Lab class 8: CTMC Modeling

Izzy Grosof

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1 CTMC

Consider a barbershop that can accommodate a maximum of 2 customers at any given time. Customers arrive at the shop following a Poisson process with an arrival rate of $\lambda = 1$ customer per unit time. Each customer receives service with service times that are exponentially distributed with a rate of $\mu = 2$ customers per unit time. One customer receives service at a time. However, when there are exactly 2 customers, the customer in waiting leaves the shop after waiting for an exponential distributed amount of time with rate 1.

1. Model this system as a Continuous-Time Markov Chain with states representing the number of customers in the shop. Find the rate matrix Q of this CTMC.

   Solution. We model the barbershop as a Continuous-Time Markov Chain with states 0, 1, and 2 representing the number of customers in the shop. Customers arrive at rate $\lambda = 1$, service is provided at rate $\mu = 2$, and when there are two customers, the waiting customer leaves at rate 1, so the rate from state 3 to state 2 is $2 + 1 = 3$. The rate matrix Q is then:

   $$Q = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -3 & 1 \\ 0 & 3 & -3 \end{pmatrix}$$

   □

2. Find the limiting distribution of this CTMC.

   Solution. To find the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2)$ of the CTMC, we solve the following system of equations:

   $$\pi Q = 0, \quad \pi_0 + \pi_1 + \pi_2 = 1$$

   Expanding $\pi Q = 0$, we get:

   $$\begin{aligned} -\pi_0 + 2\,\pi_1 &= 0 \\ \pi_0 - 3\,\pi_1 + 3\,\pi_2 &= 0 \\ \pi_1 - 3\,\pi_2 &= 0 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

   Solving this equations we get $\pi = (6/10, 3/10, 1/10)$. Alternatively, one can solve for the limiting distribution using the local balance equation since this is a birth-death process. □

3. Given that 100 customers arrive at the barbershop, what is the expected number of customers who leave without receiving a haircut, including both those who arrive when the shop is full and leave immediately and those who become impatient and leave before receiving service?

Solution. Given the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2) = (6/10, 3/10, 1/10)$, we want to find the expected number of customers who leave without receiving a haircut out of 100 arrivals. There are two scenarios where customers are lost:

- Immediate Loss: a customer arrives when the shop is full (state 2) and leaves immediately. The probability of this event is $\pi_2 = \frac{1}{10}$.
- Impatience Loss: when a customer arrives when the shop is in state 1, the customer starts waiting. This happens with a probability of 3/10. The waiting time of the waiting customer is $W \sim Exp(1)$ and the service is $S \sim Exp(2)$, therefore $P(W < S) = 1/(1+2) = 1/3$. Thus $\frac{3}{10} \cdot \frac{1}{3} = \frac{1}{10}$ of the people leaves after waiting in the barbershop.

Therefore, a total of 20 people leaves without receiving any service. $\square$

2 Birth-death Processes

Consider a barbershop that can accommodate a maximum of 3 customers at any given time. Customers arrive at the shop following a Poisson process with an arrival rate of $\lambda = 1$ customer per unit time. Each customer receives service with service times that are exponentially distributed with a rate of $\mu = 2$ customers per unit time. Again, one customer receives service at a time. If a customer arrives and finds fewer than 2 customers in the shop, they enter immediately. If there are exactly 2 customers, they enter with a probability of 0.5. Model this system as a Continuous-Time Markov Chain with states representing the number of customers in the shop. Using the local balance equations,

$$\pi_i Q_{ij} = \pi_j Q_{ji},$$

determine the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ of the number of customers in the barbershop.

Solution. The rate matrix is given by,

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & 2 & -2.5 & 0.5 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

Local balance equations are give by

$$\begin{aligned} \pi_0 \cdot 1 &= \pi_1 \cdot 2 \\ \pi_1 \cdot 1 &= \pi_2 \cdot 2 \\ \pi_2 \cdot 0.5 &= \pi_3 \cdot 2 \end{aligned}$$

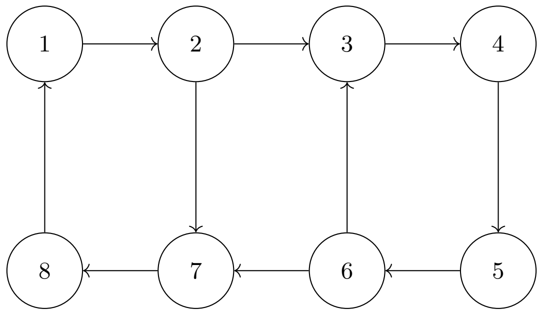
Therefore,

$$\begin{aligned} \pi_1 &= \frac{1}{2}\pi_0 \\ \pi_2 &= \frac{1}{4}\pi_0 \\ \pi_3 &= \frac{1}{16}\pi_0 \end{aligned}$$

Normalizing them $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$, we get, $\pi = (16/29, 8/29, 4/29, 1/29)$. □

3 Cut Equations

Consider a Continuous-Time Markov Chain with eight states given by the following transition diagram. All transitions occur with a unit rate of 1. Using the cut balancing approach, determine the limiting distribution of this CTMC.



The cut balance approach states that, for any subset S of the states, the stationary distribution π must satisfy the following equation:

$$\sum_{i\in S}\sum_{j\notin S}\pi_i Q_{ij} = \sum_{i\notin S}\sum_{j\in S}\pi_i Q_{ij}$$

This equation states that the rate at which the system transitions from inside the set S to outside S must equal the rate from outside S to inside S.

Solution. We determine the stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_8)$ using the cut balancing approach. We make a single cut of each state. Setting the rate of incoming transitions equal to outgoing transitions for each state, we have by the cut balancing equation,

$$\begin{cases} \pi_8 = \pi_1, \\ \pi_1 = 2\pi_2, \\ \pi_2 + \pi_6 = \pi_3, \\ \pi_3 = \pi_4, \\ \pi_4 = \pi_5, \\ \pi_5 = 2\pi_6, \\ \pi_7 = \pi_1. \end{cases}$$

From these equations,

$$\pi_8 = \pi_1, \quad \pi_2 = \frac{\pi_1}{2}, \quad \pi_6 = \frac{\pi_5}{2} = \frac{\pi_1}{2}, \quad \pi_3 = \pi_4 = \pi_5 = \pi_7 = \pi_1.$$

We have this additional equation

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 + \pi_8 = 1.$$

Substituting the values, we have

$$\pi_1 + \frac{\pi_1}{2} + \pi_1 + \pi_1 + \pi_1 + \frac{\pi_1}{2} + \pi_1 + \pi_1 = 7\pi_1 = 1 \quad \Rightarrow \quad \pi_1 = \frac{1}{7}.$$

Solving this we get,

$$\pi = \left(\frac{1}{7}, \frac{1}{14}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{14}, \frac{1}{7}, \frac{1}{7}\right).$$

□