

1. Introduction

- Encryption process:
 - Alice has a message (**plaintext**) which is **encrypted** using an **encryption key** to produce the **ciphertext**, which is sent to Bob.
 - Bob uses a **decryption key** (which depends on the encryption key) to **decrypt** the ciphertext and recover the original plaintext.
 - It should be computationally infeasible to determine the plaintext without knowing the decryption key.

- **Caesar cipher:**

- Add constant k to each letter in plaintext to produce ciphertext:

$$\text{ciphertext letter} = \text{plaintext letter} + k \pmod{26}$$

- To decrypt,

$$\text{plaintext letter} = \text{ciphertext letter} - k \pmod{26}$$

- The key is $k \pmod{26}$.
- Cryptosystem objectives:
 - **Secrecy**: an intercepted message is not able to be decrypted
 - **Integrity**: it is impossible to alter a message without the receiver knowing
 - **Authenticity**: receiver is certain of identity of sender
 - **Non-repudiation**: sender cannot claim they sent a message; the receiver can prove they did.
- **Kerckhoff's principle**: a cryptographic system should be secure even if the details of the system are known to an attacker.
- Types of attack:
 - **Ciphertext-only**: the plaintext is deduced from the ciphertext.
 - **Known-plaintext**: intercepted ciphertext and associated stolen plaintext are used to determine the key.
 - **Chosen-plaintext**: an attacker tricks a sender into encrypting various chosen plaintexts and observes the ciphertext, then uses this information to determine the key.
 - **Chosen-ciphertext**: an attacker tricks the receiver into decrypting various chosen ciphertexts and observes the resulting plaintext, then uses this information to determine the key.

2. Symmetric key ciphers

- **Converting letters to numbers**: treat letters as integers modulo 26, with $A = 1, Z = 0 \equiv 26 \pmod{26}$. Treat string of text as vector of integers modulo 26.
- **Symmetric key cipher**: one in which encryption and decryption keys are equal.
- **Key size**: $\log_2(\text{number of possible keys})$.
- Caesar cipher is a **substitution cipher**. A stronger substitution cipher is this: key is permutation of $\{a, \dots, z\}$. But vulnerable to plaintext attacks and ciphertext-only attacks, since different letters (and letter pairs) occur with different frequencies in English.

- **One-time pad:** key is uniformly, independently random sequence of integers mod 26, (k_1, k_2, \dots) , known to sender and receiver. If message is (m_1, m_2, \dots, m_r) then ciphertext is $(c_1, c_2, \dots, c_r) = (k_1 + m_1, k_2 + m_2, \dots, k_r + m_r)$. To decrypt the ciphertext, $m_i = c_i - k_i$. Once (k_1, \dots, k_r) have been used, they must never be used again.
 - One-time pad is information-theoretically secure against ciphertext-only attack: $\mathbb{P}(M = m \mid C = c) = \mathbb{P}(M = m)$.
 - Disadvantage is keys must never be reused, so must be as long as message.
 - Keys must be truly random.
- **Chinese remainder theorem:** let $m, n \in \mathbb{N}$ coprime, $a, b \in \mathbb{Z}$. Then exists unique solution $x \bmod mn$ to the congruences

$$\begin{aligned} x &\equiv a \pmod{m} \\ x &\equiv b \pmod{n} \end{aligned}$$

- **Block cipher:** group characters in plaintext into blocks of n (the **block length**) and encrypt each block with a key. So plaintext $p = (p_1, p_2, \dots)$ is divided into blocks P_1, P_2, \dots where $P_1 = (p_1, \dots, p_n)$, $P_2 = (p_{n+1}, \dots, p_{2n})$, Then ciphertext blocks are given by $C_i = f(\text{key}, P_i)$ for some encryption function f .
- **Hill cipher:**
 - Plaintext divided into blocks P_1, \dots, P_r of length n .
 - Each block represented as vector $P_i \in (\mathbb{Z}/26\mathbb{Z})^n$
 - Key is invertible $n \times n$ matrix M with elements in $\mathbb{Z}/26\mathbb{Z}$.
 - Ciphertext for block P_i is

$$C_i = MP_i$$

It can be decrypted with $P_i = M^{-1}C_i$.

- Let $P = (P_1, \dots, P_r)$, $C = (C_1, \dots, C_r)$, then $C = MP$.
- **Confusion:** each character of ciphertext depends on many characters of key.
- **Diffusion:** each character of ciphertext depends on many characters of plaintext. Ideal diffusion is when changing single character of plaintext changes a proportion of $(S - 1)/S$ of the characters of the ciphertext, where S is the number of possible symbols.
- For Hill cipher, i th character of ciphertext depends on i th row of key - this is medium confusion. If j th character of plaintext changes and $M_{ij} \neq 0$ then i th character of ciphertext changes. M_{ij} is non-zero with probability roughly 25/26 so good diffusion.
- Hill cipher is susceptible to known plaintext attack:
 - If $P = (P_1, \dots, P_n)$ are n blocks of plaintext with length n such that P is invertible and we know P and the corresponding C , then we can recover M , since $C = MP \implies M = CP^{-1}$.
 - If enough blocks of ciphertext are intercepted, it is very likely that n of them will produce an invertible matrix P .

3. Public key encryption and RSA

- **Public key cryptosystem:**

- Bob produces encryption key, k_E , and decryption key, k_D . He publishes k_E and keeps k_D secret.
- To encrypt message m , Alice sends ciphertext $c = f(m, k_E)$ to Bob.
- To decrypt ciphertext c , Bob computes $g(c, k_D)$, where g satisfies

$$g(f(m, k_E), k_D) = m$$

for all messages m and all possible keys.

- Computing m from $f(m, k_E)$ should be hard without knowing k_D .
- **Converting between messages and numbers:**
- To convert message $m_1 m_2 \dots m_r$, $m_i \in \{0, \dots, 25\}$ to number, compute

$$m = \sum_{i=1}^r m_i 26^{i-1}$$

- To convert number m to message, add character $m \bmod 26$ to message. If $m < 26$, stop. Otherwise, floor divide m by 26 and repeat.
- **Fermat's little theorem:** let p prime, $a \in \mathbb{Z}$ coprime to p , then $a^{p-1} \equiv 1 \pmod{p}$.
- **Euler φ function:**

$$\varphi : \mathbb{N} \rightarrow \mathbb{N}, \varphi(n) = |\{1 \leq a \leq n : \gcd(a, n) = 1\}| = |(\mathbb{Z}/n\mathbb{Z})^\times|$$

- $\varphi(p^r) = p^r - p^{r-1}$, $\varphi(mn) = \varphi(m)\varphi(n)$ for $\gcd(m, n) = 1$.
- **Euler's theorem:** if $\gcd(a, n) = 1$, $a^{\varphi(n)} \equiv 1 \pmod{n}$.
- **RSA algorithm:**
 - k_E is pair (n, e) where $n = pq$, the **RSA modulus**, is product of two distinct primes and $e \in \mathbb{Z}$, the **encryption exponent**, is coprime to $\varphi(n)$.
 - k_D , the **decryption exponent**, is integer d such that $de \equiv 1 \pmod{\varphi(n)}$.
 - m is an integer modulo n , m and n are coprime.
 - Encryption: $c = m^e \pmod{n}$.
 - Decryption: $m = c^d \pmod{n}$.
 - It is recommended that n have at least 2048 bits. A typical choice of e is $2^{16} + 1$.
- **RSA problem:** given $n = pq$ a product of two unknown primes, e and $m^e \pmod{n}$, recover m . If n can be factored, the RSA is solved.
- **Factorisation problem:** given $n = pq$ for large distinct primes p and q , find p and q .
- **RSA signatures:**
 - Public key is (n, e) and private key is d .
 - When sending a message m , message is **signed** by also sending $s = m^d \bmod n$, the **signature**.
 - (m, s) is received, **verified** by checking if $m = s^e \bmod n$.
 - Forging a signature on a message m would require finding s with $m = s^e \bmod n$. This is the RSA problem.

- However, choosing signature s first then taking $m = s^e \bmod n$ produces valid pairs.
- To solve this, (m, s) is sent where $s = h(m)^d$, h is **hash function**. Then the message receiver verifies $h(m) = s^e \bmod n$.
- Now, for a signature to be forged, an attacker would have to find m with $h(m) = s^e \bmod n$.
- **Hash function** is function $h : \{\text{messages}\} \rightarrow \mathcal{H}$ that:
 - Can be computed efficiently
 - Is **preimage-resistant**: can't quickly find m given $h(m)$.
 - Is **collision-resistant**: can't quickly find m, m' such that $h(m) = h(m')$.
- **Attacks on RSA**:
 - If you can factor n , you can compute d , so can break RSA (as then you know $\varphi(n)$ so can compute $e^{-1} \bmod \varphi(n)$).
 - If $\varphi(n)$ is known, then we have $pq = n$ and $(p-1)(q-1) = \varphi(n)$ so $p+q = n - \varphi(n) + 1$. Hence p and q are roots of $x^2 - (n - \varphi(n) + 1)x + n$.
 - **Known d attack**:
 - $de - 1$ is multiple of $\varphi(n)$ so $p, q \mid x^{de-1} - 1$.
 - Look for factor K of $de - 1$ with $x^K - 1$ divisible by p but not q (or vice versa) (equivalently, $(p-1) \mid K$ but $(q-1) \nmid K$).
 - Let $de - 1 = 2^r s$, $\gcd(2, s) = 1$, choose random $x \bmod n$. Let $y = x^s$, then $y^{2^r} = x^{2^r s} = x^{de-1} \equiv 1 \bmod n$.
 - If $y \equiv 1 \bmod n$, restart with new random x . Find first occurrence of 1 in y, y^2, \dots, y^{2^r} : $y^{2^j} \not\equiv 1 \bmod n$, $y^{2^{j+1}} \equiv 1 \bmod n$ for some $j \geq 0$.
 - Let $a := y^{2^j}$, then $a^2 \equiv 1 \bmod n$, $a \not\equiv 1 \bmod n$. If $a \equiv -1 \bmod n$, restart with new random x .
 - Now $n = pq \mid a^2 - 1 = (a+1)(a-1)$ but $n \nmid (a+1), (a-1)$. So p divides one of $a+1, a-1$ and q divides the other. So $\gcd(a-1, n), \gcd(a+1, n)$ are prime factors of n .
- **Theorem**: it is no easier to find $\varphi(n)$ than to factorise n .
- **Theorem**: it is no easier to find d than to factor n .
- **Miller-Rabin algorithm** for probabilistic primality testing of n :
 1. Let $n-1 = 2^r s$, $\gcd(2, s) = 1$.
 2. Choose random $x \bmod n$, compute $y = x^s \bmod n$.
 3. Compute $y, y^2, \dots, y^{2^r} \bmod n$.
 4. If 1 isn't in this list, n is **composite** (with witness x).
 5. If 1 is in list preceded by number other than ± 1 , n is **composite** (with witness x).
 6. Other, n is **possible prime** (to base x).
- **Theorem**:
 - If n prime then it is possible prime to every base.
 - If n composite then it is possible prime to $\leq 1/4$ of possible bases.

In particular, if k random bases are chosen, probability of composite n being possible prime for all k bases is $\leq 4^{-k}$.

3.1. Factorisation

- **Trial division algorithm:** for $p = 2, 3, 5, \dots$ test whether $p \mid n$.
- If $x^2 \equiv y^2 \pmod n$ but $x \not\equiv \pm y \pmod n$, then $x - y$ is divisible by factor of n but not by n itself, so $\gcd(x - y, n)$ gives proper factor of n (or 1).
- **Fermat's method:**
 - Let $a = \lceil \sqrt{n} \rceil$. Compute $a^2 \pmod n$, $(a + 1)^2 \pmod n$ until a square $x^2 \equiv (a + i)^2 \pmod n$ appears. Then compute $\gcd(a + i - x, n)$.
 - Works well under special conditions on the factors: if $|p - q| \leq 2\sqrt{2}\sqrt[4]{n}$ then Fermat's method takes one step: $x = \lceil \sqrt{n} \rceil$ works.
- **Definition:** an integer is **B -smooth** if all its prime factors are $\leq B$.
- **Quadratic sieve:**
 - Choose B and let m be number of primes $\leq B$.
 - Look at integers $x = \lceil \sqrt{n} \rceil + k$, $k = 1, 2, \dots$ and check whether $y = x^2 - n$ is B -smooth.
 - Once $y_1 = x_1^2 - n, \dots, y_t = x_t^2 - n$ are all B -smooth with $t > m$, find some product of them that is a square.
 - Deduce a congruence between the squares.
 - Time complexity is $\exp(\sqrt{\log n \log \log n})$.

4. Diffie-Hellman key exchange

- **Primitive root theorem:** let p prime, then there exists $g \in \mathbb{F}_p^\times$ such that $1, g, \dots, g^{p-2}$ is complete set of residues mod p .
- Let p prime, $g \in \mathbb{F}_p^\times$. **Order** of g is smallest $a \in \mathbb{N}_0$ such that $g^a = 1$. g is **primitive root** if its order is $p - 1$ (equivalently, $1, g, \dots, g^{p-2}$ is complete set of residues mod p).
- Let p prime, $g \in \mathbb{F}_p^\times$ primitive root. If $x \in \mathbb{F}_p^\times$ then $x = g^L$ for some $0 \leq L < p - 1$. Then L is **discrete logarithm** of x to base g . Write $L = L_g(x)$.
- **Proposition:**
 - $g^{L_g(x)} \equiv x \pmod p$ and $g^a \equiv x \pmod p \iff a \equiv L_g(x) \pmod{p-1}$.
 - $L_g(1) = 0$, $L_g(g) = 1$.
 - $L_g(xy) \equiv L_g(x) + L_g(y) \pmod{p-1}$.
 - $L_g(x^{-1}) = -L_g(x) \pmod{p-1}$.
 - $L_g(g^a \pmod p) \equiv a \pmod{p-1}$.
 - h is primitive root mod p iff $L_g(h)$ coprime to $p - 1$. So number of primitive roots mod p is $\varphi(p - 1)$.
- **Discrete logarithm problem:** given p, g, x , compute $L_g(x)$.
- **Diffie-Hellman key exchange:**
 - Alice and Bob publicly choose prime p and primitive root $g \pmod p$.
 - Alice chooses secret $\alpha \pmod{p-1}$ and sends $g^\alpha \pmod p$ to Bob publicly.
 - Bob chooses secret $\beta \pmod{p-1}$ and sends $g^\beta \pmod p$ to Alice publicly.
 - Alice and Bob both compute shared secret $\kappa = g^{\alpha\beta} = (g^\alpha)^\beta = (g^\beta)^\alpha \pmod p$.
- **Diffie-Hellman problem:** given p, g, g^α, g^β , compute $g^{\alpha\beta}$.

- If discrete logarithm problem can be solved, so can Diffie-Hellman problem (since could compute $\alpha = L_g(g^a)$ or $\beta = L_g(g^b)$).
- **Elgamal public key encryption:**
 - Alice chooses prime p , primitive root g , private key $\alpha \bmod (p-1)$.
 - Her public key is $y = g^\alpha$.
 - Bob chooses random $k \bmod (p-1)$
 - To send message m (integer mod p), he sends the pair $(r, m') = (g^k, my^k)$.
 - To decrypt message, Alice computes $r^\alpha = g^{\alpha k} = y^k$ and then $m' r^{-\alpha} = m' y^{-k} = m g^{\alpha k} g^{-\alpha k} m$.
 - If Diffie-Hellman problem is hard, then Elgamal encryption is secure against known plaintext attack.
 - Key k must be random and different each time.
- **Decision Diffie-Hellman problem:** given g^a, g^b, c in \mathbb{F}_p^\times , decide whether $c = g^{ab}$.
 - This problem is not always hard, as can tell if g^{ab} is square or not. Can fix this by taking g to have large prime order $q \mid (p-1)$. $p = 2q + 1$ is a good choice.
- **Elgamal signatures:**
 - Public key is (p, g) , $y = g^\alpha$ for private key α .
 - **Valid Elgamal signature** on $m \in \{0, \dots, p-2\}$ is pair (r, s) , $0 \leq r, s \leq p-1$ such that

$$y^r r^s = g^m \pmod{p}$$
- Alice computes $r = g^k$, $k \in (\mathbb{Z}/(p-1))^\times$ random. k should be different each time.
- Then $g^{\alpha r} g^{ks} \equiv g^m \pmod{p}$ so $\alpha r + ks \equiv m \pmod{p-1}$ so $s = k^{-1}(m - \alpha r) \pmod{p-1}$.
- **Elgamal signature problem:** given p, g, y, m , find r, s such that $y^r r^s = m$.
- **Discrete logarithm problem:** given prime p , primitive root $g \bmod p$, $x \in \mathbb{F}_p^\times$, calculate $L_g(x)$.
- **Baby-step giant-step algorithm** for solving DLP:
 - Let $N = \lceil \sqrt{p-1} \rceil$.
 - Baby-steps: compute $g^j \bmod p$ for $0 \leq j < N$.
 - Giant-steps: compute $xg^{-Nk} \bmod p$ for $0 \leq k < N$.
 - Look for a match between baby-steps and giant-steps lists: $g^j = xg^{-Nk} \implies x = g^{j+Nk}$.
 - Always works since if $x = g^L$ for $0 \leq L < p-1 \leq N^2$, L can be written as $j + Nk$ with $j, k \in \{0, \dots, N-1\}$.
- **Index calculus** method for solving DLP $x = g^L$:
 - Fix smoothness bound B .
 - Find many multiplicative relations between B -smooth numbers and powers of $g \bmod p$.
 - Solve these relations to find discrete logarithms of primes $\leq B$.
 - For $i = 1, 2, \dots$ compute $xg^i \bmod p$ until one is B -smooth, then use result from previous step.

- **Pohlig-Hellman algorithm** computes discrete logarithms mod p with approximate complexity $\log(p)\sqrt{\ell}$ where ℓ is largest prime factor of $p - 1$, so is fast if $p - 1$ is B -smooth. Therefore p is chosen so that $p - 1$ has large prime factor, e.g. choose **Germain prime** $p = 2q + 1$, with q prime.

5. Elliptic curves

- **Definition: abelian group** (G, \circ) satisfies:
 - **Associativity:** $\forall a, b, c, \in G, a \circ (b \circ c) = (a \circ b) \circ c.$
 - **Identity:** $\exists e \in G : \forall g \in G, e \times g = g.$
 - **Inverses:** $\forall g \in G, \exists h \in G : g \circ h = h \circ g = e$
 - **Commutativity:** $\forall a, b \in G, a \circ b = b \circ a.$
- **Definition:** $H \subseteq G$ is **subgroup** of G if (H, \circ) is group.
- To show H is subgroup, sufficient to show $g, h \in H \Rightarrow g \circ h \in H$ and $h^{-1} \in H.$
- **Notation:** for $g \in G$, write $[n]g$ for $g \circ \dots \circ g$ n times if $n > 0$, e if $n = 0$, $[-n]g^{-1}$ if $n < 0$.
- **Definition:** subgroup generated by g is

$$\langle g \rangle = \{[n]g : n \in \mathbb{Z}\}$$

If $\langle g \rangle$ finite, it has **order** n , and g has **order** n . If $G = \langle g \rangle$ for some $g \in G$, G is **cyclic** and g is **generator**.

- **Lagrange's theorem:** let G finite group, H subgroup of G , then $|H| \mid |G|.$
- **Corollary:** if G finite, $g \in G$ has order n , then $n \mid |G|.$
- **DLP for abelian groups:** given G a cyclic abelian group, $g \in G$ a generator of G , $x \in G$, find L such that $[L]g = x$. L is well-defined modulo $|G|.$
- **Definition:** let $(G, \circ), (H, \bullet)$ abelian groups, **homomorphism** between G and H is $f : G \rightarrow H$ with

$$\forall g, g' \in G, \quad f(g \circ g') = f(g) \bullet f(g')$$

Isomorphism is bijective homomorphism. G and H are **isomorphic**, $G \cong H$, if there is isomorphism between them.

- **Fundamental theorem of finite abelian groups:** let G finite abelian group, then there exist unique integers $2 \leq n_1, \dots, n_r$ with $n_i \mid n_{i+1}$ for all i , such that

$$G \simeq (\mathbb{Z}/n_1) \times \dots \times (\mathbb{Z}/n_r)$$

In particular, G is isomorphic to product of cyclic groups.

- **Definition:** let K field, $\text{char}(K) > 3$. An **elliptic curve** over K is defined by the equation

$$y^2 = x^3 + ax + b$$

where $a, b \in K$, $\Delta_E := 4a^3 + 27b^2 \neq 0$.

- **Remark:** $\Delta_E \neq 0$ is equivalent to $x^3 + ax + b$ having no repeated roots (i.e. E is smooth).

- **Definition:** for elliptic curve E defined over K , a **K -point (point)** on E is either:
 - A **normal point**: $(x, y) \in K^2$ satisfying the equation defining E .
 - The **point at infinity** \overline{O} which can be thought of as infinitely far along the y -axis (in either direction).

Denote set of all K -points on E as $E(K)$.

- Any elliptic curve $E(K)$ is an abelian group, with group operation \oplus is defined as:
 - We should have $P \oplus Q \oplus R = \overline{O}$ iff P, Q, R lie on straight line.
 - In this case, $P \oplus Q = -R$.
 - To find line ℓ passing through $P = (x_0, y_0)$ and $Q = (x_1, y_1)$:
 - If $x_0 \neq x_1$, then equation of ℓ is $y = \lambda x + \mu$, where

$$\lambda = \frac{y_1 - y_0}{x_1 - x_0}, \quad \mu = y_0 - \lambda x_0$$

Now

$$\begin{aligned} y^2 &= x^3 + ax + b = (\lambda x + \mu)^2 \\ \implies 0 &= x^3 - \lambda^2 x^2 + (a - 2\lambda\mu)x + (b - \mu^2) \end{aligned}$$

Since sum of roots of monic polynomial is equal to minus the coefficient of the second highest power, and two roots are x_0 and x_1 , the third root is $x_2 = \lambda^2 - x_0 - x_1$. Then $y_2 = \lambda x_2 + \mu$ and $R = (x_2, y_2)$.

- If $x_0 = x_1$, then using implicit differentiation:

$$\begin{aligned} y^2 &= x^3 + ax + b \\ \implies \frac{dy}{dx} &= \frac{3x^2 + a}{2y} \end{aligned}$$

and the rest is as above, but instead with $\lambda = \frac{3x_0^2 + a}{2y_0}$.

- **Definition: group law** of elliptic curves: let $E : y^2 = x^3 + ax + b$. For all normal points $P = (x_0, y_0), Q = (x_1, y_1) \in E(K)$, define
 - \overline{O} is group identity: $P \oplus \overline{O} = \overline{O} \oplus P = P$.
 - If $P = -Q = (x_0, -y_0)$, $P \oplus Q = \overline{O}$.
 - Otherwise, $P \oplus Q = (x_2, -y_2)$, where

$$\begin{aligned} x_2 &= \lambda^2 - (x_0 + x_1), \\ y_2 &= \lambda x_2 + \mu, \\ \lambda &= \begin{cases} \frac{y_1 - y_0}{x_1 - x_0} & \text{if } x_0 \neq x_1 \\ \frac{3x_0^2 + a}{2y_0} & \text{if } x_0 = x_1 \end{cases}, \\ \mu &= y_0 - \lambda x_0 \end{aligned}$$

- **Example:**
 - Let E be given by $y^2 = x^3 + 17$ over \mathbb{Q} , $P = (-1, 4) \in E(\mathbb{Q})$, $Q = (2, 5) \in E(\mathbb{Q})$. To find $P \oplus Q$,

$$\lambda = \frac{5-4}{2-(-1)} = \frac{1}{3}, \quad \mu = 4 - \lambda(-1) = \frac{13}{3}$$

So $x_2 = \lambda^2 - (-1) - 2 = -\frac{8}{9}$ and $y_2 = -(\lambda x_2 + \mu) = -\frac{109}{27}$ hence

$$P \oplus Q = \left(-\frac{8}{9}, -\frac{109}{27} \right)$$

To find $[2]P$,

$$\lambda = \frac{3(-1)^2 + 0}{2 \cdot 4} = \frac{3}{8}, \quad \mu = 4 - \frac{3}{8} \cdot (-1) = \frac{35}{8}$$

so $x_3 = \lambda^2 - 2 \cdot (-1) = \frac{137}{64}$, $y_3 = -(\lambda x_3 + \mu) = -\frac{2651}{512}$ hence

$$[2]P = (x_3, y_3) = \left(\frac{137}{64}, -\frac{2651}{512} \right)$$

- **Hasse's theorem:** let $|E(\mathbb{F}_p)| = N$, then

$$|N - (p + 1)| \leq 2\sqrt{p}$$

- **Theorem:** $E(\mathbb{F}_p)$ is isomorphic to either \mathbb{Z}/k or $\mathbb{Z}/m \times \mathbb{Z}/n$ with $m \mid n$.
- **Elliptic curve Diffie-Hellman:**
 - Alice and Bob publicly choose elliptic curve $E(\mathbb{F}_p)$ and $P \in \mathbb{F}_p$ with order a large prime n .
 - Alice chooses random $\alpha \in \{0, \dots, n-1\}$ and publishes $Q_A = [\alpha]P$.
 - Bob chooses random $\beta \in \{0, \dots, n-1\}$ and publishes $Q_B = [\beta]P$.
 - Alice computes $[\alpha]Q_B = [\alpha\beta]P$, Bob computes $[\beta]Q_A = [\beta\alpha]P$.
 - Shared key is $K = [\alpha\beta]P$.
- **Elliptic curve Elgamal signatures:**
 - Use agreed elliptic curve E over \mathbb{F}_p , point $P \in E(\mathbb{F}_p)$ of prime order n .
 - Alice wants to sign message m , encoded as integer mod n .
 - Alice generates private key $\alpha \in \mathbb{Z}/n$ and public key $Q = [\alpha]P$.
 - Valid signature is (R, s) where $R = (x_R, y_R) \in E(\mathbb{F}_p)$, $s \in \mathbb{Z}/n$, $[\tilde{x}_R]Q \oplus [s]R = [m]P$.
 - To generate a valid signature, Alice chooses random $0 \neq k \in \mathbb{Z}/n$ and sets $R = [k]P$, $s = k^{-1}(m - \tilde{x}_R\alpha)$.
 - k must be randomly generated for each message.
- **Baby-step giant-step algorithm for elliptic curve DLP:** given P and $Q = [\alpha]P$, find α :
 - Let $N = \lceil \sqrt{n} \rceil$, n is order of P .
 - Compute $P, [2]P, \dots, [N-1]P$.
 - Compute $Q \oplus [-N]P, Q \oplus [-2N]P, \dots, Q \oplus [-(N-1)N]P$ and find a match between these two lists: $[i]P = Q \oplus [-jN]P$, then $[i+jN]P = Q$ so $\alpha = i + jN$.
- For well-chosen elliptic curves, the best algorithm for solving DLP is the baby-step giant-step algorithm, with run time $O(\sqrt{n}) \approx O(\sqrt{p})$. This is much slower than the index-calculus method for the DLP in \mathbb{F}_p^\times .

- **Pollard's $p - 1$ algorithm** to factorise $n = pq$:
 - Choose smoothness bound B .
 - Choose random $2 \leq a \leq n - 2$. Set $a_1 = a$, $i = 1$.
 - Compute $a_i = a_{i-1}^i \bmod n$. Find $d = \gcd(a_i - 1, n)$. If $1 < d < n$, we have found a nontrivial factor of n . If $d = n$, pick new a and retry. If $d = 1$, increment i by 1 and repeat this step.
 - A variant is instead of computing $a_i = a_{i-1}^i$, compute $a_i = a_{i-1}^{m_i}$ where m_1, \dots, m_r are the prime powers $\leq B$ (each prime power is the maximal prime power $\leq B$ for that prime).
 - The algorithm works if $p - 1$ is **B -powersmooth** (all prime power factors are $\leq B$), since if b is order of $a \bmod p$, then $b \mid (p - 1)$ so $b \mid B!$ (also $b \mid m_1 \cdots m_r$). If the first i for which $i!$ (or $m_1 \cdots m_i$) is divisible by d and order of $a \bmod q$, then $a_i - 1 = a^{i!} - 1 \bmod n$ is divisible by both p and q , so must retry with different a .
- Let $n = pq$, p, q prime, $a, b \in \mathbb{Z}$, $\gcd(4a^3 + 27b^2, n) = 1$. Then $E : y^2 = x^3 + ax + b$ defines elliptic curve over \mathbb{F}_p and over \mathbb{F}_q . If $(x, y) \in \mathbb{Z}/n$ is solution to $E \bmod n$ then can reduce coordinates $\bmod p$ to obtain non-infinite point of $E(\mathbb{F}_p)$ and $\bmod q$ to obtain non-infinite point of $E(\mathbb{F}_q)$.
- **Proposition:** let $P_1, P_2 \in E \bmod n$, with

$$\begin{aligned} (P_1 \bmod p) \oplus (P_2 \bmod p) &= \overline{O} \\ (P_1 \bmod q) \oplus (P_2 \bmod q) &\neq \overline{O} \end{aligned}$$

Then $\gcd(x_1 - x_2, n)$ (or $\gcd(2x_1, n)$ if $P_1 = P_2$) is factor of n .

- **Lenstra's algorithm** to factorise n :
 - Choose smoothness bound B .
 - Choose random elliptic curve E over \mathbb{Z}/n with $\gcd(\Delta_E, n) = 1$ and $P = (x, y)$ a point on E .
 - Set $P_1 = P$, attempt to compute P_i , $2 \leq i \leq B$ by $P_i = [i]P_{i-1}$. If one of these fails, a divisor of n has been found (by failing to compute an inverse $\bmod n$). If this divisor is trivial, restart with new curve and point.
 - If $i = B$ is reached, restart with new curve and point.
 - Again, a variant is calculating $P_i = [m_i]P_{i-1}$ instead of $[i]P_{i-1}$ where m_1, \dots, m_r are the prime powers $\leq B$.
- Lenstra's algorithm works if $|E(\mathbb{Z}/p)|$ is B -powersmooth but $|E(\mathbb{Z}/q)|$ isn't. Since we can vary E , it is very likely to work eventually.
- Running time depends on p (the smaller prime factor):

$$O\left(\exp\left(\sqrt{2 \log(p) \log \log(p)}\right)\right)$$

Compare this to the general number field sieve running time:

$$O\left(\exp\left(C(\log n)^{1/3}(\log \log n)^{2/3}\right)\right)$$

5.1. Torsion points

- **Definition:** let G abelian group. $g \in G$ is a **torsion** if it has finite order. If order divides n , then $[n]g = e$ and g is **n -torsion**.
- **Definition:** **n -torsion subgroup** is

$$G[n] := \{g \in G : [n]g = e\}$$

- **Definition:** **torsion subgroup** of G is

$$G_{\text{tors}} = \{g \in G : g \text{ is torsion}\} = \bigcup_{n \in \mathbb{N}} G[n]$$

- **Example:**
 - In \mathbb{Z} , only 0 is torsion.
 - In $(\mathbb{Z}/10)^\times$, by Lagrange's theorem, every point is 4-torsion.
 - For finite groups G , $G_{\text{tors}} = G = G[|G|]$ by Lagrange's theorem.

5.2. Rational points

- **Note:** for elliptic curve $E : y^2 = x^3 + ax + b$ over \mathbb{Q} , can assume that $a, b \in \mathbb{Z}$.
- **Nagell-Lutz theorem:** let E elliptic curve, let $P = (x, y) \in E(\mathbb{Q})_{\text{tors}}$. Then $x, y \in \mathbb{Z}$, and either $y = 0$ (in which case P is 2-torsion) or $y^2 \mid \Delta_E$.
- **Corollary:** $E(\mathbb{Q})_{\text{tors}}$ is finite.
- **Example:** can use Nagell-Lutz to show a point is not torsion.
 - $P = (0, 1)$ lies on elliptic curve $y^2 = x^3 - x + 1$. $[2]P = (\frac{1}{4}, -\frac{7}{8}) \notin \mathbb{Z}^2$. Then $[2]P$ is not torsion, hence P is not torsion. So $E(\mathbb{Q})$ contains distinct points $\dots, [-2]P, -P, \overline{O}, P, [2]P, \dots$, hence E has infinitely many solutions in \mathbb{Q} .
- **Mazur's theorem:** let E be elliptic curve over \mathbb{Q} . Then $E(\mathbb{Q})_{\text{tors}}$ is either:
 - cyclic of order $1 \leq N \leq 10$ or order 12, or
 - of the form $\mathbb{Z}/2 \times \mathbb{Z}/2N$ for $1 \leq N \leq 4$.
- **Definition:** let $E : y^2 = x^3 + ax + b$ defined over \mathbb{Q} , $a, b \in \mathbb{Z}$. For odd prime p , taking reductions $\bar{a}, \bar{b} \bmod p$ gives curve over \mathbb{F}_p :

$$\overline{E} : y^2 = x^3 + \bar{a}x + \bar{b}$$

This is elliptic curve if $\Delta_E \not\equiv 0 \bmod p$, in which case p is **prime of good reduction** for E .

- **Theorem:** let $E : y^2 = x^3 + ax + b$ defined over \mathbb{Q} , $a, b \in \mathbb{Z}$, p be odd prime of good reduction for E . Then $f : E(\mathbb{Q})_{\text{tors}} \rightarrow \overline{E}(\mathbb{F}_p)$ defined by

$$f(x, y) := (\bar{x}, \bar{y}), \quad f(\overline{O}) := \overline{O}$$

is injective (note $x, y \in \mathbb{Z}$ by Nagell-Lutz).

- So $E(\mathbb{Q})_{\text{tors}}$ can be thought of as subgroup of $E(\mathbb{F}_p)$ for any prime p of good reduction, so by Lagrange's theorem, $|E(\mathbb{Q})_{\text{tors}}|$ divides $|E(\mathbb{F}_p)|$.
- **Mordell's theorem:** if E is elliptic curve over \mathbb{Q} , then

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$$

for some $r \geq 0$ the **rank** of E . So for some $P_1, \dots, P_r \in E(\mathbb{Q})$,

$$E(\mathbb{Q}) = \{n_1 P_1 + \dots + n_r P_r + T : n_i \in \mathbb{Z}, T \in E(\mathbb{Q})_{\text{tors}}\}$$

P_1, \dots, P_r, T are generators for $E(\mathbb{Q})$.

6. Basic coding theory

6.1. First definitions

- **Definition:**
 - **Alphabet** A is finite set of symbols.
 - A^n is set of all lists of n symbols from A - these are **words of length n** .
 - **Code of block length n on A** is subset of A^n .
 - **Codeword** is element of a code.
- **Definition:** if $|A| = 2$, codes on A are **binary** codes. If $|A| = 3$, codes on A are **ternary** codes. If $|A| = q$, codes on A are **q -ary** codes. Generally, use $A = \{0, 1, \dots, q-1\}$.
- **Definition:** let $x = x_1 \dots x_n, y = y_1 \dots y_n \in A^n$. **Hamming distance** between x and y is number of indices where x and y differ:

$$d : A^n \times A^n \rightarrow \{0, \dots, n\}, \quad d(x, y) := |\{i \in [n] : x_i \neq y_i\}|$$

So $d(x, y)$ is minimum number of changes needed to change x to y . If x transmitted and y received, then $d(x, y)$ **symbol-errors** have occurred.

- **Proposition:** let x, y words of length n .
 - $0 \leq d(x, y) \leq n$.
 - $d(x, y) = 0 \iff x = y$.
 - $d(x, y) = d(y, x)$.
 - $\forall z \in A^n, d(x, y) \leq d(x, z) + d(z, y)$.
- **Definition:** **minimum distance** of code C is

$$d(C) := \min\{d(x, y) : x, y \in C, x \neq y\} \in \mathbb{N}$$

- **Notation:** code of block length n with M codewords and minimum distance d is called (n, M, d) (or (n, M)) code. A q -ary code is called an $(n, M, d)_q$ code.
- **Definition:** let $C \subseteq A^n$ code, x word of length n . A **nearest neighbour** of x is codeword $c \in C$ such that $d(x, c) = \min\{d(x, y) : y \in C\}$.

6.2. Nearest-neighbour decoding

- **Definition:** **nearest-neighbour decoding (NND)** means if word x received, it is decoded to a nearest neighbour of x in a code C .
- **Proposition:** let C be code with minimum distance d , let word x be received with t symbol errors. Then
 - If $t \leq d - 1$, then we can detect that x has some errors.
 - If $t \leq \lfloor \frac{d-1}{2} \rfloor$, then NND will correct the errors.

6.3. Probabilities

- **Definition:** **q -ary symmetric channel with symbol-error probability p** is channel for q -ary alphabet A such that:
 - For every $a \in A$, probability that a is changed in channel is p .
 - For every $a \neq b \in A$, probability that a is changed to b in channel is

$$\mathbb{P}(b \text{ received} \mid a \text{ sent}) = \frac{p}{q-1}$$

i.e. symbol-errors in different positions are independent events.

- **Proposition:** let c codeword in q -ary code $C \subseteq A^n$ sent over q -ary symmetric channel with symbol-error probability p . Then

$$\mathbb{P}(x \text{ received} \mid c \text{ sent}) = \left(\frac{p}{q-1}\right)^t (1-p)^{n-t}, \quad \text{where } t = d(c, x)$$

- **Example:** let $C = \{000, 111\} \subset \{0, 1\}^3$.

x	$t = d(000, x)$	chance 000 received as x	chance if $p = 0.01$	NND decodes correctly?
000	0	$(1-p)^3$	0.970299	yes
100	1	$p(1-p)^2$	0.009801	yes
010	1	$p(1-p)^2$	0.009801	yes
001	1	$p(1-p)^2$	0.009801	yes
110	2	$p^2(1-p)$	0.000099	no
101	2	$p^2(1-p)$	0.000099	no
011	2	$p^2(1-p)$	0.000099	no
111	3	p^3	0.000001	no

- **Corollary:** if $p < \frac{q-1}{q}$ then $P(x \text{ received} \mid c \text{ sent})$ increases as $d(x, c)$ decreases.
- **Remark:** by Bayes' theorem,

$$\mathbb{P}(c \text{ sent} \mid x \text{ received}) = \frac{\mathbb{P}(c \text{ sent and } x \text{ received})}{\mathbb{P}(x \text{ received})} = \frac{\mathbb{P}(c \text{ sent})\mathbb{P}(x \text{ received} \mid c \text{ sent})}{\mathbb{P}(x \text{ received})}$$

- **Proposition:** let C be q -ary (n, M, d) code used over q -ary symmetric channel with symbol-error probability $p < (q-1)/q$, and each codeword $c \in C$ is equally likely to be sent. Then for any word x , $\mathbb{P}(c \text{ sent} \mid x \text{ received})$ increases as $d(x, c)$ decreases.

6.4. Bounds on codes

- **Proposition (singleton bound):** for q -ary code (n, M, d) code, $M \leq q^{n-d+1}$.
- **Definition:** code which saturates singleton bound is called **maximum distance separable (MDS)**.
- **Example:** let C_n be **binary repetition code** of block length n ,

$$C_n := \{\underbrace{00\dots 0}_n, \underbrace{11\dots 1}_n\} \subset \{0, 1\}^n$$

C_n is $(n, 2, n)_2$ code, and $2 = 2^{n-n+1}$ so C_n is MDS code.

- **Definition:** let A be alphabet, $|A| = q$. Let $n \in \mathbb{N}$, $0 \leq t \leq n$, $t \in \mathbb{N}$, $x \in A^n$.
 - **Ball of radius t around x** is

$$S(x, t) := \{y \in A^n : d(y, x) \leq t\}$$

- Code $C \subseteq A^n$ is **perfect** if

$$\exists t \in \mathbb{N} : A^n = \coprod_{c \in C} S(c, t)$$

where \coprod is disjoint union.

- **Example:** for $C = \{000, 111\} \subset \{0, 1\}^3$, $S(000, 1) = \{000, 100, 010, 001\}$ and $S(111, 1) = \{111, 011, 101, 110\}$. These are disjoint and $S(000, 1) \cup S(111, 1) = \{0, 1\}^3$, so C is perfect.
- **Example:** let $C = \{111, 020, 202\} \subset \{0, 1, 2\}^3$. $\forall c \in C, d(c, 012) = 2$. So 012 is not in any $S(c, 1)$ but is in every $S(c, 2)$, so C is not perfect.
- **Lemma:** let $|A| = q$, $x \in A^n$, then

$$|S(x, t)| = \sum_{k=0}^t \binom{n}{k} (q-1)^k$$

- **Example:** let $C = \{111, 020, 202\} \subset \{0, 1, 2\}^3$, so $q = 3$, $n = 3$. So $|S(x, 1)| = \binom{3}{0} + \binom{3}{1}(3-1) = 7$, $|S(x, 2)| = \binom{3}{0} + \binom{3}{1}(3-1) + \binom{3}{2}(3-1)^2 = 19$. But $|\{0, 1, 2\}^3| = 27$ and $7 \nmid 27$, $19 \nmid 27$, so $\{0, 1, 2\}^3$ can't be partitioned by balls of either size. So C can't be perfect. $|S(x, 3)| = 27$, but then C must contain only one codeword to be perfect, and $|S(x, 0)| = 1$, but then $C = A^n$ to be perfect. These are trivial, useless codes.
- **Proposition (Hamming/sphere-packing bound):** q -ary (n, M, d) code satisfies

$$M \sum_{k=0}^t \binom{n}{k} (q-1)^k \leq q^n, \quad \text{where } t = \left\lfloor \frac{d-1}{2} \right\rfloor$$

- **Corollary:** code saturates Hamming bound iff it is perfect.

7. Linear codes

7.1. Finite vector spaces

- **Definition:** **linear code** of block length n is subspace of \mathbb{F}_q^n .
- **Example:** let $\mathbf{x} = (0, 1, 2, 0)$, $\mathbf{y} = (1, 1, 1, 1)$, $\mathbf{z} = (0, 2, 1, 0) \in \mathbb{F}_3^4$. $C_1 = \{\mathbf{x}, \mathbf{y}, \mathbf{0}\}$ is not linear code since e.g. $\mathbf{x} + \mathbf{y} = (1, 2, 0, 1) \notin C_1$. $C_2 = \{\mathbf{x}, \mathbf{z}, \mathbf{0}\}$ is linear code.
- **Notation:** spanning set of S is $\langle S \rangle$.
- **Proposition:** if linear code $C \subseteq \mathbb{F}_q^n$ has $\dim(C) = k$, then $|C| = q^k$.