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1. Monochromatic sets

1.1. Ramsey's theorem

Notation 1.1 \mathbb{N} denotes the set of positive integers, $[n] = \{1, \dots, n\}$, and $X^{(r)} = \{A \subseteq X : |A| = r\}$. Elements of a set are written in ascending order, e.g. $\{i, j\}$ means $i < j$. Write e.g. ijk to mean the set $\{i, j, k\}$ with the ordering (unless otherwise stated) $i < j < k$.

Definition 1.2 A k -colouring on $A^{(r)}$ is a function $c : A^{(r)} \rightarrow [k]$.

Example 1.3

- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ is even and blue if $i + j$ is odd. Then $M = 2\mathbb{N}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$ is even and blue otherwise. $M = \{4^n : n \in \mathbb{N}\}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

Theorem 1.4 (Ramsey's Theorem for Pairs) Let $\mathbb{N}^{(2)}$ be 2-coloured by $c : \mathbb{N}^{(2)} \rightarrow \{1, 2\}$. Then there exists an infinite monochromatic subset M .

Proof.

- Let $a_1 \in A_0 := \mathbb{N}$. There exists an infinite set $A_1 \subseteq A_0$ such that $c(a_1, i) = c_1$ for all $i \in A_1$.
- Let $a_2 \in A_1$. There exists infinite $A_2 \subseteq A_1$ such that $c(a_2, i) = c_2$ for all $i \in A_2$.
- Repeating this inductively gives a sequence $a_1 < a_2 < \dots < a_k < \dots$ and $A_1 \supseteq A_2 \supseteq \dots$ such that $c(a_i, j) = c_i$ for all $j \in A_i$.
- One colour appears infinitely many times: $c_{i_1} = c_{i_2} = \dots = c_{i_k} = \dots = c$.
- $M = \{a_{i_1}, a_{i_2}, \dots\}$ is a monochromatic set.

□

Remark 1.5

- The same proof works for any $k \in \mathbb{N}$ colours.
- The proof is called a “2-pass proof”.
- An alternative proof for k colours is split the k colours $1, \dots, k$ into 2 colours: 1 and “2 or ... or k ”, and use induction.

Note 1.6 An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

Example 1.7 Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, etc. Let $\{i, j\}$ be red if $i, j \in A_k$ for some k . There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

Example 1.8 Colour $\{i < j < k\}$ red iff $i \mid (j + k)$. A monochromatic subset $M = \{2^n : n \in \mathbb{N}_0\}$ is a monochromatic set.

Theorem 1.9 (Ramsey's Theorem for r -sets) Let $\mathbb{N}^{(r)}$ be finitely coloured. Then there exists a monochromatic infinite set.

Proof.

- $r = 1$: use pigeonhole principle.
- $r = 2$: Ramsey's theorem for pairs.
- For general r , use induction.
- Let $c : \mathbb{N}^r \rightarrow [k]$ be a k -colouring. Let $a_1 \in \mathbb{N}$, and consider all $r - 1$ sets of $\mathbb{N} \setminus \{a_1\}$, induce colouring $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \rightarrow [k]$ via $c'(F) = c(F \cup \{a_1\})$.
- By inductive hypothesis, there exists $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that c' is constant on it (taking value c_1).
- Now pick $a_2 \in A_1$ and induce a colouring $c' : (A_1 \setminus \{a_2\})^{(r-1)} \rightarrow [k]$ such that $c'(F) = c(F \cup \{a_2\})$. By inductive hypothesis, there exists $A_2 \subseteq A_1 \setminus \{a_2\}$ such that c' is constant on it (taking value c_2).
- Repeating this gives a_1, a_2, \dots and A_1, A_2, \dots such that $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$ and $c(F \cup \{a_i\}) = c_i$ for all $F \subseteq A_{i+1}$, for $|F| = r - 1$.
- One colour must appear infinitely many times: $c_{i_1} = c_{i_2} = \dots = c$.
- $M = \{a_{i_1}, a_{i_2}, \dots\}$ is a monochromatic set.

□

1.2. Applications of Ramsey's theorem

Example 1.10 In a totally ordered set, any sequence has monotonic subsequence.

Proof.

- Let (x_n) be a sequence, colour $\{i, j\}$ red if $x_i \leq x_j$ and blue otherwise.
- By Ramsey's theorem for pairs, $M = \{i_1 < i_2 < \dots\}$ is monochromatic. If M is red, then the subsequence x_{i_1}, x_{i_2}, \dots is increasing, and is strictly decreasing otherwise.
- We can insist that (x_{i_j}) is either concave or convex: 2-colour $\mathbb{N}^{(3)}$ by colouring $\{j < k < \ell\}$ **red** if $(i, x_{i_j}), (j, x_{i_k}), (k, x_{i_\ell})$ form a convex triple, and **blue** if they form a concave triple. Then by Ramsey's theorem for r -sets, there is an infinite convex or concave subsequence.

□

Theorem 1.11 (Finite Ramsey) Let $r, m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k -coloured, we can find a monochromatic set of size (at least) m .

Proof.

- Assume not, i.e. $\forall n \in \mathbb{N}$, there exists colouring $c_n : [n]^{(r)} \rightarrow [k]$ with no monochromatic m -sets.
- There are only finitely many (k) ways to k -colour $[r]^{(r)}$, so there are infinitely many of colourings c_r, c_{r+1}, \dots that agree on $[r]^{(r)}$: $c_i|_{[r]^{(r)}} = d_r$ for all i in some infinite set A_1 , where d_r is a k -colouring of $[r]^{(r)}$.
- Similarly, $[r+1]^{(r)}$ has only finitely many possible k -colourings. So there exists infinite $A_2 \subseteq A_1$ such that for all $i \in A_2$, $c_i|_{[r+1]^{(r)}} = d_{r+1}$, where d_{r+1} is a k -colouring of $[r+1]^{(r)}$.
- Continuing this process inductively, we obtain $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n$. There is no monochromatic m -set for any $d_n : [n]^{(r)} \rightarrow [k]$ (because $d_n = c_i|_{[n]^{(r)}}$ for some i).
- These d_n 's are nested: $d_\ell|_{[n]^{(r)}} = d_n$ for $\ell > n$.

- Finally, we colour $\mathbb{N}^{(r)}$ by the colouring $c : \mathbb{N}^{(r)} \rightarrow [k]$, $c(F) = d_n(F)$ where $n = \max(F)$ (or in fact $n \geq \max(F)$, which is well-defined by above). So c has no monochromatic m -set (since M was a monochromatic m -set, then taking $\ell = \max(M)$, d_ℓ has a monochromatic m -set), which contradicts Ramsey's Theorem for r -sets.

□

Remark 1.12

- This proof gives no bound on $n = n(k, m)$, there are other proofs that give a bound.
- It is a proof by compactness (essentially, we proved that $\{0, 1\}^{\mathbb{N}}$ with the product topology, i.e. the topology derived from the metric $d(f, g) = \frac{1}{\min\{n \in \mathbb{N} : f(n) \neq g(n)\}}$, is sequentially compact).

Remark 1.13 Now consider a colouring $c : \mathbb{N}^{(2)} \rightarrow X$ with X potentially infinite. This does not necessarily admit an infinite monochromatic set, as we could colour each edge a different colour. Such a colouring would be injective. We can't guarantee either the colouring being constant or injective though, as $c(ij) = i$ satisfies neither.

Theorem 1.14 (Canonical Ramsey) Let $c : \mathbb{N}^{(2)} \rightarrow X$ be a colouring with X an arbitrary set. Then there exists an infinite set $M \subseteq \mathbb{N}$ such that:

1. c is constant on $M^{(2)}$, or
2. c is injective on $M^{(2)}$, or
3. $c(ij) = c(kl)$ iff $i = k$ for all $i < j$ and $k < l$, $i, j, k, l \in M$, or
4. $c(ij) = c(kl)$ iff $j = l$ for all $i < j$ and $k < l$, $i, j, k, l \in M$.

Proof (Hints).

- First consider the 2-colouring c_1 of $\mathbb{N}^{(4)}$ where $ijkl$ is coloured SAME if $c(ij) = c(kl)$ and DIFF otherwise. Show that an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ (why does this exist?) coloured SAME leads to case 1.
- Assume M_1 is coloured DIFF, consider the 2-colouring of $M_1^{(4)}$, which colours $ijkl$ SAME if $c(il) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic $M_2 \subseteq M_1$ (why does this exist?) must be coloured DIFF by contradiction.
- Consider the 2-colouring of $M_2^{(4)}$ where $ijkl$ is coloured SAME if $c(ik) = c(jl)$ and DIFF otherwise. Show an infinite monochromatic set $M_3 \subseteq M_2$ (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour $M_3^{(3)}$ by: ijk is coloured SAME if $c(ij) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic set $M_4 \subseteq M_3$ (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour $M_4^{(3)}$ by the other two similar colourings to above, obtaining monochromatic $M_6 \subseteq M_5 \subseteq M_4$.
- Consider 4 combinations of these colourings on M_6 , show 3 lead to one of the cases in the theorem, and the other leads to contradiction.

□

Proof.

- 2-colour $\mathbb{N}^{(4)}$ by: $ijkl$ is red if $c(ij) = c(kl)$ and blue otherwise. By Ramsey's Theorem for 4-sets, there is an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ for this colouring.
- If M_1 is red, then c is constant on $M_1^{(2)}$: for all pairs $ij, i'j' \in M_1^{(2)}$, pick $m < n$ with $j, j' < m$, then $c(ij) = c(mn) = c(i'j')$.
- So assume M_1 is blue.
- Colour $M_1^{(4)}$ by giving $ijkl$ colour green if $c(il) = c(jk)$ and purple otherwise. By Ramsey's theorem for 4-sets, there exists an infinite monochromatic $M_2 \subseteq M_1$ for this colouring.
- Assume M_2 is coloured green: if $i < j < k < l < m < n \in M_2$, then $c(jk) = c(in) = c(lm)$ (consider $ijkn$ and $ilmn$): contradiction, since M_1 is blue.
- Hence M_2 is purple, i.e. for $ijkl \in M_2^{(4)}$, $c(il) \neq c(jk)$.
- Colour M_2 by: $ijkl$ is orange if $c(ik) = c(jl)$, and pink otherwise.
- By Ramsey's theorem for 4-sets, there exists infinite monochromatic $M_3 \subseteq M_2$ for this colouring.
- Assume M_3 is orange, then for $i < j < k < l < m < n \in M_3$, we have $c(jm) = c(ln)$ (consider $jlmn$) and $c(jm) = c(ik)$ (consider $ijkm$): contradiction, since $M_3 \subseteq M_1$.
- Hence M_3 is pink, i.e. for $ijkl$, $c(ik) \neq c(jl)$.
- Colour $M_3^{(3)}$ by: ijk is yellow if $c(ij) = c(jk)$ and grey otherwise. By Ramsey's theorem for 3-sets, there exists infinite monochromatic $M_4 \subseteq M_3$ for this colouring.
- Assume M_4 is yellow: then (considering $ijkl \in M_4^{(4)}$) $c(ij) = c(jk) = c(kl)$: contradiction, since $M_4 \subseteq M_1$.
- So for any $ijk \in M_4^{(3)}$, $c(ij) \neq c(jk)$.
- Finally, colour $M_4^{(3)}$ by: ijk is gold if $c(ij) = c(ik)$ and $c(ik) = c(jk)$, silver if $c(ij) = c(ik)$ and $c(ik) \neq c(jk)$, bronze if $c(ij) \neq c(ik)$ and $c(ik) = c(jk)$, and platinum if $c(ij) \neq c(ik)$ and $c(ik) \neq c(jk)$.
- By Ramsey's theorem for 3-sets, there exists monochromatic $M_5 \subseteq M_4$. M_5 cannot be gold, since then $c(ij) = c(jk)$: contradiction, since $M_5 \subseteq M_4$. If silver, then we have case 3 in the theorem. If bronze, then we have case 4 in the theorem. If platinum, then we have case 2 in the theorem.

□

Remark 1.15

- A more general result of the above theorem states: let $\mathbb{N}^{(r)}$ be arbitrarily coloured. Then we can find an infinite M and $I \subseteq [r]$ such that for all $x_1 \dots x_r \in M^{(r)}$ and $y_1 \dots y_r \in M^{(r)}$, $c(x_1 \dots x_r) = c(y_1 \dots y_r)$ iff $x_i = y_i$ for all $i \in I$.
- In canonical Ramsey, $I = \emptyset$ is case 1, $I = \{1, 2\}$ is case 2, $I = \{1\}$ is case 3 and $I = \{2\}$ is case 4.
- These 2^r colourings are called the **canonical colourings** of $\mathbb{N}^{(r)}$.

Exercise 1.16 Prove the general statement.

1.3. Van der Waerden's theorem

Remark 1.17 We want to show that for any 2-colouring of \mathbb{N} , we can find a monochromatic arithmetic progression of length m for any $m \in \mathbb{N}$. By compactness, this is equivalent to showing that for all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any 2-colouring of $[n]$, there exists a monochromatic arithmetic progression of length m . (If not, then for each $n \in \mathbb{N}$, there is a colouring $c_n : [n] \rightarrow \{1, 2\}$ with no monochromatic arithmetic progression of length m . Infinitely many of these colourings agree on $[1]$, infinitely many of those agreeing in $[1]$ agree on $[2]$, and so on - we obtain a 2-colouring of \mathbb{N} with no monochromatic arithmetic progression of length m).

We will prove a slightly stronger result: whenever \mathbb{N} is k -coloured, there exists a length m monochromatic arithmetic progression, i.e. for any $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k -coloured, we have a length m monochromatic progression.

Definition 1.18 Let A_1, \dots, A_k be length m arithmetic progressions: $A_i = \{a_i, a_i + d_i, \dots, a_i + (m-1)d_i\}$. A_1, \dots, A_k are **focussed** at f if $a_i + md_i = f$ for all i .

Example 1.19 $\{4, 8\}$ and $\{6, 9\}$ are focussed at 12.

Definition 1.20 If length m arithmetic progressions A_1, \dots, A_k are focused at f and are monochromatic, each with a different colour (for a given colouring), they are called **colour-focussed** at f .

Remark 1.21 We use the idea that if A_1, \dots, A_k are colour-focussed at f (for a k -colouring) and of length $m-1$, then some $A_i \cup \{f\}$ is a length m monochromatic arithmetic progression.

Theorem 1.22 Whenever \mathbb{N} is k -coloured, there exists a monochromatic arithmetic progression of length 3, i.e. for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any k -colouring of $[n]$ admits a length 3 monochromatic progression.

Proof (Hints).

- Prove by induction the claim: $\forall r \leq k, \exists n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there exists a monochromatic arithmetic progression of length 3, or r colour-focussed arithmetic progressions of length 2.
 - $r = 1$ case is straightforward.
 - Let claim be true for $r-1$ with witness n , let $N = 2n(k^{2n} + 1)$.
 - Partition N into blocks of equal size, show that two of these blocks must have the same colouring.
 - Using the inductive hypothesis, merge the $r-1$ colour-focussed arithmetic progressions from these two blocks into a new set of $r-1$ colour-focussed arithmetic progressions.
 - Find another length 2 monochromatic arithmetic progression, reason that this is of different colour.
- Reason that this claim implies the result.

□

Proof.

- We claim that for all $r \leq k$, there exists an $n \in \mathbb{N}$ such that if $[n]$ is k -coloured, then either:
 - There exists a monochromatic arithmetic progression of length 3.
 - There exist r colour-focussed arithmetic progressions of length 2.
- This claim implies the result by the above remark.
- We prove the claim by induction on r :
 - $r = 1$: take $n = k + 1$, then by pigeonhole, some two elements of $[n]$ have the same colour, so form a length two arithmetic progression.
 - Assume true for $r - 1$ with witness n . We claim that $N = 2n(k^{2n} + 1)$ works for r .
 - Let $c : [2n(k^{2n} + 1)] \rightarrow [k]$ be a colouring. We partition $[N]$ into $k^{2n} + 1$ blocks of size $2n$: $B_i = \{2n(i - 1) + 1, \dots, 2ni\}$ for $i = 1, \dots, k^{2n} + 1$.
 - Assume there is no length 3 monochromatic progression for c . By inductive hypothesis, each block B_i has $r - 1$ colour-focussed arithmetic progressions of length 2.
 - Since $|B_i| = 2n$, each block also contains their focus. For a set M with $|M| = 2n$, there are k^{2n} ways to k -colour M . So by pigeonhole, there are blocks B_s and B_{s+t} that have the same colouring.
 - Let $\{a_i, a_i + d_i\}$ be the $r - 1$ arithmetic progressions in B_s colour-focussed at f , then $\{a_i + 2nt, a_i + d_i + 2nt\}$ is the corresponding set of arithmetic progressions in B_{s+t} , each colour-focussed at $f + 2nt$.
 - Now $\{a_i, a_i + d_i + 2nt\}$, $i \in [r - 1]$, are $r - 1$ arithmetic progresions colour-focused at $f + 4nt$. Also, $\{f, f + 2nt\}$ is monochromatic of a different colour to the $r - 1$ colours used (since there is no length 3 monochromatic progression for c). Hence, there are r arithmetic progressions of length 2 colour-focussed at $f + 4nt$.

□

Remark 1.23 The idea of looking at all possible colourings of a set is called a **product argument**.

Definition 1.24 The **Van der Waerden** number $W(k, m)$ is the smallest $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there exists a monochromatic arithmetic progression in $[n]$ of length m .

Remark 1.25 The above theorem gives a **tower-type** upper bound $W(k, 3) \leq k^{k^{(\cdot)^{k^{4k}}}}$.

Theorem 1.26 (Van der Waerden's Theorem) For all $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there is a length m monochromatic arithmetic progression.

Proof (Hints).

- Use induction on m .

- Given induction hypothesis on $m - 1$, prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, we have either a monochromatic length m arithmetic progression, or r colour-focussed arithmetic progressions of length $m - 1$. Reason that this claim implies the result.
- Use induction on r . Give an explicit n for $r = 1$.
- Let n be the witness for $r - 1$, let $N = W(k^{2n}, m - 1) \cdot 2n$. Assume a k -colouring of $[N]$, $c : [N] \rightarrow [k]$, has no arithmetic progressions of length m .
- Partition $[N]$ into the obvious choice of $W(k^{2n}, m - 1)$ blocks B_i , each of length $2n$.
- Colour the indices $1 \leq i \leq W(k^{2n}, m - 1)$ of the blocks by

$$c'(i) = (c(2n(i - 1) + 1), c(2n(i - 1) + 2), \dots, c(2ni))$$

- Reason that we can find monochromatic arithmetic progression $s, s + t, \dots, s + (m - 2)t$ of length $m - 1$ (w.r.t c'), and that this corresponds to sequence of blocks $B_s, B_{s+t}, \dots, B_{s+(m-2)t}$, each identically coloured.
- Reason that B_s contains $r - 1$ colour-focussed length $m - 1$ arithmetic progressions A_i together with their focus f .
- Let A'_i be the same arithmetic progression but with common difference $2nt$ larger than that of A_i . Show the A'_i are colour-focussed at some focus in terms of f .
- Find another length $m - 1$ arithmetic progression, show this must be monochromatic and of different colour to all A'_i . Show it also has same focus as all A'_i .

□

Proof.

- By induction on m . $m = 1$ is trivial, $m = 2$ is by pigeonhole principle. $m = 3$ is the statement of the previous theorem.
- Assume true for $m - 1$ and all $k \in \mathbb{N}$.
- For fixed k , we prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, either:
 - There is a monochromatic arithmetic progression of length m , or
 - There are r colour-focussed arithmetic progressions of length $m - 1$.
- We will then be done (by considering the focus).
- To prove the claim, we use induction on r .
- $r = 1$ is the claim of the first inductive hypothesis: take $n = W(k, m - 1)$.
- Assume the claim holds for $r - 1$ with witness n , and assume there is no monochromatic arithmetic progression of length m . We will show that $N = W(k^{2n}, m - 1)2n$ is sufficient for r .
- Partition $[N]$ into $W(k^{2n}, m - 1)$ blocks of length $2n$: $B_i = \{2n(i - 1) + 1, \dots, 2ni\}$ for $i = 1, \dots, W(k^{2n}, m - 1)$.
- Each block has k^{2n} possible colourings. Colour the blocks as

$$c'(i) = (c(2n(i - 1) + 1), c(2n(i - 1) + 2), \dots, c(2ni))$$

By definition of W , there exists a monochromatic arithmetic progression of length $m - 1$ (w.r.t. to c'): $\{\alpha, \alpha + t, \dots, \alpha + (m - 2)t\}$. The respective blocks $B_\alpha, \dots, B_{\alpha + (m-2)t}$ are identically coloured.

- B_α has length $2n$, so by induction B_α contains $r - 1$ colour-focussed arithmetic progressions of length $m - 1$, together with their focus (as length of block is $2n$).
- Let A_1, \dots, A_{r-1} , $A_i = \{a_i, a_i + d_i, \dots, a_i + (m - 2)d_i\}$, be colour-focussed at f .
- Let $A'_i = \{a_i, a_i + (d_i + 2nt), \dots, a_i + (m - 2)(d_i + 2nt)\}$ for $i = 1, \dots, r - 1$. The A'_i are monochromatic as the blocks are identically coloured and the A_i are monochromatic. Also, A_i and A'_i have the same colouring, and the A_i are colour-focussed, hence the A'_i have pairwise distinct colours.
- The A_i are focussed at f and the colour of f is different than the colour of all A_i . $f = a_i + (m - 1)d_i$ for all i .
- Now $\{f, f + 2nt, f + 4nt, \dots, f + 2n(m - 2)t\}$ is an arithmetic progression of length $m - 1$, is monochromatic and of a different colour to all the A'_i .
- It is enough to show that $a_i + (m - 1)(d_i + 2nt) = f + 2n(m - 1)t$ for all i , but this is equivalent to $a_i + (m - 1)d_i = f$, which is true as all A_i were focussed at f .

□

Corollary 1.27 For any k -colouring of \mathbb{N} , there exists a colour class containing arbitrarily long arithmetic progressions.

Remark 1.28 We can't guarantee infinitely long arithmetic progressions, e.g.

- 2-colour \mathbb{N} by 1 red, 2, 3 blue, 4, 5, 6 red, etc.
- The set of infinite arithmetic progressions in \mathbb{N} is countable (since described by two integers: the start term and step). Enumerate them by $(A_k)_{k \in \mathbb{N}}$. Pick $x_1 < y_1 \in A_1$, colour x_1 red and y_1 blue. Then pick $x_2, y_2 \in A_2$ with $y_1 < x_2 < y_2$, colour x_2 red, y_2 blue. Continue inductively.

Theorem 1.29 (Strengthened Van der Waerden) Let $m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that for any k -colouring of $[n]$, there exists a monochromatic length m arithmetic progression whose common difference is the same colour (i.e. there exists $a, a + d, \dots, a + (m - 1)d$ all of the same colour).

Proof (Hints).

- Use induction on k .
- If n is the witness for $k - 1$ colours, show that $N = W(k, n(m - 1) + 1)$ is a witness for k colours, by considering n different multiples of the step of a suitable arithmetic progression.

□

Proof.

- Fix $m \in \mathbb{N}$. We use induction on k . $k = 1$ case is trivial.
- Let n be witness for $k - 1$ colours.
- We will show that $N = W(k, n(m - 1) + 1)$ is suitable for k colours.
- If $[N]$ is k -coloured, there exists a monochromatic (say red) arithmetic progression of length $n(m - 1) + 1$: $a, a + d, \dots, a + n(m - 1)d$.

- If rd is red for any $1 \leq r \leq n$, then we are done (consider $a, a + rd, \dots, a + (m - 1)rd$).
- If not, then $\{d, 2d, \dots, nd\}$ is $k - 1$ -coloured, which induces a $k - 1$ colouring on $[n]$. Therefore, there exists a monochromatic arithmetic progression $b, b + s, \dots, b + (m - 1)s$ (with s the same colour) by induction, which translates to $db, db + ds, \dots, db + d(m - 1)s$ and ds being monochromatic.

□

Remark 1.30 The case $m = 2$ of strengthened Van der Waerden is **Schur's theorem**: for any k -colouring of \mathbb{N} , there are monochromatic x, y, z such that $x + y = z$. This can be proved directly from Ramsey's theorem for pairs: let $c : \mathbb{N} \rightarrow [k]$ be a k -colouring, then induce $c' : \mathbb{N}^{(2)} \rightarrow [k]$ by $c'(ij) = c(j - i)$. By Ramsey, there exist $i < j < k$ such that $c'(ij) = c'(ik) = c'(jk)$, i.e. $c(j - i) = c(k - i) = c(k - j)$. So take $x = j - i$, $z = k - i$, $y = k - j$.

1.4. The Hales-Jewett theorem

Definition 1.31 Let X be finite set. We say X^n consists of **words of length n on alphabet X** .

Definition 1.32 Let X be finite. A **(combinatorial) line** in X^n is a set $L \subseteq X^n$ of the form

$$L = \{(x_1, \dots, x_n) \in X^n : \forall i \notin I, x_i = a_i \text{ and } \forall i, j \in I, x_i = x_j\}$$

for some non-empty set $I \subseteq [n]$ and $a_i \in X$ (for each $i \notin I$). I is the set of **active coordinates** for L .

Note that a combinatorial line is invariant under permutations of X .

Example 1.33 Let $X = [3]$. Some lines in X^2 are:

- $I = \{1\}$: $\{(1, 1), (2, 1), (3, 1)\}$ (with $a_2 = 1$), $\{(1, 2), (2, 2), (3, 2)\}$ (with $a_2 = 2$), $\{(1, 3), (2, 3), (3, 3)\}$ (with $a_2 = 3$).
- $I = \{2\}$: $\{(1, 1), (1, 2), (1, 3)\}$ (with $a_1 = 1$), $\{(2, 1), (2, 2), (2, 3)\}$ (with $a_1 = 2$), $\{(3, 1), (3, 2), (3, 3)\}$ (with $a_1 = 3$).
- $I = \{1, 2\}$: $\{(1, 1), (2, 2), (3, 3)\}$.

Note that $\{(1, 3), (2, 2), (3, 1)\}$ is **not** a combinatorial line.

Example 1.34 Some sets of lines in $[3]^3$ are:

- $I = \{1\}$: $\{(1, 2, 3), (2, 2, 3), (3, 2, 3)\}$ (with $a_2 = 2, a_3 = 3$).
- $I = \{1, 3\}$: $\{(1, 3, 1), (2, 3, 2), (3, 3, 3)\}$ (with $a_2 = 3$).

Definition 1.35 In a line L , write L^- and L^+ for the smallest and largest points in L (with respect to the ordering on $[m]^n$ where $x \leq y$ if $x_i \leq y_i$ for all i).

Definition 1.36 Lines L_1, \dots, L_k are **focussed** at f if $L_i^+ = f$ for all $i \in [k]$. They are **colour-focussed** if they are focussed and $L_i \setminus \{L_i^+\}$ is monochromatic for all $i \in [k]$, with each $L_i \setminus \{L_i^+\}$ a different colour.

Theorem 1.37 (Hales-Jewett) Let $m, k \in \mathbb{N}$ (we use alphabet $X = [m]$), then there exists $n \in \mathbb{N}$ such that for any k -colouring of $[m]^n$, there exists a monochromatic combinatorial line.

Notation 1.38 Denote the smallest such n by $\text{HJ}(m, k)$.

Proof (Hints).

- Induction on m . Prove by induction the claim that for all $1 \leq r \leq k$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[m]^n$, we have either a monochromatic line, or r colour-focussed lines (reason that this claim implies the result).
- State why claim holds for $r = 1$.
- Let n be witness for $r - 1$, $n' = \text{HJ}(m - 1, k^{m^n})$. Want to show that $n + n'$ is witness for r .
- Write $[m]^{n+n'} = [m]^n \times [m]^{n'}$.
- For a colouring $c : [m]^{n+n'} \rightarrow [k]$, induce a suitable colouring $c' : [m]^{n'} \rightarrow [k]^{m^n}$ and consider what the definition of n' implies. Use this to induce a colouring $c'' : [m]^n \rightarrow [k]$.
- Using the inductive hypothesis and the previous point, construct $r - 1$ lines in $[m]^{n+n'}$ which are colour-focussed. Find another line in $[m]^{n+n'}$ (which should have first n coordinates constant) of different colour which has the same focus point.

□

Proof. By induction on m . The case $m = 1$ is trivial as $|[m]^n| = 1$. Assume that $\text{HJ}(m - 1, k')$ exists for all $k' \in \mathbb{N}$. We claim that for all $1 \leq r \leq k$, there exists $n \in \mathbb{N}$ such that for any k -colouring of $[m]^n$, we have either:

- a monochromatic line, or
- r colour-focussed lines.

We can then take $r = k$ and consider the focus.

We prove the claim by induction on r . For $r = 1$, $n = \text{HJ}(m - 1, k)$ suffices. Let n be a witness for $r - 1$. Let $n' = \text{HJ}(m - 1, k^{m^n})$. We will show $N = n + n'$ is a witness for r . Let $c : [m]^N \rightarrow [k]$ be a k -colouring with no monochromatic lines. Writing $[m]^N = [m]^n \times [m]^{n'}$, colour $[m]^{n'}$ by $c' : [m]^{n'} \rightarrow [k]^{m^n}$, $c'(b) = (c(a_1, b), \dots, c(a_{m^n}, b))$ (where $[m]^n = \{a_1, \dots, a_{m^n}\}$). By the inductive hypothesis, there exists a line L in $[m]^{n'}$ with active coordinates I such that

$$\forall a \in [m]^n, \forall b, b' \in L \setminus \{L^+\}, \quad c(a, b) = c(a, b').$$

But now this induces a (well-defined) colouring $c'' : [m]^n \rightarrow [k]$, $c''(a) = c(a, b)$ for any $b \in L \setminus \{L^+\}$. By definition of n , there exist $r - 1$ lines L_1, \dots, L_{r-1} colour-focussed (w.r.t c'') at f , with active coordinates I_1, \dots, I_{r-1} .

Finally, consider the $r - 1$ lines L'_i , $1 \leq i \leq r - 1$ in $[m]^N$ that start at (L_i^-, L^-) with active coordinates $I_i \cup I$, and the line L' in $[m]^N$ that starts at (f, L^-) with active coordinates I . By the construction of c'' , the colour of each point in L'_i is determined by the first n coordinates which form a point lying in L_i . Hence, since the L_i are

colour-focussed, the L'_i are colour-focussed. As for L' , the first n coordinates are constant (always equal to f), and so again by the construction of c'' , the colour of each point in L' is equal to $c''(f)$, which is a different colour to each colour of the L'_i . Hence all $L'_1, \dots, L'_{r-1}, L'$ colour-focussed at (f, L^+) , so we are done. \square

Corollary 1.39 Hales-Jewett implies Van der Waerden's theorem.

Proof (Hints). For a colouring $c : \mathbb{N} \rightarrow [k]$, consider the induced colouring $c'(x_1, \dots, x_n) = c(x_1 + \dots + x_n)$ of $[m]^n$. \square

Proof. Let c be a k -colouring of \mathbb{N} . For sufficiently large n (i.e. $n \geq \text{HJ}(m, k)$), induce a k -colouring c' of $[m]^n$ by $c'(x_1, \dots, x_n) = c(x_1 + \dots + x_n)$. By Hales-Jewett, a monochromatic (with respect to c') combinatorial line L exists. This gives a monochromatic (with respect to c) length m arithmetic progression in \mathbb{N} . The step is equal to the number of active coordinates. The first term in the arithmetic progression corresponds to the point in L with all active coordinates equal to 1, the last term corresponds to the point in L with all active coordinates equal to m . \square

Exercise 1.40 Show that the m -in-a-row noughts and crosses game cannot be a draw in sufficiently high dimensions, and that the first player can always win.

Definition 1.41 A **d -dimensional subspace** (or **d -point parameter set**) $S \subseteq X^n$ is a set such that there exist pairwise disjoint $I_1, \dots, I_d \subseteq [n]$ and $a_i \in X$ for all $i \in [n] - (I_1 \cup \dots \cup I_d)$, such that

$$S = \{x \in X^n : x_i = a_i \quad \forall i \in [n] - (I_1 \cup \dots \cup I_d), \\ \text{and } x_i = x_j \quad \forall i, j \in I_k \text{ for some } k \in [d]\}.$$

Example 1.42 Two 2-dimensional subspaces in X^3 are $\{(x, y, 2) : x, y \in X\}$ ($I_1 = \{1\}, I_2 = \{2\}$) and $\{(x, x, y) : x, y \in X\}$ ($I_1 = \{1, 2\}, I_2 = \{3\}$).

Theorem 1.43 (Extended Hales-Jewett) For all $m, k, d \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any colouring of $[m]^n$, there exists a monochromatic d -dimensional subspace.

Proof (Hints). Use Hales-Jewett on m^d and k . \square

Proof. We can view $X^{dn'}$ as $(X^d)^{n'}$. A line in $(X^d)^{n'}$ (on alphabet $Y = X^d$) corresponds to a d -dimensional subspace in $X^{dn'}$ (on alphabet X). (Each inactive coordinate in the line corresponds to d adjacent inactive coordinates in the subspace, and each active coordinate in the line corresponds to d adjacent active coordinates in the subspace). Hence, we can take $n = d \cdot \text{HJ}(m^d, k)$. \square

Definition 1.44 Let $S \subseteq \mathbb{N}^d$ be finite. A **homothetic copy** of S is a set of the form $a + \lambda S$ where $a \in \mathbb{N}^d$ and $\lambda \in \mathbb{N}$ ($\lambda \neq 0$).

Theorem 1.45 (Gallai) Let $S \subseteq \mathbb{N}^d$ be finite. For every k -colouring of \mathbb{N}^d , there exists a monochromatic homothetic copy of S .

Proof (Hints). Let $S = \{S_1, \dots, S_m\}$, consider colouring $c' : [m]^n \rightarrow [k]$ (for suitable n) given by $c'(x_1, \dots, x_n) = c(S_{x_1}, \dots, S_{x_m})$. \square

Proof. Let $S = \{S_1, \dots, S_m\}$. Let $c : \mathbb{N}^d \rightarrow [k]$ be a k -colouring. For n large enough (i.e. $n \geq \text{HJ}(m, k)$), colour $[m]^n$ by $c'(x_1, \dots, x_n) = c(S_{x_1} + \dots + S_{x_n})$. By Hales-Jewett, there exists a monochromatic line (with respect to c') in $[m]^n$ with active coordinates I . So $c\left(\sum_{i \notin I} S_i + |I|S_j\right)$ is the same colour for all $j \in [m]$. So we are done, as $\sum_{i \notin I} S_i + |I|S$ is a homothetic copy of S . \square

Remark 1.46

- Gallai's theorem can also be proven with a focussing + product colouring argument.
- For $S = \{(x, y) \in \mathbb{N}^2 : x, y \in \{1, 2\}\}$, Gallai's theorem proves the existence of a monochromatic square whereas extended Hales-Jewett only guarantees a monochromatic rectangle.

2. Partition regular systems

2.1. Rado's theorem

Strengthened Van der Waerden says that the system $x_1 + x_2 = y_1, x_1 + 2x_2 = y_2, \dots, x_1 + mx_2 = y_m$ has a monochromatic solution in $x_1, x_2, y_1, \dots, y_m$. We want to find when a general system of equations is partition regular.

Definition 2.1 Let $A \in \mathbb{Q}^{m \times n}$ be a $m \times n$ matrix. A is **partition regular (PR)** if for any finite colouring of \mathbb{N} , there exists a monochromatic $\mathbf{x} \in \mathbb{N}^n$ such that $A\mathbf{x} = \mathbf{0}$.

Example 2.2

- Schur's theorem says that $x + y = z$ has a monochromatic solution for any finite colouring of \mathbb{N} , and so that $(1, 1, -1)$ is PR.
- Strengthened Van der Waerden states that

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

is PR.

- $(a, b, -(a + b))$ is PR for any a, b (a monochromatic solution is $x = y = z$).
- $(2, -1)$ is not PR: colour \mathbb{N} by n is **red** if $\max\{m \in \mathbb{N} : 2^m \mid n\}$ is even, and **blue** otherwise. Then if $2x = y$, x and y must have different colours.

Definition 2.3 A rational matrix A with columns $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{Q}^m$ has the **column property (CP)** if there exists a partition $B_1 \sqcup \dots \sqcup B_r$ of $[n]$ such that:

1. $\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0}$.
2. For all $s \in \{2, \dots, r\}$, $\sum_{i \in B_s} \mathbf{c}_i \in \text{span}\{\mathbf{c}_j : j \in B_1 \sqcup \dots \sqcup B_{s-1}\}$ (note we can take the linear span over \mathbb{R} or over \mathbb{Q} here, as if a rational vector is a real linear combination of rational vectors, then it is also a rational linear combination of them).

Example 2.4

- $(1, 1, -1)$ has CP, with $B_1 = \{1, 3\}$, $B_2 = \{2\}$.
- The matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

from Strengthened Van der Waerden has CP, with $B_1 = \{1, 3, \dots, n\}$ and $B_2 = \{2\}$.

- $(3, 4, -7)$ has CP with $B_1 = \{1, 2, 3\}$.
- $(\lambda, -1)$ has CP iff $\lambda = 1$.
- $(3, 4, -6)$ doesn't have CP.

Example 2.5

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & a \\ 4 & -4 & b \end{bmatrix}$$

has CP iff $(a, b) = (6, 12)$.

Remark 2.6 $\mathbf{x} = (a_1, \dots, a_n)$ is PR iff $\lambda \mathbf{x}$ is PR (for any $\lambda \in \mathbb{Q}^\times$), so we can assume that each $a_i \in \mathbb{Z}$. Also, \mathbf{x} has CP iff there exists $\emptyset \neq I \subseteq [n]$ such that $\sum_{i \in I} a_i = 0$.

We may also assume WLOG each $a_i \neq 0$. We will first show that if \mathbf{x} is PR, then it has CP. Even in the $1 \times n$ matrix case of Rado's theorem, neither direction is easy.

Notation 2.7 For p prime and $x = (a_k \dots a_0)_p \in \mathbb{N}$, write $e(x)$ for the rightmost non-zero digit in the base- p expansion of x , i.e. $e(x) = a_{t(x)}$, where $t(x) = \min\{i : a_i \neq 0\}$.

Proposition 2.8 Let $a_1, \dots, a_n \in \mathbb{Q}^*$. If (a_1, \dots, a_n) is PR, then it has CP.

Proof (Hints). For p large enough (determine later a bound for p), colour \mathbb{N} by giving x colour $e(x)$, and consider $\min\{t(x_1), \dots, t(x_n)\}$. \square

Proof. Let p be a large prime ($p > \sum_{i=1}^n |a_i|$). Define a $(p-1)$ -colouring of \mathbb{N} giving x colour $e(x)$. By assumption, there are x_1, \dots, x_n of the same colour d such that $\sum_{i=1}^n a_i x_i = 0$. Let $t = \min\{t(x_1), \dots, t(x_n)\}$, and let $I = \{i \in [n] : t(x_i) = t\}$ (note I is non-empty). So when summing $\sum_{i=1}^n a_i x_i = 0$ and considering the last digit in the base p expansion, we have $\sum_{i=1}^n a_i x_i = 0 \pmod{p^{t+1}}$ and so obtain $\sum_{i \in I} a_i d = 0 \pmod{p}$, so $\sum_{i \in I} a_i = 0$ (since p is prime and was chosen large enough). \square

Remark 2.9 There is no other known proof of this proposition.

Lemma 2.10 Let $\lambda \in \mathbb{Q}$. Then $(1, \lambda, -1)$ is partition regular, i.e. for any finite colouring of \mathbb{N} , there exists monochromatic $(x, y, z) \in \mathbb{N}^3$ such that $x + \lambda y = z$.

Proof (Hints).

- Reason that we can assume $\lambda > 0$. Write $\lambda = r/s$, $r, s \in \mathbb{N}$.
- Use induction on number of colours k : given n such that any $(k-1)$ -colouring of $[n]$ admits monochromatic solution, show that $N = W(k, nr+1)ns$ works for k colours, by considering the definition of W and isd for each $i \in [n]$.

□

Proof. The case $\lambda = 0$ is trivial, and if $\lambda < 0$, we may rewrite the equation as $z - \lambda y = x$, so we may assume that $\lambda > 0$, so let $\lambda = \frac{r}{s}$ for $r, s \in \mathbb{N}$. In fact, we show that for any k -colouring of $[n]$ (for some n depending on k), there is a monochromatic solution.

We seek a monochromatic solution to $x + \frac{r}{s}y = z$ for some finite colouring $c : \mathbb{N} \rightarrow [k]$. We use induction on the number of colours k . For $k = 1$, $n = \max\{s, r + 1\}$ is sufficient, with monochromatic solution $(1, s, r + 1)$. Assume n is a witness for $k - 1$ colours. We will show $N = nsW(k, nr + 1)$ is suitable for k colours. By definition of W , given a k -colouring of $[N]$, there is a monochromatic AP inside $[W(k, nr + 1)] \subseteq [N]$ of length $nr + 1$: $a, a + d, \dots, a + nrd$, coloured red.

Consider isd for each $i \in [n]$. Note that $isd \leq nsW(k, nr + 1)$ so each isd does indeed have a colour. If some isd is also red, then $(a, isd, a + ird)$ is a monochromatic solution. If no isd is red, then $\{sd, \dots, nsd\}$ is $(k - 1)$ -coloured, so by the inductive hypothesis, there exists $i, j, k \in [n]$ such that $\{isd, jsd, ksd\}$ is monochromatic and $isd + \lambda jsd = ksd$, so (isd, jsd, ksd) is a monochromatic solution. □

Remark 2.11

- Note the similarity to the proof of Strengthened Van der Waerden.
- The case $\lambda = 1$ is Schur's theorem, which can be proven directly by Ramsey's theorem; however, there is no known proof using Ramsey's theorem for general $\lambda \in \mathbb{Q}$.

Theorem 2.12 (Rado's Theorem for Single Equations) Let $a_1, \dots, a_n \in \mathbb{Q} \setminus \{0\}$. (a_1, \dots, a_n) is PR iff it has CP.

Proof (Hints). For \Leftarrow : for the obvious choice of $I \subseteq [n]$, fix $i_0 \in I$, and define $\mathbf{x} \in \mathbb{N}^n$ componentwise:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}.$$

Show that \mathbf{x} is a solution to $\sum_{i=1}^n a_i x_i = 0$. □

Proof. \Rightarrow is by [Proposition 2.8](#). For \Leftarrow : we have that $\sum_{i \in I} a_i = 0$ for some $\emptyset \neq I \subseteq [n]$. Given a colouring $c : \mathbb{N} \rightarrow [k]$, we need to show that there are monochromatic x_1, \dots, x_n such that $\sum_{i=1}^n a_i x_i = 0$.

Fix $i_0 \in I$. We construct the following vector $\mathbf{x} \in \mathbb{N}^n$ by defining its components:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}$$

for some fixed suitable x, y, z . We need x, y, z to be monochromatic and

$$\begin{aligned}
& a_{i_0}x + \sum_{i \notin I} a_i y + \sum_{i \in I \setminus \{i_0\}} a_i z = 0 \\
& \iff a_{i_0}x - za_{i_0} + \sum_{i \notin I} a_i y = 0 \\
& \iff x + \frac{\sum_{i \notin I} a_i}{a_{i_0}} y - z = 0
\end{aligned}$$

and this holds, since x, y, z exist by the above lemma. \square

Conjecture 2.13 (Rado's Boundedness Conjecture) Let A be an $m \times n$ matrix that is not PR (so there exists a “bad” colouring, i.e. a k -colouring with no monochromatic solution to $A\mathbf{x} = \mathbf{0}$ for some $k \in \mathbb{N}$). Is k bounded (for given m, n)?

This is known for 1×3 matrices: 24 colours suffice.

Proposition 2.14 Let $A \in \mathbb{Q}^{m \times n}$. If A is PR, then it has CP.

Proof (Hints).

- Let $\mathbf{x} \in \mathbb{N}^n$ be the monochromatic solution to $A\mathbf{x} = \mathbf{0}$. For fixed prime p , partition $[n]$ into B_1, \dots, B_r by grouping $i, j \in [n]$ by $t(x_i), t(x_j)$ (and preserving the ordering).
- Reason that the same partition exists for infinitely many p .
- Considering $\sum_{i=1}^n x_i \mathbf{c}_i = \mathbf{0} \pmod{p}$ for infinitely many p , show that $\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0}$, and

$$p^t \sum_{i \in B_k} \mathbf{c}_i + \sum_{i \in B_1, \dots, B_{k-1}} x_i d^{-1} \mathbf{c}_i \equiv \mathbf{0} \pmod{p^{t+1}}.$$

- By taking the dot product with $\mathbf{u} \in \mathbb{N}^m$ for appropriate u , show by contradiction that $\sum_{i \in B_k} \mathbf{c}_i \in \text{span}\{\mathbf{c}_i : i \in B_1, \dots, B_{k-1}\}$.

\square

Proof. Let $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{Q}^m$ be the columns of A . For fixed prime p , colour \mathbb{N} as before by $c(x) = e(x)$. By assumption, there exists a monochromatic $\mathbf{x} \in \mathbb{N}^n$ such that $\sum_{i=1}^n x_i \mathbf{c}_i = \mathbf{0}$. We partition the columns (by partitioning $[n] = B_1 \sqcup \dots \sqcup B_r$) as follows:

- $i, j \in B_k$ iff $t(x_i) = t(x_j)$.
- $i \in B_k, j \in B_\ell$ for $k < \ell$ iff $t(x_i) < t(x_j)$.

We do this for infinitely many primes p . Since there are finitely many partitions of $[n]$, for infinitely many p , we will have the same blocks B_1, \dots, B_r .

Consider $\sum_{i=1}^n x_i \mathbf{c}_i = \mathbf{0}$ performed in base p . Each $i \in [n]$ has the same colour $d = e(x_i) \in [1, p-1]$. So $\sum_{i \in B_1} d \mathbf{c}_i = \mathbf{0} \pmod{p}$ (by collecting the rightmost terms in base p), hence $\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0} \pmod{p}$. But this holds for infinitely many p , hence

$$\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0}.$$

Now $\sum_{i \in B_k} p^t d c_i + \sum_{i \in B_1, \dots, B_{k-1}} x_i c_i = \mathbf{0} \pmod{p^{t+1}}$ for some t . So

$$p^t \sum_{i \in B_k} c_i + \sum_{i \in B_1, \dots, B_{k-1}} x_i d^{-1} c_i \equiv \mathbf{0} \pmod{p^{t+1}}.$$

We claim that $\sum_{i \in B_k} c_i \in \text{span}\{c_i : i \in B_1, \dots, B_{k-1}\}$. Suppose not, then there exists $\mathbf{u} \in \mathbb{N}^m$ such that $\mathbf{u} \cdot c_i = 0$ for all $i \in B_1, \dots, B_{k-1}$, but $\mathbf{u} \cdot \left(\sum_{i \in B_k} c_i\right) \neq 0$. Then dotting with \mathbf{u} , we obtain $p^t \mathbf{u} \cdot \left(\sum_{i \in B_k} c_i\right) \equiv 0 \pmod{p^{t+1}}$, so $\mathbf{u} \cdot \sum_{i \in B_k} c_i \equiv 0 \pmod{p}$. But this holds for infinitely many p , so $\mathbf{u} \cdot \sum_{i \in B_k} c_i = 0$: contradiction. \square

Definition 2.15 For $m, p, c \in \mathbb{N}$, an (m, p, c) -set $S \subseteq \mathbb{N}$ with generators $x_1, \dots, x_m \in \mathbb{N}$ is of the form

$$S = \left\{ \sum_{i=1}^m \lambda_i x_i : \exists j \in [m] : \lambda_j = c, \lambda_i = 0 \ \forall i < j, \text{ and } \lambda_k \in [-p, p] \ \forall k > j \right\}$$

where $[-p, p] = \{-p, -(p-1), \dots, p\}$. So S consists of

$$\begin{aligned} cx_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots + \lambda_m x_m, & \quad \lambda_i \in [-p, p], \\ cx_2 + \lambda_3 x_3 + \dots + \lambda_m x_m, & \quad \lambda_i \in [-p, p], \\ & \quad \vdots \\ cx_m. & \end{aligned}$$

These are the **rows** of S . We can think of S as a “progression of progressions”.

Example 2.16

- A $(2, p, 1)$ -set with generators x_1, x_2 is of the form $\{x_1 - px_2, x_1 - (p-1)x_2, \dots, x_1 + px_2, x_2\}$, so is an AP of length $2p+1$ together with its step.
- A $(2, p, 3)$ -set with generators x_1, x_2 is of the form $\{3x_1 - px_2, 3x_1 - (p-1)x_2, \dots, 3x_1 + px_2, 3x_2\}$, so is an AP of length $2p+1$, whose middle term is divisible by 3, together with three times its step.

Theorem 2.17 Let $m, p, c \in \mathbb{N}$. For any finite colouring of \mathbb{N} , there exists a monochromatic (m, p, c) -set.

Proof (Hints).

- Reason that an (m', p, c) -set contains an (m, p, c) -set for $m' \geq m$. With $M = k(m-1) + 1$, reason that if we can find an (M, p, c) -set with each row monochromatic, then we can find an monochromatic (m, p, c) -set.
- Let $A_1 = \{c, 2c, \dots, \lfloor n/c \rfloor c\}$, reason that A_1 contains a set of the form $R_1 = \{cx_1 - n_1 d_1, cx_1 - (n_1 - 1)d_1, \dots, cx_1 + n_1 d_1\}$ for some large n_1 .
- Let $B_1 = \left\{d_1, 2d_1, \dots, \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1\right\}$. We have $cx_1 + \lambda_1 b_1 + \dots + \lambda_{M-1} b_{M-1} \in R_1$, explain why these are monochromatic.
- Inside B_1 , define

$$A_2 = \left\{cd_1, 2cd_1, \dots, \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1\right\}.$$

and apply the argument as before, where the divisor in the $\lfloor \cdot \rfloor$ expression in the new B_2 is $(M-2)p$.

- Argue that after a certain number of steps, we have formed an (M, p, c) -set with each row monochromatic.

□

Proof. Let $c : \mathbb{N} \rightarrow [k]$ be the colouring of \mathbb{N} with k colours. Note that an (m', p, c) -set with $m' \geq m$ contains an (m, p, c) -set (by taking any m rows, and setting some suitable λ_i to 0). Let $M = k(m-1) + 1$. It is enough to find a (M, p, c) -set such that each row is monochromatic.

Let n be large (large enough to apply the argument that follows). Let $A_1 = \{c, 2c, \dots, \lfloor n/c \rfloor c\}$. By Van der Waerden, A_1 contains a monochromatic AP R_1 of length $2n_1 + 1$ where n_1 is large enough:

$$R_1 = \{cx_1 - n_1d_1, cx_1 - (n_1 - 1)d_1, \dots, cx_1 + n_1d_1\}.$$

has colour k_1 . Now we restrict our attention to

$$B_1 = \left\{ d_1, 2d_1, \dots, \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1 \right\}.$$

Observe that

$$cx_1 + \lambda_1 b_1 + \dots + \lambda_{M-1} b_{M-1} \in R_1$$

for all $\lambda_i \in [-p, p]$ and $b_i \in B_1$, so all these sums have colour k_1 . Inside B_1 , look at

$$A_2 = \left\{ cd_1, 2cd_1, \dots, \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1 \right\}.$$

By Van der Waerden, A_2 contains a monochromatic AP R_2 of length $2n_2 + 1$ with colour k_2 :

$$R_2 = \{cx_2 - n_2d_2, cx_2 - (n_2 - 1)d_2, \dots, cx_2 + n_2d_2\}.$$

Note that $x_2 \subseteq B_1$. Now we restrict our attention to

$$B_2 = \left\{ d_2, 2d_2, \dots, \left\lfloor \frac{n_2}{(M-2)p} \right\rfloor d_2 \right\}.$$

Again, note that for all $\lambda_i \in [-p, p]$ and $b_i \in B_2$, we have

$$cx_2 + \lambda_1 b_1 + \dots + \lambda_{M-2} b_{M-2} \in R_2$$

so has colour k_2 .

We iterate this process M times, and obtain M generators x_1, \dots, x_M such that each row of the (M, p, c) -set generated by x_1, \dots, x_M is monochromatic. But now $M = k(m-1) + 1$, so m of the rows have the same colour. □

Remark 2.18 Being extremely precise in this proofs (such as considering $\lfloor \cdot \rfloor$) is much less important than the ideas in the proof. (Won't be penalised in the exam for small details like this).

Corollary 2.19 (Folkman's Theorem) Let $m \in \mathbb{N}$ be fixed. For every finite colouring of \mathbb{N} , there exists $x_1, \dots, x_m \in \mathbb{N}$ such that

$$\text{FS}(x_1, \dots, x_m) := \left\{ \sum_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}$$

is monochromatic.

Proof (Hints). A specific case of [Theorem 2.17](#). □

Proof. By the $(m, 1, 1)$ case of [Theorem 2.17](#). □

Remark 2.20

- The case $n = 2$ of Folkman's theorem is Schur's theorem.
- For a colouring $c : \mathbb{N} \rightarrow [k]$, we induce a colouring $c' : \mathbb{N} \rightarrow [k]$ by $c'(n) = c(2^n)$. Then by Folkman's theorem for c' , there exists x_1, \dots, x_m such that

$$\text{FP}(x_1, \dots, x_m) = \left\{ \prod_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}.$$

- It is not known whether the same result holds for $\text{FS}(x_1, \dots, x_m) \cup \text{FP}(x_1, \dots, x_m)$. However, it does not hold for infinite sets $\{x_n : n \in \mathbb{N}\}$, and does hold for colourings of \mathbb{Q} .

Proposition 2.21 Let A have CP. Then there exist $m, p, c \in \mathbb{N}$ such that every (m, p, c) -set contains a solution \mathbf{y} to $A\mathbf{y} = \mathbf{0}$, i.e. all y_i belong to the (m, p, c) -set.

Proof. Let $\mathbf{c}_1, \dots, \mathbf{c}_n$ be the columns of A . By assumption, there is a partition $B_1 \sqcup \dots \sqcup B_r$ of $[n]$ such that $\forall k \in [r]$,

$$\begin{aligned} \sum_{i \in B_k} \mathbf{c}_i &\in \text{span}\{\mathbf{c}_i : i \in B_1 \cup \dots \cup B_{k-1}\} \\ \Rightarrow \sum_{i \in B_k} \mathbf{c}_i &= \sum_{i \in B_1 \cup \dots \cup B_{k-1}} q_{ik} \mathbf{c}_i \quad \text{for some } q_{ik} \in \mathbb{Q} \\ \Rightarrow \sum_{i=1}^n d_{ik} \mathbf{c}_i &= \mathbf{0} \end{aligned}$$

where

$$d_{ik} = \begin{cases} 0 & \text{if } i \notin B_1 \cup \dots \cup B_{k-1} \\ 1 & \text{if } i \in B_k \\ -q_{ik} & \text{if } i \in B_1 \cup \dots \cup B_{k-1} \end{cases}.$$

Take $m = r$. Let $x_1, \dots, x_r \in \mathbb{N}$, and let $y_i = \sum_{k=1}^r d_{ik} x_k$ for each $i \in [n]$. Now $\mathbf{y} = (y_1, \dots, y_n)$ is a solution to $A\mathbf{y} = \mathbf{0}$: we have

$$\begin{aligned}
\sum_{i=1}^n y_i c_i &= \sum_{i=1}^n \sum_{k=1}^r d_{ik} x_k c_i \\
&= \sum_{k=1}^r x_k \sum_{i=1}^n d_{ik} c_i = \mathbf{0}.
\end{aligned}$$

Let c be the LCD of all the q_{ik} . Now $cy_i = \sum_{k=1}^r cd_{ik}x_k$ is an integral linear combination of the x_k , and $c\mathbf{y}$ is a solution since \mathbf{y} is. Let p be c times maximum of the absolute values of the numerators of the q_{ik} . By definition of the d_{ik} , $c\mathbf{y}$ is in the (m, p, c) -set generated by x_1, \dots, x_r . \square

Theorem 2.22 (Rado) $A \in \mathbb{Q}^{m \times n}$ is PR iff it has CP.

Proof. \Rightarrow is by [Proposition 2.14](#). For \Leftarrow , let $c' : \mathbb{N} \rightarrow [k]$ be a finite colouring of \mathbb{N} . Also, by the above proposition, since A has CP, there exists $m, p, c \in \mathbb{N}$ such that $A\mathbf{x} = \mathbf{0}$ has a solution \mathbf{x} in any (m, p, c) -set by the above theorem. By [Theorem 2.17](#), there is a monochromatic (m, p, c) -set with respect to c' . This gives a monochromatic solution \mathbf{x} to $A\mathbf{x} = \mathbf{0}$. \square

Remark 2.23 From the proof of [Rado's Theorem](#), we obtain that if A is PR for the “mod p ” colourings, then it is PR for *any* colouring. There is no proof of this fact that is more direct than using Rado’s theorem.

Theorem 2.24 (Consistency) Let A and B be rational PR matrices. Then the matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

is PR.

Proof (Hints). [Rado's Theorem](#). \square

Proof. This is a trivial check of the CP given the CP of A and B , then we are done by [Rado's Theorem](#). \square

Remark 2.25 The [Consistency Theorem](#) says that if we can find monochromatic solutions \mathbf{x} and \mathbf{x}' to $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{y} = \mathbf{0}$, then we can find monochromatic solutions \mathbf{x}' and \mathbf{y}' , of the same colour, to $A\mathbf{x}' = \mathbf{0}$ and $B\mathbf{y}' = \mathbf{0}$.

Theorem 2.26 For any finite colouring of \mathbb{N} , some colour class contains solutions to *all* PR equations.

Proof (Hints). Use the [Consistency Theorem](#). \square

Proof. For a given k -colouring of \mathbb{N} , let $\mathbb{N} = C_1 \sqcup \dots \sqcup C_k$ be the colour classes. Assume the contrary, so for each $1 \leq i \leq k$, there exists a PR matrix A_i such that $A_i\mathbf{x} = \mathbf{0}$ has no monochromatic solution of the same colour as C_i . But then by inductively applying the consistency theorem, the matrix

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

has a monochromatic solution of the same colour as some C_j . But then $C_j \mathbf{x} = \mathbf{0}$ has a solution \mathbf{x} of the same colour as C_j : contradiction. \square

2.2. Ultrafilters

Theorem 2.27 (Hindman) For any finite colouring of \mathbb{N} , there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$\text{FS}(\{x_n : n \in \mathbb{N}\}) = \left\{ \sum_{i \in I} : I \subseteq \mathbb{N} \text{ finite}, I \neq \emptyset \right\}.$$

Definition 2.28 A **filter** on \mathbb{N} is a non-empty collection \mathcal{F} of subsets of \mathbb{N} such that:

- $\emptyset \notin \mathcal{F}$,
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$, i.e. \mathcal{F} is an **up-set**.
- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$, i.e. \mathcal{F} is closed under finite intersections.

A filter is a notion of “large” subsets of \mathbb{N} .

Example 2.29

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$ is a filter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$ is a filter.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$ is a filter, called the **cofinite filter**.
- $\mathcal{F}_4 = \{A \subseteq \mathbb{N} : A \text{ infinite}\}$ is not a filter, since it contains $2\mathbb{N}$ and $2\mathbb{N} + 1$ but not $\emptyset = (2\mathbb{N}) \cap (2\mathbb{N} + 1)$.
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$ is a filter.

Definition 2.30 An **ultrafilter** is a maximal filter.

Definition 2.31 For $x \in \mathbb{N}$, the **principal ultrafilter at x** is

$$\mathcal{U}_x := \{A \subseteq \mathbb{N} : x \in A\}.$$

Proposition 2.32 The principal ultrafilter at x is indeed an ultrafilter.

Proof (Hints). Straightforward. \square

Proof. If $B \notin \mathcal{U}_x$, then $x \in B^c$ so $B^c \in \mathcal{U}_x$, but $B^c \cap B = \emptyset$, so $\mathcal{U}_x \cup \{B\}$ is not a filter. \square

Example 2.33

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$ is an ultrafilter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$ is not an ultrafilter as \mathcal{F}_1 extends it.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$ is not an ultra filter, as \mathcal{F}_5 extends it.
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$ is not an ultrafilter, as $\{A \subseteq \mathbb{N} : 4\mathbb{N} \setminus A \text{ finite}\}$ extends it.

Proposition 2.34 A filter \mathcal{F} is an ultrafilter iff for all $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Proof (Hints). \Leftarrow : straightforward. \Rightarrow : show if $A \notin \mathcal{F}$, then $\exists B \in \mathcal{F}$ such that $A \cap B = \emptyset$. □

Proof. \Leftarrow : since $A \cap A^c = \emptyset \notin \mathcal{F}$.

\Rightarrow : let \mathcal{F} is an ultrafilter. We cannot have $A, A^c \in \mathcal{F}$ as $A \cap A^c = \emptyset \notin \mathcal{F}$. Suppose there is $A \subseteq \mathbb{N}$ such that $A, A^c \notin \mathcal{F}$. By maximality of \mathcal{F} , since $A \notin \mathcal{F}$, then $\exists B \in \mathcal{F}$ such that $A \cap B = \emptyset$ (suppose not, then $\mathcal{F}' = \{S \subseteq \mathbb{N} : S \supseteq A \cap B \text{ for some } B \in \mathcal{F}\}$ extends \mathcal{F}). Similarly, $\exists C \in \mathcal{F}$ such that $A^c \cap C = \emptyset$. So we have $C \subseteq A$, so $B \cap C = \emptyset \notin \mathcal{F}$: contradiction (or also $C \subseteq A \Rightarrow A \in \mathcal{F}$: contradiction). □

Corollary 2.35 Let \mathcal{U} be an ultrafilter and $A = B \cup C \in \mathcal{U}$. Then $B \in \mathcal{U}$ or $C \in \mathcal{U}$.

Proof (Hints). Straightforward. □

Proof. If not, then $B^c, C^c \in \mathcal{U}$ by [Proposition 2.34](#), hence $B^c \cap C^c = (B \cup C)^c = A^c \in \mathcal{U}$: contradiction. □

Proposition 2.36 Every filter is contained in an ultrafilter.

Proof (Hints). Use Zorn's Lemma. □

Proof. By Zorn's Lemma, it is enough to show that every non-empty chain of filters has an upper bound. Let $\{\mathcal{F}_i : i \in I\}$ be a chain of filters, and set $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$.

- $\emptyset \notin \mathcal{F}$ since $\emptyset \notin \mathcal{F}_i$ for each $i \in I$.
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $A \in \mathcal{F}_i$ for some $i \in I$, so $B \in \mathcal{F}_i$, so $B \in \mathcal{F}$.
- Let $A, B \in \mathcal{F}$, so $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ for some i, j . WLOG, $\mathcal{F}_i \subseteq \mathcal{F}_j$, so $A \cap B \in \mathcal{F}_j$, so $A \cap B \in \mathcal{F}$.

\mathcal{F} is an upper bound for the chain, so we are done. □

Proposition 2.37 Let \mathcal{U} be an ultrafilter. Then \mathcal{U} is non-principal iff \mathcal{U} extends the cofinite filter \mathcal{F}_C .

Proof (Hints). \Leftarrow : straightforward. \Rightarrow : use [Corollary 2.35](#). □

Proof. \Leftarrow : if $\mathcal{U} = \mathcal{U}_x$ is principal, then we have $\{x\} \in \mathcal{U}$ so $\{x\}^c \notin \mathcal{U}$ by [Proposition 2.34](#), but also $\{x\}^c \in \mathcal{F}_C$: contradiction.

\Rightarrow : let $A \in \mathcal{F}_C$, so $A^c = \{a_1, \dots, a_k\}$ is finite. Assume $A \notin \mathcal{U}$, then $A^c \in \mathcal{U}$, so by [Corollary 2.35](#), some $a_i \in \mathcal{U}$. But then by definition of a filter, each set containing a_i is in \mathcal{U} , so \mathcal{U} is principal: contradiction. □

Notation 2.38 Let $\beta\mathbb{N}$ denote the set of all ultrafilters on \mathbb{N} .

Definition 2.39 Define a topology on $\beta\mathbb{N}$ by its base (basis), which consists of

$$C_A := \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$$

for each $A \subseteq \mathbb{N}$. The sets above indeed form a base: we have $\bigcup_{A \subseteq \mathbb{N}} C_A = \beta\mathbb{N}$, and $C_A \cap C_B = C_{A \cap B}$, since $A \cap B \in \mathcal{U}$ iff $A, B \in \mathcal{U}$. The open sets are of the form $\bigcup_{i \in I} C_{A_i}$ and the closed sets are of the form $\bigcap_{i \in I} C_{A_i}$.

Remark 2.40 $\beta\mathbb{N} \setminus C_A = C_{A^c}$, since $A \notin \mathcal{U}$ iff $A^c \in \mathcal{U}$. We can view \mathbb{N} as being embedded in $\beta\mathbb{N}$ by identifying $n \in \mathbb{N}$ with $\tilde{n} := \mathcal{U}_n$, the principal ultrafilter at n . Each point in \mathbb{N} under this correspondence is isolated in $\beta\mathbb{N}$, since $C_{\{n\}} = \{\tilde{n}\}$ is an open neighbourhood of \tilde{n} . Also, \mathbb{N} is dense in $\beta\mathbb{N}$, since for every $n \in A$, $\tilde{n} \in C_A$, so every non-empty open set in $\beta\mathbb{N}$ intersects \mathbb{N} .

Theorem 2.41 $\beta\mathbb{N}$ is a compact Hausdorff topological space.

3. Euclidean Ramsey theory