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1. The Khinchin axioms for entropy

Note all random variables we deal with will be discrete, unless otherwise stated. We use $\log = \log_2$.

1.1. Entropy axioms

Definition 1.1 The **entropy** of a discrete random variable X is a quantity $H(X)$ that takes real values and satisfies the **Khinchin axioms**: [Normalisation](#), [Invariance](#), [Extendability](#), [Maximality](#), [Continuity](#) and [Additivity](#).

Axiom 1.2 (Normalisation) If X is uniform on $\{0, 1\}$ (i.e. $X \sim \text{Bern}(1/2)$), then $H(X) = 1$.

Axiom 1.3 (Invariance) If $Y = f(X)$ for some bijection f , then $H(Y) = H(X)$.

Axiom 1.4 (Extendability) If X takes values on a set A , B is disjoint from A , Y takes values in $A \sqcup B$, and for all $a \in A$, $\mathbb{P}(Y = a) = \mathbb{P}(X = a)$, then $H(Y) = H(X)$.

Axiom 1.5 (Maximality) If X takes values in a finite set A and Y is uniformly distributed in A , then $H(X) \leq H(Y)$.

Definition 1.6 The **total variance distance** between X and Y is

$$\sup_E |\mathbb{P}(X \in E) - \mathbb{P}(Y \in E)|.$$

Axiom 1.7 (Continuity) H depends continuously on X (with respect to total variation distance).

Definition 1.8 Let X and Y be random variables. The **conditional entropy** of X given Y is

$$H(X | Y) := \sum_y \mathbb{P}(Y = y) H(X | Y = y).$$

Axiom 1.9 (Additivity) $H(X, Y) := H((X, Y)) = H(Y) + H(X | Y)$.

1.2. Properties of entropy

Lemma 1.10 If X and Y are independent, then $H(X, Y) = H(X) + H(Y)$.

Proof (Hints). Straightforward. □

Proof. $H(X | Y) = \sum_y \mathbb{P}(Y = y) H(X | Y = y)$ Since X and Y are independent, the distribution of X is unaffected by knowing Y , so $H(X | Y = y) = H(X)$ for all y , which gives the result. (Note we have implicitly used [Invariance](#) here). □

Corollary 1.11 If X_1, \dots, X_n are independent, then

$$H(X_1, \dots, X_n) = H(X_1) + \dots + H(X_n).$$

Proof (Hints). Straightforward. □

Proof. By Lemma [1.10](#) and induction. □

Lemma 1.12 (Chain Rule) Let X_1, \dots, X_n be RVs. Then

$$H(X_1, \dots, X_n) = H(X_1) + H(X_2 \mid X_1) + H(X_3 \mid X_1, X_2) + \dots + H(X_n \mid X_1, \dots, X_{n-1}).$$

Proof (Hints). Straightforward. \square

Proof. The case $n = 2$ is [Additivity](#). In general,

$$H(X_1, \dots, X_n) = H(X_1, \dots, X_{n-1}) + H(X_n \mid X_1, \dots, X_{n-1}),$$

so the result follows by induction. \square

Lemma 1.13 Let X and Y be RVs. If $Y = f(X)$, then $H(X, Y) = H(X)$. Also, $H(Z \mid X, Y) = H(Z \mid X)$.

Proof (Hints). Consider an appropriate bijection. \square

Proof. The map $g : x \mapsto (x, f(x))$ is a bijection, and $(X, Y) = g(X)$, so the first statement follows from [Invariance](#). Also,

$$\begin{aligned} H(Z \mid X, Y) &= H(Z, X, Y) - H(X, Y) \quad \text{by additivity} \\ &= H(Z, X) - H(X) \quad \text{by first part} \\ &= H(Z \mid X) \quad \text{by additivity} \end{aligned}$$

\square

Lemma 1.14 If X takes only one value, then $H(X) = 0$.

Proof (Hints). Use that X and X are independent. \square

Proof. X and X are independent (verify). So by Lemma [1.10](#), $H(X, X) = 2H(X)$. But by [Invariance](#), $H(X, X) = H(X)$. So $H(X) = 0$. \square

Proposition 1.15 If X is uniformly distributed on a set of size 2^n , then $H(X) = n$.

Proof (Hints). Straightforward. \square

Proof. Let X_1, \dots, X_n be independent RVs, uniformly distributed on $\{0, 1\}$. By Corollary [1.11](#) and [Normalisation](#), $H(X_1, \dots, X_n) = n$. So the result follows by [Invariance](#). \square

Proposition 1.16 If X is uniformly distributed on a set A of size n , then $H(X) = \log n$.

Proof (Hints). Straightforward. \square

Proof. Let $r \in \mathbb{N}$ and let X_1, \dots, X_r be independent copies of X . Then (X_1, \dots, X_r) is uniform on A^r , and $H(X_1, \dots, X_r) = rH(X)$. Now pick k such that $2^k \leq n^r \leq 2^{k+1}$. Then by Proposition [1.15](#), [Invariance](#) and [Maximality](#), $k \leq rH(X) \leq k+1$. So $\frac{k}{r} \leq \log n \leq \frac{k+1}{r}$ and $\frac{k}{r} \leq H(X) \leq \frac{k+1}{r}$ for all $r \in \mathbb{N}$. So $H(X) = \log n$, as claimed. \square

Theorem 1.17 (Khinchin) If H satisfies the Khinchin axioms and X takes values in a finite set A , then

$$H(X) = \sum_{a \in A} p_a \log(1/p_a) = \mathbb{E} \left[\log \frac{1}{P_X(X)} \right],$$

where $p_a = \mathbb{P}(X = a)$.

Proof (Hints).

- Explain why it is enough to prove for when the p_a are rational.
- Pick $n \in \mathbb{N}$ such that $p_a = \frac{m_a}{n}$, $m_a \in \mathbb{N}_0$. Let Z be uniform on $[n]$. Let $\{E_a : a \in A\}$ be a partition of $[n]$ into sets with $|E_a| = m_a$.

□

Proof. First we do the case where all $p_a \in \mathbb{Q}$. Pick $n \in \mathbb{N}$ such that $p_a = \frac{m_a}{n}$, $m_a \in \mathbb{N}_0$. Let Z be uniform on $[n]$. Let $\{E_a : a \in A\}$ be a partition of $[n]$ into sets with $|E_a| = m_a$. By **Invariance**, we may assume that $X = a \Leftrightarrow Z \in E_a$. Then

$$\begin{aligned}
 \log n = H(Z) &= H(Z, X) = H(X) + H(Z | X) \\
 &= H(X) + \sum_{a \in A} p_a H(Z | X = a) \\
 &= H(X) + \sum_{a \in A} p_a \log m_a \\
 &= H(X) + \sum_{a \in A} p_a (\log p_a + \log n) \\
 &= H(X) + \sum_{a \in A} p_a \log p_a + \log n.
 \end{aligned}$$

Hence $H(X) = -\sum_{a \in A} p_a \log p_a$.

The general result follows by **Continuity**.

□

Corollary 1.18 Let X and Y be random variables. Then $0 \leq H(X)$ and $0 \leq H(X | Y)$.

Proof (Hints). Trivial.

□

Proof. Immediate consequence of **Khinchin**.

□

Corollary 1.19 If $Y = f(X)$, then $H(Y) \leq H(X)$.

Proof (Hints). Straightforward.

□

Proof. $H(X) = H(X, Y) = H(Y) + H(X | Y)$. But $H(X | Y) \geq 0$.

□

Proposition 1.20 (Subadditivity) Let X and Y be RVs. Then $H(X, Y) \leq H(X) + H(Y)$.

Proof (Hints).

- Let $p_{ab} = \mathbb{P}(X = a, Y = b)$. Explain why it is enough to show for the case when the p_{ab} are rational.
- Pick n such that $p_{ab} = m_{ab}/n$ with each $m_{ab} \in \mathbb{N}_0$. Partition $[n]$ into sets E_{ab} of size m_{ab} . Let Z be uniform on $[n]$.
- Show that if X (or Y) is uniform, then $H(X | Y) \leq H(X)$ and $H(X, Y) \leq H(X) + H(Y)$.
- Let $E_b = \cup_a E_{ab}$ for each b . So $Y = b$ iff $Z \in E_b$. Now define an RV W as follows: if $Y = b$, then W is uniformly distributed in E_b . Use conditional independence to conclude the result.

□

Proof. Note that for any two RVs X, Y ,

$$\begin{aligned} H(X, Y) &\leq H(X) + H(Y) \\ \Leftrightarrow H(X | Y) &\leq H(X) \\ \Leftrightarrow H(Y | X) &\leq H(Y) \end{aligned}$$

by [Additivity](#). Next, observe that $H(X | Y) \leq H(X)$ if X is uniform on a finite set, since $H(X | Y) = \sum_y \mathbb{P}(Y = y) H(X | Y = y) \leq \sum_y \mathbb{P}(Y = y) H(X) = H(X)$ by [Maximality](#). By the above equivalence, we also have $H(X | Y) \leq H(X)$ if Y is uniform on a finite set. Now let $p_{ab} = \mathbb{P}(X = a, Y = b)$, and assume that all p_{ab} are rational. Pick n such that $p_{ab} = m_{ab}/n$ with each $m_{ab} \in \mathbb{N}_0$. Partition $[n]$ into sets E_{ab} of size m_{ab} . Let Z be uniform on $[n]$. WLOG (by [Invariance](#)), $(X, Y) = (a, b)$ iff $Z \in E_{ab}$.

Let $E_b = \cup_a E_{ab}$ for each b . So $Y = b$ iff $Z \in E_b$. Now define an RV W as follows: if $Y = b$, then $W \in E_b$, but then W is uniformly distributed in E_b and independent of X (and Z). So W and X are conditionally independent given Y , and W is uniform on $[n]$. Then $H(X | Y) = H(X | Y, W) = H(X | W)$ by conditional independence and by Lemma [1.13](#) (since W determines Y). Since W is uniform, $H(X | W) \leq H(X)$.

The general result follows by [Continuity](#). □

Corollary 1.21 $H(X) \geq 0$ for any X .

Proof (Hints). (Without using the formula) straightforward. □

Proof. (Without using the formula). By subadditivity, $H(X | X) \leq H(X)$. But $H(X | X) = 0$. □

Corollary 1.22 Let X_1, \dots, X_n be RVs. Then

$$H(X_1, \dots, X_n) \leq H(X_1) + \dots + H(X_n).$$

Proof (Hints). Trivial. □

Proof. Trivial by induction. □

Proposition 1.23 (Submodularity) Let X, Y, Z be RVs. Then

$$H(X | Y, Z) \leq H(X | Z).$$

Proof (Hints). Use that $H(X | Y, Z = z) \leq H(Z | Z = z)$. □

Proof. $H(X | Y, Z) = \sum_z \mathbb{P}(Z = z) H(X | Y, Z = z) \leq \sum_z \mathbb{P}(Z = z) H(X | Z = z) = H(X | Z)$. □

Remark 1.24 [Submodularity](#) can be expressed in several equivalent ways. Expanding using [Additivity](#) gives

$$H(X, Y, Z) - H(Y, Z) \leq H(X, Z) - H(Z)$$

and

$$H(X, Y, Z) \leq H(X, Z) + H(Y, Z) - H(Z)$$

and

$$H(X, Y, Z) + H(Z) \leq H(X, Z) + H(Y, Z).$$

Lemma 1.25 Let X, Y, Z be RVs with $Z = f(Y)$. Then $H(X | Y) \leq H(X | Z)$.

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned} H(X | Y) &= H(X, Y) - H(Y) = H(X, Y, Z) - H(Y, Z) \\ &\leq H(X, Z) - H(Z) = H(X | Z) \end{aligned}$$

by Submodularity. □

Lemma 1.26 Let X, Y, Z be RVs with $Z = f(X) = g(Y)$. Then

$$H(X, Y) + H(Z) \leq H(X) + H(Y).$$

Proof (Hints). Straightforward. □

Proof. By Submodularity, we have $H(X, Y, Z) + H(Z) \leq H(X, Z) + H(Y, Z)$, which implies the result, since Z depends on X and Y . □

Lemma 1.27 Let X be an RV taking values in a finite set A and let Y be uniform on A . If $H(X) = H(Y)$, then X is uniform.

Proof (Hints). Use Jensen's inequality. □

Proof. Let $p_a = \mathbb{P}(X = a)$. Then

$$H(X) = \sum_{a \in A} p_a \log(1/p_a) = |A| \cdot \mathbb{E}_{a \in A} p_a \log\left(\frac{1}{p_a}\right).$$

The function $x \mapsto x \log(1/x)$ is concave on $[0, 1]$. So by Jensen's inequality,

$$H(X) \leq |A| \cdot (\mathbb{E}_{a \in A} p_a) \cdot \log\left(\frac{1}{\mathbb{E}_{a \in A} p_a}\right) = \log|A| = H(Y),$$

with equality iff $a \mapsto p_a$ is constant, i.e. X is uniform. □

Corollary 1.28 If $H(X, Y) = H(X) + H(Y)$, then X and Y are independent.

Proof (Hints). Go through the proof of Subadditivity and check when equality holds. □

Proof. We go through the proof of subadditivity and check when equality holds. Suppose that X is uniform on A . Then

$$H(X | Y) = \sum_y \mathbb{P}(Y = y) H(X | Y = y) \leq H(X),$$

with equality iff $H(X \mid Y = y)$ is uniform on A for all y (by Lemma 1.27), which implies that X and Y are independent.

At the last stage of the proof, we said $H(X \mid Y) = H(X \mid Y, W) = H(X \mid W) \leq H(X)$, where W was uniform. So equality holds only if X and W are independent, which implies (since Y depends on W), that X and Y are independent. \square

Definition 1.29 Let X and Y be RVs. The **mutual information**

$$\begin{aligned} I(X : Y) &:= H(X) + H(Y) - H(X, Y) \\ &= H(X) - H(X \mid Y) \\ &= H(Y) - H(Y \mid X). \end{aligned}$$

Remark 1.30 Subadditivity is equivalent to the statement that $I(X : Y) \geq 0$, and Corollary 1.28 implies that $I(X : Y) = 0$ iff X and Y are independent.

Note that $H(X, Y) = H(X) + H(Y) - I(X : Y)$ (note the similarity to the inclusion-exclusion formula for two sets).

Definition 1.31 Let X, Y, Z be RVs. The **conditional mutual information** of X and Y given Z is

$$\begin{aligned} I(X : Y \mid Z) &:= \sum_z \mathbb{P}(Z = z) I(X \mid Z = z : Y \mid Z = z) \\ &= \sum_z \mathbb{P}(Z = z) (H(X \mid Z = z) + H(Y \mid Z = z) - H(X, Y \mid Z = z)) \\ &= H(X \mid Z) + H(Y \mid Z) - H(X, Y \mid Z) \\ &= H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z). \end{aligned}$$

Submodularity is equivalent to the statement that $I(X : Y \mid Z) \geq 0$.

2. A special case of Sidorenko's conjecture

Definition 2.1 Let G be a bipartite graph with (finite) vertex sets X and Y and density α (defined to be $\frac{|E(G)|}{|X| \cdot |Y|}$). Let H be another (think of it as small) bipartite graph with vertex sets U and V and m edges. Now let $\varphi : U \rightarrow X$ and $\psi : V \rightarrow Y$. We say that (φ, ψ) is a **homomorphism** if $\varphi(x)\varphi(y) \in E(G)$ for every edge $xy \in E(H)$.

Conjecture 2.2 (Sidorenko's Conjecture) For every G, H , for random $\varphi : U \rightarrow X, \psi : V \rightarrow Y$,

$$\mathbb{P}((\varphi, \psi) \text{ is a homomorphism}) \geq \alpha^m.$$

Remark 2.3 Sidorenko's Conjecture is not hard to prove when H is the complete bipartite graph $K_{r,s}$ (the case $K_{2,2}$ can be proved using Cauchy-Schwarz: exercise).

Theorem 2.4 Sidorenko's Conjecture is true if H is a path of length 3.

Proof (Hints).

- Let (X_1, Y_1) be a random edge of G (with $X_1 \in X, Y_1 \in Y$). Now let X_2 be a random neighbour of Y_1 and Y_2 be a random neighbour of X_2 . Explain why it suffices to prove that $H(X_1, Y_1, X_2, Y_2) \geq \log(\alpha^3 m^2 n^2)$.
- Find an equivalent way of choosing a uniformly random edge (X_1, Y_1) of G (in terms of vertices). Use this to reaons that $X_2 Y_1$ and $X_2 Y_2$ are uniformly random in $E(G)$.
- Find the lower bound for $H(X_1, Y_1, X_2, Y_2)$ using the [Chain Rule](#) and [Maximality](#).

□

Proof. We want to show that if G is a bipartite graph of density α with vertex sets X, Y of size m and n , and we choose $x_1, x_2 \in X, y_1, y_2 \in Y$ independently at random, then $\mathbb{P}(x_1 y_1, y_1 x_2, x_2 y_2 \in E(G)) \geq \alpha^3$.

It would be enough to let P be a path of length 3 chosen uniformly at random and show that $H(P) \geq \log(\alpha^3 m^2 n^2)$ (by Proposition [1.16](#)). Instead, we shall define a different RV taking values in the set of all paths of length 3 (including degenerate paths). To do this, let (X_1, Y_1) be a random edge of G (with $X_1 \in X, Y_1 \in Y$). Now let X_2 be a random neighbour of Y_1 and Y_2 be a random neighbour of X_2 . It will be enough to prove that

$$H(X_1, Y_1, X_2, Y_2) \geq \log(\alpha^3 m^2 n^2).$$

We can choose X_1, Y_1 in three equivalent ways:

1. Pick an edge uniformly from all edges
2. Pick a vertex x with probability proportional to its degree $\deg(x)$, and then a random neighbour Y of x .
3. Same as above with x and y exchanged.

By the equivalence, it follows that $Y_1 = y$ with probability $\deg(y)/|E(G)|$, so $X_2 Y_1$ is uniform in $E(G)$, so $X_2 = x'$ with probability $d(x')/|E(G)|$, so $X_2 Y_2$ is uniform in $E(G)$.

Let U_A be the uniform distribution on A . Therefore, by the [Chain Rule](#),

$$\begin{aligned} H(X_1, Y_1, X_2, Y_2) &= H(X_1) + H(Y_1 \mid X_1) + H(X_2 \mid X_1, Y_1) + H(Y_2 \mid X_1, Y_1, X_2) \\ &= H(X_1) + H(Y_1 \mid X_1) + H(X_2 \mid Y_1) + H(Y_2 \mid X_2) \\ &= H(X_1) + H(X_1, Y_1) - H(X_1) + H(X_2, Y_1) - H(Y_1) + H(X_2, Y_2) - H(Y_2) \\ &= 3H(U_{E(G)}) - H(Y_1) - H(X_2) \\ &\geq 3H(U_{E(G)}) - H(U_Y) - H(U_X) \\ &= 3\log(\alpha mn) - \log n - \log m \\ &= \log(\alpha^3 m^2 n^2). \end{aligned}$$

So we are done, by [Maximality](#). Alternative finish to the proof: let X', Y' be uniform in X, Y and independent of each other and X_1, Y_1, X_2, Y_2 . Then by the above inequality and Corollary [1.11](#),

$$H(X_1, Y_1, X_2, Y_2, X', Y') = H(X_1, Y_1, X_2, Y_2) + H(U_X) + H(U_Y)$$

$$\geq 3H(U_{E(G)}).$$

So by Maximality, the number of paths of length 3 times $|X|$ times $|Y|$ is $\geq |E(G)|^3$. \square

3. Brigner's theorem

Definition 3.1 Let A be an $n \times n$ matrix over \mathbb{R} . The **permanent** of A is

$$\text{per}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^n A_{i\sigma(i)},$$

i.e. “the determinant without the signs”.

Proposition 3.2 Let G be a bipartite graph with vertex sets X, Y of size n . Given $(x, y) \in X \times Y$, let

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E(G) \\ 0 & \text{if } xy \notin E(G) \end{cases},$$

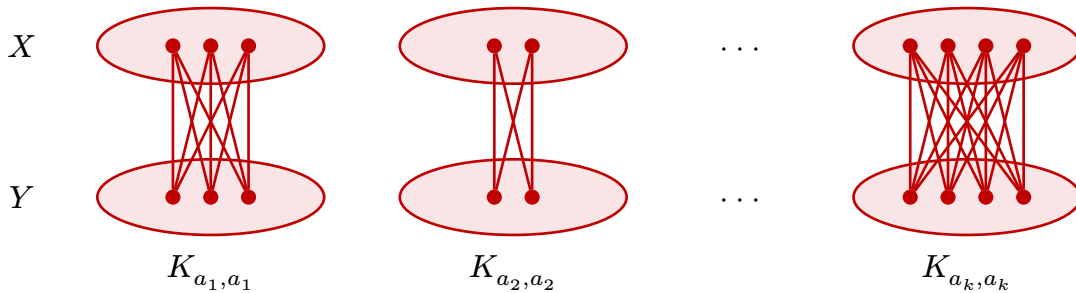
i.e. A is the bipartite adjacency matrix of G . Then $\text{per}(A)$ is the number of perfect matchings in G . (Note that $\text{per}(A)$ is well-defined as it is invariant under reordering of the vertices.)

Proof (Hints). Straightforward. \square

Proof. Each (perfect) matching corresponds to a bijection $\sigma : X \rightarrow Y$ such that $x\sigma(x) \in E(G)$ for all $x \in X$. $\sigma \in S_n$ contributes 1 to the sum iff $x\sigma(x)$ is an edge of G for all $x \in X$ (i.e. iff σ corresponds to a perfect matching), and 0 otherwise. \square

Bregman's theorem concerns how large $\text{per}(A)$ can be if A is a 0, 1 matrix and the sum of the entries in the i -th row is d_i (i.e. if the degree of $x_i \in X$ is d_i).

Example 3.3 Let G be a disjoint union of K_{a_i, a_i} 's, $i = 1, \dots, k$, with $a_1 + \dots + a_k = n$. Then the number of perfect matchings in G is $\prod_{i=1}^k a_i!$.



Theorem 3.4 (Bregman) Let G be a bipartite graph with vertex sets X, Y of size n . Then the number of perfect matchings in G is at most

$$\prod_{x \in X} (\deg(x)!)^{1/\deg(x)}.$$

Proof (Hints).

- For an enumeration x_1, \dots, x_n of X and random matching (a bijection) σ , show that $H(\sigma) \leq \log \deg(x_1) + \mathbb{E}_\sigma \log \deg_{x_1}^\sigma(x_2) + \dots + \mathbb{E}_\sigma \log \deg_{x_1, \dots, x_{n-1}}^\sigma(x_n)$ (find a suitable expression for $\deg_{x_1, \dots, x_{i-1}}^\sigma(x_i)$).
- Find another expression for $\deg_{x_1, \dots, x_{i-1}}^\sigma(x_i)$ in terms of $\deg(x)$.
- Show that the average of $\log \deg_{x_1, \dots, x_{i-1}}^\sigma(x_i)$ is $\frac{1}{d(x)}(\log(d(x)!))$.

□

Proof (by Radhakrishnan). Each (perfect) matching corresponds to a bijection $\sigma : X \rightarrow Y$ such that $x\sigma(x) \in E(G)$ for all $x \in X$. Let σ be chosen uniformly from all such bijections. Then by the **Chain Rule**,

$$\begin{aligned} H(\sigma) &= H(\sigma(x_1), \dots, \sigma(x_n)) \\ &= H(\sigma(x_1)) + H(\sigma(x_2) \mid \sigma(x_1)) + \dots + H(\sigma(x_n) \mid \sigma(x_1), \dots, \sigma(x_{n-1})), \end{aligned}$$

where x_1, \dots, x_n is some enumeration of X . We have $H(\sigma(x_1)) \leq \log \deg(x_1)$ by **Maximality**, and

$$H(\sigma(x_2) \mid \sigma(x_1)) \leq \mathbb{E}_\sigma \log \deg_{x_1}^\sigma(x_2),$$

where $\deg_{x_1}^\sigma(x_2) = |N(x_2) \setminus \{\sigma(x_1)\}|$, by the definition of conditional entropy and **Maximality**. In general,

$$H(\sigma(x_i) \mid \sigma(x_1), \dots, \sigma(x_{i-1})) \leq \mathbb{E}_\sigma \log \deg_{x_1, \dots, x_{i-1}}^\sigma(x_i),$$

where $\deg_{x_1, \dots, x_{i-1}}^\sigma(x_i) = |N(x_i) \setminus \{\sigma(x_1), \dots, \sigma(x_{i-1})\}|$.

Key idea: we now regard x_1, \dots, x_n as a *random* enumeration of X and take the average. For each $x \in X$, define the **contribution** of x to be $\log(d_{x_1, \dots, x_{i-1}}^\sigma(x_i))$, where $x_i = x$. We shall now fix σ and $x \in X$. Let the neighbours of x be y_1, \dots, y_k . Then one of the y_j will be $\sigma(x)$, say y_h . Then $d_{x_1, \dots, x_{i-1}}^\sigma(x_i)$ (given that $x_i = x$) is

$$d(x) - |\{j : \sigma^{-1}(y_j) \text{ comes earlier than } x = \sigma^{-1}(y_h)\}|.$$

All positions of $\sigma^{-1}(y_h)$ are equally likely, so the average contribution of x is

$$\begin{aligned} &\frac{1}{d(x)}(\log d(x) + \log(d(x) - 1) + \dots + \log(1)) \\ &= \frac{1}{d(x)} \log d(x)!. \end{aligned}$$

By linearity of expectation,

$$H(\sigma) \leq \sum_{x \in X} \frac{1}{d(x)} \log(d(x)!)$$

So the number of matchings is at most $\prod_{x \in X} (d(x)!)^{1/d(x)}$. □

Definition 3.5 Let G be a graph with $2n$ vertices. A **1-factor** in G is a collection of n disjoint edges.

Theorem 3.6 (Kahn-Lovasz) Let G be a graph with $2n$ vertices. Then the number of 1-factors in G is at most

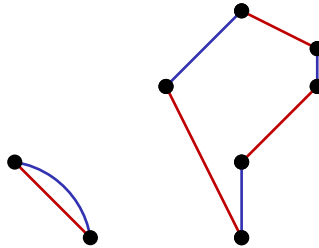
$$\prod_{x \in V(G)} (d(x)!)^{1/2d(x)}.$$

Proof (Hints).

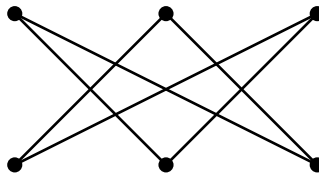
- Let M be the set of 1-factors of G and let (M_1, M_2) be a uniformly random element of $M \times M$.
- Given a cover of G by M_1 and M_2 , find an expression for the number of pairs (M'_1, M'_2) that could give rise to it, in terms of the number of even cycles.
- Let G_2 be the bipartite graph with two vertex sets V_1, V_2 , which are both copies of $V(G)$. Join $x \in V_1$ to $y \in V_2$ iff $xy \in E(G)$.
- Explain why each perfect matching of G_2 gives a cover of $V(G)$ by isolated vertices, edges and cycles, and find an expression for the number of such perfect matchings that could give rise to it.

□

Proof (by Alon, Friedman). Let M be the set of 1-factors of G and let (M_1, M_2) be a uniformly random element of $M \times M$. For each M_1, M_2 , the union $M_1 \cup M_2$ is a collection of disjoint edges and even cycles that covers all the vertices of G .



Call such a union a **cover of G by edges and even cycles**. If we are given such a cover, then the number of pairs (M_1, M_2) that could give rise to it is 2^k , where k is the number of even cycles. Now let's build a bipartite graph G_2 out of G . G_2 has two vertex sets V_1, V_2 , which are both copies of $V(G)$. Join $x \in V_1$ to $y \in V_2$ iff $xy \in E(G)$.



G_2 if G is the triangle graph

By [Bregman](#), the number of perfect matchings in G_2 is at most $\prod_{x \in V(G)} (d(x)!)^{1/d(x)}$. Each matching gives a permutation σ of $V(G)$ such that $x\sigma(x) \in E(G)$ for all $x \in V(G)$. Each such σ has a cycle decomposition, and each cycle gives a cycle in G . So σ gives a cover of $V(G)$ by isolated vertices, edges and cycles (not necessarily all even). Given such a cover with k cycles, each cycle can be directed in two ways, so the number of σ that give rise to it is $= 2^k$. So there is an injection from $M \times M$ to the set of matchings

of G_2 , since every cover by edges and even cycles is a cover by vertices, edges and cycles. So $|M|^2 \leq \prod_{x \in V(G)} (d(x)!)^{1/d(x)}$. \square

4. Shearer's lemma and applications

Notation 4.1 Given a random variable $X = (X_1, \dots, X_n)$ and $A \subseteq [n]$, $A = \{a_1 < \dots < a_k\}$, write X_A for the random variable $(X_{a_1}, \dots, X_{a_k})$.

Lemma 4.2 (Shearer) Let $X = (X_1, \dots, X_n)$ be an RV and let \mathcal{A} be a family of subsets of $[n]$ such that every $i \in [n]$ belongs to at least r of the sets $A \in \mathcal{A}$. Then

$$H(X_1, \dots, X_n) \leq \frac{1}{r} \sum_{A \in \mathcal{A}} H(X_A).$$

Proof (Hints). For each $a \in [n]$, write $X_{<a}$ for (X_1, \dots, X_{a-1}) . Show that $H(X_A) \geq \sum_{a \in A} H(X_a \mid X_{<a})$. \square

Proof. For each $a \in [n]$, write $X_{<a}$ for (X_1, \dots, X_{a-1}) . For each $A \in \mathcal{A}$, $A = \{a_1 < \dots < a_k\}$, by the [Chain Rule](#) and [Submodularity](#),

$$\begin{aligned} H(X_A) &= H(X_{a_1}) + H(X_{a_2} \mid X_{a_1}) + \dots + H(X_{a_k} \mid X_{a_1}, \dots, X_{a_{k-1}}) \\ &\geq H(X_{a_1} \mid X_{<a_1}) + H(X_{a_2} \mid X_{<a_2}) + \dots + H(X_{a_k} \mid X_{<a_k}) \\ &= \sum_{a \in A} H(X_a \mid X_{<a}). \end{aligned}$$

Therefore, $\sum_{A \in \mathcal{A}} H(X_A) \geq r \sum_{a=1}^n H(X_a \mid X_{<a}) = rH(X)$. \square

Example 4.3 $H(X_1, X_2, X_3) \leq \frac{1}{2}(H(X_1, X_2) + H(X_1, X_3) + H(X_2, X_3))$.

Lemma 4.4 Let $X = (X_1, \dots, X_n)$ be an RV and let $A \subseteq [n]$ be a randomly chosen subset of $[n]$, according to some probability distribution. Suppose that for each $i \in [n]$, $\mathbb{P}(i \in A) \geq \mu$. Then

$$H(X) \leq \mu^{-1} \cdot \mathbb{E}_A[H(X_A)].$$

Proof (Hints). Very similar to proof of [Shearer](#). \square

Proof. As in [Shearer](#),

$$H(X_A) \geq \sum_{a \in A} H(X_a \mid X_{<a}).$$

So

$$\mathbb{E}_A[H(X_A)] \geq \mathbb{E}_A \left[\sum_{a \in A} H(X_a \mid X_{<a}) \right] \geq \mu \cdot \sum_{a=1}^n H(X_a \mid X_{<a}) = \mu \cdot H(X).$$

\square

Definition 4.5 Let $E \subseteq \mathbb{Z}^n$ and let $A \subseteq [n]$. Then we write $P_A E$, if $A = \{a_1, \dots, a_k\}$, for the set of $u \in \mathbb{Z}^A$ such that there exists $v \in \mathbb{Z}^{[n] \setminus A}$ such that $[u, v] \in E$, where $[u, v]$ is u suitably intertwined with v .

Corollary 4.6 Let $E \subseteq \mathbb{Z}^n$ and let \mathcal{A} be a family of subsets of $[n]$ such that every $i \in [n]$ is contained in at least r sets in \mathcal{A} . Then

$$|E| \leq \prod_{A \in \mathcal{A}} |P_A E|^{1/r}.$$

Proof (Hints). Straightforward. □

Proof. Let X be a uniformly random element of E . Then by [Shearer](#),

$$\log |E| = H(X) \leq \frac{1}{r} \cdot \sum_{A \in \mathcal{A}} H(X_A).$$

But X_A takes values in $P_A E$, so $H(X_A) \leq \log |P_A E|$ by [Maximality](#). Hence,

$$\log |E| \leq \frac{1}{r} \sum_{A \in \mathcal{A}} |P_A E|.$$

□

Corollary 4.7 (Discrete Loomis-Whitney Theorem) If $\mathcal{A} = \{[n] \setminus \{i\} : i = 1, \dots, n\}$, we get

$$|E| \leq \prod_{i=1}^n |P_{[n] \setminus \{i\}} E|^{1/(n-1)}.$$

Theorem 4.8 Let G be a graph with m edges. Then G has at most $\frac{1}{6}(2m)^{3/2}$ triangles.

Remark 4.9 If $m = \binom{n}{2}$, then this bound is fairly sharp.

Proof (Hints). Consider a uniformly random triangle with an ordering on the vertices, and use [Shearer](#). □

Proof. Let (X_1, X_2, X_3) be a random triple of vertices such that $X_1 X_2$, $X_1 X_3$ and $X_2 X_3$ are all edges (so pick a random triangle with an ordering of the vertices). Let t be the number of triangles in G . By [Shearer](#),

$$\log(6t) = H(X_1, X_2, X_3) \leq \frac{1}{2}(H(X_1, X_2) + H(X_1, X_3) + H(X_2, X_3)).$$

Each (X_i, X_j) (for $i \neq j$) is supported in the set of edges of G , given a direction, so $H(X_i, X_j) \leq \log(2m)$ by [Maximality](#). □

Definition 4.10 Let V be a set of size n and let \mathcal{G} be a set of graphs, all with vertex set V . Then \mathcal{G} is **Δ -intersecting** (triangle-intersecting) if $G_1 \cap G_2$ contains a triangle for all $G_1, G_2 \in \mathcal{G}$.

Theorem 4.11 If $|V| = n$, then a Δ -intersecting family of graphs with vertex set V has size at most $2^{\binom{n}{2}-2}$.

Proof (Hints).

- Let \mathcal{G} be a Δ -intersecting family. View $G \in \mathcal{G}$ as a characteristic function from $V^{(2)}$ to $\{0, 1\}$. Let $X = (X_e : e \in V^{(2)})$ be chosen uniformly at random from \mathcal{G} .
- Let $G_R = K_R \cup K_{V \setminus R}$, explain why G_R is an intersecting family, use this to give upper bound on $|G_R|$.
- Give an expression for the probability that an edge e is in a random G_R . By considering X_{G_R} taking values in the above family, conclude.

□

Proof. Let \mathcal{G} be a Δ -intersecting family and let X be chosen uniformly at random from \mathcal{G} . We write $V^{(2)}$ for the set of (unordered) pairs of elements of V . We think of any $G \in \mathcal{G}$ as a characteristic function from $V^{(2)}$ to $\{0, 1\}$. So $X = (X_e : e \in V^{(2)})$, $X_e \in \{0, 1\}$ (where we fix an ordering of $V^{(2)}$). For each $R \subseteq V$, let G_R be the graph $K_R \cup K_{V \setminus R}$. For each R , we shall look at the projection X_{G_R} , which we can think of as taking values in the set $\{G \cap G_R : G \in \mathcal{G}\} =: \mathcal{G}_R$.

Note that if $G_1, G_2 \in \mathcal{G}$, $R \subseteq [n]$, then $G_1 \cap G_2 \cap G_R \neq \emptyset$, since $G_1 \cap G_2$ contains a triangle, which must intersect G_R by the pigeonhole principle (the triangle contains 3 vertices, one of which is contained in one of the two components of G_R). Thus, \mathcal{G}_R is an intersecting family, so has size at most $2^{|E(G_R)|-1}$. By Lemma 4.4,

$$H(X) \leq 2 \cdot \mathbb{E}_R [H(X_{G_R})] \leq 2 \cdot \mathbb{E}_R [|E(G_R)| - 1] = 2 \cdot \left(\frac{1}{2} \binom{n}{2} - 1 \right) = \binom{n}{2} - 2,$$

since each e belongs to G_R with probability $1/2$ (and so $\mathbb{E}_R [|E(G_R)|] = \frac{1}{2} \binom{n}{2}$). □

Definition 4.12 Let G be a graph and let $A \subseteq V(G)$. The **edge-boundary** ∂A of A is the set of edges xy such that $x \in A$, $y \notin A$. If $G = \mathbb{Z}^n$ or $\{0, 1\}^n$ and $i \in [n]$, the **i -th boundary** $\partial_i A$ is the set of edges $xy \in \partial A$ such that $x - y = \pm e_i$, i.e. $\partial_i A$ consists of edges in direction i .

Theorem 4.13 (Edge-isoperimetric Inequality in \mathbb{Z}^n) Let $A \subseteq \mathbb{Z}^n$ be a finite set. Then

$$|\partial A| \geq 2n \cdot |A|^{(n-1)/n}.$$

Proof (Hints). Use [Discrete Loomis-Whitney Theorem](#) and a suitable lower bound on $|\partial_i A|$. □

Proof. By the [Discrete Loomis-Whitney Theorem](#),

$$\begin{aligned} |A| &\leq \prod_{i=1}^n |P_{[n] \setminus \{i\}} A|^{1/(n-1)} \\ &= \left(\prod_{i=1}^n |P_{[n] \setminus \{i\}} A|^{1/n} \right)^{n/(n-1)} \\ &\leq \left(\frac{1}{n} \sum_{i=1}^n |P_{[n] \setminus \{i\}} A| \right)^{n/(n-1)} \quad \text{by AM-GM inequality} \end{aligned}$$

But $|\partial_i A| \geq 2|P_{[n]\setminus\{i\}} A|$ since each fibre contributes at least 2. So

$$\begin{aligned} |A| &\leq \left(\frac{1}{2n} \sum_{i=1}^n |\partial_i A| \right)^{n/(n-1)} \\ &= \left(\frac{1}{2n} |\partial A| \right)^{n/(n-1)} \end{aligned}$$

□

Theorem 4.14 (Edge-isoperimetric Inequality in the Cube) Let $A \subseteq \{0, 1\}^n$ (where we take usual graph on $\{0, 1\}^n$). Then

$$|\partial A| \geq |A|(n - \log |A|).$$

Proof (Hints).

- Let $X = (X_1, \dots, X_n)$ be a uniformly random element of A . Write $X_{\setminus i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$.
- Use [Shearer](#) to show that $\sum_{i=1}^n H(X_i | X_{\setminus i}) \leq H(X)$.
- What are the possible values of $|P_{[n]\setminus\{i\}}^{-1}(u)|$, and what is $H(X_i | X_{\setminus i} = u)$ in each case? How many u satisfy $|P_{[n]\setminus\{i\}}^{-1}(u)| = 1$? Use this to deduce an expression for $H(X_i | X_{\setminus i})$.

□

Proof. Let X be a uniformly random element of A and write $X = (X_1, \dots, X_n)$. Write $X_{\setminus i}$ for $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$. By [Shearer](#),

$$\begin{aligned} H(X) &\leq \frac{1}{n-1} \sum_{i=1}^n H(X_{\setminus i}) \\ &= \frac{1}{n-1} \sum_{i=1}^n (H(X) - H(X_i | X_{\setminus i})). \end{aligned}$$

Hence, $\sum_{i=1}^n H(X_i | X_{\setminus i}) \leq H(X)$. But

$$H(X_i | X_{\setminus i} = u) = \begin{cases} 1 & \text{if } |P_{[n]\setminus\{i\}}^{-1}(u)| = 2 \\ 0 & \text{if } |P_{[n]\setminus\{i\}}^{-1}(u)| = 1 \end{cases}$$

(Note that we always have $|P_{[n]\setminus\{i\}}^{-1}(u)| \in \{0, 1, 2\}$). The number of points of the second kind is $|\partial_i A|$. So

$$\begin{aligned} H(X_i | X_{\setminus i}) &= \sum_u \mathbb{P}(X_{\setminus i} = u) H(X_i | X_{\setminus i} = u) \\ &= \sum_{u \notin \partial_i A} \mathbb{P}(X_{\setminus i} = u) \\ &= 1 - \sum_{u \in \partial_i A} \mathbb{P}(X_{\setminus i} = u) \end{aligned}$$

$$= 1 - \frac{|\partial_i A|}{|A|}.$$

So

$$\begin{aligned} H(X) &\geq \sum_{i=1}^n \left(1 - \frac{|\partial_i A|}{|A|} \right) \\ &= n - \frac{|\partial A|}{|A|}. \end{aligned}$$

Also, $H(X) = \log|A|$. So we are done. \square

Definition 4.15 Let \mathcal{A} be a family of sets of size d . The **lower shadow** of \mathcal{A} is

$$\partial\mathcal{A} = \{B : |B| = d-1, \exists A \in \mathcal{A} \text{ s.t. } B \subseteq A\}.$$

Theorem 4.16 (Kruskal-Katona) If $|\mathcal{A}| = \binom{t}{d} = \frac{t(t-1)\cdots(t-d+1)}{d!}$ for some real number t , then

$$|\partial_i \mathcal{A}| \geq \binom{t}{d-1}.$$

Proof (Hints).

- Let $X = (X_1, \dots, X_d)$ be a random ordering of the elements of a uniformly random $A \in \mathcal{A}$. Give an expression for $H(X)$.
- Explain why it is enough to show $H(X_1, \dots, X_{d-1}) \geq \log((d-1)! \binom{t}{d-1})$.
- Let $T \sim \text{Bern}(p)$ be independent of X_1, \dots, X_{k-1} , and given X_1, \dots, X_{k-1} , let

$$X^* = \begin{cases} X_{k+1} & \text{if } T = 0 \\ X_k & \text{if } T = 1 \end{cases}.$$

- Show that $H(X_k | X_{<k}) \geq H(X^*, T | X_{\leq k}) = h(p) + pH(X_{k+1} | X_{\leq k})$, and so that $H(X_k | X_{<k}) \geq \log(2^{H(X_{k+1} | X_{\leq k})} + 1)$.
- Using the chain rule, show that $r + d - 1 \leq t$, and use this to conclude the desired bound on $H(X_{<d})$.

\square

Proof. Let $X = (X_1, \dots, X_d)$ be a random ordering of the elements of a uniformly random $A \in \mathcal{A}$. Then $H(X) = \log(d!|A|) = \log(d! \binom{t}{d})$. Note that (X_1, \dots, X_{d-1}) is an ordering of the elements of some $B \in \partial_i \mathcal{A}$, so

$$H(X_1, \dots, X_{d-1}) \leq \log((d-1)!|\partial_i \mathcal{A}|)$$

So it's enough to show $H(X_1, \dots, X_{d-1}) \geq \log((d-1)! \binom{t}{d-1})$. Also, $H(X) = H(X_1, \dots, X_{d-1}) + H(X_d | X_1, \dots, X_{d-1})$ and $H(X) = H(X_1) + H(X_2 | X_1) + \dots + H(X_d | X_1, \dots, X_{d-1})$. We would like an upper bound for $H(X_d | X_{<d})$. Our strategy will be to obtain a lower bound for $H(X_k | X_{<k})$ in terms of $H(X_{k+1} | X_{<k+1})$. We shall prove that $2^{H(X_k | X_{<k})} \geq 2^{H(X_{k+1} | X_{<k+1})} + 1$ for all k .

Let T be chosen independently of X . Let $T \sim \text{Bern}(1 - p)$ ($T = 0$ with probability p , p is to be chosen later). Given X_1, \dots, X_{k-1} , let

$$X^* = \begin{cases} X_{k+1} & \text{if } T = 0 \\ X_k & \text{if } T = 1 \end{cases}$$

Note that X_k and X_{k+1} have the same distribution (given X_1, \dots, X_{k-1}), so X^* does as well. Then

$$\begin{aligned} H(X_k \mid X_{<k}) &= H(X^* \mid X_{<k}) \text{ since } X_k \sim X^* \\ &\geq H(X^* \mid X_{\leq k}) \text{ by Submodularity} \\ &= H(X^*, T \mid X_{\leq k}) \text{ since } X_{\leq k} \text{ and } X^* \text{ determine } T \text{ (since } X_{k+1} \neq X_k) \\ &= H(T \mid X_{\leq k}) + H(X^* \mid T, X_{\leq k}) \text{ by Additivity} \\ &= H(T) + pH(X^* \mid X_{\leq k}, T = 0) + (1 - p)H(X^* \mid X_{\leq k}, T = 1) \\ &= H(T) + pH(X_{k+1} \mid X_{\leq k}) + (1 - p)H(X_k \mid X_{\leq k}) \\ &= h(p) + ps. \end{aligned}$$

where $s = H(X_{k+1} \mid X_{\leq k})$ and $h(p) = p \log \frac{1}{p} + (1 - p) \log \frac{1}{1-p}$. This is maximised when $p = \frac{2^s}{2^s + 1}$. Then we get

$$\frac{2^s}{2^s + 1} (\log(2^s + 1) - \log(2^s)) + \frac{1}{2^s + 1} (\log(2^s + 1)) + \frac{s2^s}{2^s + 1} = \log(2^s + 1).$$

This proves the claim.

Let $r = 2^{H(X_d \mid X_{<d})}$. Then by the claim,

$$\begin{aligned} H(X) &= H(X_1) + \dots + H(X_d \mid X_{<d}) \\ &\geq \log(r + d - 1) + \dots + \log(r) \\ &= \log \left(\frac{(r + d - 1)!}{(r - 1)!} \right) = \log \left(d! \binom{r + d - 1}{d} \right). \end{aligned}$$

Since $H(X) = \log(d! \binom{t}{d})$ is an increasing function (for $t \geq d$), it follows that $r + d - 1 \leq t$, i.e. $r \leq t + 1 - d$. It follows that

$$\begin{aligned} H(X_{<d}) &= \log \left(d! \binom{t}{d} \right) - \log r \\ &\geq \log \left(d! \frac{t!}{d!(t - d)!(t + 1 - d)} \right) \\ &= \log \left((d - 1)! \binom{t}{d - 1} \right). \end{aligned}$$

□

5. The union-closed conjecture

Definition 5.1 Let \mathcal{A} be a finite family of sets. \mathcal{A} is **union-closed** if $A \cup B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.

Conjecture 5.2 (Union-closed Conjecture) If \mathcal{A} is a non-empty union-closed family, then there exists x that belongs to at least $\frac{1}{2}|\mathcal{A}|$ sets in \mathcal{A} .

Theorem 5.3 (Gilmer) There exists a constant $c > 0$ such that if \mathcal{A} is any union-closed family, then there exists x that belongs to at least $c|\mathcal{A}|$ of the sets in \mathcal{A} .

Example 5.4 Let $\mathcal{A} = [n]^{(pn)} \cup [n]^{(\geq (2p-p^2-o(1))n)}$. Then with high probability, if A, B are random elements of $[n]^{(pn)}$, then $|A \cup B| \geq (2p - p^2 - o(1))n$ (since the intersect is likely of size at most p^2n). If $1 - (2p - p^2 - o(1)) = p$, then almost all of \mathcal{A} is contained in $[n]^{(pn)}$. The solutions of p occur roughly when $1 - 3p + p^2 = 0$, which has solutions $p = \frac{1}{2}(3 \pm \sqrt{5})$.

If we want to prove [Gilmer](#), it is natural to let A, B be independent uniformly random elements of \mathcal{A} and to consider $H(A \cup B)$. Since \mathcal{A} is union-closed, $A \cup B \in \mathcal{A}$, so $H(A \cup B) \leq \log|\mathcal{A}|$. Now we would like to get a lower bound for $H(A \cup B)$ assuming that no x belongs to more than $p|\mathcal{A}|$ sets in \mathcal{A} .

Lemma 5.5 Suppose $c > 0$ is such that $h(xy) \geq c(xh(y) + yh(x))$ for every $x, y \in [0, 1]$. Let \mathcal{A} be a family of sets such that every element of $\cup \mathcal{A}$ belongs to fewer than $p|\mathcal{A}|$ members of \mathcal{A} . Let A, B be independent uniformly members of \mathcal{A} . Then

$$H(A \cup B) > c(1 - p)(H(A) + H(B)).$$

Proof (Hints).

- Think of A, B as characteristic functions. Write $A_{<k}$ for (A_1, \dots, A_{k-1}) .
- Explain why it is enough to prove that $H((A \cup B)_k \mid A_{<k}, B_{<k}) > c(1 - p)(H(A_k \mid A_{<k}) + H(B_k \mid B_{<k}))$ for all k .
- For each $u, v \in \{0, 1\}^{k-1}$, write $p(u) = \mathbb{P}(A_k = 0 \mid A_{<k} = u)$ and $q(v) = \mathbb{P}(B_k = 0 \mid B_{<k} = v)$. Find a (simple) expression for $H((A \cup B)_k \mid A_{<k} = u, B_{<k} = v)$.
- Expand $H((A \cup B)_k \mid A_{<k}, B_{<k})$, give an upper bound, then simplify it (hint: law of total probability).

□

Proof. Think of A, B as characteristic functions. Write $A_{<k}$ for (A_1, \dots, A_{k-1}) . By the [Chain Rule](#), it is enough to prove for every k that

$$H((A \cup B)_k \mid (A \cup B)_{<k}) > c(1 - p)(H(A_k \mid A_{<k}) + H(B_k \mid B_{<k})).$$

By Lemma [1.25](#),

$$H((A \cup B)_k \mid (A \cup B)_{<k}) \geq H((A \cup B)_k \mid A_{<k}, B_{<k})$$

For each $u, v \in \{0, 1\}^{k-1}$, write $p(u) = \mathbb{P}(A_k = 0 \mid A_{<k} = u)$ and $q(v) = \mathbb{P}(B_k = 0 \mid B_{<k} = v)$. Then, since A and B are independent,

$$H((A \cup B)_k \mid A_{<k} = u, B_{<k} = v)$$

$$\begin{aligned}
&= - \sum_{i=0}^1 \mathbb{P}((A \cup B)_k = i \mid A_{<k} = u, B_{<k} = v) \log \mathbb{P}((A \cup B)_k = i \mid A_{<k} = u, B_{<k} = v) \\
&= h(p(u)q(v)).
\end{aligned}$$

which by hypothesis is at least $c(p(u)h(q(v)) + q(v)h(p(u)))$. So

$$\begin{aligned}
H((A \cup B)_k \mid (A \cup B)_{<k}) &\geq c \sum_{u,v} \mathbb{P}(A_{<k} = u) \mathbb{P}(B_{<k} = v) (p(u)h(q(v)) + q(v)h(p(u))) \\
&= c \cdot \sum_u \mathbb{P}(A_{<k} = u) p(u) \cdot \sum_v \mathbb{P}(B_{<k} = v) h(q(v)) \\
&\quad + c \cdot \sum_u \mathbb{P}_{A_{<k}=u} h(p(u)) \cdot \sum_v \mathbb{P}(B_{<k} = v) q(v)
\end{aligned}$$

But by law of total probability,

$$\sum_u \mathbb{P}(A_{<k} = u) \mathbb{P}(A_k = 0 \mid A_{<k} = u) = \mathbb{P}(A_k = 0),$$

and

$$\sum_v \mathbb{P}(B_{<k} = v) h(q(v)) = \sum_v \mathbb{P}(B_{<k} = v) H(B_k \mid B_{<k} = v) = H(B_k \mid B_{<k})$$

Similarly for the other term, so the RHS of the inequality equals

$$c(\mathbb{P}(A_k = 0)H(B_k \mid B_{<k}) + \mathbb{P}(B_k = 0)H(A_k \mid A_{<k})),$$

which by hypothesis (since $\mathbb{P}(A_k = 0) = \mathbb{P}(B_k = 0) > 1 - p$) is greater than

$$c(1 - p)(H(A_k \mid A_{<k}) + H(B_k \mid B_{<k}))$$

as required. \square

Corollary 5.6 Let \mathcal{A} , p and c be as in Lemma 5.5. If \mathcal{A} is union-closed, then we must have $p \geq 1 - 1/2c$.

Proof (Hints). Straightforward. \square

Proof. Let A and B be independent uniformly random elements of \mathcal{A} . Since \mathcal{A} is union-closed, $A \cup B \in \mathcal{A}$, so $H(A \cup B) \leq \log |\mathcal{A}|$. Also, $H(A) = H(B) = \log |\mathcal{A}|$. Hence, by Lemma 5.5, $2c(1 - p) \leq 1$. \square

Corollary 5.6 gives a non-trivial bound as long as $c > 1/2$. We shall obtain $1/(\sqrt{5} - 1)$.

We start by proving the diagonal case, i.e. where $x = y$.

Lemma 5.7 (Boppana) For every $x \in [0, 1]$,

$$h(x^2) \geq \varphi \cdot x \cdot h(x),$$

where $\varphi = \frac{1}{2}(\sqrt{5} + 1)$.

Proof (Hints).

- Let $\psi = 1/\varphi$. Show that equality holds when $x = \psi, 0, 1$.

- Let $f(x) = h(x^2) - \varphi \cdot x \cdot h(x)$. Show that $f'''(x) = 0$ iff $-\varphi x^3 - 4x^2 + 3\varphi x - 4 + 2\varphi = 0$. (Advice: use natural logs and find expressions for $h'(x)$, $h''(x)$ and $h'''(x)$ first).
- Explain why f''' has at most two roots in $(0, 1)$ and so f has at most five roots in $[0, 1]$.
- Show that f has a double root at 0 and at ψ .
- Explain why f must have constant sign on $[0, 1]$, and by considering small x , show that there is x with $f(x) > 0$.

□

Proof. Write $\psi = 1/\varphi = \frac{1}{2}(\sqrt{5} - 1)$. Then $\psi^2 = 1 - \psi$. So $h(\psi^2) = h(1 - \psi) = h(\psi)$ and $\varphi\psi = 1$, so $h(\psi^2) = \varphi \cdot \psi \cdot h(\psi)$. So equality holds when $x = \psi$, and also when $x = 0, 1$.

Toolkit: $\ln(2) \cdot h(x) = -x \ln x - (1 - x) \ln(1 - x)$. Then

$$\ln(2) \cdot h'(x) = -\ln x - 1 + \ln(1 - x) + 1 = \ln(1 - x) - \ln(x)$$

and

$$\ln(2) \cdot h''(x) = -\frac{1}{x} - \frac{1}{1 - x} = -\frac{1}{x(1 - x)}$$

and

$$\ln(2) \cdot h'''(x) = \frac{1}{x^2} - \frac{1}{(1 - x)^2} = \frac{1 - 2x}{x^2(1 - x)^2}.$$

Let $f(x) = h(x^2) - \varphi \cdot x \cdot h(x)$. Then

$$\begin{aligned} f'(x) &= 2xh'(x^2) - \varphi h(x) - \varphi xh'(x) \\ f''(x) &= 2h'(x^2) + 4x^2h''(x^2) - 2\varphi h'(x) - \varphi xh''(x) \\ f'''(x) &= 4xh''(x^2) + 8xh''(x^2) + 8x^3h'''(x^2) - 3\varphi h''(x) - \varphi xh'''(x) \\ &= 12xh''(x^2) + 8x^3h'''(x^2) - 3\varphi h''(x) - \varphi xh'''(x) \end{aligned}$$

So

$$\begin{aligned} \ln(2)f'''(x) &= \frac{-12x}{x^2(1 - x^2)} + \frac{8x^3(1 - 2x^2)}{x^4(1 - x^2)^2} + \frac{3\varphi}{x(1 - x)} - \frac{\varphi x(1 - 2x)}{x^2(1 - x)^2} \\ &= \frac{-12}{x(1 - x^2)} + \frac{8(1 - 2x^2)}{x(1 - x^2)^2} + \frac{3\varphi}{x(1 - x)} - \frac{\varphi(1 - 2x)}{x(1 - x)^2} \\ &= \frac{-12(1 - x^2) + 8(1 - 2x^2) + 3\varphi(1 - x)(1 + x)^2 - \varphi(1 - 2x)(1 + x)^2}{x(1 - x)^2(1 + x)^2} \end{aligned}$$

which is zero iff

$$-12 + 12x + 8 - 16x^2 + 3\varphi(1 + x - x^2 - x^3) - \varphi(1 - 3x^2 - 2x^3)$$

$$= -\varphi x^3 - 4x^2 + 3\varphi x - 4 + 2\varphi = 0.$$

So the numerator of $f'''(x)$ is a cubic with negative leading coefficient and constant term, so it has a negative root, so it has at most two roots in $(0, 1)$. It follows (by Rolle's theorem) that f has at most five roots in $[0, 1]$, up to multiplicity. But

$$f'(x) = 2x(\log(1-x^2) - \log(x^2)) + \varphi(x \log x + (1-x) \log(1-x)) - \varphi x(\log(1-x) - \log x)$$

So $f'(0) = 0$, so f has a double root at 0. Now

$$\begin{aligned} f'(\psi) &= 2\psi(\log \psi - 2 \log \psi) + \varphi(\psi \log \psi + 2(1-\psi) \log \psi) - (2 \log \psi - \log \psi) \\ &= -2\psi \log \psi + \log \psi + 2\varphi \log \psi - 2 \log \psi \\ &= 2 \log \psi(-\psi + \varphi - 1) \\ &= 2\varphi \log \psi(-\psi^2 - 1 - \psi) = 0 \end{aligned}$$

So there is a double root at ψ . Also, $f(1) = 0$. So f is either non-negative on all of $[0, 1]$ or non-positive on all of $[0, 1]$. If x is small,

$$\begin{aligned} f(x) &= x^2 \log \frac{1}{x^2} + (1-x^2) \log \frac{1}{1-x^2} - \varphi x \left(x \log \frac{1}{x} + (1-x) \log \frac{1}{1-x} \right) \\ &= 2x^2 \log \frac{1}{x} - \varphi x^2 \log \frac{1}{x} + O(x^2). \end{aligned}$$

So, because $2 > \varphi$, there exists x such that $f(x) > 0$. □

Lemma 5.8 The function $f(x, y) = \frac{h(xy)}{xh(y) + yh(x)}$ is minimised on $(0, 1)^2$ at a point where $x = y$.

Proof (Hints).

- Show that we can extend f continuously to the boundary by setting $f(x, y) = 1$ whenever x or y is 0 or 1 (for the case when x or y tend to 0 separately, consider an expansion for xy small, and for the case when x and y tend to 1, consider when one of x or y is 1).
- Pick any point in $(0, 1)^2$ to show that f is minimised somewhere in that region.
- Let (x^*, y^*) be a minimum with $f(x^*, y^*) = \alpha$. Let $g(x) = h(x)/x$.
- By considering the expression $g(xy) - \alpha(g(x) + g(y))$ and partial derivatives, show that $x^*g'(x^*) = y^*g'(y^*)$.
- Show that $xg'(x)$ is an injection by considering its derivative.

□

Proof. We can extend f continuously to the boundary by setting $f(x, y) = 1$ whenever x or y is 0 or 1. To see this, note first that it is easy if neither x nor y is 0. If either x or y is small then $h(xy) = -xy(\log x + \log y) + O(xy)$, and

$$\begin{aligned} xh(y) + yh(x) &= -x(y \log y + O(y)) - y(x \log x + O(x)) \\ &= h(xy) \quad \text{up to } O(xy) \end{aligned}$$

So it tends to 1 again.

We can check that $f(1/2, 1/2) < 1$, so f is minimised somewhere in $(0, 1)^2$. Let (x^*, y^*) be a minimum with $f(x^*, y^*) = \alpha$. For convenience, let $g(x) = h(x)/x$ and note that $f(x, y) = \frac{g(xy)}{g(x)+g(y)}$. Also, $g(xy) - \alpha(g(x) + g(y)) \geq 0$ with equality at (x^*, y^*) . So the partial derivatives of the LHS are both 0 at (x^*, y^*) :

$$\begin{aligned} y^* g'(x^* y^*) - \alpha g'(x^*) &= 0 \\ x^* g'(x^* y^*) - \alpha g'(y^*) &= 0. \end{aligned}$$

So $x^* g'(x^*) = y^* g'(y^*)$. So it is enough to prove that $xg'(x)$ is an injection. We have

$$g'(x) = \frac{h'(x)}{x} - \frac{h(x)}{x^2}$$

so

$$\begin{aligned} xg'(x) &= h'(x) - \frac{h(x)}{x} \\ &= \log(1-x) - \log x + \frac{x \log x + (1-x) \log(1-x)}{x} \\ &= \frac{\log(1-x)}{x}. \end{aligned}$$

Differentiating gives

$$-\frac{1}{x(1-x)} - \frac{\log(1-x)}{x^2} = \frac{-x - (1-x) \log(1-x)}{x^2(1-x)}$$

The numerator differentiates to $-1 + 1 + \log(1-x)$ which is negative. Also, it equals 0 at 0, so it has a constant sign. Thus, $xg'(x)$ is indeed an injection. \square

Combining this with [Boppana](#) we get that

$$h(xy) \geq \frac{\varphi}{2}(xh(y) + yh(x))$$

This allows us to take $p = 1 - \frac{1}{\varphi} = \frac{3-\sqrt{5}}{2}$.

6. Entropy in additive combinatorics

We shall need two “simple” results from additive combinatorics due to Imre Ruzsa.

Definition 6.1 Let G be an abelian group and let $A, B \subseteq G$. The **sumset** $A + B$ of A and B is the set

$$\{x + y : x \in A, y \in B\}$$

and the **difference set** $A - B$ is the set

$$\{x - y : x \in A, y \in B\}.$$

Write $2A$ for $A + A$, $3A$ for $A + A + A$, etc.

Definition 6.2 The **Ruzsa distance** $d(A, B)$ is

$$\frac{|A - B|}{|A|^{1/2} \cdot |B|^{1/2}}.$$

Lemma 6.3 (Ruzsa Triangle Inequality) $d(A, C) \leq d(A, B) \cdot d(B, C)$.

Proof (Hints). Expand the stated inequality and consider an appropriate injection. \square

Proof. This is equivalent to the statement

$$|A - C| \cdot |B| \leq |A - B| \cdot |B - C|.$$

For each $x \in A - C$, pick $a(x) \in A$, $c(x) \in C$ such that $x = a(x) - c(x)$. Define the map

$$\begin{aligned} \varphi : (A - C) \times B &\rightarrow (A - B) \times (B - C), \\ (x, b) &\mapsto (a(x) - b, b - c(x)). \end{aligned}$$

Adding the coordinates of $\varphi(x, b)$ gives x , so we can calculate $a(x)$ and $c(x)$ from $\varphi(x, b)$, and hence b . So φ is an injection. \square

Lemma 6.4 (Ruzsa Covering Lemma) Let G be an abelian group and let $A, B \subseteq G$ be finite. Then A can be covered by at most $|A + B|/|B|$ translates of $B - B$.

Proof (Hints). Consider a maximal subset $\{x_1, \dots, x_k\} \subseteq A$ such that the $x_i + B$ are disjoint. \square

Proof. Let $\{x_1, \dots, x_k\}$ be a maximal subset of A such that the sets $x_i + B$ are disjoint. Then for all, $a \in A$, there exists i such that $(a + B) \cap (x_i + B) \neq \emptyset$, i.e. $a \in (x_i + (B - B))$. So A can be covered by k translates of $B - B$. But since the $x_i + B$ are disjoint,

$$|B|k = |\{x_1, \dots, x_k\} + B| \leq |A + B|.$$

\square

Let X, Y be discrete random variables taking values in an abelian group. What is $X + Y$ when X and Y are independent? For each z , $\mathbb{P}(X + Y = z) = \sum_{x+y=z} \mathbb{P}(X = x)\mathbb{P}(Y = y)$. Writing p_x and q_y for $\mathbb{P}(X = x)$ and $\mathbb{P}(Y = y)$, this gives

$$\sum_{x+y=z} p_x p_y = (p * q)(z)$$

where $p(x) = p_x$, $q(y) = q_y$. So sums of independent random variables correspond to convolutions.

Definition 6.5 Let G be an abelian group and let X, Y be G -valued random variables. The **(entropic) Ruzsa distance** between X and Y is

$$\begin{aligned} d(X; Y) &= H(X' - Y') - \frac{1}{2}H(X) - \frac{1}{2}H(Y) \\ &= H(X' - Y') - \frac{1}{2}H(X') - \frac{1}{2}H(Y'). \end{aligned}$$

where X', Y' are independent copies of X, Y .

Lemma 6.6 If A, B are finite subsets of G and X, Y are uniform on A, B respectively, then

$$d(X; Y) \leq \log d(A, B).$$

Proof (Hints). Straightforward. □

Proof. WLOG X, Y are independent. Then

$$\begin{aligned} d(X, Y) &= H(X - Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) \\ &\leq \log|A - B| - \frac{1}{2}\log|A| - \frac{1}{2}\log|B| = \log d(A, B). \end{aligned}$$

□

Lemma 6.7 Let X, Y be G -valued random variables. Then

$$H(X - Y) \geq \max\{H(X), H(Y)\} - I(X : Y).$$

Proof (Hints). Use that $H(X - Y) \geq H(X - Y | Y)$ and $H(X - Y) \geq H(X - Y | X)$. □

Proof. We have

$$\begin{aligned} H(X - Y) &\geq H(X - Y | Y) \text{ by Subadditivity} \\ &= H(X - Y, Y) - H(Y) \\ &= H(X, Y) - H(Y) \text{ by Invariance} \\ &= H(X) + H(Y) - H(Y) - I(X : Y) \\ &= H(X) - I(X : Y). \end{aligned}$$

We use Invariance with the bijection $(x, y) \mapsto (x - y, y)$. By symmetry, we also have $H(X - Y) \geq H(Y) - I(X : Y)$. □

Corollary 6.8 If X, Y are G -valued RVs, then $d(X; Y) \geq 0$.

Proof (Hints). Straightforward. □

Proof. WLOG X and Y are independent. Then $I(X : Y) = 0$, so $H(X - Y) \geq \max\{H(X), H(Y)\} \geq \frac{1}{2}(H(X) + H(Y))$. □

Lemma 6.9 If X, Y are G -valued RVs, then $d(X; Y) = 0$ iff there is some (finite) subgroup H of G such that X and Y are uniform on cosets of H .

Proof (Hints).

- \Leftarrow : straightforward.
- \Rightarrow : assume WLOG that X and Y are independent. By considering entropy, explain why $X - Y$ and Y are independent.
- Deduce that for X supported on A and Y supported on B , for all $z \in A - B$ and $y_1, y_2 \in B$, $\mathbb{P}(X = y_1 + z) = \mathbb{P}(X = y_2 + z)$, and show that this implies that $z + B \subseteq A$.

- Deduce that $A = B + z$ for all $z \in A - B$, and so that $A - x$ is constant over $x \in A$.
- Deduce that $A - A$ is a subgroup.

□

Proof. \Leftarrow : If X, Y are uniform on $x + H, y + H$ then $X' - Y'$ is uniform on $(x - y) + H$, so $H(X' - Y') = H(X) = H(Y)$.

\Rightarrow : WLOG X and Y are independent. We have $H(X - Y) = \frac{1}{2}(H(X) + H(Y))$. So equality must hold throughout the proof of Lemma 6.7 and Corollary 6.8, thus $H(X - Y | Y) = H(X - Y)$. Therefore, $X - Y$ and Y are independent. So for every $z \in A - B$ and $y_1, y_2 \in B$,

$$\mathbb{P}(X - Y = z | Y = y_1) = \mathbb{P}(X - Y = z | Y = y_2),$$

where $A = \{x : \mathbb{P}(X = x) \neq 0\}$ and $B = \{y : \mathbb{P}(Y = y) \neq 0\}$. We can write this as

$$\mathbb{P}(X = y_1 + z) = \mathbb{P}(X = y_2 + z)$$

So $\mathbb{P}(X = x)$ is constant on $z + B$. In particular, $z + B \subseteq A$ ($\mathbb{P}(X = x)$ must be non-zero on $z + B$, as otherwise $(z + B) \cap A = \emptyset$, i.e. $z \notin A - B$). By the same argument, $A - z \subseteq B$. So $A = B + z$ for all $z \in A - B$. So for every $x \in A$ and $y \in B$, $A = B + x - y$, so $A - x = B - y$. Hence, $A - x$ is the same for every $x \in A$. Therefore, $A - x = \bigcup_{x \in A} (A - x) = A - A$ for all $x \in A$. It follows that

$$A - A + A - A = (A - A) - (A - A) = A - x - (A - x) = A - A.$$

So $A - x = A - A$ is a subgroup, and so A is a coset of $A - A$. $B = A + x$, so B is also a coset of $A - A$. Also, as stated above, X is uniform on $z + B = A$ and Y is uniform on $A - z = B$. □

Lemma 6.10 (Entropic Ruzsa Triangle Inequality) Let X, Y, Z be G -valued random variables. Then $d(X; Z) \leq d(X; Y) + d(Y; Z)$.

Proof (Hints). Simplify the desired inequality and use Lemma 1.26 (where $X - Z$ depends on two different (pairs of) random variables). □

Proof. We must show (assuming WLOG that X, Y, Z are independent) that

$$\begin{aligned} & H(X - Z) - \frac{1}{2}H(X) - \frac{1}{2}H(Z) \\ & \leq H(X - Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) + H(Y - Z) - \frac{1}{2}H(Y) - \frac{1}{2}H(Z), \end{aligned}$$

i.e. that $H(X - Z) + H(Y) \leq H(X - Y) + H(Y - Z)$. Since $X - Z$ depends on $(X - Y, Y - Z)$ and on (X, Z) , by Lemma 1.26,

$$H(X - Y, Y - Z, X, Z) + H(X - Z) \leq H(X - Y, Y - Z) + H(X, Z)$$

i.e. $H(X, Y, Z) + H(X - Z) \leq H(X, Z) + H(X - Y, Y - Z)$. By independence and Subadditivity, we get $H(X - Z) + H(Y) \leq H(X - Y) + H(Y - Z)$. □

Lemma 6.11 (Submodularity for Sums) If X, Y, Z are independent G -valued RVs, then

$$H(X + Y + Z) + H(Z) \leq H(X + Z) + H(Y + Z).$$

Proof (Hints). Use Lemma 1.26. □

Proof. $X + Y + Z$ is a function of $(X + Z, Y)$ and of $(X, Y + Z)$. Therefore, by Lemma 1.26,

$$H(X + Z, Y, X, Y + Z) + H(X + Y + Z) \leq H(X + Z, Y) + H(X, Y + Z),$$

thus $H(X, Y, Z) + H(X + Y + Z) \leq H(X + Z) + H(Y) + H(X) + H(Y + Z)$. By independence and cancelling equal terms, we get the desired inequality. □