Push-Pull Block Puzzles [**title**]

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1 Introduction

This work builds on a long line of work on the mathematics of games and puzzles, primarily, the recent work in the computational complexity of video games. We also give a new result showing that block pushing puzzles, like Sokaban, which also include the ability to pull blocks are NP-Hard. This result also proves a number of other games, which have embedded push-pull block puzzles, are NP-Hard.

Games have and continue to be a source of interesting problems for mathematics and computer science. Game playing continues to be a benchmark in the field of AI. The study of economic games has become a large field with applications in online auctions [**?**], network analysis [**?**], and voting [**?**]. Combinatorial games have lead to algebraic insights such as Conway's Surreal Numbers [**?**] and the Sprague-Grundy Theorem [**?**] [**?**]. More recently, games have been studied from an algorithmic perspective and can be seen as models of computation in examples like Constraint Logic [**?**] and Conway's Game of Life [**?**]. For a survey of algorithmic combinatorial game theory see Demaine and Hearn's paper [**?**]. Understanding the computational and algorithmic aspects of games may lead to better understanding of algorithms and problems that share similar properties. For example, certain asymmetries or algebraic properties may be more intuitively obvious in games, such as the relationship between two-player games and quantified boolean formulas []. In this vein, we are particularly interested in the push-pull block model because it is fully reversible, meaning any sequence of moves in the puzzle can be undone. Reversibility is fundamentally linked to quantum computation and the thermodynamics of computation; thus understanding the difference between reversible and irreversible computation is very relevant.

Finally, many games and puzzles can also be seen as simplified models of problems in the real world. Thus, solving these problems gives us partial understanding and tools to attack more difficult questions. For example, push-pull block puzzles can be seen as a simplified model of robotic forklifts operating in a warehouse. Complexity results in these very simplified models help us understand what aspects of problems makes them hard and allows us to start differentiating between the complexity that arises from the combinatorics vs the geometry of path planning problems. Similarly, path-planning in dynamic graphs is a complicated problem which occurs every day from traffic to packet routing.

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1.1 Sliding Blocks Related Work

A significant amount of research has gone into characterizing the complexity of sliding block puzzles. This includes PSpace-Completeness for well known puzzles like Sokaban [**?**] and Rush Hour [**?**]. We are specifically interested in block pushing puzzles, which involve a 'robot' that is able to move on a grid and is able to move blocks adjacent to it, typically by 'pushing' or 'pulling' the block. Further, we only address the path planning problem, in which the robot wants to get from a given location A to a target location B, rather than the storage problem in which the movable blocks must reach some final configuration. Figure 1 gives a summary of results on block pushing puzzles.

The problem of motion planning in an environment where blocks may be pushed and pulled is modeled in a very general form in Gordon Wilfong's Motion Planning in the Presence of Movable Obstacles, where he shows a polynomial time algorithm for motion planning with one movable object, NP-Hardness for the general planning problem, and PSpace-Hardness for the storage problem. [**?**] His problem, however, deals with continuous motion and polygonal walls and blocks; making the model significantly different from the 1x1 blocks on a grid that is considered for most block pushing puzzles. Marcus Ritt paper [**?**] addresses the model in which blocks can only be pulled. We primarily build off of Dor and Zwick's work which generalizes Sokoban to include block pulling. They show the block storage problem which includes 2x1 blocks which can be pushed or pulled is PSpace-Complete, and the unit-size block storage where the robot can push at five or more blocks at a time and pull one is NP-Hard [**?**]. We introduce *thin walls*, which prevent motion between two adjacent empty squares. We prove that all path planning problemsn 2D with thin wall or in 3Ds, in which the robot can push k blocks and pull l blocks for all $k, l \in \mathbb{Z}^+$ are NP-Hard. As with many of these problems, closing the gap between NP and PSpace remains open.

Name	*Push*	*Pull*	*Block Size*	*Fixed?*	*Path?*	*Sliding*	*Complexity*
Push-k	k	0	Unit	No	Path	min	NP-Hard [**?**]
Push-*	*	0	Unit	No	Path	min	NP-Hard [**?**]
PushPush-k	k	0	Unit	No	Path	Max	PSpace-Comp. [**?**]
PushPush-*	*	0	Unit	No	Path	Max	NP-Hard [**?**]
Push-kX	k	0	Unit	No	No-Cross	min	NP-Comp. [**?**]
Push-*X	*	0	Unit	No	No-Cross	min	NP-Comp. [**?**]
Push-1F	1	0	Unit	Yes	Path	min	NP-Hard [**?**]
Push-kF	k≥ 2	0	Unit	Yes	Path	min	PSpace-Comp. [**?**]
Push-*F	*	0	Unit	Yes	Path	min	PSpace-Comp. [**?**]
Sokoban	1	0	Unit	Yes	Storage	min	PSpace-Comp. [**?**]
Sokoban$^+$	k≥ 2	1	2x1	Yes	Storage	min	PSpace-Comp. [**?**]
	k[1]	1	Polygon	Yes	Storage	min	NP-Hard [**?**]
Sokoban(k,1)	k≥ 5	1	Unit	Yes	Storage	min	NP-Hard [**?**]
Pull-1	0	1	Unit	No	Storage	min	NP-Hard [**?**]
Pull-kF	0	k	Unit	Yes	Storage	min	NP-Hard [**?**]
PullPull-kF	0	k	Unit	Yes	Storage	Max	NP-Hard [**?**]
Push-1G[2]	1	0	Unit	Path	min	Yes	NP-Hard [**?**]
PushPull-kW	k	l	Unit	Wall	Path	min	**NP-Hard?**
3DPushPull-k	k	l	Unit	Yes	Path	min	**NP-Hard?**

Table 1: Summary of Block Pushing Puzzle Results

[might be more updating, check 6.890] xxx

2 Push-Pull Block Results [rough draft - real diagrams and better arguments for gadget safety pending]

In this section we prove that Push-1 Pull-1 with fixed blocks is NP-Hard in 3D or if we include *thin walls*. Thin walls are a new, but natural, notion for pushing block puzzles, which prevent blocks or the robot from passing between two adjacent, empty squares, as though there were a thin wall blocking the path. These were needed because many of the gadgets depended greatly on having very tight corridors to ensure limited behavior. We also note that being able to push or pull more blocks does not impact the gadgets, thus proving hardness for Push-k Pull-l for all k and l.
We will prove hardness by a reduction to Planar 3SAT. Clauses can be formed by splitting hallways, each with a check that the corresponding variable was set true. Our literals are composed of the Set-Verify gadget listed below. A variable is a split hallway with one side going to all the true literals and the other going to all of the false literals. Program flow, and the fact that going backwards removes the pass-ability from the Set-Verify gadgets ensures variables are only set to be true or false.

2.1 Reversible One-Way Gadget

When in the initial configuration A the gadget only permits a traversal from s_0 to s leaving the gadget in configuration B. Configuration B only permits traversals in the opposite direction, from s to s_0. Note, when the robot leave this gadget, the only possible configurations are A and B. These gadgets or similar structures will be used several times within more complicated structures.

2.2 Set-Verify Gadgets

For the Set-Verify gadget, the S entrance is the only one which allows the robot to move any blocks. From the S entrance they can traverse to S_0, and they can also pull the middle block down behind them. Doing so will allow a traversal from V to V_0. If the robot every travels back from S_0 to S, they must push the middle block back, ensuring the V to V_0 traversal is impossible. Further, access to any sequence of entrances will not allow the robot to alter the system to allow traversals between the V and S entrances.

The 3D Set-Verify is essentially the same design. All of the labeled entrances, S, S_0, V, V_0 now come into the gadget from above. All of these vertical hallways can continue straight up or in whatever desired direction except for S which must go up one square and right one square before continuing. This is to prevent any blocks from being pushed or pulled into the S hallway from the 3D Set-Verify gadget. It can easily be seen that non of these are adjacent, so there is no longer a need for thin walls.

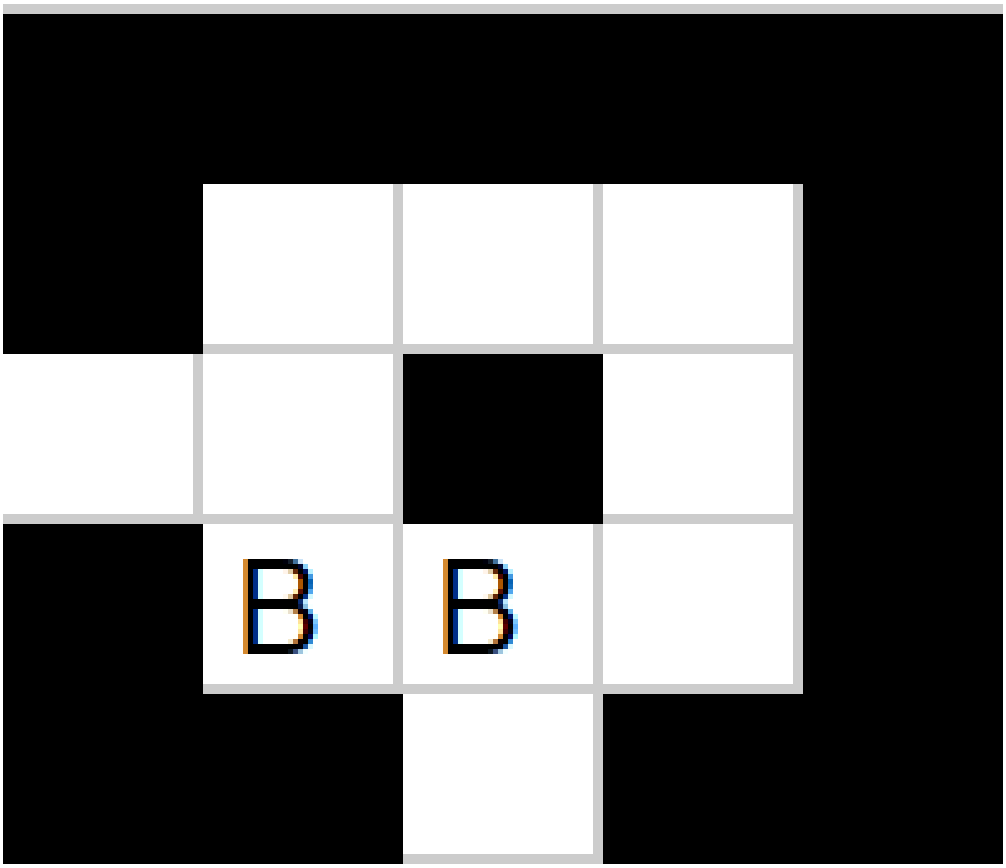


Figure 1: Reversible one-way gadget.

2.3 One-way Destructive Crossover

Initially this gadget allows either a traversal from A to A' or B to B'. Once a traversal has occurred, that path may be traversed freely, but the other is impassable unless the gadget is reset. **[include an initial and crossed gadget]** xxx

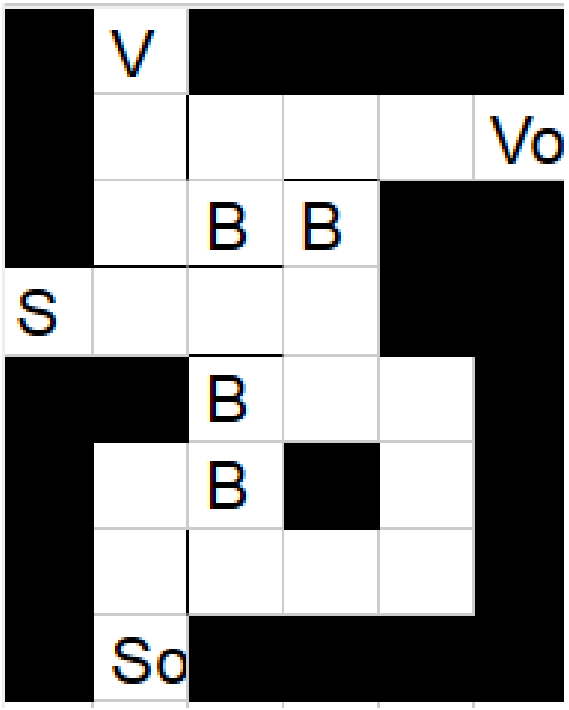
2.4 In-order One-way Crossover

This gadget allows a traversal from 1 to 1' and then 2 to 2'. Note, the 2 to 2' traversal cannot be made unless the 1 to 1' traversal has already occurred; since it is necessary for the top block and the block closest to 1 to be pulled out. Likewise, a traversal back from 1' to 1 will push these back in, preventing subsequent 2 to 2' traversals. A successful 2 to 2' traversal will also prevent 1 to 1' from being passable.

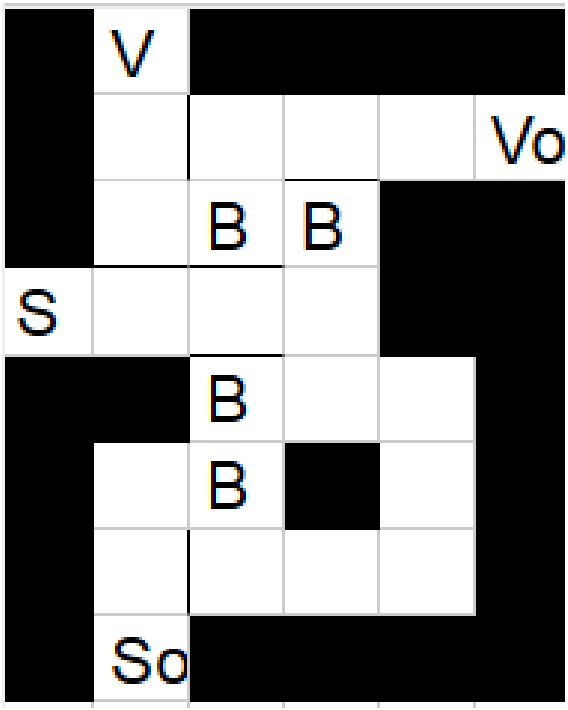
2.5 Directed Crossover

Using the One-way Destructive Crossover and the In-order One-way Crossover, we can construct a Directed Crossover which allows Arbitrary traversals from A to A' and B to B'. Notice, that in either path, one will activate one destructive crossover, but in doing so will toggle the in-order

Figure 2: Set-Verify Gadgets



(a) 3D Set-Verify gadget.



(b) 2D Set-Verify with walls.

crossover, always ensuring that there is a path in the other direction, no matter which one is crossed first. This allows us to cross each way once, in either order, which is sufficient for our hardness proof.

2.6 Crossover

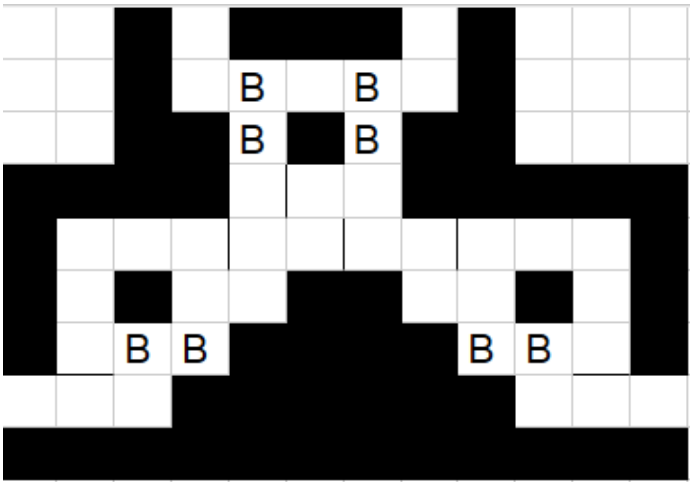
Four Directed Crossovers can be combined, as shown below, to create a crossover that can be traversed in any direction.

2.7 3-SAT Construction

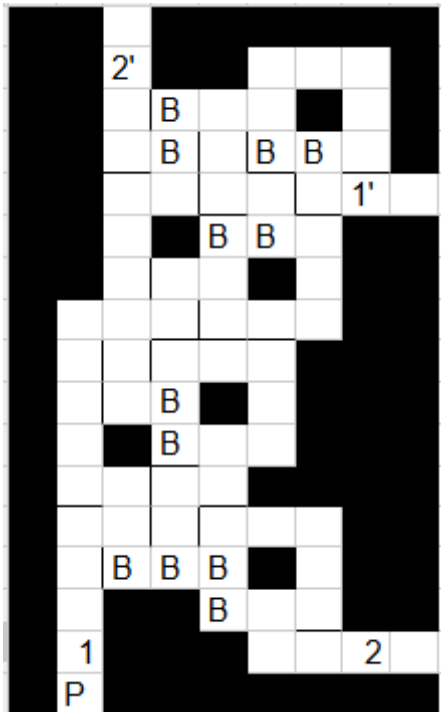
We plan to reduce from Planar 3-SAT and will be making use of the Set-Verify gadget to produce our literals. One significant difficulty with this model is the complete reversibility of all actions. Thus we need to take care to ensure that going backward at any point does not allow the robot to cheat in solving our 3-SAT instance. The directional properties of the Reversible One-Way and the Set-Verify allow us to create sections where we know if the robot exits, it must have either reset everything to the initial configuration or put everything in another known state.

Our literals will be represented by Set-Verify gadgets. They are considered true when the V to V_0 traversal is possible, and false otherwise. Thus we can set literals to true by allowing the robot to run through the S to S_0 passage of the gadget. This implies a very simple clause gadget, consisting of splitting the path into three hallways each with the corresponding verify side of our literal. We

Figure 3: Two types of crossover gadgets



(a) The one-way destructive crossover



(b) The in-order one-way crossover

can then pass through if any of the literals is set true and cannot be passed otherwise.

The variables will be encoded by a series of passages which split to allow either the true or negated literals to be set. We begin each hall with a Reversible One-Way gadget, primarily to handle the case of having no literals of that type to set. Once

Variables split into two hallways. We have to show that when you enter or exit a side of a hallway, all the literals are set true at one end and false at the other. Also, we cannot go back through the other hallway, so if we go backward, everything gets reset. Thus we cannot exploit the reversibility to set anything extra true.

3 PSpace

- 4 Togle also do -k
- Lock Chain
- Clause
- Binary Counter
- Existential and Universal setting (augmenting the counter)
- Initial setup

3.1 Toggles

We define an n-toggle to be a gadget which has n internal pathways and can be in one of two internal states, A or B. Each pathway has a side labled A and another labled B. When the toggle is in the A state, the pathways can only be traversed from A to B and similarly in the B state they can only be traversed from B to A. Whenever a pathway is traversed the state of the Toggle flips.

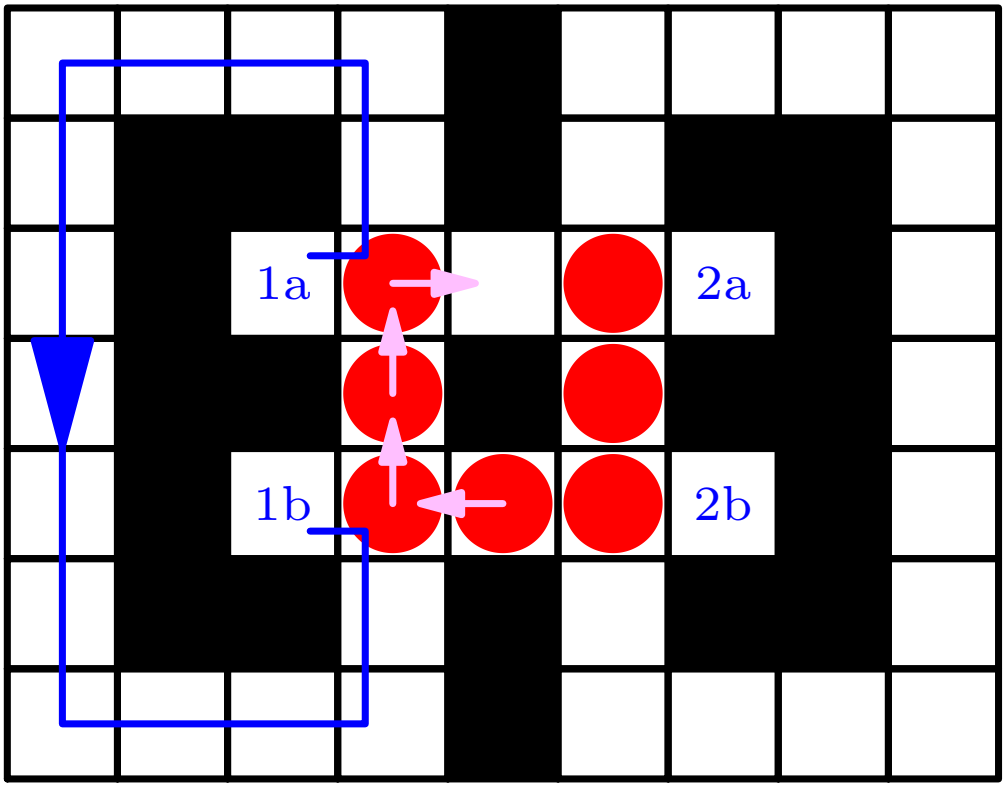


Figure 4: 2-Toggle in state A

Figure 4 acts as a 2-Toggle. Notice that there is a single block missing from the ring of eight blocks. When the missing block is on top, as diagrammed, it will represent state A, and when it is on the opposite side, we call it state B. Notice that in state A, it is impossible to enter through entries $1b$ or $2b$. When we enter in the $1a$ or $2a$ sides, we can follow the moves in the searies of diagrams to exit the corresponding $1b$ or $2b$ side and leaving the gadget in the B state. One can easily check that the gadget can only be left in either state A, B, or a broken state as seen in Figure **??**. Notice, in the broken state, every pathway except the one just exited is blocked. If we enter through that path, it is in exactly the same state as if it had been in an allowed state and entered through the corresponding pathway normally. For example, in the diagram one can only enter through $1b$ and after doing so it is the same as entering in path $1b$ on a 2-Toggle in state B. Thus the broken state is never useful for solving the puzzle and can be safely ignored.

To construct a 4-Toggle we essentially take two copies of the two toggle, rotate them perpendicular to each other in 3D, and let them overlap on the central axis. See Figure **??**. We still interpret the lack of blocks in the same positions as the 2-Toggle as states A or B. Now we have four different paths which function the same as the ones described above. Similar arguments show the broken states of the 4-Toggle also don't matter.

3.2 Locks

A lock is a gadget consisting of a 2-toggle and a separate pathway. Traversing the separate pathway can only be done in a single direction, based on whether the 2-toggle is in state A or B, and the traversal does not change the internal state of the 2-toggle. The 2-toggle functions exactly as described above. This gadget can be implemented using a 4-toggle, by connecting the $3B$ and $4B$ entrances of the 4-toggle with an additional corridor, as shown in **??**. Traversing the resultant full pathway, from $3A$ to $3B$ to $4B$ to $4A$, is possible only if the initial state of the 4-toggle is A, and will leave the 4-toggle in state A. In addition, a partial traversal, such as from $3A$ to $3B$ and back to $3A$, does not change the internal state. The two unaffected pathways of the toggle, 1 and 2, continue to function as a 2-toggle.

A lock chain is a gadget consisting of a 2-toggle and any number of separate pathways. As in the lock, the 2-toggle functions as described above, and each pathway is passable in either direction if the toggle is in state A, or impassable in either direction if the toggle is in state B. This is implemented using one lock per non-toggling pathway needed. As shown in **??**, each lock's $1A$ entrance is connected to the next lock's $1B$ entrance, and likewise between $2A$ and $2B$. Thus, the entire string of 1 pathways form the lock chain's 1 pathway, and the string of 2 pathways for the lock chain's 2 pathway. Due to this connection system, the internal states of every underlying lock will be the same whenever the robot is outside the lock chain. This state is the lock chain's internal state, and appropriately determines the passability or impassibility of the separate pathways.

3.3 Binary Counter

We now define a binary counter. The binary counter has a fixed number of internal bits. Whenever the binary counter is traversed in the forwards direction, the binary number formed by the internal bits increases by one and the robot leaves via one of the exits. If the binary counter is traversed in the reverse direction, the internal value is reduced by one. If the binary counter is partial traversed, the internal value does not change. **[I don't think this is correct, we can certainly have the value change but the robot is still stuck inside the gadget.]** xxx

The binary counter is implemented as a series of 2-toggles, as shown in **??**. The entrance pathway is connected to the 2-toggle's $1A$ and $2B$ entrances. The $1B$ exit from the 2-toggle will exit from the entire binary counter. The $2A$ exit will continue on to the next 2-toggle, attaching to that toggle's $1A$ and $2B$ entrances. This will continue for every toggle down the line, except that the last toggle's $2A$ exit signals an overflow, and exits from the counter.

To see that this produces the desired effect, identify a toggle in state A as a 0 bit, and a toggle in state B as a 1 bit. Let the entrance toggle's bit be the least significant bit, and the final toggle be the most significant. When the robot enters the binary counter in the forwards direction, it will flip the state of every toggle it passes through. When it enters a toggle that is initially in state B, and thus whose bit is 1, it will flip the state/bit and proceed to the next toggle, via the $2B - 2A$ pathway. When it encounters a toggle that is initial in state A / bit 0, it will flip the state/bit and exit, via the $1A - 1B$ pathway. Thus, the overall effect on the bits of the binary counter is to change a sequence of bits ending at the least significant bit from 01..11 to 10..00. This has the effect of increasing the value of the binary counter by one.

4 Conclusion

5 Open Questions

5.1 Block Pushing

- Close NP, PSpace gap.
- No thin walls
- All movable blocks
- Push-Push Pull (seen in Catherin-not quite since you can get on them, also Zelda?)

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