CS 1187 – Homework 03

Solutions and Grading Key – 106 Points

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Part 01 – Functions, Sequences, Summations, and Matrices (36 Points)

Exercise DMA 2.3.4 – (1 point)

Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function

a. the function that assigns to each nonnegative integer its last digit
b. the function that assigns the next largest integer to a positive integer
c. the function that assigns to a bit string the number of one bits in the string
d. the function that assigns to a bit string the number of bits in the string

Solution:

Freebie

Exercise DMA 2.3.8 – (8 points)

Find these values

a. $\lfloor 1.1 \rfloor$
b. $\lceil 1.1 \rceil$
c. $\lfloor -0.1 \rfloor$
d. $\lceil -0.1 \rceil$
e. $\lceil 2.99 \rceil$
f. $\lceil -2.99 \rceil$
g. $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$
h. $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

Solution:

a. 1
b. 2
c. -1
d. 0

e. 3
f. -2
g. 1
h. 2

Exercise DMA 2.3.10 – (3 points)

Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a. $f(a) = b, f(b) = a, f(c) = c, f(d) = d$
b. $f(a) = b, f(b) = b, f(c) = d, f(d) = c$
c. $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

Solution:

a. True
b. False
c. False

Exercise DMA 2.4.2 – (4 points)

What is the term a_8 of the sequence $\{a_n\}$ if a_n equals

a. 2^{n-1}?
b. 7?
c. $1 + (-1)^n$?
d. $-(-2)^n$?

Solution:

a. 128
b. 7
c. 2
d. -256

Exercise DMA 2.4.8 – (1 point)

Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

Solution:

Freebie

Exercise DMA 2.4.10 (a - c)

Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a. $a_n = -2a_{n-1}$, $a_0 = -1$
b. $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$
c. $a_n = 3a_{n-1}^2$, $a_0 = 1$

Solution:

a. $-1, 2, -4, 8, -16, 32$
b. $2, -1, -3, -2, 1, 3$
c. $1, 3, 27, 2187, 14348907, 6.1767 \times 10^{14}$

Exercise DMA 2.4.26 (a - c) – (3 points)

For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a. 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...
b. 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ...
c. 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...

Solution:

a. $a_0 = 3, \;\; a_n = a_{n-1} + 2n + 1$
b. $a_0 = 7, \;\; a_n = a_{n-1} + 4$
c. $a_n = 2^n$ in binary

Exercise DMA 2.4.30 – (4 points)

What are the values of these sums, where $S = \{1, 3, 5, 7\}$?

a. $\sum_{j \in S} j$
b. $\sum_{j \in S} j^2$
c. $\sum_{j \in S} (1/j)$
d. $\sum_{j \in S} 1$

Solution:

a. 16
b. 84
c. 176/105
d. 4

Exercise DMA 2.5.2 – (4 points)

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a. the integers greater than 10
b. the odd negative integers
c. the integers with absolute value less than 1,000,000
d. the real numbers between 0 and 2
e. the set $A \times \mathbb{Z}^{+}$ where $A = \{2, 3\}$
f. the integers that are multiples of 10

Solution:

a. countably infinite, enumerate starting at 11 towards infinity
b. countably infinite, enumerate starting at -1 and counting each negative odd number towards negative infinity
c. countably infinite, enumerate starting at 999,999 and working towards negative infinity by 1
d. uncountable e. countably infinite by mapping the pair of resulting numbers to each element in $\mathbb{Z}^{+}$ f. countably infinite starting at 0 and adding pairs $(a \cdot 10, -a \cdot 10)$

Exercise DMA 2.6.2 (a) – (2 points)

Find **A** + **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$$

Solution:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}$$

Exercise DMA 2.6.4 (b) – (2 points)

Find the product **AB**, where

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 0 & 3 & -1 \\ -3 & -2 & 0 & 2 \end{bmatrix}$$

Solution:

$$\mathbf{AB} = \begin{bmatrix} 4 & -1 & -7 & 6 \\ -7 & -3 & 8 & 5 \\ 4 & 0 & 7 & 3 \end{bmatrix}$$

Part 02 – Algorithms (49 Points)

Exercise DMA 3.1.2 (c, d) – (4 points)

Determine which characteristics of an algorithm described in the text (after Algorithm 1) the following procedures have and which they lack.

c.

procedure SUM(n: positive integer))
    $sum := 0$
    while $i < 10$ **do**
        $sum := sum + i$

d.

procedure CHOOSE(a, b: integers)
    $x :=$ either a or b

Solution:

c. **has:** Input, Definiteness, Effectiveness, Finiteness
lacks: Output, Correctness, Generality

d. **has:** Input, Correctness, Finiteness, Effectiveness, Generality
lacks: Output, Definiteness

Exercise DMA 3.1.9 – (5 points)

A **palindrome** is a string that reads the same forward and backward. Describe an algorithm for determining whether a string of n characters is a palindrome.

Solution:

procedure PALINDROME(s: string $s_0 s_1 \ldots s_n$)
    $i := 0$
    $j := n$
    while $i < j$ **do**
        if $s_i \neq s_j$ **then**
            return false
    return true

Exercise DMA 3.1.27 – (10 points)

The **ternary search algorithm** locates an element in a list of increasing integers by successively splitting the list into three sublists of equal (or as close to equal as possible) size, and restricting the search to the appropriate piece. Specify the steps of this algorithm

Solution:

procedure TERNARYSEARCH(x, integer to find, s: sequence of integers $s_0, s_1, \ldots, s_n$)
    $start := 0$
    $end := n$
    $i := \lfloor \frac{n}{3} \rfloor$
    $j := \lfloor \frac{2n}{3} \rfloor$
    while $start < end$ **do**
        if $s_i = x$ **then return** i
        else if $s_j = x$ **then return** j
        else
            if $x < s_i$ **then** $end := i - 1$

else if $x > s_j$ **then** $start := j + 1$
$i := \lfloor \frac{end - start}{3} \rfloor + start$
$j := \lfloor \frac{2 \cdot (end - start)}{3} \rfloor + start$
return -1 {could not find x}

Exercise DMA 3.1.31 – (10 points)

Two strings are **anagrams** if each can be formed from the other string by rearranging its characters. Devise an algorithm to determine whether two strings are anagrams

a. by first finding the frequency of each character that appears in the strings
b. by first sorting the characters in both strings

Solution:

a.

procedure ANAGRAMSFREQ(s, t: Strings $s_0 s_1 \ldots s_n$ and $t_0 t_1 \ldots t_m$)
$S := \varnothing$ {a list of pairs of characters and counts for string s}
$T := \varnothing$ {a list of pairs of characters and counts for string t}
for $i := 0$ **to** n **do**
if (**then**$s_i \in S$)
$S_{s_i} := S_{s_i} + 1$ {add one to the count for character s_i}
else
$S_{s_i} := 1$
for $i := 0$ **to** m **do**
if (**then**$t_i \in T$)
$T_{t_i} := T_{t_i} + 1$ {add one to the count for character t_i}
else
$T_{t_i} := 1$
for $c \in S$ **do**
if (**then**$c \notin T \vee S_c \neq T_c$)
return False
return True

b.

procedure ANAGRAMSSORT(s, t: Strings $s_0 s_1 \ldots s_n$ and $t_0 t_1 \ldots t_m$)
$S :=$ SORT(s) {sort lexicographically using merge or quick sort}
$T :=$ SORT(t) {sort lexicographically using merge or quick sort}
for $i := 0$ **to** n **and** $j := 0$ **to** m **and do**
if $S_i \neq S_j$ **then**
return False
return True

Exercise DMA 3.2.2 (a - c) – (3 points)

Determine whether each of these functions is $O(x^2)$

a. $f(x) = 17x + 11$
b. $f(x) = x^2 + 1000$
c. $f(x) = x \log x$

Solution:

a. $k > 1,\ \ x > k$
$0 \leq 17x + 11 \leq x^2 + 11x^2 = 12x^2$ whenever $x > 1$
$\therefore f(x)$ is $O(x^2)$

b. $k > 1,\ \ x > k$
$0 \leq x^2 + 1000 \leq x^2 + 1000x^2 = 1001x^2$
$f(x) = x^2 + 1000 < 1001x^2$ whenever $x > 1$
$\therefore f(x)$ is $O(x^2)$

c. $m = \log x,\ \ x > 1$
$0 \leq xm \leq x^2$
$f(x) = xm < x^2$ whenever $x > 1$
$\therefore f(x)$ is $O(x^2)$

Exercise DMA 3.2.26 (b) – (1 point)

Give a big-O estimate for each of these functions. For the function g in your estimate $f(x)$ is $O(g(x)$, use a simple function g of smallest order.

b. $(2^n + n^2)(n^3 + 3^n)$

Solution:

$2^n n^3 + 6^{2n} + n^5 + 3^n n^2 = O(6^{2n})$

Exercise DMA 3.2.30 (a, b) – (2 points)

Show that each of these pairs of functions are of the same order.

a. $3x + 7$, x
b. $2x^2 + x - 7$, x^2

Solution:

a. $x > 1$
$0 \leq 3x + 7 \leq x + 7x$
$f(x) = 3x + 7 \leq 8x$ whenever $x > 1 \therefore f(x)$ is $O(x)$

b. $x > 1$
$0 \leq 2x^2 + x - 7 \leq 2x^2 + x^2 + 7x^2 = 10x^2$
$f(x) = 2x^2 + x - 7 \leq 10x^2$ whenever $x > 1$
$\therefore f(x)$ is $O(x^2)$

Exercise DMA 3.3.2 – (1 point)

Give a big-O estimate for the number of additions used in this segment of an algorithm

$t := 0$
for $i := 1$ **to** n **do**
  for $j := 1$ **to** n **do**
    $t := t + i + j$

Solution:

$O(n^2)$

Exercise DMA 3.3.4 – (1 point)

Give a big-O estimate for the number of operations, where an operation is an addition or a multiplication, used in this segment of an algorithm (ignoring comparisons used to test the conditions in the **while** loop).

$i := 1$
$t := 0$
while $i \leq n$ **do**
  $t := t + i$
  $i := 2i$

Solution:

$O(n)$

Exercise DMA 3.3.16 – (8 points)

What is the largest n for which one can solve within a day using an algorithm that requires $f(n)$ bit operations where each bit operation is carried out in 10^{-11} seconds, with these functions $f(n)$?

a. $\log n$
b. $1000n$
c. n^2
d. $1000n^2$
e. n^3
f. 2^n
g. 2^{2n}
h. 2^{2^n}

Solution:

a. $2^{8.64 \times 11}$
b. 8.64×10^8
c. 9.2952×10^5
d. 9.2952×10^3
e. 9.52×10^3
f. 39.65
g. 19.825
h. 5.309

Exercise DMA 3.3.18 – (4 points)

How much time does an algorithm take to solve a problem of size n if this algorithm uses $2n^2 + 2^n$ operations, each requiring 10^{-9} seconds, with these values of n?

a. 10
b. 20
c. 50
d. 100

Solution:

a. 1.224×10^{-12}
b. 2.448×10^{-12}
c. 6.120×10^{-12}
d. 1.224×10^{-13}

Part 03 – Recursion and Induction (21 Points)

Exercise DMA 5.1.4 – (5 points)

Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a. What is the statement $P(1)$? b. Show that $P(1)$ is true, completing the basis step of the proof of $P(n)$ for all positive integers n c. What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers$n$? d. What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n? e. Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis. f. Explain why these steps show that this formula is true whenever n is a positive integer.

Solution:

a, b. $P(1) = \left(\frac{1(1+1)}{2}\right)^2 = 1^2 = 1^3$

c. $P(n) : n^3 = \left(\frac{n(n+1)}{2}\right)^2$

d. to show that $p(n+1) = (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + \left(\frac{(n+1)(n+2)}{2}\right)^2$

e. assuming the inductive hypothesis

$$\begin{aligned}
\left(\frac{(n+1)(n+2)}{2}\right)^2 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
\left(\frac{n^2+3n+2}{2}\right)^2 &= \left(\frac{n^2+n}{2}\right)^2 + (n^2+2n+1)(n+1) \\
\left(\frac{(n^2+3n+2)(n^2+3n+2)}{4}\right) &= \frac{n^4+2n^3+n^2}{4} + (n^3+n^2+3n+1) \\
\frac{n^4+6n^3+13n^2+12n+4}{4} &= \frac{n^4+6n^3+13n^2+12n+4}{4}
\end{aligned}$$

f. Thus, as we have shown the base case and the inductive case, and via the inductive hypothesis and the principle of induction we have prove this for all cases.

Exercise DMA 5.1.52 – (1 point)

Suppose that m and n are positive integers with $m > n$ and f is a function from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Use mathematical induction on the variable n to show that f is not one-to-one.

Solution:

Freebie

Exercise DMA 5.3.2 – (4 points)

Find $f(1), f(2), f(3), f(4)$, and $f(5)$ if $f(n)$ is defined recursively by $f(0) = 3$ and for $n = 0, 1, 2, \ldots$.

a. $f(n+1) = -2f(n)$
b. $f(n+1) = 3f(n) + 7$
c. $f(n+1) = f(n)^2 - 2f(n) - 2$
d. $f(n+1) = 3^{f(n)/3}$

Solution:

a. $f(1) = -6, f(2) = 12, f(3) = -24, f(4) = 48, f(5) = -96$
b. $f(1) = 16, f(2) = 55, f(3) = 172, f(4) = 523, f(5) = 1576$
c. $f(1) = 1, f(2) = -3, f(3) = 13, f(4) = 141, f(5) = 19547$
d. $f(1) = 3, f(2) = 3, f(3) = 3, f(4) = 3, f(5) = 3$

Exercise DMA 5.3.30 (b) – (1 point)

Give a recursive definition of each of these sets of ordered pairs of positive integers. [*Hint:* Plot the points in the set in the plane and look for lines containing points in the set.]

b. $S = \{(a, b) \mid a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } a \mid b\}$

Solution:

Freebie

Exercise DMA 5.4.4 – (5 points)

Trace Algorithm 3 when it finds gcd(12, 17). That is, show all the steps used by Algorithm 3 to find gcd(12, 17)

Solution:

```
gcd (12, 17)

x = 12, y = 17

while:
  1. r = x mod y = 12 mod 17 = 12, x = 17, y = 12
  2. r = 17 mod 17 = 5, x = 12, y = 5
  3. r = 12 mod 5 = 2, x = 5, y = 2
  4. r = 5 mod 2 = 1, x = 2, y = 1
  5. r = 2 mod 1 = 0, x = 1, y = 0

return 1
```

Exercise DMA 5.4.50 – (5 points)

Sort 3, 5, 7, 8, 1, 9, 2, 4, 6 using the quick sort.

Solution:

```
initial list: 3 5 7 8 1 9 2 4 6

pivot: 3
  qsort: [1 2]
    pivot: 1
    return: [1 2]

  qsort: [5 7 8 9 4 6]
    pivot 5
    qsort: [4] return [4]
    qsort: [7 8 9 6]
      pivot: 7
      qsort: [6] return [6]
      qsort: [8 9]
        pivot: 8
        qsort: [9] return [9]
      return [8 9]
    return [6 7 8 9]
  return [4 5 6 7 8 9]
return [1 2 3 4 5 6 7 8 9]
```