Contents

1. Basic theory

Example. Let $f(x_1, ..., x_r) \in \mathbb{Z}[x_1, ..., x_r]$, a Diophantine equation asks to solve $f(x_1, ..., x_r) = 0$. Easier questions are when is $f(x_1, ..., x_r) \equiv 0 (\operatorname{mod} p)$ and $f(x_1, ..., x_r) \equiv 0 (\operatorname{mod} p^n)$. Local fields "package" all this information together for all n.

1.1. Absolute values

Definition. Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that $\forall x, y \in K$:

- $|x| = 0 \iff x = 0$.
- $|xy| = |x| \cdot |y|$ (multiplicative).
- $|x + y| \leq |x| + |y|$ (triangle inequality).

$(K, |\cdot|)$ is a **valued field**.

Example.

- $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ with usual absolute value $|a + ib| = \sqrt{a^2 + b^2}$. We write $|\cdot|_\infty$ for this absolute value.
- The **trivial** absolute value is $|x| = 0$ if $x = 0$ and $|x| = 1$ otherwise.

Definition. Let $K = \mathbb{Q}$, p be prime. For $0 \neq x \in \mathbb{Q}$, write $x = p^n \frac{a}{b}$ where $p \nmid a, b$. The **p-adic absolute value** $|\cdot|_p$ is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-n} & \text{if } x = p^n \frac{a}{b} \end{cases}.$$

Proposition. The p-adic absolute value is an absolute value.

Proof.

- The first axiom is trivial.
- Let $y = p^m \frac{c}{d}$.
- $|xy|_p = |p^{m+n} \frac{ac}{bd}|_p = p^{-m-n} = |x|_p \cdot |y|_p$.
- WLOG, assume that $m \geq n$. $|x + y|_p = |p^n \frac{ad + p^{m-n} bc}{bd}|_p \leq p^{-n} = \max\{|x|_p, |y|_p\}$.

□

Proposition. An absolute value $|\cdot|$ on K induces a metric $d(x, y) = |x - y|$ (and hence a topology) on K.

Proof. Exercise. □

Definition. Two absolute values on K are **equivalent** if they induce the same topology.

A **place** is an equivalence class of absolute values.

Proposition. Let $|\cdot|$ and $|\cdot|'$ be non-trivial absolute values on K. Then TFAE:

1. $|\cdot|$ and $|\cdot|'$ are equivalent.
2. $|x| < 1$ iff $|x|' < 1$ for all $x \in K$.
3. There exists $c > 0$ such that $|x|^c = |x|'$ for all $x \in K$.

Proof.
- $(1 \Rightarrow 2)$:
  - $|x| < 1$ iff $x^n \to 0$ w.r.t $|\cdot|$ iff $x^n \to 0$ w.r.t $|\cdot|'$ iff $|x|' < 1$.
- $(2 \Rightarrow 3)$:
  - Note $|x|^c = |x|'$ iff $c \log|x| = \log|x|'$.
  - Let $a \in K^\times$ such that $|a| > 1$ (this exists since $|\cdot|$ is non-trivial).
  - We show that $\log|x| \ / \log|a| = \log|x|' \ / \log|a|'$ for all $x \in K^\times$.
  - Assume not, then $\log|x| \ / \log|a| < \log|x|' \ / \log|a|'$.
  - Choose $m, n \in \mathbb{Z}$ such that $\log|x| \ / \log|a| < \frac{m}{n} < \log|x| \ / \log|a|$.
  - Then $n \log|x| < m \log|a|$ and $n \log|x|' > m \log|a|'$, so $|\frac{x^n}{a^m}| < 1$ but $|\frac{x^n}{a^m}|' > 1$: contradiction.
  - Similarly for $\log|x| \ / \log|a| > \log|x|' \ / \log|a|'$.
- $(3 \Rightarrow 1)$:
  - Trivial, as open balls they define are the same.

□

Remark. $|\cdot|_\infty^2$ on $\mathbb{C}$ is not an absolute value by out definition since it violates the triangle inequality. Note some authors replace the triangle inequality axiom with $|x + y|^\beta \leq |x|^\beta + |y|^\beta$ for some fixed $\beta > 0$.

Definition. An absolute value $|\cdot|$ on K is **non-Archimedean** if it satisfies the **ultrametric inequality**:

$$|x + y| \leq \max\{|x|, |y|\}.$$

Otherwise, it is **Archimedean**.

Example.
- $|\cdot|_\infty$ on $\mathbb{R}$ is Archimedean.
- $|\cdot|_p$ on $\mathbb{Q}$ is non-Archimedean.

Lemma. Let $(K, |\cdot|)$ be non-Archimedean and $x, y \in K$. If $|x| < |y|$, then $|x - y| = |y|$ (i.e. all triangles are isosceles).

Proof. For $\leq$, use ultrametric inequality. For $\geq$, use that $|y| = |x - y - x|$. □

Proposition. Let $(K, |\cdot|)$ be non-Archimedean. Let (x_n) be a sequence in K. If $|x_n - x_{n+1}| \to 0$, then x_n is Cauchy. In particular, if K is complete with respect to $|\cdot|$, then (x_n) converges.

Proof.
- For $\varepsilon > 0$, choose N such that $|x_n - x_{n+1}| < \varepsilon$ for all $n > N$.
- Then for $N < n < m$, $|x_n - x_m| = |(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{m-1} - x_m)| < \varepsilon$.

□

Example. Let $p = 5$ and consider the sequence (x_n) in $\mathbb{Z}$ satisfying:
- $x_n^2 + 1 \equiv 0 \bmod 5^n$.
- $x_n \equiv x_{n+1} \bmod 5^n$.

Take $x_1 = 2$. Suppose we have constructed $x_1, ..., x_n$. Then write $x_n^2 + 1 = a5^n$ and set $x_{n+1} = x_n + b5^n$. Then $x_{n+1}^2 + 1 = x_n^2 + 2bx_n 5^n + b^2 5^{2n} + 1 = a5^n + 2bx_n 5^n + b^2 5^{2n}$. We choose b such that $a + 2bx_n \equiv 0 \bmod 5$ (this congruence is solvable). Then we have $x_{n+1}^2 + 1 = 0 \bmod 5^{n+1}$.

Hence (x_n) is Cauchy. Suppose $x_n \to l \in \mathbb{Q}$. Then $x_n^2 \to l^2 \in \mathbb{Q}$. But the first condition implies that $x_n^2 \to -1 = l^2$, contradiction. So (x_n) doesn't converge in $\mathbb{Q}$. So $(\mathbb{Q}, |\cdot|_5)$ is not complete.

Definition. The set of ***p*-adic numbers** $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Remark. There is an analogy with the construction of $\mathbb{R}$ with respect to $|\cdot|_\infty$.