1. Metric spaces

1.1. Metrics

- **Metric space**: (X, d), X is set, $d : X \times X \to [0, \infty)$ is **metric** satisfying:
  - $d(x, y) = 0 \iff x = y$
  - **Symmetry**: $d(x, y) = d(y, x)$
  - **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- Examples of metrics:
  - p-adic metric:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

  - Extension of the p-adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

  - Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

  - Discrete metric:

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Open ball of radius r around x**:

$$B(x; r) = \{y \in X : d(x, y) < r\}$$

- **Closed ball of radius r around x**:

$$D(x; r) = \{y \in X : d(x, y) \leq r\}$$

1.2. Open and closed sets

- $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

- $A \subseteq X$ is **closed** if $X - A$ is open.
- Sets can be neither closed nor open, or both.
- Any singleton $\{x\} \in \mathbb{R}$ is closed and not open.
- Let X be metric space, $x \in N \subseteq X$. N is **neighbourhood** of x if

$$\exists \text{ open } V \subseteq X : x \in V \subseteq N$$

- **Corollary**: let $x \in X$, then $N \subseteq X$ neighbourhood of x iff $\exists \varepsilon > 0 : x \in B(x; \varepsilon) \subseteq N$.
- **Proposition**: open balls are open, closed balls are closed.
- **Lemma**: let (X, d) metric space.
  - X and $\emptyset$ are both open and closed.
  - Arbitrary unions of open sets are open.
  - Finite intersections of open sets are open.

- Finite unions of closed sets are closed.
- Arbitrary intersections of closed sets are closed.

1.3. Continuity

- **Sequence** in X: $a : \mathbb{N} \to X$, written $(a_n)_{n \in \mathbb{N}}$.
- (a_n) converges to a if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, d(a, a_n) < \varepsilon$$

- **Proposition**: let X, Y metric spaces, $a \in X$, $f : X \to Y$. The following are equivalent
  - $\forall \varepsilon > 0, \exists \delta > 0 : d_X(a, x) < \delta \Longrightarrow d_Y(f(a), f(x)) < \varepsilon$.
  - For every sequence (a_n) in X with $a_n \to a$, $f(a_n) \to f(a)$.
  - For every open $U \subseteq Y$ with $f(a) \in U$, $f^{-1}(U)$ is a neighbourhood of a.

If f satisfies these, it is **continuous at a**.

- f **continuous** if continuous at every $a \in X$.
- **Proposition**: $f : X \to Y$ continuous iff $f^{-1}(U)$ open for every open $U \subseteq Y$.

2. Topological spaces

2.1. Topologies

- **Power set** of X: $\mathcal{P}(X) := \{A : A \subseteq X\}$.
- **Topology** on set X is $\tau \subseteq \mathcal{P}(X)$ with:
  - $\emptyset \in \tau$, $X \in \tau$.
  - If $\forall i \in I, U_i \in \tau$, then

$$\bigcup_{i \in I} U_i \in \tau$$

  - $U_1, U_2 \in \tau \Longrightarrow U_1 \cap U_2 \in \tau$.
- (X, τ) is **topological space**. Elements of τ are **open** subsets of X.