Mathematical Physics Course Notes

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1 The action principle

1.1 Calculus of variatons

Definition 1.1.1. A **functional** is a map from a set of functions to $\mathbb{R}$, e.g. $f : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$.

Definition 1.1.2. Let $y(t)$ be a function with fixed values at endpoints a and b. y is **stationary** for a functional S if

$$\frac{dS(y(t) + \epsilon z(t))}{d\epsilon}\bigg|_{\epsilon=0} = 0$$

for every smooth (continuous derivative to every order) $z(t)$ such that $z(a) = z(b) = 0$.

Remark. Functions $y(t)$ may be referred to as **paths** and so functions that satisfy the above definition are referred to as **stationary paths**.

Definition 1.1.3. Let S be an **action functional** (or just **action**). **The action principle** states that the paths described by particles are stationary paths of S.

Mathematically, given a particle moving in one dimension with position given by $x(t)$, for arbitrary smooth small deformations $\delta x(t)$ around the true path $x(t)$ (the path the particle follows):

$$\delta S := S(x + \delta x) - S(x) = O((\delta x)^2)$$

Lemma 1.1.4. (Fundamental lemma of the calculus of variations) Let $f(x)$ be a continuous function in the interval $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = 0$$

for every smooth function $g(x)$ in $[a, b]$ such that $g(a) = g(b) = 0$. Then $f(x) = 0\ \forall x \in [a, b]$.

Definition 1.1.5. Let $L(r, s)$ be a function of two real variables. If a functional S can be expressed as the time integral of L, i.e. if

$$S(x) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))dt$$

then L is called a **Lagrangian**.

Definition 1.1.6. For a Lagrangian L, the **Euler-Lagrange equation** is given by

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

where

$$\frac{\partial L}{\partial x} = \frac{\partial L(r, s)}{\partial r}\bigg|_{(r,s)=(x(t),\dot{x}(t))} \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}} = \frac{\partial L(r, s)}{\partial s}\bigg|_{(r,s)=(x(t),\dot{x}(t))}$$

Remark. $\dot{x}$ does not depend on x:

$$\frac{\partial x}{\partial \dot{x}} = \frac{\partial \dot{x}}{\partial x} = 0$$

Remark. The Euler-Lagrange equation only applies to one-dimensional cases.

1.2 Configuration space and generalised coordinates

Definition 1.2.1. **Configuration space**, denoted C, is the set of all possible (in principle) instantaneous configurations for a given a physical system.

Remark. This definition includes positions, but does not include velocities.

Remark. A configuration space must be constructed before a Lagrangian is constructed. The Lagrangian describes the dynamics of this configuration space.

Example 1.2.2. A particle moving in $\mathbb{R}^d$ has configuration space $\mathbb{R}^d$.

Example 1.2.3. N distinct particles moving in $\mathbb{R}^d$ have configuration space $(\mathbb{R}^d)^N = \mathbb{R}^{dN}$. The configuration space would still be $\mathbb{R}^{dN}$ if the particles were electrically charged, as the charge of the particles does not affect their positions, at least initially.

Example 1.2.4. Two distinct particles joined by a rigid rod have configuration space $\mathbb{R}^{2d-1}$. One particle has configuration space $\mathbb{R}^d$ and there are $d-1$ angles that must specified to choose the position of the second particle relative to the other.

Definition 1.2.5. Let S be a physical system with configuration space C. Then S has $\dim(C)$ **degrees of freedom**.

Remark. For every configuration space, any choice of coordinate system is valid, and the Lagrangian formalism holds regardless of this choice.

Definition 1.2.6. For a configuration space C, a set of coordinates in this space is called a set of **generalised coordinates**. Often generalized coordinates are represented with q_i, $i \in \{1, \ldots, \dim(C)\}$ where $\underline{q}$ is the coordinate vector with components q_i.

Example 1.2.7. A particle moving in $\mathbb{R}^2$, with configuration space $\mathbb{R}^2$. We could use Cartesian or polar coordinates to describe the position of the particle in this space (both are equally valid).

Definition 1.2.8. Let C be a configuration space and let $\underline{q}(t) \in C$ be a path. For a Lagrangian function $L(\underline{q}, \dot{\underline{q}})$, the **Euler-Lagrange equations** state that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \forall i \in \{1, \ldots, \dim(C)\}$$

Remark. The Euler-Lagrange equations are valid in any coordinate system.

Remark. Similarly to the one-dimensional case:

$$\frac{\partial q_i}{\partial \dot{q}_j} = \frac{\partial \dot{q}_i}{\partial q_j} = 0$$

and

$$\frac{\partial q_i}{\partial q_j} = \frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \delta_{ij}$$

1.3 Lagrangians for classical mechanics

Definition 1.3.1. In a system with kinetic energy $T(\underline{q}, \dot{\underline{q}})$ and potential energy $V(\underline{q})$, the Lagrangian that describes the equations of motion in that system is given by

$$L(\underline{q}, \dot{\underline{q}}) = T(\underline{q}, \dot{\underline{q}}) - V(\underline{q})$$

1.4 Ignorable coordinates and conservation of generalised momenta

Definition 1.4.1. Let $\{q_1, \ldots, q_N\}$ be a set of generalised coordinates. A specific coordinates q_i is **ignorable** if the Lagrangian function expressed in these generalised coordinates does not depend on q_i, i.e. if

$$\frac{\partial L}{\partial q_i} = 0$$

Definition 1.4.2. The **generalised momentum** p_i associated with a generalised coordinate q_i is given by

$$p_i := \frac{\partial L}{\partial \dot{q}_i}$$

Proposition 1.4.3. The generalised momentum associated to an ignorable coordinate is conserved.

Proof. From the Euler-Lagrange equation for q_i,

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \frac{dp_i}{dt} - 0 = \frac{dp_i}{dt}$$

□

Example 1.4.4. For a free particle moving in d dimensions, in Cartesian coordinates we have

$$L = T - V = \frac{1}{2}m\sum_{i=1}^{d} \dot{x}_i^2$$

so every coordinate is ignorable. The generalised momenta are

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

So here the conservation of generalised momenta is the conservation of the linear momenta.

2 Symmetries, Noether's theorem and conservation laws

2.1 Ordinary symmetries

Definition 2.1.1. For a uniparametric family of smooth maps $\phi(\epsilon) : C \to C$ from configuration space to itself, with $\phi(0)$ the identity map, this family of maps is called a **transformation depending on** ϵ. In any coordinates system this transformation can be written as

$$q_i \to \phi_i(q_1, \dots, q_N, \epsilon)$$

where the ϕ_i's are a set of $N := \dim(C)$ functions representing the transformation in the coordinate system. The change in velocities is defined as

$$\dot{q}_i \to \frac{d}{dt}\phi_i$$

Remark. q_i' is used to denote $\phi(q_i, \epsilon)$, so often we write $q_i \to q_i' = \dots$, where $\dots$ is a function of q_i and ϵ.

Definition 2.1.2. The **generator** of ϕ is

$$\left.\frac{d\phi(\epsilon)}{d\epsilon}\right|_{\epsilon=0} := \lim_{\epsilon \to 0} \frac{\phi(\epsilon) - \phi(0)}{\epsilon}$$

In any coordinate system,

$$q_i \to \phi_i(\underline{q}, \epsilon) = q_i + \epsilon a_i(\underline{q}) + O(\epsilon^2)$$

where

$$a_i = \left.\frac{\partial \phi_i(\underline{q}, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}$$

is a function of the generalised coordinates. Hence the transformation generator is a_i. For the velocities the transformation is

$$\dot{q}_i \to \dot{q}_i + \epsilon \dot{a}_i(q_1, \dots, q_N, \dot{q}_1, \dots, \dot{q}_N) + O(\epsilon^2)$$

where the generator is $\dot{a}_i$.

3 Hamiltonian Formalism

Definition 3.0.1. The classical **state** of a system at a given instant in time is a **complete** set of data that fully specifies the future evolution of the system.

Remark. **Any** set of data that fully fixes future evolution is valid.

Definition 3.0.2. The **phase (or state) space** is the set of all possible states for a system at a given time.

Example 3.0.3. A free particle moving in $\mathbb{R}$. The phase space is $\mathbb{R}^2$ ($\mathbb{R}$ for position, $\mathbb{R}$ for velocity).

Definition 3.0.4. The **Hamiltonian formalism** studies dynamics in a phase space, parameterised by $\underline{q}(t)$ and $\underline{p}(t)$, where $p_i = \frac{\partial L}{\partial \underline{\dot{q}_i}}$, the momentum.

Example 3.0.5. A particle moving in $\mathbb{R}$, with $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2$.

Then $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ so $\dot{x}(x, p_x) = \frac{p_x}{m}$.

In the Hamltonian formalism, $L(x, p_x) = \frac{p_x^2}{2m}$.

Example 3.0.6. A particle moving in $\mathbb{R}^2$ (in polar coordinates).

$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. So $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$.

So $\dot{r}(r, \theta, p_r, p_\theta) = \frac{p_r}{m}$, $\dot{\theta}(r, \theta, p_r, p_\theta) = \frac{p_\theta}{mr^2}$.

$L(r, \theta, \dot{r}, \dot{\theta}) = L(r, \theta, p_r, p_\theta) = \frac{1}{2}(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2})$.

Definition 3.0.7. Given two functions $f(\underline{q}, \underline{p}, t)$ and $g(\underline{q}, \underline{p}, t)$ in phase space their **Poisson bracket** is:

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

where n is the dimension of the configuration space.

Remark. In the Hamiltonian formalism, $\frac{\partial q_i}{\partial p_j} = \frac{\partial p_j}{\partial q_i} = 0$.

Similarly, $\frac{\partial q_i}{\partial q_j} = \frac{\partial p_i}{\partial p_j} = \delta_{i,j}$

Example 3.0.8. Let $f = q_i$, $g = q_j$. $\{q_i, q_j\} = 0$, and $\{p_i, p_j\} = 0$. $\{q_i, p_j\} = \sum_{k=1}^{n} \delta_{i,j}\delta_{j,k} = \delta_{i,j}$.

Definition 3.0.9. Let $\mathbb{F}$ be the set functions from a phase space P to $\mathbb{R}$

Definition 3.0.10. The Hamiltonian flow $\Phi_f^{(s)}$, with $(s) \in \mathbb{R}$, $f \in F$ operator maps $\mathbb{F}$ to $\mathbb{F}$ and is defined as

$$\Phi_f^{(s)}(g) := e^{s\{\cdot, f\}}g := g + s\{g, f\} + \frac{s^2}{2}\{\{g, f\}, f\} + \cdots$$

Remark. The transformation generated by f has generator $a_i = \{q_i, f\}$ where $q_i \to q_i + \epsilon a_i$.

Infinitesimally, $\Phi_f^{(s)}(g) := g + \epsilon\{g, f\} + O(\epsilon^2)$

TODO: properties on poisson bracket

Example 3.0.11. (Rotation in $\mathbb{R}^2$ in Cartesian coordinates) As a guess, choose $f = q_1\dot{q_2} - \dot{q_1}q_2$, the angular momentum.

$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(q_1, q_2)$ so $p_1 = \frac{\partial L}{\partial \dot{q_1}} = \dot{q_1}$ and $p_2 = \frac{\partial L}{\partial \dot{q_2}} = \dot{q_2} \Rightarrow f = q_1 p_2 - q_2 p_1$.

Then $q_1 \to q_1 + \epsilon\{q_1, f\} + O(\epsilon^2) = q_1 + \epsilon\{q_1, q_1p_2 - q_2p_1\} = q_1 + \epsilon\{q_1, q_1p_2\} - \epsilon\{q_1, q_2p_1\} = q_1 + \epsilon\{q_1, q_1\}p_2 + \epsilon\{q_1, p_2\}q_1 - \epsilon\{q_1, q_2\}p_1 - \epsilon\{q_1, p_1\}q_2 = q_1 - \epsilon q_2$

Similarly, $q_2 \to q_2 + \epsilon q_1$ so $(q_1, q_2) \to (q_1, q_2) + \epsilon((0, -1), (1, 0))(q_1, q_2)$ TODO make into matrices and column vectors.

Definition 3.0.12. The **Hamiltonian** is the energy expressed in Hamiltonian coordinates:

$$H = \sum_{i=1}^{n} \dot{q_i(\underline{q}, \underline{p})} p_i - L(\underline{q}, \underline{\dot{q}}(\underline{q}, \underline{p}))$$

Example 3.0.13. (Harmonic oscillator in one dimension) Let $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \Rightarrow p = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$.

$H = \dot{x}p - L = \frac{p^2}{m} - (\frac{1}{2}\frac{p^2}{m} - \frac{1}{2}kx^2) = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2$

Theorem 3.0.14. The time evolution of the phase space coordinates $\underline{q}, \underline{p}$ is generated by Hamiltonian flow Φ_H:

$$q_i(t+a) = \Phi_H^{(a)} q_i(t), p_i(t+a) = \Phi_H^{(a)} p_i(t)$$

Infinitesimally, $q_i(t) + \epsilon\dot{q}_i(t) + O(\epsilon^2) = q_i(t+\epsilon) = q_i(t) + \epsilon\{q_i, H\} + O(\epsilon^2) \Leftrightarrow \dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}$ and similarly, $\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$.

These equations are called **Hamilton's equations**.

Proof. $\frac{\partial H}{\partial q_i}$. TODO: complete this proof, finish rest of notes from lecture. □

Corollary 3.0.15. The time evolution of any function $f(\underline{q}, \underline{p})$ in phase space is generated by Φ_H:

$$\frac{df}{dt} = \{f, H\}$$

If $f(\underline{q}, \underline{p}, t)$ depends explicitly on time then

$$\frac{df}{dt} = \{f, h\} + \frac{\partial f}{\partial t}$$

Proof. $\frac{df}{dt} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) + \frac{\partial f}{\partial t} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t}$. □