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1. Monochromatic sets

1.1. Ramsey's theorem

Notation 1.1 \mathbb{N} denotes the set of positive integers, $[n] = \{1, \dots, n\}$, and $X^{(r)} = \{A \subseteq X : |A| = r\}$. Elements of a set are written in ascending order, e.g. $\{i, j\}$ means $i < j$. Write e.g. ijk to mean the set $\{i, j, k\}$ with the ordering (unless otherwise stated) $i < j < k$.

Definition 1.2 A k -colouring on $A^{(r)}$ is a function $c : A^{(r)} \rightarrow [k]$.

Example 1.3

- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ is even and blue if $i + j$ is odd. Then $M = 2\mathbb{N}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$ is even and blue otherwise. $M = \{4^n : n \in \mathbb{N}\}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

Theorem 1.4 (Ramsey's Theorem for Pairs) Let $\mathbb{N}^{(2)}$ be 2-coloured by $c : \mathbb{N}^{(2)} \rightarrow \{1, 2\}$. Then there exists an infinite monochromatic subset M .

Proof.

- Let $a_1 \in A_0 := \mathbb{N}$. There exists an infinite set $A_1 \subseteq A_0$ such that $c(a_1, i) = c_1$ for all $i \in A_1$.
- Let $a_2 \in A_1$. There exists infinite $A_2 \subseteq A_1$ such that $c(a_2, i) = c_2$ for all $i \in A_2$.
- Repeating this inductively gives a sequence $a_1 < a_2 < \dots < a_k < \dots$ and $A_1 \supseteq A_2 \supseteq \dots$ such that $c(a_i, j) = c_i$ for all $j \in A_i$.
- One colour appears infinitely many times: $c_{i_1} = c_{i_2} = \dots = c_{i_k} = \dots = c$.
- $M = \{a_{i_1}, a_{i_2}, \dots\}$ is a monochromatic set.

□

Remark 1.5

- The same proof works for any $k \in \mathbb{N}$ colours.
- The proof is called a “2-pass proof”.
- An alternative proof for k colours is split the k colours $1, \dots, k$ into 2 colours: 1 and “2 or ... or k ”, and use induction.

Note 1.6 An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

Example 1.7 Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, etc. Let $\{i, j\}$ be red if $i, j \in A_k$ for some k . There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

Example 1.8 Colour $\{i < j < k\}$ red iff $i \mid (j + k)$. A monochromatic subset $M = \{2^n : n \in \mathbb{N}_0\}$ is a monochromatic set.

Theorem 1.9 (Ramsey's Theorem for r -sets) Let $\mathbb{N}^{(r)}$ be finitely coloured. Then there exists a monochromatic infinite set.

Proof.

- $r = 1$: use pigeonhole principle.
- $r = 2$: Ramsey's theorem for pairs.
- For general r , use induction.
- Let $c : \mathbb{N}^r \rightarrow [k]$ be a k -colouring. Let $a_1 \in \mathbb{N}$, and consider all $r - 1$ sets of $\mathbb{N} \setminus \{a_1\}$, induce colouring $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \rightarrow [k]$ via $c'(F) = c(F \cup \{a_1\})$.
- By inductive hypothesis, there exists $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that c' is constant on it (taking value c_1).
- Now pick $a_2 \in A_1$ and induce a colouring $c' : (A_1 \setminus \{a_2\})^{(r-1)} \rightarrow [k]$ such that $c'(F) = c(F \cup \{a_2\})$. By inductive hypothesis, there exists $A_2 \subseteq A_1 \setminus \{a_2\}$ such that c' is constant on it (taking value c_2).
- Repeating this gives a_1, a_2, \dots and A_1, A_2, \dots such that $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$ and $c(F \cup \{a_i\}) = c_i$ for all $F \subseteq A_{i+1}$, for $|F| = r - 1$.
- One colour must appear infinitely many times: $c_{i_1} = c_{i_2} = \dots = c$.
- $M = \{a_{i_1}, a_{i_2}, \dots\}$ is a monochromatic set.

□

1.2. Applications of Ramsey's theorem

Example 1.10 In a totally ordered set, any sequence has monotonic subsequence.

Proof.

- Let (x_n) be a sequence, colour $\{i, j\}$ red if $x_i \leq x_j$ and blue otherwise.
- By Ramsey's theorem for pairs, $M = \{i_1 < i_2 < \dots\}$ is monochromatic. If M is red, then the subsequence x_{i_1}, x_{i_2}, \dots is increasing, and is strictly decreasing otherwise.
- We can insist that (x_{i_j}) is either concave or convex: 2-colour $\mathbb{N}^{(3)}$ by colouring $\{j < k < \ell\}$ **red** if $(i, x_{i_j}), (j, x_{i_k}), (k, x_{i_\ell})$ form a convex triple, and **blue** if they form a concave triple. Then by Ramsey's theorem for r -sets, there is an infinite convex or concave subsequence.

□

Theorem 1.11 (Finite Ramsey) Let $r, m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k -coloured, we can find a monochromatic set of size (at least) m .

Proof.

- Assume not, i.e. $\forall n \in \mathbb{N}$, there exists colouring $c_n : [n]^{(r)} \rightarrow [k]$ with no monochromatic m -sets.
- There are only finitely many (k) ways to k -colour $[r]^{(r)}$, so there are infinitely many of colourings c_r, c_{r+1}, \dots that agree on $[r]^{(r)}$: $c_i|_{[r]^{(r)}} = d_r$ for all i in some infinite set A_1 , where d_r is a k -colouring of $[r]^{(r)}$.
- Similarly, $[r+1]^{(r)}$ has only finitely many possible k -colourings. So there exists infinite $A_2 \subseteq A_1$ such that for all $i \in A_2$, $c_i|_{[r+1]^{(r)}} = d_{r+1}$, where d_{r+1} is a k -colouring of $[r+1]^{(r)}$.
- Continuing this process inductively, we obtain $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n$. There is no monochromatic m -set for any $d_n : [n]^{(r)} \rightarrow [k]$ (because $d_n = c_i|_{[n]^{(r)}}$ for some i).
- These d_n 's are nested: $d_\ell|_{[n]^{(r)}} = d_n$ for $\ell > n$.

- Finally, we colour $\mathbb{N}^{(r)}$ by the colouring $c : \mathbb{N}^{(r)} \rightarrow [k]$, $c(F) = d_n(F)$ where $n = \max(F)$ (or in fact $n \geq \max(F)$, which is well-defined by above). So c has no monochromatic m -set (since M was a monochromatic m -set, then taking $\ell = \max(M)$, d_ℓ has a monochromatic m -set), which contradicts Ramsey's Theorem for r -sets.

□

Remark 1.12

- This proof gives no bound on $n = n(k, m)$, there are other proofs that give a bound.
- It is a proof by compactness (essentially, we proved that $\{0, 1\}^{\mathbb{N}}$ with the product topology, i.e. the topology derived from the metric $d(f, g) = \frac{1}{\min\{n \in \mathbb{N} : f(n) \neq g(n)\}}$, is sequentially compact).

Remark 1.13 Now consider a colouring $c : \mathbb{N}^{(2)} \rightarrow X$ with X potentially infinite. This does not necessarily admit an infinite monochromatic set, as we could colour each edge a different colour. Such a colouring would be injective. We can't guarantee either the colouring being constant or injective though, as $c(ij) = i$ satisfies neither.

Theorem 1.14 (Canonical Ramsey) Let $c : \mathbb{N}^{(2)} \rightarrow X$ be a colouring with X an arbitrary set. Then there exists an infinite set $M \subseteq \mathbb{N}$ such that:

1. c is constant on $M^{(2)}$, or
2. c is injective on $M^{(2)}$, or
3. $c(ij) = c(kl)$ iff $i = k$ for all $i < j$ and $k < l$, $i, j, k, l \in M$, or
4. $c(ij) = c(kl)$ iff $j = l$ for all $i < j$ and $k < l$, $i, j, k, l \in M$.

Proof (Hints).

- First consider the 2-colouring c_1 of $\mathbb{N}^{(4)}$ where $ijkl$ is coloured SAME if $c(ij) = c(kl)$ and DIFF otherwise. Show that an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ (why does this exist?) coloured SAME leads to case 1.
- Assume M_1 is coloured DIFF, consider the 2-colouring of $M_1^{(4)}$, which colours $ijkl$ SAME if $c(il) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic $M_2 \subseteq M_1$ (why does this exist?) must be coloured DIFF by contradiction.
- Consider the 2-colouring of $M_2^{(4)}$ where $ijkl$ is coloured SAME if $c(ik) = c(jl)$ and DIFF otherwise. Show an infinite monochromatic set $M_3 \subseteq M_2$ (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour $M_3^{(3)}$ by: ijk is coloured SAME if $c(ij) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic set $M_4 \subseteq M_3$ (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour $M_4^{(3)}$ by the other two similar colourings to above, obtaining monochromatic $M_6 \subseteq M_5 \subseteq M_4$.
- Consider 4 combinations of these colourings on M_6 , show 3 lead to one of the cases in the theorem, and the other leads to contradiction.

□

Proof.

- 2-colour $\mathbb{N}^{(4)}$ by: $ijkl$ is red if $c(ij) = c(kl)$ and blue otherwise. By Ramsey's Theorem for 4-sets, there is an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ for this colouring.
- If M_1 is red, then c is constant on $M_1^{(2)}$: for all pairs $ij, i'j' \in M_1^{(2)}$, pick $m < n$ with $j, j' < m$, then $c(ij) = c(mn) = c(i'j')$.
- So assume M_1 is blue.
- Colour $M_1^{(4)}$ by giving $ijkl$ colour green if $c(il) = c(jk)$ and purple otherwise. By Ramsey's theorem for 4-sets, there exists an infinite monochromatic $M_2 \subseteq M_1$ for this colouring.
- Assume M_2 is coloured green: if $i < j < k < l < m < n \in M_2$, then $c(jk) = c(in) = c(lm)$ (consider $ijkn$ and $ilmn$): contradiction, since M_1 is blue.
- Hence M_2 is purple, i.e. for $ijkl \in M_2^{(4)}$, $c(il) \neq c(jk)$.
- Colour M_2 by: $ijkl$ is orange if $c(ik) = c(jl)$, and pink otherwise.
- By Ramsey's theorem for 4-sets, there exists infinite monochromatic $M_3 \subseteq M_2$ for this colouring.
- Assume M_3 is orange, then for $i < j < k < l < m < n \in M_3$, we have $c(jm) = c(ln)$ (consider $jlmn$) and $c(jm) = c(ik)$ (consider $ijkm$): contradiction, since $M_3 \subseteq M_1$.
- Hence M_3 is pink, i.e. for $ijkl$, $c(ik) \neq c(jl)$.
- Colour $M_3^{(3)}$ by: ijk is yellow if $c(ij) = c(jk)$ and grey otherwise. By Ramsey's theorem for 3-sets, there exists infinite monochromatic $M_4 \subseteq M_3$ for this colouring.
- Assume M_4 is yellow: then (considering $ijkl \in M_4^{(4)}$) $c(ij) = c(jk) = c(kl)$: contradiction, since $M_4 \subseteq M_1$.
- So for any $ijk \in M_4^{(3)}$, $c(ij) \neq c(jk)$.
- Finally, colour $M_4^{(3)}$ by: ijk is gold if $c(ij) = c(ik)$ and $c(ik) = c(jk)$, silver if $c(ij) = c(ik)$ and $c(ik) \neq c(jk)$, bronze if $c(ij) \neq c(ik)$ and $c(ik) = c(jk)$, and platinum if $c(ij) \neq c(ik)$ and $c(ik) \neq c(jk)$.
- By Ramsey's theorem for 3-sets, there exists monochromatic $M_5 \subseteq M_4$. M_5 cannot be gold, since then $c(ij) = c(jk)$: contradiction, since $M_5 \subseteq M_4$. If silver, then we have case 3 in the theorem. If bronze, then we have case 4 in the theorem. If platinum, then we have case 2 in the theorem.

□

Remark 1.15

- A more general result of the above theorem states: let $\mathbb{N}^{(r)}$ be arbitrarily coloured. Then we can find an infinite M and $I \subseteq [r]$ such that for all $x_1 \dots x_r \in M^{(r)}$ and $y_1 \dots y_r \in M^{(r)}$, $c(x_1 \dots x_r) = c(y_1 \dots y_r)$ iff $x_i = y_i$ for all $i \in I$.
- In canonical Ramsey, $I = \emptyset$ is case 1, $I = \{1, 2\}$ is case 2, $I = \{1\}$ is case 3 and $I = \{2\}$ is case 4.
- These 2^r colourings are called the **canonical colourings** of $\mathbb{N}^{(r)}$.

Exercise 1.16 Prove the general statement.

1.3. Van der Waerden's theorem

Remark 1.17 We want to show that for any 2-colouring of \mathbb{N} , we can find a monochromatic arithmetic progression of length m for any $m \in \mathbb{N}$. By compactness, this is equivalent to showing that for all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any 2-colouring of $[n]$, there exists a monochromatic arithmetic progression of length m . (If not, then for each $n \in \mathbb{N}$, there is a colouring $c_n : [n] \rightarrow \{1, 2\}$ with no monochromatic arithmetic progression of length m . Infinitely many of these colourings agree on $[1]$, infinitely many of those agreeing in $[1]$ agree on $[2]$, and so on - we obtain a 2-colouring of \mathbb{N} with no monochromatic arithmetic progression of length m).

We will prove a slightly stronger result: whenever \mathbb{N} is k -coloured, there exists a length m monochromatic arithmetic progression, i.e. for any $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k -coloured, we have a length m monochromatic progression.

Definition 1.18 Let A_1, \dots, A_k be length m arithmetic progressions: $A_i = \{a_i, a_i + d_i, \dots, a_i + (m-1)d_i\}$. A_1, \dots, A_k are **focussed** at f if $a_i + md_i = f$ for all i .

Example 1.19 $\{4, 8\}$ and $\{6, 9\}$ are focussed at 12.

Definition 1.20 If length m arithmetic progressions A_1, \dots, A_k are focused at f and are monochromatic, each with a different colour (for a given colouring), they are called **colour-focussed** at f .

Remark 1.21 We use the idea that if A_1, \dots, A_k are colour-focussed at f (for a k -colouring) and of length $m-1$, then some $A_i \cup \{f\}$ is a length m monochromatic arithmetic progression.

Theorem 1.22 Whenever \mathbb{N} is k -coloured, there exists a monochromatic arithmetic progression of length 3, i.e. for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any k -colouring of $[n]$ admits a length 3 monochromatic progression.

Proof (Hints).

- Prove by induction the claim: $\forall r \leq k, \exists n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there exists a monochromatic arithmetic progression of length 3, or r colour-focussed arithmetic progressions of length 2.
 - $r = 1$ case is straightforward.
 - Let claim be true for $r-1$ with witness n , let $N = 2n(k^{2n} + 1)$.
 - Partition N into blocks of equal size, show that two of these blocks must have the same colouring.
 - Using the inductive hypothesis, merge the $r-1$ colour-focussed arithmetic progressions from these two blocks into a new set of $r-1$ colour-focussed arithmetic progressions.
 - Find another length 2 monochromatic arithmetic progression, reason that this is of different colour.
- Reason that this claim implies the result.

□

Proof.

- We claim that for all $r \leq k$, there exists an $n \in \mathbb{N}$ such that if $[n]$ is k -coloured, then either:
 - There exists a monochromatic arithmetic progression of length 3.
 - There exist r colour-focussed arithmetic progressions of length 2.
- This claim implies the result by the above remark.
- We prove the claim by induction on r :
 - $r = 1$: take $n = k + 1$, then by pigeonhole, some two elements of $[n]$ have the same colour, so form a length two arithmetic progression.
 - Assume true for $r - 1$ with witness n . We claim that $N = 2n(k^{2n} + 1)$ works for r .
 - Let $c : [2n(k^{2n} + 1)] \rightarrow [k]$ be a colouring. We partition $[N]$ into $k^{2n} + 1$ blocks of size $2n$: $B_i = \{2n(i - 1) + 1, \dots, 2ni\}$ for $i = 1, \dots, k^{2n} + 1$.
 - Assume there is no length 3 monochromatic progression for c . By inductive hypothesis, each block B_i has $r - 1$ colour-focussed arithmetic progressions of length 2.
 - Since $|B_i| = 2n$, each block also contains their focus. For a set M with $|M| = 2n$, there are k^{2n} ways to k -colour M . So by pigeonhole, there are blocks B_s and B_{s+t} that have the same colouring.
 - Let $\{a_i, a_i + d_i\}$ be the $r - 1$ arithmetic progressions in B_s colour-focussed at f , then $\{a_i + 2nt, a_i + d_i + 2nt\}$ is the corresponding set of arithmetic progressions in B_{s+t} , each colour-focussed at $f + 2nt$.
 - Now $\{a_i, a_i + d_i + 2nt\}$, $i \in [r - 1]$, are $r - 1$ arithmetic progressions colour-focussed at $f + 4nt$. Also, $\{f, f + 2nt\}$ is monochromatic of a different colour to the $r - 1$ colours used (since there is no length 3 monochromatic progression for c). Hence, there are r arithmetic progressions of length 2 colour-focussed at $f + 4nt$.

□

Remark 1.23 The idea of looking at all possible colourings of a set is called a **product argument**.

Definition 1.24 The **Van der Waerden** number $W(k, m)$ is the smallest $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there exists a monochromatic arithmetic progression in $[n]$ of length m .

Remark 1.25 The above theorem gives a **tower-type** upper bound $W(k, 3) \leq k^{k^{(\cdot)^{k^{4k}}}}$.

Theorem 1.26 (Van der Waerden's Theorem) For all $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there is a length m monochromatic arithmetic progression.

Proof (Hints).

- Use induction on m .

- Given induction hypothesis on $m - 1$, prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, we have either a monochromatic length m arithmetic progression, or r colour-focussed arithmetic progressions of length $m - 1$. Reason that this claim implies the result.
- Use induction on r . Give an explicit n for $r = 1$.
- Let n be the witness for $r - 1$, let $N = W(k^{2n}, m - 1) \cdot 2n$. Assume a k -colouring of $[N]$, $c : [N] \rightarrow [k]$, has no arithmetic progressions of length m .
- Partition $[N]$ into the obvious choice of $W(k^{2n}, m - 1)$ blocks B_i , each of length $2n$.
- Colour the indices $1 \leq i \leq W(k^{2n}, m - 1)$ of the blocks by

$$c'(i) = (c(2n(i - 1) + 1), c(2n(i - 1) + 2), \dots, c(2ni))$$

- Reason that we can find monochromatic arithmetic progression $s, s + t, \dots, s + (m - 2)t$ of length $m - 1$ (w.r.t c'), and that this corresponds to sequence of blocks $B_s, B_{s+t}, \dots, B_{s+(m-2)t}$, each identically coloured.
- Reason that B_s contains $r - 1$ colour-focussed length $m - 1$ arithmetic progressions A_i together with their focus f .
- Let A'_i be the same arithmetic progression but with common difference $2nt$ larger than that of A_i . Show the A'_i are colour-focussed at some focus in terms of f .
- Find another length $m - 1$ arithmetic progression, show this must be monochromatic and of different colour to all A'_i . Show it also has same focus as all A'_i .

□

Proof.

- By induction on m . $m = 1$ is trivial, $m = 2$ is by pigeonhole principle. $m = 3$ is the statement of the previous theorem.
- Assume true for $m - 1$ and all $k \in \mathbb{N}$.
- For fixed k , we prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, either:
 - There is a monochromatic arithmetic progression of length m , or
 - There are r colour-focussed arithmetic progressions of length $m - 1$.
- We will then be done (by considering the focus).
- To prove the claim, we use induction on r .
- $r = 1$ is the claim of the first inductive hypothesis: take $n = W(k, m - 1)$.
- Assume the claim holds for $r - 1$ with witness n , and assume there is no monochromatic arithmetic progression of length m . We will show that $N = W(k^{2n}, m - 1)2n$ is sufficient for r .
- Partition $[N]$ into $W(k^{2n}, m - 1)$ blocks of length $2n$: $B_i = \{2n(i - 1) + 1, \dots, 2ni\}$ for $i = 1, \dots, W(k^{2n}, m - 1)$.
- Each block has k^{2n} possible colourings. Colour the blocks as

$$c'(i) = (c(2n(i - 1) + 1), c(2n(i - 1) + 2), \dots, c(2ni))$$

By definition of W , there exists a monochromatic arithmetic progression of length $m - 1$ (w.r.t. to c'): $\{\alpha, \alpha + t, \dots, \alpha + (m - 2)t\}$. The respective blocks $B_\alpha, \dots, B_{\alpha + (m-2)t}$ are identically coloured.

- B_α has length $2n$, so by induction B_α contains $r - 1$ colour-focussed arithmetic progressions of length $m - 1$, together with their focus (as length of block is $2n$).
- Let A_1, \dots, A_{r-1} , $A_i = \{a_i, a_i + d_i, \dots, a_i + (m - 2)d_i\}$, be colour-focussed at f .
- Let $A'_i = \{a_i, a_i + (d_i + 2nt), \dots, a_i + (m - 2)(d_i + 2nt)\}$ for $i = 1, \dots, r - 1$. The A'_i are monochromatic as the blocks are identically coloured and the A_i are monochromatic. Also, A_i and A'_i have the same colouring, and the A_i are colour-focussed, hence the A'_i have pairwise distinct colours.
- The A_i are focussed at f and the colour of f is different than the colour of all A_i . $f = a_i + (m - 1)d_i$ for all i .
- Now $\{f, f + 2nt, f + 4nt, \dots, f + 2n(m - 2)t\}$ is an arithmetic progression of length $m - 1$, is monochromatic and of a different colour to all the A'_i .
- It is enough to show that $a_i + (m - 1)(d_i + 2nt) = f + 2n(m - 1)t$ for all i , but this is equivalent to $a_i + (m - 1)d_i = f$, which is true as all A_i were focussed at f .

□

Corollary 1.27 For any k -colouring of \mathbb{N} , there exists a colour class containing arbitrarily long arithmetic progressions.

Remark 1.28 We can't guarantee infinitely long arithmetic progressions, e.g.

- 2-colour \mathbb{N} by 1 red, 2, 3 blue, 4, 5, 6 red, etc.
- The set of infinite arithmetic progressions in \mathbb{N} is countable (since described by two integers: the start term and step). Enumerate them by $(A_k)_{k \in \mathbb{N}}$. Pick $x_1 < y_1 \in A_1$, colour x_1 red and y_1 blue. Then pick $x_2, y_2 \in A_2$ with $y_1 < x_2 < y_2$, colour x_2 red, y_2 blue. Continue inductively.

Theorem 1.29 (Strengthened Van der Waerden) Let $m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there exists a monochromatic length m arithmetic progression whose common difference is the same colour (i.e. there exists $a, a + d, \dots, a + (m - 1)d$ all of the same colour).

Proof (Hints).

- Use induction on k .
- If n is the witness for $k - 1$ colours, show that $N = W(k, n(m - 1) + 1)$ is a witness for k colours, by considering n different multiples of the step of a suitable arithmetic progression.

□

Proof.

- Fix $m \in \mathbb{N}$. We use induction on k . $k = 1$ case is trivial.
- Let n be witness for $k - 1$ colours.
- We will show that $N = W(k, n(m - 1) + 1)$ is suitable for k colours.
- If $[N]$ is k -coloured, there exists a monochromatic (say red) arithmetic progression of length $n(m - 1) + 1$: $a, a + d, \dots, a + n(m - 1)d$.

- If rd is red for any $1 \leq r \leq n$, then we are done (consider $a, a + rd, \dots, a + (m - 1)rd$).
- If not, then $\{d, 2d, \dots, nd\}$ is $k - 1$ -coloured, which induces a $k - 1$ colouring on $[n]$. Therefore, there exists a monochromatic arithmetic progression $b, b + s, \dots, b + (m - 1)s$ (with s the same colour) by induction, which translates to $db, db + ds, \dots, db + d(m - 1)s$ and ds being monochromatic.

□

Remark 1.30 The case $m = 2$ of strengthened Van der Waerden is **Schur's theorem**: for any k -colouring of \mathbb{N} , there are monochromatic x, y, z such that $x + y = z$. This can be proved directly from Ramsey's theorem for pairs: let $c : \mathbb{N} \rightarrow [k]$ be a k -colouring, then induce $c' : \mathbb{N}^{(2)} \rightarrow [k]$ by $c'(ij) = c(j - i)$. By Ramsey, there exist $i < j < k$ such that $c'(ij) = c'(ik) = c'(jk)$, i.e. $c(j - i) = c(k - i) = c(k - j)$. So take $x = j - i, z = k - i, y = k - j$.

1.4. The Hales-Jewett theorem

Definition 1.31 Let X be finite set. We say X^n consists of **words of length n on alphabet X** .

Definition 1.32 Let X be finite. A **(combinatorial) line** in X^n is a set $L \subseteq X^n$ of the form

$$L = \{(x_1, \dots, x_n) \in X^n : \forall i \notin I, x_i = a_i \text{ and } \forall i, j \in I, x_i = x_j\}$$

for some non-empty set $I \subseteq [n]$ and $a_i \in X$ (for each $i \notin I$). I is the set of **active coordinates** for L .

Note that a combinatorial line is invariant under permutations of X .

Example 1.33 Let $X = [3]$. Some lines in X^2 are:

- $I = \{1\}$: $\{(1, 1), (2, 1), (3, 1)\}$ (with $a_2 = 1$), $\{(1, 2), (2, 2), (3, 2)\}$ (with $a_2 = 2$), $\{(1, 3), (2, 3), (3, 3)\}$ (with $a_2 = 3$).
- $I = \{2\}$: $\{(1, 1), (1, 2), (1, 3)\}$ (with $a_1 = 1$), $\{(2, 1), (2, 2), (2, 3)\}$ (with $a_1 = 2$), $\{(3, 1), (3, 2), (3, 3)\}$ (with $a_1 = 3$).
- $I = \{1, 2\}$: $\{(1, 1), (2, 2), (3, 3)\}$.

Note that $\{(1, 3), (2, 2), (3, 1)\}$ is **not** a combinatorial line.

Example 1.34 Some sets of lines in $[3]^3$ are:

- $I = \{1\}$: $\{(1, 2, 3), (2, 2, 3), (3, 2, 3)\}$ (with $a_2 = 2, a_3 = 3$).
- $I = \{1, 3\}$: $\{(1, 3, 1), (2, 3, 2), (3, 3, 3)\}$ (with $a_2 = 3$).

Definition 1.35 In a line L , write L^- and L^+ for the smallest and largest points in L (with respect to the ordering on $[m]^n$ where $x \leq y$ if $x_i \leq y_i$ for all i).

Definition 1.36 Lines L_1, \dots, L_k are **focussed** at f if $L_i^+ = f$ for all $i \in [k]$. They are **colour-focussed** if they are focussed and $L_i \setminus \{L_i^+\}$ is monochromatic for all $i \in [k]$.

Example 1.37 In $[4]^2$, three lines colour-focussed at f are shown below: TODO insert diagram.

Theorem 1.38 (Hales-Jewett) Let $m, k \in \mathbb{N}$ (we use alphabet $X = [m]$), then there exists $n \in \mathbb{N}$ such that for any k -colouring of $[m]^n$, there exists a monochromatic combinatorial line.

Proof. By induction on m . The case $m = 1$ is trivial as $|[m]^n| = 1$. Assume that $\text{HJ}(m-1, k')$ exists for all $k' \in \mathbb{N}$. We claim that for all $1 \leq r \leq k$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[m]^n$, we have either:

- a monochromatic line, or
- r colour-focussed lines.

We can then take $r = k$ and consider the focus.

We prove the claim by induction on r . For $r = 1$, $n = \text{HJ}(m-1, k)$ suffices. Let n be a witness for $r-1$. Let $n' = \text{HJ}(m-1, k^{m^n})$. We will show $N = n + n'$ is suitable for r . Let $c : [m]^N \rightarrow [k]$ be a k -colouring with no monochromatic lines. Writing $[m]^N = [m]^n \times [m]^{n'}$, colour $[m]^{n'}$ by $c' : [m]^{n'} \rightarrow [k]^{m^n}$, $c'(b) = (c(a_1, b), \dots, c(a_{m^n}, b))$ (where $[m]^n = \{a_1, \dots, a_{m^n}\}$). By the inductive hypothesis, there exists a line L with active coordinates I such that $c(a, b) = c'(a, b')$ for all $a \in [m]^n$ and for all $b, b' \in L \setminus \{L^+\}$. But now this induces a colouring $c'' : [m]^n \rightarrow [k]$, $c''(a) = c(a, b)$ for any $b \in L \setminus \{L^+\}$. By definition of n , there exist $r-1$ lines L_1, \dots, L_{r-1} colour-focussed (w.r.t c'') at f , with active coordinates I_1, \dots, I_{r-1} .

Finally, look at the lines that start at (L_i^-, L^-) with active coordinates $I_i \cup I$, and at the line that starts at (f, L^-) with active coordinate I . These are all focussed at (f, L^+) , so we are done. \square

Notation 1.39 Denote the smallest such n by $\text{HJ}(m, k)$.

Corollary 1.40 Hales-Jewett implies Van der Waerden's theorem.

Proof (Hints). For a colouring $c : \mathbb{N} \rightarrow [k]$, consider the induced colouring $c'(x_1, \dots, x_n) = c(x_1 + \dots + x_n)$ of $[m]^n$. \square

Proof. Let c be a k -colouring of \mathbb{N} . For sufficiently large n (i.e. $n \geq \text{HJ}(m, k)$), induce a k -colouring c' of $[m]^n$ by $c'(x_1, \dots, x_n) = c(x_1 + \dots + x_n)$. By Hales-Jewett, a monochromatic (with respect to c') combinatorial line L exists. This gives a monochromatic (with respect to c) length m arithmetic progression in \mathbb{N} . The step is equal to the number of active coordinates. The first term in the arithmetic progression corresponds to the point in L with all active coordinates equal to 1, the last term corresponds to the point in L with all active coordinates equal to m . \square

Exercise 1.41 Show that the m -in-a-row noughts and crosses game cannot be a draw in sufficiently high dimensions, and that the first player can always win.

Definition 1.42 A d -dimensional subspace (or d -point parameter set) $S \subseteq X^n$ is a set such that there exist pairwise disjoint $I_1, \dots, I_d \subseteq [n]$ and $a_i \in X$ for all $i \in [d]$, such that

$$S = \{x \in X^n : x_i = a_i \quad \forall i \in [n] - (I_1 \cup \dots \cup I_d), \\ \text{and } x_i = x_j \quad \forall i, j \in I_k \text{ for some } k \in [d]\}.$$

Theorem 1.43 (Extended Hales-Jewett) For all $m, k, d \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any colouring of $[m]^n$, there exists a monochromatic d -dimensional subspace.

Proof. We can view $X^{dn'}$ as $(X^d)^{n'}$. A line in $(X^d)^{n'}$ (on alphabet $Y = X^d$) corresponds to a d -dimensional subspace in $X^{dn'}$ (on alphabet X). Hence, we can take $n = \text{HJ}(n^d, k)$. \square

Example 1.44 Let $X = [3]$, $d = 3$. $Y = [3]^3$. Up to rearranging the active coordinate indices, a line in Y^n is (TODO look at scan).

Definition 1.45 Let $S \subseteq \mathbb{N}^d$ be finite. A **homothetic copy** of S is a set of the form $a + \lambda S$ where $a \in \mathbb{N}^d$ and $\lambda \in \mathbb{N}$.

Theorem 1.46 (Gallai) Let $S \subseteq \mathbb{N}^d$ be finite. For every k -colouring of \mathbb{N}^d , there exists a monochromatic homothetic copy of S .

Proof. Let $S = \{S_1, \dots, S_m\}$. Let $c : \mathbb{N}^d \rightarrow [k]$ be a k -colouring. For n large enough (i.e. $n \geq \text{HJ}(m, k)$), colour $[m]^n$ by $c(x_1, \dots, x_n) = c(S_{x_1} + \dots + S_{x_m})$. By Hales-Jewett, there exists a monochromatic line in $[m]^n$ with active coordinates I . So $c(\sum_{i \notin I} S_i + |I|S_j)$ is the same colour for all $j \in [m]$. So we are done, as $\sum_{i \notin I} S_i + |I|S$ is a homothetic copy of S . \square

2. Partition regular systems

3. Euclidean Ramsey theory