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Definition. A **category** $\mathcal{C}$ consists of:

1. a collection $\mathrm{ob}(\mathcal{C})$ of objects $A, B, C, \ldots$,
2. a collection $\mathrm{mor}(\mathcal{C})$ of morphisms $f, g, h, \ldots$,
3. two operations dom and cod from $\mathrm{mor}(\mathcal{C})$ to $\mathrm{ob}(\mathcal{C})$. We write $f : A \to B$ to mean f is a morphism and $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$.
4. an operation from $\mathrm{ob}(\mathcal{C})$ to $\mathrm{mor}(\mathcal{C})$ sending A to $\mathbb{1}_A : A \to A$.
5. a partial binary operation $(f, g) \mapsto fg$ on $\mathrm{mor}(C)$, such that fg is defined iff $\mathrm{dom}(f) = \mathrm{cod}(g)$, and in this case $\mathrm{dom}(fg) = \mathrm{dom}(g)$ and $\mathrm{cod}(fg) = \mathrm{cod}(f)$.

and satisfies the following:

1. $f\mathbb{1}_A = f$ and $\mathbb{1}_A g = g$ when the composites are defined.
2. $f(gh) = (fg)h$ whenever fg and gh are defined.

Remark.

- $\mathrm{ob}(\mathcal{C})$ and $\mathrm{mor}(\mathcal{C})$ are not necessarily sets. If they are, then $\mathcal{C}$ is called a **small** category.
- An equivalent definition exists without using objects.
- fg means first apply g, then f.

Example.

1. **Set** = the category of all sets and functions between them. (Formally, a morphism of **Set** is a pair (f, B) where f is a set-theoretic function and B is its codomain).
2. Algebraic categories:
   - **Gp** is the category of groups and group homomorphisms.
   - **Rng** is the category of rings and ring homomorphisms.
   - $\mathbf{Vect}_K$ is the category of vector spaces over a field K with linear maps.
3. Topological categories:
   - **Top** is the category of topological spaces and continuous maps.
   - **Met** is the category of metric spaces and non-expansive maps (i.e. $d(f(x), f(y)) \leq d(x, y)$).
   - **Mfd** is the category of smooth manifolds and smooth (C^∞) maps.
   - **TopGp** is the category of topological groups and continuous homomorphisms.
4. Quotient categories:
   - **Htpy** is the category with same objects as **Top** but morphisms are homotopy classes of continuous maps.
   - In general, given a category $\mathcal{C}$ and an equivalence relation $\sim$ on $\mathrm{mor}(\mathcal{C})$ such that $f \sim g \Rightarrow (\mathrm{dom}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(f) = \mathrm{cod}(g))$, and $f \sim g \Rightarrow fh \sim gh$ when the composites fh and gh are defined, we can form a **quotient** category $\mathcal{C}/\sim$.
5. **Rel** is the category with the same objects as **Set** but with morphisms that are relations $R \subseteq A \times B$, with composition defined by $R \circ S = \{(a, c) : \exists b : (a, b) \in S \wedge (b, c) \in R\}$.
6. **Part** is the category of sets and partial functions.

Definition. For every category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{\text{op}}$ has the same objects and morphisms as $\mathcal{C}$, but dom and cod are interchanged and composition is reversed. This yields a **duality principle**: if P is a true statement about categories, then so is P^* (which is obtained by reversing arrows in P).

Definition. A **monoid** is a small category with one object $*$ (a semigroup with an identity element). In particular, a group is a 1-object where all morphisms are isomorphisms.

Definition. A **groupoid** is a category where every morphism is an isomorphism.

Example. The **fundamental groupoid** of a space X, $\pi_1(X)$, is the category where objects are the points of X, and morphisms $x \to y$ are homotopy classes of path from x to y. (Note this depends only on X, whereas the fundamental group depends on X and a point $x \in X$).

Definition. A category is **discrete** if the only morphisms are identities.

Definition. A category $\mathcal{C}$ is a **preorder** if for every pair of objects (A, B), there exists at most 1 morphism $A \to B$, then $\text{mor}(C)$ becomes a reflexive and transitive relation on $\text{ob}(C)$.

In particular, a poset is a small preorder where the only isomorphisms are identities.

Example. For a field K, the category $\mathbf{Mat}_K$ has natural numbers as objects, morphisms $n \to m$ are $m \times n$ matrices with entries from K, and composition is matrix multiplication.