Main class 11: Infinite DTMCs

Izzy Grosof

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1 Takeaways and Summary

- DTMCs can have infinitely many states. For instance, this is useful to represent queueing systems with unbounded waiting room.
- In an infinite DTMC, aperiodicity and irreducibility aren't sufficient to guarantee a stationary distribution exists, or that a limiting distribution exists, or that they're equal if they do exist.
- The return probability of a state i is

  $$r_i = \min(n \geq 1 : X_n = i \mid X_0 = i).$$

  A state is "transient" if this probability is less than 1, and "recurrent" if it equals 1. If two states are connected, they are either both transient or both recurrent. In infinite DTMCs, all states can be transient – this can't happen in finite DTMCs.
- The first return time T_i of a recurrent state i is a random variable:

  $$T_i := \min_{n \geq 1}(X_n = i \mid X_0 = i)$$

- The mean return time of a recurrent state is the mean of the recurrence time: $m_{ii} = E[T_i]$.
- If a state i has finite mean recurrence time, $m_{ii} < \infty$, we call the state "positive recurrent". If it has an infinite mean recurrence time, $m_{ii} = \infty$, we call the state "null recurrent".
- If two states are connected, they are either both positive recurrent, or both null recurrent.
- If a Markov chain is null recurrent (e.g. if all of its states are null recurrent), then all states' limiting probabilities are 0, there is no limiting distribution, and there is no stationary distribution.
- If a Markov chain is positive recurrent, irreducible, and aperiodic, then it has a stationary distribution, it has a limiting distribution, and the stationary and limiting distributions are equal.

2 Recap

If finite state space, irreducible, and aperiodic then stationary distribution equals limiting distribution:

$$\pi P = \pi$$
$$\pi_j = \lim_{n \to \infty} (P^n)_{ij}$$

3 Examples of limiting distributions

3.1 Balanced random walk

State: {…, -2, -1, 0, 1, 2, …}

$P_{i,i+1} = 1/3$, $P_{i,i} = 1/3$, $P_{i,i-1} = 1/3$.

Draw transition diagram.

Stationary distribution: All equal, no solution.

Limiting probabilities: Use Central Limit Theorem! $X_n = \sum_{i=1}^n Y_i$, where $Y \sim \{-1, 0, 1\}$, equally likely.

By the central limit theorem, $X_n \approx Normal(n\mathbb{E}[Y], \sqrt{n\text{Var}[Y]}$.

$\mathbb{E}[Y] = 0, \text{Var}[Y] = \mathbb{E}[Y^2] = \frac{1}{3}(1+0+1) = \frac{2}{3}$.

$X_n \approx Normal(0, \sqrt{2n/3})$

Draw some pictures – bell curves get wider and wider, and shallower and shallower.

Limiting probability is 0, for every state – more and more spread out forever.

No limiting distribution – limiting probabilities don't add to 1.

3.2 Unbalanced random walk

$P_{i,i+1} = 1/4$, $P_{i,i} = 1/4$, $P_{i,i-1} = 1/2$.

No solution, limiting probabilities still 0.

3.3 Queue

Floor at 0: $P_{0,1} = 1/4.P_{0,0} = 3/4$.

This is a queueing model! Each time step, there's a chance someone comes in, or someone gets helped.

Write stationary equations.

You try: equation relating π_1 and π_0, equation rating π_2 and π_1, etc.

I show: Substitute into $\sum_i \pi_i = 1$, find stationary distribution. Ratio formula:

$$a + ar + ar^2 + ar^3 \ldots = \frac{a}{1-r} \text{ if } |r| < 1$$

Solution: $\pi_i = (1/2)^i$.

Turns out: This is its limiting distribution.

What's the rule?

4 Recurrence: Null and positive recurrence

Recurrent: Probability of return. For balanced and queue, probability 1. For unbalanced, less than 1.

First visit to i: $T_i := \min(n \geq 1 : X_n = i)$. This is a random variable!

Mean return time m_i: $E[T_i \mid X_0 = i]$.

Positive recurrent: Finite mean return time.

Null recurrent: Infinite mean return time. Limiting probability of 0.

Theorem: If positive recurrent, and irreducible, then stationary distribution has unique solution, matches long run probabilities.

Theorem: If positive recurrent, and irreducible, and aperiodic, then stationary distribution matches limiting probabilities.

Ergodic: positive recurrent, and irreducible, and aperiodic.

If ergodic, $m_i = 1/\pi_i$.