Contents

1. Set systems

1.1. Chains and antichains

Note 1.1 The ideas in combinatorics often occur in the proofs, so it is advisable to learn the techniques used in proofs, rather than just learning the results and not their proofs.

Definition 1.2 Let X be a set. A **set system** on X (also called a **family of subsets of X**) is a collection $\mathcal{F} \subseteq \mathbb{P}(X)$.

Notation 1.3 $X^{(r)} := \{A \subseteq X : |A| = r\}$ denotes the family of subsets of X of size r.

Remark 1.4 Usually, we take $X = [n] = \{1, ..., n\}$, so $|X^{(r)}| = \binom{n}{r}$.

Notation 1.5 For brevity, we write e.g. $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$.

Definition 1.6 We can visualise $\mathbb{P}(A)$ as a graph by joining nodes $A \in \mathbb{P}(X)$ and $B \in \mathbb{P}(X)$ if $|A \Delta B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa.

This graph is the **discrete cube Q_n**.

Alternatively, we can view Q_n as an n-dimensional unit cube $\{0, 1\}^n$ by identifying e.g. $\{1, 3\} \subseteq [5]$ with 10100 (i.e. identify A with $\mathbb{1}_A$, the characteri stic/indicator function of A).

Definition 1.7 $\mathcal{F} \subseteq \mathbb{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{F}$, $A \subseteq B$ or $B \subseteq A$.

Example 1.8
- $\mathcal{F} = \{23, 1235, 123567\}$ is a chain.
- $\mathcal{F} = \{\emptyset, 1, 12, ..., [n]\} \subseteq \mathbb{P}([n])$ is a chain.

Definition 1.9 $\mathcal{F} \subseteq \mathbb{P}(X)$ is an **antichain** if $\forall A \neq B \in \mathcal{F}$, $A \nsubseteq B$.

Example 1.10
- $\mathcal{F} = \{23, 137\}$ is an antichain.
- $\mathcal{F} = \{1, ..., n\} \subseteq \mathbb{P}([n])$ is an antichain.
- More generally, $\mathcal{F} = X^{(r)}$ is an antichain for any r.

Proposition 1.11 A chain and an antichain can meet at most once.

Proof (Hints). Trivial. □

Proof. By definition. □

Proposition 1.12 A chain $\mathcal{F} \subseteq \mathbb{P}([n])$ can have at most $n + 1$ elements.

Proof (Hints). Trivial. □

Proof. For each $0 \le r \le n$, $\mathcal{F}$ can contain at most 1 r-set (set of size r). □

Theorem 1.13 (Sperner's Lemma) Let $\mathcal{F} \subseteq \mathbb{P}(X)$ be an antichain. Then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$, i.e. the maximum size of an antichain is achieved by the set of $X^{(\lfloor n/2 \rfloor)}$.

Proof (Hints).
- Let $r < \frac{n}{2}$.

- Let G be bipartite subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$.
- By considering an expression and upper bound for number of S-$\Gamma(S)$ edges in G for each $S \subseteq X^{(r)}$, show that there is a matching from $X^{(r)}$ to $X^{(r+1)}$.
- Reason that this induces a matching from $X^{(r)}$ to $X^{(r-1)}$ for each $r > \frac{n}{2}$.
- Reason that joining these matchings together, together with length 1 chains of subsets of $X^{(\lfloor n/2 \rfloor)}$ not included in a matching, result in a partition of $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, and conclude result from here.

□

Proof.

- We use the idea: from "a chain meets each layer in ≤ 1 points, because a layer is an antichain", we try to decompose the cube into chains.
- We partition $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, so each subset of X appears exactly once in one chain. Then we are done (since to form an antichain, we can pick at most one element from each chain).
- To achieve this, it is sufficient to find:
  - For each $r < \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r+1)}$ (a matching is a set of disjoint edges, one for each point in $X^{(r)}$).
  - For each $r > \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r-1)}$.
- Then put these matchings together to form a set of chains, each passing through $X^{(\lfloor n/2 \rfloor)}$. If a subset $X^{(\lfloor n/2 \rfloor)}$ has a chain passing through it, then this chain is unique. The subsets with no chain passing through form their own one-element chain.
- By taking complements, it is enough to construct the matchings just for $r < \frac{n}{2}$ (since a matching from $X^{(r)}$ to $X^{(r+1)}$ induces a matching from $X^{(n-r-1)}$ to $X^{(n-r)}$: there is a correspondence between $X^{(r)}$ and $X^{(n-r)}$ by taking complements, and taking complements reverse inclusion, so edges in the induced matching are guaranteed to exist).
- Let G be the (bipartite) subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$.
- For any $S \subseteq X^{(r)}$, the number of S-$\Gamma(S)$ edges in G is $|S|(n-r)$ (counting from below) since there are $n-r$ ways to add an element.
- This number is $\le |\Gamma(S)|\ (r+1)$ (counting from above), since $r+1$ ways to remove an element.
- Hence $|\Gamma(S)| \ge \frac{|S|\ (n-r)}{r+1} \ge |S|$ as $r < \frac{n}{2}$.
- So by Hall's theorem, since there is a matching from S to $\Gamma(S)$, there is a matching from $X^{(r)}$ to $X^{(r+1)}$.

□

Remark 1.14 The proof above doesn't tell us when we have equality in Sperner's Lemma.

Definition 1.15 For $\mathcal{F} \subseteq X^{(r)}$ $(1 \le r \le n)$, the **shadow** of $\mathcal{F}$ is the set of subsets which can be obtained by removing one element from a subset in $\mathcal{F}$:

$$\partial\mathcal{F} = \partial^{-}\mathcal{F} := \left\{B \in X^{(r-1)} : B \subseteq \mathcal{F} \text{ for some } A \in \mathcal{F}\right\}.$$

Example 1.16 Let $\mathcal{F} = \{123, 124, 134, 137\} \in [7]^{(3)}$. Then $\partial\mathcal{F} = \{12, 13, 23, 14, 24, 34, 17, 37\}$.

Proposition 1.17 (Local LYM) Let $\mathcal{F} \subseteq X^{(r)}$, $1 \le r \le n$. Then

$$\frac{|\partial\mathcal{F}|}{\binom{n}{r-1}} \ge \frac{|\mathcal{F}|}{\binom{n}{r}}.$$

i.e. the proportion of the level occupied by $\partial\mathcal{F}$ is at least the proportion of the level occupied by $\mathcal{F}$.

Proof (Hints). Find equation and upper bound for number of $\mathcal{F}$-$\partial\mathcal{F}$ edges in Q_n. □

Proof.
- The number of $\mathcal{F}$-$\partial\mathcal{F}$ edges in Q_n is $|A|r$ (counting from above, since we can remove any of r elements from $|A|$ sets) and is $\le |\partial\mathcal{F}|\ (n - r + 1)$ (since adding one of the $n - r + 1$ elements not in $A \in \partial\mathcal{F}$ to A may not result in a subset of $\mathcal{F}$).
- So $\frac{|\partial\mathcal{F}|}{|\mathcal{F}|} \ge \frac{r}{n-r+1} = \binom{n}{r-1}/\binom{n}{r}$.

□

Remark 1.18 For equality in Local LYM, we must have that $\forall A \in \mathcal{F}$, $\forall i \in A$, $\forall j \notin A$, we must have $(A - \{i\}) \cup \{j\} \in \mathcal{F}$, i.e. $\mathcal{F} = \emptyset$ or $X^{(r)}$ for some r.

Notation 1.19 Write $\mathcal{F}_r$ for $\mathcal{F} \cap X^{(r)}$.

Theorem 1.20 (LYM Inequality) Let $\mathcal{F} \subseteq \mathbb{P}(X)$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{F} \cap X^{(r)}|}{\binom{n}{r}} \le 1.$$

Proof (Hints).
- Method 1: show the result for the sum $\sum_{r=k}^{n}$ by induction, starting with $k = n$. Use local LYM, and that $\partial\mathcal{F}_n$ and $\mathcal{F}_{n-1}$ are disjoint (and analogous results for lower levels).
- Method 2: let $\mathcal{C}$ be uniformly random maximal chain, find an expression for $\Pr(\mathcal{C}$ meets $\mathcal{F})$.
- Method 3: determine number of maximal chains in X, determine number of maximal chains passing through a fixed r-set, deduce maximal number of chains passing through $\mathcal{F}$.

□

Proof.
- Method 1: "bubble down with local LYM".
  - We trivially have that $\mathcal{F}_n/\binom{n}{n} \le 1$.
  - $\partial\mathcal{F}_n$ and $\mathcal{F}_{n-1}$ are disjoint, as $\mathcal{F}$ is an antichain.
  - So

$$\frac{|\partial \mathcal{F}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} \leq 1.$$

- So by local LYM,

$$\frac{|\mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} \leq 1.$$

- Now, $\partial(\partial \mathcal{F}_n \cup \mathcal{F}_{n-1})$ and $\mathcal{F}_{n-2}$ are disjoint, as $\mathcal{F}$ is an antichain.
- So

$$\frac{|\partial(\partial \mathcal{F}_n \cup \mathcal{F}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- So by local LYM,

$$\frac{|\partial \mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- So

$$\frac{|\mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- Continuing inductively, we obtain the result.

- Method 2:
  - Choose uniformly at random a maximal chain $\mathcal{C}$ (i.e. $C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_n$ with $|C_r| = r$ for all r).
  - For any r-set A, $\Pr(A \in \mathcal{C}) = 1/\binom{n}{r}$, since all r-sets are equally likely.
  - So $\Pr(\mathcal{C} \text{ meets } \mathcal{F}_r) = |\mathcal{F}_r|/\binom{n}{r}$, since events are disjoint.
  - So $\Pr(\mathcal{C} \text{ meets } \mathcal{F}) = \sum_{r=0}^{n} |\mathcal{F}_r|/\binom{n}{r} \leq 1$ since events are disjoint (since $\mathcal{F}$ is an antichain).
- Method 3: equivalently, the number of maximal chains is $n!$, and the number through any fixed r-set is $r!(n-r)!$, so $\sum_r |\mathcal{F}_r| r!(n-r)! \leq n!$.

□

Remark 1.21 To have equality in LYM, we must have equality in each use of local LYM in proof method 1. In this case, the maximum r with $\mathcal{F}_r \neq \emptyset$ has $\mathcal{F}_r = X^{(r)}$. So equality holds iff $\mathcal{F} = X^{(r)}$ for some r. Hence equality in Sperner's Lemma holds iff $\mathcal{F} = X^{(\lfloor n/2 \rfloor)}$ or $\mathcal{F} = X^{(\lceil n/2 \rceil)}$.

1.2. Two total orders on $X^{(r)}$

Definition 1.22 Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **lexicographic (lex)** ordering if for some j, we have $a_i = b_i$ for all $i < j$, and $a_j < b_j$. "use small elements".

Example 1.23 The elements of $[4]^{(2)}$ in lexicographic order are $12, 13, 14, 23, 24, 34$. The elements of $[6]^{(3)}$ in lexicographic order are
$123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456$.

Definition 1.24 Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **colexicographic (colex)** order if for some j, we have $a_i = b_i$ for all $i > j$, and $a_j < b_j$. "avoid large elements".

Example 1.25 The elements of $[4]^{(2)}$ in colex order are $12, 13, 23, 14, 24, 34$. The elements of $[6]^{(3)}$ are
$123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 146, 246, 346, 156, 256, 356, 456$.

Remark 1.26 Lex and colex are both total orders. Note that in colex, $[n-1]^{(r)}$ is an initial segment of $[n]^{(r)}$ (this does not hold for lex). So we can view colex as an enumeration of $\mathbb{N}^{(r)}$.

Remark 1.27 $A < B$ in colex iff $A^c < B^c$ in lex with ground set order reversed.

Remark 1.28 By Local LYM, we know that $|\partial \mathcal{F}| \geq |\mathcal{F}| r / (n - r + 1)$. Equality is rare (only for $\mathcal{F} = X^{(r)}$ for $0 \leq r \leq n$). What happens in between, i.e., given $|\mathcal{F}|$, how should we choose $\mathcal{F}$ to minimise $|\partial \mathcal{F}|$?

You should be able to convince yourself that if $|\mathcal{F}| = \binom{k}{r}$, then we should take $\mathcal{F} = [k]^{(r)}$. If $\binom{k}{r} < |\mathcal{F}| < \binom{k+1}{r}$, then convince yourself that we should take some $[k]^{(r)}$ plus some r-sets in $[k+1]^{(r)}$.

E.g. for $\mathcal{F} \subseteq X^{(r)}$ with $|\mathcal{F}| = \binom{8}{3} + \binom{4}{2}$, take $\mathcal{F} = [8]^{(3)} \cup \{9 \cup B : B \in [4]^{(2)}\}$.

Remark 1.29 We want to show that if $\mathcal{F} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{F}|$, then $|\partial \mathcal{C}| \leq |\partial \mathcal{F}|$. In particular, if $|\mathcal{F}| = \binom{k}{r}$ (so $\mathcal{C} = [k]^{(r)}$), then $|\partial \mathcal{F}| \geq \binom{k}{r-1}$.

1.3. Compressions

Remark 1.30 We want to transform $\mathcal{F} \subseteq X^{(r)}$ into some $\mathcal{F}' \subseteq X^{(r)}$ such that:
- $|\mathcal{F}'| = |\mathcal{F}|$,
- $|\partial \mathcal{F}'| \leq |\partial \mathcal{F}|$.

Ideally, we want a family of such "compressions" $\mathcal{F} \to \mathcal{F}' \to ... \to \mathcal{B}$ such that either $\mathcal{B} = \mathcal{C}$, or $\mathcal{B}$ is similar enough to $\mathcal{C}$ that we can directly check that $|\partial \mathcal{C}| \leq |\partial \mathcal{B}|$.

Definition 1.31 Let $1 \leq i < j \leq n$. The ***ij*-compression** C_{ij} is defined as:
- For $A \in X^{(r)}$,

$$C_{ij}(A) = \begin{cases} (A \cup i) - j & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}.$$

- For $\mathcal{F} \subseteq X^{(r)}$, $C_{ij}(A) = \{C_{ij}(A) : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : C_{ij}(A) \in \mathcal{F}\}$.

"replace j by i where possible". This definition is inspired by "colex prefers $i < j$ to j". Note that $C_{ij}(\mathcal{F}) \subseteq X^{(r)}$ and $|C_{ij}(\mathcal{F})| = |\mathcal{F}|$.

Definition 1.32 $\mathcal{F}$ is ***ij*-compressed** if $C_{ij}(\mathcal{F}) = \mathcal{F}$.

Example 1.33 Let $\mathcal{F} = \{123, 134, 234, 235, 146, 567\}$, then $C_{12}(\mathcal{F}) = \{123, 134, 234, 135, 146, 567\}$.

Lemma 1.34 Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq i < j \leq n$. Then $|\partial C_{ij}(\mathcal{F})| \leq |\partial \mathcal{F}|$.

Proof (Hints).

- Let $\mathcal{F}' = C_{ij}(\mathcal{F})$, $B \in \partial\mathcal{F}' - \partial\mathcal{F}$.
- Show that $i \in B$ and $j \notin B$.
- Reason that $B \cup j - i \in \partial\mathcal{F}'$.
- Show that $B \cup j - i \notin \partial\mathcal{F}'$ by contradiction.
- Conclude the result.

□

Proof.

- Let $\mathcal{F}' = C_{ij}(\mathcal{F})$. Let $B \in \partial\mathcal{F}' - \partial\mathcal{F}$.
- We'll show that $i \in B$, $j \notin B$, $(B \cup j) - i \in \partial\mathcal{F} - \partial\mathcal{F}'$.
- $B \cup x \in \mathcal{F}'$ and $B \cup x \notin \mathcal{F}$ (since $B \notin \partial\mathcal{F}$) for some x.
- So $i \in B \cup x$, $j \notin B \cup x$, $(B \cup x \cup j) - i \in \mathcal{F}$.
- We can't have $x = i$, since otherwise $(B \cup x \cup j) - i = B \cup j$, which gives $B \in \partial\mathcal{F}$, a contradiction.
- So $i \in B$ and $j \notin B$.
- Also, $B \cup j - i \in \partial\mathcal{F}$, since $B \cup x \cup j - i \in \mathcal{F}$.
- Suppose $B \cup j - i \in \partial\mathcal{F}'$: so $(B \cup j - i) \cup y \in \mathcal{F}'$ for some y.
- We cannot have $y = i$, since otherwise $B \cup j \in \mathcal{F}'$, so $B \cup j \in \mathcal{F}$ (as $j \in B \cup j$), contradicting $B \notin \partial\mathcal{F}$.
- Hence $j \in (B \cup j - i) \cup y$ and $i \notin (B \cup j - i) \cup y$.
- Thus, both $(B \cup j - i) \cup y$ and $B \cup y = C_{ij}((B \cup j - i) \cup y)$ belong to $\mathcal{F}$ (by definition of $\mathcal{F}'$), contradicting $B \notin \partial\mathcal{F}$.

□

Remark 1.35 In the above proof, we actually showed that $\partial C_{ij}(\mathcal{F}) \subseteq C_{ij}(\partial\mathcal{F})$.

Definition 1.36 $\mathcal{F} \subseteq X^{(r)}$ is **left-compressed** if $C_{ij}(\mathcal{F}) = \mathcal{F}$ for all $i < j$.

Corollary 1.37 Let $\mathcal{F} \subseteq X^{(r)}$. Then there exists a left-compressed $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{F}|$ and $|\partial\mathcal{B}| \le |\partial\mathcal{F}|$.

Proof (Hints). Define a sequence $\mathcal{F}_0, \mathcal{F}_1, ...$ of subsets of $X^{(r)}$ with $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} i$ strictly decreasing. □

Proof.

- Define a sequence $\mathcal{F}_0, \mathcal{F}_1, ...$ as follows:
- $\mathcal{F}_0 = \mathcal{F}$. Having defined $\mathcal{F}_0, ..., \mathcal{F}_k$, if $\mathcal{F}_k$ is left-compressed the end the sequence with $\mathcal{F}_k$.
- If not, choose $i < j$ such that $\mathcal{F}_k$ is not ij-compressed, and set $\mathcal{F}_{k+1} = C_{ij}(\mathcal{F}_k)$.
- This must terminate after a finite number of steps, e.g. since $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} i$ is strictly decreasing with k.
- The final term $\mathcal{B} = \mathcal{F}_k$ satisfies $|\mathcal{B}| = |\mathcal{F}|$, and $|\partial\mathcal{B}| \le |\partial\mathcal{F}|$ by the above lemma.

□

Remark 1.38

- Another way of proving this is: among all $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{F}| = |\mathcal{F}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{F}|$, choose one with minimal $\sum_{A\in\mathcal{B}} \sum_{i\in A} i$.
- We can choose an order of the C_{ij} so that no C_{ij} is applied twice.
- Any initial segment of colex is left-compressed, but the converse is false, e.g. $\{123, 124, 125, 126\}$ is left-compressed.

Definition 1.39 Let $U, V \subseteq X$, $|U| = |V|$, $U \cap V = \emptyset$ and $\max U < \max V$. Define the ***UV*-compression** C_{UV} as:

- For $A \subseteq X$,

$$C_{UV}(A) = \begin{cases} (A - V) \cup U & \text{if } V \subseteq A, U \cap A = \emptyset \\ A & \text{otherwise} \end{cases}.$$

- For $\mathcal{F} \subseteq X^{(r)}$,

$$C_{UV}(\mathcal{F}) = \{C_{UV}(A) : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : C_{UV}(A) \in \mathcal{F}\}.$$

We have $C_{UV}(\mathcal{F}) \subseteq X^{(r)}$ and $|C_{UV}(\mathcal{F})| = |\mathcal{F}|$. This definition is inspired by "colex prefers 23 to 14".

Definition 1.40 $\mathcal{F}$ is ***UV*-compressed** if $C_{UV}(\mathcal{F}) = \mathcal{F}$.

Example 1.41 Let $\mathcal{F} = \{123, 124, 147, 237, 238, 149\}$, then $C_{23,14}(\mathcal{F}) = \{123, 124, 147, 237, 238, 239\}$.

Example 1.42 We can have $|\partial C_{UV}(\mathcal{F})| > |\partial\mathcal{F}|$. E.g. $\mathcal{F} = \{147, 157\}$ has $|\partial\mathcal{F}| = 5$, but $C_{23,14}(\mathcal{F}) = \{237, 157\}$ has $\left|\partial C_{23,14}(\mathcal{F})\right| = 6$.

Lemma 1.43 Let $\mathcal{F} \subseteq X^{(r)}$ be UV-compressed for all $U, V \subseteq X$ with $|U| = |V|$, $U \cap V = \emptyset$ and $\max U < \max V$. Then $\mathcal{F}$ is an initial segment of colex.

Proof (Hints). Suppose not, consider a compression for appropriate U and V. □

Proof.

- Suppose not, then there exists $A, B \in X^{(r)}$ with $B < A$ in colex but $A \in \mathcal{F}$, $B \notin \mathcal{F}$.
- Let $V = A \setminus B$, $U = B \setminus A$. Then $|V| = |U|$, $U \cap V = \emptyset$, and $\max V > \max U$ (since $\max(A\Delta B) \in A$, by definition of colex).
- Since $\mathcal{F}$ is UV-compressed, we have $C_{UV}(A) = B \in C_{UV}(\mathcal{F}) = \mathcal{F}$, contradiction.

□

Lemma 1.44 Let $U, V \subseteq X$, $|U| = |V|$, $U \cap V = \emptyset$, $\max U < \max V$. For $\mathcal{F} \subseteq X^{(r)}$, suppose that

$$\forall u \in U, \exists v \in V : \quad \mathcal{F} \text{ is } (U - u, V - v)\text{-compressed.}$$

Then $|\partial C_{UV}(\mathcal{F})| \leq |\partial\mathcal{F}|$.

Proof (Hints).

- Let $\mathcal{F}' = C_{UV}(\mathcal{F})$, $B \in \partial\mathcal{F}' - \partial\mathcal{F}$.
- Show that $U \subseteq B$ and $V \cap B = \emptyset$.
- Reason that $(B - U) \cup V \in \partial\mathcal{F}$.

- Show that $(B-U) \cup V \notin \partial \mathcal{F}'$ by contradiction.

$\square$

Proof.

- Let $\mathcal{F}' = C_{UV}(\mathcal{F})$. For $B \in \partial \mathcal{F}' - \partial \mathcal{F}$, we will show that $U \subseteq B$, $V \cap B = \emptyset$ and $B \cup V - U \in \partial \mathcal{F} - \partial \mathcal{F}'$, then we will be done.
- We have $B \cup x \in \mathcal{F}'$ for some $x \in X$, and $B \cup x \notin \mathcal{F}$.
- So $U \subseteq B \cup x$, $V \cap (B \cup x) = \emptyset$, and $(B \cup x \cup V) - U \in \mathcal{F}$, by definition of C_{UV}.
- If $x \in U$, then $\exists y \in V$ such that $\mathcal{F}$ is $(U - x, V - y)$-compressed, so from $(B \cup x \cup V) - U \in \mathcal{F}$, we have $B \cup y \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$.
- Thus $x \notin U$, so $U \subseteq B$ and $V \cap B = \emptyset$.
- Certainly $B \cup V - U \in \partial \mathcal{F}$ (since $(B \cup x \cup V) - U \in \mathcal{F}$), so we just need to show that $B \cup V - U \notin \partial \mathcal{F}'$.
- Assume the opposite, i.e. $(B - U) \cup V \in \partial \mathcal{F}'$, so $(B - U) \cup V \cup w \in \mathcal{F}'$ for some $w \in X$. (This also belongs to $\mathcal{F}$, since it contains V).
- If $w \in U$, then since $\mathcal{F}$ is $(U - w, V - z)$-compressed for some $z \in V$, we have $B \cup z = C_{U-w,V-z}((B - U) \cup V \cup w) \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$.
- So $w \notin U$, and since $V \subseteq (B - U) \cup V \cup w$ and $U \cap ((B - U) \cup V \cup w) = \emptyset$, by definition of C_{UV}, we must have that both $(B - U) \cup V \cup w$ and $B \cup w = C_{UV}((B - U) \cup V \cup w) \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$.

$\square$

Theorem 1.45 (Kruskal-Katona) Let $\mathcal{F} \subseteq X^{(r)}$, $1 \le r \le n$, let $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial \mathcal{C}| \le |\partial \mathcal{F}|$.

In particular, if $|\mathcal{F}| = \binom{k}{r}$, then $|\partial \mathcal{F}| \ge \binom{k}{r-1}$.

Proof (Hints).

- Let $\Gamma = \{(U, V) \in \mathbb{P}(X) \times \mathbb{P}(X) : |U| = |V| > 0, U \cap V = \emptyset, \max U < \max V\} \cup \{(\emptyset, \emptyset)\}$.
- Define a sequence $\mathcal{F}_0, \mathcal{F}_1, ...$ of UV-compressions where $(U, V) \in \Gamma$, choosing $|U| = |V| > 0$ minimal each time. Show that this (U, V) satisfies condition of above lemma.
- Reason that sequence terminates by considering $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} 2^i$.

$\square$

Proof.

- Let $\Gamma = \{(U, V) \in \mathbb{P}(X) \times \mathbb{P}(X) : |U| = |V| > 0, U \cap V = \emptyset, \max U < \max V\} \cup \{(\emptyset, \emptyset)\}$.
- Define a sequence $\mathcal{F}_0, \mathcal{F}_1, ...$ of set systems in $X^{(r)}$ as follows:
  - Let $\mathcal{F}_0 = \mathcal{F}$. Having chosen $\mathcal{F}_0, ..., \mathcal{F}_k$, if $\mathcal{F}_k$ is (UV)-compressed for all $(U, V) \in \Gamma$ then stop.
  - Otherwise, choose $(U, V) \in \Gamma$ with $|U| = |V| > 0$ minimal, such that $\mathcal{F}_k$ is not (UV)-compressed.
  - Note that $\forall u \in U$, $\exists v \in V$ such that $(U - u, V - v) \in \Gamma$ (namely $v = \min(V)$).

- So by the above lemma, $|\partial C_{UV}(\mathcal{F}_k)| \leq |\partial \mathcal{F}_k|$. Set $\mathcal{F}_{k+1} = C_{UV}(\mathcal{F}_k)$, and continue.

- The sequence must terminate, as $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} 2^i$ is strictly decreasing with k.
- The final term $\mathcal{B} = \mathcal{F}_k$ satisfies $|\mathcal{B}| = |\mathcal{F}|$, $|\partial \mathcal{B}| \leq |\partial \mathcal{F}|$, and is (UV)-compressed for all $(U, V) \in \Gamma$.
- So $\mathcal{B} = \mathcal{C}$ by lemma before previous lemma.

□

Remark 1.46
- Equivalently, if $|\mathcal{F}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \cdots + \binom{k_s}{s}$ where each $k_i > k_{i-1}$ and $s \geq 1$, then

$$|\partial \mathcal{F}| \geq \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \cdots + \binom{k_s}{s-1}.$$

- Equality in Kruskal-Katona: if $|\mathcal{F}| = \binom{k}{r}$ and $|\partial \mathcal{F}| = \binom{k}{r-1}$, then $\mathcal{F} = Y^{(r)}$ for some $Y \subseteq X$ with $|Y| = k$. However, it is not true in general that if $|\partial \mathcal{F}| = |\partial C|$, then $\mathcal{F}$ is isomorphic to $\mathcal{C}$ (i.e. there is a permutation of the ground set X sending $\mathcal{F}$ to $\mathcal{C}$).

Definition 1.47 For $\mathcal{F} \subseteq X^{(r)}$, $0 \leq r \leq n-1$, the **upper shadow** of $\mathcal{F}$ is

$$\partial^+ \mathcal{F} := \{A \cup x : A \in \mathcal{F}, x \notin A\} \subseteq X^{(r+1)}.$$

Corollary 1.48 Let $\mathcal{F} \subseteq X^{(r)}$, $0 \leq r \leq n-1$, let $\mathcal{C}$ be the initial segment of lex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial^+ \mathcal{C}| \leq |\partial^+ \mathcal{F}|$.

Proof (Hints). By Kruskal-Katona. □

Proof. By Kruskal-Katona, since $A < B$ in colex iff $A^c < B^c$ in lex with ground-set (X) order reversed, and if $\mathcal{F}' = \{A^c : A \in \mathcal{F}\}$, then $|\partial^+ \mathcal{F}'| = |\partial \mathcal{F}|$. □

Remark 1.49 The fact that the shadow of an initial segment of colex on $X^{(r)}$ is an initial segment of colex on $X^{(r-1)}$ (since if $\mathcal{C} = \left\{A \in X^{(r)} : A \leq a_1 ... a_r \text{ in colex}\right\}$, then $\partial \mathcal{C} = \left\{B \in X^{(r-1)} : B \leq a_2 ... a_r \text{ in colex}\right\}$) gives:

Corollary 1.50 Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq r \leq n$, $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial^t \mathcal{C}| \leq |\partial^t \mathcal{F}|$ for all $1 \leq t \leq r$ (where ∂^t is shadow applied t times).

Proof (Hints). Straightforward. □

Proof. If $|\partial^t \mathcal{C}| \leq |\partial^t \mathcal{F}|$, then $\left|\partial^{t+1} \mathcal{C}\right| \leq \left|\partial^{t+1} \mathcal{F}\right|$, since $\partial^t \mathcal{C}$ is an initial segment of colex. So we are done by induction (base case is Kruskal-Katona). □

Remark 1.51 So if $|\mathcal{F}| = \binom{k}{r}$, then $|\partial^t \mathcal{F}| \geq \binom{k}{r-t}$.

1.4. Intersecting families

Definition 1.52 A family $\mathcal{F} \in \mathbb{P}(X)$ is **intersecting** if for all $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

We are interested in finding intersecting families of maximum size.

Proposition 1.53 For all intersecting families $\mathcal{F} \subseteq \mathbb{P}(X)$, $|\mathcal{F}| \leq 2^{n-1} = \frac{1}{2}|\mathbb{P}(X)|$.

Proof (Hints). Straightforward. □

Proof. Given any $A \subseteq X$, at most one of A and A^c can belong to $\mathcal{F}$. □

Example 1.54
- $\mathcal{F} = \{A \subseteq X : 1 \in A\}$ is intersecting, and $|\mathcal{F}| = 2^{k-1}$.
- $\mathcal{F} = \left\{A \subseteq X : |A| > \frac{n}{2}\right\}$ for n odd.

Example 1.55 Let $\mathcal{F} \subseteq X^{(r)}$:
- If $r > \frac{n}{2}$, then $\mathcal{F} = X^{(r)}$ is intersecting.
- If $r = \frac{n}{2}$, then choose one of A and A^c for all $A \in X^{(r)}$. This gives $|\mathcal{F}| = \frac{1}{2}\binom{n}{r}$.
- If $r < \frac{n}{2}$, then $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$ has size $\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$ (since the probability of a random r-set containing 1 is $\frac{r}{n}$). If $(n, r) = (8, 3)$, then $|\mathcal{F}| = \binom{7}{2} = 21$.
- Let $\mathcal{F} = \left\{A \in X^{(r)} : |A \cap \{1, 2, 3\}| \geq 2\right\}$. If $(n, r) = (8, 3)$, then $|\mathcal{F}| = 1 + \binom{3}{2}\binom{5}{1} = 16 < 21$ (since 1 set A has $|B \cap [3]| = 3$, 15 sets A have $|A \cap [3]| = 2$).

Theorem 1.56 (Erdos-Ko-Rado) Let $\mathcal{F} \subseteq X^{(r)}$ be an intersecting family, where $r < \frac{n}{2}$. Then $|\mathcal{F}| \leq \binom{n-1}{r-1}$.

Proof (Hints).
- Method 1:
  - Let $\overline{\mathcal{F}} = \{A^c : A \in \mathcal{F}\}$. Show that $\partial^{n-2r}\overline{\mathcal{F}}$ and $\mathcal{F}$ are disjoint families of r-sets.
  - Assume the opposite, show that the size of the union of these two sets is greater than the size of $X^{(r)}$.
- Method 2:
  - Let $c : [n] \to \mathbb{Z}/n$ be bijection, i.e. cyclic ordering of $[n]$. Show there at most r sets in $\mathcal{F}$ that are intervals (sets with r consecutive elements) under this ordering.
  - Find expression for number of times an r-set in $\mathcal{F}$ is an interval all possible orderings, and find an upper bound for this using the above.

□

Proof. Proof 1 ("bubble down with Kruskal-Katona"): note that $A \cap B \neq \emptyset$ iff $A \not\subseteq B^c$. Let $\overline{\mathcal{F}} = \{A^c : A \in \mathcal{F}\} \subseteq X^{(n-r)}$. We have $\partial^{n-2r}\overline{\mathcal{F}}$ and $\mathcal{F}$ are disjoint families of r-sets (if not, then there is some $A \in \mathcal{F}$ such that $A \subseteq B^c$ for some $B \in \mathcal{F}$, but then $A \cap B = \emptyset$). Suppose $|\mathcal{F}| > \binom{n-1}{r-1}$. Then $\left|\overline{\mathcal{F}}\right| = |\mathcal{F}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$. So by Kruskal-Katona, we have $\left|\partial^{n-2r}\overline{\mathcal{F}}\right| \geq \binom{n-1}{r}$. So $|\mathcal{F}| + \left|\partial^{n-2r}\overline{\mathcal{F}}\right| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r} = \left|X^{(r)}\right|$, a contradiction, since $\mathcal{F}, \partial^{n-2r}\overline{\mathcal{F}} \subseteq X^{(r)}$.

Proof 2: pick a cyclic ordering of $[n]$, i.e. a bijection $c : [n] \to \mathbb{Z}/n$. There are at most r sets in $\mathcal{F}$ that are intervals (r consecutive elements) under this ordering: for $c_1 ... c_r \in \mathcal{F}$, for each $2 \leq i \leq r$, at most one of the two intervals $c_i ... c_{i+r-1}$ and $c_{i-r} ... c_{i-1}$ can belong to $\mathcal{F}$, since they are disjoint and $\mathcal{F}$ is intersecting (the indices of c are taken $\bmod n$). For each r-set A, out of the $n!$ cyclic orderings, there are $n \cdot r!(n-r)!$ which map A to an interval ($r!$ orderings inside A, $(n-r)!$ orderings outside A, n choices for the start of the interval). Hence, by counting the number of times an r-set in $\mathcal{F}$ is an interval under a given ordering (over all r-sets in $\mathcal{F}$ and all cyclic orderings), we obtain $|\mathcal{F}|nr!(n-r)! \le n!r$, i.e. $|\mathcal{F}| \le \binom{n-1}{r-1}$. □

Remark 1.57

- The calculation at the end of proof method 1 had to give the correct answer, as the shadow calculations would all be exact if $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$ (in this case, $\mathcal{F}$ and $\partial^{n-2r}\overline{\mathcal{F}}$ partition $X^{(r)}$).
- The calculations at the end of proof method 2 had to work out, given equality for the family $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$.
- In method 2, equivalently, we are double-counting the edges in the bipartite graph, where the vertex classes (partition sets) are $\mathcal{F}$ and all cyclic orderings, with A joined to c if A is an interval under c. This method is called **averaging** or **Katona's method**.
- Equality in Erdos-Ko-Rado holds iff $\mathcal{F} = \left\{A \in X^{(r)} : i \in A\right\}$, for some $1 \le i \le n$. This can be obtained from proof 1 and equality in Kruskal-Katona, or from proof 2.

2. Isoperimetric inequalities

We seek to answer questions of the form "how do we minimise the boundary of a set of given size?"

Example 2.1 In the continuous setting:

- Among all subsets of $\mathbb{R}^2$ of a given fixed area, the disc minimises the perimeter.
- Among all subsets of $\mathbb{R}^3$ of a given fixed volume, the solid sphere minimises the surface area.
- Among all subsets of S^2 of given fixed surface area, the circular cap minimises the perimeter.

Definition 2.2 For a A of vertices of a graph G, the **boundary** of A is

$$b(A) = \{x \in G : x \notin A, xy \in E \text{ for some } y \in A\}.$$

Definition 2.3 An **isoperimetric inequality** on a graph G is an inequality of the form

$$\forall A \subseteq G, \quad |b(A)| \ge f(|A|)$$

for some function $f : \mathbb{N} \to \mathbb{R}$.

Definition 2.4 The **neighbourhood** of $A \subseteq V(G)$ is $N(A) := A \cup b(A)$, i.e.

$$N(A) = \{x \in G : d(x, A) \le 1\}.$$

Example 2.5 A good (and natural) example for A that minimises $|b(A)|$ in the discrete cube Q_n might be a ball $B(x,r) = \{y \in G : d(x,y) \le r\}$. Let $A \subseteq \mathbb{P}(X) = V(Q_3)$, $|A| = 4$.

A good guess is that balls are best, i.e. sets of the form $B(\emptyset, r) = X^{(\leq r)} = X^{(0)} \cup \dots \cup X^{(r)}$. What if $\left|X^{(\leq r)}\right| \leq |A| \leq \left|X^{(\leq r+1)}\right|$? A good guess is take A with $X^{(\leq r)} \subsetneq A \subsetneq X^{(\leq r+1)}$. If $A = X^{(\leq r)} \cup B$, where $B \subseteq X^{(r+1)}$, then $b(A) = \left(X^{(r+1)} - B\right) \cup \partial^{+} B$, so we would take B to be an initial segment of lex by Kruskal-Katona. This motivates the following definition.

Definition 2.6 The **simplicial ordering** on $\mathbb{P}(X)$ defines $x < y$ if either $|x| < |y|$, or both $|x| = |y|$ and $x < y$ in lex.

We want to show the initial segments of the simplicial ordering minimise the boundary.

Definition 2.7 For $A \subseteq \mathbb{P}(X)$ and $1 \leq i \leq n$, the ***i*-sections** of A are the families $A_{-}^{(i)}, A_{+}^{(i)} \subseteq \mathbb{P}(X \setminus i)$, given by

$$\begin{aligned} A_{-}^{(i)} &= A_{-} := \{x \in A : i \notin x\}, \\ A_{+}^{(i)} &= A_{+} := \{x - i : x \in A, i \in x\} \end{aligned}$$

Note that $A = A_{-}^{(i)} \cup \{x \cup i : x \in A_{+}^{(i)}\}$, so we can define a family by its i-sections.

Remark 2.8 When viewing $\mathbb{P}(X)$ as the n-dimensional cube Q_n, we view the i-sections as subgraphs of the $(n-1)$-dimensional cube Q_{n-1} (which we view $\mathbb{P}(X \setminus i)$ as).

Definition 2.9 The ***i*-compression** of $A \subseteq \mathbb{P}(X)$ is the family $C_i(A) \subseteq \mathbb{P}(X)$ given by its i-sections:
- $(C_i(A))_{-}^{(i)}$ is the first $\left|A_{-}^{(i)}\right|$ elements of the simplicial order on $\mathbb{P}(X - i)$, and
- $(C_i(A))_{+}^{(i)}$ is the first $\left|A_{+}^{(i)}\right|$ elements of the simplicial order on $\mathbb{P}(X - i)$.

Note that $|C_i(A)| = |A|$, and $C_i(A)$ "looks more like" a Hamming ball than A does.

Definition 2.10 $A \subseteq \mathbb{P}(X)$ is ***i*-compressed** if $C_i(A) = A$.

Definition 2.11 A **Hamming ball** is a family $A \subseteq \mathbb{P}(X)$ with $X^{(\leq r)} \subseteq A \subseteq X^{(\leq r+1)}$ for some r.

Example 2.12 Note that a set that is i-compressed for all $i \in [n]$ is not necessarily an initial segment of simplicial, e.g. take $\{\emptyset, 1, 2, 12\}$ in Q_3. However…

Lemma 2.13 Let $B \subseteq Q_n$ be i-compressed for all $i \in [n]$ but not an initial segment of the simplicial order. Then either:
- n is odd (say $n = 2k+1$) and

$$B = X^{\leq k} \setminus \underbrace{\{k+2, k+3, \dots, 2k+1\}}_{\text{last } k\text{-set}} \cup \underbrace{\{1, 2, \dots, k+1\}}_{\text{first } (k+1)\text{-set}},$$

- or n is even (say $n = 2k$), and

$$B = X^{(<k)} \cup \left\{x \in X^{(k)} : 1 \in x\right\} \setminus \underbrace{\{1, k+2, k+3, \dots, 2k\}}_{\text{last } k\text{-set with } 1} \cup \underbrace{\{2, 3, \dots, k+1\}}_{\text{first } k\text{-set without } 1}.$$

Proof. Since B is not an initial segment of simplicial, so there exist $x < y$ (in simplicial) with $y \in B$ but $x \notin B$. For each $1 \le i \le n$, we cannot have $i \in x$ and $i \in y$ (as B is i-compressed). For the same reason, we cannot have $i \notin x$ and $i \notin y$. So $x = y^c$. Thus for each $y \in B$, there is at most one $x < y$ with $x \notin B$ (namely $x = y^c$), and for each $x \notin B$, there is at most one $y > x$ with $y \in B$ (namely $y = x^c$). So no sets lie between x and y in the simplicial ordering. So $B = \{z : z \le y\} \setminus \{x\}$, with x the predecessor of y, and $x = y^c$. Hence if $n = 2k + 1$, then x is the last k-set (otherwise sizes of x and $y = x^c$ don't match), and if $n = 2k$, then x is the last k-set containing 1. □

Theorem 2.14 (Harper) Let $A \subseteq V(Q_n)$ and let C be the initial segment of the simplicial order on $\mathbb{P}(X) = V(Q_n)$, with $|C| = |A|$. Then $|N(A)| \ge |N(C)|$. So initial segments of the simplicial order minimise the boundary. In particular, if $|A| = \sum_{i=0}^{r} \binom{n}{i}$, then $|N(A)| \ge \sum_{i=0}^{r+1} \binom{n}{i}$.

Proof (Hints).
- Using induction, prove the claim that $|N(C_i(A))| \le |N(A)|$:
  - Find expressions for $N(A)_-$ as union of two sets, similarly for $N(A)_+$, same for $N(B)_-$ and $N(B)_+$.
  - Explain why $N(B_-)$ and B_+ are nested, use this to show $|N(B_-) \cup B_+| \le |N(A_-) \cup A_+|$.
  - Do the same with the $+$ and $-$ switched.

□

Proof. By induction on n. $n = 1$ is trivial. Given $n > 1$, $A \subseteq Q_n$ and $1 \le i \le n$, we claim that $|N(C_i(A))| \le |N(A)|$.

Proof of claim. Write $B = C_i(A)$. We have $N(A)_- = N(A_-) \cup A_+$, and $N(A)_+ = N(A_+) \cup A_-$. Similarly, $N(B)_- = N(B_-) \cup B_+$, and $N(B)_+ = N(B_+) \cup B_-$.

Now $|B_+| = |A_+|$ by definition of B, and by the inductive hypothesis, $|N(B_-)| \le |N(A_-)|$ (since $C_i(A_-) = B_-$). But B_+ is an initial segment of the simplicial ordering, and $N(B_-)$ is as well (since the neighbourhood of an initial segment of the simplicial ordering is also an initial segment). So B_+ and $N(B_-)$ are nested (one is contained in the other). Hence, $|N(B_-) \cup B_+| \le |N(A_-) \cup A_+|$.

Similarly, $|B_-| = |A_-|$ by definition of B. Since B_+ and $C_i(A_+)$ are both initial segments of size $|B_+| = |A_+|$, we have $B_+ = C_i(A_+)$, hence by the inductive hypothesis, $|N(B_+)| \le |N(A_+)|$. B_- and $N(B_+)$ are initial segments, so are nested. Hence $|N(B_+) \cup B_-| \le |N(A_+) \cup A_-|$.

This gives $|N(B)| = |N(B)_-| + |N(B)_+| \le |N(A)_-| + |N(A)_+| = |N(A)|$, which proves the claim.

Define a sequence $A_0, A_1, \ldots \subseteq Q_n$ as follows:
- Set $A_0 = A_1$.

- having chosen $A_0, ..., A_k$, if A_k is i-compressed for all $i \in [n]$, then end the sequence with A_k. If not, pick i with $C_i(A_k) \neq A_k$ and set $A_{k+1} = C_i(A_k)$, and continue.

The sequence must terminate, since $\sum_{x \in A_k}$(position of x in simplicial order) is strictly decreasing. The final family $B = A_k$ satisfies $|B| = |A|$, $|N(B)| \leq |N(A)|$, and is i-compressed for all $i \in [n]$.

So we are done by above lemma, since in each case certainly we have $|N(B)| > |N(C)|$. □

Remark 2.15
- If A was a Hamming ball, then we would be already done by Kruskal-Katona.
- Conversely, Harper's theorem implies Kruskal-Katona: given $B \subseteq X^{(r)}$, apply Harper's theorem to $A = X^{(\leq r-1)} \cup B$.
- We could also prove Harper's theorem using UV-compressions.
- Conversely, we can also prove Kruskal-Katona using these "codimension 1" compressions.

Definition 2.16 For $A \subseteq Q_n$ and $t \in \mathbb{N}$, the ***t*-neighbourhood** of A is

$$A_{(t)} = N^t(A) := \{x \in Q_n : d(x, A) \leq t\}.$$

Corollary 2.17 Let $A \subseteq Q_n$ wiht $|A| \geq \sum_{i=0}^{r} \binom{n}{i}$. Then

$$\forall t \leq n-r, \quad |N^t(A)| \geq \sum_{i=0}^{r-t} \binom{n}{i}.$$

Proof. By Harper's theorem and induction on t. □

Remark 2.18 To get a feeling for the strength of the above corollary, we'll need some estimates on quantities such as $\sum_{i=0}^{r} \binom{n}{i}$. Note that $i = n/2$ maximises $\binom{n}{i}$, while $i = (1/2 - \varepsilon)n$ makes it small: we are going $\varepsilon\sqrt{n}$ standard deviations away from the mean $n/2$.

Proposition 2.19 Let $0 < \varepsilon < 1/k$. Then

$$\sum_{i=0}^{\lfloor (1/2-\varepsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2} \cdot 2^n.$$

For ε fixed and $n \to \infty$, the upper bound is an exponentially small fraction of 2^n.

Proof. For $0 \leq i \leq \lfloor (1/2 - \varepsilon)n \rfloor$,

$$\binom{n}{i-1} / \binom{n}{i} = \frac{i}{n-i+1} \leq \frac{(1/2-\varepsilon)n}{(1/2+\varepsilon)n} = \frac{1/2-\varepsilon}{1/2+\varepsilon} = 1 - \frac{2\varepsilon}{1/2+\varepsilon} \leq 1 - 2\varepsilon.$$

Hence

$$\sum_{i=0}^{\lfloor(1/2-\varepsilon)n\rfloor}\binom{n}{i} \le \frac{1}{2\varepsilon}\binom{n}{\lfloor(1/2-\varepsilon)n\rfloor}$$

(since this is the sum of geometric progression). The same argument tells us that

$$\binom{n}{\lfloor(1/2-\varepsilon)n\rfloor} \le \binom{n}{\lfloor 1/2-\varepsilon/2\rfloor n}\left(1-2\frac{\varepsilon}{2}\right)^{\varepsilon n/2-1} \le 2^n\cdot 2(1-\varepsilon)^{\varepsilon n/2} \le 2^n\cdot 2e^{-\varepsilon^2 n/2}$$

since $1-\varepsilon\le e^{-\varepsilon}$ (we include -1 in the exponent due to taking floors). Then

$$\sum_{i=0}^{\lfloor(1/2-\varepsilon)n\rfloor}\binom{n}{i} \le \frac{1}{2\varepsilon}\cdot 2e^{-\varepsilon^2 n/2}\cdot 2^n.$$

□

Theorem 2.20 Let $0<\varepsilon<1/4$, $A\subseteq Q_n$. If $|A|/2^n\ge 1/2$, then

$$|N^{\varepsilon t}(A)| \ge 1-\frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}.$$

So $\frac{1}{2}$-sized sets have exponentially large εn-neighbourhoods.

Proof. It is enough to show that if εn is an integer then $|N^{\varepsilon n}(A)|/2^n \ge 1-\frac{1}{\varepsilon}e^{-\varepsilon^2 n/2}$. We have $|A|\ge\sum_{i=0}^{\lceil n/2-1\rceil}\binom{n}{i}$, so by Harper's theorem, we have $|A_{\varepsilon n}|\ge \sum_{i=0}^{\lceil n/2-1+\varepsilon n\rceil}\binom{n}{i}$, so $|N^{\varepsilon n}(A)^c| \le \sum_{i=\lceil n/2+\varepsilon n\rceil}\binom{n}{i} = \sum_{i=0}^{\lceil n/2-\varepsilon n\rceil}\binom{n}{i} \le \frac{1}{\varepsilon}e^{-\varepsilon^2 n/2}\cdot 2^n$. □

Remark 2.21 The same argument would give a result for "small" sets: if $|A|/2^n \ge \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$, then $\left|N^{2\varepsilon n}(A)\right|/2^n \ge 1-\frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$.

Definition 2.22 $f:Q_n\to\mathbb{R}$ is **Lipschitz** if for all adjacent $x,y\in Q_n$, $|f(x)-f(y)|\le 1$.

Definition 2.23 For $f:Q_n\to\mathbb{R}$, we say $M\in\mathbb{R}$ is a **Levy mean** (or **median**) of f if $|\{x\in Q_n : f(x)\le M\}|\ge 2^{n-1}$ and $|\{x\in Q_n : f(x)\ge M\}|\ge 2^{n-1}$.

Theorem 2.24 (Concentration of Measure Phenomenon) Let $f:Q_n\to\mathbb{R}$ be Lipschitz with median M. Then for all $0<\varepsilon<\frac{1}{4}$,

$$\frac{|\{x\in Q_n : |f(x)-M|\le\varepsilon n\}|}{2^n} \ge 1-\frac{4}{\varepsilon}e^{-\varepsilon^2 n/2}.$$

So "every well-behaved function on the cube Q_n is roughly constant nearly everywhere".

Proof. Let $A=\{x\in Q_n : f(x)\le M\}$. Then by definition, $|A|/2^n\ge\frac{1}{2}$, so by the above theorem, $|N^{\varepsilon n}(A)|/2^n\ge 1-\frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$. But f is Lipschitz, so $x\in N^{\varepsilon n}(A)\implies f(x)\le M+\varepsilon n$. Thus $|\{x\in Q_n : f(x)\le M+\varepsilon n\}|/2^n\ge 1-\frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$.

Similarly, $|\{x\in Q_n : f(x)\ge M-\varepsilon n\}|/2^b\ge 1-\frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$. Hence, taking the intersection, we have

$$\frac{|\{x \in Q_n : M - \varepsilon n \le f(x) \le M + \varepsilon n\}|}{2^n} \ge 1 - \frac{4}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

□

Definition 2.25 The **diameter** of a graph $G = (V, E)$ is $\max\{d(x, y) : x, y \in V\}$.

Definition 2.26 Let G be a graph of diameter D. Write

$$\alpha(G, \varepsilon) = \max\left\{1 - \frac{|N^{\varepsilon D}(A)|}{|G|} : A \subseteq G, \frac{|A|}{|G|} \ge \frac{1}{2}\right\}.$$

So if $\alpha(G, \varepsilon)$ is small, then "$\frac{1}{2}$-sized sets have large εD-neighbourhoods".

Definition 2.27 A sequence of graphs $(G_n)_{n \in \mathbb{N}}$ is a **Levy family** if for all $\varepsilon > 0$, $\alpha(G_n, \varepsilon) \to 0$ as $n \to \infty$. It is a **normal Levy family** if for all $\varepsilon > 0$, $\alpha(G_n, \varepsilon)$ decays exponentially with n.

Example 2.28 By the above theorem, the sequence (Q_n) is a normal Levy family. More generally, we have concentration of measure for any Levy family.

Example 2.29 Many naturally-occurring families of graphs are Levy families, e.g. (S_n), where the permutation group S_n is made into a graph by including an edge between σ and τ if $\tau\sigma^{-1}$ is a transposition.

Example 2.30 Similarly, we can define $\alpha(X, \varepsilon)$ for any metric measure space X (of finite measure and finite diameter). E.g. the sequence of spheres (S^n) is a Levy family. To prove this, we have:
1. An isoperimetric inequality on S^n: for $A \subseteq S^n$ and C a circular cap with $|C| = |A|$, we have $|N^\varepsilon(A)| \ge |N^\varepsilon(C)|$.
2. An estimate: a circular cap C of measure $1/2$ if the cap of angle $\pi/2$. So $N^\varepsilon(C)$ is the circular cap of angle $\pi/2 + \varepsilon$. This has measure roughly equal to $\int_\varepsilon^{\pi/2} \cos^{n-1}(t) \,\mathrm{d}t \to 0$ as $n \to \infty$.

Remark 2.31 We deduced concentration of measure from an isoperimetric inequality. Conversely, we have:

Proposition 2.32 Let G be a graph such that for any Lipschitz function $f : G \to \mathbb{R}$ with median M, we have $|\{x \in G : |f(x) - M| > t\}|/|G| \le \alpha$ for some given t, α. Then for all $A \subseteq G$ with $|A|/|G| \ge 1/2$, we have $|N^t(A)|/|G| \ge 1 - \alpha$.

Proof. The function $f(x) = d(x, A)$ is Lipschitz, and has 0 as a median. So $|\{x \in G : x \notin N^t(A)\}|/|G| \le \alpha$. □

2.1. Concentration of measure

3. Intersecting families