Contents

1. Set systems

Definition. Let X be a set. A **set system** on X (also called a **family of subsets of** X) is a collection $\mathcal{A} \subseteq \mathbb{P}(X)$.

Notation. $X^{(r)} := \{A \subseteq X : |A| = r\}$ denotes the family of subsets of X of size r.

Remark. Usually, we take $X = [n] = \{1, ..., n\}$, so $|X^{(r)}| = \binom{n}{r}$.

Notation. For brevity, we write e.g. $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$.

Definition. We can visualise $\mathbb{P}(A)$ as a graph by joining nodes $A \in \mathbb{P}(X)$ and $B \in \mathbb{P}(X)$ if $|A \Delta B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa.

This graph is the **discrete cube** Q_n.

Alternatively, we can view Q_n as an n-dimensional unit cube $\{0, 1\}^n$ by identifying e.g. $\{1, 3\} \subseteq [5]$ with 10100 (i.e. identify A with $\mathbb{1}_A$, the characteri stic/indicator function of A).

Definition. $\mathcal{A} \subseteq \mathbb{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{A}$, $A \subseteq B$ or $B \subseteq A$.

Example.
- $\mathcal{A} = \{23, 1235, 123567\}$ is a chain.
- $\mathcal{A} = \{\emptyset, 1, 12, ..., [n]\} \subseteq \mathbb{P}([n])$ is a chain.

Definition. $\mathcal{A} \subseteq \mathbb{P}(X)$ is an **antichain** if $\forall A \neq B \in \mathcal{A}$, $A \nsubseteq B$.

Example.
- $\mathcal{A} = \{23, 137\}$ is an antichain.
- $\mathcal{A} = \{1, ..., n\} \subseteq \mathbb{P}([n])$ is an antichain.
- More generally, $\mathcal{A} = X^{(r)}$ is an antichain for any r.

Proposition. A chain $\mathcal{A} \subseteq \mathbb{P}([n])$ can have at most $n + 1$ elements.

Proof. For each $0 \leq r \leq n$, $\mathcal{A}$ can contain at most 1 r-set (set of size r). □

2. Isoperimetric inequalities

3. Intersecting families