Lab class 6: Approximations

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October 17, 2024

1 Binomial distribution approximates Poisson distribution

The Poisson distribution is often used as an approximation to the binomial distribution when the number of trials n is large, and the probability of success p is small, such that $\lambda = np$ remains constant. In this problem you will show that the binomial distribution $\text{Bin}(n, p)$ converges to the Poisson distribution with mean λ as $n \to \infty$ and $p \to 0$.

1. Given that $\lambda = np$ is a constant, show that as $n \to \infty$, the probability p tends to zero.

   Solution. Since $\lambda = np$, we have $p = \dfrac{\lambda}{n}$. As $n \to \infty$, $p = \dfrac{\lambda}{n} \to 0$ because λ is constant. □

2. Recall the PMF of the binomial distribution,

$$P_n(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

   Express this formula in terms of n, k, and λ only.

   Solution. Substituting $p = \dfrac{\lambda}{n}$ into the PMF,

$$P_n(k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

   □

3. Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

   Argue that for large n and fix k, we have the following approximation,

$$\binom{n}{k} \approx \frac{n^k}{k!}.$$

   Solution. For large n and fixed k:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} \approx \frac{n^k}{k!}$$

   Because n is much larger than k, the terms $(n-i)$ for $i < k$ are approximately n. □

4. Show that

$$\left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda} \quad \text{as} \quad n \to \infty.$$

   Hint: Use the limit definition of the exponential function $e^x = \lim_{n\to\infty}(1 + x/n)^n$.

Solution. We have,

$$\left(1-\frac{\lambda}{n}\right)^n = \left(1+\frac{-\lambda}{n}\right)^n.$$

As $n \to \infty$,

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

□

5. Using your results from Parts (2), (3), and (4), show that:

$$P_n(X=k) \to \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{as} \quad n\to\infty$$

Which is the PMF of the Poisson distribution with mean λ.

Solution. Starting with,

$$P_n(X=k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

Using the approximation from Part (3)

$$\binom{n}{k} \approx \frac{n^k}{k!},$$

so

$$P_n(k) \approx \frac{n^k}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}.$$

Simplify, we get

$$\frac{n^k}{k!}\left(\frac{\lambda}{n}\right)^k = \frac{\lambda^k}{k!}.$$

Next, use the result from Part (4),

$$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}$$

And since k is fixed and n is large:

$$\left(1-\frac{\lambda}{n}\right)^{-k} \to 1.$$

Therefore,

$$\left(1-\frac{\lambda}{n}\right)^{n-k} = \left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} \approx e^{-\lambda}\times 1 = e^{-\lambda}$$

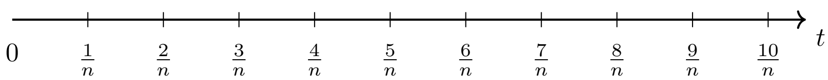
Putting it all together,

$$P_n(X=k) \approx \frac{\lambda^k}{k!}e^{-\lambda},$$

which is the PMF of the Poisson distribution with mean λ. □

2 Poisson Process as the Limit of a Bernoulli Process

Suppose we want to model the arrival of events that occur randomly at a rate λ per unit time. At time $t = 0$, there are no arrivals yet, so $N(0) = 0$. Now, we divide the timeline $[0, t)$ into tiny subintervals of length $\frac{1}{n}$, as shown below,



Each subinterval represents a time slot of length $1/n$. Thus, the intervals are $(0, 1/n]$, $(1/n, 2/n]$, $(2/n, 3/n]$, $\dots$. More generally, the k-th interval is $((k-1)/n, k/n]$. In each time slot, we flip a coin with probability of heads $P(\mathrm{H}) = \lambda/n$. If the coin lands on heads, we record an arrival in that subinterval. Otherwise, we record no arrival. Now, let $N(t)$ be defined as the number of arrivals (number of heads) from time 0 to time t.

1. What distribution does $N(t)$, the number of arrivals by time t, follow? For simplicity, assume nt is an integer.

   Solution. Since each subinterval has a small probability of arrival, $P(H) = \lambda \cdot \frac{t}{n}$, and we have n subintervals, $N(t)$ follows a Binomial distribution:

   $$N(t) \sim \text{Binomial}\left(nt, \frac{\lambda}{n}\right).$$

   □

2. What happens to this distribution in the limit as $n \to \infty$? **Hint:** use your result from question 1.

   Solution. As $n \to \infty$, the Binomial distribution $N(t) \sim \text{Binomial}\left(nt, \frac{\lambda}{n}\right)$ converges to a Poisson distribution with rate $nt \times \frac{\lambda}{n}$. We have,

   $$N(t) \sim \text{Poisson}(\lambda t).$$

   □

3 Geometric distribution approximates Exponential distribution

Recall that in a Poisson process, the interarrival time between events is exponentially distributed. In this setup, however, the interarrival time is $1/n$ times a geometric random variable: each arrival corresponds to a success of a Bernoulli trial. As $n \to \infty$, this geometric distribution, scaled by $1/n$, will converge to an exponential distribution, aligning with the behavior of a Poisson process.

1. Let E denote the time between two successes, in the same setup. Find $P(E > t)$ in terms of p, n, and t.

   Solution. $E > t$ implies that there are no success for the first $n \times t$ number of trials, therefore

   $$P(E > t) = (1-p)^{nt} = \left(1 - \frac{\lambda}{n}\right)^{nt}.$$

   □

2. Assume that $n \to \infty$ such that $np = \lambda$ remains constant. Show that,

$$P(E > t) \to e^{-\lambda t} = P(Exp(\lambda) > t) \quad \text{as} \quad n \to \infty.$$

Hint: Use Question 1 part (4).

Solution. Substitute $p = \dfrac{\lambda}{n}$ into $P(E > t)$,

$$P(E > t) = (1 - p)^{nt} = \left(1 - \frac{\lambda}{n}\right)^{nt}.$$

As $n \to \infty$,

$$P(E > t) = \left(\left(1 - \frac{\lambda}{n}\right)^{n}\right)^{t} \to \left(e^{-\lambda}\right)^{t} = e^{-\lambda t}$$

Therefore,

$$P(E > t) \to e^{-\lambda t}$$

□