Main class 5: Conditional Expectation

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1 Takeaways and Summary

- We define conditional distributions. This is the same as before in the discrete case, but we define a new object, a conditional density, in the continuous case.
- We define conditional expectation, conditioning on an event.
- We define conditioning on a specific value of a random variable.
- We explain the Law of Total Expectation: The relationship between conditional expectation and overall expectation.
- We can define recursive random variables, where some of the time the same thing happens again.

2 Recap

Joint random variables:

- Discrete: Joint pmf $P(X = x, Y = y)$.
- Continuous: Joint density $f_{X,Y}(x, y)$.

3 Conditional Distribution

Conditioning on an event: $[X \mid A]$, where X is a random variable and A is an event.

- Discrete: $P(X = x \mid A) = \frac{P(X=x \& A)}{P(A)}$
- Continuous: $f_{X|A} = \frac{f_{X \& A}(x)}{P(A)}$

  Conditioning on a specific value of a random variable:
  Define the random variable $[X \mid Y = y]$ as follows:

- Both discrete: Already seen: $P(X = x \mid Y = y) = \frac{P(X=x \& Y=y)}{P(Y=y)}$
- Both continuous: New! Define $f_{X|Y}(x \mid y) = \frac{f_{X|Y}(x,y)}{f_Y(y)}$

  Conditional PDF is a new thing!

4 Conditional Expectation

Define $\mathbb{E}[X \mid A]$:

- Discrete: $\sum_x xP(X = x \mid A)$
- Continuous: $\int_x xf_{X|A}(x)dx$

Example: Random events with random rates.
People show up to our shop at different rates on different days.
Maybe it's discrete, either $\{1, 2, 3\}$, equally likely. Call that rate R.
If someone's just shown up, what's the time until the next person?
Time until next person: $T \sim Exp(R)$
Q: What's $\mathbb{E}[T \mid R = 1]$?
A: $\mathbb{E}[Exp(1)] = 1$

5 Unconditional expectation, from conditional expectation

Law of total expectation!
Discrete conditioning:

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y = y]P(Y = y)$$

If R is uniformly sampled from $\{1, 2, 3\}$, and $T \sim Exp(R)$, what's $\mathbb{E}[T]$?
Step one: What's $\mathbb{E}[T \mid R = r]$, for a general r? A: $\mathbb{E}[T] = \mathbb{E}[Exp(r)] = 1/r$
Step two: Add them together:

$$\mathbb{E}[T] = \sum_{r=1}^{3} \mathbb{E}[T \mid R = r]\mathbb{P}\{R = r\} = \frac{1}{1}\frac{1}{3} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{3} = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$$

6 Recursive random variables

Let's say you're repairing a machine. There's 3 possibilities:

- Everything only needs light repairs: Takes 10 minutes, done. 1/3.
- Something needs medium repairs: Takes 30 minutes, might need more work. 1/3.
- Something needs heavy repairs: Takes 50 minutes, might need more work. 1/3.

X: Time to complete
Y: Type of repair needed (L, M, H).
Z: Additional time after first repair.
X = 10 if Y=L, 30 + X if Y=M, 50 + X if Y= H.
$Z \sim X$
Exercise: What is $\mathbb{E}[X]$?
Answer:
$\mathbb{E}[X \mid Y = L] = 10$
$\mathbb{E}[X \mid Y = M] = 30 + \mathbb{E}[Z] = 30 + \mathbb{E}[X]$
$\mathbb{E}[X \mid Y = H] = 50 + \mathbb{E}[Z] = 50 + \mathbb{E}[X]$
$\mathbb{E}[X] = \frac{90+2\mathbb{E}[X]}{3} = 30 + \frac{2}{3}\mathbb{E}[X]$
$\frac{1}{3}\mathbb{E}[X] = 30$
$\mathbb{E}[X] = 90$

7 Continuous conditioning – Continuous Conditioning

(If time permits - otherwise, next class)

$$\mathbb{E}[X] = \int_y \mathbb{E}[X \mid Y = y] f_Y(y) dy$$

Example: Random events with random rates (continuous!).
People show up to our shop at different rates on different days.
Suppose the rate of arrivals is uniformly distributed, $R \sim Uniform(1, 3)$.
Again, time between arrivals $T \sim Exp(R)$
Again, the conditional expectation is $\mathbb{E}[T \mid R = r] = 1/r$
Now, we integrate!

$$\mathbb{E}[T] = \int_{r=1}^{3} \mathbb{E}[T \mid R = r] f_R(r) dr = \int_{r=1}^{3} \frac{1}{r}\frac{1}{2} dr = \frac{1}{2}[\ln r]_{r=1}^{3} = \frac{1}{2}(\ln 3 - \ln 1) = \frac{1}{2}\ln 3 \approx 0.549$$

What if $R \sim Uniform(0, 2)$?

8 Recap

- Conditional random variables
- Conditional expectation
- Conditioning on random variables
- Recursive random variables