# 1. The complex plane and Riemann sphere

- $\mathbb{C}^* = \mathbb{C} - \{0\}$
- $z_1 z_2 = 0 \iff z_1 = 0$ or $z_2 = 0$.
- $|z| = \sqrt{z\overline{z}}$.
- $\mathrm{Re}(z) = (z + \overline{z}) / 2$, $\mathrm{Im}(z) = (z - \overline{z}) / 2i$.
- $z^{-1} = \overline{z} / |z|^2$.
- **Principal value of $\arg(z)$**: in interval $(-\pi, \pi]$, written $\mathrm{Arg}(z)$.
- $\arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2) \pmod{2\pi}$.
- $\arg(1 / z) = -\arg(z) \pmod{2\pi}$.
- $\arg(\overline{z}) = -\arg(z) \pmod{2\pi}$.
- Multiplication by $z_1 = r_1 e^{i\theta_1}$ represents rotation by $\theta_1$ followed by dilation by factor $r_1$.
- Addition represents translation.
- Conjugation represents reflection in the real axis.
- Taking the real (imaginary) part represents projection onto the real (imaginary) axis.
- $|z_1 z_2| = |z_1||z_2|$.
- **De Moivre's formula**: $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.
- **Triangle inequality**: $|z_1 + z_2| \leq |z_1| + |z_2|$.
- $|z| \geq 0$ and $|z| = 0 \iff z = 0$.
- $\max\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\} \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.
- **Complex exponential function**:

$$\exp(z) := e^x(\cos(y) + i\sin(y))$$

.

- $\forall z \in \mathbb{C}, e^z \neq 0$.
- $e^{z_1 + z_2} = e^{z_1} e^{z_2}$.
- $e^z = 1 \iff z = 2\pi i k$ for some $k \in \mathbb{Z}$.
- $e^{-z} = 1 / e^z$.
- $|e^z| = e^{\mathrm{Re}(z)}$.
- $\forall k \in \mathbb{Z}, \exp(z) = \exp(z + 2k\pi i)$.
- $$\sin(z) := \frac{1}{2i}(e^{iz} - e^{-iz}), \quad \cos(z) := \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sinh(z) := \frac{1}{2}(e^z + e^{-z}), \quad \cosh(z) := \frac{1}{2}(e^z + e^{-z})$$
- For every $w \in \mathbb{C}^*$,

$$e^z = w = |w|e^{i\varphi}$$

has solutions

$$z = \log(|w|) + i(\varphi + 2k\pi), \quad k \in \mathbb{Z}$$

- Let $\theta_2 - \theta_1 = 2\pi$, let arg be the argument function in $(\theta_1, \theta_2]$. Then

$$\log(z) := \log(|z|) + i\arg(z)$$

is a **branch of logarithm**. Jump discontinuity on **branch cut**, the ray $R_{\theta_1} = R_{\theta_2}$.
- **Principal branch of log**: where $\arg(z) = \mathrm{Arg}(z) \in (-\pi, \pi]$.

- $e^{\log(z)} = z$.
- Generally, $\log(zw) \neq \log(z) + \log(w)$.
- Generally, $\log(e^z) \neq z$.
- Given a branch of log, **power function** is

$$z^w := \exp(w \log(z))$$

- $\hat{\mathbb{C}} = C \cup \{\infty\}$.
- Unit sphere: $S^2 = \{(x, y, s) \in \mathbb{R}^3 : x^2 + y^2 + s^2 = 1\}$, north pole: $N = (0, 0, 1) \in S^2$. **Stereographic projection**: map that takes $v \in S^2 - \{N\}$ to $x + iy \in \mathbb{C}$, where $(x, y)$ is where the line from $N$ through $v$ intersects the $(x, y)$-plane.
- **Stereographic projection formula**:

$$P(x, y, s) = \frac{x}{1 - s} + \frac{iy}{1 - s}$$

  North pole is mapped to $\infty$.
- Inverse of stereographic projection found by intersection of line (from $z \in \mathbb{C}$ to $N$) and $S^2$ .
- **Riemann sphere**: unit sphere $S^2$ with stereographic projections from north and south pole.

# 2. Metric spaces

- **Metric space**: set $X$ and **metric** function $d : X \times X \to \mathbb{R}_{\geq 0}$, for every $x, y, z \in X$
  - **positivity**: $d(x, y) \geq 0$ and $d(x, y) = 0 \Longleftrightarrow x = y$
  - **symmetry**: $d(x, y) = d(y, x)$
  - **triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- **norm** on vector space $V$:
  - $\|v\| \geq 0$ and $\|v\| = 0 \Longleftrightarrow v = 0$
  - $\|\lambda v\| = |\lambda| \cdot \|v\|$
  - $\|v + w\| \leq \|v\| + \|w\|$
- $d(v, w) = \|v - w\|$ always defines a metric
- $d(v, w) = \sqrt{\langle v - w, v - w \rangle}$
- **$l_p$ norm**:

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

- **Taxicab norm**: $l_1$ norm
- **$l_\infty$ norm (sup-norm)**: $\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$
- **Riemannian (chordal) metric on $\hat{\mathbb{C}}$**: $d(z, w) = \|f(z) - f(w)\|_2$ where $f : \hat{\mathbb{C}} \to S^2$ is the inverse stereographic projection.
- **Discrete metric**:

$$d(x, y) := \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Open ball of radius $r$ centred at $x$**: $B_r(x) := \{y \in X : d(x, y) < r\}$

- **Closed ball of radius $r$ centred at $x$**: $\overline{B}_r(x) := \{y \in X : d(x, y) \leq r\}$
- **$U \subseteq X$ open** if $\forall x \in U, \exists \varepsilon > 0, B_\varepsilon(x) \subset U$
- **$U \subseteq X$ closed** if $X - U$ open
- **clopen**: open and closed, e.g. empty set and $X$
- Open balls are open
- Closed balls are closed
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open
- Finite unions of closed sets are closed
- Arbitrary intersections of closed sets are closed
- **Interior of $A$**: $A^0 := \{x \in A : \text{for some open } U \subseteq A, x \in U\}$. It is the largest open set in $A$.
- **Closure of $A$**: complement of interior of complement: $\overline{A} := \{x \in X : U \cup A \neq \emptyset \text{ for every open set } U \text{ with } x \in U\} = X - (X - A)^0$. It is the smallest closed set containing $A$.
- **Boundary of $A$**: closure without interior: $\partial A := \overline{A} - A^0$
- **Exterior of $A$**: complement of closure: $A^e := X - \overline{A} = (X - A)^0$
- $A$ is open $\iff \partial A \cap A = \emptyset \iff A = A^0$
- $A$ is closed $\iff \partial A \subseteq A \iff A = \overline{A}$
- For simple sets in $\mathbb{R}^n$ or $\mathbb{C}^n$, closure (or interior) is obtained by replacing by replacing strict inequality with equality (or vice versa).
- Sequence $\{x_n\}$ **converges to** $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$ or equivalently,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, d(x_n, x) < \varepsilon$$

- Limits in the complex plane follow COLT rules
- $\{z_n\}$ converges iff $\{\mathrm{Re}(z_n)\}$ and $\{\mathrm{Im}(z_n)\}$ converge.
- $\lim_{n\to\infty} x_n = x \iff \forall$ open $U$ with $x \in U, \exists N \in \mathbb{N}, \forall n > N, x_n \in U$
- $f : (X_1, d_1) \to (X_2, d_2)$ is **continuous at $x_0 \in X_1$** if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X_1, d_1(x, x_0) < \delta \implies d_2(f(x), f(x_0)) < \varepsilon$$

- $f$ is **continuous on $X_1$** if continuous at every $x_0 \in X_1$
- Products, sums and quotients of real/complex continuous functions are continuous
- Compositions of continuous functions are continuous
- **Preimage**: $f^{-1}(U) := \{x \in X_1 : f(x) \in U\}$
- **Properties of preimage**:
  - $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
  - $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
  - $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$
- $$\begin{aligned} f : X_1 \to X_2 \text{ continuous} &\iff f^{-1}(U) \text{ open in } X_1 \forall \text{ open } U \subseteq X_2 \\ &\iff f^{-1}(F) \text{ closed in } X_1 \forall \text{ closed } F \subseteq X_2 \end{aligned}$$
- $f : X_1 \to X_2$ continuous at $x \in X_1 \iff f^{-1}(U)$ open in $X_1 \forall$ open $U \subseteq X_2$ containing $f(x)$
- Non-empty $K \subseteq X$ **compact** if for every sequence $\{x_k\}$ in $K$, there exists a convergent subsequence $\{x_{n_k}\}$ with limit in $K$.

- If $\{x_k\}$ is a convergent sequence in $X$ then every subsequence $\left\{x_{n_k}\right\}$ converges to the same limit.
- $F \subseteq X$ is closed iff every sequence in $F$ converging in $X$ also converges in $F$.
- Compact sets are closed
- Every closed subset of a compact set is compact
- $A \subseteq X$ **bounded** if for some $R > 0$, $x \in X$, $A \subseteq B_R(x)$
- Compact sets are bounded
- **Heine-Borel for** $\mathbb{C}$: $K \subseteq \mathbb{C}$ is compact iff $K$ is closed and bounded.
- $f : X \to Y$ is continuous at $x \in X$ iff

  $$\lim_{n \to \infty} f(x_n) = f(x)$$

  for every convergent sequence $\{x_n\}$ in $X$ with $x_n \to x$.
- If $K \subseteq X$ is compact and $f : X \to Y$ is continuous, then $f(K)$ is compact in $Y$. So for $Y = \mathbb{R}$, any continuous real-valued function attains maxima and minima on compact sets.

# 3. Complex differentiation

- $f : U \to \mathbb{C}$ for open $U$ is **complex differentiable at $z_0 \in U$** if

  $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

  exists. Limit is the **derivative of $f$ at $z_0$**:

  $$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

  . $h \in \mathbb{C}$ so limit must exist from every direction.
- Complex differentiability at $z_0$ implies continuity at $z_0$.
- Sums, products and quotients of complex differentiable functions are complex differentiable.
- Compositions of complex differentiable functions are complex differentiable.
- The product, quotient and chain rules hold for complex differentiable functions.
- Most non-constant purely real/imaginary functions are not complex differentiable.
- If $f = u + iv$ is complex differentiable at $z_0$ then $u_x, u_y, v_x, v_y$ exist at $z_0$ and satisfy **Cauchy-Riemann equations**:

  $$u_x(z_0) = v_y(z_0), \quad u_y(z_0) = -v_x(z_0)$$

  . Also,

  $$f'(z_0) = u_x(z_0) + i v_x(z_0)$$
- Let $f : U \to \mathbb{C}$, $U$ open, $f = u + iv$. If $u_x, u_y, v_x, v_y$ exist and are continuous at $z_0$ and satisfy the Cauchy-Riemann equations at $z_0$, then $f$ is complex differentiable at $z_0$.
- Let $U \subseteq C$ open, $f : U \to \mathbb{C}$. $f$ is **holomorphic on $U$** if $f$ is complex differentiable at every $z_0 \in U$.
- $f$ is **holomorphic at $z_0 \in U$** if $f$ is complex differentiable on some $B_\varepsilon(z_0)$.
- Affine linear maps $z \to az + b$, $a \neq 0$ are holomorphic.

- **Path (curve) from $a$ to $b$**: continuous function $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = a$ and $\gamma(1) = b$. Path **closed** if $a = b$.
- **Smooth** path: continuously differentiable.
- $U \subseteq \mathbb{C}$ **path-connected** if for every $a, b \in U$, there exists a path $\gamma$ from $a$ to $b$ with $\gamma(t) \in U$ for every $t \in [0,1]$.
- **Domain (region)**: open and path-connected.
- **Chain rule**: Let $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ holomoprhic, $\gamma : [0,1] \to U$ smooth path. Then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0)$$

- Let $D$ domain, $f : D \to \mathbb{C}$ holomorphic on $D$. If $\forall z \in D$, $f'(z) = 0$, or $f$ is purely real/imaginary, or $f$ has constant real/imaginary part, or $f$ has constant modulus, then $f$ is constant on $D$.
- Let $D$ domain, $f : D \to \mathbb{C}$ **conformal at $z_0$** if $f$ preserves angle and orientation of any two tangent vectors at $z_0$. Equivalently, $f$ preserves angle and orientation of any two smooth paths through $z_0$. $f$ **conformal** if conformal at every $z_0 \in D$.
- If $f$ holomorphic, $f'(z_0) \neq 0$ then $f$ conformal at $z_0$.
- $f$ transforms the tangent vector $\gamma'(t_0)$ by multiplying it by $f'(\gamma(t_0))$.
- If $f$ is conformal at $z_0$, then $f$ is complex differentiable at $z_0$ and $f'(z_0) \neq 0$.
- $f$ is conformal on domain $D$ iff $f$ is holomorphic on $D$ and $\forall z \in D$, $f'(z) \neq 0$.
- Conformal maps map orthogonal grids in the $(x, y)$-plane to orthogonal grids. (Grids can be made of arbitrary smooth curves, not necessarily straight lines).
- For $D$ and $D'$ domains, $f : D \to D'$ is **biholomorphic** if $f$ holomorphic, bijection and $f^{-1} : D' \to D$ holomorphic. $f$ is a **biholomorphism**. $D$ and $D'$ are **biholomorphic** if such an $f$ exists and write $f : D \sim_{\to} D'$
- Affine linear maps $z \to az + b$, $a \neq 0$, are biholomorphic from $\mathbb{C}$ to $\mathbb{C}$.
- For $D$ domain, set of all biholomorphic maps from $D$ to $D$ forms a group under composition, called **automorphism group of $D$**, $\operatorname{Aut}(D)$.

# 4. Mobius transformations

- $\mathrm{GL}_2(\mathbb{C}) := \{A \in M_2(\mathbb{C}) : \det(A) \neq 0\}$.
- Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$, then **Mobius transformation** is $M_T(z) = \infty$ if $cz + d = 0$, else

$$M_T(z) = \frac{az + b}{cz + d}$$

  Also

$$M_T(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}$$

  So $M_T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
- Let $k^2 = \det(T)$ then

$$M_{\frac{1}{k}T}(z) = \frac{\frac{az}{k} + \frac{b}{k}}{\frac{cz}{k} + \frac{d}{k}} = \frac{az+b}{cz+d} = M_T(z)$$

so any $T$ can be scaled to $T' = \frac{1}{k}T$ so that $\det(T') = \det\left(\frac{1}{k}T\right) = \frac{1}{k^2}\det(T) = 1$.

- **Cayley map**: $M_T(z) = \frac{z-i}{z+i}$ where $T = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$.
- Cayley map maps $\mathbb{H} \to \mathbb{D}$.
- Set of Mobius transformations forms group under composition:
  - $M_{T_1} \circ M_{T_2} = M_{T_1 T_2}$.
  - $(M_T)^{-1} = M_{T^{-1}}$.
  - $M_T = \mathrm{Id} \Longleftrightarrow T = t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, t \in \mathbb{C}^*$.
- Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$. If $c = 0$, $M_T$ is biholomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. If $c \neq 0$, $M_T$ is biholomorphic from $\mathbb{C} - \left\{-\frac{d}{c}\right\}$ to $\mathbb{C} - \left\{\frac{a}{c}\right\}$.
- $M_T$ conformal at every $z \in \mathbb{C}$ where $M_T(z) \neq \infty$.
- $M_T$ is bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.
- $z$ is **fixed point** of $M_T$ if $M_T(z) = z$.
- If $M_T$ is not identity map, then it has at most 2 fixed points in $\hat{\mathbb{C}}$. So if $M_T$ has 3 fixed points in $\hat{\mathbb{C}}$, it is identity map.
- **Cross ratio** of distinct $z_0, z_1, z_2, z_3 \in \mathbb{C}$:

$$(z_0, z_1; z_2, z_3) := \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

  If some $z_i = \infty$ then same definition but remove all differences involving $z_i$, so

$$(\infty, z_1; z_2, z_3) := \frac{(z_1 - z_3)}{(z_1 - z_2)}$$

- **Three points theorem**: Let $\{z_1, z_2, z_3\}$, $\{w_1, w_2, w_3\}$ be sets of distinct ordered points in $\hat{\mathbb{C}}$. Then exists unique Mobius transformation $f$ such that $f(z_i) = w_i$, $i = 1, 2, 3$, given by $F^{-1} \circ G$, where

$$F(z) = (z, w_1; w_2, w_3), \quad G(z) = (z, z_1; z_2, z_3)$$

- **Mobius transformations preserve cross ratio**: For Mobius transformation $f$,

$$(f(z_0), f(z_1); f(z_2), f(z_3)) = (z_0, z_1; z_2, z_3)$$

- **Strategy to find Mobius transformation from how it acts on three points**: since cross-ratio preserved, rearrange the equation

$$(f(z), w_1; w_2, w_3) = (z, z_1; z_2, z_3)$$

- **Strategy to find image of domain $D$ under $M_T$**:
  - Find image of boundary: $M_T(\partial D)$.
  - Find image of point $z_0 \in D$ in interior: $M_T(z_0)$.
  - Image $D'$ is domain bounded by $M_T(\partial D)$ and containing $M_T(z_0)$.
- **Circline**: circle or line.
- Mobius transformations map circlines in $\hat{\mathbb{C}}$ to circlines in $\hat{\mathbb{C}}$.
- **Equations of circles and lines in $\mathbb{C}$**:

$$\gamma z\overline{z} - \alpha\overline{z} - \overline{\alpha}z + \beta = 0$$

is equation of circle if $\gamma = 1$ and $|\alpha|^2 - \beta > 0$, and equation of line if $\gamma = 0$ and $\alpha \neq 0$. Also, any circle or line can be described by this equation.

- Circle uniquely determined by three points, line determined by two points, so to determine how Mobius transformation maps circle, check where three points on circle are mapped.
- Circles through $N$ in $S^2$ correspond to lines in $\hat{\mathbb{C}}$. Circles not through $N$ correspond to circles in $\hat{\mathbb{C}}$ (via stereographic projection).
- For domain $D$, $\text{Mob}(D)$ is set of Mobius transformations that map $D$ to $D$.
- **H2H**:

$$f \in \text{Mob}(\mathbb{H}) \Longleftrightarrow f = M_T, \quad T \in \text{SL}_2(\mathbb{R}) := \{A \in M_2(\mathbb{R}) : \det(A) = 1\}$$

- **D2D**:

$$f \in \text{Mob}(\mathbb{D}) \Longleftrightarrow f = M_T, \quad T \in \text{SU}(1,1) := \left\{A = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C}, \det(A) = 1\right\}$$

- **D2D***:
  - Every $f \in \text{Mob}(\mathbb{D})$ is of form

$$f(z) = e^{i\theta}\frac{z - z_0}{\overline{z_0}z - 1}$$

    where $z_0 \in \mathbb{D}$ is unique point such that $f(z_0) = 0$.
  - Every $f \in \text{Mob}(\mathbb{D})$ where $f(0) = 0$ is a rotation about 0.
- **Strategy to find biholomorphic map between two domains**: build up biholomorphic map from simpler known ones, e.g. Mobius transformations, Cayley map, translations.

# 5. Notions of convergence in complex analysis and power series

- For $X$ and $Y$ metric spaces, $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ **converges pointwise on $X$ to $f$** if

$$\forall x \in X, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \quad d_Y\big(f_n(x), f(x)\big) < \varepsilon$$

  $f(x) = \lim_{n\to\infty} f_n(x)$ is **limit function**.
- $\{f_n\}_{n\in\mathbb{N}}$ **converges uniformly on $X$ to $f$** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \forall x \in X, \quad d_Y\big(f_n(x), f(x)\big) < \varepsilon$$

- Uniform convergence implies pointwise convergence.
- **Uniform limits of continuous functions are continuous**: let $\{f_n\}_{n\in\mathbb{N}}$ be all continuous on $X$ and converge uniformly to $f$ on $X$. Then $f$ is continuous on $X$.
- **Test for uniform convergence**: let $\{f_n\} : X \to \mathbb{C}$ converge pointwise to $f$.
  - If $\forall x \in X, \left|f_n(x) - f(x)\right| \leq s_n$, $\{s_n\}$ is sequence with $\lim_{n\to\infty} s_n = 0$, then $\{f_n\}$ converges uniformly to $f$ on $X$.

- If for some sequence $\{x_n\} \subset X$, $\left|f_n(x_n) - f(x_n)\right| \geq c$ for some $c > 0$, then $f_n$ does not converge uniformly to $f$ on $X$.
- **Weierstrass M-test**: Let $\left\{f_n\right\} : X \to \mathbb{C}$ satisfy

$$\forall x \in X, \left|f_n(x)\right| \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty$$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly to some $f$ on $X$.
- Let $\left\{f_n\right\} : [a, b] \to \mathbb{R}$ be continuous and converge uniformly to $f$ on $[a, b]$. Then

$$\forall c \in [a, b], \quad \lim_{n \to \infty} \int_a^c f_n(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$$

- $\left\{f_n\right\}$ **converges locally uniformly on $X$ to $f$** if $\forall x \in X$, exists open $U \subset X$ containing $x$ such that $\left\{f_n\right\}$ converges uniformly to $f$ on $U$.
- Let $\left\{f_n\right\}$ be continuous on $X$ and converge locally uniformly to $f$ on $X$. Then $f$ is continuous on $X$.
- **Local M-test**: let $\left\{f_n\right\} : X \to \mathbb{C}$ be continuous, such that $\forall y \in X$, exists open $U \subset X$ containing $y$ and $M_n > 0$ with

$$\forall x \in U, \left|f_n(x)\right| \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty$$

Then $\sum_{n=1}^{\infty} f_n$ converges locally uniformly to continuous function on $X$.
- **Complex power series**:

$$\sum_{n=0}^{\infty} a_n (z - c)^n, \quad a_n, c \in \mathbb{C}$$

- Either:
  - Series converges only for $z = c$ ($R = 0$).
  - Series converges absolutely for $|z - c| < R \Longleftrightarrow z \in B_R(c)$. $R$ is **radius of convergence**, $B_R(c)$ is **disc of convergence** and diverges for $|z - c| > R$.
  - Series converges absolutely for all $z$ ($R = \infty$).
- Power series with radius of convergence $R$ converges absolutely on $B_r(c)$ for every $0 < r < R$. So series is locally uniformly convergent (but not uniformly convergent) on disc of convergence.
- **Term-by-term differentiation and integration preserve radius of convergence**: let $\sum_{n=0}^{\infty} a_n (z - c)^n$ have radius of convergence $R$. Then formal derivative and antiderivative

$$\sum_{n=1}^{\infty} n a_n (z - c)^{n-1}, \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - c)^{n+1}$$

have radius of convergence $R$.
- **Power series can be differentiated term-by-term in disc of convergence**: let $\sum_{n=0}^{\infty} a_n (z - c)^n$ have radius of convergence $R$ and converge to $f : B_R(c) \to \mathbb{C}$. Then $f$ is holomorphic on $B_R(c)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-c)^{n-1}$$

- Power series with $R > 0$ can be differentiated infinitely many times and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} k! \binom{n}{k} a_n (z-c)^{n-k}$$

  So $f^{(k)}(c) = k! a_k$.
- **Power series can be integrated term-by-term in disc of convergence**: power series with $R > 0$ has holomorphic antiderivative $F : B_R(c) \to \mathbb{C}$ given by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-c)^{n+1}$$

# 6. Complex integration over contours

- Let $f : [a, b] \to \mathbb{C}$, $f = u + iv$, then

$$\int_a^b f(t) \, \mathrm{d}t = \int_a^b u(t) \, \mathrm{d}t + i \int_a^b v(t) \, \mathrm{d}t$$

- Let $f_1, f_2 : [a, b] \to \mathbb{C}$, $c \in \mathbb{C}$, then

$$\int_a^b \big(f_1(t) + f_2(t)\big) \, \mathrm{d}t = \int_a^b f_1(t) \, \mathrm{d}t + \int_a^b f_2(t) \, \mathrm{d}t, \quad \int_a^b c f_1(t) \, \mathrm{d}t = c \int_a^b f_1(t) \, \mathrm{d}t$$

- Curve $\gamma : [a, b] \to \mathbb{C}$ is $C^1$ if **continuously differentiable** (derivative exists and is continuous).
- **Integral of continuous $f : U \to \mathbb{C}$ along curve $\gamma : [a, b] \to U$, $\gamma \in C^1$**:

$$\int_\gamma f(z) \, \mathrm{d}z := \int_a^b f(\gamma(t)) \gamma'(t) \, \mathrm{d}t$$

- Let $f_1, f_2 : [a, b] \to \mathbb{C}$, $c \in \mathbb{C}$, then

$$\int_\gamma \big(f_1(z) + f_2(z)\big) \, \mathrm{d}z = \int_\gamma f_1(z) \, \mathrm{d}z + \int_\gamma f_2(z) \, \mathrm{d}z, \quad \int_\gamma c f_1(z) \, \mathrm{d}z = c \int_\gamma f_1(z) \, \mathrm{d}z$$

- $(-\gamma) : [-b, -a] \to \mathbb{C}$, $(-\gamma)(t) := \gamma(-t)$, then

$$\int_{-\gamma} f(z) \, \mathrm{d}z = -\int_\gamma f(z) \, \mathrm{d}z$$

- Let $\varphi : [a', b'] \to [a, b]$ be continuously differentiable, $\varphi(a') = a$, $\varphi(b') = b$, $\delta : [a', b'] \to \mathbb{C}$, $\delta = \gamma \circ \varphi$. Then

$$\int_\gamma f(z) \, \mathrm{d}z = \int_\delta f(z) \, \mathrm{d}z$$

- Let $\gamma : [a, b] \to \mathbb{C}$, $a = a_0 < a_1 < \cdots < a_n = b$, $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$ be $C^1$, $\gamma_i(t) := \gamma(t)$ for $t \in [a_{i-1}, a_i]$. Then $\gamma$ is **piecewise $C^1$ curve**, or **contour**.

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f(z)\,\mathrm{d}z$$

is a **contour integral**.

- **Contour union**: let $\gamma : [a,b] \to \mathbb{C}$, $\delta : [c,d] \to \mathbb{C}$, then

$$(\gamma \cup \delta) : [a, b+d-c] \to \mathbb{C}, \quad (\gamma \cup \delta)(t) := \begin{cases} \gamma(t) & \text{if } t \in [a,b] \\ \delta(t+c-b) & \text{if } t \in [b, b+d-c] \end{cases}$$

Then

$$\int_{\gamma\cup\delta} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z + \int_\delta f(z)\,\mathrm{d}z$$

- **Complex Fundamental Theorem of Calculus (FTC)** Let $U \subseteq \mathbb{C}$ open, $F : U \to \mathbb{C}$ holomorphic with derivative $f$, $\gamma : [a,b] \to U$ contour. Then

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

So if $\gamma$ closed, then $\int_\gamma f(z)\,\mathrm{d}z = 0$. Also, if $\gamma_1$ and $\gamma_2$ have same endpoints, then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$$

- If $F' = f$, $F$ is **antiderivative** or **primitive** of $f$.
- **Length** of contour $\gamma$:

$$L(\gamma) := \int_a^b |\gamma'(t)|\,\mathrm{d}t$$

- **Estimation lemma**: Let $f : U \to \mathbb{C}$ continuous, $\gamma : [a,b] \to U$ contour. Then

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le L(\gamma) \cdot \sup_\gamma |f|$$

where $\sup_\gamma |f| := \sup\{|f(z)| : z \in \gamma\}$
- **Converse to FTC**: Let $D$ domain, $f : D \to \mathbb{C}$ continuous, $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed contour $\gamma \in D$. Then exists holomorphic antiderivative $F : D \to \mathbb{C}$ (unique up to addition of constant) such that

$$F'(z) = f(z)$$

- Domain $D$ **starlike** if for some $a_0 \in D$, for every $a_0 \ne b \in D$, straight line from $a_0$ to $b$ contained in $D$.
- **Cauchy's theorem for starlike domains**: let $D$ starlike domain, $f : D \to \mathbb{C}$ holomorphic, $\gamma \in D$ closed contour. Then

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Same holds if $f$ holomorphic on $D - S$, $S$ finite set of points, and $f$ continuous on $D$.

- Let $U$ open, $f : U \to C$ holomorphic, $\Delta \in U$ be triangle. Then

$$\int_{\partial \Delta} f(z) \, \mathrm{d}z = 0$$

Same holds if $f$ holomorphic on $U - S$, $S$ finite set of points, and $f$ continuous on $U$.

- By default, always use **anti-clockwise** parameterisation of contour.
- Let $D$ starlike domain, $f : D \to \mathbb{C}$ continuous, $\int_{\partial \Delta} f(z) \, \mathrm{d}z = 0$ for every triangle $\Delta \in D$. Then exists holomorphic $F : D \to \mathbb{C}$ such that $F' = f$.
- **Cauchy's integral formula (CIF)**: let $B = B_r(a)$, $f : B \to \mathbb{C}$ holomorphic. Then for every $w \in B$, $\rho$ such that $|w - a| < \rho < r$,

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z - w} \, \mathrm{d}z$$

# 7. Features of holomorphic functions

- **Cauchy-Taylor theorem**: let $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ holomorphic, $r > 0$, $B_r(a) \subset U$. Then $f$ is given by power series (**Taylor series of $f$ around $a$**) that converges on $B_r(a)$:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n, \quad z \in B_r(a)$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z - a)^{n+1}} \, \mathrm{d}z$$

for any $0 < \rho < r$.

- Function with Taylor series expansion on $B_r(a)$, $r > 0$, is **analytic at $a$**.
- Function **analytic** if analytic at every point in domain.
- Holomorphic $\Longleftrightarrow$ analytic.
- **Cauchy's integral formula (CIF) for derivatives**: let $B = B_r(a)$, $f : B \to \mathbb{C}$ holomorphic. For every $0 < \rho < r$,

$$\int_{|z-a|=\rho} \frac{f(z)}{(z - a)^{n+1}} \, \mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(a)$$

- So $f$ has Taylor series expansion on $B_r(a)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n$$

- Equivalent of Cauchy-Taylor doesn't hold for real analysis, e.g.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

has derivatives of all orders and $f^{(n)}(0) = 0$. But Taylor series around $x = 0$ would be

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad x \in (0 - \varepsilon, 0 + \varepsilon)$$

for some $\varepsilon > 0$. But then $c_n = \frac{f^{(n)}}{n!} = 0$ but $f$ isn't identically zero in any neighbourhood of the origin. So $f$ doesn't have a Taylor series.

- **Holomorphic functions have infinitely many derivatives**: let $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ holomorphic. Then $f$ has derivatives of all orders on $U$ which are all holomorphic.
- **Morera's theorem**: let $D$ domain, $f : D \to \mathbb{C}$ continuous. If for every closed contour $\gamma$ in $D$,

$$\int_{\gamma} f(z) \, \mathrm{d}z = 0$$

  then $f$ holomorphic.
- $f : \mathbb{C} \to \mathbb{C}$ **entire** if holomorphic on $\mathbb{C}$.
- $f : \mathbb{C} \to \mathbb{C}$ **bounded** if for some $M > 0$, $|f(z)| \leq M$ for every $z \in \mathbb{C}$.
- **Liouville's theorem**: every bounded entire function is constant.
- **Fundamental theorem of algebra**: every non-constant polynomial with complex coefficients has complex root.
- **Local maximum modulus principle**: let $f : B_r(a) \to \mathbb{C}$ holomorphic. If

$$\forall z \in B_r(a), |f(z)| \leq |f(a)|$$

  then $f$ constant on $B_r(a)$.
- **Maximum modulus principle**: let $D$ domain, $f : D \to \mathbb{C}$ holomorphic. If for some $a \in D$,

$$\forall z \in D, |f(z)| \leq |f(a)|$$

  then $f$ constant on $D$.
- If $U \subset \mathbb{C}$ path-connected and open, then not possible to write $U = U_1 \cup U_2$, where $U_1, U_2$ disjoint, open, non-empty. So domains are connected.
- $f : B_r(a) \to \mathbb{C}$ has **zero of order $m$ at $a$** if for some $m > 0$, exists holomorphic $h : B_r(a) \to \mathbb{C}$ such that $f(z) = (z - a)^m h(z)$, $h(a) \neq 0$.
- $f$ has zero of order $m$ at $a$ iff

$$f(a) = f^{(1)}(a) = \cdots = f^{(m-1)}(a) = 0$$

  and $f^{(m)}(a) \neq 0$.
- **Principle of isolated zeros**: let $f : B_r(a) \to \mathbb{C}$ holomorphic, $f \neq 0$. Then for some $0 < \rho \leq r$,

$$\forall z \in B_\rho(a) - \{a\}, \quad f(z) \neq 0$$

  Holds for $f(a) = 0$, i.e. zeros of holomorphic functions are isolated.
- **Uniqueness of analytic continuation theorem**: let $D' \subset D$ non-empty domains, $f : D' \to \mathbb{C}$ holomorphic. Then exists at most one holomorphic $g : D \to \mathbb{C}$ such that

$$\forall z \in D', \quad f(z) = g(z)$$

If $g$ exists, it is **analytic continuation of $f$ to $D$**.

- Let $D$ domain, $f, g : D \to \mathbb{C}$ holomorphic, $B_r(a) \subset D$. If $f(z) = g(z)$ on $B_r(a)$ then $f(z) = g(z)$ on $D$.
- Let $S \subset C$, $w \in S$.
  - $w$ **isolated point of $S$** if for some $\varepsilon > 0$, $B_\varepsilon(w) \cap S = \{w\}$.
  - $w$ **non-isolated point of $S$** if $\forall \varepsilon > 0$, exists $w \neq z \in S$ such that $z \in B_\varepsilon(w)$.
- **Identity theorem**: Let $f, g : D \to \mathbb{C}$ holomorphic on domain $D$. If $S := \{z \in D : f(z) = g(z)\}$ contains non-isolated point, then $f(z) = g(z)$ on $D$.
- Let $D \subseteq \mathbb{C}$ domain, $u : D \to \mathbb{R}$ **harmonic** if has continuous second order partial derivatives and satisfies **Laplace's equation**:

$$u_{xx} + u_{yy} = 0$$

- Let $f = u + iv : D \to \mathbb{C}$ holomorphic on domain $D$. Then $u$ and $v$ harmonic.
- **Existence of harmonic conjugates theorem**: let $D$ starlike domain, $u : D \to \mathbb{R}$ harmonic. Then exists harmonic $v : D \to \mathbb{R}$ such that $f = u + iv$ holomorphic on $D$. $v$ is **harmonic conjugate of $u$**, unique up to addition of real constant. **Note**: condition of $D$ being starlike is removed when Cauchy's theorem is proved in generality.
- Let $f : D \to \mathbb{C}$ holomorphic on domain $D$. Then $f$ has holomorphic antiderivative on $D$.
- **Dirichlet problem**: let $D \subseteq \mathbb{C}$ domain with closure $\overline{D}$, boundary $\partial D$, $g : \partial D \to \mathbb{R}$ continuous. Find continuous $\mu : \overline{D} \to \mathbb{R}$ such that $\mu$ harmonic on $D$ and $\mu = g$ on $\partial D$.
- Let $f = u + iv : D \to \mathbb{C}$ holomorphic on domain $D$, $\mu$ harmonic on $f(D)$. Then $\tilde{\mu} := \mu \circ f$ harmonic on $D$.
- So if $\mu$ harmonic on $D'$ and want to find a harmonic $\tilde{\mu}$ on $D$, find holomorphic $f$ mapping $D$ to $D'$ so $f(D) = D'$. Then $\tilde{\mu} = \mu \circ f$ is solution.

# 8. General form of Cauchy's theorem and C.I.F.

- Let curve $\gamma : [a, b] \to \mathbb{C}$, $\gamma(t) = w + r(t)e^{i\theta(t)}$, $w \in \mathbb{C}$, $r, \theta : [a, b] \to \mathbb{R}$, piecewise $C^1$, $r(t) > 0$. **Winding number (index)** of $\gamma$ around $w$ is

$$I(\gamma; w) := \frac{\theta(b) - \theta(a)}{2\pi}$$

- Let contour $\gamma : [a, b] \to \mathbb{C}$, $w \in \mathbb{C}$, $w \notin \gamma$. Then exists $r, \theta : [a, b] \to \mathbb{R}$ piecewise $C^1$, $r(t) > 0$ such that

$$\gamma(t) = w + r(t)e^{i\theta(t)}$$

. Here, $r(t) = |\gamma(t) - w|$.

- Let $\gamma : [a, b] \to \mathbb{C}$ closed contour, $w \notin \gamma$. Then

$$I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w} \, \mathrm{d}z$$

- Let $D$ starlike domain, $\gamma$ closed contour in $D$. If $w \notin D$, then $I(\gamma; w) = 0$.
- Let $U \subseteq \mathbb{C}$ open.
  - Closed contour $\gamma$ in $U$ **homologous to zero in $U$** if $I(\gamma; w) = 0$ for every $w \notin U$.

- $U$ is **simply connected** if every closed contour in $U$ homologous to zero in $U$.
- **Cycle**: finite collection of closed contours in $U$, denoted as formal sum

$$\Gamma := \gamma_1 + \cdots + \gamma_n$$

$w$ **does not lie in Γ** if $w \notin \gamma_i$ for all $i$. Define

$$I(\Gamma; w) := \sum_{i=1}^{n} I(\gamma_i; w)$$

and

$$\int_\Gamma f(z)\, \mathrm{d}z := \sum_{i=1}^{n} \int_{\gamma_i} f(z)\, \mathrm{d}z$$

$\Gamma$ **homologous to zero in $U$** if $I(\Gamma; w) = 0$ for every $w \notin U$.
- Closed curve $\gamma : [a, b] \to \mathbb{C}$ **simple** if for any $t_1 < t_2$, $\gamma(t_1) = \gamma(t_2) \Longrightarrow t_1 = a$ and $t_2 = b$ (no self-crossing or backtracking).
- **Jordan curve theorem**: Let $\gamma$ closed curve. Then $\mathbb{C} - \gamma$ is disjoint union of two domains, exactly one of which is bounded. Bounded domain is **interior** of $\gamma$, $D_\gamma^{\text{int}}$. Unbounded domain is **exterior**, $D_\gamma^{\text{ext}}$. $w$ lies inside $\gamma$ if $w \in D_\gamma^{\text{int}}$ and outside $\gamma$ if $w \in D_\gamma^{\text{ext}}$.
- Let $\gamma$ simple closed contour. Then possible to put orientation on $\gamma$ such that $\forall w \in \mathbb{C} - \gamma$,

$$I(\gamma; w) = \begin{cases} 1 \text{ if } w \in D_\gamma^{\text{int}} \\ 0 \text{ if } w \in D_\gamma^{\text{ext}} \end{cases}$$

Then $\gamma$ is **positively oriented** (interior always on left of curve - anticlockwise).
- Let $D$ domain, $f : D \to \mathbb{C}$ holomorphic, $\Gamma$ cycle in $D$, homologous to zero in $D$.
  - **General form of Cauchy's theorem**:

$$\int_\Gamma f(z)\, \mathrm{d}z = 0$$

  - **General form of CIF**:

$$\forall w \in D - \Gamma, \quad \int_\Gamma \frac{f(z)}{z - w}\, \mathrm{d}z = 2\pi i I(\Gamma; w) f(w)$$

- For simple closed curve $\gamma$, $f$ holomorphic on $D_\gamma^{\text{int}} \cup \gamma$ if exists domain $D$ such that $D_\gamma^{\text{int}} \cup \gamma \subset D$ and $f$ holomorphic on $D$.
- Let $\gamma$ simple closed, positively oriented contour and $f$ holomorphic on $D_\gamma^{\text{int}} \cup \gamma$.
  - **Cauchy's theorem for simple closed contours**:

$$\int_\gamma f(z)\, \mathrm{d}z = 0$$

  - **CIF for simple closed contours**:

$$\forall w \in D_\gamma^{\text{int}}, \quad \int_\gamma \frac{f(z)}{z - w}\, \mathrm{d}z = 2\pi i f(w)$$

# 9. Holomorphic functions on punctured domains

- **Laurent series**:

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

  **Principal part**: $\sum_{n=-\infty}^{-1} c_n (z-a)^n$. **Analytic part**: $\sum_{n=0}^{\infty} c_n (z-a)^n$.
- Laurent series converges at $z$ iff principal and analytic parts converge at $z$.
- **Annulus centre $a$, internal/external radii $r$ and $R$**:

$$A_{r,R}(a) := \{z \in \mathbb{C} : r < |z-a| < R\}$$

- If Laurent series isn't power series ($c_n \neq 0$ for some $n < 0$) then either:
  - It never converges or
  - Exists $0 \leq r < R \leq \infty$ such that it converges on $A_{r,R}(a)$ and diverges for $|z-a| < r$ or $|z-a| > R$. $A_{r,R}(a)$ is **annulus of convergence**.
- If Laurent series has annulus of convergence $A_{r,R}(a)$ then it converges uniformly on any $A_{\rho_1,\rho_2}$ with $r < \rho_1 < \rho_2 < R$. So it converges locally uniformly on $A_{r,R}(a)$ so represents holomorphic function on $A_{r,R}(a)$.
- If Laurent series has annulus of convergence containing $A_{r,R}(a)$, then $c_n$ are unique and given by, for any $\rho \in (r, R)$

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

  So Laurent series in $A_{r,R}(a)$ unique.
- **Holomorphic functions on annuli have Laurent series**: let $f : A_{r,R}(a) \to \mathbb{C}$ holomorphic, then exist unique $c_n \in \mathbb{C}$ such that

$$\forall z \in A_{r,R}(a), \quad f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

  and annulus of convergence of Laurent series contains $A_{r,R}(a)$. Series is **Laurent series of $f$ on $A$**.
- **Punctured ball**: $B_R^*(a) := B_R(a) - \{a\} = A_{0,R}(a)$.
- If $f$ holomorphic on $B_R^*(a)$, $f$ has **isolated singularity** at $a$.
- Types of isolated singularity:
  - $f$ has **removable singularity** at $z = a$ if $c_n = 0$ for all $n \leq -1$ (principal part is zero).
  - $f$ has **pole of order $k$** at $z = a$ if $c_{-k} \neq 0$ and $c_n = 0$ for all $n < -k$.
  - $f$ has **essential singularity** at $z = a$ if exist infinitely many $n < 0$ such that $c_n \neq 0$.
- $f : B_R^*(a) \to \mathbb{C}$ has removable singularity at $z = a$ iff $f$ extends to holomorphic function on $B_R(a)$ ($f$ has analytic continuation to $B_R(a)$).
- Let $f : B_R^*(a) \to \mathbb{C}$ holomorphic, $R > 0$. Then $f$ has removable singularity at $z = a$ iff

$$\lim_{z \to a} (z-a) f(z) = 0$$

- **Riemann extension theorem**: Let $f : B_R^*(a) \to \mathbb{C}$ holomorphic and bounded, then $f$ has removable singularity at $z = a$.

- Let $f : B_R^*(a) \to \mathbb{C}$ holomorphic. The following are equivalent:
  - $f$ has pole of order $k$ at $z = a$.
  - $f(z) = (z - a)^{-k} g(z)$, $g : B_R(a) \to \mathbb{C}$ holomorphic, $g(a) \neq 0$.
  - Exists $0 < r \leq R$ and $h : B_r(a) \to \mathbb{C}$ holomorphic with zero of order $k$ at $z = a$ such that $f(z) = 1 / h(z)$ for $z \in B_r^*(a)$.
- Let $f : B_R^*(a) \to \mathbb{C}$ holomorphic. Then $f$ has pole at $z = a$ iff

$$\lim_{z \to a} |f(z)| = \infty$$

- **Casorati-Weierstrass theorem**: let $f : B_R^*(a) \to \mathbb{C}$ holomorphic with essential singularity at $z = a$. Then

$$\forall w \in \mathbb{C}, \forall 0 < r < R, \forall \varepsilon > 0, \exists z \in B_r^*(a), \quad f(z) \in B_\varepsilon(w)$$

- **Big Picard theorem**: let $f : B_R^*(a) \to \mathbb{C}$ holomorphic with essential singularity at $z = a$. Then for some $b \in \mathbb{C}$,

$$\forall 0 < r < R, \quad \mathbb{C} - \{b\} \subseteq f(B_r^*(a))$$

# 10. Cauchy's residue theorem

- $f$ **meromorphic** on domain $D$ if $f$ holomorphic on $D - S$, $S \subset D$ has no non-isolated points and $f$ has pole at every $s \in S$.
- $f$ meromorphic on $D_\gamma^{\text{int}} \cup \gamma$ if exists domain $D$ containing $D_\gamma^{\text{in}} \cup \gamma$ and $f$ meromorphic on $D$.
- Let $f$ meromorphic on domain $D$ with pole at $a$, with Laurent series

$$f(z) = \sum_{n=-k}^{\infty} c_n (z - a)^n$$

  **Residue of $f$ at $a$** is

$$\text{Res}_{z=a}(f) := c_{-1}$$

- **Cauchy's residue theorem**: Let $f$ meromorphic on $D_\gamma^{\text{int}} \cup \gamma$, $\gamma$ positively oriented simple closed contour, $f$ has no poles on $\gamma$ and finite number of poles inside $\gamma$, $\{a_1, ..., a_m\}$. Then

$$\int_\gamma f(z) \, \mathrm{d}z = 2\pi i \sum_{j=1}^{m} \text{Res}_{z=a_j}(f)$$

- **Simple pole**: pole of order 1.
- **Rules for calculating residues**:
  - **Linear combinations**: $\text{Res}_{z=a}(Af + Bg) = A\text{Res}_{z=a}(f) + B\text{Res}_{z=a}(g)$.
  - **Cover up rule for poles of order 1**: if $z = a$ is pole of order 1,

$$\text{Res}_{z=a}(f) = \lim_{z \to a} (z - a) f(z)$$

  - **Simple zero on denominator**: if $f(z) = g(z) / h(z)$, $g, h$ holomorphic at $a$, $g(a) \neq 0$, $z = a$ is zero of order 1 of $h$, then

$$\mathrm{Res}_{z=a}(f) = \frac{g(a)}{h'(a)}$$

- **Poles of higher orders**: if $f(z) = g(z) \mathbin{/} (z-a)^k$, $k > 0$, $g$ holomorphic at $a$, then

$$\mathrm{Res}_{z=a}(f) = \frac{g^{(k-1)}(a)}{(k-1)!}$$

- To calculate

$$\int_0^{2\pi} F(\sin(\theta), \cos(\theta))\,\mathrm{d}\theta$$

where $F$ is rational function, use change of variable $z = e^{i\theta}$, and use

$$\int_0^{2\pi} F(\sin(\theta), \cos(\theta))\,\mathrm{d}\theta = \int_{|z|=1} F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{\mathrm{d}z}{iz}$$

- To calculate

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{p(x)}{q(x)}\,\mathrm{d}x$$

where $\deg(q) \geq \deg(p) + 2$ and $q$ has no real roots, integrate $f(z) = p(z) \mathbin{/} q(z)$ over $\gamma_R = L_R \cup C_R$ where $R$ greater than maximum modulus of roots of $q$. Use e.g. Estimation Lemma or Jordan's lemma to show $\lim_{R \to \infty} \int_{C_R} f(z)\,\mathrm{d}z = 0$.

- $$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{r \to \infty} \int_0^r f(x)\,\mathrm{d}x + \lim_{s \to \infty} \int_{-s}^0 f(x)\,\mathrm{d}x$$
- **Cauchy principal value** of $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$:

$$P.V. \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{r \to \infty} \int_{-r}^{r} f(x)\,\mathrm{d}x$$

- If $f$ even, $P.V. \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$
- **Jordan's lemma**: let $f$ holomorphic on $D = \{z \in \mathbb{C} : |z| > r\}$ for some $r > 0$, $zf(z)$ bounded on $D$. Then for every $\alpha > 0$,

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{i\alpha z}\,\mathrm{d}z = 0$$

where $C_R = Re^{i\theta}, \theta \in [0, \pi]$.
- To calculate

$$P.V. \int_{-\infty}^{\infty} f(x) \sin(\alpha x)\,\mathrm{d}x \quad \text{or} \quad P.V. \int_{-\infty}^{\infty} f(x) \cos(\alpha x)\,\mathrm{d}x$$

where $f$ meromorphic in $\mathbb{C}$ with no real poles and $f$ satisfies Jordan's lemma, calculate integral

$$\int_{\gamma_R} f(z)e^{i\alpha z}\,\mathrm{d}z$$

with CRT, where $\gamma_R = L_R \cup C_R$. Then use

$$\int_{L_R} f(z)e^{i\alpha z}\,\mathrm{d}z = \int_{-R}^{R} f(x)\cos(\alpha x)\,\mathrm{d}x + i\int_{-R}^{R} f(x)\sin(\alpha x)\,\mathrm{d}x$$

and equate real/imaginary parts. Use Jordan's lemma to show $\lim_{R\to\infty} \int_{C_R} f(z)e^{i\alpha z}\,\mathrm{d}z = 0$.

- **Indentation lemma**: Let $g$ meromorphic on $\mathbb{C}$ with simple pole at 0, $C_\varepsilon(\theta) = \varepsilon e^{i\theta}, \theta \in [0, \pi]$. Then

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} g(z)\,\mathrm{d}z = \pi i \mathrm{Res}_{z=0}(g)$$

- To calculate

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

where $f$ has simple pole at $z = 0$, let $\gamma_{\rho,R} = L_2 \cup \left(-C_\rho\right) \cup L_1 \cup C_R$ where $L_2$ is line from $-R$ to $-\rho$, $L_1$ is line from $\rho$ to $R$. Take $\rho \to 0$ and $R \to \infty$, use indentation lemma and Jordan's lemma. **Note**: may have to choose appropriate branch cut so that $f$ holomorphic on $D$.
- Let $f$ meromorphic with zero or pole order $k > 0$ at $a$. Then $f' / f$ has simple pole at $a$ and

$$\mathrm{Res}_{z=a}(f' / f) = \begin{cases} k & \text{if f has zero at } z = a \\ -k & \text{if f has pole at } z = a \end{cases}$$

- **Argument principle**: let $\gamma$ positively oriented simple closed contour, $f$ meromorphic on $D_\gamma^{\text{int}} \cup \gamma$, $f$ has no zeros or poles on $\gamma$, $Z_f$ be number of zeros of $f$ in $D_\gamma^{\text{int}}$ (counted with multiplicity), $P_f$ be number of poles of $f$ in $D_\gamma^{\text{int}}$ (counted with multiplicity). Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = Z_f - P_f = I\left(\Gamma_f; 0\right), \quad \Gamma_f = f \circ \gamma$$

(Counted with multiplicity means zero/pole of order $k$ counts $k$ times).
- **Rouche's theorem**: let $\gamma$ simple closed contour, $f, g$ holomorphic on $D_\gamma^{\text{int}} \cup \gamma$, with

$$\forall z \in \gamma, |f(z) - g(z)| < |g(z)|$$

Then $f$ and $g$ have same number of zeros (counted with multiplcity) inside $\gamma$.
- **Open mapping theorem**: let $f$ holomorphic, non-constant on domain $D$. Then if $U \subset D$ open, $f(U)$ is open.