CS 1187 – Homework 04

Solutions and Grading Key – 90 Points

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Part 01 – Number Theory (20 Points)

Exercise DMA 4.1.14 (a,c,e) – (3 points)

What are the quotient and remainder when

a. 44 is divided by 8?
c. -123 is divided by 19?
e. -2002 is divided by 87?

Solution:

a. 5 r 4
c. -6 r 9
e. -23 r 1

Exercise DMA 4.1.36 – (2 points)

Find each of these values.

a. $(177 \mod 21 + 270 \mod 31) \mod 31$
b. $(177 \mod 31 \cdot 270 \mod 31) \mod 31$

Solution:

a. 13
b. 19

Exercise DMA 4.1.40 – (2 points)

Show that if $a \equiv b(\mod m)$ and $c \equiv d(\mod m)$, where a, b, c, d, and m are integers with $m \geq 2$, then $a - c \equiv b - d(\mod m)$

Solution:

Freebie

Exercise DMA 4.2.12 – (1 point)

Convert $(1\ 1000\ 0110\ 0011)_2$ from its binary expansion to its hexadecimal expansion.

Solution:

$(1863)_{16}$

Exercise DMA 4.2.34 – (3 points)

Determine how we can use the decimal expansion of an integer n to determine whether n is divisible by

a. 2
b. 5
c. 10

Solution:

a. right most digit is 0, 2, 4, 6, or 8
b. right most digit is 0 or 5
c. right most digit is 0

Exercise DMA 4.3.14 – (1 point)

Which positive integers less than 12 are relatively prime to 12?

Solution:

5, 7, 11

Exercise DMA 4.3.28 – (3 points)

Find $\gcd(1000, 625)$ and $\operatorname{lcm}(1000, 625)$ and verify $\gcd(1000, 625) \cdot \operatorname{lcm}(1000, 625) = 1000 \cdot 625$

Solution:

$\gcd(1000, 625) = 125$

$\operatorname{lcm}(1000, 625) = 5000$

$125 \cdot 5000 = 1000 \cdot 625$

$625000 = 625000$

Exercise DMA 4.3.32 (a, c, e) – (3 points)

Use the Euclidean algorithm to find

a. $\gcd(1, 5)$
c. $\gcd(123, 277)$
e. $\gcd(1529, 14038)$

Solution:

a. 1
c. 1
e. 1

Exercise DMA 4.4.6 (b, c) – (2 points)

Find an inverse of a modulo m for each of these pairs of relatively prime intgers using the method followed in Example 2.

b. $a = 34$, $m = 89$
c. $a = 144$, $m = 233$

Solution:

freebie

Part 02 – Counting (33 Points)

Exercise DMA 6.1.2 – (1 point)

An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

Solution:

$27 \cdot 37 = 999$

Exercise DMA 6.1.12 – (1 point)

How many bit strings are there of length six or less, not counting the empty string?

Solution:

$2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2 + 4 + 8 + 16 + 32 + 64 = 126$

Exercise DMA 6.1.26 – (3 points)

How many strings of four decimal digits

a. do not contain the same digit twice?
b. end with an even digit?
c. have exactly three digits that are 9s?

Solution:

a. $10 \cdot 9 \cdot 8 \cdot 7 = 5040$
b. $10^3 \cdot 5 = 5000$
c. 36

Exercise DMA 6.2.2 – (2 points)

Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Solution:

There are 26 letters in the alphabet, thus by the Pigeonhole principle we would need at least 27 students in a class to guarantee at least two have a last name that starts with the same letter.

Exercise DMA 6.2.18 – (2 points)

How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

Solution:

There are 8 numbers in the set, thus to guarantee that a pair selected adds to 16 ould require selecting a minimum of 5 numbers.

Exercise DMA 6.2.46 – (3 points)

There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

Solution:

There are 100 possible addresses meaning that 50 houses is the maximum to ensure that there are no consecutive addresses, thus by the Pigeonhole principle if we have 51 houses we can guarantee that at least two addresses are consecutive.

Exercise DMA 6.3.4 – (5 points)

Let $S = \{1, 2, 3, 4, 5\}$

a. List all the 3-permutations of S
b. List all the 3-combinations of S

Solution:

a. (1,2,3) (1,2,4) (1,2,5) (1,3,4) (1,3,5) (1,4,5) (1,3,2) (1,4,2) (1,5,2) (1,4,3) (1,5,3) (1,5,4) (2,1,3) (2,1,4) (2,1,5) (2,3,4) (2,4,3) (2,3,5) (2,5,3) (2,4,5) (2,5,4) (2,3,1) (2,4,1) (2,5,1) (3,1,2) (3,1,4) (3,1,5) (3,2,4) (3,2,5) (3,4,5) (3,2,1) (3,4,1) (3,5,1) (3,4,2) (3,5,2) (3,5,4) (4,1,2) (4,1,3) (4,1,5) (4,2,3) (4,3,5) (4,5,2) (4,2,1) (4,3,1) (4,5,1) (4,3,2) (4,5,3) (4,2,5) (5,1,2) (5,2,1) (5,1,3) (5,3,1) (5,1,4) (5,4,1) (5,2,3) (5,3,2) (5,2,4) (5,4,2) (5,3,4) (5,4,3)

b. (1,2,3) (1,2,4) (1,2,5) (1,3,4) (1,3,5) (1,4,5) (2,3,5) (2,3,4) (2,4,5) (3,4,5)

Exercise DMA 6.3.6 (a, c, e) – (3 points)

Find the values of each of these quantities.

a. $C(5,1)$
c. $C(8,4)$
e. $C(8,0)$

Solution:

a. 5
c. 70
e. 1

Exercise DMA 6.3.20 (a, c) – (2 points)

How many bit strings of length 10 have

a. exactly three 0s?
c. at least seven 1s?

Solution:

a. C(10, 3) = 120
c. C(10, 7) = 120

Exercise DMA 6.4.2 – (10 points)

Find the expansion of $(x+y)^5$

a. using combinatorial reasoning, as in Example 1
b. using the binomial theorem

Solution:

a.

$$\begin{aligned}
(x+y)^5 &= (x+y)(x+y)(x+y)(x+y)(x+y) \\
&= (xx+yy+yx+yy)(xx+xy+yx+yy)(x+y) \\
&= (xxxx+xxxy+xxxy+xxyy+xxxy+xxyy+xxyy+xyyy+xxxy \\
&\quad +xxyy+xxyy+xyyy+xxyy+xyyy+xyyy+yyyy)(x+y) \\
&= xxxxx+xxxxy+xxxxy+xxxyy+xxxxy+xxxyy+xxxyy+xxyyy \\
&\quad +xxxxy+xxxyy+xxxyy+xxyyy+xxxyy+xxyyy+xxyyy+xyyyy \\
&\quad +xxxxy+xxxyy+xxxyy+xyyyy+xxxyy+xxyyy+xxyyy+xyyyy \\
&\quad +xxxyy+xxyyy+xxyyy+xyyyy+xxyyy+xyyyy+xyyyy+yyyyy \\
&= x^5+x^4y+x^4y+x^3y^2+x^4y+x^3y^2+x^3y^2+x^2y^3 \\
&\quad +x^4y+x^3y^2+x^3y^2+x^2y^3+x^3y^2+x^2y^3+x^2y^3+x^4y \\
&\quad +x^4y+x^3y^2+x^3y^2+x^2y^3+x^3y^2+x^2y^3+x^2y^3+xy^4+x^3y^2 \\
&\quad +x^2y^3+x^2y^3+xy^4+x^2y^3+xy^4+xy^4+y^5 \\
&= x^5+6x^4y+10x^3y^2+10x^2y^3+6xy^4+y^5
\end{aligned}$$

b.

$$\begin{aligned}
(x+y)^5 &= \sum_{j=0}^{5}\binom{n}{j}x^{n-j}y^j \\
&= \binom{5}{0}x^5+\binom{5}{1}x^4y+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}xy^4+\binom{5}{5}y^4 \\
&= x^5+6x^4y+10x^3y^2+10x^2y^3+6xy^4+y^5
\end{aligned}$$

Exercise DMA 6.4.16 – (1 point)

The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

$$1 \ 10 \ 45 \ 120 \ 210 \ 252 \ 210 \ 120 \ 45 \ 10 \ 1$$

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle

Solution:

1 11 55 165 330 462 462 330 165 55 11 1

Part 03 – Discrete Probability (13 Points)

Exercise DMA 7.1.2 – (1 point)

What is the probability that a fair die comes up six when it is rolled?

Solution:

$P(6) = \frac{1}{6}$

Exercise DMA 7.1.16 – (1 point)

What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit?

Solution:

$$\frac{C(13,5)C(4,1)}{C(52,5)} = \frac{1287 \cdot 4}{2,598,960} = \frac{5148}{2,598,960} \sim 0.001981$$

Exercise DMA 7.1.24 – (4 points)

Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selecte does not matter, from the positive integers not exceeding

a. 30
b. 36
c. 42
d. 48

Solution:

a. $1/C(30,6) = 1/593,775$
b. $1/C(36,6) = 1/1,947,792$
c. $1/C(42,6) = 1/5,245,786$
d. $1/C(48,6) = 1/12,271,512$

Exercise DMA 7.2.2 – (2 points)

Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.

Solution:

$P(1) = P(2) = P(4) = P(5) = P(6) = \frac{1}{7}$

$P(3) = \frac{2}{7}$

Exercise DMA 7.2.6 – (3 points)

What is the probability of these events when we randomly select a permutation of $[1, 2, 3]$?

a. 1 precedes 3
b. 3 precedes 1
c. 3 precedes 1 and 3 precedes 2

Solution:

Possible outcomes: (1,2,3) (1,3,2) (2,3,1) (2,1,3) (3,1,2) (3,2,1)

a. $P(1 \prec 3) = \frac{|E|}{|S|} = \frac{3}{6} = \frac{1}{2}$
b. $P(3 \prec 1) = \frac{|E|}{|S|} = \frac{3}{6} = \frac{1}{2}$
c. $P(3 \prec 1 \wedge 3 \prec 2) = \frac{|E|}{|S|} = \frac{2}{6} = \frac{1}{3}$

Exercise DMA 7.2.10 (a, c, e) – (2 points)

What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

a. The first 13 letters of the permutation are in alphabetical order
c. a and z are next to each other in the permutation
e. a and z are separated by at least 23 letters in the permutation

Solution:

Freebie

Part 04 – Advanced Counting (7 Points)

Exercise DMA 8.1.2 – (2 points)

a. Find a recurrence relation for the number of permutations of a set with n elements
b. Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

Solution:

Freebie

Exercise DMA 8.1.8 – (2 points)

a. Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s
b. What are the initial conditions?
c. How many bit strings of length seven contain two consecutive 0s?

Solution:

Freebie

Exercise DMA 8.3.2 – (1 point)

How many comparisons are needed to locate the maximum and minimum elements in a sequence with 128 elements using the algorithm in Example 2?

Solution:

Number of comparisons would be ~ 382

Exercise DMA 8.3.22 – (2 points)

Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$

a. Find $f(16)$
b. Find a big-O estimate for $f(n)$. [*Hint:* Make the substitution $m = \log n$]

Solution:

a. 12
b. $O(\log n)$?

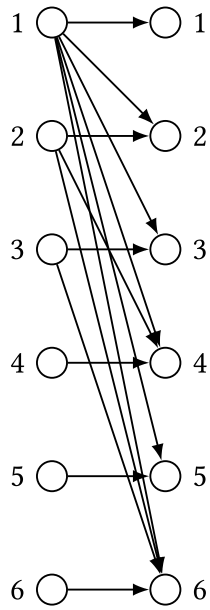
Part 05 – Relations (17 Points)

Exercise DMA 9.1.2 – (3 points)

a. List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$
b. Display this relation graphically, as was done in Example 4.
c. Display this relation in tabular form, as was done in Example 4.

Solution:

a. $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

b. 

c.

R	1	2	3	4	5	6
1	X	X	X	X	X	X
2		X		X		X
3			X			X
4				X		
5					X	
6						X

Exercise DMA 9.1.4 – (4 points)

Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a. a is taller than b
b. a and b were born on the same day
c. a has the same first name as b
d. a and b have a common grandparent

Solution:

a. transitive, antisymetric
b. symmetric, reflexive, transitive
c. symmetric, transitive, reflexive
d. transitive, symmetric, reflexive

Exercise DMA 9.1.34 (a, c, e, g) – (2 points)

Relations on the set of real numbers:

$R_1 = \{(a, b) \in \mathbb{R}^2 \mid a > b\}$, the greater than relation

$R_2 = \{(a, b) \in \mathbb{R}^2 \mid a \geq b\}$, the greater than or equal to relation

$R_3 = \{(a, b) \in \mathbb{R}^2 \mid a < b\}$, the less than relation

$R_4 = \{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$, the less than or equal to relation

Find

a. $R_1 \cup R_3$
c. $R_2 \cap R_4$
e. $R_1 - R_2$
g. $R_1 \oplus R_3$

Solution:

Freebie

Exercise DMA 9.3.2 (a, c) – (2 points)

Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).

a. $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
c. $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$

Solution:

a. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Exercise DMA 9.3.4 (a, c) – (2 points)

List the ordered pairs int he relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Solution:

a. $\{(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$
b. $\{(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1), (4, 3)\}$

Exercise DMA 9.3.14 (a, c, e) – (3 points)

Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{M}_{R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices that represent

a. $R_1 \cup R_2$
c. $R_2 \circ R_1$
e. $R_1 \oplus R_2$

Solution:

a. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise DMA 9.3.22 – (1 point)

Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$

Solution: