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1. Hidden subgroup problem

1.1. Review of Shor's algorithm

Problem: Factoring

Problem 1.1 (Factoring)

Input a positive integer N.

Promise N is composite.

Task Find a non-trivial factor of N in $O(\text{poly}(n))$ time, where $n = \log N$.

Definition: Efficient Problem

Definition 1.2 We say an **efficient problem** is one that can be solved in polynomial time.

Remark 1.3 Clasically, the best known factoring algorithm runs in $e^{O\left(n^{1/3}(\log n)^{2/3}\right)}$. Shor's algorithm runs in $O\left(n^3\right)$ by converting factoring into period finding:

- Given input N, choose $a < N$ which is coprime to N.
- Define $f : \mathbb{Z} \to \mathbb{Z}/N$, $f(x) = a^x \bmod N$. f is periodic with period r (the order of $a \bmod N$), i.e. $f(x + r) = f(x)$ for all $x \in \mathbb{Z}$. Finding r allows us to factor N.

1.2. Period finding

Problem: Periodicity Determination

Problem 1.4 (Periodicity Determination Problem)

Input An oracle for a function $f : \mathbb{Z}/M \to \mathbb{Z}/N$.

Promise

- f is periodic with period $r < M$ (i.e. $\forall x \in \mathbb{Z}/M$, $f(x + r) = f(x)$), and
- f is injective in each period (i.e. if $0 \leq x < y < r$, then $f(x) \neq f(y)$).

Task Determine the period r.

Remark 1.5 Solving the periodicity determination problem classically requires takes time $O(\sqrt{M})$.

Definition: Quantum Oracle

Definition 1.6 Let $f : \mathbb{Z}/M \to \mathbb{Z}/N$. Let H_M and H_N be quantum state spaces with orthonormal state bases $\{|i\rangle : i \in \mathbb{Z}/N\}$ and $\{|j\rangle : j \in \mathbb{Z}/M\}$. Define the unitary **quantum oracle** for f by U_f by

$$U_f|x\rangle|z\rangle = |x\rangle|z + f(x)\rangle.$$

The first register $|x\rangle$ is the **input register**, the last register $|z\rangle$ is the **output register**.

Definition: Quantum Query Complexity

Definition 1.7 The **query complexity** of an algorithm is the number of times it queries f (i.e. for the quantum case, the number of times it uses U_f).

Definition: Quantum Fourier Transform

Definition 1.8 The **quantum Fourier transform** over $\mathbb{Z}/M$ is the unitary QFT defined by its action on the computational basis:

$$\mathrm{QFT}|x\rangle = \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} \omega^{xy} |y\rangle,$$

where $\omega = e^{2\pi i/M}$ is an M-th root of unity. Note that QFT requires only $O((\log M)^2)$ gates to implement, whereas a general $M \times M$ unitary requires $O(4^M/M)$ elementary gates.

Lemma: Sum Of Geometric Series Of Phases

Lemma 1.9 Let $\alpha = e^{2\pi i y/M}$. Then

$$\sum_{j=0}^{k-1} \alpha^j = \begin{cases} \frac{1-\alpha^k}{1-\alpha} = 0 & \text{if } \alpha \neq 1 \text{ i.e. } M \nmid y \\ k & \text{if } \alpha = 1 \text{ i.e. } M \mid y \end{cases}.$$

Proof (Hints). Trivial. □

Proof. The sum is a geometric series with common ratio α. $\square$

Lemma: Boosting Success Probability

Lemma 1.10 (Boosting success probability) If a process succeeds with probability p on one trial, then

$$\Pr(\text{at least one success in } t \text{ trials}) = 1 - (1-p)^t > 1 - \delta$$

for $t = O\left(\frac{\log(1/d)}{p}\right)$.

Proof (Hints). Trivial. □

Proof. Trivial. □

Theorem: Coprimality Theorem

Theorem 1.11 (Co-primality Theorem) The number of integers less than r that are coprime to r is $O(r/\log\log r)$.

Algorithm: Quantum Period Finding

Algorithm 1.12 (Quantum Period Finding) The algorithm solves the Periodicity Determination Problem: Let $f : \mathbb{Z}/M \to \mathbb{Z}/N$ be periodic with period $r < M$ and one-to-one in each period. Let $A = \frac{M}{r}$ be the number of periods. We work over the state space $H_M \otimes H_N$.

1. Construct the state $\frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle|0\rangle$ and query U_f on it.
2. Measure second register in computational basis and discard the second register.
3. Apply the quantum Fourier transform to the input state.
4. Measure the input state, yielding outcome c.
5. Compute the denominator r_0 of the simplified fraction $\frac{c}{M}$.

6. Repeat the previous steps $O(\log\log r) = O(\log\log M) = O(\log m)$ times, halting if at any iteration, $f(0) = f(r_0)$.

Theorem: Correctness Of Period Finding Algorithm

Theorem 1.13 (Correctness of Quantum Period Finding Algorithm) When repeated, $O(\log \log r) = O(\log \log M)$ times, the quantum period finding algorithm obtains the correct value of r with high probability.

Proof (Hints). Straightforward. □

Proof. After querying U_f, we have the state $\frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle |f(i)\rangle$. Upon measuring the second register in the computational basis, the input state collapses to $|\text{per}\rangle = \frac{1}{\sqrt{A}} \sum_{j=0}^{A-1} |x_0 + jr\rangle$, where $f(x_0) = y$ and $0 \leq x_0 < r$. Applying the quantum Fourier transform to $|\text{per}\rangle$ then gives Quantum Fourier Transform to $|\text{per}\rangle$:

$$\begin{aligned} \text{QFT}|\text{per}\rangle &= \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} \frac{1}{\sqrt{A}} \sum_{j=0}^{A-1} \omega^{(x_0+jr)y} |y\rangle \\ &= \frac{1}{\sqrt{MA}} \sum_{y=0}^{M-1} \omega^{x_0 y} \sum_{j=0}^{A-1} \omega^{jry} |y\rangle \end{aligned}$$

$$= \sqrt{\frac{A}{M}} \sum_{k=0}^{r-1} \omega^{x_0 kM/r} |kM/r\rangle$$

Importantly, now the outcomes and probabilities are independent of x_0, so carry useful information about r. The outcome after the measuring the input state is $c = k_0 M/r$ for some $0 \leq k_0 < r$ (so $c/M = k_0/r$). If k_0 is coprime to r, then the denominator r_0 of the simplified fraction $\frac{c}{M}$ is equal to r. By the coprimality theorem, the probability that k_0 is coprime to r is $O(1/\log\log r)$. Checking if $f(0) = f(r_0)$ tells us if $r_0 = r$, since f is periodic and one-to-one in each period, and $r_0 \leq r$. $\square$

1.3. Analysis of QFT part of period finding algorithm

Notation 1.14 For $R = \{0, r, ..., (A-1)r\} \subseteq \mathbb{Z}/M$ $(Ar = M)$, write $|R\rangle$ for the uniform superposition of all computational basis states in R:

$$|R\rangle = \frac{1}{\sqrt{A}} \sum_{k=0}^{A-1} |kr\rangle.$$

Definition: Period Finding Shift Operator

Definition 1.15 For each $x_0 \in \mathbb{Z}/M$, define the lienar map by its action on the computational basis states:

$$U(x_0) : H_M \to H_M,$$
$$|k\rangle \mapsto |x_0 + k\rangle.$$

Definition: Quantum Fourier Characters

Definition 1.16 Note that since $(\mathbb{Z}/M, +)$ is abelian, all $U(x_i)$ commute: $U(x_1)U(x_2) = U(x_1 + x_2) = U(x_2)U(x_1)$. Hence, they have a simultaneous basis of eigenvectors $\{|\chi_k\rangle : k \in \mathbb{Z}/M\}$, i.e. for all $k, x_0 \in \mathbb{Z}/M$, $U(x_0)|\chi_k\rangle = w(x_0, k)|\chi_k\rangle$, where $|w(x_0, k)| = 1$. The $|\chi_k\rangle$ are called **shift-invariant states** and form an orthonormal basis for H_M. The $|\chi_k\rangle$ are given explicitly by

$$|\chi_k\rangle = \frac{1}{\sqrt{M}} \sum_{\ell=0}^{M-1} e^{-2\pi i k\ell/M} |\ell\rangle.$$

Definition: Fourier Basis State Are Shift Invariant

Proposition 1.17 The explicit definition of the $|\chi_k\rangle$ indeed satisfies the property $\forall k, x_0 \in \mathbb{Z}/M,\ U(x_0)|\chi_k\rangle = w(x_0,k)|\chi_k\rangle$, and we have $w(x_0,k) = \omega^{kx_0}$, where $\omega = e^{2\pi i/M}$.

Proof (Hints). Straightforward. □

Proof. We have that

$$\begin{aligned} U(x_0)|\chi_k\rangle &= \frac{1}{\sqrt{M}} \sum_{\ell=0}^{M-1} e^{-2\pi i k\ell/M} |x_0 + \ell\rangle \\ &= \frac{1}{\sqrt{M}} \sum_{\tilde{l}=0}^{M-1} e^{-2\pi i \left(\tilde{l}-x_0\right)k/M} \left|\tilde{l}\right\rangle \\ &= e^{2\pi i k x_0/M} |\chi_k\rangle \\ &=: w(x_0, k)|\chi_k\rangle \end{aligned}$$

□

Remark 1.18 Let $U : H_M \to H_M$ be the unitary mapping the shift-invariant basis to the computational basis: $U : |\chi_k\rangle \mapsto |k\rangle$. The matrix representation of U^{-1} with respect to the computational basis has entries

$$\left(U^{-1}\right)_{jk} = \langle j|U^{-1}|k\rangle = \langle j|\chi_k\rangle = \frac{1}{\sqrt{M}} e^{-2\pi ijk/M}$$

So the matrix representation of U with respect to the same basis has entries $U_{kj} = \overline{\left(U^{-1}\right)_{jk}} = \frac{1}{\sqrt{M}} e^{2\pi ijk/M}$. Hence, we have

$$U|k\rangle = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{2\pi i j k/M} |j\rangle,$$

and so U is precisely the QFT $\bmod M$.

1.4. The hidden subgroup problem (HSP)

Problem: Dlp

Problem 1.19 (Discrete Logarithm Problem (DLP))

Input $g, x \in G$ for an abelian group G.

Promise g is a generator of G.

Task Find $\log_g x$, i.e. find $L \in \mathbb{Z}/|G|$ such that $x = g^L$.

Notation 1.20 Write $[n]$ for $\{1, ..., n\}$. Write e.g. ij for the set $\{i, j\}$.

Definition: Graph.Isomorphic

Definition 1.21 Let $\Gamma_1 = ([n], E_1)$ and $\Gamma_2 = ([n], E_2)$ be (undirected) graphs. Γ_1 and Γ_2 are **isomorphic** if there exists a permutation $\pi \in S_n$ such that for all $1 \leq i, j < n$, $ij \in E_1$ iff $\pi(i)\pi(j) \in E_2$.

Definition: Graph Automorphism Group

Definition 1.22 Let $\Gamma = ([n], E)$ be a graph. The **automorphism group** of Γ is

$$\mathrm{Aut}(\Gamma) = \{\pi \in S_n : ij \in E \text{ iff } \pi(i)\pi(j) \in E \quad \forall i, j \in [n]\}.$$

$\mathrm{Aut}(\Gamma)$ is a subgroup of S_n, and $\pi \in \mathrm{Aut}(\Gamma)$ iff π leaves Γ invariant as a labelled graph.

Definition: Adjacency Matrix

Definition 1.23 The **adjacency matrix** of a graph $\Gamma = (V, E)$ is the $n \times n$ matrix M_A defined by its entries:

$$(M_A)_{ij} := \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}.$$

Problem: Graph Isomorphism

Problem 1.24 (Graph Isomorphism Problem)

Input Adjacency matrices M_1 and M_2 of graphs $\Gamma_1 = ([n], E_1)$ and $\Gamma_2 = ([n], E_2)$.

Task Determine whether Γ_1 and Γ_2 are isomorphic.

Remark 1.25 The best known classical algorithm for solving the graph isomorphism problem has quasi-polynomial time complexity $n^{O((\log n)^2)}$.

Problem: Hsp

Problem 1.26 (Hidden Subgroup Problem (HSP)) Let G be a finite group.

Input An oracle for a function $f : G \to X$.

Promise There is a subgroup $K < G$ such that:

1. f is constant on the (left) cosets of K in G.
2. f takes a different value on each coset.

Task Determine (a set of generators of) K.

Remark 1.27

- To find K, we either find a generating set for K, or sample uniformly random elements from K.
- We want to determine K with high probability in $O(\operatorname{poly}\log|G|)$ queries. Using $O(|G|)$ queries is easy, as we just query all values $f(g)$ and find the "level sets" (sets where f is constant).

Example 1.28 The following problems are special cases of HSP:

- The Periodicity Determination Problem: $G = \mathbb{Z}/M$, $K = \langle r \rangle = \{0, r, ..., (A-1)r\}$. The cosets are $x_0 + K = \{x_0, x_0 + r, ..., x_0 + (A-1)r\}$ for each $0 \leq x_0 < r$.
- The DLP on $(\mathbb{Z}/p)^\times$: let $f : \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1) \to (\mathbb{Z}/p)^\times$ be defined by $f(a,b) = g^a x^{-b} = g^{a-Lb}$. $G = \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$, the hidden subgroup is $K = \{\lambda(L, 1) : \lambda \in \mathbb{Z}/(p-1)\}$. (Note that if we know K, we can pick any $(c,d) = (\lambda L, \lambda) \in K$ and compute $L = \frac{c}{d}$ to find L.)
- The Graph Isomorphism Problem: $G = S_n$, hidden subgroup is $K = \mathrm{Aut}(G)$. Let $f_\Gamma : S_n \to X$ where X is set of adjacency matrices of

labelled graphs on $[n]$, defined by $f_\Gamma(\pi) = \pi(A)$. Note $|S_n| = |G| = n!$, so $\log|G| \approx n \log n$, so $O(\text{poly} \log|G|) = O(\text{poly}\, n)$.

Definition: Irrep

Definition 1.29 An **irreducible representation (irrep)** of a finite abelian group G is a homomorphism $\chi : G \to \mathbb{C}^\times$.

Theorem: Basic Properties Of Irreps

Theorem 1.30

- Let $\chi : G \to \mathbb{C}^\times$ be an irrep. For all $g \in G$, $\chi(g)$ is a $|G|$-th root of unity.
- There are always exactly $|G|$ distinct irreps. In particular, we can label each irrep uniquely by some $g \in G$.

Theorem: Schurs Lemma

Theorem 1.31 (Schur's Lemma) Let χ_i and χ_j be irreps of G. Then

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \delta_{ij}.$$

Proof. Omitted. □

Example 1.32 $\chi_0 : G \to \mathbb{C}^\times$, $\chi_0(g) = 1$ is the **trivial irrep**. Note that for any $\chi_i \neq \chi_0$, $\sum_{g \in G} \chi_i(g) = 0$ by Schur's lemma.

Definition: Shift Operator

Definition 1.33 For finite abelian G, we define the **shift operators** on $H_{|G|}$ for each $k \in G$ by

$$\begin{aligned} U(k) : H_{|G|} &\to H_{|G|}, \\ |g\rangle &\mapsto |k+g\rangle. \end{aligned}$$

Note that since G is abelian, the $U(k)$ commute:

$$U(k)U(l) = U(k+l) = U(l)U(k) \quad \forall k, l \in G.$$

Hence, they have simultaneous eigenstates, which gives an orthonormal basis for $H_{|G|}$.

Proposition: Shift Invariant States

Proposition 1.34 For each $k \in G$, consider the state

$$|\chi_k\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \overline{\chi_k(g)} |g\rangle.$$

The $|\chi_k\rangle$ are shift-invariant (invariant up to a phase under the action of all $U(g)$, $g \in G$).

Proof (Hints). Straightforward. □

Proof. Since χ_k is a homomorphism, we have $\overline{\chi_k(g)} = \chi_k(-g)$. Now

$$\begin{aligned} U(g_0)|\chi_k\rangle &= \frac{1}{\sqrt{|G|}} \sum_{g \in G} \overline{\chi_k(g)} |g_0 + g\rangle \\ &= \frac{1}{\sqrt{|G|}} \sum_{g' \in G} \overline{\chi_k(g' - g_0)} |g'\rangle \\ &= \frac{1}{\sqrt{|G|}} \sum_{g' \in G} \overline{\chi_k(g')} \chi_k(g_0) |g'\rangle \\ &= \chi_k(g_0) |\chi_k\rangle. \end{aligned}$$

□

Definition: Quantum Fourier Transform Abelian

Definition 1.35 The **quantum Fourier transform (QFT)** on $H_{|G|}$ is the unitary implementing the change of basis from the shift-invariant states $\{|\chi_g\rangle : g \in G\}$ to the computational basis $\{|g\rangle : g \in G\}$.

Note that $\mathrm{QFT}^{-1}|g\rangle = |\chi_g\rangle$. So $\left(\mathrm{QFT}^{-1}\right)_{kg} = \langle k|\chi_g\rangle = \frac{1}{\sqrt{|G|}}\overline{\chi_g(k)}$, so $\mathrm{QFT}_{kg} = \frac{1}{\sqrt{|G|}}\chi_k(g)$. So the explicit form is

$$\mathrm{QFT}|g\rangle = \frac{1}{\sqrt{|G|}} \sum_{k \in G} \chi_k(g)|k\rangle.$$

Example 1.36

- For $G = \mathbb{Z}/M$, we can check that $\chi_a(b) = e^{2\pi iab/M}$ are irreps. So the irreps of $\mathbb{Z}/M$ are naturally labelled by $a \in \mathbb{Z}/M$ and this gives the usual QFT mod M as defined earlier.
- Similarly, for $G = \mathbb{Z}/(M_1) \times \cdots \times \mathbb{Z}/(M_r)$, $\chi_g(h) = e^{2\pi i(g_1h_1/M_1 + \cdots + g_rh_r/M_r)}$ are the irreps.

Algorithm: Quantum Hsp Solver

Algorithm 1.37 (Quantum HSP solver for finite abelian G)

1. Working in the state space $H_{|G|} \otimes H_{|X|}$, prepare the uniform superposition state

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |0\rangle$$

   and query U_f on it.
2. Measure the output register, then discard this register.
3. Apply QFT mod $|G|$ to the input register, then measure this register.
4. Repeat the above steps $O(\log |G|)$ times.

Theorem: Correctness Of Quantum Hsp Solver

Theorem 1.38 (Correctness of Quantum HSP Solver) The quantum HSP solver algorithm solves the HSP for finite abelian groups with high probability.

Proof (Hints). Use that irreps of G restricted to K are irreps of K, and that if $O(\log|G|)$ measurements are made, then with probability at least $2/3$, we have enough equations to determine the generators of K. $\square$

Proof. Querying U_f on the state gives

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |f(g)\rangle$$

Upon measurement of the output register, we obtain a uniformly random value $f(g_0)$ from $f(G)$, and the state collapses to a **coset state**

$$|g_0 + K\rangle = \frac{1}{\sqrt{|K|}} \sum_{k \in K} |g_0 + k\rangle.$$

We have $|K\rangle = \sum_{g \in G} a_g |\chi_g\rangle$, so $|g_0 + K\rangle = U(g_0)|K\rangle = \sum_{g \in G} a_g \chi_g(g_0) |\chi_g\rangle$. So applying QFT to the input state gives

$\sum_{g\in G} a_g \chi_g(g_0)|g\rangle$, so the probability of measuring outcome k is $|a_k \chi_k(g_0)|^2 = |a_k|^2$. Now

$$\begin{aligned}\mathrm{QFT}|K\rangle &= \frac{1}{\sqrt{|K|}} \sum_{k\in K} \mathrm{QFT}|k\rangle \\ &= \frac{1}{\sqrt{|G||K|}} \sum_{g\in G} \left(\sum_{k\in K} \chi_g(k) \right) |g\rangle\end{aligned}$$

Note that irreps of G restricted to K are irreps of K. The trivial irrep $\chi_0 : G \to \mathbb{C}$ remains the trivial irrep χ_0 for K. But there may be other irreps that become the trivial irrep on restriction to K. Hence

$$\sum_{k \in K} \chi_g(k) = \begin{cases} |K| & \text{if } \chi_g|_K = \chi_0|_K \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\mathrm{QFT}|K\rangle = \sqrt{\frac{|K|}{|G|}} \sum_{\substack{g \in G \\ \chi_g|_K = \chi_0|_K}} |g\rangle$$

and measuring in the computational basis on this state yields random $g \in G$ such that $\forall k \in K$, $\chi_g(k) = 1$.

If K has generators $k_1, ..., k_m$ (note that for an arbitrary group, we have $m = O(\log|G|)$), then we have a set of equations $\chi_g(k_i) = 1$ for all $i \in [m]$. We can show that if $O(\log|G|)$ such g are drawn uniformly at random, then with probability at least $2/3$, we have enough equations to determine $k_1, ..., k_m$. $\square$

Example 1.39 Let $G = \mathbb{Z}/M_1 \times \cdots \times \mathbb{Z}/M_r$. The irreps are

$$\chi_g(h) = e^{2\pi i(g_1 h_1/M_1 + \cdots + g_r h_r/M_r)}.$$

For $k \in K$, $\chi_g(k) = 1$ iff $\frac{g_1 k_1}{M_1} + \cdots + \frac{g_r k_r}{M_r} = 0 \bmod 1$. This is a homogenous linear equation in k, and $O(\log|G|)$ independent such equations determine K as the nullspace.

Remark 1.40 We can implement QFT over abelian groups (and some non-abelian groups, including S_n) using circuits with $O((\log|G|)^2)$ elementary gates.

In the non-abelian case, we can still easily prepare coset states with one query to f. But the shift operators $U(g_0)$ no longer commute, so we don't have a (canonical) shift-invariant basis.

Definition: Unitary Representation

Definition 1.41 A ***d*-dimensional unitary representation** of a finite group G is a homomorphism

$$\chi : G \to U(d)$$

where $U(d)$ is the group of $d \times d$ unitary matrices.

Definition: Unitary Representation.Irreducible

Definition 1.42 A d-dimensional unitary representation χ of G is **irreducible** if no non-trivial subspace of $\mathbb{C}^d$ is invariant under the action of $\{\chi(g_1), ..., \chi(g_{|G|})\}$ (i.e. we cannot simultaneously block diagonalise all the $\chi(g)$ matrices by a basis change).

Definition: Complete Irreps

Definition 1.43 A set of irreps $\{\chi_1, ..., \chi_m\}$ is a **complete set of irreps** for every irrep χ of G, there exists $1 \leq i \leq m$ such that χ is unitarily equivalent to χ_i, i.e. for some $V \in U(d)$, $\forall g \in G, \chi(g) = V\chi_i(g)V^\dagger$.

Theorem: Squares Of Dimensions Of Complete Set Of Irreps Sum To Size Of Group

Theorem 1.44 Let the dimensions of a complete set of irreps $\chi_1, ..., \chi_m$ be $d_1, ..., d_m$. Then $d_1^2 + \cdots + d_m^2 = |G|$.

Notation 1.45 Write $\chi_{i,jk}(g)$ for the (j,k)-th entry of the matrix $\chi_i(g)$.

Theorem: Schur Orthogonality

Theorem 1.46 (Schur Orthogonality) Let $\chi_1, ..., \chi_m$ be a complete set of irreps for G with respective dimensions $d_1, ..., d_m$, and let $i \in [m]$, $j, k \in [d_i]$. Then

$$\frac{1}{|G|} \sum_{g \in G} \chi_{i,jk}(g) \overline{\chi_{i',j'k'}(g)} = \frac{1}{d_i} \delta_{ii'} \delta_{jj'} \delta_{kk'}.$$

Definition: Fourier Basis

Definition 1.47 The **Fourier basis** for a group G consists of

$$|\chi_{i,jk}\rangle = \sqrt{\frac{d_i}{|G|}} \sum_{g \in G} \overline{\chi_{i,jk}(g)} |g\rangle$$

for each $i \in [n]$ and $j, k \in [d_i]$. Note that by Schur orthogonality, this is an orthonormal basis.

Remark 1.48 Note that these states are not shift invariant for every $U(g_0) : |g\rangle \mapsto |g_0 g\rangle$. So measurement of the coset state $|g_0 K\rangle$ yields an output distribution that is not independent of g_0.

Definition: Qft Non Abelian

Definition 1.49 The **Quantum Fourier transform** over $H_{|G|}$ is the unitary mapping the Fourier basis to the computational basis:

$$\mathrm{QFT}|\chi_{i,jk}\rangle = |i, jk\rangle.$$

$|i, jk\rangle$ is a relabelling of the states $|g\rangle$ for $g \in G$ (note this is valid by Theorem 1.44).

Remark 1.50

- Measuring $\mathrm{QFT}|g_0 K\rangle$ does **not** give g_0-independent outcomes. A complete measurement in the computational basis gives an outcome i, j, k.
- However, there is an incomplete measurement which projects into the d_i^2-dimensional subspaces

$$S_i = \mathrm{span}\big\{|\chi_{i,jk}\rangle : j, k \in [d_i]\big\}.$$

  for each $i \in [n]$. Call this measurement operator M_{rep}. Note that this distinguishes only between the irreps.

- Measuring only the representation labels of $\mathrm{QFT}|g_0 K\rangle$ gives an outcome distribution of the i values that i independent of the random shift g_0, since the χ_i are homomorphisms.
- Note this only gives partial information about K. If K is a normal subgroup, then in fact we can then determine K with $O(\log|G|)$ queries.

2. Quantum phase estimation (QPE)

Quantum phase estimation is a unifying algorithmic primitive, e.g. there is an alternative factoring algorithm based on QPE, and has many important applications in physics.

Problem: Qpe

Problem 2.1 (Quantum Phase Estimation)

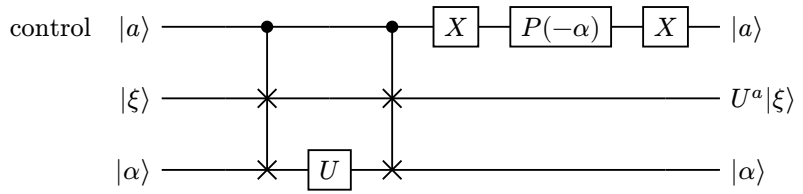
Input Unitary $U \in U(d)$ acting on $\mathbb{C}^d$; state $|v_\varphi\rangle \in \mathbb{C}^d$; level of precision $n \in \mathbb{N}$.

Promise $|v_\varphi\rangle$ is an eigenstate of U with **phase** (eigenvalue) $e^{2\pi i\varphi}$, $\varphi \in [0, 1)$ (i.e. $U|v_\varphi\rangle = e^{2\pi i\varphi}|v_\varphi\rangle$).

Task Output an estimate $\tilde{\varphi}$ of φ, accurate to n binary bits of precision.

Remark 2.2 If U is given as a circuit, we can implement the controlled-U operation, C–U, by controlling each elementary gate in the circuit of U.

If U is given as a black box, we need more information. Note that U is equivalent to $U' = e^{i\theta}U$ and $|\psi\rangle$ is equivalent to $e^{i\theta}|\psi\rangle$, but C–U is not equivalent to C–U'. It suffices to be given an eigenstate $|\alpha\rangle$ with known phase $e^{i\alpha}$ (so $U|\alpha\rangle = e^{i\alpha}|\alpha\rangle$): we have $U'|\alpha\rangle = e^{i(\theta+\alpha)}|\alpha\rangle$. so U and U' can be distinguished using this additional information. The following circuit implements C–U (the top two lines end in state C–$U|a\rangle|\xi\rangle$):



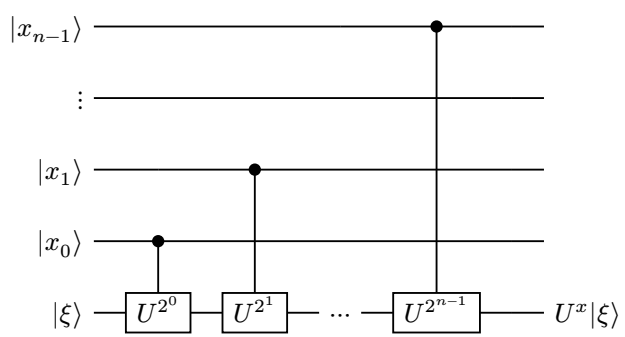
where $P(-\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$, and $\bullet\!-\!\times\!-\!\times$ denotes the controlled SWAP operation. We can prove the top two lines end in the state $C\text{–}U|a\rangle|\xi\rangle$ by checking the action of the circuit for $|a\rangle = |0\rangle$ and $|a\rangle = |1\rangle$.

Definition: Generalised Control

Definition 2.3 For a unitary U, the **generalised control** unitary C–U is defined linearly by

$$\forall x \in \{0,1\}^n, \quad C\text{–}U|x\rangle|\xi\rangle = |x\rangle U^x|\xi\rangle,$$

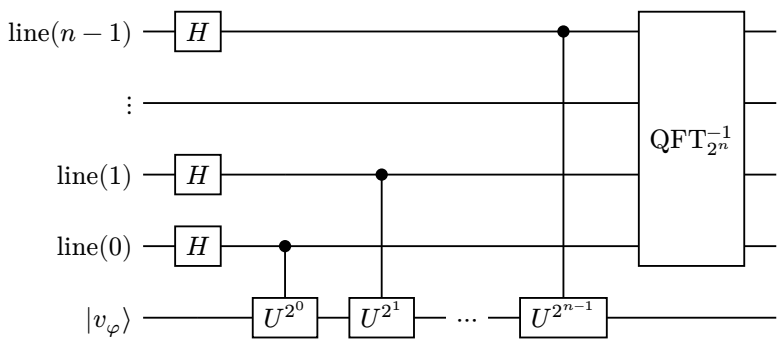
where U^x denotes U applied x times (e.g. $C\text{–}U|11\rangle|\xi\rangle = |11\rangle U^3|\xi\rangle$). Note that $C\text{–}U^k = (C\text{–}U)^k$. The following circuit implements C–U:



Algorithm: Qpe

Algorithm 2.4 (Quantum Phase Estimation) Work over the space $(\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^d$, where $(\mathbb{C}^2)^{\otimes n}$ is the n-qubit register, $\mathbb{C}^d$ is the "qudit" register.

1. Apply the following circuit to $|0...0\rangle|v_\varphi\rangle$:



We write U_{PE} for this unitary part of the circuit.

2. Discard the qudit register holding $|v_\varphi\rangle$, and measure the input qubits, yielding outcome $y_0 ... y_{n-1}$ from lines $0, ..., n-1$.
3. The estimate of φ is $\tilde{\varphi} = y/2^n = y_0/2 + \cdots + y_{n-1}/2^n$.

Remark 2.5 After C–$U^{2^{n-1}}$, the input qubits are in the state

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} e^{2\pi i \varphi x} |x\rangle.$$

If φ had an exact n-bit expansion $0.i_1 i_2 ... i_n = (i_1 ... i_n)/2^n =: \varphi_n/2^n$, then this state is precisely $\mathrm{QFT}_{2^n} |\varphi_n\rangle$, in which case, after applying QFT^{-1}, we have $|\varphi_n\rangle$, so measuring the input bits gives φ_n, and so φ, exactly.

Lemma: Qpe Complex Modulus Inequalities

Lemma 2.6 For all $\alpha \in \mathbb{R}$,

1. If $|\alpha| \leq \pi$, then $\left|1 - e^{i\alpha}\right| = 2|\sin(\alpha/2)| \geq \frac{2}{\pi}|\alpha|$.
2. If $\alpha \geq 0$, then $\left|1 - e^{i\alpha}\right| \leq \alpha$.

Proof (Hints). For both, think graphically. □

Proof.

1. The line $y = \frac{2}{\pi}\alpha$ lies below $2\sin(\alpha/2)$ for $0 \leq \alpha \leq \pi$).
2. On the complex unit circle, the arc length α from 1 to $e^{i\alpha}$ is at least the chord length from 1 to $e^{i\alpha}$.

□

Theorem: Phase Estimation

Theorem 2.7 (Phase Estimation Theorem) Let $\tilde{\varphi}$ be the estimate of φ from the quantum phase estimation algorithm. Then

1. $\Pr(\tilde{\varphi}$ is closest n-bit approximation of $\varphi) \geq \frac{4}{\pi^2} \approx 0.4$.
2. For all $\varepsilon > 0$, $\Pr(|\tilde{\varphi} - \varphi| > \varepsilon) = O\left(\frac{1}{2^n \varepsilon}\right)$. So for any desired accuracy ε, the probability of failure decays exponentially with the number of bits of precision (lines in the circuit).

Proof (Hints). Let $\delta(y) = \varphi - y/2^n + \varepsilon_y$, where $\varepsilon_y \in \{-1, 0, 1\}$ is such that $\delta(y) \in [-1/2, 1/2]$. Show the probability of the measuring yielding outcome y is

$$p_y = \frac{1}{2^{2n}} \left| \frac{1 - e^{2^n 2\pi i \delta(y)}}{1 - e^{2\pi i \delta(y)}} \right|^2.$$

1. Find an upper bound on $\delta(a)$ where a is the closest n-bit approximation of φ.
2. Show that

$$p_y \leq \frac{1}{2^{2n}} \left(\frac{2}{4\delta(y)} \right)^2 = \frac{1}{2^{2n+2}\delta(y)^2}.$$

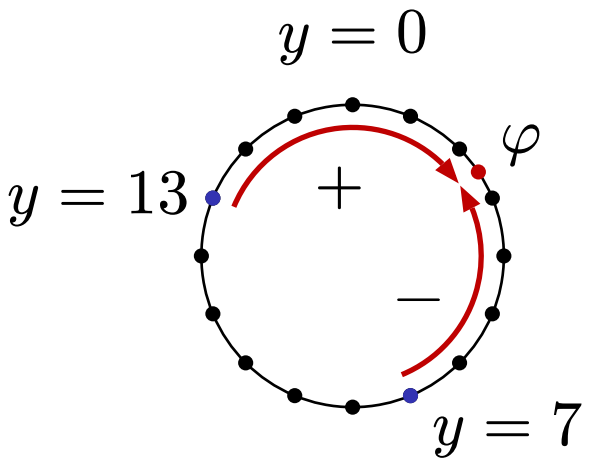
Let $B = \{y \in \{0,1\}^n : |\delta(y)| > \varepsilon\}$. Show that for each $y \in B$, $|\delta(y)| \leq \varepsilon + k_y/2^n$ for some $k_y \in \mathbb{N}$, and that each k_y occurs at most twice here. Conclude the upper bound using an integral.

□

Proof. Let

$$|A\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{2\pi i \varphi x} |x\rangle.$$

Let $\delta(y) = \varphi - y/2^n + \varepsilon_y$, where $\varepsilon_y \in \{-1, 0, 1\}$ is such that $\delta(y) \in [-1/2, 1/2]$. $\delta(y)$ can be thought of as the signed (positive if clockwise, negative if anticlockwise) arc distance between the points $y/2^n$ and φ on a circle of circumference 1.



Example when $n = 4$: the 16 possible values of y are equally spaced around the circle. $\delta(13)$ is positive, $\delta(7)$ is negative.

Since $\mathrm{QFT}^{-1}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2\pi i x y/2^n} |y\rangle$, we have

$$\mathrm{QFT}^{-1}|A\rangle = \frac{1}{2^n} \sum_{y=0}^{2^n-1} \sum_{x=0}^{2^n-1} e^{2\pi i x \delta(y)} |y\rangle$$

so the probability of measuring outcome y is

$$p_y = \Pr\left(\tilde{\varphi} = \frac{y}{2^n}\right) = \frac{1}{2^{2n}} \left| \frac{1 - e^{2^n 2\pi i \delta(y)}}{1 - e^{2\pi i \delta(y)}} \right|^2 .$$

1. Let $\alpha = 2^n 2\pi\delta(a)$, where a is the closest n-bit approximation of φ. Note we can imagine the possible values of $\tilde{\varphi}$ as lying on the unit

circle, spaced by angle $\frac{2\pi}{2^n}$. This gives a visual intuition to the fact that $|\delta(a)| \leq \frac{1}{2^{n+1}}$. Hence $|\alpha| \leq \pi$, and so by the above lemma,

$$p_a = \Pr(\tilde{\varphi} = a) \geq \frac{1}{2^{2n}} \left(\frac{2^{n+2}\delta(a)}{2\pi\delta(a)} \right)^2 = \frac{4}{\pi^2}.$$

2. Note that $\left|1 - e^{2^n 2\pi i \delta(y)}\right| \leq 2$ by the triangle inequality. Let $B = \{y \in \{0,1\}^n : |\delta(y)| > \varepsilon\}$ denote the set of "bad" values of y. For all $y \in \{0,1\}^n$, we have $\delta(y) \in [-1/2, 1/2]$. So by Lemma 2.6, we have $\left|1 - e^{2\pi i \delta(y)}\right| \geq 4|\delta(y)|$, thus

$$p_y \leq \frac{1}{2^{2n}} \left(\frac{2}{4\delta(y)} \right)^2 = \frac{1}{2^{2n+2}\delta(y)^2}.$$

Let $\delta^+ = \min\{\delta(y) : y \in B, \delta(y) > 0\}$ be the smallest $\delta(y)$ such that $\delta(y) > \varepsilon$, and $\delta^- = \max\{\delta(y) : y \in B : \delta(y) < 0\}$ be the largest $\delta(y)$ such that $\delta(y) < -\varepsilon$. For all $y \in B$, we have $\delta(y) = \delta^+ + k_y/2^n$ or $\delta(y) = \delta^- - k_y/2^n$ for some $k_y \in \mathbb{N}$, so $|\delta(y)| > \varepsilon + k_y/2^n$. Note that each $k \in \mathbb{N}$, $k = k_y$ for at most 2 values of $y \in B$. Hence,

$$\Pr(|\delta(y)| > \varepsilon) = \Pr(y \in B) = \sum_{y \in B} p_y$$

$$
\begin{aligned}
&\leq \sum_{y \in B} \frac{1}{2^{2n+2}\left(\varepsilon + k_y/2^n\right)^2} \\
&< 2 \sum_{k=0}^{\infty} \frac{1}{2^{2n+2}} \frac{1}{\left(\varepsilon + k/2^n\right)^2} \\
&\leq \frac{1}{2^{2n+1}\varepsilon^2} + \sum_{k=1}^{\infty} \frac{1}{2^{2n+1}} \frac{1}{\left(\varepsilon + k/2^n\right)^2} \\
&= \frac{1}{2^{2n+1}\varepsilon^2} + \int_0^{\infty} \frac{1}{2^{2n+1}} \frac{1}{\left(\varepsilon + x/2^n\right)^2} \, \mathrm{d}x
\end{aligned}
$$

$$= \frac{1}{2^{2n+1}\varepsilon^2} + \int_{2^n\varepsilon}^{\infty} \frac{1}{2u^2}\,\mathrm{d}u = \frac{1}{2^{2n+1}\varepsilon^2} + \frac{1}{2^{n+1}\varepsilon}.$$

□

Remark 2.8 The QPE algorithm excluding the measurement is a unitary - call this unitary U_{PE}. If we apply U_{PE} to an arbitrary state $|\psi\rangle = \sum_j c_j |v_j\rangle$ where $|v_j\rangle$ are the eigenstates of U with eigenvalue $e^{2\pi i \varphi_j}$, then we have

$$U_{\text{PE}}|\psi\rangle = \sum_j c_j |\tilde{\varphi}_j\rangle |v_j\rangle$$

If every φ_j has an exact n-bit representation, then this is exact. Otherwise, we have $|\tilde{\varphi}_j\rangle = \sqrt{1-\eta}|\tilde{\varphi}_1\rangle + \sqrt{\eta}|\tilde{\varphi}_0\rangle$, where $|\tilde{\varphi}_1\rangle$ is a superposition of all n-bit strings that are correct to the first n-bits of

φ, and $|\tilde{\varphi}_0\rangle$ is a superposition of strings with the first n bits not all correct.

Remark 2.9 Complexity of QPE: we use C–$U, ..., C$–$U^{2^{n-1}}$, so the number of uses of C–U is $\approx 2^n$. So this initially looks like exponential time, but there are special cases of U where by repeated squaring, this can be implemented with poly(n) gates.

If we want to estimate φ accurate to m bits of precision with probability $1 - \eta$, then by the phase estimation theorem with $\varepsilon = \frac{1}{2^m}$, we need $n = O(m + \log(1/\eta))$ lines. Note this is a modest, polynomial increase in the number of lines of the circuit for an exponential reduction in η.

3. Amplitude amplification

Amplitude amplification is an extension of the key insights in Grover's algorithm.

Notation 3.1 Given $|\alpha\rangle \in H_d$, write $L_{|\alpha\rangle} = \text{span}\{|\alpha\rangle\}$ for the one-dimensional subspace generated by $|\alpha\rangle$.

Notation 3.2 Given a subspace $A \leq H_d$, denote the projector onto A by P_A. Note that

$$P_A = \sum_{i=1}^{k} |a_i\rangle\langle a_i|$$

for any orthonormal basis $\{|a_1\rangle, ..., |a_k\rangle\}$ of A.

Notation 3.3 Given a subspace $A \leq H_d$, define the unitary $I_A = I - 2P_A$, which is the reflection in the "mirror" $A^\perp$: indeed, note that for all $|\varphi\rangle \in A$, $I_A = -|\varphi\rangle$, and for all $|\psi\rangle \in A^\perp$, $I_A|\psi\rangle = |\psi\rangle$, since $P_A|\psi\rangle = 0$.

In the case that A is one-dimensional and spanned by $|\alpha\rangle$, we have $P_A = |\alpha\rangle\langle\alpha|$, and write $I_{|\alpha\rangle} = I - 2|\alpha\rangle\langle\alpha|$.

Proposition: Expression For Conjugation Of Reflection Operator

Proposition 3.4 Let $|\alpha\rangle \in H_d$. For any unitary $U \in U(d)$, we have

$$U I_{|\alpha\rangle} U^\dagger = I_{U|\alpha\rangle}.$$

Proof (Hints). Trivial. □

Proof. $U I_{|\alpha\rangle} U^\dagger = UU^\dagger - 2U|\alpha\rangle\langle\alpha|U^\dagger = I_{U|\alpha\rangle}$. □

Problem: Unstructured Search

Problem 3.5 (Unstructured Search Problem)

Input An oracle for a function $f : \{0,1\}^n \to \{0,1\}$.

Promise There is a unique $x_0 \in \{0,1\}^n$ such that $f(x_0) = 1$.

Task Find x_0.

Remark 3.6 The unstructured search problem is closely related to the complexity class NP and to Boolean satisfiability.

Definition: Grover Iteration Operator

Definition 3.7 For fixed $|x_0\rangle \in H_2^{\otimes n}$, the **Grover iteration operator** Q is defined as

$$Q := -H^{\otimes n} I_{|0\rangle} H^{\otimes n} I_{|x_0\rangle} = -I_{H^{\otimes n}|0\rangle} I_{|x_0\rangle}.$$

Remark 3.8 Note that for a function $f : \{0,1\}^n \to \{0,1\}$ fulfilling the promise of the Unstructured Search Problem, we can implement $I_{|x_0\rangle}$ without knowing x_0: we have $U_f|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{f(x)}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = I_{|x_0\rangle}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Hence, implementing Q requires only one query to f.

Theorem: Grover

Theorem 3.9 (Grover) In the 2-dimensional subspace spanned by $|\psi\rangle = H^{\otimes n}|0\rangle$ and $|x_0\rangle$, the action of Q is a rotation by angle 2α, where $\sin(\alpha) = \frac{1}{\sqrt{2^n}} = \langle x_0|\psi\rangle$.

Algorithm: Grovers

Algorithm 3.10 (Grover's Algorithm) Work in the state space $H_2^{\otimes n}$.

1. Prepare $|\psi\rangle = H^{\otimes n}|0\rangle$.
2. Apply Q^m to $|\psi\rangle$, where m is closest integer to $\frac{\arccos\left(1/\sqrt{N}\right)}{2\arcsin\left(1/\sqrt{N}\right)} = \frac{\theta}{2\alpha}$ and $\cos(\theta) = \sin(\alpha) = \langle x_0|\psi\rangle = 1/\sqrt{2^n}$. This rotates $|\psi\rangle$ to be close to $|x_0\rangle$ (within angle $\pm\alpha$ of $|x_0\rangle$).
3. Measure to get x_0 with probability $p = |\langle x_0|Q^m|\psi\rangle|^2 = 1 - \frac{1}{N}$. For large N, $\arccos\left(1/\sqrt{N}\right) \approx \frac{\pi}{2}$, and $\arcsin\left(1/\sqrt{N}\right) \approx 1/\sqrt{N}$. The number of iterations is $m = \frac{\pi}{4}\sqrt{N} = O\left(\sqrt{N}\right)$. So we need $O\left(\sqrt{N}\right)$ queries to U_f. In contrast, clasically we need $\Omega(N)$ queries to f to find x_0 with any desired constant probability. Note that $\Omega(N)$ queries are both necessary and sufficient.

Notation 3.11 Write G for the subspace of the state space H whose associated amplitudes in a given state we wish to amplify. G is called the "good" subspace. We call the subspace $G^{\perp}$ the "bad" subspace. Note that $H = G \oplus G^{\perp}$, and for any state $|\varphi\rangle \in H$, there is a unique decomposition with real, positive coefficients $|\varphi\rangle = \sin(\theta)|g\rangle + \cos(\theta)|b\rangle$, where $|g\rangle = P_G|\varphi\rangle$ and $|b\rangle = P_{G^{\perp}}|\varphi\rangle$.

Theorem: Amplitude Amplification

Theorem 3.12 (Amplitude Amplification Theorem/2D-subspace Lemma) Let $G \leq H_2^{\otimes n}$ be a subspace and $|g\rangle = P_G|\psi\rangle$, $|b\rangle = P_{G^\perp}|\psi\rangle$. In the 2-dimensional subspace $\text{span}\{|\psi\rangle, |g\rangle\} = \text{span}\{|b\rangle, |g\rangle\}$, the unitary

$$Q = -I_{|\psi\rangle} I_G$$

is a rotation by angle 2θ, where $\sin(\theta) = \|P_G|\psi\rangle\|_2$ is the length of the "good" projection of $|\psi\rangle$.

Proof (Hints). Consider the matrix representation of Q in the $\text{span}\{|b\rangle, |g\rangle\}$ basis. $\square$

Proof. By definition, we have $I_G|g\rangle = -|g\rangle$, and $I_G|b\rangle = |b\rangle$. Hence $Q|g\rangle = I_{|\psi\rangle}|g\rangle$ and $Q|b\rangle = -I_{|\psi\rangle}|b\rangle$. The matrix representation of $I_{|\psi\rangle}$ in the $\{|b\rangle, |g\rangle\}$ basis is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}[\cos(\theta) \;\; \sin(\theta)] = \begin{bmatrix} 1 - 2\cos(\theta)^2 & -2\sin(\theta)\cos(\theta) \\ -2\sin(\theta)\cos(\theta) & 1 - 2\sin(\theta)^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}.$$

So $Q|b\rangle = \cos(2\theta)|b\rangle + \sin(2\theta)|g\rangle$, and $Q|g\rangle = -\sin(2\theta)|b\rangle + \cos(2\theta)|g\rangle$. So the matrix representation of Q in the $\{|b\rangle, |g\rangle\}$ basis is

$$\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

which indeed is a rotation by angle 2θ. $\square$

Corollary: Action Of Iterated Rotation Operator

Corollary 3.13 We have $Q^m|\psi\rangle = \cos((2m+1)\theta)|b\rangle + \sin((2m+1)\theta)|g\rangle$.

Proof (Hints). Trivial. □

Proof. Induction on m. $\square$

Notation 3.14 Denote by m_{opt} the $m \in \mathbb{Z}$ which maximises the probability of measuring (in the $\{|b\rangle, |g\rangle\}$ basis) an outcome in G on the state $Q^m|\psi\rangle$. Note that this probability is equal to $\sin((2m+1)\theta)^2$, which is maximised when

$$(2m+1)\theta = \pi/2 \quad \Longrightarrow m = \pi/4\theta - 1/2.$$

So m_{opt} is the nearest integer to $\pi/4\theta - 1/2$.

Example 3.15 Let $\theta = \pi/6$, then $m_{\text{opt}} = 1$ and $Q|\psi\rangle = |g\rangle$. So we obtain a good outcome with certainty on measurement.

Remark 3.16 Note that since Q is a rotation by angle 2θ, $Q^{m_{\text{opt}}}|\psi\rangle$ lies within angle $\pm\theta$ of $|g\rangle$, hence the $|g\rangle$ component of $Q^{m_{\text{opt}}}|\psi\rangle$ has amplitude $\geq \cos(\theta)^2$. So for small θ,

$$\Pr(\text{measuring good outcome}) \geq \cos(\theta)^2 \approx 1 - O(\theta^2).$$

Also, for small θ,

$$m_{\text{opt}} = O(1/\theta) \approx O(1/\sin(\theta)).$$

Remark 3.17 Q can be implemented (efficiently) if $I_{|\psi\rangle}$ and I_G can be implemented (efficiently). For an efficient implementation of I_G, it suffices for G to be spanned by some subset of computational basis states, and the indicator function $\mathbb{1}_G$ is efficiently computable. In this case, we have

$$U_{\mathbb{1}_G}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{\mathbb{1}_G(x)}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Since I_G is defined by its action $|g\rangle \mapsto -|g\rangle$ for $g \in G$ and $|b\rangle \mapsto |b\rangle$ for $b \in G^{\perp}$, the first register holds the value $I_G|x\rangle$.

For an efficient implementation of $I_{|\psi\rangle}$, we usually have $|\psi\rangle = H^{\otimes n}|0...0\rangle$, and then $I_{|\psi\rangle} = H^{\otimes n} I_{|0\rangle} H^{\otimes n}$ can be implemented with $O(n)$ gates.

Remark 3.18 In the amplitude amplification process, the relative amplitudes of basis states inside $|g\rangle$ and $|b\rangle$ won't change. So amplitude amplification boosts the overall amplitude of $|g\rangle$ at the expense of the amplitude of $|b\rangle$.

3.1. Applications of amplitude amplification

Example 3.19 We can generalise Grover search from 1 marked item to k marked items out of $N = 2^n$ items. In this case,

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \\
&= \sqrt{\frac{k}{N}} \frac{1}{\sqrt{k}} \sum_{x \in G} |x\rangle + \sqrt{\frac{N-k}{N}} \frac{1}{\sqrt{N-k}} \sum_{x \in G^\perp} |x\rangle \\
&=: \sqrt{\frac{k}{N}} |g\rangle + \sqrt{\frac{N-k}{N}} |b\rangle
\end{aligned}
$$

so $\sin(\theta) = \sqrt{k/N}$. For $k \ll N$, $\sin(\theta) \approx \theta$, so $m_{\text{opt}} = O(\sqrt{N/k})$ uses of Q required. E.g. $N = 4 = 2^2$ items and $k = 1$ marked item, we have $\sin(\theta) = 1/2$, so $\theta = \pi/6$, so Grover search is exact, and requires exactly one application of Q.

Exm: Quadratic Speed Up Of General Quantum Algorithm

Example 3.20 (Quadratic speedup of general quantum algorithms) Let U be a unitary representing a quantum algorithm/circuit, with $U|0...0\rangle = \alpha|g\rangle + \beta|b\rangle$ where $|g\rangle$ is a superposition of good/correct outcomes, and $|b\rangle$ is a superposition of bad/incorrect outcomes. Note $|g\rangle = \sum_{x\in\{0,1\}^n} c_x|x\rangle$ is generally a non-uniform superposition. We have

$$\Pr(\text{measuring } U|0...0\rangle \text{ yields good outcome}) = |\alpha|^2.$$

Thus, we need to run U about $O(1/|\alpha|^2)$ times to succeed with high probability.

Now define $|\psi\rangle = U|0...0\rangle$ and $Q = -I_{|\psi\rangle} I_G$. We can implement Q if we have a method to verify the output of U; so in particular, we can use this method for any NP problem. This will mean we can efficiently implement the indicator function $\mathbb{1}_G$ of good labels and therefore also I_G. So by the Amplitude Amplification Theorem, Q performs a rotation of 2θ where $\sin(\theta) = |\alpha|$. So after approximately

$$m_{\text{opt}} \approx \pi/4\theta = O(1/\theta) = O(1/\sin(\theta)) = O(1/|\alpha|)$$

(for θ small) uses of Q, we get a good outcome with high probability.

Problem: Counting

Problem 3.21 (Counting Problem)

Input $f : \{0,1\}^n \to \{0,1\}$.

Task Estimate the number $k = \left|f^{-1}(\{1\})\right|$ of inputs that evaluate to 1.

Exm: Quantum Counting

Example 3.22 (Quantum Counting) This combines amplitude amplification and quantum phase estimation. Let the "good" subspace G be the subspace with basis $f^{-1}(\{1\})$. As usual, let

$$|g\rangle := \frac{1}{\sqrt{k}} \sum_{x \in f^{-1}(\{1\})} |x\rangle, \quad |b\rangle := \frac{1}{\sqrt{2^n - k}} \sum_{x \in f^{-1}(\{0\})} |x\rangle,$$

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle = \sqrt{\frac{k}{N}} \frac{1}{\sqrt{k}} \sum_{x \in f^{-1}(\{1\})} |x\rangle + \sqrt{\frac{N-k}{N}} \frac{1}{\sqrt{N-k}} \sum_{x \in f^{-1}(\{0\})} |x\rangle.$$

Recall that Q has matrix representation

$$Q = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

in the orthonormal basis $\{|b\rangle, |g\rangle\}$ where $\sin(\theta) = \|P_G|\psi\rangle\|$. The eigenvalues and eigenstates of Q are $\lambda_\pm = e^{\pm 2i\theta}$ and $|e_\pm\rangle = \frac{1}{\sqrt{2}}(|b\rangle \mp i|g\rangle)$. So we can write $|\psi\rangle = \sin(\theta)|g\rangle + \cos(\theta)|b\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\theta}|e_+\rangle + e^{i\theta}|e_-\rangle\right)$. So $|\psi\rangle$ is an equally-weighted superposition of eigenstates of Q. Write $e^{\pm 2i\theta} = e^{2\pi i\varphi_\pm}$ with $\varphi_\pm \in (0, 1)$. We have $\varphi_+ = \theta/\pi$ and $\varphi_- = (-2\theta + 2\pi)/2\pi = 1 - \theta/\pi$. When $k \ll N$, $\sin(\theta) = \sqrt{k/N} \approx \theta$, so using U_{PE} with m qubits of precision

$$U_{\mathrm{PE}}|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\theta}|e_+\rangle|\tilde{\varphi}_+\rangle + e^{i\theta}|e_-\rangle|\tilde{\varphi}_-\rangle\right)$$

Measuring the QPE output gives (with probability $1/2$) an estimate of $\varphi_+ = \theta/\pi \approx \frac{1}{\pi}\sqrt{k/N}$ or (with probability $1/2$) an estimate of $\varphi_- = 1 - \theta/\pi \approx 1 - \frac{1}{\pi}\sqrt{k/N}$. So in either case, we get an estimate of $\sqrt{k/N}$ (since we can tell when $k \ll N$ which case we are in). By the Phase Estimation Theorem, with probability at least $4/\pi^2$, QPE with m lines gives us an approximation of $\sqrt{k/N}$ to precision $O(1/2^m)$, using $O(2^m)$ C–Q operations, each of which requires one query to f. Write $\delta/\sqrt{2^n} = 1/2^m$ for some $\delta > 0$. So we can estimate $\sqrt{k}$ to precision δ,

and since $\Delta\left(x^2\right) = 2x\Delta(x)$, we estimate k to additive error (precision) $O\left(\delta\sqrt{k}\right)$ using $O(2^m) = O\left(\sqrt{N}/\delta\right)$ queries to f.

Remark 3.23 The quantum counting algorithm is quadratically faster than the best possible classical algorithm, which is:

- Sample random x from $\{0,1\}^n$, then $\Pr(f(x)=1) = k/N$.
- Draw m samples $x_1, ..., x_m$, then the estimate is $\tilde{k} = \frac{\ell}{m}N$, where $\ell = |\{i \in [m] : f(x_i) = 1\}|$.

We need $m = O(N/\delta^2)$ to estimate k to high precision.

4. Hamiltonian simulation

We want to use a quantum system to simulate the evolution/dynamics of another quantum system, given its Hamiltonian H. For an n-qubit system, in general this requires $O(2^n)$ time on a classical computer. For some physically interesting classes of H, we have quantum algorithms that run in $O(\text{poly}(n))$ time.

Definition: Matrix Exponential

Definition 4.1 The **exponential** of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\exp(A) = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Note that if $[A, B] = 0$, then $\exp(A)\exp(B) = \exp(A + B)$, but generally this does not hold when $[A, B] \neq 0$.

Theorem: Evolution Operator Is Unitary For Hermitian Matrices

Theorem 4.2 If H is Hermitian, then e^{-iHt} is unitary for all $t \in \mathbb{R}$.

Proof (Hints). Straightforward. □

Proof. We have $(e^{-iHt})^* = e^{iHt}$ since $H = H^*$, and since $[-iHt, iHt] = 0$, $e^{-iHt}e^{iHt} = e^0 = I$. □

Proposition: Exponential Of Matrix Is Exponential Applied To Spectrum

Proposition 4.3 If $A \in \mathbb{C}^{n\times n}$ has orthonormal spectrum $\{(\lambda_i, |e_i\rangle) : i \in [n]\}$, then

$$\exp(A) = \sum_{i=1}^{n} \exp(\lambda_i)|e_i\rangle\langle e_i|.$$

Proof (Hints). Straightforward. □

Proof. By orthonormality, we have

$$A^k = \left(\sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i| \right)^k = \sum_{i=1}^{n} \lambda_i^k |e_i\rangle\langle e_i|.$$

Hence,

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} |e_i\rangle\langle e_i|.$$

□

Definition: Evolution Operator

Definition 4.4 Given a Hamiltonian H, the unitary $U(t) = e^{-iHt}$ is called the **evolution operator** of H. Given H and $t > 0$, we want to simulate $U(t)$ accurately.

Proposition: Solution To Finite Dimensional Time Independent Schrodinger Equation

Proposition 4.5 The Schrodinger equation which governs the time evolution of a physical state $|\psi(t)\rangle$, given by (assuming $\hbar = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -iH|\psi(t)\rangle,$$

has solution

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

when H is time-independent.

Proof (Hints). Use the spectral decomposition. □

Proof. Let H have orthonormal spectrum $\{(\lambda_j, |e_j\rangle) : j \in [n]\}$. By Proposition 4.3, we have

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \sum_{j=1}^{n} e^{-i\lambda_j t}|e_j\rangle\langle e_j|\psi(0)\rangle.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \sum_{j=1}^{n} -i\lambda_j t e^{-i\lambda_j t}|e_j\rangle\langle e_j|\psi(0)\rangle$$

$$= \left(\sum_{j=1}^{n} -i\lambda_j |e_j\rangle\langle e_j| \right) \sum_{k=1}^{n} e^{-i\lambda_k t} |e_k\rangle\langle e_k|\psi(0)\rangle$$

by orthonormality. □

Definition: Operator Norm

Definition 4.6 The **operator norm** (**spectral norm**) of an operator $A : H \to H$ acting on the space H of states is

$$\|A\| := \max\{\|A|\psi\rangle\| : \psi \in H, \|\psi\| = 1\}.$$

Theorem: Norm Of Matrix Is Max Abs Value Of Its Eigenvalues

Theorem 4.7 If A is diagonalisable with eigenvalues $\lambda_1, ..., \lambda_n$, then

$$\|A\| = \max\{|\lambda_1|, ..., |\lambda_n|\}.$$

Proof (Hints). Straightforward. □

Proof. Let $\{(\lambda_j, |e_j\rangle) : j \in [n]\}$ be the orthonormal spectrum of A. For $\geq$, we have

$$A\| \geq \max\{\|A|e_1\rangle\|, ...\|A|e_n\rangle\|\} = \max\{\|\lambda_1|e_1\rangle\|, ..., \|\lambda_n|e_n\rangle\|\} = \max\{|\lambda_1|, ..., |\lambda_n|\}$$

For $\leq$, let $|\psi\rangle \in H$ with $\|\psi\| = 1$. The $|e_j\rangle$ form an orthonormal basis, so $|\psi\rangle = \sum_{j=1}^{n} a_j|e_j\rangle$ for some coefficients $a_j \in [-1, 1]$. Let $\lambda' = \max\{|\lambda_1|, ..., |\lambda_n|\}$. Then

$$\|A|\psi\rangle\|^2 = \left\|\sum_{j=1}^{n} a_j\lambda_j|e_j\rangle\right\|^2$$

$$= \sum_{j=1}^{n} |a_j|^2 |\lambda_j|^2 \qquad \text{by orthonormality}$$

$$\leq \sum_{j=1}^{n} |a_j|^2 \lambda'^2 = \lambda'^2.$$

□

Proposition: Operator Norm Is Submultiplicative And Subadditive

Proposition 4.8 The operator norm satisfies the following properties:

1. **Submultiplicative**: $\|AB\| \leq \|A\| \cdot \|B\|$.
2. **Triangle inequality**: $\|A + B\| \leq \|A\| + \|B\|$.

Proof (Hints). Straightforward. □

Proof.

1. Let $|\psi_0\rangle$ achieve the maximum in $\|AB\| = \max\{\|AB|\psi\rangle\| : |\psi\rangle \in H, \||\psi\rangle\| = 1\}$. Let $b = \|B|\psi_0\rangle\|$. We have

$$\frac{1}{b}\|AB\| = \frac{1}{b}\|AB|\psi_0\rangle\| = \left\|A\frac{B|\psi_0\rangle}{b}\right\| \leq \|A\|,$$

   hence $\|AB\| \leq \|A\| \cdot b$, but also $b \leq \|B\|$.

2. We have

$$\begin{aligned}\|A+B\| &= \max\{\|A|\psi\rangle + B|\psi\rangle\| : |\psi\rangle \in H, \|\psi\| = 1\} \\ &\leq \max\{\|A|\psi\rangle\| + \|B|\psi\rangle\| : |\psi\rangle \in H, \|\psi\| = 1\}\end{aligned}$$

$$\leq \max\{\|A|\psi\rangle\| : |\psi\rangle \in H, \|\psi\| = 1\} + \max\{\|B|\psi\rangle\| : |\psi\rangle \in H, \|\psi\| = 1\}.$$

□

Definition: Operator Approximation

Definition 4.9 Let $U, \tilde{U} : H \to H$ be operators. $\tilde{U}$ **ε-approximates** U if

$$\left\| U - \tilde{U} \right\| \leq \varepsilon,$$

i.e. for all normalised states $|\psi\rangle$, $\left\| U|\psi\rangle - \tilde{U}|\psi\rangle \right\| \leq \varepsilon$.

Lemma: Approximation Of Unitary Product Grows Linearly

Lemma 4.10 Let $U_1, ..., U_m, \tilde{U}_1, ..., \tilde{U}_m$ be unitaries. Suppose $\tilde{U}_i$ ε-approximates U_i for each $1 \leq i \leq m$. Then

$$\left\| U_m \cdots U_1 - \tilde{U}_m \cdots \tilde{U}_1 \right\| \leq m\varepsilon.$$

So the error increases at most linearly.

Proof (Hints). Straightforward. □

Proof. By induction on m. The case $m = 1$ is true by assumption. Assume it is true for $m = k$. We have

$$\begin{aligned}
&\left\|U_{k+1} \cdots U_1 - \tilde{U}_{k+1} \cdots \tilde{U}_1\right\| \\
&= \left\|U_{k+1}U_k \cdots U_1 - \tilde{U}_{k+1}U_k \cdots U_1 + \tilde{U}_{k+1}U_k \cdots U_1 - \tilde{U}_{k+1}\tilde{U}_k \cdots \tilde{U}_1\right\| \\
&\leq \left\|\left(U_{k+1} - \tilde{U}_{k+1}\right)U_k \cdots U_1\right\| + \left\|\tilde{U}_{k+1}\left(U_k \cdots U_1 - \tilde{U}_k \cdots \tilde{U}_1\right)\right\| \\
&\leq \varepsilon \cdot 1 + 1 \cdot k\varepsilon = (k+1)\varepsilon
\end{aligned}$$

by assumption, Proposition 4.8 and the inductive hypothesis. □

Definition: K Local Hamiltonian

Definition 4.11 H is a ***k*-local Hamiltonian on *n* qubits** if we can write

$$H = \sum_{j=1}^{m} H_j$$

where each H_j acts non-trivially on at most k qubits, in which case we write $H_j = \widetilde{H}_j \otimes I$ (note these qubits need not be adjacent).

Note that $m \leq \binom{n}{k} = O\big(n^k\big)$, and we usually take k to be a constant.

Notation 4.12 Write $U_{(i)}$ for the unitary

$$I \otimes \cdots \otimes I \otimes U \otimes I \otimes \cdots \otimes I$$

where U is in the i-th position, i.e. $U_{(i)}$ is the unitary acting on the i-th qubit on n-qubits.

Example 4.13

- $H = X \otimes I \otimes I - 5Z \otimes Y \otimes I$ is 2-local on 3 qubits.
- For the **Ising model** on an $n \times n$ grid, where each qubit acts non-trivially only with its neighbours, the Hamiltonian is

$$H = J \sum_{i,j=1}^{n} \left(Z_{(i,j)} Z_{(i,j+1)} + Z_{(i,j)} Z_{(i+1,j)} \right)$$

  where $J \in \mathbb{R}$ is a coupling constant.
- For the **Heisenberg model** on a line, the Hamiltonain is

$$H = \sum_{i=1}^{n-1} \Big(J_X X_{(i)} X_{(i+1)} + J_Y Y_{(i)} Y_{(i+1)} + J_Z Z_{(i)} Z_{(i+1)} \Big),$$

where $J_X, J_Y, J_Z \in \mathbb{R}$ are constants.

Theorem: Solovay Kitaev

k is constant

Theorem 4.14 (Solovay-Kitaev) Let U be a unitary on k-qubits, and S be a universal set of elementary gates. Then U can be ε-approximated using $O((\log(1/\varepsilon))^c)$ gates from S, where $c < 4$ is a constant.

Proof. Omitted. □

Proposition: Simulation For K Local Hamiltonian With Commuting Local Terms

Proposition 4.15 Let $H = \sum_{j=1}^{m} H_j$ be a k-local Hamiltonian where all the local terms H_j commute. Then for all $t > 0$ and $\varepsilon > 0$, the evolution operator $U(t) = e^{-iHt}$ can be ε-approximated by a circuit with $O(m \operatorname{polylog}(m/\varepsilon))$ gates from any universal gate set.

Note that $m = O(n^k)$, so the time-complexity is polynomial in n.

Proof (Hints). Straightforward. □

Proof. Fix $t > 0$ and $\varepsilon > 0$. We have

$$U(t) = e^{-iHt} = e^{-i\sum_{j=1}^{m} H_j t} = \prod_{j=1}^{m} e^{-iH_j t}.$$

Each $e^{-iH_j t}$ is a unitary that acts non-trivially on at most k qubits. So we have a circuit for e^{-iHt} in terms of some set of k-qubit gates. By Solovay-Kitaev, each $e^{-iH_j t}$ can be δ-approximated by a unitary $\tilde{U}_j(t)$ circuit with $O(\text{polylog}(1/\delta))$ gates. By Lemma 4.10, we have

$$\left\| U(t) - \prod_{i=1}^{m} \tilde{U}_j(t) \right\| < m\delta.$$

So choosing $\delta = \varepsilon/m$, we obtain a circuit of size $O(m \operatorname{polylog}(m/\varepsilon))$ which ε-approximates $U(t)$. $\square$

Notation 4.16 For $N \times N$ matrices X and Y, write $X = Y + O(\varepsilon)$ to mean $X = Y + E$ where $\|E\| \leq \varepsilon$.

Lemma: Lie Trotter Product Formula

Lemma 4.17 (Lie-Trotter Product Formula) Let A and B be $N \times N$ matrices with $\|A\|, \|B\| \leq \delta < 1$. Then

$$e^{-iA}e^{-iB} = e^{-i(A+B)} + O\big(\delta^2\big).$$

Proof (Hints). Write $e^{-iA} = I - iA + E_A$ and show that $\|E_A\| = O(\delta^2)$, do the same for two other matrices. $\square$

Proof. We have

$$e^{-iA} = I - iA + \sum_{j=2}^{\infty} \frac{(-iA)^j}{j!} =: I - iA + E_A.$$

Now

$$\begin{aligned} \|E_A\| &= \left\| (-iA)^2 \sum_{j=0}^{\infty} \frac{(-iA)^j}{(j+2)!} \right\| \\ &\leq \|(-iA)^2\| \cdot \left\| \sum_{j=0}^{\infty} \frac{(-iA)^j}{(j+2)!} \right\| \qquad \text{by submultiplicativity} \end{aligned}$$

$$\leq \left\|(-iA)^2\right\| \cdot \sum_{j=0}^{\infty} \frac{\left\|(-iA)^j\right\|}{(j+2)!} \qquad \text{by triangle inequality and continuity}$$

$$\leq \|A\|^2 \sum_{j=0}^{\infty} \frac{\|(-iA)\|^j}{j!} \qquad \text{by submultiplicativity}$$

$$= \delta^2 e^{\delta} \leq \delta^2.$$

So $e^{-iA} = I - iA + O(\delta^2)$. By the same argument, we have

$$e^{-iB} = I - iB + O(\delta^2),$$

$$e^{-i(A+B)} = I - i(A+B) + O(4\delta^2) = I - i(A+B) + O(\delta^2)$$

since $\|A + B\| \leq \|A\| + \|B\| = 2\delta$. Hence,

$$\begin{aligned} e^{-iA}e^{-iB} &= (I - iA + O(\delta^2))(I - iB + O(\delta^2)) \\ &= I - i(A + B) + O(\delta^2) \\ &= e^{-i(A+B)} + O(\delta^2), \end{aligned}$$

since $\|AB\| \leq \delta^2$ by submultiplicativity. $\square$

Proposition: Simulation For General K Local Hamiltonian

Proposition 4.18 There is a $\mathrm{poly}(n, 1/\varepsilon, t)$-time quantum algorithm for simulating the evolution operators of k-local Hamiltonians.

Proof (Hints).

- Let $\widetilde{H}_j = H_j t/M$ for some constant M to be determined later. Show that

$$e^{-i\widetilde{H}_1} \ldots e^{-i\widetilde{H}_m} = e^{-i\left(\widetilde{H}_1+\cdots+\widetilde{H}_m\right)} + O\Big(m^3\tilde{\delta}^2\Big),$$

  assuming a bound on $\left\|\widetilde{H}_j\right\|$, which you should determine.
- Using that

$$U(t) = e^{-i(H_1+\cdots+H_m)t} = \left(e^{-i(H_1+\cdots+H_m)t/M}\right)^M$$

  and the above, find an $O(...)$ lower bound for M.

Proof. Let $H = \sum_{j=1}^{m} H_j$ be a k-local Hamiltonian and $U(t) = e^{-iHt}$ be its evolution operator. If all the H_j commute, then we are done by Proposition 4.15. Otherwise, define $\widetilde{H}_j = H_j t/M$ for each $j \in [m]$, for some fixed constant M to be determined later. We want each $\left\|\widetilde{H}_j\right\| \leq \tilde{\delta}$ with $\tilde{\delta} \leq 1/m$, since then $\left\|\widetilde{H}_1 + \cdots + \widetilde{H}_\ell\right\| \leq \ell\tilde{\delta}$, and we need the Lie-Trotter approximation to hold for all $\ell \in [m]$. We have

$$\left(e^{-i\widetilde{H}_1} e^{-i\widetilde{H}_2}\right) \cdots e^{-i\widetilde{H}_m}$$

$$= \left(e^{-i\left(\widetilde{H}_1+\widetilde{H}_2\right)} + O\left(\tilde{\delta}^2\right)\right) e^{-i\widetilde{H}_3} \cdots e^{-i\widetilde{H}_m} \qquad \text{by Lie-Trotter}$$

$$= e^{-i\left(\widetilde{H}_1+\widetilde{H}_2\right)}e^{-i\widetilde{H}_3}\ldots e^{-i\widetilde{H}_m} + O\left(\tilde{\delta}^2\right) \qquad \text{by submultiplicativity}$$

$$= \left(e^{-i\left(\widetilde{H}_1+\widetilde{H}_2+\widetilde{H}_3\right)} + O\left(\left(2\tilde{\delta}\right)^2\right)\right)e^{-i\widetilde{H}_4}\ldots e^{-i\widetilde{H}_m} + O\left(\tilde{\delta}^2\right).$$

since each $e^{-i\widetilde{H}_j}$ is unitary, so has unit norm. Repeatedly applying Lie-Trotter, we obtain

$$e^{-i\widetilde{H}_1}\ldots e^{-i\widetilde{H}_m} = e^{-i\left(\widetilde{H}_1+\cdots+\widetilde{H}_m\right)} + O\left(\tilde{\delta}^2\right) + \cdots + O\left(\left((m-1)\tilde{\delta}\right)^2\right)$$

$$= e^{-i\left(\widetilde{H}_1+\cdots+\widetilde{H}_m\right)} + O\left(m^3\tilde{\delta}^2\right).$$

Let the $O\left(m^3\tilde{\delta}^2\right)$ error be $Em^3\tilde{\delta}^2$. Now

$$U(t) = e^{-i(H_1+\cdots+H_m)t} = \left(e^{-i(H_1+\cdots+H_m)t/M}\right)^M = \left(e^{-i\left(\widetilde{H}_1+\cdots+\widetilde{H}_m\right)}\right)^M.$$

So we need the error in approximating $e^{-iHt/M}$ to be at most ε/M, so we require $Em^3\tilde{\delta}^2 < \varepsilon/M$. Let $\delta = \tilde{\delta}M/t$ (so that $\left\|H_j\right\| \leq \delta$ for each j): we want $\left\|H_j\right\| \leq \delta = \tilde{\delta}M/t \leq M/(mt)$, i.e. $M \geq mt\left\|H_j\right\|$ for all j. So any

$$M > \max\left\{Em^3(\delta t)^2/\varepsilon, mt\|H_1\|, \ldots mt\|H_m\|\right\} = O(m^3(\delta t)^2/\varepsilon)$$

suffices. We have

$$\left\|e^{-iH_1t/M} \ldots e^{-iH_mt/M} - e^{-i(H_1+\cdots+H_m)t/M}\right\| \leq \varepsilon/M.$$

and so by Lemma 4.10,

$$\left\| e^{-iH_1 t} \dots e^{-iH_m t} - e^{-i(H_1+\cdots+H_m)t} \right\| \leq \varepsilon.$$

The circuit is composed of Mm gates of the form $e^{-iH_j t/M}$, so the entire circuit consists of $O\big(m^4(\delta t)^2/\varepsilon\big)$ of these gates. Recall that if H is k-local, then $m \leq \binom{n}{k} = O\big(n^k\big)$. So we have a circuit with $C = O\big(n^{4k}(\delta t)^2/\varepsilon\big)$ gates of the form $e^{-iH_j t/M}$ approximating e^{-iHt} to precision ε. By Solovay-Kitaev, each of these gates can be (ε/C)-approximated by $O\big(\log^4(C/\varepsilon)\big)$ gates from an elementary universal gate set. So the final complexity is $\tilde{O}\big(n^{4k}(\delta t)^2/\varepsilon\big)$ which is $\mathrm{poly}(n, 1/\varepsilon, t)$.$\square$

Remark 4.19

- The time dependence is quadratic, but there are improved product formulae that allow the dependence of the circuit size on t to be $O(t^{1+\alpha})$ for any $\alpha > 0$.
- The ε-dependence is $\text{poly}(1/\varepsilon)$ whereas in the commuting case it was $\text{polylog}(1/\varepsilon)$. However, there are methods that decrease this to $(1/\varepsilon)^{1/2k}$.

5. The Harrow-Hassidim-Lloyd (HHL) algorithm

Definition: Condition Number

Definition 5.1 The **condition number** of a square matrix $A \in \mathbb{C}^{N\times N}$ is defined as

$$\kappa(A) := \begin{cases} \|A^{-1}\| \cdot \|A\| & \text{if } A \text{ is invertible} \\ \infty & \text{otherwise} \end{cases}.$$

We say A is **well-conditioned** if $\kappa(A)$ is small, i.e. if $\kappa(A) = O(\text{polylog}(N))$.

Remark 5.2 $\kappa(A)$ can be thought of a measure of "how invertible" A is. As $\kappa(A)$ increases, numerically computing A^{-1} becomes more unstable.

Proposition: Condition Number Of Hermitian Matrix Is Ratio Of Max Abs Eigenvalue And Min Abs Eigenvalue

Proposition 5.3 If $A \in \mathbb{C}^{N\times N}$ is Hermitian with non-zero eigenvalues $\lambda_1, ..., \lambda_N$, then

$$\kappa(A) = \frac{\max\{|\lambda_i| : i \in [N]\}}{\min\{|\lambda_i| : i \in [N]\}}.$$

Proof (Hints). Straightforward. □

Proof. We have

$$A = \sum_{i=1}^{N} \lambda_i |v_i\rangle\langle v_i|$$

where $|v_1\rangle, ..., |v_n\rangle$ are the eigenvalues of A. Hence,

$$A^{-1} = \sum_{i=1}^{N} \lambda_i^{-1} |v_i\rangle\langle v_i|,$$

so A^{-1} has eigenvalues $\lambda_1^{-1}, ..., \lambda_N^{-1}$. So by Theorem 4.7, we have

$$\kappa(A) = \left\|A^{-1}\right\| \cdot \|A\| = \max\left\{\left|\lambda_i^{-1}\right| : i \in [N]\right\} \cdot \max\{|\lambda_i| : i \in [N]\}$$

$$= \frac{\max\{|\lambda_i| : i \in [N]\}}{\min\{|\lambda_i| : i \in [N]\}}.$$

□

Problem: Linear System Solution

Problem 5.4 (Linear System Solution Problem)

Input matrix $A \in \mathbb{C}^{N \times N}$, vector $b \in \mathbb{C}^N$.

Task find a vector $x \in \mathbb{C}^N$ such that $Ax = b$.

Remark 5.5 The best known classical algorithms for solving linear systems require $O(\text{poly}(N) \cdot \log(1/\varepsilon))$ time. Note that even just reading the inputs A and b, or writing the solution x requires $O(\text{poly}(N))$ time.

Instead of computing the full solution x, the HHL algorithm estimates properties of x of the form $\mu = x^T M x$ (i.e. quadratic forms), where M is Hermitian, e.g. the total weight assigned by x to a subset of indices/components.

Algorithm: Hhl

Algorithm 5.6 (HHL) We are given $|b\rangle = \frac{1}{\|b\|_2} \sum_{i=0}^{N-1} b_i |i\rangle$ and a Hermitian matrx $A \in \mathbb{C}^{N \times N}$. Let $n = \log N$.

1. Apply the unitary part U_{PE} of Quantum Phase Estimation for the unitary e^{-iA} with m bits of precision to the state $|b\rangle|0\rangle^{\otimes m}$.
2. Append an ancillary qubit (in state $|0\rangle$) to the state, then apply a controlled rotation (acting on the last m qubits plus the ancillary qubit) to it.
3. Perform a **post-selection** step: measure the last qubit, and if the outcome is 0, reject and go back to step 1, otherwise accept.
4. Apply U_{PE}^{-1} (the inverse of phase estimation) to the resulting state.

5. Perform a measurement in the M basis on the first n qubits of the resulting state.
6. Repeat all of the above $O\left(\log\left(\frac{1}{\eta}\right)/\delta^2\right)$ times and compute the empirical mean of the measurements, where η controls the probability of success and δ controls the approximation error.

Remark 5.7

- We can also use HHL for non-Hermitian A: double the system size and set

$$\tilde{A} = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ b \end{bmatrix}.$$

  Then run HHL on $\tilde{A}$ and $\tilde{b}$: if $Ax = b$, then $\tilde{A}\tilde{x} = \tilde{b}$ where $\tilde{x} = \begin{bmatrix} x \\ 0 \end{bmatrix}$.
- We can also use HHL for non-Hermitian M: run HHL on $M_1 = \frac{1}{2}(M + M^\dagger)$ and $M_2 = \frac{1}{2i}(M - M^\dagger)$ (which are Hermitian) to give estimates $\hat{\mu}_1$ and $\hat{\mu}_2$, then combine to give $\hat{\mu} := \hat{\mu}_1 + i\hat{\mu}_2$.

Theorem: Chernoff Hoeffding

Theorem 5.8 (Chernoff-Hoeffding) Let $X_1, ..., X_k$ be IID RVs on $[a, b]$ with mean μ, and let $\overline{X} = \frac{1}{k}(X_1 + \cdots + X_k)$. Then

$$\Pr(|\overline{X} - \mu| > \delta) \leq e^{-2k\delta^2/(b-a)^2}.$$

Proof. Omitted. □

Theorem 5.9 If Hamiltonian simulation and phase estimation are exact, then the HHL algorithm computes a estimate $\hat{\mu}$ of $\mu = x^T M x$ to accuracy δ, with probability at least $1 - \eta$.

Proof (Sketch). Let

$$A = \sum_{i=1}^{N} \lambda_i |v_i\rangle\langle v_i|$$

be the spectral decomposition of A. For simplicity, we assume that Hamiltonian simulation and phase estimation are exact; in particular, for each $i \in [N]$, $\lambda_i = \ell_i/2^n$ for some $\ell_i \in \{0, ...N-1\}$. Assume that $\max\{|\lambda_1|, ..., |\lambda_n|\} = 1$ and that $\kappa(A)$ is known or bounded above by some value $\kappa_{\max}$. Writing $\kappa = \kappa(A)$, we have by Proposition 5.3 that $|\lambda_i| \in [1/\kappa, 1]$ for each $i \in [N]$. Work in the n-qubit Hilbert space spanned by $\{|1\rangle, ..., |N\rangle\}$. Write

$$|b\rangle = \sum_{i=1}^{N} b_i |i\rangle = \sum_{j=1}^{N} \beta_j |v_j\rangle$$

Due to the bounds on the $|\lambda_i|$, A^{-1} exists and is equal to $\sum_{i=1}^{N} \frac{1}{\lambda_j} |v_j\rangle\langle v_j|$. Thus, the (unnormalised) solution to $A|x\rangle = |b\rangle$ is

$$|x\rangle = A^{-1}|b\rangle = \sum_{j=1}^{N} \beta_j \cdot \frac{1}{\lambda_j} |v_j\rangle.$$

1. Applying U_{PE} on the state $|\psi\rangle := |b\rangle|0\rangle^{\otimes m}$ gives the state

$$U_{\mathrm{PE}}|\psi\rangle = \sum_{i=1}^{N} \beta_i |v_i\rangle|\ell_i\rangle$$

(assuming e^{-iA} and U_{PE} are exact and error-free). Consider the controlled rotation C–Rot acting on $n+1$ qubits, defined by the following action: for all $j \in [N]$,

$$C\text{–Rot } |\ell_j\rangle|0\rangle = |\ell_j\rangle\big(\cos(\theta_j)|0\rangle + \sin(\theta_j)|1\rangle\big) = |\ell_j\rangle\left(\sqrt{1-\frac{c^2}{\lambda_j^2}}|0\rangle + \frac{c}{\lambda_j}|1\rangle\right),$$

where $\theta = \arcsin(c/\lambda)$ and $c \le \min\{|\lambda_j| : j \in [N]\}$, and e.g. C–Rot $|y\rangle|0\rangle = |y\rangle|0\rangle$ for all other y (this part is not important

and could be chosen to be something else which is consistent with C–Rot being unitary). By linearity, this fully determines C–Rot. By the above bounds on the $|\lambda_j|$, we can choose $c = 1/\kappa$. So in C–Rot, the angle depends on the first register but not on A or b (so we're not sneaking extra info in here). C–Rot can be implemented efficiently using $O(\text{poly}(n))$ one and two-qubit gates (by e.g. Solovay-Kitaev).

2. Applying C–Rot to the state $U_{\text{PE}}|\psi\rangle \otimes |0\rangle$ produces the state

$$\sum_{j=1}^{N} \beta_j \sqrt{1 - c^2/\lambda_j^2}|v_j\rangle|\ell_j\rangle|0\rangle + \beta_j \frac{c}{\lambda_j}|v_j\rangle|\ell_j\rangle|1\rangle$$

3. At the post-selection step, the probability of success (i.e. measuring outcome 1) is (Probability of successfully preparing state $|x\rangle$ is

$$p = \left\| \sum_{j=1}^{N} \beta_j \frac{c}{\lambda_j} |v_j\rangle |\ell_j\rangle \right\|^2 = \sum_{j=1}^{N} \left| \beta_j c/\lambda_j \right|^2 \geq c^2 \sum_{j=1}^{N} \left| \beta_j \right|^2 = c^2 = \frac{1}{\kappa^2}$$

In this case, the post-measurement state has collapsed to (ignoring the ancillary qubit)

$$|\varphi\rangle = \frac{1}{\sqrt{p}} \sum_{j=1}^{N} \beta_j \frac{c}{\lambda_j} |v_j\rangle |\ell_j\rangle$$

To boost the success probability to at least $1-\eta$, we can either repeat the post-selection step $O\left(\frac{\log(1/\eta)}{p}\right) = O(\log(1/\eta)\kappa^2)$ times.

4. Applying U_{PE}^{-1} to $|\varphi\rangle$ "uncomputes" the register holding the $|\ell_j\rangle$, resetting it to $|0\rangle$, so gives

$$|\tilde{x}\rangle = \frac{c}{\sqrt{p}} \sum_{j=1}^{N} \frac{\beta_j}{\lambda_j} |v_j\rangle = \frac{c}{\sqrt{p}} |x\rangle.$$

5. Performing a measurement on $|\tilde{x}\rangle$ in the M basis on the first n qubits (i.e. measuring in the $M \otimes I_m$ basis on the entire state, where I_m is the identity operator on m qubits) yields an eigenvalue of M.

6. We have $\mu = \langle x|M|x\rangle = \frac{p}{c^2}\langle \tilde{x}|M \otimes I|\tilde{x}\rangle$. Computing the empirical average of the measurements gives us an estimate of $\tilde{\mu} = \langle \tilde{x}|M \otimes I_m|\tilde{x}\rangle$. By the Chernoff-Hoeffding bound, to estimate $\tilde{\mu}$ with probability at least $1-\eta$ to accuracy δ, we need $O\left(\frac{\log(1/\eta)}{\delta^2}\right)$ of the measurements in the M basis. Since c is known, we just need an estimate of p to be able to estimate μ.

   To estimate p: the post-selection step is a Bernoulli trial, with probability of outcome 1 being p. So the mean is $0 \cdot (1-p) + 1 \cdot p = p$. This can also be estimated by the empirical average of the proportion of post-selection steps performed that succeed, and we again use the Chernoff-Hoeffding bound.

Remark 5.10 Note that in reality HHL outputs a state $|\tilde{x}'\rangle$ that is ε-close to $|\tilde{x}\rangle$, rather than $|\tilde{x}\rangle$ itself.

Remark 5.11 Instead of repeating the post-selection step (step 3 of HHL) until we succeed, we can use amplitude amplification to the state before we measure, i.e. to the state

$$U_{\text{HHL}}|b\rangle = \sqrt{1-p}|\text{junk}\rangle|0\rangle + \sqrt{p}|\tilde{x}\rangle|1\rangle,$$

which gives a quadratic improvement on the $O(\kappa^2)$ measurements needed for success in the post-selection step.

Similarly, instead of estimating the probability p using the Chernoff-Hoeffding bound, we could use amplitude amplification in the form of Quantum Counting.

Remark 5.12 In HHL, we want to be able to apply the transformation $|b\rangle \mapsto A^{-1}|b\rangle = |x\rangle$. However, this is generally non-unitary so cannot be directly implemented. Instead, we implemented it probabilistically using Quantum Phase Estimation, performed on the unitary $U = e^{-iA}$, which in turn is implemented by Hamiltonian simulation. At the end, the measurement in M basis re-introduces the non-unitarity.

Definition: Matrix.Row Sparse

Definition 5.13 We say a matrix $A \in \mathbb{C}^{N \times N}$ is **row sparse** if each row of A contains $O(\text{polylog}(N))$ non-zero entries.

More generally, A is **row s-sparse** if each row contains at most s non-zero entries.

Definition: Matrix.Row Computable

Definition 5.14 We say an s-sparse matrix A is **row-computable** if the entries of A can be efficiently computed in the following sense: there is a (classical) $O(s)$-time algorithm C which, given a row index $1 \leq i \leq N$ and an integer k, outputs the k-th non-zero entry A_{ij} of row i and its column index j:

$$C(i,k) = \big(j, A_{ij}\big).$$

Theorem: Conditions For Hhl Polynomial Time Complexity

Theorem 5.15 Under the following assumptions, for row s-sparse A, HHL runs in time $O\big((\log N)\kappa^2 s^2 \cdot \frac{1}{\varepsilon}\big)$:

- There is an efficient Hamiltonian simulation algorithm for A.
- $\|b\|_2$ is 1 (or is efficiently computable), and the state $|b\rangle$ can be prepared exactly and efficiently.
- The unitary C–Rot can be implemented exactly and efficiently.
- Measurements in the M basis can be performed efficiently.

Proof. Non-examinable. □

Remark 5.16 The condition that there is an efficient Hamiltonian simulation algorithm for A holds for local Hamiltonians and for row-computable matrices.

Since reading A and b would take $\text{poly}(N)$ time if they were stored as their components, there needs to be a more efficient presentation of A (e.g. row-computable matrices) and b (stored as a quantum state). The only time the presentation of A features in HHL is in the Hamiltonian simulation step, so requiring this to run in $\text{poly}(n, t)$ time is sufficient.

Remark 5.17 Given s-sparse A, the best known classical algorithm for computing an ε-approximation of $\mu = x^T M x$ runs in $O\big(Ns\sqrt{\kappa} \cdot \log\big(\frac{1}{\varepsilon}\big)\big)$ time, even with assumptions comparable to our assumptions for HHL. Classically, there is no known method of estimating the quadratic forms $\mu = x^T M x$ which is faster than computing x first. Thus, when ε is constant (or even $\varepsilon = O\Big(\frac{1}{\text{poly}(n)}\Big)$) and A is well-conditioned and row-sparse, the HHL runtime is an exponential speedup over the classical algorithm runtime.

6. Clifford computations and classical simulation of quantum computation

6.1. Classical simulation of quantum computation

We want to know whether there is a "key quantum effect or resource" that gives quantum computing its (potential) benefits over classical computing.

To formalise this comparison of quantum vs classical computing, we will define a precise mathematical framework of classical simulation of quantum computation.

Problem: Classical Simulation

Problem 6.1 (Classical Simulation)

Input

- Description of a quantum circuit C as a list of 1- and 2- qubit gates acting on n qubit lines.
- Description of an input product state $|\alpha_1\rangle \otimes \cdots \otimes |\alpha_n\rangle$ (note this also has a poly(n)-sized classical description).
- Description of the designated output qubit(s) line(s).

Promise

- C has size $N = |C| = \text{poly}(n)$.
- We only measure one qubit, for a decision answer.

Task By (randomised) classical means only, perform in poly(n) time one of the following:

- **Weak simulation**: sample a bit from the output distribution of $C|\alpha_1\rangle ... |\alpha_n\rangle$ with the output qubit measured in the computational basis.
- **Strong simulation**: calculate the output probabilities $\Pr(\text{output is } 0) = p$.

Remark 6.2 Note that the ability to perform strong simulation implies the ability to perform weak simulation.

Definition: No Quantum Advantage

Definition 6.3 If C is classical simulable (in poly(n) time), then we say there is no **quantum advantage** (up to polynomial overheads).

Remark 6.4

- Any quantum process performs a weak simulation of itself, i.e. the final measurement gives a sample from the output distribution.
- Strong simulability is a much stronger property.
- "Direct" strong simulation is always possible, but generally not in polynomial time: the action of successive gates is simply matrix-vector multiplication in a 2^n-dimensional space, and so we can compute all amplitudes of the output state in $O(2^n)$ time. Although this direct simulation isn't efficient, it shows that any quantum-computable function is also classically computable.

Theorem: Circuits Where Each Stage Is Product State Are Efficiently Strongly Simulable

Theorem 6.5 If the state (including the input state) at each stage of a poly(n)-sized circuit C is a product of (single-qubit) states, then direct strong simulation can be efficiently performed for C with that input state.

Proof (Hints). Assume that the gate acting on a state in a given stage acts non-trivially on the first two qubits (why can we assume this?).□

Proof. We can assume that each gate of the circuit acts two qubits (by extending the 1-qubit gates to act trivially on another qubit). Say at a given stage of the circuit, the state is

$$|\psi\rangle = \sum_{i_1,\ldots,i_n \in \{0,1\}} c_{i_1 \ldots i_n} |i_1 \ldots i_n\rangle,$$

and is transformed by a gate U. Suppose for simplicity that U acts on the first two qubits (the same argument works for any qubit indices pair). Let U_i^j denote the (i,j)-th entry of the matrix representation of U in the computational basis. The action of U on $|\psi\rangle$ produces the state

$$U|\psi\rangle = \sum_{i_1,\ldots,i_n \in \{0,1\}} d_{i_1 \ldots i_n} |i_1 \ldots i_n\rangle,$$

$$\text{where} \quad d_{i_1 \ldots i_n} = \sum_{k_1,k_2 \in \{0,1\}} U^{k_1 k_2}_{i_1 i_2} c_{k_1 k_2 i_3 \ldots i_n}.$$

But now since $|\psi\rangle$ is a product state, it can be expressed as

$$|\psi\rangle = \bigotimes_{j=1}^{n} \Big(\alpha_j^{(0)} |0\rangle + \alpha_j^{(1)} |1\rangle \Big),$$

where $c_{i_1 \ldots i_n} = \prod_{j=1}^{n} \alpha_j^{(i_j)}$. Hence,

$$d_{i_1 \ldots i_n} = \underbrace{\sum_{k_1, k_2 \in \{0,1\}} U_{i_1 i_2}^{k_1 k_2} \alpha_1^{(k_1)} \alpha_2^{(k_2)}}_{=: \gamma_{i_1 i_2}} \cdot \prod_{j=3}^{n} \alpha_j^{(i_j)}.$$

The first term in the product is simply an inner product of two 4-dimensional vectors, so can be computed in $O(1)$ time. Finally, since by assumption $U|\psi\rangle$ is a product state, it can be expressed as

$$U|\psi\rangle = \bigotimes_{j=1}^{n} \Big(\beta_j^{(0)} |0\rangle + \beta_j^{(1)} |1\rangle \Big),$$

where $d_{i_1 \ldots i_n} = \prod_{j=1}^{n} \beta_j^{(i_j)}$. We have $\beta_j^{(k)} = \alpha_j^{(k)}$ for all $k \in \{0, 1\}$ and $j \geq 3$ since U acts non-trivially only on the first two qubits. $\beta_0^{(0)}, \beta_0^{(1)}, \beta_1^{(0)}, \beta_1^{(1)}$ can be determined from $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$ in $O(1)$ time.

So, inductively, the amplitudes of each single-qubit state which appears in each product state in the circuit can be computed, given those of the previous product state, in constant time, so the final state's amplitudes can be computed in $O(|C|)$ time from the initial state's amplitudes.$\square$

Remark 6.6 Recall that a state is a product state iff it has no entanglement. So it is sometimes claimed that entanglement is the source of quantum speedups. However, the converse is not true, i.e. entanglement is necessary but not sufficient.

6.2. Clifford computations

Definition: Pauli Group

Definition 6.7 The **1-qubit Pauli group** is the subgroup of $U(2)$ defined as

$$\mathcal{P}_1 := \{\pm i, \pm 1\} \cdot \{I, X, Y, Z\}$$

with the usual rules $XY = iZ$, etc. The ***n*-qubit Pauli group** is the finite subgroup of $U(2^n)$ defined as

$$\mathcal{P}_n := \mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_1 = \{P_1 \otimes \cdots \otimes P_n : P_i \in \mathcal{P}_1 \ \forall i\}.$$

Definition: Clifford Group

Definition 6.8 A **Clifford operation** on n-qubits is a unitary $C \in U(2^n)$ which preserves the Pauli group under conjugation, i.e.

$$\forall P \in \mathcal{P}_n, \quad CPC^{-1} \in \mathcal{P}_n.$$

The **Clifford group** is the set of all Clifford operations - it is the normaliser of the subgroup $\mathcal{P}_n$ in $U(2^n)$.

Remark 6.9 Clifford groups are important in applications:

- Quantum error correction (e.g. stabiliser codes) and fault-tolerance.
- They give insights into the power of quantum vs classical computing.

Example 6.10

- All Pauli matrices are Clifford operations.
- 1-qubit Clifford operations include H (e.g. $HZH = X$) and $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.
- 2-qubit Clifford operations include $C\text{–}X$ and SWAP, e.g. $C\text{–}X_{12}Z_1C\text{–}X_{12} = Z_1Z_2$, and $C\text{–}X_{12}C\text{–}X_{21}C\text{–}X_{12} = \mathrm{SWAP}_{12}$.

Theorem: Clifford Iff Decomposable Into Hadamard S And C Not Gates

Theorem 6.11 $C \in U(2^n)$ is Clifford iff it can be decomposed into a circuit of H, S and C–X gates.

Proof. Omitted. □

Definition: Clifford Circuit

Definition 6.12 A **Clifford computation/circuit** is a circuit consisting only of Clifford gates, with a measurement in the computational basis at the end, and possibly with intermediate (for our purposes, 1-qubit) measurements in the Z basis.

We treat each intermediate measurement as an extra elementary computational step (called a "measurement gate"). Note we can apply unitary gates to the post-intermediate-measurement states.

Example 6.13

$$|0\rangle|0\rangle \xrightarrow{H} |+\rangle|0\rangle \xrightarrow{C\text{–}X} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is a Clifford circuit. More generally, the family of **cat/GHZ** circuits are Clifford circuits:

$$|0\rangle^{\otimes n} \xrightarrow{H} |+\rangle|0\rangle^{\otimes n-1} \xrightarrow{C\text{–}X_{12}} \cdots \xrightarrow{C\text{–}X_{1n}} \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}).$$

Theorem: Gottesman Knill

Theorem 6.14 (Gottesman-Knill) Let C be any $M = \text{poly}(n)$-size Clifford circuit on n qubits with no intermediate measurements, let any product state $|\alpha_1\rangle ... |\alpha_n\rangle$ be the input state and let the output be a measurement on the (WLOG, by applying SWAP gates) first qubit line. Then the output can always be classically strongly (and so also weakly) simulated efficiently.

Proof (Hints). Denote by p_i the probability of the measurement yielding outcome i for $i = 0, 1$. Express $p_0 - p_1$ in bra-ket notation, the rest is straightforward. $\square$

Proof. Idea: instead of evolving the input state $|\psi_{\text{in}}\rangle$ to $C|\psi_{\text{in}}\rangle =: |\psi_{\text{end}}\rangle$, we "backpropagate" the final measurement to $|\psi_{\text{in}}\rangle$.

Noting that $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, write $Z_1 = \Pi_0 - \Pi_1$, where Π_0 and Π_1 are projectors onto the subspaces where first qubit is 0 and 1 respectively. Let the unitary part of C be $C_M \cdots C_1$. Denoting by p_i the probability of the measurement yielding outcome i for $i = 0, 1$, we have

$$\begin{aligned} p_0 - p_1 &= \langle\psi_{\text{end}}|Z_1|\psi_{\text{end}}\rangle = \langle\psi_{\text{in}}|C^\dagger Z_1 C|\psi_{\text{in}}\rangle \\ &= \langle\psi_{\text{in}}|C_1^\dagger \cdots C_M^\dagger Z_1 C_M \cdots C_1|\psi_{\text{in}}\rangle \end{aligned}$$

$C_1^\dagger \cdots C_M^\dagger Z_1 C_M \cdots C_1$ is a successive conjugation of Z_1 by Clifford operations (and each is a 1- or 2- qubit conjugation, hence each conjugation is a constant size computation).

By the definition of Clifford operations, we obtain

$$p_0 - p_1 = \langle \psi_{\text{in}} | \tilde{P}_1 \otimes \cdots \otimes \tilde{P}_n | \psi_{\text{in}} \rangle$$

where each $\tilde{P}_i$ is in the Pauli group. Hence, since $|\psi_{\text{in}}\rangle = |\alpha_1\rangle ... |\alpha_n\rangle$, the above factorises:

$$p_0 - p_1 = \prod_{i=1}^{n} \langle \alpha_i | \tilde{P}_i | \alpha_i \rangle.$$

Each $\langle \alpha_i | \tilde{P}_i | \alpha_i \rangle$ is a 2×2 matrix computation. So computing $p_0 - p_1$ takes $O(n)$ time classically. We also need $O(M)$ time for computing $\tilde{P}_1 \otimes \cdots \otimes \tilde{P}_n = C^\dagger Z_1 C$. Since we also know $p_0 + p_1 = 1$, we can obtain p_0 and p_1. $\square$

Definition: Clifford Circuit.Adaptive

Definition 6.15 A Clifford circuit with intermediate measurements is **non-adaptive** if it does not depend on the intermediate measurement outcomes, and **adaptive** otherwise.

Theorem: Cliffords And T Are Universal Gate Set

Theorem 6.16 The set of Clifford gates together with the gate $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$ are a universal gate set for quantum computation.

Proof. Omitted. □

Theorem: Adaptivity Determines Classical Strong Simulation Or Universal Quantum Computation

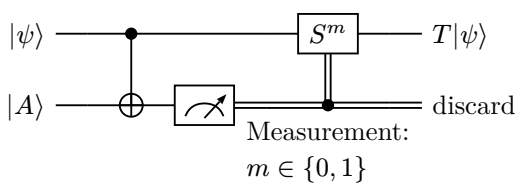
Theorem 6.17 Let C be any poly(n)-sized Clifford circuit consisting of intermediate measurements, let $|\psi_{\text{in}}\rangle = |\alpha_1\rangle ... |\alpha_n\rangle$ be the input state, and say we measure on the (WLOG) first qubit.

1. If C is non-adaptive, then the output is classically strongly simulable efficiently.
2. Adaptive Clifford circuits are sufficient for fully universal quantum computation.

Proof (Hints). See proof sketch. □

Proof sketch.

1. Non-adaptive Clifford circuits are reducible to fully unitary Clifford circuits, using ancillary qubits and C–X gates (then done by Gottesman-Knill).
2. Use the Brayvi-Kitaev idea of "magic states" and Theorem 6.16. Let $|A\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + e^{i\pi/4}|1\rangle\big)$ denote the 1-qubit "magic state". We'll implement the T gate using the following "T-gadget":



□

Remark 6.18 Theorem 6.16 and Theorem 6.17 show we can achieve an increase in power from classically simulable to quantum universal by either:

- Allowing Clifford circuits to use non-Clifford gates, e.g. T, or
- Allowing only Clifford operations and intermediate measurements, but also the exotic resource "magic" states.

Remark 6.19 If C is a poly(n)-sized adaptive Clifford circuit whose only inputs are computational basis states, then the output of C is classically *weakly* simulable efficiently.