

1. Introduction

- Basic encryption process:
 - A has a message (**plaintext**) which is **encrypted** using an **encryption key** to produce the **ciphertext**, which is sent to *B*.
 - B uses a **decryption key** (which depends on the encryption key) to **decrypt** the ciphertext and recover the original plaintext.
 - It should be computationally infeasible to determine the plaintext without knowing the decryption key.
- **Caesar cipher**:
 - Add a constant to each letter in the plaintext to produce the ciphertext:

$$\text{ciphertext letter} = \text{plaintext letter} + k \pmod{26}$$

- To decrypt,
$$\text{plaintext letter} = \text{ciphertext letter} - k \pmod{26}$$
 - The key is $k \pmod{26}$.
- Cryptosystem objectives:
 - **Secrecy**: the intercepted message should be not able to be decrypted
 - **Integrity**: a message should not allowed to be altered without the receiver knowing
 - **Authenticity**: the receiver should be certain of the identity of the sender
 - **Non-repudiation**: the sender should not be able to claim they sent a message; the receiver should be able to prove they did.
- **Kerckhoff's principle**: a cryptographic system should be secure even if the details of the system are known to an attacker.
- Types of attack:
 - **Ciphertext-only**: the plaintext is deduced from the ciphertext.
 - **Known-plaintext**: intercepted ciphertext and associated stolen plaintext are used to determine the key.
 - **Chosen-plaintext**: an attacker tricks a sender into encrypting various chosen plaintexts and observes the ciphertext, then uses this information to determine the key.
 - **Chosen-ciphertext**: an attacker tricks the receiver into decrypting various chosen ciphertexts and observes the resulting plaintext, then uses this information to determine the key.

2. Symmetric key ciphers

- **Converting letters to numbers**: treat letters as integers modulo 26, with $A = 1$, $Z = 0 \equiv 26 \pmod{26}$. Treat a string of text as a vector of integers modulo 26.
- **Symmetric key cipher**: one in which encryption and decryption keys are equal.
- **Key size**: $\log_2(\text{number of possible keys})$.

- **Substitution cipher:** key is permutation of $\{a, \dots, z\}$. Key size is $\log_2(26!)$. It is vulnerable to plaintext attacks and ciphertext-only attacks, since different letters (and letter pairs) occur with different frequencies in English.
- **Stirling's formula:**

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

- **One-time pad:** key is uniformly, independently random sequence of integers mod 26, (k_1, k_2, \dots) , it is known to the sender and receiver. If message is (m_1, m_2, \dots, m_r) then ciphertext is $(c_1, c_2, \dots, c_r) = (k_1 + m_1, k_2 + m_2, \dots, k_r + m_r)$. To decrypt the ciphertext, $m_i = c_i - k_i$. Once (k_1, \dots, k_r) have been used, they must never be used again.
 - One-time pad is information-theoretically secure against ciphertext-only attack: $\mathbb{P}(M = m \mid C = c) = \mathbb{P}(M = m)$.
 - Disadvantage is keys must never be reused, so must be as long as message.
 - Keys must be truly random.
- **Chinese remainder theorem:** let $m, n \in \mathbb{N}$ coprime, $a, b \in \mathbb{Z}$. Then exists unique solution $x \bmod mn$ to the congruences

$$\begin{aligned} x &\equiv a \pmod{m} \\ x &\equiv b \pmod{n} \end{aligned}$$

- **Block cipher:** group characters in plaintext into blocks of n (the **block length**) and encrypt each block with a key. So plaintext $p = (p_1, p_2, \dots)$ is divided into blocks P_1, P_2, \dots where $P_1 = (p_1, \dots, p_n)$, $P_2 = (p_{n+1}, \dots, p_{2n})$. Then ciphertext blocks are given by $C_i = f(\text{key}, P_i)$ for some encryption function f .
- **Hill cipher:**
 - Plaintext divided into blocks P_1, \dots, P_r of length n .
 - Each block represented as vector $P_i \in (\mathbb{Z}/26\mathbb{Z})^n$
 - Key is invertible $n \times n$ matrix M with elements in $\mathbb{Z}/26\mathbb{Z}$.
 - Ciphertext for block P_i is

$$C_i = MP_i$$

It can be decrypted with $P_i = M^{-1}C_i$.

- Let $P = (P_1, \dots, P_r)$, $C = (C_1, \dots, C_r)$, then $C = MP$.
- **Confusion:** each character of ciphertext depends on many characters of key.
- **Diffusion:** each character of ciphertext depends on many characters of plaintext. Ideal diffusion changes a proportion of $(S - 1)/S$ of the characters of the ciphertext, where S is the number of possible symbols.
- For Hill cipher, i th character of ciphertext depends on i th row of key - this is medium confusion. If j th character of plaintext changes and $M_{ij} \neq 0$ then i th character of ciphertext changes. M_{ij} is non-zero with probability roughly $25/26$ so good diffusion.
- Hill cipher is susceptible to known plaintext attack:

- If $P = (P_1, \dots, P_n)$ are n blocks of plaintext with length n such that P is invertible and we know P and the corresponding C , then we can recover M , since $C = MP \implies M = CP^{-1}$.
- If enough blocks of ciphertext are intercepted, it is very likely that n of them will produce an invertible matrix P .

3. Public key cryptography and the RSA algorithm

- **Euler φ function:**

$$\varphi : \mathbb{N} \rightarrow \mathbb{N}, \varphi(n) = |\{1 \leq a \leq n : \gcd(a, n) = 1\}| = |(\mathbb{Z}/n\mathbb{Z})^\times|$$

- $\varphi(p^r) = p^r - p^{r-1}$, $\varphi(mn) = \varphi(m)\varphi(n)$ for $\gcd(m, n) = 1$.
- **Euler's theorem:** if $\gcd(a, n) = 1$, $a^{\varphi(n)} \equiv 1 \pmod{n}$.
- **Public key cryptography:**
 - Create two keys, k_D and k_E . k_E is public, k_D is private.
 - Plaintext m is encrypted as $c = f(m, k_E)$.
 - Ciphertext decrypted by $m = g(c, k_D)$.
- **RSA:**
 - k_E is pair (n, e) where $n = pq$ is product of two distinct primes and $e \in \mathbb{Z}$ is coprime to $\varphi(n)$.
 - k_D is integer d such that $de \equiv 1 \pmod{\varphi(n)}$.
 - m is an integer modulo n , m and n are coprime.
 - Encryption: $c = m^e \pmod{n}$.
 - Decryption: $m = c^d \pmod{n}$.
- **RSA problem:** given $n = pq$ a product of two unknown primes, e and $m^e \pmod{n}$, recover m . If n can be factored, the RSA is solved.
- It is recommended that n have at least 2048 bits. A typical choice of e is $2^{16} + 1$.
- **Attacks on RSA:**
 - If you can factor n , you can compute d , so can break RSA (as then you know $\varphi(n)$ so can compute $e^{-1} \pmod{\varphi(n)}$).
 - If $\varphi(n)$ is known, then we have $pq = n$ and $(p-1)(q-1) = \varphi(n)$ so $p + q = n - \varphi(n) + 1$. Hence p and q are roots of $x^2 - (n - \varphi(n) + 1)x + n = 0$.
 - **Known d :** we have $de - 1$ is multiple of $\varphi(n)$. Look for a factor A of $de - 1$ such that $(p-1) \mid A$, $(q-1) \nmid A$. Then try $x^A - 1$ for random x , this satisfies $x^A - 1$ is divisible by p , hence $\gcd(x^A - 1, n) = p$.
- **RSA signatures:**
 - Public key is (n, e) and private key is d .
 - When sending a message m , message is **signed** by also sending $s = m^d \pmod{n}$.
 - (m, s) is received, **verified** by checking if $m = s^e \pmod{n}$.
 - Forging a signature on a message m would require finding s with $m = s^e \pmod{n}$. This is the RSA problem.
 - However, choosing signature s first then taking $m = s^e \pmod{n}$.
 - To solve this, (m, s) is sent where $s = h(m)^d$, h is **hash function**. Then the message receiver verifies $h(m) = s^e \pmod{n}$.

- Now, for a signature to be forged, an attacker would have to find m with $h(m) = s^e \bmod n$.
- **Hash function** is function $h : \{\text{messages}\} \rightarrow \mathcal{H}$ that:
 - Can be computed efficiently
 - Is preimage-resistant: can't quickly find m with given $h(m)$.
 - Is collision-resistant: can't quickly find m, m' with $h(m) = h(m')$.

Example is SHA-256.

- **Theorem:** it is no easier to find $\varphi(n)$ than to factorise n .
- **Theorem:** it is no easier to find d than to factor n .
- **Miller-Rabin algorithm:**
 1. Choose random $x \bmod n$.
 2. Let $n - 1 = 2^r s$, $y = x^s$.
 3. Compute $y, y^2, \dots, y^{2^r} \bmod n$.
 4. If 1 isn't in this list, n is **composite** (with witness x).
 5. If 1 is in list preceded by number other than ± 1 , n is **composite** (with witness a).
 6. Other, n is **possible prime** (to base x).

3.1. Factorisation

- **Trial division algorithm:** for $p = 2, 3, 5, \dots$ test whether $p \mid n$.
- **Fermat's method:**
 - Let $a = \lceil \sqrt{n} \rceil$. Compute $a^2 \bmod n$, $(a + 1)^2 \bmod n$ until a square $x^2 \equiv (a + i)^2 \bmod n$ appears. Then compute $\gcd(a + i - x, n)$.
 - Works well under special conditions on the factors: if $|p - q| \leq 2\sqrt{2}\sqrt[4]{n}$ then Fermat's method takes one step: $x = \lceil \sqrt{n} \rceil$ works.
- An integer is B -smooth if all its prime factors are $\leq B$.
- **Quadratic sieve:**
 - Choose B and let m be number of primes $\leq B$.
 - Look at integers $x = \lceil \sqrt{n} \rceil + k$, $k = 1, 2, \dots$ and check whether $y = x^2 - n$ is B -smooth.
 - Once $y_1 = x_1^2 - n, \dots, y_t = x_t^2 - n$ are all B -smooth with $t > m$, find some product of them that is a square.
 - Deduce a congruence between the squares.
- **Other factorisation algorithms:**
 - Pollard's ρ algorithm.
 - Pollard's $p - 1$ algorithm.
 - Lenstra's algorithm using elliptic curves.
 - General number field sieve
 - Shor's algorithm: $\ln(N)^2 \ln(\ln(N))$.

3.2. Primitive roots

- Let p prime, $g \in \mathbb{F}_p^\times$. **Order** of g is smallest $a \in \mathbb{N}_0$ such that $g^a = 1$. g is **primitive root** if its order is $p - 1$.

- Let p prime, $g \in \mathbb{F}_p^\times$ primitive root. If $x \in \mathbb{F}_p^\times$ then $x = g^L$ for some $0 \leq L < p - 1$. Then L is **discrete logarithm** of x to base g . Write $L = L_g(x)$. It satisfies:
 - $g^{L_g(x)} \equiv x \pmod{p}$ and $g^a \equiv x \pmod{p} \iff a \equiv L_g(x) \pmod{p-1}$.
 - $L_g(1) = 0, L_g(g) = 1$.
 - $L_g(xy) \equiv L_g(x) + L_g(y) \pmod{p-1}$.
 - h is primitive root mod p iff $L_g(h)$ coprime to $p-1$. So number of primitive roots mod p is $\varphi(p-1)$.
- **Discrete logarithm problem:** given p, g, x , compute $L_g(x)$.
- **Diffie-Hellman key exchange:**
 - Two parties agree on prime p and primitive root $g \bmod p$.
 - Alice chooses secret $\alpha \bmod(p-1)$ and sends $g^\alpha \bmod p$ to Bob.
 - Bob chooses secret $\beta \bmod(p-1)$ and sends $g^\beta \bmod p$ to Alice.
 - Alice and Bob both compute $\kappa = g^{\alpha\beta} = (g^\alpha)^\beta = (g^\beta)^\alpha \bmod p$.
- **Diffie-Hellman problem:** given p, g, g^α, g^β , compute $g^{\alpha\beta}$.
- If discrete logarithm problem can be solved, so can Diffie-Hellman problem (since could compute $\alpha = L_g(g^\alpha)$ or $\beta = L_g(g^\beta)$).
- **Elgamal public key encryption:**
 - Alice chooses prime p , primitive root g , private key $\alpha \bmod(p-1)$.
 - Her public key is $y = g^\alpha$.
 - Bob chooses random $k \bmod(p-1)$
 - To send message m (integer mod p), he sends the pair $(r, m') = (g^k, my^k)$.
 - To decrypt the message, Alice computes $r^\alpha = g^{\alpha k} = y^k$ and then $m = m'y^{-k} = m'r^{-\alpha}$.
 - If Diffie-Hellman problem is hard, then Elgamal encryption is secure against known plaintext attack.
 - Key k must be random and different each time.
- **Decision Diffie-Hellman problem:** given g^a, g^b, c in \mathbb{F}_p^\times , decide whether $c = g^{ab}$.
 - This problem is not always hard, as can tell if g^{ab} is square or not. Can fix this by taking g to have large prime order $q \mid (p-1)$. $p = 2q + 1$ is a good choice.
- **Elgamal signatures:**
 - Public key is (p, g) , $y = g^\alpha$ for private key α .
 - **Valid Elgamal signature** on m is pair (r, s) , $r \geq 0, s < p-1$ such that

$$y^r r^s = g^m \pmod{p}$$
 - Alice computes $r = g^k, k \in (\mathbb{Z}/(p-1))^\times$ random.
 - Then $g^{\alpha r} g^{ks} \equiv g^m \pmod{p}$ so $\alpha r + ks \equiv m \pmod{p-1}$ so $s = k^{-1}(m - \alpha r) \pmod{p-1}$.
- **Elgamal signature problem:** given p, g, y, m , find r, s such that $y^r r^s = m$.
- **Discrete logarithm problem:** given prime p , primitive root $g \bmod p$, $x \in \mathbb{F}_p^\times$, calculate $L_g(x)$.

- **Baby-step giant-step algorithm** for solving DLP:
 - Let $N = \lceil \sqrt{p-1} \rceil$.
 - Baby-steps: compute $g^j \bmod p$ for $0 \leq j < N$.
 - Giant-steps: compute $xg^{-Nk} \bmod p$ for $0 \leq k < N$.
 - Look for a match between baby-steps and giant-steps lists:
 $g^j = xg^{-Nk} \implies x = g^{j+Nk}$.
 - Always works since if $x = g^L$ for $0 \leq L < p-1 \leq N^2$, so L can be written as $j + Nk$ with $j, k \in \{0, \dots, N-1\}$.

4. Elliptic curves

- **Definition: abelian group** (G, \circ) satisfies:
 - **Associativity:** $\forall a, b, c \in G, a \circ (b \circ c) = (a \circ b) \circ c$.
 - **Identity:** $\exists e \in G : \forall g \in G, e \times g = g$.
 - **Inverses:** $\forall g \in G, \exists h \in G : g \circ h = h \circ g = e$
 - **Commutativity:** $\forall a, b \in G, a \circ b = b \circ a$.
- **Notation:** for $g \in G$, write $[n]g$ for $g \circ \dots \circ g$ n times if $n > 0$, e if $n = 0$, $[-n]g$ if $n < 0$.
- **DLP for abelian groups:** given G a cyclic abelian group, $g \in G$ a generator of G , $x \in G$, find L such that $[L]g = x$. L is well-defined modulo $|G|$.
- **Fundamental theorem of finite abelian groups:** let G finite abelian group, then there exist unique integers $2 \leq n_1, \dots, n_r$ with $n_i \mid n_{i+1}$ for all i , such that

$$G \simeq (\mathbb{Z}/n_1) \times \dots \times (\mathbb{Z}/n_r)$$

In particular, G is isomorphic to a product of cyclic groups.

- **Definition:** let K field, $\text{char}(K) > 3$. An **elliptic curve** over K is defined by the equation

$$y^2 = x^3 + ax + b$$

where $a, b \in K$, $\Delta_E = 4a^3 + 27b^2 \neq 0$.

- **Remark:** elliptic curve over \mathbb{Q} is also elliptic curve over \mathbb{R} or \mathbb{C} .
- **Remark:** $\Delta_E \neq 0$ is equivalent to $x^3 + ax + b$ having no repeated roots (i.e. E is smooth).
- **Definition:** for elliptic curve E defined over K , a **K -point (point)** on E is either:
 - A **normal point:** $(x, y) \in K^2$ satisfying the equation defining E .
 - The **point at infinity** \overline{O} which can be thought of as infinitely far along the y -axis (in either direction).

Denote set of all K -points on E as $E(K)$.

- Any elliptic curve $E(K)$ is an abelian group, with group operation \oplus is defined as:
 - We should have $P \oplus Q \oplus R = \overline{O}$ iff P, Q, R lie on straight line.
 - In this case, $P \oplus Q = -R$.
 - To find line ℓ passing through $P = (x_0, y_0)$ and $Q = (x_1, y_1)$:
 - If $x_0 \neq x_1$, then equation of ℓ is $y = \lambda x + \mu$, where

$$\lambda = \frac{y_1 - y_0}{x_1 - x_0}, \quad \mu = y_0 - \lambda x_0$$

Now

$$\begin{aligned} y^2 &= x^3 + ax + b = (\lambda x + \mu)^2 \\ \implies 0 &= x^3 - \lambda^2 x^2 + (a - 2\lambda\mu)x + (b - \mu^2) \end{aligned}$$

Since sum of roots of monic polynomial is equal to minus the coefficient of the second highest power, and two roots are x_0 and x_1 , the third root is $x_2 = \lambda^2 - x_0 - x_1$. Then $y_2 = \lambda x_2 + \mu$ and $R = (x_2, y_2)$.

- If $x_0 = x_1$, then using implicit differentiation:

$$\begin{aligned} y^2 &= x^3 + ax + b \\ \implies \frac{dy}{dx} &= \frac{3x^2 + a}{2y} \end{aligned}$$

and the rest is as above, but instead with $\lambda = \frac{3x_0^2 + a}{2y_0}$.

- **Definition: group law** of elliptic curves: let $E : y^2 = x^3 + ax + b$. For all normal points $P = (x_0, y_0), Q = (x_1, y_1) \in E(K)$, define
 - \bar{O} is group identity: $P \oplus \bar{O} = \bar{O} \oplus P = P$.
 - If $P = -Q$, $P \oplus Q = \bar{O}$.
 - Otherwise, $P \oplus Q = (x_2, -y_2)$, where

$$\begin{aligned} x_2 &= \lambda^2 - (x_0 + x_1), \\ y_2 &= \lambda x_2 + \mu, \\ \lambda &= \begin{cases} \frac{y_1 - y_0}{x_1 - x_0} & \text{if } x_0 \neq x_1 \\ \frac{3x_0^2 + a}{2y_0} & \text{if } x_0 = x_1 \end{cases}, \\ \mu &= y_0 - \lambda x_0 \end{aligned}$$

- **Example:**
 - Let E be given by $y^2 = x^3 + 17$ over \mathbb{Q} , $P = (-1, 4) \in E(\mathbb{Q})$, $Q = (2, 5) \in E(\mathbb{Q})$. To find $P \oplus Q$,

$$\lambda = \frac{5 - 4}{2 - (-1)} = \frac{1}{3}, \quad \mu = 4 - \lambda(-1) = \frac{13}{3}$$

So $x_2 = \lambda^2 - (-1) - 2 = -\frac{8}{9}$ and $y_2 = -(\lambda x_2 + \mu) = -\frac{109}{27}$ hence

$$P \oplus Q = \left(-\frac{8}{9}, -\frac{109}{27} \right)$$

To find $[2]P$,

$$\lambda = \frac{3(-1)^2 + 0}{2 \cdot 4} = \frac{3}{8}, \quad \mu = 4 - \frac{3}{8} \cdot (-1) = \frac{35}{8}$$

so $x_3 = \lambda^2 - 2 \cdot (-1) \frac{137}{64}$, $y_3 = -(\lambda x_3 + \mu) = -\frac{2651}{512}$ hence

$$[2]P = (x_3, y_3) = \left(\frac{137}{64}, -\frac{2651}{512} \right)$$

- **Hasse's theorem:** let $|E(\mathbb{F}_p)| = N$, then

$$|N - (p + 1)| \leq 2\sqrt{p}$$