Lab class 7: CTMC Modeling

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1 Truck Route

There's a truck. The driver goes back and forth between locations A and B, as needed. Requests to go from A to B occur according to a Poisson process with rate 2 requests/hour, independent of the current location of the truck. If the truck is at location A when a request comes in, it will start driving to location B. Otherwise, the request is ignored. Similarly, requests to go from B to A occur according to a Poisson process with rate 1 request/hour, and if the truck is at location B when a request comes in, it starts driving to location A. The time to drive from A to B or B to A is an Exponential distribution with mean 20 minutes.

We will model this situation as a Continuous-Time Markov Chain (CTMC).

1. What is the state space? Note that the truck isn't always at locations A and B, it can also be driving between locations.

   Solution. States: Truck at A, truck driving from A to B, truck at B, truck driving from B to A. □

2. Draw the transition diagram for this CTMC.

   Solution. At A $\rightarrow^2$ Driving to B $\rightarrow^3$ At B $\rightarrow^1$ Driving to A $\rightarrow^3$ At A. □

3. What is the rate matrix Q for this CTMC? Remember that $Q_{i,j}$ is the rate of the exponential timer from i to j if i and j are different, and $Q_{i,i} = -\sum_{j \neq i} Q_{i,j}$.

   Solution. With the state ordering of [At A, Driving to B, At B, Driving to A], the matrix is:

$$Q = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -1 & 1 \\ 3 & 0 & 0 & -3 \end{pmatrix}$$

□

2 Car-counting game

You're on a long drive on a highway, and it's very boring. To have something to do, you decide to play a game: If you see a red car, you get +1 point. If you see a blue car, you get -2 points.

Along this highway, you're seeing cars according to a Poisson process with a rate of 10 cars/minute. The color of each car is independent and identically distributed to the color of all the other cars. Each car is red with probability 20%, and blue with probability 10%.

We will model your current score as a CTMC.

1. What is the state space of this CTMC?

   Solution. All integers, $\{\dots, -2, -1, 0, 1, 2, \dots\}$ □

2. For a particular current score i, what are the scores that you could move to? What are the rates of the timers associated with those transitions?

   Solution. We'll use Poisson Splitting. A Poisson process with rate 10 cars/minute, where we select 20% of the cars independently, is simply a Poisson process with rate 2 red cars/minute. Likewise, the blue cars for a Poisson process with rate 1 blue car/minute.

   As a result, from score i, you can move to score $i + 1$, at rate 2 transitions per minute, or to score $i - 2$, at rate 1 transition per minute. □

3. Draw the transition diagram, focusing on the part of the diagram near state 0.

   Solution. From state 0, there are two transitions: To state 1, with rate 2, and to state -2, with rate 1. To state 0, there are two transitions: From state -1, with rate 2, and from state 2, with rate 1. All states have two transitions going out, and two coming in. □

3 Restaurant

You're operating a counter-serve restaurant. You'd like to track the number of people waiting in line to order. Groups arrive to the restaurant according to a Poisson process with rate 1 group per 2 minutes. A group has 1, 2, or 3 people, each with equal probability. Each person takes an exponential amount of time to order, with mean 15 seconds, independent of group size. Only the person at the front of the line can make their order – there's only one employee taking orders. There's no limit on how many people can be in line.

We'll model the number of people waiting in line as a Poisson Process.

1. What is the state space of this CTMC?

   Solution. All nonnegative integer number of people, $\{0, 1, 2, \ldots\}$. □

2. Suppose that there are currently 2 people waiting in line. What are the states that you could move to? What are the rates of the timers associated with those transitions?

   Solution. We do the Poisson splitting operation when it comes to arrivals of size-1 groups, size-2 groups, and size-3 groups.

   The possible transitions from state 2 are to:

   - state 1, at rate 4 events/minute,
   - state 3, at rate 1/6 events/minute,
   - state 4, at rate 1/6 events/minute,
   - state 5, at rate 1/6 events/minute.

   □

3. Draw the transition diagram.

   Solution. Every state is the same as state 2, except that from state 0, it's impossible for anyone to leave - people can only arrive. □