Lab class 9: Queues and Little's Law

Izzy Grosof

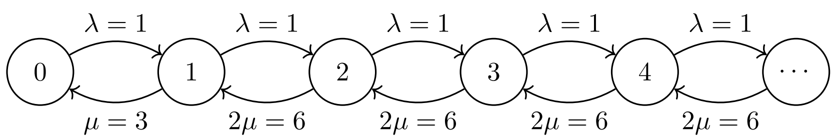
November, 2024

1 Cafeteria M/M/2

You're in a cafeteria. There's two checkout stations, and a shared line for both of them. People arrive to the line according to a Poisson process with rate 1 person per minute, and take $Exp(3)$ minutes to check out (an average of $E[Exp(3)] = 1/3$ minute $= 20$ seconds).

This is an M/M/2 queue, and we'll model it as a CTMC, where the state is the number of people present, in the queue or at a checkout station.

1. Draw the transition diagram for this CTMC.



Solution. □

2. Using the Birth-Death rule $\alpha_i Q_{i,i+1} = \alpha_{i+1} Q_{i+1,i}$ for the stationary distribution α, find the relationship between α_i and α_{i+1}.

Solution. The transition rates are:

$$Q_{i,i+1} = \lambda = 1$$

$$Q_{i+1,i} = \begin{cases} 3 & \text{if } i = 0, \\ 6 & \text{if } i \geq 1. \end{cases}$$

Substituting into the Birth-Death condition:

$$\alpha_i \cdot 1 = \alpha_{i+1} \cdot Q_{i+1,i}$$

$$\alpha_{i+1} = \frac{\alpha_i}{Q_{i+1,i}}$$

Therefore, the relationship between α_i and α_{i+1} is:

$$\alpha_{i+1} = \begin{cases} \frac{\alpha_i}{3} & \text{for } i = 0, \\ \frac{\alpha_i}{6} & \text{for } i \geq 1. \end{cases}$$

□

3. Find the relationship between α_i and α_0. Start by relating α_1 and α_0, then α_2 and α_0, then generalize.

Solution. From part 2, we see that $\alpha_1 = \frac{\alpha_0}{3}$ and $\alpha_2 = \frac{\alpha_1}{6} = \frac{\alpha_0}{3 \cdot 6}$. Generalizing this, we have,

$$\alpha_i = \frac{\alpha_0}{3 \cdot 6^{i-1}}, \quad i \geq 1.$$

□

4. Using the fact that $\sum_{i=0}^{\infty} \alpha_i = 1$, find α_0.

   Solution. Plugging in $\alpha_i = \dfrac{\alpha_0}{3 \cdot 6^{i-1}}$ into the equation $\sum_{i=0}^{\infty} = 1$, we have,

$$\sum_{i=0}^{\infty} \alpha_i = \alpha_0 + \sum_{i=1}^{\infty} \frac{\alpha_0}{3 \cdot 6^{i-1}} = \alpha_0 + \frac{\alpha_0}{3} \sum_{i=0}^{\infty} \frac{1}{6^i} = \alpha_0 + \frac{\alpha_0}{3} \left(\frac{1}{1 - \frac{1}{6}} \right) = \frac{7}{5} \alpha_0.$$

   Setting the above to 1 we get $\alpha_0 = \frac{5}{7}$. □

5. What is the stationary distribution α?

   Solution. The stationary distribution is $\alpha = [\alpha_0, \alpha_1, \alpha_2, \cdots]$ where $\alpha_0 = \frac{5}{7}$ and $\alpha_1 = \frac{1}{3} \cdot \frac{5}{7}$ and $\alpha_2 = \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{5}{7}$ and so on. □

6. Suppose you arrive and see a given number of people n in the system? What is your mean time in queue $E[T_Q \mid N = n]$? What is your mean response time $E[T \mid N = n]$?

   Solution. When there are less than 2 people in the system, at least one of the checkout station is free, there is no waiting time, so $E[T_Q \mid N = n] = 0$.

   If there are two or more people in the system, both checkout stations are busy. You must wait for one of the servers to become available. The service rate for two servers is $\mu = 6$ per minute.

   The expected waiting time in the queue is the time until one server becomes free, multiplied by the number of people you are effectively behind. Therefore if $N \geq 2$,

$$E[T_Q \mid N = n] = \frac{1}{6} \times (n - 1).$$

   For the mean response time, we have $E[T \mid N = n] = E[T_Q \mid N = n] + 1/3$ in both cases. □

7. Applying PASTA (Poisson Arrivals See Time Averages), what is $E[T]$ in terms of $E[N]$?

   Solution. Recall that $E[T \mid N = n] = \frac{n-1}{6} + \frac{1}{3} = \frac{n+1}{6}$ if $n > 1$ and $E[T \mid N = n] = \frac{1}{3}$ if $n = 0$ or $n = 1$. Therefore,

$$E[T \mid N = n] = \begin{cases} \frac{1}{3}, & n = 0, \\ \frac{n+1}{6}, & n > 0. \end{cases} = \frac{n + 1 + \mathbb{1}_{\{n=0\}}}{6}.$$

   Therefore, using law of total expectation,

$$E[T] = E[E[T \mid N]] = \frac{1}{6}(E[N] + 1) + \frac{1}{6}\mathbb{P}(N = 0).$$

   By PASTA, $\mathbb{P}(N = 0) = \alpha_0 = \frac{5}{7}$, therefore, $E[T] = \frac{1}{6}E[N] + \frac{2}{7}$. □

8. Applying Little's Law, solve for $E[T]$ and $E[N]$.

   Solution. The Little's law state that $E[N] = \lambda E[T]$ where $\lambda = 1$ in this case. Therefore solving the following system of equations,

$$\begin{aligned} E[T] &= E[N], \\ E[T] &= \frac{1}{6}E[N] + \frac{2}{7}. \end{aligned}$$

   We get $E[T] = E[N] = \frac{12}{35}$. □

9. If you have extra time, replace 1 person/minute with λ people/minute and $Exp(3)$ with $Exp(\mu)$, and solve for the general result.

   Solution. We start with the relationships between α_i and α_{i+1}, and then relate them to α_0.

$$\lambda\alpha_0 = \mu\alpha_1$$
$$\lambda\alpha_i = 2\mu\alpha_{i+1}\ \forall i \geq 1$$
$$\alpha_i = \alpha_0\frac{\lambda^i}{2^{i-1}\mu^i}\ \forall i \geq 1$$

   Next, we solve for α_0.

$$1 = \alpha_0 + \sum_{i=1}^{\infty}\alpha_0\frac{\lambda^i}{2^{i-1}\mu^i}$$

   This summation has initial value $a = \alpha_0\lambda/\mu$, and has ratio $r = \frac{\lambda}{2\mu}$. This system only has a stationary distribution if $\frac{\lambda}{2\mu} < 1$. The sum is $\frac{a}{1-r}$.

$$\sum_{i=1}^{\infty}\alpha_0\frac{\lambda^i}{2^{i-1}\mu^i} = \alpha_0\frac{\lambda}{\mu}\frac{1}{1-\frac{\lambda}{2\mu}} = \alpha_0\frac{\lambda}{\mu-\lambda/2} = \alpha_0\frac{2\lambda}{2\mu-\lambda}$$
$$1 = \alpha_0 + \alpha_0\frac{2\lambda}{2\mu-\lambda} = \alpha_0(1+\frac{2\lambda}{2\mu-\lambda}) = \alpha_0\frac{2\mu+\lambda}{2\mu-\lambda}$$
$$\alpha_0 = \frac{2\mu-\lambda}{2\mu+\lambda}$$

   The stationary distribution is

$$\alpha_i = \frac{2\mu-\lambda}{2\mu+\lambda}\frac{\lambda^i}{2^{i-1}\mu^i}\ \forall i \geq 1$$

   Now, we calculate conditional mean waiting time and mean response time:

$$E[T_Q \mid N = n] = \frac{n-1}{2\mu}\forall n \geq 1.\ E[T_Q \mid N = 0] = 0$$
$$E[T_Q \mid N = n] = \frac{n-1+\mathbb{1}_{n=0}}{2\mu}$$
$$E[T \mid N = n] = E[T_Q \mid N = n] + \frac{1}{\mu} = \frac{n+1+\mathbb{1}_{n=0}}{2\mu}$$

   We now apply PASTA:

$$E[T] = \frac{E[N]+1+P(N=0)}{2\mu} = \frac{E[N]+1+\frac{2\mu-\lambda}{2\mu+\lambda}}{2\mu} = \frac{E[N]}{2\mu} + \frac{\frac{4\mu}{2\mu+\lambda}}{2\mu} = \frac{E[N]}{2\mu} + \frac{2}{2\mu+\lambda}$$

   We now solve our system of equations using Little's Law, $\lambda E[T] = E[N]$.

$$E[T] = \frac{\lambda E[T]}{2\mu} + \frac{2}{2\mu+\lambda}$$
$$(1-\frac{\lambda}{2\mu})E[T] = \frac{2}{2\mu+\lambda}$$
$$\frac{2\mu-\lambda}{2\mu}E[T] = \frac{2}{2\mu+\lambda}$$
$$E[T] = \frac{4\mu}{(2\mu-\lambda)(2\mu+\lambda)}$$
$$E[N] = \frac{4\lambda\mu}{(2\mu-\lambda)(2\mu+\lambda)}$$

Note that for $\lambda = 1$, $\mu = 3$, we find that $E[T] = \frac{12}{35}$ minutes, and $E[N] = \frac{12}{35}$ people, which matches our earlier result. $\square$