1. Motivation

1.1. Plane curves

- Curves mainly parametrised: $\alpha : I \to \mathbb{R}^2$, $I \subset \mathbb{R}$ interval, with a direction.
- **Four vertex theorem**: every closed plane curve has at least 4 vertices.

1.2. Surfaces

- Surfaces are 2-dimensional subsets of $\mathbb{R}^3$.

2. Regular curves in $\mathbb{R}^n$

2.1. Regular curves, length and tangent vectors

- Let I be open interval, then $\underline{\alpha} : I \to \mathbb{R}^n$ is **parametrised curve**.
- $\underline{\alpha}$ is **smooth** if $\underline{\alpha}(u) = (\alpha_1(u), ..., \alpha_n(u))$ where all $\alpha_i : I \to \mathbb{R}$ are smooth maps.
- Image $\underline{\alpha}(I) \subset \mathbb{R}^n$ is the **trace**.
- **Tangent vector of $\boldsymbol{\alpha}$ at $\boldsymbol{u}$** is

$$\underline{\alpha}'(u) = (\alpha_1'(u), ..., \alpha_n'(u))$$

- $\underline{\alpha}$ is **regular** if $\forall u \in I, \underline{\alpha}'(u) \neq 0$. $\underline{\alpha}$ is **singular at $\boldsymbol{u}$** if $\underline{\alpha}'(u) = 0$.
- If $\underline{\alpha}$ is regular, **unit tangent vector of $\boldsymbol{\alpha}$ at $\boldsymbol{u}$** is

$$\underline{t}(u) = \underline{\alpha}' \frac{u}{\|\underline{\alpha}'(u)\|}$$

- If $\forall u \in I, \|\underline{\alpha}'(u)\| = 1$ then $\underline{\alpha}$ is a **unit speed curve**. If $\forall u \in I, \|\underline{\alpha}'(u)\| = c$, $\underline{\alpha}$ is **constant speed curve**.
- **Example**: unit circle $\underline{\alpha}(u) = (\cos u, \sin u)$ is regular: $\alpha'(u) = (-\sin u, \cos u) \neq 0$. It is unit speed as $\|\alpha'(u)\| = 1$.
- **Example**: helix $\underline{\alpha}(u) = (\cos u, \sin u, u)$, $\underline{\alpha}'(u) = (-\sin u, \cos u, 1)$, $\|\underline{\alpha}'(u)\| = \sqrt{2}$ so constant speed.
- **Example**: **cusp** $\underline{\alpha}(u) = (u^3, u^2)$, $\underline{\alpha}'(u) = (3u^2, 2u)$ so $\underline{\alpha}'(u) = 0 \iff u = 0$ so $\underline{\alpha}$ singular at 0.
- **Example**: **node** $\underline{\alpha}(u) = \left(u^3 - u, u^2 - 1\right)$. So $\underline{\alpha}(-1) = \underline{\alpha}(1) = (0, 0)$ so it has a self-intersection at the origin. $\underline{\alpha}'(u) = \left(3u^2 - 1, 2u\right)$ so is regular.
- **Definition**: let $\underline{\alpha} : I \to \mathbb{R}^n$, $[a, b] \subset I$. $\underline{\alpha}$ is **rectifiable** on $[a, b]$ if

$$L\left(\underline{\alpha}|_{[a,b]}\right) := \sup\left\{\sum_{i=0}^{n-1} \left\|\underline{\alpha}(u_{i+1}) - \underline{\alpha}(u_i)\right\| : n \in \mathbb{N}, a = u_0 < \cdots < u_m = b\right\}$$

  is finite. Then $L\left(\underline{\alpha}|_{[a,b]}\right)$ is the **(arc) length** of $\underline{\alpha} : [a, b] \to \mathbb{R}^n$.
- **Proposition**: let $\underline{\alpha} : I \to \mathbb{R}^n$ smooth, $[a, b] \subset I$. Then

$$L\left(\underline{\alpha}|_{[a,b]}\right) = \int_a^b \|\underline{\alpha}'(u)\| \,\mathrm{d}u$$

2.2. Reparametrisation

- **Definition**: let $\underline{\alpha} : I \to \mathbb{R}^n$ be smooth regular curve. A **parameter change** for α is a smooth map $h : J \to I$, $J \subset \mathbb{R}$ is open interval, where
  - $\forall t \in J, h'(t) \neq 0$
  - $h(J) = I$.

$\underline{\tilde{\alpha}} = \underline{\alpha} \circ h : J \to \mathbb{R}^n$ is a **reparametrisation** of $\underline{\alpha}$. If $h' > 0$, h is **orientation preserving**, otherwise it is **orientation reversing**.

- **Proposition**: let $\underline{\alpha} : I \to \mathbb{R}^n$ be smoooth, regular curve, $u_0 \in I$, $\ell : I \to \mathbb{R}$ defined by

$$\ell(u) = \int_{u_0}^{u} \|\underline{\alpha}'(t)\| \,\mathrm{d}t$$

  Let $J = \ell(I)$. Then ℓ is strictly monotone increasing and $\underline{\tilde{\alpha}} = \underline{\alpha} \circ \ell^{-1} : J \to \mathbb{R}^n$ is unit speed.
- **Proposition**: let $\underline{\alpha} : I \to \mathbb{R}^n$ be smooth regular curve and $\underline{\tilde{\alpha}} \coloneqq \underline{\alpha} \circ h : J \to \mathbb{R}^n$ be reparametrisation of $\underline{\alpha}$ with parameter change $h : J \to I$. Let $[a, b] \subset I$ and $[c, d] = h^{-1}([a, b])$. Then

$$L\left(\underline{\alpha}|_{[a,b]}\right) = L\left(\underline{\tilde{\alpha}}|_{[c,d]}\right)$$

  i.e. length is independent of parametrisation.

3. Plane curves

3.1. Unit normal vectors and curvature

- **Definition**: let $\alpha : I \to \mathbb{R}^2$ be smooth regular plane curve. The **unit normal vector** of α at u is

$$\underline{n}_\alpha(u) = \underline{t}(u) \begin{bmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = (-t_2(u), t_1(u))$$

- **Lemma**: let $\alpha : I \to \mathbb{R}^2$ be smooth unit speed plane curve. Then $\underline{t}'(s) = \alpha''(s)$ is parallel to $\underline{n}(s)$.
- **Definition**: **(signed) curvature** $\kappa(s)$ of unit speed plane curve $\alpha : I \to \mathbb{R}^2$ at $s \in I$ is defined by

$$\underline{t}'(s) = \kappa(s)\underline{n}(s)$$

  Note that we can compute $\kappa(s)$ by

$$\underline{t}'(s) \cdot \underline{n}(s) = \kappa(s)\underline{n}(s) \cdot \underline{n}(s) = \kappa(s)\|\underline{n}(s)\|^2 = \kappa(s)$$

- **Rule for sign of curvature**: if curve turns clockwise, curvature is negative, if curve turns anti-clockwise, its curvature is positive.
- **Proposition**: let $\alpha : I \to \mathbb{R}^2$ be any smooth regular plane curve, $\alpha(u) = (x(u), y(u))$. Then its curvative is

$$\kappa(u) = \frac{x'(u)y''(u) - x''(u)y'(u)}{\left((x'(u))^2 + (y'(u))^2\right)^{3/2}}$$

- **Definition**: let $\alpha : I \to \mathbb{R}^2$ regular and smooth, $\kappa : I \to \mathbb{R}$ be its curvature, $n : I \to \mathbb{R}^2$ its unit normal vector. Assume $\kappa(u) \neq 0$. Then **radius of curvature** of α at $\alpha(u)$ is

  $$r(u) = \frac{1}{|\kappa(u)|}$$

  The **centre of curvature** of α at $\alpha(u)$ is

  $$e(u) = \alpha(u) + \frac{1}{|\kappa(u)|} n(u)$$

  Corresponding **curvature circle** of α at $\alpha(u)$ is

  $$\{P \in \mathbb{R}^2 : \|P - e(u)\| = r(u)\}$$

3.2. Four vertex theorem and fundamental theorem of plane curves

- **Definition**: let $\alpha : I \to \mathbb{R}^2$ regular and smooth, $\kappa : I \to \mathbb{R}$ its curvature. Then
  - $\alpha(u)$ is **inflection point** of α if $\kappa(u) = 0$.
  - $\alpha(u)$ is **vertex** of α if $\kappa'(u) = 0$.
- **Example**: for parabola $\alpha(u) = (u, u^2)$,

  $$\kappa(u) = \frac{2}{(1 + 4u^2)^{3/2}}, \quad \kappa'(u) = -\frac{24u}{(1 + 4u^2)^{5/2}}$$

  So there are no inflection points, and there is one vertex at the origin ($u = 0$).
- **Jordan Curve Theorem**: a simple closed continuous curve $\alpha : [a, b] \to \mathbb{R}^2$ divides $\mathbb{R}^2$ into two regions: one bounded and one unbounded.
- **Four vertex theorem**: let $\alpha : [a, b] \to \mathbb{R}^2$ smooth, regular, closed, simple plane curve. Then α has at least 4 vertices.
- **Theorem (Isoperimetric inequality)**: let $\alpha : [a, b] \to \mathbb{R}^2$ smooth, regular, simple plane curve of length $L = l(\alpha)$ and A be area of bounded region enclosed by a. Then

  $$L^2 \geq 4\pi A$$

  with equality iff α describes a circle.
- **Theorem (Fundamental theorem of local theory of plane curves)**: let $I \subset \mathbb{R}$ open interval, smooth $\kappa : I \to \mathbb{R}$, $s_o \in I$, $a \in \mathbb{R}^2$, $v_0 \in \mathbb{R}^2$, $\|v_0\| = 1$. Then exists unique smooth unit speed curve $\alpha : I \to \mathbb{R}^2$ with curvature $\kappa_\alpha = \kappa$ satisfying $\alpha(s_0) = a, \alpha'(s_0) = t_\alpha(s_0) = v_0$
- **Remark**: all orientation preserving isometries of $\mathbb{R}^2$ are of the form

  $$f(x) = f_{A,b}(x) = x \cdot A + b, \quad A \in \mathrm{SO}(2), b \in \mathbb{R}^2$$