Main class 14: Poisson Process Intuition & Exponential Deep Dive

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1 Takeaways and Summary

1.1 Poisson Intuition

- A Poisson process N_t is a continuous-time stochastic process. It counts the number of events that have happened by time t, starting with $N_0 = 0$. We can also write N_t as $N(t)$.
- An "increment" is the number of arrivals in an interval of time: $[N_u - N_t]$.
- Poisson processes have independent increments: For any set of times $t_0 \leq t_1 \leq t_2 \leq t_3$, the number of events during the intervals $[t_0, t_1]$ and $[t_2, t_3]$ are independent:

$$[N(t_1) - N(t_0)] \perp [N(t_3) - N(t_2)]$$
$$P(N(t_1) - N(t_0) = x \wedge N(t_3) - N(t_2) = y) = P((N(t_1) - N(t_0) = x)P(N(t_3) - N(t_2) = y).$$

- Poisson processes have stationary increments. For any interval of length s, the distribution of the number of arrivals during that interval is the same:

$$\forall t,\ [N(t+s) - N(t)] \sim [N(s) - N(0)] = N(s)$$

- Poisson process increments have Poisson distributions. Every Poisson process has a rate λ, and

$$N_t \sim Poisson(\lambda t)$$

- Nonoverlapping increments corresponding to intervals of the same length are i.i.d.
- The arrival time process $\{T_n\}$ is a discrete-time stochastic process, where the nth arrival occurs at time T_n.

$$N_t = n \implies T_n \leq t \leq T_{n+1}$$
$$T_n = t \implies N_t = n$$

- The interarrival times $[T_{n+1} - T_n] \sim Exp(\lambda)$ are i.i.d.

1.2 Exponential

- If $X \sim Exp(\lambda)$,
  - $f_X(x) = \lambda e^{-\lambda x}$, $\forall x \geq 0$
  - $F_X(x) = 1 - e^{-\lambda x}$
  - $\mathbb{E}[X] = \frac{1}{\lambda}$
  - $\text{Var}[X] = \frac{1}{\lambda^2} \Leftrightarrow \mathbb{E}[X^2] = \frac{2}{\lambda^2}$

- Exponential random variables are memoryless:

$$\mathbb{P}\{X > s+t \mid X > t\} = P(X > s), \forall s, t$$

- This is true even if the cutoffs are random variables, as long as X, S, and T are mutually independent.:

$$\mathbb{P}\{X > S+T \mid X > T\} = P(X > S), \forall S, T$$

- The "failure rate" or "hazard rate" of a random variable Y is defined as:

$$r_Y(t) := \frac{f_Y(t)}{1 - F_Y(t)}$$

- Exponential distributions have constant failure rate, which is the infinitesimal equivalent of memorylessness: $r_X(t) = \lambda$.

- Let $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2)$, and $X_n \sim Exp(\lambda_n)$, and let all of the X_i's be independent. The minimum of the exponentials is distributed as follows:

$$\min(X_1, X_2) \sim Exp(\lambda_1 + \lambda_2)$$
$$\min(X_1, X_2, \ldots, X_n) \sim Exp(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$$

  The probability of one exponential being less than another can be calculated as follows:

$$\mathbb{P}\{X_1 \leq X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$
$$\mathbb{P}\{X_1 \leq X_2 \,\&\, X_1 \leq X_2 \,\&\, \ldots \,\&\, X_1 \leq X_n\} = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}$$

2 Recap: Stochastic Process

A stochastic process is a list of random variables, one for each time t in a set of times T.

- Discrete-time process: Times T are all nonnegative integers, $\{0, 1, 2, \ldots\}$. Random variables $X_0, X_1, X_2, \ldots$.
- Continuous-time process: Times T are all nonnegative reals, $[0, \infty)$.

A Poisson process is a continuous time stochastic process.

3 Poisson Process Intuition

Model for arrivals into a system. Customers or users showing up, packets arriving, etc. Works well when you're aggregating the behavior of lots of independent users.

Poisson process is about a sequence of events. Timeline, with a bunch of times when events happen.

Let N_t be the number of events that occurred by time t. This is a random variable, and $\{N_t\}$ forms a continuous time stochastic process, with integer values. We say that $N_0 = 0$.

Draw increasing sequence.

Value is number of arrivals, subscript is time.

Another useful concept: "Arrivals time" stochastic process.

Let T_n be the time when the n event happens. This is a random variable, and $\{T_n\}$ forms a discrete time stochastic process, with real values.

For all times t, if $N_t = n$, then $T_n \leq t \leq T_{n+1}$. Likewise, if $T_n = t$, then $N_t = n$.

Same information, two different formats. $\{N_t\}$ is a stepwise function. $\{T_n\}$ is a integer-input function.

4 Poisson Memoryless

Over any interval of time, number of arrivals only depends on the length of the interval, not what happened previously.

How do we write the number of arrivals in an interval of time? Interval $[t, u]$:

$$[N_u - N_t]$$

Number of arrivals during an interval is a random variable, called an "increment".

Independent increments:

$$[N_u - N_t] \perp N_t$$

Doesn't matter how many people arrived previously.

Stationary increments:

$$[N_u - N_t] \sim N_{u-t}$$

Number of arrivals only depends on the length of the interval, not when it happens.

$$N_1 \sim [N_2 - N_1] \sim [N_3 - N_2] \sim \dots$$

We can be more specific about that distribution. Every Poisson process has a rate λ. $N_t \sim Poisson(\lambda t)$. Poisson distribution with mean λt.

Nonoverlapping intervals of the same legnth are i.i.d.

Remember: Poisson distribution is a single random variable. Poisson process is an infinite sequence of random variables.

$$PP(\lambda)$$

5 Exponential interarrival times

Corresponding to memorylessnes for $\{N_t\}$, we also have memorylessness for $\{T_n\}$, the arrival times. Interarrival times are exponential:

$$\forall n \geq 0, [T_{n+1} - T_n] \sim Exp(\lambda), i.i.d.$$

6 Exponential review

6.1 Recap

If $X \sim Exp(\lambda)$,

- $f_X(x) = \lambda e^{-\lambda x}$, $\forall x \geq 0$
- $F_X(x) = 1 - e^{-\lambda x}$
- $\mathbb{E}[X] = \frac{1}{\lambda}$
- $\text{Var}[X] = \frac{1}{\lambda^2} \Leftrightarrow \mathbb{E}[X^2] = \frac{2}{\lambda^2}$

6.2 Recap: Memorylessness

$$\mathbb{P}\{X > s + t \mid X > t\} = P(X > s), \forall s, t\mathbb{P}\{X > 100 \mid X > 60\} = \mathbb{P}\{X > 40\}$$

Think of X as a randomized timer. Q: "If X hasn't gone off by 60 seconds, what's the chance that it's still going at 100 seconds?"

A: "Same as the chance it makes it 40 seconds."

All 40-second intervals have the same chance. In particular, memorylessness still holds if times are random variables:

$$\mathbb{P}\{X > S + T \mid X > T\} = P(X > S), \forall S, T$$

6.3 Failure rate

The infinitesimal version of memorylessness.

Failure rate/hazard rate

$$r_Y(t) := \frac{f_Y(t)}{1 - F_Y(t)}$$

For exponential:

$$r_X(t) = \frac{f_X(t)}{1 - F_X(t)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda$$

6.4 Min of exponentials

Let $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2)$.

$$\min(X_1, X_2) \sim Exp(\lambda_1 + \lambda_2)$$
$$\min(X_1, X_2, \ldots, X_n) \sim Exp(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$$

Probability of one exponential less than another:

$$\mathbb{P}\{X_1 \leq X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$
$$\mathbb{P}\{X_1 \leq X_2 \,\&\, X_1 \leq X_2 \,\&\, \ldots \,\&\, X_1 \leq X_n\} = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}$$

6.5 Example: post office

A, B, C at servers, D, E in queue.

When a server frees up, next person in line goes in.

Each server complete jobs with duration distribution Exp(1/2).

6.5.1 Expected time until D completes:

Time until someone frees up a spot for someone + Time for D.

$$\mathbb{E}[\min(X_A, X_B, X_C) + X_D] = \mathbb{E}[Exp(3/2)] + \mathbb{E}[Exp(1/2)] = 2/3 + 2 = 8/3$$

6.5.2 Expected time until everyone's done:

Time until first completion, time until second completion, time until third completion, time until fourth, time until fifth. T_1, T_2, T_3, T_4, T_5. Want sum of all, $T = T_1 + T_2 + T_3 + T_4 + T_5$.

$$\begin{aligned}
&T_1 \sim \min(Exp(1/2), Exp(1/2), Exp(1/2)) \\
&T_2 \sim \min(Exp(1/2), Exp(1/2), Exp(1/2)) \\
&T_3 \sim \min(Exp(1/2), Exp(1/2), Exp(1/2)) \\
&T_4 \sim \min(Exp(1/2), Exp(1/2) \\
&T_5 \sim Exp(1/2)
\end{aligned}$$

Q: What's $\mathbb{E}[T]$?
A:

$$\begin{aligned}
T_1 &\sim Exp(3/2), \mathbb{E}[T_1] = 2/3 \\
T_2 &\sim Exp(3/2), \mathbb{E}[T_2] = 2/3 \\
T_3 &\sim Exp(3/2), \mathbb{E}[T_3] = 2/3 \\
T_4 &\sim Exp(3/2), \mathbb{E}[T_4] = 1 \\
T_5 &\sim Exp(3/2), \mathbb{E}[T_5] = 2 \\
\mathbb{E}[T] &= 5
\end{aligned}$$