0.1. Integration and measure

- Dirichlet's function: $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

1. The real numbers

- $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \leq a$.
- $c \in \mathbb{R}$ is a **least upper bound (supremum)** if $c \leq a$ for every upper bound a.
- $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \geq a$.
- $c \in \mathbb{R}$ is a **greatest lower bound (supremum)** if $c \geq a$ for every upper bound a.
- **Completeness axiom of the real numbers**: every subset E with an upper bound has a least upper bound. Every subset E with a lower bound has a greatest lower bound.
- **Archimedes' principle**:

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

- Every non-empty subset of $\mathbb{N}$ has a minimum.
- **The rationals are dense in the reals**:

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

1.1. Conventions on sets and functions

- For $f : X \to Y$, **preiamge** of $Z \subseteq Y$ is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

- $f : X \to Y$ **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

- $f : X \to Y$ **surjective** if $Y = f(X)$.

1.2. Open and closed sets

- $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon : (x - \varepsilon, x + \varepsilon) \subseteq U$$

- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.
- $x \in E \subseteq \mathbb{R}$ is **point of closure** if

$$\forall \delta > 0, \exists y \in E : |x - y| < \delta$$

  Equivalently, x is point of closure if every open interval containing x contains another point of E.
- **Closure** of E, $\overline{E}$, is set of points of closure.
- If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$. **Exercise (todo)**: prove this.
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2. Further analysis of subsets of $\mathbb{R}$

TODO: up to here, check that all notes are made from these topics

2.1. Countability and uncountability

- A is **countable** if $A = \emptyset$, A is finite or there is a bijection $\varphi : \mathbb{N} \to A$ (in which case A is **countably infinite**). Otherwise A is **uncountable**. φ is called an **enumeration**.
- Examples of countable sets:
  - $\mathbb{N}$ ($\varphi(n) = n$)
  - $2\mathbb{N}$ ($\varphi(n) = 2n$)
- **Exercise (todo)**: show that any subset of $\mathbb{N}$ is countable.
- **Exercise (todo)**: show that if there is an injection from A to $\mathbb{N}$, or if there is a surjection from $\mathbb{N}$ to A, then A is countable.
- $\mathbb{Q}$ is countable.
- **Exercise (todo)**: show that $\mathbb{N}^k$ is countable for any $k \in \mathbb{N}$.
- **Exercise (todo)**: show that if a_n is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Exercise (todo)**: show that if $a_{n,k}$ is a nonnegative sequence and $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- $f : X \to Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.
- Let f be monotone on (a, b). Then it is discountinuous on a countable set.
- Set of sequences in $\{0, 1\}$, $\{((x_n))_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.
- $\mathbb{R}$ is uncountable.

TODO: problem sheet one