

1. The real numbers

1.1. Conventions on sets and functions

- **Definition:** for $f : X \rightarrow Y$, **preimage** of $Z \subseteq Y$ is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

- **Definition:** $f : X \rightarrow Y$ **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

- **Definition:** $f : X \rightarrow Y$ **surjective** if $Y = f(X)$.
- **Proposition:** let $f : X \rightarrow Y$, $A, B \subseteq X$, then

$$f(A \cap B) \subseteq f(A) \cap f(B),$$

$$f(A \cup B) = f(A) \cup f(B),$$

$$f(X) - f(A) \subseteq f(X - A)$$

- **Proposition:** let $f : X \rightarrow Y$, $C, D \subseteq Y$, then

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D),$$

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D),$$

$$f^{-1}(Y - C) = X - f^{-1}(C)$$

1.2. The real numbers

- **Definition:** $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \leq a$.
- **Definition:** $c \in \mathbb{R}$ is a **least upper bound (supremum)** of E , $c = \sup(E)$, if $c \leq a$ for every upper bound a .
- **Definition:** $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \geq a$.
- **Definition:** $c \in \mathbb{R}$ is a **greatest lower bound (infimum)**, $c = \inf(E)$, if $c \geq a$ for every lower bound a .
- **Completeness axiom of the real numbers:** every $E \subseteq \mathbb{R}$ with an upper bound has a least upper bound. Every $E \subseteq \mathbb{R}$ with a lower bound has a greatest lower bound.
- **Archimedes' principle:**

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

- **Remark:** every non-empty subset of \mathbb{N} has a minimum.
- **Proposition:** \mathbb{Q} is dense in \mathbb{R} :

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

1.3. Sequences, limits and series

- **Definition:** $l \in \mathbb{R}$ is **limit** of (x_n) ((x_n) converges to l) if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$$

A sequence **converges in \mathbb{R} (is convergent)** if it has a limit $l \in \mathbb{R}$. Limit $l = \lim_{n \rightarrow \infty} x_n$ is unique.

- **Definition:** (x_n) **tends to infinity** if

$$\forall K > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad x_n > K$$

- **Definition:** **subsequence** of (x_n) is sequence (x_{n_j}) , $n_1 < n_2 < \dots$.
- **Definition:** **limit inferior** of sequence x_n is

$$\liminf_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \left(\inf_{m \geq n} x_m \right) = \sup_{n \in \mathbb{N}} \inf_{m \geq n} x_m$$

- **Definition:** **limit superior** of sequence x_n is

$$\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \left(\sup_{m \geq n} x_m \right) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} x_m$$

- **Proposition:** let (x_n) bounded, $l \in \mathbb{R}$. The following are equivalent:
 - $l = \limsup x_n$.
 - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < l + \varepsilon$.
 - $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n > l - \varepsilon$.
- **Proposition:** let (x_n) bounded, $l \in \mathbb{R}$. The following are equivalent:
 - $l = \liminf x_n$.
 - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > l - \varepsilon$.
 - $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : x_n < l + \varepsilon$.
- **Theorem (Bolzano-Weierstrass):** every bounded sequence has a convergent subsequence.
- **Proposition:** let (x_n) bounded. There exists convergent subsequence with limit $\limsup x_n$ and convergent subsequence with limit $\liminf x_n$.
- **Proposition:** let (x_n) bounded, then (x_n) is convergent iff $\limsup x_n = \liminf x_n$.
- **Monotone convergence theorem for sequences:** monotone sequence converges in \mathbb{R} or tends to either ∞ or $-\infty$.
- **Definition:** (x_n) is **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad |x_n - x_m| < \varepsilon$$

- **Theorem:** every Cauchy sequence in \mathbb{R} is convergent.

1.4. Open and closed sets

- **Definition:** $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subseteq U$$

- **Proposition:** arbitrary unions of open sets are open. Finite intersections of open sets are open.
- **Definition:** $x \in \mathbb{R}$ is **point of closure (limit point)** for $E \subseteq \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists y \in E : |x - y| < \varepsilon$$

Equivalently, x is point of closure of E if every open interval containing x contains another point of E .

- **Definition:** **closure** of E , \overline{E} , is set of points of closure. Note $E \subseteq \overline{E}$.
- **Definition:** F is **closed** if $F = \overline{F}$.
- **Proposition:** $\overline{A \cup B} = \overline{A} \cup \overline{B}$. If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$.

- **Proposition:** for any set E , \overline{E} is closed, i.e. $\overline{\overline{E}} = \overline{E}$.
- **Proposition:** let $E \subseteq \mathbb{R}$. The following are equivalent:
 - E is closed.
 - $\mathbb{R} - E$ is open.
- **Proposition:** arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.
- **Definition:** collection C of subsets of \mathbb{R} **covers** (is a **covering** of) $F \subseteq \mathbb{R}$ if $F \subseteq \cup_{S \in C} S$. If each S in C open, C is **open covering**. If C is finite, C is **finite covering**.
- **Definition:** covering C of F **contains a finite subcover** if exists $\{S_1, \dots, S_n\} \subseteq C$ with $F \subseteq \cup_{i=1}^n S_i$ (i.e. a finite subset of C covers F).
- **Definition:** F is **compact** if any open covering of F contains a finite subcover.
- **Example:** \mathbb{R} is not compact, $[a, b]$ is compact.
- **Heine-Borel theorem:** F compact iff F closed and bounded.

1.5. Continuity, pointwise and uniform convergence of functions

- **Definition:** let $E \subseteq \mathbb{R}$. $f : E \rightarrow \mathbb{R}$ is **continuous at** $a \in E$ if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

f is **continuous** if continuous at all $y \in E$.

- **Definition:** $\lim_{x \rightarrow a} f(x) = l$ if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

- **Proposition:** $\lim_{x \rightarrow a} f(x) = l$ iff for every sequence (a_n) with $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} f(a_n) = l$.
- **Proposition:** f is continuous at $a \in E$ iff $\lim_{x \rightarrow a} f(x) = f(a)$ (and this limit exists).
- **Definition:** $f : E \rightarrow \mathbb{R}$ is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in E, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

- **Proposition:** let F closed and bounded, $f : F \rightarrow \mathbb{R}$ continuous. Then f is uniformly continuous.
- **Definition:** let $f_n : E \rightarrow \mathbb{R}$ sequence of functions, $f : E \rightarrow \mathbb{R}$. (f_n) **converges pointwise** to f if

$$\forall \varepsilon > 0, \forall x \in E, \exists N \in \mathbb{N} : \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

(f_n) **converges uniformly** to f is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$$

- **Theorem:** let $f_n : E \rightarrow \mathbb{R}$ sequence of continuous functions converging uniformly to $f : E \rightarrow \mathbb{R}$. Then f is continuous.
- **Definition:** $P = \{x_0, \dots, x_n\}$ is **partition** of $[a, b]$ if $a = x_0 < \dots < x_n = b$.
- **Definition:** $f : [a, b] \rightarrow \mathbb{R}$ is **piecewise linear** if there exists partition $P = \{x_0, \dots, x_n\}$ and $m_i, c_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad f(x) = m_i x + c_i$$

f is continuous on $[a, b] - P$.

- **Definition:** $g : [a, b] \rightarrow \mathbb{R}$ is **step function** if there exists partition $P = \{x_0, \dots, x_n\}$ and $m_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad g(x) = m_i$$

g is continuous on $[a, b] - P$.

- **Theorem:** let $f : E \rightarrow \mathbb{R}$ continuous, E closed and bounded. Then there exist continuous piecewise linear f_n with $f_n \rightarrow f$ uniformly, and step functions g_n with $g_n \rightarrow f$ uniformly.
- **Definition:** $f : E \rightarrow \mathbb{R}$ is **Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad |f(x) - f(y)| \leq C|x - y|$$

- **Definition:** $f : E \rightarrow \mathbb{R}$ is **bi-Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

1.6. The extended real numbers

- **Definition:** **extended reals** are $\mathbb{R} \cup \{-\infty, \infty\}$ with the order relation $-\infty < \infty$ and $\forall x \in \mathbb{R}, -\infty < x < \infty$. ∞ is an upper bound and $-\infty$ is a lower bound for every $x \in \mathbb{R}$, so $\sup(\mathbb{R}) = \infty$, $\inf(\mathbb{R}) = -\infty$.
 - Addition: $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$. $\infty + \infty = \infty - (-\infty) = \infty$. $\infty - \infty$ is undefined.
 - Multiplication: $\forall a > 0, a \cdot \infty = \infty$, $\forall a < 0, a \cdot \infty = -\infty$. Also $\infty \cdot \infty = \infty$.
 - \limsup and \liminf are defined as

$$\limsup x_n := \inf_{n \in \mathbb{N}} \left\{ \sup_{k \geq n} x_k \right\}, \quad \liminf x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{k \geq n} x_k \right\}$$

- **Definition:** extended real number l is **limit** of (x_n) if either
 - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$. Then (x_n) **converges to l** . or
 - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$ (limit is ∞) or
 - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$ (limit is $-\infty$).

(x_n) **converges in the extended reals** if it has a limit in the extended reals.

2. Further analysis of subsets of \mathbb{R}

2.1. Countability and uncountability

- **Definition:** A is **countable** if $A = \emptyset$, A is finite or there is a bijection $\varphi : \mathbb{N} \rightarrow A$ (in which case A is **countably infinite**). Otherwise A is **uncountable**.
Enumeration is bijection from A to $[n]$ or \mathbb{N} .
- **Proposition:** if surjection from countable set to A , or injection from A to countable set, then A is countable.
- **Proposition:** any subset of \mathbb{N} is countable.
- **Proposition:** \mathbb{Q} is countable.

- **Proposition:** show that if (a_n) is a nonnegative sequence and $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Proposition:** show that if $(a_{n,k})$ is a nonnegative sequence and $\varphi : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Definition:** $f : X \rightarrow Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \leq f(y)$.
- **Proposition:** let f be monotone on (a, b) . Then it is discontinuous on a countable set.
- **Lemma:** set of sequences in $\{0, 1\}$, $\{(x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.
- **Theorem:** \mathbb{R} is uncountable.

2.2. The structure theorem for open sets

- Collection $\{A_i : i \in I\}$ of sets is **(pairwise) disjoint** if $n \neq m \Rightarrow A_n \cap A_m = \emptyset$.
- **Structure theorem for open sets:** let $U \subseteq \mathbb{R}$ open. Then exists countable collection of disjoint open intervals $\{I_n : n \in \mathbb{N}\}$ such that $U = \bigcup_{n \in \mathbb{N}} I_n$.

2.3. Accumulation points and perfect sets

- **Definition:** $x \in \mathbb{R}$ is **accumulation point** of $E \subseteq \mathbb{R}$ if x is point of closure of $E - \{x\}$. Equivalently, x is a point of closure if

$$\forall \varepsilon > 0, \exists y \in E : y \neq x \wedge |x - y| < \varepsilon$$

Equivalently, there exists a sequence of distinct $y_n \in E$ with $y_n \rightarrow x$ as $n \rightarrow \infty$.

- **Proposition:** set of accumulation points of \mathbb{Q} is \mathbb{R} .
- **Proposition:** set of accumulation points E' of E is closed.
- **Definition:** $E \subseteq \mathbb{R}$ is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

- **Proposition:** E is isolated iff it has no accumulation points.
- **Definition:** bounded set E is **perfect** if it equals its set of accumulation points.
- **Theorem:** every non-empty perfect set is uncountable.

2.4. The middle-third Cantor set

- **Proposition:** let $\{F_n : n \in \mathbb{N}\}$ be collection of non-empty nested closed sets (so $F_{n+1} \subseteq F_n$), one of which is bounded. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$$

- **Definition:** the **middle third Cantor set** is defined by:
 - Define $C_0 := [0, 1]$

- Given $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$, $a_1 < b_1 < a_2 < \dots < a_{2^n} < b_{2^n}$, with $|b_i - a_i| = 3^{-n}$, define

$$C_{n+1} := \cup_{i=1}^{2^{n+1}} [a_i, a_i + 3^{-(n+1)}] \cup [b_i - 3^{-(n+1)}, b_i]$$

which is a union of 2^{n+1} disjoint intervals, with all differences in endpoints equalling $3^{-(n+1)}$.

- The **middle third Cantor set** is

$$C := \bigcap_{n \in \mathbb{N}} C_n$$

Observe that if a is an endpoint of an interval in C_n , it is contained in C .

- **Proposition:** the middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and so uncountable.
- **Definition:** let $k \in \mathbb{N} - \{1\}$, $x \in [0, 1)$. $0.a_1a_2\dots$, $a_i \in \{0, \dots, k-1\}$, is a **k -ary expansion** of x if

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{k^i}$$

- **Remark:** the k -ary expansion may not be unique, but there is a countable set $E \subseteq [0, 1)$ such that every $x \in [0, 1) - E$ has a unique k -ary expansion.
- **Remark:** for every $x \in C$, the ternary ($k = 3$) expansion of x is unique and

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}$$

Moreover, every choice of sequence (a_i) , $a_i \in \{0, 2\}$, gives $x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i} \in C$.

- **Definition: Cantor-Lebesgue function**, $g : [0, 1] \rightarrow [0, 1]$, is defined by

$$g(x) := \begin{cases} \sum_{i \in \mathbb{N}} \frac{a_i/2}{2^i} & \text{if } x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, a_i \in \{0, 2\} \\ \sup\{f(y) : y \in C, y \leq x\} & \text{if } x \notin C \end{cases}$$

g is a surjection, monotone and continuous.

2.5. G_δ, F_σ

- **Definition:** $E \subseteq \mathbb{R}$ is **G_δ** if $E = \cap_{n \in \mathbb{N}} U_n$ with U_n open.
- **Definition:** $E \subseteq \mathbb{R}$ is **F_σ** if $E = \cup_{n \in \mathbb{N}} F_n$ with F_n closed.
- **Lemma:** set of points where $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous is G_δ .

3. Construction of Lebesgue measure

3.1. Lebesgue outer measure

- **Definition:** let I non-empty interval with endpoints $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$ and $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$. The **length** of I is

$$\ell(I) := b - a$$

and set $\ell(\emptyset) = 0$.

- **Definition:** let $A \subseteq \mathbb{R}$. **Lebesgue outer measure** of A is infimum of all sums of lengths of intervals covering A :

$$\mu^*(A) := \inf \left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subseteq \bigcup_{k \in \mathbb{N}} I_k, I_k \text{ intervals} \right\}$$

It satisfies **monotonicity**: $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

- **Proposition:** outer measure is **countably subadditive**:

$$\mu^* \left(\bigcup_{k \in \mathbb{N}} E_k \right) \leq \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

This implies **finite subadditivity**:

$$\mu^* \left(\bigcup_{k=1}^n E_k \right) \leq \sum_{k=1}^n \mu^*(E_k)$$

- **Lemma:** we have

$$\mu^*(A) = \inf \left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subset \bigcup_{k \in \mathbb{N}} I_k, I_k \neq \emptyset \text{ open intervals} \right\}$$

- **Proposition:** outer measure of interval is its length: $\mu^*(I) = \ell(I)$.

3.2. Measurable sets

- **Notation:** $E^c = \mathbb{R} - E$.
- **Proposition:** let $E = (a, \infty)$. Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

- **Definition:** $E \subseteq \mathbb{R}$ is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Collection of such sets is \mathcal{F}_{μ^*} .

- **Lemma (excision property):** let E Lebesgue measurable set with finite measure and $E \subseteq B$, then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

- **Proposition:** if E_1, \dots, E_n Lebesgue measurable then $\bigcup_{k=1}^n E_k$ is Lebesgue measurable. If E_1, \dots, E_n disjoint then

$$\mu^* \left(A \cap \bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(A \cap E_k)$$

for any $A \subseteq \mathbb{R}$. In particular, for $A = \mathbb{R}$,

$$\mu^* \left(\bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(E_k)$$

- **Remark:** not every set is Lebesgue measurable.
- **Definition:** collection of subsets of \mathbb{R} is an **algebra** if contains \emptyset and closed under taking complements and finite unions: if $A, B \in \mathcal{A}$ then $\mathbb{R} - A, A \cup B \in \mathcal{A}$.
- **Remark:** a union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if $\{A_k\}_{k \in \mathbb{N}}$ is countable collection of Lebesgue measurable sets, then let $A_1' := A_1$ and for $k > 1$, define

$$A_{k'} := A_k - \bigcup_{i=1}^{k-1} A_i$$

then $\{A_{k'}\}_{k \in \mathbb{N}}$ is disjoint union of Lebesgue measurable sets.

- **Proposition:** if E is countable union of Lebesgue measurable sets, then E is Lebesgue measurable. Also, if $\{E_k\}_{k \in \mathbb{N}}$ is countable disjoint collection of Lebesgue measurable sets then

$$\mu^* \left(\bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

3.3. Abstract definition of a measure

- **Definition:** let $X \subseteq \mathbb{R}$. Collection of subsets of \mathcal{F} of X is **σ -algebra** if
 - $\emptyset \in \mathcal{F}$
 - $E \in \mathcal{F} \implies E^c \in \mathcal{F}$
 - $E_1, \dots, E_n \in \mathcal{F} \implies \bigcup_{k \in \mathbb{N}} E_k \in \mathcal{F}$.
- **Example:**
 - Trivial examples are $\mathcal{F} = \{\emptyset, \mathbb{R}\}$ and $\mathcal{F} = \mathcal{P}(\mathbb{R})$.
 - Countable intersections of σ -algebras are σ -algebras.
- **Definition:** let \mathcal{F} σ -algebra of X . $\nu : \mathcal{F} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is **measure** satisfying
 - $\nu(\emptyset) = 0$
 - $\forall E \in \mathcal{F}, \nu(E) \geq 0$
 - **Countable additivity:** if $E_1, E_2, \dots \in \mathcal{F}$ are disjoint then

$$\nu \left(\bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu(E_k)$$

Elements of \mathcal{F} are **measurable** (as they are the only sets on which the measure ν is defined).

- **Proposition:** if ν is measure then it satisfies:
 - **Monotonicity:** $A \subseteq B \implies \nu(A) \leq \nu(B)$.
 - **Countable subadditivity:** $\nu(\bigcup_{k \in \mathbb{N}} E_k) \leq \sum_{k \in \mathbb{N}} \nu(E_k)$.
 - **Excision:** if A has finite measure, then $A \subseteq B \implies \nu(B - A) = \nu(B) - \nu(A)$.

3.4. Lebesgue measure

- **Lemma:** F_{μ^*} is σ -algebra and contains every interval.
- **Theorem (Carathéodory extension):** restriction of the μ^* to F_{μ^*} is a measure.

- **Hahn extension theorem:** there exists unique measure μ defined on \mathcal{F}_{μ^*} for which $\mu(I) = \ell(I)$ for any interval I .
- **Definition:** the measure μ of μ^* restricted to \mathcal{F}_{μ^*} is the **Lebesgue measure**. It satisfies $\mu(I) = \ell(I)$ for any interval I and is translation invariant.

3.5. Sets of measure 0

- **Proposition:** middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.
- **Proposition:** any countable set is Lebesgue measurable and has Lebesgue measure 0.
- **Proposition:** any E with $\mu^*(E) = 0$ is Lebesgue measurable and has $\mu(E) = 0$.
- **Lemma:** let E Lebesgue measurable set with $\mu(E) = 0$, then $\forall E' \subseteq E$, E' is Lebesgue measurable.

3.6. Continuity of measure

- **Definition:** countable collection $\{E_k\}_{k \in \mathbb{N}}$ is **ascending** if $\forall k \in \mathbb{N}, E_k \subseteq E_{k+1}$ and **descending** if $\forall k \in \mathbb{N}, E_{k+1} \subseteq E_k$.
- **Theorem:** every measure m satisfies:
 - If $\{A_k\}_{k \in \mathbb{N}}$ is ascending collection of measurable sets, then

$$m\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \lim_{k \rightarrow \infty} m(A_k)$$

- If $\{B_k\}_{k \in \mathbb{N}}$ is descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k \in \mathbb{N}} B_k\right) = \lim_{k \rightarrow \infty} m(B_k)$$

3.7. An approximation result for Lebesgue measure

- **Definition:** **Borel σ -algebra** $\mathcal{B}(\mathbb{R})$ is smallest σ -algebra containing all intervals: for any other σ -algebra \mathcal{F} containing all intervals, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

$$\mathcal{B}(\mathbb{R}) := \bigcap \{\mathcal{F} : \mathcal{F} \text{ } \sigma \text{-algebra containing all intervals}\}$$

$E \in \mathcal{B}(\mathbb{R})$ is **Borel** or **Borel measurable**.

- **Lemma:** all open subsets of \mathbb{R} , closed subsets of \mathbb{R} , G_δ sets and F_σ sets are Borel.
- **Proposition:** the following are equivalent:
 - E is Lebesgue measurable
 - $\forall \varepsilon > 0, \exists \text{ open } G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
 - $\forall \varepsilon > 0, \exists \text{ closed } F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
 - $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
 - $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

4. Measurable functions

4.1. Definition of a measurable function

- **Proposition:** let $f : \mathbb{R} \rightarrow \mathbb{R}$. f continuous iff \forall open $U \subseteq \mathbb{R}$, $f^{-1}(U) \subseteq \mathbb{R}$ is open.
- **Lemma:** let $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ with E Lebesgue measurable. The following are equivalent:
 - $\forall c \in \mathbb{R}$, $\{x \in E : f(x) > c\}$ is Lebesgue measurable.
 - $\forall c \in \mathbb{R}$, $\{x \in E : f(x) \geq c\}$ is Lebesgue measurable.
 - $\forall c \in \mathbb{R}$, $\{x \in E : f(x) < c\}$ is Lebesgue measurable.
 - $\forall c \in \mathbb{R}$, $\{x \in E : f(x) \leq c\}$ is Lebesgue measurable.

The same statement holds for Borel measurable sets.

- **Definition:** $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is **(Lebesgue) measurable** if it satisfies any of the above properties and if E is Lebesgue measurable. f being **Borel measurable** is defined similarly.
- **Corollary:** if f is measurable then for every $B \in \mathcal{B}(\mathbb{R})$, $f^{-1}(B)$ is measurable. In particular, if f is measurable, preimage of any interval is measurable.
- **Definition: indicator function** on set A , $\mathbb{1}_A : \mathbb{R} \rightarrow \{0, 1\}$, is

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

- **Definition:** $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is **simple (measurable) function** if φ is measurable function that has finite codomain.

4.2. Fundamental aspects of measurable functions

- **Definition:** let $E \subseteq F \subseteq \mathbb{R}$, let $f : F \rightarrow \mathbb{R}$. **Restriction** f_E is function with domain E and for which $\forall x \in E$, $f_E(x) = f(x)$.
- **Definition:** real-valued function which is increasing or decreasing is **monotone**.
- **Definition:** sequence (f_n) on domain E is increasing if $f_n \leq f_{n+1}$ on E for all $n \in \mathbb{N}$.
- **Example:** continuous functions are measurable.
- **Definition:** for $f_1 : E \rightarrow \mathbb{R}, \dots, f_n : E \rightarrow \mathbb{R}$, define

$$\max\{f_1, \dots, f_n\}(x) := \max\{f_1(x), \dots, f_n(x)\}$$

$\min\{f_1, \dots, f_n\}$ is defined similarly.

- **Proposition:** for finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E , $\max\{f_1, \dots, f_n\}$ and $\min\{f_1, \dots, f_n\}$ are measurable.
- **Definition:** for $f : E \rightarrow \mathbb{R}$, functions $|f|, f^+, f^-$ defined on E are

$$|f|(x) := \max\{f(x), -f(x)\}, \quad f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}$$

- **Corollary:** if f measurable on E , so are $|f|, f^+$ and f^- .
- **Proposition:** let $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$. For measurable $D \subseteq E$, f measurable on E iff restrictions of f to D and $E - D$ are measurable.
- **Theorem:** let $f, g : E \rightarrow \mathbb{R}$ measurable.
 - **Linearity:** $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is measurable.
 - **Products:** fg is measurable.
- **Proposition:** let $f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be sequence of measurable functions that converges pointwise to $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$. Then f is measurable.

- **Simple approximation lemma:** let $f : E \rightarrow \mathbb{R}$ measurable and bounded, so $\exists M \geq 0 : \forall x \in E, |f|(x) < M$. Then $\forall \varepsilon > 0$, there exist simple measurable functions $\varphi_\varepsilon, \psi_\varepsilon : E \rightarrow \mathbb{R}$ such that

$$\forall x \in E, \quad \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \wedge 0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$$

- **Simple approximation theorem:** let $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$, E measurable. Then f is measurable iff there exists sequence (φ_n) of simple functions on E which converge pointwise on E to f and satisfy

$$\forall n \in \mathbb{N}, \forall x \in E, |\varphi_n|(x) \leq |f|(x)$$

If f is nonnegative, (φ_n) can be chosen to be increasing.

- **Definition:** let $f, g : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$. Then $f = g$ **almost everywhere** if $\{x \in E : f(x) \neq g(x)\}$ has measure 0.
- **Proposition:** let $f_1, f_2, f_3 : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ measurable. If $f_1 = f_2$ almost everywhere and $f_2 = f_3$ almost everywhere then $f_1 = f_3$ almost everywhere.
- **Remark:** Lebesgue measurable functions can be modified arbitrarily on a set of measure 0 without affecting measurability.
- **Proposition:** let $f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ sequence of measurable functions, $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ measurable. Set of points where (f_n) converges pointwise to f is measurable.
- **Proposition:** let $f, g : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ measurable and finite almost everywhere on E .
 - **Linearity:** $\forall \alpha, \beta \in \mathbb{R}$, there exists function equal to $\alpha f + \beta g$ almost everywhere on E (any such function is measurable).
 - **Products:** there exists function equal to fg almost everywhere on E (any such function is measurable).
- **Definition:** sequence of functions (f_n) with domain E **converge in measure** to f if (f_n) and f are finite almost everywhere and

$$\forall \varepsilon > 0, \quad \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

5. The Lebesgue integral

5.1. The integral of a simple measurable function

- **Definition:** let φ be real-valued function taking finitely many values $\alpha_1 < \dots < \alpha_n$, then **standard representation** of φ is

$$\varphi = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}, \quad A_i = \varphi^{-1}(\{\alpha_i\})$$

- **Lemma:** let $\varphi = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$, B_i disjoint measurable collection, $\beta_i \in \mathbb{R}$, then φ is simple measurable. If φ takes value 0 outside a set of finite measure then

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i)$$

where A_i in standard representation.

- **Definition:** let φ be simple nonnegative measurable function or simple measurable function taking value 0 outside set of finite measure. **Integral** of φ with respect to μ is

$$\int \varphi = \sum_{i=1}^n \alpha_i \mu(A_i)$$

where $\varphi = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$ is standard representation. Here, use convention $0 \cdot \infty = 0$. For measurable $E \subseteq \mathbb{R}$, define

$$\int_E \varphi = \int \mathbb{1}_E \varphi$$

- **Example:**
 - Let $\varphi_2 = \mathbb{1}_{[0,2]} + \mathbb{1}_{[1,3]} = \mathbb{1}_{[0,1) \cup (2,3]} + 2\mathbb{1}_{[1,2]}$ so $\int \varphi_2 = 4$.
 - Let $\varphi_3 = \mathbb{1}_{\mathbb{R}}$, then $\int \varphi_3 = 1 \cdot \infty = \infty$.
 - Let $\varphi_4 = \mathbb{1}_{(0,\infty)} + (-1)\mathbb{1}_{(-\infty,0)}$. This can't be integrated.
 - Let $\varphi_5 = \mathbb{1}_{(-1,0)} + (-1)\mathbb{1}_{(0,1)}$.
- **Lemma:** let B_1, \dots, B_m be measurable sets, $\beta_1, \dots, \beta_m \in \mathbb{R} - \{0\}$. Then $\varphi = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$ is simple measurable function. Also,

$$\mu\left(\bigcup_{i=1}^m B_i\right) < \infty \implies \sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i)$$

where A_i in standard representation.

- **Proposition:** let φ, ψ be simple measurable functions:
 - If φ, ψ take value 0 outside a set of finite measure, then $\forall \alpha, \beta \in \mathbb{R}$,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- If φ, ψ nonnegative, then $\forall \alpha, \beta \geq 0$,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- **Monotonicity:**

$$0 \leq \varphi \leq \psi \implies 0 \leq \int \varphi \leq \int \psi$$

- **Corollary:** let φ nonnegative simple function, then

$$\int \varphi = \sup \left\{ \int \psi : 0 \leq \psi \leq \varphi, \psi \text{ simple measurable} \right\}$$

- **Lemma:** let φ simple measurable nonnegative function. φ takes value 0 outside a set of finite measure iff $\int \varphi < \infty$. Also, $\int \varphi = \infty$ iff there exist $\alpha > 0$, measurable A with $\mu(A) = \infty$ and $\forall x \in A, \varphi(x) \geq \alpha$.
- **Lemma:** let $\{E_n\}$ be ascending collection of measurable sets, $\bigcup_{n \in \mathbb{N}} E_n = \mathbb{R}$. Let φ be simple nonnegative measurable function. Then

$$\int_{E_n} \varphi \rightarrow \int \varphi \quad \text{as } n \rightarrow \infty$$

5.2. The integral of a nonnegative function

- **Notation:** let \mathcal{M}^+ denote collection of nonnegative measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$.
- **Definition: support** of measurable function f with domain E is $\text{supp}(f) := \{x \in E : f(x) \neq 0\}$.
- **Definition:** let $f \in \mathcal{M}^+$. **Integral of f with respect to μ** is

$$\int f := \sup \left\{ \int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple measurable} \right\} \in \mathbb{R} \cup \{\infty\}$$

For measurable set E , define

$$\int_E f := \int \mathbb{1}_E f$$

- **Proposition:** let f, g measurable. If $g \leq f$ then $\int g \leq \int f$. Let E, F measurable. If $E \subseteq F$ then $\int_E f \leq \int_F f$.
- **Monotone convergence theorem:** let (f_n) be sequence in \mathcal{M}^+ . If (f_n) is increasing on measurable set E and converges pointwise to f on E then

$$\int_E f_n \rightarrow \int_E f \quad \text{as } n \rightarrow \infty$$

- **Corollary:** restriction of integral to nonnegative functions is linear: $\forall f, g \in \mathcal{M}^+, \forall \alpha \geq 0$,

$$\begin{aligned} \int (f + g) &= \int f + \int g \\ \int \alpha f &= \alpha \int f \end{aligned}$$

- **Fatou's lemma:** let (f_n) be sequence in \mathcal{M}^+ , then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$$

- **Lemma:** let $(f_n) \subset \mathcal{M}^+$, then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

- **Proposition (Chebyshev's inequality):** let f be nonnegative measurable function on E . Then

$$\forall \lambda > 0, \quad \mu(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f$$

- **Proposition:** let f be nonnegative measurable function on E . Then

$$\int_E f = 0 \iff f = 0 \text{ almost everywhere on } E$$

5.3. Integration of measurable functions

- **Notation:** let \mathcal{M} denote set of measurable functions.
- **Definition:** $f \in \mathcal{M}^+$ is **integrable** if $\int f < \infty$.
- **Definition:** let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ measurable function. f is **integrable** if $\int f^+$ and $\int f^-$ are finite. In this case, for any measurable set E , define

$$\int_E f := \int_E f^+ - \int_E f^-$$

Note that if f integrable then $f^+ - f^-$ is well-defined.

- **Proposition:** if $f = f_1 - f_2$, $f_1, f_2 \in \mathcal{M}^+$, f_1, f_2 integrable, then

$$\int f^+ - \int f^- = \int f_1 - \int f_2$$

- **Definition:** $f \in \mathcal{M}$ is **integrable over E** (E is measurable) if $\int_E f^+$ and $\int_E f^-$ are finite (i.e. $f \cdot \mathbb{1}_E$ is integrable).
- **Theorem:** $f \in \mathcal{M}$ is integrable iff $|f|$ is integrable. If f integrable, then

$$\left| \int f \right| \leq \int |f|$$

- **Corollary:** let $f, g \in \mathcal{M}$, $|f| \leq |g|$. If g integrable then $|f|$ is integrable, and $\int |f| \leq \int |g|$.
- **Example:** \sin is not integrable over \mathbb{R} , but is integrable over $[0, 2\pi]$, since $|f_{[0, 2\pi]}| \leq \mathbb{1}_{[0, 2\pi]}$.
- **Theorem (Linearity of Integration):** let $f, g \in \mathcal{M}$ integrable. Then $f + g$ is integrable and $\forall \alpha \in \mathbb{R}$, αf is integrable. The integral is linear:

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

- **Dominated Convergence Theorem:** let (f_n) be sequence of integrable functions. If there exists an integrable g with $\forall n \in \mathbb{N}$, $|f_n| \leq g$, and $f_n \rightarrow f$ pointwise almost everywhere then f is integrable and

$$\int f = \lim_{n \rightarrow \infty} \int f_n$$

5.4. Integrability: Riemann vs Lebesgue

- **Proposition:** let f bounded function on bounded measurable domain E . Then f is measurable and $\int_E |f| < \infty$ iff

$$\sup \left\{ \int_E \varphi : \varphi \leq f, \varphi \text{ simple measurable} \right\} = \inf \left\{ \int_E \psi : f \leq \psi : \psi \text{ simple measurable} \right\}$$

(If f satisfies either condition then $\int_E f$ is equal to the two above expressions).

- **Definition:** bounded function f is **Lebesgue integrable** if it satisfies either of the equivalences in the above proposition.
- **Definition:** let $P = \{x_1, \dots, x_n\}$ partition of $[a, b]$, $f : [a, b] \rightarrow \mathbb{R}$ bounded. **Lower and upper Darboux sums** for f with respect to P are

$$L(f, P) := \sum_{i=1}^n m_i(x_i - x_{i-1}), \quad U(f, P) := \sum_{i=1}^n M_i(x_i - x_{i-1})$$

where

$$m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

If $P \subseteq Q$ (Q is a **refinement of P**), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

- **Definition:** lower and upper Riemann integrals of f over $[a, b]$ are

$$\underline{\mathcal{J}}_a^b(f) := \sup\{L(f, P) : P \text{ partition of } [a, b]\}$$

$$\overline{\mathcal{J}}_a^b(f) := \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

- **Definition:** let $f : [a, b] \rightarrow \mathbb{R}$ bounded, then f is **Riemann integrable** ($f \in \mathcal{R}$), if

$$\underline{\mathcal{J}}_a^b(f) = \overline{\mathcal{J}}_a^b(f)$$

and common value $\mathcal{J}_a^b(f) = \int_a^b f(x) dx$ is **Riemann integral** of f .

- Let $g : [a, b] \rightarrow \mathbb{R}$ step function with discontinuities at $P = \{x_0, \dots, x_n\}$, so $g = \sum_{i=1}^n \alpha_i \mathbb{1}_{(x_{i-1}, x_i)}$ almost everywhere. So g is simple measurable and

$$L(g, P) = \sum_{i=1}^n \alpha_i(x_i - x_{i-1}) = U(g, P) = \int g = \mathcal{J}_a^b(g)$$

Hence for any bounded $f : [a, b] \rightarrow \mathbb{R}$,

$$\underline{\mathcal{J}}_a^b(f) = \sup \left\{ \int \varphi : \varphi \leq f, \varphi \text{ step function} \right\},$$

$$\overline{\mathcal{J}}_a^b(f) = \inf \left\{ \int \psi : f \leq \psi, \psi \text{ step function} \right\}$$

- **Theorem:** let $f : [a, b] \rightarrow \mathbb{R}$ bounded, $a, b \neq \pm\infty$. If f Riemann integrable over $[a, b]$ then f Lebesgue integrable over $[a, b]$ and the two integrals are equal.
- **Theorem:** let $f : [a, b] \rightarrow \mathbb{R}$ bounded, $a, b \neq \pm\infty$. Then f is Riemann integrable on $[a, b]$ iff f is continuous on $[a, b]$ except on a set of measure zero.
- **Lemma:** let $(\varphi_n), (\psi_n)$ be sequences of functions, all integrable over E , (φ_n) increasing on E , (ψ_n) decreasing on E . Let $f : E \rightarrow \mathbb{R}$ with

$$\forall n \in \mathbb{N}, \varphi_n \leq f \leq \psi_n \text{ on } E, \quad \lim_{n \rightarrow \infty} \int_E (\psi_n - \varphi_n) = 0$$

Then $\varphi_n, \psi_n \rightarrow f$ pointwise almost everywhere on E , f is integrable over E and

$$\lim_{n \rightarrow \infty} \int_E \varphi_n = \lim_{n \rightarrow \infty} \int_E \psi_n = \int_E f$$

- **Definition:** for partition $P = \{x_0, \dots, x_n\}$, **gap** of P is

$$\text{gap}(P) := \max\{|x_i - x_{i-1}| : i \in \{1, \dots, n\}\}$$

- **Lemma:** let $f : [a, b] \rightarrow \mathbb{R}$, $E \subseteq [a, b]$ be set where f is continuous. Let (P_n) be sequence of partitions of $[a, b]$ with $P_{n+1} \subseteq P_n$ and $\text{gap}(P_n) \rightarrow 0$ as $n \rightarrow \infty$. Let $\varphi_n, \psi_n : [a, b] \rightarrow \mathbb{R}$ step functions with

$$\varphi_n(x) := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad \psi_n(x) := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

for $P_n = \{x_0, \dots, x_n\}$. Then $\forall x \in E - \cup_{n \in \mathbb{N}} P_n$,

$$\varphi_n(x), \psi_n(x) \rightarrow f(x) \quad \text{as } n \rightarrow \infty$$

- **Definition:** let $f : (a, b) \rightarrow \mathbb{R}$, $-\infty \leq a < b < \infty$, f bounded and Riemann integrable on all closed bounded sub-intervals of $(a, b]$. If

$$\lim_{t \rightarrow a, t > a} \mathcal{J}_t^b(f)$$

exists then this is defined as the **improper Riemann integral** $\mathcal{J}_a^b(f)$. Similar definitions exist for $f : (a, b) \rightarrow \mathbb{R}$ and $f : [a, b) \rightarrow \mathbb{R}$.

- **Note:** improper Riemann integral may exist without function being Lebesgue integral.
- **Proposition:** if f is integrable, the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.
- **Definition:** let $\alpha : [a, b] \rightarrow \mathbb{R}$ monotonically increasing (and so bounded). For partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ and bounded $f : [a, b] \rightarrow \mathbb{R}$, define

$$L(f, P, \alpha) := \sum_{i=1}^n m_i(\alpha(x_i) - \alpha(x_{i-1})), \quad U(f, P, \alpha) := \sum_{i=1}^n M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

where $m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}$, $M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$. Then f is **integrable with respect to α** , $f \in \mathcal{R}(\alpha)$, if

$$\inf\{U(f, P, \alpha) : P \text{ partition of } [a, b]\} = \sup\{L(f, P, \alpha) : P \text{ partition of } [a, b]\}$$

and the common value $\int_a^b f d\alpha$ is the **Riemann-Stieltjes integral** of f with respect to α .

- **Proposition:** let $f : (a, b) \rightarrow \mathbb{R}$, then set of points where f is differentiable is measurable.
- **Remark:** if $\alpha : [0, 1] \rightarrow [a, b]$ bijection, then

$$\int_0^1 f \circ \alpha \, d\alpha = \int_a^b f(x) \, dx$$

- **Proposition:** let α be monotonically increasing and differentiable with $\alpha' \in \mathcal{R}$. Then $g \in \mathcal{R}(\alpha)$ iff $g\alpha' \in \mathcal{R}$, and in that case,

$$\int_a^b g \, d\alpha = \int_a^b g(x)\alpha'(x) \, dx$$

- **Remark:** when $g = 1$, this says $\int_a^b 1 \, d\alpha = \alpha(b) - \alpha(a) = \int \alpha'(x) \, dx$, similar to the fundamental theorem of calculus.

6. Lebesgue spaces

6.1. Normed linear spaces

- **Definition:** let X be **complex linear space** (vector space over \mathbb{C}). $\|\cdot\| : X \rightarrow \mathbb{R}_{\geq 0}$ is **norm on X** if
 - $\forall x \in X, \|x\| = 0 \iff x = 0$.
 - $\forall x \in X, \forall \lambda \in \mathbb{C}, \|\lambda x\| = |\lambda| \|x\|$.
 - $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.

X equipped with norm $\|\cdot\|$, $(X, \|\cdot\|)$, is called **complex normed linear space**.

- **Example:**
 - $\|x\| = \sqrt{x\bar{x}}$ is norm on \mathbb{C} .
 - Let $C[a, b]$ denote linear space of continuous real-valued functions on $[a, b]$. Then

$$\|f\|_{\max} := \max\{|f(x)| : x \in [a, b]\}$$

is norm on $C[a, b]$.

- **Proposition:** norm induces metric on X : $d(x, y) = \|x - y\|$.
- **Definition:** let $(X, \|\cdot\|)$ be normed linear space.
 - Sequence (f_n) in X is **Cauchy sequence** in X if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \|f_n - f_m\| < \varepsilon$$
 - Sequence (f_n) in X **converges in X** , $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$, if

$$\exists f \in X : \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \|f_n - f\| < \varepsilon$$
 - $(X, \|\cdot\|)$ is **complete** if every Cauchy sequence converges in X .
 - **Banach space** is complete normed linear space.
- **Proposition:** let $(X, \|\cdot\|)$ be normed linear space.
 - If (x_n) converges in X , (x_n) is Cauchy sequence in X .
 - Let (x_n) be Cauchy sequence in X . If (x_n) has convergent subsequence in X then (x_n) converges in X .

6.2. Lebesgue spaces L^p , $p \in [1, \infty)$

- **Definition:** let $p \in [1, \infty)$, $E \subseteq \mathbb{R}$.
 - Linear space $L^p(E)$ is defined as

$$L^p(E) := \left\{ f : E \rightarrow \mathbb{C} : f \text{ is measurable and } \int_E |f|^p < \infty \right\} / \cong$$

where $f \cong g$ iff $f = g$ almost everywhere:

$$f \cong g \iff \exists F \subseteq E : \mu(F) = 0 \wedge \forall x \in E - F, f(x) = g(x)$$

- Define $\|\cdot\|_{L^p} : L^p(E) \rightarrow \mathbb{R}$ as

$$\|f\|_{L^p} := \left(\int_E |f|^p \right)^{1/p}$$

• **Remark:**

- We often consider space $L^p(E)$ of real-valued measurable functions $f : E \rightarrow \mathbb{R}$ such that $\int_E |f|^p < \infty$.
- For $f : E \rightarrow \mathbb{C}$, $f = f_1 + if_2$, f is measurable iff $f_1 : E \rightarrow \mathbb{R}$ and $f_2 : E \rightarrow \mathbb{R}$ are measurable. Also,

$$\int_E |f|^p < \infty \iff \left(\int_E |f_1|^p < \infty \wedge \int_E |f_2|^p < \infty \right)$$

- **Example:** let $E = \mathbb{R}$, $f(x) = \mathbf{1}_{\mathbb{R}-\mathbb{Q}}(x) + i\mathbf{1}_{\mathbb{Q}}(x)$ and $g(x) = 1$. Then $\mu(\mathbb{Q}) = 0$ so $f \cong g$.
- **Proposition:** let $(f_n), (g_n)$ sequences of measurable functions, $\forall n \in \mathbb{N}, f_n \cong g_n$, $\lim_{n \rightarrow \infty} f_n = f$ and $\lim_{n \rightarrow \infty} g_n = g$. Then $f \cong g$.
- **Definition:** $p, q \in \mathbb{R}$ are **conjugate exponents** if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
- **Lemma (Young's inequality):** let p, q conjugate exponents, then

$$\forall A, B \in \mathbb{R}_{\geq 0}, \quad AB \leq \frac{A^p}{p} + \frac{B^q}{q}$$

with equality iff $A^p = B^q$.

- **Lemma (Hölder's inequality):** let p, q conjugate exponents. If $f \in L^p(E)$, $g \in L^q(E)$, then

$$\int_E |fg| \leq \|f\|_{L^p} \|g\|_{L^q}$$

- **Corollary (Cauchy-Schwarz inequality for $L^2(E)$):** if $f, g \in L^2(E)$, then

$$\left| \int_E f \bar{g} \right| \leq \int_E |fg| \leq \|f\|_{L^2} \|g\|_{L^2}$$

- **Lemma (Minkowski's inequality):** let $p \in [1, \infty)$. If $f, g \in L^p(E)$ then $f + g \in L^p(E)$ and

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

- **Theorem:** for $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is normed linear space.
- **Proposition:** let $1 \leq p < q < \infty$. If $\mu(E) < \infty$ then $L^q(E) \subseteq L^p(E)$ and

$$\|f\|_{L^p} \leq \mu(E)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q}$$

- **Remark:**
 - Convergence in L^p is also called convergence in the mean of order p .
 - This notion of convergence is different to pointwise convergence, uniform convergence and convergence in measure.
- **Riesz-Fischer theorem:** for $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is complete.

6.3. Lebesgue space L^∞

- **Definition:**
 - Let $f : E \rightarrow \mathbb{C}$ measurable. f is **essentially bounded** if

$$\exists M \geq 0 : |f(x)| \leq M \quad \text{almost everywhere on } E$$

- $L^\infty(E)$ is collection of equivalence classes of essentially bounded functions where $f \cong g$ iff $f = g$ almost everywhere.
- For $f \in L^\infty(E)$, define

$$\|f\|_{L^\infty} := \text{ess sup} |f| := \inf\{M \in \mathbb{R} : \mu(\{x \in E : |f(x)| > M\}) = 0\}$$

- **Proposition:**
 - $0 \leq |f(x)| \leq \|f\|_{L^\infty}$ almost everywhere.
 - $\|f\|_{L^\infty}$ is norm on $L^\infty(E)$.
 - If $f \in L^1(E)$, $g \in L^\infty(E)$, then

$$\int_E |fg| \leq \|f\|_{L^1} \|g\|_{L^\infty}$$

- **Proposition:** let (f_n) sequence of functions in $L^\infty(E)$. Then (f_n) converges to $f \in L^\infty(E)$ iff there exists $G \subseteq E$ with $\mu(G) = 0$ and (f_n) converges to f uniformly on $E - G$.
- **Theorem:** $(L^\infty(E), \|\cdot\|_{L^\infty})$ is complete.
- **Remark:** if $\mu(E) < \infty$, then $L^\infty(E) \subset L^p(E)$ for $p \in [1, \infty)$ and

$$\|f\|_{L^p} \leq \mu(E)^{1/p} \|f\|_{L^\infty}$$

since

$$\|f\|_{L^p}^p = \int_E |f|^p \leq \int_E \|f\|_{L^\infty}^p \cdot \mathbb{1}_E = \|f\|_{L^\infty}^p \mu(E)$$

6.4. Approximation and separability

- **Definition:** let $(X, \|\cdot\|)$ be normed linear space. Let $F \subseteq G \subseteq X$. F is **dense in G** if

$$\forall g \in G, \forall \varepsilon > 0, \exists f \in F : \|f - g\| < \varepsilon$$

- **Proposition:**
 - F is dense in G iff for every $g \in G$, there exists sequence (f_n) in F such that $\lim_{n \rightarrow \infty} f_n = g$ in X .
 - For $F \subseteq G \subseteq H \subseteq X$, if F dense in G and G dense in H , then F dense in H .

- **Proposition:** let $p \in [1, \infty]$. Then subspace of simple functions in $(L^p(E), \|\cdot\|_{L^p})$ is dense in $(L^p(E), \|\cdot\|_{L^p})$.
- **Definition:** $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is **step function** if it can be written as

$$\psi = \sum_{k=1}^N \tilde{a}_k \mathbb{1}_{(a_k, b_k)}$$

where the intervals (a_k, b_k) are disjoint.

- **Proposition:** let $[a, b]$ be bounded, $p \in [1, \infty)$. Then subspace of step functions on $[a, b]$ is dense in $(L^p([a, b]), \|\cdot\|_{L^p})$.
- **Definition:** normed linear space $(X, \|\cdot\|)$ is **separable** if there exists countable, dense subset $X' \subseteq X$.
- **Example:** \mathbb{R} is separable, since \mathbb{Q} is countable and dense in \mathbb{R} .
- **Theorem:** let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$. Then $(L^p(E), \|\cdot\|_{L^p})$ is separable.
- **Proposition:** let $\varepsilon > 0$, $f \in L^p(E)$, $p \in [1, \infty)$. There exists continuous $g \in L^p(E)$ such that $\|f - g\|_{L^p} < \varepsilon$.
- **Remark:** linear space of continuous functions that vanish outside bounded set is dense in $(L^p(E), \|\cdot\|_{L^p})$ for $p \in [1, \infty)$.
- **Remark:** differentiable functions are also dense in $(L^p(E), \|\cdot\|_{L^p})$ for $p \in [1, \infty)$.
- **Remark:** step functions and continuous functions are not dense in $(L^\infty(E), \|\cdot\|_{L^\infty})$.
- **Example:** in general, $(L^\infty(E), \|\cdot\|_{L^\infty})$ is not separable. Let $[a, b]$ be bounded, $a \neq b$. Assume there is countable $\{f_n : n \in \mathbb{N}\}$ which is dense in $(L^\infty([a, b]), \|\cdot\|_{L^\infty})$. Then for every $x \in [a, b]$, can choose $g(x) \in \mathbb{N}$ such that

$$\|\mathbb{1}_{[a, x]} - f_{g(x)}\|_{L^\infty} < \frac{1}{2}$$

Also, for $x_1 \leq x_2$,

$$\|\mathbb{1}_{[a, x_1]} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} = \begin{cases} 1 & \text{if } a \leq x_1 < x_2 \leq b \\ 0 & \text{if } x_1 = x_2 \end{cases}$$

and

$$\begin{aligned} \|\mathbb{1}_{[a, x_1]} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} &\leq \|\mathbb{1}_{[a, x_1]} - f_{g(x_1)}\|_{L^\infty} + \|f_{g(x_1)} - f_{g(x_2)}\|_{L^\infty} + \|f_{g(x_2)} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} \\ &< 1 + \|f_{g(x_1)} - f_{g(x_2)}\|_{L^\infty} \end{aligned}$$

If $g(x_1) = g(x_2)$ then $\|\mathbb{1}_{[a, x_1]} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} = 0$ so $g : [a, b] \rightarrow \mathbb{N}$ is injective. But \mathbb{N} is countable and $[a, b]$ is not countable: contradiction.

6.5. Riesz representation theorem for $L^p(E)$, $p \in [1, \infty)$

- **Definition:** let X be linear space. $T : X \rightarrow \mathbb{R}$ is **linear functional** if

$$\forall f, g \in X, \forall a, b \in \mathbb{R}, \quad T(af + bg) = aT(f) + bT(g)$$

Any linear combination of linear functionals is linear, so set of linear functionals on linear space is also linear space.

- **Definition:** let $(X, \|\cdot\|)$ be normed linear space. $T : X \rightarrow \mathbb{R}$ is **bounded functional** if

$$\exists M \geq 0 : \forall f \in X, \quad |T(f)| \leq M\|f\|$$

Norm of T , $\|T\|_*$, is the smallest such M .

- **Remark:** for bounded linear functional T on normed linear space $(X, \|\cdot\|)$,

$$|T(f) - T(g)| \leq \|T\|_* \|f - g\|$$

This gives the following continuity property: if $f_n \rightarrow f \in X$, then $T(f_n) \rightarrow T(f)$.

- **Example:** let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, q conjugate to p . Let $h \in L^q(E)$. Define $T : L^p(E) \rightarrow \mathbb{R}$ by

$$T(f) = \int_E h \cdot f$$

By Holder's inequality,

$$|T(f)| = \left| \int_E hf \right| \leq \int_E |hf| \leq \|h\|_{L^q} \|f\|_{L^p}$$

So T is bounded linear functional.

- **Remark:** we can write $\|\cdot\|_*$ as

$$\|T\|_* := \inf\{M \in \mathbb{R} : \forall f \in X, |T(f)| \leq M\|f\|\} = \sup\{|T(f)| : f \in X, \|f\| \leq 1\}$$

- **Definition: dual space** of X , X^* , is set of bounded linear functionals on X with norm $\|\cdot\|_*$.
- **Proposition:** let $(X, \|\cdot\|)$ be normed linear space, then dual space of X is linear space.
- **Remark:** bounded linear functional is special case of **bounded linear transformation** between normed spaces. $T : X \rightarrow Y$ is bounded linear transformation if $T(af + bg) = aT(f) + bT(g)$ and $\exists M \geq 0 : \|T(f)\|_Y \leq M\|f\|_X$.
- **Proposition:** let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, q conjugate to p , $h \in L^q(E)$. Define $T : L^p(E) \rightarrow \mathbb{R}$ by

$$T(f) = \int_E hf$$

Then $\|T\|_* = \|h\|_{L^q}$.

- **Riesz representation theorem for L^p :** let $p \in [1, \infty)$, q conjugate to p , $E \subseteq \mathbb{R}$ measurable. For $h \in L^q(E)$, define bounded linear functional $R_h : L^p(E) \rightarrow \mathbb{R}$ by

$$R_h(f) = \int_E hf$$

Then for every bounded linear functional $T : L_p(E) \rightarrow \mathbb{R}$, there is unique $h \in L^q(E)$ such that

$$R_h = T \quad \wedge \quad \|T\|_* = \|h\|_{L^q}$$

- **Theorem:** let $[a, b]$ be non-degenerate, bounded interval, $p \in [1, \infty)$, q conjugate to p . If T is bounded linear functional on $L^p([a, b])$ then there exists $h \in L^q([a, b])$ such that

$$T(f) = \int_a^b h f$$

7. Hilbert spaces

7.1. Inner product spaces

- **Definition:** let H be complex linear space. **Inner product** on H is function $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$ such that $\forall a, b \in \mathbb{C}, \forall x, y, z \in H$,
 - **Linear in first variable:** $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
 - **Conjugate symmetric:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
 - **Positive:** $x \neq 0 \implies \langle x, x \rangle \in (0, \infty)$
 - $\langle x, x \rangle = 0 \iff x = 0$.

These imply that $\langle 0, x \rangle = 0$ and inner product is conjugate linear in second variable: $\langle z, ax + by \rangle = \bar{a}\langle z, x \rangle + \bar{b}\langle z, y \rangle$.

- **Example:**
 - \mathbb{R}^n has inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$.
 - \mathbb{C}^n has inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i$.
 - Inner product induces metric on H :

$$d(x, y) = \langle x - y, x - y \rangle^{1/2}$$

- **Definition:** complex linear space H with inner product $\langle \cdot, \cdot \rangle$ is called **pre-Hilbert space** or **inner product space**.
- **Definition:** let H inner product space. For $x \in H$, define the norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

- **Proposition:** $\|x \pm y\|^2 = \|x\|^2 \pm 2 \operatorname{Re}(\langle x, y \rangle) + \|y\|^2$.
- **Theorem (Cauchy-Schwarz inequality):** let $(H, \langle \cdot, \cdot \rangle)$ be pre-Hilbert space. Then

$$\forall x, y \in H, \quad |\langle x, y \rangle| \leq \|x\| \|y\|$$

with equality iff x and y linearly dependent.

- **Theorem (Parallelogram Identity):** a normed linear space X is an inner product space with norm derived from the inner product (i.e. $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$) iff

$$\forall x, y \in X, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

- **Definition:** let $(X, \langle \cdot, \cdot \rangle_X)$, $(Y, \langle \cdot, \cdot \rangle_Y)$ be inner product spaces.

- An inner product on $X \times Y$ is

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{X \times Y} = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y$$

- The associated norm on $X \times Y$ is

$$\|(x, y)\|_{X \times Y} = \sqrt{\langle (x, y), (x, y) \rangle_{X \times Y}} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$$