

# 1. The complex plane and Riemann sphere

- $\mathbb{C}^* = \mathbb{C} - \{0\}$
- $z_1 z_2 = 0 \iff z_1 = 0 \text{ or } z_2 = 0$ .
- $|z| = \sqrt{z \bar{z}}$ .
- $\operatorname{Re}(z) = (z + \bar{z}) / 2$ ,  $\operatorname{Im}(z) = (z - \bar{z}) / 2i$ .
- $z^{-1} = \bar{z} / |z|^2$ .
- **Principal value of  $\arg(z)$** : in interval  $(-\pi, \pi]$ , written  $\operatorname{Arg}(z)$ .
- $\arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2) \pmod{2\pi}$ .
- $\arg(1/z) = -\arg(z) \pmod{2\pi}$ .
- $\arg(\bar{z}) = -\arg(z) \pmod{2\pi}$ .
- Multiplication by  $z_1 = r_1 e^{i\theta_1}$  represents rotation by  $\theta_1$  followed by dilation by factor  $r_1$ .
- Addition represents translation.
- Conjugation represents reflection in the real axis.
- Taking the real (imaginary) part represents projection onto the real (imaginary) axis.
- $|z_1 z_2| = |z_1| |z_2|$ .
- $(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$ .
- **Triangle inequality**:  $|z_1 + z_2| \leq |z_1| + |z_2|$ .
- $|z| \geq 0$  and  $|z| = 0 \iff z = 0$ .
- $\max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\} \leq |z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$ .
- **Complex exponential function**:

$$\exp(z) := e^x (\cos(y) + i \sin(y))$$

- $\forall z \in \mathbb{C}, e^z = 0$ .
- $e^{z_1 + z_2} = e^{z_1} e^{z_2}$ .
- $e^z = 1 \iff z = 2\pi i k$  for some  $k \in \mathbb{Z}$ .
- $e^{-z} = 1 / e^z$ .
- $|e^z| = e^{\operatorname{Re}(z)}$ .
- $\forall k \in \mathbb{Z}, \exp(z) = \exp(z + 2k\pi i)$ .
- $$\sin(z) := \frac{1}{2i}(e^{iz} - e^{-iz}), \quad \cos(z) := \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sinh(z) := \frac{1}{2}(e^z + e^{-z}), \quad \cosh(z) := \frac{1}{2}(e^z + e^{-z})$$
- For every  $w \in \mathbb{C}^*$ ,

$$e^z = w = |w| e^{i\varphi}$$

has solutions

$$z = \log(|w|) + i(\varphi + 2k\pi), \quad k \in \mathbb{Z}$$

- Let  $\theta_2 - \theta_1 = 2\pi$ , let  $\arg$  be the argument function in  $(\theta_1, \theta_2]$ . Then

$$\log(z) := \log(|z|) + i \arg(z)$$

is a **branch of logarithm**. Jump discontinuity on **branch cut**, the ray  $R_{\theta_1} = R_{\theta_2}$ .

- **Principal branch of log**: where  $\arg(z) = \operatorname{Arg}(z) \in (-\pi, \pi]$ .

- $e^{\log(z)} = z$ .
- Generally,  $\log(zw) \neq \log(z) + \log(w)$ .
- Generally,  $\log(e^z) \neq z$ .
- Given a branch of  $\log$ , **power function** is

$$z^w := \exp(w \log(z))$$

- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ .
- Unit sphere:  $S^2 = \{(x, y, s) \in \mathbb{R}^3 : x^2 + y^2 + s^2 = 1\}$ , north pole:  $N = (0, 0, 1) \in S^2$ .  
**Stereographic projection**: map that takes  $v \in S^2 - \{N\}$  to  $x + iy \in \mathbb{C}$ , where  $(x, y)$  is where the line from  $N$  through  $v$  intersects the  $(x, y)$ -plane.
- **Stereographic projection formula**:

$$P(x, y, s) = \frac{x}{1-s} + \frac{iy}{1-s}$$

North pole is mapped to  $\infty$ .

- Inverse of stereographic projection found by intersection of line (from  $z \in \mathbb{C}$  to  $N$ ) and  $S^2$ .
- **Riemann sphere**: unit sphere  $S^2$  with stereographic projections from north and south pole.

## 2. Metric spaces

- **Metric space**: set  $X$  and **metric** function  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ , for every  $x, y, z \in X$ 
  - **positivity**:  $d(x, y) \geq 0$  and  $d(x, y) = 0 \iff x = y$
  - **symmetry**:  $d(x, y) = d(y, x)$
  - **triangle inequality**:  $d(x, y) \leq d(x, z) + d(z, y)$
- **norm** on vector space  $V$ :
  - $\|v\| \geq 0$  and  $\|v\| = 0 \iff v = 0$
  - $\|\lambda v\| = |\lambda| \cdot \|v\|$
  - $\|v + w\| \leq \|v\| + \|w\|$
- $d(v, w) = \|v - w\|$  always defines a metric
- $d(v, w) = \sqrt{\langle v - w, v - w \rangle}$
- **$l_p$  norm**:

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$$

- **Taxicab norm**:  $l_1$  norm
- **$l_\infty$  norm (sup-norm)**:  $\|x\|_\infty := \max_{i=1, \dots, n} |x_i|$
- **Riemannian (chordal) metric on  $\hat{\mathbb{C}}$** :  $d(z, w) = \|f(z) - f(w)\|_2$  where  $f : \hat{\mathbb{C}} \rightarrow S^2$  is the inverse stereographic projection.
- **Discrete metric**:

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

- **Open ball of radius  $r$  centred at  $x$** :  $B_r(x) := \{y \in X : d(x, y) < r\}$

- **Closed ball of radius  $r$  centred at  $x$ :**  $\overline{B}_r(x) := \{y \in X : d(x, y) \leq r\}$
- $U \subseteq X$  **open** if  $\forall x \in U, \exists \varepsilon > 0, B_\varepsilon(x) \subset U$
- $U \subseteq X$  **closed** if  $X - U$  open
- **clopen**: open and closed, e.g. empty set and  $X$
- Open balls are open
- Closed balls are closed
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open
- Finite unions of closed sets are closed
- Arbitrary intersections of closed sets are closed
- **Interior of  $A$ :**  $A^0 := \{x \in A : \text{for some open } U \subseteq A, x \in U\}$ . It is the largest open set in  $A$ .
- **Closure of  $A$ :** complement of interior of complement:  
 $\overline{A} := \{x \in X : U \cup A \neq \emptyset \text{ for every open set } U \text{ with } x \in U\} = X - (X - A)^0$ . It is the smallest closed set containing  $A$ .
- **Boundary of  $A$ :** closure without interior:  $\partial A := \overline{A} - A^0$
- **Exterior of  $A$ :** complement of closure:  $A^e := X - \overline{A} = (X - A)^0$
- $A$  is open  $\iff \partial A \cap A = \emptyset \iff A = A^0$
- $A$  is closed  $\iff \partial A \subseteq A \iff A = \overline{A}$
- For simple sets in  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , closure (or interior) is obtained by replacing  $<$  by  $\leq$  (or vice versa).
- Sequence  $\{x_n\}$  **converges to**  $x \in X$  if  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$  or equivalently,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, d(x_n, x) < \varepsilon$$

- Limits in the complex plane follow COLT rules
- $\{z_n\}$  converges iff  $\{\operatorname{Re}(z_n)\}$  and  $\{\operatorname{Im}(z_n)\}$  converge.
- $\lim_{n \rightarrow \infty} x_n = x \iff \forall \text{ open } U \text{ with } x \in U, \exists N \in \mathbb{N}, \forall n > N, x_n \in U$
- $f : (X_1, d_1) \rightarrow (X_2, d_2)$  is **continuous at**  $x_0 \in X_1$  if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X_1, d_1(x, x_0) < \delta \implies d_2(f(x), f(x_0)) < \varepsilon$$

- $f$  is **continuous on**  $X_1$  if continuous at every  $x_0 \in X_1$
- Products, sums and quotients of real/complex continuous functions are continuous
- Compositions of continuous functions are continuous
- **Preimage:**  $f^{-1}(U) := \{x \in X_1 : f(x) \in U\}$
- **Properties of preimage:**
  - $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
  - $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
  - $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$
- $f : X_1 \rightarrow X_2$  continuous  $\iff f^{-1}(U)$  open in  $X_1 \forall$  open  $U \subseteq X_2$   
 $\iff f^{-1}(F)$  closed in  $X_1 \forall$  closed  $F \subseteq X_2$
- $f : X_1 \rightarrow X_2$  continuous at  $x \in X_1 \iff f^{-1}(U)$  open in  $X_1 \forall$  open  $U \subseteq X_2$  containing  $f(x)$
- Non-empty  $K \subseteq X$  **compact** if for every sequence  $\{x_k\}$  in  $K$ , there exists a convergent subsequence  $\{x_{n_k}\}$  with limit in  $K$ .

- If  $\{x_k\}$  is a convergent sequence in  $X$  then every subsequence  $\{x_{n_k}\}$  converges to the same limit.
- $F \subseteq X$  is closed iff every sequence in  $F$  converging in  $X$  also converges in  $F$ .
- Compact sets are closed
- Every closed subset of a compact set is compact
- $A \subseteq X$  **bounded** if for some  $R > 0, x \in X, A \subseteq B_R(x)$
- Compact sets are bounded
- **Heine-Borel for  $\mathbb{C}$ :**  $K \subseteq \mathbb{C}$  is compact iff  $K$  is closed and bounded.
- $f : X \rightarrow Y$  is continuous at  $x \in X$  iff

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

for every convergent sequence  $\{x_n\}$  in  $X$  with  $x_n \rightarrow x$ .

- If  $K \subseteq X$  is compact and  $f : X \rightarrow Y$  is continuous, then  $f(K)$  is compact in  $Y$ . So for  $Y = \mathbb{R}$ , any continuous real-valued function attains maxima and minima on compact sets.

### 3. Complex differentiation

- $f : U \rightarrow \mathbb{C}$  for open  $U$  is **complex differentiable at  $z_0 \in U$**  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Limit is the **derivative of  $f$  at  $z_0$** :

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

.  $h \in \mathbb{C}$  so limit must exist from every direction.

- Complex differentiability at  $z_0$  implies continuity at  $z_0$ .
- Sums, products and quotients of complex differentiable functions are complex differentiable.
- Compositions of complex differentiable functions are complex differentiable.
- The product, quotient and chain rules hold for complex differentiable functions.
- Most non-constant purely real/imaginary functions are not complex differentiable.
- If  $f = u + iv$  is complex differentiable at  $z_0$  then  $u_x, u_y, v_x, v_y$  exist at  $z_0$  and satisfy **Cauchy-Riemann equations**:

$$u_x(z_0) = v_y(z_0), \quad u_y(z_0) = -v_x(z_0)$$

. Also,

$$f'(z_0) = u_x(z_0) + iv_x(z_0)$$

- Let  $f : U \rightarrow \mathbb{C}, U$  open,  $f = u + iv$ . If  $u_x, u_y, v_x, v_y$  exist and are continuous at  $z_0$  and satisfy the Cauchy-Riemann equations at  $z_0$ , then  $f$  is complex differentiable at  $z_0$ .
- Let  $U \subseteq \mathbb{C}$  open,  $f : U \rightarrow \mathbb{C}$ .  $f$  is **holomorphic on  $U$**  if  $f$  is complex differentiable at every  $z_0 \in U$ .
- $f$  is **holomorphic at  $z_0 \in U$**  if  $f$  is complex differentiable on some  $B_\varepsilon(z_0)$ .
- Affine linear maps  $z \rightarrow az + b, a \neq 0$  are holomorphic.

- **Path (curve) from  $a$  to  $b$ :** continuous function  $\gamma : [0, 1] \rightarrow \mathbb{C}$  with  $\gamma(0) = a$  and  $\gamma(1) = b$ . Path **closed** if  $a = b$ .
- **Smooth** path: continuously differentiable.
- $U \subseteq \mathbb{C}$  **path-connected** if for every  $a, b \in U$ , there exists a path  $\gamma$  from  $a$  to  $b$  with  $\gamma(t) \in U$  for every  $t \in [0, 1]$ .
- **Domain (region):** open and path-connected.
- **Chain rule:** Let  $U \subseteq \mathbb{C}$  open,  $f : U \rightarrow \mathbb{C}$  holomorphic,  $\gamma : [0, 1] \rightarrow U$  smooth path. Then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0)$$

- Let  $D$  domain,  $f : D \rightarrow \mathbb{C}$  holomorphic on  $D$ . If  $\forall z \in D, f'(z) = 0$ , or  $f$  is purely real/imaginary, or  $f$  has constant real/imaginary part, or  $f$  has constant modulus, then  $f$  is constant on  $D$ .
- Let  $D$  domain,  $f : D \rightarrow \mathbb{C}$  **conformal at  $z_0$**  if  $f$  preserves angle and orientation of any two tangent vectors at  $z_0$ . Equivalently,  $f$  preserves angle and orientation of any two smooth paths through  $z_0$ .  $f$  **conformal** if conformal at every  $z_0 \in D$ .
- If  $f$  holomorphic,  $f'(z_0) \neq 0$  then  $f$  conformal at  $z_0$ .
- $f$  transforms the tangent vector  $\gamma'(t_0)$  by multiplying it by  $f'(\gamma(t_0))$ .
- If  $f$  is conformal at  $z_0$ , then  $f$  is complex differentiable at  $z_0$  and  $f'(z_0) \neq 0$ .
- $f$  is conformal on domain  $D$  iff  $f$  is holomorphic on  $D$  and  $\forall z \in D, f'(z) \neq 0$ .
- Conformal maps map orthogonal grids in the  $(x, y)$ -plane to orthogonal grids. (Grids can be made of arbitrary smooth curves, not necessarily straight lines).
- For  $D$  and  $D'$  domains,  $f : D \rightarrow D'$  is **biholomorphic** if  $f$  holomorphic, bijection and  $f^{-1} : D' \rightarrow D$  holomorphic.  $f$  is a **biholomorphism**.  $D$  and  $D'$  are **biholomorphic** if such an  $f$  exists and write  $f : D \sim_{\rightarrow} D'$
- Affine linear maps  $z \rightarrow az + b, a \neq 0$ , are biholomorphic from  $\mathbb{C}$  to  $\mathbb{C}$ .
- For  $D$  domain, set of all biholomorphic maps from  $D$  to  $D$  forms a group under composition, called **automorphism group of  $D$** ,  $\text{Aut}(D)$ .

## 4. Möbius transformations

- $\text{GL}_2(\mathbb{C}) := \{A \in M_2(\mathbb{C}) : \det(A) \neq 0\}$ .
- Let  $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{C})$ , then **Möbius transformation** is  $M_T(z) = \infty$  if  $cz + d = 0$ , else

$$M_T(z) = \frac{az + b}{cz + d}$$

Also

$$M_T(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}$$

So  $M_T : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ .

- Let  $k^2 = \det(T)$  then

$$M_{\frac{1}{k}T}(z) = \frac{\frac{az}{k} + \frac{b}{k}}{\frac{cz}{k} + \frac{d}{k}} = \frac{az + b}{cz + d} = M_T(z)$$

so any  $T$  can be scaled to  $T' = \frac{1}{k}T$  so that  $\det(T') = \det(\frac{1}{k}T) = \frac{1}{k^2} \det(T) = 1$ .

- **Cayley map:**  $M_T(z) = \frac{z-i}{z+i}$  where  $T = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ .
- Cayley map maps  $\mathbb{H} \rightarrow \mathbb{D}$ .
- Set of Mobius transformations forms group under composition:
  - $M_{T_1} \circ M_{T_2} = M_{T_1 T_2}$ .
  - $(M_T)^{-1} = M_{T^{-1}}$ .
  - $M_T = \text{Id} \iff T = t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, t \in \mathbb{C}^*$ .
- Let  $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{C})$ . If  $c = 0$ ,  $M_T$  is biholomorphic from  $\hat{\mathbb{C}}$  to  $\hat{\mathbb{C}}$ . If  $c \neq 0$ ,  $M_T$  is biholomorphic from  $\mathbb{C} - \{-\frac{d}{c}\}$  to  $\mathbb{C} - \{\frac{a}{c}\}$ .
- $M_T$  conformal at every  $z \in \mathbb{C}$  where  $M_T(z) \neq \infty$ .
- $M_T$  is bijection from  $\hat{\mathbb{C}}$  to  $\hat{\mathbb{C}}$ .
- $z$  is **fixed point** of  $M_T$  if  $M_T(z) = z$ .
- If  $M_T$  is not identity map, then it has at most 2 fixed points in  $\hat{\mathbb{C}}$ . So if  $M_T$  has 3 fixed points in  $\hat{\mathbb{C}}$ , it is identity map.
- **Cross ratio** of distinct  $z_0, z_1, z_2, z_3 \in \mathbb{C}$ :

$$(z_0, z_1; z_2, z_3) := \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

If some  $z_i = \infty$  then same definition but remove all differences involving  $z_i$ , so

$$(\infty, z_1; z_2, z_3) := \frac{(z_1 - z_3)}{(z_1 - z_2)}$$

- **Three points theorem:** Let  $\{z_1, z_2, z_3\}, \{w_1, w_2, w_3\}$  be sets of distinct ordered points in  $\hat{\mathbb{C}}$ . Then exists unique Mobius transformation  $f$  such that  $f(z_i) = w_i, i = 1, 2, 3$ , given by  $F^{-1} \circ G$ , where

$$F(z) = (z, w_1; w_2, w_3), \quad G(z) = (z, z_1; z_2, z_3)$$

- **Mobius transformations preserve cross ratio:** For Mobius transformation  $f$ ,

$$(f(z_0), f(z_1); f(z_2), f(z_3)) = (z_0, z_1; z_2, z_3)$$

- **Strategy to find Mobius transformation from how it acts on three points:** since cross-ratio preserved, rearrange the equation

$$(f(z), w_1; w_2, w_3) = (z, z_1; z_2, z_3)$$

- **Strategy to find image of domain  $D$  under  $M_T$ :**
  - Find image of boundary:  $M_T(\partial D)$ .
  - Find image of point  $z_0 \in D$  in interior:  $M_T(z_0)$ .
  - Image  $D'$  is domain bounded by  $M_T(\partial D)$  and containing  $M_T(z_0)$ .
- **Circline:** circle or line.
- Mobius transformations map circlines in  $\hat{\mathbb{C}}$  to circlines in  $\hat{\mathbb{C}}$ .
- **Equations of circles and lines in  $\mathbb{C}$ :**

$$\gamma z \bar{z} - \alpha \bar{z} - \bar{\alpha} z + \beta = 0$$

is equation of circle if  $\gamma = 1$  and  $|\alpha|^2 - \beta > 0$ , and equation of line if  $\gamma = 0$  and  $\alpha \neq 0$ . Also, any circle or line can be described by this equation.

- Circle uniquely determined by three points, line determined by two points, so to determine how Mobius transformation maps circle, check where three points on circle are mapped.
- Circles through  $N$  in  $S^2$  correspond to lines in  $\hat{\mathbb{C}}$ . Circles not through  $N$  correspond to circles in  $\hat{\mathbb{C}}$  (via stereographic projection).
- For domain  $D$ ,  $\text{Mob}(D)$  is set of Mobius transformations that map  $D$  to  $D$ .
- **H2H:**

$$f \in \text{Mob}(\mathbb{H}) \iff f = M_T, \quad T \in \text{SL}_2(\mathbb{R}) := \{A \in M_2(\mathbb{R}) : \det(A) = 1\}$$

- **D2D:**

$$f \in \text{Mob}(\mathbb{D}) \iff f = M_T, \quad T \in \text{SU}(1, 1) := \left\{ A = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C}, \det(A) = 1 \right\}$$

- **D2D\*:**

- Every  $f \in \text{Mob}(\mathbb{D})$  is of form

$$f(z) = e^{i\theta} \frac{z - z_0}{\bar{z}_0 z - 1}$$

where  $z_0 \in \mathbb{D}$  is unique point such that  $f(z_0) = 0$ .

- Every  $f \in \text{Mob}(\mathbb{D})$  where  $f(0) = 0$  is a rotation about 0.
- **Strategy to find biholomorphic map between two domains:** build up biholomorphic map from simpler known ones, e.g. Mobius transformations, Cayley map, translations.

## 5. Notions of convergence in complex analysis and power series

- For  $X$  and  $Y$  metric spaces,  $\{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$  **converges pointwise on  $X$  to  $f$**  if

$$\forall x \in X, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \quad d_Y(f_n(x), f(x)) < \varepsilon$$

$f(x) = \lim_{n \rightarrow \infty} f_n(x)$  is **limit function**.

- $\{f_n\}_{n \in \mathbb{N}}$  **converges uniformly on  $X$  to  $f$**  if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \forall x \in X, \quad d_Y(f_n(x), f(x)) < \varepsilon$$

- Uniform convergence implies pointwise convergence.
- **Uniform limits of continuous functions are continuous:** let  $\{f_n\}_{n \in \mathbb{N}}$  be all continuous on  $X$  and converge uniformly to  $f$  on  $X$ . Then  $f$  is continuous on  $X$ .
- **Test for uniform convergence:** let  $\{f_n\} : X \rightarrow \mathbb{C}$  converge pointwise to  $f$ .
  - If  $\forall x \in X, |f_n(x) - f(x)| \leq s_n$ ,  $\{s_n\}$  is sequence with  $\lim_{n \rightarrow \infty} s_n = 0$ , then  $\{f_n\}$  converges uniformly to  $f$  on  $X$ .

- If for some sequence  $\{x_n\} \subset X$ ,  $|f_n(x_n) - f(x_n)| \geq c$  for some  $c > 0$ , then  $f_n$  does not converge uniformly to  $f$  on  $X$ .
- **Weierstrass M-test:** Let  $\{f_n\} : X \rightarrow \mathbb{C}$  satisfy

$$\forall x \in X, |f_n(x)| \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty$$

Then  $\sum_{n=1}^{\infty} f_n$  converges uniformly to some  $f$  on  $X$ .

- Let  $\{f_n\} : [a, b] \rightarrow \mathbb{R}$  be continuous and converge uniformly to  $f$  on  $[a, b]$ . Then

$$\forall c \in [a, b], \quad \lim_{n \rightarrow \infty} \int_a^c f_n(x) dx = \int_a^c f(x) dx$$

- $\{f_n\}$  **converges locally uniformly on  $X$  to  $f$**  if  $\forall x \in X$ , exists open  $U \subset X$  containing  $x$  such that  $\{f_n\}$  converges uniformly to  $f$  on  $U$ .
- Let  $\{f_n\}$  be continuous on  $X$  and converge locally uniformly to  $f$  on  $X$ . Then  $f$  is continuous on  $X$ .
- **Local M-test:** let  $\{f_n\} : X \rightarrow \mathbb{C}$  be continuous, such that  $\forall y \in X$ , exists open  $U \subset X$  containing  $y$  and  $M_n > 0$  with

$$\forall x \in U, |f_n(x)| \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty$$

Then  $\sum_{n=1}^{\infty} f_n$  converges locally uniformly to continuous function on  $X$ .

- **Complex power series:**

$$\sum_{n=0}^{\infty} a_n(z - c)^n, \quad a_n, c \in \mathbb{C}$$

- Either:
  - Series converges only for  $z = c$  ( $R = 0$ ).
  - Series converges absolutely for  $|z - c| < R \iff z \in B_R(c)$ .  $R$  is **radius of convergence**,  $B_R(c)$  is **disc of convergence** and diverges for  $|z - c| > R$ .
  - Series converges absolutely for all  $z$  ( $R = \infty$ ).
- Power series with radius of convergence  $R$  converges absolutely on  $B_r(c)$  for every  $0 < r < R$ . So series is locally uniformly convergent (but not uniformly convergent) on disc of convergence.
- **Term-by-term differentiation and integration preserve radius of convergence:** let  $\sum_{n=0}^{\infty} a_n(z - c)^n$  have radius of convergence  $R$ . Then formal derivative and antiderivative

$$\sum_{n=1}^{\infty} n a_n(z - c)^{n-1}, \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - c)^{n+1}$$

have radius of convergence  $R$ .

- **Power series can be differentiated term-by-term in disc of convergence:** let  $\sum_{n=0}^{\infty} a_n(z - c)^n$  have radius of convergence  $R$  and converge to  $f : B_R(c) \rightarrow \mathbb{C}$ . Then  $f$  is holomorphic on  $B_R(c)$  and



$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - c)^{n-1}$$

- Power series with  $R > 0$  can be differentiated infinitely many times and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} k! \binom{n}{k} a_n (z - c)^{n-k}$$

So  $f^{(k)}(c) = k! a_k$ .

- **Power series can be integrated term-by-term in disc of convergence:** power series with  $R > 0$  has holomorphic antiderivative  $F : B_R(c) \rightarrow \mathbb{C}$  given by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - c)^{n+1}$$

## 6. Complex integration over contours

- Let  $f : [a, b] \rightarrow \mathbb{C}$ ,  $f = u + iv$ , then

$$\int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

- Let  $f_1, f_2 : [a, b] \rightarrow \mathbb{C}$ ,  $c \in \mathbb{C}$ , then

$$\int_a^b (f_1(t) + f_2(t)) dt = \int_a^b f_1(t) dt + \int_a^b f_2(t) dt, \quad \int_a^b c f_1(t) dt = c \int_a^b f_1(t) dt$$

- Curve  $\gamma : [a, b] \rightarrow \mathbb{C}$  is  $C^1$  if **continuously differentiable** (derivative exists and is continuous).
- **Integral of continuous  $f : U \rightarrow \mathbb{C}$  along curve  $\gamma : [a, b] \rightarrow U$ ,  $\gamma \in C^1$ :**

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt$$

- Let  $f_1, f_2 : [a, b] \rightarrow \mathbb{C}$ ,  $c \in \mathbb{C}$ , then

$$\int_{\gamma} (f_1(z) + f_2(z)) dz = \int_{\gamma} f_1(z) dz + \int_{\gamma} f_2(z) dz, \quad \int_{\gamma} c f_1(z) dz = c \int_{\gamma} f_1(z) dz$$

- $(-\gamma) : [-b, -a] \rightarrow \mathbb{C}$ ,  $(-\gamma)(t) := \gamma(-t)$ , then

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$$

- Let  $\varphi : [a', b'] \rightarrow [a, b]$  be continuously differentiable,  $\varphi(a') = a$ ,  $\varphi(b') = b$ ,  $\delta : [a', b'] \rightarrow \mathbb{C}$ ,  $\delta = \gamma \circ \varphi$ . Then

$$\int_{\gamma} f(z) dz = \int_{\delta} f(z) dz$$

- Let  $\gamma : [a, b] \rightarrow \mathbb{C}$ ,  $a = a_0 < a_1 < \dots < a_n = b$ ,  $\gamma_i : [a_{i-1}, a_i] \rightarrow \mathbb{C}$  be  $C^1$ ,  $\gamma_i(t) := \gamma(t)$  for  $t \in [a_{i-1}, a_i]$ . Then  $\gamma$  is **piecewise  $C^1$  curve**, or **contour**.

$$\int_{\gamma} f(z) dz = \sum_{i=1}^n \int_{\gamma_i} f(z) dz$$

is a **contour integral**.

- **Contour union:** let  $\gamma : [a, b] \rightarrow \mathbb{C}$ ,  $\delta : [c, d] \rightarrow \mathbb{C}$ , then

$$(\gamma \cup \delta) : [a, b + d - c] \rightarrow \mathbb{C}, \quad (\gamma \cup \delta)(t) := \begin{cases} \gamma(t) & \text{if } t \in [a, b] \\ \delta(t + c - b) & \text{if } t \in [b, b + d - c] \end{cases}$$

Then

$$\int_{\gamma \cup \delta} f(z) dz = \int_{\gamma} f(z) dz + \int_{\delta} f(z) dz$$

- **Complex Fundamental Theorem of Calculus (FTC)** Let  $U \subseteq \mathbb{C}$  open,  $F : U \rightarrow \mathbb{C}$  holomorphic with derivative  $f$ ,  $\gamma : [a, b] \rightarrow U$  contour. Then

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a))$$

So if  $\gamma$  closed, then  $\int_{\gamma} f(z) dz = 0$ . Also, if  $\gamma_1$  and  $\gamma_2$  have same endpoints, then

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

- If  $F' = f$ ,  $F$  is **antiderivative** or **primitive** of  $f$ .
- **Length** of contour  $\gamma$ :

$$L(\gamma) := \int_a^b |\gamma'(t)| dt$$

- **Estimation lemma:** Let  $f : U \rightarrow \mathbb{C}$  continuous,  $\gamma : [a, b] \rightarrow U$  contour. Then

$$\left| \int_{\gamma} f(z) dz \right| \leq L(\gamma) \cdot \sup_{\gamma} |f|$$

where  $\sup_{\gamma} |f| := \sup\{|f(z)| : z \in \gamma\}$

- **Converse to FTC:** Let  $D$  domain,  $f : D \rightarrow \mathbb{C}$  continuous,  $\int_{\gamma} f(z) dz = 0$  for every closed contour  $\gamma \in D$ . Then exists holomorphic antiderivative  $F : D \rightarrow \mathbb{C}$  (unique up to addition of constant) such that

$$F'(z) = f(z)$$

- Domain  $D$  **starlike** if for some  $a_0 \in D$ , for every  $a_0 \neq b \in D$ , straight line from  $a_0$  to  $b$  contained in  $D$ .
- **Cauchy's theorem for starlike domains:** let  $D$  starlike domain,  $f : D \rightarrow \mathbb{C}$  holomorphic,  $\gamma \in D$  closed contour. Then

$$\int_{\gamma} f(z) dz = 0$$

Same holds if  $f$  holomorphic on  $D - S$ ,  $S$  finite set of points, and  $f$  continuous on  $D$ .

- Let  $U$  open,  $f : U \rightarrow \mathbb{C}$  holomorphic,  $\Delta \in U$  be triangle. Then

$$\int_{\partial\Delta} f(z) dz = 0$$

Same holds if  $f$  holomorphic on  $U - S$ ,  $S$  finite set of points, and  $f$  continuous on  $U$ .

- By default, always use **anti-clockwise** parameterisation of contour.
- Let  $D$  starlike domain,  $f : D \rightarrow \mathbb{C}$  continuous,  $\int_{\partial\Delta} f(z) dz = 0$  for every triangle  $\Delta \in D$ . Then exists holomorphic  $F : D \rightarrow \mathbb{C}$  such that  $F' = f$ .
- **Cauchy's integral formula (CIF)**: let  $B = B_r(a)$ ,  $f : B \rightarrow \mathbb{C}$  holomorphic. Then for every  $w \in B$ ,  $\rho$  such that  $|w - a| < \rho < r$ ,

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz$$

## 7. Features of holomorphic functions

- **Cauchy-Taylor theorem**: let  $U \subseteq \mathbb{C}$  open,  $f : U \rightarrow \mathbb{C}$  holomorphic,  $r > 0$ ,  $B_r(a) \subset U$ . Then  $f$  is given by power series (**Taylor series of  $f$  around  $a$** ) that converges on  $B_r(a)$ :

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad z \in B_r(a)$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

for any  $0 < \rho < r$ .

- Function with Taylor series expansion on  $B_r(a)$ ,  $r > 0$ , is **analytic at  $a$** .
- Function **analytic** if analytic at every point in domain.
- Holomorphic  $\iff$  analytic.
- **Cauchy's integral formula (CIF) for derivatives**: let  $B = B_r(a)$ ,  $f : B \rightarrow \mathbb{C}$  holomorphic. For every  $0 < \rho < r$ ,

$$\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

- So  $f$  has Taylor series expansion on  $B_r(a)$ :

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

- Equivalent of Cauchy-Taylor doesn't hold for real analysis, e.g.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

has derivatives of all orders and  $f^{(n)}(0) = 0$ . But Taylor series around  $x = 0$  would be

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad x \in (0 - \varepsilon, 0 + \varepsilon)$$

for some  $\varepsilon > 0$ . But then  $c_n = \frac{f^{(n)}}{n!} = 0$  but  $f$  isn't identically zero in any neighbourhood of the origin. So  $f$  doesn't have a Taylor series.

- **Holomorphic functions have infinitely many derivatives:** let  $U \subseteq \mathbb{C}$  open,  $f : U \rightarrow \mathbb{C}$  holomorphic. Then  $f$  has derivatives of all orders on  $U$  which are all holomorphic.
- **Morera's theorem:** let  $D$  domain,  $f : D \rightarrow \mathbb{C}$  continuous. If for every closed contour  $\gamma$  in  $D$ ,

$$\int_{\gamma} f(z) dz = 0$$

then  $f$  holomorphic.

- $f : \mathbb{C} \rightarrow \mathbb{C}$  **entire** if holomorphic on  $\mathbb{C}$ .
- $f : \mathbb{C} \rightarrow \mathbb{C}$  **bounded** if for some  $M > 0$ ,  $|f(z)| \leq M$  for every  $z \in \mathbb{C}$ .
- **Liouville's theorem:** every bounded entire function is constant.
- **Fundamental theorem of algebra:** every non-constant polynomial with complex coefficients has complex root.
- **Local maximum modulus principle:** let  $f : B_r(a) \rightarrow \mathbb{C}$  holomorphic. If

$$\forall z \in B_r(a), |f(z)| \leq |f(a)|$$

then  $f$  constant on  $B_r(a)$ .

- **Maximum modulus principle:** let  $D$  domain,  $f : D \rightarrow \mathbb{C}$  holomorphic. If for some  $a \in D$ ,

$$\forall z \in D, |f(z)| \leq |f(a)|$$

then  $f$  constant on  $D$ .

- If  $U \subset \mathbb{C}$  path-connected and open, then not possible to write  $U = U_1 \cup U_2$ , where  $U_1, U_2$  disjoint, open, non-empty. So domains are connected.
- $f : B_r(a) \rightarrow \mathbb{C}$  has **zero of order  $m$  at  $a$**  if for some  $m > 0$ , exists holomorphic  $h : B_r(a) \rightarrow \mathbb{C}$  such that  $f(z) = (z - a)^m h(z)$ ,  $h(a) \neq 0$ .
- $f$  has zero of order  $m$  at  $a$  iff

$$f(a) = f^{(1)}(a) = \dots = f^{(m-1)}(a) = 0$$

and  $f^{(m)}(a) \neq 0$ .

- **Principle of isolated zeros:** let  $f : B_r(a) \rightarrow \mathbb{C}$  holomorphic,  $f \neq 0$ . Then for some  $0 < \rho \leq r$ ,

$$\forall z \in B_{\rho}(a) - \{a\}, \quad f(z) \neq 0$$

Holds for  $f(a) = 0$ , i.e. zeros of holomorphic functions are isolated.

- **Uniqueness of analytic continuation theorem:** let  $D' \subset D$  non-empty domains,  $f : D' \rightarrow \mathbb{C}$  holomorphic. Then exists at most one holomorphic  $g : D \rightarrow \mathbb{C}$  such that

$$\forall z \in D', \quad f(z) = g(z)$$

If  $g$  exists, it is **analytic continuation of  $f$  to  $D$** .

- Let  $D$  domain,  $f, g : D \rightarrow \mathbb{C}$  holomorphic,  $B_r(a) \subset D$ . If  $f(z) = g(z)$  on  $B_r(a)$  then  $f(z) = g(z)$  on  $D$ .
- Let  $S \subset \mathbb{C}$ ,  $w \in S$ .
  - $w$  **isolated point of  $S$**  if for some  $\varepsilon > 0$ ,  $B_\varepsilon(w) \cap S = \{w\}$ .
  - $w$  **non-isolated point of  $S$**  if  $\forall \varepsilon > 0$ , exists  $w \neq z \in S$  such that  $z \in B_\varepsilon(w)$ .
- **Identity theorem:** Let  $f, g : D \rightarrow \mathbb{C}$  holomorphic on domain  $D$ . If  $S := \{z \in D : f(z) = g(z)\}$  contains non-isolated point, then  $f(z) = g(z)$  on  $D$ .
- Let  $D \subseteq \mathbb{C}$  domain,  $u : D \rightarrow \mathbb{R}$  **harmonic** if has continuous second order partial derivatives and satisfies **Laplace's equation**:

$$u_{xx} + u_{yy} = 0$$

- Let  $f = u + iv : D \rightarrow \mathbb{C}$  holomorphic on domain  $D$ . Then  $u$  and  $v$  harmonic.
- **Existence of harmonic conjugates theorem:** let  $D$  starlike domain,  $u : D \rightarrow \mathbb{R}$  harmonic. Then exists harmonic  $v : D \rightarrow \mathbb{R}$  such that  $f = u + iv$  holomorphic on  $D$ .  $v$  is **harmonic conjugate of  $u$** , unique up to addition of real constant. **Note:** condition of  $D$  being starlike is removed when Cauchy's theorem is proved in generality.
- Let  $f : D \rightarrow \mathbb{C}$  holomorphic on domain  $D$ . Then  $f$  has holomorphic antiderivative on  $D$ .
- **Dirichlet problem:** let  $D \subseteq \mathbb{C}$  domain with closure  $\overline{D}$ , boundary  $\partial D$ ,  $g : \partial D \rightarrow \mathbb{R}$  continuous. Find continuous  $\mu : \overline{D} \rightarrow \mathbb{R}$  such that  $\mu$  harmonic on  $D$  and  $\mu = g$  on  $\partial D$ .
- Let  $f = u + iv : D \rightarrow \mathbb{C}$  holomorphic on domain  $D$ ,  $\mu$  harmonic on  $f(D)$ . Then  $\tilde{\mu} := \mu \circ f$  harmonic on  $D$ .
- So if  $\mu$  harmonic on  $D'$  and want to find a harmonic  $\tilde{\mu}$  on  $D$ , find holomorphic  $f$  mapping  $D$  to  $D'$  so  $f(D) = D'$ . Then  $\tilde{\mu} = \mu \circ f$  is solution.

## 8. General form of Cauchy's theorem and C.I.F.

- Let curve  $\gamma : [a, b] \rightarrow \mathbb{C}$ ,  $\gamma(t) = w + r(t)e^{i\theta(t)}$ ,  $w \in \mathbb{C}$ ,  $r, \theta : [a, b] \rightarrow \mathbb{R}$ , piecewise  $C^1$ ,  $r(t) > 0$ . **Winding number (index)** of  $\gamma$  around  $w$  is

$$I(\gamma; w) := \frac{\theta(b) - \theta(a)}{2\pi}$$

- Let contour  $\gamma : [a, b] \rightarrow \mathbb{C}$ ,  $w \in \mathbb{C}$ ,  $w \notin \gamma$ . Then exists  $r, \theta : [a, b] \rightarrow \mathbb{R}$  piecewise  $C^1$ ,  $r(t) > 0$  such that

$$\gamma(t) = w + r(t)e^{i\theta(t)}$$

. Here,  $r(t) = |\gamma(t) - w|$ .

- Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  closed contour,  $w \notin \gamma$ . Then

$$I(\gamma; w) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - w} dz$$

- Let  $D$  starlike domain,  $\gamma$  closed contour in  $D$ . If  $w \notin D$ , then  $I(\gamma; w) = 0$ .
- Let  $U \subseteq \mathbb{C}$  open.
  - Closed contour  $\gamma$  in  $U$  **homologous to zero in  $U$**  if  $I(\gamma; w) = 0$  for every  $w \notin U$ .

- $U$  is **simply connected** if every closed contour in  $U$  homologous to zero in  $U$ .
- **Cycle**: finite collection of closed contours in  $U$ , denoted as formal sum

$$\Gamma := \gamma_1 + \cdots + \gamma_n$$

$w$  **does not lie in**  $\Gamma$  if  $w \notin \gamma_i$  for all  $i$ . Define

$$I(\Gamma; w) := \sum_{i=1}^n I(\gamma_i; w)$$

and

$$\int_{\Gamma} f(z) dz := \sum_{i=1}^n \int_{\gamma_i} f(z) dz$$

$\Gamma$  **homologous to zero in**  $U$  if  $I(\Gamma; w) = 0$  for every  $w \notin U$ .

- Closed curve  $\gamma : [a, b] \rightarrow \mathbb{C}$  **simple** if for any  $t_1 < t_2$ ,  
 $\gamma(t_1) = \gamma(t_2) \implies t_1 = a$  and  $t_2 = b$  (no self-crossing or backtracking).
- **Jordan curve theorem**: Let  $\gamma$  closed curve. Then  $\mathbb{C} - \gamma$  is disjoint union of two domains, exactly one of which is bounded. Bounded domain is **interior** of  $\gamma$ ,  $D_{\gamma}^{\text{int}}$ . Unbounded domain is **exterior**,  $D_{\gamma}^{\text{ext}}$ .  $w$  lies inside  $\gamma$  if  $w \in D_{\gamma}^{\text{int}}$  and outside  $\gamma$  if  $w \in D_{\gamma}^{\text{ext}}$ .
- Let  $\gamma$  simple closed contour. Then possible to put orientation on  $\gamma$  such that  $\forall w \in \mathbb{C} - \gamma$ ,

$$I(\gamma; w) = \begin{cases} 1 & \text{if } w \in D_{\gamma}^{\text{int}} \\ 0 & \text{if } w \in D_{\gamma}^{\text{ext}} \end{cases}$$

Then  $\gamma$  is **positively oriented** (interior always on left of curve - anticlockwise).

- Let  $D$  domain,  $f : D \rightarrow \mathbb{C}$  holomorphic,  $\Gamma$  cycle in  $D$ , homologous to zero in  $D$ .
- **General form of Cauchy's theorem**:

$$\int_{\Gamma} f(z) dz = 0$$

- **General form of CIF**:

$$\forall w \in D - \Gamma, \quad \int_{\Gamma} \frac{f(z)}{z - w} dz = 2\pi i I(\Gamma; w) f(w)$$

- For simple closed curve  $\gamma$ ,  $f$  holomorphic on  $D_{\gamma}^{\text{int}} \cup \gamma$  if exists domain  $D$  such that  $D_{\gamma}^{\text{int}} \cup \gamma \subset D$  and  $f$  holomorphic on  $D$ .
- Let  $\gamma$  simple closed curve and  $f$  holomorphic on  $D_{\gamma}^{\text{int}} \cup \gamma$ .
- **Cauchy's theorem for simple closed curves**:

$$\int_{\gamma} f(z) dz = 0$$

- **CIF for simple closed contours**:

$$\forall w \in D_{\gamma}^{\text{int}}, \quad \int_{\gamma} \frac{f(z)}{z - w} dz = 2\pi i f(w)$$

## 9. Holomorphic functions on punctured domains

- **Laurent series:**

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

**Principal part:**  $\sum_{n=-\infty}^{-1} c_n (z-a)^n$ . **Analytic part:**  $\sum_{n=0}^{\infty} c_n (z-a)^n$ .

- Laurent series converges at  $z$  iff principal and analytic parts converge at  $z$ .
- **Annulus centre  $a$ , internal/external radii  $r$  and  $R$ :**

$$A_{r,R}(a) := \{z \in \mathbb{C} : r < |z-a| < R\}$$

- If Laurent series isn't power series ( $c_n \neq 0$  for some  $n < 0$ ) then either:
  - It never converges or
  - Exists  $0 \leq r < R \leq \infty$  such that it converges on  $A_{r,R}(a)$  and diverges for  $|z-a| < r$  or  $|z-a| > R$ .  $A_{r,R}(a)$  is **annulus of convergence**.
- If Laurent series has annulus of convergence  $A_{r,R}(a)$  then it converges uniformly on any  $A_{\rho_1, \rho_2}$  with  $r < \rho_1 < \rho_2 < R$ . So it converges locally uniformly on  $A_{r,R}(a)$  so represents holomorphic function on  $A_{r,R}(a)$ .
- If Laurent series has annulus of convergence containing  $A_{r,R}(a)$ , then  $c_n$  are unique and given by, for any  $\rho \in (r, R)$

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

So Laurent series in  $A_{r,R}(a)$  unique.

- **Holomorphic functions on annuli have Laurent series:** let  $f : A_{r,R}(a) \rightarrow \mathbb{C}$  holomorphic, then exist unique  $c_n \in \mathbb{C}$  such that

$$\forall z \in A_{r,R}(a), \quad f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

and annulus of convergence of Laurent series contains  $A_{r,R}(a)$ . Series is **Laurent series of  $f$  on  $A$** .

- **Punctured ball:**  $B_R^*(a) := B_R(a) - \{a\} = A_{0,R}(a)$ .
- If  $f$  holomorphic on  $B_R^*(a)$ ,  $f$  has **isolated singularity** at  $a$ .
- Types of isolated singularity:
  - $f$  has **removable singularity** at  $z = a$  if  $c_n = 0$  for all  $n \leq -1$  (principal part is zero).
  - $f$  has **pole of order  $k$**  at  $z = a$  if  $c_{-k} \neq 0$  and  $c_n = 0$  for all  $n < -k$ .
  - $f$  has **essential singularity** at  $z = a$  if exist infinitely many  $n < 0$  such that  $c_n \neq 0$ .
- $f : B_R^*(a) \rightarrow \mathbb{C}$  has removable singularity at  $z = a$  iff  $f$  extends to holomorphic function on  $B_R(a)$ .
- Let  $f : B_R^*(a) \rightarrow \mathbb{C}$  holomorphic,  $R > 0$ . Then  $f$  has removable singularity at  $z = a$  iff

$$\lim_{z \rightarrow a} (z-a)f(z) = 0$$

- **Riemann extension theorem:** Let  $f : B_R^*(a) \rightarrow \mathbb{C}$  holomorphic and bounded, then  $f$  has removable singularity at  $z = a$ .

- Let  $f : B_R^*(a) \rightarrow \mathbb{C}$  holomorphic. The following are equivalent:
  - $f$  has pole of order  $k$  at  $z = a$ .
  - $f(z) = (z - a)^{-k} g(z)$ ,  $g : B_R(a) \rightarrow \mathbb{C}$  holomorphic,  $g(a) \neq 0$ .
  - Exists  $0 < r \leq R$  and  $h : B_r(a) \rightarrow \mathbb{C}$  holomorphic with zero of order  $k$  at  $z = a$  such that  $f(z) = 1 / h(z)$  for  $z \in B_r^*(a)$ .
- Let  $f : B_R^*(a) \rightarrow \mathbb{C}$  holomorphic. Then  $f$  has pole at  $z = a$  iff

$$\lim_{z \rightarrow a} |f(z)| = \infty$$

- **Casorati-Weierstrass theorem:** let  $f : B_R^*(a) \rightarrow \mathbb{C}$  holomorphic with essential singularity at  $z = a$ . Then

$$\forall w \in \mathbb{C}, \forall 0 < r < R, \forall \varepsilon > 0, \exists z \in B_r^*(a), \quad f(z) \in B_\varepsilon(w)$$

- **Big Picard theorem:** let  $f : B_R^*(a) \rightarrow \mathbb{C}$  holomorphic with essential singularity at  $z = a$ . Then for some  $b \in \mathbb{C}$ ,

$$\forall 0 < r < R, \quad \mathbb{C} - \{b\} \subseteq f(B_r^*(a))$$

## 10. Cauchy's residue theorem

- $f$  **meromorphic** on domain  $D$  if  $f$  holomorphic on  $D - S$ ,  $S \subset D$  has no non-isolated points and  $f$  has pole at every  $s \in S$ .
- $f$  meromorphic on  $D_\gamma^{\text{in}} \cup \gamma$  if exists domain  $D$  containing  $D_\gamma^{\text{in}} \cup \gamma$  and  $f$  meromorphic on  $D$ .
- Let  $f$  meromorphic on domain  $D$  with pole at  $a$ , with Laurent series

$$f(z) = \sum_{n=-k}^{\infty} c_n (z - a)^n$$

**Residue of  $f$  at  $a$  is**

$$\text{Res}_{z=a}(f) := c_{-1}$$

- **Cauchy's residue theorem:** Let  $f$  meromorphic on  $D_\gamma^{\text{in}} \cup \gamma$ ,  $\gamma$  positively oriented simple closed contour,  $f$  has no poles on  $\gamma$  and finite number of poles inside  $\gamma$ ,  $\{a_1, \dots, a_m\}$ . Then

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{j=1}^m \text{Res}_{z=a_j}(f)$$

- **Simple pole:** pole of order 1.
- **Rules for calculating residues:**
  - **Linear combinations:**  $\text{Res}_{z=a}(Af + Bg) = A\text{Res}_{z=a}(f) + B\text{Res}_{z=a}(g)$ .
  - **Cover up rule for poles of order 1:** if  $z = a$  is pole of order 1,

$$\text{Res}_{z=a}(f) = \lim_{z \rightarrow a} (z - a)f(z)$$

- **Simple zero on denominator:** if  $f(z) = g(z) / h(z)$ ,  $g, h$  holomorphic at  $a$ ,  $g(a) \neq 0$ ,  $z = a$  is zero of order 1 of  $h$ , then

$$\text{Res}_{z=a}(f) = \frac{g(a)}{h'(a)}$$



- **Poles of higher orders:** if  $f(z) = g(z) / (z - a)^k$ ,  $k > 0$ ,  $g$  holomorphic at  $a$ , then

$$\text{Res}_{z=a}(f) = \frac{g^{(k-1)}(a)}{(k-1)!}$$

- To calculate

$$\int_0^{2\pi} F(\sin(\theta), \cos(\theta)) d\theta$$

where  $F$  is rational function, use change of variable  $z = e^{i\theta}$ , and use

$$\int_0^{2\pi} F(\sin(\theta), \cos(\theta)) d\theta = \int_{|z|=1} F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz}$$

- To calculate

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{p(x)}{q(x)} dx$$

where  $\deg(q) \geq \deg(p) + 2$  and  $q$  has no real roots, integrate  $f(z) = p(z) / q(z)$  over  $\gamma_R = L_R \cup C_R$  where  $R$  greater than maximum modulus of roots of  $q$ . Use e.g. Estimation Lemma or Jordan's lemma to show  $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$ .

- $$\int_{-\infty}^{\infty} f(x) dx = \lim_{r \rightarrow \infty} \int_0^r f(x) dx + \lim_{s \rightarrow \infty} \int_{-s}^0 f(x) dx$$
- **Cauchy principal value** of  $\int_{-\infty}^{\infty} f(x) dx$ :

$$P.V. \int_{-\infty}^{\infty} f(x) dx = \lim_{r \rightarrow \infty} \int_{-r}^r f(x) dx$$

- If  $f$  even,  $P.V. \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} f(x) dx$
- **Jordan's lemma:** let  $f$  holomorphic on  $D = \{z \in \mathbb{C} : |z| > r\}$  for some  $r > 0$ ,  $zf(z)$  bounded on  $D$ . Then for every  $\alpha > 0$ ,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{i\alpha z} dz = 0$$

where  $C_R = Re^{i\theta}$ ,  $\theta \in [0, \pi]$ .

- To calculate

$$P.V. \int_{-\infty}^{\infty} f(x) \sin(\alpha x) dx \quad \text{or} \quad P.V. \int_{-\infty}^{\infty} f(x) \cos(\alpha x) dx$$

where  $f$  meromorphic in  $\mathbb{C}$  with no real poles and  $f$  satisfies Jordan's lemma, calculate integral

$$\int_{\gamma_R} f(z) e^{i\alpha z} dz$$

with CRT, where  $\gamma_R = L_R \cup C_R$ . Then use

$$\int_{L_R} f(z)e^{i\alpha z} dz = \int_{-R}^R f(x)\cos(\alpha x) dx + i \int_{-R}^R f(x)\sin(\alpha x) dx$$

and equate real/imaginary parts. Use Jordan's lemma to show

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z)e^{i\alpha z} dz = 0.$$

- **Indentation lemma:** Let  $g$  meromorphic on  $\mathbb{C}$  with simple pole at 0,  $C_\varepsilon(\theta) = \varepsilon e^{i\theta}, \theta \in [0, \pi]$ . Then

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon} g(z) dz = \pi i \text{Res}_{z=0}(g)$$

- To calculate

$$\int_{-\infty}^{\infty} f(x) dx$$

where  $f$  has simple pole at  $z = 0$ , let  $\gamma_{\rho,R} = L_2 \cup (-C_\rho) \cup L_1 \cup C_R$  where  $L_2$  is line from  $-R$  to  $-\rho$ ,  $L_1$  is line from  $\rho$  to  $R$ . Take  $\rho \rightarrow 0$  and  $R \rightarrow \infty$ , use indentation lemma and Jordan's lemma. **Note:** may have to choose appropriate branch cut so that  $f$  holomorphic on  $D$ .

- Let  $f$  meromorphic with zero or pole order  $k > 0$  at  $a$ . Then  $f' / f$  has simple pole at  $a$  and

$$\text{Res}_{z=a}(f' / f) = \begin{cases} k & \text{if } f \text{ has zero at } z = a \\ -k & \text{if } f \text{ has pole at } z = a \end{cases}$$

- **Argument principle:** let  $\gamma$  positively oriented simple closed contour,  $f$  meromorphic on  $D_\gamma^{\text{int}} \cup \gamma$ ,  $f$  has no zeros or poles on  $\gamma$ ,  $Z_f$  be number of zeros of  $f$  in  $D_\gamma^{\text{int}}$  (counted with multiplicity),  $P_f$  be number of poles of  $f$  in  $D_\gamma^{\text{int}}$  (counted with multiplicity). Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z_f - P_f = I(\Gamma_f; 0), \quad \Gamma_f = f \circ \gamma$$

(Counted with multiplicity means zero/pole of order  $k$  counts  $k$  times).

- **Rouche's theorem:** let  $\gamma$  simple closed contour,  $f, g$  holomorphic on  $D_\gamma^{\text{int}} \cup \gamma$ , with

$$\forall z \in \gamma, |f(z) - g(z)| < |g(z)|$$

Then  $f$  and  $g$  have same number of zeros (counted with multiplicity) inside  $\gamma$ .

- **Open mapping theorem:** let  $f$  holomorphic, non-constant on domain  $D$ . Then if  $U \subset D$  open,  $f(U)$  is open.