Contents

1. Combinatorial group theory

1.1. Free groups and presentations

Definition 1.1 Let $A = \{a_1, a_2, a_3, ...\}$ be an alphabet. A group F is **free on A** if:

- There exists a map of sets $A \to F$, and
- The **universal property of free groups** holds: for any group G and any map of sets $A \to G$, there is a unique homomorphism $F \to G$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & F \arrow[d, dashed, "\exists!"'] \\ & G \end{tikzcd}$$

F is unique up to unique isomorphism (proof: exercise). We thus write $F = F(A)$.

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ F_1 \arrow[rr, dashed, leftrightarrow, "\cong"] & & F_2 \end{tikzcd}$$

Remark 1.2 This leaves open the question of *existence*. We may resolve this question in two different ways:

- **Topologically**: let $X = \bigvee_{a \in A} S^1$, where all the S^1 are disjoint except at the distinguished point. Then $\pi_1(X) \cong F(A)$ by the Seifert-van Kampen (SVK) theorem.
- **Combinatorially**: let $A^* = \{\text{words in } a, a^{-1} \text{ for } a \in A\}$, e.g. $A = \{a, b\}$. Some examples of words are $1 = \emptyset$, $aba^{-1}b^{-1}$, $a^{100}a^{-100}b$.

Definition 1.3 A word w is **reducible** if $w = ...aa^{-1}...$ or $w = ...a^{-1}a...$ for some $a \in A$. Otherwise, w is **reduced**.

Definition 1.4 We may define the **free group on A** as $F(A) = \{w \in A^* : w \text{ is reduced}\}$. The identity is $1 = \emptyset$ (the empty word). Multiplication is given by "concatenate, then reduce", e.g. $(aba^{-1}b^{-1})(b^2a) = aba^{-1}b^{-1}b^2a = aba^{-1}ba$.

Definition 1.5 A **presentation** consists of a set of **generators** A and a set of **relations** $R \subseteq F(A)$.

We write $\langle A \mid R \rangle$ or $\langle a_1, a_2, ... \mid r_1, r_2, ... \rangle$ or $\langle a_1, a_2, ... \mid r_1, r_2, ... = 1 \rangle$ for the presentation of the group $F(A)/\langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ denotes the normal closure of R (the smallest normal subgroup of $F(A)$ containing R).

Definition 1.6 Given $a, b \in A$, the **commutator** of a and b is $[a, b] = aba^{-1}b^{-1}$.

Example 1.7

- $\langle a \mid a^n \rangle \cong \mathbb{Z}_n$.
- $\langle r, s \mid r^n, s^2, srsr \rangle \cong D_{2n}$.
- $\langle A \mid \rangle \cong F(A)$.

- $\left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g}[a_i, b_i] \right\rangle \cong \pi_1(\Sigma_g)$, where Σ_g is the orientable surface of genus g.
- $\langle x, y \mid x^2 y^{-3} \rangle \cong \pi_1(M_T)$, where $M_T = \mathbb{R}^3 \setminus T$-trefoil.

Remark 1.8 A corollary of the SVK theorem: for $G = \langle a_1, a_2, \ldots \mid r_1, r_2, \ldots \rangle$, let X be the "presentation complex" space constructed as follows: take the space $\bigvee_{a \in A} S^1$, where all the S^1 are disjoint except at one point, and consider disc for each $r \in R$ (these discs are disjoint). Then map the boundary of the each relation disc via the word the relation makes. Then we have $\pi_1(X) \cong G$.

We have G is finitely presented iff X is compact.

Every group appears as a quotient/presentation of a free group, all of which appear as a fundamental group.

Problem 1.9 (Word Problem) For A, R finite, determine whether or not $w \in A^*$ represents 1 in $\langle A \mid R \rangle$ (equivalently, whether $u \underset{G}{=} v$, for $u, v \in A^*$).

Problem 1.10 (Conjugacy Problem) For A, R finite, determine whether or not $u, v \in A^*$ represent conjugate elements in $\langle A \mid R \rangle$.

Problem 1.11 (Isomorphism Problem) Determine if $\langle A \mid R \rangle \cong \langle A' \mid R' \rangle$ or not (given that they are both finite).

Remark 1.12
- The conjugacy problem is stronger than the word problem.
- All these problems turn out to be independent of the choice of finite presentation $\langle A \mid R \rangle$. (Proof: exercise…).
- Dehn was motivated by topology. All these problems ask for algorithms (in 1911!).
- All these problems are undecidable in full generality. Norikov (1955) and Boone (1959) unsolved the word (and hence conjugacy) problem. Adyan (1955) and Rabin (1958) unsolved the isomorphism problem.
- Nevertheless, positive solutions exist for "reasonable" classes of groups.

Example 1.13 (Word problem in finitely-generated free groups) Let $w \in A^*$. If w is reduced, then $w \underset{F(A)}{=} 1$ iff w is the empty word. Otherwise, w contains a cancelling pair aa^{-1} (or $a^{-1}a$): $w = uaa^{-1}v$. Cancelling aa^{-1} gives $w' = uv$. Note that $w' \underset{F(A)}{=} w$ and the length of w' is shorter. Continuing inductively, we eventually arrive in the reduced case (note that A is finite).

What about the conjugacy problem in free groups?

Definition 1.14 There is a natural action of $\mathbb{Z}$ on A^*, given by cyclically permuting words:

$$1.a_1 \ldots a_n = a_2 \ldots a_n a_1, \quad a_i \in A \cup A^{-1}.$$

The orbits of this action are called **cyclic words**.

Example 1.15 The cyclic word defined by $aba^{-1}b^{-1}$ can be represented as

$$\begin{array}{ccc} & a & \\ b^{-1} & & b \\ & a^{-1} & \end{array}$$

Definition 1.16 If $u, v \in A^*$ define the same cyclic word, we say that u and v are **cyclic conjugates**.

Definition 1.17 $w \in A^*$ is **cyclically reduced** if all its cyclic conjugates are reduced.

Example 1.18 $aba^{-1} \rightsquigarrow ba^{-1}a \underset{F(A)}{=} b$. So aba^{-1} is reduced but not cyclically reduced.

Lemma 1.19 If $u, v \in F(A)$ are cyclically reduced, then u is conjugate to v iff u and v are cyclic conjugates.

Proof (Hints).
- $\Longleftarrow$: straightforward.
- $\Longrightarrow$: explain why we can assume $g \in A \cup A^{-1}$.

$\square$

Proof. $\Longleftarrow$: suppose $u = a_1 ... a_n$, $a_i \in A \cup A^{-1}$. Then $v = a_k ... a_n (a_1 ... a_{k-1})$ for some k. Let $g = a_1 ... a_{k-1}$, then $u = gvg^{-1}$, as required.

$\Longrightarrow$: suppose $u = gvg^{-1}$. By induction on the length of g, we may assume that $g \in A \cup A^{-1}$. Since u is cyclically reduced, v decomposes as one of:
- $v = g^{-1}v'$ or
- $v = v'g$.

In the first case, we obtain $u = v'g^{-1}$ and in the second case $u = gv'$. In either case, u is a cyclic conjugate of v as required. $\square$

Example 1.20 (Conjugacy problem in free groups) Consider $F(A)$ for A finite. If $w \in A^*$ is reduced but not cyclically reduced, then $w = aw'a^{-1}$ for some $a \in A \cup A^{-1}$. Note that w' is conjugate to w and shorter than w. Therefore, continuing inductively, we may assume that w is cyclically reduced.

So Lemma 1.19 solves the problem (since each word of finite length has a finite number of cyclic conjugates).

2. Historical case study

We need to understand the state of topology in the early 20th century. Poincaré knew that 2D compact surfaces are classified by their homology groups. He wondered if the same could be true in dimension 3.

Conjecture 2.1 (Poincaré Conjecture (version 1)) Let M be a closed 3-manifold. If $H_*(M) = \begin{cases} \mathbb{Z} \text{ if } *=0,3 \\ 0 \text{ otherwise} \end{cases}$, then $M \cong S^3$. Such a 3-manifold is called a **homology sphere**.

Theorem 2.2 (Poincaré) There is a 3-dimensional homology sphere P such that $\pi_1(P) \twoheadrightarrow A_5$ ($\twoheadrightarrow$ means surjects). In particular, $P \ncong S^3$.

So the Poincaré Conjecture (version 1) is false and homology is not enough in dimension 3.

Conjecture 2.3 (Poincaré Conjecture (version 2)) Let M be a closed, connected 3-manifold. If $\pi_1(M) \cong \{e\}$, then $M \cong S^3$.

This was proven in 2003 by Perelman.

Theorem 2.4 (Dehn) There are infinitely many pairwise non-homeomorphic homology spheres in dimension 3.

Dehn's construction is as follows: let K be the trefoil knot and $N = S^3 \setminus N^o(K)$ where $N^o(K)$ is a small open tubular neighbourhood of K. We have $\mathbb{T} \cong \partial N$. $\pi_1(N) = \langle x,\, y,\, z \mid x^2 = y^3 = z \rangle$. The homology sphere is $\pi_1(N)_{\text{ab}} = \mathbb{Z}^2/\langle (2, -3) \rangle \cong \mathbb{Z}$, the abelianisation of the fundamental group. In general,

$$H_*(N) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

It turns out that $\pi_1(\mathbb{T}) \cong \mathbb{Z}^2 = \langle xy,\, z \rangle \leq \pi_1(N)$. We have

$$\begin{array}{rcl} H_1(\mathbb{T}) \cong \mathbb{Z}^2 & \longrightarrow & H_1(N) \cong \mathbb{Z} \\ xy & \mapsto & 5 \\ z & \mapsto & 6 \end{array}$$

We now build infinitely many manifolds using "Dehn filling". Let $U = D^2 \times S^1$ be the solid sphere. For any homeomorphism $\varphi : \partial U \to \partial N$, define $M_\varphi = (U \sqcup N)/\{x \sim \varphi(x) : x \in \partial U\}$. By SVK theorem, if $g = \varphi_*(\mu) \in \pi_1(\mathbb{T}) \leq \pi_1(N)$, then $\pi_1\left(M_\varphi\right) = \pi_1(N)/\langle\langle g \rangle\rangle$ and $H_1\left(M_\varphi\right) = \mathbb{Z}/\langle [g] \rangle$.

So, to produce homology spheres, we need $[g] = \pm 1$ in $H_1(N)$. If $g = (xy)^a z^b$ in $\pi_1(\mathbb{T})$, then $[g] = 5a + 6b$. Dehn chooses $a = 6n + 5$, $b = -5n - 4$ for all $n \in \mathbb{N}$. So we define $g_n = (xy)^{6n+5} z^{-5n-4}$. For these cases, M_φ is a homology sphere:

- $H_0\left(M_\varphi\right) \cong \mathbb{Z}$
- $H_1\left(M_\varphi\right) \cong \{0\}$
- $H_2\left(M_\varphi\right) \cong \{0\}$ by Poincare duality
- $H_3\left(M_\varphi\right) \cong \mathbb{Z}$ by Poincare duality.

For φ_n that sends $\mu \to g_n = 5a + 6b$, write $M_n = M_{\varphi_n}$. Then $G_n := \pi_1(M_n) = \langle x,\, y,\, z \mid x^2 = y^3 = z,\, (xy)^{6n+5} = z^{5n+4} \rangle$. To prove that the M_n are pairwise distinct, we are left with the challenge of proving that $G_m \cong G_n \implies m = n$.

Also, note that if $g_n = g_m$, then $g_n \underset{\text{conj}}{\sim} g_m$ which implies $G_n \cong G_m$.

2.1. Van Kampen diagrams

Definition 2.5 A map of cell complexes $Y \to X$ is called **combinatorial** if, for all k, and for every k-cell e^k of Y, f maps the interior of e^k homeomorphically to the interior of a k-cell of X.

Consider a presentation $\langle a_i \mid r_j \rangle \cong G$ and let X be the associated presentation complex.

Definition 2.6 A **(singular) disc diagram** is a compact contractible 2-complex D with an embedding $D \hookrightarrow \mathbb{R}^2$.

Definition 2.7 A disc diagram D is said to be **over $\boldsymbol{X}$** if it is equipped with a combinatorial map $D \to X$. Equivalently, each edge of D is oriented and labelled by some a_i, so that the boundary of each 2-cell reads some $r_j^{\pm 1}$, thought of as a cyclic word.

The **boundary cycle** reads a word $w' \in A^*$, which reduces to some $w \in \langle\langle R \rangle\rangle \leq F(A)$.

D is said to be a **van Kampen diagram** for w'.

Lemma 2.8 (van Kampen's Lemma) If $w \in \langle\langle R \rangle\rangle$, then a van Kampen diagram exists for w.

Proof. Since $w \in \langle\langle R \rangle\rangle$, there are $k_i \in F(A)$ and $r_i \in R$ such that

$$w \underset{F(A)}{=} \prod_{i=1}^{\ell} k_i r_i^{\pm 1} k_i^{-1} =: w_0 \in A^*.$$

It is easy to write down a van Kampen diagram D_0 for w_0: (see diagram). If w_0 is reduced, we are done, because $w \underset{A^*}{=} w_0$. Otherwise, w_0 contains a "cancelling pair of consecutive edges" e_1, e_2, labelled a, a^{-1} for some $a \in A \cup A^{-1}$. We now simplify D_0 to produce a new disc diagram D_1 with shorter boundary word w_1. There are two cases:

- If the origin of e_1 is also the terminus of e_2: (see diagram) in this case, $D_0 = D_1 \vee D'$, where $\partial D' = aa^{-1}$.
- Otherwise, (see diagram).

In either case, $w_1 = \partial D_1$ is obtained from w_0 by cancelling a pair. Therefore we may proceed by induction on the length of w_0. Eventually, we build D_n with $w_n = \partial D_n$ reduced, so $w_n = w$ and D_n is the required van Kampen diagram. □

Remark 2.9 The proof shows that the minimal number of 2-cells in a van Kampen diagram for w is equal to the minimal ℓ such that $w = \prod_{i=1}^{\ell} k_i r_i^{\pm 1} k_i^{-1}$. This number is called the **area** of w.

Example 2.10 Let $G = \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$. Let $w = a^n b^n a^{-n} b^{-n} = [a^n, b^n]$. (see diagram) We shall see on the first example sheet that D is minimal, so the area of w is n^2.

Definition 2.11 Let $P = \langle A \mid R \rangle$ be a presentation. Define

$$\begin{aligned} d_P &: \mathbb{N} \to \mathbb{N}, \\ &\ell \mapsto \max\{\mathrm{Area}(w) : w \in \langle\langle R \rangle\rangle, \mathrm{length}(w) = \ell\}. \end{aligned}$$

d_P is called the **Dehn function** of P.

3. Basics of geometric group theory

3.1. Cayley graphs

Definition 3.1 A graph is a one-dimensional cell-complex.

Example 3.2 (see diagram)

Definition 3.3 Let G be a group with a generating set S. A **Cayley graph** $\mathrm{Cay}_S(G)$ is defined as follows:

- The vertex set is G.
- The edges are as follows: there is an edge between g and gs for each $g \in G$ and each $s \in S$.