Observables

- Observables/measurements represented by Hermitian operators $\hat{A}$.
- Hermitian conjugate is defined as $< A\psi, \chi >=< \psi, A^\dagger \chi >$. Matrix representation is $(A^*)^t$.
- outcome of measurement is eigenvalue of $\hat{A}$. Spectral representation is $\hat{A} = \sum_i \alpha_i\ |i >< i|$ for eigenvalues α_i and eigenstates $|i >$.
- If $\hat{B} = \sum_j \beta_j\ |j >< j|$ and $\left[\hat{A}, \hat{B}\right] \neq 0$ then measuring $\hat{A}$ and $\hat{B}$ will not result in state collapsing to simultaneous eigenstate.

Mixed states vs pure states

- Pure state: definite quantum mechanical state.
- Mixed state is ensemble of pure states, each with associated classical probability of system being in that state.
- e.g. $|\psi >= \frac{1}{\sqrt{2}}(|0 > + |1 >) = | + >$ is pure, mixed state is $\left\{\left(\frac{1}{2}, |0 >\right), \left(\frac{1}{2}, |1 >\right)\right\}$. If $\hat{B}$ has eigenstates $| + >$ and $| - >$ with eigenvalues $1, -1$ (i.e. σ_1), then measuring $\hat{B}$ on $| + >$ then result will always be 1. But measuring $\hat{B}$ on mixed state will randomly yield 1 or -1 with equal probability.
- Double slit experiment analog: mixed state is closing one slit in half of the cases, closing the other in the other half. Pure state: electron goes through both slits at the same time.

Density matrix

- $\hat{\rho} = |\psi >< \psi|$ for pure state $|\psi >$.
- $\hat{\rho} = \sum_i p_i\ |\psi_i >< \psi_i|$ for mixed state $\{(p_i, |\psi_i >)\}$.
- For any density matrix, $\mathrm{tr}(\hat{\rho}) = 1$ but $\mathrm{tr}(\hat{\rho}^2) = 1$ iff $\hat{\rho}$ is pure state.
- Spectral decomposition: $\hat{A} = \sum_i \alpha_i\ |i >< i| = \sum_i \alpha_i P_i$. Probability of measuring result α_i is $\mathrm{tr}(\rho P_i)$.

Bloch sphere not as important to revise.

Bipartite systems

- Entangled vs separable states.
- Separable states: $|\psi > \otimes |\varphi >$.
- Entangled: sum of separable states which cannot be factorised into a separable state.
- If $|\psi >= \frac{1}{\sqrt{2}}(|0 > \otimes |1 > + |1 > \otimes |0 >)$, it has density matrix $\frac{1}{2}(|01 >< 01| + |01 >< 10| + |10 >< 01| + |10 >< 10|)$.
  - If Alice measures with operator $A = |0 >< 0| - |1 >< 1|$ then if her measurement is 1 then state collapses to $|01 >$. If her measurement is -1 then state collapses to $|10 >$.
  - Density matrix afterwards is $\frac{1}{2}\ |01 >< 01| + \frac{1}{2}\ |10 >< 10|$ if know Alice has measured but we don't know what she measured.
  - Reduced density matrix for Bob:
    - Before measurement: $\mathrm{tr}_A(\rho_{\text{before}}) = \frac{1}{2}(|1 >< 1| + |0 >< 0|)$ (only states kept are those where A and B bits agree). This is the density matrix of a mixed state.
    - After measurement: $\mathrm{tr}_B(\rho_{\text{after}}) = \frac{1}{2}(|1 >< 1| + |0 >< 0|)$ i.e. same as before measurement.
    - So Bob's reduced density matrix is not affected by Alice's measurement.

Entanglement applications

- Don't learn QKD etc. by heart, learn the key ingredients. They are:
- 1. We can entangled information into an existing state: e.g. start with product state $|\psi > \otimes |\beta_{00} >$, then apply CNOT on first two qubits, this is used in teleportation.

- 2. Quantum systems do not have local realism: since it is possible to have measurement operators which do not commute.

Don’t expect heavy computations about entropy chapter.