Isaac Physics Skills

Linking concepts in pre-university physics

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TABLE OF PHYSICAL CONSTANTS

Quantity & Symbol		**Magnitude**	**Unit**
Permittivity of free space	ϵ_0	8.85×10^{-12}	$\mathrm{F\,m^{-1}}$
Electrostatic force constant	$1/4\pi\epsilon_0$	8.99×10^{9}	$\mathrm{N\ m^{2}\ C^{-2}}$
Speed of light in vacuum	c	3.00×10^{8}	$\mathrm{m\,s^{-1}}$
Specific heat capacity of water	c_{water}	4180	$\mathrm{J\,kg^{-1}\,K^{-1}}$
Charge of proton	e	1.60×10^{-19}	C
Gravitational field strength on Earth	g	9.81	$\mathrm{N\,kg^{-1}}$
Universal gravitational constant	G	6.67×10^{-11}	$\mathrm{N\ m^{2}\ kg^{-2}}$
Planck constant	h	6.63×10^{-34}	J s
Boltzmann constant	k_{B}	1.38×10^{-23}	$\mathrm{J\,K^{-1}}$
Mass of electron	m_{e}	9.11×10^{-31}	kg
Mass of neutron	m_{n}	1.67×10^{-27}	kg
Mass of proton	m_{p}	1.67×10^{-27}	kg
Mass of Earth	M_{Earth}	5.97×10^{24}	kg
Mass of Sun	M_{Sun}	2.00×10^{30}	kg
Avogadro constant	N_{A}	6.02×10^{23}	$\mathrm{mol^{-1}}$
Gas constant	R	8.31	$\mathrm{J\ mol^{-1}\ K^{-1}}$
Radius of Earth	R_{Earth}	6.37×10^{6}	m

OTHER INFORMATION YOU MAY FIND USEFUL

Electron volt	1 eV	=	1.60×10^{-19} J
Unified mass unit	1 u	=	1.66×10^{-27} kg
Absolute zero	0 K	=	-273 °C
Year	1 yr	=	3.16×10^{7} s
Light year	1 ly	=	9.46×10^{15} m
Parsec	1 pc	=	3.09×10^{16} m

PREFIXES

1 km = 1000 m	1 Mm = 10^{6} m	1 Gm = 10^{9} m	1 Tm = 10^{12} m
1 mm = 0.001 m	1 μm = 10^{-6} m	1 nm = 10^{-9} m	1 pm = 10^{-12} m

3 Momentum and kinetic energy

It is helpful to be able to calculate a momentum from a kinetic energy without first working out the speed.

Example context: In particle physics, the wavelength of a particle is related to its momentum. In a question you are more likely to be told its energy (eg. a 50 keV electron) than its speed.

Quantities:	p momentum $\left(\text{kg m s}^{-1}\right)$	E kinetic energy (J)
	m mass (kg)	λ wavelength (m)
	v speed $\left(\text{m s}^{-1}\right)$	q charge (C)
	V accelerating voltage (V)	h Planck constant (J s)

Equations: $p = mv \quad E = \frac{1}{2}mv^2 \quad E = qV \quad \lambda = \dfrac{h}{p}$

3.1 Use the equations to derive expressions without v for

a) the kinetic energy E in terms of p and m,

b) the momentum p in terms of E and m,

c) the momentum of an accelerated particle in terms of V, m and q,

d) the wavelength of an accelerated particle in terms of V and q.

Example 1 – *Calculate the kinetic energy of a 9 kg pumpkin with a momentum of 150 kg m s^{-1}.*

$$E = \frac{m}{2}v^2 = \frac{m}{2}\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m} = \frac{150^2}{2 \times 9} = 1250 \text{ J}$$

Example 2 – *calculate the wavelength of a 1 keV electron.*
Kinetic energy $E = qV$ where q is the charge on one electron and $V = 1000$ V. As $E = \frac{1}{2}mv^2$, the momentum will be

$$p = mv = m\sqrt{\frac{2E}{m}} = \sqrt{2mE} = \sqrt{2mqV}$$, so we calculate $\lambda = \dfrac{h}{p}$ as

$$\lambda = \frac{h}{\sqrt{2mqV}} = \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19} \times 10^3}} = 3.89 \times 10^{-11} \text{ m}$$

3.2 Calculate the kinetic energy of a $p = 23\,700$ kg m s^{-1}, 720 kg car.

3.3 Fill in the missing entries in the table below.

Mass / kg	Momentum / kg m s^{-1}	Kinetic energy / J
32	(a)	0.040
5.6	252	(b)
4.6×10^{-3}	(c)	980
12 000	168 000	(d)

3.4 Calculate the momentum of a 200 g orange with 54 J of kinetic energy.

3.5 Calculate the momentum of a proton accelerated by 20 kV.

3.6 Calculate the kinetic energy of a neutron with a wavelength of 2.4 nm.

3.7 Calculate the wavelength of an 80 keV electron.

3.8 Calculate the accelerating voltage needed to produce protons with a wavelength of 3.5 pm.

3.9 Calculate the wavelength of a 50 MeV proton.

3.10 Calculate the wavelength of a 10 MeV alpha particle.

3.11 A 10 MeV particle in a particle detector travels on a curved path in a magnetic field. Its charge is 1.60×10^{-19} C. From the curvature, the momentum of the particle is calculated to be 7.31×10^{-20} kg m s^{-1}.

a) What is the mass of the particle?

b) What is the particle?

3.12 A 15 g bullet hits and stops within a 1.500 kg sandbag, which then swings up by a height of 5.1 cm. Work out the initial speed of the bullet. Hint: the height can be used to work out the gravitational potential energy, and hence the initial kinetic energy of the bag. The momentum of the bag just after the collision will be equal to the momentum of the bullet before it.

7 Photon flux for an LED

Photon flux (the number of photons per second) is closely related to intensity of light. Understanding how light is quantised and how current and photon flux are related in devices like LEDs and solar cells can be useful.

Example context: The energy levels in the material cause Light Emitting Diodes (LEDs) to emit light of particular wavelengths. The energy band levels correspond to the emitted photon energies, and therefore the wavelength of the emitted light. The potential difference across a component is how much energy per unit charge has been transferred by the component as the charge flows through it. Here we will assume that the drop in potential difference across an LED is entirely due to an electron changing energy state in the LED, releasing a photon in the process.

Quantities:

- Φ_q photon flux $\left(s^{-1}\right)$
- E photon energy (J)
- λ wavelength of light (m)
- I electric current (A)
- t duration (s)
- V potential difference (V)
- e electron charge (magnitude) (C)
- P LED power (W)
- n number of electrons or photons

Equations: $E = eV \quad E = \frac{hc}{\lambda} \quad \Phi_q = \frac{n}{t} \quad ne = It \quad P = IV$

7.1 Use the equations to derive expressions for

a) the current I in terms of Φ_q and e,

b) the potential difference across a conducting LED V in terms of h, c, e, and λ,

c) The power of the LED P in terms of h, c, λ, and Φ_q.

Example – *Calculate the current through an LED of power rating* 88.8 *mW that produces light of wavelength* 700 *nm.*

$$P = IV = I\frac{hc}{e\lambda} \text{ so } \frac{e\lambda}{hc}P = I$$

$$I = \frac{Pe\lambda}{hc} = \frac{(8.88 \times 10^{-2})(1.60 \times 10^{-19})(7.00 \times 10^{-7})}{(6.63 \times 10^{-34})(3.00 \times 10^{8})} = 50.0\ \text{mA}$$

7.2 A 1.50 W Infra-Red LED produces electromagnetic radiation of wavelength 850 nm. Calculate

a) the potential difference across the LED,

b) the current that passes through the LED,

c) the photon flux emitted by the LED.

7.3 A UV-C (ultra-violet) LED emits 1.00×10^{19} photons per second when there is a potential difference of 6.22 V across it. Calculate

a) the current passing through the LED,

b) the LED's power,

c) the wavelength of electromagnetic radiation emitted.

7.4 Fill in the missing entries in the table below for different LEDs.

power P/mW	current I/mA	potential difference V/V	photon flux $\Phi_q/(10^{17}\,\mathrm{s}^{-1})$	wavelength λ/nm
	52.2		(a)	
75.0	(b)	2.26		(c)
	18.1	(d)	(e)	450
250	(f)	(g)	3.02	(h)

7.5 An LED has a power rating of 500 mW and produces blue light of wavelength 400 nm. Calculate

a) the potential difference across the LED,

b) the amount of charge that flows through the LED in one minute,

c) the number of photons emitted in one minute.

7.6 An LED has a potential difference across it of 2.07 V and emits 2.72×10^{17} photons each second. Calculate

a) the power of the LED,

b) the amount of charge that flows through the LED each second,

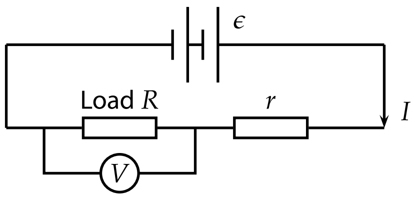
c) the amount of energy transferred by the LED in one hour.

10 Power in a potential divider

It is helpful to be able to calculate the power (or fraction of the total power) dissipated in one part of a potential divider circuit.

Example context: Electrical generators have internal resistance. A power supply company wishes to maximise the efficiency of the system by ensuring that as much of the electricity generated as possible is passed on to customers.

Quantities: I current (A) — P load power (W)
R load resistance (Ω) — V voltage or p.d. across load (V)
r internal resistance (Ω) — η efficiency (no unit)
ϵ electromotive force (emf) of supply (V)



Equations: $P = IV = I^2R = \dfrac{V^2}{R}$ $\quad V = IR \quad \epsilon = V + Ir \quad \eta = \dfrac{P}{I\epsilon}$

10.1 Use the equations to derive expressions for

a) the current I in terms of ϵ, R and r,

b) the load voltage V in terms of ϵ, R and r,

c) the load power P in terms of ϵ, R and r,

d) the efficiency η in terms of ϵ, R and r.

Example 1 – *Calculate the efficiency if a* $20\,\Omega$ *resistor is supplied from a* 12 *V battery with an internal resistance of* $4\,\Omega$.

Total resistance is $20 + 4 = 24\,\Omega$, so current $I = \dfrac{12\text{ V}}{24\,\Omega} = 0.50$ A.

Power in load $P = IV = I \times IR = I^2R = 0.50^2 \times 20 = 5.0$ W.

Power supplied $I\epsilon = 0.50 \times 12 = 6.0$ W. Efficiency $= \dfrac{5.0\text{ W}}{6.0\text{ W}} = 0.83$

10.2 Calculate the load power P for an $\epsilon = 240$ V generator with internal resistance $2.5\,\Omega$ when it is supplying 4.2 A. Hint: use $\epsilon = V + Ir$

10.3 Calculate the efficiency η of the generator in question 10.2.

10.4 An $\epsilon = 12$ V battery has an internal resistance $r = 4.0\,\Omega$. Fill in the missing entries in the table below.

R / Ω	V / V	I / A	P / W	Efficiency η
0.10	(a)	(b)	(c)	(d)
2.0	(e)	(f)	(g)	(h)
4.0	(i)	(j)	(k)	(l)
6.0	(m)	(n)	(o)	(p)
50	(q)	(r)	(s)	(t)

10.5 Use your answers to question 10.4 to state the value of r/R which gives the greatest load power P for given, fixed values of ϵ and r.

10.6 Use your answers to question 10.4 (or other reasoning) to state the value of r/R which gives the greatest efficiency for given values of ϵ and r.

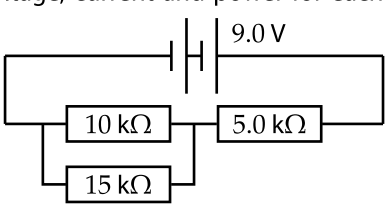
10.7 Calculate r if $P = 500$ MW, $V = 23$ kV and $\eta = 0.99$.

Example 2 – *The load resistor R in the circuit shown is replaced by $30\,\Omega$ and $60\,\Omega$ heaters wired in parallel. Calculate the power dissipated in the $30\,\Omega$ heater if $\epsilon = 230$ V and $r = 3.0\,\Omega$.*
Resistance of the two heaters in parallel $R = \left(30^{-1} + 60^{-1}\right)^{-1} = 20\,\Omega$.
Total circuit resistance $= 20 + 3 = 23\,\Omega$, so current $= \dfrac{230\text{ V}}{23\,\Omega} = 10$ A.
Voltage across the heaters is $V = IR = 10\text{ A} \times 20\,\Omega = 200$ V.
Power in $30\,\Omega$ heater is given by $\dfrac{\text{Voltage}^2}{\text{Resistance}} = \dfrac{200^2}{30} = 1300$ W to 2sf.

10.8 An $\epsilon = 5.4$ V power supply (with $r = 8.0\,\Omega$) powers a $50\,\Omega$ phone. A voltmeter (with resistance $200\,\Omega$) is connected to measure V.

a) How much voltage V is measured across the phone?
b) Calculate the power delivered to the phone.

10.9 Calculate the voltage, current and power for each of the resistors in the circuit below.

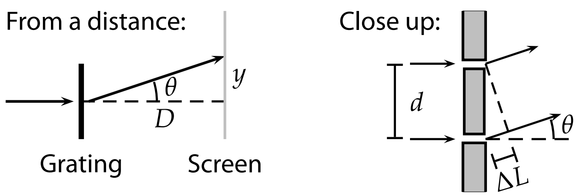


11 Path and phase difference

When waves of the same frequency arrive at a position from more than one source or route, it is helpful to calculate how they will interfere. The path and phase difference tell us whether they will interfere constructively or destructively.

Example context: a microphone placed between two speakers can receive either a strong or weak signal depending on where it is placed.

Quantities:	λ wavelength (m)	f frequency (Hz)
	v wave speed $\left(\mathrm{m\,s^{-1}}\right)$	ΔL path difference (m)
	D distance to screen (m)	$\Delta\phi$ phase difference $(^\circ)$
	d slit separation (m)	N slits per mm $\left(\mathrm{mm^{-1}}\right)$
	θ angle from axis $(^\circ)$	y distance from axis (m)
	n order of interference (no unit) $n = 0, 1, 2, 3...$ if constructive	



Equations: $v = f\lambda$ $\quad \Delta\phi = \dfrac{\Delta L}{\lambda} \times 360^\circ$ $\quad y = D\tan\theta$ $\quad d = \dfrac{1\,\mathrm{mm}}{N}$

For slits: $\Delta L = d\sin\theta$ $\quad$ Small angles: $\tan\theta \approx \sin\theta$

11.1 Use the equations to derive expressions for

a) the phase difference $\Delta\phi$ in terms of d, θ and λ,

b) $\sin\theta$ for constructive interference in terms of λ, n and d,

c) $\sin\theta$ for constructive interference in terms of λ, n and N,

d) $\sin\theta$ for constructive interference in terms of n, N, f and v,

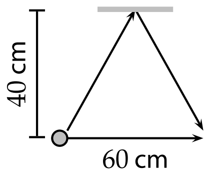
e) y for $n = 1$ in terms of λ, D and d if θ is small,

f) y for $n = 5$ in terms of f, v, D and d if θ is small,

g) ΔL for a microphone placed between two speakers connected to the same signal. The speakers are a distance D apart, and the microphone is a distance y from the mid point.

11.2 Calculate $\Delta\phi$ (as an angle less than 360°) for $\Delta L = 40.0$ cm if $\lambda = 3.6$ cm.

Example 1 – *Calculate $\Delta\phi$ between the two routes below. $\lambda = 12$ cm*



Longer route: $L = 2 \times \sqrt{30^2 + 40^2} = 100$ cm.
Direct route: $L = 60$ cm. $\Delta L = 100 - 60 = 40$ cm.
$\Delta\phi = \frac{\Delta L}{\lambda} \times 360^\circ = \frac{40}{12} \times 360^\circ = \left(3\frac{1}{3}\right) \times 360^\circ$.
Ignoring the 3 whole rotations, which do not affect the interference, $\Delta\phi = \frac{1}{3} \times 360^\circ = 120^\circ$.

11.3 A 440 Hz sound wave reaches a microphone by two routes. The sound travels 2.50 m directly and travels 4.00 m if it reflects off a wall on the way. Calculate the phase difference on arrival. Assume that $v = 330$ m s^{-1}.

Example 2 – *Light from a sodium lamp passes a grating with 650 lines mm^{-1} and then strikes a wall which is $D = 50.0$ cm from the grating. The grating and wall are both at right angles to the original ray. The first order ($n = 1$) interference hits the wall $y = 20.7$ cm from the centre. Calculate the wavelength.*

$$d = \frac{1.0 \times 10^{-3}\text{ m}}{650} = 1.538 \times 10^{-6}\text{ m}; \theta = \tan^{-1}\left(\frac{0.207}{0.500}\right) = 22.49^\circ$$

$n = 1$ so $\lambda = d \sin\theta = 1.538 \times 10^{-6}$ m $\times \sin(22.49^\circ) = 5.88 \times 10^{-7}$ m.

11.4 A grating with 450 lines mm^{-1} is 75.0 cm from a wall. Light shines perpendicular to both grating and wall onto the centre of the grating.

a) Calculate the angle θ for the $n = 1$ diffraction of 450 nm blue light.

b) Calculate the $n = 1$ distance y for 633 nm light.

c) Calculate λ for $n = 1$ light with $y = 27.8$ cm.

d) Visible light has wavelengths in the range 400 nm $< \lambda <$ 700 nm. How wide is the colourful $n = 1$ pattern on the wall?

11.5 20 GHz microwaves pass through a pair of narrow slits 10 cm apart. Calculate the fringe spacing (y when $n = 1$) on a screen 2.00 m behind the slits.

11.6 Calculate the smallest angle θ at which you would get destructive ($\Delta\phi = 180^\circ$) interference when 550 nm light passes two slits 50 μm apart.

11.7 Using $v = 330$ m s^{-1}, calculate $\Delta\phi$ for a microphone placed between two speakers which are 1.5 m apart if

a) $f = 440$ Hz and the microphone is 37.5 cm from one speaker,

b) $f = 660$ Hz and the microphone is 65 cm from one speaker.

11.8 Two synchronised 10 GHz microwave transmitters face each other. How far from the mid point is the first place with $\Delta\phi = 60^\circ$?

15 Standing waves on a string

Standing waves (also known as stationary waves) appear in many places in physics, so it is useful to be able to work out the frequency of a stationary wave for different harmonics.

Example context: On a string bound at both ends and held under tension, standing waves can exist at certain discrete frequencies. The lowest frequency of standing wave is called the fundamental mode or the first harmonic. The second harmonic has double the frequency of the first harmonic.

Quantities:

- f frequency (Hz)
- n harmonic (no unit)
- λ wavelength (m)
- T tension in string (N)
- ℓ length of vibrating string (m)
- μ linear mass density $\left(\mathrm{kg\,m^{-1}}\right)$
- c speed of progressive wave $\left(\mathrm{m\,s^{-1}}\right)$
- M mass of vibrating string (kg)

Equations: $c^2 = \dfrac{T}{\mu}$ $\quad \mu = \dfrac{M}{\ell}$ $\quad \lambda = \dfrac{2\ell}{n}$ $\quad c = f\lambda$

15.1 Use the equations to derive expressions for

a) the fundamental frequency f_1 in terms of λ, μ and T *(Hint: $n = 1$)*,

b) the fundamental frequency f_1 in terms of ℓ, μ and T,

c) the frequency of the n^{th} harmonic f_n in terms of ℓ, n, μ and T.

15.2 Fill in the missing entries in the table below for different standing waves.

Tension T / N	Linear mass density μ / $\mathrm{g\,m^{-1}}$	Frequency f / Hz	Wavelength λ / cm
(a)	5.00	50.0	28.3
5.00	(b)	50.0	28.3
5.00	5.00	(c)	28.3
5.00	5.00	50.0	(d)

Example – *Calculate the frequency of the 5th harmonic on a vibrating string of length* 1.00 *m under* 5.00 *N tension with a linear mass density of* 1.00 *g m*$^{-1}$

$$f_n = \frac{c}{\lambda} = \frac{\sqrt{T/\mu}}{2\ell/n} = \frac{n}{2\ell}\sqrt{\frac{T}{\mu}} = \frac{5}{2 \times 1.00}\sqrt{\frac{5.00}{0.001}} = 177 \text{ Hz}$$

15.3 Fill in the missing entries in the table below for different standing waves.

Frequency f / Hz	Harmonic n	Length ℓ / cm	Tension T / N	Linear mass density μ / g m^{-1}
82.4	1	64.8	(a)	5.78
313	3	64.8	56.4	(b)
523	(c)	33.0	98.0	3.29
824	4	(d)	650	36.4

15.4 A standing wave has 4 nodes including the two at each end. The length of the vibrating string is 85.0 cm, the tension in the string is 75.0 N, and it vibrates at a frequency of 50 Hz. Calculate the linear mass density μ of the string.

15.5 A 2.00 m long string has a mass of 10.9 g. It is used in an experiment where two bridges are placed horizontally 90 cm apart. The string is kept under tension by suspending an unknown mass on the end of the string, which passes over a low-friction pulley wheel. The other end of the string is clamped in place. A large speaker nearby produces vibrations of 50.0 Hz, which causes the string to resonate with 3 nodes between the bridges. Calculate the mass suspended on the string.

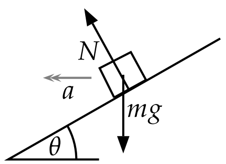
15.6 Two strings (*string A* and *string B*) are set up on a benchtop alongside each other. Each string has two bridges along its length the same distance apart. The two strings have equal tension. A nearby loudspeaker produces sound at 440 Hz. *String A* shows three nodes between the bridges, *string B* shows two nodes between the bridges. Calculate μ_A / μ_B where μ_A and μ_B are the linear mass densities of *string A* and *string B*.

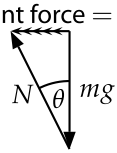
15.7 A string is held under 5.00 N of tension, with a distance between two bridges of 50.0 cm. A signal generator can produce vibrations in the string, but is broken and does not work for frequencies below 100 Hz. Resonance is observed at 125 Hz, 187.5 Hz, and 250 Hz. Calculate the speed of the progressive wave along the string.

17 Banked tracks for turning

For an object travelling in a circular path, there must be a force pushing it inwards towards the centre of the circle. On a sloped (banked) track, the force from the inwards push of the bank (plus any friction) can provide this centripetal force.

Example context: A car travels along a smooth sloping (banked) track at constant speed and height. For a track of radius r, at the right speed and angle of slope the resultant force is the centripetal force needed.
We draw two diagrams: the forces on the car, and a vector diagram of the forces in which the resultant force is required to be horizontal. We can resolve (a) in directions H and V (so $mg = N\cos\theta$), or (b) $\parallel$ and $\perp$ to the slope (then $N = mg\cos\theta$). These are not both correct; this is not in equilibrium - the force diagram requires a horizontal resultant; so (a) (H and V) is the correct choice.





Quantities:
- m mass (kg)
- r radius of path (m)
- ω angular velocity $\left(\text{rad s}^{-1}\right)$
- t_p period of orbit (s)
- N Normal contact force on car from track (N)
- a acceleration inwards $\left(\text{m s}^{-2}\right)$
- v speed $\left(\text{m s}^{-1}\right)$
- θ angle of track $(^\circ)$

Equations: $F = ma$ $\quad a_\text{centripetal} = \dfrac{v^2}{r}$ $\quad v = r\omega$ $\quad t_\text{p} = \dfrac{2\pi r}{v}$

17.1 A car of weight mg travels at constant speed v around a smooth, banked track of radius r and slope θ above the horizontal, and remains at a constant height up the slope. Use diagrams to write down expressions for

a) N in terms of m, g and θ,

b) v in terms of g, r and θ,

c) t_p in terms of r, g and θ,

d) N in terms of m, g, r and v,

e) a in terms of v and ω,

f) ω in terms of g, r and θ.

17.2 A motorbike of mass 160 kg moves in a circular path of radius 120 m at a speed of $15\,\text{m s}^{-1}$.

a) What is the resultant centripetal force on the motorbike?

b) Calculate the centripetal force as a fraction of the weight of the bike.

c) If the motor bike is driven along the slope of a smooth, banked circular track, what is the angle of the track to the horizontal that would provide this centripetal force?

17.3 A fairground experience consists of lying on a flat surface in a freely pivoted carriage being swung in a circle of radius $12\,\mathrm{m}$. If the maximum force they can cope with is $6mg$, what is the minimum period of rotation t_p?

Example – *A smooth banked track is constructed around a flat circular service area of diameter* 150 *m. The width of the track surface is* 30 *m and the slope is at* 15° *to the horizontal. What is the highest constant speed that a motorbike can travel around the track so that it does not drive over the top of the bank?*

$$\text{Radius of track at bank top} = 75 + (30\cos 15^\circ) = 104\text{ m}$$
$$\text{Now, } v^2 = rg\tan\theta = 104 \times 9.81 \times \tan 15^\circ$$
$$\text{So, } v = 17\,\mathrm{m\,s^{-1}} \text{ to 2sf.}$$

17.4 A small car travels around a rough circular track of radius 44 m, banked at 20° to the horizontal. The magnitude of the frictional force on the car pointing down the slope is equal to the weight of the car. At what speed v is the car travelling?

17.5 Car **A** travels at a speed of $20\,\mathrm{m\,s^{-1}}$ around a smooth banked circular track at an angle of 20° to the horizontal. It overtakes car **B** on the same track, also travelling in a circle of constant radius but at a speed of $22\,\mathrm{m\,s^{-1}}$. What is the minimum separation of the car centres when they pass?

17.6 A spherically shaped bowl of inner radius $16\,\mathrm{cm}$ is $8.0\,\mathrm{cm}$ deep at its centre, and contains a small marble. The bowl is rotated about a vertical axis. At what angular velocity ω will the marble leave the bowl?

17.7 A particle of mass m slides round in a circle of radius r on the inside of a smooth conical surface at a constant vertical height h above the apex, which is of angle 2α. What is the speed of the particle in terms of r, g and α?

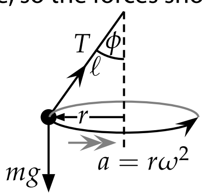
17.8 A particle of mass $m = 0.40\,\mathrm{kg}$ slides round the inside of a smooth conical surface (with the apex below) in a horizontal circle of radius $r = 20\,\mathrm{cm}$ at speed $v = 3.2\,\mathrm{m\,s^{-1}}$. The angle of the apex is $2\alpha = 30^\circ$ and the particle is attached to the apex by a light wire, which prevents it rising up the conical surface. What is the tension T in the wire?

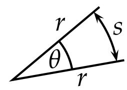
17.9 The standard railway gauge has tracks separated by $1435\,\mathrm{mm}$ on banked sleepers. Calculate the vertical displacement between the two tracks such that a train travelling at $180\,\mathrm{km\,h^{-1}}$ along a curve of radius $1500\,\mathrm{m}$ will experience a normal reaction force on the wheels only.

18 Conical pendulum

A particle of mass m at the end of a light string fixed to a point can be set in motion so that it moves in a horizontal circle centred below the point of suspension.

Example context: Fairground rides, mechanical speed controllers; examples are closely related to those on smooth banked tracks, as the normal reaction force of the track is replaced by tension in a string or rod. In the diagram, the tension in the string and the weight are not aligned, so the object is not in equilibrium. A resultant force of constant magnitude is directed horizontally towards the centre of the circle, so the forces should be resolved horizontally and vertically.





(the view from above)

$$s = r\theta \qquad \frac{s}{t} = r\frac{\theta}{t} = r\omega$$

Quantities:

T tension in the string (N)	a acceleration inwards $\left(\text{m}\,\text{s}^{-2}\right)$
r radius of orbit (m)	ω angular velocity $\left(\text{rad s}^{-1}\right)$
ϕ angle to vertical $(^\circ)$	f frequency $\left(\text{s}^{-1}, \text{Hz}\right)$
ℓ length of string (m)	t_p period (s)
v speed of object $\left(\text{m s}^{-1}\right)$	θ angle of rotation $\left(\text{rad s}^{-1}\right)$

Equations: $F = ma \quad a_\text{centripetal} = r\omega^2 \quad v = r\omega \quad \omega = 2\pi f \quad t_\text{p} = \dfrac{1}{f}$

18.1 A metal ball of mass m is attached to a light string of length ℓ and moves in a horizontal circular path at an angular velocity ω. Use diagrams to write down expressions for

a) the angular velocity ω of the ball in terms of ϕ, r and g,

b) the period of orbit, t_p, in terms of ϕ, r and g,

c) the [horizontal] acceleration of the ball, a in terms of ϕ and g,

d) the acceleration of the ball, a, in terms of m, T and ϕ,

e) the tension in the string, T, in terms of m, g, r and ω,

f) $\cos\phi$ in terms of ℓ and r,

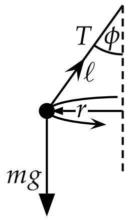
g) the angular velocity ω in terms of g, ℓ and r,

h) $\cos\phi$ in terms of g, r and ω,

i) v in terms of ϕ, r and g,

j) t_p in terms of v and a.

Example – *A small ball of mass* 0.60 *kg is suspended at the end of a light string of length* 0.80 *m attached to the ceiling. The ball travels in a horizontal circle about a vertical axis* 1.3 *times per second. How far below the ceiling is the ball?*
Resolving the forces on the sphere H and V, we obtain the two equations



$$T\sin\phi = mr\omega^2 \quad \text{and} \quad T\cos\phi = mg$$

$$\text{Dividing,} \quad \tan\phi = \frac{r\omega^2}{g} = \frac{r}{g}4\pi^2 f^2$$

$$\text{But also,} \quad \tan\phi = \frac{r}{h} = \frac{r}{g}4\pi^2 f^2$$

$$\text{Hence,} \quad h = \frac{9.81}{4\pi^2 \times 1.3^2} = 0.15\text{ m}$$

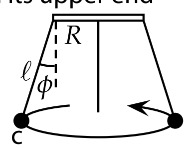
18.2 A small sphere of mass 2.0 kg, attached to the end of a light string of length 90 cm at 24° to the vertical, moves in a horizontal circle. Calculate

a) the tension T in the string, and

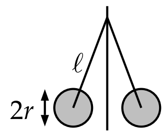
b) the height h by which the mass is raised above its position at rest.

18.3 A lead ball of mass 45 g is attached to the end of an 80 cm long light string and swung around in a horizontal circle at high speed. If the string snaps at a tension of 195 N, what is the maximum frequency of rotation f possible?

18.4 A fairground ride consists of several small carriages (**c**) each supported at its centre of mass by a light cable of length $\ell = 2.20$ m with its upper end attached to a supporting ring of radius $R = 3.40$ m from the axis of rotation. What is the period when the carriages are rotating so that the cables are inclined at $\phi = 30.0°$ to the vertical?



18.5 A mechanical governor consists of a narrow central axle to which are hinged to two light rods of length ℓ, each attached to the centres of spherical masses of radius r. At what angular velocity ω, in terms of g, ℓ and r, will the spheres lose contact with the axle?



18.6 A conical pendulum on Earth produces a period of 0.34 s for a 30° semi-angle of the cone. When the same pendulum is used on the Moon where $g = 1.6$ m s^{-2}, what would be the period for double the semi-angle?

18.7 An aircraft travelling at 160 knots maintains its altitude during a circular banked "rate one turn", which is a 3.0° s^{-1} turning rate. At what angle to the horizontal are the wings of the plane? (1 knot = 0.514 m s^{-1})

26 Orbits

An orbit is the path that an object follows in a gravitational or electromagnetic field. This includes the paths of the planets around the Sun.

Example context: The planets of the solar system orbit around the Sun due to the gravitational force of attraction between the planets and the Sun. To accelerate a particle in orbit in a particle accelerator the magnetic field strength must be increased so that the radius of the particles orbit remains the same and the particles do not collide with the walls of the accelerator.

Quantities:
- G Newton's gravitational constant $\left(\mathrm{N\,m^2\,kg^{-2}}\right)$
- g gravitational field strength $\left(\mathrm{N\,kg^{-1}}\right)$
- E electric field strength $\left(\mathrm{N\,C^{-1}}\right)$
- B magnetic flux density (T)
- a centripetal acceleration $\left(\mathrm{m\,s^{-2}}\right)$
- ϵ_0 permittivity of free space $\left(\mathrm{F\,m^{-1}}\right)$
- q, Q charge (C)
- r radius of orbit (m)
- F centripetal force (N)
- T orbital period (s)
- m, M mass (kg)
- v velocity $\left(\mathrm{m\,s^{-1}}\right)$

Equations:

$$g = \frac{GM}{r^2} \qquad E = \frac{Q}{4\pi\epsilon_0 r^2} \qquad a = \frac{v^2}{r} \qquad F = ma \qquad v = \frac{2\pi r}{T}$$

$$F = mg \qquad F = qE \qquad F = qvB \qquad r^3 \propto T^2$$

26.1 A moon of mass m moves at speed v in a circular orbit around a planet of mass M

a) Use the equations above to obtain v in terms of G, M and r.

b) Use the equations above to derive Kepler's Third Law: $r^3 \propto T^2$,

c) What is the constant of proportionality r^3/T^2 in terms of G and M?

26.2 A positron of charge $+q$ and mass m enters a magnetic field B travelling at a speed v perpendicular to the direction of the magnetic field.

a) Derive an expression for r in terms of q, B, m and v.

b) If we now change the particle from a positron to a proton, keeping the magnetic field and the velocity of the particle the same, what would happen?

26.3 Calculate the radius of the Moon's orbit around the Earth given that Moon takes approximately 27 days to orbit the Earth and the mass of the Earth is 6.0×10^{24} kg.

26.4 Astronauts on the International Space Station appear weightless because both they and the space station have the same centripetal acceleration and therefore there is no contact force between the astronauts and the floor of the space station. They are in free-fall. What is the centripetal acceleration of the international space station in orbit at a height $h = 400$ km above the surface of the Earth?

> **Example** – *Venus takes* 225 *Earth days to orbit the Sun at an average distance of* 1.08×10^8 *km. What is the mass of the Sun according to this data?*
>
> $$r^3 = \frac{GM}{4\pi^2}T^2 \text{ therefore } M = \frac{4\pi^2 r^3}{GT^2}$$
>
> $$M = \frac{4\pi^2(1.08 \times 10^{11})^3}{6.67 \times 10^{-11}\ (225 \times 24 \times 3600)^2} \approx 1.97 \times 10^{30} \text{ kg}$$

26.5 Calculate the orbital period of Jupiter in units of Earth years given that the mass of the Sun, $M = 2.0 \times 10^{30}$ kg, the mass of Jupiter, $m = 1.9 \times 10^{27}$ kg and the average radius of Jupiter's orbit around the sun is $R = 7.8 \times 10^8$ km.

26.6 Calculate the ratio of the radii of the orbits of Phobos and Deimos, which are the moons of Mars. The mass of Mars is $M = 6.4 \times 10^{23}$ kg, the mass of Phobos $m_1 = 11 \times 10^{15}$ kg and the mass of Deimos $m_2 = 1.5 \times 10^{15}$ kg. The period of Phobos's orbit is $T_1 = 7.7$ hours and of Deimos's orbit is $T_2 = 30.4$ hours.

26.7 61 Cygni is a wide binary star system. It contains two stars of nearly equal mass which orbit once around their mid point every 659 years. They are 1.26×10^{13} m apart. Assuming that the two stars have equal mass, calculate

a) the speed of the stars,

b) the total mass of the system.

26.8 Find an expression for the the ratio of the gravitational field to the electric field, g/E, for an electron that is in orbit at a radius r around the central proton of a hydrogen atom.

26.9 In a particle accelerator protons are accelerated in the $+x$-direction until they have a velocity of $v = 6.5 \times 10^6$ m s^{-1}. They then pass into a magnetic field of strength 0.1 T that is oriented in the $+y$-direction.

a) In which direction do the protons accelerate when they first enter the magnetic field?

b) What is the radius of the orbital path that the protons take?

31 Deriving kinetic theory

We create a mathematical model using Newton's laws for the particles in a gas. When we have done this, we find it predicts many aspects of bulk gas behaviour correctly. To do this, we assume that the gas is an **ideal gas**.

Example context: explaining how the volume, pressure and temperature of a gas change by considering the collisions of the particles in the gas with each other and the walls of the container. This allows you to predict the thermodynamic behaviour of a gas without having to do an experiment.

Quantities:

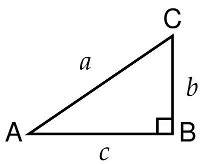
- Δ "A change in"
- F force (N)
- t time taken (s)
- s distance travelled (m)
- A area of face (m^2)
- N number of molecules
- m mass of a particle (kg)
- p momentum of a particle $(\mathrm{kg\,m\,s^{-1}})$
- c speed of a particle $(\mathrm{m\,s^{-1}})$
- P pressure of a gas $(\mathrm{N\,m^{-2}}$ or Pa$)$
- V volume of gas (m^3)
- $\overline{c^2}$ mean-square speed $(\mathrm{m^2\,s^{-2}})$
- u, v, w components of velocity in the x, y, z directions $(\mathrm{m\,s^{-1}})$

Equations:

$$F = \frac{\Delta m v}{\Delta t} \qquad P = \frac{F}{A}$$

$$v = \frac{s}{t} \qquad p = m \times \text{velocity}$$

$$\Delta p = p_{\text{after}} - p_{\text{before}}$$

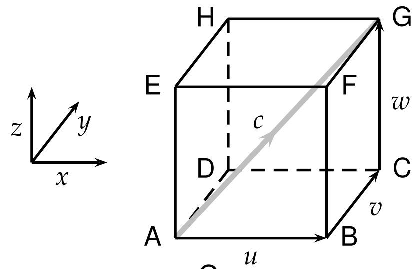


$a^2 = b^2 + c^2$ (Pythagoras)

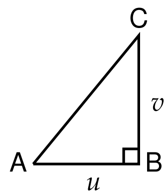
Assumptions about ideal gases:

1. The volume of a particle is so small compared to the volume of the gas, we can ignore it.
2. There are no attractive forces between particles, only collision forces.
3. Particle movement is continuous and random.
4. Particle collisions are perfectly elastic, so there is no loss of kinetic energy.
5. Collision time is very short in comparison with the time between impacts.
6. There are enough molecules for statistics to be applied.

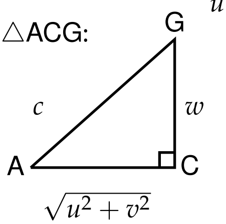
Example – *Consider a gas particle of mass m with speed c. We can write c in terms of 3 velocity components, u, v and w in the x, y and z directions respectively. Prove that $c^2 = u^2 + v^2 + w^2$ using Pythagoras' Theorem.*



In $\triangle$ABC:



In $\triangle$ACG:

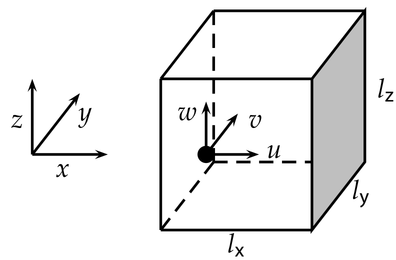


$$\text{AC}^2 = u^2 + v^2$$

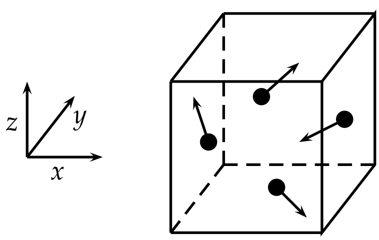
$$\text{AG}^2 = \left(\sqrt{u^2+v^2}\right)^2 + w^2 = u^2 + v^2 + w^2$$

So $c^2 = u^2 + v^2 + w^2$

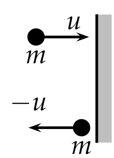
31.A The particle is in a box of dimensions l_{x}, l_{y}, l_{z}. The box represents the volume of the gas. The shaded faces represent the collisions. Write down the formula for the volume of the box V in terms of l_{x}, l_{y} and l_{z}.



We can think of the gas as a group of N particles moving around randomly, hitting the sides of the container. As the motion is random, we expect the average speeds in different directions to be the same.



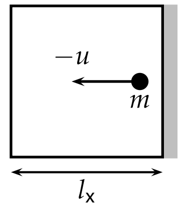
Let's consider one particle moving in the positive x direction. The particle collides with the container wall.



31.B Write an expression for the change in momentum of the particle, Δp, in terms of m and u. Pay attention to which direction is positive.

31.C Write an expression for the average force F_{particle} on the particle (from the wall), to cause the change in momentum of the particle. The time between collisions with the wall is Δt.

31.D Use Newton's Third Law of Motion to write down an expression for the average force, F_{wall}, of the particle on the wall over time Δt. Pay attention to the sign.



31.E Between collisions the particle will travel to the other side of the container and back again. Find an expression for Δt in terms of u and l_{x}.

31.F Now substitute your expression for Δt from 31.E into your equation in 31.D and simplify it. This will give you a new expression for the force of the particle on the wall, F_{wall}, in terms of m, u, and l_{x}.

31.G The average pressure exerted by the particle on the wall may be written as F_{wall}/A, where A is the area of the wall. Use your answer to 31.F to find an expression for the average pressure P_1 due to this one molecule in terms of u, m and:

a) l_{x}, l_{y}, and l_{z}

b) the volume, V, of the container. Use your answer from 31.A.

We now have an expression for the pressure P_1 on the container due to the collision of one particle. From here on we refer to this particle as 'particle 1' and label its speed as c_1 and its velocity components as u_1, v_1 and w_1. There are actually N particles in the gas. They each have the same mass m, but will have different velocities. For example, 'particle 2' has velocity components u_2, v_2 and w_2, has speed c_2, and will cause a pressure P_2.

31.H Up until now, we have assumed that our particle was only moving in the x direction. Does the expression for P_1 derived in question 31.G change if v_1 and w_1 are not necessarily zero?

31.I By looking at your reasoning for particle 1, write down an expression for the pressure P_2 on the same wall in terms of m, u_2, v_2, w_2 and V.

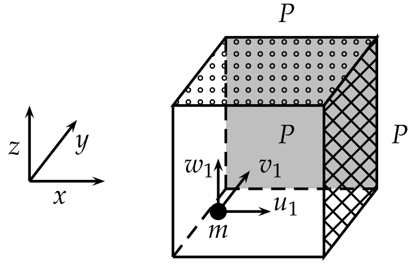
The total pressure on this wall will be the sum of the pressures due to all of the individual particles: $P = P_1 + P_2 + \ldots$.

31.J Use your expression from 31.G (b) to write the equation for total pressure P in terms of m, V, u_1, u_2 and the other x components of velocity. Assume that all the particles have the same mass, m.

31.K Find an expression for the average squared x component of velocity $\overline{u^2}$ if there are N molecules whose squared velocity components are u_1^2, u_2^2 and so on.

31.L Use your answer to 31.K to re-write the pressure from 31.J in terms of m, V, N and $\overline{u^2}$.

We now have an expression for the pressure of the particles on the right hand wall. As the particles are moving randomly, they exert the same pressure on the other walls as well.



We now take into account the fact that the molecules are not just moving in the x direction.

31.M The y components of each molecule's velocity are written v_1, v_2, v_3 and so on. Use your answer to 31.K to write expressions (when there are N particles) for:

a) the average squared y velocity component $\overline{v^2}$ and

b) the average squared z velocity component $\overline{w^2}$.

31.N In question 31.L, you wrote an equation linking P and $\overline{u^2}$. By thinking of collisions with the back wall causing an equal pressure, write a similar equation linking P and $\overline{v^2}$. Then, by thinking of collisions with the top wall, write a similar equation linking P and $\overline{w^2}$.

31.O In the example, we saw that $c^2 = u^2 + v^2 + w^2$, where c^2 is the square speed of one molecule. Applied to particle 1 this means that $c_1^2 = u_1^2 + v_1^2 + w_1^2$. Use this information to write an equation relating $\overline{c^2}$ (the mean square speed), to $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ (the mean square velocity components).

31.P Use your answers to questions 31.N and 31.O to write an equation for the pressure P in terms of the mean square speed $\overline{c^2}$.

This equation beautifully links the macroscopic behaviour of a gas (PV) with the average (square) speed of the N microscopic gas particles.

Exercise – *Copy the diagram of the box and see how far you can progress with the proof without looking at all the steps. Remember:*

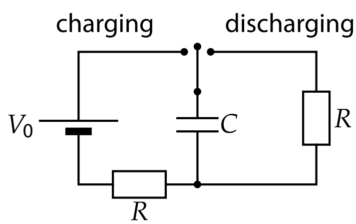
1. *1 particle*
2. *N particles*
3. *3 dimensions.*

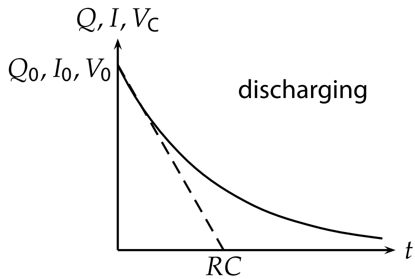
The parts of this section are lettered A, B, C…to match with the implementation of this question online.

33 Capacitors and resistors

A capacitor can be charged or discharged gradually by connecting it in series with a resistor (and if charging, a voltage source). The voltages and currents in the circuit are decaying exponential functions of time.

Example context: Circuits containing capacitors and resistors in series are important in electronics applications, including signal processing and timing. You can calculate the capacitor charge, voltages and current in the circuit at any time.





Quantities:	R resistance (Ω)	V_C voltage across capacitor (V)
	C capacitance (F)	V_R voltage across resistor (V)
	t time (s)	V_0 initial or max voltage (V)
	Q charge on capacitor (C)	Q_0 initial or max charge (C)
	I current in circuit (A)	I_0 initial current (A)

Equations:	$Q = CV_C$	$V_R = IR$	$I = I_0 e^{-t/RC}$
	When discharging:	$V_C = V_R$	$V_C = V_0 e^{-t/RC}$
	When charging:	$V_C + V_R = V_0$	$V_C = V_0 (1 - e^{-t/RC})$

33.1 Use the equations to write down expressions for

a) the charge Q versus time, when discharging;

b) the charge Q versus time, when charging;

c) the initial charge Q_0 in terms of V_0 and C;

d) the voltage V_R across the resistor versus time, when discharging;

e) the voltage V_R across the resistor versus time, when charging;

f) Q in terms of I when discharging;

g) Q in terms of $\mathrm{d}Q/\mathrm{d}t = -I$ when discharging;

h) I_0 in terms of V_0 and R when discharging;

i) I_0 in terms of Q_0, R and C when discharging;

j) the time to completely discharge if the current were constant at I_0;

k) the fraction of Q_0 still on the capacitor after a time RC.

33.2 Find Q_0 and I_0 if $R = 200\ \Omega$, $C = 0.0010$ F and $V_0 = 5.0$ V when discharging.

33.3 A $47\ \mu$F capacitor discharges through a resistor. The initial charge on the capacitor is $9.4\ \mu$C and the initial current is 8.0 mA. Find the value of the resistor.

Example – *A $600\ \mu$F capacitor is charged to 9.0 V and then discharged through a 70 kΩ resistor. How much charge is on the capacitor after 100 s?*

$$Q = CV_C = CV_0 e^{-t/RC} = (6 \times 10^{-4}) \times 9 \times e^{-100/(7\times10^4\times6\times10^{-4})} = 500\ \mu\text{C}$$

33.4 Fill in the missing entries in the table below for a capacitor discharging from an initial voltage of 10 V.

R / Ω	C / F	t / s	Q / C	V / V	I / A
550	8.0×10^{-3}	5.0		(a)	
550	8.0×10^{-3}	15		(b)	
2.2×10^6	3.0×10^{-5}	10	(c)		(d)
2.2×10^6	3.0×10^{-5}	20	(e)		(f)
2.2×10^6	3.0×10^{-5}	50	(g)		(h)

33.5 A capacitor with $C = 10$ nF and initial charge 1.5×10^{-7} C is discharged through a resistor with $R = 10$ MΩ. What is the current after 0.25 s?

33.6 An initially uncharged 0.0020 F capacitor is connected to a 6.0 V battery via a $9.0\ \Omega$ resistor. How much charge has entered the capacitor after the first 0.02 s?

33.7 In a timing circuit, an initially uncharged 0.10 mF capacitor is connected to a 4.5 V source through a $80\ \Omega$ resistor.

a) What is the voltage across the capacitor after 5 ms?

b) After 5 ms, the capacitor is then disconnected from the source and connected across another $80\ \Omega$ resistor to discharge. What is the voltage across the resistor after another 5 ms?

(e) $$\begin{aligned} E_{\mathrm{GP}} + E_{\mathrm{EP}} &= -mgx + \tfrac{1}{2}kx^2 = -mg\left(x_{\mathrm{B}} + y\right) + \tfrac{1}{2}k\left(x_{\mathrm{B}} + y\right)^2 \\ &= -mg\left(\frac{mg}{k} + y\right) + \frac{k}{2}\left(\frac{mg}{k} + y\right)^2 \\ &= -\frac{m^2g^2}{k} - mgy + \frac{m^2g^2}{2k} + mgy + \frac{ky^2}{2} \\ &= \frac{ky^2}{2} - \frac{m^2g^2}{2k} = \frac{ky^2}{2} + E_{\mathrm{B}} \end{aligned}$$

3 Momentum and kinetic energy

(a) $p = mv$ so $v = \dfrac{p}{m}$. Therefore $E = \dfrac{m}{2}v^2 = \dfrac{m}{2}\left(\dfrac{p}{m}\right)^2 = \dfrac{p^2}{2m}$

(b) $E = \dfrac{mv^2}{2}$ so $v = \sqrt{\dfrac{2E}{m}}$. Now $p = mv = m\sqrt{\dfrac{2E}{m}} = \sqrt{\dfrac{2Em^2}{m}} = \sqrt{2mE}$

(c) $p = \sqrt{2mE} = \sqrt{2mqV}$ as $E = qV$

(d) $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2mqV}}$

4 Elastic collisions

(a) $p_0 + P_0 = p_1 + P_1$ so $mv_0 + 0 = mv_1 + MV_1$ and $V_1 = \dfrac{m\left(v_0 - v_1\right)}{M}$

(b) $p_0 + P_0 = p_1 + P_1$ so $mv_0 + 0 = 0 + mV_1$ and $V_1 = v_0$

Part (b) could also be completed using energy conservation.

For the third and optional part (c), the algebra is much more complicated, but we show it so that you can see why approach and separation speeds are the same in elastic collisions. Remember that r is defined as the approach speed ($v - V = r$), so $v = V + r$.

(c) $$\begin{aligned} P + p &= MV + mv = MV + m\left(V + r\right) = \left(M + m\right)V + mr \\ \left(P + p\right)^2 &= \left(M + m\right)^2 V^2 + 2\left(M + m\right)mrV + m^2r^2 \\ K + k &= \frac{MV^2}{2} + \frac{mv^2}{2} = \frac{M^2V^2 + MmV^2 + m^2v^2 + Mmv^2}{2\left(M + m\right)} \end{aligned}$$

$$\begin{aligned}
K+k &= \frac{M^2V^2 + MmV^2 + m^2(V+r)^2 + Mm(V+r)^2}{2(M+m)} \\
&= \frac{M^2V^2 + 2MmV^2 + m^2V^2 + 2m^2Vr + m^2r^2 + 2MmVr + Mmr^2}{2(M+m)} \\
&= \frac{(M+m)^2 V^2 + 2(M+m)mVr + m^2r^2 + Mmr^2}{2(M+m)} \\
&= \frac{(P+p)^2 + Mmr^2}{2(M+m)} \\
&= \frac{(P+p)^2}{2(M+m)} + \frac{Mm}{2(M+m)}r^2
\end{aligned}$$

In an elastic collision $k+K$ will be the same before and after the collision. As the total momentum $p+P$ will also be conserved, it follows that r^2 will not change either. Therefore $|r_1| = |r_0|$, so for a one-dimensional collision, $r_1 = \pm r_0$. In the $r_1 = r_0$ case, nothing has changed (there has been no collision), so in collisions $r_1 = -r_0$. In other words, when an elastic collision is viewed from the perspective of one object, the other object bounces off it at the same speed as it arrived.

5 Vectors and motion – relative motion

(a) $v_{\mathsf{REL}} = v_{\mathsf{A}} - v_{\mathsf{T}}$

(b) $v_{\mathsf{REL}} = \dfrac{s_0}{T} \longrightarrow T = \dfrac{s_0}{v_{\mathsf{REL}}} = \dfrac{s_0}{v_{\mathsf{A}} - v_{\mathsf{T}}}$

(c) $s = s_0 - (v_{\mathsf{A}} - v_{\mathsf{T}})\, t$

6 Vectors and motion – projectiles

(a) $v_{\mathsf{y}}^2 = u_{\mathsf{y}}^2 + 2a_{\mathsf{y}}s_{\mathsf{y}}$ using the vertical components of the vectors.

$$\begin{aligned}
s_{\mathsf{y}} &= -h \text{ when } v_{\mathsf{y}} = 0 \text{ and } a_{\mathsf{y}} = g \text{ (downwards is positive)} \\
u_{\mathsf{y}} &= -u\sin\theta \text{ (as upwards is negative)} \\
-h &= \frac{v_{\mathsf{y}}^2 - u_{\mathsf{y}}^2}{2a_{\mathsf{y}}} = \frac{0 - u^2\sin^2\theta}{2g} \\
h &= \frac{u^2\sin^2\theta}{2g}
\end{aligned}$$

(b) Analysing motion from high point to end $s_{\mathsf{y}} = h + D,\ u_{\mathsf{y}} = 0,\ a_{\mathsf{y}} = g$

$v_{\mathsf{y,final}}^2 = u_{\mathsf{y}}^2 + 2a_{\mathsf{y}}s_{\mathsf{y}} = 0 + 2g\,(h+D)$, so $v_{\mathsf{y,final}} = \sqrt{2g\,(h+D)}$

(c) Using $v_{\mathsf{y}} = u_{\mathsf{y}} + a_{\mathsf{y}}t$ over the whole motion, $v_{\mathsf{y,final}} = -u\sin\theta + gT$

$$T = \frac{v_{\mathsf{y,final}} + u\sin\theta}{g} = \frac{\sqrt{2g\,(h+D)} + u\sin\theta}{g}$$
$$= \frac{\sqrt{u^2\sin^2\theta + 2gD} + u\sin\theta}{g}$$

(d) $u_{\mathsf{x}} = v_{\mathsf{x}}$ because $a_{\mathsf{x}} = 0$

$$R = u_{\mathsf{x}}T = u\cos\theta \times \frac{\sqrt{u^2\sin^2\theta + 2gD} + u\sin\theta}{g}$$

7 Photon flux for an LED

(a) $I = \dfrac{\text{charge}}{t} = \dfrac{ne}{t} = \dfrac{n}{t}\cdot e = \Phi_{\mathsf{q}}e$

(b) $V = \dfrac{E}{e} = \dfrac{hc}{\lambda}\cdot\dfrac{1}{e} = \dfrac{hc}{e\lambda}$

(c) $P = IV = \Phi_{\mathsf{q}}e\cdot\dfrac{hc}{e\lambda} = \Phi_{\mathsf{q}}\dfrac{hc}{\lambda}$

8 Potential dividers with LEDs

(a) $V = IR$ so $R = \dfrac{V}{I} = \dfrac{\varepsilon - V_{\mathsf{LED}}}{I}$

(b) $R = \dfrac{\varepsilon - V_{\mathsf{LED}}}{I} = \dfrac{\varepsilon}{I} - \dfrac{hc}{Ie\lambda}$

9 Current division

(a) $V = I_{\mathsf{C}}R_{\mathsf{parallel}} = I_{\mathsf{C}}\left(R_1^{-1} + R_2^{-1}\right)^{-1} = \dfrac{I_{\mathsf{C}}}{R_1^{-1} + R_2^{-1}}$

(b) $I_1 = \dfrac{V}{R_1} = VR_1^{-1} = \dfrac{I_{\mathsf{C}}R_1^{-1}}{R_1^{-1} + R_2^{-1}}$

(c) $\frac{I_1}{I_C} = I_1 \times \frac{1}{I_C} = \frac{I_C R_1^{-1}}{R_1^{-1} + R_2^{-1}} \times \frac{1}{I_C} = \frac{R_1^{-1}}{R_1^{-1} + R_2^{-1}}$

(d) $G_1 = \frac{I_1}{V} = \frac{1}{R_1} = R_1^{-1}$

(e) $G_C = G_1 + G_2 = R_1^{-1} + R_2^{-1}$

(f) $\frac{G_1}{G_C} = \frac{R_1^{-1}}{R_1^{-1} + R_2^{-1}}$

We hope you noticed that $\frac{G_1}{G_C} = \frac{I_1}{I_C}$. If one resistor has two thirds of the conductance, it will carry two thirds of the current.

10 Power in a potential divider

(a) $I = \frac{\epsilon}{\text{Circuit resistance}} = \frac{\epsilon}{R+r}$

(b) $V = IR = \frac{\epsilon}{R+r} \times R = \frac{\epsilon R}{R+r}$

(c) $P = IV = \frac{\epsilon}{R+r} \times \frac{\epsilon R}{R+r} = \frac{\epsilon^2 R}{(R+r)^2}$

(d) $\eta = \frac{P}{I\epsilon} = P \times \frac{1}{\epsilon I} = \frac{\epsilon^2 R}{(R+r)^2} \times \frac{1}{\epsilon \times \epsilon / (R+r)} = \frac{R}{R+r}$

11 Path and phase difference

(a) $\Delta\phi = \frac{\Delta L}{\lambda} \times 360° = \frac{d \sin\theta}{\lambda} \times 360°$

(b) $\sin\theta = \frac{\Delta L}{d} = \frac{n\lambda}{d}$ for constructive interference

(c) $\sin\theta = \frac{n\lambda}{d} = \frac{n\lambda}{1\,\text{mm}/N} = \frac{nN\lambda}{1 \times 10^{-3}\,\text{m}}$

(d) $\sin\theta = \frac{nN\lambda}{1 \times 10^{-3}\,\text{m}} = \frac{nN\,(v/f)}{1 \times 10^{-3}\,\text{m}} = \frac{nNv}{1 \times 10^{-3}\,\text{m} \times f}$

(e) $y = D\tan\theta \approx D\sin\theta = D \times \frac{n\lambda}{d} = \frac{1 \times \lambda D}{d} = \frac{\lambda D}{d}$

(f) $y = D\tan\theta \approx D\sin\theta = D \times \frac{n\lambda}{d} = \frac{5 \times (v/f)\,D}{d} = \frac{5vD}{df}$

(g) $\Delta L = \left(\frac{1}{2}D + y\right) - \left(\frac{1}{2}D - y\right) = 2y$

12 Diffraction, interference and multiple slits

(a) $L_2^2 = D^2 + \left(\frac{1}{2}d\right)^2$ therefore $L_2 = \sqrt{D^2 + \frac{d^2}{4}}$

$L_1^2 = D^2 + \left(\frac{3}{2}d\right)^2$ therefore $L_1 = \sqrt{D^2 + \frac{9d^2}{4}}$

(b) $L_2 - L_1 = \frac{1}{2}\lambda$

(c) $\sqrt{D^2 + \frac{9d^2}{4}} - \sqrt{D^2 + \frac{d^2}{4}} = \frac{1}{2}\lambda$

13 Reflection and refraction – angle of acceptance and prisms

(a) $n_\mathrm{A} \sin\theta_1 = n_\mathrm{G} \sin\theta_2$ therefore

$$\theta_2 = \sin^{-1}\left(\frac{n_\mathrm{A}}{n_\mathrm{G}} \sin\theta_1\right)$$

(b) $n_\mathrm{A} \sin\theta_1 = n_\mathrm{G} \sin\theta_2$ and $\theta_3 = \alpha - \theta_2$ therefore

$$\theta_3 = \alpha - \sin^{-1}\left(\frac{n_\mathrm{A}}{n_\mathrm{G}} \sin\theta_1\right)$$

(c) $n_\mathrm{A} \sin\theta_4 = n_\mathrm{G} \sin\theta_3$

$$= n_\mathrm{G} \sin\left[\alpha - \sin^{-1}\left(\frac{n_\mathrm{A}}{n_\mathrm{G}} \sin\theta_1\right)\right] \quad \text{therefore}$$

$$\theta_4 = \sin^{-1}\left\{\frac{n_\mathrm{G}}{n_\mathrm{A}} \sin\left[\alpha - \sin^{-1}\left(\frac{n_\mathrm{A}}{n_\mathrm{G}} \sin\theta_1\right)\right]\right\}$$

(Or, more sensibly, do it in three stages, as it is done in the example.)

14 Optical path

(a) $\Delta\phi = \frac{\Delta\ell}{\lambda} \times 360^\circ = \frac{\ell - x}{\lambda} \times 360^\circ$

(b) $\Delta\phi = \dfrac{\Delta\ell}{\lambda} \times 360^\circ = \dfrac{\ell - x}{\lambda} \times 360^\circ = \dfrac{n \cdot x - x}{\lambda} \times 360^\circ = \dfrac{(n-1)\,x}{\lambda} \times 360^\circ$

(c) $180^\circ = \dfrac{(n-1)\,x}{\lambda} \times 360^\circ$ therefore $\dfrac{1}{2} = \dfrac{(n-1)\,x}{\lambda}$ so $x = \dfrac{\lambda}{2\,(n-1)}$

(d) $\lambda' = \dfrac{v}{f} = \dfrac{c/n}{f} = \dfrac{c/f}{n} = \dfrac{\lambda}{n}$

(e) $\dfrac{x}{\lambda'} = \dfrac{x}{\lambda/n} = \dfrac{nx}{\lambda} = \dfrac{\ell}{\lambda}$

Parts (d) and (e) here show that comparing optical paths $\ell = nx$ takes into account the different wavelengths in the different media without having to calculate those wavelengths λ' separately.

15 Standing waves on a string

(a) $f_1 = \dfrac{c}{\lambda} = \dfrac{\sqrt{\dfrac{T}{\mu}}}{\lambda} = \dfrac{1}{\lambda}\sqrt{\dfrac{T}{\mu}}$

(b) $f_1 = \dfrac{1}{\lambda}\sqrt{\dfrac{T}{\mu}} = \dfrac{1}{2\ell}\sqrt{\dfrac{T}{\mu}}$

(c) $f_n = \dfrac{c}{\lambda} = \dfrac{\sqrt{\dfrac{T}{\mu}}}{\dfrac{2\ell}{n}} = \dfrac{n}{2\ell}\sqrt{\dfrac{T}{\mu}}$

16 Inverse square intensity

(a) Area illuminated is $A_{\text{sphere}} = 4\pi r^2$, so $I = \dfrac{P}{A_{\text{sphere}}} = \dfrac{P}{4\pi r^2}$

(b) Using (a) $I = \dfrac{P}{4\pi r^2}$ so $r^2 = \dfrac{P}{4\pi I}$ and $r = \sqrt{\dfrac{P}{4\pi I}}$

(c) $P = I_1 A_1 = I_2 A_2$, so $I_1 \times 4\pi r_1^2 = I_2 \times 4\pi r_2^2$, so $I_2 = \dfrac{I_1\, r_1^2}{r_2^2}$

In (c) we assumed that the radiation spread equally in all directions (so $A = 4\pi r^2$). The reasoning is also true for radiation which spreads in all **relevant** directions. In this case, P will not be the power of the source, but the power of a source which could shine this brightly in all directions.

17 Banked tracks for turning

(a) Resolving vertically, $N\cos\theta = mg$, so $N = \dfrac{mg}{\cos\theta}$

(b) $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{mv^2/r}{mg} = \dfrac{v^2}{rg}$. So, $v = \sqrt{rg\tan\theta}$

(c) $t_{\text{p}} = \dfrac{2\pi r}{v} = \dfrac{2\pi r}{\sqrt{rg\tan\theta}} = 2\pi\sqrt{\dfrac{r}{g\tan\theta}}$

(d) By Pythagoras: $N = \sqrt{(mg)^2 + \left(\dfrac{mv^2}{r}\right)^2} = mg\sqrt{1 + \dfrac{v^4}{r^2g^2}}$

(e) $a = \dfrac{v^2}{r} = \dfrac{v}{r} \times v = \omega v$

(f) Resolving horizontally, $N\sin\theta = mv^2/r = m\,(r\omega)^2\,/r = mr\omega^2$

Resolving vertically, $N\cos\theta = mg$.

Dividing, $\tan\theta = \dfrac{N\sin\theta}{N\cos\theta} = \dfrac{r\omega^2}{g}$. Hence, $\omega = \sqrt{\dfrac{g}{r}\tan\theta}$

18 Conical pendulum

(a) Resolve (H): $T\sin\phi = mr\omega^2$ and (V): $T\cos\phi = mg$. Divide the equations,

$\dfrac{T\sin\phi}{T\cos\phi} = \dfrac{mr\omega^2}{mg}$. So, $\tan\phi = \dfrac{r\omega^2}{g}$. $\omega = \sqrt{\dfrac{g}{r}\tan\phi}$

(b) $t_{\text{p}} = \dfrac{2\pi r}{v} = \dfrac{2\pi}{\omega}$. So, $t_{\text{p}} = 2\pi\sqrt{\dfrac{r}{g\tan\phi}}$

(c) Resolving, $T\sin\phi = ma$ and $T\cos\phi = mg$. Then $a = g\tan\phi$

(d) From the horizontal equation in (c), $a = \dfrac{T\sin\phi}{m}$

(e) From the equations in (a), squaring and adding,

$(T\cos\phi)^2 + (T\sin\phi)^2 = T^2(\cos^2\phi + \sin^2\phi) = T^2 = (mg)^2 + (mr\omega)^2$.

Then, $T = mg\sqrt{1 + \dfrac{r^2\omega^4}{g^2}}$

(f) Using Pythagoras, $\cos\phi = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{\ell^2 - r^2}}{\ell} = \sqrt{1 - \dfrac{r^2}{\ell^2}}$

(g) $\tan\phi = \dfrac{r\omega^2}{g}$ and also $\tan\phi = \dfrac{r}{\sqrt{\ell^2 - r^2}}$. Equating, $\omega^2 = \dfrac{g}{\sqrt{\ell^2 - r^2}}$

(h) $\tan^2\phi = \dfrac{\sin^2\phi}{\cos^2\phi} = \dfrac{1 - \cos^2\phi}{\cos^2\phi} = \dfrac{1}{\cos^2\phi} - 1$ so $\cos^2\phi = \dfrac{1}{1 + \tan^2\phi}$

As $\tan\phi = \dfrac{r\omega^2}{g}$, then $\cos\phi = \dfrac{1}{\sqrt{1 + \dfrac{r^2\omega^4}{g^2}}}$

(i) Resolving (H) and (V), and dividing $\dfrac{T\sin\phi}{T\cos\phi} = \dfrac{mv^2/r}{mg}$ so $\tan\phi = \dfrac{v^2}{rg}$

Therefore $v = \sqrt{rg\tan\phi}$

(j) $a = \dfrac{v^2}{r}$, so $r = \dfrac{v^2}{a}$. Hence $t_{\text{p}} = \dfrac{2\pi r}{v} = \dfrac{2\pi}{v}\dfrac{v^2}{a} = \dfrac{2\pi v}{a}$

19 Vertical circles

(a) Acceleration is $\uparrow$ towards centre. $N - W = ma$, so $N = W + ma$

(b) $N = W + ma = mg + \dfrac{mu^2}{r} = m\left(g + \dfrac{u^2}{r}\right)$

(c) Acceleration is $\downarrow$ towards centre. $W - N = ma$, so $N = W - ma$

(d) $N = W - ma = mg - \dfrac{mv^2}{r} = m\left(g - \dfrac{v^2}{r}\right)$

(e) $N = mg - \dfrac{mv^2}{r}$, but $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - 2mgr$, so $mv^2 = mu^2 - 4gr$

$N = mg - \dfrac{mu^2 - 4mgr}{r} = mg - \left(\dfrac{mu^2}{r} - 4mg\right) = 5mg - \dfrac{mu^2}{r}$

(f) Using (d) with $N = 0$, $mg = \dfrac{mv^2}{r}$, so $v^2 = gr$ and $v = \sqrt{gr}$

(g) Using (e) with $N = 0$, $5mg = \dfrac{mu^2}{r}$, so $u^2 = 5gr$ and $u = \sqrt{5gr}$

20 Simple pendulum

(a) $x = l\theta$ From the definition of the radian.

(b) $60^\circ = 60 \times \dfrac{2\pi}{360} = 1.047$ rad So, $x = l\theta = 30$ cm $\times\ 1.047 = 31.4$ cm

(c) Resultant force perpendicular to string has magnitude $=$
component of weight perpendicular to string $= mg\sin\theta$

(d) $ma = -mg\sin\theta$ so $a = -g\sin\theta$

(e) $a = -g\sin\theta \approx -g\theta$

(f) $\theta = \dfrac{x}{l}$ so $a \approx -g\theta = -\dfrac{gx}{l}$

(g) $a = -\dfrac{g}{l}x$ so if $a = -\omega^2 x$ then $\omega^2 = \dfrac{g}{l}$

21 Electromagnetic induction – moving wire

(a) $A = Lw = Lut$

(b) $BA = BLut$

(c) $\dfrac{\mathrm{d}(BA)}{\mathrm{d}t} = \dfrac{BA}{t} = BLu$

(d) $V = \dfrac{\mathrm{d}(BA)}{\mathrm{d}t} = BLu$

(e) Force $F_{\mathrm{B}} = quB$

(f) Electric field $E = \dfrac{\text{Force}}{q} = uB$

(g) $V = EL = (uB)L = BLu$ – i.e. the same as part (d)

22 Electromagnetic induction – rotating coil

(a) $\phi = BA = BA_0 \cos \omega t$

(b) $$\varepsilon = -N\frac{d\phi}{dt} = -N\frac{d}{dt}\left(BA_0 \cos \omega t\right) = -NBA_0\frac{d}{dt}\cos \omega t$$
$$= NBA_0\omega \sin \omega t$$

(c) maximum value $\sin \omega t$ can take is 1, so $\varepsilon_{\text{max}} = NBA_0\omega$

(d) $$\varepsilon^2 = N^2B^2A_0^2\omega^2 \sin^2 \omega t$$
$$\left(\varepsilon^2\right)_{\text{mean}} = N^2B^2A_0^2\omega^2\left(\sin^2 \omega t\right)_{\text{mean}} = N^2B^2A_0^2\omega^2 \times \tfrac{1}{2}$$
$$\sqrt{(\varepsilon^2)_{\text{mean}}} = \varepsilon_{\text{rms}} = NBA_0\omega \times \sqrt{0.5} = \frac{1}{\sqrt{2}}NBA_0\omega \quad \text{hence,}$$
$$\varepsilon_{\text{rms}} = \frac{1}{\sqrt{2}}\varepsilon_{\text{max}}$$

23 Energy and fields – accelerator

(a) $$p = mv = m\sqrt{\frac{2K}{m}} = \sqrt{2mK} = \sqrt{2mqV}$$

(b) $$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2qV}{m}}$$

(c) $$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2}{m}\left(\frac{mu^2}{2} + qV\right)} = \sqrt{u^2 + \frac{2qV}{m}}$$

(d) $\Delta K = FL = qEL$

(e) $$E = \frac{F}{q} = \frac{\Delta K}{qL} = \frac{V}{L}$$

(f) $$p = mv = m\sqrt{\frac{2K}{m}} = \sqrt{2mK} = \sqrt{2mqV} = \sqrt{2mqEL}$$

(g) $$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{m\sqrt{2K/m}} = \frac{h}{\sqrt{2Km}} = \frac{h}{\sqrt{2mqV}}$$

24 Energy and fields – relativistic accelerator

(a) $\gamma = \frac{E}{mc^2} = \frac{K + mc^2}{mc^2} = \frac{K}{mc^2} + 1 = \frac{qV}{mc^2} + 1$

(b) $\gamma^{-2} = 1 - \frac{v^2}{c^2}$ so $\frac{v}{c} = \sqrt{1 - \gamma^{-2}}$ and $v = c\sqrt{1 - \gamma^{-2}}$

(c) $v = c\sqrt{1 - \gamma^{-2}} = c\sqrt{1 - \left(1 + \frac{qV}{mc^2}\right)^{-2}}$

(d) $p^2 = \gamma^2 m^2 v^2 = \frac{m^2c^2\left(v^2/c^2\right)}{1 - v^2/c^2} = \frac{m^2c^2\left(v^2/c^2 - 1 + 1\right)}{1 - v^2/c^2}$

$= -m^2c^2 + \frac{m^2c^2}{1 - v^2/c^2} = -m^2c^2 + \gamma^2 m^2 c^2$

therefore $p^2c^2 = -m^2c^4 + \gamma^2 m^2 c^4 = E^2 - m^2c^4$

(e) $p^2 = \frac{E^2}{c^2} - m^2c^2 = \frac{\left(K + mc^2\right)^2 - m^2c^4}{c^2} = \frac{K^2 + 2Kmc^2}{c^2}$

$= \frac{K^2}{c^2} + 2Km = \frac{q^2V^2}{c^2} + 2qVm$

25 Energy and fields – closest approach

(a) $U = qV = \frac{Qq}{4\pi\epsilon_0 r}$, so $r = \frac{Qq}{4\pi\epsilon_0 U}$

(b) $r = \frac{Qq}{4\pi\epsilon_0 U}$ and $U = \frac{mv^2}{2}$, so $r = \frac{Qq}{4\pi\epsilon_0}\frac{2}{mv^2} = \frac{Qq}{2\pi\epsilon_0 mv^2}$

(c) $r = \frac{Qq}{4\pi\epsilon_0 U}$ and $U = \frac{3k_\mathrm{B}T}{2}$, so $r = \frac{Qq}{4\pi\epsilon_0}\frac{2}{3k_\mathrm{B}T} = \frac{Qq}{6\pi\epsilon_0 k_\mathrm{B}T}$

26 Orbits

(1a) Newton's Second Law: $m\frac{v^2}{r} = \frac{GMm}{r^2}$ so $v^2 = \frac{GM}{r}$

(1b) From (a) $v^2 = \frac{GM}{r}$ We also know $v = \frac{2\pi r}{T}$ so $v^2 = \frac{4\pi^2 r^2}{T^2}$

Therefore $\frac{GM}{r} = \frac{4\pi^2 r^2}{T^2}$ and so $4\pi^2 r^3 = GMT^2$

(1c) From (b) $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$

(2a) Newton's Second Law: $m\frac{v^2}{r} = qvB$ so $r = \frac{mv}{Bq}$

27 Vectors and fields – between a planet and a moon

(a) $F_\mathsf{M} = mg_\mathsf{M} = +\frac{GM_\mathsf{M}m}{r_\mathsf{M}^2}$

(b) $F_\mathsf{P} = mg_\mathsf{P} = -\frac{GM_\mathsf{P}m}{r_\mathsf{P}^2}$

(c) $F = F_\mathsf{M} + F_\mathsf{P} = +\frac{GM_\mathsf{M}m}{r_\mathsf{M}^2} - \frac{GM_\mathsf{P}m}{r_\mathsf{P}^2}$

(d) $g = \frac{F}{m} = +\frac{GM_\mathsf{M}}{r_\mathsf{M}^2} - \frac{GM_\mathsf{P}}{r_\mathsf{P}^2}$

(e) $g = 0$ so $\frac{GM_\mathsf{M}}{r_\mathsf{M}^2} = \frac{GM_\mathsf{P}}{r_\mathsf{P}^2}$ therefore $\frac{M_\mathsf{M}}{r_\mathsf{M}^2} = \frac{M_\mathsf{P}}{r_\mathsf{P}^2}$ and $\frac{r_\mathsf{P}}{r_\mathsf{M}} = \sqrt{\frac{M_\mathsf{P}}{M_\mathsf{M}}}$

28 Vectors and fields – electric deflection

(a) $a_\mathsf{y} = \frac{F_\mathsf{E}}{m} = \frac{qE}{m} = \frac{qV}{dm}$

(b) $s_\mathsf{y} = \frac{1}{2}\left(\frac{qV}{dm}\right)t^2 = \frac{1}{2}\left(\frac{qV}{dm}\right)\left(\frac{s_\mathsf{x}}{v_\mathsf{x}}\right)^2 = \frac{qVs_\mathsf{x}^2}{2dmv_\mathsf{x}^2}$

(c) $\theta = \tan^{-1}\left(\frac{v_\mathsf{y}}{v_\mathsf{x}}\right) = \tan^{-1}\left(\frac{a_\mathsf{y}t}{v_\mathsf{x}}\right) = \tan^{-1}\left(\frac{qVs_\mathsf{x}}{dmv_\mathsf{x}^2}\right)$

(d) $s_\mathsf{y} = \frac{1}{2}\left(\frac{qE}{m}\right)t^2 = \frac{qEt^2}{2m}$

(e) $\theta = \tan^{-1}\left(\frac{a_\mathsf{y}t}{v_\mathsf{x}}\right) = \tan^{-1}\left(\frac{qEt}{mv_\mathsf{x}}\right)$

29 Vectors and fields – helix in magnetic field

(a) $F = ma$, so $qv_\perp B = \dfrac{mv_\perp^2}{r}$ and $qvB\sin\theta = \dfrac{mv^2\sin^2\theta}{r}$.

Rearranging gives $r = \dfrac{mv\sin\theta}{qB}$

(b) From a): $r = \dfrac{mv\sin\theta}{qB}$ and $v_\perp = \dfrac{2\pi r}{T} = v\sin\theta$.

So $r = \dfrac{mv\sin\theta}{qB} = \dfrac{T}{2\pi}v\sin\theta$. Therefore $T = \dfrac{2\pi m}{qB}$

(c) $v_\perp^2 + v_\parallel^2 = v^2\sin^2\theta + v^2\cos^2\theta = v^2\left(\sin^2\theta + \cos^2\theta\right) = v^2$.

Thus $v = \sqrt{v_\perp^2 + v_\parallel^2}$.

(d) From c): $T = \dfrac{2\pi m}{qB}$ and $s_\mathrm{p} = v_\parallel T = v\cos\theta\dfrac{2\pi m}{qB}$.

Re-arranging gives $q/m = \dfrac{2\pi}{Bs_\mathrm{p}}v\cos\theta$.

30 Vectors and fields – mass spectrometer

(a) $F_\mathrm{B} = ma$ so $Bqv = \dfrac{mv^2}{r}$. Rearranging gives $r = \dfrac{mv}{Bq}$

(b) $qV_\mathrm{a} = \frac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2qV_\mathrm{a}}{m}}$. Now using our result for r from (a),

$$r = \frac{mv}{Bq} = \frac{m}{Bq}\sqrt{\frac{2qV_\mathrm{a}}{m}} = \sqrt{\frac{2mV_\mathrm{a}}{B^2q}}$$

(c) From (a): $r = \dfrac{mv}{Bq}$ so $\dfrac{q}{m} = \dfrac{v}{Br}$

(d) From (b): $r^2 = \dfrac{2mV_\mathrm{a}}{B^2q}$ so $\dfrac{q}{m} = \dfrac{2V_\mathrm{a}}{B^2r^2}$

(e) $F_\mathrm{E} = F_\mathrm{B}$ so $qE = qvB$ and $E = vB$. So $V_\mathrm{s} = Ed = vBd$

31 Deriving kinetic theory

(A) $V = l_x\, l_y\, l_z$

(B) $\Delta p = -mu - mu = -2mu$

(C) $F_{\text{particle}} = \dfrac{\Delta p}{\Delta t} = -\dfrac{2mu}{\Delta t}$

(D) $F_{\text{wall}} = -F_{\text{particle}} = \dfrac{2mu}{\Delta t}$

(E) Using velocity $= \dfrac{\text{displacement}}{\text{time}}$, time $= \dfrac{\text{displacement}}{\text{velocity}}$, $\Delta t = \dfrac{2l_x}{u}$

(F) $F_{\text{wall}} = \dfrac{2mu}{2l_{\text{x}}/u} = \dfrac{mu^2}{l_x}$

(G) a) $P_1 = \dfrac{F_{\text{wall}}}{l_y l_z} = \dfrac{mu^2}{l_x l_y l_z}$ b) $V = l_x l_y l_z$ so $P_1 = \dfrac{mu^2}{V}$

(H) No, as v and w do not change in the collision

(I) $P_2 = \dfrac{mu_2^2}{V}$

(J) $P = P_1 + P_2 + \ldots = \dfrac{mu_1^2}{V} + \dfrac{mu_2^2}{V} + \ldots = \dfrac{m}{V}\left(u_1^2 + u_2^2 + \ldots\right)$

(K) $\overline{u^2} = \dfrac{u_1^2 + u_2^2 + u_3^2 + ... + u_N^2}{N}$

(L) $\dfrac{PV}{m} = u_1^2 + u_2^2 + \ldots = N\overline{u^2}$ so $P = \dfrac{Nm\overline{u^2}}{V}$

(M) a) $\overline{v^2} = \dfrac{v_1^2 + v_2^2 + v_3^2 + \ldots}{N}$ b) $\overline{w^2} = \dfrac{w_1^2 + w_2^2 + w_3^2 + \ldots}{N}$

(N) $P = \dfrac{Nm}{V}\overline{v^2}$ using y components of velocity on back wall

$P = \dfrac{Nm}{V}\overline{w^2}$ using z components of velocity on top wall

(O)
$$\begin{aligned}\overline{c^2} &= \frac{c_1^2 + c_2^2 + \ldots}{N} = \frac{\left(u_1^2 + v_1^2 + w_1^2\right) + \left(u_2^2 + v_2^2 + w_2^2\right) + \ldots}{N} \\ &= \frac{u_1^2 + u_2^2 + \ldots}{N} + \frac{v_1^2 + v_2^2 + \ldots}{N} + \frac{w_1^2 + w_2^2 + \ldots}{N} \\ &= \overline{u^2} + \overline{v^2} + \overline{w^2}\end{aligned}$$

(P) $\overline{u^2} = \dfrac{PV}{Nm} = \overline{v^2} = \overline{w^2}$ so $\overline{c^2} = \overline{u^2} + \overline{v^2} + \overline{w^2} = \dfrac{3PV}{Nm}$ and so $PV = \dfrac{Nm\overline{c^2}}{3}$

32 Gas laws, density and kinetic energy

(a) $PV = nRT$ and $n = \dfrac{M}{M_\mathrm{M}}$ so $PV = \dfrac{MRT}{M_\mathrm{M}}$ and $P = \dfrac{MRT}{M_\mathrm{M}V}$

(b) From (a) $V = \dfrac{MRT}{M_\mathrm{M}P}$ so $\rho = \dfrac{M}{V} = \dfrac{M}{MRT/M_\mathrm{M}P} = \dfrac{M_\mathrm{M}P}{RT}$

(c) $\rho = \dfrac{M}{V} = \dfrac{Nm}{V} = \dfrac{Nm}{Nk_\mathrm{B}T/P} = \dfrac{mP}{k_\mathrm{B}T}$

(d) $PV = \dfrac{Nm\overline{c^2}}{3}$ so $P = \dfrac{Nm}{V} \cdot \dfrac{\overline{c^2}}{3} = \dfrac{\rho\overline{c^2}}{3}$ and $\rho = \dfrac{3P}{\overline{c^2}}$

(e) $PV = Nk_\mathrm{B}T = \frac{1}{3}Nm\overline{c^2}$ so $m\overline{c^2} = 3k_\mathrm{B}T$ and $\bar{K} = \dfrac{m\overline{c^2}}{2} = \dfrac{3k_\mathrm{B}T}{2}$

33 Capacitors and resistors

(a) $Q = CV_0e^{-t/RC}$ (or $Q_0e^{-t/RC}$)

(b) $Q = CV_0(1 - e^{-t/RC})$ or $Q_0(1 - e^{-t/RC})$

(c) $Q_0/V_0 = C$ so $Q_0 = CV_0$

(d) $V_\mathrm{R} = V_\mathrm{C} = V_0e^{-t/RC}$

(e) $V_\mathrm{R} = V_0 - V_\mathrm{C} = V_0e^{-t/RC}$ i.e. the same as when discharging

(f) $Q = CV_\mathrm{C} = CV_\mathrm{R} = C(IR) = I \times RC$

(g) $Q = -RC\dfrac{\mathrm{d}Q}{\mathrm{d}t}$ so the differential equation is $\dfrac{\mathrm{d}Q}{\mathrm{d}t} = \dfrac{-Q}{RC}$

(h) $I_0 = \dfrac{V_0}{R}$

(i) $I_0 = \dfrac{(Q_0/C)}{R} = \dfrac{Q_0}{RC}$

(j) Time to discharge at constant current $= \dfrac{Q_0}{I_0} = RC$

(k) $t = RC$, so $\dfrac{Q}{Q_0} = e^{-1} = 0.37$