# Lab class 4: Limiting and Stationary Distributions

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## 1 A 3-state DTMC

Consider a discrete-time Markov chain (DTMC) that models the daily status of a machine in a factory. The machine can be in one of three states:

- **State A**: Operating normally.
- **State B**: Under maintenance.
- **State C**: Broken down.

The machine transitions between these states with the following probabilities: If the machine is **Operating normally (State A)** today:

- It continues to operate normally tomorrow with a probability of 0.5.
- It goes under maintenance tomorrow with a probability of 0.5.
- It breaks down tomorrow with a probability of 0.

If the machine is **Under maintenance (State B)** today:

- It returns to operating normally tomorrow with a probability of 0.5.
- It remains under maintenance tomorrow with a probability of 0.4.
- It breaks down tomorrow with a probability of 0.1.

If the machine is **Broken down (State C)** today:

- It goes under maintenance tomorrow with a probability of 0.5.
- It remains broken down tomorrow with a probability of 0.5.
- It returns to operating normally tomorrow with a probability of 0.

1. What is the transition matrix for this Markov chain?

*Solution.* We have the following transition matrix,

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.4 & 0.1 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

□

2. What is the probability that the machine will be operating normally two days from now if it is under maintenance today?

*Solution.* You can compute the probabilities of all possible trajectory or simply calculate $P^2$ and $P^2_{21} = 0.45$. □

3. Explain why the limiting distribution exist for this Markov chain. (Hint: Theorem 25.6)

*Solution.* The limiting distribution exists because the chain is irreducible and aperiodic. Therefore by theorem 25.6 we know that the limiting distribution exits. □

4. Calculate the limiting distribution.

*Solution.* We compute the stationary distribution, if it exists and is unique then it must be the limiting distribution. Denote $\pi = [\pi_1, \pi_2, \pi_3]$ then we want to solve,

$$\pi P = \pi, \quad \sum_{i=1}^{3} \pi_3 = 1.$$

These equations are,

$$\begin{aligned}
&0.5\pi_1 + 0.5\pi_2 = \pi_1 \\
&0.5\pi_1 + 0.4\pi_2 + 0.5\pi_3 = \pi_3 \\
&0.1\pi_2 + 0.5\pi_3 = \pi_3 \\
&\pi_1 + \pi_2 + \pi_3 = 1.
\end{aligned}$$

Solving this equations we have $\pi = [5/11, 5/11, 1/11]$. □

5. In the long run, what fraction of days is the machine expected to be broken down?

*Solution.* The long run proportion is 1/11. □

6. Suppose that when the machine is broken down, it incurs a loss of \$1,000 per day; when it is under maintenance, it earns \$500 per day; and when it is operating normally, it earns \$1,500 per day. How much money do you make on average every week?

*Solution.* This is given by $(-1000 \cdot \frac{1}{11} + 500 \cdot \frac{5}{11} + 1500 \cdot \frac{5}{11}) \cdot 7 \approx 5727.27$. □

## 2 An Infinite DTMC

In a hospital waiting room, there are two types of patients: Urgent patients and standard patients.

Each time step, two things happen: Potential arrivals, and care.

Arrivals:

- Urgent: Bernoulli($p_U$) urgent arrivals/step
- Standard: Bernoulli($p_S$) standard arrivals/step. Independent of Urgent.

Care: After arrivals,

- If $\geq 1$ urgent person is waiting, 1 urgent person receives care.
- Otherwise, if $\geq 1$ standard person is waiting, 1 standard person receives care.

Model this waiting room as a DTMC.

Q1: What is the state space? What are all possibilities for the set of people waiting, at the end of a time step, after any amount of time has gone by?

*Solution.* At the end of each time step, the only people who can be waiting are standard patients. Standard patients can accumulate each time step indefinitely, so the state space is all positive integer numbers of standard patients. □

Q2: Draw the transition diagram.

If your Markov chain is reducible, use a smaller state space and try again.

*Solution.* There is a state for every number of standard patients. For states $i \geq 1$, there are three arrows: $i$ to $i+1$, with probability $p_S p_U$, from $i$ to $i-1$ with probability $(1-p_S)(1-p_U)$, and from $i$ to $i$ with probability $p_S(1-p_U)+p_U(1-p_S)$. For state 0, there are two arrows: to 1 with probability $p_S p_U$, and to 0 with probability $1-p_S p_U$. □