

# 1. Introduction, the natural numbers

- $\mathbb{N} = \{1, 2, 3, \dots\}$
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$
- **Peano's axioms:** three primitive terms:  $\mathbb{N}_0$ , 0 and **successor function**,  $S$ .
  - $0 \in \mathbb{N}_0$ .
  - $\forall a \in \mathbb{N}_0, S(a) \neq 0$ .
  - $S(a) = S(b) \Rightarrow a = b$ .
  - If  $X \subseteq \mathbb{N}_0$  and
    - $0 \in X$  and
    - $\forall a \in X, S(a) \in X$then  $X = \mathbb{N}_0$ .
- Last axiom applied to  $X = \{n \in \mathbb{N}_0 : P(n) \text{ true}\}$  gives **Principle of Mathematical Induction (PMI)**: for statement  $P(n)$ , if  $P(0)$  true and  $\forall n \in \mathbb{N}_0, P(n) \Rightarrow P(n+1)$  then  $P(n)$  true for every  $n \in \mathbb{N}_0$ .
- **PMI variants:**
  - If  $P(0)$  true and if for every  $n \in \mathbb{N}_0, P(x)$  for every  $x < n$  implies  $P(n)$ , then  $P(n)$  true for every  $n \in \mathbb{N}_0$ .
  - Same as two variants above but with base case  $P(1)$  true leading to  $P(n)$  true for every  $n \in \mathbb{N}$ .
- **Addition of natural numbers:** let  $a \in \mathbb{N}_0$ .
  - $a + 0 = a$ .
  - $a + S(b) = S(a + b)$
- **Well ordering principle (WOP):** let  $S \subseteq \mathbb{N}_0, S \neq \emptyset$ , then  $S$  has a smallest element.

# 2. Divisibility

- $a$  **divides**  $b$ ,  $a \mid b$  if  $\exists d \in \mathbb{Z}, b = ad$ . If not, write  $a \nmid b$ .
- **Properties of divisibility:**
  - $a \mid 0$ .
  - If  $a \neq 0, 0 \nmid a$ .
  - $1 \mid a$  and  $a \mid a$ .
  - $a \mid b \Rightarrow a \mid bc$ .
  - $a \mid b$  and  $b \mid c \Rightarrow a \mid c$ .
  - $a \mid b$  and  $a \mid c \Rightarrow a \mid (bx + cy)$  for any  $x, y \in \mathbb{Z}$ .
  - $a \mid b$  and  $b \mid a \Rightarrow a = \pm b$ .
  - $a \mid b, a > 0, b > 0 \Rightarrow a \leq b$ .
  - $a \mid b \Rightarrow ac \mid bc$ .
- **Division algorithm:** let  $a \in \mathbb{Z}, b \in \mathbb{N}$ . Then exist unique  $q$  and  $r$  such that

$$a = qb + r, \quad 0 \leq r < b$$

- **Common divisor**  $d$  of  $a$  and  $b$  is such that  $d \mid a$  and  $d \mid b$ .
- **Greatest common divisor (gcd)** of  $a$  and  $b$  is maximal common divisor.
- $\gcd(0, 0)$  doesn't exist.
- **Properties of gcd:**

- $\gcd(a, b) = \gcd(b, a)$ .
- If  $a > 0$ ,  $\gcd(a, 0) = a$ .
- $\gcd(a, b) = \gcd(-a, b)$ .
- If  $a > 0, b > 0$ ,  $\gcd(a, b) \leq \min\{a, b\}$ .
- For every  $a, b, q \in \mathbb{Z}$ ,

$$\gcd(a, b) = \gcd(a, b - a) = \dots = \gcd(a, b - qa)$$

- **Euclidean algorithm:** let  $a, b \in \mathbb{N}$ . Repeating the division algorithm:

$$a = q_1 b + r_1$$

$$b = q_2 r_1 + r_2$$

$$r_1 = q_3 r_2 + r_3$$

$$\vdots$$

$$r_{n-2} = q_n r_{n-1} + r_n$$

Then exists smallest  $n$  such that  $r_n = 0$ . Then if  $n = 1$ ,  $\gcd(a, b) = b$ , else  $\gcd(a, b) = r_{n-1}$ . Also, exists  $x, y \in \mathbb{Z}$  such that

$$\gcd(a, b) = ax + by$$

### 3. Primes, composite numbers, and the FTA

- $n \in \mathbb{N}$  **prime** if  $n > 1$  and if  $d \mid n$  then  $d = n$  or  $d = 1$ . If  $n > 1$  not prime,  $n$  **composite**.
- There are infinitely many primes.
- There are infinitely many primes of form  $4n - 1$ .
- **Dirichlet's theorem:** Let  $a, b$  coprime. Then exist infinitely many primes of form  $an + b$ .
- **Euclid's lemma:** Let  $p > 1$ .  $p$  prime iff for every  $a, b \in \mathbb{Z}$ ,  $p \mid ab \implies p \mid a$  or  $p \mid b$ .
- Let  $p$  prime. If  $p \mid a_1 \cdots a_n$  then  $p \mid a_i$  for some  $i$ .
- **Fundamental theorem of arithmetic (FTA):** let  $n > 1$ , then  $n$  can be written as product of primes, unique up to reordering. So exist distinct primes  $p_1, \dots, p_r$  and  $e_1, \dots, e_r \in \mathbb{N}$  such that

$$n = p_1^{e_1} \cdots p_r^{e_r}$$

and if  $n = q_1^{f_1} \cdots q_s^{f_s}$  for distinct primes  $q_i$ , then  $r = s$  and up to reordering,  $p_i = q_i$  and  $e_i = f_i$  for every  $i$ .

### 4. Classical equations and integer solutions

- **Pythagorean triple**  $(x, y, z)$ : solution to  $x^2 + y^2 = z^2$ . **Primitive** if  $\gcd(x, y, z) = 1$ .
- Every Pythagorean triple  $(x, y, z)$ , with  $2 \mid x$ , given by

$$\begin{cases} x = 2st \\ y = s^2 - t^2 \\ z = s^2 + t^2 \end{cases}$$

with  $s > t \geq 1$ ,  $\gcd(s, t) = 1$  and  $s \not\equiv t \pmod{2}$ .

- **Fermat's theorem:** no integer solutions exist to  $x^4 + y^4 = z^2$ .

## 5. Modular arithmetic and congruences

- **Residue (congruence) class:** set of integers congruent mod  $n$ .
- $a \equiv b \pmod{n}$  if  $n \mid (a - b)$ .
- If  $a \equiv a' \pmod{n}$  and  $a' \equiv b' \pmod{n}$  then:
  - $a + a' \equiv b + b' \pmod{n}$  and
  - $aa' \equiv bb' \pmod{n}$ .
- If  $\gcd(c, n) = 1$ , then  $ac \equiv bc \pmod{n}$  implies  $a \equiv b \pmod{n}$ .
- **Complete set of residues mod  $n$ :** subset  $R \subset \mathbb{Z}$  of size  $n$  whose remainders mod  $n$  are distinct.
- Let  $R$  be complete set of residues mod  $n$  and let  $\gcd(a, n) = 1$ , then

$$aR := \{ax : x \in R\}$$

is also complete set of residues mod  $n$ .

- **Linear congruence:**  $ax \equiv b \pmod{n}$ .
- If  $\gcd(a, n) = 1$ , linear congruence has solution, unique up to adding multiples of  $n$  (solutions lie in same congruence class).
- **Method for solving linear congruence** (if  $\gcd(a, n) = 1$ ):
  - Use Euclidean algorithm to find  $u, v$  such that  $1 = au + nv$ .
  - $au \equiv 1 \pmod{n}$  so  $a(ub) \equiv b \pmod{n}$  so solutions are

$$x \equiv ub \pmod{n}$$

- Linear congruence solvable iff  $\gcd(a, n) \mid b$ .
- **Chinese remainder theorem (CRT):** let  $m, n \in \mathbb{N}$  coprime,  $a, b \in \mathbb{Z}$ . Then exists solution to

$$x \equiv a \pmod{m}$$

$$x \equiv b \pmod{n}$$

Any solutions are congruent mod  $mn$  and exists unique solution  $x$  with  $0 \leq x < mn$ .

- **Method to solve CRT problem:**
  - Use Euclidean algorithm to find  $r, s$  such that  $1 = rm + sn$ , so  $rm \equiv 1 \pmod{n}$  and  $sn \equiv 1 \pmod{m}$ .
  - So  $brm \equiv b \pmod{n}$  and  $asn \equiv a \pmod{m}$ .
  - So  $asn + brm \equiv b \pmod{n}$  and  $asn + brm \equiv a \pmod{m}$ .
  - So  $x = brm + asn$  is solution.
- **Euler  $\varphi$ -function:**  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ :

$$\varphi(n) := |\{r \in \mathbb{N} : r \leq n \text{ and } \gcd(r, n) = 1\}|$$

- $\varphi(p) = p - 1$  for prime  $p$ .
- For prime  $p$ ,  $\varphi(p^n) = p^n - p^{n-1} = p^{n-1}(p - 1)$ .
- If  $\gcd(m, n) = 1$ , then  $\varphi(mn) = \varphi(m)\varphi(n)$ .
- Let  $n$  have prime factorisation  $n = \prod_{i=1}^r p_i^{e_i}$ . Then

$$\varphi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right)$$

- Let  $n \in \mathbb{N}$ , then

$$\sum_{d|n} \varphi(d) = n$$

where sum is over positive divisors of  $n$ .

- **Euler's theorem:** For  $a \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ ,  $\gcd(a, n) = 1$ ,

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

- **Fermat's little theorem:** let  $p$  prime,  $a \in \mathbb{Z}$ ,  $p \nmid a$ . Then

$$a^{p-1} \equiv 1 \pmod{p}$$

## 6. Primitive roots

- Let  $n \in \mathbb{N}$ ,  $a \in \mathbb{Z}$ ,  $\gcd(a, n) = 1$ . **(Multiplicative) order** of  $a \bmod n$ ,  $\text{ord}_n(a) = \text{ord}(a)$ , is smallest  $d \in \mathbb{N}$  such that

$$a^d \equiv 1 \pmod{n}$$

- If  $a^d \equiv 1 \pmod{n}$  for some  $d$ , then  $\text{ord}(a) \mid d$ . So  $\text{ord}(a) \mid \varphi(n)$ .
- Let  $\gcd(a, n) = 1$ ,  $a$  is **primitive root mod  $n$**  if  $\text{ord}_n(a) = \varphi(n)$ .
- Let  $p$  prime,  $f(x)$  polynomial with integer coefficients of degree  $n$ . Then  $f$  has at most  $n$  roots mod  $p$ , so  $f(x) \equiv 0 \pmod{p}$  has at most  $n$  solutions mod  $p$ .
- Let  $p$  prime,  $d \mid p-1$ . Then  $x^d - 1 \equiv 0 \pmod{p}$  has exactly  $d$  solutions mod  $p$ .
- Let  $p$  prime, then there are  $\varphi(p-1)$  primitive roots mod  $p$ .
- Let  $g$  primitive root mod  $p$ ,  $\gcd(a, p) = 1$ . Then for some  $r \in \mathbb{N}$ ,

$$a \equiv g^r \pmod{p}$$

$(g, g^2, \dots, g^{p-1})$  are distinct).

- Primitive roots mod  $n$  exist iff  $n = 2, 4, p^k$  or  $2p^k$  for odd prime  $p$ ,  $k \in \mathbb{N}$ .

## 7. Quadratic residues

- Let  $p$  prime,  $a \in \mathbb{Z}$ ,  $\gcd(a, p) = 1$ .  $a$  is **quadratic residue (QR) mod  $p$**  if for some  $x \in \mathbb{Z}$ ,  $x^2 \equiv a \pmod{p}$ . If not,  $a$  is **quadratic non-residue (NQR) mod  $p$** .
- For  $p$  odd prime, there are  $\frac{p-1}{2}$  QR's and QNR's mod  $p$ .
- Products of QR's and NQR's satisfy:

$$QR \times QR = QR$$

$$QR \times NR = NR$$

$$NR \times NR = QR$$

- Let  $p$  prime,  $a \in \mathbb{Z}$ , **Legendre symbol** is

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ QR} \\ -1 & \text{if } a \text{ NQR} \\ 0 & \text{if } p \mid a \end{cases}$$

- $$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$
- $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$  if  $a \equiv b \pmod{p}$ .
- **Euler's criterion:** Let  $p$  odd prime,  $a \in \mathbb{Z}$ ,  $\gcd(a, p) = 1$ , then

$$a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \pmod{p}$$

- $-1$  is QR if  $p \equiv 1 \pmod{4}$  and is NQR if  $p \equiv 3 \pmod{4}$ .
- **Quadratic reciprocity law (QRL):** let  $p \neq q$  odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

If  $p = 2$ ,

$$\left(\frac{2}{q}\right) = (-1)^{\frac{q^2-1}{8}}$$

- **Algorithm for computing Legendre symbol  $\left(\frac{a}{p}\right)$ :**
  - Divide  $a$  by  $p$  to get  $a = tp + r$  so  $\left(\frac{a}{p}\right) = \left(\frac{r}{p}\right)$ .
  - If  $r = 0$ ,  $\left(\frac{r}{p}\right) = 0$  so stop.
  - If  $r = 1$ ,  $\left(\frac{r}{p}\right) = 1$  so stop.
  - If  $r \neq 0, 1$  factorise into primes  $r = q_1^{e_1} \dots q_k^{e_k}$  so  $\left(\frac{r}{p}\right) = \prod_{i=1}^k \left(\frac{q_i}{p}\right)^{e_i}$ .
  - $r < p$  so  $q_i < p$ , so calculate  $\left(\frac{q_i}{p}\right)$  for each  $i$ .
    - If  $q_i = 2$ , use  $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$ .
    - If  $q_i > 2$ , use  $\left(\frac{q_i}{p}\right) = (-1)^{\frac{(q_i-1)(p-1)}{4}} \left(\frac{p}{q_i}\right)$  and go to step 1 to calculate  $\left(\frac{p}{q_i}\right)$ .
- **Note:** to evaluate  $\left(\frac{-1}{p}\right)$ , easier to use Euler's criterion.
- There are infinitely many primes of form  $4n + 1$ .

## 8. Sums of two squares

- If  $m$  and  $n$  are sums of two squares, then so is  $mn$  since  $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$ .
- Let  $p$  odd prime. Then  $p$  sum of two squares iff  $p \equiv 1 \pmod{4}$ .
- Let  $n > 1$ ,  $n = p_1 p_2 \dots p_k N^2$ ,  $p_i$  distinct primes,  $N \in \mathbb{N}$ . Then  $n$  sum of two squares iff  $p_i = 2$  or  $p_i \equiv 1 \pmod{4}$  for all  $i$ .

## 9. Continued fractions

- **Finite continued fraction (CF):**

$$[a_0; a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}$$

- **Simple CF:**  $a_0 \in \mathbb{Z}$ ,  $a_1, \dots, a_n \in \mathbb{N}$ .
- Any rational number can be written as finite simple continued fraction.
- **$k$ th convergent** of CF  $[a_0; a_1, \dots, a_n]$ :

$$C_k := [a_0; a_1, \dots, a_k]$$

- $C_n = p_n / q_n$ , where

$$\begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix} \dots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}$$

so  $p_1 = a_0 a_1 + 1, p_0 = a_0, q_1 = a_1, q_0 = 1$  and  $p_k = a_k p_{k-1} + p_{k-2}, q_k = a_k q_{k-1} + q_{k-2}$

- If  $[a_0; a_1, \dots, a_n]$  is simple CF, then  $q_{k-1} \leq q_k$  and  $q_{k-1} < q_k$  if  $k > 1$ .

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$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k+1}$$

- $\gcd(p_k, q_k) = 1$ .
- Let  $\alpha = [a_0; a_1, \dots, a_n], k = 0, \dots, n-1$ , then even numbered convergents increasing:  $C_0 < C_2 < \dots < C_{2m}$ , odd numbered convergents decreasing  $C_{2m+1} < \dots < C_3 < C_1$  and for every  $k$  with  $2k+1 \leq n$ ,

$$\frac{p_{2k}}{q_{2k}} < \alpha \leq \frac{p_{2k+1}}{q_{2k+1}}$$

and

$$\left| \alpha - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k q_{k+1}}$$

- **Infinite CF**  $[a_0; a_1, \dots]$  is limit of convergents  $C_n = [a_0; a_1, \dots, a_n]$ .
- For simple infinite CF, limit always exists.
- **Pell's equation:**  $x^2 - dy^2 = 1, d \in \mathbb{N}$  not square.
- **Negative Pell's equation:**  $x^2 - dy^2 = -1$ .
- Infinite CF **periodic** if of form

$$[a_0; a_1, \dots, a_m, a_{m+1}, \dots, a_{m+n}, a_{m+1}, \dots, a_{m+n}, \dots]$$

$a_0; a_1, \dots, a_m$  is initial part,  $a_{m+1}, \dots, a_{m+n}, a_{m+1}, \dots, a_{m+n}, \dots$  is periodic part. In periodic part,  $a_i = a_j$  iff  $i \equiv j \pmod{n}$ . Write as

$$[a_0; a_1, \dots, a_m, \overline{a_{m+1}, \dots, a_{m+n}}]$$

$n$  is **period**.

- If  $d$  not square, CF of  $\sqrt{d}$  is periodic with initial part only  $a_0$ .
- Let  $p_k / q_k$  be convergents of simple CF expansion of  $\sqrt{d}$  with period  $n$ , then for all  $k \geq 1$ ,

$$p_{kn-1}^2 - dq_{kn-1}^2 = (-1)^{kn}$$

- So if  $n$  even or  $k$  even,  $(x, y) = (p_{kn-1}, q_{kn-1})$  are solution to Pell's equation. Else  $(x, y) = (p_{kn-1}, q_{kn-1})$  are solution to negative Pell's equation. **All** positive solutions to (negative) Pell equation given by above.