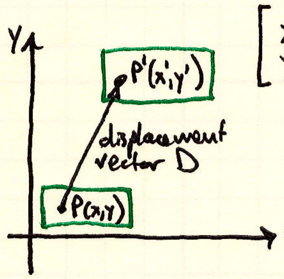
Outcomes

- Understand the basic theory of transformations in 2-space
- Be capable of applying the transformation by hand
- Understand why homogenous coordinates and affine transformations are used by OpenGL

• Geometric Transformations in $\mathbb{R}^2$

– **Translation**: specified by a displacement vector $D = [d_x \; d_y]^T$ added to the location vector ~~specified~~ of a point

A point $P = [x \; y]^T$ is translated to $P' = [x' \; y']^T$ as



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} x + d_x \\ y + d_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

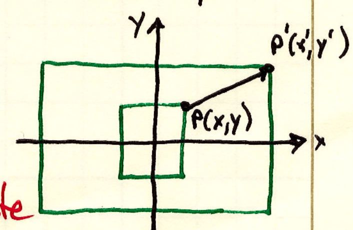
$$\Downarrow$$

$$P' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P + D$$

– **Scaling**: Specified by a scaling factor s_x along the x-axis and a scaling factor s_y along the y-axis.

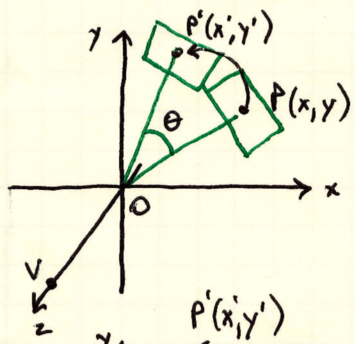
A point P by this scaling is one where the x coordinate value is multiplied by s_x and the y coordinate is multiplied by s_y, or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow P' = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} P$$



• If either s_x or s_y are zero, then the scaling is degenerate

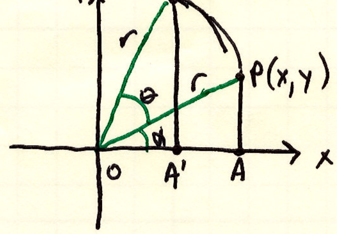
– **Rotation**: specified by an angle θ measured CCW as seen by a viewer V located on the positive side of the Z-axis



We calculate $P' = [x' \; y']^T$ of $P = [x \; y]^T$ as a rotation about the origin O using

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\downarrow$$



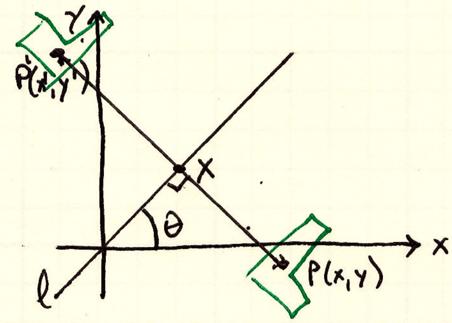
$$P' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} P$$

Translate: $P = (5, 1)$ by $D = (3, 2)$ Rotate: $P = (10, 8)$ by $15°$

Scale: $P = (3, 2)$ by $S = (2, -1)$

- <u>Reflection</u>: A point $P=[x\,y]^T$ is transformed by reflection about a straight line ℓ, called the mirror, to $P'=[x'\,y']^T$, s.t.
  - if P lies on ℓ, then $P'=P$
  - If P does not lie on ℓ, then P' is a point on the otherside of ℓ s.t., PP' is perpendicular to ℓ, and P' is equidistant from ℓ as P
  - P maps to P' across ℓ as follows:

$$P' = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} P$$



- <u>Shear</u>: A distortion ~~const~~ uniquely determined by two parameters

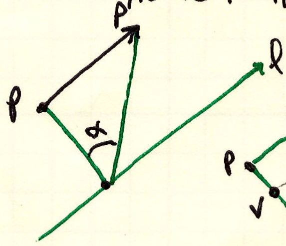
  ℓ – the line of shear
  α – the angle of shear

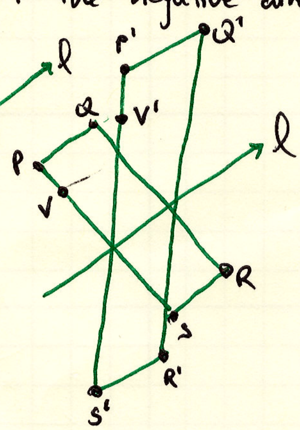
  A point $P \in \mathbb{R}^2$ is mapped to $P' \in \mathbb{R}^2$ by s as follows

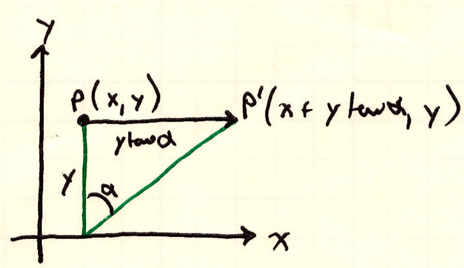
  (a) If P lies on ℓ, then it is unchanged

  (b) If P lies a distance, h, left of ℓ, then it it moves parallel to ℓ in the ~~pos~~ positive direction of ℓ a distance of $h \tan\alpha$

  (c) If P lies a distance of h right of ℓ, then it moves parallel to ℓ in the negative direction of ℓ a distance $h \tan\alpha$







Reflect: $P=(2,1)$ ~~across~~ by $\theta = 45°$

~~Shear: P = (10, 3)~~

- Affine Transformations

Linear Transformation f^M of $\mathbb{R}^M$ with M defined as

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mm} \end{bmatrix}$$

Then $f^M: \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$f^M(P) = MP, \; P \in \mathbb{R}^m$ → which is the linear combination

An affine transformation of $\mathbb{R}^m$ is defined as $g: \mathbb{R}^m \rightarrow \mathbb{R}^m$

where: $g(P) = f^M(P) + D = MP + D$

- f^M is the non-singular linear transformation of $\mathbb{R}^M$
- D is the transformational component ~ arbitrary
- M is the defining matrix of g ~ non-singular

Prop 5.1

An affine transform g of $\mathbb{R}^2$ maps straight lines to straight lines

" " " " maps convex sets to convex sets

" " " g of $\mathbb{R}^3$ maps straight lines to straight lines and planes to planes

" " " " maps a convex set on one plane of $\mathbb{R}^3$ to a convex set on another plane

Affine transformations are the most general class of transformations of degree 1, and preserve straightness, flatness, and convexity in both $\mathbb{R}^2$ + $\mathbb{R}^3$

Using homogeneous coordinates i.e., $P = [x, y\ z\ 1]^T$ provides the capability to define any Affine transformation as

$$\begin{bmatrix} g(P) \\ 1 \end{bmatrix} = \begin{bmatrix} P' \\ 1 \end{bmatrix} = \begin{bmatrix} M & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix}$$

which then simplifies composition of transformations to

$$\begin{bmatrix} (g_3 \circ g_2 \circ g_1)(P) \\ 1 \end{bmatrix} = \left(\begin{bmatrix} M_3 & D_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_2 & D_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 & D_1 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} P \\ 1 \end{bmatrix}$$

Affine Geometric Transformations

Prop 5.2: Let g be either a translation, a scaling, a rotation or a reflection. Then:

- g maps straight lines to straight lines
- g maps convex sets to convex sets

Geometric Transformation Equations

Translation: by D

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Scaling: by s_x and s_y

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Rotation: by θ about the origin

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Reflection: across l at angle θ

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Prop 5.3: Any affine transformation of $\mathbb{R}^2$ is the composition in some order of translations, scalings, and rotations about the origin

Thus, it can be factored into a composition:

$g = g_4 \circ g_3 \circ g_2 \circ g_1$ where g_1 is a rotation about O, g_2 a scaling, g_3 another rotation about O, and g_4 a translation

Q: Why then does OpenGL support Affine Transformations?

- **Euclidean and Rigid Transformations**

  - When animating rigid objects (i.e., ball, person, etc) it is wise to use distance preserving transformations

  - A Euclidean Transformation (also called isometry) of $\mathbb{R}^2$ is one that preserves distance. That is,

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is Euclidean if } |f(P)f(Q)| = |PQ| \text{ for any two points } P, Q \in \mathbb{R}^2$$

  - An Euclidean Transformation f of $\mathbb{R}^2$ is said to be orientation-reversing if there exists three non-collinear points P, Q and $R \in \mathbb{R}^2$ such that, looking at $\mathbb{R}^2$ from a fixed side, one of the two sequences PQR and $f(P)f(Q)f(R)$ appears clockwise and the other counter-clockwise

  - A Euclidean transform that is not orientation-~~preserving~~ reversing is said to be orientation-preserving, also called a rigid transformation

  - These are so called because they model the physical motion of a rigid object restricted always to a plane — such motion can never reverse orientation.

  - Prop 5.5 : A translation or a rotation about an arbitrary point is a rigid transformation of 2-space. A reflection about an abitrary mirror is an orientation-reversing Euclidean transformation of 2-space

Prop 5.7

  - Affine, Euclidean and rigid transformations of 2-space are related by the following inclusions, which are each proper

$$\text{rigid transformations} \subset \text{Euclidean transformations} \subset \text{Affine transformations}$$

rigid transformations: Translation, Rotation

Euclidean transformations: Translation, Rotation, Reflection

Affine transformations: Translation, Rotation, Reflection, Scaling

Q: What is the take-away on these different types of transformations and their application to animation?

For Next Time

- Read Ch: 5 sect 4
- Read Ch. 6
- Review Notes
- Start HW 02

Additional Notes