Perfect subsets of divisor multisets

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1 Result

We define a *divisor multiset* S of size n to be a multiset of positive integers such that $|S| = n$, and such that for all $s \in S$, s divides n.

For instance, if $n = 6$, then $S = [1, 1, 1, 2, 2, 3]$ is a divisor multiset.

Let's define a *perfect subset* S' of a divisor multiset S with $|S| = n$ to be a subset $S' \subset S$ such that the sum of the elements of S' is exactly n.

For instance, for the divisor multiset $S = [1, 1, 1, 2, 2, 3]$, the subset $S' = [1, 1, 1, 3]$ is a perfect subset. Two more perfect subsets of S are $[1, 1, 2, 2]$ and $[1, 2, 3]$.

Now, we are ready to state our main theorem:

Theorem 1. *Each divisor multiset has a perfect subset.*

Proof. We will proceed via strong induction on n, the size of the divisor multiset.

As a base case, the only size-1 divisor multiset is $[1]$. It is a perfect subset of itself.

We must prove that if each divisor multiset of size $< n$ has a perfect subset, then each divisor multiset of size n has a perfect subset as well.

We will now split into cases based on the size n.

1.1 Odd n

First, consider the case where n is odd. Let k be the smallest prime factor of n. Note that $k \geq 3$. Let $d = n/k$ be the largest factor of n less than n itself.

Let us divide S into three disjoint subsets:

- S_k: The elements of S which are multiples of k,
- S_1: The elements of S which are the integer 1,
- S_r: The remaining elements of S.

Note that every element of S_r is a divisor of d. Note also that every element of S_r is larger than k. In particular, every element of S_r is at least 5, because $k \geq 3$ and n is odd.

Suppose $|S_k| \geq d$. Then consider the set S'_k:

$$S'_k := \{s/k \mid s \in S_k\}$$

All elements of S'_k divide $n/k = d$. Thus, S'_k is a divisor multiset of size d. By the inductive hypothesis, S'_k has a perfect subset, with sum exactly d. Multiplying each element by k, we get a subset of $S_k \subset S$ with sum $dk = n$.

We may therefore proceed under the assumption that $|S_k| < d$.

Next, suppose that $|S_1| \geq d$. Let $S_{>1} = S \setminus S_1$. Consider two possibilities: $\text{sum}(S_{>1}) \geq n$, and $\text{sum}(S_{>1}) < n$.

In the case that $\text{sum}(S_{>1}) < n$, note that $\text{sum}(S_{>1}) \geq |S_{>1}|$, because each element of $S_{>1}$ is at least 1. As a result,

$$n - \text{sum}(S_{>1}) \leq n - |S_{>1}| = |S_1|.$$

Thus, the union of $S_{>1}$ and $n - \text{sum}(S_{<1})$ ones is a perfect subset, having sum exactly n.

We therefore proceed under the assumption that $\text{sum}(S_{>1}) \geq n$. Let $m = |S_{>1}|$. Order the elements of $S_{>1}$ arbitrarily as $S_{>1}(1), S_{>1}(2), \ldots, S_{>1}(m)$, and form the prefix sequence P_i of incrementally growing subsets of S:

$$P_i := \{S_{>1}(j) \mid 1 \leq j \leq i\}.$$

For some index i^*,

$$\text{sum}(P_{i^*-1}) < n \text{ and } \text{sum}(P_{i^*}) \geq n.$$

If $\text{sum}(P_{i^*}) = n$, then P_{i^*} is our perfect subset. Otherwise, $\text{sum}(P_{i^*}) > n$. Consider the element $S_{>1}(i^*)$. If $S_{>1}(i^*) = n$, then $[n]$ is our perfect subset. Otherwise, $S_{>1}(i^*) \leq d$, because d is the largest divisor of n less than n. Note that

$$\begin{aligned}
&\text{sum}(P_{i^*-1}) = \text{sum}(P_{i^*}) - S_{>1}(i^*) \\
&\text{sum}(P_{i^*-1}) \geq n - d
\end{aligned}$$

By assumption, $|S_1| \geq d$. Thus, the union of P_{i^*-1} and $n - \text{sum}(P_{i^*-1})$ ones is a perfect subset.

We may therefore proceed under the assumption that $|S_1| < d$.

Because S_k, S_1, and S_r are disjoint subsets of S,

$$|S_r| \geq n - |S_k| - |S_1| > n - 2d = d(k-2)$$

Recall that by assumption, $k \geq 3$. Thus, $|S_r| \geq d$. Recall also that all elements of S_r are divisors of d. Thus, any size d subset of S_r is a divisor multiset.

By the inductive hypothesis, there is some perfect subset $S'_r \subset S_r$. Note that $\text{sum}(S'_r) = d$, and that all elements of S'_r have value at least $k+2$, because they are odd integers that are not divisible by k. Thus, $|S'_r| \leq d/(k+2)$.

Let us set the elements of S'_r aside. Our goal is to find k disjoint subsets of S that each sum to d. The union of these subsets will be our perfect subset.

To form further divisor multisets, we will take d unused elements from S_r, supplemented by ones from S_1 if S_r does not have d remaining unused elements. We must track the remaining unused elements from S_r and S_1 to ensure that they together have at least d remaining unused elements for each of the k iterations. We will write S_r^i and S_1^i to represent the remaining unused elements from each subset.

Let us start by focusing on the case where $k = 3$.

$$\begin{aligned} |S_r^0| &\geq d \\ |S_r^0| + |S_1^0| &\geq 2d \\ |S_r^1| &\geq 4d/5 \\ |S_r^1| + |S_1^1| &\geq 9d/5 \end{aligned}$$

Now, we form our size-d divisor multiset from at least $4d/5$ elements of S_r^1 and at most $d/5$ elements of S_1^1. Let $S_r^{1'}$ be the resulting perfect subset guaranteed by the inductive hypothesis.

$S_r^{1'}$ contains at most $d/5$ ones, and hence at most $d/5 + 4d/25$ elements in total. Thus,

$$|S_r^2| + |S_1^2| \geq 36d/25$$

Because $36/25 \geq 1$, we can use to the inductive hypothesis form our third disjoint subset of sum d. The union of these disjoint subsets forms our perfect subset.

If $k > 3$, the same argument holds, albeit simpler. $|S_r| \geq (k-2)d$. Each iteration removes at most $d/(k+2)$ elements from S_r. To guarantee that all k disjoint subsets can be formed purely as subsets of S_r, we need it to be the case that

$$|S_r| \geq \frac{d(k-1)}{k+2} + d$$

After we remove subsets of size $d/(k+2)$ in each of the first $k-1$ iterations, S_r must still contain at least d elements. A sufficient condition is that $|S_r| \geq 2d$, which always holds for $k > 3$.

We have now proven that a perfect subset must exist, under the inductive hypothesis, if n is odd.

We now consider the case where n is even.

1.2 Even n: Bounding $|S_1|$

As before, let us form the disjoint subsets S_1, the ones, S_2, the even elements of S, and S_r, the odd elements of S greater than 1.

As before, note that if $|S_2| \geq n/2$, then we can form a perfect subset of S which is a subset of S_2. We therefore proceed assuming that $|S_2| < n/2$.

Next, note that if $|S_1| \geq \max(S)$, it is simple to form a perfect subset using the prefix argument above. To constrain $|S_1|$, we will constrain $\max(S)$. If $n \in S$, then $[n]$ is a perfect subset. If $n/2 \in S$, then there are at least $n/2$ other elements of $S \setminus S_2$, all of which divide $n/2$. This forms a divisor multiset, which contains a perfect subset with sum $n/2$ by the inductive hypothesis. Adding in the element $n/2$ itself, we get a perfect subset of S. We may therefore assume that $\max(S) < n/2$, which implies that $\max(S) \leq n/3$.

Now, we are ready for a more precise version of the prefix argument.

This time, index the m elements of $S_{>1}$ in descending order:

$$S_{>1}(1) \geq S_{>1}(2) \geq \ldots \geq S_{>1}(m)$$

We define P_i and i^* as before. Again, we may assume that $\mathrm{sum}(S_{>1}) \geq n$.

Now, we want to bound how large $n - \mathrm{sum}(P_{i^*-1})$ can be. Note that if $|S_1| \geq n - \mathrm{sum}(P_{i^*-1})$, then we can add ones to $P_{i^*-1})$ to form our perfect subset.

We will show that $n - \mathrm{sum}(P_{i^*-1}) \leq n/6$, implying that if $|S_1| \geq n/6$, a perfect subset exists. For convenience, let g^* denote the gap $n - \mathrm{sum}(P_{i^*-1})$. Note that $g^* \leq S_{>1}(i^*)$. Thus, to show that $g^* \leq n/6$, it suffices to only consider cases in which $S_{>1}(i^*) > n/6$. Because $S_{>1}$ is indexed in decreasing order of size, it must be the case that

$$S_{>1}(i) > n/6 \quad \forall i \leq i^*$$

On the other hand, we are assuming that $\max(S) \leq n/3$. Thus, the first i^* elements of the ordering must consist of only the elements $n/3, n/4$, and $n/5$. We now enumerate all possibilities for the first i^* elements, giving rise to 8 different values of g^*.

$$\begin{array}{ll}
[n/3, n/3, n/4, n/4]: & g^* = n/12 \\
[n/3, n/3, n/5, n/5]: & g^* = 2n/15 \\
[n/3, n/4, n/4, n/4]: & g^* = n/6 \\
[n/3, n/4, n/5, n/5, n/5]: & g^* = n/60 \\
[n/3, n/5, n/5, n/5, n/5]: & g^* = n/15 \\
[n/4, n/4, n/4, n/5, n/5]: & g^* = n/20 \\
[n/4, n/4, n/5, n/5, n/5]: & g^* = n/10 \\
[n/4, n/5, n/5, n/5, n/5]: & g^* = 3n/20
\end{array}$$

We omitted possibilities that only differ in the value of $S_{>1}(i^*)$, as this does not affect the value of g^*. We also omitted possibilities in which $P_{i^*} = n$, leading to a perfect subset.

In each case, $g^* \leq n/6$, as desired. Thus, if $|S_1| \geq n/6$, we can form a perfect subset. We therefore proceed under the assumption that $|S_1| < n/6$.

1.3 n with large factors

Now we are ready to handle the case where n is divisible by a prime $k \geq 5$, even when k is not the least prime factor of n.

As always, form the disjoint subsets S_k, S_1, and S_r. By the same argument as above, if $|S_k| \geq n/k \geq n/5$, a perfect subset exists which is a subset of S_k. By assumption, $|S_1| \geq n/6$. As a result, $|S_r| \geq n - n/6 - n/5 = 19n/30$. As above, all elements of S_r are divisors of n/k. One important difference is that elements of S_r can be as small as 2, as we are not assuming that k is the least prime factor of k.

Again, we will remove disjoint subsets from S_r that sum to exactly n/k. Each such subset will have size at most $n/2k$, because each element of S_r is at least 2.

To repeat this process k times, we must ensure that we can remove $n/2k$ elements from S_r $k-1$ times and still have n/k elements remaining. In other words, we need it to be the case that

$$|S_r| \geq \frac{(k-1)n}{2k} + \frac{n}{k}$$
$$\frac{(k-1)n}{2k} + \frac{n}{k} = \frac{(k+1)n}{2k}$$

But note that because $k \geq 5$,

$$\frac{(k+1)n}{2k} \leq \frac{3n}{5} < \frac{19n}{30}$$

As a result, S_r is always large enough, and a perfect subset always exists.

Now, we may proceed under the assumption that the only prime factors of n are 2 and 3.

Suppose that n is divisible by 4.

As always, form the disjoint subsets S_4, S_1, and S_r. Note that the elements of S_r are not guaranteed to divide $n/4$, but they are guaranteed to divide $n/2$, because n is divisible by 4 and the elements of S_r are not.

As always, we may assume that $|S_4| < n/4$, and we have already assumed that $|S_1| \leq n/6$. Thus, $|S_r| \geq 7n/12$. We form a divisor multiset from S_4 of size $n/2$, removing a perfect subset summing to $n/2$ with at most $n/4$ elements. Focusing only on the remaining unused elements, $|S_r| + |S_1| \geq n/2$. We form another size-$n/2$ divisor multiset and another disjoint perfect subset with sum $n/2$. Combining these subsets, we form the perfect subset of S. We may therefore assume that n is not divisible by 4.

At this point, n must be of the form $2 \cdot 3^i$ for some integer i.

Let S_2 denote the even elements of S, let S_1 denote the ones, and let S_3 denote the odd elements of S, all of which are multiples of 3. By assumption, we know that $|S_1| \leq n/6$. As a result, it is either the case that $|S_2| \geq n/2$, or $|S_3| \geq n/3$. In either case, we can form a perfect subset that is a subset of S_2 or S_3 respectively, as described above.

$\square$