Handout for Lab class 1: More on Axioms of Probability

Izzy Grosof

September 2024

1 Recap

First, let's recap the three axioms of probability (Kolmogorov's definition of probability)

Definition 1. A probability $\mathbb{P}$ is defined to be a function mapping *events* (subsets of sample space $E \subset \Omega$) to *real numbers* that satisfies the following three properties (axioms):

Non-negativity $\mathbb{P}\{E\} \geq 0$, for any event E.

Normalization $\mathbb{P}\{\Omega\} = 1$, where Ω is the entire sample space of all possible outcomes.

Countably-infinite Additivity If $E_1, E_2, E_3 \ldots$ is a finite or countably-infinite sequence of events, where each pair of events is disjoint ($E_i \cap E_j = \emptyset$, $\forall i \neq j$), then

$$\mathbb{P}\{E_1 \cup E_2 \cup E_3 \cup \ldots\} = \mathbb{P}\{E_1\} + \mathbb{P}\{E_2\} + \mathbb{P}\{E_3\} + \ldots.$$

2 Infinite Coin Flips

Our experiment consists of an infinite sequence of coin flips, each either heads (H) or tails (T).
1a. What is our *sample space* Ω?

1b. What's an example of a *possible sample*?

1c. What's an example of an *event*?

2. What's the probability of an event consisting of a single sample?
Let event E_1 be a single-outcome event, $\{HTHTHTHTHTHTHTHTHTHTH\ldots\}$
What's $P(E_1)$, or in other words $P(outcome = HTHTHTHTHTHTHTHTHTHTH\ldots)$?

3. What scenario does this remind you of?

3 One-to-one correspondences

We say two sets are in one-to-one correspondence if each element of one set is paired with an element of the other set, and vice versa. We call two sets *the same size* (same cardinality) if we can put them in one-to-one correspondence.

For instance, we can pair up the positive integers and the rationals:
$(1, 0/1), (2, 1/1), (3, 1/2), (4, 1/3), (5, 2/3), (6, 1/4) \ldots$.

We'll just go through all the rationals in increasing order of denominator, skipping any repeats. This will reach every positive integer and every rational, eventually.

On the other hand, we can pair up the reals in $(0, 1)$ and all reals.

Pair x with $\frac{1}{x} + \frac{1}{x-1}$. For inputs in $(0, 1)$, covers all real numbers as outputs.

Conclusion: Rationals and positive integers are both countably infinite (small infinite) and reals in $(0, 1)$ and all reals are both uncountably infinite (big infinite).

Theorem (Cantor): Integers and reals *are not* the same size.

4. Can you put the sample space Ω into one-to-one correspondence with the positive integers? One-to-one with the reals in $[0, 1]$?

Try one, if it doesn't seem to work then try the other.

5. Is Ω countably infinite (small infinite)? Uncountably infinite (big infinite)?

4 Exactly one heads

Let event E_2 be the event where exactly one heads comes up, ever.

So, $E_2 = \{HTTTTTTTT\ldots, THTTTTTT\ldots, TTHTTTTTTT\ldots, \ldots\}$

6. Does E_2 consist of countably infinitely many outcomes? Uncountably infinitely many outcomes?

7. What is $P(E_2)$?