1. Metric spaces

1.1. Metrics

- **Metric space**: (X, d), X is set, $d : X \times X \to [0, \infty)$ is **metric** satisfying:
  - $d(x, y) = 0 \iff x = y$
  - **Symmetry**: $d(x, y) = d(y, x)$
  - **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- Examples of metrics:
  - p-adic metric:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

  - Extension of the p-adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

  - Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

  - Discrete metric:

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Open ball of radius r around x**:

$$B(x; r) = \{y \in X : d(x, y) < r\}$$

- **Closed ball of radius r around x**:

$$D(x; r) = \{y \in X : d(x, y) \leq r\}$$

1.2. Open and closed sets

- $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

- $A \subseteq X$ is **closed** if $X - A$ is open.
- Sets can be neither closed nor open, or both.
- Any singleton $\{x\} \in \mathbb{R}$ is closed and not open.
- Let X be metric space, $x \in N \subseteq X$. N is **neighbourhood** of x if

$$\exists \text{ open } V \subseteq X : x \in V \subseteq N$$

- **Corollary**: let $x \in X$, then $N \subseteq X$ neighbourhood of x iff $\exists \varepsilon > 0 : x \in B(x; \varepsilon) \subseteq N$.
- **Proposition**: open balls are open, closed balls are closed.
- **Lemma**: let (X, d) metric space.
  - X and $\emptyset$ are both open and closed.
  - Arbitrary unions of open sets are open.
  - Finite intersections of open sets are open.

- Finite unions of closed sets are closed.
- Arbitrary intersections of closed sets are closed.

1.3. Continuity

- **Sequence** in X: $a : \mathbb{N} \to X$, written $(a_n)_{n \in \mathbb{N}}$.
- (a_n) converges to a if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, d(a, a_n) < \varepsilon$$

- **Proposition**: let X, Y metric spaces, $a \in X$, $f : X \to Y$. The following are equivalent
  - $\forall \varepsilon > 0, \exists \delta > 0 : d_X(a, x) < \delta \Longrightarrow d_Y(f(a), f(x)) < \varepsilon$.
  - For every sequence (a_n) in X with $a_n \to a$, $f(a_n) \to f(a)$.
  - For every open $U \subseteq Y$ with $f(a) \in U$, $f^{-1}(U)$ is a neighbourhood of a.

If f satisfies these, it is **continuous at a**.

- f **continuous** if continuous at every $a \in X$.
- **Proposition**: $f : X \to Y$ continuous iff $f^{-1}(U)$ open for every open $U \subseteq Y$.

2. Topological spaces

2.1. Topologies

- **Power set** of X: $\mathcal{P}(X) := \{A : A \subseteq X\}$.
- **Topology** on set X is $\tau \subseteq \mathcal{P}(X)$ with:
  - $\emptyset \in \tau$, $X \in \tau$.
  - If $\forall i \in I, U_i \in \tau$, then

$$\bigcup_{i \in I} U_i \in \tau$$

  - $U_1, U_2 \in \tau \Longrightarrow U_1 \cap U_2 \in \tau$ (this is equivalent to $U_1, ..., U_n \in \tau \Longrightarrow \cap_{i \in [n]} U_i \in \tau$).
- (X, τ) is **topological space**. Elements of τ are **open** subsets of X.
- $A \subseteq X$ **closed** if $X - A$ is open.
- Let X be a set. Then $\tau = \mathcal{P}(X)$ is the **discrete topology** on X.
- $\tau = \{\emptyset, X\}$ is the **indiscrete topology** on X.
- **Examples**:
  - For metric space (M, d), find the open sets with respect to metric d. Let $\tau_d \subseteq \mathcal{P}(M)$ exactly contain these open sets. Then (M, τ_d) is a topological space. The metric d **induces** the topology τ_d.
  - Let $X = \mathbb{N}_0$ and $\tau = \{\emptyset\} \cup \{U \subseteq X : X - U \text{ is finite}\}$.
- **Proposition**: for topological space X:
  - X and $\emptyset$ are closed
  - Arbitrary intersections of closed sets are closed
  - Finite unions of closed sets are closed
- **Proposition**: for topological space (X, τ) and $A \subseteq X$, the **induced (subspace) topology on A**

$$\tau_A = \{A \cap U : U \in \tau\}$$

is a topology on A.

- **Example**: let $X = \mathbb{R}$ with standard topology induced by metric $d(x, y) = |x - y|$. Let $A = [1, 5]$. Then $[1, 3) = A \cap (0, 3)$ and $[1, 5] = A \cap (0, 6)$ are open in A.
- **Example**: consider $\mathbb{R}$ with standard topology τ. Then
  - $\tau_{\mathbb{Z}}$ is the discrete topology on $\mathbb{Z}$.
  - $\tau_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.
- **Proposition**: the metrics d_p for $p \in [1, \infty)$ and d_∞ all induce the same topology on $\mathbb{R}^n$.
- **Definition**: (X, τ) is **Hausdorff** if

$$\forall x \neq y \in X, \exists U, V \in \tau : U \cap V = \emptyset \wedge x \in U, y \in V$$

- **Lemma**: any metric space (M, d) is Hausdorff.
- **Example**: let $|X| \geq 2$ with the indiscrete topology. Then X is not Hausdorff, since $\tau = \{X, \emptyset\}$ and if $x \neq y \in X$, the only open set containing x is X (same for y). But $X \cap X = X \neq \emptyset$.
- **Furstenberg's topology on** $\mathbb{Z}$: define $U \subseteq \mathbb{Z}$ to be open if

$$\forall a \in U, \exists 0 \neq d \in \mathbb{Z} : a + d\mathbb{Z} =: \{a + dn : n \in \mathbb{Z}\} \subseteq U$$

- Furstenberg's topology is Hausdorff.

2.2. Continuity

- **Definition**: let X, Y topological spaces.
  - $f : X \to Y$ is **continuous** if

$$\forall V \text{ open in } Y, f^{-1}(V) \text{ open in } X$$

  - f is **continuous at $a \in X$** if

$$\forall V \text{ open in } Y, f(a) \in V, \exists U \text{ open in } X : a \in U \subseteq f^{-1}(V)$$

- **Lemma**: $f : X \to Y$ continuous iff f continuous at every $a \in X$. (Key idea for proof: $\cup_{a \in f^{-1}(V)} U_a \subseteq f^{-1}(V) = \cup_{a \in f^{-1}(V)} \{a\} \subseteq \cup_{a \in f^{-1}(V)} U_a$)
- **Example**: inclusion $i : (A, \tau_A) \to (X, \tau_X)$, $A \subseteq X$, is always continuous.
- **Lemma**: a composition of continuous functions is continuous.
- **Lemma**: let $f : X \to Y$ be function between topological spaces. Then f is continuous iff

$$\forall A \text{ closed in } Y, \quad f^{-1}(A) \text{ closed in } X$$

- **Remark**: we can use continuous functions decide that sets are open or closed.
- **Definition**: **n-sphere** is

$$S^n := \left\{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 = 1\right\}$$

- **Example**: in the standard topology, the n-sphere is a closed subset of $\mathbb{R}^{n+1}$. (Consider the preimage of $\{1\}$ which is closed in $\mathbb{R}$).
- Can consider set of square matrices $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and give it the standard topology.

- **Example**:
  - Note

$$\det(A) = \sum_{\sigma \in \text{ sym}(n)} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

  is a polynomial in the entries of A so is continuous function from $M_n(\mathbb{R})$ to $\mathbb{R}$.
  - $\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} = \det^{-1}(\mathbb{R} - \{0\})$ is open.
  - $\mathrm{SL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} = \det^{-1}(\{1\})$ is closed.
  - $O(n) = \left\{A \in M_n(\mathbb{R}) : AA^T = I\right\}$ is closed - consider $f_{i,j}(A) = \left(AA^T\right)_{i,j}$ then

$$O(n) = \bigcap_{1 \leq i,j \leq n} \left(f_{i,j}\right)^{-1}\left(\left\{\delta_{i,j}\right\}\right)$$

  - $\mathrm{SO}(n) = O(n) \cap \mathrm{SL}_n(\mathbb{R})$ is closed.
- **Definition**: for X, Y topological spaces, $h : X \to Y$ is **homeomorphism** if h is bijective, continuous and h^{-1} is continuous. X and Y are **homeomorphic**. A homeomorphism gives bijection between τ_X and τ_Y and satisfies

$$h(A \cap B) = h(A) \cap h(B), \quad h(A \cup B) = h(A) \cup h(B)$$

- **Example**: in standard topology, $(0, 1)$ is homeomorphic to $\mathbb{R}$. (Consider $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$, $f = \tan$, $g : (0, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $g(x) = \pi\left(x - \frac{1}{2}\right)$ and $f \circ g$).
- **Example**: $\mathbb{R}$ with standard topology τ_{st} is not homoeomorphic to $\mathbb{R}$ with the discrete topology τ_d. (Consider $h^{-1}(\{a\}) = \left\{h^{-1}(a)\right\}$, $\{a\} \in \tau_{\text{st}}$ but $\left\{h^{-1}(a)\right\} \notin \tau_{\text{st}}$).
- **Example**: let $X = \mathbb{R} \cup \left\{\overline{0}\right\}$. Define $f_0 : \mathbb{R} \to X$, $f_0(a) = a$ and $f_{\overline{0}} : \mathbb{R} \to X$, $f_{\overline{0}}(a) = a$ for $a \neq 0$, $f_{\overline{0}}(0) = \overline{0}$. Topology on X has $A \subseteq X$ open iff $f_0^{-1}(A)$ and $f_{\overline{0}}^{-1}(A)$ open. Every point in X lies in open set: for $a \notin \left\{0, \overline{0}\right\}$, $a \in \left(a - \frac{|a|}{2}, a + \frac{|a|}{2}\right)$ and both pre-images of this are same open interval, for 0, set $U_0 = (-1, 0) \cup \{0\} \cup (0, 1) \subseteq X$ then $f_0^{-1}(U_0) = (-1, 1)$ and $f_{\underline{0}}^{-1}(U_0) = (-1, 0) \cup (0, 1)$ are both open. For $\overline{0}$, set $U_{\overline{0}} = (-1, 0) \cup \left\{\overline{0}\right\} \cup (0, 1) \subseteq X$, then $f_{\overline{0}}^{-1}(U_{\overline{0}}) = (-1, 1)$ and $f_0^{-1}(U_{\overline{0}}) = (-1, 0) \cup (0, 1)$ are both open. So U_0 and $U_{\overline{0}}$ both open in X. X is not Hausdorff since any open sets containing 0 and $\overline{0}$ must contain "open intervals" such as U_0 and $U_{\overline{0}}$.
- **Example (Furstenberg's proof of infinitude of primes)**: since $a + d\mathbb{Z}$ is infinite, any nonempty finite set is not open, so any set with finite complement is not closed. For fixed d, sets $d\mathbb{Z}$, $1 + d\mathbb{Z}, ..., (d - 1) + d\mathbb{Z}$ partition $\mathbb{Z}$. So the complement of each is the union of the rest, so each is open and closed. Every $n \in \mathbb{Z} - \{-1, 1\}$ is prime or product of primes, so $\mathbb{Z} - \{-1, 1\} = \cup_{p \text{ prime}} p\mathbb{Z}$, but finite unions of closed sets are closed, and since $\mathbb{Z} - \{-1, 1\}$ has finite complement, the union must be infinite.

2.3. Limits, bases and products

2.4. Limit points, interiors and closures

- **Definition**: for topological space X, $x \in X$, $A \subseteq X$:

  - **Open neighbourhood of x** is open set N, $x \in N$.
  - $x \in X$ is **limit point** of A if every open neighbourhood N of x satisfies

$$(N - \{x\}) \cap A \neq \emptyset$$

- **Corollary**: x is not limit point of A iff exists neighbourhood N of x with

$$A \cap N = \begin{cases} \{x\} & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

- **Example**: let $X = \mathbb{R}$ with standard topology.
  - $0 \in X$, then $(-1/2, 1/2)$ is open neighbourhood of 0.
  - If $U \subseteq X$ open, U is open neighbourhood for any $x \in U$.
  - Let $A = \left\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\right\}$, then only limit point in A is 0.
- **Definition**: let $A \subseteq X$.
  - **Interior** of A is largest open set contained in A:

$$A^\circ = \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U$$

  - **Closure** of A is smallest closed set containing A:

$$\overline{A} = \bigcap_{\substack{F \text{ closed} \\ A \subseteq F}}$$

  If $A^\circ = X$, A is **dense** in X.
- **Lemma**:
  - $\overline{X - A} = X - A^\circ$
  - $\overline{A} = X - (X - A)^\circ$
- **Example**:
  - Let $\mathbb{Q} \subset \mathbb{R}$ with standard topology. Then $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$ (since every nonempty open set in $\mathbb{R}$ contains rational and irrational numbers).
- **Lemma**: $\overline{A} = A \cup L$ where L is the set of limit points of A.