Main class 10: Stationary Distribution

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October 8, 2025

1 Takeaways and Summary

- Limiting distribution: If, for a given state j and all initial states i, the limit $\lim_{n\to\infty}(P^n)_{ij}$ and takes the same value for all initial states, we call it the limiting probability P_j^∞ of state j.
- If limiting probabilities exist for all states and sum to 1, we call it the limiting distribution.
- The stationary distribution π is a probability distribution that doesn't change from step to step: If $X_0 \sim \pi$, then $X_1 \sim \pi$.
- The stationary distribution is the solution to the stationary equations:

$$\sum_i \pi_i P_{ij} = \pi_j$$

$$\sum_i \pi_i = 1$$

  In vector notation, that's $\pi P = \pi, ||\pi||_1 = 1$.

- If a Markov chain has a finite state space and is irreducible, the stationary distribution exists, and it matches the long-run probability distribution:

$$\pi_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{P}\{X_m = i\}, \forall i$$

- If a Markov chain has a finite state space and is irreducible *and aperiodic*, then limiting distribution exists, and it matches the stationary distribution:

$$\lim_{n\to\infty} (P^n)_{ij} = \pi_j, \forall i, j$$

2 Recap

- $i \to j$: Path in transition diagram. $i \leftrightarrow j$: Path both ways.
- Irreducible: Everything's connected.
- Period: GCD of return times – lengths of paths from i back to i.
- Aperiodic: All states have period 1.

2.1 Recap recurrence

Clarify recurrent and transient. Use example: $A \to B$, $B \to C$, $C \to A, B$, $1/2$ each. Prob of A returning to A in 3 steps? $1/2$. ≤ 5 steps? $3/4$. ≤ 7 steps? $7/8$. Etc. Limit is 1. Recurrent! Transient means limit is < 1.

Finite Markov chains: Recurrent means can always get back. Transient means can't always get back. If there's somewhere we could go where we won't be able to get back, transient. If not, recurrent. Infinite is different - we'll see on Friday.

2.2 Recap communicating classes

Strongly connected components. Maximal set of states that are all connected to each other.

3 Recap example

Draw transition diagram:

W to W 1/2, W to B 1/2, B to W 1

Transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$$

Large power ($n = 10$)

$$P^{=} \begin{pmatrix} 0.667 & 0.334 \\ 0.666 & 0.333 \end{pmatrix}$$

P^n_{WW} and P^n_{BW} are both converging towards $\frac{2}{3}$ – initial state doesn't matter, only final state matters!
P^n_{WB} and P^n_{BB} are both converging towards $\frac{1}{3}$ – initial state doesn't matter, only final state matters!
This is a limiting distribution: $P^\infty_W = \frac{2}{3}$, and $P^\infty_B = \frac{1}{3}$.

4 Stationary Distribution

A probability distribution $\pi = (\pi_1, \pi_2, \pi_3, \ldots)$ is a stationary distribution of a Markov chain if, after one step, the distribution is the same.

If $X_0 \sim \pi$ ($P(X_0 = i) = \pi_i$ for all i) then $X_1 \sim \pi$ ($P(X_1 = i)$ is also π_i).

In other words

$$\sum_i \pi_i P_{ij} = \pi_j,$$

for all j. Or, in vector notation, $\pi P = \pi$.

We call this the stationary equation (along with $\sum_i \pi_i = 1$)

Note: If we take more steps, distribution never changes. $\pi P^2 = \pi, \pi P^n = \pi$. Stationary!

4.1 Example

For the working/broken example, what's the stationary distribution?

Solve the following system of equations (stationary equations):

$$\begin{aligned} \pi_W P_{WW} + \pi_B P_{BW} &= \pi_W \\ \pi_W P_{WB} + \pi_B P_{BB} &= \pi_B \\ \pi_W + \pi_B &= 1 \end{aligned}$$

Solution: $\pi_W = 2/3$, $\pi_B = 1/3$.

That's the same as the limiting distribution! Stationary distribution and limiting distribution are the same!

5 Sufficient conditions for stationary distribution to match limiting distribution

Thm: Suppose $\{X_t\}$ is a DTMC with

1. Finite state space
2. Irreducible

Then the stationary equations have a unique solution π, and that unique solution matches the long-run probability distribution:

$$\pi_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{P}\{X_m = i\} \forall i$$

Suppose also that $\{X_t\}$ is aperiodic.

Then the limiting distribution exists, and it matches the solution to the stationary equations:

$$\begin{aligned} \lim_{n\to\infty} (P^n)_{ij} = \pi_j \\ \lim_{t\to\infty} \mathbb{P}\{X_n = j \mid X_0 = i\} = \pi_j \\ \forall i, j \end{aligned}$$

Takeaway: You don't need to raise matrices to big powers to find their limits! You can just solve the stationary equations instead!

6 Recap

- Stationary distribution: Distribution that doesn't change from step to step. Solution to stationary equations.
- If irreducible and aperiodic, unique stationary distribution, matches the limiting distribution.