Lab class 10: Final Review

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1 Variability of rain

Jory is a farmer, and it's important for them to estimate how much rain will fall in the next month, which is 30 days long.

The rainfall is concentrated into storms. These storms arrive according to a Poisson Process with a random rate. Some months are chaotic, and the arrival rate of storms is $\frac{1}{2}$ storms per day. Some months are more peaceful, and the arrival rate is $\frac{1}{4}$ per day. Jory estimates that there's a $\frac{1}{3}$ chance that the next month will be chaotic, and a $\frac{2}{3}$ chance that it will be peaceful.

Call the number of storms in the next month N.

a. What is the mean number of storms in the next month, $E[N]$?

Solution. If it's chaotic, there'll be an average of $30 \cdot \frac{1}{2} = 15$ storms. If it's peaceful, there'll be an average of $30 \cdot \frac{1}{4} = \frac{15}{2}$ storms.

Overall, the expected number of storms is

$$15 \cdot \frac{1}{3} + \frac{15}{2} \cdot \frac{2}{3} = 5 + 5 = 10$$

□

b. What is the mean squared number of storms in the next month, $E[N^2]$?
Note: If $X \sim Poisson(\lambda)$, $E[X^2] = \lambda^2 + \lambda$.

Solution. If it's chaotic, the number of storms is distributed as $Poisson(15)$, so $E[N^2 \mid chaotic] = 15^2 + 15 = 240$. If it's peaceful, the number of storms is distributed as $Poisson(15/2)$, so $E[N^2 \mid peaceful] = (\frac{15}{2})^2 + \frac{15}{2} = \frac{225}{4} + \frac{30}{4} = \frac{255}{4}$.

Overall, the mean squared number of storms is

$$E[N^2] = \frac{1}{3}E[N^2 \mid chaotic] + \frac{2}{3}E[N^2 \mid peaceful] = 240 \cdot \frac{1}{3} + \frac{255}{4} \cdot \frac{2}{3} = 80 + \frac{85}{2} = \frac{245}{2}$$

□

c. What is the variance of the number of storms in the next month, $Var(N)$?

Solution.

$$Var(N) = E[N^2] - E[N]^2 = \frac{245}{2} - 10^2 = \frac{45}{2}$$

□

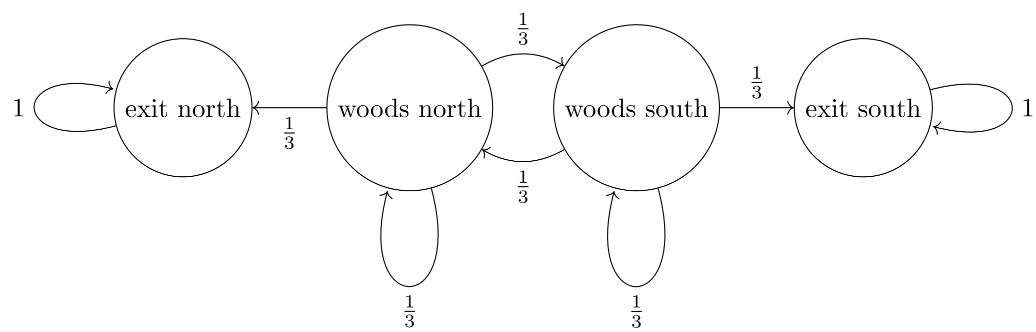
2 Wandering

Sam is on a hike in the woods, and he's a bit lost. There's four possible locations he could be: In the woods on the north side, in the woods on the south side, exited to the north, and exited to the south. On each time step, if he's in the woods, he walks north with probability 1/3, stays where he is with probability 1/3, and walks south with probability 1/3, independent of previous steps. If he's out of the woods, he stays where he is.

Let's model this situation as a DTMC.

a. Draw the transition diagram for this DTMC.

Solution.



□

b. What is the transition matrix for this DTMC?

Solution. We'll use the above ordering of the states: exit north, woods north, woods south, exit south. Remember, rows are where we're coming from, columns are where we're going to.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

c. In this DTMC, which states are transient, which states are absorbing, and which states are neither?

Solution. Exit north and exit south are absorbing, and woods north and woods south are transient.

□

d. Suppose that Sam is initially in the woods on the north side. What are the distributions of Sam's position after 1, 2, and 3 time steps?

Solution. After 0 time steps, Sam's position distribution is $[0, 1, 0, 0]$, using the same state ordering as above.

After 1 time steps, Sam's position distribution is $[1/3, 1/3, 1/3, 0]$.

After 2 time steps, Sam's position distribution is $[4/9, 2/9, 2/9, 1/9]$.

After 3 time steps, Sam's position distribution is $[14/27, 4/27, 4/27, 5/27]$. □

e. Does this DTMC has no stationary distribution, a unique stationary distribution, or many stationary distributions?

Solution. Let's solve the stationary equations. Let's call the state probabilities $\pi_{EN}, \pi_{WN}, \pi_{WS}, \pi_{ES}$.

$$\begin{aligned}
\pi_{EN} &= \pi_{EN} + \frac{1}{3}\pi_{WN} \\
\pi_{WN} &= \frac{1}{3}\pi_{WN} + \frac{1}{3}\pi_{WS} \\
\pi_{WS} &= \frac{1}{3}\pi_{WN} + \frac{1}{3}\pi_{WS} \\
\pi_{ES} &= \frac{1}{3}\pi_{WS} + \pi_{ES} \\
1 &= \pi_{EN} + \pi_{WN} + \pi_{WS} + \pi_{ES}
\end{aligned}$$

From the first equation, we find that $\pi_{WN} = 0$. From the fourth equation, we find that $\pi_{WS} = 0$. From the final equation, we find that $\pi_{EN} + \pi_{ES} = 1$. We have no further information about π_{EN} or π_{ES}.

For any probability p in the range from 0 to 1, the distribution $\pi_{EN} = p, \pi_{ES} = 1 - p, \pi_{WN} = 0, \pi_{WS} = 0$ is a stationary distribution, which can be verified by checking all of the equations. There are infinitely many stationary distributions. □

3 Running

Alex is out for a run. Their run takes them through several regions near where they live. They start in region A, then go to regions B, C, D, and E, before they go back to region A. Region A is their favorite, so they stay there the longest, $Exp(1/10)$ minutes. Region B is their next favorite, where they stay for $Exp(2/10)$ minutes. Then region C, $Exp(3/10)$ minutes, region D, $Exp(4/10)$ minutes, and region E, $Exp(5/10)$ minutes.

Model Alex's run as a CTMC.

1. Draw the transition diagram for this CTMC.

   Solution. Drawing in Latex is hard. It's a five state cycle, with rates $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}$ □

2. How long, on average, is each cycle of the Alex's run, from starting in region A to next returning to region A?

   Solution. Time from A to B is 10 minutes on average, B to C is 5 minutes on average, C to D is $\frac{10}{3}$, D to E is $\frac{10}{4}$, and E to A is 2. In total, that's an average of

$$10 + 5 + \frac{10}{3} + \frac{10}{4} + 2 = \frac{137}{6} \approx 22.83 \text{ minutes}$$

□

3. In the long run, what fraction of time does Alex spend in each region? Let's solve the stationary equations.

$$\begin{aligned}
(1/10)\pi_A &= (5/10)\pi_E \\
(2/10)\pi_B &= (1/10)\pi_A \\
(3/10)\pi_C &= (2/10)\pi_B \\
(4/10)\pi_D &= (3/10)\pi_C \\
(5/10)\pi_E &= (4/10)\pi_D \\
\pi_A &= 5\pi_E \\
\pi_B &= (1/2)\pi_A \\
\pi_C &= (2/3)\pi_B \\
\pi_D &= (3/4)\pi_C \\
\pi_E &= (4/5)\pi_D \\
\pi_B &= (1/2)\pi_A \\
\pi_C &= (1/3)\pi_A \\
\pi_D &= (1/4)\pi_A \\
\pi_E &= (1/5)\pi_A
\end{aligned}$$

   Now, we solve for π_A.

$$\begin{aligned}
&\pi_A + \frac{1}{2}\pi_A + \frac{1}{3}\pi_A + \frac{1}{4}\pi_A + \frac{1}{5}\pi_A = 1 \\
&\pi_A = \frac{60}{137}, \pi_B = \frac{30}{137}, \pi_C = \frac{20}{137}, \pi_D = \frac{15}{137}, \pi_E = \frac{12}{137}
\end{aligned}$$