Contents

1. Non-classical logic

1.1. Intuitionistic logic

Idea: a statement is true if there is a proof of it. A proof of $\varphi \Rightarrow \psi$ is a "procedure" that can convert a proof of φ to a proof of ψ. A proof of $\neg\varphi$ is a proof that there is no proof of φ.

In particular, $\neg\neg\varphi$ is not always the same as φ.

Fact. The Law of Excluded Middle (LEM) ($\varphi \vee \neg\varphi$) is not (generally) intuitionistically valid.

Moreover, the Axiom of Choice is incompatible with intuitionistic set theory.

In intuitionistic logic, $\exists$ means an explicit element can be found.

Why bother with intuitionistic logic?
- Intuitionistic mathematics is more general, as we assume less (no LEM or AC).
- Several notions that are conflated in classical mathematics are genuinely different constructively.
- Intuitionistic proofs have a computable content that may be absent in classical proofs.
- Intuitionistic logic is the internal logic of an arbitrary topos.

We will inductively define a provability relation by enforcing rules that implement the BHK-interpretation.

Definition. A set is **inhabited** if there is a proof that it is non-empty.

Axiom (Choice - Intutionistic Version). Any family of inhabited sets admits a choice function.

Theorem (Diaconescu). The Law of Excluded Middle can be intutionistically deduced from the Axiom of Choice.

Proof (Hints).
- Proof should use Axioms of Separation, Extensionality and Choice.
- For proposition φ, consider $A = \{x \in \{0,1\} : \varphi \vee (x = 0)\}$ and $B = \{x \in \{0,1\} : \varphi \vee (x = 1)\}$.
- Show that we have a proof of $f(A) = 0 \vee f(A) = 1$, similarly for $f(B)$.
- Consider the possibilities that arise from above, show that they lead to either a proof of φ or a proof of $\neg\varphi$.

$\square$

Proof.
- Let φ be a proposition. By the Axiom of Separation, the following are sets:

$$A = \{x \in \{0,1\} : \varphi \vee (x = 0)\},$$
$$B = \{x \in \{0,1\} : \varphi \vee (x = 1)\}.$$

- Since $0 \in A$ and $1 \in B$, we have a proof that $\{A, B\}$ is a family of inhabited sets, thus admits a choice function $f : \{A, B\} \to A \cup B$ by the Axiom of Choice.
- f satisfies $f(A) \in A$ and $f(B) \in B$ by definition.
- So we have $f(A) = 0$ or φ is true, and $f(B) = 1$ or φ is true. Also, $f(A), f(B) \in \{0, 1\}$.
- Now $f(A) \in \{0, 1\}$ means we have a proof of $f(A) = 0 \vee f(A) = 1$ and similarly for $f(B)$.
- There are the following possibilities:
  1. We have a proof that $f(A) = 1$, so $\varphi \vee (1 = 0)$ has a proof, so we must have a proof of φ.
  2. We have a proof that $f(B) = 0$, so $\varphi \vee (0 = 1)$ has a proof, so we must have a proof of φ.
  3. We have a proof that $f(A) = 0 \wedge f(B) = 1$, in which case we can prove $\neg\varphi$: assume there is a proof of φ, we can prove that $A = B$ (by the Axiom of Extensionality), in which case $0 = f(A) = f(B) = 1$: contradiction.
- So we can always specify a proof of φ or a proof of $\neg\varphi$.

$\square$

Notation. We write $\Gamma \vdash \varphi$ to mean that φ is a consequence of the formulae in the set Γ. Γ is called the **set of hypotheses or open assumptions**.

Notation. Notation for assumptions and deduction.

Definition. The rules of the **intuitionistic propositional calculus (IPC)** are:
- conjunction introduction,
- conjunction elimination,
- disjunction introduction,
- disjunction elimination,
- implication introduction,
- implication elimination,
- assumption,
- weakening,
- construction,
- and for any A,

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}.$$

as defined below.

Definition. The **conjunction introduction ($\wedge$-I)** rule:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}.$$

Definition. The **conjunction elimination ($\wedge$-E)** rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}, \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}.$$

Definition. The **disjunction introduction ($\vee$-I)** rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}, \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}.$$

Definition. The **disjunction elimination (proof by cases) ($\vee$-E)** rule:

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C \quad \Gamma \vdash A \vee B}{\Gamma \vdash C}.$$

Definition. The **implication/arrow introduction ($\to$-I)** rule (note the similarity to the deduction theorem):

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}.$$

Definition. The **implication/arrow elimination ($\to$-E)** rule (note the similarity to modus ponens):

$$\frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}.$$

Definition. The **assumption (Ax)** rule: for any A,

$$\frac{}{\Gamma, A \vdash A}$$

Definition. The **weakening** rule:

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}.$$

Definition. The **construction** rule:

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}.$$

Remark. We obtain classical propositional logic (CPC) from IPC by adding either:

- $\Gamma \vdash A \vee \neg A$:

$$\frac{}{\Gamma \vdash A \vee \neg A},$$

  or

- If $\Gamma, \neg A \vdash \perp$, then $\Gamma \vdash A$:

$$\frac{\Gamma, \neg A \vdash \perp}{\Gamma \vdash A}.$$

Notation. see scan

Definition. We obtain **intuitionistic first-order logic (IQC)** by adding the following rules to IPC for quantification:

- existental inclusion,
- existential elimination,
- universal inclusion,
- universal elimination

as defined below.

Definition. The **existential inclusion (∃-I)** rule: for any term t,

$$\frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x.\varphi(x)}.$$

Definition. The **existential elimination (∃-I)** rule:

$$\frac{\Gamma \vdash \exists x.\varphi \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi},$$

where x is not free in Γ or ψ.

Definition. The **universal inclusion (∀-I)** rule:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x.\varphi},$$

where x is not free in Γ.

Definition. The **universal exclusion (∀-E)** rule:

$$\frac{\Gamma \vdash \forall x.\varphi(x)}{\Gamma \vdash \varphi[t/x]},$$

where t is a term.

Definition. We define the notion of **discharging/closing** open assumptions, which informally means that we remove them as open assumptions, and append them to the consequence by adding implications. We enclose discharged assumptions in square brackets [] to indicate this, and add discharged assumptions in parentheses to the right of the proof. For example, →-I is written as

$$\begin{array}{c} \Gamma, [A] \\ \vdots \\ \dfrac{B}{\Gamma \vdash A \to B}(A) \end{array}$$

Example. A natural deduction proof that $A \wedge B \to B \wedge A$ is given below:

$$\dfrac{\dfrac{\frac{[A\wedge B]}{A} \quad \frac{[A\wedge B]}{B}}{B\wedge A}}{A\wedge B \to B\wedge A}(A \wedge B)$$

Example. A natural deduction proof of $\varphi \to (\psi \to \varphi)$ is given below (note clearly we must use $\to$-I):

$$\begin{array}{c} [\varphi] \quad [\psi] \\ \hline \psi \to \varphi \\ \hline \varphi \to (\psi \to \varphi) \end{array}$$

Example. A natural deduction proof of $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ (note clearly we must use $\to$-I):

$$\begin{array}{c} [\varphi \to (\psi \to \chi)] \quad [\varphi \to \psi] \quad [\varphi] \\ \hline \psi \to \chi \qquad \psi \\ \hline \chi \\ \hline \varphi \to \chi \\ \hline (\varphi \to \psi) \to (\varphi \to \chi) \\ \hline (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \end{array}$$

Notation. If Γ is a set of propositions, φ is a proposition and $L \in \{\text{IPC}, \text{IQC}, \text{CPC}, \text{CQC}\}$, write $\Gamma \underset{L}{\vdash} \varphi$ if there is a proof of φ from Γ in the logic L.

Lemma. If $\Gamma \underset{\text{IPC}}{\vdash} \varphi$, then $\Gamma, \psi \underset{\text{IPC}}{\vdash} \varphi$ for any proposition ψ. If p is a primitive proposition (doesn't contain any logical connectives or quantifiers) and ψ is any proposition, then $\Gamma[\psi/p] \underset{\text{IPC}}{\vdash} \varphi[\psi/p]$.

Proof. Induction on number of lines of proof (exercise). $\square$

1.2. The simply typed λ-calculus

Definition. The set Π of **simple types** is generated by the grammar

$$\Pi \coloneqq U \mid \Pi \to \Pi$$

where U is a countable set of **type variables (primitive types)** together with an infinite set of V of **variables**. So Π consists of U and is closed under $\to$: for any $a, b \in \Pi$, $a \to b \in \Pi$.

Definition. The set Λ_Π of **simply typed λ-terms** is defined by the grammar

$$\Lambda_\Pi \coloneqq V \mid \lambda V : \Pi \,.\, \Lambda_\Pi \mid \Lambda_\Pi \; \Lambda_\Pi$$

In the term $\lambda x : \tau.M$, x is a variable, τ is type and M is a λ-term. Forming terms of this form is called **λ-abstraction**. Forming terms of the form $\Lambda_\Pi \Lambda_\Pi$ is called **λ-application**.

Example. The λ-term $\lambda x : \mathbb{Z}.x^2$ should represent the function $x \mapsto x^2$ on $\mathbb{Z}$.

Definition. A **context** is a set of pairs $\Gamma = \{x_1 : \tau_1, ..., x_n : \tau_n\}$ where the x_i are distinct variables and each τ_i is a type. So a context is an assignment of a type to each variable in a given set. Write C for the set of all possible contexts. Given a context $\Gamma \in C$, write $\Gamma, x : \tau$ for the context $\Gamma \cup \{x : \tau\}$ (if x does not appear in Γ).

The **domain** of Γ is the set of variables $\{x_1, ..., x_n\}$ that occur in it, and its **range**, $|\Gamma|$, is the set of types $\{\tau_1, ..., \tau_n\}$ that it manifests.

Definition. Recursively define the **typability relation** $\Vdash \subseteq C \times \Lambda_\Pi \times \Pi$ via:

1. For every context Γ, variable x not occurring in Γ and type τ, we have $\Gamma, x : \tau \Vdash x : \tau$.
2. For every context Γ, variable x not occurring in Γ, types $\sigma, \tau \in \Pi$, and λ-term M, if $\Gamma, x : \sigma \Vdash M : \tau$, then $\Gamma \Vdash (\lambda x : \sigma.M) : (\sigma \to t)$.
3. For all contexts Γ, types $\sigma, \tau \in \Pi$, and terms $M, N \in \Lambda_\Pi$, if $\Gamma \Vdash M : (\sigma \to t)$ and $\Gamma \Vdash N : \sigma$, then $\Gamma \Vdash (MN) : \tau$.

Notation. We will refer to the λ-calculus of Λ_Π with this typability relation as $\lambda(\to)$.

Definition. A variable x occurring in a λ-abstraction $\lambda x : \sigma.M$ is **bound** and is **free** otherwise. A term with no free variables is called **closed**.

Definition. Terms M and N are **α-equivalent** if they differ only in the names of their bound variables.

Definition. If M and N are λ-terms and x is a variable, then we define the **substitution of N for x in M** by the following rules:

- $x[x := N] = N$.
- $y[x := N] = y$ for $y \neq x$.
- $(PQ)[x := N] = P[x := N]Q[x := N]$ for λ-terms P, Q.
- $(\lambda y : \sigma.P)[x := N] = \lambda y : \sigma.(P[x := N])$ for $x \neq y$ and y not free in N.

Definition. The **β-reduction** relation is the smallest relation $\underset{\beta}{\longrightarrow}$ on Λ_Π closed under the following rules:

- $(\lambda x : \sigma.P)Q \underset{\beta}{\longrightarrow} P[x := Q]$. The term being reduced is called a **β-redex**, and the result is called its **β-contraction**.
- If $P \underset{\beta}{\longrightarrow} P'$, then for all variables x and types $\sigma \in \Pi$, we have $\lambda x : \sigma.P \underset{\beta}{\longrightarrow} \lambda x : \sigma.P'$.
- If $P \underset{\beta}{\longrightarrow} P'$ and Z is a λ-term, then $PZ \underset{\beta}{\longrightarrow} P'Z$ and $ZP \underset{\beta}{\longrightarrow} ZP'$.

Definition. We define **β-equivalence**, $\underset{\beta}{\equiv}$, as the smallest equivalence relation containing $\underset{\beta}{\longrightarrow}$.

Example. We have $(\lambda x : \mathbb{Z}.(\lambda y : \tau.x))2 \underset{\beta}{\longrightarrow} (\lambda y : \tau.2)$.

Lemma (Free Variables Lemma). Let $\Gamma \Vdash M : \sigma$. Then

- If $\Gamma \subseteq \Gamma'$, then $\Gamma' \Vdash M : \sigma$.
- The free variables of M occur in Γ.

- There is a context $\Gamma^* \subseteq \Gamma$ whose variables are exactly the free variables in M, with $\Gamma^* \Vdash M : \sigma$.

Lemma (Generation Lemma).

1. For every variable $x \in V$, context Γ and type $\sigma \in \Pi$: if $\Gamma \Vdash x : \sigma$, then $x : \sigma \in \Gamma$.
2. If $\Gamma \Vdash (MN) : \sigma$, then there is a type $\tau \in \Pi$ such that $\Gamma \Vdash M : \tau \to \sigma$ and $\Gamma \Vdash N : \tau$.
3. If $\Gamma \Vdash (\lambda x.M) : \sigma$, then there are types $\tau, \rho \in \Pi$ such that $\Gamma, x : \tau \Vdash M : \rho$ and $\sigma = (\tau \to \rho)$.

Proof. By induction (exercise). $\square$

Lemma (Substitution Lemma).

1. If $\Gamma \Vdash M : \sigma$ and $\alpha \in U$ is a type variable, then $\Gamma[\alpha := \tau] \Vdash M : \sigma[\alpha := \tau]$.
2. If $\Gamma, x : \tau \Vdash M : \sigma$ and $\Gamma \Vdash N : \tau$, then $\Gamma \Vdash M[x := N] : \sigma$.

Proposition (Subject Reduction). If $\Gamma \Vdash M : \sigma$ and $M \underset{\beta}{\longrightarrow} N$, then $\Gamma \Vdash N : \sigma$.

Proof.

- By induction on the derivation of $M \underset{\beta}{\longrightarrow} N$, using Generation and Substitution Lemmas (exercise).

$\square$

Definition. A λ-term $M \in \Lambda_\Pi$ is an **β-normal form (β-NF)** if there is no term N such that $M \underset{\beta}{\longrightarrow} N$.

Notation. Write $M \underset{\beta}{\twoheadrightarrow} N$ if M reduces to N after (potentially multiple) β-reductions.

Theorem (Church-Rosser for $\lambda(\to)$). Suppose that $\Gamma \Vdash M : \sigma$. If $M \underset{\beta}{\twoheadrightarrow} N_1$ and $M \underset{\beta}{\twoheadrightarrow} N_2$, then there is a λ-term L such that $N_1 \underset{\beta}{\twoheadrightarrow} L$ and $N_2 \underset{\beta}{\twoheadrightarrow} L$, and $\Gamma : L : \sigma$.

Corollary (Uniqueness of normal form). If a simply-typed λ-term admits a β-NF, then this form is unique.

Proposition (Uniqueness of types).

1. If $\Gamma \Vdash M : \sigma$ and $\Gamma \Vdash M : \tau$, then $\sigma = \tau$.
2. If $\Gamma \Vdash M : \sigma$ and $\Gamma \Vdash N : \tau$, and $M \underset{\beta}{\equiv} N$, then $\sigma = \tau$.

Proof.

1. Induction (exercise).
2. By Church-Rosser, there is a λ-term L which both M and N reduce to. By Subject Reduction, we have $\Gamma \Vdash L : \sigma$ and $\Gamma \Vdash L : \tau$, so $\sigma = \tau$ by 1.

$\square$

Example. There is no way to assign a type to $\lambda x.xx$: let x be of type τ, then by the Generation Lemma, in order to apply x to x, x must be of type $\tau \to \sigma$ for some type σ. But $\tau \neq \tau \to \sigma$, which contradicts Uniqueness of Types.

Definition. The **height function** is the recursively defined map $h : \Pi \to \mathbb{N}$ that maps all type variables $u \in U$ to 0, and a function type $\sigma \to \tau$ to $1 + \max\{h(\sigma), h(\tau)\}$:

$$\begin{aligned} h : \Pi &\to \mathbb{N}, \\ h(u) &= 0 \quad \forall u \in U, \\ h(\sigma \to \tau) &= 1 + \max\{h(\sigma), h(\tau)\} \quad \forall \sigma, \tau \in \Pi. \end{aligned}$$

We extend the height function from types to redexes by taking the height of its λ-abstraction.

Notation. $(\lambda x : \sigma . P^{\tau})^{\sigma \to \tau}$ denotes that P has type τ and the λ-abstraction has type $\sigma \to \tau$.

Theorem (Weak normalisation for $\lambda(\to)$). Let $\Gamma \Vdash M : \sigma$. Then there is a finite reduction path $M := M_0 \underset{\beta}{\longrightarrow} M_1 \underset{\beta}{\longrightarrow} \ldots \underset{\beta}{\longrightarrow} M_n$, where M_n is in β-normal form.

Proof. ("Taming the Hydra")

- Idea is to apply induction on the complexity of M.
- Define a function $m : \Lambda_{\Pi} \to \mathbb{N} \times \mathbb{N}$ by

$$m(M) := \begin{cases} (0,0) & \text{if } M \text{ is in } \beta\text{-NF} \\ (h(M), \mathrm{redex}(M)) & \text{otherwise} \end{cases}$$

  where $h(M)$ is the maximal height of a redex in M, and $\mathrm{redex}(M)$ is the number of redexes in M of that height.
- We use induction over $\omega \times \omega$ to show that if M is typable, then it admits a reduction to β-NF.
- The problem is that inductions can copy redexes or create new ones, so our strategy is to always reduce the right-most redex of maximal height.
- We will argue that, by following this strategy, any new redexes that we generate have a strictly lower height than the height of the redex we chose to reduce.
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$\square$