CS 1187 – Homework 02

Solutions and Grading Key – 200 Points

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Part 01 – Logic (101 Points)

1.1 - Propositional Logic

Exercise DMUC 6.2 – (5 points)

Check your understanding of *or*, *and*, and *not* by deciding what the value each of these expressions has and then evaluating it with the computer.

Solution:

a. `False`
c. `False`
f. `True`
h. `True`
k. `False`

Grading:

Exercise DMUC 6.21 – (2 points)

Notice that in the proof of $\vdash$ True $\wedge$ False $\rightarrow$ True we used $\{\wedge E_R\}$ to obtain False from (False $\rightarrow$ False) $\wedge$ False, and everything worked fine. However, we could have used $\{\wedge E_L\}$ instead to infer False $\rightarrow$ False, which is True. What would happen if that choice is made? Would it result in calculating the wrong value of True $\wedge$ False? Is it possible to show that True $\wedge$ False is *not* logically equivalent to True?

Solution:

Freebie

Exercise DMUC 6.24 – (3 points)

Translate each of the following Haskell expressions into the conventional mathematical notation.

```
(a) And P Q
(b) Imply (Not P) (Or R S)
(c) Equ (Imply P Q) (Or (Not P) Q)
```

Solution:

a. $P \wedge Q$
b. $\neg P \longrightarrow R \vee S$
c. $(P \longrightarrow Q) \leftrightarrow (\neg P \vee Q)$

Grading:

Exercise DMUC 6.26 – (4 points)

Prove the equation $(P \wedge \text{False}) \wedge \text{True} = \text{False}$

Solution:

$$\begin{array}{rcll} (P \wedge \text{False}) & \wedge & \text{True} & \\ & = & P \wedge \text{False} & \{\wedge \text{ Identity}\} \\ & = & \text{False} & \{\wedge \text{ Null}\} \end{array}$$

Grading:

Exercise DMUC 6.40 – (4 points)

Prove the following using inference rules; natural deduction and proof-checker notation:

$A \vee (B \vee C) \vdash (A \vee B) \vee C$

Solution:

Freebie

Grading:

Exercise DMA 1.1.12 – (8 points)

Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.

a. $\neg p$
b. $p \vee q$
c. $\neg p \wedge q$
d. $q \longrightarrow p$
e. $\neg q \longrightarrow \neg p$
f. $\neg p \longrightarrow \neg q$
g. $p \leftrightarrow q$
h. $\neg q \vee (\neg p \wedge q)$

Solution:

a. The election is not decided.
b. The election is decided or the votes have been counted.
c. The election is not decided, but the votes have been counted.
d. If the votes have been counted, then the election is decided.
e. If the votes have not been counted, then the election is not decided.
f. If the election is not decided, then the votes have not been counted.
g. The election is decided if and only if the votes have been counted.
h. Either the votes have not been counted or the election is not decided and the votes have been counted.

Grading:

Exercise DMA 1.1.16 – (6 points)

Let p, q, and r be the propositions

p: You get an A on the final exam
q: You do every exercise in the book
r: You get an A in the class

Write these propositions using p, q, and r and logical connectives (including negations)

a. You get an A in this class, but you do not do every exercise in this book.
b. You get an A on the final, you do every exercise in this book, and you get an A in this class.
c. To get an A in this class, it is necessary for you to get an A on the final.
d. You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.
e. Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
f. You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Solution:

a. $r \land \neg q$
b. $p \land q \land r$
c. $r \longrightarrow q$
d. $(p \land \neg q) \longrightarrow r$
e. $(p \land q) \longrightarrow r$
f. $r \leftrightarrow (q \lor p)$

Grading:

Exercise DMA 1.1.20 – (4 points)

Determine whether each of these conditional statements is true or false.

a. If $1 + 1 = 3$, then unicorns exist.
b. If $1 + 1 = 3$, then dogs can fly.

c. If 1 + 1 = 2, then dogs can fly.
d. If 2 + 2 = 4, then 1 + 2 = 3.

Solution:

a. $F \longrightarrow F \vdash T$
b. $F \longrightarrow F \vdash T$
c. $T \longrightarrow F \vdash F$
d. $T \longrightarrow T \vdash T$

Grading:

Exercise DMA 1.1.34 – (3 points)

Construct a truth table for each of these compound propositions.

a. $(p \vee q) \longrightarrow (p \oplus q)$
c. $(p \vee q) \oplus (p \wedge q)$
e. $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

Solution:

a.

p	q	p ∨ q	p ⊕ q	(p ∨ q) ⟶ (p ⊕ q)
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

b.

p	q	p ∨ q	p ∧ q	(p ∨ q) ⊕ (p ∧ q)
T	T	T	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

c.

p	q	r	p ↔ q	¬p ↔ ¬r	(p ↔ q) ⊕ (¬p ↔ ¬r)
T	T	T	T	T	F
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	T	F	T
F	F	F	T	T	F

Grading:

Exercise DMA 1.3.22 – (3 points)

Show that $p \longrightarrow q$ and $\neg q \longrightarrow \neg p$ are logically equivalent (using Boolean Algebra)

Solution:

$$\begin{aligned}
p \quad \longrightarrow \quad & q \\
= \quad & \neg p \land q && \{\text{Implication}\} \\
= \quad & q \lor \neg p && \{\lor \text{ Commutative}\} \\
= \quad & \neg q \longrightarrow \neg p && \{\text{Implication}\}
\end{aligned}$$

Grading:

Exercise DMA 1.3.28 – (5 points)

Show that $(p \longrightarrow q) \lor (p \longrightarrow r)$ and $p \longrightarrow (q \lor r)$ are logically equivalent (using Boolean Algebra)

Solution:

$$\begin{aligned}
(p \longrightarrow q) \quad \lor \quad & (p \longrightarrow r) \\
= \quad & (\neg p \lor q) \lor (p \longrightarrow r) && \{\text{implication}\} \\
= \quad & (\neg p \lor q) \lor (\neg p \lor r) && \{\text{implication}\} \\
= \quad & (\neg p \lor \neg p) \lor (q \lor r) && \{\lor \text{ associative}\} \\
= \quad & \neg p \lor (q \lor r) && \{\lor \text{ idempotent}\} \\
= \quad & p \longrightarrow (q \lor r) && \{\text{implication}\}
\end{aligned}$$

Grading:

Exercise DMA 1.3.66 – (3 points)

Determine whether each of these compound propositions is satisfiable (a only)

a. $(p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg s) \land (p \lor \neg r \lor \neg s) \land (\neg p \lor \neg q \lor \neg s) \land (p \lor q \lor \neg s)$
b. $(\neg p \lor \neg q \lor r) \land (\neg p \lor q \lor \neg s) \land (p \lor \neg q \neg s) \land (\neg p \lor \neg r \lor \neg s) \land (p \lor q \lor \neg r) \land (p \lor \neg r \lor \neg s)$
c. $(p \lor q \lor r) \land (p \lor \neg q \lor \neg s) \land (q \lor \neg r \lor s) \land (\neg p \lor r \lor s) \land (\neg p \lor q \lor \neg s) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor \neg q \lor s) \land (\neg p \lor \neg r \lor \neg s)$

Solution:

a. Satisfiable
b. Satisfiable
c. Satisfiable

Grading:

1.2 - Predicate Logic

Exercise DMUC 7.14 – (5 points)

Prove $(\exists x. f(x)) \lor (\exists x. g(x)) \vdash \exists x. (f(x) \lor g(x))$

Solution:

$$\dfrac{\dfrac{(\exists x\ f(x)) \vee (\exists x\ g(x)) \qquad \dfrac{f(c) \qquad \{c \text{ arbitrary}\}}{f(c)}\{ID\}}{f(c) \vee (\exists x\ g(x))}\{\exists E\} \qquad \dfrac{g(c) \qquad \{c \text{ arbitrary}\}}{g(c)}\{ID\}}{\dfrac{f(c) \vee g(c)}{\exists x\ f(x) \vee g(x)}\{\exists I\}}\{\exists E\}$$

Exercise DMUC 7.15 – (5 points)

Prove $(\forall x.f(x)) \vee (\forall x.g(x)) \vdash \forall x.(f(x) \vee g(x))$

Solution:

$$\dfrac{\dfrac{(\forall x\ f(x)) \vee (\forall x\ g(x)) \qquad \dfrac{\dfrac{\cancel{\forall x\ f(x)}}{f(p)}\{\forall E\}}{f(p) \vee g(p)}\{\vee I_R\} \qquad \dfrac{\dfrac{\cancel{\forall x\ g(x)}}{g(p)}\{\forall E\}}{f(p) \vee g(p)}\{\vee I_L\}}{f(p) \vee g(p)}\{\vee E\} \qquad \{p \text{ arbitrary}\}}{\forall x\ (f(x) \vee g(x))}\{\forall I\}$$

Exercise DMUC 7.18 – (9 points)

Prove the following implication:

$$(\forall x.f(x) \longrightarrow h(x) \wedge \forall x.g(x) \longrightarrow h(x)) \longrightarrow \forall x.(f(x) \vee g(x) \longrightarrow h(x))$$

Solution:

$$\begin{array}{rcll}
\forall x\ f(x) \longrightarrow h(x) & \vee & \forall x\ g(x) \longrightarrow h(x) & \\
& = & \forall x\ (f(x) \longrightarrow h(x)) \vee (g(x) \longrightarrow h(x)) & \{7.11\} \\
& = & \forall x\ (\neg f(x) \vee h(x)) \vee (\neg g(x) \vee h(x)) & \{Implication\} \\
& = & \forall x\ (h(x) \vee \neg f(x)) \vee (h(x) \vee \neg g(x)) & \{\vee \text{ Commutative}\} \\
& = & \forall x\ (h(x) \vee (\neg f(x) \wedge \neg g(x)) & \{\vee \text{ Ditributes Over } \wedge\} \\
& = & \forall x\ ((\neg f(x) \wedge \neg g(x)) \vee h(x)) & \{\vee \text{ Commutative}\} \\
& = & \forall x\ (\neg(\neg f(x) \wedge \neg g(x)) \longrightarrow h(x)) & \{\text{Implication}\} \\
& = & \forall x\ \neg\neg f(x) \vee \neg\neg g(x) \longrightarrow h(x) & \{\text{DeMorgan's}\} \\
& = & \forall x\ f(x) \vee \neg\neg g(x) \longrightarrow h(x) & \{\text{Double Negation}\} \\
& = & \forall x\ f(x) \vee g(x) \longrightarrow h(x) & \{\text{Double Negation}\}
\end{array}$$

Exercise DMA 1.4.8 – (4 points)

Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

a. $\forall x.(R(x) \longrightarrow H(x))$
b. $\forall x.(R(x) \wedge H(x))$
c. $\exists x.(R(x) \longrightarrow H(x))$
d. $\exists x.(R(x) \wedge H(x))$

Solution:

a. Rabbits can hop
b. Rabbits hop
c. Some rabbits can hop
d. Some rabbits hop

Exercise DMA 1.5.2 – (3 points)

Translate these statements into English, where the domain for each variable consists of all real numbers.

a. $\exists x.\forall y.(xy = y)$
b. $\forall x.\forall y.(((x \geq 0) \land (y < 0)) \longrightarrow (x - y > 0))$
c. $\forall x.\forall y.\exists z.(x = y + z)$

Solution:

a. There exists a real number x such that for all y that are real numbers the product of xy is y
b. For any two real numbers x and y, if $x \geq 0$ and $y < 0$ then $x - y > 0$.
c. For all real numbers x and y there exists a real number z such that $x = y + z$

Exercise DMA 1.5.32 – (4 points)

Express the negations of each of these statements so that all negation symbols immediately precede predicates

a. $\exists z.\forall y.\forall x.T(x, y, z)$
b. $\forall x.\exists y.P(x, y) \lor \forall x.\exists y.Q(x, y)$
c. $\forall x.\exists y.(P(x, y) \land \exists z.R(x, y, z))$
d. $\forall x.\exists y.(P(x, y) \longrightarrow Q(x, y))$

Solution:

a. $\forall z\, \exists y\, \exists x\ \neg T(x, y, z)$
b. $\exists x\, \forall y\, \neg P(x, y) \land \exists x\, \forall y\, \neg Q(x, y)$
c. $\exists x\, \forall y\, (\neg P(x, y) \lor \forall z\, \neg R(x, y, z)$
d. $\exists x\, \forall y\, P(x, y) \land \neg Q(x, y)$

Exercise DMA 1.6.28 – (13 points)

Use rules of inference to show that if $\forall x.(P(x) \lor Q(x))$ and $\forall x.((\neg P(x) \land Q(x)) \longrightarrow R(x))$ are true, then $\forall x.(\neg R(x) \longrightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

Solution:

$$
\begin{aligned}
\forall x(P(x) \lor Q(x)) \land \forall x\ ((\neg P(x) \land Q(x)) \longrightarrow R(x)) & \\
&= \forall x\ (P(x) \lor Q(x)) \land ((\neg P(x) \land Q(x)) \longrightarrow R(x)) && \{7.11\} \\
&= \forall x\ (P(x) \lor Q(x)) \land (\neg(\neg P(x) \land Q(x)) \lor R(x)) && \{\text{Implication}\} \\
&= \forall x\ (P(x) \lor Q(x)) \land ((\neg\neg P(x) \lor \neg Q(x)) \lor R(x)) && \{\text{DeMorgan's}\} \\
&= \forall x\ (P(x) \lor Q(x)) \land ((P(x) \lor \neg Q(x)) \lor R(x)) && \{\text{Double Negation}\} \\
&= \forall x\ ((P(x) \lor Q(x)) \lor R(x)) \land ((P(x) \lor \neg Q(x)) \lor R(x)) && \{\text{Disjunctive Implication}\} \\
&= \forall x\ (R(x) \lor (P(x) \lor Q(x)) \land ((P(x) \lor \neg Q(x)) \lor R(x)) && \{\lor \text{ Associative}\} \\
&= \forall x\ (R(x) \lor ((P(x) \lor Q(x)) \land (R(x) \lor (P(x) \lor \neg Q(x)) && \{\lor \text{ Associative}\} \\
&= \forall x\ (R(x) \lor ((P(x) \lor Q(x)) \land (P(x) \lor \neg Q(x))) && \{\lor \text{ Distributes Over } \land\} \\
&= \forall x\ (R(x) \lor (P(x) \lor (Q(x) \land \neg Q(x))) && \{\lor \text{ Distributes Over } \land\} \\
&= \forall x\ (R(x) \lor (P(x) \lor \text{False})) && \{\land \text{ Complement}\} \\
&= \forall x\ (R(x) \lor P(x)) && \{\lor \text{ Identity}\} \\
&= \forall x\ (\neg R(x) \longrightarrow P(x)) && \{\text{Implication}\}
\end{aligned}
$$

Exercise DMA 1.7.19 – (4 points)

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a. a proof by contraposition
b. a proof by contradiction

Solution:

a. Suppose n is odd, then $n = 2k + 1$

$$
\begin{aligned}
n^3 + 5 &= (2k+1)^3 + 5 = 8k^3 + 12k^2 + 8k + 2 \\
&= 2(4k^3 + 6k^2 + 8k + 2)
\end{aligned}
$$

which is even

$\therefore$ by contraposition if $n^3 + 5$ is odd, then n is even

b. Suppose n is not even and $n^3 + 5$ is odd then there is an integer k such that $n = 2k + 1$, then

$$
\begin{aligned}
n^3 + 5 &= (2k+1)^3 + 5 = 8k^3 + 12k^2 + 8k + 2 \\
&= 2(4k^3 + 6k^2 + 8k + 2)
\end{aligned}
$$

which is even and a contradiction

$\therefore$ by contradiction if $n^3 + 5$ is odd, then n is even

Exercise DMA 1.7.27 – (4 points)

Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$ [*Hint*: Assume that $r = a/b$ is a root, where a and b are integers and a/b is in the lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]

Solution:

Freebie

Part 02 – Sets, Recursion, and Induction (99 Points)

2.1 - Sets

Exercise DMUC 8.2 – (6 points)

Work out the values of the following set expressions, and then check you answer using the Haskell expression that follows.

```
(a) [1,2,3] +++ [3]
(d) [] *** [1,3,5]
(f) [2,3] ~~~ [1,2,3]
(l) subset [1,3] [4,1,3,6]
(o) setEq [3,4,6] [2,3,5]
(q) [] ~~~ [1,2]
```

Solution:

```
(a) [1,2,3]
(d) []
(f) []
(l) True
(o) False
(q) []
```

Exercise DMUC 8.4 – (2 points)

The *cross product* of two sets A and B is defined as $A \times B = \{(a, b) \mid a \in A, b \in B\}$. The function

```
crossproduct :: (Eq a, Show a, Eq b, Show b) =>
              Set a -> Set b -> Set (a, b)
```

takes two sets and returns their cross product. Evaluate these expressions

```
crossproduct [1,2,3] ['a', 'b']
crossproduct [1] ['a', 'b']
```

Solution:

```
crossproduct [1,2,3] ['a','b']
  -> [(1,'a'),(1,'b'),(2,'a'),(2,'b'),(3,'a'),(3,'b')]

crossproduct [1] ['a','b']
  -> [(1,'a'),(1,'b')]
```

Exercise DMUC 8.6 – (1 point)

What are the elements of the set $\{x + y \mid x \in \{1, 2, 3\} \wedge y \in \{4, 5\}\}$?

Solution:

$\{5, 6, 7, 8\}$

Exercise DMUC 8.7 – (1 point)

Write and evaluate a list comprehension that expresses the set

$\{x \mid x \in \{1, 2, 3, 4, 5\} \land x < 0\}$

Solution:

```
[x | x <- [1..5], x < 0]
   -> []
```

Exercise DMUC 8.9 – (1 point)

Write and evaluate a list comprehension that expresses the set

$\{x \mid x \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \land \mathit{even}\ x\}$

Solution:

```
[x | x <- [1..10], x `mod` 2 == 0]
   -> [2,4,6,8,10]
```

Exercise DMUC 8.10 – (4 points)

What is the value of each of the following expressions?

a. `subset [1,3,4] [4,3]`
b. `subset [] [2,3,4]`
c. `setEq [2,3] [4,5,6]`
d. `setEq [1,2] [1,2,3]`

Solution:

a. `False`
b. `True`
c. `False`
d. `False`

Exercise DMUC 8.11 – (7 points)

Let A, B, and C be sets. Prove that if $A \subset B$ and $B \subset C$, then $A \subset C$.

Solution:

Let x be an element in the universe

1. $A \subset B$ {premise}
2. $x \in A \longrightarrow x \in B$ {def. $\subset$}
3. $B \subset C$ {premise}
4. $x \in B \longrightarrow x \in C$ {def $\subset$}
5. $x \in A \longrightarrow x \in C$ {chain rule (2), (4)}
6. $\forall x\ (x \in A \longrightarrow x \in C)$ {$\forall$ Introduction}
7. $A \subset C$ {def. $\subset$}

Exercise DMA 2.1.4 – (3 points)

For each of these intervals, list all its elements or explain why it is empty

a. $[a, a]$
c. $(a, a]$
e. (a, b), where $a > b$

Solution:

a. a
b. $\varnothing$, there are no value both greater than a and less than or equal to a simultaneously
c. $\varnothing$, there is no value that is greater than a and less than b while $a > b$

Exercise DMA 2.1.34 – (4 points)

Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a. $A \times B \times C$
b. $C \times B \times A$
c. $C \times A \times B$
d. $B \times B \times B$

Solution:

a. $\{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$
b. $\{(0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c)\}$
c. $\{(0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x), (0, c, y), (1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1, c, y)\}$
d. $\{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$

Exercise DMA 2.2.4 – (4 points)

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a. $A \cup B$
b. $A \cap B$
c. $A - B$
d. $B - A$

Solution:

a. $\{0, 1, 2, 3, 4, 5, 6\}$
b. $\{3\}$
c. $\{1, 2, 4, 5\}$
d. $\{0, 6\}$

Exercise DMA 2.2.16 – (3 points)

Let A and B be sets. Show that

a. $(A \cap B) \subseteq A$
c. $A - B \subseteq A$
e. $A \cup (B - A) = A \cup B$

Solution:

a. $$\begin{aligned} (A \cap B) &= \{x \mid x \in A \wedge x \in B\} && \{\text{def. } \cap\} \\ &= \{x \mid x \in A\} && \{\text{conjunctive implication}\} \\ &\subseteq A && \{\text{def. } \subseteq\} \end{aligned}$$

b. $$\begin{aligned} A - B &= \{x \mid x \in A \wedge x \notin B\} && \{\text{def. } -\} \\ &= \{x \mid x \in A\} && \{\text{conjunctive implication}\} \\ &\subseteq A && \{\text{def. } \subseteq\} \end{aligned}$$

c. $$\begin{aligned} A \cup (B - A) &= \{x \mid x \in A \vee (x \in B \wedge x \notin A)\} && \{\text{def. } A \cup (B - A)\} \\ &= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \notin B)\} && \{\vee \text{ distributes over } \wedge\} \\ &= \{x \mid (x \in A \vee x \in B) \wedge \text{ True}\} && \{\vee \text{ complement}\} \\ &= \{x \mid x \in A \vee x \in B\} && \{\wedge \text{ identity}\} \\ &= A \cup B && \{\text{def. } \cup\} \end{aligned}$$

Exercise DMA 2.2.54 – (2 points)

Let $A_i = \{\dots, -2, -1, 0, 1, \dots, i$. Find

a. $\bigcup_{i=1}^{n} A_i$
b. a. $\bigcap_{i=1}^{n} A_i$

Solution:

a. $\{\dots, -2, -2, 0, \dots, n\}$
b. $\{n\}$

Exercise DMA 2.2.58 – (3 points)

Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise

a. $\{3, 4, 5\}$
b. $\{1, 3, 6, 10\}$
c. $\{2, 3, 4, 7, 8, 9\}$

Solution:

a. 0011100000
b. 1010010001
c. 0111001110

2.2 - Recursion

Exercise DMUC 3.19 – (3 points)

Using recursion, define `last`, a function that takes a list and returns a `Maybe` type that is `Nothing` if the list is empty.

Solution:

```
last :: [a] -> Maybe a
last [] = Nothing
last x:xs = let y = last xs
            in if y == Nothing then Just x
               else y
```

Exercise DMUC 3.20 – (6 points)

Using recursion, write two functions that expect a string containing a number that contains a decimal point (for example, `23.455`). The first function returns the whole part of the number (i.e., the part to the left of the decimal point). The second function returns the fractional part (the part to the right of the decimal point).

Solution:

```
whole :: [Char] -> [Char]
whole [] = []
whole x:xs if x == '.' then []
           else x : whole xs
```

```
fractional :: [Char] -> [Char]
fractional [] = []
fractional x:xs = if x == '.' then foldr (:) [] xs
                  else fractional xs
```

Exercise DMA 5.4.8 – (3 points)

Give a recursive algorithm for finding the sum of the first n positive integers.

Solution:

procedure SUMN(n: positive integer)
  if $n = 0$ **then**
    return n
  else
    return $n +$ SUMN($n - 1$)

Exercise DMA 5.4.14 – (4 points)

Give a recursive algorithm for finding a **mode** of a list of integers. (A **mode** is an element in the list that occurs at least as often as every other element.)

Solution:

Freebie

Exercise DMA 5.4.44 – (5 points)

Use a merge sort to sort b, d, a, f, g, h, z, p, o, k into alphabetic order. Show all the steps used by the algorithm.

Solution:

```
      b d a f g | h z p o k

  b d | a f g       h z | p o k

b  d    a | f g   h  z    p | o k

  b d   a  f | g    h z   p   o   k

  b d   a  f g      h z   p   k o

  b d   a f g       h z   k o p

    a b d f g     h k o p z

      a b d f g h k o p z
```

Exercise DMA 5.4.50 – (5 points)

Sort 3, 5, 7, 8, 1, 9, 2, 4, 6 using the quick sort.

Solution:

```
initial list: 3 5 7 8 1 9 2 4 6

pivot: 3
  qsort: [1 2]
    pivot: 1
    return: [1 2]

  qsort: [5 7 8 9 4 6]
    pivot 5
    qsort: [4] return [4]
    qsort: [7 8 9 6]
      pivot: 7
      qsort: [6] return [6]
      qsort: [8 9]
        pivot: 8
        qsort: [9] return [9]
      return [8 9]
    return [6 7 8 9]
  return [4 5 6 7 8 9]
return [1 2 3 4 5 6 7 8 9]
```

2.3 - Induction

Exercise DMUC 4.3 – (4 points)

Prove that $\sum_{i=1}^{6} a^i = (a^{n+1} - 1)/(a - 1$, where a is a real number and $a \neq 1$

Solution:

Freebie

Exercise DMUC 4.10 – (4 points)

Prove Theorem 25.

Theorem 25: $map\ f\ .\ concat = concat\ (map\ (map\ f))$

Solution:

Freebie

Exercise DMUC 4.27 – (4 points)

Given a list `xs` of type `Bool`, prove that `and ([False] ++ xs) = False`

Solution:

Freebie

Exercise DMUC 9.1 – (1 point)

Is the following a chain? You can test your conclusions by evaluating `s` in each case.

```
imp1 :: Integer -> Integer
imp1 1 = 2
imp1 x = error "imp1: premise does not apply"

imp2 :: Integer -> Integer
imp2 2 = 3
imp2 x = error "imp2: premise does not apply"

imp3 :: Integer -> Integer
imp3 3 = 4
imp3 x = error "imp3: premise does not apply"
s :: [Integer]
s = [1, imp1 (s !! 0), imp2 (s !! 1), imp3 (s !! 2)]
```

Solution:

Yes this is a chain.

Exercise DMUC 9.2 – (1 point)

Is the following a chain?

```
imp1 :: Integer -> Integer
imp1 1 = 2
imp1 x = error "imp1: premise does not apply"

imp2 :: Integer -> Integer
imp2 3 = 4
imp2 x = error "imp2: premise does not apply"

s :: [Integer]
s = [0, imp1 (s !! 0), imp2 (s !! 1)]
```

Solution:

No this is not a chain

Exercise DMUC 9.5 – (1 point)

Given the base case $0 \in n$ and the induction rule $x \in n \longrightarrow x+1 \in n$, fix the following calculation so that 3 is in set n:

```
fun :: Integer -> Integer
fun x = x - 1

n :: [Integer]
n = 0 : map fun n
```

Solution:

```haskell
fun :: Integer -> Integer
fun x = x + 1
```

Exercise DMUC 9.16 – (2 points)

Use the computer to evaluate the first 10 elements of the set and describe the result.

```
nextIntegers5 :: Integer -> [Integer]
nextIntegers5 x
  = if x > 0 \/ x == 0
       then [x + 1, -(x + 1)]
       else []
```

Solution:

```
[[1,-1],[2,-2],[3,-3],[4,-4],[5,-5],[6,-6],[7,-7],[8,-8],[9,-9],[10,-10]]
```

Exercise DMUC 9.18 – (1 point)

Does twos enumerate the set of even natural numbers?

```
twos :: [Integer]
twos = build 0 (2 *)
```

Solution:

No it only enumerates the number 0

Exercise DMA 5.1.4 – (5 points)

Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a. What is the statement $P(1)$?
b. Show that $P(1)$ is true, completing the basis step of the proof of $P(n)$ for all positive integers n
c. What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers$n$?
d. What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?
e. Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.
f. Explain why these steps show that this formula is true whenever n is a positive integer.

Solution:

a, b. $P(1) = \left(\frac{1(1+1)}{2}\right)^2 = 1^2 = 1^3$

c. $P(n) \, : \, n^3 = \left(\frac{n(n+1)}{2}\right)^2$

d. to show that $p(n+1) = (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + \left(\frac{(n+1)(n+2)}{2}\right)^2$

e. assuming the inductive hypothesis

$$
\begin{aligned}
\left(\frac{(n+1)(n+2)}{2}\right)^2 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
\left(\frac{n^2+3n+2}{2}\right)^2 &= \left(\frac{n^2+n}{2}\right)^2 + (n^2+2n+1)(n+1) \\
\left(\frac{(n^2+3n+2)(n^2+3n+2)}{4}\right) &= \frac{n^4+2n^3+n^2}{4} + (n^3+n^2+3n+1) \\
\frac{n^4+6n^3+13n^2+12n+4}{4} &= \frac{n^4+6n^3+13n^2+12n+4}{4}
\end{aligned}
$$

f. Thus, as we have shown the base case and the inductive case, and via the inductive hypothesis and the principle of induction we have prove this for all cases.

Exercise DMA 5.1.20 – (4 points)

Prove that $3^n < n!$ if n is an integer greater than 6

Solution:

Freebie

Exercise DMA 5.2.8 – (4 points)

Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

Solution:

Freebie

Exercise DMA 5.2.32 – (1 point)

Find the flaw with the following "proof" that every postage of three cents or more can be formed using just 3-cent and 4-cent stamps.

Basis Step: We can form postage of three cents with a single 3-cent stamp and we can form postage of four cents using a single 4-cent stamp.

Inductive Step: assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just 3-cent and 4-cent stamps. We can then form postage of $k + 1$ cents by replacing one 3-cent stamp with a 4-cent stamp or by replacing two 4-cent stamps by three 3-cent stamps.

Solution:

k = 5 cents cannot be formed using any combination of stamps, thus the base case is incorrect.