Lab class 1: More on Axioms of Probability

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1 Recap

First, let's recap the three axioms of probability (Kolmogorov's definition of probability)

Definition 1. A probability $\mathbb{P}$ is defined to be a function mapping *events* (subsets of sample space $E \subset \Omega$) to *real numbers* that satisfies the following three properties (axioms):

Non-negativity $\mathbb{P}\{E\} \geq 0$, for any event E.

Normalization $\mathbb{P}\{\Omega\} = 1$, where Ω is the entire sample space of all possible outcomes.

Countably-infinite Additivity If $E_1, E_2, E_3 \ldots$ is a finite or countably-infinite sequence of events, where each pair of events is disjoint ($E_i \cap E_j = \emptyset$, $\forall i \neq j$), then

$$\mathbb{P}\{E_1 \cup E_2 \cup E_3 \cup \ldots\} = \mathbb{P}\{E_1\} + \mathbb{P}\{E_2\} + \mathbb{P}\{E_3\} + \ldots.$$

We saw an example where limiting things to countable additivity really helped, where we had a continuous sample space Ω consisting of the real interval $[0, 1]$. Uncountable (more than countable) additivity fails in this context – the probability of an event consisting of a single real number being sampled is 0.

2 Infinite Coin Flips

Now let's do a different example. Infinite sequence of events - we'll look at a lot of models with infinite sequences of events, over the course of the class.

Let's say our experiment consists of performing infinitely many coin flips.

Q: What is our sample space Ω?

All infinite sequences of H and T.

Q: What's a possible sample?

$HTHTHHHTHTHHTHTHHTHHHH\ldots$

Q: What's an event?

1st, 3rd, 6th all heads.

All even flips are heads.

Fraction of heads is always over 50%.

Q: What's the probability of an event consisting of a single sample?

For instance, what's the probability of $HTHTHTHTHTHTHTHT\ldots$? (alternative odd or even)

It's 0!

Wait, is that a problem, with countable additivity?

Are there countably many infinite sequences of coin flips?

Is the number of sequences countably infinite (small infinite) (like the integers) or uncountably infinite (big infinite) (like the reals)?

Key idea: one-to-one correspondences. A bijection between the two sets - every element of one set is matched with exactly one member of the other set, and vice versa. If a set can be paired up with the integers, it's small infinite. If a set can be paired up with the reals, (reals in $[0, 1]$ is fine) it's big infinite.

Cantor, thm: Integers can't be paired up with the reals – too many reals, not enough integers.
Example: Number of rational numbers between 0 and 1 is small infinite.
Reason: Here's the matching:

1. 0/1
2. 1/1
3. 1/2
4. 1/3
5. 2/3
6. 1/4 ...

Q: Can you put infinite sequences of coin flips into one-to-one correspondence with the integers? With the reals in $[0, 1]$?

We say two sets are in one-to-one correspondence if each element of one set is paired with an element of the other set, and vice versa. We call two sets *the same size* (same cardinality) if we can put them in one-to-one correspondence.

For instance, we can pair up the positive integers and the rationals:
$(1, 0/1), (2, 1/1), (3, 1/2), (4, 1/3), (5, 2/3), (6, 1/4) \ldots$.

We'll just go through all the rationals in increasing order of denominator, skipping any repeats. This will reach every positive integer and every rational, eventually.

On the other hand, we can pair up the reals in $(0, 1)$ and all reals.

Pair x with $\frac{1}{x} + \frac{1}{x-1}$. For inputs in $(0, 1)$, covers all real numbers as outputs.

Conclusion: Rationals and positive integers are both countably infinite (small infinite) and reals in $(0, 1)$ and all reals are both uncountably infinite (big infinite).

Theorem (Cantor): Integers and reals *are not* the same size.

4. Can you put the sample space Ω into one-to-one correspondence with the positive integers? One-to-one with the reals in $[0, 1]$?

Try one, see if it feels like it's working. If not, try the other one.
Think of it as binary!
So $HTHTHTHTH\ldots$ is paired with $0.0101010\ldots = 1/3$.
This basically works, there's countably main duplicates, not a problem.

5. Is Ω countably infinite (small infinite)? Uncountably infinite (big infinite)?

Conclusion: The set of infinite sequences of coin flips is uncountably infinite (big infinite)!

3 Exactly one heads

Let event E_2 be the event where exactly one heads comes up, ever.

So, $E_2 = \{HTTTTTTTT\ldots, THTTTTTT\ldots, TTHTTTTTTT\ldots, \ldots\}$

6. Does E_2 consist of countably infinitely many outcomes? Uncountably infinitely many outcomes?

Countably many! Correspondence is where the H goes.

7. What is $P(E_2)$?

0, by countable additivity