1. The complex plane and Riemann sphere

- $\mathbb{C}^* = \mathbb{C} - \{0\}$
- $z_1 z_2 = 0 \iff z_1 = 0$ or $z_2 = 0$.
- $|z| = \sqrt{z\overline{z}}$.
- $\mathrm{Re}(z) = (z + \overline{z}) / 2$, $\mathrm{Im}(z) = (z - \overline{z}) / 2i$.
- $z^{-1} = \overline{z} / |z|^2$.
- **Principal value of** $\arg(\boldsymbol{z})$: in interval $(-\pi, \pi]$, written $\mathrm{Arg}(z)$.
- $\arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2) \pmod{2\pi}$.
- $\arg(1 / z) = -\arg(z) \pmod{2\pi}$.
- $\arg(\overline{z}) = -\arg(z) \pmod{2\pi}$.
- Multiplication by $z_1 = r_1 e^{i\theta_1}$ represents rotation by θ_1 followed by dilation by factor r_1.
- Addition represents translation.
- Conjugation represents reflection in the real axis.
- Taking the real (imaginary) part represents projection onto the real (imaginary) axis.
- $|z_1 z_2| = |z_1||z_2|$.
- $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.
- **Triangle inequality**: $|z_1 + z_2| \leq |z_1| + |z_2|$.
- $|z| \geq 0$ and $|z| = 0 \iff z = 0$.
- $\max\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\} \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.
- **Complex exponential function**:

$$\exp(z) := e^x(\cos(y) + i\sin(y))$$

.

- $\forall z \in \mathbb{C}, e^z = 0$.
- $e^{z_1 + z_2} = e^{z_1} e^{z_2}$.
- $e^z = 1 \iff z = 2\pi i k$ for some $k \in \mathbb{Z}$.
- $e^{-z} = 1 / e^z$.
- $|e^z| = e^{\mathrm{Re}(z)}$.
- $\forall k \in \mathbb{Z}, \exp(z) = \exp(z + 2k\pi i)$.
- $$\sin(z) := \frac{1}{2i}(e^{iz} - e^{-iz}), \quad \cos(z) := \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sinh(z) := \frac{1}{2}(e^z + e^{-z}), \quad \cosh(z) := \frac{1}{2}(e^z + e^{-z})$$
- For every $w \in \mathbb{C}^*$,

$$e^z = w = |w|e^{i\varphi}$$

has solutions

$$z = \log(|w|) + i(\varphi + 2k\pi), \quad k \in \mathbb{Z}$$

- Let $\theta_2 - \theta_1 = 2\pi$, let arg be the argument function in $(\theta_1, \theta_2]$. Then

$$\log(z) := \log(|z|) + i\arg(z)$$

is a **branch of logarithm**. Jump discontinuity on **branch cut**, the ray $R_{\theta_1} = R_{\theta_2}$.
- **Principal branch of log**: where $\arg(z) = \mathrm{Arg}(z) \in (-\pi, \pi]$.

- $e^{\log(z)} = z$.
- Generally, $\log(zw) \neq \log(z) + \log(w)$.
- Generally, $\log(e^z) \neq z$.
- Given a branch of log, **power function** is

$$z^w := \exp(w \log(z))$$

- $\hat{\mathbb{C}} = C \cup \{\infty\}$.
- Unit sphere: $S^2 = \left\{(x, y, s) \in \mathbb{R}^3 : x^2 + y^2 + s^2 = 1\right\}$, north pole: $N = (0, 0, 1) \in S^2$. **Stereographic projection**: map that takes $v \in S^2 - \{N\}$ to $x + iy \in \mathbb{C}$, where (x, y) is where the line from N through v intersects the (x, y)-plane.
- **Stereographic projection formula**:

$$P(x, y, s) = \frac{x}{1 - s} + \frac{iy}{1 - s}$$

  North pole is mapped to ∞.
- Inverse of stereographic projection found by intersection of line (from $z \in \mathbb{C}$ to N) and S^2 .
- **Riemann sphere**: unit sphere S^2 with stereographic projections from north and south pole.

2. Metric spaces

- **Metric space**: set X and **metric** function $d : X \times X \to \mathbb{R}_{\geq 0}$, for every $x, y, z \in X$
  - **positivity**: $d(x, y) \geq 0$ and $d(x, y) = 0 \Longleftrightarrow x = y$
  - **symmetry**: $d(x, y) = d(y, x)$
  - **triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- **norm** on vector space V:
  - $\|v\| \geq 0$ and $\|v\| = 0 \Longleftrightarrow v = 0$
  - $\|\lambda v\| = |\lambda| \cdot \|v\|$
  - $\|v + w\| \leq \|v\| + \|w\|$
- $d(v, w) = \|v - w\|$ always defines a metric
- $d(v, w) = \sqrt{\langle v - w, v - w \rangle}$
- $\boldsymbol{l_p}$ **norm**:

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

- **Taxicab norm**: l_1 norm
- $\boldsymbol{l_\infty}$ **norm (sup-norm)**: $\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$
- **Riemannian (chordal) metric on** $\hat{\mathbb{C}}$: $d(z, w) = \|f(z) - f(w)\|_2$ where $f : \hat{\mathbb{C}} \to S^2$ is the inverse stereographic projection.
- **Discrete metric**:

$$d(x, y) := \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Open ball of radius r centred at x**: $B_r(x) := \{y \in X : d(x, y) < r\}$

- **Closed ball of radius r centred at x**: $\overline{B}_r(x) := \{y \in X : d(x, y) \leq r\}$
- **$U \subseteq X$ open** if $\forall x \in U, \exists \varepsilon > 0, B_\varepsilon(x) \subset U$
- **$U \subseteq X$ closed** if $X - U$ open
- **clopen**: open and closed, e.g. empty set and X
- Open balls are open
- Closed balls are closed
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open
- Finite unions of closed sets are closed
- Arbitrary intersections of closed sets are closed
- **Interior of A**: $A^0 := \{x \in A : \text{for some open } U \subseteq A, x \in U\}$. It is the largest open set in A.
- **Closure of A**: complement of interior of complement: $\overline{A} := \{x \in X : U \cup A \neq \emptyset \text{ for every open set } U \text{ with } x \in U\} = X - (X - A)^0$. It is the smallest closed set containing A.
- **Boundary of A**: closure without interior: $\partial A := \overline{A} - A^0$
- **Exterior of A**: complement of closure: $A^e := X - \overline{A} = (X - A)^0$
- A is open $\iff \partial A \cap A = \emptyset \iff A = A^0$
- A is closed $\iff \partial A \subseteq A \iff A = \overline{A}$
- For simple sets in $\mathbb{R}^n$ or $\mathbb{C}^n$, closure (or interior) is obtained by replacing by replacing strict inequality with equality (or vice versa).
- Sequence $\{x_n\}$ **converges to** $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$ or equivalently,
$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, d(x_n, x) < \varepsilon$$
- Limits in the complex plane follow COLT rules
- $\{z_n\}$ converges iff $\{\mathrm{Re}(z_n)\}$ and $\{\mathrm{Im}(z_n)\}$ converge.
- $\lim_{n\to\infty} x_n = x \iff \forall$ open U with $x \in U, \exists N \in \mathbb{N}, \forall n > N, x_n \in U$
- $f : (X_1, d_1) \to (X_2, d_2)$ is **continuous at $x_0 \in X_1$** if
$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X_1, d_1(x, x_0) < \delta \implies d_2(f(x), f(x_0)) < \varepsilon$$
- f is **continuous on X_1** if continuous at every $x_0 \in X_1$
- Products, sums and quotients of real/complex continuous functions are continuous
- Compositions of continuous functions are continuous
- **Preimage**: $f^{-1}(U) := \{x \in X_1 : f(x) \in U\}$
- **Properties of preimage**:
  - $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
  - $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
  - $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$
- $$\begin{aligned} f : X_1 \to X_2 \text{ continuous} &\iff f^{-1}(U) \text{ open in } X_1 \forall \text{ open } U \subseteq X_2 \\ &\iff f^{-1}(F) \text{ closed in } X_1 \forall \text{ closed } F \subseteq X_2 \end{aligned}$$
- $f : X_1 \to X_2$ continuous at $x \in X_1 \iff f^{-1}(U)$ open in $X_1 \forall$ open $U \subseteq X_2$ containing $f(x)$
- Non-empty $K \subseteq X$ **compact** if for every sequence $\{x_k\}$ in K, there exists a convergent subsequence $\left\{x_{n_k}\right\}$ with limit in K.

- If $\{x_k\}$ is a convergent sequence in X then every subsequence $\left\{x_{n_k}\right\}$ converges to the same limit.
- $F \subseteq X$ is closed iff every sequence in F converging in X also converges in F.
- Compact sets are closed
- Every closed subset of a compact set is compact
- $A \subseteq X$ **bounded** if for some $R > 0$, $x \in X$, $A \subseteq B_R(x)$
- Compact sets are bounded
- **Heine-Borel for** $\mathbb{C}$: $K \subseteq \mathbb{C}$ is compact iff K is closed and bounded.
- $f : X \to Y$ is continuous at $x \in X$ iff

  $$\lim_{n \to \infty} f(x_n) = f(x)$$

  for every convergent sequence $\{x_n\}$ in X with $x_n \to x$.
- If $K \subseteq X$ is compact and $f : X \to Y$ is continuous, then $f(K)$ is compact in Y.

3. Complex differentiation

- $f : U \to \mathbb{C}$ for open U is **complex differentiable at $z_0 \in U$** if

  $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

  exists. Limit is the **derivative of f at z_0**:

  $$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

  . $h \in \mathbb{C}$ so limit must exist from every direction.
- Complex differentiability at z_0 implies continuity at z_0.
- Sums, products and quotients of complex differentiable functions are complex differentiable.
- Compositions of complex differentiable functions are complex differentiable.
- The product, quotient and chain rules hold for complex differentiable functions.
- Most non-constant purely real/imaginary functions are not complex differentiable.
- If $f = u + iv$ is complex differentiable at z_0 then u_x, u_y, v_x, v_y exist at z_0 and satisfy **Cauchy-Riemann equations**:

  $$u_x(z_0) = v_y(z_0), \quad u_y(z_0) = -v_x(z_0)$$

  . Also,

  $$f'(z_0) = u_x(z_0) + iv_x(z_0)$$
- Let $f : U \to \mathbb{C}$, U open, $f = u + iv$. If u_x, u_y, v_x, v_y exist and are continuous at z_0 and satisfy the Cauchy-Riemann equations at z_0, then f is complex differentiable at z_0.
- Let $U \subseteq C$ open, $f : U \to \mathbb{C}$. f is **holomorphic on U** if f is complex differentiable at every $z_0 \in U$.
- f is **holomorphic at $z_0 \in U$** if f is complex differentiable on some $B_\varepsilon(z_0)$.
- Affine linear maps $z \to az + b$, $a \neq 0$ are holomorphic.

- **Path (curve) from a to b**: continuous function $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = a$ and $\gamma(1) = b$. Path **closed** if $a = b$.
- **Smooth** path: continuously differentiable.
- $U \subseteq \mathbb{C}$ **path-connected** if for every $a, b \in U$, there exists a path γ from a to b with $\gamma(t) \in U$ for every $t \in [0, 1]$.
- **Domain (region)**: open and path-connected.
- **Chain rule**: Let $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ holomoprhic, $\gamma : [0, 1] \to U$ smooth path. Then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0)$$

- Let D domain, $f : D \to \mathbb{C}$ holomorphic on D. If $\forall z \in D, f'(z) = 0$, or f is purely real/imaginary, or f has constant real/imaginary part, or f has constant modulus, then f is constant on D.
- Let D domain, $f : D \to \mathbb{C}$ **conformal at z_0** if f preserves angle and orientation of any two tangent vectors at z_0. Equivalently, f preserves angle and orientation of any two smooth paths through z_0. f **conformal** if conformal at every $z_0 \in D$.
- If f holomorphic, $f'(z_0) \neq 0$ then f conformal at z_0.
- f transforms the tangent vector $\gamma'(t_0)$ by multiplying it by $f'(\gamma(t_0))$.
- If f is conformal at z_0, then f is complex differentiable at z_0 and $f'(z_0) \neq 0$.
- f is conformal on domain D iff f is holomorphic on D and $\forall z \in D, f'(z) \neq 0$.
- Conformal maps map orthogonal grids in the (x, y)-plane to orthogonal grids. (Grids can be made of arbitrary smooth curves, not necessarily straight lines).
- For D and D' domains, $f : D \to D'$ is **biholomorphic** if f holomorphic, bijection and $f^{-1} : D' \to D$ holomorphic. f is a **biholomorphism**. D and D' are **biholomorphic** if such an f exists and write $f : D \overset{\sim}{\to} D'$
- Affine linear maps $z \to az + b$, $a \neq 0$, are biholomorphic from $\mathbb{C}$ to $\mathbb{C}$.
- For D domain, set of all biholomorphic maps from D to D forms a group under composition, called **automorphism group of D**, $\operatorname{Aut}(D)$.

4. Mobius transformations

- $\mathrm{GL}_2(\mathbb{C}) := \{A \in M_2(\mathbb{C}) : \det(A) \neq 0\}$.
- Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$, then **Mobius transformation** is $M_T(z) = \infty$ if $cz + d = 0$, else

$$M_T(z) = \frac{az + b}{cz + d}$$

Also

$$M_T(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}$$

So $M_T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
- Let $k^2 = \det(T)$ then

$$M_{\frac{1}{k}T}(z) = \frac{\frac{az}{k} + \frac{b}{k}}{\frac{cz}{k} + \frac{d}{k}} = \frac{az + b}{cz + d} = M_T(z)$$

so any T can be scaled to $T' = \frac{1}{k}T$ so that $\det(T') = \det\left(\frac{1}{k}T\right) = \frac{1}{k^2}\det(T) = 1$.

- **Cayley map**: $M_T(z) = \frac{z-i}{z+i}$ where $T = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$.
- Cayley map maps $\mathbb{H} \to \mathbb{D}$.
- Set of Mobius transformations forms group under composition:
  - $M_{T_1} \circ M_{T_2} = M_{T_1 T_2}$.
  - $(M_T)^{-1} = M_{T^{-1}}$.
  - $M_T = \mathrm{Id} \Longleftrightarrow T = t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, t \in \mathbb{C}^*$.
- Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$. If $c = 0$, M_T is biholomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. If $c \neq 0$, M_T is biholomorphic from $\mathbb{C} - \left\{-\frac{d}{c}\right\}$ to $\mathbb{C} - \left\{\frac{a}{c}\right\}$.
- M_T conformal at every $z \in \mathbb{C}$ where $M_T(z) \neq \infty$.
- M_T is bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.
- z is **fixed point** of M_T if $M_T(z) = z$.
- If M_T is not identity map, then it has at most 2 fixed points in $\hat{\mathbb{C}}$. So if M_T has 3 fixed points in $\hat{\mathbb{C}}$, it is identity map.
- **Cross ratio** of distinct $z_0, z_1, z_2, z_3 \in \mathbb{C}$:

  $$(z_0, z_1; z_2, z_3) := \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

  If some $z_i = \infty$ then same definition but remove all differences involving z_i, so

  $$(\infty, z_1; z_2, z_3) := \frac{(z_1 - z_3)}{(z_1 - z_2)}$$

- **Three points theorem**: Let $\{z_1, z_2, z_3\}$, $\{w_1, w_2, w_3\}$ be sets of distinct ordered points in $\hat{\mathbb{C}}$. Then exists unique Mobius transformation f such that $f(z_i) = w_i$, $i = 1, 2, 3$, given by $F^{-1} \circ G$, where

  $$F(z) = (z, w_1; w_2, w_3), \quad G(z) = (z, z_1; z_2, z_3)$$

- **Mobius transformations preserve cross ratio**: For Mobius transformation f,

  $$(f(z_0), f(z_1); f(z_2), f(z_3)) = (z_0, z_1; z_2, z_3)$$

- **Strategy to find Mobius transformation from how it acts on three points**: since cross-ratio preserved, rearrange the equation

  $$(f(z), w_1; w_2, w_3) = (z, z_1; z_2, z_3)$$

- **Strategy to find image of domain D under M_T**:
  - Find image of boundary: $M_T(\partial D)$.
  - Find image of point $z_0 \in D$ in interior: $M_T(z_0)$.
  - Image D' is domain bounded by $M_T(\partial D)$ and containing $M_T(z_0)$.
- **Circline**: circle or line.
- Mobius transformations map circlines in $\hat{\mathbb{C}}$ to circlines in $\hat{\mathbb{C}}$.
- **Equations of circles and lines in $\mathbb{C}$**:

$$\gamma z\overline{z} - \alpha\overline{z} - \overline{\alpha}z + \beta = 0$$

is equation of circle if $\gamma = 1$ and $|\alpha|^2 - \beta > 0$, and equation of line if $\gamma = 0$ and $\alpha \neq 0$. Also, any circle or line can be described by this equation.

- Circle uniquely determined by three points, line determined by two points, so to determine how Mobius transformation maps circle, check where three points on circle are mapped.
- Circles through N in S^2 correspond to lines in $\hat{\mathbb{C}}$. Circles not through N correspond to circles in $\hat{\mathbb{C}}$ (via stereographic projection).
- For domain D, $\text{Mob}(D)$ is set of Mobius transformations that map D to D.
- **H2H**:

$$f \in \text{Mob}(\mathbb{H}) \Longleftrightarrow f = M_T, \quad T \in \text{SL}_2(\mathbb{R}) := \{A \in M_2(\mathbb{R}) : \det(A) = 1\}$$

- **D2D**:

$$f \in \text{Mob}(\mathbb{D}) \Longleftrightarrow f = M_T, \quad T \in \text{SU}(1,1) := \left\{A = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C}, \det(A) = 1\right\}$$

- **D2D***:
  - Every $f \in \text{Mob}(\mathbb{D})$ is of form

$$f(z) = e^{i\theta}\frac{z - z_0}{\overline{z_0}z - 1}$$

    where $z_0 \in \mathbb{D}$ is unique point such that $f(z_0) = 0$.
  - Every $f \in \text{Mob}(\mathbb{D})$ where $f(0) = 0$ is a rotation about 0.
- **Strategy to find biholomorphic map between two domains**: build up biholomorphic map from simpler known ones, e.g. Mobius transformations, Cayley map, translations.

5. Notions of convergence in complex analysis and power series

- For X and Y metric spaces, $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ **converges pointwise on X to f** if

$$\forall x \in X, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > n, \quad d_Y\big(f_n(x), f(x)\big) < \varepsilon$$

  $f(x) = \lim_{n\to\infty} f_n(x)$ is **limit function**.
- $\{f_n\}_{n\in\mathbb{N}}$ **converges uniformly on X to f** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \forall x \in X, \quad d_Y\big(f_n(x), f(x)\big) < \varepsilon$$

- Uniform convergence implies pointwise convergence.
- **Uniform limits of continuous functions are continuous**: let $\{f_n\}_{n\in\mathbb{N}}$ be all continuous on X and converge uniformly to f on X. Then f is continuous on X.
- **Test for uniform convergence**: let $\{f_n\} : X \to \mathbb{C}$ converge pointwise to f.
  - If $\forall x \in X, \left|f_n(x) - f(x)\right| \leq s_n$, $\{s_n\}$ is sequence with $\lim_{n\to\infty} s_n = 0$, then $\{f_n\}$ converges uniformly to f on X.

- If for some sequence $\{x_n\} \subset X$, $\left|f_n(x_n) - f(x_n)\right| \geq c$ for some $c > 0$, then f_n does not converge uniformly to f on X.
- **Weierstrass M-test**: Let $\left\{f_n\right\} : X \to \mathbb{C}$ satisfy

$$\forall x \in X, \left|f_n(x)\right| \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty$$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly to some f on X.
- Let $\left\{f_n\right\} : [a, b] \to \mathbb{R}$ be continuous and converge uniformly to f on $[a, b]$. Then

$$\forall c \in [a, b], \quad \lim_{n \to \infty} \int_a^c f_n(x) \, \mathrm{d}x = \int_a^c f(x) \, \mathrm{d}x$$

- $\left\{f_n\right\}$ **converges locally uniformly on X to f** if $\forall x \in X$, exists open $U \subset X$ containing x such that $\left\{f_n\right\}$ converges uniformly to f on U.
- Let $\left\{f_n\right\}$ be continuous on X and converge locally uniformly to f on X. Then f is continuous on X.
- **Local M-test**: let $\left\{f_n\right\} : X \to \mathbb{C}$ be continuous, such that $\forall y \in X$, exists open $U \subset X$ containing y and $M_n > 0$ with

$$\forall x \in U, \left|f_n(x)\right| \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty$$

Then $\sum_{n=1}^{\infty} f_n$ converges locally uniformly to continuous function on X.
- **Complex power series**:

$$\sum_{n=0}^{\infty} a_n (z - c)^n, \quad a_n, c \in \mathbb{C}$$

- Either:
  - Series converges only for $z = c$ ($R = 0$).
  - Series converges absolutely for $|z - c| < R \iff z \in B_R(c)$. R is **radius of convergence**, $B_R(c)$ is **disc of convergence** and diverges for $|z - c| > R$.
  - Series converges absolutely for all z ($R = \infty$).
- Power series with radius of convergence R converges absolutely on $B_r(c)$ for every $0 < r < R$. So series is locally uniformly convergent (but not uniformly convergent) on disc of convergence.
- **Term-by-term differentiation and integration preserve radius of convergence**: let $\sum_{n=0}^{\infty} a_n (z - c)^n$ have radius of convergence R. Then formal derivative and antiderivative

$$\sum_{n=1}^{\infty} n a_n (z - c)^{n-1}, \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - c)^{n+1}$$

have radius of convergence R.
- **Power series can be differentiated term-by-term in disc of convergence**: let $\sum_{n=0}^{\infty} a_n (z - c)^n$ have radius of convergence R and converge to $f : B_R(c) \to \mathbb{C}$. Then f is holomorphic on $B_R(c)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - c)^{n-1}$$

- Power series with $R > 0$ can be differentiated infinitely many times and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} k! \binom{n}{k} a_n (z - c)^{n-k}$$

  So $f^{(k)}(c) = k! a_k$.
- **Power series can be integrated term-by-term in disc of convergence**: power series with $R > 0$ has holomorphic antiderivative $F : B_R(c) \to \mathbb{C}$ given by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - c)^{n+1}$$