Complex Analysis II Course Notes

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1 Mobius Transformations

Corollary 1.0.1. Any Mobius transformation is a bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Let $T \in GL_2(\mathbb{C})$ and M_T be a Mobius transformation, then a point z is a fixed point of M_T if $M_T(z) = z$.

Lemma 1.0.2. Let $T \in GL_2(\mathbb{C})$. If $M_T : \mathbb{C} \to \mathbb{C}$ is not the identity map, then M_T has at most two fixed points in $\mathbb{C}$. If a Mobius transformation has three fixed points then it is the identity map.

Proof. Case 1: Suppose $M_T(\infty) = \infty$. From the definition, $M_T(z) = \frac{az+b}{cz+d}$, therefore $c = 0$. So $M_T(z) = \frac{a}{d}z + \frac{b}{d}$, with $a \neq 0, d \neq 0$ (since $\det T \neq 0$).

Such an affine linear map has at most one fixed point because:

- If $a \neq d$ then $\frac{a}{d}z + \frac{b}{d} = z \iff z = \frac{b}{d-a}$ so M_T has a unique fixed point.
- If $a = d$ then $b \neq 0$ (since we assume M_T is not the identity). So $M_T(z) = z + \frac{b}{a}$ is a translation which has no fixed points.

Case 2: Suppose $M_T(\infty) \neq \infty$. Suppose $z_0 \in \mathbb{C}$ is such that $M_T(z_0) = z_0$. We have $M_T(z_0) = z_0 \iff \frac{az_0+b}{cz_0+d} = z_0 \iff cz_0^2 + (d-a)z_0 - b = 0$. This quadratic equation has at most two roots so there are at most two fixed points of M_T. □

Definition 1.0.3. Given four distinct points $z_0, z_1, z_2, z_3 \in \mathbb{C}$, the cross-ratio of these points denoted $(z_0, z_1; z_2, z_3)$ is defined by

$$\frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

We extend the definition to the case where one of the points is ∞ by removing all differences involving that point e.g. $(\infty, z_0; z_2, z_3) = \frac{z_1-z_3}{z_1-z_2}$.

Theorem 1.0.4. (Three points theorem) Let z_1, z_2, z_3 and w_1, w_2, w_3 be two sets of three ordered points in $\hat{\mathbb{C}}$. Then there exists a unique Mobius transformation f such that $f(z_i) = w_i$ for every $i \in \{1, 2, 3\}$.

Proof. Existence:

We consider the functions $F(z) = (z, w_1; w_2, w_3) = \frac{(z-w_2)(w_1-w_3)}{(z-z_3)(w_1-w_2)}$ and $G(z) = \frac{(z-z_2)(z-z_3)}{(z-z_3)(z_1-z_2)}$. These are Mobius transformations with the properties that $F(w_1) = 1$, $F(w_2) = 0$, $F(w_3) = \infty$ and similarly, $G(z_1) = 1$, $G(z_2) = 0$, $G(z_3) = \infty$. Therefore $F^{-1} \circ G$ maps each z_i to w_i.

Uniqueness:

Assume that there are two such maps, say f_1 and f_2. Then the Mobius transformation $H = f_1^{-1} \circ f_2$ satisfies $H(z_i) = z_i$.

This shows that H has three fixed points so, by Three Point Theorem, it must be the identity. Thus $f_1 = f_2$. □

Proposition 1.0.5. Mobius transformations preserve the cross ratio. That is, if z_0, z_1, z_2, z_3 are four distinct points in $\hat{\mathbb{C}}$ and f is a Mobius transformation, then $(f(z_0), f(z_1); f(z_2), f(z_3)) = (z_0, z_1; z_2, z_3)$.

Proof. Let $w_i = f(z_i)$ for every $i \in \{1, 2, 3\}$. Let $F(z) = (z, w_1; w_2, w_3)$ and $G(z) = (z, z_1; z_2, z_3)$. Recall $F^{-1} \circ G$ maps z_i to w_i like f does. Since there is a unique Mobius transformation with this property, we have

$$f = F^{-1} \circ G$$

and

$$F \circ f = G$$

That is, $(f(z_0), w_1; w_2, w_3) = F \circ f(z_0) = G(z_0) = (z_0, z_1; z_2, z_3)$. $\square$

Remark. General strategy: to find Mobius transformation, find image of 3 points and use the fact that cross ratio is preserved. Plug known points into (*) and rearrange for $f(z_0)$.

1.1 The Riemann Sphere Revisited

Circles in $\hat{\mathbb{C}}$ correspond to circles in S^2 that don't pass through N (the North pole). Lines in $\hat{\mathbb{C}}$ correspond to circle in S^2 that pass through N.

Remark. Mobius transformations give all biholomorphic maps from S^2 to S^2.

Remark. Stereographic projections are conformal.

1.2 Mobius transformations preserving the upper half plane and the unit disc

Notation: for a domain $D \subset \mathbb{C}$, let $Mob(D)$ be the set of Mobius transformations f such that $f(D) = D$.

Proposition 1.2.1. (H2H) Every Mobius transformation mapping $\mathbb{H}$ to $\mathbb{H}$ ($\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$) is of the form M_T with $T \in SL_2(\mathbb{R}) := \{T = () : a, b, c, d \in \mathbb{R}, \det T = 1\}$

Conversely, every such Mobius transformation maps $\mathbb{H}$ to $\mathbb{H}$ and hence a biholomorphism from $\mathbb{H}$ to $\mathbb{H}$.

i.e. H2H: $f \in \text{Mob}(\mathbb{H}) \Leftrightarrow f = M_T$ with $T \in SL_2(\mathbb{R})$.

Remark. $T \to M_T$ gives a group homomorphism $SL_2(\mathbb{R}) \to Aut(\mathbb{H})$

Proof. Any Mobius transformation $f : \mathbb{H} \to \mathbb{H}$ must map $\partial\mathbb{H}$ to $\partial\mathbb{H}$. As $\partial\mathbb{H}$ is the real line, $f : \mathbb{R} \cup \infty \to \mathbb{R} \cup \infty$. So f must map the ordered set $\{1, 0, \infty\}$ to $\{x_1, x_2, x_3\}$ for some $x_i \in \mathbb{R} \cup \infty$.

We know that the cross ratio is preserved under a Mobius transformation:

$$(f(z), x_1; x_2, x_3) = \frac{(f(z) - x_2)(x_1 - x_3)}{(f(z) - x_3)(x_1 - x_2)} = \frac{z - 0}{1 - 0} = (z, 1; 0, \infty)$$

$$\Leftrightarrow (f(z) - x_2)(x_1 - x_3) = z(f(z) - x_3)(x_1 - x_2)$$

$$\Leftrightarrow f(z) = \frac{x_3(x_1 - x_2)z + x_2(x_3 - x_1)}{(x_1 - x_2)z + x_3 - x_1}$$

We see that the coefficients of T are real.

If $T \in GL_2(\mathbb{R})$ and $z = x + iy$ then

$$Im(M_T(z)) = Im(\frac{az+b}{cz+d}) = Im(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2})$$

$$= Im(\frac{bc\bar{z}+adz}{(cz+d)}) = \frac{y \det T}{|cz+d|}$$

We have $z \in \mathbb{H} \Leftrightarrow y > 0$ so $M_T(z) \in H \Leftrightarrow T \in GL_2(\mathbb{R}), \det T > 0$. We can therefore replace T by a real matrix of determinant 1 by scaling T by a real number. □

Proposition 1.2.2. (D2D): Every Mobius transformation from the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to $\mathbb{D}$ is of the form $T \in SU(1,1)$

Conversely, every such Mobius transformation maps $\mathbb{D}$ to $\mathbb{D}$ and hence gives a h biholopmorhpic automorphism of $\mathbb{D}$.

i.e. $f \in Mob(\mathbb{D}) \Leftrightarrow f = M_T$, $T \in SU(1,1)$.

Proof. ($\Rightarrow$): Let $M_T : \mathbb{D} \to \mathbb{D}$ be a Mobius transformation. The Cayley map H_C maps $\mathbb{H}$ to $\mathbb{D}$. We have that $f = M_C^{-1} \circ M_T \circ M_C$ is a Mobius transformation from $\mathbb{H}$ to $\mathbb{H}$. By proposition 4.20, we have $f = M_S$ where $S \in SL_2(\mathbb{R})$.

Hence $C^{-1}TC = S \in SL_2(\mathbb{R})$ by Lemma 4.4.

Let $S \in \mathbb{M}_2(\mathbb{R})$, $\det S = 1$. Then $T = CSC^{-1}$. Evalutating this shows $T \in SU(1,1)$.

($\Leftarrow$): If $T \in SU(1,1)$, then the same calculation in reverse shows that the matrix $S = C^{-1}TC \in SL_2(\mathbb{R})$. Then $M_S : \mathbb{H} \to \mathbb{H}$ is a Mobius transformation by proposition 4.20 (H2H), and the map $M_T := M_C \circ M_S \circ M_C^{-1}$ is a Mobius transformation from $\mathbb{D}$ to $\mathbb{D}$ □

Remark. $T \to M_T$ gives a group homomorphism from $SU(1,1)$ to $Aut(\mathbb{D})$.

Corollary 1.2.3. (D2D*):

1. Every Mobius transformation f from $\mathbb{D}$ to $\mathbb{D}$ can be written as

$$f(z) = e^{i\theta} \frac{z - z_0}{\bar{z}_0 z - 1}$$

for some angle θ and $z_0 \in \mathbb{D}$ where z_0 is the unique point in $\mathbb{D}$ such that $f(z_0) = 0$.

2. Every Mobius transformation of the unit disc $\mathbb{D}$ to $\mathbb{D}$ for which $f(0) = 0$ are rotations about 0.

Proof. 1. By proposition D2D, we have

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}} = \frac{a(z+b/a)}{-\bar{a}((-\bar{b}/\bar{a})z-1)} = -\frac{a}{\bar{a}} \frac{z-(-b/a)}{(-\bar{b}/\bar{a})z-1}$$

So $z_0 = -\frac{b}{a}$. Since $|-\frac{a}{\hat{a}} = 1$, $-\frac{a}{\hat{a}} = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$.

$|z_0|^2 - 1 = |-\frac{b}{a}|^2 - 1 = \frac{|b|^2}{|a|^2} - 1$. Now $1 = |a|^2 - |b|^2$ so $|z_0|^2 - 1 = \frac{-1}{|a|^2} < 0$ so $|z_0|^2 < 1$ and so $|z_0| < 1$.

2. $f(0) = 0 \Leftrightarrow e^{i\theta}\frac{0-z_0}{\bar{z_0}\cdot 0-1} = 0 \Leftrightarrow z_0 = 0 \Leftrightarrow f(z) = e^{i\theta}\frac{z-0}{0-1} = e^{-i\theta}z$.

   So f is a rotation.

$\square$

Remark. The map $g(z) = \frac{z-z_0}{\bar{z_0}z-1}$ swaps z_0 and 0 and is an involution ($g \circ g = Id$). Also, $z \to e^{i\theta}z$ is a rotation.

So every Mobius transformation from $\mathbb{D}$ to $\mathbb{D}$ is given by an involution followed by a rotation.

1.3 Finding biholomorphic maps between domains

To find a biholomorphism f between domains, we build f in various stages using simpler known maps.

Example 1.3.1. Find biholomorphism from $D = \{z \in \mathbb{D} : Im(z) < 0\}$ to $\mathbb{H}$.

The Cayley Map M_C is a map from $\mathbb{H}$ to $\mathbb{D}$, so $M_C^{-1} : \mathbb{D} \to \mathbb{H}$, $M_C^{-1}(z) = \frac{iz+i}{-z+1}$.

To find the image of D under M_C^{-1}, consider how it acts on two segments of δD:

- Under M_C^{-1}, $-1 \to 0$, $0 \to i$ and $1 \to \infty$. Therefore the line segment from -1 to 1 through 0 is mapped to the positive imaginary axis.
- Under M_C^{-1}, $-i \to 1$, so the circular arc from -1 to 1 through $-i$ is mapped to the positive real axis.

Now $-\frac{i}{2} \in D$ and $M_C^{-1}(-\frac{i}{2}) = \frac{4+3i}{5}$. The image of D under M_C^{-1} is $\Omega = \{w \in \mathbb{C} : 0 < Arg(w) < \frac{\pi}{2}\}$.

Now we find a biholomorphic map from Ω to $\mathbb{H}$. $g(z) = z^2$ satisfies this, as it doubles the argument of z.

So the map is $f = g \circ M_C^{-1}$, $f : D \to \mathbb{H}$.

2 Notions of convergence in complex analysis and power series

2.1 Pointwise and uniform convergence

Definition 2.1.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. A sequence of functions $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ converges pointwise (on X) to f if for every $x \in X$, the limit function $f(x) := \lim_{n\to\infty} f_n(x)$ exists in Y.

In other words, we have for every $x \in X$ and for every $\epsilon > 0$, for some $N \in \mathbb{N}$, for every $n > N$, $d_Y(f_n(x), f_n(x)) < \epsilon$. (Not that N depends on x).

Remark. For every $x \in X$, $f_n(x)$ is just a sequence of points in Y. The above definition is what we get by applying definition 2.11 (in notes) to the sequence $f_n(z)$.

Example 2.1.2. Let $f_n(z) = z^n$, $f_n : \mathbb{C} \to \mathbb{C}$. There are the following cases:

1. $z \in \mathbb{D}$. Let $\epsilon > 0$. Then $|z|^N < \epsilon$ for every $N > \frac{\log \epsilon}{\log |z|}$. So for every $n > N$ we have $f_n(z) - 0 = |z|^n < |z|^N \epsilon$, hence $\lim_{n\to\infty} f_n(z) = 0 \in \mathbb{D}$.

2. $|z| = 1$. The point z rotates around the unit circle $\delta\mathbb{D}$ by $Arg(z)$ anticlockwise every iteration. For $z \neq 1$, this sequence doesn't converge. But for $z = 1$, $\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} 1 = 1$.

3. $|z| > 1$. The value of $|z|^n$ is unbounded so doesn't converge.

The sequence f_n doesn't converge pointwise on $\mathbb{C}$. But it is pointwise convergent on $\mathbb{D} \cup 1$ with limit function:

$$f(z) = \begin{cases} 0 & \text{if } z \in \mathbb{D} \\ 1 & \text{if } z = 1 \end{cases} \tag{1}$$

Definition 2.1.3. Let (X, d_X) and (Y, d_Y) be two metric spaces. A sequence of functions $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ converges uniformly (on X) to the limit function f if for every $\epsilon > 0$ for some $N \in \mathbb{N}$, for every $n > N$, $d_Y(f_n(x), f(x)) < \epsilon$ for every $x \in X$.

Theorem 2.1.4. Let (X, d_X) and (Y, d_Y) be two metric spaces and let $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ be a sequence of functions that converges uniformly to f on X.

Then f is continuous on X.

Proof. Same as in Analysis I. □

Lemma 2.1.5. let $\{f_n\}_{n\in\mathbb{N}} : X \to \mathbb{C}$ be a sequence of functions converging pointwise to a limit function f.

1. If $|f_n(x) - f(x)| \leq s_n$ for every $x \in X$ where $\{s_n\}_{n\in\mathbb{N}}$ is some sequence in $\mathbb{R} > 0$ (independent of x) with $\lim_{n\to\infty} s_n = 0$ then f_n converge uniformly to f on X.

2. If for some sequence $x_n \in X$, $|f_n(x_n) - f(x_n)| \geq c$ for some positive constant c then f_n does not converge uniformly to f on X.

Theorem 2.1.6. (Weierstrass M-test): Let $f_n : X \to \mathbb{C}$ be a sequence of fucntions such that $|f_n(x)| \leq M_n$ for every $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on X to some limit function $f : X \to \mathbb{C}$.

Proof. Similar to Analysis I. $\square$

Theorem 2.1.7. Let a sequence of functions $f_n : [a, b] \to \mathbb{R}$ converge uniformly on an interval $[a, b]$ to some function f, such that $\{f_n\}$ are all continuous. Then

$$\lim_{n\to\infty} \int_a^c f_n(x)dx = \int_a^c f(x)dx \text{ for every } c \in [a, b]$$

Definition 2.1.8. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in a metric space X. f_n converges locally uniformly (on X) to the limit function f if for every $x \in X$, for some open set $U \subset X$ containing x, f_n converges uniformly to f on U.

Theorem 2.1.9. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions which converges locally uniformly on X to a limit function f. Then f is continuous on X.

Proof. For every $x \in X$, f_n converges uniformly on some open set U containing x. Hence f is continuous on U by theorem 5.5 (in notes). So f is continuous at x for every $x \in X$. $\square$

Remark. The limit of a locally uniform convergent sequence of holomorphic functions is again holomorphic.

Example 2.1.10. For every $w \in \mathbb{D}$, for some $r < 1$, $w \in B_r(0)$ and $B_r(0)$ is open. Then for every $z \in B_r(0)$, $|z|^n < r^n$ and $\lim_{n\to\infty} r_n = 0$. So by lemma 5.6 (in notes), with $s_n = r^n$, f_n converges uniformly to f in $B_r(0)$.

Remark. To prove that the limit function is conitnuous on all of $\mathbb{D}$, it is enough to prove locally uniform convergence on every ball $B_r(0)$, $0 < r < 1$, in $\mathbb{D}$.

Theorem 2.1.11. Let X be a metric space and let $f_n : X \to \mathbb{C}$ be a sequence of continuous functions such that for any $y \in X$, there is an open $U \subset X$ containing y and constants $M_n > 0$ with $\sum_{n=1}^{\infty} M_n < \infty$ and $|f_n(x)| \le M_n$ for every $x \in U$. Then $\sum_{n=1}^{\infty} f_n$ converges locally uniformly to a continuous function on X.

Proof. Given $y \in X$, the hypotheses of the theorem imply that for some constants $M_n > 0$, $|f_n(y)| \le M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$.

$$|F_k(y)| = |\sum_{n=1}^{k} f_n(y)| \le \sum_{n=1}^{\infty} |f_n(y)| \le \sum_{n=1}^{k} M_n$$

As $k \to \infty$, the RHS $\sum_{n=1}^{k} M_n$ converges so it must be bounded, and let the upper bound by L. Thus for every k, $|F_k(y)| \le L$. So the sequence $(F_k(y))_k$ is bounded, hence it lies in some boundedd, closed ball in $\mathbb{C}$, which is compact by Heine-Borel.

Therefore there is a subsequence $(F_{k_j}(y))_{k_j}$ that converges to $F(y)$.

Now, for $k_j > k$,

$$|F_{k_j}(y) - F_k(y)| = |\sum_{n=k+1}^{k_j} f_n(y)| \le \sum_{n=k+1}^{k_j} |f_n(y)| \le \sum_{n=k+1}^{k_j} M_n$$

Taking the limit as $j \to \infty$, both the LHS and RHS converge, and we get

$$|F(y) - F_k(y)| \le \sum_{n=k+1}^{\infty} M_n$$

Now taking the limit as $k \to \infty$, we get

$$\lim_{k\to\infty} |F(y) - F_k(y)| = 0$$

since the RHS tends to zero.

Repeating this for every y, $F_k \to F$ pointwise on X.

From the hypotheses of the theorem, we have that for every $y \in X$, for some open $U \subset X$ containing y and constants $M_n > 0$ with $\sum_{n=1}^{\infty} < \infty$ and $|f_n(x)| \leq M_n$ for every $x \in U$.

Then, for every $x \in U$ and for every $L > k$,

$$|F_L(x) - F_k(x)| = |\sum_{n=k+1}^{L} f_n(x)| = \sum_{n=k+1}^{L} |f_n(x)| \leq \sum_{n=k+1}^{L} M_n$$

Taking the limit as $l \to \infty$:

$$|F(x) - F_k(x)| \leq \sum_{n=k+1}^{\infty} M_n$$

for every $x \in U$.

$\lim_{k\to\infty} \sum_{n=k+1}^{\infty} M_n = 0$. So by lemma 5.6 (in notes), $F_k \to F$ uniformly on U. $\square$

2.2 Complex power series

Theorem 2.2.1. A complex power series is an expression of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$, $a_n, c \in \mathbb{C}$. There are three cases:

1. $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges only for $z = c$ ($R = 0$).
2. There exists $R > 0$ (radius of convergence) such that
   - $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges absolutely for $|z-c| < R$ (We call $B_R(c)$ the disc of convergence).
   - $\sum_{n=0}^{\infty} a_n(z-c)^n$ diverges for $|z-c| > R$ (anything can happen on the circle $|z-c| = R$).
3. $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges absolutely for every $z \in \mathbb{C}$ ($R = \infty$).

Remark. Radius of convergence is usually determined via ratio test or root test.

Theorem 2.2.2. A power series $\sum_{n=0}^{\infty} a_n(z-c)^n$ with radius of convergence $0 < R < \infty$ converges uniformly on every ball $B_r(c)$ with $0 < r < R$. This implies that the power series is locally uniformly convergent on its disc of convergence.

Proof. Follows via the M-test. $\square$

Remark. The power series do not converge uniformly in the entire disc of conergence $B_R(c)$.

Proposition 2.2.3. Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series with radius of convergence $0 < R < \infty$. Then the formal derivatives and antiderivatives

$$\sum_{n=0}^{\infty} na_n(z-c)^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$$

have the same radius of convergence R.

Theorem 2.2.4. Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series with radius of convergence $0 < R < \infty$ and let $f : B_R(c) \to \mathbb{C}$ be the resulting limit function. Then f is holomorphic on $B_R(c)$ with

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-c)^{n-1}$$

for $z \in B_R(c)$.

Proof. Assume $c = 0$ (the general case for c is analogous).

$$f(z) - f(w) = \sum_{n=1}^{\infty} a_n(z^n - w^n) = \sum_{n=1}^{\infty} (z-w) q_n(z)$$

where $q_n(z) = \sum_{k=0}^{n-1} w^k z^{n-1-k}$.

So for $z \neq w$, let $h(z) := \frac{f(z)-f(w)}{z-w} = \sum_{n=1}^{\infty} a_n q_n(z)$

Given $z_0 \in B_R(0)$, let $r < R$ such that $w, z_0 \in B_r(0)$. To apply the local M-test, we need constants M_n for this set $B_r(0)$ that bound the terms $a_n q_n(z)$ defining h.

For $z \in B_r(0)$,

$$|a_n q_n(z)| = |a_n \sum_{k=0}^{n-1} w^k z^{n-1-k}| \leq |a_n| \sum_{k=0}^{n-1} |w|^k |z|^{n-1-k} < |a_n| \sum_{k=0}^{n-1} r^{n-1} = n|a_n| r^{n-1}$$

So let $M_n = n|a_n|r^{n-1}$, then $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} n|a_n|r^{n-1}$ which converges by proposition 5.19 (in lecture notes).

The formal derivative $\sum_{n=1}^{\infty} n a_n r^{n-1}$ has radius of convergence R so converges absolutely on its disc of convergence $B_R(0)$. In particular, it converges at $z = R$. By the local M-test, the series defining h converges locally uniformly to a continuous function on $B_R(0)$. Hence

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = \lim_{h \to w} h(z) = h(w) = \sum_{n=1}^{\infty} a_n q_n(w) = \sum_{n=1}^{\infty} n a_n w^{n-1}$$

□

Corollary 2.2.5. A power series f as theorem 5.21 (in lecture notes) with positive radius of convergence R can be differentiated infinitely many times and

$$f^{(k)} := \sum_{n=k}^{\infty} k! \binom{n}{k} a_n (z-c)^{n-k}$$

for $z \in B_R(c)$

Corollary 2.2.6. A power series f as in theorem 5.21 (in lecture notes) with positive radius of convergence R has a holomorphic antiderivative $F : B_R(c) \to \mathbb{C}$, with $F'(z) = f(z)$, defined by

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$$

3 Complex integration over contours

3.1 Definition of contour integrals

Definition 3.1.1. For a continuous function $f : [a,b] \to \mathbb{C}$, with $f(z) = u(z) + iv(z)$,

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt \in \mathbb{C}$$

Lemma 3.1.2.

1. Let f_1 and f_2 be continuous functions from $[a,b]$ to $\mathbb{C}$. Then $\int_a^b (f_1(t)+f_2(t))dt = \int_a^b f_1(t)dt + \int_a^b f_2(t)dt$.

2. For any complex number $c \in \mathbb{C}$ and continuous function $f : [a,b] \to \mathbb{C}$,

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt$$

Definition 3.1.3. A smooth curve in $\mathbb{C}$ is a continuously differentiable function $\gamma : [0,1] \to \mathbb{C}$ (i.e. differentiable with continuous derivative). More generally we can consider continuously differentiable curves $\gamma : [a,b] \to \mathbb{C}$. We say that such curves are C^1.

Remark. We write $\gamma(t) = u(t) + iv(t)$ with $u, v : [a,b] \to \mathbb{R}$. Then the derivative γ' is defined as

$$\gamma'(t) := u'(t) + iv'(t)$$

At the endpoints, we demand that the one-sided derivative exists and is continuous from the one side:

$$\gamma'(b) := \lim_{h\to 0^-} \frac{u(b+h) - u(b)}{h} + i \lim_{h\to 0^-} \frac{v(b+h) - v(b)}{h}$$

exists and

$$\lim_{t\to b^-} \gamma'(t) = \gamma'(b)$$

Definition 3.1.4. Let $U \subset \mathbb{C}$ be an open set, and $f : U \to \mathbb{C}$ be a continuous function. Let $\gamma : [a,b] \to U$ be a C^1 curve. The integral of f along the curve γ is defined as

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Corollary 3.1.5. Properties of the integral along a curve:

1. $\int_\gamma (f_1(z) + f_2(z))dz = \int_\gamma f_1(z)dz + \int_\gamma f_2(z)dz$

2. For $c \in \mathbb{C}$, $\int_\gamma cf(z)dz = c\int_\gamma f(z)dz$

Proof. Easy $\square$

Definition 3.1.6. Given $\gamma : [a,b] \to \mathbb{C}$, the curve $(-\gamma) : [-b,-a] \to \mathbb{C}$ is defined as

$$(-\gamma)(t) := \gamma(-t)$$

Then we have

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$$

Lemma 3.1.7. Let $U \subset \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be continuous and $\gamma : [a,b] \to \mathbb{C}$ be a C^1 curve. If $\phi : [a',b'] \to [a,b]$ with $\phi(a') = a$ and $\phi(b') = b$ is continuously differentiable and we define $\delta : [a',b'] \to \mathbb{C}$, $\delta := \gamma \circ \phi$, then

$$\int_{\gamma} f(z)dz = \int_{\delta} f(z)dz$$

Proof.

$$\int_{\delta} f(z)dz = \int_{a'}^{b'} f(\delta(t))\delta'(t)dt = \int_{a'}^{b'} f(\gamma(\phi(t)))(\gamma(\phi(t)))'dt$$

$$= \int_{a'}^{b'} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt$$

With a change of variables $s = \phi(t)$, $ds = \phi'(t)dt$:

$$\int_{a'}^{b'} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt = \int_{a}^{b} f(\gamma(s))\gamma'(s)ds = \int_{\gamma} f(z)dz$$

□

Definition 3.1.8. Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and suppose there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that the curves $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$, defined by $\gamma_i(t) = \gamma(t)$ for $t \in [a_{i-1}, a_i]$ are C^1 curves. Then γ is a piecewise C^1 curve or contour.

For a contour γ above, a contour integral is defined as

$$\int_{\gamma} f(z)dz = \sum_{n=1}^{n} \int_{\gamma_i} f(z)dz$$

Definition 3.1.9. If $\gamma : [a,b] \to \mathbb{C}$ and $\delta : [c,d] \to \mathbb{C}$ are two contours with $\gamma(b) = \delta(c)$ the contour $\gamma \cup \delta : [a, b+d-c] \to \mathbb{C}$ is defined as

$$(\gamma \cup \delta)(t) := \begin{cases} \gamma(t) & \text{if } a \leq t \leq b \\ \delta(t) & \text{if } c \leq t \leq d \end{cases}$$

Then

$$\int_{\gamma \cup \delta} f(z)dz = \int_{\gamma} f(z)dz + \int_{\delta} f(z)dz$$

3.2 The fundamental theorem of calculus

Theorem 3.2.1. Let $U \in \mathbb{C}$ be an open set and let $F : U \to \mathbb{C}$ be holomorphic with continuous derivative f. Then for every contour $\gamma : [a, b] \to U$,

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, if γ is closed, so $\gamma(a) = \gamma(b)$, then

$$\int_\gamma f(z)dz = 0$$

Proof. First consider the case where γ is a C^1 curve. Let $F = u + iv$. Then

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F(\gamma(t)))'dt$$

$$= \int_a^b (u(\gamma(t)))'dt + i\int_a^b (v(\gamma(t)))'dt = [u(\gamma(t))]_a^b + i[v(\gamma(t))]_a^b$$

$$= u(\gamma(b)) - u(\gamma(b)) + i(v(\gamma(b)) - v(\gamma(b))) = F(\gamma(b)) - F(\gamma(a))$$

Now extend this proof to any contour.

Let $\gamma : [a, b] \to \mathbb{C}$ be a contour, then for some $a = a_0 < a_1 < \cdots < a_n = b$, the curves $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$, $i = 1, \cdots, n$, defined by $\gamma_i(t) = \gamma(t)$ for $t \in [a_{i-1}, a_i]$ are C^1 curves. Then

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = \sum_{i=1}^n \int_{\gamma_i} F'(z)dz$$

$$= \sum_{i=1}^n (F(\gamma(a_i)) - F(\gamma(a_{i-1}))) = F(\gamma(a_n)) - F(\gamma(a_0)) = F(\gamma(b)) - F(\gamma(a))$$

□

Remark. Under the hypotheses on F, the integral only depends on the endpoints of the curve.

Theorem 3.2.2. If $f : [a, b] \to \mathbb{R}$ is continuous,

$$\int_a^b f(t)dt \le \int_a^b \max_{t\in[a,b]} f(t)dt \le (b - a) \max_{t\in[a,b]}$$

Proof. From Analysis I. □

Definition 3.2.3. Let $\gamma : [a, b] \to \mathbb{C}$ be a contour. The **length** of γ is defined as

$$L(\gamma) = \int_a^b |\gamma'(t)|dt$$

Lemma 3.2.4. **(The Estimation Lemma)** Let $f : U \to \mathbb{C}$ be continuous and $\gamma : [a, b] \to U$ be a contour. Then

$$\left|\int_\gamma f(z)dz\right| \le L(\gamma) \sup_\gamma |f|$$

where $\sup_\gamma |f| := \sup\{|f(z)| : z \in \gamma\}$.

Proof. First prove that for a continuous function $g : [a, b] \to \mathbb{C}$,

$$\left|\int_a^b g(t)dt\right| \le \int_a^b |g(t)|dt$$

If we write $\int_a^b g(t)dt = re^{i\theta}$ with $r \ge 0$, then

$$\left|\int_a^b g(t)dt\right| = |re^{i\theta}| = r = \mathrm{Re}\left(e^{-i\theta}\int_a^b g(t)dt\right)$$

$$= \mathrm{Re}\left(\int_a^b g(t)e^{-i\theta}dt\right) = \int_a^b \mathrm{Re}(g(t)e^{-i\theta})dt \le \int_a^b \left|e^{-i\theta}g(t)\right| dt = \int_a^b |g(t)|dt$$

Let $g(t) = f(\gamma(t))\gamma'(t)$, then

$$\left|\int_\gamma g(z)dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\, dt$$

Then

$$\int_a^b |f(\gamma(t))\gamma'(t)|\, dt \le \sup_\gamma |f| \int_a^b |\gamma'(t)|dt = L(\gamma) \sup_\gamma |f|$$

□

Theorem 3.2.5. (Converse to FTC) Let $f : D \to \mathbb{C}$ be continuous on a domain D. If $\int_\gamma f(z)dz = 0$ for every closed contour $\gamma \in D$, for some $F : D \to \mathbb{C}$, $F'(z) = f(z)$.

Proof. Let $a_0 \in D$. For every $a_0 \neq w \in D$, let $\gamma(w)$ be a contour connecting a_0 to w and is contained in D.

Since D is a domain, it is path-connected, i.e. there is a smooth path γ_w connecting a_0 to w, therefore the collection of contours contained in D and connecting a_0 and W is non-empty. Let

$$F(w) := \int_{\gamma(w)} f(z)dz$$

Let $\tilde{\gamma}(w)$ be another contour that connects a_0 to w and is contained in D. Then let $c(w) = \gamma(w) \cup (-\tilde{\gamma}(w))$ that is obtained by moving through γ then through $\tilde{\gamma}$ in the opposite direction. Since c is a closed contour in D, $\int_C f(z)dz = 0$.

Then $0 = \int_C f(z)dz = \int_{\gamma(w)\cup(-\tilde{\gamma}(w))} f(z)dz = \int_{\gamma(w)} f(z)dz + \int_{-\tilde{\gamma}(w)} f(z)dz = \int_{\gamma(w)} f(z)dz - \int_{\tilde{\gamma}(w)} f(z)dz$. Hence

$$\int_{\gamma(w)} f(z)dz = \int_{\tilde{\gamma}(w)} f(z)dz$$

Therefore F does not depend on the contour chosen to join a_0 to w.

Now we claim F is holomorphic and we claim that F is holomorphic and $\forall z \in D, F'(z) = f(z) \Rightarrow \lim_{h\to 0} \frac{F(w+h)-F(w)}{h} = f(w)$.

To evaluate $F(w+h)$ we need a contour joining a_0 to $w+h$ contained in D. For every $w \in D$, let $r > 0$ such that $B_r(w) \subset D$. This ball must exist since D is open. Then for every $h \in \mathbb{C}$ with $|h| < r$ consider the striaght line δ_h that connects w to $w+h$.

A parameterisation of this line is given by

$$\delta_h : [0,1] \to D, \quad \delta_h(t) = w + th$$

The contour $\gamma_w \cup \delta_h$ is contained in D. So

$$F(w+h) = \int_{\gamma_w \cup \delta_h} f(z)dz = \int_{\gamma_w} f(z)dz + \int_{\delta_h} f(z)dz = F(w) + \int_{\delta_h} f(z)dz$$

$$\int_{\delta_h} f(w)dz = f(w) \int_{\delta_h} dz = f(w) \int_0^1 hdt = hf(w)$$

We can rewrite the previous equation as

$$F(w+h) = F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w))dz$$

For $h \neq 0$,

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| = \frac{1}{|h|} \left| \int_{\delta_h} (f(z) - f(w))dz \right|$$

$\square$

3.3 First Version of Cauchy's Theorem

Definition 3.3.1. A domain D is **starlike** if for some point $a_0 \in D$, for every $b \neq a_0 \in D$, the straight line connecting a_0 and b is contained in D.

Example 3.3.2.

1. $\mathbb{C}$ is starlike.
2. The ball $B_r(a)$ is starlike.
3. Any convex set is starlike.

Example 3.3.3.

1. $\mathbb{C}^\star$ is not starlike, because a straight line between two points could through 0, and $0 \notin \mathbb{C}^\star$.
2. Similarly, $B_r^\star(a) = B_r(a) - \{a\}$ is not starlike.

Lemma 3.3.4. Let U be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Then

$$\int_{\partial\Delta} f(z)dz = 0$$

for every **triangle** Δ in U.

Remark. Here $\partial\Delta$ is the boundary of Δ, traversed anticlockwise.

Remark. Given any closed contour without a parameterisation given, we will assume that it is traversed anticlockwise.

Proof. First, split Δ into four triangles, $\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}, \Delta^{(4)}$, using the midpoints of each side. Then

$$\int_{\partial\Delta} f(z)dz = \sum_{i=1}^{4} \int_{\partial\Delta^{(i)}} f(z)dz$$

Let Δ_1 be one of these four triangles which has the largest integral, then

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq 4\left|\int_{\partial\Delta_1} f(z)dz\right|$$

We then continue this procedure to produce a sequence of triangles

$$\Delta > \Delta_1 > \cdots > \Delta_n > \cdots$$

The length of Δ_1, $L(\Delta_1)$ satisfies $L(\Delta_1) = \frac{1}{2}L(\Delta)$, therefore

$$L(\Delta_n) = \frac{1}{2}nL(\Delta) \Longrightarrow L(\Delta_n) \to \text{ as } n \to \infty$$

Also,

$$\bigcap_{n\in\mathbb{N}} \Delta_n = \{w\}$$

is a single point in D. Now, notice that

$$\int_{\partial\Delta_n} 1dz = 0 = \int_{\partial\Delta_n} zdz$$

and that $w, f(w), f'(w)$ are constants. Then sneakily,

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} (f(z) - f(w) - (z-w)f'(w))$$

Define the auxiliary function

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} - f'(w) & \text{if } z \in D \setminus \{w\} \\ 0 & \text{if } z = w \end{cases}$$

which is continuous at $z = w$, so is continuous on D. So

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} (z-w)g(z)dz$$

Now,

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq 4^n \left|\int_{\partial\Delta_n} f(z)dz\right| = 4^n \left|\int_{\partial\Delta_n} (z-w)g(z)dz\right|$$

Note

$$\sup_{z\in\partial\Delta_n} |z-w| \leq L(\partial\Delta_n)$$

so by the Estimation Lemma,

$$\begin{aligned}\left|\int_{\partial\Delta} f(z)dz\right| &\leq 4^n L(\partial\Delta_n) \sup_{z\in\partial\Delta_n} |(z-w)g(z)| \\ &\leq 4^n L(\partial\Delta_n) \sup_{z\in\partial\Delta_n} |(z-w)| \sup_{z\in\partial\Delta_n} |g(z)| \\ &\leq 4^n (L(\partial\Delta_n))^2 \sup_{z\in\partial\Delta_n} |g(z)| \\ &= L(\Delta)^2 \sup_{z\in\partial\Delta_n} |g(z)|\end{aligned}$$

As $n \to \infty$, $\sup_{z\in\partial\Delta_n} |g(z)| \to g(w) = 0$. This completes the proof. □

Lemma 3.3.5. Let D be a starlike domain and $f : D \to \mathbb{C}$ be continuous. Then, if

$$\int_{\partial\Delta} f(z)dz = 0$$

for every $\Delta \subset D$, then for some $F : D \to \mathbb{C}$,

$$F'(z) = f(z) \quad \forall z \in D$$

Proof. Similar to the proof of converse of FTC. □

Theorem 3.3.6. (**Cauchy's Theorem for Starlike Domains - CTSD**) Let D be a starlike domain and let $f : D \to \mathbb{C}$ be holomorphic. Then for every closed contour $\gamma \in D$,

$$\int_\gamma f(z)dz = 0$$

Proof. By Lemma 3.3.4,

$$\int_{\partial\Delta} f(z)dz = 0 \quad \forall \Delta \in D$$

By Lemma 3.3.5, f has a holomorphic antiderivative F. Then, by FTC,

$$\int_\gamma f(z)dz = 0 \quad \forall \text{ closed } \gamma \in D$$

□

Remark. The same result holds if f is holomorphic on $D - S$, where S is a finite set of points and f is continuous on D. We will need this in proofs but is not used much elsewhere.

Example 3.3.7. Consider

$$\int_{|z|=\frac{1}{2}} \frac{e^z(\sin z)^2}{e^{z^2}} dz$$

Because the function in the integral is holomorphic and $|z| = \frac{1}{2}$ is a closed contour, by CTSD, this integral is equal to 0.

3.4 Cauchy's integral formula

Theorem 3.4.1. (**Cauchy's integral formula - CIF**) Let $B_r(a)$ be a ball in $\mathbb{C}$ and $f : B_r(a) \to \mathbb{C}$ be holomorphic. Then for every $w \in B_r(a)$,

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz$$

where ρ is any real number with $|w-a| < \rho < r$.

Proof. Define an auxiliary function g by

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

Note that g is continuous at $z = w$, and holomorphic elsewhere. By CTSD,

$$\int_{|z-a|=\rho} g(z) dz = 0$$

Therefore

$$\int_{|z-a|=\rho} \frac{f(z)}{z-w} dz = \int_{|z-a|=\rho} \frac{f(w)}{z-w} dz$$

Now,

$$\begin{aligned} \frac{1}{z-w} &= \frac{1}{z-a+a-w} \\ &= \frac{1}{(z-a)(1-\frac{w-a}{z-a})} \\ &= \frac{1}{z-a} \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a} \right)^n \end{aligned}$$

which converges uniformly, since $|\frac{w-a}{z-a}| = |\frac{w-a}{\rho} < 1|$. So we have

$$\begin{aligned} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz &= f(w) \int_{|z-a|=\rho} \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} dz \\ &= \sum_{n=0}^{\infty} \left(f(w)(w-a)^n \int_{|z-a|=\rho} \frac{1}{(z-a)^{n+1}} dz \right) \end{aligned}$$

The inner integral is equal to 0 except when $n = 0$, when it's value is $2\pi i$. So

$$\int_{|z-a|=\rho} \frac{f(z)}{z-w} dz = f(w)(w-a)^0 \cdot 2\pi i = 2\pi i \cdot f(w)$$

$\square$

4 Features of holomorphic functions

Theorem 4.0.1. (**Cauchy-Taylor theorem**) Let U be an open set and $f : U \to \mathbb{C}$ be holomorphic on U. Then for every $r > 0$ such that $B_r(a) \subset U$, f has a power series converging on $B_r(a)$ given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

is a constant for every $0 < \rho < r$. This is the **Taylor series** of f about a.

Proof. By the CIF, for every w with $|w - a| < \rho$,

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz \\ &= \frac{1}{2\pi i} \int_{|z-a|=\rho} f(z) \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} dz \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz \right) (w-a)^n \\ &= \sum_{n=0}^{\infty} c_n (w-a)^n \end{aligned}$$

□

Theorem 4.0.2. (**CIF for derivatives**) Let $f : B_r(a) \to \mathbb{C}$ be holomorphic. Then for every $0 < \rho < r$,

$$\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Proof. By Cauchy-Taylor, we have a convergent power series such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

But we also have (corollary 5.22 in lecture notes),

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Equating these two expressions for c_n completes the proof. □

Remark. Combining theorem 7.1 (lecture notes) and theorem 7.2 (lecture notes), every holomorphic function f has power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

Remark. Cauchy-Taylor does not hold in real analysis. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

f is differentiable for $x \neq 0$. For $x = 0$,

$$f'(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

$$\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{n \to 0^-} \frac{0}{x} = 0$$

and

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{e^{-1/x}}{x} = \lim_{x \to 0^+} \frac{1/x}{e^{1/x}} = \lim_{y \to \infty} \frac{y}{e^y} = 0$$

so $f'(0) = 0$, hence f is differentiable on $\mathbb{R}$. But if f had a Taylor series at $x = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = 0$$

around $x = 0$.

Corollary 4.0.3. (Holomorphic functions have infinitely many derivatives) If $f : U \to \mathbb{C}$ is holomorphic on an open set U then f has derivatives of all orders and each derivative is also holomorphic.

Proof. Since U is open, $\exists B_r(a) \subset U$ around a point $z = a$. But then by Cauchy-Taylor, f has a power series. By theorem 5.21 (lecture notes), this power series is holomorphic. By corollary 5.22 (lecture notes) we can term-by-term differentiate to get a power series for $f'(z)$. By theorem 5.21 (lecture notes), $f'(z)$ is holomorphic. This can be repeated indefinitely. □

Remark. This is a huge difference between real and complex analysis. Let $f : \mathbb{R} \to \mathbb{R}$ by defined as

$$f_n(x) = |x| x^n$$

$$f_n'(0) = \lim_{x \to 0} \frac{f_n(x) - f_n(0)}{x - 0} = \lim_{x \to 0} |x| x^{n-1} = 0$$

$f_n'(x) = (n+1)|x|x^{n-1}$ and $f^{(n)}(x) = c|x|$ which is not differentiable.

Theorem 4.0.4. (**Morera's Theorem**) Let $f : D \to \mathbb{C}$ be continuous on a domain D. If

$$\int_\gamma f(z) dz = 0 \quad \forall \text{ closed } \gamma \subset D$$

then f is holomorphic.

Proof. By the converse FTC, f has a holomorphic antiderivative $F : D \to \mathbb{C}$ such that $F'(z) = f(z) \quad \forall z \in D$. By corollary 7.6 (lecture notes), if F is holomorphic, its derivative f must be. □

Example 4.0.5. Consider

$$\int_{|z|=3} \frac{e^z}{z^2(z-1)} dz$$

We use partial fractions:

$$\frac{1}{z^2(z-1)} = \frac{a}{z} + \frac{b}{z^2} + \frac{c}{z-1}$$

So $1 = (c+a)z^2 + (b-a)z - b$, so $b = -1, a - -1, c = 1$. Using the CIF and CIF for derivatives,

$$\begin{aligned}
\int_{|z|=3} \frac{e^z}{z^2(z-1)} dz &= -\int_{|z|=3} \frac{e^z}{z} dz - \int_{|z|=3} \frac{e^z}{z^2} dz + \int_{|z|=3} \frac{e^z}{z-1} dz \\
&= -2\pi i e^0 - 2\pi i e^0 + 2\pi i e^1 \\
&= 2\pi i(e-2)
\end{aligned}$$

4.1 Liouville's theorem

Definition 4.1.1. A function $f : \mathbb{C} \to \mathbb{C}$ is **entire** if f is holomorphic on $\mathbb{C}$.

Definition 4.1.2. A function $f : \mathbb{C} \to \mathbb{C}$ is **bounded** if for some $M > 0$, $|f(z)| \leq M \ \forall z \in \mathbb{C}$.

Theorem 4.1.3. (**Liouville's theorem**) Every bounded entire function is constant.

Proof. Let f be entire and bounded. We will show that $\forall w \in \mathbb{C},\ f(w) = f(0)$. By the CIF, for every $\rho > |w|$,

$$\begin{aligned}
|f(w) - f(0)| &= \left| \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z-w} dz - \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z} dz \right| \\
&= \frac{|w|}{2\pi} \left| \int_{|z|=\rho} f(z) \frac{1}{z(z-w)} dz \right| \\
&=
\end{aligned}$$

Using the Estimation lemma, boundedness of f and the reverse triangle inequality,

$$\begin{aligned}
|f(w) - f(0)| &\leq \frac{|w|}{2\pi} 2\pi\rho \cdot \sup_{|z|=\rho} \frac{|f(z)|}{|z||z-w|} \\
&\leq |w|\rho \frac{M}{\rho} \sup_{|z|=\rho} \frac{1}{|z-w|} \\
&\leq |w|M \sup_{|z|=\rho} \frac{1}{||z| - |w||} \\
&= \frac{|w|M}{\rho - |w|} \\
&\to 0 \quad \text{as } \rho \to \infty
\end{aligned}$$

and ρ can be arbitrarily large. □

Remark. The holomorphicity condition is essential (we can't just say that f is continuous). For example,

$$f(z) = f(x+iy) = \sin(x) + i\sin(y)$$

is continuous and bounded on $\mathbb{C}$ but is not entire.

Theorem 4.1.4. (**Fundamental theorem of Algebra**) Every non-constant polynomial with complex coefficients $p(z) = a_d z^d + \cdots + a_1 z + a_0$, $a_d \neq 0$ has a complex root: for some $z_0 \in \mathbb{C}, P(z_0) = 0$.

Proof. By assumption, $d \geq 1$, so $|p(z)| \to \infty$ as $|z| \to \infty$. In particular, $\exists R > 0, |p(z)| > 1$ if $|z| = R$. Assume the converse, that p has no roots.

Then $f(z) := \frac{1}{p(z)}$ is holomorphic on $\mathbb{C}$. On the set $|z| > R$, f is bounded, since $|f(z)| = \frac{1}{|p(z)|} < 1$. But $\overline{B_R}(0) = \{z \in \mathbb{C} : |z| \leq R\}$ is compact, so by theorem 2.30 (lecture notes), $|f(z)|$ attains a maximum on $\overline{B_R}(0)$. In particular, f is bounded on $\overline{B_R}(0)$. Thus f is bounded and entire, so by Liouville's theorem, f is constant, which is a contradiction. $\square$