Main class 8: Introduction to Discrete-Time Markov Chains

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1 Takeaways and Summary

- A *stochastic process* $\{X_t\}$ is a collection of random variables, one for every point in time. For each $t \in T$, X_t is a random variable.
- Discrete-time: The set of times T is the integers $\{0, 1, 2, \ldots\}$.
- Markov property: A stochastic process obeys the Markov property if the future state, given the present state, is independent of the past states:

$$\mathbb{P}\{X_{n+1} = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i\} = \mathbb{P}\{X_{n+1} = j \mid X_n = i\}$$

- Time-homogeneous: A stochastic process is time-homogeneous if for all times m, n, $P(X_{n+1} = j \mid X_n = i) = P(X_{m+1} = j \mid X_m = i)$.
- The focus of this module is Discrete-time Markov Chains (DTMCs), which are discrete time stochastic processes that obey the Markov property and are time-homogeneous.
- The *transition probability* $P_{ij} := P(X_{n+1} = j \mid X_n = i)$
- The transition matrix is the matrix formed from the transition probabilities. Each row corresponds to starting from a certain state, each column corresponds to going to a given state.
- We draw transition diagrams using circles for the states, and arrows between the states for transitions, with probabilities written next to the arrows.

2 Stochastic Processes

Starting the meat of the class! Module 2: Discrete-Time Markov Chains

Motivating example: Medical waiting room. Arrivals, departures, space for at most 4 people - others are turned away.

Number of people in the room: Arrivals, departures.

T is all possible times. Discrete-time: $T = [0, 1, 2, 3 \ldots]$.

Continuous-time: $T = [0, \infty)$.

Definition 1. A Stochastic Process $\{X_t\}$ is a collection of random variables, one for every point in time. For each $t \in T$, X_t is a random variable.

Discrete-time stochastic process: Sequence $X_0, X_1, X_2, \ldots$.

Continuous-time stochastic process:

$X_t \forall t \in [0, \infty)$.

This unit: Discrete-time Stochastic Processes. But those are very general. More specifically: Discrete-time Markov Chains

3 Discrete-Time Markov Chains (DTMCs)

Our focus in this unit is on time-homogeneous discrete-time Markov chains. We saw what discrete-time meant, what about the other two?

3.1 Markov Property

A Discrete-Time Markov is a discrete-time stochastic process with the following property:

$$\mathbb{P}\{X_{n+1} = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i\} = \mathbb{P}\{X_{n+1=j|X_n=i}\}$$

Interpretation: Most accurate possible prediction of the future. All available information, past and present. Actually, to make this prediction, only need the present. Given the present, past doesn't matter.

"Markov property".

We call the possible values "states", and we refer to a state space S, $X_n(\omega) \in S$.

"No hidden information" - state contains all available information.

3.2 Time-homogeneous

$$\mathbb{P}\{X_{n+1} = j \mid X_n = i\} = \mathbb{P}\{X_{m+1} = j \mid X_m = i\}$$

Doesn't change from time m to time n.

Transition probability $P_{i,j} := \mathbb{P}\{X_{n+1} = j \mid X_n = i\}$

3.3 Transition matrix

Matrix whose entries are $P_{i,j}$. Rows are constant i (initial state) Columns are constant j (next state)

Example:

1/4 chance of a new person arriving each time slot (unless turned away)

1/2 chance of a person being helped each time slot (unless no one present) (independent)

3/8: -1 person

1/2 (1/8 + 3/8): 0 person

1/8: +1 person

$$\begin{aligned}
P_{1,0} &= P_{2,1} = P_{3,2} = P_{4,3} = 3/8 \\
P_{0,0} &= P_{1,1} = P_{2,2} = P_{3,3} = P_{4,4} = 1/2 \\
P_{0,1} &= P_{1,2} = P_{2,3} = P_{3,4} = 1/8
\end{aligned}$$

Matrix:

$$\begin{pmatrix}
7/8 & 1/8 & 0 & 0 & 0 \\
3/8 & 1/2 & 1/8 & 0 & 0 \\
0 & 3/8 & 1/2 & 1/8 & 0 \\
0 & 0 & 3/8 & 1/2 & 1/8 \\
0 & 0 & 0 & 3/8 & 5/8
\end{pmatrix}$$

Arrows on rows: From 0, from 1, from 2

Arrows on cols: To 0, to 1, to 2

Note: Each row adds up to 1.

3.4 Drawing a picture

Circles for states, arrows for transitions, write probabilities next to arrows.

3.5 Probability of a path

$$
\begin{aligned}
&P(X_0 = 1, X_1 = 2, X_2 = 3, X_3 = 3, X_4 = 2)\\
&= P(X_0 = 1)P(X_1 = 2 \mid X_0 = 1)P(X_2 = 3 \mid X_1 = 2)P(X_3 = 3 \mid X_2 = 3)P(X_4 = 2 \mid X_3 = 3)\\
&= P(X_0 = 1)P_{1,2}P_{2,3}P_{3,3}P_{3,2}
\end{aligned}
$$

All needed information: Starting distribution X_0, and transition probabilities P.

4 Recap

- Stochastic process: Sequence of random variables, one for each time.
- Discrete time: The points in time are integers $0, 1, \ldots$.
- Markov property: For probability of future, all you need is the current state. Past state doesn't matter.
- Time homogeneous: Probabilities of going from i to j doesn't depend on when you take that step.
- Transition matrix: Summary of transition probabilities. Rows are where you come from, columns are where you go to.