Problem Set 3 (Main classes 8-10)

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October 8, 2025 – Due October 15th (Wednesday)

1. Alice is playing basketball. Their probability of making a shot is dependent on their last 2 most recent shots. If they made both of their last two shots, they have a 80% chance of making their next shot. If they made one of their last two shots, they have a 70% chance of making their next shot. If they made neither of their last two shots, they have a 60% chance of making their next shot. Their probability of success is independent of all shots before their last two shots.

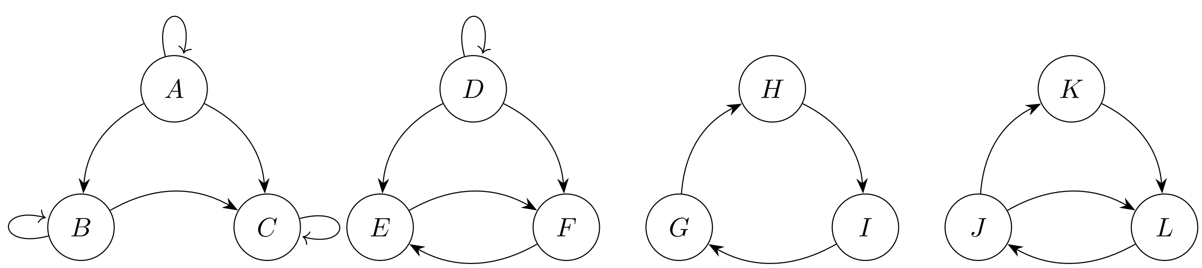
   In this problem, you will model Alice's shooting as a Markov chain. To do so, we must define the state that Alice is in after each shot. This state must be enough information to determine the probability that Alice makes their next shot, and to determine which state may Alice transition to, after that shot.

   (a) List the states of the DTMC corresponding to Alice's shooting.

   (b) Give either a transition matrix or a transition diagram for the DTMC for Alice's shooting, showing the probabilities of all possible state transitions.

   (c) Suppose that Alice's first 3 shots were [Miss, Make, Make]. What is the probability that Alice's next 3 shots will be [Make, Miss, Miss]?

2. Consider the following 4 DTMC transition diagrams:



   For each diagram, determine:

   (a) Is the diagram reducible or irreducible? If it is reducible, how many communicating classes does it have?

   (b) For each communicating class in each diagram, what is its period?

   (c) For each communicating class in each diagram, is it recurrent or transient?

3. Each day I teach class, I start the class with either 1, 2, or 3 pieces of chalk. During the class, if I started with at least 2 pieces of chalk, I'll use up 1 or 2 pieces of chalk, with 50% probability each, i.i.d. If I started the class with 1 piece of chalk, I'll use it up.

   After the class, if I have 0 pieces of chalk left, I'll refill, so I start the next class with 3 pieces of chalk. Otherwise, I'll start the next class with the same number of pieces of chalk I ended class with.

   Let X_n be the number of pieces of chalk I start the nth class with. The sequence $\{X_1, X_2, X_3, \ldots\}$ forms a DTMC.

(a) Give either the transition matrix P or the transition diagram for this DTMC.

(b) Find the stationary distribution π for this DTMC.

(c) In the long run, after what fraction of classes do I refill my chalk?

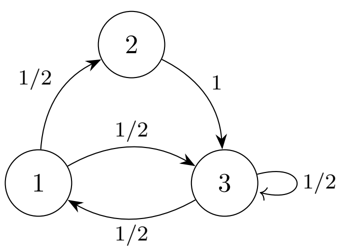
4. Raven claims to have proven a new variant of the Markov property, which we'll call "Raven's property". Specifically, she claims that if $\{X_t\}$ is a DTMC with integer states, then it satisfies the following variant of the Markov property:

$$\mathbb{P}\{X_3 \geq 1 \mid X_2 \geq 1 \& X_1 \geq 1\} = \mathbb{P}\{X_3 \geq 1 \mid X_2 \geq 1\}$$

Tara disagrees: She thinks that Raven's property doesn't always hold, and she thinks that there's a DTMC which is a counterexample to Raven's property with 3 states, and with all transition probabilities P_{ij} equal to either 0 or 1.

(a) Find the counterexample that Tara's describing. Specifically, find a transition matrix P where all entries are either 0 or 1, and a initial distribution for X_1, such that Raven's property fails.

5. Consider the following Markov chain:



(a) What is the stationary distribution of this DTMC?

(b) This is a programming problem. Write a program in Python, as either a standalone file or as a Jupyter notebook. Include your answers in your solution directly, and also submit the code you write, either in the same document or as a separate upload.

Simulate the above Markov chain for 1000 steps, starting in state 1. Over those 1000 steps, in what fraction of steps was the DTMC in each of states 1, 2, and 3?

(c) What is the difference between the empirical fractions you calculated in (b) and the exact stationary distribution you found in (a)?