### 0.1. Integration and measure

- Dirichlet's function: $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

# 1. The real numbers

- $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \le a$.
- $c \in \mathbb{R}$ is a **least upper bound (supremum)** if $c \le a$ for every upper bound $a$.
- $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \ge a$.
- $c \in \mathbb{R}$ is a **greatest lower bound (supremum)** if $c \ge a$ for every upper bound $a$.
- **Completeness axiom of the real numbers**: every subset $E$ with an upper bound has a least upper bound. Every subset $E$ with a lower bound has a greatest lower bound.
- **Archimedes' principle**:

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

- Every non-empty subset of $\mathbb{N}$ has a minimum.
- **The rationals are dense in the reals**:

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

## 1.1. Conventions on sets and functions

- For $f : X \to Y$, **preiamge** of $Z \subseteq Y$ is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

- $f : X \to Y$ **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

- $f : X \to Y$ **surjective** if $Y = f(X)$.
- **Limit inferior** of sequence $x_n$:

$$\liminf_{n \to \infty} x_n := \lim_{n \to \infty} \left( \inf_{m \ge n} x_m \right) = \sup_{n \ge 0} \inf_{m \ge n} x_m$$

- **Limit superior** of sequence $x_n$:

$$\limsup_{n \to \infty} x_n := \lim_{n \to \infty} \left( \sup_{m \ge n} x_m \right) = \inf_{n \ge 0} \sup_{m \ge n} x_m$$

## 1.2. Open and closed sets

- $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon : (x - \varepsilon, x + \varepsilon) \subseteq U$$

- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.
- $x \in \mathbb{R}$ is **point of closure (limit point)** for $E \subseteq \mathbb{R}$ if

$$\forall \delta > 0, \exists y \in E : |x - y| < \delta$$

Equivalently, $x$ is point of closure if every open interval containing $x$ contains another point of $E$.

- **Closure** of $E$, $\overline{E}$, is set of points of closure.
- $F$ is **closed** if $F = \overline{F}$.
- If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$.
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- For any set $E$, $\overline{E}$ is closed.
- Let $E \subseteq \mathbb{R}$. The following are equivalent:
  - $E$ is closed.
  - $\mathbb{R} - E$ is open.
- Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.
- **Definition**: collection $C$ of subsets of $\mathbb{R}$ **covers** (is a **covering** of) $F \subseteq \mathbb{R}$ if $F \subseteq \cup_{S \in C} S$. If each $S$ in $C$ open, $G$ is **open covering**. If $C$ is finite, $C$ is **finite cover**.
- Covering $C$ of $F$ **contains a finite subcover** if exists $\{S_1, ..., S_n\} \subseteq C$ with $F \subseteq \cup_{i=1}^n S_i$ (i.e. a finite subset of $C$ covers $F$). $F$ is **compact** if any open covering $U$ contains a finite subcover.
- **Example**: $\mathbb{R}$ is not compact, $[a, b]$ is compact.
- **Heine-Borel theorem**: if $F \subset \mathbb{R}$ closed and bounded then any open covering of $F$ has finite subcovering (so $F$ is compact). If $F$ compact then $F$ closed and bounded.

## 1.3. The extended real numbers

- **Definition**: **extended reals** are $\mathbb{R} \cup \{-\infty, \infty\}$ with the order relation $-\infty < \infty$ and $\forall x \in \mathbb{R}, -\infty < x < \infty$. $\infty$ is an upper bound and $-\infty$ is a lower bound for every $x \in \mathbb{R}$, so $\sup(\mathbb{R}) = \infty$, $\inf(\mathbb{R}) = -\infty$.
  - Addition: $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$. $\infty + \infty = \infty - (-\infty) = \infty$. $\infty - \infty$ is undefined.
  - Multiplication: $\forall a \in \mathbb{R}_{>0}, a \cdot \infty = \infty$, $\forall a \in \mathbb{R}_{<0}, a \cdot = -\infty$. $\infty \cdot \infty = \infty$ and $0 \cdot \infty = \infty$.
  - $\lim \sup$ and $\lim \inf$ are defined as

$$\limsup x_n := \inf_{n \in \mathbb{N}} \left\{ \sup_{k \geq n} x_k \right\}, \quad \liminf x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{k \geq n} x_k \right\}$$

- **Definition**: extended real number $l$ is **limit** of $(x_n)$ if either
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$. Then $(x_n)$ **converges to $l$**. or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$ (limit is $\infty$) or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$ (limit is $-\infty$).

$(x_n)$ **converges in the extended reals** if it has a limit in the extended reals.

# 2. Further analysis of subsets of $\mathbb{R}$

TODO: up to here, check that all notes are made from these topics

## 2.1. Countability and uncountability

- $A$ is **countable** if $A = \emptyset$, $A$ is finite or there is a bijection $\varphi : \mathbb{N} \to A$ (in which case $A$ is **countably infinite**). Otherwise $A$ is **uncountable**. $\varphi$ is called an **enumeration**.
- If surjection from $\mathbb{N}$ to $A$, or injection from $A$ to $\mathbb{N}$, then $A$ is countable.
- Examples of countable sets:
  - $\mathbb{N}$ ($\varphi(n) = n$)
  - $2\mathbb{N}$ ($\varphi(n) = 2n$)
- $\mathbb{Q}$ is countable.
- **Exercise (todo)**: show that $\mathbb{N}^k$ is countable for any $k \in \mathbb{N}$.
- **Exercise (todo)**: show that if $a_n$ is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Exercise (todo)**: show that if $a_{n,k}$ is a nonnegative sequence and $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- $f : X \to Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.
- Let $f$ be monotone on $(a, b)$. Then it is discountinuous on a countable set.
- Set of sequences in $\{0, 1\}$, $\{((x_n))_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.
- **Theorem**: $\mathbb{R}$ is uncountable.

## 2.2. The structure theorem for open sets

- Collection $\{A_i : i \in I\}$ of sets is **(pairwise) disjoint** if $n \neq m \implies A_n \cap A_m = \emptyset$.
- **Structure theorem for open sets**: let $U \subseteq \mathbb{R}$ open. Then exists countable collection of disjoint open intervals $\{I_n : n \in \mathbb{N}\}$ such that $U = \cup_{n \in \mathbb{N}} I_n$.

## 2.3. Accumulation points and perfect sets

- $x \in \mathbb{R}$ is **accumulation point** of $E \subseteq \mathbb{R}$ if $x$ is point of closure of $E - \{x\}$. Equivalently, $x$ is a point of closure if

$$\forall \delta > 0, \exists y \in E : y \neq x \wedge |x - y| < \delta$$

  Equivalently, there exists a sequence of distinct $y_n \in E$ with $y_n \to x$ as $n \to \infty$.
- **Exercise**: set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$.
- $E \subseteq \mathbb{R}$ is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

- **Proposition**: set of accumulation points $E'$ of $E$ is closed.
- Bounded set $E$ is **perfect** if it equals its set of accumulation points.

- **Exercise (todo)**: what is the set of accumulation points of an isolated set?
- Every non-empty perfect set is uncountable.

## 2.4. The middle-third Cantor set
- **Middle third Cantor set**:
    - Define $C_0 := [0, 1]$
    - Given $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$, $a_i < b_1 < a_2 < \cdots$, with $|b_i - a_i| = 3^{-n}$, define
    $$C_{n+1} := \cup_{i=1}^{2^n} \left[a_i, a_i + 3^{-(n+1)}\right] \cup \left[b_i - 3^{-(n+1)}, b_i\right]$$
    which is a union of $2^{n+1}$ disjoint intervals, with difference in endpoints equalling $3^{-(n+1)}$.
    - The **middle third Cantor set** is
    $$C := \bigcup_{n \in \mathbb{N}} C_n$$
    Observe that if $a$ is an endpoint of an interval in $C_n$, it is contained in $C$.
- **Proposition**: the middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and uncountable.

## 2.5. $G_s, F_\sigma$
- Set $E$ is $\boldsymbol{G_\delta}$ if $E = \cap_{n \in \mathbb{N}} U_n$ with $U_n$ open.
- Set $E$ is $\boldsymbol{F_\sigma}$ if $E = \cup_{n \in \mathbb{N}} F_n$ with $F_n$ closed.
- **Lemma**: set of points where $f : \mathbb{R} \to \mathbb{R}$ is continuous is $G_\delta$.

# 3. Construction of Lebesgue measure

## 3.1. Lebesgue outer measure
- **Definition**: let $I$ non-empty interval with endpoints $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$ and $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$. The **length** of $I$ is
$$\ell(I) := b - a$$
and set $\ell(\emptyset) = 0$.
- **Example**: if $I = (-\infty, b] = (-\infty, a] \cup [a, b]$ then $\ell(I) = \infty = \ell(-\infty, a]) + \ell([a, b])$
- **Definition**: let $A \subseteq \mathbb{R}$. **Lebesgue outer measure** of $A$ is infimum of all sums of lengths of intervals covering $A$:
$$\mu^*(A) := \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k, I_k \text{ intervals} \right\}$$
It satisfies **monotonicity**: $A \subseteq B \implies \mu^*(A) \le \mu^*(B)$.
- **Proposition**: outer measure is **countably subadditive**: if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets then

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$$

- **Lemma**: we have

$$\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k, I_k \neq \emptyset \text{ open intervals}\right\}$$

- Lebesgue outer measure of interval is its length: $\mu^*(I) = \ell(I)$.

## 3.2. Measurable sets

- **Notation**: $E^c = \mathbb{R} - E$.
- **Proposition**: let $E = (a, \infty)$. Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

- **Definition**: $E \subseteq \mathbb{R}$ is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

  Collection of such sets is $\mathcal{F}_{\mu^*}$.

- **Lemma (excision property)**: let $E$ Lebesgue measurable set with finite measure and $E \subseteq B$, then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

- **Remark**: not every set is Lebesgue measurable.
- **Definition**: collection of subsets of $\mathbb{R}$ is an **algebra** if contains $\emptyset$ and closed under taking complements and finite unions: if $A, B \in \mathcal{A}$ then $\mathbb{R} - A, A \cup B \in \mathcal{A}$.
- **Proposition**: if $E_1, ..., E_n$ Lebesgue measurable then $\cup_{k=1}^{n} E_k$ is Lebesgue measurable. If $E_1, ..., E_n$ disjoint then

$$\mu^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

  for any $A \subseteq \mathbb{R}$. In particular, for $A = \mathbb{R}$,

$$\mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k)$$