Contents

1. Monochromatic sets

1.1. Ramsey's theorem

Notation 1.1 $\mathbb{N}$ denotes the set of positive integers, $[n] = \{1, ..., n\}$, and $X^{(r)} = \{A \subseteq X : |A| = r\}$. Elements of a set are written in ascending order, e.g. $\{i, j\}$ means $i < j$. Write e.g. ijk to mean the set $\{i, j, k\}$ with the ordering (unless otherwise stated) $i < j < k$.

Definition 1.2 A ***k*-colouring** on $A^{(r)}$ is a function $c : A^{(r)} \to [k]$.

Example 1.3
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ is even and blue if $i + j$ is odd. Then $M = 2\mathbb{N}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$ is even and blue otherwise. $M = \{4^n : n \in \mathbb{N}\}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

Theorem 1.4 (Ramsey's Theorem for Pairs) Let $\mathbb{N}^{(2)}$ are 2-coloured by $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Then there exists an infinite monochromatic subset M.

Proof.
- Let $a_1 \in A_0 := \mathbb{N}$. There exists an infinite set $A_1 \subseteq A_0$ such that $c(a_1, i) = c_1$ for all $i \in A_1$.
- Let $a_2 \in A_1$. There exists infinite $A_2 \subseteq A_1$ such that $c(a_2, i) = c_2)$ for all $i \in A_2$.
- Repeating this inductively gives a sequence $a_1 < a_2 < \cdots < a_k < \cdots$ and $A_1 \supseteq A_2 \supseteq \cdots$ such that $c(a_i, j) = c_i$ for all $j \in A_i$.
- One colour appears infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c_{i_k} = \cdots = c$.
- $M = \{a_{i_1}, a_{i_2}, ...\}$ is a monochromatic set.

□

Remark 1.5
- The same proof works for any $k \in \mathbb{N}$ colours.
- The proof is called a "2-pass proof".
- An alternative proof for k colours is split the k colours $1, ..., k$ into 2 colours: 1 and "2 or ... or k", and use induction.

Note 1.6 An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

Example 1.7 Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, etc. Let $\{i, j\}$ be red if $i, j \in A_k$ for some k. There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

Example 1.8 Colour $\{i < j < k\}$ red iff $i \mid (j + k)$. A monochromatic subset $M = \{2^n : n \in \mathbb{N}_0\}$ is a monochromatic set.

Theorem 1.9 (Ramsey's Theorem for r-sets) Let $\mathbb{N}^{(r)}$ be finitely coloured. Then there exists a monochromatic infinite set.

Proof.

- $r = 1$: use pigeonhole principle.
- $r = 2$: Ramsey's theorem for pairs.
- For general r, use induction.
- Let $c : \mathbb{N}^r \to [k]$ be a k-colouring. Let $a_1 \in \mathbb{N}$, and consider all $r-1$ sets of $\mathbb{N} \setminus \{a_1\}$, induce colouring $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \to [k]$ via $c'(F) = c(F \cup \{a_1\})$.
- By inductive hypothesis, there exists $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that c' is constant on it (taking value c_1).
- Now pick $a_2 \in A_1$ and induce a colouring $c' : (A_1 \setminus \{a_2\})^{(r-1)} \to [k]$ such that $c'(F) = c(F \cup \{a_2\})$. By inductive hypothesis, there exists $A_2 \subseteq A_1 \setminus \{a_2\}$ such that c' is constant on it (taking value c_2).
- Repeating this gives $a_1, a_2, ...$ and $A_1, A_2, ...$ such that $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$ and $c(F \cup \{a_i\}) = c_i$ for all $F \subseteq A_{i+1}$, for $|F| = r - 1$.
- One colour must appear infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c$.
- $M = \left\{a_{i_1}, a_{i_2}, ...\right\}$ is a monochromatic set.

□

1.2. Applications of Ramsey's theorem

Example 1.10 In a totally ordered set, any sequence has monotonic subsequence.

Proof.

- Let (x_n) be a sequence, colour $\{i, j\}$ red if $x_i \leq x_j$ and blue otherwise.
- By Ramsey's theorem for pairs, $M = \{i_1 < i_2 < \cdots\}$ is monochromatic. If M is red, then the subsequence $x_{i_1}, x_{i_2}, ...$ is increasing, and is strictly decreasing otherwise.
- We can insist that (x_{i_j}) is either concave or convex: 2-colour $\mathbb{N}^{(3)}$ by colouring $\{j < k < \ell\}$ red if $\left(i, x_{i_j}\right), \left(j, x_{i_k}\right), \left(k, x_{i_\ell}\right)$ form a convex triple, and blue if they form a concave triple. Then by Ramsey's theorem for r-sets, there is an infinite convex or concave subsequence.

□

Theorem 1.11 (Finite Ramsey) Let $r, m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k-coloured, we can find a monochromatic set of size (at least) m.

Proof.

- Assume not, i.e. $\forall n \in \mathbb{N}$, there exists colouring $c_n : [n]^{(r)} \to [k]$ with no monochromatic m-sets.
- There are only finitely many (k) ways to k-colour $[r]^{(r)}$, so there are infinitely many of colourings $c_r, c_{r+1}, ...$ that agree on $[r]^{(r)}$: $c_i \mid_{[r]^{(r)}} = d_r$ for all i in some infinite set A_1, where d_r is a k-colouring of $[r]^{(r)}$.
- Similarly, $[r+1]^{(r)}$ has only finitely many possible k-colourings. So there exists infinite $A_2 \subseteq A_1$ such that for all $i \in A_2$, $c_i \mid_{[r+1]^{(r)}} = d_{r+1}$, where d_{r+1} is a k-colouring of $[r+1]^{(r)}$.
- Continuing this process inductively, we obtain $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$. There is no monochromatic m-set for any $d_n : [n]^{(r)} \to [k]$ (because $d_n = c_i|_{[n]^{(r)}}$ for some i).
- These d_n's are nested: $d_\ell|_{[n]^{(r)}} = d_n$ for $\ell > n$.

- Finally, we colour $\mathbb{N}^{(r)}$ by the colouring $c : \mathbb{N}^{(r)} \to [k]$, $c(F) = d_n(F)$ where $n = \max(F)$ (or in fact $n \geq \max(F)$, which is well-defined by above). So c has no monochromatic m-set (since M was a monochromatic m-set, then taking $\ell = \max(M)$, d_ℓ has a monochromatic m-set), which contradicts Ramsey's Theorem for r-sets.

□

Remark 1.12
- This proof gives no bound on $n = n(k, m)$, there are other proofs that give a bound.
- It is a proof by compactness (essentially, we proved that $\{0, 1\}^{\mathbb{N}}$ with the product topology, i.e. the topology derived from the metric $d(f, g) = \frac{1}{\min\{n \in \mathbb{N} : f(n) \neq g(n)\}}$, is sequentially compact).

Remark 1.13 Now consider a colouring $c : \mathbb{N}^{(2)} \to X$ with X potentially infinite. This does not necessarily admit an infinite monochromatic set, as we could colour each edge a different colour. Such a colouring would be injective. We can't guarantee either the colouring being constant or injective though, as $c(ij) = i$ satisfies neither.

Theorem 1.14 (Canonical Ramsey) Let $c : \mathbb{N}^{(2)} \to X$ be a colouring with X an arbitrary set. Then there exists an infinite set $M \subseteq \mathbb{N}$ such that:
1. c is constant on $M^{(2)}$, or
2. c is injective on $M^{(2)}$, or
3. $c(ij) = c(kl)$ iff $i = k$ for all $i < j$ and $k < l$, $i, j, k, l \in M$, or
4. $c(ij) = c(kl)$ iff $j = l$ for all $i < j$ and $k < l$, $i, j, k, l \in M$.

Proof (Hints).
- First consider the 2-colouring c_1 of $\mathbb{N}^{(4)}$ where $ijkl$ is coloured SAME if $c(ij) = c(kl)$ and DIFF otherwise. Show that an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ (why does this exist?) coloured SAME leads to case 1.
- Assume M_1 is coloured DIFF, consider the 2-colouring of $M_1^{(4)}$, which colours $ijkl$ SAME if $c(il) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic $M_2 \subseteq M_1$ (why does this exist?) must be coloured DIFF by contradiction.
- Consider the 2-colouring of $M_2^{(4)}$ where $ijkl$ is coloured SAME if $c(ik) = c(jl)$ and DIFF otherwise. Show an infinite monochromatic set $M_3 \subseteq M_2$ (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour $M_3^{(3)}$ by: ijk is coloured SAME if $c(ij) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic set $M_4 \subseteq M_3$ (why does this exist) must be coloured DIFF by contradiction.
- 2-colour $M_4^{(3)}$ by the other two similar colourings to above, obtaining monochromatic $M_6 \subseteq M_5 \subseteq M_4$.
- Consider 4 combinations of these colourings on M_6, show 3 lead to one of the cases in the theorem, and the other leads to contradiction.

□

Proof.

- 2-colour $\mathbb{N}^{(4)}$ by: $ijkl$ is red if $c(ij) = c(kl)$ and blue otherwise. By Ramsey's Theorem for 4-sets, there is an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ for this colouring.
- If M_1 is red, then c is constant on $M_1^{(2)}$: for all pairs $ij, i'j' \in M_1^{(2)}$, pick $m < n$ with $j, j' < m$, then $c(ij) = c(mn) = c(i'j')$.
- So assume M_1 is blue.
- Colour $M_1^{(4)}$ by giving $ijkl$ colour green if $c(il) = c(jk)$ and purple otherwise. By Ramsey's theorem for 4-sets, there exists an infinite monochromatic $M_2 \subseteq M_1$ for this colouring.
- Assume M_2 is coloured green: if $i < j < k < l < m < n \in M_2$, then $c(jk) = c(in) = c(lm)$ (consider $ijkn$ and $ilmn$): contradiction, since M_1 is blue.
- Hence M_2 is purple, i.e. for $ijkl \in M_2^{(4)}$, $c(il) \neq c(jk)$.
- Colour M_2 by: $ijkl$ is orange if $c(ik) = c(jl)$, and pink otherwise.
- By Ramsey's theorem for 4-sets, there exists infinite monochromatic $M_3 \subseteq M_2$ for this colouring.
- Assume M_3 is orange, then for $i < j < k < l < m < n \in M_3$, we have $c(jm) = c(ln)$ (consider $jlmn$) and $c(jm) = c(ik)$ (consider $ijkm$): contradiction, since $M_3 \subseteq M_1$.
- Hence M_3 is pink, i.e. for $ijkl$, $c(ik) \neq c(jl)$.
- Colour $M_3^{(3)}$ by: ijk is yellow if $c(ij) = c(jk)$ and grey otherwise. By Ramsey's theorem for 3-sets, there exists infinite monochromatic $M_4 \subseteq M_3$ for this colouring.
- Assume M_4 is yellow: then (considering $ijkl \in M_4^{(4)}$) $c(ij) = c(jk) = c(kl)$: contradiction, since $M_4 \subseteq M_1$.
- So for any $ijk \in M_4^{(3)}$, $c(ij) \neq c(jk)$.
- Finally, colour $M_4^{(3)}$ by: ijk is gold if $c(ij) = c(ik)$ and $c(ik) = c(jk)$, silver if $c(ij) = c(ik)$ and $c(ik) \neq c(jk)$, bronze if $c(ij) \neq c(ik)$ and $c(ik) = c(jk)$, and platinum if $c(ij) \neq c(ik)$ and $c(ik) \neq c(jk)$.
- By Ramsey's theorem for 3-sets, there exists monochromatic $M_5 \subseteq M_4$. M_5 cannot be gold, since then $c(ij) = c(jk)$: contradiction, since $M_5 \subseteq M_4$. If silver, then we have case 3 in the theorem. If bronze, then we have case 4 in the theorem. If platinum, then we have case 2 in the theorem.

$\square$

Remark 1.15

- A more general result of the above theorem states: let $\mathbb{N}^{(r)}$ be arbitrarily coloured. Then we can find an infinite M and $I \subseteq [r]$ such that for all $x_1 ... x_r \in M^{(r)}$ and $y_1 ... y_r \in M^{(r)}$, $c(x_1 ... x_r) = c(y_1 ... y_r)$ iff $x_i = y_i$ for all $i \in I$.
- In canonical Ramsey, $I = \emptyset$ is case 1, $I = \{1, 2\}$ is case 2, $I = \{1\}$ is case 3 and $I = \{2\}$ is case 4.
- These 2^r colourings are called the **canonical colourings** of $\mathbb{N}^{(r)}$.

Exercise 1.16 Prove the general statement.

1.3. Van der Waerden's theorem

Remark 1.17 We want to show that for any 2-colouring of $\mathbb{N}$, we can find a monochromatic arithmetic progression of length m for any $m \in \mathbb{N}$. By compactness, this is equivalent to showing that for all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any 2-colouring of $[n]$, there exists a monochromatic arithmetic progression of length m. (If not, then for each $n \in \mathbb{N}$, there is a colouring $c_n : [n] \to \{1, 2\}$ with no monochromatic arithmetic progression of length m. Infinitely many of these colourings agree on [1], infinitely many of those agreeing in [1] agree on [2], and so on - we obtain a 2-colouring of $\mathbb{N}$ with no monochromatic arithmetic progression of length m).

We will prove a slightly stronger result: whenever $\mathbb{N}$ is k-coloured, there exists a length m monochromatic arithmetic progression, i.e. for any $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k-coloured, we have a length m monochromatic progression.

Definition 1.18 Let $A_1, ..., A_k$ be length m arithmetic progressions: $A_i = \{a_i, a_i + d_i, ..., a_i + (m-1)d_i\}$. $A_1, ..., A_k$ are **focussed** at f if $a_i + md_i = f$ for all i.

Example 1.19 $\{4, 8\}$ and $\{6, 9\}$ are focussed at 12.

Definition 1.20 If length m arithmetic progressions $A_1, ..., A_k$ are focused at f and are monochromatic, each with a different colour (for a given colouring), they are called **colour-focussed** at f.

Remark 1.21 We use the idea that if $A_1, ..., A_k$ are colour-focussed at f (for a k-colouring) and of length $m - 1$, then some $A_i \cup \{f\}$ is a length m monochromatic arithmetic progression.

Theorem 1.22 Whenever $\mathbb{N}$ is k-coloured, there exists a monochromatic arithmetic progression of length 3, i.e. for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any k-colouring of $[n]$ admits a length 3 monochromatic progression.

Proof (Hints).

- Prove by induction the claim: $\forall r \leq k$, $\exists n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there exists a monochromatic arithmetic progression of length 3, or r colour-focussed arithmetic progressions of length 2.
  - $r = 1$ case is straightforward.
  - Let claim be true for $r - 1$ with witness n, let $N = 2n(k^{2n} + 1)$.
  - Partition N into blocks of equal size, show that two of these blocks must have the same colouring.
  - Using the inductive hypothesis, merge the $r - 1$ colour-focussed arithmetic progressions from these two blocks into a new set of $r - 1$ colour-focussed arithmetic progressions.
  - Find another length 2 monochromatic arithmetic progression, reason that this is of different colour.
- Reason that this claim implies the result.

□

Proof.

- We claim that for all $r \leq k$, there exists an $n \in \mathbb{N}$ such that if $[n]$ is k-coloured, then either:
  - There exists a monochromatic arithmetic progression of length 3.
  - There exist r colour-focussed arithmetic progressions of length 2.
- This claim implies the result by the above remark.
- We prove the claim by induction on r:
  - $r = 1$: take $n = k + 1$, then by pigeonhole, some two elements of $[n]$ have the same colour, so form a length two arithmetic progression.
  - Assume true for $r - 1$ with witness n. We claim that $N = 2n(k^{2n} + 1)$ works for r.
  - Let $c : [2n(k^{2n} + 1)] \to [k]$ be a colouring. We partition $[N]$ into $k^{2n} + 1$ blocks of size $2n$: $B_i = \{2n(i - 1) + 1, ..., 2ni\}$ for $i = 1, ..., k^{2n} + 1$.
  - Assume there is no length 3 monochromatic progression for c. By inductive hypothesis, each block B_i has $r - 1$ colour-focussed arithmetic progressions of length 2.
  - Since $|B_i| = 2n$, each block also contains their focus. For a set M with $|M| = 2n$, there are k^{2n} ways to k-colour M. So by pigeonhole, there are blocks B_s and B_{s+t} that have the same colouring.
  - Let $\{a_i, a_i + d_i\}$ be the $r - 1$ arithmetic progressions in B_s colour-focussed at f, then $\{a_i + 2nt, a_i + d_i + 2nt\}$ is the corresponding set of arithmetic progressions in B_{s+t}, each colour-focussed at $f + 2nt$.
  - Now $\{a_i, a_i + d_i + 2nt\}$, $i \in [r - 1]$, are $r - 1$ arithmetic progresions colour-focused at $f + 4nt$. Also, $\{f, f + 2nt\}$ is monochromatic of a different colour to the $r - 1$ colours used (since there is no length 3 monochromatic progression for c). Hence, there are r arithmetic progressions of length 2 colour-focussed at $f + 4nt$.

□

Remark 1.23 The idea of looking at all possible colourings of a set is called a **product argument**.

Definition 1.24 The **Van der Waerden** number $W(k, m)$ is the smallest $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there exists a monochromatic arithmetic progression in $[n]$ of length m.

Remark 1.25 The above theorem gives a **tower-type** upper bound $W(k, 3) \leq k^{k^{(\cdot\cdot\cdot)^{k^{4k}}}}$.

Theorem 1.26 (Van der Waerden's Theorem) For all $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there is a length m monochromatic arithmetic progression.

Proof (Hints).

- Use induction on m.

- Given induction hypothesis on $m-1$, prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, we have either a monochromatic length m arithmetic progression, or r colour-focussed arithmetic progressions of length $m-1$. Reason that this claim implies the result.
- Use induction on r. Give an explicit n for $r = 1$.
- Let n be the witness for $r-1$, let $N = W(k^{2n}, m-1) \cdot 2n$. Assume a k-colouring of $[N]$, $c : [N] \to [k]$, has no arithmetic progressions of length m.
- Partition $[N]$ into the obvious choice of $W(k^{2n}, m-1)$ blocks B_i, each of length $2n$.
- Colour the indices $1 \leq i \leq W(k^{2n}, m-1)$ of the blocks by

$$c'(i) = (c(2n(i-1)+1), c(2n(i-1)+2)...., c(2ni))$$

- Reason that we can find monochromatic arithmetic progression $s, s+t, ..., s+(m-2)t$ of length $m-1$ (w.r.t c'), and that this corresponds to sequence of blocks $B_s, B_{s+t}, ..., B_{s+(m-2)t}$, each identically coloured.
- Reason that B_s contains $r-1$ colour-focussed length $m-1$ arithmetic progressions A_i together with their focus f.
- Let A_i' be the same arithmetic progression but with common difference $2nt$ larger than that of A_i. Show the A_i' are colour-focussed at some focus in terms of f.
- Find another length $m-1$ arithmetic progression, show this must be monochromatic and of different colour to all A_i'. Show it also has same focus as all A_i'.

□

Proof.

- By induction on m. $m = 1$ is trivial, $m = 2$ is by pigeonhole principle. $m = 3$ is the statement of the previous theorem.
- Assume true for $m-1$ and all $k \in \mathbb{N}$.
- For fixed k, we prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, either:
  - There is a monochromatic arithmetic progression of length m, or
  - There are r colour-focussed arithmetic progressions of length $m-1$.
- We will then be done (by considering the focus).
- To prove the claim, we use induction on r.
- $r = 1$ is the claim of the first inductive hypothesis: take $n = W(k, m-1)$.
- Assume the claim holds for $r-1$ with witness n, and assume there is no monochromatic arithmetic progression of length m. We will show that $N = W(k^{2n}, m-1)2n$ is sufficient for r.
- Partition $[N]$ into $W(k^{2n}, m-1)$ blocks of length $2n$: $B_i = \{2n(i-1)+1, ..., 2ni\}$ for $i = 1, ..., W(k^{2n}, m-1)$.
- Each block has k^{2n} possible colourings. Colour the blocks as

$$c'(i) = (c(2n(i-1)+1), c(2n(i-1)+2)...., c(2ni))$$

By definition of W, there exists a monochromatic arithmetic progression of length $m-1$ (w.r.t. to c'): $\{\alpha, \alpha+t, ..., \alpha+(m-2)t\}$. The repsective blocks $B_\alpha, ..., B_{\alpha+(m-2)t}$ are identically coloured.

- B_α has length $2n$, so by induction B_α contains $r-1$ colour-focussed arithmetic progressions of length $m-1$, together with their focus (as length of block is $2n$).
- Let $A_1, ..., A_{r-1}$, $A_i = \{a_i, a_i+d_i, ..., a_i+(m-2)d_i\}$, be colour-focussed at f.
- Let $A_i' = \{a_i, a_i+(d_i+2nt), ..., a_i+(m-2)(d_i+2nt)\}$ for $i = 1, ..., r-1$. The A_i' are monochromatic as the blocks are identically coloured and the A_i are monochromatic. Also, A_i and A_i' have the same colouring, and the A_i are colour-focussed, hence the A_i' have pairwise distinct colours.
- The A_i are focussed at f and the colour of f of different than the colour of all A_i. $f = a_i + (m-1)d_i$ for all i.
- Now $\{f, f+2nt, f+4nt, ..., f+2n(m-2)t\}$ is an arithmetic progression of length $m-1$, is monochromatic and of a different colour to all the A_i'.
- It is enough to show that $a_i + (m-1)(d_i+2nt) = f + 2n(m-1)t$ for all i, but this is equivalent to $a_i + (m-1)d_i = f$, which is true as all A_i were focussed at f.

□

Corollary 1.27 For any k-colouring of $\mathbb{N}$, there exists a colour class containing arbitrarily long arithmetic progressions.

Remark 1.28 We can't guarantee infinitely long arithmetic progressions, e.g.

- 2-colour $\mathbb{N}$ by 1 red, $2, 3$ blue, $4, 5, 6$ red, etc.
- The set of infinite arithmetic progressions in $\mathbb{N}$ is countable (since described by two integers: the start term and step). Enumerate them by $(A_k)_{k\in\mathbb{N}}$. Pick $x_1 < y_1 \in A_1$, colour x_1 red and y_1 blue. Then pick $x_2, y_2 \in A_2$ with $y_1 < x_2 < y_2$, colour x_2 red, y_2 blue. Continue inductively.

Theorem 1.29 (Strengthened Van der Waerden) Let $m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there exists a monochromatic length m arithmetic progression whose common difference is the same colour (i.e. there exists $a, a+d, ..., a+(m-1), d$ all of the same colour).

Proof (Hints).

- Use induction on k.
- If n is the witness for $k-1$ colours, show that $N = W(k, n(m-1)+1)$ is a witness for k colours, by considering n different multiples of the step of a suitable arithmetic progression.

□

Proof.

- Fix $m \in \mathbb{N}$. We use induction on k. $k = 1$ case is trivial.
- Let n be witness for $k-1$ colours.
- We will show that $N = W(k, n(m-1)+1)$ is suitable for k colours.
- If $[N]$ is k-coloured, there exists a monochromatic (say red) arithemtic progression of length $n(m-1)+1$: $a, a+d, ..., a+n(m-1)d$.

- If rd is red for any $1 \leq r \leq n$, then we are done (consider $a, a + rd, ..., a + (m - 1)rd$).
- If not, then $\{d, 2d, ..., nd\}$ is $k - 1$-coloured, which induces a $k - 1$ colouring on $[n]$. Therefore, there exists a monochromatic arithmetic progression $b, b + s, ..., b + (m - 1)s$ (with s the same colour) by induction, which translates to $db, db + ds, ..., db + d(m - 1)s$ and ds being monochromatic.

□

Remark 1.30 The case $m = 2$ of strengthened Van der Waerden is **Schur's theorem**: for any k-colouring of $\mathbb{N}$, there are monochromatic x, y, z such that $x + y = z$. This can be proved directly from Ramsey's theorem for pairs: let $c : \mathbb{N} \to [k]$ be a k -colouring, then induce $c' : \mathbb{N}^{(2)} \to [k]$ by $c'(ij) = c(j - i)$. By Ramsey, there exist $i < j < k$ such that $c'(ij) = c'(ik) = c'(jk)$, i.e. $c(j - i) = c(k - i) = c(k - j)$. So take $x = j - i$, $z = k - i$, $y = k - j$.

1.4. The Hales-Jewett theorem

Definition 1.31 Let X be finite set. We say X^n consists of **words of length n on alphabet X**.

Definition 1.32 Let X be finite. A **(combinatorial) line** in X^n is a set $L \subseteq X^n$ of the form

$$L = \left\{(x_1, ..., x_n) \in X^n : \forall i \notin I, x_i = a_i \text{ and } \forall i, j \in I, x_i = x_j\right\}$$

for some non-empty set $I \subseteq [n]$ and $a_i \in X$ (for each $i \notin I$). I is the set of **active coordinates** for L.

Note that a combinatorial line is invariant under permutations of X.

Example 1.33 Let $X = [3]$. Some lines in X^2 are:
- $I = \{1\}$: $\{(1, 1), (2, 1), (3, 1)\}$ (with $a_2 = 1$), $\{(1, 2), (2, 2), (3, 2)\}$ (with $a_2 = 2$), $\{(1, 3), (2, 3), (3, 3)\}$ (with $a_2 = 3$).
- $I = \{2\}$: $\{(1, 1), (1, 2), (1, 3)\}$ (with $a_1 = 1$), $\{(2, 1), (2, 2), (2, 3)\}$ (with $a_1 = 2$), $\{(3, 1), (3, 2), (3, 3)\}$ (with $a_1 = 3$).
- $I = \{1, 2\}$: $\{(1, 1), (2, 2), (3, 3)\}$.

Note that $\{(1, 3), (2, 2), (3, 1)\}$ is **not** a combinatorial line.

Example 1.34 Some sets of lines in $[3]^3$ are:
- $I = \{1\}$: $\{(1, 2, 3), (2, 2, 3), (3, 2, 3)\}$ (with $a_2 = 2, a_3 = 3$).
- $I = \{1, 3\}$: $\{(1, 3, 1), (2, 3, 2), (3, 3, 3)\}$ (with $a_2 = 3$).

Definition 1.35 In a line L, write L^- and L^+ for the smallest and largest points in L (with respect to the ordering on $[m]^n$ where $x \leq y$ if $x_i \leq y_i$ for all i).

Definition 1.36 Lines $L_1, ..., L_k$ are **focussed** at f if $L_i^+ = f$ for all $i \in [k]$. They are **colour-focussed** if they are focussed and $L_i \setminus \left\{L_i^+\right\}$ is monochromatic for all $i \in [k]$, with each $L_i \setminus \left\{L_i^+\right\}$ a different colour.

Theorem 1.37 (Hales-Jewett) Let $m, k \in \mathbb{N}$ (we use alphabet $X = [m]$), then there exists $n \in \mathbb{N}$ such that for any k-colouring of $[m]^n$, there exists a monochromatic combinatorial line.

Notation 1.38 Denote the smallest such n by $\mathrm{HJ}(m, k)$.

Proof (Hints).
- Induction on m. Prove by induction the claim that for all $1 \le r \le k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[m]^n$, we have either a monochromatic line, or r colour-focussed lines (reason that this claim implies the result).
- State why claim holds for $r = 1$.
- Let n be witness for $r - 1$, $n' = \mathrm{HJ}(m - 1, k^{m^n})$. Want to show that $n + n'$ is witness for r.
- Write $[m]^{n+n'} = [m]^n \times [m]^{n'}$.
- For a colouring $c : [m]^{n+n'} \to [k]$, induce a suitable colouring $c' : [m]^{n'} \to [k]^{m^n}$ and consider what the definition of n' implies. Use this to induce a colouring $c'' : [m]^n \to [k]$.
- Using the inductive hypothesis and the previous point, construct $r - 1$ lines in $[m]^{n+n'}$ which are colour-focussed. Find another line in $[m]^{n+n'}$ (which should have first n coordinates constant) of different colour which has the same focus point.

□

Proof. By induction on m. The case $m = 1$ is trivial as $|[m]^n| = 1$. Assume that $\mathrm{HJ}(m - 1, k')$ exists for all $k' \in \mathbb{N}$. We claim that for all $1 \le r \le k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[m]^n$, we have either:
- a monochromatic line, or
- r colour-focussed lines.

We can then take $r = k$ and consider the focus.

We prove the claim by induction on r. For $r = 1$, $n = \mathrm{HJ}(m - 1, k)$ suffices. Let n be a witness for $r - 1$. Let $n' = \mathrm{HJ}(m - 1, k^{m^n})$. We will show $N = n + n'$ is a witness for r. Let $c : [m]^N \to [k]$ be a k-colouring with no monochromatic lines. Writing $[m]^N = [m]^n \times [m]^{n'}$, colour $[m]^{n'}$ by $c' : [m]^{n'} \to [k]^{m^n}$, $c'(b) = (c(a_1, b), ..., c(a_{m^n}, b))$ (where $[m]^n = \{a_1, ..., a_{m^n}\}$). By the inductive hypothesis, there exists a line L in $[m]^{n'}$ with active coordinates I such that

$$\forall a \in [m]^n, \forall b, b' \in L \setminus \{L^+\}, \quad c(a, b) = c(a, b').$$

But now this induces a (well-defined) colouring $c'' : [m]^n \to [k]$, $c''(a) = c(a, b)$ for any $b \in L \setminus \{L^+\}$. By definition of n, there exist $r - 1$ lines $L_1, ..., L_{r-1}$ colour-focussed (w.r.t c'') at f, with active coordinates $I_1, ..., I_{r-1}$.

Finally, consider the $r - 1$ lines L_i', $1 \le i \le r - 1$ in $[m]^N$ that start at (L_i^-, L^-) with active coordinates $I_i \cup I$, and the line L' in $[m]^N$ that starts at (f, L^-) with active coordinates I. By the construction of c'', the colour of each point in L_i' is determined by the first n coordinates which form a point lying in L_i. Hence, since the L_i are colour-focussed, the L'_i are colour-focussed. As for L', the first n coordinates are constant (always equal to f), and so again by the construction of c'', the colour of each point in L' is equal to $c''(f)$, which is a different colour to each colour of the L'_i. Hence all $L'_1, ..., L'_{r-1}, L'$ colour-focussed at (f, L^+), so we are done. □

Corollary 1.39 Hales-Jewett implies Van der Waerden's theorem.

Proof (Hints). For a colouring $c : \mathbb{N} \to [k]$, consider the induced colouring $c'(x_1, ..., x_n) = c(x_1 + \cdots + x_n)$ of $[m]^n$. □

Proof. Let c be a k-colouring of $\mathbb{N}$. For sufficiently large n (i.e. $n \geq \mathrm{HJ}(m, k)$), induce a k-colouring c' of $[m]^n$ by $c'(x_1, ..., x_n) = c(x_1 + \cdots + x_n)$. By Hales-Jewett, a monochromatic (with respect to c') combinatorial line L exists. This gives a monochromatic (with respect to c) length m arithmetic progression in $\mathbb{N}$. The step is equal to the number of active coordinates. The first term in the arithmetic progression corresponds to the point in L with all active coordinates equal to 1, the last term corresponds to the point in L with all active coordinates equal to m. □

Exercise 1.40 Show that the m-in-a-row noughts and crosses game cannot be a draw in sufficiently high dimensions, and that the first player can always win.

Definition 1.41 A ***d*-dimensional subspace** (or ***d*-point parameter set**) $S \subseteq X^n$ is a set such that there exist pairwise disjoint $I_1, ..., I_d \subseteq [n]$ and $a_i \in X$ for all $i \in [n] - (I_1 \cup \cdots \cup I_d)$, such that

$$\begin{aligned} S = \big\{x \in X^n : x_i = a_i \quad &\forall i \in [n] - (I_1 \cup \cdots \cup I_d), \\ \text{and } x_i = x_j \quad &\forall i, j \in I_k \text{ for some } k \in [d]\big\}. \end{aligned}$$

Example 1.42 Two 2-dimensional subspaces in X^3 are $\{(x, y, 2) : x, y \in X\}$ $(I_1 = \{1\}, I_2 = \{2\})$ and $\{(x, x, y) : x, y \in X\}$ $(I_1 = \{1, 2\}, I_2 = \{3\})$.

Theorem 1.43 (Extended Hales-Jewett) For all $m, k, d \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any colouring of $[m]^n$, there exists a monochromatic d-dimensional subspace.

Proof (Hints). Use Hales-Jewett on m^d and k. □

Proof. We can view $X^{dn'}$ as $\left(X^d\right)^{n'}$. A line in $\left(X^d\right)^{n'}$ (on alphabet $Y = X^d$) corresponds to a d-dimensional subspace in $X^{dn'}$ (on alphabet X). (Each inactive coordinate in the line corresponds to d adjacent inactive coordinates in the subspace, and each active coordinate in the line corresponds to d adjacent active coordinates in the subspace). Hence, we can take $n = d \cdot \mathrm{HJ}\left(m^d, k\right)$. □

Definition 1.44 Let $S \subseteq \mathbb{N}^d$ be finite. A **homothetic copy** of S is a set of the form $a + \lambda S$ where $a \in \mathbb{N}^d$ and $\lambda \in \mathbb{N}$ $(l \neq 0)$.

Theorem 1.45 (Gallai) Let $S \subseteq \mathbb{N}^d$ be finite. For every k-colouring of $\mathbb{N}^d$, there exists a monochromatic homothetic copy of S.

Proof (Hints). Let $S = \{S_1, ..., S_m\}$, consider colouring $c' : [m]^n \to [k]$ (for suitable n) given by $c'(x_1, ..., x_n) = c(S_{x_1}, ..., S_{x_m})$. □

Proof. Let $S = \{S_1, ..., S_m\}$. Let $c : \mathbb{N}^d \to [k]$ be a k-colouring. For n large enough (i.e. $n \geq \mathrm{HJ}(m, k)$), colour $[m]^n$ by $c'(x_1, ..., x_n) = c\left(S_{x_1} + \cdots + S_{x_m}\right)$. By Hales-Jewett, there exists a monochromatic line (with respect to c') in $[m]^n$ with active coordinates I. So $c\left(\sum_{i \notin I} S_i + |I| S_j\right)$ is the same colour for all $j \in [m]$. So we are done, as $\sum_{i \notin I} S_i + |I| S$ is a homothetic copy of S. $\square$

Remark 1.46

- Gallai's theorem can also be proven with a focussing + product colouring argument.
- For $S = \{(x, y) \in \mathbb{N}^2 : x, y \in \{1, 2\}\}$, Gallai's theorem proves the existence of a monochromatic square whereas extended Hales-Jewett only guarantees a monochromatic rectangle.

2. Partition regular systems

2.1. Rado's theorem

Strengthened Van der Waerden says that the system $x_1 + x_2 = y_1, x_1 + 2x_2 = y_2, ..., x_1 + m x_2 = y_m$ has a monochromatic solution in $x_1, x_2, y_1, ..., y_m$. We want to find when a general system of equations is partition regular.

Definition 2.1 Let $A \in \mathbb{Q}^{m \times n}$ be a $m \times n$ matrix. A is **partition regular (PR)** if for any finite colouring of $\mathbb{N}$, there exists a monochromatic $\boldsymbol{x} \in \mathbb{N}^n$ such that $A\boldsymbol{x} = \boldsymbol{0}$.

Example 2.2

- Schur's theorem says that $x + y = z$ has a monochromatic solution for any finite colouring of $\mathbb{N}$, and so that $(1, 1, -1)$ is PR.
- Strengthened Van der Waerden states that

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

  is PR.
- $(a, b, -(a + b))$ is PR for any a, b (a monochromatic solution is $x = y = z$).
- $(2, -1)$ is not PR: colour $\mathbb{N}$ by n is red if $\max\{m \in \mathbb{N} : 2^m \mid n\}$ is even, and blue otherwise. Then if $2x = y$, x and y must have different colours.

Definition 2.3 A rational matrix A with columns $\boldsymbol{c}_1, ..., \boldsymbol{c}_n \in \mathbb{Q}^m$ has the **column property (CP)** if there exists a partition $B_1 \sqcup \cdots \sqcup B_r$ of $[n]$ such that:

1. $\sum_{i \in B_1} \boldsymbol{c}_i = \boldsymbol{0}$.
2. For all $s \in \{2, ..., r\}$, $\sum_{i \in B_s} \boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{c}_j : j \in B_1 \sqcup \cdots \sqcup B_{s-1}\}$ (note we can take the linear span over $\mathbb{R}$ or over $\mathbb{Q}$ here, as if a rational vector is a real linear combination of rational vectors, then it is also a rational linear combination of them).

Example 2.4

- $(1, 1, -1)$ has CP, with $B_1 = \{1, 3\}$, $B_2 = \{2\}$.
- The matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

  from Strengthened Van der Waerden has CP, with $B_1 = \{1, 3, ..., n\}$ and $B_2 = \{2\}$.
- $(3, 4, -7)$ has CP with $B_1 = \{1, 2, 3\}$.
- $(\lambda, -1)$ has CP iff $\lambda = 1$.
- $(3, 4, -6)$ doesn't have CP.

Example 2.5

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & a \\ 4 & -4 & b \end{bmatrix}$$

has CP iff $(a, b) = (6, 12)$.

Remark 2.6 $\boldsymbol{x} = (a_1, ..., a_n)$ is PR iff $\lambda\boldsymbol{x}$ is PR (for any $\lambda \in \mathbb{Q}^\times$), so we can assume that each $a_i \in \mathbb{Z}$. Also, $\boldsymbol{x}$ has CP iff there exists $\emptyset \neq I \subseteq [n]$ such that $\sum_{i \in I} a_i = 0$. We may also assume WLOG each $a_i \neq 0$. We will first show that if $\boldsymbol{x}$ is PR, then it has CP. Even in the $1 \times n$ matrix case of Rado's theorem, neither direction is easy.

Notation 2.7 For p prime and $x = (a_k ... a_0)_p \in \mathbb{N}$, write $e(x)$ for the rightmost non-zero digit in the base-p expansion of x, i.e. $e(x) = a_{t(x)}$, where $t(x) = \min\{i : a_i \neq 0\}$.

Proposition 2.8 Let $a_1, ..., a_n \in \mathbb{Q}^*$. If $(a_1, ..., a_n)$ is PR, then it has CP.

Proof (Hints). For p large enough (determine later a bound for p), colour $\mathbb{N}$ by giving x colour $e(x)$, and consider $\min\{t(x_1), ..., t(x_n)\}$. □

Proof. Let p be a large prime ($p > \sum_{i=1}^n |a_i|$). Define a $(p-1)$-colouring of $\mathbb{N}$ giving x colour $e(x)$. By assumption, there are $x_1, ..., x_n$ of the same colour d such that $\sum_{i=1}^n a_i x_i = 0$. Let $t = \min\{t(x_1), ..., t(x_n)\}$, and let $I = \{i \in [n] : t(x_i) = t\}$ (note I is non-empty). So when summing $\sum_{i=1}^n a_i x_i = 0$ and considering the last digit in the base p expansion, we have $\sum_{i=1}^n a_i x_i = 0 \bmod p^{t+1}$ and so obtain $\sum_{i \in I} a_i d = 0 \bmod p$, so $\sum_{i \in I} a_i = 0$ (since p is prime and was chosen large enough). □

Remark 2.9 There is no other known proof of this proposition.

Lemma 2.10 Let $\lambda \in \mathbb{Q}$. Then $(1, \lambda, -1)$ is partition regular, i.e. for any finite colouring of $\mathbb{N}$, there exists monochromatic $(x, y, z) \in \mathbb{N}^3$ such that $x + \lambda y = z$.

Proof (Hints).

- Reason that we can assume $\lambda > 0$. Write $\lambda = r/s$, $r, s \in \mathbb{N}$.
- Use induction on number of colours k: given n such that any $(k-1)$-colouring of $[n]$ admits monochromatic solution, show that $N = W(k, nr + 1)ns$ works for k colours, by considering the definition of W and isd for eacah $i \in [n]$.

□

Proof. The case $\lambda = 0$ is trivial, and if $\lambda < 0$, we may rewrite the equation as $z - \lambda y = x$, so we may assume that $\lambda > 0$, so let $\lambda = \frac{r}{s}$ for $r, s \in \mathbb{N}$. In fact, we show that for any k-colouring of $[n]$ (for some n depending on k), there is a monochromatic solution.

We seek a monochromatic solution to $x + \frac{r}{s}y = z$ for some finite colouring $c : \mathbb{N} \to [k]$. We use induction on the number of colours k. For $k = 1$, $n = \max\{s, r + 1\}$ is sufficient, with monochromatic solution $(1, s, r + 1)$. Assume n is a witness for $k - 1$ colours. We will show $N = nsW(k, nr + 1)$ is suitable for k colours. By definition of W, given a k-colouring of $[N]$, there is a monochromatic AP inside $[W(k, nr + 1)] \subseteq [N]$ of length $nr + 1$: $a, a + d, ..., a + nrd$, coloured red.

Consider isd for each $i \in [n]$. Note that $isd \leq nsW(k, nr + 1)$ so each isd does indeed have a colour. If some isd is also red, then $(a, isd, a + ird)$ is a monochromatic solution. If no isd is red, then $\{sd, ..., nsd\}$ is $(k - 1)$-coloured, so by the inductive hypothesis, there exists $i, j, k \in [n]$ such that $\{isd, jsd, ksd\}$ is monochromatic and $isd + \lambda jsd = ksd$, so (isd, jsd, ksd) is a monochromatic solution. □

Remark 2.11

- Note the similarity to the proof of Strengthened Van der Waerden.
- The case $\lambda = 1$ is Schur's theorem, which can be proven directly by Ramsey's theorem; however, there is no known proof using Ramsey's theorem for general $\lambda \in \mathbb{Q}$.

Theorem 2.12 (Rado's Theorem for Single Equations) Let $a_1, ..., a_n \in \mathbb{Q} \setminus \{0\}$. $(a_1, ..., a_n)$ is PR iff it has CP.

Proof (Hints). For $\Longleftarrow$: for the obvious choice of $I \subseteq [n]$, fix $i_0 \in I$, and define $\boldsymbol{x} \in \mathbb{N}^n$ componentwise:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}.$$

Show that $\boldsymbol{x}$ is a solution to $\sum_{i=1}^{n} a_i x_i = 0$. □

Proof. $\Longrightarrow$ is by Proposition 2.8. For $\Longleftarrow$: we have that $\sum_{i \in I} a_i = 0$ for some $\emptyset \neq I \subseteq [n]$. Given a colouring $c : \mathbb{N} \to [k]$, we need to show that there are monochromatic $x_1, ..., x_n$ such that $\sum_{i=1}^{n} a_i x_i = 0$.

Fix $i_0 \in I$. We construct the following vector $\boldsymbol{x} \in \mathbb{N}^n$ by defining its components:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}$$

for some fixed suitable x, y, z. We need x, y, z to be monochromatic and

$$
\begin{aligned}
a_{i_0}x + \sum_{i\notin I} a_i y + \sum_{i\in I\setminus\{i_0\}} a_i z &= 0 \\
\iff a_{i_0}x - z a_{i_0} + \sum_{i\notin I} a_i y &= 0 \\
\iff x + \frac{\sum_{i\notin I} a_i}{a_{i_0}} y - z &= 0
\end{aligned}
$$

and this holds, since x, y, z exist by the above lemma. □

Conjecture 2.13 (Rado's Boundedness Conjecture) Let A be an $m \times n$ matrix that is not PR (so there exists a "bad" colouring, i.e. a k-colouring with no monochromatic solution to $A\boldsymbol{x} = \boldsymbol{0}$ for some $k \in \mathbb{N}$). Is k bounded (for given m, n)?

This is known for 1×3 matrices: 24 colours suffice.

Proposition 2.14 Let $A \in \mathbb{Q}^{m\times n}$. If A is PR, then it has CP.

Proof (Hints).

- Let $\boldsymbol{x} \in \mathbb{N}^n$ be the monochromatic solution to $A\boldsymbol{x} = \boldsymbol{0}$. For fixed prime p, partition $[n]$ into $B_1, ..., B_r$ by grouping $i, j \in [n]$ by $t(x_i), t(x_j)$ (and preserving the ordering).
- Reason that the same partition exists for infinitely many p.
- Considering $\sum_{i=1}^n x_i \boldsymbol{c}_i = \boldsymbol{0} \bmod p$ for infinitely many p, show that $\sum_{i\in B_1} \boldsymbol{c}_i = 0$, and

$$p^t \sum_{i\in B_k} \boldsymbol{c}_i + \sum_{i\in B_1,...,B_{k-1}} x_i d^{-1} \boldsymbol{c}_i \equiv \boldsymbol{0} \bmod p^{t+1}.$$

- By taking the dot product with $\boldsymbol{u} \in \mathbb{N}^m$ for appropriate u, show by contradiction that $\sum_{i\in B_k} \boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{c}_i : i \in B_1, ..., B_{k-1}\}$.

□

Proof. Let $\boldsymbol{c}_1, ..., \boldsymbol{c}_n \in \mathbb{Q}^m$ be the columns of A. For fixed prime p, colour $\mathbb{N}$ as before by $c(x) = e(x)$. By assumption, there exists a monochromatic $\boldsymbol{x} \in \mathbb{N}^n$ such that $\sum_{i=1}^n x_i \boldsymbol{c}_i = \boldsymbol{0}$. We partition the columns (by partitioning $[n] = B_1 \sqcup \cdots \sqcup B_r$) as follows:

- $i, j \in B_k$ iff $t(x_i) = t(x_j)$.
- $i \in B_k$, $j \in B_\ell$ for $k < \ell$ iff $t(x_i) < t(x_j)$.

We do this for infinitely many primes p. Since there are finitely many partitions of $[n]$, for infinitely many p, we will have the same blocks $B_1, ..., B_r$.

Consider $\sum_{i=1}^n x_i \boldsymbol{c}_i = \boldsymbol{0}$ performed in base p. Each $i \in [n]$ has the same colour $d = e(x_i) \in [1, p-1]$. So $\sum_{i\in B_1} d\boldsymbol{c}_i = 0 \bmod p$ (by collecting the rightmost terms in base p), hence $\sum_{i\in B_1} c_i = 0 \bmod p$. But this holds for infinitely many p, hence

$$\sum_{i\in B_1} \boldsymbol{c}_i = 0.$$

Now $\sum_{i \in B_k} p^t d \boldsymbol{c}_i + \sum_{i \in B_1, ..., B_{k-1}} x_i \boldsymbol{c}_i = \mathbf{0} \bmod p^{t+1}$ for some t. So

$$p^t \sum_{i \in B_k} \boldsymbol{c}_i + \sum_{i \in B_1, ..., B_{k-1}} x_i d^{-1} \boldsymbol{c}_i \equiv \mathbf{0} \bmod p^{t+1}.$$

We claim that $\sum_{i \in B_k} \boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{c}_i : i \in B_1, ..., B_{k-1}\}$. Suppose not, then there exists $\boldsymbol{u} \in \mathbb{N}^m$ such that $\boldsymbol{u}.\boldsymbol{c}_i = 0$ for all $i \in B_1, ..., B_{k-1}$, but $\boldsymbol{u}.\left(\sum_{i \in B_k} \boldsymbol{c}_i\right) \neq 0$. Then dotting with $\boldsymbol{u}$, we obtain $p^t \boldsymbol{u}.\left(\sum_{i \in B_k} \boldsymbol{c}_i\right) \equiv 0 \bmod p^{t+1}$, so $\boldsymbol{u}.\sum_{i \in B_k} \boldsymbol{c}_i \equiv 0 \bmod p$. But this holds for infinitely many p, so $\boldsymbol{u}.\sum_{i \in B_k} \boldsymbol{c}_i = 0$: contradiction. □

Definition 2.15 For $m, p, c \in \mathbb{N}$, an **(m, p, c)-set** $S \subseteq \mathbb{N}$ with generators $x_1, ..., x_m \in \mathbb{N}$ is of the form

$$S = \left\{ \sum_{i=1}^{m} \lambda_i x_i : \exists j \in [m] : \lambda_j = c, \lambda_i = 0\ \forall i < j, \text{and } \lambda_k \in [-p, p]\ \forall k > j \right\}$$

where $[-p, p] = \{-p, -(p-1), ..., p\}$. So S consists of

$$\begin{aligned} cx_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_m x_m, &\quad \lambda_i \in [-p, p], \\ cx_2 + \lambda_3 x_3 + \cdots + \lambda_m x_m, &\quad \lambda_i \in [-p, p], \\ \vdots \\ cx_m. \end{aligned}$$

These are the **rows** of S. We can think of S as a "progression of progressions".

Example 2.16

- A $(2, p, 1)$-set with generators x_1, x_2 is of the form $\{x_1 - px_2, x_1 - (p-1)x_2, ..., x_1 + px_2, x_2\}$, so is an AP of length $2p + 1$ together with its step.
- A $(2, p, 3)$-set with generators x_1, x_2 is of the form $\{3x_1 - px_2, 3x_1 - (p-1)x_2, ..., 3x_1, ..., 3x_1 + px_2, 3x_2\}$, so is an AP of length $2p + 1$, whose middle term is divisible by 3, together with three times its step.

Theorem 2.17 Let $m, p, c \in \mathbb{N}$. For any finite colouring of $\mathbb{N}$, there exists a monochromatic (m, p, c)-set.

Proof (Hints).

- Reason that an (m', p, c)-set contains an (m, p, c)-set for $m' \geq m$. With $M = k(m-1) + 1$, reason that if we can find an (M, p, c)-set with each row monochromatic, then we can find an monochromatic (m, p, c)-set.
- Let $A_1 = \{c, 2c, ..., \lfloor n/c \rfloor c\}$, reason that A_1 contains a set of the form $R_1 = \{cx_1 - n_1 d_1, cx_1 - (n_1 - 1)d_1, ..., cx_1 + n_1 d_1\}$ for some large n_1.
- Let $B_1 = \left\{d_1, 2d_1, ..., \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1\right\}$. We have $cx_1 + \lambda_1 b_1 + \cdots + \lambda_{M-1} b_{M-1} \in R_1$, explain why these are monochromatic.
- Inside B_1, define

$$A_2 = \left\{cd_1, 2cd_1, ..., \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1\right\}.$$

and apply the argument as before, where the divisor in the $\lfloor \cdot \rfloor$ expression in the new B_2 is $(M-2)p$.
- Argue that after a certain number of steps, we have formed an (M, p, c)-set with each row monochromatic.

$\square$

Proof. Let $c : \mathbb{N} \to [k]$ be the colouring of $\mathbb{N}$ with k colours. Note that an (m', p, c)-set with $m' \geq m$ contains an (m, p, c)-set (by taking any m rows, and setting some suitable λ_i to 0). Let $M = k(m-1)+1$. It is enough to find a (M, p, c)-set such that each row is monochromatic.

Let n be large (large enough to apply the argument that follows). Let $A_1 = \{c, 2c, ..., \lfloor n/c \rfloor c\}$. By Van der Waerden, A_1 contains a monochromatic AP R_1 of length $2n_1+1$ where n_1 is large enough:

$$R_1 = \{cx_1 - n_1 d_1, cx_1 - (n_1 - 1)d_1, ..., cx_1 + n_1 d_1\}.$$

has colour k_1. Now we restrict our attention to

$$B_1 = \left\{d_1, 2d_1, ..., \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1\right\}.$$

Observe that

$$cx_1 + \lambda_1 b_1 + \cdots + \lambda_{M-1} b_{M-1} \in R_1$$

for all $\lambda_i \in [-p, p]$ and $b_i \in B_1$, so all these sums have colour k_1. Inside B_1, look at

$$A_2 = \left\{cd_1, 2cd_1, ..., \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1\right\}.$$

By Van der Waerden, A_2 contains a monochromatic AP R_2 of length $2n_2+1$ with colour k_2:

$$R_2 = \{cx_2 - n_2 d_2, cx_2 - (n_2 - 1)d_2, ..., cx_2 + n_2 d_2\}.$$

Note that $x_2 \subseteq B_1$. Now we restrict our attention to

$$B_2 = \left\{d_2, 2d_2, ..., \left\lfloor \frac{n_2}{(M-2)p} \right\rfloor d_2\right\}.$$

Again, note that for all $\lambda_i \in [-p, p]$ and $b_i \in B_2$, we have

$$cx_2 + \lambda_1 b_1 + \cdots + \lambda_{M-2} b_{M-2} \in R_2$$

so has colour k_2.

We iterate this process M times, and obtain M generators $x_1, ..., x_M$ such that each row of the (M, p, c)-set generated by $x_1, ..., x_M$ is monochromatic. But now $M = k(m-1)+1$, so m of the rows have the same colour. $\square$

Remark 2.18 Being extremely precise in this proofs (such as considering $\lfloor\cdot\rfloor$) is much less important than the ideas in the proof. (Won't be penalised in the exam for small details like this).

Corollary 2.19 (Folkman's Theorem) Let $m \in \mathbb{N}$ be fixed. For every finite colouring of $\mathbb{N}$, there exists $x_1, ..., x_m \in \mathbb{N}$ such that

$$\mathrm{FS}(x_1, ..., x_m) := \left\{ \sum_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}$$

is monochromatic.

Proof (Hints). A specific case of Theorem 2.17. □

Proof. By the $(m, 1, 1)$ case of Theorem 2.17. □

Remark 2.20

- The case $n = 2$ of Folkman's theorem is Schur's theorem.
- For a colouring $c : \mathbb{N} \to [k]$, we induce a colouring $c' : \mathbb{N} \to [k]$ by $c'(n) = c(2^n)$. Then by Folkman's theorem for c', there exists $x_1, ..., x_m$ such that

$$\mathrm{FP}(x_1, ..., x_m) = \left\{ \prod_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}.$$

- It is not known whether the same result holds for $\mathrm{FS}(x_1, ..., x_m) \cup \mathrm{FP}(x_1, ..., x_m)$. However, it does not hold for infinite sets $\{x_n : n \in \mathbb{N}\}$, and does hold for colourings of $\mathbb{Q}$.

Proposition 2.21 Let A have CP. Then there exist $m, p, c \in \mathbb{N}$ such that every (m, p, c)-set contains a solution $\boldsymbol{y}$ to $A\boldsymbol{y} = \boldsymbol{0}$, i.e. all y_i belong to the (m, p, c)-set.

Proof. Let $\boldsymbol{c}_1, ..., \boldsymbol{c}_n$ be the columns of A. By assumption, there is a partition $B_1 \sqcup \cdots \sqcup B_r$ of $[n]$ such that $\forall k \in [r]$,

$$\begin{aligned}
&\sum_{i \in B_k} \boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{c}_i : \in B_1 \cup \cdots \cup B_{k-1}\} \\
\implies &\sum_{i \in B_k} \boldsymbol{c}_i = \sum_{i \in B_1 \cup \cdots \cup B_{k-1}} q_{ik} \boldsymbol{c}_i \quad \text{for some } q_{ik} \in \mathbb{Q} \\
\implies &\sum_{i=1}^{n} d_{ik} \boldsymbol{c}_i = \boldsymbol{0}
\end{aligned}$$

where

$$d_{ik} = \begin{cases} 0 & \text{if } i \notin B_1 \cup \cdots \cup B_{k-1} \\ 1 & \text{if } i \in B_k \\ -q_{ik} & \text{if } i \in B_1 \cup \cdots \cup B_{k-1} \end{cases}.$$

Take $m = r$. Let $x_1, ..., x_r \in \mathbb{N}$, and let $y_i = \sum_{k=1}^{r} d_{ik} x_k$ for each $i \in [n]$. Now $\boldsymbol{y} = (y_1, ..., y_n)$ is a solution to $A\boldsymbol{y} = \boldsymbol{0}$: we have

$$\begin{aligned}\sum_{i=1}^{n} y_i \boldsymbol{c}_i &= \sum_{i=1}^{n}\sum_{k=1}^{r} d_{ik} x_k \boldsymbol{c}_i \\ &= \sum_{k=1}^{r} x_k \sum_{i=1}^{n} d_{ik} \boldsymbol{c}_i = \mathbf{0}.\end{aligned}$$

Let c be the LCD of all the q_{ik}. Now $cy_i = \sum_{k=1}^{n} cd_{ik}x_k$ is an integral linear combination of the x_k, and $c\boldsymbol{y}$ is a solution since $\boldsymbol{y}$ is. Let p be c times maximum of the absolute values of the numerators of the q_{ik}. By definition of the d_{ik}, $c\boldsymbol{y}$ is in the (m, p, c)-set generated by $x_1, ..., x_r$. □

Theorem 2.22 (Rado) $A \in \mathbb{Q}^{m\times n}$ is PR iff it has CP.

Proof. $\Longrightarrow$ is by Proposition 2.14. For $\Longleftarrow$, let $c' : \mathbb{N} \to [k]$ be a finite colouring of $\mathbb{N}$. Also, by the above proposition, since A has CP, there exists $m, p, c \in \mathbb{N}$ such that $A\boldsymbol{x} = \mathbf{0}$ has a solution $\boldsymbol{x}$ in any (m, p, c)-set by the above theorem. By Theorem 2.17, there is a monochromatic (m, p, c)-set with respect to c'. This gives a monochromatic solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$. □

Remark 2.23 From the proof of Rado's Theorem, we obtain that if A is PR for the "$\bmod p$" colourings, then it is PR for *any* colouring. There is no proof of this fact that is more direct than using Rado's theorem.

Theorem 2.24 (Consistency) Let A and B be rational PR matrices. Then the matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

is PR.

Proof (Hints). Rado's Theorem. □

Proof. This is a trivial check of the CP given the CP of A and B, then we are done by Rado's Theorem. □

Remark 2.25 The Consistency Theorem says that if we can find monochromatic solutions $\boldsymbol{x}$ and $\boldsymbol{x}'$ to $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{y} = \mathbf{0}$, then we can find monochromatic solutions $\boldsymbol{x}'$ and $\boldsymbol{y}'$, of the same colour, to $A\boldsymbol{x}' = \mathbf{0}$ and $B\boldsymbol{y}' = \mathbf{0}$.

Theorem 2.26 For any finite colouring of $\mathbb{N}$, some colour class contains solutions to *all* PR equations.

Proof (Hints). Use the Consistency Theorem. □

Proof. For a given k-colouring of $\mathbb{N}$, let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_k$ be the colour classes. Assume the contrary, so for each $1 \le i \le k$, there exists a PR matrix A_i such that $A_i\boldsymbol{x} = \mathbf{0}$ has no monochromatic solution of the same colour as C_i. But then by inductively applying the consistency theorem, the matrix

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

has a monochromatic solution of the same colour as some C_j. But then $C_j\boldsymbol{x} = \mathbf{0}$ has a solution $\boldsymbol{x}$ of the same colour as C_j: contradiction. □

2.2. Ultrafilters

Definition 2.27 A **filter** on $\mathbb{N}$ is a non-empty collection $\mathcal{F}$ of subsets of $\mathbb{N}$ such that:

- $\emptyset \notin \mathcal{F}$,
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$, i.e. $\mathcal{F}$ is an **up-set**.
- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$, i.e. $\mathcal{F}$ is closed under finite intersections.

A filter is a notion of "large" subsets of $\mathbb{N}$.

Example 2.28

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$ is a filter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$ is a filter.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$ is a filter, called the **cofinite filter**.
- $\mathcal{F}_4 = \{A \subseteq \mathbb{N} : A \text{ infinite}\}$ is not a filter, since it contains $2\mathbb{N}$ and $2\mathbb{N} + 1$ but not $\emptyset = (2\mathbb{N}) \cap (2\mathbb{N} + 1)$.
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$ is a filter.

Definition 2.29 An **ultrafilter** is a maximal filter.

Definition 2.30 For $x \in \mathbb{N}$, the **principal ultrafilter at $\boldsymbol{x}$** is

$$\mathcal{U}_x \coloneqq \{A \subseteq \mathbb{N} : x \in A\}.$$

Proposition 2.31 The principal ultrafilter at x is indeed an ultrafilter.

Proof (Hints). Straightforward. □

Proof. If $B \notin \mathcal{U}_x$, then $x \in B^c$ so $B^c \in \mathcal{U}_x$, but $B^c \cap B = \emptyset$, so $\mathcal{U}_x \cup \{B\}$ is not a filter. □

Example 2.32

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$ is an ultrafilter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$ is not an ultrafilter as $\mathcal{F}_1$ extends it.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$ is not an ultra filter, as $\mathcal{F}_5$ extends it.
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$ is not an ultrafilter, as $\{A \subseteq \mathbb{N} : 4\mathbb{N} \setminus A \text{ finite}\}$ extends it.

Proposition 2.33 A filter $\mathcal{F}$ is an ultrafilter iff for all $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Proof (Hints). $\Longleftarrow$: straightforward. $\Longrightarrow$: show if $A \notin \mathcal{F}$, then $\exists B \in \mathcal{F}$ such that $A \cap B = \emptyset$. □

Proof. $\Longleftarrow$: since $A \cap A^c = \emptyset \notin \mathcal{F}$.

$\Longrightarrow$: let $\mathcal{F}$ is an ultrafilter. We cannot have $A, A^c \in \mathcal{F}$ as $A \cap A^c = \emptyset \notin \mathcal{F}$. Suppose there is $A \subseteq \mathbb{N}$ such that $A, A^c \notin \mathcal{F}$. By maximality of $\mathcal{F}$, since $A \notin \mathcal{F}$, then $\exists B \in \mathcal{F}$ such that $A \cap B = \emptyset$ (suppose not, then $\mathcal{F}' = \{S \subseteq \mathbb{N} : S \supseteq A \cap B \text{ for some } B \in \mathcal{F}\}$ extends $\mathcal{F}$). Similarly, $\exists C \in \mathcal{F}$ such that $A^c \cap C = \emptyset$. So we have $C \subseteq A$, so $B \cap C = \emptyset \notin \mathcal{F}$: contradiction (or also $C \subseteq A \Longrightarrow A \in \mathcal{F}$: contradiction). □

Corollary 2.34 Let $\mathcal{U}$ be an ultrafilter and $A = B \cup C \in \mathcal{U}$. Then $B \in U$ or $C \in U$.

Proof (Hints). Straightforward. □

Proof. If not, then $B^c, C^c \in \mathcal{U}$ by Proposition 2.33, hence $B^c \cap C^c = (B \cup C)^c = A^c \in \mathcal{U}$: contradiction. □

Proposition 2.35 Every filter is contained in an ultrafilter.

Proof (Hints). Use Zorn's Lemma. □

Proof. Let $\mathcal{F}_0$ be a filter. By Zorn's Lemma, it is enough to show that every non-empty chain of filters has an upper bound. Let $\{\mathcal{F}_i : i \in I\}$ be a chain of filters in the poset of filters containing $\mathcal{F}_0$, partially ordered by inclusion, and set $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$.

- $\emptyset \notin \mathcal{F}$ since $\emptyset \notin \mathcal{F}_i$ for each $i \in I$.
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $A \in \mathcal{F}_i$ for some $i \in I$, so $B \in \mathcal{F}_i$, so $B \in \mathcal{F}$.
- Let $A, B \in \mathcal{F}$, so $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ for some i, j. WLOG, $\mathcal{F}_i \subseteq \mathcal{F}_j$, so $A \cap B \in \mathcal{F}_j$, so $A \cap B \in \mathcal{F}$.

$\mathcal{F}$ is an upper bound for the chain, so we are done. □

Proposition 2.36 Let $\mathcal{U}$ be an ultrafilter. Then $\mathcal{U}$ is non-principal iff $\mathcal{U}$ extends the cofinite filter $\mathcal{F}_C$.

Proof (Hints). $\Longleftarrow$: straightforward. $\Longrightarrow$: use Corollary 2.34. □

Proof. $\Longleftarrow$: if $\mathcal{U} = \mathcal{U}_x$ is principal, then we have $\{x\} \in \mathcal{U}$ so $\{x\}^c \notin \mathcal{U}$ by Proposition 2.33, but also $\{x\}^c \in \mathcal{F}_C$: contradiction.

$\Longrightarrow$: let $A \in \mathcal{F}_C$, so $A^c = \{a_1, ..., a_k\}$ is finite. Assume $A \notin \mathcal{U}$, then $A^c \in \mathcal{U}$, so by Corollary 2.34, some $a_i \in \mathcal{U}$. But then by definition of a filter, each set containing a_i is in $\mathcal{U}$, so $\mathcal{U}$ is principal: contradiction. □

Notation 2.37 Let $\beta\mathbb{N}$ denote the set of all ultrafilters on $\mathbb{N}$.

Definition 2.38 Define a topology on $\beta\mathbb{N}$ by its base (basis), which consists of

$$C_A := \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$$

for each $A \subseteq \mathbb{N}$. The sets above indeed form a base: we have $\bigcup_{A \subseteq \mathbb{N}} C_A = \beta\mathbb{N}$, and $C_A \cap C_B = C_{A \cap B}$, since $A \cap B \in \mathcal{U}$ iff $A, B \in \mathcal{U}$. The open sets are of the form $\bigcup_{i \in I} C_{A_i}$ and the closed sets are of the form $\bigcap_{i \in I} C_{A_i}$.

Remark 2.39 $\beta\mathbb{N} \setminus C_A = C_{A^c}$, since $A \notin \mathcal{U}$ iff $A^c \in \mathcal{U}$. We can view $\mathbb{N}$ as being embedded in $\beta\mathbb{N}$ by identifying $n \in \mathbb{N}$ with $\tilde{n} := \mathcal{U}_n$, the principal ultrafilter at n. Each point in $\mathbb{N}$ under this correspondence is isolated in $\beta\mathbb{N}$, since $C_{\{n\}} = \{\tilde{n}\}$ is an open neighbourhood of $\tilde{n}$. Also, $\mathbb{N}$ is dense in $\beta\mathbb{N}$, since for every $n \in A$, $\tilde{n} \in C_A$, so every non-empty open set in $\beta\mathbb{N}$ intersects $\mathbb{N}$.

Theorem 2.40 $\beta\mathbb{N}$ is a compact Hausdorff topological space.

Proof. Hausdorff: let $\mathcal{U} \neq \mathcal{V}$ be ultrafilters, so there is $A \in \mathcal{U}$ such that $A \notin \mathcal{V}$. But then $A^c \in \mathcal{V}$, so $\mathcal{U} \in C_A$, $\mathcal{V} \in C_{A^c}$, and $C_A \cap C_{A^c}$ is open.

Compact: it is compact iff every open admits a finite subcover iff a collection of open sets such that no finite subcollection covers $\beta\mathbb{N}$, they don't cover $\beta\mathbb{N}$ iff for every collection of closed sets such that they have finite intersection property ($(F_i)_{i\in I}$, $\cap_{i\in J} F_i \neq \emptyset$ for all J finite), then their intersection is non-empty. We can assume each F_i is a basis set, i.e. $F_i = C_{A_i}$ for some $A_i \in \mathbb{N}$. Suppose $\left\{C_{A_i} : i \in I\right\}$ have teh finite intersection property. First, $C_{A_{i_1}} \cap \dots \cap C_{A_{i_k}} = C_{A_{i_1} \cap \dots \cap A_{i_k}} \neq \emptyset$, hence $\bigcap_{j=1}^k A_{i_j} \neq \emptyset$. So let $\mathcal{F} = \left\{A : A \supseteq A_{i_1} \cap \dots \cap A_{i_k} \text{ for some } A_{i_1}, ..., A_{i_n}\right\}$. We have $\emptyset \notin \mathcal{F}$, if $B \supseteq A \in \mathcal{F}$ then $B \in \mathcal{F}$, and if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. Hence $\mathcal{F}$ is a filter. $\mathcal{F}$ extends to an ultrafilter $\mathcal{U}$. Note that $(\forall i, A_i \in \mathcal{U}) \iff \left(\mathcal{U} \in C_{A_i} \forall i\right)$. So $U \in \cap C_{A_i}$, so $\cap C_{A_i} \neq \emptyset$. $\square$

Remark 2.41

- $\beta\mathbb{N}$ can be viewed as a subset of $\{0,1\}^{\mathbb{P}(\mathbb{N})}$ (so each ultrafilter is viewed as a function $\mathbb{P}(\mathbb{N}) \to \{0,1\}$). The topology on $\beta\mathbb{N}$ is the restriction of the product topology on $\{0,1\}^{\mathbb{P}(\mathbb{N})}$. Also, $\beta\mathbb{N}$ is a closed subset of $\{0,1\}^{\mathbb{P}(\mathbb{N})}$, so is compact by Tychonov's theorem (TODO: look up statement of this theorem).
- $\beta\mathbb{N}$ is the largest compact Hausdorff topological space in which (the embedding of) $\mathbb{N}$ is dense. In other words, if X is compact and Hausdorff, and $f : \mathbb{N} \to X$, there exists a unique continuous $\tilde{f} : \beta\mathbb{N} \to X$ extending f. TODO: insert diagram.
- $\beta\mathbb{N}$ is called the **Stone-Čech compactification** of $\mathbb{N}$.

Definition 2.42 Let p be a statement and $\mathcal{U}$ be an ultrafilter. $\forall_{\mathcal{U}} x\, p(x)$ to mean $\{x \in \mathbb{N} : p(x)\} \in \mathcal{U}$ and say $p(x)$ "for most x" or "for $\mathcal{U}$-most x".

Example 2.43

- For $\mathcal{U} = \tilde{n}$, we have $\forall_{\mathcal{U}} x\, p(x)$ iff $p(n)$.
- For non-principal $\mathcal{U}$, we have $\forall_{\mathcal{U}} x\, (x > 4)$ (if not, then $\{1,2,3\} = \{x \in \mathbb{N} : x > 4\}^c \in \mathcal{U}$, so $\{i\} \in \mathcal{U}$ for some $i = 1,2,3$, so $\mathcal{U}$ is principal: contradiction).

Proposition 2.44 Let $\mathcal{U}$ be an ultrafilter and p, q be statements. Then

1. $\forall_{\mathcal{U}} x\, (p(x) \wedge q(x))$ iff $(\forall_{\mathcal{U}} x\, p(x)) \wedge (\forall_{\mathcal{U}} x\, q(x))$.
2. $\forall_{\mathcal{U}} x\, (p(x) \vee q(x))$ iff $(\forall_{\mathcal{U}} x\, p(x)) \vee (\forall_{\mathcal{U}} x\, q(x))$.
3. $\neg(\forall_{\mathcal{U}} x\, p(x))$ iff $\forall_{\mathcal{U}} x\, (\neg p(x))$.

Proof. Let $A = \{x \in \mathbb{N} : p(x)\}$ and $B = \{x \in \mathbb{N} : q(x)\}$. We have

1. $A \cap B \in \mathcal{U}$ iff $A \in \mathcal{U}$ and $B \in \mathcal{U}$ by definition.
2. $A \cup B \in \mathcal{U}$ iff $A \in \mathcal{U}$ and $B \in \mathcal{U}$ by (find result).
3. $A \notin \mathcal{U}$ iff $A^c \in \mathcal{U}$ by (find result).

$\square$

Note 2.45 $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\, p(x,y)$ is not necessarily the same as $\forall_{\mathcal{V}}y\forall_{\mathcal{U}}x\, p(x,y)$, even when $\mathcal{U} = \mathcal{V}$. For example, let $\mathcal{U}$ be non-principal, and $p(x,y) = (x < y)$. Then $\forall_{\mathcal{U}}x(\forall_{\mathcal{U}}y\,(x < y))$ is true, as every x satisfies $\forall_{\mathcal{U}}y\,(x < y)$. But $\forall_{\mathcal{U}}y\forall_{\mathcal{U}}x\,(x < y)$ is false, as no y has $\forall_{\mathcal{U}}x\,(x < y)$. So **don't swap quantifiers!**.

Definition 2.46 Given ultrafilters $\mathcal{U}, \mathcal{V}$, define their sum to be

$$\mathcal{U} + \mathcal{V} := \{A \subseteq \mathbb{N} : \forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\,(x + y \in A)\}.$$

Example 2.47 We have $\widetilde{m} + \tilde{n} = \widetilde{m+n}$.

Proposition 2.48 For any ultrafilters $\mathcal{U}$ and $\mathcal{V}$, $\mathcal{U} + \mathcal{V}$ is an ultrafilter.

Proof. We have $\emptyset \notin \mathcal{U} + \mathcal{V}$. If $A \in \mathcal{U} + \mathcal{V}$ and $A \subseteq B$, then $B \in \mathcal{U} + \mathcal{V}$. If $A, B \in \mathcal{U} + \mathcal{V}$, then $(\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\,(x + y \in A)) \wedge (\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\,(x + y \in B))$, so by above proposition, we have $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y(x + y \in A \wedge x + y \in B)$, i.e. $\forall_{\mathcal{U}}x\forall \mathcal{V}y\,(x + y \in A \cap B)$, i.e. $A \cap B \in \mathcal{U} + \mathcal{V}$. Hence $\mathcal{U} + \mathcal{V}$ is a filter.

Suppose that $A \notin \mathcal{U} + \mathcal{V}$, i.e. $\neg(\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\,(x + y \in A))$. Then by above proposition twice, we have $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\,\neg(x + y \in A)$. So $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\,(x + y \in A^c)$, i.e. $A^c \in \mathcal{U} + \mathcal{V}$. $\square$

Proposition 2.49 Ultrafilter addition is associative.

Proof. Let $A \subseteq \mathcal{U} + (\mathcal{V} + \mathcal{W})$, so $\forall_{\mathcal{U}}x\forall_{\mathcal{V}+\mathcal{W}}\,(x + y \in A)$. So $B := \{y : x + y \in A\} \in \mathcal{V} + \mathcal{W}$, i.e. $\forall_{\mathcal{V}}y_1\forall_{\mathcal{W}}y_2\,(y_1 + y_2 \in B)$. So we have $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y_1\forall_{\mathcal{W}}y_2\,(x + y_1 + y_2 \in A)$. So

$$\mathcal{U} + (\mathcal{V} + \mathcal{W}) = \{A \subseteq \mathbb{N} : \forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\forall_{\mathcal{W}}z\,(x + y + z \in A)\} = (\mathcal{U} + \mathcal{V}) + \mathcal{W}.$$

$\square$

Proposition 2.50 Ultrafilter addition is left-continuous: for fixed $\mathcal{V}$, $\mathcal{U} \mapsto \mathcal{U} + \mathcal{V}$ is continuous.

Proof. For $A \subseteq \mathbb{N}$, we have

$$\begin{aligned}\mathcal{U} + \mathcal{V} \in C_A &\iff A \in \mathcal{U} + \mathcal{V} \\ &\iff \forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\,(x + y \in A) \\ &\iff B := \{x \in \mathbb{N} : \forall_{\mathcal{V}}y\,(x + y \in A)\} \in \mathcal{U} \\ &\iff \mathcal{U} \in C_B\end{aligned}$$

hence the preimage of C_A, which is C_B, is open. $\square$

Proposition 2.51 (Idempotent Lemma) There exists an idempotent ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ (i.e. $\mathcal{U} = \mathcal{U} + \mathcal{U}$).

Proof. For $M \subseteq \beta\mathbb{N}$, define $M + M := \{x + y : x, y \in M\}$. We seek a non-empty, compact $M \subseteq \beta\mathbb{N}$ which is minimal such that $M + M \subseteq M$, and hope to show that M is a singleton.

Such an M exists ($\beta\mathbb{N}$ is one such), so the set of all such M is non-empty. By Zorn's Lemma, it suffices to show that if $\{M_i : i \in I\}$ is a chain of such sets, then $M =$

$\bigcap_{i \in I} M_i$ (an upper bound with respect to the partial ordering $\supseteq$) is another such set. This M will be compact as an intersection of closed sets, since $\beta\mathbb{N}$ is compact and Hausdorff, so any subspace is closed iff it is compact. Also, $M + M \subseteq M$: for $x, y \in M$, we have $x, y \in M_i$ so $x + y \in M_i + M_i \subseteq M_i$ for all $i \in I$, so $x + y \in M$. Finally, M is non-empty: $\{M_i : i \in I\}$ have the finite intersection property, as they are a chain, and are closed, so their intersection is non-empty.

So by Zorn's lemma, there exists such a minimal M. Given $x \in M$, we have $M + x = M$, since $M + x \neq \emptyset$, $M + x$ is compact (as the continuous image of a compact set) and $(M + x) + (M + x) = (M + x + M) + x \subseteq (M + M + M) + x \subseteq M + x$, so by minimality of M, $M + x = M$.

In particular, there exists $y \in M$ such that $y + x = x$. Let $T = \{y \in M : y + x = x\}$. We claim that $T = M$, and since $T \subseteq M$, it is enough to show that T is compact, non-empty and $T + T \subseteq T$, by minimality of M. Indeed, $y \in T$, so $T \neq \emptyset$, T is the pre-image of a singleton which is compact, hence closed, so T is closed, so compact. Finally, for $y, z \in T$, we have $y + x = x = z + x$ so $y + z + x = y + x = x$, so $y + z \in T$, so $T + T \subseteq T$.

Hence, $y + x = x$ for all $y \in M$, hence $x + x = M$. In fact, $M = \{x\}$. □

Remark 2.52 The finite subgroup problem asks whether we can find a non-trivial subgroup of $\beta\mathbb{N}$ (e.g. find $\mathcal{U}$ with $\mathcal{U} + \mathcal{U} \neq \mathcal{U}$ but $\mathcal{U} + \mathcal{U} + \mathcal{U} = \mathcal{U}$). This was recently proven to be negative.

Remark 2.53 An open problem is whether an ultrafilter can "absorb" another ultrafilter, i.e. whether there exist $\mathcal{U} \neq \mathcal{V}$ such that $\mathcal{U} + \mathcal{U} = \mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U} = \mathcal{V} + \mathcal{V} = \mathcal{V}$.

Theorem 2.54 (Hindman) For any finite colouring of $\mathbb{N}$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$\mathrm{FS}(\{x_n : n \in \mathbb{N}\}) = \left\{\sum_{i \in I} x_i : I \subseteq \mathbb{N} \text{ finite}, I \neq \emptyset\right\}.$$

Proof. Let $\mathcal{U}$ be an idempotent ultrafilter, and partition $\mathbb{N}$ into its colour classes: $\mathbb{N} = A_1 \sqcup \cdots \sqcup A_k$. Since $\emptyset \notin \mathcal{U}$ by definition, we have $A_1 \cup \cdots \cup A_k \in \mathbb{N} \in \mathcal{U}$ by Proposition 2.33. So by Corollary 2.34, $A := A_i \in \mathcal{U}$ for some $i \in [k]$. We have $\forall_{\mathcal{U}} y\, (y \in A)$ by definition. Thus:
1. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\, (y \in A)$.
2. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\, (x \in A)$.
3. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\, (x + y \in A)$ since $A \in \mathcal{U} + \mathcal{U} = \mathcal{U}$.

Proposition 2.44 then gives that $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y\, (\mathrm{FS}(x, y) \subseteq A)$. Fix $x_1 \in A$ such that $\forall_{\mathcal{U}} y\, (\mathrm{FS}(x_1, y) \subseteq A)$.

Now assume we have found $x_1, ..., x_n$ such that $\forall_{\mathcal{U}} y\, (\mathrm{FS}(x_1, ..., x_n, y) \subseteq A)$, i.e. $B := \{y \in \mathbb{N} : \mathrm{FS}(x_1, ..., x_n, y) \subseteq A\} \in \mathcal{U} = \mathcal{U} + \mathcal{U}$, i.e. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\, (x + y \in B)$ by definition. We have:

1. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (\mathrm{FS}(x_1, ..., x_n, y) \subseteq A)$.
2. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (\mathrm{FS}(x_1, ..., x_n, x) \subseteq A)$.
3. For each $z \in \mathrm{FS}(x_1, ..., x_n, y)$, we have $\forall_{\mathcal{U}} y\ (z + y \in A)$, so $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (z + x + y \in A)$.

Proposition 2.44 then gives that

$$\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (\mathrm{FS}(x_1, ..., x_n, x, y) \subseteq A).$$

The result follows by induction. □

3. Euclidean Ramsey theory