm{M} = \dfrac{GM_\mathrm{M}}{r_\mathrm{M}^2}$ for magnitudes only

27.1 Taking the direction $\rightarrow$ as positive, write expressions (in terms of m, M_P, M_M, r_P, r_M and G)

a) for the force on m due to the moon,

b) for the force on m due to the planet,

c) for the total force F on m,

d) for the total field g at the point where m is,

e) relating r_P and r_M for the location where $g = 0$.

27.2 Now repeat the first four parts of question 27.1 for the situation where $r_\mathrm{P} > R$, and the mass m is on the far side of the moon.

27.3 Calculate g_M on the surface of the Earth nearest the Moon. The radius of the Earth is 6.37×10^6 m, the mass of the Moon is $M_\mathrm{M} = 7.38 \times 10^{22}$ kg, and $R = 3.85 \times 10^8$ m.

Example – *Calculate the distance from the centre of a planet $M_P = 5.0 \times 10^{24}$ kg at which there is no net force on an object. The planet's moon has a mass of 5.0×10^{22} kg and orbits at a radius of 4.4×10^7 m.*

At this point total field $g = -\dfrac{GM_P}{r_P^2} + \dfrac{GM_M}{r_M^2} = 0$ so $\dfrac{M_P}{r_P^2} = \dfrac{M_M}{r_M^2}$

$$\frac{r_M}{r_P} = \sqrt{\frac{M_M}{M_S}} = \sqrt{\frac{5.0 \times 10^{22}}{5.0 \times 10^{24}}} = 0.1$$

$r_P + r_M = R$, therefore $\dfrac{r_P}{r_P} + \dfrac{r_M}{r_P} = \dfrac{R}{r_P}$. So $1 + 0.1 = \dfrac{R}{r_P}$

Therefore $r_P = \dfrac{R}{1.1} = \dfrac{4.4 \times 10^7}{1.1} = 4.0 \times 10^7$ m

27.4 For the system in the example, calculate the total field g at the locations below on the same side of the planet as the moon. Take the direction from the planet to the moon as positive. Assume that the planet and moon have radii less than 10^6 m.

a) $r_P = 2.2 \times 10^7$ m

b) $r_P = 3.9 \times 10^7$ m

c) $r_P = 4.1 \times 10^7$ m

d) $r_P = 1.6 \times 10^6$ m

27.5 For the Earth - Moon system, $M_P = 81 M_M$ and $R = 3.85 \times 10^8$ m.

a) Calculate r_M / r_P for the place where $g = 0$.

b) Evaluate r_M (for the same place) as a fraction of R.

c) Evaluate r_P at this place.

27.6 Mars has a mass of 6.39×10^{23} kg, and its 1.06×10^{17} kg moon Phobos has an orbital radius of 9.38×10^6 m. Calculate the gravitational field strength 6.0 km from the centre of Phobos on its surface nearest to Mars. Does the field point towards or away from Mars?

27.7 The mass of the Sun is 2.00×10^{30} kg, and the Earth-Sun distance is 1.50×10^{11} m. You may also use data from questions 27.3 and 27.5. Work out the component of the field g due to the Sun and the Moon (separately) pointing towards the centre of the Earth at the locations below:

a) at the surface of the Earth, nearest the Moon, at a full Moon,

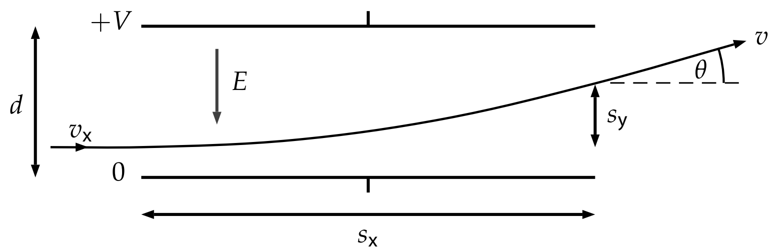
b) at the surface of the Earth, nearest the Sun, at a full Moon.

c) Given your answers, what is principally responsible for the Earth's tides? The Sun or the Moon?

28 Vectors and fields – electric deflection

When a charged particle moves in a region with a uniform electric field perpendicular to its motion, it is deflected sideways and follows a parabolic path. You can calculate the deflection distance and the angle by which its velocity changes.

Example context: Some cathode ray tubes (CRTs) deflect electrons in this way to aim them at a screen, especially those used in oscilloscopes in the 20th Century.



Quantities:

m mass (kg)	a_y acceleration $(\mathrm{m\,s^{-2}})$
q charge (C)	v_x initial speed $(\mathrm{m\,s^{-1}})$
V voltage across plates (V)	v_y new velocity component $(\mathrm{m\,s^{-1}})$
d plate separation (m)	v new speed $(\mathrm{m\,s^{-1}})$
E electric field $(\mathrm{V\,m^{-1}})$	s_x length of field region (m)
F_E electric force (N)	s_y deflection distance (m)
t time spent in field (s)	θ deflection angle (°)

Equations:

$$E = V/d \qquad F_E = qE \qquad F_E = ma_y \qquad v_y = a_y t$$
$$s_x = v_x t \qquad s_y = \tfrac{1}{2}a_y t^2 \qquad \tan\theta = v_y/v_x \qquad v = \sqrt{v_x^2 + v_y^2}$$

28.1 Use the equations to derive expressions for

a) a_y in terms of V, d, q and m,

b) s_y in terms of v_x, s_x, V, d, q and m,

c) θ in terms of v_x, s_x, V, d, q and m,

d) s_y in terms of t, E, q and m,

e) θ in terms of v_x, t, E, q and m.

28.2 Find F_E and a_y if $q = 1 \times 10^{-12}$ C, $m = 1 \times 10^{-20}$ kg, $V = 5$ V and $d = 0.1$ m.

28.3 Using the values from the previous question, find s_y and θ if $v_x = 5 \times 10^4\ \mathrm{m\,s^{-1}}$ and $s_x = 0.2$ m.

> **Example** – *A charge $q = 1.0 \times 10^{-17}$ C with mass 2.0×10^{-22} kg moving at 8000 m s^{-1} passes through a region of sideways electric field for 0.1 ms and is deflected by $750\ \mu$m. What is the electric field strength?*
>
> $$s_y = \frac{1}{2}\left(\frac{qE}{m}\right)t^2 \text{ so } E = \frac{2ms_y}{qt^2} = \frac{2(2\times10^{-22})(750\times10^{-6})}{(10^{-17})(10^{-4})^2} = 3.0\,\mathrm{V\,m^{-1}}$$

28.4 Fill in the missing entries in the table below for an electron with charge 1.60×10^{-19} C and mass 9.11×10^{-31} kg.

V / V	d / m	v_x / m s^{-1}	s_x / m	s_y / m	θ / °
0.25	0.20	4.0×10^6	1.0	(a)	(b)
(c)	0.010	7.0×10^6	0.025	0.0020	(d)
0.060	0.012	9.0×10^5	(e)	(f)	2.0

28.5 Fill in the missing entries in the table below for an α-particle with charge 3.20×10^{-19} C and mass 6.64×10^{-27} kg.

E / V m^{-1}	t / s	v_x / m s^{-1}	v_y / m s^{-1}	s_y / m	θ / °
100	5.0×10^{-6}	40 000	(a)	(b)	(c)
4.8	(d)	2.5×10^5	(e)	(f)	0.40
(g)	1.4×10^{-5}	(h)	(i)	0.0027	7.3

28.6 ^{23}Na$^+$ ions (mass 23 u; 1 u $= 1.66 \times 10^{-27}$ kg) are accelerated to 9600 m s^{-1} and deflected by an electric field region of length 3.0 cm between plates of separation 1.5 cm. Find the deflection distance if the plate voltage is

a) 500 mV, b) 3.0 V, c) 10 V.

28.7 An engineer claims electrons with initial speed 800 km s^{-1} are deflected by 45° after travelling through a 5.0 mm-wide region between plates with a voltage 150 mV between them. Find s_y and d. Is this possible?

28.8 In an oscilloscope, a deflector applies up to 1 kV between two 3.0 mm-long plates separated by 2.0 mm. Electrons with initial kinetic energy 1500 eV pass through the deflector.

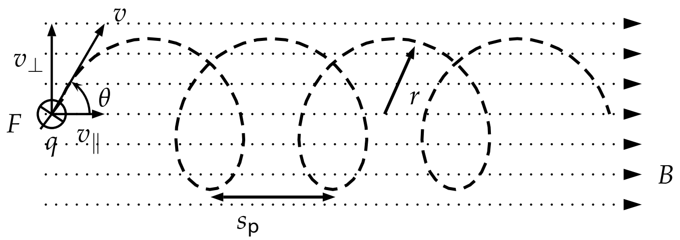
a) By what angle are electrons deflected at the maximum voltage?

b) A phosphorescent screen is placed 10 cm away so that undeflected electrons hit the centre of the screen. How far from the centre of the screen can deflected electrons land?

29 Vectors and Fields - helix in magnetic field

When a charged particle moves in a region with a uniform magnetic field , it will follow a helical path, like a corkscrew or a screw. To determine the size of the helix and its pitch, you need to consider the components of velocity parallel and perpendicular to the magnetic field.

Example context: Cosmic rays spiral along our Earth's magnetic field. They then travel to our polar regions along the magnetic field lines where they collide with other particles and produce the Northern and Southern Lights.



Quantities:	q charge (C)	v velocity $\left(\mathrm{m\,s^{-1}}\right)$
	B magnetic field (F)	F force (N)
	s_p pitch of helix (m)	r radius of helix (m)
	T period of rotation (s)	m mass (kg)
	θ angle between velocity and magnetic field $(^\circ)$	
	Subscripts $\perp$, $\parallel$ refer to perpendicular and parallel components.	

Equations:

$$F = qv_\perp B \qquad s_\mathrm{p} = v_\parallel T \qquad F = m\frac{v_\perp^2}{r}$$

$$v_\parallel = v\cos\theta \qquad v_\perp = v\sin\theta$$

29.1 Use the equations to derive expressions for

a) r in terms of m, q, v, B, and θ,

b) T in terms of m, q, and B,

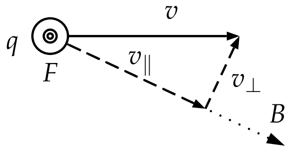
c) v in terms of $v_\parallel$ and $v_\perp$,

d) q/m, known as the charge to mass ratio, in terms of v, B, s_p, and θ.

Example – *A charge of* -1.0×10^{-17} *C with mass* 2.0×10^{-22} *kg moving at* 8000 *m s*$^{-1}$ *along the* x*-axis enters a magnetic field magnitude* 510 *mT. The magnetic field is at* 25° *to the* x*-axis. What is the force from the magnetic field on the charge?*

The magnitude of the force is

$$F = qvB\sin\theta = 1\times10^{-17}\times8000\times0.510\times\sin(25^\circ) = 1.7\times10^{-14}\text{ N}$$



To find the direction of the force on the particle, use Fleming's left-hand rule. If the charge is negative, the force will be pointing in the opposite direction to the thumb. Here, the force is pointing out of the page.

29.2 A charged particle of mass 1.05×10^{-25} kg and charge 3.2×10^{-19} C enters a 1.45 T magnetic field with a speed of 3.2×10^{6} m s^{-1}. The velocity of the particle is at an angle of 30° to the magnetic field. What is the radius and the pitch of the helix that the particle now follows?

29.3 An α-particle with charge to mass ratio $q/m = 4.82\times10^{7}$ C kg^{-1} is detected in a large cloud chamber with a 1.5 T magnetic field. It followed a helical path with a pitch of 44.6 cm and radius 19.5 cm. What was the kinetic energy of the α-particle?

29.4 A charged particle moves in a magnetic field. It follows a helical path with a radius 19 cm and pitch 1.20 m.

a) At what angle to the magnetic field was the particle travelling at when it entered the magnetic field?

b) If $q/m = 2.47\times10^{3}$ C kg^{-1} and the magnitude of the magnetic field is 1.78 T, what was the speed when the particle entered the field?

c) What is the speed of the particle now that it is in the magnetic field?

29.5 A proton of mass 1.67×10^{-27} kg and charge 1.6×10^{-19} C enters a 500 mT magnetic field parallel to its velocity. Both are along the positive x-axis. The proton is travelling with velocity 4.3×10^{7} m s^{-1}.

a) What is the magnitude and direction of the force acting on the particle?

b) Will it travel along a helical, circular or straight path? If the path of the proton is a helix or a circle what is the pitch and radius of the helix?

c) The same proton now enters a 500 mT magnetic field perpendicular to its direction of travel, i.e. the magnetic field is along the positive y-axis. What will the magnitude and direction of the force acting on the particle be now?

d) Will it travel along a helical, circular or straight path in the perpendicular field? If the path of the proton is a helix or a circle what is the pitch and radius of the helix?

24 Energy and fields – relativistic accelerator

(a) $\gamma = \frac{E}{mc^2} = \frac{K + mc^2}{mc^2} = \frac{K}{mc^2} + 1 = \frac{qV}{mc^2} + 1$

(b) $\gamma^{-2} = 1 - \frac{v^2}{c^2}$ so $\frac{v}{c} = \sqrt{1 - \gamma^{-2}}$ and $v = c\sqrt{1 - \gamma^{-2}}$

(c) $v = c\sqrt{1 - \gamma^{-2}} = c\sqrt{1 - \left(1 + \frac{qV}{mc^2}\right)^{-2}}$

(d) $p^2 = \gamma^2 m^2 v^2 = \frac{m^2 c^2 \left(v^2/c^2\right)}{1 - v^2/c^2} = \frac{m^2 c^2 \left(v^2/c^2 - 1 + 1\right)}{1 - v^2/c^2}$

$= -m^2 c^2 + \frac{m^2 c^2}{1 - v^2/c^2} = -m^2 c^2 + \gamma^2 m^2 c^2$

therefore $p^2 c^2 = -m^2 c^4 + \gamma^2 m^2 c^4 = E^2 - m^2 c^4$

(e) $p^2 = \frac{E^2}{c^2} - m^2 c^2 = \frac{\left(K + mc^2\right)^2 - m^2 c^4}{c^2} = \frac{K^2 + 2Kmc^2}{c^2}$

$= \frac{K^2}{c^2} + 2Km = \frac{q^2 V^2}{c^2} + 2qVm$

25 Energy and fields – closest approach

(a) $U = qV = \frac{Qq}{4\pi\epsilon_0 r}$, so $r = \frac{Qq}{4\pi\epsilon_0 U}$

(b) $r = \frac{Qq}{4\pi\epsilon_0 U}$ and $U = \frac{mv^2}{2}$, so $r = \frac{Qq}{4\pi\epsilon_0}\frac{2}{mv^2} = \frac{Qq}{2\pi\epsilon_0 mv^2}$

(c) $r = \frac{Qq}{4\pi\epsilon_0 U}$ and $U = \frac{3k_B T}{2}$, so $r = \frac{Qq}{4\pi\epsilon_0}\frac{2}{3k_B T} = \frac{Qq}{6\pi\epsilon_0 k_B T}$

26 Orbits

(1a) Newton's Second Law: $m\frac{v^2}{r} = \frac{GMm}{r^2}$ so $v^2 = \frac{GM}{r}$

(1b) From (a) $v^2 = \frac{GM}{r}$ We also know $v = \frac{2\pi r}{T}$ so $v^2 = \frac{4\pi^2 r^2}{T^2}$

Therefore $\frac{GM}{r} = \frac{4\pi^2 r^2}{T^2}$ and so $4\pi^2 r^3 = GMT^2$

(1c) From (b) $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$

(2a) Newton's Second Law: $m\frac{v^2}{r} = qvB$ so $r = \frac{mv}{Bq}$

27 Vectors and fields – between a planet and a moon

(a) $F_{\mathrm{M}} = mg_{\mathrm{M}} = +\frac{GM_{\mathrm{M}}m}{r_{\mathrm{M}}^2}$

(b) $F_{\mathrm{P}} = mg_{\mathrm{P}} = -\frac{GM_{\mathrm{P}}m}{r_{\mathrm{P}}^2}$

(c) $F = F_{\mathrm{M}} + F_{\mathrm{P}} = +\frac{GM_{\mathrm{M}}m}{r_{\mathrm{M}}^2} - \frac{GM_{\mathrm{P}}m}{r_{\mathrm{P}}^2}$

(d) $g = \frac{F}{m} = +\frac{GM_{\mathrm{M}}}{r_{\mathrm{M}}^2} - \frac{GM_{\mathrm{P}}}{r_{\mathrm{P}}^2}$

(e) $g = 0$ so $\frac{GM_{\mathrm{M}}}{r_{\mathrm{M}}^2} = \frac{GM_{\mathrm{P}}}{r_{\mathrm{P}}^2}$ therefore $\frac{M_{\mathrm{M}}}{r_{\mathrm{M}}^2} = \frac{M_{\mathrm{P}}}{r_{\mathrm{P}}^2}$ and $\frac{r_{\mathrm{P}}}{r_{\mathrm{M}}} = \sqrt{\frac{M_{\mathrm{P}}}{M_{\mathrm{M}}}}$

28 Vectors and fields – electric deflection

(a) $a_{\mathrm{y}} = \frac{F_{\mathrm{E}}}{m} = \frac{qE}{m} = \frac{qV}{dm}$

(b) $s_{\mathrm{y}} = \frac{1}{2}\left(\frac{qV}{dm}\right)t^2 = \frac{1}{2}\left(\frac{qV}{dm}\right)\left(\frac{s_{\mathrm{x}}}{v_{\mathrm{x}}}\right)^2 = \frac{qVs_{\mathrm{x}}^2}{2dmv_{\mathrm{x}}^2}$

(c) $\theta = \tan^{-1}\left(\frac{v_{\mathrm{y}}}{v_{\mathrm{x}}}\right) = \tan^{-1}\left(\frac{a_{\mathrm{y}}t}{v_{\mathrm{x}}}\right) = \tan^{-1}\left(\frac{qVs_{\mathrm{x}}}{dmv_{\mathrm{x}}^2}\right)$

(d) $s_{\mathrm{y}} = \frac{1}{2}\left(\frac{qE}{m}\right)t^2 = \frac{qEt^2}{2m}$

(e) $\theta = \tan^{-1}\left(\frac{a_{\mathrm{y}}t}{v_{\mathrm{x}}}\right) = \tan^{-1}\left(\frac{qEt}{mv_{\mathrm{x}}}\right)$

29 Vectors and fields – helix in magnetic field

(a) $F = ma$, so $qv_\perp B = \dfrac{mv_\perp^2}{r}$ and $qvB\sin\theta = \dfrac{mv^2\sin^2\theta}{r}$.

Rearranging gives $r = \dfrac{mv\sin\theta}{qB}$

(b) From a): $r = \dfrac{mv\sin\theta}{qB}$ and $v_\perp = \dfrac{2\pi r}{T} = v\sin\theta$.

$$\text{So } r = \frac{mv\sin\theta}{qB} = \frac{T}{2\pi}v\sin\theta. \text{ Therefore } T = \frac{2\pi m}{qB}$$

(c) $v_\perp^2 + v_\parallel^2 = v^2\sin^2\theta + v^2\cos^2\theta = v^2\left(\sin^2\theta + \cos^2\theta\right) = v^2$.

Thus $v = \sqrt{v_\perp^2 + v_\parallel^2}$.

(d) From c): $T = \dfrac{2\pi m}{qB}$ and $s_\mathrm{p} = v_\parallel T = v\cos\theta\dfrac{2\pi m}{qB}$.

Re-arranging gives $q/m = \dfrac{2\pi}{Bs_\mathrm{p}}v\cos\theta$.

30 Vectors and fields – mass spectrometer

(a) $F_\mathrm{B} = ma$ so $Bqv = \dfrac{mv^2}{r}$. Rearranging gives $r = \dfrac{mv}{Bq}$

(b) $qV_\mathrm{a} = \frac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2qV_\mathrm{a}}{m}}$. Now using our result for r from (a),

$$r = \frac{mv}{Bq} = \frac{m}{Bq}\sqrt{\frac{2qV_\mathrm{a}}{m}} = \sqrt{\frac{2mV_\mathrm{a}}{B^2q}}$$

(c) From (a): $r = \dfrac{mv}{Bq}$ so $\dfrac{q}{m} = \dfrac{v}{Br}$

(d) From (b): $r^2 = \dfrac{2mV_\mathrm{a}}{B^2q}$ so $\dfrac{q}{m} = \dfrac{2V_\mathrm{a}}{B^2r^2}$

(e) $F_\mathrm{E} = F_\mathrm{B}$ so $qE = qvB$ and $E = vB$. So $V_\mathrm{s} = Ed = vBd$