Main class 3: Random Variables – Discrete and Continuous

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1 Recap – 5 min

- Ω: Sample space. All possible outcomes.
- ω: Particular outcome. $\omega \in \Omega$.
- E: Event. Set of outcomes, subset of sample space. $E \subset \Omega$.

Ex: Flip 5 coins. Ex outcome: Sequence of heads and tails, HHTHT Ex event: Exactly 3 heads.
Events either happen or they don't – yes or no.
Events have probabilities.
Today, more general.

2 Defining random variables – 5 min

Definition 1. A real-valued random variable (r.v.) is a function mapping the sample space Ω into $\mathbb{R}$.

So for each outcome ω, $X(\omega) = x \in \mathbb{R}$.
We can have other-valued random variables too, just a different kind of output.
Coin flipping: X could be number of heads.
We can look at events corresponding to this random variable. For example:
$\mathbb{P}\{X(\omega) = 3\}$.
Prob. of exactly 3 heads. $\frac{10}{32}$.
We can also look at functions of random variables.
So, let $Y(x)$ be the function $Y(x) = x^2$.
Then $Y(X(\omega))$, the composition function, is also a random variable.
So X^2 is a random variable, just like X is a random variable.

3 Notation conventions – 2 min

Shorthand: leave out the ω.
$\mathbb{P}\{X(\omega) = 3\} \equiv \mathbb{P}\{X = 3\}$.
Convention: Random variables are upper case, numbers are lower case.
$\mathbb{P}\{X = x\}$.

4 Cummulative distribution function – 5 min

$F_X(x) := \mathbb{P}\{X \leq x\}$.
Draw picture of a CDF.
Properties:

- (weakly) increasing: Flat or goes up.
- Tops out at 1:$\lim_{x\to\infty} F_X(x) = 1$.
- Bottoms out at 0: $\lim_{x\to-\infty} F_X(x) = 0$.

5 Discrete Random Variables – 13 min

A r.v. that can only have finitely many values, or countably infinitely many values, is discrete.

Discrete random variables can be summarized by listing the probability of each possible value: $\mathbb{P}\{X = x\}$. Might also write $p_X(x)$. Probability mass function, pmf.

5.1 Examples – will be used in the class - 10 min

Bernoulli r.v.

$X \sim Ber(p)$.

One coin flip.

$X = 1$ w.p. p, else 0.

Binomial r.v.

$X \sim Bin(n, p)$.

n independent coin flips. Sum of n independent $Ber(p)$.

$\mathbb{P}\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$

Poisson r.v.

$X \sim Poisson(\lambda)$.

Event happens randomly over time, λ times per second on average. But could be more or less than λ.

$\mathbb{P}\{X = i\} = e^{-\lambda} \frac{\lambda^i}{i!}$.

Note: These examples were all integer-valued, but discrete can be any values as long as finite or countably infinite many possibilities.

6 Continuous random variables – 15 min

Based on a density function, rather than individual probabilities.

CDF is continuous, density $f_X(x)$ is derivative of CDF. Probability density function. pdf.

If CDF isn't continuous, density isn't always defined.

Can't focus on probabilities of individual values: $\mathbb{P}\{X = x\} = 0$. Like our timer example.

Instead, probabilities of intervals. Difference of CDF: $\mathbb{P}\{X \in [a, b]\} = F_X(a) - F_X(b)$.

Can also integrate density.

7 Examples

Uniform r.v. $X \sim Unif(a, b)$.

Density: If $x \in [a, b], f_X(x) = \frac{1}{b-a}$.

Flat line.

Cummulative: If $x \in [a, b], F_X(x) = \frac{x-a}{b-a}$.

Increasing line.

Normal r.v. – seeen in previous classes.

Exponential r.v. $X \sim Exp(a, b)$.

Going to use a lot.

Density: $f_X(x) = \lambda e^{-\lambda x}$.

Cummulative: $F_X(x) = 1 - e^{-\lambda x}$.

Imagine a timer.

Memoryless. Say it didn't go off by time 1, and we want to know the chance it goes off by time 2.

$$\mathbb{P}\{X < 2 \mid X > 1\} = \frac{\mathbb{P}\{X \in [1,2]\}}{\mathbb{P}\{X > 1\}} = \frac{F_X(2) - F_X(1)}{1 - F_X(1)}$$
$$\frac{\lambda e^{-\lambda} - \lambda e^{-2\lambda}}{\lambda e^{-\lambda}} = 1 - e^{-\lambda} = \mathbb{P}\{X < 1\}$$

Same as the chance it goes off by time 1.
The chance of going off in the next second is always the same, if it hasn't gone off yet.

8 Final note – 2 min

Some random variables are neither continuous nor discrete. In my research: Amount of work left to be done in the system.
Sometimes none, sometimes a general real-valued amount.
Draw picture of CDF – up at 0, then continuous.
Not discrete, because all positive number is possible.
Not continuous, because jump at 0.
Its own thing.

9 Recap. – 3 min

Random variables: Function from outcome to real value.
Discrete: Finite or countably-infinitely many outcomes.