

# Contents

1. Monochromatic sets .....	2
1.1. Ramsey's theorem .....	2
1.2. Applications of Ramsey's theorem .....	3
1.3. Van der Waerden's theorem .....	6
1.4. The Hales-Jewett theorem .....	10
2. Partition regular systems .....	13
2.1. Rado's theorem .....	13
2.2. Ultrafilters .....	21
3. Euclidean Ramsey theory .....	26

# 1. Monochromatic sets

## 1.1. Ramsey's theorem

**Notation 1.1**  $\mathbb{N}$  denotes the set of positive integers,  $[n] = \{1, \dots, n\}$ , and  $X^{(r)} = \{A \subseteq X : |A| = r\}$ . Elements of a set are written in ascending order, e.g.  $\{i, j\}$  means  $i < j$ . Write e.g.  $ijk$  to mean the set  $\{i, j, k\}$  with the ordering (unless otherwise stated)  $i < j < k$ .

**Definition 1.2** A  $k$ -colouring on  $A^{(r)}$  is a function  $c : A^{(r)} \rightarrow [k]$ .

### Example 1.3

- Colour  $\{i, j\} \in \mathbb{N}^{(2)}$  red if  $i + j$  is even and blue if  $i + j$  is odd. Then  $M = 2\mathbb{N}$  is a monochromatic subset.
- Colour  $\{i, j\} \in \mathbb{N}^{(2)}$  red if  $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$  is even and blue otherwise.  $M = \{4^n : n \in \mathbb{N}\}$  is a monochromatic subset.
- Colour  $\{i, j\} \in \mathbb{N}^{(2)}$  red if  $i + j$  has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

**Theorem 1.4** (Ramsey's Theorem for Pairs) Let  $\mathbb{N}^{(2)}$  be 2-coloured by  $c : \mathbb{N}^{(2)} \rightarrow \{1, 2\}$ . Then there exists an infinite monochromatic subset  $M$ .

*Proof.*

- Let  $a_1 \in A_0 := \mathbb{N}$ . There exists an infinite set  $A_1 \subseteq A_0$  such that  $c(a_1, i) = c_1$  for all  $i \in A_1$ .
- Let  $a_2 \in A_1$ . There exists infinite  $A_2 \subseteq A_1$  such that  $c(a_2, i) = c_2$  for all  $i \in A_2$ .
- Repeating this inductively gives a sequence  $a_1 < a_2 < \dots < a_k < \dots$  and  $A_1 \supseteq A_2 \supseteq \dots$  such that  $c(a_i, j) = c_i$  for all  $j \in A_i$ .
- One colour appears infinitely many times:  $c_{i_1} = c_{i_2} = \dots = c_{i_k} = \dots = c$ .
- $M = \{a_{i_1}, a_{i_2}, \dots\}$  is a monochromatic set.

□

### Remark 1.5

- The same proof works for any  $k \in \mathbb{N}$  colours.
- The proof is called a “2-pass proof”.
- An alternative proof for  $k$  colours is split the  $k$  colours  $1, \dots, k$  into 2 colours: 1 and “2 or ... or  $k$ ”, and use induction.

**Note 1.6** An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

**Example 1.7** Let  $A_1 = \{1, 2\}$ ,  $A_2 = \{3, 4, 5\}$ , etc. Let  $\{i, j\}$  be red if  $i, j \in A_k$  for some  $k$ . There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

**Example 1.8** Colour  $\{i < j < k\}$  red iff  $i \mid (j + k)$ . A monochromatic subset  $M = \{2^n : n \in \mathbb{N}_0\}$  is a monochromatic set.

**Theorem 1.9** (Ramsey's Theorem for  $r$ -sets) Let  $\mathbb{N}^{(r)}$  be finitely coloured. Then there exists a monochromatic infinite set.

*Proof.*

- $r = 1$ : use pigeonhole principle.
- $r = 2$ : Ramsey's theorem for pairs.
- For general  $r$ , use induction.
- Let  $c : \mathbb{N}^r \rightarrow [k]$  be a  $k$ -colouring. Let  $a_1 \in \mathbb{N}$ , and consider all  $r - 1$  sets of  $\mathbb{N} \setminus \{a_1\}$ , induce colouring  $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \rightarrow [k]$  via  $c'(F) = c(F \cup \{a_1\})$ .
- By inductive hypothesis, there exists  $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$  such that  $c'$  is constant on it (taking value  $c_1$ ).
- Now pick  $a_2 \in A_1$  and induce a colouring  $c' : (A_1 \setminus \{a_2\})^{(r-1)} \rightarrow [k]$  such that  $c'(F) = c(F \cup \{a_2\})$ . By inductive hypothesis, there exists  $A_2 \subseteq A_1 \setminus \{a_2\}$  such that  $c'$  is constant on it (taking value  $c_2$ ).
- Repeating this gives  $a_1, a_2, \dots$  and  $A_1, A_2, \dots$  such that  $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$  and  $c(F \cup \{a_i\}) = c_i$  for all  $F \subseteq A_{i+1}$ , for  $|F| = r - 1$ .
- One colour must appear infinitely many times:  $c_{i_1} = c_{i_2} = \dots = c$ .
- $M = \{a_{i_1}, a_{i_2}, \dots\}$  is a monochromatic set.

□

## 1.2. Applications of Ramsey's theorem

**Example 1.10** In a totally ordered set, any sequence has monotonic subsequence.

*Proof.*

- Let  $(x_n)$  be a sequence, colour  $\{i, j\}$  red if  $x_i \leq x_j$  and blue otherwise.
- By Ramsey's theorem for pairs,  $M = \{i_1 < i_2 < \dots\}$  is monochromatic. If  $M$  is red, then the subsequence  $x_{i_1}, x_{i_2}, \dots$  is increasing, and is strictly decreasing otherwise.
- We can insist that  $(x_{i_j})$  is either concave or convex: 2-colour  $\mathbb{N}^{(3)}$  by colouring  $\{j < k < \ell\}$  **red** if  $(i, x_{i_j}), (j, x_{i_k}), (k, x_{i_\ell})$  form a convex triple, and **blue** if they form a concave triple. Then by Ramsey's theorem for  $r$ -sets, there is an infinite convex or concave subsequence.

□

**Theorem 1.11** (Finite Ramsey) Let  $r, m, k \in \mathbb{N}$ . There exists  $n \in \mathbb{N}$  such that whenever  $[n]^{(r)}$  is  $k$ -coloured, we can find a monochromatic set of size (at least)  $m$ .

*Proof.*

- Assume not, i.e.  $\forall n \in \mathbb{N}$ , there exists colouring  $c_n : [n]^{(r)} \rightarrow [k]$  with no monochromatic  $m$ -sets.
- There are only finitely many  $(k)$  ways to  $k$ -colour  $[r]^{(r)}$ , so there are infinitely many of colourings  $c_r, c_{r+1}, \dots$  that agree on  $[r]^{(r)}$ :  $c_i|_{[r]^{(r)}} = d_r$  for all  $i$  in some infinite set  $A_1$ , where  $d_r$  is a  $k$ -colouring of  $[r]^{(r)}$ .
- Similarly,  $[r+1]^{(r)}$  has only finitely many possible  $k$ -colourings. So there exists infinite  $A_2 \subseteq A_1$  such that for all  $i \in A_2$ ,  $c_i|_{[r+1]^{(r)}} = d_{r+1}$ , where  $d_{r+1}$  is a  $k$ -colouring of  $[r+1]^{(r)}$ .
- Continuing this process inductively, we obtain  $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n$ . There is no monochromatic  $m$ -set for any  $d_n : [n]^{(r)} \rightarrow [k]$  (because  $d_n = c_i|_{[n]^{(r)}}$  for some  $i$ ).
- These  $d_n$ 's are nested:  $d_\ell|_{[n]^{(r)}} = d_n$  for  $\ell > n$ .

- Finally, we colour  $\mathbb{N}^{(r)}$  by the colouring  $c : \mathbb{N}^{(r)} \rightarrow [k]$ ,  $c(F) = d_n(F)$  where  $n = \max(F)$  (or in fact  $n \geq \max(F)$ , which is well-defined by above). So  $c$  has no monochromatic  $m$ -set (since  $M$  was a monochromatic  $m$ -set, then taking  $\ell = \max(M)$ ,  $d_\ell$  has a monochromatic  $m$ -set), which contradicts Ramsey's Theorem for  $r$ -sets.

□

**Remark 1.12**

- This proof gives no bound on  $n = n(k, m)$ , there are other proofs that give a bound.
- It is a proof by compactness (essentially, we proved that  $\{0, 1\}^{\mathbb{N}}$  with the product topology, i.e. the topology derived from the metric  $d(f, g) = \frac{1}{\min\{n \in \mathbb{N} : f(n) \neq g(n)\}}$ , is sequentially compact).

**Remark 1.13** Now consider a colouring  $c : \mathbb{N}^{(2)} \rightarrow X$  with  $X$  potentially infinite. This does not necessarily admit an infinite monochromatic set, as we could colour each edge a different colour. Such a colouring would be injective. We can't guarantee either the colouring being constant or injective though, as  $c(ij) = i$  satisfies neither.

**Theorem 1.14** (Canonical Ramsey) Let  $c : \mathbb{N}^{(2)} \rightarrow X$  be a colouring with  $X$  an arbitrary set. Then there exists an infinite set  $M \subseteq \mathbb{N}$  such that:

1.  $c$  is constant on  $M^{(2)}$ , or
2.  $c$  is injective on  $M^{(2)}$ , or
3.  $c(ij) = c(kl)$  iff  $i = k$  for all  $i < j$  and  $k < l$ ,  $i, j, k, l \in M$ , or
4.  $c(ij) = c(kl)$  iff  $j = l$  for all  $i < j$  and  $k < l$ ,  $i, j, k, l \in M$ .

*Proof (Hints).*

- First consider the 2-colouring  $c_1$  of  $\mathbb{N}^{(4)}$  where  $ijkl$  is coloured SAME if  $c(ij) = c(kl)$  and DIFF otherwise. Show that an infinite monochromatic set  $M_1 \subseteq \mathbb{N}$  (why does this exist?) coloured SAME leads to case 1.
- Assume  $M_1$  is coloured DIFF, consider the 2-colouring of  $M_1^{(4)}$ , which colours  $ijkl$  SAME if  $c(il) = c(jk)$  and DIFF otherwise. Show an infinite monochromatic  $M_2 \subseteq M_1$  (why does this exist?) must be coloured DIFF by contradiction.
- Consider the 2-colouring of  $M_2^{(4)}$  where  $ijkl$  is coloured SAME if  $c(ik) = c(jl)$  and DIFF otherwise. Show an infinite monochromatic set  $M_3 \subseteq M_2$  (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour  $M_3^{(3)}$  by:  $ijk$  is coloured SAME if  $c(ij) = c(jk)$  and DIFF otherwise. Show an infinite monochromatic set  $M_4 \subseteq M_3$  (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour  $M_4^{(3)}$  by the other two similar colourings to above, obtaining monochromatic  $M_6 \subseteq M_5 \subseteq M_4$ .
- Consider 4 combinations of these colourings on  $M_6$ , show 3 lead to one of the cases in the theorem, and the other leads to contradiction.

□

*Proof.*

- 2-colour  $\mathbb{N}^{(4)}$  by:  $ijkl$  is red if  $c(ij) = c(kl)$  and blue otherwise. By Ramsey's Theorem for 4-sets, there is an infinite monochromatic set  $M_1 \subseteq \mathbb{N}$  for this colouring.
- If  $M_1$  is red, then  $c$  is constant on  $M_1^{(2)}$ : for all pairs  $ij, i'j' \in M_1^{(2)}$ , pick  $m < n$  with  $j, j' < m$ , then  $c(ij) = c(mn) = c(i'j')$ .
- So assume  $M_1$  is blue.
- Colour  $M_1^{(4)}$  by giving  $ijkl$  colour green if  $c(il) = c(jk)$  and purple otherwise. By Ramsey's theorem for 4-sets, there exists an infinite monochromatic  $M_2 \subseteq M_1$  for this colouring.
- Assume  $M_2$  is coloured green: if  $i < j < k < l < m < n \in M_2$ , then  $c(jk) = c(in) = c(lm)$  (consider  $ijkn$  and  $ilmn$ ): contradiction, since  $M_1$  is blue.
- Hence  $M_2$  is purple, i.e. for  $ijkl \in M_2^{(4)}$ ,  $c(il) \neq c(jk)$ .
- Colour  $M_2$  by:  $ijkl$  is orange if  $c(ik) = c(jl)$ , and pink otherwise.
- By Ramsey's theorem for 4-sets, there exists infinite monochromatic  $M_3 \subseteq M_2$  for this colouring.
- Assume  $M_3$  is orange, then for  $i < j < k < l < m < n \in M_3$ , we have  $c(jm) = c(ln)$  (consider  $jlmn$ ) and  $c(jm) = c(ik)$  (consider  $ijkm$ ): contradiction, since  $M_3 \subseteq M_1$ .
- Hence  $M_3$  is pink, i.e. for  $ijkl$ ,  $c(ik) \neq c(jl)$ .
- Colour  $M_3^{(3)}$  by:  $ijk$  is yellow if  $c(ij) = c(jk)$  and grey otherwise. By Ramsey's theorem for 3-sets, there exists infinite monochromatic  $M_4 \subseteq M_3$  for this colouring.
- Assume  $M_4$  is yellow: then (considering  $ijkl \in M_4^{(4)}$ )  $c(ij) = c(jk) = c(kl)$ : contradiction, since  $M_4 \subseteq M_1$ .
- So for any  $ijk \in M_4^{(3)}$ ,  $c(ij) \neq c(jk)$ .
- Finally, colour  $M_4^{(3)}$  by:  $ijk$  is gold if  $c(ij) = c(ik)$  and  $c(ik) = c(jk)$ , silver if  $c(ij) = c(ik)$  and  $c(ik) \neq c(jk)$ , bronze if  $c(ij) \neq c(ik)$  and  $c(ik) = c(jk)$ , and platinum if  $c(ij) \neq c(ik)$  and  $c(ik) \neq c(jk)$ .
- By Ramsey's theorem for 3-sets, there exists monochromatic  $M_5 \subseteq M_4$ .  $M_5$  cannot be gold, since then  $c(ij) = c(jk)$ : contradiction, since  $M_5 \subseteq M_4$ . If silver, then we have case 3 in the theorem. If bronze, then we have case 4 in the theorem. If platinum, then we have case 2 in the theorem.

□

### Remark 1.15

- A more general result of the above theorem states: let  $\mathbb{N}^{(r)}$  be arbitrarily coloured. Then we can find an infinite  $M$  and  $I \subseteq [r]$  such that for all  $x_1 \dots x_r \in M^{(r)}$  and  $y_1 \dots y_r \in M^{(r)}$ ,  $c(x_1 \dots x_r) = c(y_1 \dots y_r)$  iff  $x_i = y_i$  for all  $i \in I$ .
- In canonical Ramsey,  $I = \emptyset$  is case 1,  $I = \{1, 2\}$  is case 2,  $I = \{1\}$  is case 3 and  $I = \{2\}$  is case 4.
- These  $2^r$  colourings are called the **canonical colourings** of  $\mathbb{N}^{(r)}$ .

**Exercise 1.16** Prove the general statement.

### 1.3. Van der Waerden's theorem

**Remark 1.17** We want to show that for any 2-colouring of  $\mathbb{N}$ , we can find a monochromatic arithmetic progression of length  $m$  for any  $m \in \mathbb{N}$ . By compactness, this is equivalent to showing that for all  $m \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that for any 2-colouring of  $[n]$ , there exists a monochromatic arithmetic progression of length  $m$ . (If not, then for each  $n \in \mathbb{N}$ , there is a colouring  $c_n : [n] \rightarrow \{1, 2\}$  with no monochromatic arithmetic progression of length  $m$ . Infinitely many of these colourings agree on  $[1]$ , infinitely many of those agreeing in  $[1]$  agree on  $[2]$ , and so on - we obtain a 2-colouring of  $\mathbb{N}$  with no monochromatic arithmetic progression of length  $m$ ).

We will prove a slightly stronger result: whenever  $\mathbb{N}$  is  $k$ -coloured, there exists a length  $m$  monochromatic arithmetic progression, i.e. for any  $k, m \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that whenever  $[n]$  is  $k$ -coloured, we have a length  $m$  monochromatic progression.

**Definition 1.18** Let  $A_1, \dots, A_k$  be length  $m$  arithmetic progressions:  $A_i = \{a_i, a_i + d_i, \dots, a_i + (m-1)d_i\}$ .  $A_1, \dots, A_k$  are **focussed** at  $f$  if  $a_i + md_i = f$  for all  $i$ .

**Example 1.19**  $\{4, 8\}$  and  $\{6, 9\}$  are focussed at 12.

**Definition 1.20** If length  $m$  arithmetic progressions  $A_1, \dots, A_k$  are focused at  $f$  and are monochromatic, each with a different colour (for a given colouring), they are called **colour-focussed** at  $f$ .

**Remark 1.21** We use the idea that if  $A_1, \dots, A_k$  are colour-focussed at  $f$  (for a  $k$ -colouring) and of length  $m-1$ , then some  $A_i \cup \{f\}$  is a length  $m$  monochromatic arithmetic progression.

**Theorem 1.22** Whenever  $\mathbb{N}$  is  $k$ -coloured, there exists a monochromatic arithmetic progression of length 3, i.e. for all  $k \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that any  $k$ -colouring of  $[n]$  admits a length 3 monochromatic progression.

*Proof (Hints).*

- Prove by induction the claim:  $\forall r \leq k, \exists n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[n]$ , there exists a monochromatic arithmetic progression of length 3, or  $r$  colour-focussed arithmetic progressions of length 2.
  - $r = 1$  case is straightforward.
  - Let claim be true for  $r-1$  with witness  $n$ , let  $N = 2n(k^{2n} + 1)$ .
  - Partition  $N$  into blocks of equal size, show that two of these blocks must have the same colouring.
  - Using the inductive hypothesis, merge the  $r-1$  colour-focussed arithmetic progressions from these two blocks into a new set of  $r-1$  colour-focussed arithmetic progressions.
  - Find another length 2 monochromatic arithmetic progression, reason that this is of different colour.
- Reason that this claim implies the result.

□

*Proof.*

- We claim that for all  $r \leq k$ , there exists an  $n \in \mathbb{N}$  such that if  $[n]$  is  $k$ -coloured, then either:
  - There exists a monochromatic arithmetic progression of length 3.
  - There exist  $r$  colour-focussed arithmetic progressions of length 2.
- This claim implies the result by the above remark.
- We prove the claim by induction on  $r$ :
  - $r = 1$ : take  $n = k + 1$ , then by pigeonhole, some two elements of  $[n]$  have the same colour, so form a length two arithmetic progression.
  - Assume true for  $r - 1$  with witness  $n$ . We claim that  $N = 2n(k^{2n} + 1)$  works for  $r$ .
    - Let  $c : [2n(k^{2n} + 1)] \rightarrow [k]$  be a colouring. We partition  $[N]$  into  $k^{2n} + 1$  blocks of size  $2n$ :  $B_i = \{2n(i - 1) + 1, \dots, 2ni\}$  for  $i = 1, \dots, k^{2n} + 1$ .
    - Assume there is no length 3 monochromatic progression for  $c$ . By inductive hypothesis, each block  $B_i$  has  $r - 1$  colour-focussed arithmetic progressions of length 2.
    - Since  $|B_i| = 2n$ , each block also contains their focus. For a set  $M$  with  $|M| = 2n$ , there are  $k^{2n}$  ways to  $k$ -colour  $M$ . So by pigeonhole, there are blocks  $B_s$  and  $B_{s+t}$  that have the same colouring.
    - Let  $\{a_i, a_i + d_i\}$  be the  $r - 1$  arithmetic progressions in  $B_s$  colour-focussed at  $f$ , then  $\{a_i + 2nt, a_i + d_i + 2nt\}$  is the corresponding set of arithmetic progressions in  $B_{s+t}$ , each colour-focussed at  $f + 2nt$ .
    - Now  $\{a_i, a_i + d_i + 2nt\}$ ,  $i \in [r - 1]$ , are  $r - 1$  arithmetic progressions colour-focussed at  $f + 4nt$ . Also,  $\{f, f + 2nt\}$  is monochromatic of a different colour to the  $r - 1$  colours used (since there is no length 3 monochromatic progression for  $c$ ). Hence, there are  $r$  arithmetic progressions of length 2 colour-focussed at  $f + 4nt$ .

□

**Remark 1.23** The idea of looking at all possible colourings of a set is called a **product argument**.

**Definition 1.24** The **Van der Waerden** number  $W(k, m)$  is the smallest  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[n]$ , there exists a monochromatic arithmetic progression in  $[n]$  of length  $m$ .

**Remark 1.25** The above theorem gives a **tower-type** upper bound  $W(k, 3) \leq k^{k^{(\cdot)^{k^{4k}}}}$ .

**Theorem 1.26** (Van der Waerden's Theorem) For all  $k, m \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[n]$ , there is a length  $m$  monochromatic arithmetic progression.

*Proof (Hints).*

- Use induction on  $m$ .

- Given induction hypothesis on  $m - 1$ , prove the claim: for all  $r \leq k$ , there exists  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[n]$ , we have either a monochromatic length  $m$  arithmetic progression, or  $r$  colour-focussed arithmetic progressions of length  $m - 1$ . Reason that this claim implies the result.
- Use induction on  $r$ . Give an explicit  $n$  for  $r = 1$ .
- Let  $n$  be the witness for  $r - 1$ , let  $N = W(k^{2n}, m - 1) \cdot 2n$ . Assume a  $k$ -colouring of  $[N]$ ,  $c : [N] \rightarrow [k]$ , has no arithmetic progressions of length  $m$ .
- Partition  $[N]$  into the obvious choice of  $W(k^{2n}, m - 1)$  blocks  $B_i$ , each of length  $2n$ .
- Colour the indices  $1 \leq i \leq W(k^{2n}, m - 1)$  of the blocks by

$$c'(i) = (c(2n(i - 1) + 1), c(2n(i - 1) + 2), \dots, c(2ni))$$

- Reason that we can find monochromatic arithmetic progression  $s, s + t, \dots, s + (m - 2)t$  of length  $m - 1$  (w.r.t  $c'$ ), and that this corresponds to sequence of blocks  $B_s, B_{s+t}, \dots, B_{s+(m-2)t}$ , each identically coloured.
- Reason that  $B_s$  contains  $r - 1$  colour-focussed length  $m - 1$  arithmetic progressions  $A_i$  together with their focus  $f$ .
- Let  $A'_i$  be the same arithmetic progression but with common difference  $2nt$  larger than that of  $A_i$ . Show the  $A'_i$  are colour-focussed at some focus in terms of  $f$ .
- Find another length  $m - 1$  arithmetic progression, show this must be monochromatic and of different colour to all  $A'_i$ . Show it also has same focus as all  $A'_i$ .

□

*Proof.*

- By induction on  $m$ .  $m = 1$  is trivial,  $m = 2$  is by pigeonhole principle.  $m = 3$  is the statement of the previous theorem.
- Assume true for  $m - 1$  and all  $k \in \mathbb{N}$ .
- For fixed  $k$ , we prove the claim: for all  $r \leq k$ , there exists  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[n]$ , either:
  - There is a monochromatic arithmetic progression of length  $m$ , or
  - There are  $r$  colour-focussed arithmetic progressions of length  $m - 1$ .
- We will then be done (by considering the focus).
- To prove the claim, we use induction on  $r$ .
- $r = 1$  is the claim of the first inductive hypothesis: take  $n = W(k, m - 1)$ .
- Assume the claim holds for  $r - 1$  with witness  $n$ , and assume there is no monochromatic arithmetic progression of length  $m$ . We will show that  $N = W(k^{2n}, m - 1)2n$  is sufficient for  $r$ .
- Partition  $[N]$  into  $W(k^{2n}, m - 1)$  blocks of length  $2n$ :  $B_i = \{2n(i - 1) + 1, \dots, 2ni\}$  for  $i = 1, \dots, W(k^{2n}, m - 1)$ .
- Each block has  $k^{2n}$  possible colourings. Colour the blocks as

$$c'(i) = (c(2n(i - 1) + 1), c(2n(i - 1) + 2), \dots, c(2ni))$$



By definition of  $W$ , there exists a monochromatic arithmetic progression of length  $m - 1$  (w.r.t. to  $c'$ ):  $\{\alpha, \alpha + t, \dots, \alpha + (m - 2)t\}$ . The respective blocks  $B_\alpha, \dots, B_{\alpha + (m-2)t}$  are identically coloured.

- $B_\alpha$  has length  $2n$ , so by induction  $B_\alpha$  contains  $r - 1$  colour-focussed arithmetic progressions of length  $m - 1$ , together with their focus (as length of block is  $2n$ ).
- Let  $A_1, \dots, A_{r-1}$ ,  $A_i = \{a_i, a_i + d_i, \dots, a_i + (m - 2)d_i\}$ , be colour-focussed at  $f$ .
- Let  $A'_i = \{a_i, a_i + (d_i + 2nt), \dots, a_i + (m - 2)(d_i + 2nt)\}$  for  $i = 1, \dots, r - 1$ . The  $A'_i$  are monochromatic as the blocks are identically coloured and the  $A_i$  are monochromatic. Also,  $A_i$  and  $A'_i$  have the same colouring, and the  $A_i$  are colour-focussed, hence the  $A'_i$  have pairwise distinct colours.
- The  $A_i$  are focussed at  $f$  and the colour of  $f$  is different than the colour of all  $A_i$ .  $f = a_i + (m - 1)d_i$  for all  $i$ .
- Now  $\{f, f + 2nt, f + 4nt, \dots, f + 2n(m - 2)t\}$  is an arithmetic progression of length  $m - 1$ , is monochromatic and of a different colour to all the  $A'_i$ .
- It is enough to show that  $a_i + (m - 1)(d_i + 2nt) = f + 2n(m - 1)t$  for all  $i$ , but this is equivalent to  $a_i + (m - 1)d_i = f$ , which is true as all  $A_i$  were focussed at  $f$ .

□

**Corollary 1.27** For any  $k$ -colouring of  $\mathbb{N}$ , there exists a colour class containing arbitrarily long arithmetic progressions.

**Remark 1.28** We can't guarantee infinitely long arithmetic progressions, e.g.

- 2-colour  $\mathbb{N}$  by 1 red, 2, 3 blue, 4, 5, 6 red, etc.
- The set of infinite arithmetic progressions in  $\mathbb{N}$  is countable (since described by two integers: the start term and step). Enumerate them by  $(A_k)_{k \in \mathbb{N}}$ . Pick  $x_1 < y_1 \in A_1$ , colour  $x_1$  red and  $y_1$  blue. Then pick  $x_2, y_2 \in A_2$  with  $y_1 < x_2 < y_2$ , colour  $x_2$  red,  $y_2$  blue. Continue inductively.

**Theorem 1.29** (Strengthened Van der Waerden) Let  $m, k \in \mathbb{N}$ . There exists  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[n]$ , there exists a monochromatic length  $m$  arithmetic progression whose common difference is the same colour (i.e. there exists  $a, a + d, \dots, a + (m - 1)d$  all of the same colour).

*Proof (Hints).*

- Use induction on  $k$ .
- If  $n$  is the witness for  $k - 1$  colours, show that  $N = W(k, n(m - 1) + 1)$  is a witness for  $k$  colours, by considering  $n$  different multiples of the step of a suitable arithmetic progression.

□

*Proof.*

- Fix  $m \in \mathbb{N}$ . We use induction on  $k$ .  $k = 1$  case is trivial.
- Let  $n$  be witness for  $k - 1$  colours.
- We will show that  $N = W(k, n(m - 1) + 1)$  is suitable for  $k$  colours.
- If  $[N]$  is  $k$ -coloured, there exists a monochromatic (say red) arithmetic progression of length  $n(m - 1) + 1$ :  $a, a + d, \dots, a + n(m - 1)d$ .

- If  $rd$  is red for any  $1 \leq r \leq n$ , then we are done (consider  $a, a + rd, \dots, a + (m - 1)rd$ ).
- If not, then  $\{d, 2d, \dots, nd\}$  is  $k - 1$ -coloured, which induces a  $k - 1$  colouring on  $[n]$ . Therefore, there exists a monochromatic arithmetic progression  $b, b + s, \dots, b + (m - 1)s$  (with  $s$  the same colour) by induction, which translates to  $db, db + ds, \dots, db + d(m - 1)s$  and  $ds$  being monochromatic.

□

**Remark 1.30** The case  $m = 2$  of strengthened Van der Waerden is **Schur's theorem**: for any  $k$ -colouring of  $\mathbb{N}$ , there are monochromatic  $x, y, z$  such that  $x + y = z$ . This can be proved directly from Ramsey's theorem for pairs: let  $c : \mathbb{N} \rightarrow [k]$  be a  $k$ -colouring, then induce  $c' : \mathbb{N}^{(2)} \rightarrow [k]$  by  $c'(ij) = c(j - i)$ . By Ramsey, there exist  $i < j < k$  such that  $c'(ij) = c'(ik) = c'(jk)$ , i.e.  $c(j - i) = c(k - i) = c(k - j)$ . So take  $x = j - i, z = k - i, y = k - j$ .

## 1.4. The Hales-Jewett theorem

**Definition 1.31** Let  $X$  be finite set. We say  $X^n$  consists of **words of length  $n$  on alphabet  $X$** .

**Definition 1.32** Let  $X$  be finite. A **(combinatorial) line** in  $X^n$  is a set  $L \subseteq X^n$  of the form

$$L = \{(x_1, \dots, x_n) \in X^n : \forall i \notin I, x_i = a_i \text{ and } \forall i, j \in I, x_i = x_j\}$$

for some non-empty set  $I \subseteq [n]$  and  $a_i \in X$  (for each  $i \notin I$ ).  $I$  is the set of **active coordinates** for  $L$ .

Note that a combinatorial line is invariant under permutations of  $X$ .

**Example 1.33** Let  $X = [3]$ . Some lines in  $X^2$  are:

- $I = \{1\}$ :  $\{(1, 1), (2, 1), (3, 1)\}$  (with  $a_2 = 1$ ),  $\{(1, 2), (2, 2), (3, 2)\}$  (with  $a_2 = 2$ ),  $\{(1, 3), (2, 3), (3, 3)\}$  (with  $a_2 = 3$ ).
- $I = \{2\}$ :  $\{(1, 1), (1, 2), (1, 3)\}$  (with  $a_1 = 1$ ),  $\{(2, 1), (2, 2), (2, 3)\}$  (with  $a_1 = 2$ ),  $\{(3, 1), (3, 2), (3, 3)\}$  (with  $a_1 = 3$ ).
- $I = \{1, 2\}$ :  $\{(1, 1), (2, 2), (3, 3)\}$ .

Note that  $\{(1, 3), (2, 2), (3, 1)\}$  is **not** a combinatorial line.

**Example 1.34** Some sets of lines in  $[3]^3$  are:

- $I = \{1\}$ :  $\{(1, 2, 3), (2, 2, 3), (3, 2, 3)\}$  (with  $a_2 = 2, a_3 = 3$ ).
- $I = \{1, 3\}$ :  $\{(1, 3, 1), (2, 3, 2), (3, 3, 3)\}$  (with  $a_2 = 3$ ).

**Definition 1.35** In a line  $L$ , write  $L^-$  and  $L^+$  for the smallest and largest points in  $L$  (with respect to the ordering on  $[m]^n$  where  $x \leq y$  if  $x_i \leq y_i$  for all  $i$ ).

**Definition 1.36** Lines  $L_1, \dots, L_k$  are **focussed** at  $f$  if  $L_i^+ = f$  for all  $i \in [k]$ . They are **colour-focussed** if they are focussed and  $L_i \setminus \{L_i^+\}$  is monochromatic for all  $i \in [k]$ , with each  $L_i \setminus \{L_i^+\}$  a different colour.

**Theorem 1.37** (Hales-Jewett) Let  $m, k \in \mathbb{N}$  (we use alphabet  $X = [m]$ ), then there exists  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[m]^n$ , there exists a monochromatic combinatorial line.

**Notation 1.38** Denote the smallest such  $n$  by  $\text{HJ}(m, k)$ .

*Proof (Hints).*

- Induction on  $m$ . Prove by induction the claim that for all  $1 \leq r \leq k$ , there exists  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[m]^n$ , we have either a monochromatic line, or  $r$  colour-focussed lines (reason that this claim implies the result).
- State why claim holds for  $r = 1$ .
- Let  $n$  be witness for  $r - 1$ ,  $n' = \text{HJ}(m - 1, k^{m^n})$ . Want to show that  $n + n'$  is witness for  $r$ .
- Write  $[m]^{n+n'} = [m]^n \times [m]^{n'}$ .
- For a colouring  $c : [m]^{n+n'} \rightarrow [k]$ , induce a suitable colouring  $c' : [m]^{n'} \rightarrow [k]^{m^n}$  and consider what the definition of  $n'$  implies. Use this to induce a colouring  $c'' : [m]^n \rightarrow [k]$ .
- Using the inductive hypothesis and the previous point, construct  $r - 1$  lines in  $[m]^{n+n'}$  which are colour-focussed. Find another line in  $[m]^{n+n'}$  (which should have first  $n$  coordinates constant) of different colour which has the same focus point.

□

*Proof.* By induction on  $m$ . The case  $m = 1$  is trivial as  $|[m]^n| = 1$ . Assume that  $\text{HJ}(m - 1, k')$  exists for all  $k' \in \mathbb{N}$ . We claim that for all  $1 \leq r \leq k$ , there exists  $n \in \mathbb{N}$  such that for any  $k$ -colouring of  $[m]^n$ , we have either:

- a monochromatic line, or
- $r$  colour-focussed lines.

We can then take  $r = k$  and consider the focus.

We prove the claim by induction on  $r$ . For  $r = 1$ ,  $n = \text{HJ}(m - 1, k)$  suffices. Let  $n$  be a witness for  $r - 1$ . Let  $n' = \text{HJ}(m - 1, k^{m^n})$ . We will show  $N = n + n'$  is a witness for  $r$ . Let  $c : [m]^N \rightarrow [k]$  be a  $k$ -colouring with no monochromatic lines. Writing  $[m]^N = [m]^n \times [m]^{n'}$ , colour  $[m]^{n'}$  by  $c' : [m]^{n'} \rightarrow [k]^{m^n}$ ,  $c'(b) = (c(a_1, b), \dots, c(a_{m^n}, b))$  (where  $[m]^n = \{a_1, \dots, a_{m^n}\}$ ). By the inductive hypothesis, there exists a line  $L$  in  $[m]^{n'}$  with active coordinates  $I$  such that

$$\forall a \in [m]^n, \forall b, b' \in L \setminus \{L^+\}, \quad c(a, b) = c(a, b').$$

But now this induces a (well-defined) colouring  $c'' : [m]^n \rightarrow [k]$ ,  $c''(a) = c(a, b)$  for any  $b \in L \setminus \{L^+\}$ . By definition of  $n$ , there exist  $r - 1$  lines  $L_1, \dots, L_{r-1}$  colour-focussed (w.r.t  $c''$ ) at  $f$ , with active coordinates  $I_1, \dots, I_{r-1}$ .

Finally, consider the  $r - 1$  lines  $L'_i$ ,  $1 \leq i \leq r - 1$  in  $[m]^N$  that start at  $(L_i^-, L^-)$  with active coordinates  $I_i \cup I$ , and the line  $L'$  in  $[m]^N$  that starts at  $(f, L^-)$  with active coordinates  $I$ . By the construction of  $c''$ , the colour of each point in  $L'_i$  is determined by the first  $n$  coordinates which form a point lying in  $L_i$ . Hence, since the  $L_i$  are

colour-focussed, the  $L'_i$  are colour-focussed. As for  $L'$ , the first  $n$  coordinates are constant (always equal to  $f$ ), and so again by the construction of  $c''$ , the colour of each point in  $L'$  is equal to  $c''(f)$ , which is a different colour to each colour of the  $L'_i$ . Hence all  $L'_1, \dots, L'_{r-1}, L'$  colour-focussed at  $(f, L^+)$ , so we are done.  $\square$

**Corollary 1.39** Hales-Jewett implies Van der Waerden's theorem.

*Proof (Hints).* For a colouring  $c : \mathbb{N} \rightarrow [k]$ , consider the induced colouring  $c'(x_1, \dots, x_n) = c(x_1 + \dots + x_n)$  of  $[m]^n$ .  $\square$

*Proof.* Let  $c$  be a  $k$ -colouring of  $\mathbb{N}$ . For sufficiently large  $n$  (i.e.  $n \geq \text{HJ}(m, k)$ ), induce a  $k$ -colouring  $c'$  of  $[m]^n$  by  $c'(x_1, \dots, x_n) = c(x_1 + \dots + x_n)$ . By Hales-Jewett, a monochromatic (with respect to  $c'$ ) combinatorial line  $L$  exists. This gives a monochromatic (with respect to  $c$ ) length  $m$  arithmetic progression in  $\mathbb{N}$ . The step is equal to the number of active coordinates. The first term in the arithmetic progression corresponds to the point in  $L$  with all active coordinates equal to 1, the last term corresponds to the point in  $L$  with all active coordinates equal to  $m$ .  $\square$

**Exercise 1.40** Show that the  $m$ -in-a-row noughts and crosses game cannot be a draw in sufficiently high dimensions, and that the first player can always win.

**Definition 1.41** A  **$d$ -dimensional subspace** (or  **$d$ -point parameter set**)  $S \subseteq X^n$  is a set such that there exist pairwise disjoint  $I_1, \dots, I_d \subseteq [n]$  and  $a_i \in X$  for all  $i \in [n] - (I_1 \cup \dots \cup I_d)$ , such that

$$S = \{x \in X^n : x_i = a_i \quad \forall i \in [n] - (I_1 \cup \dots \cup I_d), \\ \text{and } x_i = x_j \quad \forall i, j \in I_k \text{ for some } k \in [d]\}.$$

**Example 1.42** Two 2-dimensional subspaces in  $X^3$  are  $\{(x, y, 2) : x, y \in X\}$  ( $I_1 = \{1\}, I_2 = \{2\}$ ) and  $\{(x, x, y) : x, y \in X\}$  ( $I_1 = \{1, 2\}, I_2 = \{3\}$ ).

**Theorem 1.43** (Extended Hales-Jewett) For all  $m, k, d \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that for any colouring of  $[m]^n$ , there exists a monochromatic  $d$ -dimensional subspace.

*Proof (Hints).* Use Hales-Jewett on  $m^d$  and  $k$ .  $\square$

*Proof.* We can view  $X^{dn'}$  as  $(X^d)^{n'}$ . A line in  $(X^d)^{n'}$  (on alphabet  $Y = X^d$ ) corresponds to a  $d$ -dimensional subspace in  $X^{dn'}$  (on alphabet  $X$ ). (Each inactive coordinate in the line corresponds to  $d$  adjacent inactive coordinates in the subspace, and each active coordinate in the line corresponds to  $d$  adjacent active coordinates in the subspace). Hence, we can take  $n = d \cdot \text{HJ}(m^d, k)$ .  $\square$

**Definition 1.44** Let  $S \subseteq \mathbb{N}^d$  be finite. A **homothetic copy** of  $S$  is a set of the form  $a + \lambda S$  where  $a \in \mathbb{N}^d$  and  $\lambda \in \mathbb{N}$  ( $\lambda \neq 0$ ).

**Theorem 1.45** (Gallai) Let  $S \subseteq \mathbb{N}^d$  be finite. For every  $k$ -colouring of  $\mathbb{N}^d$ , there exists a monochromatic homothetic copy of  $S$ .

*Proof (Hints).* Let  $S = \{S_1, \dots, S_m\}$ , consider colouring  $c' : [m]^n \rightarrow [k]$  (for suitable  $n$ ) given by  $c'(x_1, \dots, x_n) = c(S_{x_1}, \dots, S_{x_m})$ .  $\square$

*Proof.* Let  $S = \{S_1, \dots, S_m\}$ . Let  $c : \mathbb{N}^d \rightarrow [k]$  be a  $k$ -colouring. For  $n$  large enough (i.e.  $n \geq \text{HJ}(m, k)$ ), colour  $[m]^n$  by  $c'(x_1, \dots, x_n) = c(S_{x_1} + \dots + S_{x_n})$ . By Hales-Jewett, there exists a monochromatic line (with respect to  $c'$ ) in  $[m]^n$  with active coordinates  $I$ . So  $c\left(\sum_{i \notin I} S_i + |I|S_j\right)$  is the same colour for all  $j \in [m]$ . So we are done, as  $\sum_{i \notin I} S_i + |I|S$  is a homothetic copy of  $S$ .  $\square$

**Remark 1.46**

- Gallai's theorem can also be proven with a focussing + product colouring argument.
- For  $S = \{(x, y) \in \mathbb{N}^2 : x, y \in \{1, 2\}\}$ , Gallai's theorem proves the existence of a monochromatic square whereas extended Hales-Jewett only guarantees a monochromatic rectangle.

## 2. Partition regular systems

### 2.1. Rado's theorem

Strengthened Van der Waerden says that the system  $x_1 + x_2 = y_1, x_1 + 2x_2 = y_2, \dots, x_1 + mx_2 = y_m$  has a monochromatic solution in  $x_1, x_2, y_1, \dots, y_m$ . We want to find when a general system of equations is partition regular.

**Definition 2.1** Let  $A \in \mathbb{Q}^{m \times n}$  be a  $m \times n$  matrix.  $A$  is **partition regular (PR)** if for any finite colouring of  $\mathbb{N}$ , there exists a monochromatic  $\mathbf{x} \in \mathbb{N}^n$  such that  $A\mathbf{x} = \mathbf{0}$ .

**Example 2.2**

- Schur's theorem says that  $x + y = z$  has a monochromatic solution for any finite colouring of  $\mathbb{N}$ , and so that  $(1, 1, -1)$  is PR.
- Strengthened Van der Waerden states that

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

is PR.

- $(a, b, -(a + b))$  is PR for any  $a, b$  (a monochromatic solution is  $x = y = z$ ).
- $(2, -1)$  is not PR: colour  $\mathbb{N}$  by  $n$  is **red** if  $\max\{m \in \mathbb{N} : 2^m \mid n\}$  is even, and **blue** otherwise. Then if  $2x = y$ ,  $x$  and  $y$  must have different colours.

**Definition 2.3** A rational matrix  $A$  with columns  $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{Q}^m$  has the **column property (CP)** if there exists a partition  $B_1 \sqcup \dots \sqcup B_r$  of  $[n]$  such that:

1.  $\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0}$ .
2. For all  $s \in \{2, \dots, r\}$ ,  $\sum_{i \in B_s} \mathbf{c}_i \in \text{span}\{\mathbf{c}_j : j \in B_1 \sqcup \dots \sqcup B_{s-1}\}$  (note we can take the linear span over  $\mathbb{R}$  or over  $\mathbb{Q}$  here, as if a rational vector is a real linear combination of rational vectors, then it is also a rational linear combination of them).

**Example 2.4**

- $(1, 1, -1)$  has CP, with  $B_1 = \{1, 3\}$ ,  $B_2 = \{2\}$ .
- The matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

from Strengthened Van der Waerden has CP, with  $B_1 = \{1, 3, \dots, n\}$  and  $B_2 = \{2\}$ .

- $(3, 4, -7)$  has CP with  $B_1 = \{1, 2, 3\}$ .
- $(\lambda, -1)$  has CP iff  $\lambda = 1$ .
- $(3, 4, -6)$  doesn't have CP.

### Example 2.5

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & a \\ 4 & -4 & b \end{bmatrix}$$

has CP iff  $(a, b) = (6, 12)$ .

**Remark 2.6**  $\mathbf{x} = (a_1, \dots, a_n)$  is PR iff  $\lambda \mathbf{x}$  is PR (for any  $\lambda \in \mathbb{Q}^\times$ ), so we can assume that each  $a_i \in \mathbb{Z}$ . Also,  $\mathbf{x}$  has CP iff there exists  $\emptyset \neq I \subseteq [n]$  such that  $\sum_{i \in I} a_i = 0$ .

We may also assume WLOG each  $a_i \neq 0$ . We will first show that if  $\mathbf{x}$  is PR, then it has CP. Even in the  $1 \times n$  matrix case of Rado's theorem, neither direction is easy.

**Notation 2.7** For  $p$  prime and  $x = (a_k \dots a_0)_p \in \mathbb{N}$ , write  $e(x)$  for the rightmost non-zero digit in the base- $p$  expansion of  $x$ , i.e.  $e(x) = a_{t(x)}$ , where  $t(x) = \min\{i : a_i \neq 0\}$ .

**Proposition 2.8** Let  $a_1, \dots, a_n \in \mathbb{Q}^*$ . If  $(a_1, \dots, a_n)$  is PR, then it has CP.

*Proof (Hints).* For  $p$  large enough (determine later a bound for  $p$ ), colour  $\mathbb{N}$  by giving  $x$  colour  $e(x)$ , and consider  $\min\{t(x_1), \dots, t(x_n)\}$ .  $\square$

*Proof.* Let  $p$  be a large prime ( $p > \sum_{i=1}^n |a_i|$ ). Define a  $(p-1)$ -colouring of  $\mathbb{N}$  giving  $x$  colour  $e(x)$ . By assumption, there are  $x_1, \dots, x_n$  of the same colour  $d$  such that  $\sum_{i=1}^n a_i x_i = 0$ . Let  $t = \min\{t(x_1), \dots, t(x_n)\}$ , and let  $I = \{i \in [n] : t(x_i) = t\}$  (note  $I$  is non-empty). So when summing  $\sum_{i=1}^n a_i x_i = 0$  and considering the last digit in the base  $p$  expansion, we have  $\sum_{i=1}^n a_i x_i = 0 \pmod{p^{t+1}}$  and so obtain  $\sum_{i \in I} a_i d = 0 \pmod{p}$ , so  $\sum_{i \in I} a_i = 0$  (since  $p$  is prime and was chosen large enough).  $\square$

**Remark 2.9** There is no other known proof of this proposition.

**Lemma 2.10** Let  $\lambda \in \mathbb{Q}$ . Then  $(1, \lambda, -1)$  is partition regular, i.e. for any finite colouring of  $\mathbb{N}$ , there exists monochromatic  $(x, y, z) \in \mathbb{N}^3$  such that  $x + \lambda y = z$ .

*Proof (Hints).*

- Reason that we can assume  $\lambda > 0$ . Write  $\lambda = r/s$ ,  $r, s \in \mathbb{N}$ .
- Use induction on number of colours  $k$ : given  $n$  such that any  $(k-1)$ -colouring of  $[n]$  admits monochromatic solution, show that  $N = W(k, nr+1)ns$  works for  $k$  colours, by considering the definition of  $W$  and  $isd$  for each  $i \in [n]$ .

□

*Proof.* The case  $\lambda = 0$  is trivial, and if  $\lambda < 0$ , we may rewrite the equation as  $z - \lambda y = x$ , so we may assume that  $\lambda > 0$ , so let  $\lambda = \frac{r}{s}$  for  $r, s \in \mathbb{N}$ . In fact, we show that for any  $k$ -colouring of  $[n]$  (for some  $n$  depending on  $k$ ), there is a monochromatic solution.

We seek a monochromatic solution to  $x + \frac{r}{s}y = z$  for some finite colouring  $c : \mathbb{N} \rightarrow [k]$ . We use induction on the number of colours  $k$ . For  $k = 1$ ,  $n = \max\{s, r + 1\}$  is sufficient, with monochromatic solution  $(1, s, r + 1)$ . Assume  $n$  is a witness for  $k - 1$  colours. We will show  $N = nsW(k, nr + 1)$  is suitable for  $k$  colours. By definition of  $W$ , given a  $k$ -colouring of  $[N]$ , there is a monochromatic AP inside  $[W(k, nr + 1)] \subseteq [N]$  of length  $nr + 1$ :  $a, a + d, \dots, a + nrd$ , coloured red.

Consider  $isd$  for each  $i \in [n]$ . Note that  $isd \leq nsW(k, nr + 1)$  so each  $isd$  does indeed have a colour. If some  $isd$  is also red, then  $(a, isd, a + ird)$  is a monochromatic solution. If no  $isd$  is red, then  $\{sd, \dots, nsd\}$  is  $(k - 1)$ -coloured, so by the inductive hypothesis, there exists  $i, j, k \in [n]$  such that  $\{isd, jsd, ksd\}$  is monochromatic and  $isd + \lambda jsd = ksd$ , so  $(isd, jsd, ksd)$  is a monochromatic solution. □

### Remark 2.11

- Note the similarity to the proof of Strengthened Van der Waerden.
- The case  $\lambda = 1$  is Schur's theorem, which can be proven directly by Ramsey's theorem; however, there is no known proof using Ramsey's theorem for general  $\lambda \in \mathbb{Q}$ .

**Theorem 2.12** (Rado's Theorem for Single Equations) Let  $a_1, \dots, a_n \in \mathbb{Q} \setminus \{0\}$ .  $(a_1, \dots, a_n)$  is PR iff it has CP.

*Proof (Hints).* For  $\Leftarrow$ : for the obvious choice of  $I \subseteq [n]$ , fix  $i_0 \in I$ , and define  $\mathbf{x} \in \mathbb{N}^n$  componentwise:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}.$$

Show that  $\mathbf{x}$  is a solution to  $\sum_{i=1}^n a_i x_i = 0$ . □

*Proof.*  $\Rightarrow$  is by [Proposition 2.8](#). For  $\Leftarrow$ : we have that  $\sum_{i \in I} a_i = 0$  for some  $\emptyset \neq I \subseteq [n]$ . Given a colouring  $c : \mathbb{N} \rightarrow [k]$ , we need to show that there are monochromatic  $x_1, \dots, x_n$  such that  $\sum_{i=1}^n a_i x_i = 0$ .

Fix  $i_0 \in I$ . We construct the following vector  $\mathbf{x} \in \mathbb{N}^n$  by defining its components:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}$$

for some fixed suitable  $x, y, z$ . We need  $x, y, z$  to be monochromatic and

$$\begin{aligned}
& a_{i_0}x + \sum_{i \notin I} a_i y + \sum_{i \in I \setminus \{i_0\}} a_i z = 0 \\
& \iff a_{i_0}x - za_{i_0} + \sum_{i \notin I} a_i y = 0 \\
& \iff x + \frac{\sum_{i \notin I} a_i}{a_{i_0}} y - z = 0
\end{aligned}$$

and this holds, since  $x, y, z$  exist by the above lemma.  $\square$

**Conjecture 2.13** (Rado's Boundedness Conjecture) Let  $A$  be an  $m \times n$  matrix that is not PR (so there exists a “bad” colouring, i.e. a  $k$ -colouring with no monochromatic solution to  $A\mathbf{x} = \mathbf{0}$  for some  $k \in \mathbb{N}$ ). Is  $k$  bounded (for given  $m, n$ )?

This is known for  $1 \times 3$  matrices: 24 colours suffice.

**Proposition 2.14** Let  $A \in \mathbb{Q}^{m \times n}$ . If  $A$  is PR, then it has CP.

*Proof (Hints).*

- Let  $\mathbf{x} \in \mathbb{N}^n$  be the monochromatic solution to  $A\mathbf{x} = \mathbf{0}$ . For fixed prime  $p$ , partition  $[n]$  into  $B_1, \dots, B_r$  by grouping  $i, j \in [n]$  by  $t(x_i), t(x_j)$  (and preserving the ordering).
- Reason that the same partition exists for infinitely many  $p$ .
- Considering  $\sum_{i=1}^n x_i \mathbf{c}_i = \mathbf{0} \pmod{p}$  for infinitely many  $p$ , show that  $\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0}$ , and

$$p^t \sum_{i \in B_k} \mathbf{c}_i + \sum_{i \in B_1, \dots, B_{k-1}} x_i d^{-1} \mathbf{c}_i \equiv \mathbf{0} \pmod{p^{t+1}}.$$

- By taking the dot product with  $\mathbf{u} \in \mathbb{N}^m$  for appropriate  $u$ , show by contradiction that  $\sum_{i \in B_k} \mathbf{c}_i \in \text{span}\{\mathbf{c}_i : i \in B_1, \dots, B_{k-1}\}$ .

$\square$

*Proof.* Let  $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{Q}^m$  be the columns of  $A$ . For fixed prime  $p$ , colour  $\mathbb{N}$  as before by  $c(x) = e(x)$ . By assumption, there exists a monochromatic  $\mathbf{x} \in \mathbb{N}^n$  such that  $\sum_{i=1}^n x_i \mathbf{c}_i = \mathbf{0}$ . We partition the columns (by partitioning  $[n] = B_1 \sqcup \dots \sqcup B_r$ ) as follows:

- $i, j \in B_k$  iff  $t(x_i) = t(x_j)$ .
- $i \in B_k, j \in B_\ell$  for  $k < \ell$  iff  $t(x_i) < t(x_j)$ .

We do this for infinitely many primes  $p$ . Since there are finitely many partitions of  $[n]$ , for infinitely many  $p$ , we will have the same blocks  $B_1, \dots, B_r$ .

Consider  $\sum_{i=1}^n x_i \mathbf{c}_i = \mathbf{0}$  performed in base  $p$ . Each  $i \in [n]$  has the same colour  $d = e(x_i) \in [1, p-1]$ . So  $\sum_{i \in B_1} d \mathbf{c}_i = \mathbf{0} \pmod{p}$  (by collecting the rightmost terms in base  $p$ ), hence  $\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0} \pmod{p}$ . But this holds for infinitely many  $p$ , hence

$$\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0}.$$



Now  $\sum_{i \in B_k} p^t d c_i + \sum_{i \in B_1, \dots, B_{k-1}} x_i c_i = \mathbf{0} \pmod{p^{t+1}}$  for some  $t$ . So

$$p^t \sum_{i \in B_k} c_i + \sum_{i \in B_1, \dots, B_{k-1}} x_i d^{-1} c_i \equiv \mathbf{0} \pmod{p^{t+1}}.$$

We claim that  $\sum_{i \in B_k} c_i \in \text{span}\{c_i : i \in B_1, \dots, B_{k-1}\}$ . Suppose not, then there exists  $\mathbf{u} \in \mathbb{N}^m$  such that  $\mathbf{u} \cdot c_i = 0$  for all  $i \in B_1, \dots, B_{k-1}$ , but  $\mathbf{u} \cdot \left(\sum_{i \in B_k} c_i\right) \neq 0$ . Then dotting with  $\mathbf{u}$ , we obtain  $p^t \mathbf{u} \cdot \left(\sum_{i \in B_k} c_i\right) \equiv 0 \pmod{p^{t+1}}$ , so  $\mathbf{u} \cdot \sum_{i \in B_k} c_i \equiv 0 \pmod{p}$ . But this holds for infinitely many  $p$ , so  $\mathbf{u} \cdot \sum_{i \in B_k} c_i = 0$ : contradiction.  $\square$

**Definition 2.15** For  $m, p, c \in \mathbb{N}$ , an  $(m, p, c)$ -set  $S \subseteq \mathbb{N}$  with generators  $x_1, \dots, x_m \in \mathbb{N}$  is of the form

$$S = \left\{ \sum_{i=1}^m \lambda_i x_i : \exists j \in [m] : \lambda_j = c, \lambda_i = 0 \ \forall i < j, \text{ and } \lambda_k \in [-p, p] \ \forall k > j \right\}$$

where  $[-p, p] = \{-p, -(p-1), \dots, p\}$ . So  $S$  consists of

$$\begin{aligned} cx_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots + \lambda_m x_m, & \quad \lambda_i \in [-p, p], \\ cx_2 + \lambda_3 x_3 + \dots + \lambda_m x_m, & \quad \lambda_i \in [-p, p], \\ & \quad \vdots \\ cx_m. & \end{aligned}$$

These are the **rows** of  $S$ . We can think of  $S$  as a “progression of progressions”.

**Example 2.16**

- A  $(2, p, 1)$ -set with generators  $x_1, x_2$  is of the form  $\{x_1 - px_2, x_1 - (p-1)x_2, \dots, x_1 + px_2, x_2\}$ , so is an AP of length  $2p+1$  together with its step.
- A  $(2, p, 3)$ -set with generators  $x_1, x_2$  is of the form  $\{3x_1 - px_2, 3x_1 - (p-1)x_2, \dots, 3x_1 + px_2, 3x_2\}$ , so is an AP of length  $2p+1$ , whose middle term is divisible by 3, together with three times its step.

**Theorem 2.17** Let  $m, p, c \in \mathbb{N}$ . For any finite colouring of  $\mathbb{N}$ , there exists a monochromatic  $(m, p, c)$ -set.

*Proof (Hints).*

- Reason that an  $(m', p, c)$ -set contains an  $(m, p, c)$ -set for  $m' \geq m$ . With  $M = k(m-1) + 1$ , reason that if we can find an  $(M, p, c)$ -set with each row monochromatic, then we can find an monochromatic  $(m, p, c)$ -set.
- Let  $A_1 = \{c, 2c, \dots, \lfloor n/c \rfloor c\}$ , reason that  $A_1$  contains a set of the form  $R_1 = \{cx_1 - n_1 d_1, cx_1 - (n_1 - 1)d_1, \dots, cx_1 + n_1 d_1\}$  for some large  $n_1$ .
- Let  $B_1 = \left\{d_1, 2d_1, \dots, \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1\right\}$ . We have  $cx_1 + \lambda_1 b_1 + \dots + \lambda_{M-1} b_{M-1} \in R_1$ , explain why these are monochromatic.
- Inside  $B_1$ , define

$$A_2 = \left\{cd_1, 2cd_1, \dots, \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1\right\}.$$

and apply the argument as before, where the divisor in the  $\lfloor \cdot \rfloor$  expression in the new  $B_2$  is  $(M-2)p$ .

- Argue that after a certain number of steps, we have formed an  $(M, p, c)$ -set with each row monochromatic.

□

*Proof.* Let  $c : \mathbb{N} \rightarrow [k]$  be the colouring of  $\mathbb{N}$  with  $k$  colours. Note that an  $(m', p, c)$ -set with  $m' \geq m$  contains an  $(m, p, c)$ -set (by taking any  $m$  rows, and setting some suitable  $\lambda_i$  to 0). Let  $M = k(m-1) + 1$ . It is enough to find a  $(M, p, c)$ -set such that each row is monochromatic.

Let  $n$  be large (large enough to apply the argument that follows). Let  $A_1 = \{c, 2c, \dots, \lfloor n/c \rfloor c\}$ . By Van der Waerden,  $A_1$  contains a monochromatic AP  $R_1$  of length  $2n_1 + 1$  where  $n_1$  is large enough:

$$R_1 = \{cx_1 - n_1d_1, cx_1 - (n_1 - 1)d_1, \dots, cx_1 + n_1d_1\}.$$

has colour  $k_1$ . Now we restrict our attention to

$$B_1 = \left\{ d_1, 2d_1, \dots, \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1 \right\}.$$

Observe that

$$cx_1 + \lambda_1 b_1 + \dots + \lambda_{M-1} b_{M-1} \in R_1$$

for all  $\lambda_i \in [-p, p]$  and  $b_i \in B_1$ , so all these sums have colour  $k_1$ . Inside  $B_1$ , look at

$$A_2 = \left\{ cd_1, 2cd_1, \dots, \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1 \right\}.$$

By Van der Waerden,  $A_2$  contains a monochromatic AP  $R_2$  of length  $2n_2 + 1$  with colour  $k_2$ :

$$R_2 = \{cx_2 - n_2d_2, cx_2 - (n_2 - 1)d_2, \dots, cx_2 + n_2d_2\}.$$

Note that  $x_2 \subseteq B_1$ . Now we restrict our attention to

$$B_2 = \left\{ d_2, 2d_2, \dots, \left\lfloor \frac{n_2}{(M-2)p} \right\rfloor d_2 \right\}.$$

Again, note that for all  $\lambda_i \in [-p, p]$  and  $b_i \in B_2$ , we have

$$cx_2 + \lambda_1 b_1 + \dots + \lambda_{M-2} b_{M-2} \in R_2$$

so has colour  $k_2$ .

We iterate this process  $M$  times, and obtain  $M$  generators  $x_1, \dots, x_M$  such that each row of the  $(M, p, c)$ -set generated by  $x_1, \dots, x_M$  is monochromatic. But now  $M = k(m-1) + 1$ , so  $m$  of the rows have the same colour. □

**Remark 2.18** Being extremely precise in this proofs (such as considering  $\lfloor \cdot \rfloor$ ) is much less important than the ideas in the proof. (Won't be penalised in the exam for small details like this).

**Corollary 2.19** (Folkman's Theorem) Let  $m \in \mathbb{N}$  be fixed. For every finite colouring of  $\mathbb{N}$ , there exists  $x_1, \dots, x_m \in \mathbb{N}$  such that

$$\text{FS}(x_1, \dots, x_m) := \left\{ \sum_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}$$

is monochromatic.

*Proof (Hints).* A specific case of [Theorem 2.17](#). □

*Proof.* By the  $(m, 1, 1)$  case of [Theorem 2.17](#). □

**Remark 2.20**

- The case  $n = 2$  of Folkman's theorem is Schur's theorem.
- For a colouring  $c : \mathbb{N} \rightarrow [k]$ , we induce a colouring  $c' : \mathbb{N} \rightarrow [k]$  by  $c'(n) = c(2^n)$ . Then by Folkman's theorem for  $c'$ , there exists  $x_1, \dots, x_m$  such that

$$\text{FP}(x_1, \dots, x_m) = \left\{ \prod_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}.$$

- It is not known whether the same result holds for  $\text{FS}(x_1, \dots, x_m) \cup \text{FP}(x_1, \dots, x_m)$ . However, it does not hold for infinite sets  $\{x_n : n \in \mathbb{N}\}$ , and does hold for colourings of  $\mathbb{Q}$ .

**Proposition 2.21** Let  $A$  have CP. Then there exist  $m, p, c \in \mathbb{N}$  such that every  $(m, p, c)$ -set contains a solution  $\mathbf{y}$  to  $A\mathbf{y} = \mathbf{0}$ , i.e. all  $y_i$  belong to the  $(m, p, c)$ -set.

*Proof.* Let  $\mathbf{c}_1, \dots, \mathbf{c}_n$  be the columns of  $A$ . By assumption, there is a partition  $B_1 \sqcup \dots \sqcup B_r$  of  $[n]$  such that  $\forall k \in [r]$ ,

$$\begin{aligned} \sum_{i \in B_k} \mathbf{c}_i &\in \text{span}\{\mathbf{c}_i : i \in B_1 \cup \dots \cup B_{k-1}\} \\ \Rightarrow \sum_{i \in B_k} \mathbf{c}_i &= \sum_{i \in B_1 \cup \dots \cup B_{k-1}} q_{ik} \mathbf{c}_i \quad \text{for some } q_{ik} \in \mathbb{Q} \\ \Rightarrow \sum_{i=1}^n d_{ik} \mathbf{c}_i &= \mathbf{0} \end{aligned}$$

where

$$d_{ik} = \begin{cases} 0 & \text{if } i \notin B_1 \cup \dots \cup B_{k-1} \\ 1 & \text{if } i \in B_k \\ -q_{ik} & \text{if } i \in B_1 \cup \dots \cup B_{k-1} \end{cases}.$$

Take  $m = r$ . Let  $x_1, \dots, x_r \in \mathbb{N}$ , and let  $y_i = \sum_{k=1}^r d_{ik} x_k$  for each  $i \in [n]$ . Now  $\mathbf{y} = (y_1, \dots, y_n)$  is a solution to  $A\mathbf{y} = \mathbf{0}$ : we have

$$\begin{aligned}
\sum_{i=1}^n y_i c_i &= \sum_{i=1}^n \sum_{k=1}^r d_{ik} x_k c_i \\
&= \sum_{k=1}^r x_k \sum_{i=1}^n d_{ik} c_i = \mathbf{0}.
\end{aligned}$$

Let  $c$  be the LCD of all the  $q_{ik}$ . Now  $cy_i = \sum_{k=1}^r cd_{ik}x_k$  is an integral linear combination of the  $x_k$ , and  $c\mathbf{y}$  is a solution since  $\mathbf{y}$  is. Let  $p$  be  $c$  times maximum of the absolute values of the numerators of the  $q_{ik}$ . By definition of the  $d_{ik}$ ,  $c\mathbf{y}$  is in the  $(m, p, c)$ -set generated by  $x_1, \dots, x_r$ .  $\square$

**Theorem 2.22** (Rado)  $A \in \mathbb{Q}^{m \times n}$  is PR iff it has CP.

*Proof.*  $\Rightarrow$  is by [Proposition 2.14](#). For  $\Leftarrow$ , let  $c' : \mathbb{N} \rightarrow [k]$  be a finite colouring of  $\mathbb{N}$ . Also, by the above proposition, since  $A$  has CP, there exists  $m, p, c \in \mathbb{N}$  such that  $A\mathbf{x} = \mathbf{0}$  has a solution  $\mathbf{x}$  in any  $(m, p, c)$ -set by the above theorem. By [Theorem 2.17](#), there is a monochromatic  $(m, p, c)$ -set with respect to  $c'$ . This gives a monochromatic solution  $\mathbf{x}$  to  $A\mathbf{x} = \mathbf{0}$ .  $\square$

**Remark 2.23** From the proof of [Rado's Theorem](#), we obtain that if  $A$  is PR for the “mod  $p$ ” colourings, then it is PR for *any* colouring. There is no proof of this fact that is more direct than using Rado’s theorem.

**Theorem 2.24** (Consistency) Let  $A$  and  $B$  be rational PR matrices. Then the matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

is PR.

*Proof (Hints).* [Rado's Theorem](#).  $\square$

*Proof.* This is a trivial check of the CP given the CP of  $A$  and  $B$ , then we are done by [Rado's Theorem](#).  $\square$

**Remark 2.25** The [Consistency Theorem](#) says that if we can find monochromatic solutions  $\mathbf{x}$  and  $\mathbf{x}'$  to  $A\mathbf{x} = \mathbf{0}$  and  $B\mathbf{y} = \mathbf{0}$ , then we can find monochromatic solutions  $\mathbf{x}'$  and  $\mathbf{y}'$ , of the same colour, to  $A\mathbf{x}' = \mathbf{0}$  and  $B\mathbf{y}' = \mathbf{0}$ .

**Theorem 2.26** For any finite colouring of  $\mathbb{N}$ , some colour class contains solutions to *all* PR equations.

*Proof (Hints).* Use the [Consistency Theorem](#).  $\square$

*Proof.* For a given  $k$ -colouring of  $\mathbb{N}$ , let  $\mathbb{N} = C_1 \sqcup \dots \sqcup C_k$  be the colour classes. Assume the contrary, so for each  $1 \leq i \leq k$ , there exists a PR matrix  $A_i$  such that  $A_i\mathbf{x} = \mathbf{0}$  has no monochromatic solution of the same colour as  $C_i$ . But then by inductively applying the consistency theorem, the matrix

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

has a monochromatic solution of the same colour as some  $C_j$ . But then  $C_j \mathbf{x} = \mathbf{0}$  has a solution  $\mathbf{x}$  of the same colour as  $C_j$ : contradiction.  $\square$

## 2.2. Ultrafilters

**Definition 2.27** A **filter** on  $\mathbb{N}$  is a non-empty collection  $\mathcal{F}$  of subsets of  $\mathbb{N}$  such that:

- $\emptyset \notin \mathcal{F}$ ,
- If  $A \in \mathcal{F}$  and  $A \subseteq B$ , then  $B \in \mathcal{F}$ , i.e.  $\mathcal{F}$  is an **up-set**.
- If  $A, B \in \mathcal{F}$  then  $A \cap B \in \mathcal{F}$ , i.e.  $\mathcal{F}$  is closed under finite intersections.

A filter is a notion of “large” subsets of  $\mathbb{N}$ .

### Example 2.28

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$  is a filter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$  is a filter.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$  is a filter, called the **cofinite filter**.
- $\mathcal{F}_4 = \{A \subseteq \mathbb{N} : A \text{ infinite}\}$  is not a filter, since it contains  $2\mathbb{N}$  and  $2\mathbb{N} + 1$  but not  $\emptyset = (2\mathbb{N}) \cap (2\mathbb{N} + 1)$ .
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$  is a filter.

**Definition 2.29** An **ultrafilter** is a maximal filter.

**Definition 2.30** For  $x \in \mathbb{N}$ , the **principal ultrafilter at  $x$**  is

$$\mathcal{U}_x := \{A \subseteq \mathbb{N} : x \in A\}.$$

**Proposition 2.31** The principal ultrafilter at  $x$  is indeed an ultrafilter.

*Proof (Hints).* Straightforward.  $\square$

*Proof.* If  $B \notin \mathcal{U}_x$ , then  $x \in B^c$  so  $B^c \in \mathcal{U}_x$ , but  $B^c \cap B = \emptyset$ , so  $\mathcal{U}_x \cup \{B\}$  is not a filter.  $\square$

### Example 2.32

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$  is an ultrafilter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$  is not an ultrafilter as  $\mathcal{F}_1$  extends it.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$  is not an ultrafilter, as  $\mathcal{F}_5$  extends it.
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$  is not an ultrafilter, as  $\{A \subseteq \mathbb{N} : 4\mathbb{N} \setminus A \text{ finite}\}$  extends it.

**Proposition 2.33** A filter  $\mathcal{F}$  is an ultrafilter iff for all  $A \subseteq \mathbb{N}$ , either  $A \in \mathcal{F}$  or  $A^c \in \mathcal{F}$ .

*Proof (Hints).*  $\Leftarrow$ : straightforward.  $\Rightarrow$ : show if  $A \notin \mathcal{F}$ , then  $\exists B \in \mathcal{F}$  such that  $A \cap B = \emptyset$ .  $\square$

*Proof.*  $\Leftarrow$ : since  $A \cap A^c = \emptyset \notin \mathcal{F}$ .

$\implies$ : let  $\mathcal{F}$  is an ultrafilter. We cannot have  $A, A^c \in \mathcal{F}$  as  $A \cap A^c = \emptyset \notin \mathcal{F}$ . Suppose there is  $A \subseteq \mathbb{N}$  such that  $A, A^c \notin \mathcal{F}$ . By maximality of  $\mathcal{F}$ , since  $A \notin \mathcal{F}$ , then  $\exists B \in \mathcal{F}$  such that  $A \cap B = \emptyset$  (suppose not, then  $\mathcal{F}' = \{S \subseteq \mathbb{N} : S \supseteq A \cap B \text{ for some } B \in \mathcal{F}\}$  extends  $\mathcal{F}$ ). Similarly,  $\exists C \in \mathcal{F}$  such that  $A^c \cap C = \emptyset$ . So we have  $C \subseteq A$ , so  $B \cap C = \emptyset \notin \mathcal{F}$ : contradiction (or also  $C \subseteq A \implies A \in \mathcal{F}$ : contradiction).  $\square$

**Corollary 2.34** Let  $\mathcal{U}$  be an ultrafilter and  $A = B \cup C \in \mathcal{U}$ . Then  $B \in \mathcal{U}$  or  $C \in \mathcal{U}$ .

*Proof (Hints).* Straightforward.  $\square$

*Proof.* If not, then  $B^c, C^c \in \mathcal{U}$  by [Proposition 2.33](#), hence  $B^c \cap C^c = (B \cup C)^c = A^c \in \mathcal{U}$ : contradiction.  $\square$

**Proposition 2.35** Every filter is contained in an ultrafilter.

*Proof (Hints).* Use Zorn's Lemma.  $\square$

*Proof.* Let  $\mathcal{F}_0$  be a filter. By Zorn's Lemma, it is enough to show that every non-empty chain of filters has an upper bound. Let  $\{\mathcal{F}_i : i \in I\}$  be a chain of filters in the poset of filters containing  $\mathcal{F}_0$ , partially ordered by inclusion, and set  $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$ .

- $\emptyset \notin \mathcal{F}$  since  $\emptyset \notin \mathcal{F}_i$  for each  $i \in I$ .
- If  $A \in \mathcal{F}$  and  $A \subseteq B$ , then  $A \in \mathcal{F}_i$  for some  $i \in I$ , so  $B \in \mathcal{F}_i$ , so  $B \in \mathcal{F}$ .
- Let  $A, B \in \mathcal{F}$ , so  $A \in \mathcal{F}_i$  and  $B \in \mathcal{F}_j$  for some  $i, j$ . WLOG,  $\mathcal{F}_i \subseteq \mathcal{F}_j$ , so  $A \cap B \in \mathcal{F}_j$ , so  $A \cap B \in \mathcal{F}$ .

$\mathcal{F}$  is an upper bound for the chain, so we are done.  $\square$

**Proposition 2.36** Let  $\mathcal{U}$  be an ultrafilter. Then  $\mathcal{U}$  is non-principal iff  $\mathcal{U}$  extends the cofinite filter  $\mathcal{F}_C$ .

*Proof (Hints).*  $\Leftarrow$ : straightforward.  $\Rightarrow$ : use [Corollary 2.34](#).  $\square$

*Proof.*  $\Leftarrow$ : if  $\mathcal{U} = \mathcal{U}_x$  is principal, then we have  $\{x\} \in \mathcal{U}$  so  $\{x\}^c \notin \mathcal{U}$  by [Proposition 2.33](#), but also  $\{x\}^c \in \mathcal{F}_C$ : contradiction.

$\Rightarrow$ : let  $A \in \mathcal{F}_C$ , so  $A^c = \{a_1, \dots, a_k\}$  is finite. Assume  $A \notin \mathcal{U}$ , then  $A^c \in \mathcal{U}$ , so by [Corollary 2.34](#), some  $a_i \in \mathcal{U}$ . But then by definition of a filter, each set containing  $a_i$  is in  $\mathcal{U}$ , so  $\mathcal{U}$  is principal: contradiction.  $\square$

**Notation 2.37** Let  $\beta\mathbb{N}$  denote the set of all ultrafilters on  $\mathbb{N}$ .

**Definition 2.38** Define a topology on  $\beta\mathbb{N}$  by its base (basis), which consists of

$$C_A := \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$$

for each  $A \subseteq \mathbb{N}$ . The sets above indeed form a base: we have  $\bigcup_{A \subseteq \mathbb{N}} C_A = \beta\mathbb{N}$ , and  $C_A \cap C_B = C_{A \cap B}$ , since  $A \cap B \in \mathcal{U}$  iff  $A, B \in \mathcal{U}$ . The open sets are of the form  $\bigcup_{i \in I} C_{A_i}$  and the closed sets are of the form  $\bigcap_{i \in I} C_{A_i}$ .

**Remark 2.39**  $\beta\mathbb{N} \setminus C_A = C_{A^c}$ , since  $A \notin \mathcal{U}$  iff  $A^c \in \mathcal{U}$ . We can view  $\mathbb{N}$  as being embedded in  $\beta\mathbb{N}$  by identifying  $n \in \mathbb{N}$  with  $\tilde{n} := \mathcal{U}_n$ , the principal ultrafilter at  $n$ . Each point in  $\mathbb{N}$  under this correspondence is isolated in  $\beta\mathbb{N}$ , since  $C_{\{n\}} = \{\tilde{n}\}$  is an

open neighbourhood of  $\tilde{n}$ . Also,  $\mathbb{N}$  is dense in  $\beta\mathbb{N}$ , since for every  $n \in A$ ,  $\tilde{n} \in C_A$ , so every non-empty open set in  $\beta\mathbb{N}$  intersects  $\mathbb{N}$ .

**Theorem 2.40**  $\beta\mathbb{N}$  is a compact Hausdorff topological space.

*Proof.* Hausdorff: let  $\mathcal{U} \neq \mathcal{V}$  be ultrafilters, so there is  $A \in \mathcal{U}$  such that  $A \notin \mathcal{V}$ . But then  $A^c \in \mathcal{V}$ , so  $\mathcal{U} \in C_A$ ,  $\mathcal{V} \in C_{A^c}$ , and  $C_A \cap C_{A^c}$  is open.

Compact: it is compact iff every open admits a finite subcover iff a collection of open sets such that no finite subcollection covers  $\beta\mathbb{N}$ , they don't cover  $\beta\mathbb{N}$  iff for every collection of closed sets such that they have finite intersection property ( $(F_i)_{i \in I}$ ,  $\cap_{i \in J} F_i \neq \emptyset$  for all  $J$  finite), then their intersection is non-empty. We can assume each  $F_i$  is a basis set, i.e.  $F_i = C_{A_i}$  for some  $A_i \in \mathbb{N}$ . Suppose  $\{C_{A_i} : i \in I\}$  have the finite intersection property. First,  $C_{A_{i_1}} \cap \dots \cap C_{A_{i_k}} = C_{A_{i_1} \cap \dots \cap A_{i_k}} \neq \emptyset$ , hence  $\bigcap_{j=1}^k A_{i_j} \neq \emptyset$ . So let  $\mathcal{F} = \{A : A \supseteq A_{i_1} \cap \dots \cap A_{i_k} \text{ for some } A_{i_1}, \dots, A_{i_k}\}$ . We have  $\emptyset \notin \mathcal{F}$ , if  $B \supseteq A \in \mathcal{F}$  then  $B \in \mathcal{F}$ , and if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ . Hence  $\mathcal{F}$  is a filter.  $\mathcal{F}$  extends to an ultrafilter  $\mathcal{U}$ . Note that  $(\forall i, A_i \in \mathcal{U}) \iff (\mathcal{U} \in C_{A_i} \forall i)$ . So  $\mathcal{U} \in \cap C_{A_i}$ , so  $\cap C_{A_i} \neq \emptyset$ .  $\square$

**Remark 2.41**

- $\beta\mathbb{N}$  can be viewed as a subset of  $\{0, 1\}^{\mathbb{P}(\mathbb{N})}$  (so each ultrafilter is viewed as a function  $\mathbb{P}(\mathbb{N}) \rightarrow \{0, 1\}$ ). The topology on  $\beta\mathbb{N}$  is the restriction of the product topology on  $\{0, 1\}^{\mathbb{P}(\mathbb{N})}$ . Also,  $\beta\mathbb{N}$  is a closed subset of  $\{0, 1\}^{\mathbb{P}(\mathbb{N})}$ , so is compact by Tychonov's theorem (TODO: look up statement of this theorem).
- $\beta\mathbb{N}$  is the largest compact Hausdorff topological space in which (the embedding of)  $\mathbb{N}$  is dense. In other words, if  $X$  is compact and Hausdorff, and  $f : \mathbb{N} \rightarrow X$ , there exists a unique continuous  $\tilde{f} : \beta\mathbb{N} \rightarrow X$  extending  $f$ . TODO: insert diagram.
- $\beta\mathbb{N}$  is called the **Stone-Čech compactification** of  $\mathbb{N}$ .

**Definition 2.42** Let  $p$  be a statement and  $\mathcal{U}$  be an ultrafilter.  $\forall_{\mathcal{U}} x p(x)$  to mean  $\{x \in \mathbb{N} : p(x)\} \in \mathcal{U}$  and say  $p(x)$  “for most  $x$ ” or “for  $\mathcal{U}$ -most  $x$ ”.

**Example 2.43**

- For  $\mathcal{U} = \tilde{n}$ , we have  $\forall_{\mathcal{U}} x p(x)$  iff  $p(n)$ .
- For non-principal  $\mathcal{U}$ , we have  $\forall_{\mathcal{U}} x (x > 4)$  (if not, then  $\{1, 2, 3\} = \{x \in \mathbb{N} : x > 4\}^c \in \mathcal{U}$ , so  $\{i\} \in \mathcal{U}$  for some  $i = 1, 2, 3$ , so  $\mathcal{U}$  is principal: contradiction).

**Proposition 2.44** Let  $\mathcal{U}$  be an ultrafilter and  $p, q$  be statements. Then

1.  $\forall_{\mathcal{U}} x (p(x) \wedge q(x))$  iff  $(\forall_{\mathcal{U}} x p(x)) \wedge (\forall_{\mathcal{U}} x q(x))$ .
2.  $\forall_{\mathcal{U}} x (p(x) \vee q(x))$  iff  $(\forall_{\mathcal{U}} x p(x)) \vee (\forall_{\mathcal{U}} x q(x))$ .
3.  $\neg(\forall_{\mathcal{U}} x p(x))$  iff  $\forall_{\mathcal{U}} x (\neg p(x))$ .

*Proof.* Let  $A = \{x \in \mathbb{N} : p(x)\}$  and  $B = \{x \in \mathbb{N} : q(x)\}$ . We have

1.  $A \cap B \in \mathcal{U}$  iff  $A \in \mathcal{U}$  and  $B \in \mathcal{U}$  by definition.
2.  $A \cup B \in \mathcal{U}$  iff  $A \in \mathcal{U}$  and  $B \in \mathcal{U}$  by (find result).
3.  $A \notin \mathcal{U}$  iff  $A^c \in \mathcal{U}$  by (find result).

$\square$

**Note 2.45**  $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y p(x, y)$  is not necessarily the same as  $\forall_{\mathcal{V}}y\forall_{\mathcal{U}}x p(x, y)$ , even when  $\mathcal{U} = \mathcal{V}$ . For example, let  $\mathcal{U}$  be non-principal, and  $p(x, y) = (x < y)$ . Then  $\forall_{\mathcal{U}}x(\forall_{\mathcal{U}}y (x < y))$  is true, as every  $x$  satisfies  $\forall_{\mathcal{U}}y (x < y)$ . But  $\forall_{\mathcal{U}}y\forall_{\mathcal{U}}x (x < y)$  is false, as no  $y$  has  $\forall_{\mathcal{U}}x (x < y)$ . So **don't swap quantifiers!**

**Definition 2.46** Given ultrafilters  $\mathcal{U}, \mathcal{V}$ , define their sum to be

$$\mathcal{U} + \mathcal{V} := \{A \subseteq \mathbb{N} : \forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in A)\}.$$

**Example 2.47** We have  $\tilde{m} + \tilde{n} = \widetilde{m + n}$ .

**Proposition 2.48** For any ultrafilters  $\mathcal{U}$  and  $\mathcal{V}$ ,  $\mathcal{U} + \mathcal{V}$  is an ultrafilter.

*Proof.* We have  $\emptyset \notin \mathcal{U} + \mathcal{V}$ . If  $A \in \mathcal{U} + \mathcal{V}$  and  $A \subseteq B$ , then  $B \in \mathcal{U} + \mathcal{V}$ . If  $A, B \in \mathcal{U} + \mathcal{V}$ , then  $(\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in A)) \wedge (\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in B))$ , so by above proposition, we have  $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in A \wedge x + y \in B)$ , i.e.  $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in A \cap B)$ , i.e.  $A \cap B \in \mathcal{U} + \mathcal{V}$ . Hence  $\mathcal{U} + \mathcal{V}$  is a filter.

Suppose that  $A \notin \mathcal{U} + \mathcal{V}$ , i.e.  $\neg(\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in A))$ . Then by above proposition twice, we have  $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y \neg(x + y \in A)$ . So  $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in A^c)$ , i.e.  $A^c \in \mathcal{U} + \mathcal{V}$ .  $\square$

**Proposition 2.49** Ultrafilter addition is associative.

*Proof.* Let  $A \subseteq \mathcal{U} + (\mathcal{V} + \mathcal{W})$ , so  $\forall_{\mathcal{U}}x\forall_{\mathcal{V}+\mathcal{W}} (x + y \in A)$ . So  $B := \{y : x + y \in A\} \in \mathcal{V} + \mathcal{W}$ , i.e.  $\forall_{\mathcal{V}}y_1\forall_{\mathcal{W}}y_2 (y_1 + y_2 \in B)$ . So we have  $\forall_{\mathcal{U}}x\forall_{\mathcal{V}}y_1\forall_{\mathcal{W}}y_2 (x + y_1 + y_2 \in A)$ . So

$$\mathcal{U} + (\mathcal{V} + \mathcal{W}) = \{A \subseteq \mathbb{N} : \forall_{\mathcal{U}}x\forall_{\mathcal{V}}y\forall_{\mathcal{W}}z (x + y + z \in A)\} = (\mathcal{U} + \mathcal{V}) + \mathcal{W}.$$

$\square$

**Proposition 2.50** Ultrafilter addition is left-continuous: for fixed  $\mathcal{V}$ ,  $\mathcal{U} \mapsto \mathcal{U} + \mathcal{V}$  is continuous.

*Proof.* For  $A \subseteq \mathbb{N}$ , we have

$$\begin{aligned} \mathcal{U} + \mathcal{V} \in C_A &\iff A \in \mathcal{U} + \mathcal{V} \\ &\iff \forall_{\mathcal{U}}x\forall_{\mathcal{V}}y (x + y \in A) \\ &\iff B := \{x \in \mathbb{N} : \forall_{\mathcal{V}}y (x + y \in A)\} \in \mathcal{U} \\ &\iff \mathcal{U} \in C_B \end{aligned}$$

hence the preimage of  $C_A$ , which is  $C_B$ , is open.  $\square$

**Proposition 2.51** (Idempotent Lemma) There exists an idempotent ultrafilter  $\mathcal{U} \in \beta\mathbb{N}$  (i.e.  $\mathcal{U} = \mathcal{U} + \mathcal{U}$ ).

*Proof.* For  $M \subseteq \beta\mathbb{N}$ , define  $M + M := \{x + y : x, y \in M\}$ . We seek a non-empty, compact  $M \subseteq \beta\mathbb{N}$  which is minimal such that  $M + M \subseteq M$ , and hope to show that  $M$  is a singleton.

Such an  $M$  exists ( $\beta\mathbb{N}$  is one such), so the set of all such  $M$  is non-empty. By Zorn's Lemma, it suffices to show that if  $\{M_i : i \in I\}$  is a chain of such sets, then  $M =$



$\bigcap_{i \in I} M_i$  (an upper bound with respect to the partial ordering  $\supseteq$ ) is another such set. This  $M$  will be compact as an intersection of closed sets, since  $\beta\mathbb{N}$  is compact and Hausdorff, so any subspace is closed iff it is compact. Also,  $M + M \subseteq M$ : for  $x, y \in M$ , we have  $x, y \in M_i$  so  $x + y \in M_i + M_i \subseteq M_i$  for all  $i \in I$ , so  $x + y \in M$ . Finally,  $M$  is non-empty:  $\{M_i : i \in I\}$  have the finite intersection property, as they are a chain, and are closed, so their intersection is non-empty.

So by Zorn's lemma, there exists such a minimal  $M$ . Given  $x \in M$ , we have  $M + x = M$ , since  $M + x \neq \emptyset$ ,  $M + x$  is compact (as the continuous image of a compact set) and  $(M + x) + (M + x) = (M + x + M) + x \subseteq (M + M + M) + x \subseteq M + x$ , so by minimality of  $M$ ,  $M + x = M$ .

In particular, there exists  $y \in M$  such that  $y + x = x$ . Let  $T = \{y \in M : y + x = x\}$ . We claim that  $T = M$ , and since  $T \subseteq M$ , it is enough to show that  $T$  is compact, non-empty and  $T + T \subseteq T$ , by minimality of  $M$ . Indeed,  $y \in T$ , so  $T \neq \emptyset$ ,  $T$  is the pre-image of a singleton which is compact, hence closed, so  $T$  is closed, so compact. Finally, for  $y, z \in T$ , we have  $y + x = x = z + x$  so  $y + z + x = y + x = x$ , so  $y + z \in T$ , so  $T + T \subseteq T$ .

Hence,  $y + x = x$  for all  $y \in M$ , hence  $x + x = M$ . In fact,  $M = \{x\}$ .  $\square$

**Remark 2.52** The finite subgroup problem asks whether we can find a non-trivial subgroup of  $\beta\mathbb{N}$  (e.g. find  $\mathcal{U}$  with  $\mathcal{U} + \mathcal{U} \neq \mathcal{U}$  but  $\mathcal{U} + \mathcal{U} + \mathcal{U} = \mathcal{U}$ ). This was recently proven to be negative.

**Remark 2.53** It has been recently shown that there exist  $\mathcal{U} \neq \mathcal{V}$  such that  $\mathcal{U} + \mathcal{U} = \mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U} = \mathcal{V} + \mathcal{V} = \mathcal{V}$ .

**Theorem 2.54** (Hindman) For any finite colouring of  $\mathbb{N}$ , there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  such that

$$\text{FS}(\{x_n : n \in \mathbb{N}\}) = \left\{ \sum_{i \in I} x_i : I \subseteq \mathbb{N} \text{ finite}, I \neq \emptyset \right\}.$$

*Proof.* Let  $\mathcal{U}$  be an idempotent ultrafilter, and partition  $\mathbb{N}$  into its colour classes:  $\mathbb{N} = A_1 \sqcup \dots \sqcup A_k$ . Since  $\emptyset \notin \mathcal{U}$  by definition, we have  $A_1 \cup \dots \cup A_k \in \mathcal{U}$  by [Proposition 2.33](#). So by [Corollary 2.34](#),  $A := A_i \in \mathcal{U}$  for some  $i \in [k]$ . We have  $\forall_{\mathcal{U}} y (y \in A)$  by definition. Thus:

1.  $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (y \in A)$ .
2.  $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (x \in A)$ .
3.  $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (x + y \in A)$  since  $A \in \mathcal{U} + \mathcal{U} = \mathcal{U}$ .

[Proposition 2.44](#) then gives that  $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y (\text{FS}(x, y) \subseteq A)$ . Fix  $x_1 \in A$  such that  $\forall_{\mathcal{U}} y (\text{FS}(x_1, y) \subseteq A)$ .

Now assume we have found  $x_1, \dots, x_n$  such that  $\forall_{\mathcal{U}} y (\text{FS}(x_1, \dots, x_n, y) \subseteq A)$ , i.e.  $B := \{y \in \mathbb{N} : \text{FS}(x_1, \dots, x_n, y) \subseteq A\} \in \mathcal{U} = \mathcal{U} + \mathcal{U}$ , i.e.  $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (x + y \in B)$  by definition. We have:

1.  $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (\text{FS}(x_1, \dots, x_n, y) \subseteq A)$ .

2.  $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (\text{FS}(x_1, \dots, x_n, x) \subseteq A)$ .
3. For each  $z \in \text{FS}(x_1, \dots, x_n, y)$ , we have  $\forall_{\mathcal{U}} y (z + y \in A)$ , so  $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (z + x + y \in A)$ .

[Proposition 2.44](#) then gives that

$$\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y (\text{FS}(x_1, \dots, x_n, x, y) \subseteq A).$$

The result follows by induction. □

### 3. Euclidean Ramsey theory

If we 2-colour  $\mathbb{R}^2$ , there are 2 points of distance at most 1 of the same colour (consider equilateral triangle).

If we 3-colour  $\mathbb{R}^3$ , there are 2 points of distance at most 1 of the same colour (consider regular tetrahedron)

If we  $k$ -colour  $\mathbb{R}^k$ , then by considering the regular simplex with  $k + 1$  vertices such that any 2 points have distance 1 between them, 2 points have the same colour.

**Definition 3.1**  $X'$  is an **isometric copy** of  $X$  if there exists a bijection  $\varphi : X \rightarrow X'$  which preserves distances:

$$\forall x, y \in X, \quad d(x, y) = d(\varphi(x), \varphi(y)).$$

**Definition 3.2** A finite set  $X \subseteq \mathbb{R}^m$  is **(Euclidean) Ramsey** if for all  $k \in \mathbb{N}$ , there exists a finite set  $S \subseteq \mathbb{R}^n$  ( $n$  could be very large) such that for any  $k$ -colouring of  $S$ , there exists a monochromatic isometric copy of  $X$ .

#### Example 3.3

- $\{0, 1\}$  is Ramsey, by the above simplex argument.
- The equilateral triangle of side length 1 is Ramsey, by considering the  $2k$ -dimensional unit simplex.
- Any  $\{0, a\}$  is Ramsey.
- By the same argument, any regular simplex is Ramsey.

#### Remark 3.4

- If  $X$  is infinite, then (exercise) we can construct a 2-colouring of  $\mathbb{R}^n$  with no monochromatic isometric copy of  $X$ .
- Above, we took  $S$  to be in  $\mathbb{R}^k$  for  $k$  colours. Can we do better? We can't do it for  $\{0, 1\}$  in  $\mathbb{R}$ : consider the colouring  $x \mapsto \lfloor x \rfloor \bmod 2$ . For  $\{0, 1\}$  with 3 colours, can do this in  $\mathbb{R}^2$ : look at diagram. Actually this shows  $\chi(\mathbb{R}^2) \geq 4$ . Can show  $\chi(\mathbb{R}^2) \leq 7$  by hexagonal argument. We know  $\chi(\mathbb{R}^2) \geq 5$ . In general,  $1.2^n \leq \chi(\mathbb{R}^n) \leq 3^n$ . The upper bound easily follows from a hexagonal colouring.

**Proposition 3.5**  $X$  is Euclidean Ramsey iff  $\forall k \in \mathbb{N}, \exists n \in \mathbb{N}$  such that for any  $k$ -colouring of  $\mathbb{R}^n$ , there exists a monochromatic isometric copy of  $X$ .

*Proof.* If  $X$  is Euclidean Ramsey then take  $S$  finite in  $\mathbb{R}^n$  (for  $k$  colours).

$\Leftarrow$ : we use a compactness proof. Suppose not, therefore for any finite  $S \subseteq \mathbb{R}^n$ , there is a bad  $k$ -colouring (i.e. no monochromatic isometric copy of  $X$ ). The space of all  $k$ -colourings is  $[k]^{\mathbb{R}^n}$ , which is compact by Tychonov (TODO: add this statement).

Consider the set  $C_{X'}$  of colourings under which  $X'$  is not monochromatic.  $C_{X'}$  is closed. Look at  $\{C_{X'} : X' \text{ isometric copy of } X\}$ . It has the finite intersection property, because any finite  $S$  has a bad  $k$ -colouring. Therefore,  $\bigcap C_{X'} \neq \emptyset$ , so there exists a  $k$ -colouring of  $\mathbb{R}^n$  with no monochromatic isometric copy of  $X$  in  $S$ .  $\square$

**Lemma 3.6** If  $X \subseteq \mathbb{R}^n$  and  $Y \subseteq \mathbb{R}^m$  are both Ramsey, then  $X \times Y \subseteq \mathbb{R}^{n+m}$  is also Ramsey.

*Proof.* Let  $c$  be a colouring of  $S \times T$ , where  $S$  is  $k$ -Ramsey for  $X$  and  $T$  is  $k^{|S|}$ -Ramsey for  $Y$ .  $k^{|S|}$ -colour  $T$  as follows:  $c'(t) = (c(s_1, t), \dots, c(s_{|S|}, t))$ . By choice of  $T$ , there is a monochromatic (with respect to  $c'$ ) isometric copy  $Y'$  of  $Y$ . So  $c(s, y) = c(s, y')$  for all  $y, y' \in Y$  and  $s \in S$ . Now  $k$ -colour  $S$  by  $c''(s) = c(s, y)$  for any  $y \in Y$  (note this is well-defined). By choice of  $S$ , there is a monochromatic (with respect to  $c''$ ) isometric copy  $X'$  of  $X$ , so  $X' \times Y'$  is monochromatic with respect to  $c$ .

TODO: convince yourself that this is a very standard product argument.  $\square$

**Remark 3.7** Since any  $\{0, a\}$  and  $\{0, b\}$  are Ramsey, any rectangle is Ramsey, so any right-angle triangle is Ramsey (since it is embedded in a rectangle). Similarly, any cuboid is Ramsey, and so any acute triangle (which is embedded in a cuboid) is Ramsey.

**Remark 3.8** In general, to prove sets are Ramsey, we will first embed them in “nicer” sets (with useful symmetry groups) and show instead that those sets are Ramsey. We will show:

- any triangle is Ramsey
- any regular  $n$ -gon is Ramsey
- any Platonic solid is Ramsey

**Proposition 3.9**  $X = \{0, 1, 2\}$  is not Ramsey.

*Proof.* Recall in  $\mathbb{R}^n$  we have  $\|x + y\|_2^2 + \|x - y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2$ . Every isometric copy of  $\{0, 1, 2\}$  in any  $\mathbb{R}^n$  is of the form  $\{x - y, x, x + y\}$  with  $\|y\|_2 = 1$ . So

$$\|x + y\|_2^2 + \|x - y\|_2^2 = 2\|x\|_2^2 + 2.$$

If we can find a colouring  $\varphi$  of  $\mathbb{R}_{\geq 0}$  such that there is no monochromatic solutions to  $a + b = 2c + 2$ . Colouring  $\mathbb{R}^n$  by  $c(x) = \varphi(\|x\|_2^2)$ . We 4-colour  $\mathbb{R}_{\geq 0}$  by  $\varphi(x) = \lfloor x \rfloor \bmod 4$ . Suppose  $a, b, c$  all have colour  $n \in \{0, 1, 2, 3\}$ . Then if  $a + b = 2c + 2$ , writing  $a = \lfloor a \rfloor + \{a\}$ ,  $b = \lfloor b \rfloor + \{b\}$ ,  $c = \lfloor c \rfloor + \{c\}$ , we have  $2 = a + b - 2c = 4k + \{a\} + \{b\} - 2\{c\}$  for some  $k \in \mathbb{Z}$ , which is impossible as  $-2 < \{a\} + \{b\} - 2\{c\} < 2$ .  $\square$

**Remark 3.10**

- The proof shows that for all  $n$ , there is a 4-colouring of  $\mathbb{R}^n$  which stops every isometric copy of  $X = \{0, 1, 2\}$  from being monochromatic.
- It is very important that in  $a + b = 2c + 2$ , the 2 in the equation is not 0.

- It will turn out that the property that made  $\{0, 1, 2\}$  not Ramsey is “it does not lie on a sphere”.

**Definition 3.11** A finite set  $X \subseteq \mathbb{R}^n$  is called **spherical** if it lies on the surface of a sphere.

**Example 3.12** Any triangle, rectangle, and non-degenerate simplex is spherical.

**Definition 3.13**  $x_1, \dots, x_d \in \mathbb{R}^n$  form a simplex if  $\{x_1 - x_d, x_2 - x_d, \dots, x_{d-1} - x_d\}$  are linearly independent, i.e. if  $x_1, \dots, x_d$  do not lie in a  $(d - 2)$ -dimensional affine subspace.

**Remark 3.14** We want to show that if  $X$  is Ramsey, then it is spherical.

**Lemma 3.15**  $\{x_1, \dots, x_d\} \in \mathbb{R}^n$  is not spherical iff there exist  $c_i \in \mathbb{R}$ , not all zero, such that:

1.  $\sum_{i=1}^d c_i = 0$ .
2.  $\sum_i c_i x_i = 0$ .
3.  $\sum_i c_i \|x_i\|_2^2 \neq 0$ .

*Proof.*  $\Rightarrow$ : let  $\{x_1, \dots, x_d\}$  be not spherical. Since  $\{x_1, \dots, x_d\}$  are not the vertices of a simplex, there exist  $c_1, \dots, c_d \in \mathbb{R}$  such that  $\sum_i c_i (x_i - x_d) = 0$  since  $\sum_{i=1}^{d-1} c_i x_i + (-c_1 - \dots - c_{d-1})x_d = 0$ . This gives the first two conditions. Note that all three conditions are invariant under translation by  $v \in \mathbb{R}^n$ :  $\sum_i c_i (x_i + v) = 0$ ,  $\sum_i c_i \|x_i + v\|_2^2 = \sum_i c_i \|x_i\|_2^2 + 2c_i x_i \cdot v + c_i \|v\|_2^2 = \sum_i c_i \|x_i\|_2^2$ .

Look at a minimal subset of  $\{x_1, \dots, x_d\}$  that is not spherical. If we can show the result for WLOG  $\{x_1, \dots, x_k\}$ ,  $k \leq d$ , then we can take  $c_i = 0$  for all  $i \in [k + 1, d]$ . Now  $\{x_2, \dots, x_k\}$  is spherical by minimality. Say it lies on the sphere of radius  $r$ , centred at  $v$ . By translation invariance, then we can translate the set such that  $\{x_2, \dots, x_k\}$  is centred at 0.  $\{x_1, \dots, x_d\}$  is not spherical so does not form a  $(d - 1)$ -simplex, so there exist  $c_i$  such that  $\sum_i c_i (x_i - x_k) = 0$  so  $c_1 x_1 + \dots + c_{k-1} x_{k-1} + (-c_1 - \dots - c_{k-1})x_k = 0$ . WLOG, we have  $c_1 \neq 0$  (can assume this since the same  $c_i$  work after translation). Now

$$\sum_{i=1}^k c_i \|x_i\|_2^2 = c_1 \|x_1\|_2^2 + r^2 \sum_{i=2}^k c_i \neq 0$$

as  $\|x_1\| \neq r$ , since  $\{x_1, \dots, x_k\}$  is not spherical.

$\Leftarrow$ : assume for a contradiction that  $\{x_1, \dots, x_d\}$  are spherical, and lie on the sphere of radius  $r$  centred at  $v$ . By the above argument, we can translate the set so that they are centred at the origin: this preserves all conditions and does not change the value of  $\sum_i c_i \|x_i\|_2^2$ . We have  $\|x_i\|_2^2 = r^2$  for all  $i$ , so  $\sum_i c_i \|x_i\|_2^2 = r^2 \sum_i c_i = 0$ : contradiction.  $\square$

**Remark 3.16** In the previous proof, we had  $c = (1, 1, -2)$  and  $\sum_i c_i \|x_i\|_2^2 = 2$ .

**Corollary 3.17** Let  $X = \{x_1, \dots, x_n\}$  be non-spherical. Then there exist  $c_1, \dots, c_n$  not all 0 with  $\sum_i c_i = 0$  and a  $b \neq 0$  such that  $\sum_i c_i \|x_i\|_2^2 = b$ .

**Remark 3.18** The above corollary is true for every isometric copy  $X'$  of  $X$  with *the same witnesses  $c_i$  and  $b$* : if  $\varphi(X)$  is a copy of  $X$  (for distance-preserving bijection  $\varphi$ ), we can translate (as in proof of above lemma) and the  $c_i$  and  $b$  are unaffected, in such a way that  $\varphi(0) = 0$ . After that, applying a matrix  $A$  that corresponds to rotation/reflection.

**Theorem 3.19** If  $X = \{x_1, \dots, x_n\}$  is Ramsey, then  $X$  is spherical.

*Proof.* Suppose  $X$  is not spherical. Then by above lemma, there exist  $c_i$  not all zero such that  $\sum_i c_i \|x_i\|^2 = b \neq 0$  and  $\sum_i c_i = 0$ . This is also true for any isometric copy of  $X'$ . We will split  $[0, 1)$  into  $[0, \delta)$ ,  $[\delta, 2\delta)$ , ... for small  $\delta$  and colour depending on where  $c_i \|x\|^2$  lies. It is enough to construct a colouring  $c : \mathbb{R}_+ \rightarrow [k]$  such that  $\sum_i c_i y_i = b$  does not have a monochromatic solution, where  $\sum_i c_i = 0$ . If we show this, then we define a colouring  $c' : \mathbb{R}^n \rightarrow [k]$  by  $c'(x) = c(\|x\|^2)$ .

We have  $\sum_{i=1}^{n-1} c_i (y_i - y_n) = b$ . By rescaling the  $c_i$ , we may assume that  $b = 1/2$ . Now we split  $[0, 1)$  into intervals  $[0, \delta)$ ,  $[\delta, 2\delta)$ , ... where  $\delta$  is very small. Define the colouring  $c(y) = (\text{interval where } \{c_1 y\} \text{ is, interval where } \{c_2 y\} \text{ is, } \dots)$ . This is a  $\lfloor \frac{1}{\delta} \rfloor^{n-1}$ -colouring. Assume  $y_1, \dots, y_{n-1}$  are monochromatic under  $c$  such that  $\sum_i c_i (y_i - y_n) = 1/2$ . The sum is within  $(n-1)\delta$  of an integer, which is  $\neq \frac{1}{2}$  for  $\delta$  small enough.  $\square$

What about spherical  $\Rightarrow$  Ramsey? This is open.

It is known that triangles, simplices, and any  $n$ -gon is Ramsey.

We want to show that any regular  $m$ -gon  $X = \{v_1, \dots, v_m\}$  (with side length 1) is Ramsey. Idea: first find a copy of  $X$  such that  $v_1$  and  $v_2$  are monochromatic, then use a product argument to get an isometric copy of  $X^N$ , where  $N$  is very large, such that the colouring is invariant under swapping around  $v_1$  and  $v_2$ . Use this to find copy of  $X$  on for which  $v_1, v_2, v_3$  are monochromatic.

**Definition 3.20** For a finite  $A \subseteq X$ , a colouring  $c$  of  $X^n$  is **A-invariant** if it is invariant under permuting the coordinates within  $A$ , i.e. for  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{x}' = (x'_1, \dots, x'_n)$ , if for all  $i \in [n]$ , either  $x_i = x'_i$  or  $x_i, x'_i \in A$ , then  $c(\mathbf{x}') = c(\mathbf{x})$ .

Note if  $c$  is  $X$  invariant, then  $X$  is monochromatic under  $c$ .

**Proposition 3.21** Let  $X \subseteq \mathbb{R}^d$  be finite and  $A \subseteq X$ . Suppose  $\forall k \in \mathbb{N}$ , there exists a finite  $S \subseteq \mathbb{R}^c$  such that whenever  $S$  is  $k$ -coloured, there exists an isometric copy of  $X$  that is constant on  $A$ . Then  $\forall n, k \in \mathbb{N}$ , there exists finite  $S'$  such that whenever  $S'$  is  $k$ -coloured, there exists a copy of  $X^n$  that is  $A$ -invariant.

So we are “boosting the colouring from  $A$ -constant to  $A$ -invariant”.

*Proof.* We use induction on  $n$  (and all  $k$  at once).  $n = 1$  is by assumption. Suppose it is true for  $n - 1$ . Fix  $k \in \mathbb{N}$ . Let  $S$  be a finite set such that whenever  $S$  is  $k^{|X|}$ -coloured, there exists an  $A$ -invariant copy of  $X^{n-1}$ , and let  $T$  be a finite set such that whenever  $T$  is  $k^{|S|}$ -coloured, there exists an isometric copy of  $X$  with  $A$  monochromatic. We claim that  $S \times T$  works for  $X^n$ .

Let  $c$  be a  $k$ -colouring of  $S \times T$ . By definition of  $T$ , if we look at the  $k^{|S|}$ -colouring  $c'(t) = (c(s_1, t), c(s_2, t), \dots, c(s_{|S|}, t))$ , we have an isometric copy of  $X$  with  $A$  monochromatic. This induces a colouring of  $S$  as follows:  $c''(s) = (c(s, a), c(s, x_1), \dots, c(s, x_{|X|-|A|}))$  for any  $a \in A$  (this is well-defined as  $A$  is monochromatic). This is a  $k^{|X|-|A|+1}$ -colouring. So by the choice of  $S$ , there exists an isometric copy of  $X^{n-1}$  that is  $A$ -invariant. Thus we are done, since the Cartesian product of this copy with the copy of  $X$  in  $T$  is  $A$ -invariant.  $\square$

**Theorem 3.22** (Kříž) Every regular  $m$ -gon is Ramsey.