Graphs

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Outline

The lecture is structured as follows:

- Connectivity
- Euler and Hamilton Paths
- Shortest-Path Problems
- Planar Graphs
- Graph Coloring

Connectivity

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Paths

- Paths in undirected graphs
  - In an undirected graph, a **path** of length n from u to v is a sequence of adjacent edges going from vertex u to vertex v
  - A path is a **circuit** if $u = v$
  - A path **traverses** the vertices along it.
  - A path is **simple** if it contains no edge more than once.
- Paths in directed graphs
  - Same as in undirected graphs, but the path must go in the direction of the arrows

Connectedness

- An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.

- **Connected Component:** connected subgraph

- A **cut vertex** or **cut edge** separates 1 connected component into 2 if removed

- **Theorem:** There is a *simple path* between any pair of vertices in a connected *undirected graph*

Directed Connectedness

- A directed graph is **strongly connected** iff there is a directed path from *a* to *b* for any two vertices *a* and *b*

- It is **weakly connected** iff the underlying undirected graph (i.e., with edge directions removed) is connected.

- Note strongly implies weakly but not vice-versa

- Note that connectedness, and the existence of a circuit or simple circuit of length *k* are graph invariants with respect to isomorphism.

Counting Paths Using Adjacency Matrices

- Let $\mathbf{A}$ be the adjacency matrix of graph G
- The number of paths of length k from v_i to v_j is equal to $\left(\mathbf{A}^k\right)_i j$.

Euler and Hamilton Paths

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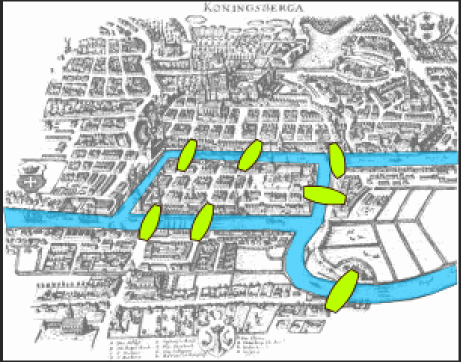
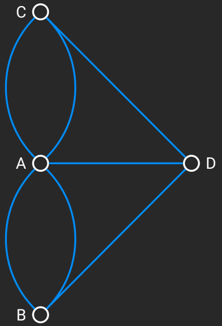
Euler Paths and Circuits

- An **Euler path** in *G* is a simple path containing every edge of *G*
- An **Euler circuit** in a graph *G* is a simple circuit containing every edge of *G*





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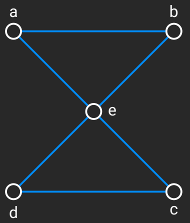
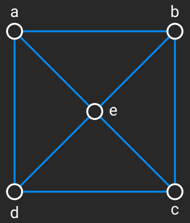
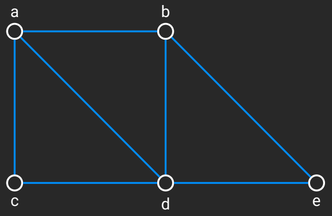
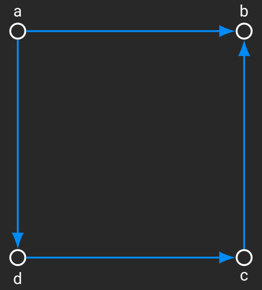
Examples

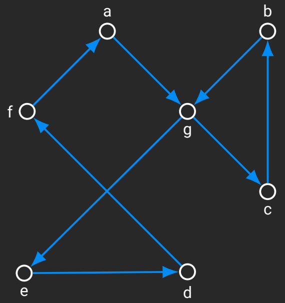


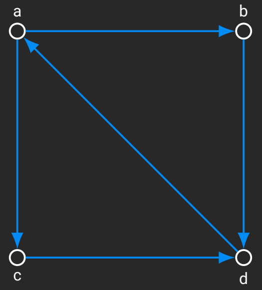




Examples







Euler Circuits and Paths

- **Theorem:** A connected multigraph has an Euler circuit iff each vertex has even degree
- **Theorem:** A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree

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Constructing Euler Circuits

procedure EULER(G: connected multigraph with all vertices of even degree)
  $circuit$:= a circuit in G
  H := G with edges of this circuit removed
  while H has edges **do**
    $subcircuit$:= a circuit in H
    H := H with the edges of $subcircuit$ removed
    $circuit$:= $circuit$ with $subcircuit$ inserted at the appropriate vertex
▷ $circuit$ is an Euler circuit

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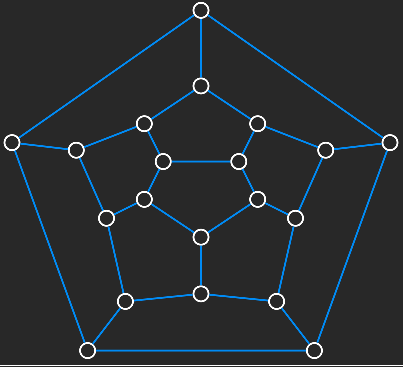
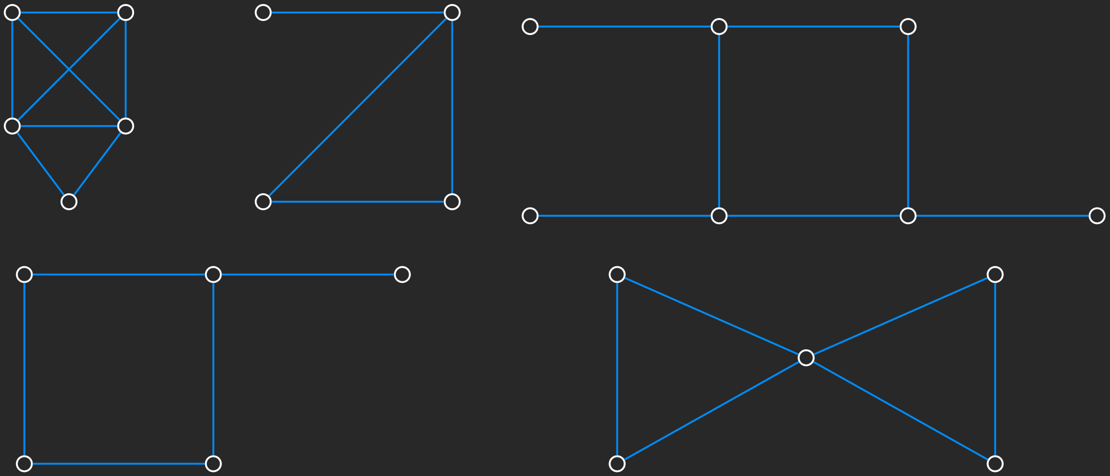
Hamilton Paths and Circuits

- A **Hamilton path** is a path that traverses each vertex in *G* exactly once
- A **Hamilton circuit** is a circuit that traverses each vertex in *G* exactly once



Examples



Some Useful Theorems

- **Dirac's Theorem:** If (but not only if) G is connected, simple, has $n \geq 3$ vertices, and $\forall v : \deg(v) \geq n/2$, then G has a Hamilton circuit

- **Ore's Theorem:** If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit

Shortest-Path Problems

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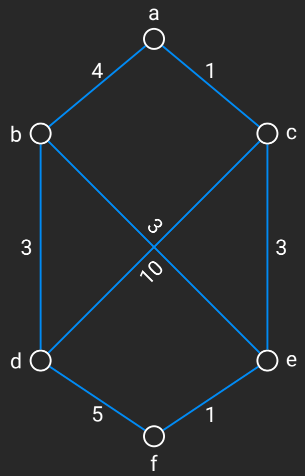
Shortest-Path Problems

- Weighted Graphs $G(V, E, w)$
  - V: a vertex set
  - E: an edge set
  - w: a weighting function on E
- The length of a path, e.g.

$$
\begin{aligned}
& w(\{a,b\},\{b,d\},\{d,f\}) \\
&= w(\{a,b\}) + w(\{b,d\}) + w(\{d,f\}) \\
&= 1 + 3 + 5 \\
&= 9
\end{aligned}
$$

- The shortest path



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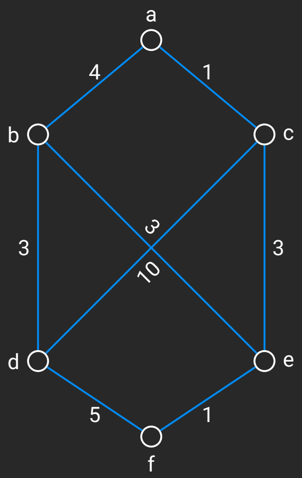
Examples

- Some paths between *a* and *f*
  - Path ({a, b}, {b, d}, {d, f}):
    length of ({a, b}, {b, d}, {d, f}) =
    w({a, b}) + w({b, d}) + w({d, f}) =
    1 + 2 + 5 = 9
  - Path ({a, b}, {b, e}, {e, f}):
    length of ({a, b}, {b, e}, {e, f}) =
    w({a, b}) + w({b, e}) + w({e, f}) =
    1 + 3 + 1 = 5
  - Path ({a, c}, {c, e}, {e, f}):
    length of ({a, c}, {c, e}, {e, f}) =
    w({a, c}) + w({c, e}) + w({e, f}) =
    4 + 3 + 1 = 8



Dijkstra's Algorithm

procedure DIJKSTRA(G: a weighted connected simple graph with all weights positive, a is the source, z is the destination)

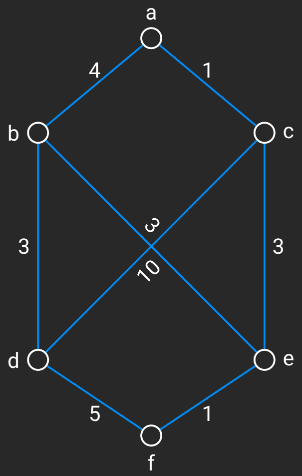
▷ there exists a path from a to z

for $i := 1$ **to** n **do**
  $L(v_i) := \infty$
$L(a) := 0$
$S := \varnothing$
while $z \notin S$ **do**
  $u :=$ a vertex not in S with $L(u)$ minimal
  $S := S \cup \{u\}$
  for all vertices v not in S **do**
    if $L(u) + w(u,v) < L(V)$ **then**
      $L(v) := L(u) + w(u,v)$

▷ $L(z) =$ length of a shortest path from a to z

Example of Dijkstra's Algorithm

- Find the shortest path between *a* and *f*
- Find the shortest path between *d* and *c*



Traveling Salesman Problem

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- The traveling salesman problem asks for the circuit of minimum total weight in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point.
- No algorithm with polynomial worst-case time complexity is known.
- c-approximation algorithms: $W \leq W' \leq cW$
  - W: total length of an exact solution
  - W': total weight of a Hamilton circuit
  - c: a constant

Planar Graphs

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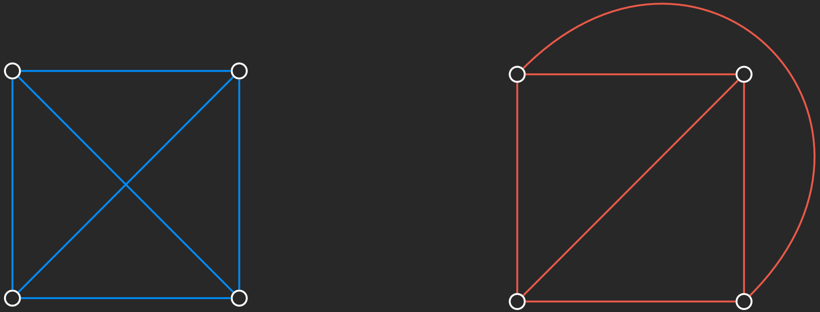
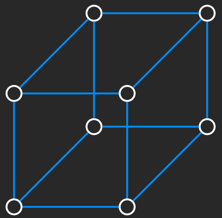
Planar Graphs

- A graph is called **planar** if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of the graph.

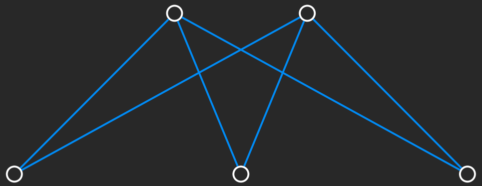
- Example:



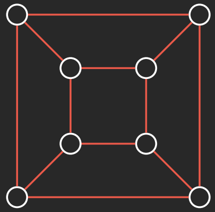
More Examples

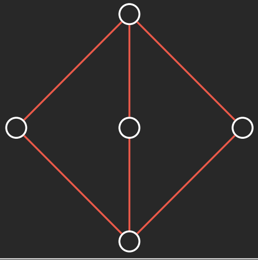


Q_3



$K_{2,3}$





Euler's Formula

- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation fo G. Then $r = e - v + 2$
- **Corollary:** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$
- **Proof:**
  1. $2e \geq 3r$
  2. $r = e - v + 2$

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Euler’s Formula

- **Corollary:** If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

- Show that K_5 is nonplanar using above corollary.

- **Exercise:** If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$

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Kuratowski's Theorem

- If a graph is planar, any graph is obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.

- The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivision.

- **Kuratowski's Theorem:** A graph is nonplanar iff it contains a subgraph *homeomorphic* to $K_{3,3}$ or K_5.

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Graph Coloring

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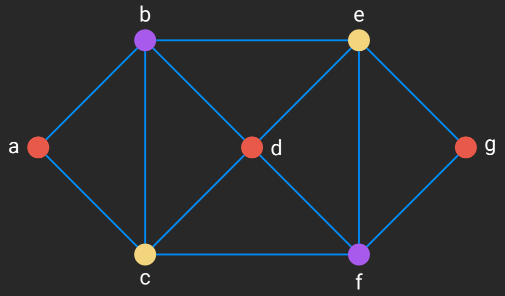
Graph Coloring

- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

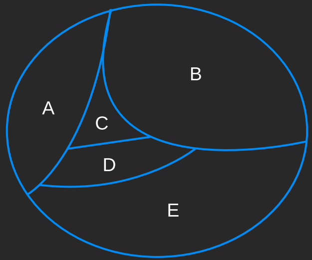
- The **chromatic number** of a graph *G*, denoted by $\chi(G)$ is the least number of colors needed for a coloring of this graph.

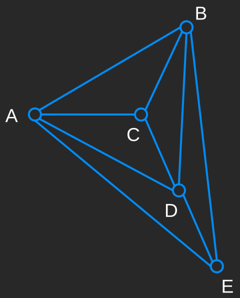


Coloring of Maps

- Color a *map* such that two adjacent regions don't have the same color

- Each map in the plane can be represented by a dual planar graph

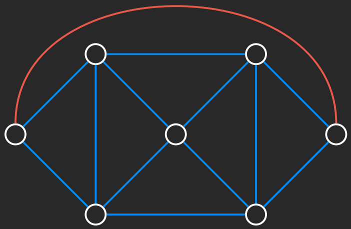


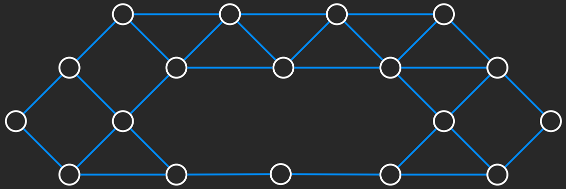


- **The Four Color Theorem:** The chromatic number of a planar graph is no greater than four.

Example

- What is the chromatic number of the graphs?





- What is the chromatic number of K_n?

- What is the chromatic number of $K_{m,n}$?

Are there any questions?