Lab class 3: Conditioning and DTMCs

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1 Random distributions with random rates

Suppose you have a coin. Let H be the indicator random variable for whether it lands on heads. 1 if heads, 0 if tails. The coin is not necessarily a fair coin – it flips heads with probability X. You're not sure which way it's biased – X is either $\{0, 1/4, 1/2, 3/4, 1\}$, with equal probability.

a. What's the conditional probability distribution $P(H = 1 \mid X = x)$, for each possible value of X?

Solution. For each possible value x, the probability $H = 1$ given $X = x$ is x. Thus,

$$P(H = 1 \mid X = x) = x$$

for $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1. □

b. What's the joint pmf $P(H = h, X = x)$, for all pairs (h, x), $h \in \{0, 1\}, x \in \{0, 1/4, 1/2, 3/4, 1\}$?

Solution. The joint pmf is given by

$$P(H = h, X = x) = P(H = h \mid X = x) \cdot P(X = x),$$

where $P(X = x) = \frac{1}{5}$ for $x \in \{0, 1/4, 1/2, 3/4, 1\}$. Note also that $P(H = 0 \mid X = x) = 1 - P(H = 1 \mid X = x) = 1 - x$. □

c. What's the chance of an unlikely outcome: Either $H = 1$ and $X < 1/2$, or $H = 0$ and $X > 1/2$?

Solution. There's four ways this could happen: $H = 1$ and $X = 0$ or $X = 1/4$, or $H = 0$ and $X = 1$ or $X = 3/4$. But actually, two of these are impossible: $H = 1$ and $X = 0$ can't happen, and $H = 0$ and $X = 1$ can't happen. So we focus on the last two cases.

As found in (b), $P(H = 1, X = 1/4) = 1/20$ and $P(H = 0, X = 3/4) = 1/20$. In total, the chance of an unlikely outcome is 1/10. □

2 Modeling as a Discrete-Time Markov Chain

You're running a manufacturing facility with two locations, A and B. You have 2 trucks that you use to carry parts and supplies between these two locations. Each hour, a call will come in. It might request parts to be taken from A to B, or from B to A. There's 1/2 probability of each, and it's independent of past calls.

If you have a truck at the starting point, you send the truck, with the parts, to the other location. The truck parks at its destination and waits for the next hour.

If you don't have a truck at the starting point, one of the trucks from the other location comes over and then goes back. The truck has to do a round trip. It's annoying, you don't like it when that happens.

All trucks are either at location A or location B, whenever the hour rolls over and a new call comes in.

We're going to model this scenario as a Discrete-Time Markov chain.

a. What's the state of the system? In other words, where can the trucks be when a call comes in each hour? How many states are there?

Solution. There are three states. Both of the trucks are at location A, one at location A and the other at location B and both of the trucks are at location B. □

b. What are the transition probabilities? Draw the Markov chain two ways: Transition diagram (circles and arrows) and Transition matrix (grid of probabilities)

Solution.

$$P = \begin{array}{c|ccc} & (A,A) & (A,B) & (B,B) \\ \hline (A,A) & \frac{1}{2} & \frac{1}{2} & 0 \\ (A,B) & \frac{1}{2} & 0 & \frac{1}{2} \\ (B,B) & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

□

c. Suppose at time 0, just before the first call comes in, we're in the state where there's one truck in each location. What's the probability of being in each state at time 1?

Solution.

$$\mathbb{P}\{X_1 = (A,A) \mid X_0 = (A,B)\} = 1/2.$$
$$\mathbb{P}\{X_1 = (A,B) \mid X_0 = (A,B)\} = 0.$$
$$\mathbb{P}\{X_1 = (B,B) \mid X_0 = (A,B)\} = 1/2.$$

□

d. What's the chance that on time step 1, a truck has to do a round trip?

Solution. At time step 0, the trucks are at different locations and at time step 1 they have to be in the same location, either state (A,A) or state (B,B). If they are at state (A,A), one of the truck does a round trip with probability $\frac{1}{2}$. Similarly for the state (B,B). Thus, the probability is $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$. □