Contents

1. Introduction

- Encryption process:
  - Alice has a message (**plaintext**) which is **encrypted** using an **encryption key** to produce the **ciphertext**, which is sent to Bob.
  - Bob uses a **decryption key** (which depends on the encryption key) to **decrypt** the ciphertext and recover the original plaintext.
  - It should be computationally infeasible to determine the plaintext without knowing the decryption key.
- **Caesar cipher**:
  - Add constant k to each letter in plaintext to produce ciphertext:

$$\text{ciphertext letter} = \text{plaintext letter} + k \mod 26$$

  - To decrypt,

$$\text{plaintext letter} = \text{ciphertext letter} - k \mod 26$$

  - The key is $k \bmod 26$.
- **Note**: Z is represented as $0 = 26 \bmod 26$, A as $1 \bmod 26$.
- Cryptosystem objectives:
  - **Secrecy**: an intercepted message is not able to be decrypted
  - **Integrity**: it is impossible to alter a message without the receiver knowing
  - **Authenticity**: receiver is certain of identity of sender (they can tell if an impersonator sent the message)
  - **Non-repudiation**: sender cannot claim they did not send a message; the receiver can prove they did.
- **Kerckhoff's principle**: a cryptographic system should be secure even if the details of the system are known to an attacker.
- Types of attack:
  - **Ciphertext-only**: the plaintext is deduced from the ciphertext.
  - **Known-plaintext**: intercepted ciphertext and associated stolen plaintext are used to determine the key.
  - **Chosen-plaintext**: an attacker tricks a sender into encrypting various chosen plaintexts and observes the ciphertext, then uses this information to determine the key.
  - **Chosen-ciphertext**: an attacker tricks the receiver into decrypting various chosen ciphertexts and observes the resulting plaintext, then uses this information to determine the key.

2. Symmetric key ciphers

- **Converting letters to numbers**: treat letters as integers modulo 26, with $A = 1$, $Z = 0 \equiv 26 \pmod{26}$. Treat string of text as vector of integers modulo 26.
- **Symmetric key cipher**: one in which encryption and decryption keys are equal.
- **Key size**: $\log_2(\text{number of possible keys})$.
- Caesar cipher is a **substitution cipher**. A stronger substitution cipher is this: key is permutation of $\{a, ..., z\}$. But vulnerable to known-plaintext attacks and

ciphertext-only attacks, since different letters (and letter pairs) occur with different frequencies in English.

- **One-time pad**: key is uniformly, independently random sequence of integers $\bmod 26$, $(k_1, k_2, ...)$, known to sender and receiver. If message is $(m_1, m_2, ..., m_r)$ then ciphertext is $(c_1, c_2, ..., c_r) = (k_1 + m_1, k_2 + m_2, ..., k_r + m_r)$. To decrypt the ciphertext, $m_i = c_i - k_i$. Once $(k_1, ..., k_r)$ have been used, they must never be used again.
  - One-time pad is information-theoretically secure against ciphertext-only attack: $\mathbb{P}(M = m \mid C = c) = \mathbb{P}(M = m)$.
  - Disadvantage is keys must never be reused, so must be as long as message.
  - Keys must be truly random.
- **Chinese remainder theorem**: let $m, n \in \mathbb{N}$ coprime, $a, b \in \mathbb{Z}$. Then exists unique solution $x \bmod mn$ to the congruences

$$\begin{aligned} x &\equiv a \mod m \\ x &\equiv b \mod n \end{aligned}$$

- **Block cipher**: group characters in plaintext into blocks of n (the **block length**) and encrypt each block with a key. So plaintext $p = (p_1, p_2, ...)$ is divided into blocks $P_1, P_2, ...$ where $P_1 = (p_1, ..., p_n)$, $P_2 = (p_{n+1}, ..., p_{2n}), ...$ Then ciphertext blocks are given by $C_i = f(\text{key}, P_i)$ for some encryption function f.
- **Hill cipher**:
  - Plaintext divided into blocks $P_1, ..., P_r$ of length n.
  - Each block represented as vector $P_i \in (\mathbb{Z}/26\mathbb{Z})^n$
  - Key is invertible $n \times n$ matrix M with elements in $\mathbb{Z}/26\mathbb{Z}$.
  - Ciphertext for block P_i is

$$C_i = MP_i$$

    It can be decrypted with $P_i = M^{-1}C_i$.
  - Let $P = (P_1, ..., P_r)$, $C = (C_1, ..., C_r)$, then $C = MP$.
- **Confusion**: each character of ciphertext depends on many characters of key.
- **Diffusion**: changing single character of plaintext changes many characters of ciphertext. Ideal diffusion is when changing single character of plaintext changes a proportion of $(S - 1)/S$ of the characters of the ciphertext, where S is the number of possible symbols.
- Confusion and diffusion make ciphertext-only attacks difficult.
- For Hill cipher, ith character of ciphertext depends on ith row of key (so depends on n characters of the key M) - this is medium confusion. If jth character of plaintext changes and $M_{ij} \neq 0$ then ith character of ciphertext changes. M_{ij} is non-zero with probability roughly $25/26$ so good diffusion.
- Hill cipher is susceptible to known plaintext attack:
  - If $P = (P_1, ..., P_n)$ are n blocks of plaintext with length n such that P is invertible and we know P and the corresponding C, then we can recover M, since $C = MP \Longrightarrow M = CP^{-1}$.

- If enough blocks of ciphertext are intercepted, it is very likely that n of them will produce an invertible matrix P.

3. Public key encryption and RSA

- **Public key cryptosystem**:
  - Bob produces encryption key, k_E, and decryption key, k_D. He publishes k_E and keeps k_D secret.
  - To encrypt message m, Alice sends ciphertext $c = f(m, k_E)$ to Bob.
  - To decrypt ciphertext c, Bob computes $g(c, k_D)$, where g satisfies
  $$g(f(m, k_E), k_D) = m$$
  for all messages m and all possible keys.
  - Computing m from $f(m, k_E)$ should be hard without knowing k_D.
- **Converting between messages and numbers**:
  - To convert message $m_1 m_2 ... m_r$, $m_i \in \{0, ..., 25\}$ to number, compute
  $$m = \sum_{i=1}^{r} m_i 26^{i-1}$$
  - To convert number m to message, append character $m \bmod 26$ to message. If $m < 26$, stop. Otherwise, floor divide m by 26 and repeat.
- **Fermat's little theorem**: let p prime, $a \in \mathbb{Z}$ coprime to p, then $a^{p-1} \equiv 1 \pmod{p}$.
- **Euler φ function**:
$$\varphi : \mathbb{N} \to \mathbb{N}, \quad \varphi(n) = |\{1 \le a \le n : \gcd(a, n) = 1\}| = |(\mathbb{Z}/n\mathbb{Z})^\times|$$
- $\varphi(p^r) = p^r - p^{r-1}$, $\varphi(mn) = \varphi(m)\varphi(n)$ for $\gcd(m, n) = 1$.
- **Euler's theorem**: if $\gcd(a, n) = 1$, $a^{\varphi(n)} \equiv 1 \pmod{n}$.
- **RSA algorithm**:
  - k_E is pair (n, e) where $n = pq$, the **RSA modulus**, is product of two distinct primes and $e \in \mathbb{Z}$, the **encryption exponent**, is coprime to $\varphi(n)$.
  - k_D, the **decryption exponent**, is integer d such that $de \equiv 1 \pmod{\varphi(n)}$.
  - m is an integer modulo n, m and n are coprime.
  - Encryption: $c = m^e \pmod{n}$.
  - Decryption: $m = c^d \pmod{n}$.
  - It is recommended that n have at least 2048 bits. A typical choice of e is $2^{16} + 1$.
- **RSA problem**: given $n = pq$ a product of two unknown primes, e and $m^e \pmod{n}$, recover m. If n can be factored, then RSA is solved.
- **Factorisation problem**: given $n = pq$ for large distinct primes p and q, find p and q.
- **RSA signatures**:
  - Public key is (n, e) and private key is d.
  - When sending a message m, message is **signed** by also sending $s = m^d \bmod n$, the **signature**.
  - (m, s) is received, **verified** by checking if $m = s^e \bmod n$.

- Forging a signature on a message m would require finding s with $m = s^e \bmod n$. This is the RSA problem.
- However, choosing signature s first then taking $m = s^e \bmod n$ produces valid pairs.
- To solve this, (m, s) is sent where $s = h(m)^d$, h is **hash function**. Then the message receiver verifies $h(m) = s^e \bmod n$.
- Now, for a signature to be forged, an attacker would have to find m with $h(m) = s^e \bmod n$.

- **Hash function** is function $h : \{\text{messages}\} \to \mathcal{H}$ that:
  - Can be computed efficiently
  - Is **preimage-resistant**: can't quickly find m given $h(m)$.
  - Is **collision-resistant**: can't quickly find m, m' such that $h(m) = h(m')$.
- **Attacks on RSA**:
  - If you can factor n, you can compute d, so can break RSA (as then you know $\varphi(n)$ so can compute $e^{-1} \bmod \varphi(n)$).
  - If $\varphi(n)$ is known, then we have $pq = n$ and $(p-1)(q-1) = \varphi(n)$ so $p + q = n - \varphi(n) + 1$. Hence p and q are roots of $x^2 - (n - \varphi(n) + 1)x + n$.
  - **Known d attack**:
    - $de - 1$ is multiple of $\varphi(n)$ so $p, q \mid x^{de-1} - 1$.
    - Look for factor K of $de - 1$ with $x^K - 1$ divisible by p but not q (or vice versa) (so likely that $(p-1) \mid K$ but $(q-1) \nmid K$).
    - Let $de - 1 = 2^r s$, $\gcd(2, s) = 1$, choose random $x \bmod n$. Let $y = x^s$, then $y^{2^r} = x^{2^r s} = x^{de-1} \equiv 1 \bmod n$.
    - If $y \equiv 1 \bmod n$, restart with new random x. Find first occurence of 1 in $y, y^2, ..., y^{2^r}$: $y^{2^j} \not\equiv 1 \bmod n$, $y^{2^{j+1}} \equiv 1 \bmod n$ for some $j \geq 0$.
    - Let $a := y^{2^j}$, then $a^2 \equiv 1 \bmod n$, $a \not\equiv 1 \bmod n$. If $a \equiv -1 \bmod n$, restart with new random x.
    - Now $n = pq \mid a^2 - 1 = (a+1)(a-1)$ but $n \nmid (a+1), (a-1)$. So p divides one of $a+1$, $a-1$ and q divides the other. So $\gcd(a-1, n)$, $\gcd(a+1, n)$ are prime factors of n.
- **Theorem**: it is no easier to find $\varphi(n)$ than to factorise n.
- **Theorem**: it is no easier to find d than to factor n.
- **Miller-Rabin algorithm** for probabilistic primality testing of n:
  1. Let $n - 1 = 2^r s$, $\gcd(2, s) = 1$.
  2. Choose random $x \bmod n$, compute $y = x^s \bmod n$.
  3. Compute $y, y^2, ..., y^{2^r} \bmod n$.
  4. If 1 isn't in this list, n is **composite** (with witness x).
  5. If 1 is in list preceded by number other than ± 1, n is **composite** (with witness x).
  6. Other, n is **possible prime** (to base x).
- **Theorem**:
  - If n prime then it is possible prime to every base.
  - If n composite then it is possible prime to $\leq 1/4$ of possible bases.

In particular, if k random bases are chosen, probability of composite n being possible prime for all k bases is $\leq 4^{-k}$.

3.1. Factorisation

- **Trial division algorithm**: for $p = 2, 3, 5, \ldots$ up to $\sqrt{n}$, test whether $p \mid n$.
- If $x^2 \equiv y^2 \bmod n$ but $x \not\equiv \pm y \bmod n$, then $x - y$ is divisible by factor of n but not by n itself, so $\gcd(x - y, n)$ gives proper factor of n (or 1).
- **Fermat's method**:
  - Let $a = \lceil \sqrt{n} \rceil$. Compute $a^2 \bmod n$, $(a+1)^2 \bmod n$ until a square $x^2 \equiv (a+i)^2 \bmod n$ appears. Then compute $\gcd(a + i - x, n)$.
  - Works well under special conditions on the factors: if $|p - q| \leq 2\sqrt{2}\sqrt[4]{n}$ then Fermat's method takes one step: $x = \lceil \sqrt{n} \rceil$ works.
- **Definition**: an integer is **B-smooth** if all its prime factors are $\leq B$.
- **Quadratic sieve**:
  - Choose B and let m be number of primes $\leq B$.
  - Look at integers $x = \lceil \sqrt{n} \rceil + k$, $k = 1, 2, \ldots$ and check whether $y = x^2 - n$ is B-smooth.
  - Once $y_1 = x_1^2 - n, \ldots, y_t = x_t^2 - n$ are all B-smooth with $t > m$, find some product of them that is a square.
  - Deduce a congruence between the squares. Use difference of two squares and gcd to factor n.
  - Time complexity is $\exp\left(\sqrt{\log n \log\log n}\right)$.

4. Diffie-Hellman key exchange

- **Primitive root theorem**: let p prime, then there exists $g \in \mathbb{F}_p^\times$ such that $1, g, \ldots, g^{p-2}$ is complete set of residues $\bmod p$.
- Let p prime, $g \in \mathbb{F}_p^\times$. **Order** of g is smallest $a \in \mathbb{N}_0$ such that $g^a = 1$. g is **primitive root** if its order is $p - 1$ (equivalently, $1, g, \ldots, g^{p-2}$ is complete set of residues $\bmod p$).
- Let p prime, $g \in \mathbb{F}_p^\times$ primitive root. If $x \in \mathbb{F}_p^\times$ then $x = g^L$ for some $0 \leq L < p - 1$. Then L is **discrete logarithm** of x to base g. Write $L = L_g(x)$.
- **Proposition**:
  - $g^{L_g(x)} \equiv x \pmod{p}$ and $g^a \equiv x \pmod{p} \Longleftrightarrow a \equiv L_g(x) \pmod{p-1}$.
  - $L_g(1) = 0$, $L_g(g) = 1$.
  - $L_g(xy) \equiv L_g(x) + L_g(y) \pmod{p-1}$.
  - $L_g(x^{-1}) = -L_g(x) \ (\bmod p - 1)$.
  - $L_g(g^a \bmod p) \equiv a \ (\bmod p - 1)$.
  - h is primitive root mod p iff $L_g(h)$ coprime to $p - 1$. So number of primitive roots mod p is $\varphi(p-1)$.
- **Discrete logarithm problem**: given p, g, x, compute $L_g(x)$.
- **Diffie-Hellman key exchange**:
  - Alice and Bob publicly choose prime p and primitive root $g \bmod p$.
  - Alice chooses secret $\alpha \bmod (p-1)$ and sends $g^\alpha \bmod p$ to Bob publicly.
  - Bob chooses secret $\beta \bmod (p-1)$ and sends $g^\beta \bmod p$ to Alice publicly.

- Alice and Bob both compute shared secret $\kappa = g^{\alpha\beta} = (g^\alpha)^\beta) = \left(g^\beta\right)^\alpha \bmod p$.

- **Diffie-Hellman problem**: given p, g, g^α, g^β, compute $g^{\alpha\beta}$.
- If discrete logarithm problem cna be solved, so can Diffie-Hellman problem (since could compute $\alpha = L_g(g^a)$ or $\beta = L_g\left(g^\beta\right)$).
- **Elgamal public key encryption**:
  - Alice chooses prime p, primitive root g, private key $\alpha \mod(p-1)$.
  - Her public key is $y = g^\alpha$.
  - Bob chooses random $k \bmod (p-1)$
  - To send message m (integer mod p), he sends the pair $(r, m') = \left(g^k, my^k\right)$.
  - To decrypt message, Alice computes $r^\alpha = g^{\alpha k} = y^k$ and then $m' r^{-\alpha} = m' y^{-k} = mg^{\alpha k} g^{-\alpha k} m$.
  - If Diffie-Hellman problem is hard, then Elgamal encryption is secure against known plaintext attack.
  - Key k must be random and different each time.
- **Decision Diffie-Hellman problem**: given g^a, g^b, c in $\mathbb{F}_p^\times$, decide whether $c = g^{ab}$.
  - This problem is not always hard, as can tell if g^{ab} is square or not. Can fix this by taking g to have large prime order $q \mid (p-1)$. $p = 2q+1$ is a good choice.
- **Elgamal signatures**:
  - Public key is (p, g), $y = g^\alpha$ for private key α.
  - **Valid Elgamal signature** on $m \in \{0, ..., p-2\}$ is pair (r, s), $0 \leq r, s \leq p-1$ such that
  $$y^r r^s = g^m \pmod p$$
  - Alice computes $r = g^k$, $k \in (\mathbb{Z}/(p-1))^\times$ random. k should be different each time.
  - Then $g^{\alpha r} g^{ks} \equiv g^m \mod p$ so $\alpha r + ks \equiv m \pmod{p-1}$ so $s = k^{-1}(m - \alpha r) \mod p-1$.
- **Elgamal signature problem**: given p, g, y, m, find r, s such that $y^r r^s = m$.
- **Discrete logarithm problem**: given prime p, primitive root $g \bmod p$, $x \in \mathbb{F}_p^\times$, calculate $L_g(x)$.
- **Baby-step giant-step algorithm** for solving DLP:
  - Let $N = \left\lceil\sqrt{p-1}\right\rceil$.
  - Baby-steps: compute $g^j \bmod p$ for $0 \leq j < N$.
  - Giant-steps: compute $xg^{-Nk} \bmod p$ for $0 \leq k < N$.
  - Look for a match between baby-steps and giant-steps lists: $g^j = xg^{-Nk} \Longrightarrow x = g^{j+Nk}$.
  - Always works since if $x = g^L$ for $0 \leq L < p-1 \leq N^2$, L can be written as $j + Nk$ with $j, k \in \{0, ..., N-1\}$.
- **Index calculus** method for solving DLP $x = g^L$:
  - Fix smoothness bound B.
  - Find many multiplicative relations between B-smooth numbers and powers of $g \bmod p$.
  - Solve these relations to find discrete logarithms of primes $\leq B$.

    - For $i = 1, 2, ...$ compute $xg^i \bmod p$ until one is B-smooth, then use result from previous step.
- **Pohlig-Hellman algorithm** computes discrete logarithms $\bmod p$ with approximate complexity $\log(p)\sqrt{\ell}$ where ℓ is largest prime factor of $p - 1$, so is fast if $p - 1$ is B-smooth. Therefore p is chosen so that $p - 1$ has large prime factor, e.g. choose **Germain prime** $p = 2q + 1$, with q prime.

5. Elliptic curves

- **Definition**: **abelian group** $(G, \circ)$ satisfies:
    - **Associativity**: $\forall a, b, c, \in G, a \circ (b \circ c) = (a \circ b) \circ c$.
    - **Identity**: $\exists e \in G : \forall g \in G, e \times g = g$.
    - **Inverses**: $\forall g \in G, \exists h \in G : g \circ h = h \circ g = e$
    - **Commutativity**: $\forall a, b \in G, a \circ b = b \circ a$.
- **Definition**: $H \subseteq G$ is **subgroup** of G if $(H, \circ)$ is group.
- To show H is subgroup, sufficient to show $g, h \in H \Rightarrow g \circ h \in H$ and $h^{-1} \in H$.
- **Notation**: for $g \in G$, write $[n]g$ for $g \circ \cdots \circ g$ n times if $n > 0$, e if $n = 0$, $[-n]g^{-1}$ if $n < 0$.
- **Definition**: **subgroup generated by** g is

$$\langle g \rangle = \{[n]g : n \in \mathbb{Z}\}$$

  If $\langle g \rangle$ finite, it has **order** n, and g has **order** n. If $G = \langle g \rangle$ for some $g \in G$, G is **cyclic** and g is **generator**.
- **Lagrange's theorem**: let G finite group, H subgroup of G, then $|H| \mid |G|$.
- **Corollary**: if G finite, $g \in G$ has order n, then $n \mid |G|$.
- **DLP for abelian groups**: given G a cyclic abelian group, $g \in G$ a generator of G, $x \in G$, find L such that $[L]g = x$. L is well-defined modulo $|G|$.
- **Definition**: let $(G, \circ)$, $(H, \bullet)$ abelian groups, **homomorphism** between G and H is $f : G \to H$ with

$$\forall g, g' \in G, \quad f(g \circ g') = f(g) \bullet f(g')$$

  Isomorphism is bijective homomorphism. G and H are **isomorphic**, $G \cong H$, if there is isomorphism between them.
- **Fundamental theorem of finite abelian groups**: let G finite abelian group, then there exist unique integers $2 \leq n_1, ..., n_r$ with $n_i \mid n_{i+1}$ for all i, such that

$$G \simeq (\mathbb{Z}/n_1) \times \cdots \times (\mathbb{Z}/n_r)$$

  In particular, G is isomorphic to product of cyclic groups.
- **Definition**: let K field, $\mathrm{char}(K) > 3$. An **elliptic curve** over K is defined by the equation

$$y^2 = x^3 + ax + b$$

  where $a, b \in K$, $\Delta_E := 4a^3 + 27b^2 \neq 0$.

- **Remark**: $\Delta_E \neq 0$ is equivalent to $x^3 + ax + b$ having no repeated roots (i.e. E is smooth).
- **Definition**: for elliptic curve E defined over K, a **K-point** (**point**) on E is either:
  - A **normal point**: $(x, y) \in K^2$ satisfying the equation defining E.
  - The **point at infinity** $\overline{O}$ which can be thought of as infinitely far along the y-axis (in either direction).

  Denote set of all K-points on E as $E(K)$.
- Any elliptic curve $E(K)$ is an abelian group, with group operation $\oplus$ is defined as:
  - We should have $P \oplus Q \oplus R = \overline{O}$ iff P, Q, R lie on straight line.
  - In this case, $P \oplus Q = -R$.
  - To find line ℓ passing through $P = (x_0, y_0)$ and $Q = (x_1, y_1)$:
    - If $x_0 \neq x_1$, then equation of ℓ is $y = \lambda x + \mu$, where

      $$\lambda = \frac{y_1 - y_0}{x_1 - x_0}, \quad \mu = y_0 - \lambda x_0$$

      Now

      $$\begin{aligned} y^2 &= x^3 + ax + b = (\lambda x + \mu)^2 \\ \Longrightarrow 0 &= x^3 - \lambda^2 x^2 + (a - 2\lambda\mu)x + (b - \mu^2) \end{aligned}$$

      Since sum of roots of monic polynomial is equal to minus the coefficient of the second highest power, and two roots are x_0 and x_1, the third root is $x_2 = \lambda^2 - x_0 - x_1$. Then $y_2 = \lambda x_2 + \mu$ and $R = (x_2, y_2)$.
    - If $x_0 = x_1$, then using implicit differentiation:

      $$\begin{aligned} y^2 &= x^3 + ax + b \\ \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2 + a}{2y} \end{aligned}$$

      and the rest is as above, but instead with $\lambda = \frac{3x_0^2 + a}{2y_0}$.
- **Definition**: **group law** of elliptic curves: let $E : y^2 = x^3 + ax + b$. For all normal points $P = (x_0, y_0), Q = (x_1, y_1) \in E(K)$, define
  - $\overline{O}$ is group identity: $P \oplus \overline{O} = \overline{O} \oplus P = P$.
  - If $P = -Q =: (x_0, -y_0)$, $P \oplus Q = \overline{O}$.
  - Otherwise, $P \oplus Q = (x_2, -y_2)$, where

    $$\begin{aligned} x_2 &= \lambda^2 - (x_0 + x_1), \\ y_2 &= \lambda x_2 + \mu, \\ \lambda &= \begin{cases} \frac{y_1 - y_0}{x_1 - x_0} & \text{if } x_0 \neq x_1 \\ \frac{3x_0^2 + a}{2y_0} & \text{if } x_0 = x_1 \end{cases}, \\ \mu &= y_0 - \lambda x_0 \end{aligned}$$
- **Example**:

- Let E be given by $y^2 = x^3 + 17$ over $\mathbb{Q}$, $P = (-1, 4) \in E(\mathbb{Q})$, $Q = (2, 5) \in E(\mathbb{Q})$. To find $P \oplus Q$,

$$\lambda = \frac{5-4}{2-(-1)} = \frac{1}{3}, \quad \mu = 4 - \lambda(-1) = \frac{13}{3}$$

So $x_2 = \lambda^2 - (-1) - 2 = -\frac{8}{9}$ and $y_2 = -(\lambda x_2 + \mu) = -\frac{109}{27}$ hence

$$P \oplus Q = \left(-\frac{8}{9}, -\frac{109}{27}\right)$$

To find $[2]P$,

$$\lambda = \frac{3(-1)^2 + 0}{2 \cdot 4} = \frac{3}{8}, \quad \mu = 4 - \frac{3}{8} \cdot (-1) = \frac{35}{8}$$

so $x_3 = \lambda^2 - 2 \cdot (-1) \frac{137}{64}$, $y_3 = -(\lambda x_3 + \mu) = -\frac{2651}{512}$ hence

$$[2]P = (x_3, y_3) = \left(\frac{137}{64}, -\frac{2651}{512}\right)$$

- **Hasse's theorem**: let $|E(\mathbb{F}_p)| = N$, then

$$|N - (p+1)| \leq 2\sqrt{p}$$

- **Theorem**: $E(\mathbb{F}_p)$ is isomorphic to either $\mathbb{Z}/k$ or $\mathbb{Z}/m \times \mathbb{Z}/n$ with $m \mid n$.
- **Elliptic curve Diffie-Hellman**:
  - Alice and Bob publicly choose elliptic curve $E(\mathbb{F}_p)$ and $P \in \mathbb{F}_p$ with order a large prime n.
  - Alice chooses random $\alpha \in \{0, ..., n-1\}$ and publishes $Q_A = [\alpha]P$.
  - Bob chooses random $\beta \in \{0, ..., n-1\}$ and publishes $Q_B = [\beta]P$.
  - Alice computes $[\alpha]Q_B = [\alpha\beta]P$, Bob computes $[\beta]Q_A = [\beta\alpha]P$.
  - Shared key is $K = [\alpha\beta]P$.
- **Elliptic curve Elgamal signatures**:
  - Use agreed elliptic curve E over $\mathbb{F}_p$, point $P \in E(\mathbb{F}_p)$ of prime order n.
  - Alice wants to sign message m, encoded as integer mod n.
  - Alice generates private key $\alpha \in \mathbb{Z}/n$ and public key $Q = [\alpha]P$.
  - Valid signature is (R, s) where $R = (x_R, y_R) \in E(\mathbb{F}_p)$, $s \in \mathbb{Z}/n$, $[\widetilde{x_R}]Q \oplus [s]R = [m]P$.
  - To generate a valid signature, Alice chooses random $0 \neq k \in \mathbb{Z}/n$ and sets $R = [k]P$, $s = k^{-1}(m - \widetilde{x_R}\alpha)$.
  - k must be randomly generated for each message.
- **Baby-step giant-step algorithm for elliptic curve DLP**: given P and $Q = [\alpha]P$, find α:
  - Let $N = \lceil\sqrt{n}\rceil$, n is order of P.
  - Compute $P, [2]P, ..., [N-1]P$.
  - Compute $Q \oplus [-N]P, Q \oplus [-2N]P, ..., Q \oplus [-(N-1)N]P$ and find a match between these two lists: $[i]P = Q \oplus [-jN]P$, then $[i + jN]P = Q$ so $\alpha = i + jN$.

- For well-chosen elliptic curves, the best algorithm for solving DLP is the baby-step giant-step algorithm, with run time $O(\sqrt{n}) \approx O(\sqrt{p})$. This is much slower than the index-calculus method for the DLP in $\mathbb{F}_p^\times$.
- **Pollard's $p - 1$ algorithm** to factorise $n = pq$:
  - Choose smoothness bound B.
  - Choose random $2 \leq a \leq n - 2$. Set $a_1 = a$, $i = 1$.
  - Compute $a_i = a_{i-1}^i \bmod n$. Find $d = \gcd(a_i - 1, n)$. If $1 < d < n$, we have found a nontrivial factor of n. If $d = n$, pick new a and retry. If $d = 1$, increment i by 1 and repeat this step.
  - A variant is instead of computing $a_i = a_{i-1}^i$, compute $a_i = a_{i-1}^{m_{i-1}}$ where $m_1, ..., m_r$ are the prime powers $\leq B$ (each prime power is the maximal prime power $\leq B$ for that prime).
  - The algorithm works if $p - 1$ is **B-powersmooth** (all prime power factors are $\leq B$), since if b is order of $a \bmod p$, then $b \mid (p-1)$ so $b \mid B!$ (also $b \mid m_1 \cdots m_r$). If the first i for which $i!$ (or $m_1 \cdots m_i$) is divisible by d and order of $a \bmod q$, then $a_i - 1 = a^{i!} - 1 \bmod n$ is divisible by both p and q, so must retry with different a.
- Let $n = pq$, p, q prime, $a, b \in \mathbb{Z}$, $\gcd(4a^3 + 27b^2, n) = 1$. Then $E : y^2 = x^3 + ax + b$ defines elliptic curve over $\mathbb{F}_p$ and over $\mathbb{F}_q$. If $(x, y) \in \mathbb{Z}/n$ is solution to $E \bmod n$ then can reduce coordinates $\bmod p$ to obtain non-infinite point of $E(\mathbb{F}_p)$ and $\bmod q$ to obtain non-infinite point of $E(\mathbb{F}_q)$.
- **Proposition**: let $P_1, P_2 \in E \bmod n$, with

  $$(P_1 \bmod p) \oplus (P_2 \bmod p) = \overline{O}$$
  $$(P_1 \bmod q) \oplus (P_2 \bmod q) \neq \overline{O}$$

  Then $\gcd(x_1 - x_2, n)$ (or $\gcd(2x_1, n)$ if $P_1 = P_2$) is factor of n.
- **Lenstra's algorithm** to factorise n:
  - Choose smoothness bound B.
  - Choose random elliptic curve E over $\mathbb{Z}/n$ with $\gcd(\Delta_E, n) = 1$ and $P = (x, y)$ a point on E.
  - Set $P_1 = P$, attempt to compute P_i, $2 \leq i \leq B$ by $P_i = [i]P_{i-1}$. If one of these fails, a divisor of n has been found (by failing to compute an inverse $\bmod n$). If this divisor is trivial, restart with new curve and point.
  - If $i = B$ is reached, restart with new curve and point.
  - Again, a variant is calculating $P_i = [m_i]P_{i-1}$ instead of $[i]P_{i-1}$ where $m_1, ..., m_r$ are the prime powers $\leq B$
- Lenstra's algorithm works if $|E(\mathbb{Z}/p)|$ is B-powersmooth but $|E(\mathbb{Z}/q)|$ isn't. Since we can vary E, it is very likely to work eventually.
- Running time depends on p (the smaller prime factor):

  $$O\left(\exp\left(\sqrt{2 \log(p) \log\log(p)}\right)\right)$$

  Compare this to the general number field sieve running time:

$$O\left(\exp\left(C(\log n)^{1/3}(\log\log n)^{2/3}\right)\right)$$

5.1. Torsion points

- **Definition**: let G abelian group. $g \in G$ is a **torsion** if it has finite order. If order divides n, then $[n]g = e$ and g is **n-torsion**.
- **Definition**: **n-torsion subgroup** is

$$G[n] := \{g \in G : [n]g = e\}$$

- **Definition**: **torsion subgroup** of G is

$$G_{\text{tors}} = \{g \in G : g \text{ is torsion}\} = \bigcup_{n \in \mathbb{N}} G[n]$$

- **Example**:
  - In $\mathbb{Z}$, only 0 is torsion.
  - In $(\mathbb{Z}/10)^{\times}$, by Lagrange's theorem, every point is 4-torsion.
  - For finite groups G, $G_{\text{tors}} = G = G[|G|]$ by Lagrange's theorem.

5.2. Rational points

- **Note**: for elliptic curve $E : y^2 = x^3 + ax + b$ over $\mathbb{Q}$, can assume that $a, b \in \mathbb{Z}$.
- **Nagell-Lutz theorem**: let E elliptic curve, let $P = (x, y) \in E(\mathbb{Q})_{\text{tors}}$. Then $x, y \in \mathbb{Z}$, and either $y = 0$ (in which case P is 2-torsion) or $y^2 \mid \Delta_E$.
- **Corollary**: $E(\mathbb{Q})_{\text{tors}}$ is finite.
- **Example**: can use Nagell-Lutz to show a point is not torsion.
  - $P = (0, 1)$ lies on elliptic curve $y^2 = x^3 - x + 1$. $[2]P = \left(\frac{1}{4}, -\frac{7}{8}\right) \notin \mathbb{Z}^2$. Then $[2]P$ is not torsion, hence P is not torsion. So $E(\mathbb{Q})$ contains distinct points $..., [-2]P, -P, \overline{O}, P, [2]P, ...$, hence E has infinitely many solutions in $\mathbb{Q}$.
- **Mazur's theorem**: let E be elliptic curve over $\mathbb{Q}$. Then $E(\mathbb{Q})_{\text{tors}}$ is either:
  - cyclic of order $1 \leq N \leq 10$ or order 12, or
  - of the form $\mathbb{Z}/2 \times \mathbb{Z}/2N$ for $1 \leq N \leq 4$.
- **Definition**: let $E : y^2 = x^3 + ax + b$ defined over $\mathbb{Q}$, $a, b \in \mathbb{Z}$. For odd prime p, taking reductions $\overline{a}$, $\overline{b} \bmod p$ gives curve over $\mathbb{F}_p$:

$$\overline{E} : y^2 = x^3 + \overline{a}x + \overline{b}$$

  This is elliptic curve if $\Delta_E \not\equiv 0 \bmod p$, in which case p is **prime of good reduction** for E.
- **Theorem**: let $E : y^2 = x^3 + ax + b$ defined over $\mathbb{Q}$, $a, b \in \mathbb{Z}$, p be odd prime of good reduction for E. Then $f : E(\mathbb{Q})_{\text{tors}} \to \overline{E}(\mathbb{F}_p)$ defined by

$$f(x, y) := (\overline{x}, \overline{y}), \quad f(\overline{O}) := \overline{O}$$

  is injective (note $x, y \in \mathbb{Z}$ by Nagell-Lutz).
- So $E(\mathbb{Q})_{\text{tors}}$ can be thought of as subgroup of $E(\mathbb{F}_p)$ for any prime p of good reduction, so by Lagrange's theorem, $|E(\mathbb{Q})_{\text{tors}}|$ divides $|E(\mathbb{F}_p)|$.
- **Mordell's theorem**: if E is elliptic curve over $\mathbb{Q}$, then

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$$

for some $r \geq 0$ the **rank** of E. So for some $P_1, ..., P_r \in E(\mathbb{Q})$,

$$E(\mathbb{Q}) = \{n_1 P_1 + \cdots + n_r P_r + T : n_i \in \mathbb{Z}, T \in E(\mathbb{Q})_{\text{tors}}\}$$

$P_1, ..., P_r$ (together with T) are **generators** for $E(\mathbb{Q})$.

6. Basic coding theory

6.1. First definitions

- **Definition**:
  - **Alphabet** A is finite set of symbols.
  - A^n is set of all lists of n symbols from A - these are **words of length** n.
  - **Code of block length** n **on** A is subset of A^n.
  - **Codeword** is element of a code.

Definition[If $|A| = 2$, codes on A are **binary** codes. If $|A| = 3$, codes on A are **ternary codes**. If $|A| = q$, codes on A are q**-ary** codes. Generally, use $A = \{0, 1, ..., q - 1\}$.]

Definition. Let $x = x_1 ... x_n, y = y_1 ... y_n \in A^n$. **Hamming distance** between x and y is number of indices where x and y differ:

$$d : A^n \times A^n \to \{0, ..., n\}, \quad d(x, y) := |\{i \in [n] : x_i \neq y_i\}|$$

So $d(x, y)$ is minimum number of changes needed to change x to y. If x transmitted and y received, then $d(x, y)$ **symbol-errors** have occurred.

Proposition. Let x, y words of length n.

- $0 \leq d(x, y) \leq n$.
- $d(x, y) = 0 \Longleftrightarrow x = y$.
- $d(x, y) = d(y, x)$.
- $\forall z \in A^n, d(x, y) \leq d(x, z) + d(z, y)$.

Definition. **Minimum distance** of code C is

$$d(C) := \min\{d(x, y) : x, y \in C, x \neq y\} \in \mathbb{N}$$

Notation. Code of block length n with M codewords and minimum distance d is called (n, M, d) (or (n, M)) code. A q-ary code is called an $(n, M, d)_q$ code.

Definition. Let $C \subseteq A^n$ code, x word of length n. A **nearest neighbour** of x is codeword $c \in C$ such that $d(x, c) = \min\{d(x, y) : y \in C\}$.

6.2. Nearest-neighbour decoding

Definition. **Nearest-neighbour decoding (NND)** means if word x received, it is decoded to a nearest neighbour of x in a code C.

Proposition. Let C be code with minimum distance d, let word x be received with t symbol errors. Then

- If $t \leq d - 1$, then we can detect that x has some errors.
- If $t \leq \left\lfloor \frac{d-1}{2} \right\rfloor$, then NND will correct the errors.

6.3. Probabilities

Definition. **q-ary symmetric channel with symbol-error probability** p is channel for q-ary alphabet A such that:

- For every $a \in A$, probability that a is changed in channel is p (i.e. symbol-errors in different positions are independent events).
- For every $a \neq b \in A$, probability that a is changed to b in channel is

$$\mathbb{P}(b \text{ received} \mid a \text{ sent}) = \frac{p}{q-1}$$

i.e. symbol-errors in different positions are independent events.

Proposition. Let c codeword in q-ary code $C \subseteq A^n$ sent over q-ary symmetric channel with symbol-error probability p. Then

$$\mathbb{P}(x \text{ received} \mid c \text{ sent}) = \left(\frac{p}{q-1}\right)^t (1-p)^{n-t}, \quad \text{where } t = d(c, x)$$

Example. Let $C = \{000, 111\} \subset \{0, 1\}^3$.

x	$t = d(000, x)$	chance 000 received as x	chance if $p = 0.01$	NND decodes correctly?
000	0	$(1-p)^3$	0.970299	yes
100	1	$p(1-p)^2$	0.009801	yes
010	1	$p(1-p)^2$	0.009801	yes
001	1	$p(1-p)^2$	0.009801	yes
110	2	$p^2(1-p)$	0.000099	no
101	2	$p^2(1-p)$	0.000099	no
011	2	$p^2(1-p)$	0.000099	no
111	3	p^3	0.000001	no

Corollary. If $p < \frac{q-1}{q}$ then $P(x \text{ received} \mid c \text{ sent})$ increases as $d(x, c)$ decreases.

Remark. By Bayes' theorem,

$$\mathbb{P}(c \text{ sent} \mid x \text{ received}) = \frac{\mathbb{P}(c \text{ sent and } x \text{ received})}{\mathbb{P}(x \text{ received})} = \frac{\mathbb{P}(c \text{ sent})\mathbb{P}(x \text{ received} \mid c \text{ sent})}{\mathbb{P}(x \text{ received})}$$

Proposition. Let C be q-ary (n, M, d) code used over q-ary symmetric channel with symbol-error probability $p < (q-1)/q$, and each codeword $c \in C$ is equally likely to be sent. Then for any word x, $\mathbb{P}(c \text{ sent} \mid x \text{ received})$ increases as $d(x, c)$ decreases.

6.4. Bounds on codes

- **Proposition (singleton bound)**: for q-ary code (n, M, d) code, $M \leq q^{n-d+1}$.

Definition. Code which saturates singleton bound is called **maximum distance separable (MDS)**.

Example. Let C_n be **binary repetition code** of block length n,

$$C_n \coloneqq \{\underbrace{00...0}_{n}, \underbrace{11...1}_{n}\} \subset \{0,1\}^n$$

C_n is $(n, 2, n)_2$ code, and $2 = 2^{n-n+1}$ so C_n is MDS code.

Definition. Let A be alphabet, $|A| = q$. Let $n \in \mathbb{N}$, $0 \le t \le n$, $t \in \mathbb{N}$, $x \in A^n$.

- **Ball of radius t around x** is

$$S(x, t) \coloneqq \{y \in A^n : d(y, x) \le t\}$$

- Code $C \subseteq A^n$ is **perfect** if

$$\exists t \in \mathbb{N}_0 : A^n = \coprod_{c \in C} S(c, t)$$

  where $\coprod$ is disjoint union.

Example. For $C = \{000, 111\} \subset \{0,1\}^3$, $S(000, 1) = \{000, 100, 010, 001\}$ and $S(111, 1) = \{111, 011, 101, 110\}$. These are disjoint and $S(000, 1) \cup S(111, 1) = \{0,1\}^3$, so C is perfect.

Example. Let $C = \{111, 020, 202\} \subset \{0,1,2\}^3$. $\forall c \in C, d(c, 012) = 2$. So 012 is not in any $S(c, 1)$ but is in every $S(c, 2)$, so C is not perfect.

Lemma. Let $|A| = q$, $x \in \mathbb{A}^n$, then

$$|S(x, t)| = \sum_{k=0}^{t} \binom{n}{k} (q-1)^k$$

Example. Let $C = \{111, 020, 202\} \subset \{0,1,2\}^3$, so $q = 3$, $n = 3$. So $|S(x, 1)| = \binom{3}{0} + \binom{3}{1}(3-1) = 7$, $|S(x, 2)| = \binom{3}{0} + \binom{3}{1}(3-1) + \binom{3}{2}(3-1)^2 = 19$. But $|\{0,1,2\}|^3 = 27$ and $7 \nmid 27$, $19 \nmid 27$, so $\{0,1,2\}^3$ can't be partioned by balls of either size. So C can't be perfect. $|S(x, 3)| = 27$, but then C must contain only one codeword to be perfect, and $|S(x, 0)| = 1$, but then $C = A^n$ to be perfect. These are trivial, useless codes.

- **Proposition (Hamming/sphere-packing bound)**: q-ary (n, M, d) code satisfies

$$M \sum_{k=0}^{t} \binom{n}{k} (q-1)^k \le q^n, \quad \text{where } t = \left\lfloor \frac{d-1}{2} \right\rfloor$$

Corollary. Code saturates Hamming bound iff it is perfect.

7. Linear codes

7.1. Finite vector spaces

Definition. **Linear code** of block length n is subspace of $\mathbb{F}_q^n$.

Example. Let $\boldsymbol{x} = (0, 1, 2, 0)$, $\boldsymbol{y} = (1, 1, 1, 1)$, $\boldsymbol{z} = (0, 2, 1, 0) \in \mathbb{F}_3^4$. $C_1 = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{0}\}$ is not linear code since e.g. $\boldsymbol{x} + \boldsymbol{y} = (1, 2, 0, 1) \notin C_1$. $C_2 = \{\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{0}\}$ is linear code.

Notation. Spanning set of S is $\langle S \rangle$.

Proposition. If linear code $C \subseteq \mathbb{F}_q^n$ has $\dim(C) = k$, then $|C| = q^k$.

Definition. A q-ary $[n, k, d]$ code is linear code: a subspace of $\mathbb{F}_q^n$ of dimension k with minimum distance d. Note: a q-ary $[n, k, d]$ code is a q-ary $\left(n, q^k, d\right)$ code.

7.2. Weight and minimum distance

Definition. **Weight** of $\boldsymbol{x} \in \mathbb{F}_q^n$, $w(\boldsymbol{x})$, is number of non-zero entries in $\boldsymbol{x}$:

$$w(\boldsymbol{x}) = |\{i \in [n] : x_i \neq 0\}|$$

Lemma. $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_q^n$, $d(\boldsymbol{x}, \boldsymbol{y}) = w(\boldsymbol{x} - \boldsymbol{y})$. In particular, $w(\boldsymbol{x}) = d(\boldsymbol{x}, \boldsymbol{0})$.

Proposition. Let $C \subseteq \mathbb{F}_q^n$ linear code, then

$$d(C) = \min\{w(\boldsymbol{c}) : \boldsymbol{c} \in C, \boldsymbol{c} \neq \boldsymbol{0}\}$$

Remark. To find $d(C)$ for linear code with q^k words, only need to consider q^k weights instead of $\binom{q^k}{2}$ distances.

8. Codes as images

8.1. Generator-matrices

Definition. Let $C \subseteq \mathbb{F}_q^n$ be linear code. Let $G \in M_{k,n}\left(\mathbb{F}_q\right)$, $f_G : \mathbb{F}_q^k \to \mathbb{F}_q^n$ be linear map defined by $f_G(\boldsymbol{x}) = \boldsymbol{x}G$. Then G is **generator-matrix** for C if

- $C = \operatorname{im}(f) = \{\boldsymbol{x}G : \boldsymbol{x} \in \mathbb{F}_q^k\} \subseteq \mathbb{F}_q^n$.
- The rows of G are linearly independent.

i.e. G is generator-matrix for C iff rows of G form basis for C (note $\boldsymbol{x}G = x_1 \boldsymbol{g_1} + \cdots + x_k \boldsymbol{g_k}$ where $\boldsymbol{g_i}$ are rows of G).

Remark. Given linear code $C = \langle \boldsymbol{a}_1, ..., \boldsymbol{a}_m \rangle$, a generator-matrix can be found for C by constructing the matrix A with rows $\boldsymbol{a}_i$, then performing elementary row operations to bring A into RREF. Once the $m - k$ bottom zero rows have been removed, the resulting matrix is a generator-matrix.

Example. Let $C = \langle \{(0, 0, 3, 1, 4), (2, 4, 1, 4, 0), (5, 3, 0, 1, 6)\} \rangle \subseteq \mathbb{F}_7^5$.

$$A = \begin{bmatrix} 2 & 4 & 1 & 4 & 0 \\ 5 & 3 & 0 & 1 & 6 \\ 0 & 0 & 3 & 1 & 4 \end{bmatrix} \underset{A_{12}(1)}{\to} \begin{bmatrix} 2 & 4 & 1 & 4 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 3 & 1 & 4 \end{bmatrix} \underset{M_1(4)}{\to} \begin{bmatrix} 1 & 2 & 4 & 2 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 3 & 1 & 4 \end{bmatrix} \underset{A_{21}(3), A_{23}(4)}{\to} \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $G = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{bmatrix}$ is generator matrix for C and $\dim(C) = 2$.

8.2. Encoding and channel decoding

- Let C be q-ary $[n, k]$ code with generator matrix $G \in M_{k,n}\left(\mathbb{F}_q\right)$. To encode a message $x \in \mathbb{F}_q^k$, multiply by G: codeword is $c = xG$.

- Note that rows of G being independent implies f_G is injective, so no two messages are mapped to same codeword.
- If we want the code to correct (and detect) errors, we must have $k < n$.
- The received word $y \in \mathbb{F}_q^n$ is decoded to the codeword $c' \in C$.
- **Channel decoding** is finding the unique word x' such that $x'G = c'$, i.e. $x' \cdot g_i = c_i'$ where g_i is ith column of G. This gives n equations in k unknowns. Since c' is a codeword, these equations are consistent, and since f_G is injective, there is a unique solution.
- To solve $x'G = c'$, either use that $G^t (x')^t = (c')^t$ and row-reduce augmented matrix $\left(G^t \mid (c')^t\right)$, or pick generator-matrix in RREF, which then picks out each x_i'.

8.3. Equivalence and standard form

Definition. Codes C_1, C_2 of block length n over alphabet A are **equivalent** if we can transform one to the other by applying sequence of the following two kinds of changes to all the codewords (simultaneously):
- Permute the n positions.
- In a particular position, permuting the $|A| = q$ symbols.

Proposition. Equivalent codes have the same parameters (n, M, d).

Definition. Linear codes $C_1, C_2 \subseteq \mathbb{F}_q^n$ are **monomially equivalent** if we can obtain one from the other by applying sequence of the following two kinds of changes to all codewords (simultaneously):
- Permuting the n positions.
- In particular position, multiply by $\lambda \in \mathbb{F}_q^\times$.

If only the first change is used, the codes are **permutation equivalent**.

Definition. $P \in M_n(\mathbb{F}_q)$ is **permutation matrix** if it has a single 1 in each row and column, and zeros elsewhere. Any permutation of n positions of row vector in $\mathbb{F}_q^n$ can be described as right multiplication by permutation matrix.

Proposition. Permutation matrices are orthogonal: $P^T = P^{-1}$.

Proposition. Let $C_1, C_2 \subseteq \mathbb{F}_q^n$ linear codes with generator matrices G_1, G_2. Then if $G_1 = G_2 P$ for permutation matrix P, then C_1 and C_2 are permutation equivalent.

Definition. $M \in M_m(\mathbb{F}_q)$ is **monomial matrix** if it has exactly one non-zero element in each row and column.

Proposition. Monomial matrix M can always be written as $M = DP$ or $M = PD'$ where P is permutation matrix and D, D' are diagonal matrices. P is **permutation part**, D and D' are **diagonal parts** of M.

Example.

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Proposition. Let $C_1, C_2 \subseteq \mathbb{F}_q^n$ be linear codes with generator-matrices G_1, G_2. Then if $G_2 = G_1 M$ for some monomial matrix M, then C_1 and C_2 are monomially equivalent.

Definition. Let $C \subseteq \mathbb{F}_q^n$ linear code. If $G = (I_k \mid A)$, with $A \in M_{k,n-k}(\mathbb{F}_q)$, is generator-matrix for C, then G is in **standard form**.

Note. Not every linear code has generator-matrix in standard form.

Proposition. Every linear code is permutation equivalent to a linear code with generator-matrix in standard form.

Example. Let $C_1 \subseteq \mathbb{F}_7^5$ have generator matrix $G_1 = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{bmatrix}$. Then applying permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow G_1 P = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 & 6 \end{bmatrix} = (I_2 \mid A)$$

9. Codes as kernels

9.1. Dual codes

Definition. Let $C \subseteq \mathbb{F}_q^n$ linear code. **Dual** of C is

$$C^\perp := \left\{ \boldsymbol{v} \in \mathbb{F}_q^n : \forall \boldsymbol{u} \in C, \boldsymbol{v} \cdot \boldsymbol{u} = 0 \right\}$$

Proposition. If G is generator matrix for linear code C then

$$C^\perp = \{ \boldsymbol{v} \in \mathbb{F}_q^n : \boldsymbol{v} G^T = \boldsymbol{0} \} = \ker(f_{G^T})$$

where $f_{G^T} : \mathbb{F}_q^n \to \mathbb{F}_q^k$, $f(x) = x G^T$ is linear map.

Proposition. Let $C \subseteq \mathbb{F}_q^n$ linear code. Then $C^\perp$ is also linear code and $\dim(C) + \dim(C^\perp) = n$.

Proposition. Let $C \subseteq \mathbb{F}_q^n$ linear code, then $\left(C^\perp\right)^\perp = C$.

Proof. Show $\dim\left(\left(C^\perp\right)^\perp\right) = \dim(C)$ and $C \subseteq \left(C^\perp\right)^\perp$. □

Proposition. Let $C \subseteq \mathbb{F}_q^n$ have generator-matrix in standard form, $G = (I_k \mid A)$, then $H = \left(-A^T \mid I_{n-k}\right)$ is generator-matrix for $C^\perp$.

Proof. Show $\forall y \in \mathbb{F}_q^{n-k}$, $yH \in C^\perp$, let $f_H(y) = yH$ so $\mathrm{im}(f_H) \subseteq C^\perp$ and show $\dim(\mathrm{im}(f_H)) = \dim(C^\perp)$. □

Proposition. Let G be generator matrix of $C \subseteq \mathbb{F}_q^n$, let $P \in M_n(\mathbb{F}_q)$ permutation matrix such that $GP = (I_k \mid A)$ for some $A \in M_{k,n-k}(\mathbb{F}_q)$. Then $H = (-A^T \mid I_{n-k}) P^T$ is generator matrix for $C^\perp$.

Proof. Similar to previous proposition, use that $P^T = P^{-1}$. □

Algorithm. To find basis for dual code $C^{\perp}$, given generator matrix $G = (g_{ij}) \in M_{k,n}(\mathbb{F}_q)$ for C in RREF:

1. Let $L = \{1 \leq j \leq n : G \text{ has leading 1 in column } j\}$.
2. For each $1 \leq j \leq n$, $j \notin L$, construct $\boldsymbol{v}_j$ as follows:
   1. For $m \notin L$, mth entry of $\boldsymbol{v}_j$ is 1 if $m = j$ and 0 otherwise.
   2. Fill in the other entries of $\boldsymbol{v}_j$ (left to right) as $-g_{1j}, ..., -g_{kj}$.
3. The $n - k$ vectors $\boldsymbol{j}$ are basis for $C^{\perp}$.

Example. Let $C \subseteq \mathbb{F}_5^7$ be linear code with generator-matrix

$$G = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Then $L = \{1, 3, 6\}$.

- $v_2 = (3, 1, 0, 0, 0, 0, 0)$
- $v_4 = (2, 0, 4, 1, 0, 0, 0)$
- $v_5 = (1, 0, 3, 0, 1, 0, 0)$
- $v_7 = (0, 0, 2, 0, 0, 1, 1)$
- So generator matrix for $C^{\perp}$ is

$$H = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 1 \end{bmatrix}$$

9.2. Check-matrices

Definition. Let C be $[n, k]_q$ code, assume there exists $H \in M_{n-k,n}(\mathbb{F}_q)$ with linearly independent rows, such that

$$C = \left\{ \boldsymbol{v} \in \mathbb{F}_q^n : \boldsymbol{v}H^t = \boldsymbol{0} \right\}$$

Then H is **check-matrix** for C.

Proposition. If code C has generator-matrix G and check-matrix H, then $C^{\perp}$ has check-matrix G and generator-matrix H.

Proof. Use Proposition 9.1.2 to show G is check-matrix for $C^{\perp}$. Show rows of H form basis for $C^{\perp}$. □

Remark. We can use above algorithm for the $G \longleftrightarrow H$ algorithm: obtain a generator-matrix for C from a check-matrix for C, or vice versa.

9.3. Minimum distance from a check-matrix

Lemma. Let C be $[n, k]_q$ code, $C = \left\{ \boldsymbol{x} \in \mathbb{F}_q^n : \boldsymbol{x}A^T = \boldsymbol{0} \right\}$ for some $A \in M_{m,n}(\mathbb{F}_q)$. The following are equivalent:

- There are d linearly dependent columns of A.
- $\exists \boldsymbol{c} \in C : 0 < w(\boldsymbol{c}) \leq d$.

Proof.

- $\Longrightarrow$: use definition of linear dependence, construct a *word* $\boldsymbol{c}$ with d at most non-zero symbols, based on the definition. Show that $\boldsymbol{c} \in C$.
- $\Longleftarrow$: use non-zero entries of $\boldsymbol{c}$ as coefficients for linear dependence between d corresponding columns of A.

$\square$

Example. Let $C = \{\boldsymbol{x} \in \mathbb{F}_7^5 : \boldsymbol{x}A^T = \boldsymbol{0}\}$ where

$$A = \begin{bmatrix} 3 & 1 & 1 & 4 & 1 \\ 2 & 2 & 5 & 1 & 4 \\ 6 & 3 & 5 & 0 & 2 \end{bmatrix} \in M_{3,5}(\mathbb{F}_7)$$

We have $(0,1,2,0,4)A^T = \boldsymbol{0}$. So $(0,1,2,0,4) \in C$, so C has codeword of weight 3. Also, $1(1,2,3) + 2(1,5,5) + 4(1,2,4) = (0,0,0)$ so A has 3 linearly dependent columns.

Theorem. Let $C = \left\{\boldsymbol{x} \in \mathbb{F}_q^n : \boldsymbol{x}A^T = \boldsymbol{0}\right\}$ for some $A \in M_{m,n}\left(\mathbb{F}_q\right)$. Then there is a linearly dependent set of $d(C)$ columns of A, but any set of $d(C) - 1$ columns of A is linearly independent.

Proof. Use Proposition 7.2.3 and above lemma. $\square$

10. Polynomials and cyclic codes

10.1. Non-prime finite fields

Theorem. Let $f(x) \in \mathbb{F}_q[x]$, then $\mathbb{F}_q[x]/\langle f(x)\rangle$ is ring. $\mathbb{F}_q[x]/\langle f(x)\rangle$ is field iff $f(x)$ irreducible in $\mathbb{F}_q[x]$.

Proposition. If $f(x) = \lambda m(x) \in \mathbb{F}_q[x]$, with $0 \neq \lambda \in \mathbb{F}_q$, then

$$\mathbb{F}_q[x]/\langle f(x)\rangle = \mathbb{F}_q[x]/\langle m(x)\rangle$$

In particular, we only need to consider monic polynomials.

Definition. $\alpha \in \mathbb{F}_q$ is **primitive** if

$$\mathbb{F}_q^\times = \left\{\alpha^j : j \in \{0, ..., q-2\}\right\}$$

Every finite field has a primitive element.

Definition. Let $f(x) \in \mathbb{F}_q[x]$ irreducible. If x is primitive in $\mathbb{F}_q[x]/\langle f(x)\rangle$, then $f(x)$ is **primitive polynomial** over $\mathbb{F}_q$.

Theorem. Let $q = p^r$, p prime, $r \geq 2$ integer. Then there exists monic, irreducible $f(x) \in \mathbb{F}_p[x]$ with $\deg(f) = r$. In particular, $\mathbb{F}_q = \mathbb{F}_p[x]/\langle f(x)\rangle$ is field with $q = p^r$ elements. Moreover, we can choose $f(x)$ to be primitive.

10.2. Cyclic codes

Definition. Code C is **cyclic** if it is linear and

$$(a_0, ..., a_{n-1}) \in C \Longleftrightarrow (a_{n-1}, a_0, ..., a_{n-2}) \in C$$

i.e. any cyclic shift of a codeword is also a codeword.

Notation. Let $R_n = \mathbb{F}_q[x]/(x^n - 1)$. Note R_n is not field. There is correspondence between elements in R_n and vectors in $\mathbb{F}_q^n$:

$$a(x) = a_0 + \cdots + a_{n-1}x^{n-1} \longleftrightarrow \boldsymbol{a} = (a_0, ..., a_{n-1})$$

Lemma. If $a(x) \longleftrightarrow \boldsymbol{a}$, then $xa(x) \longleftrightarrow (a_{n-1}, a_0, ..., a_{n-2})$.

Proposition. $C \subseteq R_n$ is cyclic iff C is ideal in R_n, i.e. $a(x), b(x) \in C \Longrightarrow a(x) + b(x) \in C$ and $a(x) \in C, r(x) \in R_n \Longrightarrow r(x)a(x) \in C$.

Proof.
- $\Longrightarrow$: use linearity of C and Lemma 10.2.3.
- $\Longleftarrow$: for linearity, use $r(x) = r_0$ constant. For cyclicity, use Lemma 10.2.3 with $r(x) = x^m$.

□

Definition. For $f(x) \in R_n$, the **code generated by** $f(x)$ is

$$\langle f(x) \rangle := \{r(x)f(x) : r(x) \in R_n\}$$

Proposition. For any $f(x) \in R_n$, $\langle f(x) \rangle$ is cyclic code.

Example. Let $R_3 = \mathbb{F}_2[x]/(x^3 - 1)$, $f(x) = x^2 + 1 \in R_3$. Then

$$\begin{aligned} \langle f(x) \rangle &= \{0, 1 + x, 1 + x^2, x + x^2\} \subseteq \mathbb{R}_3 \\ &\longleftrightarrow \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\} \subseteq \mathbb{F}_2^3 \end{aligned}$$

Theorem. Let C cyclic code in R_n over $\mathbb{F}_q$, $C \neq \{0\}$. Then
- There is unique monic polynomial $g(x)$ of smallest degree in C.
- $C = \langle g(x) \rangle$.
- $g(x) \mid x^n - 1$.

Remark. Converse of above theorem holds: every monic factor $g(x)$ of $x^n - 1$ is the unique generator polynomial of $\langle g(x) \rangle$, so distinct factors generate distinct codes. So to find all cyclic codes in R_n, find each monic divisor $g(x)$ of $x^n - 1$ to give cyclic code $\langle g(x) \rangle$.

Proof.
- First assume there are two such $g(x)$ which are different, obtain contradiction.
- Use division algorithm to show $C \subseteq \langle g(x) \rangle$ and that $g(x) \mid x^n - 1$.

□

Remark. If $C = \{0\}$, then setting $g(x) = x^n - 1$, we have $C = \langle g(x) \rangle$.

Definition. In cyclic code C, monic polynomial of minimal degree is the **generator-polynomial** of C.

Example. To find all binary cyclic codes of block-length 3, consider $R_3 = \mathbb{F}_2[x]/\langle x^3 - 1 \rangle$. In $\mathbb{F}_2[x]$, $x^3 - 1 = (x + 1)(x^2 + x + 1)$ and $x^2 + x + 1$ is irreducible. So the possible candidates for the generator-polynomial are

generator	code in R_3	code in $\mathbb{F}_2^3$
1	R_3	$\mathbb{F}_2^3$
$x+1$	$\{0, 1+x, 1+x^2, x+x^2\}$	$\{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$
x^2+x+1	$\{0, 1+x+x^2\}$	$\{(0,0,0),(1,1,1)\}$
x^3-1	$\{0\}$	$\{(0,0,0)\}$

10.3. Matrices for cyclic codes

Proposition. If C is cyclic code with generator-polynomial $g(x) = g_0 + \cdots + g_r x^r$, then $\dim(C) = n - r$ and C has generator-matrix

$$G = \begin{bmatrix} g_0 & g_1 & \cdots & g_r & 0 & \cdots & \cdots & 0 \\ 0 & g_0 & g_1 & \cdots & g_r & 0 & \cdots & 0 \\ 0 & 0 & g_0 & g_1 & \cdots & g_r & 0 & \cdots \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & g_0 & g_1 & \cdots & g_r \end{bmatrix} \in M_{n-r,n}(\mathbb{F}_q)$$

Proof.

- Show $g_0 \neq 0$, use this to show rows are linearly independent.
- Show rows of G span C by using polynomial representation of C.

□

Example. Let $C = \{(0,0,0),(1,1,0),(0,1,1),(1,0,1)\} \in \mathbb{F}_2^3$. $C = \langle 1+x \rangle$ so $\dim(C) = 3 - 1 = 2$,

$$G = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Definition. Let $C \subseteq R_n$ be $[n,k]$ cyclic code with generator polynomial $g(x)$, let $g(x)h(x) = x^n - 1 \in \mathbb{F}_q[x]$. Then $h(x)$ is the **check-polynomial** of C.

Lemma. Check-polynomial of cyclic $[n,k]$ code is monic of degree k.

Proposition. Let C be cyclic code in R_n with check-polynomial $h(x)$. Then $c(x) \in C$ iff $c(x)h(x) = 0$ in R_n.

Proof.

- $\Longrightarrow$: use that $C = \langle g(x) \rangle$.
- $\Longleftarrow$: use division algorithm.

□

Definition. The **reciprocal polynomial** of $h(x) = h_0 + h_1 x + \cdots + h_k x^k$ is

$$\overline{h}(x) = h_k + h_{k-1}x + \cdots + h_0 x^k = x^k h(x^{-1})$$

Proposition. Let C cyclic $[n,k]$ code with check-polynomial $h(x) = h_0 + \cdots + h_k x^k$. Then

- C has check-matrix

$$H = \begin{bmatrix} h_k & h_{k-1} & \cdots & h_0 & 0 & \cdots & \cdots & 0 \\ 0 & h_k & h_{k-1} & \cdots & h_0 & 0 & \cdots & 0 \\ 0 & 0 & h_k & h_{k-1} & \cdots & h_0 & 0 & \cdots \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & h_k & h_{k-1} & \cdots & h_0 \end{bmatrix}$$

- $C^\perp$ is cyclic and generated by $\overline{h}(x)$ (i.e. $h_0^{-1}\overline{h}(x)$ is generator-polynomial for $C^\perp$).

Proof.
- Show that H is generator matrix for $C^\perp$:
  - Show rows of H are linearly independent.
  - Show rows of H are in $C^\perp$:
    - Let $c(x) \in C$, use Proposition 10.3.5 to show $c(x)h(x) = b(x)x^n - b(x)$ for some $b(x) \in \mathbb{F}_q[x]$, $\deg(b) \leq k-1$.
- Show that $\overline{h}(x) \mid x^n - 1$ (hint: write $x^n = x^k x^{n-k}$).
- Show that if $\overline{h}(x)$ monic, then $\langle \overline{h}(x) \rangle$ and $C^\perp$ have a common generator-matrix.
- If $\overline{h}(x)$ not monic, show that multiplying by h_0 is row operation, and so $\langle \overline{h}(x) \rangle$ and $C^\perp$ have a common generator matrix.

□

11. MDS and perfect codes

11.1. Reed-Solomon codes

Notation. Let $\boldsymbol{P}_k = \mathbb{F}_q[z]_{<k}$ be vector space of polynomials of degree $< k$ in $\mathbb{F}_q$:

$$\mathbb{F}_q[z]_{<k} = \left\{a_0 + \cdots + a_{k-1}z^{k-1} : a_i \in \mathbb{F}_q\right\}$$

Dimension of $\mathbb{F}_q[z]_{<k}$ is k.

Definition. Let $0 \leq k \leq n \leq q$, $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in \mathbb{F}_q^n$ with all a_j distinct and all b_j non-zero. Define the linear map

$$\varphi_{\boldsymbol{a},\boldsymbol{b}} : \boldsymbol{P}_k \to \mathbb{F}_q^n, \quad \varphi_{\boldsymbol{a},\boldsymbol{b}}(f(z)) := (b_1 f(a_1), ..., b_n f(a_n)) \in \mathbb{F}_q^n$$

The q**-ary Reed-Solomon code** $\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b})$ is the image of $\varphi_{\boldsymbol{a},\boldsymbol{b}}$:

$$\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b}) = \varphi_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{P}_k) \subseteq \mathbb{F}_q^n$$

Proposition.
- $\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b})$ is a q-ary $[n, k, n-k+1]$ code. In particular, it is an MDS code.
- A generator-matrix for $\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b})$ is

$$G = (b_j a_j^{i-1})_{i,j} = \begin{bmatrix} \varphi_{\boldsymbol{a},\boldsymbol{b}}(1) \\ \vdots \\ \varphi_{\boldsymbol{a},\boldsymbol{b}}(z^{k-1}) \end{bmatrix} \in M_{k,n}(\mathbb{F}_q)$$

  where $1 \leq i \leq k$, $1 \leq j \leq n$.

Proof.

- To show dimension is k, show that $\varphi_{\boldsymbol{a},\boldsymbol{b}}$ is injective, by showing it has trivial kernel.
- To show minimum distance is $n-k+1$, show for $f(z) \neq 0$ that $w\big(\varphi_{\boldsymbol{a},\boldsymbol{b}}(z)\big) \geq n-(k-1)$.
- Use linearity and injectivity of $\varphi_{\boldsymbol{a},\boldsymbol{b}}$ and fact that $\{1, ..., z^{k-1}\}$ is basis for $\boldsymbol{P}_k$ to show G is generator-matrix for $\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b})$.

$\square$

Remark. We have

$$\{0\} = \mathrm{RS}_0(\boldsymbol{a}, \boldsymbol{b}) \subset \mathrm{RS}_1(\boldsymbol{a}, \boldsymbol{b}) \subset \cdots \subset \mathrm{RS}_n(\boldsymbol{a}, \boldsymbol{b}) = \mathbb{F}_q^n$$

(since a row is added to the generator matrix each time).

Example. Let $q = 7$, $n = 5$, $k = 3$, $\boldsymbol{a} = (0, 1, 6, 2, 3)$, $\boldsymbol{b} = (5, 4, 3, 2, 1)$. Then

$$\begin{aligned} \varphi_{\boldsymbol{a},\boldsymbol{b}} : \boldsymbol{P}_3 &\to \mathbb{F}_7^5, \\ f(z) &\mapsto (5f(0), 4f(1), 3f(0), 2f(2), 1f(3)) \end{aligned}$$

So a generator matrix for $\mathrm{RS}_3(\boldsymbol{a}, \boldsymbol{b})$ is

$$G = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 4 & 4 & 3 \\ 0 & 4 & 3 & 1 & 2 \end{bmatrix}$$

Definition. $\alpha \in \mathbb{F}_q$ is **primitive n-th root of unity** if $\alpha^n = 1$ and $\forall 0 < j < n$, $\alpha^j \neq 1$.

Proposition. Let $\alpha \in \mathbb{F}_q$ primitive n-th root of unity, $m \in \mathbb{Z}$, define

$$\boldsymbol{a}^{(m)} = \left((\alpha^0)^m, ..., (\alpha^{n-1})^m\right) \in \mathbb{F}_q^n$$

Then for $0 \leq k \leq n$, $\mathrm{RS}_k\big(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(m)}\big)$ is cyclic.

Proof.

- Show cyclic permutation of $\boldsymbol{\alpha}^{(m)}$ is equivalent to multiplying by $\alpha^{-m} \in \mathbb{F}_q$.
- Show rows of generator matrix of $\mathrm{RS}_k\big(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(m)}\big)$ has rows $\boldsymbol{\alpha}^{(m+i-1)}$ for $1 \leq i \leq k$.
- Use linearity of a permutation to conclude result.

$\square$

Example. In $\mathbb{F}_5$, $2^1 = 2$, $2^2 = 4$, $2^3 = 3$, $2^4 = 1$ so 2 is primitive 4th root of unity in $\mathbb{F}_5$ so $\boldsymbol{\alpha}^m = (1^m, 2^m, 4^m, 3^m)$. We have $\boldsymbol{\alpha}^{(1)} = (1, 2, 4, 3)$, $\boldsymbol{\alpha}^{(2)} = (1, 4, 1, 4)$, so a generator matrix for $\mathrm{RS}_2\big(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\big)$ is

$$G = \begin{bmatrix} 1 & 4 & 1 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}$$

By performing ERO's, we obtain another generator matrix

$$G' = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}$$

This is generator matrix for the cyclic code with generator polynomial $g(x) = (x - 1)(x - 3) = x^2 + x + 3$. So $\mathrm{RS}_2\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)$ is cyclic with generator polynomial $g(x)$. Note $x^4 - 1 = (x-1)(x-2)(x-3)(x-4)$ so $g(x) \mid x^4 - 1$.

Proposition. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{F}_q^n$ with a_j all distinct and b_j all non-zero,
- There exists $\boldsymbol{c}$ with all $c_j \neq 0$ such that

$$1 \leq k \leq n-1, \quad (\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b}))^{\perp} = \mathrm{RS}_{n-k}(\boldsymbol{a}, \boldsymbol{c})$$

- $\boldsymbol{c}$ is given by the $1 \times n$ check-matrix for $\mathrm{RS}_{n-1}(\boldsymbol{a}, \boldsymbol{b})$.

Proof.
- First consider $k = n-1$. Let $\boldsymbol{c}$ be the $1 \times n$ check-matrix for $\mathrm{RS}_{n-1}(\boldsymbol{a}, \boldsymbol{b})$.
  - Use that $\mathrm{RS}_{n-1}(\boldsymbol{a}, \boldsymbol{b})$ saturates singleton bound to show all $c_j \neq 0$, and so that $\mathrm{RS}_1(\boldsymbol{a}, \boldsymbol{c})$ and $(\mathrm{RS}_{n-1}(\boldsymbol{a}, \boldsymbol{b}))^{\perp}$ share a generator matrix (so are the same code).
  - $\forall f(z) \in \boldsymbol{P}_{n-1}$, since $\varphi_{\boldsymbol{a},\boldsymbol{b}}(f(z)) \in \mathrm{RS}_{n-1}(\boldsymbol{a}, \boldsymbol{b})$, and $\boldsymbol{c}$ is check-matrix for $\mathrm{RS}_{n-1}(\boldsymbol{a}, \boldsymbol{b})$,

$$\varphi_{\boldsymbol{a},\boldsymbol{b}}(f(z)) \cdot \boldsymbol{c} = 0$$

- Since $\dim(\mathrm{RS}_{n-k}(\boldsymbol{a}, \boldsymbol{c})) = n - k = \dim\left((\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b}))^{\perp}\right)$, enough to show $\mathrm{RS}_{n-k}(\boldsymbol{a}, \boldsymbol{c}) \subseteq (\mathrm{RS}_k(\boldsymbol{a}, \boldsymbol{b}))^{\perp}$:
  - By considering degrees, show that for $g(z) \in \boldsymbol{P}_k$ and $g(z) \in \boldsymbol{P}_{n-k}$, $(fg)(z) \in \boldsymbol{P}_{n-1}$. Deduce that $\varphi_{\boldsymbol{a},\boldsymbol{c}}(g(z)) \cdot \varphi_{\boldsymbol{a},\boldsymbol{b}}(f(z)) = 0$.

□

11.2. Hamming codes

Definition. Let $r \geq 2$, $n = 2^r - 1$, let $H \in M_{r,n}(\mathbb{F}_2)$ have columns corresponding to all non-zero vectors in $\mathbb{F}_2^r$. The **binary Hamming code of redundancy** r is

$$\mathrm{Ham}_2(r) = \{\boldsymbol{x} \in \mathbb{F}_2^n : \boldsymbol{x}H^t = \boldsymbol{0}\}$$

Note the order of columns is not specified, so we have a collection of permutation-equivalent codes.

Example. For $r = 2, 3$, we can take

$$H_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Proposition. For $r \geq 2$, $\mathrm{Ham}_2(r)$ is perfect $[2^r - 1, 2^r - r - 1, 3]$ code with check-matrix H.

Proof.
- $n = 2^r - 1$: count rows in H^t.
- To show H is check-matrix, verify its rows are linearly independent by considering its column rank.

- $k = 2^r - r - 1$: $k = n -$ number of rows of H.
- $d = 3$: use criterion of minimum distance from linearly (in)dependent columns. No column is multiple of another, but columns $\boldsymbol{e_1}, \boldsymbol{e_2}$ and $\boldsymbol{e_1} + \boldsymbol{e_2}$ are linearly dependent.
- $\mathrm{Ham}_2(r)$ is perfect: we have $|\mathrm{Ham}_2(r)| = 2^k = 2^{2^r-r-1}$, $t = \left\lfloor \frac{d-1}{2} \right\rfloor = 1$. $|S(c,1)| = 1 + n = 2^r$ and the $S(c,1)$ are disjoint, so $|\cup_{c\in C} S(c,1)| = 2^{2^r-r-1} \cdot 2^r = 2^n$.

□

Definition. Can define Hamming codes for $q > 2$. Consider $\mathbb{F}_q^r$ for $r \geq 2$. $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}_q^r - \{0\}$ are **equivalent** if $\boldsymbol{v} = \lambda \cdot \boldsymbol{w}$ for some $\lambda \in \mathbb{F}_q^\times$. For $\boldsymbol{v} \in \mathbb{F}_q^r - \{0\}$, set

$$L_{\boldsymbol{v}} = \left\{\boldsymbol{w} \in \mathbb{F}_q^r : \boldsymbol{w} \text{ equivalent to } \boldsymbol{v}\right\} = \left\{\lambda \boldsymbol{v} : \lambda \in \mathbb{F}_q^\times\right\}$$

Note $|L_{\boldsymbol{v}}| = q - 1$ and $w \in L_{\boldsymbol{v}}$ iff $L_{\boldsymbol{w}} = L_{\boldsymbol{v}}$. Also, if $L_{\boldsymbol{v}} \neq L_{\boldsymbol{w}}$ then $L_{\boldsymbol{v}} \cap L_{\boldsymbol{w}} = \emptyset$. Hence the $L_{\boldsymbol{v}}$ partition $\mathbb{F}_q^r - \{0\}$ and there are $(q^r - 1)/(q-1)$ of them.

Example. For $q = 3$, $r = 2$ there are $\left(3^2 - 1\right)/(3-1) = 4$ sets:

$$\begin{aligned} L_{(0,1)} &= \{(0,1),(0,2)\}, \quad L_{(1,0)} = \{(1,0),(2,0)\}, \\ L_{(1,1)} &= \{(1,1),(2,2)\}, \quad L_{(1,2)} \ = \{(1,2),(2,1)\} \end{aligned}$$

Definition. For $r \geq 2$, $n = (q^r - 1)/(q-1)$, construct $H \in M_{r,n}\left(\mathbb{F}_q\right)$ by taking one colum from each of the n different L_v. The **Hamming code of redundancy** r is

$$\mathrm{Ham}_q(r) = \left\{\boldsymbol{x} \in \mathbb{F}_q^n : \boldsymbol{x}H^t = \boldsymbol{0}\right\}$$

Note that different choices of H give monomially equivalent codes.

Example. For $\mathrm{Ham}_3(2)$, we can choose e.g.

$$H = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{or} \quad H = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Proposition. For $r \geq 2$, $\mathrm{Ham}_q(r)$ is perfect $[n, n-r, 3]$ code, with check-matrix H.