Main class 6: Continuous Conditioning

Izzy Grosof

September 29, 2025

1 Recap

Conditioning, both continuous: $f_{X|Y}(x \mid y) = \frac{f_{X|Y}(x,y)}{f_Y(y)}$.

Random parameters: $T \sim Exp(R)$, $[T \mid R = r] \sim Exp(r)$.

Conditional expectation: $\mathbb{E}[X \mid Y = y] = \int_x x f_{X|Y}(x \mid y) dx$.

2 Unconditional Expectation from Continuous Expectation

$$\mathbb{E}[X] = \int_y \mathbb{E}[X \mid Y = y] f_Y(y) dy$$

Example: Random events with random rates (continuous!).
Suppose the rate of arrivals is uniformly distributed, $R \sim Uniform(1, 3)$.
Again, time between arrivals $T \sim Exp(R)$
Again, the conditional expectation is $\mathbb{E}[T \mid R = r] = 1/r$
Now, we integrate!

$$\mathbb{E}[T] = \int_{r=1}^{3} \mathbb{E}[T \mid R = r] f_R(r) dr = \int_{r=1}^{3} \frac{1}{r}\frac{1}{2} dr = \frac{1}{2}[\ln r]_{r=1}^{3} = \frac{1}{2}(\ln 3 - \ln 1) = \frac{1}{2}\ln 3 \approx 0.549$$

Follow up:
What if $R \sim Uniform(0, 2)$?

$$\mathbb{E}[T] = \int_{r=0}^{2} \mathbb{E}[T \mid R = r] f_R(r) dr = \int_{r=0}^{2} \frac{1}{r}\frac{1}{2} dr = \frac{1}{2}[\ln r]_{r=0}^{2} = \frac{1}{2}(\ln 2 - \ln 0) = \infty$$

Infinite mean! Yes, distributions can have infinite mean! There's no rules.

3 Expectation distribution

We've seen $\mathbb{E}[X \mid Y = y]$. But you'll also see another concept: $\mathbb{E}[X \mid Y]$. No equals!

First, we'll do an example/
$T \sim Exp(R)$

$$\mathbb{E}[T \mid R = r] \sim \frac{1}{r}$$
$$\mathbb{E}[T \mid R] \sim \frac{1}{R}$$

Intuition: Substitute in R in place of r.
Mathematically precise:

- $Z \sim [X \mid Y = y]$ is a distribution with density $f_Z(z) = f_{X|Y}(z \mid y)$.
- $E[X \mid Y = y]$ is a real number: $\int_x f_{X|Y}(x \mid y) dx$.
- $W \sim E[X \mid Y]$ is a distribution with CDF: $F_W(w) = P_{y \sim Y}(E[X \mid Y = y] \leq w)$.

4 Nested Expectation (Tower Principle)

Thm: For any joint random variables (X, Y):

$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$$

Inner expectation $\mathbb{E}[X \mid Y]$ is an expectation distribution! Outer expectation is a standard expectation, gives a real value.

$T \sim Exp(R), R \sim Uniform(1, 3)$.

$$\begin{aligned}
\mathbb{E}[T \mid R = r] &= 1/r \\
\mathbb{E}[T \mid R] &\sim 1/R \\
\mathbb{E}[\mathbb{E}[T \mid R]] &\sim \mathbb{E}[1/R] = \int_{r=1}^{3} \frac{1}{r}\frac{1}{2} dr = \frac{1}{2}[\ln r]_1^3 = \frac{1}{2}\ln 3
\end{aligned}$$