Relations

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Outline

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The lecture is structured as follows:

- Binary Relations
  - Representing Relations
  - Relation Properties
  - Combining Relations
  - Relation Operations
- *n*-ary Relations
  - Databases
  - Operations
  - SQL

Binary Relations

CS 1187

Binary Relation

- **Definition:** Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$
  - A set of ordered pairs (a, b) where $a \in A$ and $b \in B$
  - Notation: aRb denotes $(a, b) \in R$, $a \not R b$ denotes $(a, b) \notin R$

ROAR

Binary Relation

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Example: Let A be the set of cities in the US, and let B be the set of the 50 states. Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the state b. Examples in R include:

- (Bolder, Colorado)
- (Bangor, Maine)
- (Ann Arbor, Michigan)
- (Cupertino, California)

Relations on a Set

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Example: Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R iff a and b are positive integers not exceeding 4 such that a divides b, we see that:

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

Representing Relations

- We can represent relations in several ways
  - As a function
  - As a matrix
  - As a digraph

Relations as Functions

- A function f from a set A to a set B assigns exactly one element of B to each element of A
  - Thus, a graph of f is the set of ordered pairs (a, b) such that $b = f(a)$
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- Relations are a generalization of graphs of functions

Relations as Matrices

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- For a relation R from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
- The relation R can be represented by the matrix $\mathbf{M_R} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

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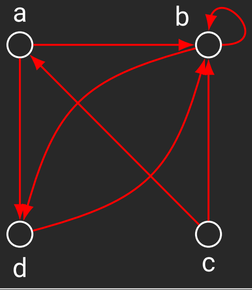
- An edge of the form (a, a) is represented using an arc from the vertex a back to itself, also called a **loop**.



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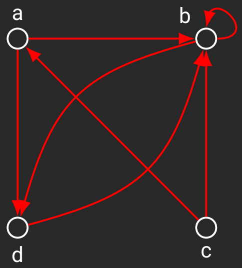
Relations as Digraphs

Example: The directed graph with vertices *a*, *b*, *c*, and *d*, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b)



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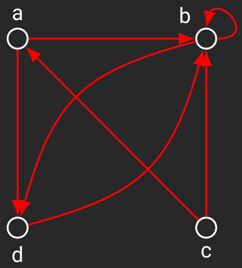
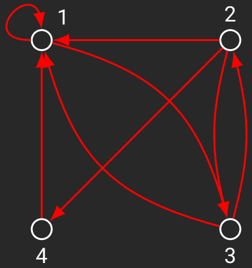


Example: The directed graph of the relation

$$R_1 = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$$

on the set $\{1,2,3,4\}$



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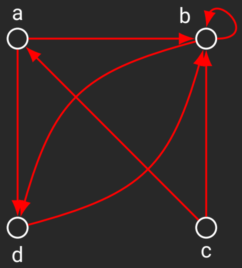

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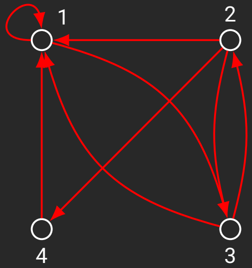

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Properties of Relations

- There are 4 basic properties of a relation
  - Reflexivity
  - Symmetry and Antisymmetry
  - Transitivity

Reflexivity

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Matrix Perspective:

- A relation R is reflexive iff $m_{ii} = 1$, for $i = 1, 2, \ldots, n$. That is, all elements on the main diagonal of $\mathbf{M_R}$ are 1.
- A relation R is irreflexive if there exists a zero on the main diagonal of $\mathbf{M_R}$.

$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

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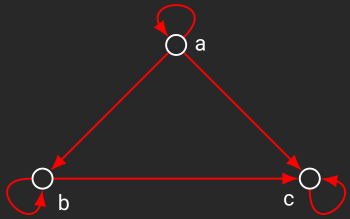
Digraph Perspective:

- A relation is reflexive iff for every vertex in the digraph, there is a loop



Reflexivity

Haskell Perspective:

- The STDM tools provide two functions to determine if a relation over a set is either reflexive or irreflexive

```
isReflexive :: (Eq a, Show a) => Digraph a -> Bool
isIrreflexive :: (Eq a, Show a) => Digraph a -> Bool
```

Which of the following digraphs are reflexive and which are irreflexive?

```
a = [1,2,3]
digraph1 = (a, [(1,1), (1,2), (2,2), (2,3), (3,3)])
digraph2 = (a, [(1,2), (2,3), (3,1)])
digraph3 = (a, [(1,1), (1,2), (2,2), (2,3)])
```

Symmetry

- **Definition:** A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

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- **Definition:** A relation R on a set A such that for all $a, b \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.
  - For R to be antisymmetric both of the following must be true:
    - $\forall x, y \in A.\ x \neq y \rightarrow \neg(xRy \wedge yRx)$
    - $\forall x, y \in A.\ x \neq y \rightarrow \neg xRy \vee \neg yRx$

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Matrix Perspective:

- A matrix $\mathbf{M_R}$ for relation R depicts symmetry when $m_{ij} = m_{ji}$, for all i and j where $0 \leq i \leq n$ and $0 \leq j \leq n$.
  - $\mathbf{M_R} = (\mathbf{M_R})^T$
- A matrix $\mathbf{M_R}$ for relation R depicts antisymmetric when $m_{ij} = 1$ then $m_{ji} = 0$ and $i \neq j$. That is either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

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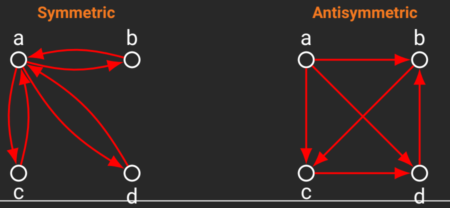
Digraph Perspective:

- A relation is symmetric iff for every edge between distinct vertices there is an edge in the opposite direction (i.e., an edge (x, y) and (y, x) both exist)
- A relation is antisymmetric iff there are never two edges in opposite directs between distinct vertices.



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Symmetry

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Haskell Perspective:

- The STDM tools provide two functions to determine if a relation over a set is either symmetric or antisymmetric

```
isSymmetric, isAntisymmetric ::
  (Eq a, Show a) => Digraph a -> Bool
```

Work out whether the relations in the following expressions are symmetric and whether they are antisymmetric, and then check your conclusions by evaluating the expressions with Haskell:

```
isSymmetric ([1,2,3], [(1,2), (2,3)])
isSymmetric ([1,2], [(2,2), (1,1)])
isAntisymmetric ([1,2,3], [(2,1), (1,2)])
isAntisymmetric ([1,2,3], [(1,2), (2,3), (3,1)])
```

Transitivity

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- **Definition:** A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$
  - That is,

$$\forall x, y, z \in A.\, xRy \land yRz \rightarrow xRz$$

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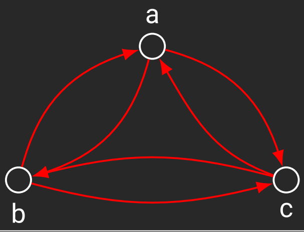
Digraph Perspective:

- A relation is transitive iff whenever there is an edge (x, y) and an edge (y, z) there is also an edge (x, z) forming a triangle with the correct direction.



Transitivity

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Haskell Perspective:

- The STDM tools provide a function to determine if a relation over a set is transitive

```
isTransitive :: (Eq a, Show a) => Digraph a -> Bool
```

Determine by hand whether the following relations are transitive, and then check your conclusion using the computer

```
isTransitive ([1,2], [(1,2), (2,1), (2,2)])
isTransitive ([1,2,3], [(1,2)])
```


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Examples

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Example: consider the following relations on $\{1, 2, 3, 4\}$

$$
\begin{aligned}
R_1 &= \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1)(4,4)\}\\
R_2 &= \{(1,1),(1,2),(2,1)\}\\
R_3 &= \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}\\
R_4 &= \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}\\
R_5 &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}\\
R_6 &= \{(3,4)\}
\end{aligned}
$$

Which of these relations are reflexive? symmetric? antisymmetric? transitive?

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$$

Which of these relations are reflexive? symmetric? antisymmetric? transitive?

Solution:

- Reflexive: R_3 and R_5
- Symmetric: R_3, R_4, and R_6
- Antisymmetric: R_1, R_2, R_4, and R_5
- Transitive: R_4, R_5, and R_6

Combining Relations

- We can combine relations in three distinct ways
  - Using set operators (as a relation from A to B is a subset of $A \times B$)
  - Through composite relations
  - Through Powers of relations

Set Operations

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- Relations are sets, thus the set operations all apply

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Set Operations

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  Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain:

$$\begin{aligned}
R_1 \cup R_2 &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\} \\
R_1 \cap R_2 &= \{(1,1)\} \\
R_1 - R_2 &= \{(2,2),(3,3)\} \\
R_2 - R_1 &= \{(1,2),(1,3),(1,4)\}
\end{aligned}$$

Set Operations

- Additionally, we can use the Boolean operations join and meet to find the union and intersection of two matrices representing relations. If we have $\mathbf{M_{R_1}}$ and $\mathbf{M_{R_2}}$ representing relations R_1 and R_2, then

$$\mathbf{M_{R_1 \cup R_2}} = \mathbf{M_{R_1}} \vee \mathbf{M_{R_2}} \quad \text{and} \quad \mathbf{M_{R_1 \cap R_2}} = \mathbf{M_{R_1}} \wedge \mathbf{M_{R_2}}$$

Matrix Example

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Example: Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{\mathbf{R_1}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{M}_{\mathbf{R_2}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

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Solution:

$$\mathbf{M_{R_1 \cup R_2}} = \mathbf{M_{R_1}} \vee \mathbf{M_{R_2}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{M_{R_1 \cap R_2}} = \mathbf{M_{R_1}} \wedge \mathbf{M_{R_2}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composite Relations

- **Defintion:** Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $RandS$ as $R \circ S$

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  Example: What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

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  Solution: $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$

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Finding Composite Relations

- Computing the composite requires we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second element

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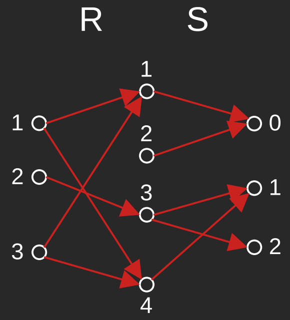

$1 \to 1 \to 0$ (0, 1)
$1 \to 4 \to 1$ (1, 1)
$2 \to 3 \to 1$ (2, 1)
$2 \to 3 \to 2$ (2, 2)
$3 \to 1 \to 0$ (3, 0)
$3 \to 4 \to 1$ (3, 1)

Finding Composite Relations

- In the case of the matrix representation, this is handled by using the Boolean product.

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S$$

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$$\mathbf{M}_{\mathbf{S} \circ \mathbf{R}} = \mathbf{M}_{\mathbf{R}} \odot \mathbf{M}_{\mathbf{S}}$$

Example: Find the matrix representing the relations $S \circ R$

$$\mathbf{M}_{\mathbf{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{M}_{\mathbf{S}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Finding Composite Relations

- In the case of the matrix representation, this is handled by using the Boolean product.

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Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{\mathbf{S}\circ\mathbf{R}} = \mathbf{M}_{\mathbf{R}} \odot \mathbf{M}_{\mathbf{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Powers of Relations

- **Definition:** Let R be a relation on the set A. The *powers* R^n, $n = 1, 2, 3, \ldots$, are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R$$

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Solution: Because $R^2 = R \circ R$, we find that

$$\begin{aligned} R^2 &= \{(1,1), (2,1), (3,1), (4,2)\} \\ R^3 = R^2 \circ R &= \{(1,1), (2,1), (3,1), (4,1)\} \\ R^4 = R^3 \circ R &= \{(1,1), (2,1), (3,1), (4,1)\} \end{aligned}$$

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The same can be said for $n = 5, 6, 7, \ldots$

- **Theorem:** The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Powers of Relations

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Solution: The matrix for R^2 is

$$\mathbf{M}_{\mathbf{R}^2} = \mathbf{M}_{\mathbf{R}}^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Relational Composition

- The STDM tools provide some functions to help us find the composition and powers of relations

```
relationalComposition ::
  (Eq a, Show a, Eq b, Show b, Eq c, Show c) =>
    Set (a,b) -> Set (b,c) -> Set(a,c)
relationalPower ::
  (Eq a, Show a) => Digraph a -> Int -> Relation a
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relationalComposition [(1,2),(2,3)] [(3,4)]
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```

Example: : Work out the values of these expressions, then evaluate using a computer

```
relationalPower ([1,2,3,4], [(1,2), (2,3), (3,4)]) 1
relationalPower ([1,2,3,4], [(1,2), (2,3), (3,4)]) 2
relationalPower ([1,2,3,4], [(1,2), (2,3), (3,4)]) 3
```

Exercises

Exercise: For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive:

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1. $\{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$
2. $\{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
3. $\{(2,4), (4,2)\}$
4. $\{(1,2), (2,3), (3,4)\}$
5. $\{(1,1), (2,2), (3,3), (4,4)\}$
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1. $\{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$
2. $\{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$
3. $\{(2,4),(4,2)\}$
4. $\{(1,2),(2,3),(3,4)\}$
5. $\{(1,1),(2,2),(3,3),(4,4)\}$
6. $\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$

Solution:

- Reflexive: 2, 5
- Symmetric: 2, 3, 5
- Antisymmetric: 4, 5
- Transitive: 1, 2, 5

Exercises

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Exercise: Represent each of these relations on $\{1, 2, 3\}$ with a matrix and a digraph

1. $\{(1, 1), (1, 2), (1, 3)\}$
2. $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

Exercises

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1. $\{(1,1),(1,2),(1,3)\}$
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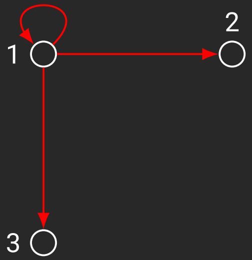
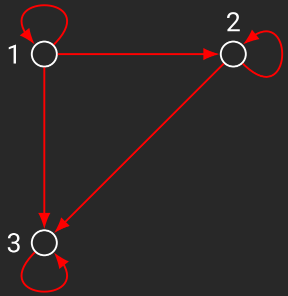
Solution:

$$\mathbf{M_1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M_2} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$





N-ary Relations

CS 1187

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- In addition to binary relations, we can consider relations between many sets.

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- **Definition:** Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. These sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and n is called its *degree*.

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- These types of relations form the basis for databases

ROAR

Examples

- **Example:** Let R be the relation consisting of 5-tuples (A, N, S, D, T), representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time, the the following is an example of a particular tuple:

  (Nadir, 963, Newark, Bangor, 15:00)

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  - The degree of this relation is 5
  - The domain of this relation is
    - The set of all airlines
    - The set of all flight numbers
    - The set of cities
    - The set of cities (again)
    - The set of times

Databases and Relations

- **Relational data model** - one of the various approaches for representing databases based on the concept of a relation
  - These methods make the operations (adding, deleting, updating, querying records) and storage of information efficient

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- **Relational data model** - one of the various approaches for representing databases based on the concept of a relation
  - These methods make the operations (adding, deleting, updating, querying records) and storage of information efficient

- Database components
  - **Records:** *n*-tuples consisting of *fields*
  - **Fields:** entries in the *n*-tuples (i.e., a student record may have fields such as name, student number, major, and gpa).
  - **Table:** another name for the relations that represent databases.
    - Each column of the table corresponds to an *attribute* or *field* of the database, and each row to a **record**

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Keys

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- Keys uniquely identify a *n*-tuple, or record, in a relation. There are two types of keys:
  - **Primary Key:** the domain of an *n*-ary relation, where the value of the *n*-tuple from this domain determines the *n*-tuple.
    - A domain is a *primary key* when no two *n*-tuples in the relation have the same value from this domain
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  - **Composite Key:** often combinations of domains can be used to uniquely identify *n*-tuples in an *n*-ary relation. The Cartesian product of these domains is called a *composite key*

- Records are often added to or deleted from databases
  - **Extension:** of a relation is the current collection of *n*-tuples in the relation
  - **Intension:** is the permanent components (i.e., table structure) of the database
  - However, a key should be time-independent
  - Thus a key should be selected which will remain one as that database changes, thus it should be valid across all possible extensions

N-ary Relation Operations

- **Definition:** Let R be an n-ary relation and C a condition that elements in R must satisfy. Then the *selection operator* s_C maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

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- **Definition:** The *projection* $P_{i_1, i_2, \ldots, i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

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- **Definition:** Let R be a relation of degree m and S a relation of degree n. The *join* $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.
  - In essence, the **join** operator allows us to combine two table into one when the tables share identical fields

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Selection Operator Example

Example: To find the records of computer science majors in the n-ary relation R shown in the table below, we use the operator s_{C_1}, where C_1 is the condition Major = "Computer Science"

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

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Solution:

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Projection Operator Example

Example: What is the table obtained when the project $P_{1,2}$ is applied to the relation in the following Table?

Student	*Major*	*Course*
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

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Solution:

Student	*Major*
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

Join Operator Example

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Example: What is the join of the following two tables?

Professor	*Department*	*Course_number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

Department	*Course_number*	*Room*	*Time*
Computer Science	518	N521	2:00 PM
Mathematics	575	N502	3:00 PM
Mathematics	611	N521	4:00 PM
Physics	544	B505	4:00 PM
Psychology	501	A100	3:00 PM
Psychology	617	A110	11:00 AM
Zoology	335	A100	9:00 AM
Zoology	412	A100	8:00 AM

Join Operator Example

Solution: Resulting Table

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 AM
Cruz	Zoology	412	A100	8:00 AM
Farber	Psychology	501	A100	3:00 PM
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Rosen	Computer Science	518	N521	2:00 PM
Rosen	Mathematics	575	N502	3:00 PM

SQL

- SQL (Structured Query Language) is a database query language used to carry out the operations we have previously discussed.

Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

```sql
SELECT Departure_time
FROM Flights
WHERE Destination='Detroit'
```

Departure_time
08:10
08:47
09:44

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Psychology	617	A110	11:00 AM
Zoology	335	A100	9:00 AM
Zoology	412	A100	8:00 AM

```sql
SELECT Professor, Time
FROM Teaching_assignments as ta,
     Class_schedule as cs
WHERE ta.Department='Mathematics' AND
      cs.Department='Mathematics' AND
      ta.Course_number = cs.Course_number;
```

Professor	*Time*
Rosen	3:00 PM

Are there any questions?