Main class 12: Mean recurrence time, Birth-Death Markov chains, Local Balance

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1 Takeaways and Summary

- Thm: If there's a stationary distribution π, and $\pi_i > 0$, then the mean recurrence time is the reciprocal of the stationary probability:
$$\mathbb{E}[T_{i,i}] = \frac{1}{\pi_i}$$
- A Markov chain satisfies "local balance" if there exists a probability distribution π such that for each pair of states, $\pi_i P_{ij} = \pi_j P_{ji}$. This is also called "detailed balance" or "time reversibility".
- Theorem: If a Markov chain has a probability distribution π that satisfies the local balance equations, then π is a stationary distribution.
- Certain classes of Markov chains always satisfy local balance, such as Birth-Death Markov chains. A Birth-Death Markov chain is one where states are numbered by integers, and each state i can only transition to states in the set $\{i-1, i, i+1\}$. The transition from i to i is optional.
- Formulas for the stationary distribution of an infinite Birth-Death Markov chain with least state 0:
$$\pi_{i+1} P_{i+1,i} = \pi_i P_{i,i+1}$$
$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{P_{j-1,j}}{P_{j,j-1}}$$
$$\frac{1}{\pi_0} = 1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{P_{j-1,j}}{P_{j,j-1}}$$
- If it's a finite Birth-Death Markov chain with largest state n, just change the final formula to
$$\frac{1}{\pi_0} = 1 + \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{P_{j-1,j}}{P_{j,j-1}}$$

2 Calculating mean recurrence times

For some infinite Markov chains, we can directly calculate the mean recurrence time.

For instance, consider the Markov chain where the state is the number of heads flipped in a row, increasing by 1 with probability 1/2 and resetting to 0 otherwise.

The recurrence time from state 0 to state 0, written $T_{0,0}$ is a $Geom(1/2)$ geometric distribution. It's mean is 2. So we know that the mean recurrence time is 2. We could do something similar to find the recurrence time $T_{1,1}$, or $T_{i,i}$, though it would be more complicated. But we have a formula. Thm: If there's a stationary distribution π, and $\pi_i > 0$, then the mean recurrence time is the reciprocal of the stationary probability:
$$\mathbb{E}[T_{i,i}] = \frac{1}{\pi_i}$$

3 Local balance (20 minutes)

Local balance is a property that some DTMCs have, but not all of them.

The local balance equations are as follows, one equation for each pair of (neighboring) states i, j:

$$\pi_i P_{ij} = \pi_j P_{ji}$$

Note: Textbook calls this the "time reversibility equations", also called "detailed balance".

Draw transition probabilities.

If the local balance equations hold for a particular distribution π, then π is the stationary distribution. In particular, the stationary equations hold:

$$\pi_j = \sum_i \pi_i P_{ij}$$

Theorem: If π satisfies local balance, then π satisfies the stationary equations. Proof:

$$\sum_i \pi_i P_{ij} = \sum_i \pi_j P_{ji} = \pi_j \sum_i P_{ji} = \pi_j$$

Some kinds of Markov chains always satisfy local balance. For example, MCMC's Markov chains always satisfy local balance, by design. See lab notes.

3.1 Birth-death chains

As another example, "Birth-death" Markov chains. Chains where i only transitions to $i-1, i$, or $i+1$. For example, common in many queueing systems.

If it's a birth-death Markov chain, then local balance is satisfied, if stationary distribution equals limiting distribution (ergodic).

Example: Cat shelter DTMC from practice midterm

$$\hookrightarrow_{2/3} 0 \leftrightarrow^{2/3}_{1/3} 1 \leftrightarrow^{2/3}_{1/3} 2 \leftrightarrow^{2/3}_{1/3} 3 \hookleftarrow_{1/3}$$

Local balance:

$$\begin{aligned}
\pi_0 \frac{1}{3} = \pi_1 \frac{2}{3} &\implies \pi_0 = 2\pi_1 \\
\pi_1 \frac{1}{3} = \pi_2 \frac{2}{3} &\implies \pi_1 = 2\pi_2 \\
\pi_2 \frac{1}{3} = \pi_3 \frac{2}{3} &\implies \pi_2 = 2\pi_3
\end{aligned}$$

Sum to 1: $\pi = \{\frac{8}{15}, \frac{4}{15}, \frac{2}{15}, \frac{1}{15}\}$

Local balance holds! Stationary distribution!

3.1.1 Infinite local balance

Infinite cat shelter:

$$\hookrightarrow_{2/3} 0 \leftrightarrow^{2/3}_{1/3} 1 \leftrightarrow^{2/3}_{1/3} 2 \leftrightarrow^{2/3}_{1/3} 3 \leftrightarrow^{2/3}_{1/3} \ldots$$

Infinite DTMCs also satisfy local balance!

Question: Using local balance equations, find stationary distribution.

3.2 General Birth-Death formulas

Here we consider infinite irreducible Birth Death formulas with states $0, 1, 2, \ldots$. If it's finite, just adjust the summation.

$$\pi_{i+1} P_{i+1,i} = \pi_i P_{i,i+1}$$

$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{P_{j-1,j}}{P_{j,j-1}}$$

$$\frac{1}{\pi_0} = 1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{P_{j-1,j}}{P_{j,j-1}}$$

If that sum is finite, a stationary distribution exists, and can be found by combining these equations, and the Markov chain is positive recurrent. If the sum is infinite, no stationary distribution exists, and the Markov chain is null recurrent or transient.