Virtual Target Selection for a Multiple-Pursuer Multiple-Evader Scenario

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Overview

Outline

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2. Optimal Control Problem
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Motivation

Background

- Pursuit assignment problems involving mobile agents represents a relevant and popular class of problems for the aerospace and defense community
- Weapon target assignment is a relevent problem in aerospace and defense
- Scalable methods for performing weapon-target-assignment are desired
- Perform the optimization for an overall fleet of vehicles rather than individually
- Delayed descirions made by the pursuers is of interest to provide agility and flexibility to operations

Reference Categories

Weapon Target Assignment

Kline (2019), and Hughes (2022)

Multi-Pursuer Multi-Evader Games

Isaacs (1965), Shishika (2020), Day (2012), Katz (2005), and Stipanović (2010)

Multi-Pursuer Multi-Evader Border/Target Defense Games

Cruz (2001), Earl (2002), Rusnak (2005), Garcia (2020), Yan (2022), Garcia (2022), Asgharnia (2022), Zepp (2022), Fu (2021), and Fu (2021)

Multi-Pursuer Multi-Evader Games of Incomplete Information

Antoniades (2003), Li (2005), Li (2008), and Wei (2007)

Multi-player Reach-Avoid Games

Chen (2017), and Zhou (2018)

Multiple Pursuer Single Evader

Awheda (2016), Zhou (2016), Bakolas (2010), Bakolas (2012), Makkapati (2019), Ibragimov (2005), Ibragimov (2023), Garcia (2021), Jin (2010), and Chen (2016)

Delayed Decision Guidance and Virtual Targets

Turetsky (2019), Weiss (2022), and Merkulov (2023)

Combinatorics

Papadimitriou (1998), Balas (1991), Pierskalla (1968), and Crama (1992)

Apollonius Circle Definition

Weintraub (2020)

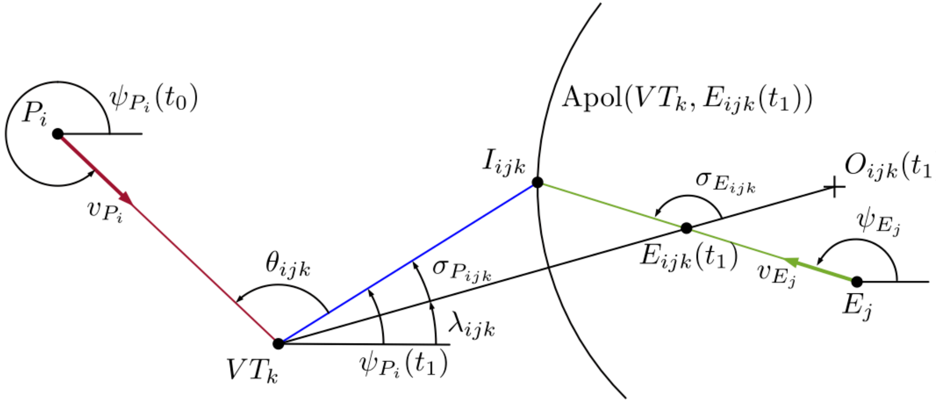
Julia Programming Language, JuMP, and CBC

Bezanson (2017), Lubin (2023), and Forrest (2023)

Introduction

Problem Setup

- Consider many pursuers and evaders
- Faster pursuers aim to capture slower evaders
- Evaders stay on fixed-course
- Pursuers exhibit simple motion
- Pursuers navigate to a virtual target prior to engaging the evaders



Generalized Geometry

Objectives

- Pursuer intercept strategies to evaders by way of a virtual target
- Obtain the pursuer-evader assignments to minimize overall *energy* required by pursuer team.

Optimal Control Problem

State Space:

$$[x_{P_i}, y_{P_i}, x_{E_j}, y_{E_j}]^\top \forall i \in [1..N] \wedge j \in [1..M]$$

Controls: $\psi_P \in \Theta^N, \Theta \in [0, 2\pi] \subset \mathbb{R}$

Initial Conditions:

$$P_i = \left(x_{P_i}, y_{P_i}\right) \in \mathbb{R}^2, P \in \mathbb{R}^{N\times 2}, \quad P_0 = P(t_0)$$

$$E_i = \left(x_{E_i}, y_{E_i}\right) \in \mathbb{R}^2, E \in \mathbb{R}^{M\times 2}, \quad E_0 = E(t_0)$$

$$VT_i = \left(x_{VT_i}, y_{VT_i}\right) \in \mathbb{R}^2, P \in \mathbb{R}^{L\times 2}, VT_0 = VT(t_0)$$

Objective:

$$\min_{\psi_P} J = \min_{\psi_P}\left\{\pi - \theta_{ijk} + \int_{t_0}^{t_f} 1 dt\right\}$$

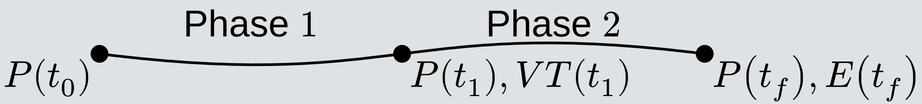
Dynamics:

$$\begin{aligned}
\dot{x}_{P_i} &= v_{P_i} \cos\psi_{P_i} & \forall i \in [1...N] \\
\dot{y}_{P_i} &= v_{P_i} \sin\psi_{P_i} & \forall i \in [1...N] \\
\dot{x}_{E_j} &= v_{E_j} \cos\psi_{E_j} & \forall j \in [1...M] \\
\dot{y}_{E_j} &= v_{E_j} \sin\psi_{E_j} & \forall j \in [1...M]
\end{aligned}$$

Equality Conditions:

$$P_f = P(t_f) = E_f = E(t_f)$$

$$P(t_1) = VT(t_1)$$



Optimal Control Problem - Indirect Method

The optimal control problem is broken into two phases: Phase 1: $t \in [t_0, t_1)$ and Phase 2: $t \in [t_1, t_f]$.

Phase 1 Hamiltonian:

$$\mathcal{H}_I = p_{x_{P_i}}(t) v_{P_i} \cos\left(\psi_{P_i}(t)\right) + p_{y_{P_i}}(t) v_{P_i} \sin\left(\psi_{P_i}(t)\right)$$

Phase 1 Costate Dynamics:

$$\dot{p}_{x_{P_i}} = -\frac{\partial \mathcal{H}_I}{\partial x_{P_i}} = 0, \quad \dot{p}_{y_{P_i}} = -\frac{\partial \mathcal{H}_I}{\partial y_{P_i}} = 0$$

Costates are constant for Phase 1.

Phase 1 Stationarity Condition:

$$\frac{\partial \mathcal{H}_I}{\partial \psi_{P_i}} = 0 \Rightarrow -p_{x_{P_i}} v_{P_i} \sin\psi_{P_i} + p_{y_{P_i}} v_{P_i} \cos\psi_{P_i} = 0$$

The optimal control is constant for Phase 1 and is:

$$\psi^*_{P_i}(t) = \left\{\text{atan2}\left(y_{VT_k} - y_{P_i}, x_{VT_k} - x_{P_i}\right) \mid t \in [t_0, t_1)\right\}$$

Phase 2 Hamiltonian:

$$\mathcal{H}_{\mathbb{I}} = p_{x_{P_i}}(t) v_{P_i} \cos\left(\psi_{P_i}(t)\right) + p_{y_{P_i}}(t) v_{P_i} \sin\left(\psi_{P_i}(t)\right)$$

Phase 2 Costate Dynamics:

$$\dot{p}_{x_{P_i}} = -\frac{\partial \mathcal{H}_{\mathbb{I}}}{\partial x_{P_i}} = 0, \quad \dot{p}_{y_{P_i}} = -\frac{\partial \mathcal{H}_{\mathbb{I}}}{\partial y_{P_i}} = 0$$

Phase 2 Stationarity Condition:

$$\frac{\partial \mathcal{H}_{\mathbb{I}}}{\partial \psi_{P_i}} = 0 \Rightarrow -p_{x_{P_i}} v_{P_i} \sin\psi_{P_i} + p_{y_{P_i}} v_{P_i} \cos\psi_{P_i} = 0.$$

Therefore the optimal control is constant for Phase 2.

Phase 2 Solution Strategy:

- Pursuer takes a straight line course
- Intercept a slower evader
- Use Apollonius circle geometry

Apollonius Circle and the Interception Point

Time for P to reach VT:

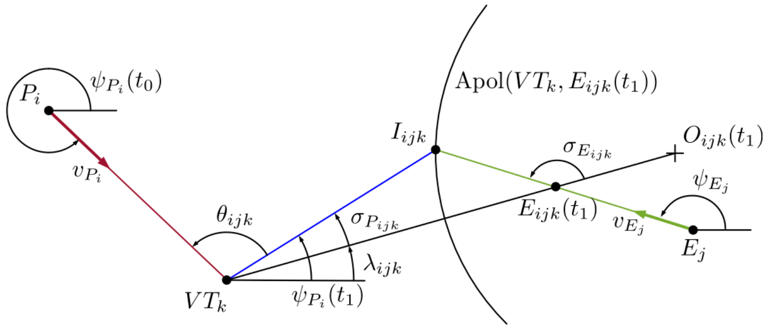
$$t_1 = \frac{1}{v_{P_i}}\sqrt{\left(x_{VT_k}-x_{P_i}(t_0)\right)^2+\left(y_{VT_k}-y_{P_i}(t_0)\right)^2}$$

Position of E when P is at VT :

$$E_{ijk}(t_1) = E_j(t_0) + t_1 v_{E_j}\hat{v}_{E_j}$$

Speed Ratio:

$$\mu_{ij} = \frac{v_{E_j}}{v_{P_i}}$$



Apollonius Circle

$$R_{ijk} = \frac{\mu_{ij}\overline{VT_kE_{ijk}(t_1)}}{1-\mu_{ij}^2}$$

$$O_{ijk}(t_1) = E_{ijk}(t_1) + \frac{\mu_{ij}\overline{VT_kE_{ijk}(t_1)}^2}{1-\mu_{ij}^2}\begin{pmatrix}\cos\lambda_{ijk}\\ \sin\lambda_{ijk}\end{pmatrix}$$

where

$$\overline{VT_kE_{ijk}} = \sqrt{\left(x_{E_{ijk}}(t_1)-x_{VT_k}\right)^2+\left(y_{E_{ijk}}(t_1)-y_{VT_k}\right)^2}$$

Interception point

$$I_{ijk} = \overline{VT_kI_{ijk}}\begin{pmatrix}\cos\left(\sigma_{P_{ijk}}+\lambda_{ijk}\right)\\ \sin\left(\sigma_{P_{ijk}}+\lambda_{ijk}\right)\end{pmatrix} + \begin{pmatrix}x_{VT_k}\\ y_{VT_k}\end{pmatrix}$$

where

$$\sigma_{E_{ijk}} = \psi_{E_j} - \lambda_{ijk}, \quad \sigma_{P_{ijk}} = \sin^{-1}\left(\mu_{ij}\sin\sigma_{E_{ijk}}\right)$$

$$\lambda_{ijk} = \mathrm{atan2}\ \left(y_{E_{ijk}}(t_1)-y_{VT_k}, x_{E_{ijk}}(t_1)-x_{VT_k}\right)$$

$$\overline{VT_kI_{ijk}} = \frac{\overline{VT_kE_{ijk}}}{1-\mu_{ij}^2}\left(\mu_{ij}\cos\sigma_{E_{ijk}}+\sqrt{1-\mu_{ij}^2\sin^2\sigma_{E_{ijk}}}\right)$$

Mixed Integer Linear Program Formulation

Solution Approach Revisited

Value Function

1. Use Optimal Control Theory to find the cost for the assignment of: $P_i \rightarrow VT_k \rightarrow E_j$
2. Optimal Control Theory allows for the utility of Apollonius circle geometry
3. Calculate the Length of $\overline{P_i VT_k}$ and $\overline{VT_k E_j}$
4. Calculate the maneuver θ_{ijk}
5. The cost of the assignment is

$$\overline{P_i VT_k} + \overline{VT_k E_j} + \pi - \theta_{ijk}$$

Assignment Problem

1. Obtain the cost for the assignment of: $P_i \rightarrow VT_k \rightarrow E_j$
2. Limit the number of VT_k cadidates to some maximum: M_V
3. Use a mixed integer linear program to find the optimal assignments.

Mixed Integer Linear Program (MILP) Formulation

- The cost for $P_i \to VT_k \to E_j$ is denoted as c_{ijk}

MILP Equation

$$\text{Objective:} \qquad \min \sum_{i\in\mathcal{P}, j\in\mathcal{E}, k\in\mathcal{V}} c_{ijk} x_{ijk},$$

subject to

$$\text{Contraint 1:} \qquad \sum_{i\in\mathcal{P}, k\in\mathcal{V}} x_{ijk} \geq 1, \forall j \in \mathcal{E},$$

$$\text{Contraint 2:} \qquad \sum_{j\in\mathcal{E}, k\in\mathcal{V}} x_{ijk} = 1, \forall i \in \mathcal{P},$$

$$\text{Contraint 3:} \qquad \sum_{i\in\mathcal{P}, j\in\mathcal{E}} x_{ijk} \leq y_k, \forall k \in \mathcal{V},$$

$$\text{Contraint 4:} \qquad \sum_{k\in\mathcal{V}} y_k \leq M_V,$$

$$\text{Contraint 5: } x_{ijk}, y_k \in \{0,1\}, \forall i \in \mathcal{P}, j \in \mathcal{E}, k \in \mathcal{V}.$$

Objective:
Find the assignment that minimizes the cost

Constraint 1:
Every E is assigned to at least one P at a VT

Constraint 2:
Every P is assigned to some VT and an E combination

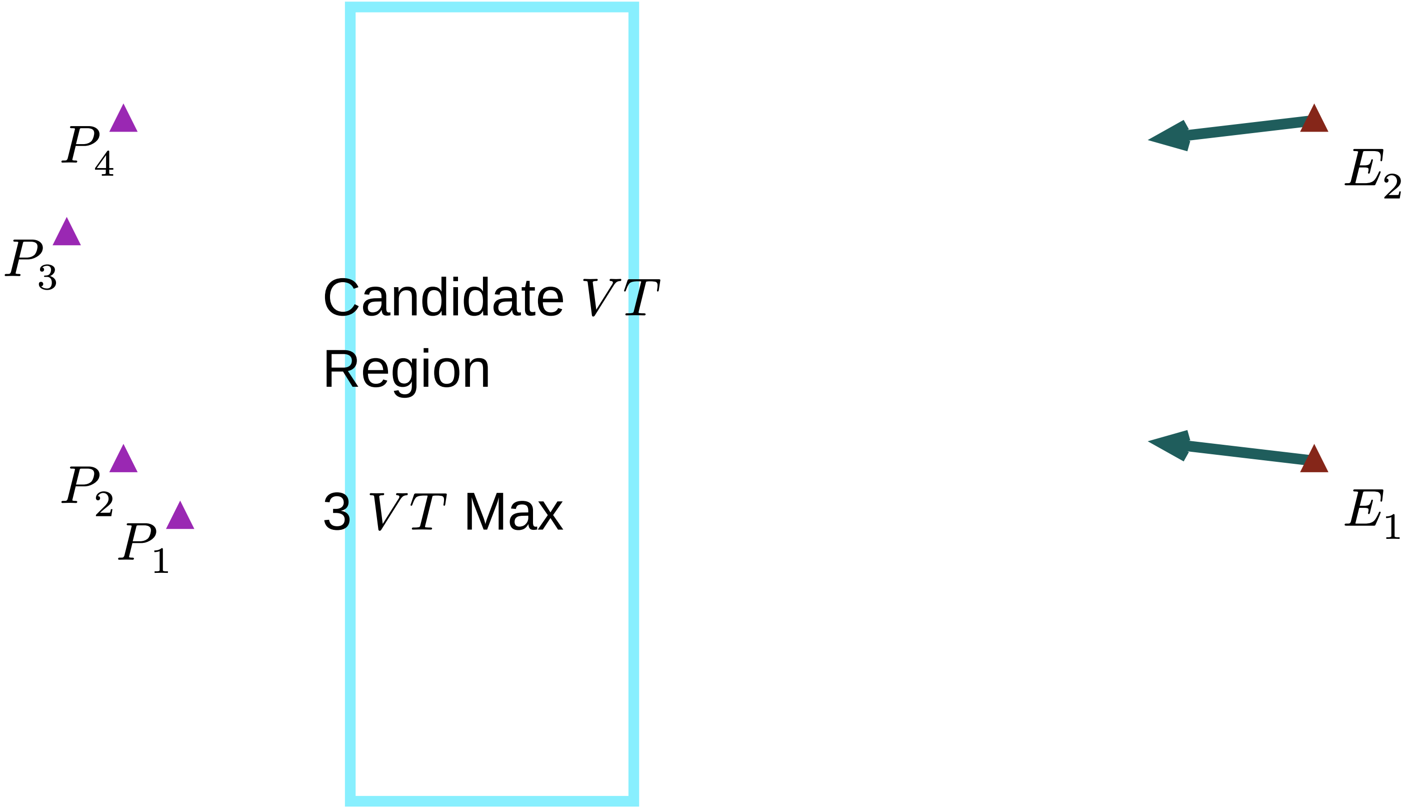
Constraint 3:
A VT_k, can only be used if the corresponding $y_k = 1$

Constraint 4:
Limit the maximum number of VT's.

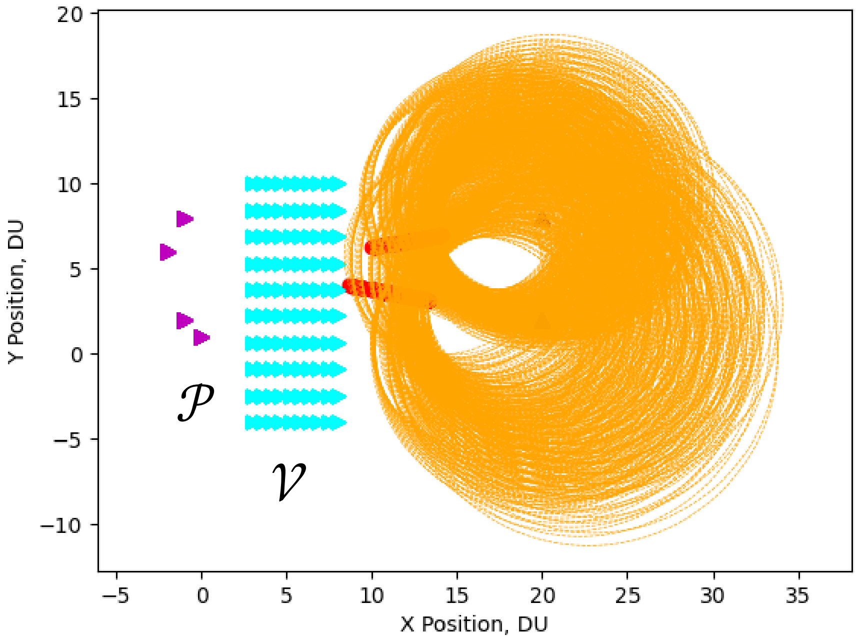
Constraint 5:
Binary constraints

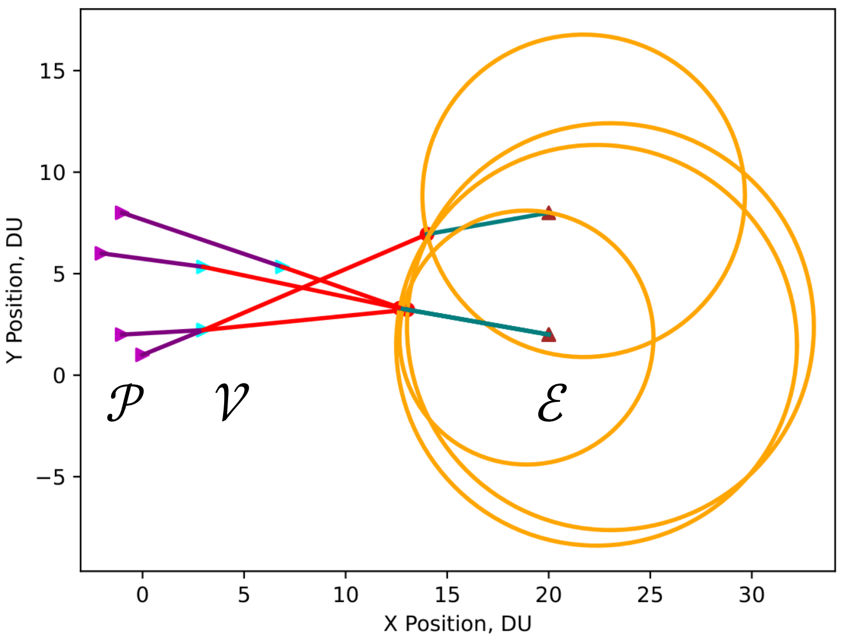
Example: Setup

Problem Setup: 4 Pursuers, 2 Evaders, 3 Virtual Targets



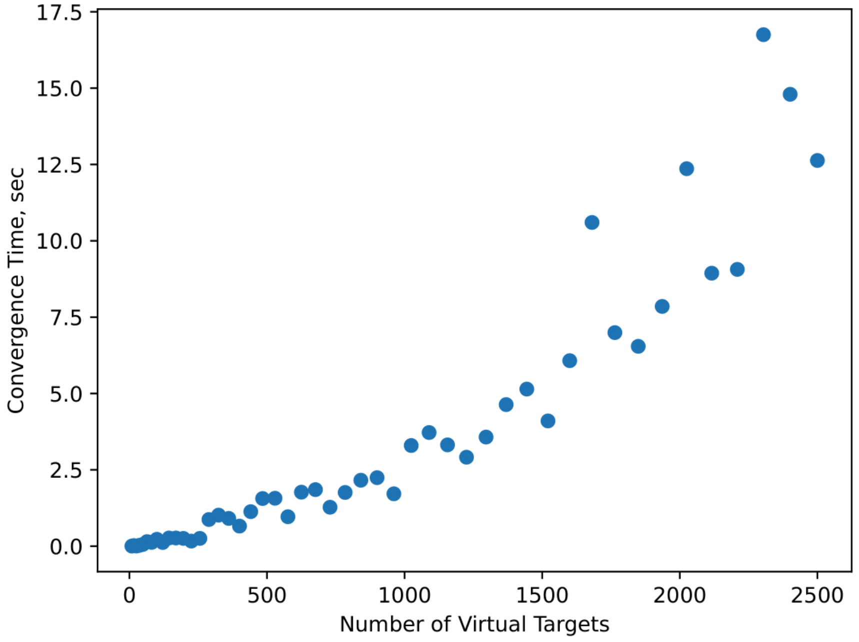
Example: Results - 100 Candidate Virtual Targets



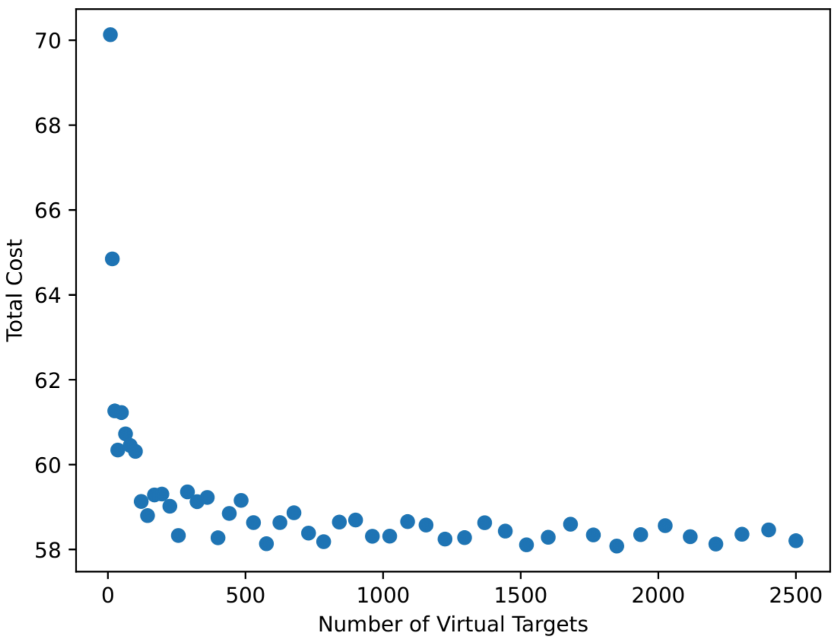


Covergence Time and Performance Tradeoff

Convergence Time

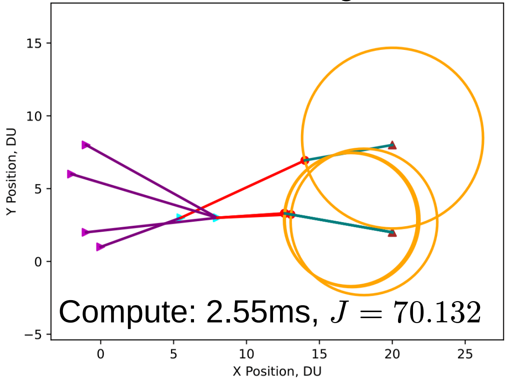


Total Cost

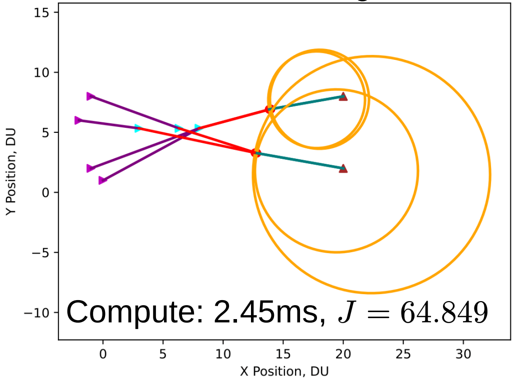


Results: Various Number of Candidate VTs

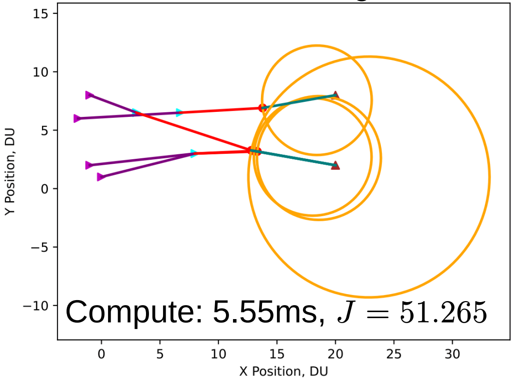
3/9 Virtual Targets



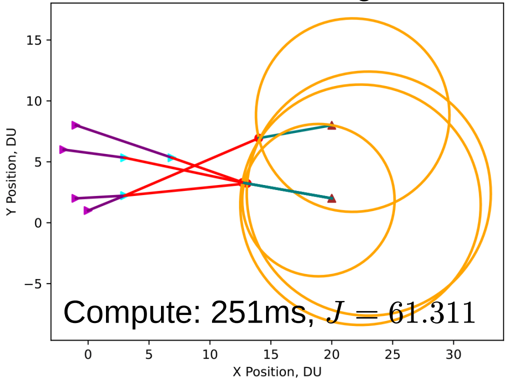
3/16 Virtual Targets



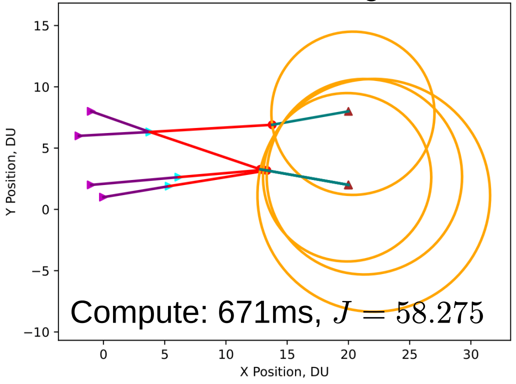
3/25 Virtual Targets



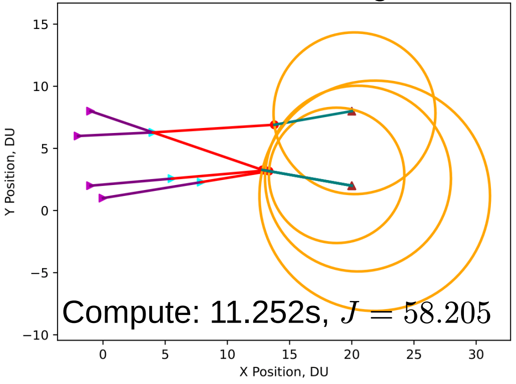
3/100 Virtual Targets



3/400 Virtual Targets



3/2500 Virtual Targets



Conclusions

Summary

- Multi-pursuer Multi-Evader Assignment Problem
- Pursuers navigate to virtual targets then to an assigned target
- Apollponius circle geomery and linear program solver leveraged
- Energy of the team is minimized

Observations

- A weighting factor on manuever or path travel changes the performance
- Solutions scale well for potential hardware applications

Future Work

- Ensuring path deconfliction of solutions
- Simulating higher-fidelity vehicles
- Performing software-in-the-loop and or hardware-in-the-loop tests
- Flight test where possible.

Questions?

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