Main class 9: Multi-step transitions and limiting probabilities

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October 6, 2025

1 Takeaways and Summary

- P^n, the nth power of the one-step transition matrix P, is the n-step transition matrix. In other words, for a DTMC $\{X_t\}$, $\mathbb{P}\{X_n = j \mid X_0 = i\} = (P^n)_{ij}$.
- I will clarify the definition of limiting probability: the limiting probability of state j is defined if and only if the following limit converges to the same value for every initial state i: $\lim_{n\to\infty}(P^n)_{ij}$.
- The long-run proportion of time spent in state j is defined as $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}\{X_k = j\}$.
- State j is accessible from state i if there is a path in the transition diagram from state i to state j. This is written $i \to j$.
- State i and state j communicate if $i \to j$ and $j \to i$. This is written $i \leftrightarrow j$.
- A Markov chain is irreducible if all states communicate.
- A state i's return probability is defined as the probability that given that the system starts in state i, there exists a time step for which the system is back in state i: $\mathbb{P}\{\exists m > 1, X_m = i \mid X_0 = i\}$.
- A state i is transient if its return probability is less than 1.
- A state i is recurrent if its return probability is 1.
- Thm: If two states communicate, they are either both recurrent or both transient.
- Thm: If a Markov chain is irreducible, then all states communicate.
- A state's return times are all time steps on which the system has a nonzero probability of returning to the original state. The set of t for which $\mathbb{P}\{X_t = i \mid X_0 = i\} > 0$.
- A state's period d_i is the greatest common divisor of its return times.
- If $d_i = 1$, we say that i is aperiodic, else periodic. If all states of a Markov chain are aperiodic, we say that the chain is aperiodic.
- Thm: If i and j communicate, then $d_i = d_j$.

2 Recap: DTMC modeling

Let's say you're operating a machine. Each hour, it's either working or broken.
States: W, B. $\{M_t\}$.
If it's working, there's a 1/2 chance it'll break in the next hour.
If it's broken, we'll repair it, and it'll definitely be working in the next hour.
Draw transition diagram.
Transition matrix:

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$$

3 Multi-step transitions

P is the *one-step* transition probabilities.

What about two steps?

For instance, what's $\mathbb{P}\{M_2 = W \mid M_0 = W\}$?

Either it breaks and repairs (WBW) or it was always working (WWW).

$\mathbb{P}\{WBW\} = P_{WB}P_{BW} = 1/2 \cdot 1 = 1/2$

$\mathbb{P}\{WWW\} = P_{WW}P_{WW} = 1/2 \cdot 1/2 = 1/4$

$\mathbb{P}\{M_2 = W \mid M_0 = W\} = 1/2 + 1/4 = 3/4.$

2-step transition matrix:

$$P^2 = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

This is matrix multiplication! $P \cdot P$ is the two-step transition matrix!

Theorem 1. *P^n is the n-step transition matrix.* $(P^n)_{ij} = P(M_n = j \mid M_1 = i)$

4 Limiting probability

What is the limiting probability of going from i to j in n steps, as $n \to \infty$?

$$P_{ij}^{\infty} := \lim_{n\to\infty} P_{ij}^{n}?$$

Does the limit exist, and if so, what's its value?

$$P^3 = \begin{pmatrix} 5/8 & 3/8 \\ 3/4 & 1/4 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 11/16 & 5/16 \\ 5/8 & 3/8 \end{pmatrix}$$

Seems to be converging – difference is getting smaller.

Long-run proportion:

$$\pi_j = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}\{X_k = j\}$$

We'll see when limiting probability exists, and when long-run proportion exists, in the next few classes. Some Markov chains yes, some no.

Need a few properties of Markov chains.

5 Connectedness

j is accessible from i if there's a path through the transition diagram from i to j, where each arrow has positive probability.

i and j communicate if there's a path both ways. Also "connected".

Irreducible: Everything communicates. Fully connected.

Return probability: When you leave, chance you ever return.

$$P(\exists m, X_m = i \mid X_0 = i)$$

Recurrent: Probability 1 to return eventually.
Transient: Probability < 1 to return eventually.
Thm: If i and j communicate, either both recurrent, or both transient.
Thm: If irreducible, all recurrent.

6 Periodic

d_i: Greatest common divisor of return times. Return time is length of cycle starting and ending at i.
i is aperiodic: $d_i = 1$
i is periodic: $d_i > 1$
Thm: If i and j communicate, same period ($d_i = d_j$).
Irreducible: All i have same period.
Thm: If irreducible and periodic, some (most) initial probabilities have no limiting distribution.