Contents

1. Set systems

1.1. Chains and antichains

Note. The ideas in combinatorics often occur in the proofs, so it is advisable to learn the techniques used in proofs, rather than just learning the results and not their proofs.

Definition. Let X be a set. A **set system** on X (also called a **family of subsets of** X) is a collection $\mathcal{A} \subseteq \mathbb{P}(X)$.

Notation. $X^{(r)} := \{A \subseteq X : |A| = r\}$ denotes the family of subsets of X of size r.

Remark. Usually, we take $X = [n] = \{1, ..., n\}$, so $|X^{(r)}| = \binom{n}{r}$.

Notation. For brevity, we write e.g. $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$.

Definition. We can visualise $\mathbb{P}(A)$ as a graph by joining nodes $A \in \mathbb{P}(X)$ and $B \in \mathbb{P}(X)$ if $|A \Delta B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa.

This graph is the **discrete cube Q_n**.

Alternatively, we can view Q_n as an n-dimensional unit cube $\{0, 1\}^n$ by identifying e.g. $\{1, 3\} \subseteq [5]$ with 10100 (i.e. identify A with $\mathbb{1}_A$, the characteri stic/indicator function of A).

Definition. $\mathcal{A} \subseteq \mathbb{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{A}$, $A \subseteq B$ or $B \subseteq A$.

Example.
- $\mathcal{A} = \{23, 1235, 123567\}$ is a chain.
- $\mathcal{A} = \{\emptyset, 1, 12, ..., [n]\} \subseteq \mathbb{P}([n])$ is a chain.

Definition. $\mathcal{A} \subseteq \mathbb{P}(X)$ is an **antichain** if $\forall A \neq B \in \mathcal{A}$, $A \nsubseteq B$.

Example.
- $\mathcal{A} = \{23, 137\}$ is an antichain.
- $\mathcal{A} = \{1, ..., n\} \subseteq \mathbb{P}([n])$ is an antichain.
- More generally, $\mathcal{A} = X^{(r)}$ is an antichain for any r.

Proposition. A chain and an antichain can meet at most once.

Proof (Hints). Trivial. □

Proof. By definition. □

Proposition. A chain $\mathcal{A} \subseteq \mathbb{P}([n])$ can have at most $n + 1$ elements.

Proof (Hints). Trivial. □

Proof. For each $0 \leq r \leq n$, $\mathcal{A}$ can contain at most 1 r-set (set of size r). □

Theorem (Sperner's Lemma). Let $\mathcal{A} \subseteq \mathbb{P}(X)$ be an antichain. Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$, i.e. the maximum size of an antichain is achieved by the set of $X^{(\lfloor n/2 \rfloor)}$.

Proof (Hints).
- Let $r < \frac{n}{2}$.

- Let G be bipartite subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$.
- By considering an expression and upper bound for number of S-$\Gamma(S)$ edges in G for each $S \subseteq X^{(r)}$, show that there is a matching from $X^{(r)}$ to $X^{(r+1)}$.
- Reason that this induces a matching from $X^{(r)}$ to $X^{(r-1)}$ for each $r > \frac{n}{2}$.
- Reason that joining these matchings together, together with length 1 chains of subsets of $X^{(\lfloor n/2 \rfloor)}$ not included in a matching, result in a partition of $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, and conclude result from here.

□

Proof.

- We use the idea: from "a chain meets each layer in ≤ 1 points, because a layer is an antichain", we try to decompose the cube into chains.
- We partition $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, so each subset of X appears exactly once in one chain. Then we are done (since to form an antichain, we can pick at most one element from each chain).
- To achieve this, it is sufficient to find:
  - For each $r < \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r+1)}$ (a matching is a set of disjoint edges, one for each point in $X^{(r)}$).
  - For each $r > \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r-1)}$.
- Then put these matchings together to form a set of chains, each passing through $X^{(\lfloor n/2 \rfloor)}$. If a subset $X^{(\lfloor n/2 \rfloor)}$ has a chain passing through it, then this chain is unique. The subsets with no chain passing through form their own one-element chain.
- By taking complements, it is enough to construct the matchings just for $r < \frac{n}{2}$ (since a matching from $X^{(r)}$ to $X^{(r+1)}$ induces a matching from $X^{(n-r-1)}$ to $X^{(n-r)}$: there is a correspondence between $X^{(r)}$ and $X^{(n-r)}$ by taking complements, and taking complements reverse inclusion, so edges in the induced matching are guaranteed to exist).
- Let G be the (bipartite) subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$.
- For any $S \subseteq X^{(r)}$, the number of S-$\Gamma(S)$ edges in G is $|S|(n-r)$ (counting from below) since there are $n-r$ ways to add an element.
- This number is $\leq |\Gamma(S)|\ (r+1)$ (counting from above), since $r+1$ ways to remove an element.
- Hence $|\Gamma(S)| \geq \frac{|S|\ (n-r)}{r+1} \geq |S|$ as $r < \frac{n}{2}$.
- So by Hall's theorem, since there is a matching from S to $\Gamma(S)$, there is a matching from $X^{(r)}$ to $X^{(r+1)}$.

□

Remark. The proof above doesn't tell us when we have equality in Sperner's Lemma.

Definition. For $\mathcal{A} \subseteq X^{(r)}$ $(1 \leq r \leq n)$, the **shadow** of $\mathcal{A}$ is the set of subsets which can be obtained by removing one element from a subset in $\mathcal{A}$:

$$\partial\mathcal{A} = \partial^{-}\mathcal{A} := \left\{B \in X^{(r-1)} : B \subseteq \mathcal{A} \text{ for some } A \in \mathcal{A}\right\}.$$

Example. Let $\mathcal{A} = \{123, 124, 134, 137\} \in [7]^{(3)}$. Then $\partial\mathcal{A} = \{12, 13, 23, 14, 24, 34, 17, 37\}$.

Proposition (Local LYM). Let $\mathcal{A} \subseteq X^{(r)}$, $1 \leq r \leq n$. Then

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

i.e. the proportion of the level occupied by $\partial\mathcal{A}$ is at least the proportion of the level occupied by $\mathcal{A}$.

Proof (Hints). Find equation and upper bound for number of $\mathcal{A}$-$\partial\mathcal{A}$ edges in Q_n. □

Proof.

- The number of $\mathcal{A}$-$\partial\mathcal{A}$ edges in Q_n is $|A|r$ (counting from above, since we can remove any of r elements from $|A|$ sets) and is $\leq |\partial\mathcal{A}|\ (n - r + 1)$ (since adding one of the $n - r + 1$ elements not in $A \in \partial\mathcal{A}$ to A may not result in a subset of $\mathcal{A}$).
- So $\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1} = \binom{n}{r-1}/\binom{n}{r}$.

□

Remark. For equality in Local LYM, we must have that $\forall A \in \mathcal{A}$, $\forall i \in A$, $\forall j \notin A$, we must have $(A - \{i\}) \cup \{j\} \in \mathcal{A}$, i.e. $\mathcal{A} = \emptyset$ or $X^{(r)}$ for some r.

Notation. Write $\mathcal{A}_r$ for $\mathcal{A} \cap X^{(r)}$.

Theorem (LYM Inequality). Let $\mathcal{A} \subseteq \mathbb{P}(X)$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

Proof (Hints).

- Method 1: show the result for the sum $\sum_{r=k}^{n}$ by induction, starting with $k = n$. Use local LYM, and that $\partial\mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are disjoint (and analogous results for lower levels).
- Method 2: let $\mathcal{C}$ be uniformly random maximal chain, find an expression for $\Pr(\mathcal{C}$ meets $\mathcal{A})$.
- Method 3: determine number of maximal chains in X, determine number of maximal chains passing through a fixed r-set, deduce maximal number of chains passing through $\mathcal{A}$.

□

Proof.

- Method 1: "bubble down with local LYM".
  - We trivially have that $\mathcal{A}_n/\binom{n}{n} \leq 1$.
  - $\partial\mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are disjoint, as $\mathcal{A}$ is an antichain.
  - So

$$\frac{|\partial \mathcal{A}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1.$$

- So by local LYM,

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1.$$

- Now, $\partial(\partial A_n \cup A_{n-1})$ and $\mathcal{A}_{n-2}$ are disjoint, as $\mathcal{A}$ is an antichain.
- So

$$\frac{|\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- So by local LYM,

$$\frac{|\partial A_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- So

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- Continuing inductively, we obtain the result.

- Method 2:
  - Choose uniformly at random a maximal chain $\mathcal{C}$ (i.e. $C_0 \subsetneq C_1 \subseteq \cdots \subsetneq C_n$ with $|C_r| = r$ for all r).
  - For any r-set A, $\Pr(A \in \mathcal{C}) = 1/\binom{n}{r}$, since all r-sets are equally likely.
  - So $\Pr(\mathcal{C} \text{ meets } \mathcal{A}_r) = |\mathcal{A}_r|/\binom{n}{r}$, since events are disjoint.
  - So $\Pr(\mathcal{C} \text{ meets } \mathcal{A}) = \sum_{r=0}^{n} |\mathcal{A}_r|/\binom{n}{r} \leq 1$ since events are disjoint (since $\mathcal{A}$ is an antichain).
- Method 3: equivalently, the number of maximal chains is $n!$, and the number through any fixed r-set is $r!(n-r)!$, so $\sum_r |\mathcal{A}_r| r!(n-r)! \leq n!$.

□

Remark. To have equality in LYM, we must have equality in each use of local LYM in proof method 1. In this case, the maximum r with $\mathcal{A}_r \neq \emptyset$ has $\mathcal{A}_r = X^{(r)}$. So equality holds iff $\mathcal{A} = X^{(r)}$ for some r. Hence equality in Sperner's Lemma holds iff $\mathcal{A} = X^{(\lfloor n/2 \rfloor)}$ or $\mathcal{A} = X^{(\lceil n/2 \rceil)}$.

1.2. Two total orders on $X^{(r)}$

Definition. Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **lexicographic (lex)** ordering if for some j, we have $a_i = b_i$ for all $i < j$, and $a_j < b_j$.

Example. The elements of $[4]^{(2)}$ in lexicographic order are $12, 13, 14, 23, 24, 34$. The elements of $[6]^{(3)}$ in lexicographic order are
$123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456$.

Definition. Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **colexicographic (colex)** order if for some j, we have $a_i = b_i$ for all $i > j$, and $a_j < b_j$. “avoid large elements”.

Example. The elements of $[4]^{(2)}$ in colex order are $12, 13, 23, 14, 24, 34$. The elements of $[6]^{(3)}$ are
$123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 146, 246, 346, 156, 256, 356, 456.$

Remark. Lex and colex are both total orders. Note that in colex, $[n-1]^{(r)}$ is an initial segment of $[n]^{(r)}$ (this does not hold for lex). So we can view colex as an enumeration of $\mathbb{N}^{(r)}$.

Remark. $A < B$ in colex iff $A^c < B^c$ in lex with ground set order reversed.

Remark. We want to show that if $\mathcal{A} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{A}|$, then $|\partial\mathcal{C}| \leq |\partial\mathcal{A}|$. In particular, if $|\mathcal{A}| = \binom{k}{r}$, then $|\partial\mathcal{A}| \geq \binom{k}{r-1}$.

1.3. Compressions

Remark. We want to transform $\mathcal{A} \subseteq X^{(r)}$ into some $\mathcal{A}' \subseteq X^{(r)}$ such that:
- $|\mathcal{A}'| = |\mathcal{A}|$,
- $|\partial\mathcal{A}'| \leq |\partial\mathcal{A}|$.

Ideally, we want a family of such “compressions” $\mathcal{A} \to \mathcal{A}' \to ... \to \mathcal{B}$ such that either $\mathcal{B} = \mathcal{C}$, or $\mathcal{B}$ is similar enough to $\mathcal{C}$ that we can directly check that $|\partial\mathcal{B}| \geq |\partial\mathcal{C}|$.

1.4. Shadows

Remark. By Local LYM, we know that $|\partial\mathcal{A}| \geq |\mathcal{A}|r/(n-r+1)$. Equality is rare (only for $\mathcal{A} = X^{(r)}$ for $0 \leq r \leq n$). What happens in between, i.e., given $|\mathcal{A}|$, how should we choose $\mathcal{A}$ to minimise $|\partial A|$?

You should be able to convince yourself that if $|\mathcal{A}| = \binom{k}{r}$, then we should take $\mathcal{A} = [k]^{(r)}$. If $\binom{k}{r} < |\mathcal{A}| < \binom{k+1}{r}$, then convince yourself that we should take some $[k]^{(r)}$ plus some r-sets in $[k+1]^{(r)}$.

E.g. for $\mathcal{A} \subseteq X^{(r)}$ with $|\mathcal{A}| = \binom{8}{3} + \binom{4}{2}$, take $\mathcal{A} = [8]^{(3)} \cup \{9 \cup B : B \in [4]^{(2)}\}$.

2. Isoperimetric inequalities

3. Intersecting families