Additional Problems: **Wind–driven yachts, sand yachts and ice boats.**

Introduction

We explore the physics of sailing across the wind where the sail acts as an aerofoil (think of an aircraft wing). A question throws this matter into focus: how is it possible that any wind–driven craft can travel faster than the wind?

As typical in physics, we break the question into sub-questions [each requiring its own physics and maths].

(A) How is the propulsion force generated from the wind? [Conservation of momentum, momentum flux.]

(B) How can a vehicle propelled by the wind travel faster than the wind? ["Apparent wind", vectors.]

(C) What is the maximum speed that a wind–driven vehicle can attain? At what angle must you sail to the wind to attain this maximum speed? [Newton's laws, trigonometric identities.]

(D) What limits the actual speed of a wind–driven craft? [Resistance, graphs, stability.]

Figure 1: Racing yacht [photo: Ogden Trust]

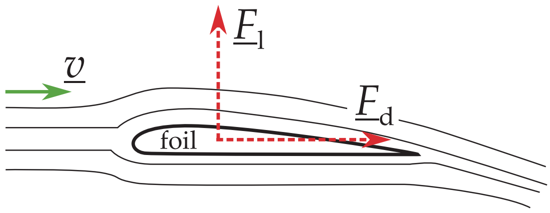


Figure 2: An aerofoil seen in section with the lift and drag forces, $\underline{F_l}$ and $\underline{F_d}$. Note that the flow is deflected downwards so that the overall momentum change of the flow and forces on the wing add up.

1 - Preliminary

The aerofoil offers lift (and suffers some drag) when it harnesses the momentum flux of the wind flowing around it; see Figure 2.

The *flux* of mass, momentum, energy, . . . is how much of these quantities is intercepted by unit surface in unit time from an incoming flow.

Momentum flux is the rate of flow of momentum per unit area, and is a force per unit area; see Figure 3.

The cuboid of fluid passing through the area A in time $\mathrm{d}t$ has volume $\mathrm{d}V = Av\,\mathrm{d}t$ and mass $\mathrm{d}m = \rho Av\,\mathrm{d}t$, where v is the wind speed, and ρ is the fluid density.

It has speed v, thus the momentum is:

$$dp = v\,\mathrm{d}m = \rho A v^2\,\mathrm{d}t.$$

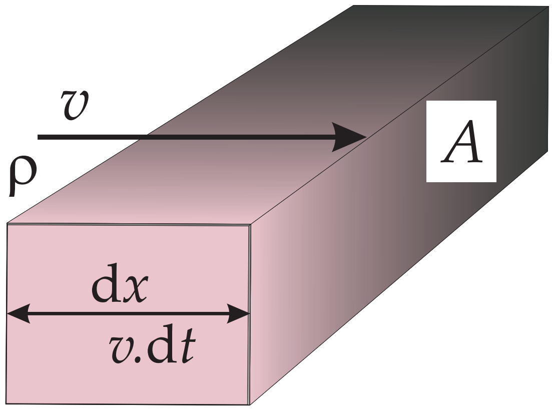


Figure 3: Fluid of density ρ moves with speed v. The cuboid of fluid of end area A and thickness $\mathrm{d}x = v\,\mathrm{d}t$ passes through A in time $\mathrm{d}t$.

Recall Newton's second law: force is the rate of change of momentum. The momentum *flux* (that is, momentum per unit time per unit area), on dividing by $\mathrm{d}t$ and A, is then ρv^2.

Forces from such flows are of the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\rho A v^2\,\mathrm{d}t}{\mathrm{d}t} = \rho A v^2. \tag{1}$$

The drag and lift forces on an *aerofoil* depend on the momentum flux, with directions as in Figure 2:

$$F_l = \tfrac{1}{2} c_l \rho_a A_f v^2 \qquad F_d = \tfrac{1}{2} c_d \rho_a A_f v^2. \tag{2}$$

A_f is the frontal projected area of the foil, c_l and c_d are lift and drag coefficients of the foil, the $\frac{1}{2}$ is conventional, v is the wind speed *seen by the foil* – very important in what follows – and ρ_a is now the density of air.

(a) Sketch Figure 2 for yourself, add an arrow for the resultant force $\underline{F}$ to your diagram and give an expression for γ, the angle $\underline{F}$ makes with $\underline{F}_l$.

(b) For a yacht with values[1] of $c_l = 2.5$ and $c_d = 1.0$, determine a value for the angle γ. Clearly the smaller drag is compared with lift, the smaller γ is (a desirable characteristic).

2 - The "apparent wind" direction on a moving vehicle.

Consider a wind with velocity $\underline{w}$ and a wind–driven craft such as a yacht proceeding with velocity $\underline{u}$ (see Figure 4). The faster you sail, the closer the "apparent wind" $\underline{v}$ is to your direction of motion, that is $\beta < \alpha$.

The aerofoil is aligned around the direction $\underline{v}$ to generate the forces discussed in section 1. Masts often have a pennant at their tip; it points in the direction of $\underline{v}$. For a given fixed α, one can then align[2] the sail close to $\underline{v}$ as this apparent velocity changes direction due to changes in speed u or direction α.

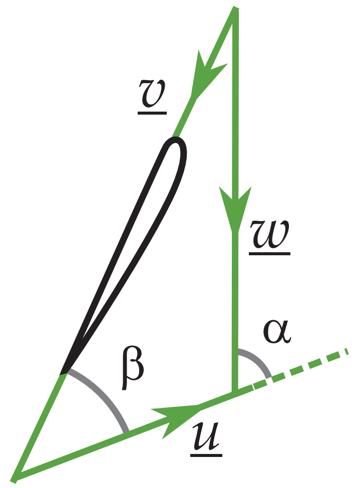


Figure 4: Motion $\underline{u}$ in a wind $\underline{w}$ gives an apparent incident wind velocity of $\underline{v} = \underline{w} - \underline{u}$ at an angle β to $\underline{u}$. We simplify by aligning the aerofoil in the direction of $\underline{v}$.

(a) Prove the equivalent relations:

$$v\cos(\beta) = w\cos(\alpha) + u \quad \text{and} \quad v\sin(\beta) = w\sin(\alpha).$$

[Hint: for the former consider components in the direction of $\underline{u}$ (or equivalently consider $\underline{u} \cdot \underline{v}$), and for the latter, the sine rule.]
Use the cosine rule to prove that these relations are identical.

(b) Show that the aerofoil's force on the yacht along the direction of its motion is $F_u = F\sin(\beta - \gamma)$ where $F^2 = F_d^2 + F_l^2$.
[Hint: See Figure 5; note that this force includes elements of lift and drag.]

(c) Figure 5 is in the frame of the craft (the frame of reference in which the craft is at rest. Stall is when the force in the direction of $\underline{u}$ drops to zero when at zero speed. How close (α_{min}) can one sail to the wind before one approaches stall conditions?

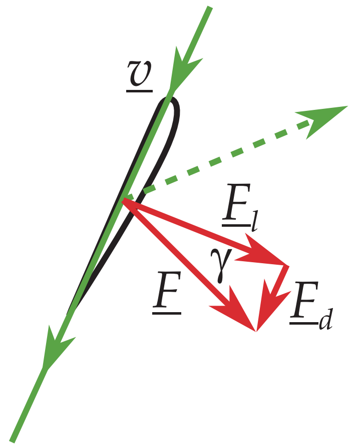


Figure 5: Wind and forces $\underline{F}_l$ and $\underline{F}_d$, perpendicular and parallel to $\underline{v}$ respectively, drawn in the frame of the boat. We assume that motion perpendicular to the direction of travel (dotted) is suppressed - e.g. by a keel or centreboard.

3 - The maximum speed of the craft.

The craft will reach its maximum velocity when the *net* force on the craft along $\underline{u}$ is zero: $F_u = F_f$ where F_f is the resistive force, other than from the sail, impeding the motion.

(a) Show that the force along the direction of travel, reduced by dividing by the scale of force expected simply in the wind, $\frac{1}{2}\rho_a w^2 Ac$, is $F_r = \frac{F_u}{\frac{1}{2}\rho_a w^2 Ac}$, is:

$$F_r = \frac{F_u}{\frac{1}{2}\rho_a w^2 Ac} = \frac{v}{w}\left(\sin(\alpha - \gamma) - \frac{u}{w}\sin(\gamma)\right) \tag{3}$$

where $c = \sqrt{c_l^2 + c_d^2}$. The speed of the craft reduced by the wind speed is $\frac{u}{w}$, and F_r gives a reduced measure of the force F_u. Note that both quantities are unitless.

Give an expression for the reduced apparent wind speed $\frac{v}{w}$.

(b) Consider the case where there is no resistance to motion, other than F_d from the sail itself. Show that the maximum reduced speed is given by $\frac{u}{w}|_{max} = \frac{\sin(\alpha-\gamma)}{\sin\gamma}$.
[Hint: see Figure 5]

(c) Find the angle of motion, α, at which $\frac{u}{w}|_{max}$ is itself maximised.

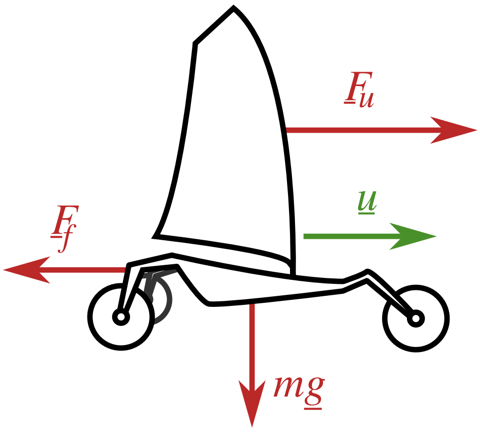
(d) Using the relation above for $\frac{u}{w}|_{max}$ verify your previous calculation for the angle at which stall occurs.

[1] The lift coefficient, used to rate the efficiency of a wing is around 2 to 2.5 for a yacht aerofoil. The lift coefficient for a state-of-the-art, traditional-style sail set-up is approximately 1.5 to 2.

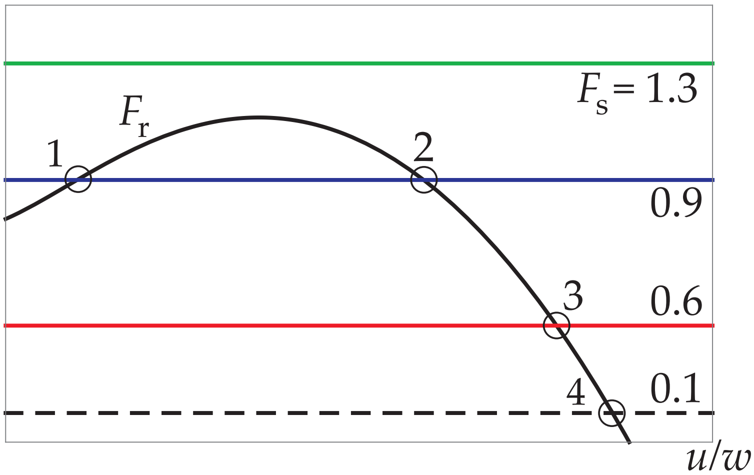
[2] Choosing the precise alignment, and the shape of the sail, is the skill of the sailor – the choices effect c_l, c_d and hence γ, and can differ according to the direction of travel, α, with respect to the wind – something ignored here.

D - What determines the actual speed u?

Resistive forces from motion cause the maximum velocity to be limited at $F_u = F_f$. An ice boat and a sand yacht are both wind–driven craft; the ice boat sits on runners and the sand yacht on wheels. Both crafts experience resistive forces caused by friction given by $F_f = \mu_d mg$ where μ_d is the coefficient of dynamic friction and m their mass; see Figure 6a. One reduces this force as $F_s = \frac{F_f}{\frac{1}{2}\rho_a w^2 Ac}$.



(a) The weight mg, the resistive force $F_f = \mu_k mg$ and the driving force F_u acting on a sand yacht



(b) F_r against $\frac{u}{w}$ with $\gamma = 10°$ (a typical value for a sand yacht) and $\alpha = 60°$. The solid lines show typical values of F_s for a sand yacht and the dotted line shows a typical value of F_s for an ice boat.

Figure 6: The forces experienced by a sand yacht and an ice boat.

(a) Looking at Figure 6b, F_r against $\frac{u}{w}$, determine whether points 1, 2 and 3 of $F_u = F_f$ are stable.

(b) What is F_r at $u = 0$ for the γ and α values of Figure 6b?

(c) What is the maximum reduced speed, $\frac{u}{w}|_{max}$, of this ice boat? Ignore frictional forces and use values of $\gamma = 10°$ and $\alpha = 60°$, as in Figure 6b.

(d) For each friction curve, *estimate* the reduced speed that needs to be reached before the craft moves under the force of the wind only. [Hint: see figure 6b, establishing the u/w scale from the $F_r = 0$ intercept, and using measurement.]

(e) What is the smallest value of the reduced frictional force F_s for which no motion is possible? [Hint: use your answer in (b) for the γ and α values of Figure 6b and a measurement on Figure 6b]

(f) *Estimate* the maximum speed of the ice boat without ignoring frictional forces for $F_s = 0.1$.
Estimate the speed of the sand yacht at point 3.

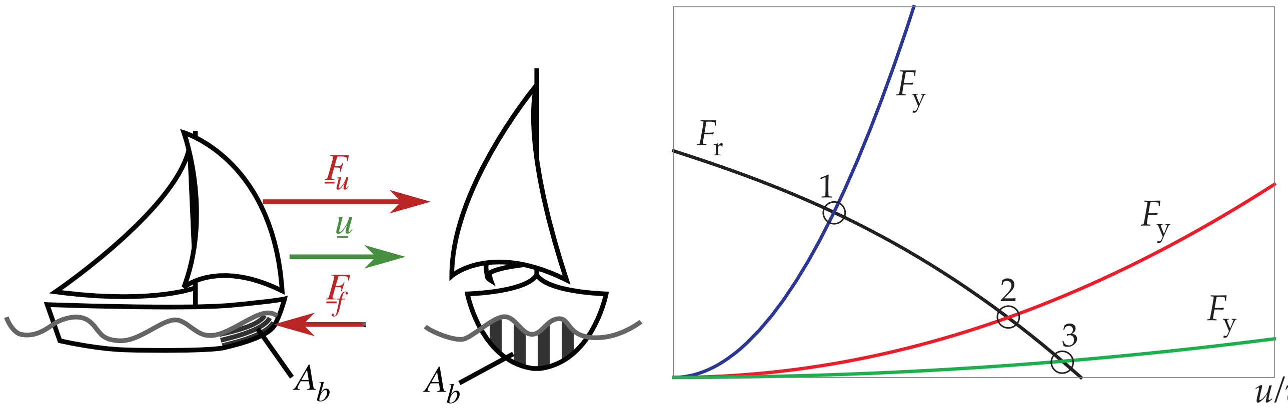
For a normal yacht, the resistive forces $F_f = \frac{1}{2}\rho_w u^2 A_b c_f$ are caused by the inertial drag from moving through the water at speed u with respect to the water, A_b being the area of the boat presented to the water, c_f the drag coefficient, and ρ_w the density of water; see Figure 7a. We can reduce the force as before by dividing F_f by the wind force scale $\frac{1}{2}\rho_a w^2 Ac$ to give the reduced resistance $F_y = \frac{F_f}{\frac{1}{2}\rho_a w^2 Ac}$. For normal yachts:

(g) In Figure 7b, determine whether points 1, 2 and 3 are stable.

(h) What is the maximum reduced speed[3] of this yacht? Ignore inertial drag from the water and use values of γ calculated from c_l and c_d and $\alpha = 60°$ as in Figure 7b.

(i) Assuming the lowest inertial drag F_y, *estimate* the maximum reduced speed of the yacht.

[3] In late 2012 the Vestas Sailrocket 2 skippered by Paul Larsen achieved a new outright world speed record of around 2.5 times the speed of the wind. Also in 2009, the world land speed record for a wind-powered vehicle was set by the sand yacht Greenbird sailing at about three times the speed of the wind. On ice it is accepted that craft often travel at as much as five times the speed of the wind. Compare these values with your calculations.



(a) Inertial drag $F_f = \frac{1}{2}\rho_w u^2 A_b c_f$ and driving force F_u.

(b) F_r varying with u/w for γ based on $c_l = 2.5$ and $c_d = 1.0$, and for $\alpha = 60\,^\circ$, with inertial drag curves F_y for various c_f.

Figure 7: The forces experienced by a yacht.

[MW and RP; September 2014]

Figure 8: Racing yacht with physicist at the helm!