1. The real numbers

1.1. Conventions on sets and functions

- **Definition**: for $f : X \to Y$, **preimage** of $Z \subseteq Y$ is
$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$
- **Definition**: $f : X \to Y$ **injective** if
$$\forall y \in f(X), \exists! x \in X : y = f(x)$$
- **Definition**: $f : X \to Y$ **surjective** if $Y = f(X)$.
- **Proposition**: let $f : X \to Y$, $A, B \subseteq X$, then
$$\begin{aligned} f(A \cap B) &\subseteq f(A) \cap f(B), \\ f(A \cup B) &= f(A) \cup f(B), \\ f(X) - f(A) &\subseteq f(X - A) \end{aligned}$$
- **Proposition**: let $f : X \to Y$, $C, D \subseteq Y$, then
$$\begin{aligned} f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D), \\ f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D), \\ f^{-1}(Y - C) &= X - f^{-1}(C) \end{aligned}$$

1.2. The real numbers

- **Definition**: $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \leq a$.
- **Definition**: $c \in \mathbb{R}$ is a **least upper bound (supremum)** of E, $c = \sup(E)$, if $c \leq a$ for every upper bound a.
- **Definition**: $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \geq a$.
- **Definition**: $c \in \mathbb{R}$ is a **greatest lower bound (supremum)**, $c = \inf(E)$, if $c \geq a$ for every upper bound a.
- **Completeness axiom of the real numbers**: every $E \subseteq \mathbb{R}$ with an upper bound has a least upper bound. Every $E \subseteq \mathbb{R}$ with a lower bound has a greatest lower bound.
- **Archimedes' principle**:
$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$
- **Remark**: every non-empty subset of $\mathbb{N}$ has a minimum.
- **Proposition**: $\mathbb{Q}$ is dense in $\mathbb{R}$:
$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

1.3. Sequences, limits and series

- **Definition**: $l \in \mathbb{R}$ is **limit** of (x_n) ((x_n) converges to l) if
$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad |x_n - l| < \varepsilon$$

  A sequence **converges in $\mathbb{R}$ (is convergent)** if it has a limit $l \in \mathbb{R}$. Limit $l = \lim_{n\to\infty} x_n$ is unique.

- **Definition**: (x_n) **tends to infinity** if

$$\forall K > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad x_n > K$$

- **Definition**: **subsequence** of (x_n) is sequence (x_{n_j}), $n_1 < n_2 < \cdots$.
- **Definition**: **limit inferior** of sequence x_n is

$$\liminf_{n \to \infty} x_n := \lim_{n \to \infty} \left(\inf_{m \geq n} x_m \right) = \sup_{n \in \mathbb{N}} \inf_{m \geq n} x_m$$

- **Definition**: **limit superior** of sequence x_n is

$$\limsup_{n \to \infty} x_n := \lim_{n \to \infty} \left(\sup_{m \geq n} x_m \right) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} x_m$$

- **Proposition**: let (x_n) bounded, $l \in \mathbb{R}$. The following are equivalent:
  - $l = \limsup x_n$.
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < l + \varepsilon$.
  - $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n > l - \varepsilon$.
- **Proposition**: let (x_n) bounded, $l \in \mathbb{R}$. The following are equivalent:
  - $l = \liminf x_n$.
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > l - \varepsilon$.
  - $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n < l + \varepsilon$.
- **Theorem (Bolzano-Weierstrass)**: every bounded sequence has a convergent subsequence.
- **Proposition**: let (x_n) bounded. There exists convergent subsequence with limit $\limsup x_n$ and convergent subsequence with limit $\liminf x_n$.
- **Proposition**: let (x_n) bounded, then (x_n) is convergent iff $\limsup x_n = \liminf x_n$.
- **Monotone convergence theorem for sequences**: monotone sequence converges in $\mathbb{R}$ or tends to either ∞ or $-\infty$.
- **Definition**: (x_n) is **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad |x_n - x_m| < \varepsilon$$

- **Theorem**: every Cauchy sequence in $\mathbb{R}$ is convergent.

1.4. Open and closed sets

- **Definition**: $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subseteq U$$

- **Proposition**: arbitrary unions of open sets are open. Finite intersections of open sets are open.
- **Definition**: $x \in \mathbb{R}$ is **point of closure (limit point)** for $E \subseteq \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists y \in E : |x - y| < \varepsilon$$

  Equivalently, x is point of closure of E if every open interval containing x contains another point of E.
- **Definition**: **closure** of E, $\overline{E}$, is set of points of closure. Note $E \subseteq \overline{E}$.
- **Definition**: F is **closed** if $F = \overline{F}$.
- **Proposition**: $\overline{A \cup B} = \overline{A} \cup \overline{B}$. If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$.

- **Proposition**: for any set E, $\overline{E}$ is closed, i.e. $\overline{E} = \overline{\overline{E}}$.
- **Proposition**: let $E \subseteq \mathbb{R}$. The following are equivalent:
  - E is closed.
  - $\mathbb{R} - E$ is open.
- **Proposition**: arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.
- **Definition**: collection C of subsets of $\mathbb{R}$ **covers** (is a **covering** of) $F \subseteq \mathbb{R}$ if $F \subseteq \cup_{S \in C} S$. If each S in C open, C is **open covering**. If C is finite, C is **finite covering**.
- **Definition**: covering C of F **contains a finite subcover** if exists $\{S_1, ..., S_n\} \subseteq C$ with $F \subseteq \cup_{i=1}^n S_i$ (i.e. a finite subset of C covers F).
- **Definition**: F is **compact** if any open covering of F contains a finite subcover.
- **Example**: $\mathbb{R}$ is not compact, $[a, b]$ is compact.
- **Heine-Borel theorem**: F compact iff F closed and bounded.

1.5. Continuity, pointwise and uniform convergence of functions

- **Definition**: let $E \subseteq \mathbb{R}$. $f : E \to \mathbb{R}$ is **continuous at $\boldsymbol{a} \in \boldsymbol{E}$** if

  $$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon$$

  f is **continuous** if continuous at all $y \in E$.
- **Definition**: $\lim_{x \to a} f(x) = l$ if

  $$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \Longrightarrow |f(x) - l| < \varepsilon$$
- **Proposition**: $\lim_{x \to a} f(x) = l$ iff for every sequence (a_n) with $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} f(a_n) = l$.
- **Proposition**: f is continuous at $a \in E$ iff $\lim_{x \to a} f(x) = f(a)$ (and this limit exists).
- **Definition**: $f : E \to \mathbb{R}$ is **uniformly continuous** if

  $$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in E, |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$$
- **Proposition**: let F closed and bounded, $f : F \to \mathbb{R}$ continuous. Then f is uniformly continuous.
- **Definition**: let $f_n : E \to \mathbb{R}$ sequence of functions, $f : E \to \mathbb{R}$. (f_n) **converges pointwise** to f if

  $$\forall \varepsilon > 0, \forall x \in E, \exists N \in \mathbb{N} : \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

  (f_n) **converges uniformly** to f is

  $$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$$
- **Theorem**: let $f_n : E \to \mathbb{R}$ sequence of continuous functions converging uniformly to $f : E \to \mathbb{R}$. Then f is continuous.
- **Definition**: $P = \{x_0, ..., x_n\}$ is **partition** of $[a, b]$ if $a = x_0 < \cdots < x_n = b$.
- **Definition**: $f : [a, b] \to \mathbb{R}$ is **piecewise linear** if there exists partition $P = \{x_0, ..., x_n\}$ and $m_i, c_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad f(x) = m_i x + c_i$$

f is continuous on $[a, b] - P$.

- **Definintion**: $g : [a, b] \to \mathbb{R}$ is **step function** if there exists partition $P = \{x_0, ..., x_n\}$ and $m_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i)), \quad g(x) = m_i$$

g is continuous on $[a, b] - P$.

- **Theorem**: let $f : E \to \mathbb{R}$ continuous, E closed and bounded. Then there exist continuous piecewise linear f_n with $f_n \to f$ uniformly, and step functions g_n with $g_n \to f$ uniformly.
- **Definition**: $f : E \to \mathbb{R}$ is **Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad |f(x) - f(y)| \leq C|x - y|$$

- **Definition**: $f : E \to \mathbb{R}$ is **bi-Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

1.6. The extended real numbers

- **Definition**: **extended reals** are $\mathbb{R} \cup \{-\infty, \infty\}$ with the order relation $-\infty < \infty$ and $\forall x \in \mathbb{R}, -\infty < x < \infty$. ∞ is an upper bound and $-\infty$ is a lower bound for every $x \in \mathbb{R}$, so $\sup(\mathbb{R}) = \infty$, $\inf(\mathbb{R}) = -\infty$.
  - Addition: $\forall a \in \mathbb{R}, a + \infty = \infty \land a + (-\infty) = -\infty$. $\infty + \infty = \infty - (-\infty) = \infty$. $\infty - \infty$ is undefined.
  - Multiplication: $\forall a > 0, a \cdot \infty = \infty$, $\forall a < 0, a \cdot \infty = -\infty$. Also $\infty \cdot \infty = \infty$.
  - $\limsup$ and $\liminf$ are defined as

$$\limsup x_n := \inf_{n \in \mathbb{N}} \left\{ \sup_{k \geq n} x_k \right\}, \quad \liminf x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{k \geq n} x_k \right\}$$

- **Definition**: extended real number l is **limit** of (x_n) if either
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$. Then (x_n) **converges to l**. or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$ (limit is ∞) or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$ (limit is $-\infty$).

  (x_n) **converges in the extended reals** if it has a limit in the extended reals.

2. Further analysis of subsets of $\mathbb{R}$

2.1. Countability and uncountability

- **Definition**: A is **countable** if $A = \emptyset$, A is finite or there is a bijection $\varphi : \mathbb{N} \to A$ (in which case A is **countably infinite**). Otherwise A is **uncountable**. **Enumeration** is bijection from A to $[n]$ or $\mathbb{N}$.
- **Proposition**: if surjection from countable set to A, or injection from A to countable set, then A is countable.
- **Proposition**: any subset of $\mathbb{N}$ is countable.
- **Proposition**: $\mathbb{Q}$ is countable.

- **Proposition**: show that if (a_n) is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Proposition**: show that if $(a_{n,k})$ is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Definition**: $f : X \to Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.
- **Proposition**: let f be monotone on (a, b). Then it is discountinuous on a countable set.
- **Lemma**: set of sequences in $\{0, 1\}$, $\{(x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.
- **Theorem**: $\mathbb{R}$ is uncountable.

2.2. The structure theorem for open sets

- Collection $\{A_i : i \in I\}$ of sets is **(pairwise) disjoint** if $n \neq m \Longrightarrow A_n \cap A_m = \emptyset$.
- **Structure theorem for open sets**: let $U \subseteq \mathbb{R}$ open. Then exists countable collection of disjoint open intervals $\{I_n : n \in \mathbb{N}\}$ such that $U = \cup_{n \in \mathbb{N}} I_n$.

2.3. Accumulation points and perfect sets

- **Definition**: $x \in \mathbb{R}$ is **accumulation point** of $E \subseteq \mathbb{R}$ if x is point of closure of $E - \{x\}$. Equivalently, x is a point of closure if

$$\forall \varepsilon > 0, \exists y \in E : y \neq x \land |x - y| < \varepsilon$$

  Equivalently, there exists a sequence of distinct $y_n \in E$ with $y_n \to x$ as $n \to \infty$.
- **Proposition**: set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$.
- **Proposition**: set of accumulation points E' of E is closed.
- **Definition**: $E \subseteq \mathbb{R}$ is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

- **Proposition**: E is isolated iff it has no accumulation points.
- **Definition**: bounded set E is **perfect** if it equals its set of accumulation points.
- **Theorem**: every non-empty perfect set is uncountable.

2.4. The middle-third Cantor set

- **Proposition**: let $\{F_n : n \in \mathbb{N}\}$ be collection of non-empty nested closed sets (so $F_{n+1} \subseteq F_n$), one of which is bounded. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$$

- **Definition**: the **middle third Cantor set** is defined by:
  - Define $C_0 := [0, 1]$

- Given $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$, $a_1 < b_1 < a_2 < \cdots < a_{2^n} < b_{2^n}$, with $|b_i - a_i| = 3^{-n}$, define

$$C_{n+1} := \cup_{i=1}^{2^n} \left[a_i, a_i + 3^{-(n+1)}\right] \cup \left[b_i - 3^{-(n+1)}, b_i\right]$$

which is a union of 2^{n+1} disjoint intervals, with all differences in endpoints equalling $3^{-(n+1)}$.
- The **middle third Cantor set** is

$$C := \bigcap_{n \in \mathbb{N}} C_n$$

Observe that if a is an endpoint of an interval in C_n, it is contained in C.

- **Proposition**: the middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and so uncountable.
- **Definition**: let $k \in \mathbb{N} - \{1\}$, $x \in [0, 1)$. $0.a_1 a_2 ...$, $a_i \in \{0, ..., k-1\}$, is a ***k*-ary expansion** of x if

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{k^i}$$

- **Remark**: the k-ary expansion may not be unique, but there is a countable set $E \subseteq [0, 1)$ such that every $x \in [0, 1) - E$ has a unique k-ary expansion.
- **Remark**: for every $x \in C$, the ternary ($k = 3$) expansion of x is unique and

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}$$

Moreover, every choice of sequence (a_i), $a_i \in \{0, 2\}$, gives $x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i} \in C$.
- **Definition**: **Cantor-Lebesgue function**, $g : [0, 1] \to [0, 1]$, is defined by

$$g(x) := \begin{cases} \sum_{i \in \mathbb{N}} \frac{a_i/2}{2^i} & \text{if } x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, a_i \in \{0, 2\} \\ \sup\{f(y) : y \in C, y \leq x\} & \text{if } x \notin C \end{cases}$$

g is a surjection, monotone and continuous.

2.5. G_δ, F_σ

- **Definition**: $E \subseteq \mathbb{R}$ is $\boldsymbol{G_\delta}$ if $E = \cap_{n \in \mathbb{N}} U_n$ with U_n open.
- **Definition**: $E \subseteq \mathbb{R}$ is $\boldsymbol{F_\sigma}$ if $E = \cup_{n \in \mathbb{N}} F_n$ with F_n closed.
- **Lemma**: set of points where $f : \mathbb{R} \to \mathbb{R}$ is continuous is G_δ.

3. Construction of Lebesgue measure

3.1. Lebesgue outer measure

- **Definition**: let I non-empty interval with endpoints $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$ and $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$. The **length** of I is

$$\ell(I) := b - a$$

and set $\ell(\emptyset) = 0$.

- **Definition**: let $A \subseteq \mathbb{R}$. **Lebesgue outer measure** of A is infimum of all sums of lengths of intervals covering A:

$$\mu^*(A) := \inf\left\{\sum_{k\in\mathbb{N}} \ell(I_k) : A \subseteq \bigcup_{k\in\mathbb{N}} I_k, I_k \text{ intervals}\right\}$$

  It satisfies **monotonicity**: $A \subseteq B \Longrightarrow \mu^*(A) \leq \mu^*(B)$.
- **Proposition**: outer measure is **countably subadditive**:

$$\mu^*\left(\bigcup_{k\in\mathbb{N}} E_k\right) \leq \sum_{k\in\mathbb{N}} \mu^*(E_k)$$

  This implies **finite subadditivity**:

$$\mu^*\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} \mu^*(E_k)$$

- **Lemma**: we have

$$\mu^*(A) = \inf\left\{\sum_{k\in\mathbb{N}} \ell(I_k) : A \subset \bigcup_{k\in\mathbb{N}} I_k, I_k \neq \emptyset \text{ open intervals}\right\}$$

- **Proposition**: outer measure of interval is its length: $\mu^*(I) = \ell(I)$.

3.2. Measurable sets

- **Notation**: $E^c = \mathbb{R} - E$.
- **Proposition**: let $E = (a, \infty)$. Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

- **Definition**: $E \subseteq \mathbb{R}$ is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

  Collection of such sets is $\mathcal{F}_{\mu^*}$.
- **Lemma (excision property)**: let E Lebesgue measurable set with finite measure and $E \subseteq B$, then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

- **Proposition**: if $E_1, ..., E_n$ Lebesgue measurable then $\cup_{k=1}^n E_k$ is Lebesgue measurable. If $E_1, ..., E_n$ disjoint then

$$\mu^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

  for any $A \subseteq \mathbb{R}$. In particular, for $A = \mathbb{R}$,

$$\mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k)$$

- **Remark**: not every set is Lebesgue measurable.
- **Definition**: collection of subsets of $\mathbb{R}$ is an **algebra** if contains $\emptyset$ and closed under taking complements and finite unions: if $A, B \in \mathcal{A}$ then $\mathbb{R} - A, A \cup B \in \mathcal{A}$.
- **Remark**: a union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if $\{A_k\}_{k \in \mathbb{N}}$ is countable collection of Lebesgue measurable sets, then let $A_{1'} := A_1$ and for $k > 1$, define

$$A_{k'} := A_k - \cup_{i=1}^{k-1} A_i$$

then $\{A_{k'}\}_{k \in \mathbb{N}}$ is disjoint union of Lebesgue measurable sets.
- **Proposition**: if E is countable union of Lebesgue measurable sets, then E is Lebesgue measurable. Also, if $\{E_k\}_{k \in \mathbb{N}}$ is countable disjoint collection of Lebesgue measurable sets then

$$\mu^* \left(\bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

3.3. Abstract definition of a measure

- **Definition**: let $X \subseteq \mathbb{R}$. Collection of subsets of $\mathcal{F}$ of X is **σ-algebra** if
  - $\emptyset \in \mathcal{F}$
  - $E \in \mathcal{F} \Longrightarrow E^c \in \mathcal{F}$
  - $E_1, ..., E_n \in \mathcal{F} \Longrightarrow \cup_{k \in \mathbb{N}} E_k \in \mathcal{F}$.
- **Example**:
  - Trivial examples are $\mathcal{F} = \{\emptyset, \mathbb{R}\}$ and $\mathcal{F} = \mathcal{P}(\mathbb{R})$.
  - Countable intersections of σ-algebras are σ-algebras.
- **Definition**: let $\mathcal{F}$ σ-algebra of X. $\nu : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is **measure** satisfying
  - $\nu(\emptyset) = 0$
  - $\forall E \in \mathcal{F}, \nu(E) \geq 0$
  - **Countable additivity**: if $E_1, E_2, ... \in \mathcal{F}$ are disjoint then

$$\nu \left(\bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu(E_k)$$

  Elements of $\mathcal{F}$ are **measurable** (as they are the only sets on which the measure ν is defined).
- **Proposition**: if ν is measure then it satisfies:
  - **Monotonicity**: $A \subseteq B \Longrightarrow \nu(A) \leq \nu(B)$.
  - **Countable subadditivity**: $\nu(\cup_{k \in \mathbb{N}} E_k) \leq \sum_{k \in \mathbb{N}} \nu(E_k)$.
  - **Excision**: if A has finite measure, then $A \subseteq B \Longrightarrow m(B - A) = m(B) - m(A)$.

3.4. Lebesgue measure

- **Lemma**: F_{μ^*} is σ-algebra and contains every interval.
- **Theorem (Carathéodory extension)**: restriction of the μ^* to F_{μ^*} is a measure.

- **Hahn extension theorem**: there exists unique measure μ defined on $\mathcal{F}_{\mu^*}$ for which $\mu(I) = \ell(I)$ for any interval I.
- **Definition**: the measure μ of μ^* restricted to $\mathcal{F}_{\mu^*}$ is the **Lebesgue measure**. It satisfies $\mu(I) = \ell(I)$ for any interval I and is translation invariant.

3.5. Sets of measure 0

- **Proposition**: middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.
- **Proposition**: any countable set is Lebesgue measurable and has Lebesgue measure 0.
- **Proposition**: any E with $\mu^*(E) = 0$ is Lebesgue measurable and has $\mu(E) = 0$.
- **Lemma**: let E Lebesgue measurable set with $\mu(E) = 0$, then $\forall E' \subseteq E$, E' is Lebesgue measurable.

3.6. Continuity of measure

- **Definition**: countable collection $\{E_k\}_{k\in\mathbb{N}}$ is **ascending** if $\forall k \in \mathbb{N}, E_k \subseteq E_{k+1}$ and **descending** if $\forall k \in \mathbb{N}, E_{k+1} \subseteq E_k$.
- **Theorem**: every measure m satisfies:
  - If $\{A_k\}_{k\in\mathbb{N}}$ is ascending collection of measurable sets, then

$$m\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \lim_{k\to\infty} m(A_k)$$

  - If $\{B_k\}_{k\in\mathbb{N}}$ is descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k\in\mathbb{N}} B_k\right) = \lim_{k\to\infty} m(B_k)$$

3.7. An approximation result for Lebesgue measure

- **Definition**: **Borel σ-algebra** $\mathcal{B}(\mathbb{R})$ is smallest σ-algebra containing all intervals: for any other σ-algebra $\mathcal{F}$ containing all intervals, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

$$\mathcal{B}(\mathbb{R}) := \bigcap\{\mathcal{F} : \mathcal{F}\ \sigma\text{ -algebra containing all intervals}\}$$

  $E \in \mathcal{B}(\mathbb{R})$ is **Borel** or **Borel measurable**.
- **Lemma**: all open subsets of $\mathbb{R}$, closed subsets of $\mathbb{R}$, G_δ sets and F_σ sets are Borel.
- **Proposition**: the following are equivalent:
  - E is Lebesgue measurable
  - $\forall \varepsilon > 0, \exists$ open $G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
  - $\forall \varepsilon > 0, \exists$ closed $F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
  - $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
  - $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

4. Measurable functions

4.1. Definition of a measurable function

- **Proposition**: let $f : \mathbb{R} \to \mathbb{R}$. f continuous iff $\forall$ open $U \subseteq \mathbb{R}, f^{-1}(U) \subseteq \mathbb{R}$ is open.
- **Lemma**: let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ with E Lebesgue measurable. The following are equivalent:
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) > c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) \geq c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) < c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) \leq c\}$ is Lebesgue measurable.

  The same statement holds for Borel measurable sets.
- **Definition**: $f : E \to \mathbb{R} \cup \{\pm\infty\}$ is **(Lebesgue) measurable** if it satisfies any of the above properties and if E is Lebesgue measurable. f being **Borel measurable** is defined similarly.
- **Corollary**: if f is measurable then for every $B \in \mathcal{B}(\mathbb{R})$, $f^{-1}(B)$ is measurable. In particular, if f is measurable, preimage of any interval is measurable.
- **Definition**: **indicator function** on set A, $\mathbb{1}_A : \mathbb{R} \to \{0, 1\}$, is

$$\mathbb{1}_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

- **Definition**: $\varphi : \mathbb{R} \to \mathbb{R}$ is **simple (measurable) function** if φ is measurable function that has finite codomain.

4.2. Fundamental aspects of measurable functions

- **Definition**: let $E \subseteq F \subseteq \mathbb{R}$, let $f : F \to \mathbb{R}$. **Restriction** f_E is function with domain E and for which $\forall x \in E, f_E(x) = f(x)$.
- **Definition**: real-valued function which is increasing or decreasing is **monotone**.
- **Definition**: sequence (f_n) on domain E is increasing if $f_n \leq f_{n+1}$ on E for all $n \in \mathbb{N}$.
- **Example**: continuous functions are measurable.
- **Definition**: for $f_1 : E \to \mathbb{R}, ..., f_n : E \to \mathbb{R}$, define

$$\max\{f_1, ..., f_n\}(x) := \max\{f_1(x), ..., f_n(x)\}$$

  $\min\{f_1, ..., f_n\}$ is defined similarly.
- **Proposition**: for finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E, $\max\{f_1, ..., f_n\}$ and $\min\{f_1, ..., f_n\}$ are measurable.
- **Definition**: for $f : E \to \mathbb{R}$, functions $|f|, f^+, f^-$ defined on E are

$$|f|(x) := \max\{f(x), -f(x)\}, \quad f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}$$

- **Corollary**: if f measurable on E, so are $|f|$, f^+ and f^-.
- **Proposition**: let $f : E \to \mathbb{R} \cup \{\pm\infty\}$. For measurable $D \subseteq E$, f measurable on E iff restrictions of f to D and $E - D$ are measurable.
- **Theorem**: let $f, g : E \to \mathbb{R}$ measurable.
  - **Linearity**: $\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g$ is measurable.
  - **Products**: fg is measurable.
- **Proposition**: let $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ be sequence of measurable functions that converges pointwise to $f : E \to \mathbb{R} \cup \{\pm\infty\}$. Then f is measurable.

- **Simple approximation lemma**: let $f : E \to \mathbb{R}$ measurable and bounded, so $\exists M \geq 0 : \forall x \in E, |f|(x) < M$. Then $\forall \varepsilon > 0$, there exist simple measurable functions $\varphi_\varepsilon, \psi_\varepsilon : E \to \mathbb{R}$ such that

$$\forall x \in E, \quad \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \wedge 0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$$

- **Simple approximation theorem**: let $f : E \to \mathbb{R} \cup \{\pm\infty\}$, E measurable. Then f is measurable iff there exists sequence (φ_n) of simple functions on E which converge pointwise on E to f and satisfy

$$\forall n \in \mathbb{N}, \forall x \in E, |\varphi_n|(x) \leq |f|(x)$$

  If f is nonnegative, (φ_n) can be chosen to be increasing.
- **Definition**: let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$. Then $f = g$ **almost everywhere** if $\{x \in E : f(x) \neq g(x)\}$ has measure 0.
- **Proposition**: let $f_1, f_2, f_3 : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable. If $f_1 = f_2$ almost everywhere and $f_2 = f_3$ almost everywhere then $f_1 = f_3$ almost everywhere.
- **Remark**: Lebesgue measurable functions can be modified arbitrarily on a set of measure 0 without affecting measurability.
- **Proposition**: let $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ sequence of measurable functions, $f : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable. Set of points where (f_n) converges pointwise to f is measurable.
- **Proposition**: let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable and finite almost everywhere on E.
  - **Linearity**: $\forall \alpha, \beta \in \mathbb{R}$, there exists function equal to $\alpha f + \beta g$ almost everywhere on E (any such function is measurable).
  - **Products**: there exists function equal to fg almost everywhere on E (any such function is measurable).
- **Definition**: sequence of functions (f_n) with domain E **converge in measure** to f if (f_n) and f are finite almost everywhere and

$$\forall \varepsilon > 0, \quad \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \to 0 \text{ as } n \to \infty$$

5. The Lebesgue integral

5.1. The integral of a simple measurable function

- **Definition**: let φ be real-valued function taking finitely many values $\alpha_1 < \cdots < \alpha_n$, then **standard representation** of φ is

$$\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}, \quad A_i = \varphi^{-1}(\{\alpha_i\})$$

- **Lemma**: let $\varphi = \sum_{i=1}^{m} \beta_i \mathbb{1}_{B_i}$, B_i disjoint measurable collection, $\beta_i \in \mathbb{R}$, then φ is simple measurable. If φ takes value 0 outside a set of finite measure then

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \beta_i \mu(B_i)$$

where A_i in standard representation.

- **Definition**: let φ be simple nonnegative measurable function or simple measurable function taking value 0 outside set of finite measure. **Integral** of φ with respect to μ is

$$\int \varphi = \sum_{i=1}^{n} \alpha_i \mu(A_i)$$

where $\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$ is standard representation. Here, use convention $0 \cdot \infty = 0$. For measurable $E \subseteq \mathbb{R}$, define

$$\int_E \varphi = \int \mathbb{1}_E \varphi$$

- **Example**:
  - Let $\varphi_2 = \mathbb{1}_{[0,2]} + \mathbb{1}_{[1,3]} = \mathbb{1}_{[0,1) \cup (2,3]} + 2\mathbb{1}_{[1,2]}$ so $\int \varphi_2 = 4$.
  - Let $\varphi_3 = \mathbb{1}_{\mathbb{R}}$, then $\int \varphi_3 = 1 \cdot \infty = \infty$.
  - Let $\varphi_4 = \mathbb{1}_{(0,\infty)} + (-1)\mathbb{1}_{(-\infty,0)}$. This can't be integrated.
  - Let $\varphi_5 = \mathbb{1}_{(-1,0)} + (-1)\mathbb{1}_{(0,1)}$.
- **Lemma**: let $B_1, ..., B_m$ be measurable sets, $\beta_1, ..., \beta_m \in \mathbb{R} - \{0\}$. Then $\varphi = \sum_{i=1}^{m} \beta_i \mathbb{1}_{B_i}$ is simple measurable function. Also,

$$\mu\left(\bigcup_{i=1}^{m} B_i\right) < \infty \Longrightarrow \sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \beta_i \mu(B_i)$$

where A_i in standard representation.

- **Proposition**: let φ, ψ be simple measurable functions:
  - If φ, ψ take value 0 outside a set of finite measure, then $\forall \alpha, \beta \in \mathbb{R}$,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

  - If φ, ψ nonnegative, then $\forall \alpha, \beta \geq 0$,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

  - **Monotonicity**:

$$0 \leq \varphi \leq \psi \Longrightarrow 0 \leq \int \varphi \leq \int \psi$$

- **Corollary**: let φ nonnegative simple function, then

$$\int \varphi = \sup\left\{\int \psi : 0 \leq \psi \leq \varphi,\, \psi \text{ simple measurable}\right\}$$

- **Lemma**: let φ simple measurable nonnegative function. φ takes value 0 outside a set of finite measure iff $\int \varphi < \infty$. Also, $\int \varphi = \infty$ iff there exist $\alpha > 0$, measurable A with $\mu(A) = \infty$ and $\forall x \in A, \varphi(x) \geq \alpha$.
- **Lemma**: let $\{E_n\}$ be ascending collection of measurable sets, $\cup_{n \in \mathbb{N}} E_n = \mathbb{R}$. Let φ be simple nonnegative measurable function. Then

$$\int_{E_n} \varphi \to \int \varphi \quad \text{as } n \to \infty$$

5.2. The integral of a nonnegative function

- **Notation**: let $\mathcal{M}^+$ denote collection of nonnegative measurable functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.
- **Definition**: **support** of measurable function f with domain E is $\operatorname{supp}(f) := \{x \in E : f(x) \neq 0\}$.
- **Definition**: let $f \in \mathcal{M}^+$. **Integral of $\boldsymbol{f}$ with respect to $\boldsymbol{\mu}$** is

$$\int f := \sup\left\{\int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple measurable}\right\} \in \mathbb{R} \cup \{\infty\}$$

  For measurable set E, define

$$\int_E f := \int \mathbb{1}_E f$$

- **Proposition**: let f, g measurable. If $g \leq f$ then $\int g \leq \int f$. Let E, F measurable. If $E \subseteq F$ then $\int_E f \leq \int_F f$.
- **Monotone convergence theorem**: let (f_n) be sequence in $\mathcal{M}^+$. If (f_n) is increasing on measurable set E and converges pointwise to f on E then

$$\int_E f_n \to \int_E f \quad \text{as } n \to \infty$$

- **Corollary**: restriction of integral to nonnegative functions is linear: $\forall f, g \in \mathcal{M}^+$, $\forall \alpha \geq 0$,

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

- **Fatou's lemma**: let (f_n) be sequence in $\mathcal{M}^+$, then

$$\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n$$

- **Lemma**: let $(f_n) \subset \mathcal{M}^+$, then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

- **Proposition (Chebyshev's inequality)**: let f be nonnegative measurable function on E. Then

$$\forall \lambda > 0, \quad \mu(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f$$

- **Proposition**: let f be nonnegative measurable function on E. Then

$$\int_E f = 0 \Longleftrightarrow f = 0 \text{ almost everywhere on } E$$

5.3. Integration of measurable functions

- **Notation**: let $\mathcal{M}$ denote set of measurable functions.
- **Definition**: $f \in \mathcal{M}^+$ is **integrable** if $\int f < \infty$.
- **Definition**: let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ measurable function. f is **integrable** if $\int f^+$ and $\int f^-$ are finite. In this case, for any measurable set E, define

$$\int_E f := \int_E f^+ - \int_E f^-$$

  Note that if f integrable then $f^+ - f^-$ is well-defined.
- **Proposition**: if $f = f_1 - f_2$, $f_1, f_2 \in \mathcal{M}^+$, f_1, f_2 integrable, then

$$\int f^+ - \int f^- = \int f_1 - \int f_2$$

- **Definition**: $f \in \mathcal{M}$ is **integrable over $\boldsymbol{E}$** (E is measurable) if $\int_E f^+$ and $\int_E f^-$ are finite (i.e. $f \cdot \mathbb{1}_E$ is integrable).
- **Theorem**: $f \in \mathcal{M}$ is integrable iff $|f|$ is integrable. If f integrable, then

$$\left|\int f\right| \leq \int |f|$$

- **Corollary**: let $f, g \in \mathcal{M}$, $|f| \leq |g|$. If g integrable then $|f|$ is integrable, and $\int |f| \leq \int |g|$.
- **Example**: sin is not integrable over $\mathbb{R}$, but is integrable over $[0, 2\pi]$, since $|f_{[0,2\pi]}| \leq \mathbb{1}_{[0,2\pi]}$.
- **Theorem (Linearity of Integration)**: let $f, g \in \mathcal{M}$ integrable. Then $f + g$ is integrable and $\forall \alpha \in \mathbb{R}$, αf is integrable. The integral is linear:

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

- **Dominated Convergence Theorem**: let (f_n) be sequence of integrable functions. If there exists an integrable g with $\forall n \in \mathbb{N}, |f_n| \leq g$, and $f_n \to f$ pointwise almost everywhere then f is integrable and

$$\int f = \lim_{n \to \infty} \int f_n$$

5.4. Integrability: Riemann vs Lebesgue

- **Proposition**: let f bounded function on bounded measurable domain E. Then f is measurable and $\int_E |f| < \infty$ iff

$$\sup\left\{\int_E \varphi : \varphi \leq f, \varphi \text{ simple measurable}\right\} = \inf\left\{\int_E \psi : f \leq \psi : \psi \text{ simple measurable}\right\}$$

(If f satisfies either condition then $\int_E f$ is equal to the two above expressions).

- **Definition**: bounded function f is **Lebesgue integrable** if it satisfies either of the equivalences in the above proposition.
- **Definition**: let $P = \{x_1, ..., x_n\}$ partition of $[a, b]$, $f : [a, b] \to \mathbb{R}$ bounded. **Lower and upper Darboux sums** for f with respect to P are

$$L(f, P) := \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(f, P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where

$$m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

If $P \subseteq Q$ (Q is a **refinement of $\boldsymbol{P}$**), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

- **Definition**: **lower and upper Riemann integrals** of f over $[a, b]$ are

$$\underline{\mathcal{I}}_a^b(f) := \sup\{L(f, P) : P \text{ partition of } [a, b]\}$$

$$\overline{\mathcal{I}}_a^b(f) := \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

- **Definition**: let $f : [a, b] \to \mathbb{R}$ bounded, then f is **Riemann integrable** ($f \in \mathcal{R}$), if

$$\underline{\mathcal{I}}_a^b(f) = \overline{\mathcal{I}}_a^b(f)$$

and common value $\mathcal{I}_a^b(f) = \int_a^b f(x) \, \mathrm{d}x$ is **Riemann integral** of f.

- Let $g : [a, b] \to \mathbb{R}$ step function with discontinuities at $P = \{x_0, ..., x_n\}$, so $g = \sum_{i=1}^n \alpha_i \mathbb{1}_{(x_{i-1}, x_i)}$ almost everywhere. So g is simple measurable and

$$L(g, P) = \sum_{i=1}^{n} \alpha_i(x_i - x_{i-1}) = U(g, P) = \int g = \mathcal{I}_a^b(g)$$

Hence for any bounded $f : [a, b] \to \mathbb{R}$,

$$\underline{\mathcal{I}}_a^b(f) = \sup\left\{\int \varphi : \varphi \leq f, \varphi \text{ step function}\right\},$$

$$\overline{\mathcal{I}}_a^b(f) = \inf\left\{\int \psi : f \leq \psi, \psi \text{ step function}\right\}$$

- **Theorem**: let $f : [a, b] \to \mathbb{R}$ bounded, $a, b \neq \pm\infty$. If f Riemann integrable over $[a, b]$ then f Lebesgue integrable over $[a, b]$ and the two integrals are equal.
- **Theorem**: let $f : [a, b] \to \mathbb{R}$ bounded, $a, b \neq \pm\infty$. Then f is Riemann integrable on $[a, b]$ iff f is continuous on $[a, b]$ except on a set of measure zero.
- **Lemma**: let (φ_n), (ψ_n) be sequences of functions, all integrable over E, (φ_n) increasing on E, (ψ_n) decreasing on E. Let $f : E \to \mathbb{R}$ with

$$\forall n \in \mathbb{N}, \varphi_n \leq f \leq \psi_n \text{ on } E, \quad \lim_{n \to \infty} \int_E (\psi_n - \varphi_n) = 0$$

Then $\varphi_n, \psi_n \to f$ pointwise almost everywhere on E, f is integrable over E and

$$\lim_{n \to \infty} \int_E \varphi_n = \lim_{n \to \infty} \int_E \psi_n = \int_E f$$

- **Definition**: for partition $P = \{x_0, ..., x_n\}$, **gap** of P is

$$\text{gap}(P) := \max\{|x_i - x_{i-1}| : i \in \{1, ..., n\}\}$$

- **Lemma**: let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$ be set where f is continuous. Let (P_n) be sequence of partitions of $[a, b]$ with $P_{n+1} \subseteq P_n$ and $\text{gap}(P_n) \to 0$ as $n \to \infty$. Let $\varphi_n, \psi_n : [a, b] \to \mathbb{R}$ step functions with

$$\varphi_n(x) := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad \psi_n(x) := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

for $P_n = \{x_0, ..., x_n\}$. Then $\forall x \in E - \cup_{n \in \mathbb{N}} P_n$,

$$\varphi_n(x), \psi_n(x) \to f(x) \quad \text{as } n \to \infty$$

- **Definition**: let $f : (a, b] \to \mathbb{R}$, $-\infty \leq a < b < \infty$, f bounded and Riemann integrable on all closed bounded sub-intervals of $(a, b]$. If

$$\lim_{t \to a, t > a} \mathcal{I}_t^b(f)$$

exists then this is defined as the **improper Riemann integral** $\mathcal{I}_a^b(f)$. Similar definitions exist for $f : (a, b) \to \mathbb{R}$ and $f : [a, b) \to \mathbb{R}$.

- **Note**: improper Riemann integral may exist without function being Lebesgue integral.
- **Proposition**: if f is integrable, the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.
- **Definition**: let $\alpha : [a, b] \to \mathbb{R}$ monotonically increasing (and so bounded). For partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ and bounded $f : [a, b] \to \mathbb{R}$, define

$$L(f, P, \alpha) := \sum_{i=1}^n m_i(\alpha(x_i) - \alpha(x_{i-1})), \quad U(f, P, \alpha) := \sum_{i=1}^n M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

where $m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}$, $M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$. Then f is **integrable with respect to $\boldsymbol{\alpha}$**, $f \in \mathcal{R}(\alpha)$, if

$$\inf\{U(f, P, \alpha) : P \text{ partition of } [a, b]\} = \sup\{L(f, P, \alpha) : P \text{ partition of } [a, b]\}$$

and the common value $\int_a^b f \, \mathrm{d}\alpha$ is the **Riemann-Stieltjes integral** of f with respect to α.

- **Proposition**: let $f : (a, b) \to \mathbb{R}$, then set of points where f is differentiable is measurable.
- **Remark**: if $\alpha : [0, 1] \to [a, b]$ bijection, then

$$\int_0^1 f \circ \alpha \, \mathrm{d}\alpha = \int_a^b f(x) \, \mathrm{d}x$$

- **Proposition**: let α be monotonically increasing and differentiable with $\alpha' \in \mathcal{R}$. Then $g \in \mathcal{R}(\alpha)$ iff $g\alpha' \in \mathcal{R}$, and in that case,

$$\int_a^b g \, \mathrm{d}\alpha = \int_a^b g(x)\alpha'(x) \, \mathrm{d}x$$

- **Remark**: when $g = 1$, this says $\int_a^b 1 \, \mathrm{d}\alpha = \alpha(b) - \alpha(a) = \int \alpha'(x) \, \mathrm{d}x$, similar to the fundamental theorem of calculus.

6. Lebesgue spaces

6.1. Normed linear spaces

- **Definition**: let X be **complex linear space** (vector space over $\mathbb{C}$). $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ is **norm on X** if
  - $\forall x \in X, \|x\| = 0 \iff x = 0$.
  - $\forall x \in X, \forall \lambda \in \mathbb{C}, \|\lambda x\| = |\lambda| \; \|x\|$.
  - $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.

  X equipped with norm $\|\cdot\|$, $(X, \|\cdot\|)$, is called **complex normed linear space**.
- **Example**:
  - $\|x\| = \sqrt{x\overline{x}}$ is norm on $\mathbb{C}$.
  - Let $C[a, b]$ denote linear space of continuous real-valued functions on $[a, b]$. Then

$$\|f\|_{\max} := \max\{|f(x)| : x \in [a, b]\}$$

    is norm on $C[a, b]$.
- **Proposition**: norm induces metric on X: $d(x, y) = \|x - y\|$.
- **Definition**: let $(X, \|\cdot\|)$ be normed linear space.
  - Sequence (f_n) in X is **Cauchy sequence** in X if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad \|f_n - f_m\| < \varepsilon$$

  - Sequence (f_n) in X **converges in X**, $\|f_n - f\| \to 0$ as $n \to \infty$, if

$$\exists f \in X : \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad \|f_n - f\| < \varepsilon$$

  - $(X, \|\cdot\|)$ is **complete** if every Cauchy sequence converges in X.
  - **Banach space** is complete normed linear space.
- **Proposition**: let $(X, \|\cdot\|)$ be normed linear space.
  - If (x_n) converges in X, (x_n) is Cauchy sequence in X.
  - Let (x_n) be Cauchy sequence in X. If (x_n) has convergent subsequence in X then (x_n) converges in X.

6.2. Lebesgue spaces L^p, $p \in [1, \infty)$

- **Definition**: let $p \in [1, \infty)$, $E \subseteq \mathbb{R}$.
  - Linear space $L^p(E)$ is defined as

$$L^p(E) := \left\{ f : E \to \mathbb{C} : f \text{ is measurable and } \int_E |f|^p < \infty \right\} / \cong$$

where $f \cong g$ iff $f = g$ almost everywhere:

$$f \cong g \Longleftrightarrow \exists F \subseteq E : \mu(F) = 0 \land \forall x \in E - F, f(x) = g(x)$$

- Define $\|\cdot\|_{L^p} : L^p(E) \to \mathbb{R}$ as

$$\|f\|_{L^p} := \left(\int_E |f|^p \right)^{1/p}$$

- **Remark**:
  - We often consider space $L^p(E)$ of real-valued measurable functions $f : E \to \mathbb{R}$ such that $\int_E |f|^p < \infty$.
  - For $f : E \to \mathbb{C}$, $f = f_1 + i f_2$, f is measurable iff $f_1 : E \to \mathbb{R}$ and $f_2 : E \to \mathbb{R}$ are measurable. Also,

$$\int_E |f|^p < \infty \Longleftrightarrow \left(\int_E |f_1|^p < \infty \land \int_E |f_2|^p < \infty \right)$$

- **Example**: let $E = \mathbb{R}$, $f(x) = \mathbb{1}_{\mathbb{R}-\mathbb{Q}}(x) + i\mathbb{1}_{\mathbb{Q}}(x)$ and $g(x) = 1$. Then $\mu(\mathbb{Q}) = 0$ so $f \cong g$.
- **Proposition**: let $(f_n), (g_n)$ sequences of measurable functions, $\forall n \in \mathbb{N}, f_n \cong g_n$, $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$. Then $f \cong g$.
- **Definition**: $p, q \in \mathbb{R}$ are **conjugate exponents** if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
- **Lemma (Young's inequality)**: let p, q conjugate exponents, then

$$\forall A, B \in \mathbb{R}_{\geq 0}, \quad AB \leq \frac{A^p}{p} + \frac{B^q}{q}$$

with equality iff $A^p = B^q$.
- **Lemma (Hölder's inequality)**: let p, q conjugate exponents. If $f \in L^p(E)$, $g \in L^q(E)$, then

$$\int_E |fg| \leq \|f\|_{L^p} \|g\|_{L^q}$$

- **Corollary (Cauchy-Schwarz inequality for $\boldsymbol{L^2(E)}$)**: if $f, g \in L^2(E)$, then

$$\left| \int_E f\overline{g} \right| \leq \int_E |fg| \leq \|f\|_{L^2} \|g\|_{L^2}$$

- **Lemma (Minkowski's inequality)**: let $p \in [1, \infty)$. If $f, g \in L^p(E)$ then $f + g \in L^p(E)$ and

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

- **Theorem**: for $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is normed linear space.
- **Proposition**: let $1 \leq p < q < \infty$. If $\mu(E) < \infty$ then $L^q(E) \subseteq L^p(E)$ and

$$\|f\|_{L^p} \le \mu(E)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q}$$

- **Remark**:
  - Convergence in L^p is also called convergence in the mean of order p.
  - This notion of convergence is different to pointwise convergence, uniform convergence and convergence in measure.
- **Riesz-Fischer theorem**: for $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is complete.

6.3. Lebesgue space L^∞

- **Definition**:
  - Let $f : E \to \mathbb{C}$ measurable. f is **essentially bounded** if

$$\exists M \ge 0 : |f(x)| \le M \quad \text{almost everywhere on } E$$

  - $L^\infty(E)$ is collection of equivalence classes of essentially bounded functions where $f \cong g$ iff $f = g$ almost everywhere.
  - For $f \in L^\infty(E)$, define

$$\|f\|_{L^\infty} := \operatorname{ess\,sup}|f| := \inf\{M \in \mathbb{R} : \mu(\{x \in E : |f(x)| > M\}) = 0\}$$

- **Proposition**:
  - $0 \le |f(x)| \le \|f\|_{L^\infty}$ almost everywhere.
  - $\|f\|_{L^\infty}$ is norm on $L^\infty(E)$.
  - If $f \in L^1(E)$, $g \in L^\infty(E)$, then

$$\int_E |fg| \le \|f\|_{L^1} \|g\|_{L^\infty}$$

- **Proposition**: let (f_n) sequence of functions in $L^\infty(E)$. Then (f_n) converges to $f \in L^\infty(E)$ iff there exists $G \subseteq E$ with $\mu(G) = 0$ and (f_n) converges to f uniformly on $E - G$.
- **Theorem**: $(L^\infty(E), \|\cdot\|_{L^\infty})$ is complete.
- **Remark**: if $\mu(E) < \infty$, then $L^\infty(E) \subset L^p(E)$ for $p \in [1, \infty)$ and

$$\|f\|_{L^p} \le \mu(E)^{1/p} \|f\|_{L^\infty}$$

  since

$$\|f\|_{L^p}^p = \int_E |f|^p \le \int_E \|f\|_{L^\infty}^p \cdot \mathbb{1}_E = \|f\|_{L^\infty}^p \mu(E)$$

6.4. Approximation and separability

- **Definition**: let $(X, \|\cdot\|)$ be normed linear space. Let $F \subseteq G \subseteq X$. F is **dense in G** if

$$\forall g \in G, \forall \varepsilon > 0, \exists f \in F : \quad \|f - g\| < \varepsilon$$

- **Proposition**:
  - F is dense in G iff for every $g \in G$, there exists sequence (f_n) in F such that $\lim_{n\to\infty} f_n = g$ in X.
  - For $F \subseteq G \subseteq H \subseteq X$, if F dense in G and G dense in H, then F dense in H.

- **Proposition**: let $p \in [1, \infty]$. Then subspace of simple functions in $(L^p(E), \|\cdot\|_{L^p})$ is dense in $(L^p(E), \|\cdot\|_{L^p})$.
- **Definition**: $\psi : \mathbb{R} \to \mathbb{R}$ is **step function** if it can be written as

$$\psi = \sum_{k=1}^{N} \tilde{a}_k \mathbb{1}_{(a_k, b_k)}$$

  where the intervals (a_k, b_k) are disjoint.
- **Proposition**: let $[a, b]$ be bounded, $p \in [1, \infty)$. Then subspace of step functions on $[a, b]$ is dense in $(L^p([a, b]), \|\cdot\|_{L^p})$.
- **Definition**: normed linear space $(X, \|\cdot\|)$ is **separable** if there exists countable, dense subset $X' \subseteq X$.
- **Example**: $\mathbb{R}$ is separable, since $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.
- **Theorem**: let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$. Then $(L^p(E), \|\cdot\|_{L^p})$ is separable.
- **Proposition**: let $\varepsilon > 0$, $f \in L^p(E)$, $p \in [1, \infty)$. There exists continuous $g \in L^p(E)$ such that $\|f - g\|_{L^p} < \varepsilon$.
- **Remark**: linear space of continuous functions that vanish outside bounded set is dense in $(L^p(E), \|\cdot\|_{L^p})$ for $p \in [1, \infty)$.
- **Remark**: differentiable functions are also dense in $(L^p(E), \|\cdot\|_{L^p})$ for $p \in [1, \infty)$.
- **Remark**: step functions and continuous functions are not dense in $(L^\infty(E), \|\cdot\|_{L^\infty})$.
- **Example**: in general, $(L^\infty(E), \|\cdot\|_{L^\infty})$ is not separable. Let $[a, b]$ be bounded, $a \neq b$. Assume there is countable $\{f_n : n \in \mathbb{N}\}$ which is dense in $(L^\infty([a, b]), \|\cdot\|_{L^\infty})$. Then for every $x \in [a, b]$, can choose $g(x) \in \mathbb{N}$ such that

$$\left\| \mathbb{1}_{[a,x]} - f_{g(x)} \right\|_{L^\infty} < \frac{1}{2}$$

  Also, for $x_1 \leq x_2$,

$$\left\| \mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]} \right\|_{L^\infty} = \begin{cases} 1 & \text{if } a \leq x_1 < x_2 \leq b \\ 0 & \text{if } x_1 = x_2 \end{cases}$$

  and

$$\begin{aligned} \left\| \mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]} \right\|_{L^\infty} &\leq \left\| \mathbb{1}_{[a,x_1]} - f_{g(x_1)} \right\|_{L^\infty} + \left\| f_{g(x_1)} - f_{g(x_2)} \right\|_{L^\infty} + \left\| f_{g(x_2)} - \mathbb{1}_{[a,x_2]} \right\|_{L^\infty} \\ &< 1 + \left\| f_{g(x_1)} - f_{g(x_2)} \right\|_{L^\infty} \end{aligned}$$

  If $g(x_1) = g(x_2)$ then $\left\| \mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]} \right\|_{L^\infty} = 0$ so $g : [a, b] \to \mathbb{N}$ is injective. But $\mathbb{N}$ is countable and $[a, b]$ is not countable: contradiction.

6.5. Riesz representation theorem for $L^p(E)$, $p \in [1, \infty)$

- **Definition**: let X be linear space. $T : X \to \mathbb{R}$ is **linear functional** if

$$\forall f, g \in X, \forall a, b \in \mathbb{R}, \quad T(af + bg) = aT(f) + bT(g)$$

Any linear combination of linear functionals is linear, so set of linear functionals on linear space is also linear space.

- **Definition**: let $(X, \|\cdot\|)$ be normed linear space. $T : X \to \mathbb{R}$ is **bounded functional** if

$$\exists M \geq 0 : \forall f \in X, \quad |T(f)| \leq M\|f\|$$

Norm of T, $\|T\|_*$, is the smallest such M.
- **Remark**: for bounded linear functional T on normed linear space $(X, \|\cdot\|)$,

$$|T(f) - T(g)| \leq \|T\|_* \|f - g\|$$

This gives the following continuity property: if $f_n \to f \in X$, then $T(f_n) \to T(f)$.
- **Example**: let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, q conjugate to p. Let $h \in L^q(E)$. Define $T : L^p(E) \to \mathbb{R}$ by

$$T(f) = \int_E h \cdot f$$

By Holder's inequality,

$$|T(f)| = \left|\int_E hf\right| \leq \int_E |hf| \leq \|h\|_{L^q} \|f\|_{L^p}$$

So T is bounded linear functional.
- **Remark**: we can write $\|\cdot\|_*$ as

$$\|T\|_* := \inf\{M \in \mathbb{R} : \forall f \in X, |T(f)| \leq M\|f\|\} = \sup\{|T(f)| : f \in X, \|f\| \leq 1\}$$
- **Definition**: **dual space** of X, X^*, is set of bounded linear functionals on X with norm $\|\cdot\|_*$.
- **Proposition**: let $(X, \|\cdot\|)$ be normed linear space, then dual space of X is linear space.
- **Remark**: bounded linear functional is special case of **bounded linear transformation** between normed spaces. $T : X \to Y$ is bounded linear transformation if $T(af + bg) = aT(f) + bT(g)$ and $\exists M \geq 0 : \|T(f)\|_Y \leq M\|f\|_X$.
- **Proposition**: let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, q conjugate to p, $h \in L^q(E)$. Define $T : L^p(E) \to \mathbb{R}$ by

$$T(f) = \int_E hf$$

Then $\|T\|_* = \|h\|_{L^q}$.