Lab class 2:

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1 Alternative expression of expectation.

1. Let X be a discrete random variable taking values in $\{1, 2, 3, \cdots\}$. Show that $\mathbb{E}[X] = \sum_{x=1}^{\infty} \mathbb{P}(X \geq x)$.

Solution. In the discrete case, if X is nonnegative, the expected value $E[X]$ is given by:

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} xP(X = x).$$

This implies that we sum $P(X = 0)$ zero times, $P(X = 1)$ once, $P(X = 2)$ twice, and so on. This process can be represented in an array form, where the addition happens column by column,

$$\begin{array}{ccccc} P(X=1) & P(X=2) & P(X=3) & P(X=4) & \cdots \\ & P(X=2) & P(X=3) & P(X=4) & \cdots \\ & & P(X=3) & P(X=4) & \cdots \\ & & & P(X=4) & \cdots \end{array}$$

We can also sum them up row by row, the first row is just $P(X \geq 1)$ and the second row is just $P(X \geq 2)$ and the third row is just $P(X \geq 3)$ and so on, this can be written as $\sum_{x=1}^{\infty} \mathbb{P}(X \geq x)$, which thus must also be equal to $\mathbb{E}[X]$. □

2. Sometimes, this alternative expression may be more convenient for calculating expectations. One such example is the geometric distribution, which you read about in Section 3.2.3. Let X denote a geometric random variable. Use the above expression for the expectation to compute $\mathbb{E}[X]$. *Hint: You might need the geometric series $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ if $|r| < 1$.*

Solution. We need to calculate $\mathbb{P}(X \geq x)$ for a geometric random variable, which represents the probability that the first success occurs on or after the x-th trial. This means that the first $(x-1)$-th trials are all failures, with the probability of this event being $(1-p)^{x-1}$, since they are independent.

Plugging this in the alternative formula,

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} \mathbb{P}(X \geq x) = \sum_{x=1}^{\infty} (1-p)^{x-1} = 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots$$

matching this with the hint, we find $a = 1$ and $r = (1-p)$. So the expectation is $\frac{1}{1-(1-p)} = \frac{1}{p}$. □

2 Generating Random Variables Using Computers

In this problem, we study how to generate random variables programmatically. One basic approach for generating random variables is the inverse transform method. Suppose you want to generate random variables X with CDF $F_X(x)$ (for simplicity we assume X is continuous and $F_X(x)$ is also continuous) the inverse transform sampling method works as follows.

1. Find the inverse CDF $F_X^{-1}(x)$
2. Generate a random number $u \sim U[0, 1]$.

3. Plug u into the inverse of the CDF $F_X^{-1}(u)$. This random variable has the same distribution as X.
4. Repeat step 2 and 3.

1. Find the inverse CDF of an exponential distribution with rate λ.

Solution. The CDF of exponential random variable is $F_X(x) = 1 - e^{-\lambda x}$. Denote $y = F_X(x)$, to find the inverse, we swap x and y and solve for y. That is,

$$x = 1 - e^{-\lambda y} \implies y = -\frac{1}{\lambda}\log(1-x).$$

□

2. Suppose you want to generate a sample from an exponential distribution with rate 1, and your random number generator outputs 1/2. What value does this procedure return?

Solution. We have $\lambda = 1$ and $u = 1/2$ so $-\frac{1}{1}\log(1-u) = \log(2)$. □

3. Show that $Y := F_X^{-1}(U)$ has the CDF as X, i.e., show $F_Y(x) = F_X(x)$.

Solution. By definition of CDF, we have $\mathbb{P}(Y \leq x) = \mathbb{P}(F^{-1}(U) \leq x)$. Next we apply $F_X(x)$ to both sides of the inequality $F_X^{-1}(U) \leq x$.

This is allowed because if $a \leq b$ then for any increasing function g we have $g(a) \leq g(b)$, in particular, the CDF of a random variable is increasing. Why? Because if x increases $P(X \leq x)$ also increases. Therefore,

$$\mathbb{P}(F_X^{-1}(U) \leq x) = \mathbb{P}(F_X(F_X^{-1}(U)) \leq F_X^{-1}(x)) = \mathbb{P}(U \leq F_X^{-1}(x)) = F_X(x).$$

The second last equality holds because $g(g^{-1}(x)) = x$ by definition of inverse. □

3 Expectation of Indicators

1. Let X be a discrete random variable. The indicator function $\mathbb{I}\{X = x\}$ is defined as follows:

$$\mathbb{I}\{X = x\} = \begin{cases} 1, & \text{if } X = x, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the expectation of the indicator function $\mathbb{I}\{X = x\}$ is equal to the probability that $X = x$, i.e., show that

$$\mathbb{E}[\mathbb{I}\{X = x\}] = \mathbb{P}(X = x).$$

Solution. The expectation of $\mathbb{I}\{X = x\}$ is given by:

$$\mathbb{E}[\mathbb{I}\{X = x\}] = \sum_{x'} \mathbb{I}\{x' = x\}\mathbb{P}(X = x'),$$

where x' ranges over all possible values of X. Since $\mathbb{I}\{x' = x\} = 1$ only when $x' = x$ and 0 otherwise, the summation reduces to:

$$\mathbb{E}[\mathbb{I}\{X = x\}] = 1 \cdot \mathbb{P}(X = x) + 0 = \mathbb{P}(X = x).$$

Thus, we have shown that:

$$\mathbb{E}[\mathbb{I}\{X = x\}] = \mathbb{P}(X = x).$$

□