

# 1. The real numbers

## 1.1. Conventions on sets and functions

- **Definition:** for  $f : X \rightarrow Y$ , **preimage** of  $Z \subseteq Y$  is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

- **Definition:**  $f : X \rightarrow Y$  **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

- **Definition:**  $f : X \rightarrow Y$  **surjective** if  $Y = f(X)$ .
- **Proposition:** let  $f : X \rightarrow Y$ ,  $A, B \subseteq X$ , then

$$f(A \cap B) \subseteq f(A) \cap f(B),$$

$$f(A \cup B) = f(A) \cup f(B),$$

$$f(X) - f(A) \subseteq f(X - A)$$

- **Proposition:** let  $f : X \rightarrow Y$ ,  $C, D \subseteq Y$ , then

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D),$$

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D),$$

$$f^{-1}(Y - C) = X - f^{-1}(C)$$

## 1.2. The real numbers

- **Definition:**  $a \in \mathbb{R}$  is an **upper bound** of  $E \subseteq \mathbb{R}$  if  $\forall x \in E, x \leq a$ .
- **Definition:**  $c \in \mathbb{R}$  is a **least upper bound (supremum)** of  $E$ ,  $c = \sup(E)$ , if  $c \leq a$  for every upper bound  $a$ .
- **Definition:**  $a \in \mathbb{R}$  is an **lower bound** of  $E \subseteq \mathbb{R}$  if  $\forall x \in E, x \geq a$ .
- **Definition:**  $c \in \mathbb{R}$  is a **greatest lower bound (infimum)**,  $c = \inf(E)$ , if  $c \geq a$  for every lower bound  $a$ .
- **Completeness axiom of the real numbers:** every  $E \subseteq \mathbb{R}$  with an upper bound has a least upper bound. Every  $E \subseteq \mathbb{R}$  with a lower bound has a greatest lower bound.
- **Archimedes' principle:**

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

- **Remark:** every non-empty subset of  $\mathbb{N}$  has a minimum.
- **Proposition:**  $\mathbb{Q}$  is dense in  $\mathbb{R}$ :

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

## 1.3. Sequences, limits and series

- **Definition:**  $l \in \mathbb{R}$  is **limit** of  $(x_n)$  ( $(x_n)$  converges to  $l$ ) if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$$

A sequence **converges in  $\mathbb{R}$  (is convergent)** if it has a limit  $l \in \mathbb{R}$ . Limit  $l = \lim_{n \rightarrow \infty} x_n$  is unique.

- **Definition:**  $(x_n)$  **tends to infinity** if

$$\forall K > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad x_n > K$$

- **Definition:** **subsequence** of  $(x_n)$  is sequence  $(x_{n_j})$ ,  $n_1 < n_2 < \dots$ .
- **Definition:** **limit inferior** of sequence  $x_n$  is

$$\liminf_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \left( \inf_{m \geq n} x_m \right) = \sup_{n \in \mathbb{N}} \inf_{m \geq n} x_m$$

- **Definition:** **limit superior** of sequence  $x_n$  is

$$\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} x_m \right) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} x_m$$

- **Proposition:** let  $(x_n)$  bounded,  $l \in \mathbb{R}$ . The following are equivalent:
  - $l = \limsup x_n$ .
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < l + \varepsilon$ .
  - $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n > l - \varepsilon$ .
- **Proposition:** let  $(x_n)$  bounded,  $l \in \mathbb{R}$ . The following are equivalent:
  - $l = \liminf x_n$ .
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > l - \varepsilon$ .
  - $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n < l + \varepsilon$ .
- **Theorem (Bolzano-Weierstrass):** every bounded sequence has a convergent subsequence.
- **Proposition:** let  $(x_n)$  bounded. There exists convergent subsequence with limit  $\limsup x_n$  and convergent subsequence with limit  $\liminf x_n$ .
- **Proposition:** let  $(x_n)$  bounded, then  $(x_n)$  is convergent iff  $\limsup x_n = \liminf x_n$ .
- **Monotone convergence theorem for sequences:** monotone sequence converges in  $\mathbb{R}$  or tends to either  $\infty$  or  $-\infty$ .
- **Definition:**  $(x_n)$  is **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad |x_n - x_m| < \varepsilon$$

- **Theorem:** every Cauchy sequence in  $\mathbb{R}$  is convergent.

## 1.4. Open and closed sets

- **Definition:**  $U \subseteq \mathbb{R}$  is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subseteq U$$

- **Proposition:** arbitrary unions of open sets are open. Finite intersections of open sets are open.
- **Definition:**  $x \in \mathbb{R}$  is **point of closure (limit point)** for  $E \subseteq \mathbb{R}$  if

$$\forall \varepsilon > 0, \exists y \in E : |x - y| < \varepsilon$$

Equivalently,  $x$  is point of closure of  $E$  if every open interval containing  $x$  contains another point of  $E$ .

- **Definition:** **closure** of  $E$ ,  $\overline{E}$ , is set of points of closure. Note  $E \subseteq \overline{E}$ .
- **Definition:**  $F$  is **closed** if  $F = \overline{F}$ .
- **Proposition:**  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ . If  $A \subset B \subseteq \mathbb{R}$  then  $\overline{A} \subset \overline{B}$ .

- **Proposition:** for any set  $E$ ,  $\overline{E}$  is closed, i.e.  $\overline{\overline{E}} = \overline{E}$ .
- **Proposition:** let  $E \subseteq \mathbb{R}$ . The following are equivalent:
  - $E$  is closed.
  - $\mathbb{R} - E$  is open.
- **Proposition:** arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.
- **Definition:** collection  $C$  of subsets of  $\mathbb{R}$  **covers** (is a **covering** of)  $F \subseteq \mathbb{R}$  if  $F \subseteq \cup_{S \in C} S$ . If each  $S$  in  $C$  open,  $C$  is **open covering**. If  $C$  is finite,  $C$  is **finite covering**.
- **Definition:** covering  $C$  of  $F$  **contains a finite subcover** if exists  $\{S_1, \dots, S_n\} \subseteq C$  with  $F \subseteq \cup_{i=1}^n S_i$  (i.e. a finite subset of  $C$  covers  $F$ ).
- **Definition:**  $F$  is **compact** if any open covering of  $F$  contains a finite subcover.
- **Example:**  $\mathbb{R}$  is not compact,  $[a, b]$  is compact.
- **Heine-Borel theorem:**  $F$  compact iff  $F$  closed and bounded.

## 1.5. Continuity, pointwise and uniform convergence of functions

- **Definition:** let  $E \subseteq \mathbb{R}$ .  $f : E \rightarrow \mathbb{R}$  is **continuous at**  $a \in E$  if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

$f$  is **continuous** if continuous at all  $y \in E$ .

- **Definition:**  $\lim_{x \rightarrow a} f(x) = l$  if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

- **Proposition:**  $\lim_{x \rightarrow a} f(x) = l$  iff for every sequence  $(a_n)$  with  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\lim_{n \rightarrow \infty} f(a_n) = l$ .
- **Proposition:**  $f$  is continuous at  $a \in E$  iff  $\lim_{x \rightarrow a} f(x) = f(a)$  (and this limit exists).
- **Definition:**  $f : E \rightarrow \mathbb{R}$  is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in E, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

- **Proposition:** let  $F$  closed and bounded,  $f : F \rightarrow \mathbb{R}$  continuous. Then  $f$  is uniformly continuous.
- **Definition:** let  $f_n : E \rightarrow \mathbb{R}$  sequence of functions,  $f : E \rightarrow \mathbb{R}$ .  $(f_n)$  **converges pointwise** to  $f$  if

$$\forall \varepsilon > 0, \forall x \in E, \exists N \in \mathbb{N} : \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

$(f_n)$  **converges uniformly** to  $f$  is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$$

- **Theorem:** let  $f_n : E \rightarrow \mathbb{R}$  sequence of continuous functions converging uniformly to  $f : E \rightarrow \mathbb{R}$ . Then  $f$  is continuous.
- **Definition:**  $P = \{x_0, \dots, x_n\}$  is **partition** of  $[a, b]$  if  $a = x_0 < \dots < x_n = b$ .
- **Definition:**  $f : [a, b] \rightarrow \mathbb{R}$  is **piecewise linear** if there exists partition  $P = \{x_0, \dots, x_n\}$  and  $m_i, c_i \in \mathbb{R}$  such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad f(x) = m_i x + c_i$$

$f$  is continuous on  $[a, b] - P$ .

- **Definition:**  $g : [a, b] \rightarrow \mathbb{R}$  is **step function** if there exists partition  $P = \{x_0, \dots, x_n\}$  and  $m_i \in \mathbb{R}$  such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad g(x) = m_i$$

$g$  is continuous on  $[a, b] - P$ .

- **Theorem:** let  $f : E \rightarrow \mathbb{R}$  continuous,  $E$  closed and bounded. Then there exist continuous piecewise linear  $f_n$  with  $f_n \rightarrow f$  uniformly, and step functions  $g_n$  with  $g_n \rightarrow f$  uniformly.
- **Definition:**  $f : E \rightarrow \mathbb{R}$  is **Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad |f(x) - f(y)| \leq C|x - y|$$

- **Definition:**  $f : E \rightarrow \mathbb{R}$  is **bi-Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

## 1.6. The extended real numbers

- **Definition:** **extended reals** are  $\mathbb{R} \cup \{-\infty, \infty\}$  with the order relation  $-\infty < \infty$  and  $\forall x \in \mathbb{R}, -\infty < x < \infty$ .  $\infty$  is an upper bound and  $-\infty$  is a lower bound for every  $x \in \mathbb{R}$ , so  $\sup(\mathbb{R}) = \infty$ ,  $\inf(\mathbb{R}) = -\infty$ .
  - Addition:  $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$ .  $\infty + \infty = \infty - (-\infty) = \infty$ .  $\infty - \infty$  is undefined.
  - Multiplication:  $\forall a > 0, a \cdot \infty = \infty$ ,  $\forall a < 0, a \cdot \infty = -\infty$ . Also  $\infty \cdot \infty = \infty$ .
  - $\limsup$  and  $\liminf$  are defined as

$$\limsup x_n := \inf_{n \in \mathbb{N}} \left\{ \sup_{k \geq n} x_k \right\}, \quad \liminf x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{k \geq n} x_k \right\}$$

- **Definition:** extended real number  $l$  is **limit** of  $(x_n)$  if either
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$ . Then  $(x_n)$  **converges to  $l$** . or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$  (limit is  $\infty$ ) or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$  (limit is  $-\infty$ ).

$(x_n)$  **converges in the extended reals** if it has a limit in the extended reals.

## 2. Further analysis of subsets of $\mathbb{R}$

### 2.1. Countability and uncountability

- **Definition:**  $A$  is **countable** if  $A = \emptyset$ ,  $A$  is finite or there is a bijection  $\varphi : \mathbb{N} \rightarrow A$  (in which case  $A$  is **countably infinite**). Otherwise  $A$  is **uncountable**.  
**Enumeration** is bijection from  $A$  to  $[n]$  or  $\mathbb{N}$ .
- **Proposition:** if surjection from countable set to  $A$ , or injection from  $A$  to countable set, then  $A$  is countable.
- **Proposition:** any subset of  $\mathbb{N}$  is countable.
- **Proposition:**  $\mathbb{Q}$  is countable.

- **Proposition:** show that if  $(a_n)$  is a nonnegative sequence and  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Proposition:** show that if  $(a_{n,k})$  is a nonnegative sequence and  $\varphi : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  is a bijection then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Definition:**  $f : X \rightarrow Y$  is **monotone** if  $x \geq y \Rightarrow f(x) \geq f(y)$  or  $x \leq y \Rightarrow f(x) \leq f(y)$ .
- **Proposition:** let  $f$  be monotone on  $(a, b)$ . Then it is discontinuous on a countable set.
- **Lemma:** set of sequences in  $\{0, 1\}$ ,  $\{(x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$  is uncountable.
- **Theorem:**  $\mathbb{R}$  is uncountable.

## 2.2. The structure theorem for open sets

- Collection  $\{A_i : i \in I\}$  of sets is **(pairwise) disjoint** if  $n \neq m \Rightarrow A_n \cap A_m = \emptyset$ .
- **Structure theorem for open sets:** let  $U \subseteq \mathbb{R}$  open. Then exists countable collection of disjoint open intervals  $\{I_n : n \in \mathbb{N}\}$  such that  $U = \bigcup_{n \in \mathbb{N}} I_n$ .

## 2.3. Accumulation points and perfect sets

- **Definition:**  $x \in \mathbb{R}$  is **accumulation point** of  $E \subseteq \mathbb{R}$  if  $x$  is point of closure of  $E - \{x\}$ . Equivalently,  $x$  is a point of closure if

$$\forall \varepsilon > 0, \exists y \in E : y \neq x \wedge |x - y| < \varepsilon$$

Equivalently, there exists a sequence of distinct  $y_n \in E$  with  $y_n \rightarrow x$  as  $n \rightarrow \infty$ .

- **Proposition:** set of accumulation points of  $\mathbb{Q}$  is  $\mathbb{R}$ .
- **Proposition:** set of accumulation points  $E'$  of  $E$  is closed.
- **Definition:**  $E \subseteq \mathbb{R}$  is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

- **Proposition:**  $E$  is isolated iff it has no accumulation points.
- **Definition:** bounded set  $E$  is **perfect** if it equals its set of accumulation points.
- **Theorem:** every non-empty perfect set is uncountable.

## 2.4. The middle-third Cantor set

- **Proposition:** let  $\{F_n : n \in \mathbb{N}\}$  be collection of non-empty nested closed sets (so  $F_{n+1} \subseteq F_n$ ), one of which is bounded. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$$

- **Definition:** the **middle third Cantor set** is defined by:
  - Define  $C_0 := [0, 1]$

- Given  $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$ ,  $a_i < b_1 < a_2 < \dots < a_{2^n} < b_{2^n}$ , with  $|b_i - a_i| = 3^{-n}$ , define

$$C_{n+1} := \cup_{i=1}^{2^{n+1}} [a_i, a_i + 3^{-(n+1)}] \cup [b_i - 3^{-(n+1)}, b_i]$$

which is a union of  $2^{n+1}$  disjoint intervals, with all differences in endpoints equalling  $3^{-(n+1)}$ .

- The **middle third Cantor set** is

$$C := \bigcup_{n \in \mathbb{N}} C_n$$

Observe that if  $a$  is an endpoint of an interval in  $C_n$ , it is contained in  $C$ .

- **Proposition:** the middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and so uncountable.
- **Definition:** let  $k \in \mathbb{N} - \{1\}$ ,  $x \in [0, 1)$ .  $0.a_1a_2\dots$ ,  $a_i \in \{0, \dots, k-1\}$ , is a  **$k$ -ary expansion** of  $x$  if

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{k^i}$$

- **Remark:** the  $k$ -ary expansion may not be unique, but there is a countable set  $E \subseteq [0, 1)$  such that every  $x \in [0, 1) - E$  has a unique  $k$ -ary expansion.
- **Remark:** for every  $x \in C$ , the ternary ( $k = 3$ ) expansion of  $x$  is unique and

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}$$

Moreover, every choice of sequence  $(a_i)$ ,  $a_i \in \{0, 2\}$ , gives  $x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i} \in C$ .

- **Definition: Cantor-Lebesgue function**,  $g : [0, 1] \rightarrow [0, 1]$ , is defined by

$$g(x) := \begin{cases} \sum_{i \in \mathbb{N}} \frac{a_i/2}{2^i} & \text{if } x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, a_i \in \{0, 2\} \\ \sup\{f(y) : y \in C, y \leq x\} & \text{if } x \notin C \end{cases}$$

$g$  is a surjection, monotone and continuous.

## 2.5. $G_\delta, F_\sigma$

- **Definition:**  $E \subseteq \mathbb{R}$  is  **$G_\delta$**  if  $E = \cap_{n \in \mathbb{N}} U_n$  with  $U_n$  open.
- **Definition:**  $E \subseteq \mathbb{R}$  is  **$F_\sigma$**  if  $E = \cup_{n \in \mathbb{N}} F_n$  with  $F_n$  closed.
- **Lemma:** set of points where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous is  $G_\delta$ .

## 3. Construction of Lebesgue measure

### 3.1. Lebesgue outer measure

- **Definition:** let  $I$  non-empty interval with endpoints  $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$  and  $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$ . The **length** of  $I$  is

$$\ell(I) := b - a$$

and set  $\ell(\emptyset) = 0$ .

- **Definition:** let  $A \subseteq \mathbb{R}$ . **Lebesgue outer measure** of  $A$  is infimum of all sums of lengths of intervals covering  $A$ :

$$\mu^*(A) := \inf \left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subseteq \bigcup_{k \in \mathbb{N}} I_k, I_k \text{ intervals} \right\}$$

It satisfies **monotonicity**:  $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$ .

- **Proposition:** outer measure is **countably subadditive**:

$$\mu^* \left( \bigcup_{k \in \mathbb{N}} E_k \right) \leq \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

This implies **finite subadditivity**:

$$\mu^* \left( \bigcup_{k=1}^n E_k \right) \leq \sum_{k=1}^n \mu^*(E_k)$$

- **Lemma:** we have

$$\mu^*(A) = \inf \left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subset \bigcup_{k \in \mathbb{N}} I_k, I_k \neq \emptyset \text{ open intervals} \right\}$$

- **Proposition:** outer measure of interval is its length:  $\mu^*(I) = \ell(I)$ .

### 3.2. Measurable sets

- **Notation:**  $E^c = \mathbb{R} - E$ .
- **Proposition:** let  $E = (a, \infty)$ . Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

- **Definition:**  $E \subseteq \mathbb{R}$  is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Collection of such sets is  $\mathcal{F}_{\mu^*}$ .

- **Lemma (excision property):** let  $E$  Lebesgue measurable set with finite measure and  $E \subseteq B$ , then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

- **Proposition:** if  $E_1, \dots, E_n$  Lebesgue measurable then  $\bigcup_{k=1}^n E_k$  is Lebesgue measurable. If  $E_1, \dots, E_n$  disjoint then

$$\mu^* \left( A \cap \bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(A \cap E_k)$$

for any  $A \subseteq \mathbb{R}$ . In particular, for  $A = \mathbb{R}$ ,

$$\mu^* \left( \bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(E_k)$$

- **Remark:** not every set is Lebesgue measurable.
- **Definition:** collection of subsets of  $\mathbb{R}$  is an **algebra** if contains  $\emptyset$  and closed under taking complements and finite unions: if  $A, B \in \mathcal{A}$  then  $\mathbb{R} - A, A \cup B \in \mathcal{A}$ .
- **Remark:** a union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if  $\{A_k\}_{k \in \mathbb{N}}$  is countable collection of Lebesgue measurable sets, then let  $A_1' := A_1$  and for  $k > 1$ , define

$$A_{k'} := A_k - \bigcup_{i=1}^{k-1} A_i$$

then  $\{A_{k'}\}_{k \in \mathbb{N}}$  is disjoint union of Lebesgue measurable sets.

- **Proposition:** if  $E$  is countable union of Lebesgue measurable sets, then  $E$  is Lebesgue measurable. Also, if  $\{E_k\}_{k \in \mathbb{N}}$  is countable disjoint collection of Lebesgue measurable sets then

$$\mu^* \left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

### 3.3. Abstract definition of a measure

- **Definition:** let  $X \subseteq \mathbb{R}$ . Collection of subsets of  $\mathcal{F}$  of  $X$  is  **$\sigma$ -algebra** if
  - $\emptyset \in \mathcal{F}$
  - $E \in \mathcal{F} \implies E^c \in \mathcal{F}$
  - $E_1, \dots, E_n \in \mathcal{F} \implies \bigcup_{k \in \mathbb{N}} E_k \in \mathcal{F}$ .
- **Example:**
  - Trivial examples are  $\mathcal{F} = \{\emptyset, \mathbb{R}\}$  and  $\mathcal{F} = \mathcal{P}(\mathbb{R})$ .
  - Countable intersections of  $\sigma$ -algebras are  $\sigma$ -algebras.
- **Definition:** let  $\mathcal{F}$   $\sigma$ -algebra of  $X$ .  $\nu : \mathcal{F} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is **measure** satisfying
  - $\nu(\emptyset) = 0$
  - $\forall E \in \mathcal{F}, \nu(E) \geq 0$
  - **Countable additivity:** if  $E_1, E_2, \dots \in \mathcal{F}$  are disjoint then

$$\nu \left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu(E_k)$$

Elements of  $\mathcal{F}$  are **measurable** (as they are the only sets on which the measure  $\nu$  is defined).

- **Proposition:** if  $\nu$  is measure then it satisfies:
  - **Monotonicity:**  $A \subseteq B \implies \nu(A) \leq \nu(B)$ .
  - **Countable subadditivity:**  $\nu(\bigcup_{k \in \mathbb{N}} E_k) \leq \sum_{k \in \mathbb{N}} \nu(E_k)$ .
  - **Excision:** if  $A$  has finite measure, then  $A \subseteq B \implies \nu(B - A) = \nu(B) - \nu(A)$ .

### 3.4. Lebesgue measure

- **Lemma:**  $F_{\mu^*}$  is  $\sigma$ -algebra and contains every interval.
- **Theorem (Carathéodory extension):** restriction of the  $\mu^*$  to  $F_{\mu^*}$  is a measure.



- **Hahn extension theorem:** there exists unique measure  $\mu$  defined on  $\mathcal{F}_{\mu^*}$  for which  $\mu(I) = \ell(I)$  for any interval  $I$ .
- **Definition:** the measure  $\mu$  of  $\mu^*$  restricted to  $\mathcal{F}_{\mu^*}$  is the **Lebesgue measure**. It satisfies  $\mu(I) = \ell(I)$  for any interval  $I$  and is translation invariant.

### 3.5. Sets of measure 0

- **Proposition:** middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.
- **Proposition:** any countable set is Lebesgue measurable and has Lebesgue measure 0.
- **Proposition:** any  $E$  with  $\mu^*(E) = 0$  is Lebesgue measurable and has  $\mu(E) = 0$ .
- **Lemma:** let  $E$  Lebesgue measurable set with  $\mu(E) = 0$ , then  $\forall E' \subseteq E$ ,  $E'$  is Lebesgue measurable.

### 3.6. Continuity of measure

- **Definition:** countable collection  $\{E_k\}_{k \in \mathbb{N}}$  is **ascending** if  $\forall k \in \mathbb{N}, E_k \subseteq E_{k+1}$  and **descending** if  $\forall k \in \mathbb{N}, E_{k+1} \subseteq E_k$ .
- **Theorem:** every measure  $m$  satisfies:
  - If  $\{A_k\}_{k \in \mathbb{N}}$  is ascending collection of measurable sets, then

$$m\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \lim_{k \rightarrow \infty} m(A_k)$$

- If  $\{B_k\}_{k \in \mathbb{N}}$  is descending collection of measurable sets and  $m(B_1) < \infty$ , then

$$m\left(\bigcap_{k \in \mathbb{N}} B_k\right) = \lim_{k \rightarrow \infty} m(B_k)$$

### 3.7. An approximation result for Lebesgue measure

- **Definition:** **Borel  $\sigma$ -algebra**  $\mathcal{B}(\mathbb{R})$  is smallest  $\sigma$ -algebra containing all intervals: for any other  $\sigma$ -algebra  $\mathcal{F}$  containing all intervals,  $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$ .

$$\mathcal{B}(\mathbb{R}) := \bigcap \{\mathcal{F} : \mathcal{F} \text{ } \sigma \text{-algebra containing all intervals}\}$$

$E \in \mathcal{B}(\mathbb{R})$  is **Borel** or **Borel measurable**.

- **Lemma:** all open subsets of  $\mathbb{R}$ , closed subsets of  $\mathbb{R}$ ,  $G_\delta$  sets and  $F_\sigma$  sets are Borel.
- **Proposition:** the following are equivalent:
  - $E$  is Lebesgue measurable
  - $\forall \varepsilon > 0, \exists \text{ open } G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
  - $\forall \varepsilon > 0, \exists \text{ closed } F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
  - $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
  - $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

## 4. Measurable functions

### 4.1. Definition of a measurable function

- **Proposition:** let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .  $f$  continuous iff  $\forall$  open  $U \subseteq \mathbb{R}$ ,  $f^{-1}(U) \subseteq \mathbb{R}$  is open.
- **Lemma:** let  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  with  $E$  Lebesgue measurable. The following are equivalent:
  - $\forall c \in \mathbb{R}$ ,  $\{x \in E : f(x) > c\}$  is Lebesgue measurable.
  - $\forall c \in \mathbb{R}$ ,  $\{x \in E : f(x) \geq c\}$  is Lebesgue measurable.
  - $\forall c \in \mathbb{R}$ ,  $\{x \in E : f(x) < c\}$  is Lebesgue measurable.
  - $\forall c \in \mathbb{R}$ ,  $\{x \in E : f(x) \leq c\}$  is Lebesgue measurable.

The same statement holds for Borel measurable sets.

- **Definition:**  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is **(Lebesgue) measurable** if it satisfies any of the above properties and if  $E$  is Lebesgue measurable.  $f$  being **Borel measurable** is defined similarly.
- **Corollary:** if  $f$  is measurable then for every  $B \in \mathcal{B}(\mathbb{R})$ ,  $f^{-1}(B)$  is measurable. In particular, if  $f$  is measurable, preimage of any interval is measurable.
- **Definition: indicator function** on set  $A$ ,  $\mathbb{1}_A : \mathbb{R} \rightarrow \{0, 1\}$ , is

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

- **Definition:**  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is **simple (measurable) function** if  $\varphi$  is measurable function that has finite codomain.

## 4.2. Fundamental aspects of measurable functions

- **Definition:** let  $E \subseteq F \subseteq \mathbb{R}$ , let  $f : F \rightarrow \mathbb{R}$ . **Restriction**  $f_E$  is function with domain  $E$  and for which  $\forall x \in E$ ,  $f_E(x) = f(x)$ .
- **Definition:** real-valued function which is increasing or decreasing is **monotone**.
- **Definition:** sequence  $(f_n)$  on domain  $E$  is increasing if  $f_n \leq f_{n+1}$  on  $E$  for all  $n \in \mathbb{N}$ .
- **Example:** continuous functions are measurable.
- **Definition:** for  $f_1 : E \rightarrow \mathbb{R}, \dots, f_n : E \rightarrow \mathbb{R}$ , define

$$\max\{f_1, \dots, f_n\}(x) := \max\{f_1(x), \dots, f_n(x)\}$$

$\min\{f_1, \dots, f_n\}$  is defined similarly.

- **Proposition:** for finite family  $\{f_k\}_{k=1}^n$  of measurable functions with common domain  $E$ ,  $\max\{f_1, \dots, f_n\}$  and  $\min\{f_1, \dots, f_n\}$  are measurable.
- **Definition:** for  $f : E \rightarrow \mathbb{R}$ , functions  $|f|, f^+, f^-$  defined on  $E$  are

$$|f|(x) := \max\{f(x), -f(x)\}, \quad f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}$$

- **Corollary:** if  $f$  measurable on  $E$ , so are  $|f|, f^+$  and  $f^-$ .
- **Proposition:** let  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . For measurable  $D \subseteq E$ ,  $f$  measurable on  $E$  iff restrictions of  $f$  to  $D$  and  $E - D$  are measurable.
- **Theorem:** let  $f, g : E \rightarrow \mathbb{R}$  measurable.
  - **Linearity:**  $\forall \alpha, \beta \in \mathbb{R}$ ,  $\alpha f + \beta g$  is measurable.
  - **Products:**  $fg$  is measurable.
- **Proposition:** let  $f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  be sequence of measurable functions that converges pointwise to  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . Then  $f$  is measurable.

- **Simple approximation lemma:** let  $f : E \rightarrow \mathbb{R}$  measurable and bounded, so  $\exists M \geq 0 : \forall x \in E, |f|(x) < M$ . Then  $\forall \varepsilon > 0$ , there exist simple measurable functions  $\varphi_\varepsilon, \psi_\varepsilon : E \rightarrow \mathbb{R}$  such that

$$\forall x \in E, \quad \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \wedge 0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$$

- **Simple approximation theorem:** let  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ ,  $E$  measurable. Then  $f$  is measurable iff there exists sequence  $(\varphi_n)$  of simple functions on  $E$  which converge pointwise on  $E$  to  $f$  and satisfy

$$\forall n \in \mathbb{N}, \forall x \in E, |\varphi_n|(x) \leq |f|(x)$$

If  $f$  is nonnegative,  $(\varphi_n)$  can be chosen to be increasing.

- **Definition:** let  $f, g : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . Then  $f = g$  **almost everywhere** if  $\{x \in E : f(x) \neq g(x)\}$  has measure 0.
- **Proposition:** let  $f_1, f_2, f_3 : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable. If  $f_1 = f_2$  almost everywhere and  $f_2 = f_3$  almost everywhere then  $f_1 = f_3$  almost everywhere.
- **Remark:** Lebesgue measurable functions can be modified arbitrarily on a set of measure 0 without affecting measurability.
- **Proposition:** let  $f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  sequence of measurable functions,  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable. Set of points where  $(f_n)$  converges pointwise to  $f$  is measurable.
- **Proposition:** let  $f, g : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable and finite almost everywhere on  $E$ .
  - **Linearity:**  $\forall \alpha, \beta \in \mathbb{R}$ , there exists function equal to  $\alpha f + \beta g$  almost everywhere on  $E$  (any such function is measurable).
  - **Products:** there exists function equal to  $fg$  almost everywhere on  $E$  (any such function is measurable).
- **Definition:** sequence of functions  $(f_n)$  with domain  $E$  **converge in measure** to  $f$  if  $(f_n)$  and  $f$  are finite almost everywhere and

$$\forall \varepsilon > 0, \quad \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

## 5. The Lebesgue integral

### 5.1. The integral of a simple measurable function

- **Definition:** let  $\varphi$  be real-valued function taking finitely many values  $\alpha_1 < \dots < \alpha_n$ , then **standard representation** of  $\varphi$  is

$$\varphi = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}, \quad A_i = \varphi^{-1}(\{\alpha_i\})$$

- **Lemma:** let  $\varphi = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$ ,  $B_i$  disjoint measurable collection,  $\beta_i \in \mathbb{R}$ , then  $\varphi$  is simple measurable. If  $\varphi$  takes value 0 outside a set of finite measure then

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i)$$

where  $A_i$  in standard representation.

- **Definition:** let  $\varphi$  be simple nonnegative measurable function or simple measurable function taking value 0 outside set of finite measure. **Integral** of  $\varphi$  with respect to  $\mu$  is

$$\int \varphi = \sum_{i=1}^n \alpha_i \mu(A_i)$$

where  $\varphi = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$  is standard representation. Here, use convention  $0 \cdot \infty = 0$ . For measurable  $E \subseteq \mathbb{R}$ , define

$$\int_E \varphi = \int \mathbb{1}_E \varphi$$

- **Example:**
  - Let  $\varphi_2 = \mathbb{1}_{[0,2]} + \mathbb{1}_{[1,3]} = \mathbb{1}_{[0,1) \cup (2,3]} + 2\mathbb{1}_{[1,2]}$  so  $\int \varphi_2 = 4$ .
  - Let  $\varphi_3 = \mathbb{1}_{\mathbb{R}}$ , then  $\int \varphi_3 = 1 \cdot \infty = \infty$ .
  - Let  $\varphi_4 = \mathbb{1}_{(0,\infty)} + (-1)\mathbb{1}_{(-\infty,0)}$ . This can't be integrated.
  - Let  $\varphi_5 = \mathbb{1}_{(-1,0)} + (-1)\mathbb{1}_{(0,1)}$ .
- **Lemma:** let  $B_1, \dots, B_m$  be measurable sets,  $\beta_1, \dots, \beta_m \in \mathbb{R} - \{0\}$ . Then  $\varphi = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$  is simple measurable function. Also,

$$\mu\left(\bigcup_{i=1}^m B_i\right) < \infty \implies \sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i)$$

where  $A_i$  in standard representation.

- **Proposition:** let  $\varphi, \psi$  be simple measurable functions:
  - If  $\varphi, \psi$  take value 0 outside a set of finite measure, then  $\forall \alpha, \beta \in \mathbb{R}$ ,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- If  $\varphi, \psi$  nonnegative, then  $\forall \alpha, \beta \geq 0$ ,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- **Monotonicity:**

$$0 \leq \varphi \leq \psi \implies 0 \leq \int \varphi \leq \int \psi$$

- **Corollary:** let  $\varphi$  nonnegative simple function, then

$$\int \varphi = \sup \left\{ \int \psi : 0 \leq \psi \leq \varphi, \psi \text{ simple measurable} \right\}$$

- **Lemma:** let  $\varphi$  simple measurable nonnegative function.  $\varphi$  takes value 0 outside a set of finite measure iff  $\int \varphi < \infty$ . Also,  $\int \varphi = \infty$  iff there exist  $\alpha > 0$ , measurable  $A$  with  $\mu(A) = \infty$  and  $\forall x \in A, \varphi(x) \geq \alpha$ .
- **Lemma:** let  $\{E_n\}$  be ascending collection of measurable sets,  $\bigcup_{n \in \mathbb{N}} E_n = \mathbb{R}$ . Let  $\varphi$  be simple nonnegative measurable function. Then

$$\int_{E_n} \varphi \rightarrow \int \varphi \quad \text{as } n \rightarrow \infty$$

## 5.2. The integral of a nonnegative function

- **Notation:** let  $\mathcal{M}^+$  denote collection of nonnegative measurable functions  $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ .
- **Definition: support** of measurable function  $f$  with domain  $E$  is  $\text{supp}(f) := \{x \in E : f(x) \neq 0\}$ .
- **Definition:** let  $f \in \mathcal{M}^+$ . **Integral of  $f$  with respect to  $\mu$**  is

$$\int f := \sup \left\{ \int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple measurable} \right\} \in \mathbb{R} \cup \{\infty\}$$

For measurable set  $E$ , define

$$\int_E f := \int \mathbb{1}_E f$$

- **Proposition:** let  $f, g$  measurable. If  $g \leq f$  then  $\int g \leq \int f$ . Let  $E, F$  measurable. If  $E \subseteq F$  then  $\int_E f \leq \int_F f$ .
- **Monotone convergence theorem:** let  $(f_n)$  be sequence in  $\mathcal{M}^+$ . If  $(f_n)$  is increasing on measurable set  $E$  and converges pointwise to  $f$  on  $E$  then

$$\int_E f_n \rightarrow \int_E f \quad \text{as } n \rightarrow \infty$$

- **Corollary:** restriction of integral to nonnegative functions is linear:  $\forall f, g \in \mathcal{M}^+, \forall \alpha \geq 0$ ,

$$\begin{aligned} \int (f + g) &= \int f + \int g \\ \int \alpha f &= \alpha \int f \end{aligned}$$

- **Fatou's lemma:** let  $(f_n)$  be sequence in  $\mathcal{M}^+$ , then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$$

- **Lemma:** let  $(f_n) \subset \mathcal{M}^+$ , then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

- **Proposition (Chebyshev's inequality):** let  $f$  be nonnegative measurable function on  $E$ . Then

$$\forall \lambda > 0, \quad \mu(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f$$

- **Proposition:** let  $f$  be nonnegative measurable function on  $E$ . Then

$$\int_E f = 0 \iff f = 0 \text{ almost everywhere on } E$$

### 5.3. Integration of measurable functions

- **Notation:** let  $\mathcal{M}$  denote set of measurable functions.
- **Definition:**  $f \in \mathcal{M}^+$  is **integrable** if  $\int f < \infty$ .
- **Definition:** let  $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable function.  $f$  is **integrable** if  $\int f^+$  and  $\int f^-$  are finite. In this case, for any measurable set  $E$ , define

$$\int_E f := \int_E f^+ - \int_E f^-$$

Note that if  $f$  integrable then  $f^+ - f^-$  is well-defined.

- **Proposition:** if  $f = f_1 - f_2$ ,  $f_1, f_2 \in \mathcal{M}^+$ ,  $f_1, f_2$  integrable, then

$$\int f^+ - \int f^- = \int f_1 - \int f_2$$

- **Definition:**  $f \in \mathcal{M}$  is **integrable over  $E$**  ( $E$  is measurable) if  $\int_E f^+$  and  $\int_E f^-$  are finite (i.e.  $f \cdot \mathbb{1}_E$  is integrable).
- **Theorem:**  $f \in \mathcal{M}$  is integrable iff  $|f|$  is integrable. If  $f$  integrable, then

$$\left| \int f \right| \leq \int |f|$$

- **Corollary:** let  $f, g \in \mathcal{M}$ ,  $|f| \leq |g|$ . If  $g$  integrable then  $|f|$  is integrable, and  $\int |f| \leq \int |g|$ .
- **Example:**  $\sin$  is not integrable over  $\mathbb{R}$ , but is integrable over  $[0, 2\pi]$ , since  $|f_{[0, 2\pi]}| \leq \mathbb{1}_{[0, 2\pi]}$ .
- **Theorem (Linearity of Integration):** let  $f, g \in \mathcal{M}$  integrable. Then  $f + g$  is integrable and  $\forall \alpha \in \mathbb{R}$ ,  $\alpha f$  is integrable. The integral is linear:

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

- **Dominated Convergence Theorem:** let  $(f_n)$  be sequence of integrable functions. If there exists an integrable  $g$  with  $\forall n \in \mathbb{N}$ ,  $|f_n| \leq g$ , and  $f_n \rightarrow f$  pointwise almost everywhere then  $f$  is integrable and

$$\int f = \lim_{n \rightarrow \infty} \int f_n$$

### 5.4. Integrability: Riemann vs Lebesgue

- **Proposition:** let  $f$  bounded function on bounded measurable domain  $E$ . Then  $f$  is measurable and  $\int_E |f| < \infty$  iff

$$\sup \left\{ \int_E \varphi : \varphi \leq f, \varphi \text{ simple measurable} \right\} = \inf \left\{ \int_E \psi : f \leq \psi : \psi \text{ simple measurable} \right\}$$

(If  $f$  satisfies either condition then  $\int_E f$  is equal to the two above expressions).

- **Definition:** bounded function  $f$  is **Lebesgue integrable** if it satisfies either of the equivalences in the above proposition.
- **Definition:** let  $P = \{x_1, \dots, x_n\}$  partition of  $[a, b]$ ,  $f : [a, b] \rightarrow \mathbb{R}$  bounded. **Lower and upper Darboux sums** for  $f$  with respect to  $P$  are

$$L(f, P) := \sum_{i=1}^n m_i(x_i - x_{i-1}), \quad U(f, P) := \sum_{i=1}^n M_i(x_i - x_{i-1})$$

where

$$m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

If  $P \subseteq Q$  ( $Q$  is a **refinement of  $P$** ), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

- **Definition:** lower and upper Riemann integrals of  $f$  over  $[a, b]$  are

$$\underline{\mathcal{J}}_a^b(f) := \sup\{L(f, P) : P \text{ partition of } [a, b]\}$$

$$\overline{\mathcal{J}}_a^b(f) := \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

- **Definition:** let  $f : [a, b] \rightarrow \mathbb{R}$  bounded, then  $f$  is **Riemann integrable** ( $f \in \mathcal{R}$ ), if

$$\underline{\mathcal{J}}_a^b(f) = \overline{\mathcal{J}}_a^b(f)$$

and common value  $\mathcal{J}_a^b(f) = \int_a^b f(x) dx$  is **Riemann integral** of  $f$ .

- Let  $g : [a, b] \rightarrow \mathbb{R}$  step function with discontinuities at  $P = \{x_0, \dots, x_n\}$ , so  $g = \sum_{i=1}^n \alpha_i \mathbb{1}_{(x_{i-1}, x_i)}$  almost everywhere. So  $g$  is simple measurable and

$$L(g, P) = \sum_{i=1}^n \alpha_i(x_i - x_{i-1}) = U(g, P) = \int g = \mathcal{J}_a^b(g)$$

Hence for any bounded  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$\underline{\mathcal{J}}_a^b(f) = \sup \left\{ \int \varphi : \varphi \leq f, \varphi \text{ step function} \right\},$$

$$\overline{\mathcal{J}}_a^b(f) = \inf \left\{ \int \psi : f \leq \psi, \psi \text{ step function} \right\}$$

- **Theorem:** let  $f : [a, b] \rightarrow \mathbb{R}$  bounded,  $a, b \neq \pm\infty$ . If  $f$  Riemann integrable over  $[a, b]$  then  $f$  Lebesgue integrable over  $[a, b]$  and the two integrals are equal.
- **Theorem:** let  $f : [a, b] \rightarrow \mathbb{R}$  bounded,  $a, b \neq \pm\infty$ . Then  $f$  is Riemann integrable on  $[a, b]$  iff  $f$  is continuous on  $[a, b]$  except on a set of measure zero.
- **Lemma:** let  $(\varphi_n), (\psi_n)$  be sequences of functions, all integrable over  $E$ ,  $(\varphi_n)$  increasing on  $E$ ,  $(\psi_n)$  decreasing on  $E$ . Let  $f : E \rightarrow \mathbb{R}$  with

$$\forall n \in \mathbb{N}, \varphi_n \leq f \leq \psi_n \text{ on } E, \quad \lim_{n \rightarrow \infty} \int_E (\psi_n - \varphi_n) = 0$$

Then  $\varphi_n, \psi_n \rightarrow f$  pointwise almost everywhere on  $E$ ,  $f$  is integrable over  $E$  and

$$\lim_{n \rightarrow \infty} \int_E \varphi_n = \lim_{n \rightarrow \infty} \int_E \psi_n = \int_E f$$

- **Definition:** for partition  $P = \{x_0, \dots, x_n\}$ , **gap** of  $P$  is

$$\text{gap}(P) := \max\{|x_i - x_{i-1}| : i \in \{1, \dots, n\}\}$$

- **Lemma:** let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $E \subseteq [a, b]$  be set where  $f$  is continuous. Let  $(P_n)$  be sequence of partitions of  $[a, b]$  with  $P_{n+1} \subseteq P_n$  and  $\text{gap}(P_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\varphi_n, \psi_n : [a, b] \rightarrow \mathbb{R}$  step functions with

$$\varphi_n(x) := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad \psi_n(x) := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

for  $P_n = \{x_0, \dots, x_n\}$ . Then  $\forall x \in E - \cup_{n \in \mathbb{N}} P_n$ ,

$$\varphi_n(x), \psi_n(x) \rightarrow f(x) \quad \text{as } n \rightarrow \infty$$

- **Definition:** let  $f : (a, b) \rightarrow \mathbb{R}$ ,  $-\infty \leq a < b < \infty$ ,  $f$  bounded and Riemann integrable on all closed bounded sub-intervals of  $(a, b]$ . If

$$\lim_{t \rightarrow a, t > a} \mathcal{J}_t^b(f)$$

exists then this is defined as the **improper Riemann integral**  $\mathcal{J}_a^b(f)$ . Similar definitions exist for  $f : (a, b) \rightarrow \mathbb{R}$  and  $f : [a, b) \rightarrow \mathbb{R}$ .

- **Note:** improper Riemann integral may exist without function being Lebesgue integral.
- **Proposition:** if  $f$  is integrable, the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.
- **Definition:** let  $\alpha : [a, b] \rightarrow \mathbb{R}$  monotonically increasing (and so bounded). For partition  $P = \{x_0, \dots, x_n\}$  of  $[a, b]$  and bounded  $f : [a, b] \rightarrow \mathbb{R}$ , define

$$L(f, P, \alpha) := \sum_{i=1}^n m_i(\alpha(x_i) - \alpha(x_{i-1})), \quad U(f, P, \alpha) := \sum_{i=1}^n M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

where  $m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}$ ,  $M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$ . Then  $f$  is **integrable with respect to  $\alpha$** ,  $f \in \mathcal{R}(\alpha)$ , if

$$\inf\{U(f, P, \alpha) : P \text{ partition of } [a, b]\} = \sup\{L(f, P, \alpha) : P \text{ partition of } [a, b]\}$$

and the common value  $\int_a^b f d\alpha$  is the **Riemann-Stieltjes integral** of  $f$  with respect to  $\alpha$ .

- **Proposition:** let  $f : (a, b) \rightarrow \mathbb{R}$ , then set of points where  $f$  is differentiable is measurable.
- **Remark:** if  $\alpha : [0, 1] \rightarrow [a, b]$  bijection, then



$$\int_0^1 f \circ \alpha \, d\alpha = \int_a^b f(x) \, dx$$

- **Proposition:** let  $\alpha$  be monotonically increasing and differentiable with  $\alpha' \in \mathcal{R}$ . Then  $g \in \mathcal{R}(\alpha)$  iff  $g\alpha' \in \mathcal{R}$ , and in that case,

$$\int_a^b g \, d\alpha = \int_a^b g(x)\alpha'(x) \, dx$$

- **Remark:** when  $g = 1$ , this says  $\int_a^b 1 \, d\alpha = \alpha(b) - \alpha(a) = \int \alpha'(x) \, dx$ , similar to the fundamental theorem of calculus.