1. The real numbers

1.1. Conventions on sets and functions

Definition. For $f : X \to Y$, **preimage** of $Z \subseteq Y$ is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

- asdf

Definition. $f : X \to Y$ **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

Definition. $f : X \to Y$ **surjective** if $Y = f(X)$.

Proposition. Let $f : X \to Y$, $A, B \subseteq X$, then

$$\begin{aligned} f(A \cap B) &\subseteq f(A) \cap f(B), \\ f(A \cup B) &= f(A) \cup f(B), \\ f(X) - f(A) &\subseteq f(X - A) \end{aligned}$$

Proposition. Let $f : X \to Y$, $C, D \subseteq Y$, then

$$\begin{aligned} f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D), \\ f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D), \\ f^{-1}(Y - C) &= X - f^{-1}(C) \end{aligned}$$

1.2. The real numbers

Definition. $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \leq a$.

Definition. $c \in \mathbb{R}$ is a **least upper bound (supremum)** of E, $c = \sup(E)$, if $c \leq a$ for every upper bound a.

Definition. $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \geq a$.

Definition. $c \in \mathbb{R}$ is a **greatest lower bound (supremum)**, $c = \inf(E)$, if $c \geq a$ for every upper bound a.

Theorem (Completeness axiom of the real numbers). Every $E \subseteq \mathbb{R}$ with an upper bound has a least upper bound. Every $E \subseteq \mathbb{R}$ with a lower bound has a greatest lower bound.

Proposition (Archimedes' principle).

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

Remark. Every non-empty subset of $\mathbb{N}$ has a minimum.

Proposition. $\mathbb{Q}$ is dense in $\mathbb{R}$:

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

1.3. Sequences, limits and series

Definition. $l \in \mathbb{R}$ is **limit** of (x_n) ((x_n) converges to l) if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad |x_n - l| < \varepsilon$$

A sequence **converges in $\mathbb{R}$ (is convergent)** if it has a limit $l \in \mathbb{R}$. Limit $l = \lim_{n\to\infty} x_n$ is unique.

Definition. (x_n) **tends to infinity** if

$$\forall K > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad x_n > K$$

Definition. **Subsequence** of (x_n) is sequence (x_{n_j}), $n_1 < n_2 < \cdots$.

Definition. **Limit inferior** of sequence x_n is

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \left(\inf_{m\geq n} x_m \right) = \sup_{n\in\mathbb{N}} \inf_{m\geq n} x_m$$

Definition. **Limit superior** of sequence x_n is

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \left(\sup_{m\geq n} x_m \right) = \inf_{n\in\mathbb{N}} \sup_{m\geq n} x_m$$

Proposition. Let (x_n) bounded, $l \in \mathbb{R}$. The following are equivalent:

- $l = \limsup x_n$.
- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < l + \varepsilon$.
- $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n > l - \varepsilon$.

Proposition. Let (x_n) bounded, $l \in \mathbb{R}$. The following are equivalent:

- $l = \liminf x_n$.
- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > l - \varepsilon$.
- $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n < l + \varepsilon$.

Theorem (Bolzano-Weierstrass). Every bounded sequence has a convergent subsequence.

Proposition. Let (x_n) bounded. There exists convergent subsequence with limit $\limsup x_n$ and convergent subsequence with limit $\liminf x_n$.

Proposition. Let (x_n) bounded, then (x_n) is convergent iff $\limsup x_n = \liminf x_n$.

Theorem (Monotone convergence theorem for sequences). Monotone sequence converges in $\mathbb{R}$ or tends to either ∞ or $-\infty$.

Definition. (x_n) is **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad |x_n - x_m| < \varepsilon$$

Theorem. Every Cauchy sequence in $\mathbb{R}$ is convergent.

1.4. Open and closed sets

Definition. $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subseteq U$$

Proposition. Arbitrary unions of open sets are open. Finite intersections of open sets are open.

Definition. $x \in \mathbb{R}$ is **point of closure (limit point)** for $E \subseteq \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists y \in E : |x - y| < \varepsilon$$

Equivalently, x is point of closure of E if every open interval containing x contains another point of E.

Definition. **Closure** of E, $\overline{E}$, is set of points of closure. Note $E \subseteq \overline{E}$.

Definition. F is **closed** if $F = \overline{F}$.

Proposition. $\overline{A \cup B} = \overline{A} \cup \overline{B}$. If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$.

Proposition. For any set E, $\overline{E}$ is closed, i.e. $\overline{E} = \overline{\overline{E}}$.

Proposition. Let $E \subseteq \mathbb{R}$. The following are equivalent:

- E is closed.
- $\mathbb{R} - E$ is open.

Proposition. Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.

Definition. Collection C of subsets of $\mathbb{R}$ **covers** (is a **covering** of) $F \subseteq \mathbb{R}$ if $F \subseteq \cup_{S \in C} S$. If each S in C open, C is **open covering**. If C is finite, C is **finite covering**.

Definition. Covering C of F **contains a finite subcover** if exists $\{S_1, ..., S_n\} \subseteq C$ with $F \subseteq \cup_{i=1}^n S_i$ (i.e. a finite subset of C covers F).

Definition. F is **compact** if any open covering of F contains a finite subcover.

Example. $\mathbb{R}$ is not compact, $[a, b]$ is compact.

Theorem (Heine Borel). F compact iff F closed and bounded.

1.5. Continuity, pointwise and uniform convergence of functions

Definition. Let $E \subseteq \mathbb{R}$. $f : E \to \mathbb{R}$ is **continuous at $a \in E$** if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon$$

f is **continuous** if continuous at all $y \in E$.

Definition. $\lim_{x \to a} f(x) = l$ if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \Longrightarrow |f(x) - l| < \varepsilon$$

Proposition. $\lim_{x \to a} f(x) = l$ iff for every sequence (a_n) with $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} f(a_n) = l$.

Proposition. f is continuous at $a \in E$ iff $\lim_{x \to a} f(x) = f(a)$ (and this limit exists).

Definition. $f : E \to \mathbb{R}$ is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in E, |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$$

Proposition. Let F closed and bounded, $f : F \to \mathbb{R}$ continuous. Then f is uniformly continuous.

Definition. Let $f_n : E \to \mathbb{R}$ sequence of functions, $f : E \to \mathbb{R}$. (f_n) **converges pointwise** to f if

$$\forall \varepsilon > 0, \forall x \in E, \exists N \in \mathbb{N} : \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

(f_n) **converges uniformly** to f is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$$

Theorem. Let $f_n : E \to \mathbb{R}$ sequence of continuous functions converging uniformly to $f : E \to \mathbb{R}$. Then f is continuous.

Definition. $P = \{x_0, ..., x_n\}$ is **partition** of $[a, b]$ if $a = x_0 < \cdots < x_n = b$.

Definition. $f : [a, b] \to \mathbb{R}$ is **piecewise linear** if there exists partition $P = \{x_0, ..., x_n\}$ and $m_i, c_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad f(x) = m_i x + c_i$$

f is continuous on $[a, b] - P$.

Definition. $g : [a, b] \to \mathbb{R}$ is **step function** if there exists partition $P = \{x_0, ..., x_n\}$ and $m_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i)), \quad g(x) = m_i$$

g is continuous on $[a, b] - P$.

Theorem. Let $f : E \to \mathbb{R}$ continuous, E closed and bounded. Then there exist continuous piecewise linear f_n with $f_n \to f$ uniformly, and step functions g_n with $g_n \to f$ uniformly.

Definition. $f : E \to \mathbb{R}$ is **Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad |f(x) - f(y)| \leq C|x - y|$$

Definition. $f : E \to \mathbb{R}$ is **bi-Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

1.6. The extended real numbers

Definition. **Extended reals** are $\mathbb{R} \cup \{-\infty, \infty\}$ with the order relation $-\infty < \infty$ and $\forall x \in \mathbb{R}, -\infty < x < \infty$. ∞ is an upper bound and $-\infty$ is a lower bound for every $x \in \mathbb{R}$, so $\sup(\mathbb{R}) = \infty$, $\inf(\mathbb{R}) = -\infty$.

- Addition: $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$. $\infty + \infty = \infty - (-\infty) = \infty$. $\infty - \infty$ is undefined.
- Multiplication: $\forall a > 0, a \cdot \infty = \infty$, $\forall a < 0, a \cdot \infty = -\infty$. Also $\infty \cdot \infty = \infty$.
- lim sup and lim inf are defined as

$$\limsup x_n := \inf_{n \in \mathbb{N}} \left\{ \sup_{k \geq n} x_k \right\}, \quad \liminf x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{k \geq n} x_k \right\}$$

Definition. Extended real number l is **limit** of (x_n) if either

- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$. Then (x_n) **converges to *l***. or

- $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$ (limit is ∞) or
- $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$ (limit is $-\infty$).

(x_n) **converges in the extended reals** if it has a limit in the extended reals.

2. Further analysis of subsets of $\mathbb{R}$

2.1. Countability and uncountability

Definition. A is **countable** if $A = \emptyset$, A is finite or there is a bijection $\varphi : \mathbb{N} \to A$ (in which case A is **countably infinite**). Otherwise A is **uncountable**. **Enumeration** is bijection from A to $[n]$ or $\mathbb{N}$.

Proposition. If surjection from countable set to A, or injection from A to countable set, then A is countable.

Proposition. Any subset of $\mathbb{N}$ is countable.

Proposition. $\mathbb{Q}$ is countable.

Proposition. Show that if (a_n) is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

Proposition. Show that if $(a_{n,k})$ is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

Definition. $f : X \to Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.

Proposition. Let f be monotone on (a, b). Then it is discountinuous on a countable set.

Lemma. Set of sequences in $\{0, 1\}$, $\{(x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.

Theorem. $\mathbb{R}$ is uncountable.

2.2. The structure theorem for open sets

Definition. Collection $\{A_i : i \in I\}$ of sets is **(pairwise) disjoint** if $n \neq m \Longrightarrow A_n \cap A_m = \emptyset$.

Theorem (Structure theorem for open sets). Let $U \subseteq \mathbb{R}$ open. Then exists countable collection of disjoint open intervals $\{I_n : n \in \mathbb{N}\}$ such that $U = \cup_{n \in \mathbb{N}} I_n$.

2.3. Accumulation points and perfect sets

Definition. $x \in \mathbb{R}$ is **accumulation point** of $E \subseteq \mathbb{R}$ if x is point of closure of $E - \{x\}$. Equivalently, x is a point of closure if

$$\forall \varepsilon > 0, \exists y \in E : y \neq x \land |x - y| < \varepsilon$$

Equivalently, there exists a sequence of distinct $y_n \in E$ with $y_n \to x$ as $n \to \infty$.

Proposition. Set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$.

Proposition. Set of accumulation points E' of E is closed.

Definition. $E \subseteq \mathbb{R}$ is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

Proposition. E is isolated iff it has no accumulation points.

Definition. Bounded set E is **perfect** if it equals its set of accumulation points.

Theorem. Every non-empty perfect set is uncountable.

2.4. The middle-third Cantor set

Proposition. Let $\{F_n : n \in \mathbb{N}\}$ be collection of non-empty nested closed sets (so $F_{n+1} \subseteq F_n$), one of which is bounded. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$$

Definition. The **middle third Cantor set** is defined by:

- Define $C_0 := [0, 1]$
- Given $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$, $a_1 < b_1 < a_2 < \cdots < a_{2^n} < b_{2^n}$, with $|b_i - a_i| = 3^{-n}$, define

  $$C_{n+1} := \cup_{i=1}^{2^n} \left[a_i, a_i + 3^{-(n+1)}\right] \cup \left[b_i - 3^{-(n+1)}, b_i\right]$$

  which is a union of 2^{n+1} disjoint intervals, with all differences in endpoints equalling $3^{-(n+1)}$.
- The **middle third Cantor set** is

  $$C := \bigcap_{n \in \mathbb{N}} C_n$$

  Observe that if a is an endpoint of an interval in C_n, it is contained in C.

Proposition. The middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and so uncountable.

Definition. Let $k \in \mathbb{N} - \{1\}$, $x \in [0, 1)$. $0.a_1 a_2 \ldots$, $a_i \in \{0, \ldots, k - 1\}$, is a ***k*-ary expansion** of x if

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{k^i}$$

Remark. The k-ary expansion may not be unique, but there is a countable set $E \subseteq [0, 1)$ such that every $x \in [0, 1) - E$ has a unique k-ary expansion.

Remark. For every $x \in C$, the ternary ($k = 3$) expansion of x is unique and

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}$$

Moreover, every choice of sequence (a_i), $a_i \in \{0, 2\}$, gives $x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i} \in C$.

Definition. **Cantor-Lebesgue function**, $g : [0, 1] \to [0, 1]$, is defined by

$$g(x) := \begin{cases} \sum_{i \in \mathbb{N}} \frac{a_i/2}{2^i} & \text{if } x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, a_i \in \{0, 2\} \\ \sup\{f(y) : y \in C, y \leq x\} & \text{if } x \notin C \end{cases}$$

g is a surjection, monotone and continuous.

2.5. G_δ, F_σ

Definition. $E \subseteq \mathbb{R}$ is $\boldsymbol{G_\delta}$ if $E = \bigcap_{n \in \mathbb{N}} U_n$ with U_n open.

Definition. $E \subseteq \mathbb{R}$ is $\boldsymbol{F_\sigma}$ if $E = \bigcup_{n \in \mathbb{N}} F_n$ with F_n closed.

Lemma. Set of points where $f : \mathbb{R} \to \mathbb{R}$ is continuous is G_δ.

3. Construction of Lebesgue measure

3.1. Lebesgue outer measure

Definition. Let I non-empty interval with endpoints $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$ and $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$. The **length** of I is

$$\ell(I) := b - a$$

and set $\ell(\emptyset) = 0$.

Definition. Let $A \subseteq \mathbb{R}$. **Lebesgue outer measure** of A is infimum of all sums of lengths of intervals covering A:

$$\mu^*(A) := \inf\left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subseteq \bigcup_{k \in \mathbb{N}} I_k, I_k \text{ intervals} \right\}$$

It satisfies **monotonicity**: $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

Proposition. Outer measure is **countably subadditive**:

$$\mu^*\left(\bigcup_{k \in \mathbb{N}} E_k \right) \leq \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

This implies **finite subadditivity**:

$$\mu^*\left(\bigcup_{k=1}^{n} E_k \right) \leq \sum_{k=1}^{n} \mu^*(E_k)$$

Lemma. We have

$$\mu^*(A) = \inf\left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subset \bigcup_{k \in \mathbb{N}} I_k, I_k \neq \emptyset \text{ open intervals} \right\}$$

Proposition. Outer measure of interval is its length: $\mu^*(I) = \ell(I)$.

3.2. Measurable sets

Notation. $E^c = \mathbb{R} - E$.

Proposition. Let $E = (a, \infty)$. Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Definition. $E \subseteq \mathbb{R}$ is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Collection of such sets is $\mathcal{F}_{\mu^*}$.

Lemma (Excision Property). Let E Lebesgue measurable set with finite measure and $E \subseteq B$, then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

Proposition. If $E_1, ..., E_n$ Lebesgue measurable then $\cup_{k=1}^n E_k$ is Lebesgue measurable. If $E_1, ..., E_n$ disjoint then

$$\mu^*\left(A \cap \bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu^*(A \cap E_k)$$

for any $A \subseteq \mathbb{R}$. In particular, for $A = \mathbb{R}$,

$$\mu^*\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu^*(E_k)$$

Remark. Not every set is Lebesgue measurable.

Definition. Collection of subsets of $\mathbb{R}$ is an **algebra** if contains $\emptyset$ and closed under taking complements and finite unions: if $A, B \in \mathcal{A}$ then $\mathbb{R} - A, A \cup B \in \mathcal{A}$.

Remark. A union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if $\{A_k\}_{k\in\mathbb{N}}$ is countable collection of Lebesgue measurable sets, then let $A_{1'} := A_1$ and for $k > 1$, define

$$A_{k'} := A_k - \cup_{i=1}^{k-1} A_i$$

then $\{A_{k'}\}_{k\in\mathbb{N}}$ is disjoint union of Lebesgue measurable sets.

Proposition. If E is countable union of Lebesgue measurable sets, then E is Lebesgue measurable. Also, if $\{E_k\}_{k\in\mathbb{N}}$ is countable disjoint collection of Lebesgue measurable sets then

$$\mu^*\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \mu^*(E_k)$$

3.3. Abstract definition of a measure

Definition. Let $X \subseteq \mathbb{R}$. Collection of subsets of $\mathcal{F}$ of X is $\boldsymbol{\sigma}$**-algebra** if

- $\emptyset \in \mathcal{F}$

- $E \in \mathcal{F} \implies E^c \in \mathcal{F}$
- $E_1, ..., E_n \in \mathcal{F} \implies \cup_{k\in\mathbb{N}} E_k \in \mathcal{F}$.

Example.
- Trivial examples are $\mathcal{F} = \{\emptyset, \mathbb{R}\}$ and $\mathcal{F} = \mathcal{P}(\mathbb{R})$.
- Countable intersections of σ-algebras are σ-algebras.

Definition. Let $\mathcal{F}$ σ-algebra of X. $\nu : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is **measure** satisfying
- $\nu(\emptyset) = 0$
- $\forall E \in \mathcal{F}, \nu(E) \geq 0$
- **Countable additivity**: if $E_1, E_2, ... \in \mathcal{F}$ are disjoint then

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k)$$

Elements of $\mathcal{F}$ are **measurable** (as they are the only sets on which the measure ν is defined).

Proposition. If ν is measure then it satisfies:
- **Monotonicity**: $A \subseteq B \implies \nu(A) \leq \nu(B)$.
- **Countable subadditivity**: $\nu(\cup_{k\in\mathbb{N}} E_k) \leq \sum_{k\in\mathbb{N}} \nu(E_k)$.
- **Excision**: if A has finite measure, then $A \subseteq B \implies m(B - A) = m(B) - m(A)$.

3.4. Lebesgue measure

Lemma. F_{μ^*} is σ-algebra and contains every interval.

Theorem (Carathéodory Extension). Restriction of the μ^* to F_{μ^*} is a measure.

Theorem (Hahn extension theorem). There exists unique measure μ defined on $\mathcal{F}_{\mu^*}$ for which $\mu(I) = \ell(I)$ for any interval I.

Definition. The measure μ of μ^* restricted to $\mathcal{F}_{\mu^*}$ is the **Lebesgue measure**. It satisfies $\mu(I) = \ell(I)$ for any interval I and is translation invariant.

3.5. Sets of measure 0

Proposition. Middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.

Proposition. Any countable set is Lebesgue measurable and has Lebesgue measure 0.

Proposition. Any E with $\mu^*(E) = 0$ is Lebesgue measurable and has $\mu(E) = 0$.

Lemma. Let E Lebesgue measurable set with $\mu(E) = 0$, then $\forall E' \subseteq E$, E' is Lebesgue measurable.

3.6. Continuity of measure

Definition. Countable collection $\{E_k\}_{k\in\mathbb{N}}$ is **ascending** if $\forall k \in \mathbb{N}, E_k \subseteq E_{k+1}$ and **descending** if $\forall k \in \mathbb{N}, E_{k+1} \subseteq E_k$.

Theorem. Every measure m satisfies:
- If $\{A_k\}_{k\in\mathbb{N}}$ is ascending collection of measurable sets, then

$$m\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \lim_{k\to\infty} m(A_k)$$

- If $\{B_k\}_{k\in\mathbb{N}}$ is descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k\in\mathbb{N}} B_k\right) = \lim_{k\to\infty} m(B_k)$$

3.7. An approximation result for Lebesgue measure

Definition. **Borel $\boldsymbol{\sigma}$-algebra** $\mathcal{B}(\mathbb{R})$ is smallest σ-algebra containing all intervals: for any other σ-algebra $\mathcal{F}$ containing all intervals, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

$$\mathcal{B}(\mathbb{R}) := \bigcap\{\mathcal{F} : \mathcal{F}\ \sigma\text{ -algebra containing all intervals}\}$$

$E \in \mathcal{B}(\mathbb{R})$ is **Borel** or **Borel measurable**.

Lemma. All open subsets of $\mathbb{R}$, closed subsets of $\mathbb{R}$, G_δ sets and F_σ sets are Borel.

Proposition. The following are equivalent:

- E is Lebesgue measurable
- $\forall \varepsilon > 0, \exists$ open $G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
- $\forall \varepsilon > 0, \exists$ closed $F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
- $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
- $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

4. Measurable functions

4.1. definition of a measurable function

Proposition. Let $f : \mathbb{R} \to \mathbb{R}$. f continuous iff $\forall$ open $U \subseteq \mathbb{R}, f^{-1}(U) \subseteq \mathbb{R}$ is open.

Lemma. Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ with E Lebesgue measurable. The following are equivalent:

- $\forall c \in \mathbb{R}, \{x \in E : f(x) > c\}$ is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) \geq c\}$ is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) < c\}$ is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) \leq c\}$ is Lebesgue measurable.

The same statement holds for Borel measurable sets.

Definition. $f : E \to \mathbb{R} \cup \{\pm\infty\}$ is **(Lebesgue) measurable** if it satisfies any of the above properties and if E is Lebesgue measurable. f being **Borel measurable** is defined similarly.

Corollary. If f is measurable then for every $B \in \mathcal{B}(\mathbb{R})$, $f^{-1}(B)$ is measurable. In particular, if f is measurable, preimage of any interval is measurable.

Definition. **Indicator function** on set A, $\mathbb{1}_A : \mathbb{R} \to \{0, 1\}$, is

$$\mathbb{1}_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Definition. $\varphi : \mathbb{R} \to \mathbb{R}$ is **simple (measurable) function** if φ is measurable function that has finite codomain.

4.2. Fundamental aspects of measurable functions

Definition. Let $E \subseteq F \subseteq \mathbb{R}$, let $f : F \to \mathbb{R}$. **Restriction** f_E is function with domain E and for which $\forall x \in E, f_E(x) = f(x)$.

Definition. Real-valued function which is increasing or decreasing is **monotone**.

Definition. Sequence (f_n) on domain E is increasing if $f_n \leq f_{n+1}$ on E for all $n \in \mathbb{N}$.

Example. Continuous functions are measurable.

Definition. For $f_1 : E \to \mathbb{R}, ..., f_n : E \to \mathbb{R}$, define

$$\max\{f_1, ..., f_n\}(x) := \max\{f_1(x), ..., f_n(x)\}$$

$\min\{f_1, ..., f_n\}$ is defined similarly.

Proposition. For finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E, $\max\{f_1, ..., f_n\}$ and $\min\{f_1, ..., f_n\}$ are measurable.

Definition. For $f : E \to \mathbb{R}$, functions $|f|, f^+, f^-$ defined on E are

$$|f|(x) := \max\{f(x), -f(x)\}, \quad f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}$$

Corollary. If f measurable on E, so are $|f|$, f^+ and f^-.

Proposition. Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$. For measurable $D \subseteq E$, f measurable on E iff restrictions of f to D and $E - D$ are measurable.

Theorem. Let $f, g : E \to \mathbb{R}$ measurable.
- **Linearity**: $\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g$ is measurable.
- **Products**: fg is measurable.

Proposition. Let $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ be sequence of measurable functions that converges pointwise to $f : E \to \mathbb{R} \cup \{\pm\infty\}$. Then f is measurable.

Lemma (Simple approximation lemma). Let $f : E \to \mathbb{R}$ measurable and bounded, so $\exists M \geq 0 : \forall x \in E, |f|(x) < M$. Then $\forall \varepsilon > 0$, there exist simple measurable functions $\varphi_\varepsilon, \psi_\varepsilon : E \to \mathbb{R}$ such that

$$\forall x \in E, \quad \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \land 0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$$

Theorem (Simple approximation theorem). Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$, E measurable. Then f is measurable iff there exists sequence (φ_n) of simple functions on E which converge pointwise on E to f and satisfy

$$\forall n \in \mathbb{N}, \forall x \in E, |\varphi_n|(x) \leq |f|(x)$$

If f is nonnegative, (φ_n) can be chosen to be increasing.

Definition. Let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$. Then $f = g$ **almost everywhere** if $\{x \in E : f(x) \neq g(x)\}$ has measure 0.

Proposition. Let $f_1, f_2, f_3 : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable. If $f_1 = f_2$ almost everywhere and $f_2 = f_3$ almost everywhere then $f_1 = f_3$ almost everywhere.

Remark. Lebesgue measurable functions can be modified arbitrarily on a set of measure 0 without affecting measurability.

Proposition. Let $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ sequence of measurable functions, $f : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable. Set of points where (f_n) converges pointwise to f is measurable.

Proposition. Let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable and finite almost everywhere on E.

- **Linearity**: $\forall \alpha, \beta \in \mathbb{R}$, there exists function equal to $\alpha f + \beta g$ almost everywhere on E (any such function is measurable).
- **Products**: there exists function equal to fg almost everywhere on E (any such function is measurable).

Definition. Sequence of functions (f_n) with domain E **converge in measure** to f if (f_n) and f are finite almost everywhere and

$$\forall \varepsilon > 0, \quad \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \to 0 \text{ as } n \to \infty$$

5. The Lebesgue integral

5.1. The integral of a simple measurable function

Definition. Let φ be real-valued function taking finitely many values $\alpha_1 < \cdots < \alpha_n$, then **standard representation** of φ is

$$\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}, \quad A_i = \varphi^{-1}(\{\alpha_i\})$$

Lemma. Let $\varphi = \sum_{i=1}^{m} \beta_i \mathbb{1}_{B_i}$, B_i disjoint measurable collection, $\beta_i \in \mathbb{R}$, then φ is simple measurable. If φ takes value 0 outside a set of finite measure then

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \beta_i \mu(B_i)$$

where A_i in standard representation.

Definition. Let φ be simple nonnegative measurable function or simple measurable function taking value 0 outside set of finite measure. **Integral** of φ with respect to μ is

$$\int \varphi = \sum_{i=1}^{n} \alpha_i \mu(A_i)$$

where $\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$ is standard representation. Here, use convention $0 \cdot \infty = 0$. For measurable $E \subseteq \mathbb{R}$, define

$$\int_E \varphi = \int \mathbb{1}_E \varphi$$

Example.
- Let $\varphi_2 = \mathbb{1}_{[0,2]} + \mathbb{1}_{[1,3]} = \mathbb{1}_{[0,1)\cup(2,3]} + 2\mathbb{1}_{[1,2]}$ so $\int \varphi_2 = 4$.
- Let $\varphi_3 = \mathbb{1}_{\mathbb{R}}$, then $\int \varphi_3 = 1 \cdot \infty = \infty$.
- Let $\varphi_4 = \mathbb{1}_{(0,\infty)} + (-1)\mathbb{1}_{(-\infty,0)}$. This can't be integrated.
- Let $\varphi_5 = \mathbb{1}_{(-1,0)} + (-1)\mathbb{1}_{(0,1)}$.

Lemma. Let $B_1, ..., B_m$ be measurable sets, $\beta_1, ..., \beta_m \in \mathbb{R} - \{0\}$. Then $\varphi = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$ is simple measurable function. Also,

$$\mu\left(\bigcup_{i=1}^m B_i\right) < \infty \Longrightarrow \sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i)$$

where A_i in standard representation.

Proposition. Let φ, ψ be simple measurable functions:
- If φ, ψ take value 0 outside a set of finite measure, then $\forall \alpha, \beta \in \mathbb{R}$,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- If φ, ψ nonnegative, then $\forall \alpha, \beta \geq 0$,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- **Monotonicity**:

$$0 \leq \varphi \leq \psi \Longrightarrow 0 \leq \int \varphi \leq \int \psi$$

Corollary. Let φ nonnegative simple function, then

$$\int \varphi = \sup\left\{\int \psi : 0 \leq \psi \leq \varphi, \psi \text{ simple measurable}\right\}$$

Lemma. Let φ simple measurable nonnegative function. φ takes value 0 outside a set of finite measure iff $\int \varphi < \infty$. Also, $\int \varphi = \infty$ iff there exist $\alpha > 0$, measurable A with $\mu(A) = \infty$ and $\forall x \in A, \varphi(x) \geq \alpha$.

Lemma. Let $\{E_n\}$ be ascending collection of measurable sets, $\cup_{n\in\mathbb{N}} E_n = \mathbb{R}$. Let φ be simple nonnegative measurable function. Then

$$\int_{E_n} \varphi \to \int \varphi \quad \text{as } n \to \infty$$

5.2. The integral of a nonnegative function

Notation. Let $\mathcal{M}^+$ denote collection of nonnegative measurable functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Definition. **Support** of measurable function f with domain E is $\operatorname{supp}(f) := \{x \in E : f(x) \neq 0\}$.

Definition. Let $f \in \mathcal{M}^+$. **Integral of f with respect to μ** is

$$\int f := \sup\left\{\int \varphi : 0 \le \varphi \le f, \varphi \text{ simple measurable}\right\} \in \mathbb{R} \cup \{\infty\}$$

For measurable set E, define

$$\int_E f := \int \mathbb{1}_E f$$

Proposition. Let f, g measurable. If $g \le f$ then $\int g \le \int f$. Let E, F measurable. If $E \subseteq F$ then $\int_E f \le \int_F f$.

Theorem (Monotone convergence theorem). Let (f_n) be sequence in $\mathcal{M}^+$. If (f_n) is increasing on measurable set E and converges pointwise to f on E then

$$\int_E f_n \to \int_E f \quad \text{as } n \to \infty$$

Corollary. Restriction of integral to nonnegative functions is linear: $\forall f, g \in \mathcal{M}^+$, $\forall \alpha \ge 0$,

$$\int (f+g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

Lemma (Fatou's Lemma). Let (f_n) be sequence in $\mathcal{M}^+$, then

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

Lemma. Let $(f_n) \subset \mathcal{M}^+$, then

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$$

Proposition (Chebyshev's inequality). Let f be nonnegative measurable function on E. Then

$$\forall \lambda > 0, \quad \mu(\{x \in E : f(x) \ge \lambda\}) \le \frac{1}{\lambda} \int_E f$$

Proposition. Let f be nonnegative measurable function on E. Then

$$\int_E f = 0 \iff f = 0 \text{ almost everywhere on } E$$

5.3. Integration of measurable functions

Notation. Let $\mathcal{M}$ denote set of measurable functions.

Definition. $f \in \mathcal{M}^+$ is **integrable** if $\int f < \infty$.

Definition. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ measurable function. f is **integrable** if $\int f^+$ and $\int f^-$ are finite. In this case, for any measurable set E, define

$$\int_E f := \int_E f^+ - \int_E f^-$$

Note that if f integrable then $f^+ - f^-$ is well-defined.

Proposition. If $f = f_1 - f_2$, $f_1, f_2 \in \mathcal{M}^+$, f_1, f_2 integrable, then

$$\int f^+ - \int f^- = \int f_1 - \int f_2$$

Definition. $f \in \mathcal{M}$ is **integrable over E** (E is measurable) if $\int_E f^+$ and $\int_E f^-$ are finite (i.e. $f \cdot \mathbb{1}_E$ is integrable).

Theorem. $f \in \mathcal{M}$ is integrable iff $|f|$ is integrable. If f integrable, then

$$\left|\int f\right| \leq \int |f|$$

Corollary. Let $f, g \in \mathcal{M}$, $|f| \leq |g|$. If g integrable then $|f|$ is integrable, and $\int |f| \leq \int |g|$.

Example. sin is not integrable over $\mathbb{R}$, but is integrable over $[0, 2\pi]$, since $|f_{[0,2\pi]}| \leq \mathbb{1}_{[0,2\pi]}$.

Theorem (Linearity of Integration). Let $f, g \in \mathcal{M}$ integrable. Then $f + g$ is integrable and $\forall \alpha \in \mathbb{R}$, αf is integrable. The integral is linear:

$$\int (f+g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

Theorem (Dominated Convergence Theorem). Let (f_n) be sequence of integrable functions. If there exists an integrable g with $\forall n \in \mathbb{N}, |f_n| \leq g$, and $f_n \to f$ pointwise almost everywhere then f is integrable and

$$\int f = \lim_{n \to \infty} \int f_n$$

5.4. Integrability: Riemann vs Lebesgue

Proposition. Let f bounded function on bounded measurable domain E. Then f is measurable and $\int_E |f| < \infty$ iff

$$\sup\left\{\int_E \varphi : \varphi \leq f, \varphi \text{ simple measurable}\right\} = \inf\left\{\int_E \psi : f \leq \psi : \psi \text{ simple measurable}\right\}$$

(If f satisfies either condition then $\int_E f$ is equal to the two above expressions).

Definition. Bounded function f is **Lebesgue integrable** if it satisfies either of the equivalences in the above proposition.

Definition. Let $P = \{x_1, ..., x_n\}$ partition of $[a, b]$, $f : [a, b] \to \mathbb{R}$ bounded. **Lower and upper Darboux sums** for f with respect to P are

$$L(f,P) := \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(f,P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where

$$m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

If $P \subseteq Q$ (Q is a **refinement of P**), then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

Definition. **Lower and upper Riemann integrals** of f over $[a,b]$ are

$$\underline{\mathcal{I}}_a^b(f) := \sup\{L(f,P) : P \text{ partition of } [a,b]\}$$
$$\overline{\mathcal{I}}_a^b(f) := \inf\{U(f,P) : P \text{ partition of } [a,b]\}$$

Definition. Let $f : [a,b] \to \mathbb{R}$ bounded, then f is **Riemann integrable** ($f \in \mathcal{R}$), if

$$\underline{\mathcal{I}}_a^b(f) = \overline{\mathcal{I}}_a^b(f)$$

and common value $\mathcal{I}_a^b(f) = \int_a^b f(x)\,\mathrm{d}x$ is **Riemann integral** of f.

Remark. Let $g : [a,b] \to \mathbb{R}$ step function with discontinuities at $P = \{x_0, ..., x_n\}$, so $g = \sum_{i=1}^n \alpha_i \mathbb{1}_{(x_{i-1}, x_i)}$ almost everywhere. So g is simple measurable and

$$L(g,P) = \sum_{i=1}^{n} \alpha_i(x_i - x_{i-1}) = U(g,P) = \int g = \mathcal{I}_a^b(g)$$

Hence for any bounded $f : [a,b] \to \mathbb{R}$,

$$\underline{\mathcal{I}}_a^b(f) = \sup\left\{\int \varphi : \varphi \leq f, \varphi \text{ step function}\right\},$$
$$\overline{\mathcal{I}}_a^b(f) = \inf\left\{\int \psi : f \leq \psi, \psi \text{ step function}\right\}$$

Theorem. Let $f : [a,b] \to \mathbb{R}$ bounded, $a, b \neq \pm\infty$. If f Riemann integrable over $[a,b]$ then f Lebesgue integrable over $[a,b]$ and the two integrals are equal.

Theorem. Let $f : [a,b] \to \mathbb{R}$ bounded, $a, b \neq \pm\infty$. Then f is Riemann integrable on $[a,b]$ iff f is continuous on $[a,b]$ except on a set of measure zero.

Lemma. Let (φ_n), (ψ_n) be sequences of functions, all integrable over E, (φ_n) increasing on E, (ψ_n) decreasing on E. Let $f : E \to \mathbb{R}$ with

$$\forall n \in \mathbb{N}, \varphi_n \leq f \leq \psi_n \text{ on } E, \quad \lim_{n\to\infty} \int_E (\psi_n - \varphi_n) = 0$$

Then $\varphi_n, \psi_n \to f$ pointwise almost everywhere on E, f is integrable over E and

$$\lim_{n\to\infty} \int_E \varphi_n = \lim_{n\to\infty} \int_E \psi_n = \int_E f$$

Definition. For partition $P = \{x_0, ..., x_n\}$, **gap** of P is

$$\text{gap}(P) \coloneqq \max\{|x_i - x_{i-1}| : i \in \{1, ..., n\}\}$$

Lemma. Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$ be set where f is continuous. Let (P_n) be sequence of partitions of $[a,b]$ with $P_{n+1} \subseteq P_n$ and $\text{gap}(P_n) \to 0$ as $n \to \infty$. Let $\varphi_n, \psi_n : [a,b] \to \mathbb{R}$ step functions with

$$\varphi_n(x) \coloneqq \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad \psi_n(x) \coloneqq \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

for $P_n = \{x_0, ..., x_n\}$. Then $\forall x \in E - \cup_{n \in \mathbb{N}} P_n$,

$$\varphi_n(x), \psi_n(x) \to f(x) \quad \text{as } n \to \infty$$

Definition. Let $f : (a,b] \to \mathbb{R}$, $-\infty \leq a < b < \infty$, f bounded and Riemann integrable on all closed bounded sub-intervals of $(a,b]$. If

$$\lim_{t \to a, t > a} \mathcal{I}_t^b(f)$$

exists then this is defined as the **improper Riemann integral** $\mathcal{I}_a^b(f)$. Similar definitions exist for $f : (a,b) \to \mathbb{R}$ and $f : [a,b) \to \mathbb{R}$.

Note. Improper Riemann integral may exist without function being Lebesgue integral.

Proposition. If f is integrable, the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.

Definition. Let $\alpha : [a,b] \to \mathbb{R}$ monotonically increasing (and so bounded). For partition $P = \{x_0, ..., x_n\}$ of $[a,b]$ and bounded $f : [a,b] \to \mathbb{R}$, define

$$L(f, P, \alpha) \coloneqq \sum_{i=1}^n m_i(\alpha(x_i) - \alpha(x_{i-1})), \quad U(f, P, \alpha) \coloneqq \sum_{i=1}^n M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

where $m_i \coloneqq \inf\{f(x) : x \in (x_{i-1}, x_i)\}$, $M_i \coloneqq \sup\{f(x) : x \in (x_{i-1}, x_i)\}$. Then f is **integrable with respect to $\boldsymbol{\alpha}$**, $f \in \mathcal{R}(\alpha)$, if

$$\inf\{U(f, P, \alpha) : P \text{ partition of } [a,b]\} = \sup\{L(f, P, \alpha) : P \text{ partition of } [a,b]\}$$

and the common value $\int_a^b f \, \mathrm{d}\alpha$ is the **Riemann-Stieltjes integral** of f with respect to α.

Proposition. Let $f : (a,b) \to \mathbb{R}$, then set of points where f is differentiable is measurable.

Remark. If $\alpha : [0,1] \to [a,b]$ bijection, then

$$\int_0^1 f \circ \alpha \, \mathrm{d}\alpha = \int_a^b f(x) \, \mathrm{d}x$$

Proposition. Let α be monotonically increasing and differentiable with $\alpha' \in \mathcal{R}$. Then $g \in \mathcal{R}(\alpha)$ iff $g\alpha' \in \mathcal{R}$, and in that case,

$$\int_a^b g\,\mathrm{d}\alpha = \int_a^b g(x)\alpha'(x)\,\mathrm{d}x$$

Remark. When $g = 1$, this says $\int_a^b 1\,\mathrm{d}\alpha = \alpha(b) - \alpha(a) = \int \alpha'(x)\,\mathrm{d}x$, similar to the fundamental theorem of calculus.

6. Lebesgue spaces

6.1. Normed linear spaces

Definition. Let X be **complex linear space** (vector space over $\mathbb{C}$). $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ is **norm on X** if

- $\forall x \in X, \|x\| = 0 \Longleftrightarrow x = 0$.
- $\forall x \in X, \forall \lambda \in \mathbb{C}, \|\lambda x\| = |\lambda|\ \|x\|$.
- $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.

X equipped with norm $\|\cdot\|$, $(X, \|\cdot\|)$, is called **complex normed linear space**.

Example.

- $\|x\| = \sqrt{x\overline{x}}$ is norm on $\mathbb{C}$.
- Let $C[a, b]$ denote linear space of continuous real-valued functions on $[a, b]$. Then

$$\|f\|_{\max} := \max\{|f(x)| : x \in [a, b]\}$$

  is norm on $C[a, b]$.

Proposition. Norm induces metric on X: $d(x, y) = \|x - y\|$.

Definition. Let $(X, \|\cdot\|)$ be normed linear space.

- Sequence (f_n) in X is **Cauchy sequence** in X if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad \|f_n - f_m\| < \varepsilon$$

- Sequence (f_n) in X **converges in X**, $\|f_n - f\| \to 0$ as $n \to \infty$, if

$$\exists f \in X : \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad \|f_n - f\| < \varepsilon$$

- $(X, \|\cdot\|)$ is **complete** if every Cauchy sequence converges in X.
- **Banach space** is complete normed linear space.

Proposition. Let $(X, \|\cdot\|)$ be normed linear space.

- If (x_n) converges in X, (x_n) is Cauchy sequence in X.
- Let (x_n) be Cauchy sequence in X. If (x_n) has convergent subsequence in X then (x_n) converges in X.

6.2. Lebesgue spaces L^p, $p \in [1, \infty)$

Definition. Let $p \in [1, \infty)$, $E \subseteq \mathbb{R}$.

- Linear space $L^p(E)$ is defined as

$$L^p(E) := \left\{ f : E \to \mathbb{C} : f \text{ is measurable and } \int_E |f|^p < \infty \right\} / \cong$$

  where $f \cong g$ iff $f = g$ almost everywhere:

$$f \cong g \iff \exists F \subseteq E : \mu(F) = 0 \land \forall x \in E - F, f(x) = g(x)$$

- Define $\|\cdot\|_{L^p} : L^p(E) \to \mathbb{R}$ as

$$\|f\|_{L^p} := \left(\int_E |f|^p\right)^{1/p}$$

Remark.

- We often consider space $L^p(E)$ of real-valued measurable functions $f : E \to \mathbb{R}$ such that $\int_E |f|^p < \infty$.
- For $f : E \to \mathbb{C}$, $f = f_1 + i f_2$, f is measurable iff $f_1 : E \to \mathbb{R}$ and $f_2 : E \to \mathbb{R}$ are measurable. Also,

$$\int_E |f|^p < \infty \iff \left(\int_E |f_1|^p < \infty \land \int_E |f_2|^p < \infty\right)$$

Example. Let $E = \mathbb{R}$, $f(x) = \mathbb{1}_{\mathbb{R}-\mathbb{Q}}(x) + i\mathbb{1}_{\mathbb{Q}}(x)$ and $g(x) = 1$. Then $\mu(\mathbb{Q}) = 0$ so $f \cong g$.

Proposition. Let $(f_n), (g_n)$ sequences of measurable functions, $\forall n \in \mathbb{N}, f_n \cong g_n$, $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$. Then $f \cong g$.

Definition. $p, q \in \mathbb{R}$ are **conjugate exponents** if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma (Young's inequality). Let p, q conjugate exponents, then

$$\forall A, B \in \mathbb{R}_{\geq 0}, \quad AB \leq \frac{A^p}{p} + \frac{B^q}{q}$$

with equality iff $A^p = B^q$.

Lemma (Hölder's inequality). Let p, q conjugate exponents. If $f \in L^p(E)$, $g \in L^q(E)$, then

$$\int_E |fg| \leq \|f\|_{L^p} \|g\|_{L^q}$$

Corollary (Cauchy-Schwarz inequality for $L^2(E)$). If $f, g \in L^2(E)$, then

$$\left|\int_E f\overline{g}\right| \leq \int_E |fg| \leq \|f\|_{L^2} \|g\|_{L^2}$$

Lemma (Minkowski's inequality). Let $p \in [1, \infty)$. If $f, g \in L^p(E)$ then $f + g \in L^p(E)$ and

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Theorem. For $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is normed linear space.

Proposition. Let $1 \leq p < q < \infty$. If $\mu(E) < \infty$ then $L^q(E) \subseteq L^p(E)$ and

$$\|f\|_{L^p} \leq \mu(E)^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q}$$

Remark.
- Convergence in L^p is also called convergence in the mean of order p.
- This notion of convergence is different to pointwise convergence, uniform convergence and convergence in measure.

Theorem (Riesz-Fischer). For $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is complete.

6.3. Lebesgue space L^∞

Definition.
- Let $f : E \to \mathbb{C}$ measurable. f is **essentially bounded** if

$$\exists M \geq 0 : |f(x)| \leq M \quad \text{almost everywhere on } E$$

- $L^\infty(E)$ is collection of equivalence classes of essentially bounded functions where $f \cong g$ iff $f = g$ almost everywhere.
- For $f \in L^\infty(E)$, define

$$\|f\|_{L^\infty} := \operatorname{ess\,sup} |f| := \inf\{M \in \mathbb{R} : \mu(\{x \in E : |f(x)| > M\}) = 0\}$$

Proposition.
- $0 \leq |f(x)| \leq \|f\|_{L^\infty}$ almost everywhere.
- $\|f\|_{L^\infty}$ is norm on $L^\infty(E)$.
- If $f \in L^1(E)$, $g \in L^\infty(E)$, then

$$\int_E |fg| \leq \|f\|_{L^1} \|g\|_{L^\infty}$$

Proposition. Let (f_n) sequence of functions in $L^\infty(E)$. Then (f_n) converges to $f \in L^\infty(E)$ iff there exists $G \subseteq E$ with $\mu(G) = 0$ and (f_n) converges to f uniformly on $E - G$.

Theorem. $(L^\infty(E), \|\cdot\|_{L^\infty})$ is complete.

Remark. If $\mu(E) < \infty$, then $L^\infty(E) \subset L^p(E)$ for $p \in [1, \infty)$ and

$$\|f\|_{L^p} \leq \mu(E)^{1/p} \|f\|_{L^\infty}$$

since

$$\|f\|_{L^p}^p = \int_E |f|^p \leq \int_E \|f\|_{L^\infty}^p \cdot \mathbb{1}_E = \|f\|_{L^\infty}^p \mu(E)$$

6.4. Approximation and separability

Definition. Let $(X, \|\cdot\|)$ be normed linear space. Let $F \subseteq G \subseteq X$. F is **dense in G** if

$$\forall g \in G, \forall \varepsilon > 0, \exists f \in F : \quad \|f - g\| < \varepsilon$$

Proposition.
- F is dense in G iff for every $g \in G$, there exists sequence (f_n) in F such that $\lim_{n\to\infty} f_n = g$ in X.
- For $F \subseteq G \subseteq H \subseteq X$, if F dense in G and G dense in H, then F dense in H.

Proposition. Let $p \in [1,\infty]$. Then subspace of simple functions in $(L^p(E), \|\cdot\|_{L^p})$ is dense in $(L^p(E), \|\cdot\|_{L^p})$.

Definition. $\psi : \mathbb{R} \to \mathbb{R}$ is **step function** if it can be written as

$$\psi = \sum_{k=1}^{N} \tilde{a}_k \mathbb{1}_{(a_k,b_k)}$$

where the intervals (a_k, b_k) are disjoint.

Proposition. Let $[a,b]$ be bounded, $p \in [1,\infty)$. Then subspace of step functions on $[a,b]$ is dense in $(L^p([a,b]), \|\cdot\|_{L^p})$.

Definition. Normed linear space $(X, \|\cdot\|)$ is **separable** if there exists countable, dense subset $X' \subseteq X$.

Example. $\mathbb{R}$ is separable, since $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

Theorem. Let $E \subseteq \mathbb{R}$ measurable, $p \in [1,\infty)$. Then $(L^p(E), \|\cdot\|_{L^p})$ is separable.

Proposition. Let $\varepsilon > 0$, $f \in L^p(E)$, $p \in [1,\infty)$. There exists continuous $g \in L^p(E)$ such that $\|f - g\|_{L^p} < \varepsilon$.

Remark. Linear space of continuous functions that vanish outside bounded set is dense in $(L^p(E), \|\cdot\|_{L^p})$ for $p \in [1,\infty)$.

Remark. Differentiable functions are also dense in $(L^p(E), \|\cdot\|_{L^p})$ for $p \in [1,\infty)$.

Remark. Step functions and continuous functions are not dense in $(L^\infty(E), \|\cdot\|_{L^\infty})$.

Example. In general, $(L^\infty(E), \|\cdot\|_{L^\infty})$ is not separable. Let $[a,b]$ be bounded, $a \neq b$. Assume there is countable $\{f_n : n \in \mathbb{N}\}$ which is dense in $(L^\infty([a,b]), \|\cdot\|_{L^\infty})$. Then for every $x \in [a,b]$, can choose $g(x) \in \mathbb{N}$ such that

$$\left\|\mathbb{1}_{[a,x]} - f_{g(x)}\right\|_{L^\infty} < \frac{1}{2}$$

Also, for $x_1 \leq x_2$,

$$\left\|\mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]}\right\|_{L^\infty} = \begin{cases} 1 & \text{if } a \leq x_1 < x_2 \leq b \\ 0 & \text{if } x_1 = x_2 \end{cases}$$

and

$$\begin{aligned}\left\|\mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]}\right\|_{L^\infty} &\leq \left\|\mathbb{1}_{[a,x_1]} - f_{g(x_1)}\right\|_{L^\infty} + \left\|f_{g(x_1)} - f_{g(x_2)}\right\|_{L^\infty} + \left\|f_{g(x_2)} - \mathbb{1}_{[a,x_2]}\right\|_{L^\infty} \\ &< 1 + \left\|f_{g(x_1)} - f_{g(x_2)}\right\|_{L^\infty}\end{aligned}$$

If $g(x_1) = g(x_2)$ then $\left\|\mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]}\right\|_{L^\infty} = 0$ so $g : [a,b] \to \mathbb{N}$ is injective. But $\mathbb{N}$ is countable and $[a,b]$ is not countable: contradiction.

6.5. Riesz representation theorem for $L^p(E)$, $p \in [1,\infty)$

Definition. Let X be linear space. $T : X \to \mathbb{R}$ is **linear functional** if

$$\forall f, g \in X, \forall a, b \in \mathbb{R}, \quad T(af + bg) = aT(f) + bT(g)$$

Any linear combination of linear functionals is linear, so set of linear functionals on linear space is also linear space.

Definition. Let $(X, \|\cdot\|)$ be normed linear space. $T : X \to \mathbb{R}$ is **bounded functional** if

$$\exists M \geq 0 : \forall f \in X, \quad |T(f)| \leq M\|f\|$$

Norm of T, $\|T\|_*$, is the smallest such M.

Remark. For bounded linear functional T on normed linear space $(X, \|\cdot\|)$,

$$|T(f) - T(g)| \leq \|T\|_* \|f - g\|$$

This gives the following continuity property: if $f_n \to f \in X$, then $T(f_n) \to T(f)$.

Example. Let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, q conjugate to p. Let $h \in L^q(E)$. Define $T : L^p(E) \to \mathbb{R}$ by

$$T(f) = \int_E h \cdot f$$

By Holder's inequality,

$$|T(f)| = \left| \int_E hf \right| \leq \int_E |hf| \leq \|h\|_{L^q} \|f\|_{L^p}$$

So T is bounded linear functional.

Remark. We can write $\|\cdot\|_*$ as

$$\|T\|_* := \inf\{M \in \mathbb{R} : \forall f \in X, |T(f)| \leq M\|f\|\} = \sup\{|T(f)| : f \in X, \|f\| \leq 1\}$$

Definition. **Dual space** of X, X^*, is set of bounded linear functionals on X with norm $\|\cdot\|_*$.

Proposition. Let $(X, \|\cdot\|)$ be normed linear space, then dual space of X is linear space.

Remark. Bounded linear functional is special case of **bounded linear transformation** between normed spaces. $T : X \to Y$ is bounded linear transformation if $T(af + bg) = aT(f) + bT(g)$ and $\exists M \geq 0 : \|T(f)\|_Y \leq M\|f\|_X$.

Proposition. Let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, q conjugate to p, $h \in L^q(E)$. Define $T : L^p(E) \to \mathbb{R}$ by

$$T(f) = \int_E hf$$

Then $\|T\|_* = \|h\|_{L^q}$.

Theorem (Riesz representation theorem for L^p). Let $p \in [1, \infty)$, q conjugate to p, $E \subseteq \mathbb{R}$ measurable. For $h \in L^q(E)$, define bounded linear functional $R_h : L^p(E) \to \mathbb{R}$ by

$$R_h(f) = \int_E hf$$

Then for every bounded linear functional $T : L_p(E) \to \mathbb{R}$, there is unique $h \in L^q(E)$ such that

$$R_h = T \quad \land \quad \|T\|_* = \|h\|_{L^q}$$

Theorem. Let $[a, b]$ be non-degenerate, bounded interval, $p \in [1, \infty)$, q conjugate to p. If T is bounded linear functional on $L^p([a, b])$ then there exists $h \in L^q([a, b])$ such that

$$T(f) = \int_a^b hf$$

7. Hilbert spaces

7.1. Inner product spaces

Definition. Let H be complex linear space. **Inner product** on H is function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ such that $\forall a, b \in \mathbb{C}, \forall x, y, z \in H$,

- **Linear in first variable**: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
- **Conjugate symmetric**: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- **Positive**: $x \neq 0 \Longrightarrow \langle x, x \rangle \in (0, \infty)$
- $\langle x, x \rangle = 0 \Longleftrightarrow x = 0$.

These imply that $\langle 0, x \rangle = 0$ and inner product is conjugate linear in second variable: $\langle z, ax + by \rangle = \overline{a}\langle z, x \rangle + \overline{b}\langle z, y \rangle$.

Example.

- $\mathbb{R}^n$ has inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$.
- $\mathbb{C}^n$ has inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$.
- Inner product induces metric on H:

$$d(x, y) = \langle x - y, x - y \rangle^{1/2}$$

Definition. Complex linear space H with inner product $\langle \cdot, \cdot \rangle$ is called **pre-Hilbert space** or **inner product space**.

Definition. Let H inner product space. For $x \in H$, define the norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Proposition. $\|x \pm y\|^2 = \|x\|^2 \pm 2 \operatorname{Re}(\langle x, y \rangle) + \|y\|^2$.

Theorem (Cauchy-Schwarz inequality). Let $(H, \langle \cdot, \cdot \rangle)$ be pre-Hilbert space. Then

$$\forall x, y \in H, \quad |\langle x, y \rangle| \leq \|x\|\|y\|$$

with equality iff x and y linearly dependent.

Theorem (Parallelogram Identity). A normed linear space X is an inner product space with norm derived from the inner product (i.e. $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$) iff

$$\forall x, y \in X, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Definition. Let $\left(X, \langle\cdot,\cdot\rangle_X\right)$, $\left(Y, \langle\cdot,\cdot\rangle_Y\right)$ be inner product spaces.
- An inner product on $X \times Y$ is

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{X \times Y} = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y$$

- The associated norm on $X \times Y$ is

$$\|(x,y)\|_{X\times Y} = \sqrt{\langle (x_1, y_1), (x_2, y_2) \rangle_{X \times Y}} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$$

Theorem. Let X inner product space, $x_n \to x$, $y_n \to y$ in X. Then $\langle x_n, y_n \rangle_X \to \langle x, y \rangle_X$.

7.2. Hilbert spaces

Definition. Hilbert space is inner product space which is complete with respect to norm induced by inner product.

Example. $\mathbb{R}^n$ with standard inner product is Hilbert space.

Example. Define inner product on $L^2(E)$

$$\langle f, g \rangle_{L^2} := \int_E f\overline{g}$$

Induced norm is the L^2 norm. So by Riesz-Fischer theorem, $(L^2(E), \langle\cdot,\cdot\rangle_{L^2})$ is Hilbert space.

Definition. Let H Hilbert space with inner product $\langle\cdot,\cdot\rangle$.
- $x, y \in H$ are **orthogonal**, $x \perp y$ if $\langle x, y \rangle = 0$.
- $A, B \subseteq H$ are **orthogonal**, $A \perp B$ if $\forall x \in A, \forall y \in B, \quad x \perp y$.
- **Orthogonal complement** of $A \subseteq H$ is

$$A^\perp := \{x \in H : \forall y \in A,\ x \perp y\}$$

Theorem (Pythagorean Theorem). If $x_1, ..., x_n \in H$, $x_i \perp x_j$ for $i \neq j$, then

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \sum_{i=1}^n \|x_i\|^2$$

Theorem. Let H Hilbert space, $A \subseteq H$, then $A^\perp$ is closed subspace of H.

Theorem (Projection). Let M closed subspace of Hilbert space H.
- For every $x \in H$, there exists unique closest point $y \in M$:

$$\forall x \in H, \exists! y \in M : \quad \|x - y\| = \min\{\|x - z\| : z \in M\}$$

  We say y is “the best approximation” to x in M.

- The point $y \in M$ closest to $x \in H$ is unique element of M such that $(x - y) \perp M$.

Definition. **Direct sum** of subspaces M and N of linear space is

$$M \oplus N := \{y + z : y \in M, z \in N\}$$

Corollary. If M closed subspace of Hilbert space H, then $H = M \oplus M^{\perp}$.

Definition. Let H Hilbert space. $\{u_\alpha\}_{\alpha \in I}$ is **orthonormal** if it is **orthogonal**: $u_\alpha \perp u_\beta$ for $\alpha \neq \beta$, and **normalised**: $\forall \alpha \in I, \|u_\alpha\| = 1$.

Definition. Let X Banach space, $\{x_\alpha \in X : \alpha \in I\}$ be indexed set where I is countable or uncountable.

- For each finite $J \subseteq I$, define **partial sum** as

$$S_J := \sum_{\alpha \in J} x_\alpha$$

- Unordered sum of $\{x_\alpha \in X : \alpha \in I\}$ **converges unconditionally** to $x \in X$, written $x = \sum_{\alpha \in I} x_\alpha$, if $\forall \varepsilon > 0$, there exists finite $J \subseteq I$ such that $\|S_K - x\| < \varepsilon$ for every finite $J \subseteq K \subseteq I$.
- Unordered sum $\sum_{\alpha \in I} x_\alpha$ is **Cauchy** if $\forall \varepsilon > 0$, there exists finite $J \subseteq I$ such that $\|S_L\| < \varepsilon$ for every finite $L \subseteq I - J$. Note that

$$\|S_L\| = \left\| \sum_{\alpha \in L \cup J} x_\alpha - \sum_{\alpha \in J} x_\alpha \right\|$$

- Unordered sum of $\{x_\alpha \in X : \alpha \in I\}$ **converges absolutely** if $\sum_{\alpha \in I} \|x_\alpha\|$ converges unconditionally in $\mathbb{R}$.

Proposition. Unordered sum in Banach space converges unconditionally iff it is Cauchy.

Definition. Let $\{c_\alpha : \alpha \in I\} \subseteq [0, \infty]$. Define

$$\sum_{\alpha \in I} c_\alpha = \sup \left\{ \sum_{\alpha \in J} c_\alpha : J \subseteq I, J \text{ finite} \right\}$$

Proposition. Let $\{c_\alpha : \alpha \in I\} \subseteq [0, \infty]$, $K = \{\alpha \in I : c_\alpha > 0\}$. If $\sum_{\alpha \in I} c_\alpha < \infty$, then K is countable.

Theorem (Bessel's inequality). Let $U = \{u_\alpha : \alpha \in I\}$ orthonormal in Hilbert space H. Then

$$\forall x \in H, \quad \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

In particular, $\forall x \in H$, $\{\alpha \in I : \langle x, u_\alpha \rangle \neq 0\}$ is countable.

Theorem. If $U = \{u_\alpha : \alpha \in I\}$ is orthonormal subset of Hilbert space H then the following are equivalent:

- If $\forall \alpha \in I, \langle x, u_\alpha \rangle = 0$, then $x = 0$.
- $\forall x \in H$, $x = \sum_{\alpha \in I} \langle x, u_\alpha \rangle u_\alpha$ where sum converges unconditionally in H and only has countably many non-zero terms.

- **Parseval's identity**:

$$\forall x \in H, \quad \|x\|^2 = \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2$$