Main class 7: Law of Large Numbers and Central Limit Theorem

Izzy Grosof

October 1, 2025

1 Law of Large Numbers

Clarify: *weak* law of large numbers. Strong LLN in another class.

Just a fancy word for "sample average converges to expectation".

Setting: We have n i.i.d. r.v.s *with finite mean.* $X_1, X_2, \ldots, X_n$. We examine their sum $S_n = \sum_{i=1}^n X_i$.

Intuition: Sum S_n converges to $n\mathbb{E}[X]$. Another way of looking at it: Sample average S_n/n converges to $\mathbb{E}[X]$.

Interpretation: Show sequences of heads and tails. Convert to 0s and 1s - indicator of heads. Sums. Averages.

Mathematically precise: What do we mean by converges?

$$\mathbb{P}\{\lim_{n\to\infty} \frac{S_n}{n} = E[X]\} = 1$$

Interpretation: So, so, so many sequences converge to $E[X] - 1/2$, in this case. In fact, it happens with probability 1 – exceptions so rare, happen with probabilitiy 0. "Almost always". Like different random seeds, averages converge on (almost) every single seed.

Almost sure convergence. $\frac{S_n}{n} \to_{a.s.} E[X]$.

2 Normal distribution

Draw a sketch of a bell curve. Mean: Where's the center. Standard deviation: Spread.

Standard deviation = square root of variance.

Define. Mean and standard deviation.

Definition 1. A continuous r.v. X follows a Normal distribution with mean μ and standard deviation σ, written $X \sim Normal(\mu, \sigma)$, if X has PDF:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Abbreviation: $N(\mu, \sigma)$.

Sometimes, we focus on *standard normal*: $N(0, 1)$.

To get CDF: Integrate the PDF.

$$P(X \le x) = F_{N(\mu,\sigma)}(x) = \int_{x'=-\infty}^{x} f_X(x')dx'$$

Examples:

$$F_{N(0,1)}(0) = 50\%,$$
$$F_{N(0,1)}(1) = 68\%,$$
$$F_{N(0,1)}(2) = 95\%,$$
$$F_{N(0,1)}(3) = 99.7\%$$

2.1 Properties of Normal

Linear transformation of a Normal is a Normal:

$$X \sim N(\mu, \sigma) \implies [aX + b] \sim N(a\mu + b, a\sigma)$$

Sum of independent Normals is a Normal:

$$X \sim N(\mu_x, \sigma_x), Y \sim N(\mu_y, \sigma_y), X \perp Y$$
$$[X + Y] \sim N(\mu_x + \mu_y, \sqrt{\sigma_x^2 + \sigma_y^2})$$

Remember: Variances of sum of independent r.v.s is sum of underlying variances.
$Var(X + Y) = Var(X) + Var(Y)$
Generalizes to any number of Normals added together.

3 Central limit theorem

Setting: We have n i.i.d. r.v.s *with finite mean and variance.* $X_1, X_2, \ldots, X_n$. We examine their sum $S_n = \sum_{i=1}^{n} X_i$.

Intuition: Sum converges to Normal with mean $nE[X]$ and standard deviation $\sigma_X \sqrt{n}$.

Mathematically precise: what does "converges" mean here?

$$\forall z, \lim_{n\to\infty} \mathbb{P}\{S_n \le nE[X] + z\sigma_X \sqrt{n}\} = F_{N(0,1)}(z)$$

"Chance of being at most z standard deviations above the mean is the same as the chance that a normal deviation is at most z standard deviations above the mean."

Alternate formulation: Normalize random variable first. Like an average.

$$Z_n := \frac{S_n - nE[X]}{\sigma_X \sqrt{n}}$$
$$\lim_{n\to\infty} \mathbb{P}\{Z_n \le z\} = F_{N(0,1)}(z)$$

This is an example of convergence in distribution. $Z_n \to_d N(0, 1)$.

3.1 Example

Machine undergoes inspections every 900 days. Number of cracks per day C_i is i.i.d. Distribution of C_i:

$$C_i = \begin{cases} 0 \text{ with probability } 0.96 \\ 5 \text{ with probability } 0.04 \end{cases}$$

Q: If more than 240 cracks before inspection, major problem. What's the chance of a major problem, roughly?

Parts: Mean? Standard deviation?

Average of 0.2 cracks per day, variance of 0.96 crack per day, standard deviation of ≈ 0.98 crack per day. Round off to 1.

Mean of total cracks after 900 days: 180 cracks. Standard deviation: ≈ 30 cracks.

Change of being at least 2 standard deviations over mean: About same probability as $1 - F_{N(0,1)}(2) \approx 5\%$.