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Question: toss a fair coin $n = 10000$ times. How many heads?

$X = \sum_{i=1}^n X_i$, $X_i \sim \text{Bern}(1/2)$. $\mathbb{E}[X] = 5000$. But $\mathbb{P}(X = 5000) = \binom{10^4}{5000} \cdot 2^{-10^4} \approx 0.008$.

By WLLN, $\mathbb{P}(X \in [5000 - n\varepsilon, 5000 + n\varepsilon]) \approx 1$.

Theorem 0.1 (Central Limit Theorem) Let X_1, \dots, X_n be IID RVs with mean $\mathbb{E}[X_1] = \mu$. Let $\text{Var}(X_1) = \sigma^2 < \infty$. Then $\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n (X_i - \mu) \xrightarrow{D} N(0, 1)$, i.e.

$$\mathbb{P}\left(\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n (X_i - \mu) \in A\right) \rightarrow \int_A \frac{1}{\sqrt{2n}} e^{-x^2/2} dx$$

for all A .

So $\mathbb{P}\left(X \in \left[\frac{n}{2} - \frac{\sqrt{n}}{2} Q^{-1}(\delta), \frac{n}{2} + \frac{\sqrt{n}}{2} Q^{-1}(\delta)\right]\right) \geq 1 - \delta$, for n large enough, where $Q(\delta) = \int_{\delta}^{\infty} \frac{1}{\sqrt{2n}} e^{-x^2/2} dx$. We have $Q^{-1}(x) \propto \sqrt{\log \frac{1}{x}}$. So interval has length $\propto \sqrt{n} \sqrt{\log \frac{1}{\delta}}$.

Theorem 0.2 (Chebyshev's Inequality) $\mathbb{P}(|X - \mu| \geq \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}$ for all $\varepsilon > 0$.

Corollary 0.3 $\mathbb{P}\left(\left|\sum_{i=1}^n (X_i) - \frac{n}{2}\right| \geq t\right) \leq \frac{\text{Var}(\sum_{i=1}^n X_i)}{t^2} = n \frac{\sigma^2}{t^2} \leq \delta$ where $t = \sqrt{n}\sigma/\sqrt{\delta}$.

So $\mathbb{P}(X \in [\frac{n}{2} - \frac{n}{2}\sqrt{\delta}, \frac{n}{2} + \frac{n}{2}\sqrt{\delta}]) \geq 1 - \delta$.

Question 2: we have N coupons. Each day receive one uniformly at random independent of the past. How many days until all coupons received?

We have $X = \sum_{i=1}^n X_i$, $X_i \sim \text{Geom}(\frac{1}{n})$. $\mathbb{E}[X] = \sum_i \mathbb{E}[X_i] \approx n \log n$ (verify this).

Question 3: Let $(X_1, \dots, X_n), (Y_1, \dots, Y_n)$ be IID. What is the longest common subsequence, i.e. $f(X_1, \dots, X_n, Y_1, \dots, Y_n) = \max\{k : \exists i_1, \dots, i_k, j_1, \dots, j_k \text{ s.t. } X_{i_j} = Y_{j_j} \forall j \in [k]\}$. Computing f is NP-hard. f is smooth.

Principle: a smooth function of many independent random variables concentrates around its mean.

Theorem 0.4 (Law of Total Expectation) We have $\mathbb{E}_Y[\mathbb{E}_X[X | Y]] = \mathbb{E}_X[X]$.

Theorem 0.5 (Tower Property of Conditional Expectation) We have $\mathbb{E}[\mathbb{E}[Z | X, Y] | Y] = \mathbb{E}[Z | Y]$.

Theorem 0.6 We have $\mathbb{E}[f(Y)X | Y] = f(Y)\mathbb{E}[X | Y]$.

Theorem 0.7 (Holder's Inequality) Let $p \geq 1$ and $1/p + 1/q = 1$. Then

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{1/p} \cdot \mathbb{E}[|Y|^q]^{1/q}.$$

Theorem 0.8 (Cauchy-Schwarz) A special case of Holder's inequality:

$$\mathbb{E}[|XY|] \leq \mathbb{E}[X^2]^{1/2} \cdot \mathbb{E}[Y^2]^{1/2}.$$

Definition 0.9 The **conditional variance** of Y given X is the random variable

$$\text{Var}(Y | X) := \mathbb{E}[(Y - \mathbb{E}[Y | X])^2 | X].$$

1. The Chernoff-Cramer method

1.1. The Chernoff bound and Cramer transform

Theorem 1.1 (Weak Law of Large Numbers) Let X_1, \dots, X_n be IID RVs with mean $\mathbb{E}[X_1] = \mu$. Then, for all $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mu\right| > \varepsilon\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Theorem 1.2 (Markov's Inequality) Let Y be a non-negative RV. For any $t \geq 0$,

$$\mathbb{P}(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}.$$

Proof (Hints). Split Y using indicator variables. □

Proof. We have $Y = Y \cdot \mathbb{I}_{\{Y \geq t\}} + Y \cdot \mathbb{I}_{\{Y < t\}} \geq t \cdot \mathbb{I}_{\{Y \geq t\}}$. Taking expectations gives the result. □

Corollary 1.3 Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ be non-decreasing, then

$$\mathbb{P}(Y \geq t) \leq \mathbb{P}(\varphi(Y) \geq \varphi(t)) \leq \frac{\mathbb{E}[\varphi(Y)]}{\varphi(t)}.$$

For $\varphi(t) = t^2$, we can use $\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i)$.

Corollary 1.4 (Chebyshev's Inequality) For any RV Y and $t > 0$,

$$\mathbb{P}(|Y - \mathbb{E}[Y]| \geq t) \leq \frac{\text{Var}(Y)}{t^2}.$$

Proof (Hints). Straightforward. □

Proof. Take $Z = |Y - \mathbb{E}[Y]|$ and use Corollary 1.3 with $\varphi(t) = t^2$. □

Exercise 1.5 Prove WLLN, assuming that $\text{Var}(X_1) < \infty$, using Chebyshev's inequality.

Remark 1.6 If higher moments exist, we can use them in a similar way: let $\varphi(t) = t^q$ for $q > 0$, then for all $t > 0$,

$$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq t) \leq \frac{\mathbb{E}[|Z - \mathbb{E}[Z]|^q]}{t^q}.$$

We can then optimise over q to pick the lowest bound on $\mathbb{P}(|Z - \mathbb{E}[Z]| \geq t)$. Note that **Chebyshev's Inequality** is the most popular form of this bound due to the additivity of variance.

Definition 1.7 The **moment generating function (MGF)** of F is

$$F(\lambda) := \mathbb{E}[e^{\lambda Z}] = \sum_{k=0}^{\infty} \frac{\lambda^k \mathbb{E}[Z^k]}{k!}.$$

Definition 1.8 The **log-MGF** of Z is $\psi_Z(\lambda) = \log F(\lambda)$.

Note that $\psi_Z(\lambda)$ is additive: if $Z = \sum_{i=1}^n Z_i$, with Z_1, \dots, Z_n independent, then

$$\psi_Z(\lambda) = \log(\mathbb{E}[e^{\lambda Z}]) = \sum_{i=1}^n \log \mathbb{E}[e^{\lambda Z_i}] = \sum_{i=1}^n \psi_{Z_i}(\lambda).$$

Definition 1.9 The **Cramer transform** of Z is

$$\psi_Z^*(t) = \sup\{\lambda t - \psi_Z(\lambda) : \lambda > 0\}.$$

Proposition 1.10 (Chernoff Bound) Let Z be an RV. For all $t > 0$,

$$\mathbb{P}(Z \geq t) \leq e^{-\psi_Z^*(t)}.$$

Proof (Hints). Use Corollary 1.3. □

Proof. By Corollary 1.3, we have

$$\mathbb{P}(Z \geq t) \leq \frac{\mathbb{E}[e^{\lambda Z}]}{e^{\lambda t}} = e^{-(\lambda t - \psi_Z(\lambda))}.$$

Taking the infimum over all $\lambda > 0$ gives $\mathbb{P}(Z \geq t) \leq \inf\{e^{-(\lambda t - \psi_Z(\lambda))} : \lambda > 0\}$, which gives the result. □

Remark 1.11 Our goal is to obtain an upper bound on $\psi_Z(\lambda)$, as this will give exponential concentration. The function $\psi_{Z - \mathbb{E}[Z]}(\lambda)$ gives upper bounds on $\mathbb{P}(Z - \mathbb{E}[Z] \geq t)$, the function $\psi_{-Z + \mathbb{E}[Z]}(\lambda)$ gives upper bounds on $\mathbb{P}(Z - \mathbb{E}[Z] \leq -t)$.

Proposition 1.12

1. $\psi_Z(\lambda)$ is convex and infinitely differentiable on $(0, b)$, where $b = \sup\{\lambda > 0 : \psi_Z(\lambda) < \infty\}$.
2. $\psi_Z^*(t)$ is non-negative and convex.
3. If $t > \mathbb{E}[Z]$, then $\psi_Z^*(t) = \sup_{\lambda \in \mathbb{R}}\{\lambda t - \psi_Z(\lambda)\}$, the **Fenchel-Legendre** dual.

Proof (Hints).

1. Differentiability proof omitted. For convexity, use Holder's Inequality.
2. Straightforward (note that each $t \mapsto \lambda t - \psi_Z(\lambda)$ is linear).
3. Straightforward. □

Proof.

1. $\psi_Z(\alpha\lambda_1 + (1 - \alpha)\lambda_2) = \log \mathbb{E}[e^{\alpha\lambda_1 Z} \cdot e^{(1 - \alpha)\lambda_2 Z}] \leq \alpha \log \mathbb{E}[e^{\lambda_1 Z}] + (1 - \alpha) \log \mathbb{E}[e^{\lambda_2 Z}]$ by Holder's inequality. The differentiability proof is omitted.
2. $\lambda t - \psi_Z(\lambda)|_{\lambda=0} = 0$, so $\psi_Z^*(t) \geq 0$ by definition. Convexity follows since it is a supremum of linear functions.
3. By convexity and Jensen's inequality, $\mathbb{E}[e^{\lambda Z}] \geq e^{\lambda \mathbb{E}[Z]}$. So for $\lambda < 0$, $\lambda t - \psi_Z(\lambda) \leq \lambda(t - \mathbb{E}[Z]) < 0 = \lambda t - \psi_Z(\lambda)|_{\lambda=0}$. □

Example 1.13 Let $Z \sim N(0, \sigma^2)$. Then the MGF of Z is

$$\begin{aligned}
\mathbb{E}[e^{\lambda Z}] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} e^{\lambda x} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x^2 - 2\lambda\sigma^2 x + \lambda^2\sigma^4)/2\sigma^2} e^{\lambda^2\sigma^2/2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x - \lambda\sigma^2)^2/2\sigma^2} e^{\lambda^2\sigma^2/2} dx \\
&= e^{\lambda^2\sigma^2/2}.
\end{aligned}$$

So $\psi_Z(\lambda) = \frac{\lambda^2\sigma^2}{2}$. By Proposition 1.12, for $t > 0 = \mathbb{E}[Z]$, the Cramer transform is

$$\psi_Z^*(t) = \sup_{\lambda \in \mathbb{R}} \{\lambda t - \lambda^2\sigma^2/2\} =: \sup_{\lambda \in \mathbb{R}} g(\lambda).$$

We have $g'(\lambda) = t - \lambda\sigma^2 = 0$ iff $\lambda = t/\sigma^2$. So $\psi_Z^*(t) = t^2/\sigma^2 - \sigma^2 t^2/2\sigma^4 = t^2/2\sigma^2$. So Chernoff Bound gives

$$\mathbb{P}(Z \geq t) \leq e^{-t^2/2\sigma^2}.$$

Definition 1.14 Let X be an RV with $\mathbb{E}[X] = 0$. X is **sub-Gaussian** with variance parameter ν if

$$\psi_X(\lambda) \leq \frac{\lambda^2\nu}{2} \quad \forall \lambda \in \mathbb{R},$$

i.e. if its log MGF is less than that of a normally distributed random variable with mean 0 and variance ν . The set of all such sub-Gaussian variables is denoted $\mathcal{G}(\nu)$.

Proposition 1.15 For any sub-Gaussian RV X ,

1. If $X \in \mathcal{G}(\nu)$, then $\mathbb{P}(X \geq t), \mathbb{P}(X \leq -t) \leq e^{-t^2/2\nu}$ for all $t > 0$.
2. If X_1, \dots, X_n are independent with each $X_i \in \mathcal{G}(\nu_i)$ then $\sum_{i=1}^n X_i \in \mathcal{G}(\sum_{i=1}^n \nu_i)$.
3. If $X \in \mathcal{G}(\nu)$, then $\text{Var}(X) \leq \nu$.

Proof. Exercise. □

Definition 1.16 The **Gamma function** is defined as

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt.$$

Theorem 1.17 Let $\mathbb{E}[X] = 0$. TFAE for suitable choices of ν, b, c, d :

1. $X \in \mathcal{G}(\nu)$.
2. $\mathbb{P}(X \geq t), \mathbb{P}(X \leq -t) \leq e^{-t^2/2b}$ for all $t > 0$.
3. $\mathbb{E}[X^{2q}] \leq q!c^q$ for all $q \geq \mathbb{N}$.
4. $\mathbb{E}[e^{dX^2}] \leq 2$.

Proof (Hints).

- (1 \Rightarrow 2): straightforward.
- (2 \Rightarrow 3): Explain why we can assume $b = 1$. Use that $\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y > t) dt$ for $Y \geq 0$, and the Γ function.

- (3 \Rightarrow 1): show that $\mathbb{E}[e^{\lambda X}] \leq \mathbb{E}[e^{\lambda(X-X')}]$ where X' is an IID copy of X . Show that $\mathbb{E}[(X - X')^{2q}] \leq 2^{2q} \cdot \mathbb{E}[X^{2q}]$. Expand $\mathbb{E}[e^{\lambda(X-X')}]$ as a series. Conclude that $X \in \mathcal{G}(4c)$.
- (3 \Leftrightarrow 4): exercise.

□

Proof. (1 \Rightarrow 2) instantly follows (with $b = \nu$) by Proposition [1.15](#).

(2 \Rightarrow 3): WLOG, $b = 1$. Otherwise consider $\tilde{X} = X/\sqrt{b}$. Recall that for $Y \geq 0$, $\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y > t) dt$. Now

$$\begin{aligned}
\mathbb{E}[X^{2q}] &= \int_0^\infty \mathbb{P}(X^{2q} > t) dt = \int_0^\infty \mathbb{P}(|X| > t^{1/2q}) dt \\
&\leq 2 \int_0^\infty e^{-t^{1/q}/2} dt \\
&= 2 \cdot 2^q \cdot q \int_0^\infty u^{q-1} e^{-u} du \\
&= 2 \cdot 2^q \cdot q \cdot \Gamma(q) \\
&= 2^{q+1} \cdot q! \leq c^q q!
\end{aligned}$$

for some constant c , where we use the substitution $t^{1/q}/2 = u$, so $t = (2u)^q$, so $dt = 2^q q u^{q-1} du$.

(3 \Rightarrow 1): $\mathbb{E}[e^{-\lambda X}] \cdot \mathbb{E}[e^{\lambda X}] = \mathbb{E}[e^{\lambda(X-X')}]$, where X' is an IID copy of X . By Jensen's inequality, $\mathbb{E}[e^{-\lambda X}] \geq e^{-\lambda \mathbb{E}[X]} = 1$. So

$$\mathbb{E}[e^{\lambda X}] \leq \mathbb{E}[e^{\lambda(X-X')}] = \sum_{q=0}^\infty \frac{\lambda^{2q} \mathbb{E}[(X - X')^{2q}]}{(2q)!}$$

(we can ignore odd powers since $X - X'$ is a symmetric RV: $X - X'$ has the same distribution as $X' - X$). Now

$$\mathbb{E}[(X - X')^{2q}] = \sum_{k=0}^{2q} \binom{2q}{k} \mathbb{E}[X^k] \mathbb{E}[(X')^{2q-k}] \leq \sum_{k=0}^{2q} \binom{2q}{k} \mathbb{E}[X^{2q}] = 2^{2q} \cdot \mathbb{E}[X^{2q}],$$

by Holder's inequality with $p = 2q/k$ and $q = 2q/(2q - k)$ for each k . Thus,

$$\begin{aligned}
\mathbb{E}[e^{\lambda X}] &\leq \sum_{q=0}^\infty \frac{\lambda^{2q} \mathbb{E}[X^{2q}] \cdot 2^{2q}}{(2q)!} \\
&\leq \sum_{q=0}^\infty \frac{\lambda^{2q} c^q q! 2^{2q}}{(2q)!} \\
&\leq \sum_{q=0}^\infty \frac{\lambda^{2q} \cdot c^q 2^q}{q!} = \sum_{q=0}^\infty \frac{(\lambda^2 \cdot 2c)^q}{q!} = e^{2\lambda^2 c},
\end{aligned}$$

where we used that $(2q)!/q! = \prod_{j=1}^q (q+1)! \geq 2^q \cdot q!$. Hence $\psi_X(\lambda) = 2\lambda^2 c = \frac{\lambda^2 \cdot 4c}{2}$, hence $X \in \mathcal{G}(4c)$.

(3 \Leftrightarrow 4): exercise. □

1.2. Hoeffding's and related inequalities

Lemma 1.18 (Hoeffding's Lemma) Let Y be a RV with $\mathbb{E}[Y] = 0$ and $Y \in [a, b]$ almost surely. Then $\psi_Y''(\lambda) \leq (b-a)^2/4$ and $Y \in \mathcal{G}((b-a)^2/4)$.

Proof (Hints).

- Define a new distribution based on λ , which should be obvious after expanding $\psi_Y'(\lambda)$.
 - Show that $\psi_Y''(\lambda)$ is equal to the variance of this distribution, and obtain the upper bound on $\psi_Y''(\lambda)$ by using the shift-invariance of the variance.
 - To conclude the result, use a Taylor expansion at 0 of $\psi_Y(\lambda)$.
-

Proof. Let Y have distribution P . We have

$$\psi_Y'(\lambda) = \frac{\mathbb{E}_{Y \sim P}[Y e^{\lambda Y}]}{\mathbb{E}_{Y \sim P}[e^{\lambda Y}]} = \mathbb{E}_{Y \sim P} \left[Y \cdot \frac{e^{\lambda Y}}{\mathbb{E}[e^{\lambda Y}]} \right] = \mathbb{E}_{Y \sim P_\lambda}[Y],$$

where if P is discrete, then P_λ is the discrete distribution with PMF

$$P_\lambda(y) = \frac{e^{\lambda y} P(y)}{\sum_z P(z) e^{\lambda z}} = \frac{e^{\lambda y} P(y)}{\mathbb{E}[e^{\lambda Y}]},$$

and if P is continuous with PDF f , then P_λ is the continuous distribution with PDF

$$f_\lambda(y) = \frac{e^{\lambda y} f(y)}{\int_{-\infty}^{\infty} f(z) e^{\lambda z} dz} = \frac{e^{\lambda y} f(y)}{\mathbb{E}[e^{\lambda Y}]},$$

Now

$$\begin{aligned} \psi_Y''(\lambda) &= \frac{\mathbb{E}_{Y \sim P}[Y^2 e^{\lambda Y}] \cdot \mathbb{E}_{Y \sim P}[e^{\lambda Y}] - \mathbb{E}_{Y \sim P}[Y e^{\lambda Y}]^2}{\mathbb{E}_{Y \sim P}[e^{\lambda Y}]^2} \\ &= \mathbb{E}_{Y \sim P} \left[Y^2 \frac{e^{\lambda Y}}{\mathbb{E}_{Y \sim P}[e^{\lambda Y}]} \right] - \mathbb{E} \left[Y \frac{e^{\lambda Y}}{\mathbb{E}_{Y \sim P}[e^{\lambda Y}]} \right]^2 \\ &= \mathbb{E}_{Y \sim P_\lambda}[Y^2] - \mathbb{E}_{Y \sim P_\lambda}[Y]^2 = \text{Var}_{Y \sim P_\lambda}(Y). \end{aligned}$$

Note that if $Y \in [a, b]$, then $|Y - \frac{b-a}{2}|^2 \leq (b-a)^2/4$. So we have

$$\text{Var}_{Y \sim P_\lambda}(Y) = \text{Var}_{Y \sim P_\lambda}(Y - (b-a)/2) \leq \mathbb{E}_{Y \sim P_\lambda} \left[\left(Y - \frac{b-a}{2} \right)^2 \right] \leq \frac{(b-a)^2}{4}.$$

Finally, using a Taylor expansion at 0, we obtain

$$\psi_Y(\lambda) = \psi_Y(0) + \lambda'_Y(0)\lambda + \psi''_Y(\xi)\frac{\lambda^2}{2} = \psi''_Y(\xi)\frac{\lambda^2}{2} \leq \lambda^2 \frac{(b-a)^2}{8},$$

for some $\xi \in [0, \lambda]$, since $\mathbb{E}_{Y \sim P}[Y] = \mathbb{E}_{Y \sim P_0}[Y] = 0$. \square

Remark 1.19 The distribution P_λ in the above proof is called the **exponentially tilted** distribution.

Theorem 1.20 (Hoeffding's Inequality) Let X_1, \dots, X_n be independent RVs where each X_i takes values in $[a_i, b_i]$. Then for all $t \geq 0$,

$$\mathbb{P}\left(\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \geq t\right) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

Proof (Hints). Straightforward. \square

Proof. By Hoeffding's Lemma, $X_i - \mathbb{E}[X_i] \in \mathcal{G}((b_i - a_i)^2/4)$ for all i . By Proposition 1.15 (part 2), we have

$$\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \in \mathcal{G}\left(\frac{1}{4} \sum_{i=1}^n (b_i - a_i)^2\right).$$

Hence, by Proposition 1.15 (part 1), we are done. \square

Remark 1.21 A drawback of Hoeffding's Inequality is that the bound does not involve $\text{Var}(X_i)$, and the variances could be much smaller than the upper bound of $(b_i - a_i)^2/4$. This is addressed by Bennett's inequality:

Theorem 1.22 (Bennett's Inequality) Let X_1, \dots, X_n be independent RVs with $\mathbb{E}[X_i] = 0$ and $|X_i| \leq c$ for all i . Let $\nu = \text{Var}(X_1) + \dots + \text{Var}(X_n)$. Then for all $t \geq 0$,

$$\mathbb{P}\left(\sum_{i=1}^n X_i \geq t\right) \leq \exp\left(-\frac{\nu}{c^2} \cdot h_1\left(\frac{ct}{\nu}\right)\right),$$

where $h_1(x) = (1+x)\log(1+x) - x$ for $x > 0$.

Proof (Hints).

- Show that $\mathbb{E}[e^{\lambda X_i}] \leq 1 + \frac{\text{Var}(X_i)}{c^2}(e^{\lambda c} - \lambda c - 1)$.
- Deduce that $\psi_{\sum_i X_i} \leq \frac{\nu}{c^2}(e^{\lambda c} - \lambda c - 1)$.
- Find a lower bound for $\psi_{\sum_i X_i}^*(t)$.

\square

Proof. Denote $\sigma_i^2 = \text{Var}(X_i) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = \mathbb{E}[X_i^2]$. The MGF of X_i is

$$\begin{aligned} \mathbb{E}[e^{\lambda X_i}] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}[X_i^k] = 1 + \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}[X_i^{k-2} X_i^2] \\ &\leq 1 + c^{k-2} \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}[X_i^2] = 1 + \frac{\sigma_i^2}{c^2} \sum_{k=2}^{\infty} \frac{\lambda^k c^k}{k!} \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{\sigma_i^2}{c^2} \left(\sum_{k=0}^{\infty} \frac{\lambda^k c^k}{k!} - \lambda c - 1 \right) \\
&= 1 + \frac{\sigma_i^2}{c^2} (e^{\lambda c} - \lambda c - 1).
\end{aligned}$$

(We can apply the inequality since $\mathbb{E}[X_i^k] \geq \mathbb{E}[X_i]^k = 0$ by Jensen's inequality.) So $\psi_{X_i}(\lambda) = \log\left(1 + \frac{\sigma_i^2}{c^2}(e^{\lambda c} - \lambda c - 1)\right) \leq \frac{\sigma_i^2}{c^2}(e^{\lambda c} - \lambda c - 1)$. So by additivity of ψ , we have

$$\psi_{\sum_{i=1}^n X_i}(\lambda) \leq \frac{\nu}{c^2} e^{\lambda c} - \frac{\nu}{c^2} \lambda c - \frac{\nu}{c^2}.$$

So for $t \geq 0 = \mathbb{E}[\sum_i X_i]$, by Proposition 1.12,

$$\psi_{\sum_i X_i}^*(t) \geq \sup_{\lambda \in \mathbb{R}} \left\{ \lambda t - \frac{\nu}{c^2} e^{\lambda c} + \frac{\nu}{c} \lambda + \frac{\nu}{c^2} \right\} =: \sup_{\lambda \in \mathbb{R}} \{g(\lambda)\}$$

We have $g'(\lambda) = t - \frac{\nu}{c} e^{\lambda c} + \frac{\nu}{c}$ which is 0 iff $t + \frac{\nu}{c} = \frac{\nu}{c} e^{\lambda c}$, i.e. iff $\lambda = \frac{1}{c} \log(1 + t \frac{c}{\nu}) =: \lambda^*$. So

$$\begin{aligned}
\psi_{\sum X_i}^*(t) &\geq \frac{1}{c} t \log\left(1 + \frac{tc}{\nu}\right) - \frac{\nu}{c^2} \left(1 + \frac{tc}{\nu}\right) + \frac{\nu}{c^2} \log\left(1 + \frac{tc}{\nu}\right) + \frac{\nu}{c^2} \\
&= \frac{\nu}{c^2} \left(\left(1 + \frac{tc}{\nu}\right) \log\left(1 + \frac{tc}{\nu}\right) - \frac{tc}{\nu} \right) \\
&= \frac{\nu}{c^2} h_1\left(\frac{tc}{\nu}\right).
\end{aligned}$$

So we are done by the Chernoff Bound. □

Remark 1.23 We can show that $h_1(x) \geq \frac{x^2}{2(x/3+1)}$ for $x \geq 0$. So by Bennett's Inequality, we obtain

$$\mathbb{P}\left(\sum_{i=1}^n X_i \geq t\right) \leq \exp\left(-\frac{t^2}{2(ct/3 + \nu)}\right),$$

which is **Bernstein's inequality**. If $\nu \gg ct$, then this yields a sub-Gaussian tail bound, and if $\nu \ll ct$, then this yields an exponential bound. So Bernstein misses a log factor.

Remark 1.24 If $Z \sim \text{Pois}(\lambda)$, then $\psi_{Z-\nu}(\lambda) = \nu(e^\lambda - \lambda - 1)$.

2. The variance method

2.1. The Efron-Stein inequality

Notation 2.1 Denote $X^{(i)} = (X_{1:(i-1)}, X_{(i+1):n})$ and for $i < j$, denote $X_{i:j} = (X_i, \dots, X_j)$.

Notation 2.2 Denote $E_i Z = \mathbb{E}[Z \mid X_{1:i}]$, $E_0 Z = \mathbb{E}[Z]$, $E^{(i)} = \mathbb{E}[Z \mid X^{(i)}]$, and $\text{Var}^{(i)}(Z) = \text{Var}(Z \mid X^{(i)})$.

We want to study the concentration of $Z = f(X_1, \dots, X_n)$ for independent X_i . If $Z = \sum_i X_i$, then $\text{Var}(\sum_i X_i) = \sum_i \text{Var}(X_i)$ if $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i] \mathbb{E}[X_j]$ for all $i \neq j$, which holds if the X_i are independent.

Theorem 2.3 (Efron-Stein Inequality) Let X_1, \dots, X_n be independent and let $Z = f(X_1, \dots, X_n)$. Then

$$\text{Var}(Z) \leq \sum_{i=1}^n \mathbb{E}[(Z - E^{(i)} Z)^2] = \mathbb{E} \left[\sum_{i=1}^n \text{Var}^{(i)}(Z) \right].$$

Proof (Hints).

- The [Law of Total Expectation](#) and [Tower Property of Conditional Expectation](#) will come in handy a lot...
- Let $\Delta_i = E_i Z - E_{i-1} Z$. Show that $\mathbb{E}[\Delta_i] = 0$.
- Show that the Δ_i are uncorrelated, i.e. $\mathbb{E}[\Delta_i \Delta_j] = \mathbb{E}[\Delta_i] \mathbb{E}[\Delta_j]$.
- Show that $\Delta_i = E_i(Z - E^{(i)} Z)$.

□

Proof. Let $\Delta_i = E_i Z - E_{i-1} Z$. By the [Law of Total Expectation](#), we have

$$\mathbb{E}[\Delta_i] = \mathbb{E}[\mathbb{E}[Z \mid X_{1:i}]] - \mathbb{E}[\mathbb{E}[Z \mid X_{1:(i-1)}]] = \mathbb{E}[Z] - \mathbb{E}[Z] = 0.$$

Also, note that $Z - \mathbb{E}[Z] = \mathbb{E}[Z \mid X_{1:n}] - \mathbb{E}[Z] = \sum_{i=1}^n \Delta_i$. We claim that the Δ_i are uncorrelated, i.e. $\mathbb{E}[\Delta_i \Delta_j] = \mathbb{E}[\Delta_i] \mathbb{E}[\Delta_j] = 0$ for $i \neq j$. Indeed, for $i < j$, by the [Law of Total Expectation](#), we can write

$$\mathbb{E}[\Delta_i \Delta_j] = \mathbb{E}[\mathbb{E}[\Delta_i \Delta_j \mid X_{1:i}]] = \mathbb{E}[\Delta_i \mathbb{E}[\Delta_j \mid X_{1:i}]],$$

since Δ_i is a function of $X_{1:i}$. But

$$\begin{aligned} \mathbb{E}[\Delta_j \mid X_{1:i}] &= \mathbb{E}(E_j Z - E_{j-1} Z \mid X_{1:i}) \\ &= \mathbb{E}[\mathbb{E}[Z \mid X_{1:j}] \mid X_{1:i}] - \mathbb{E}[\mathbb{E}[Z \mid X_{1:(j-1)}] \mid X_{1:i}] \\ &= \mathbb{E}[Z \mid X_{1:i}] - \mathbb{E}[Z \mid X_{1:i}] = E_i Z - E_i Z = 0, \end{aligned}$$

where on the third line we used the [Tower Property of Conditional Expectation](#). Hence, the Δ_i are uncorrelated, which implies

$$\text{Var}(Z) = \text{Var}(Z - \mathbb{E}[Z]) = \sum_{i=1}^n \text{Var}(\Delta_i) = \sum_{i=1}^n \mathbb{E}[\Delta_i^2] - \mathbb{E}[\Delta_i]^2 = \sum_{i=1}^n \mathbb{E}[\Delta_i^2].$$

Now

$$\begin{aligned} E_i(E^{(i)} Z) &= \mathbb{E}[E^{(i)} Z \mid X_{1:i}] \\ &= \mathbb{E}[E^{(i)} Z \mid X_{1:(i-1)}, X_i] \\ &= \mathbb{E}[\mathbb{E}[Z \mid X^{(i)}] \mid X_{1:(i-1)}] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}[Z \mid X_{1:(i-1)}] \\
&= E_{i-1}Z,
\end{aligned}$$

where on the third line we used that X_i and $X^{(i)}$ are independent, and on the fourth line we used the [Tower Property of Conditional Expectation](#). So we can rewrite $\Delta_i = E_i Z - E_{i-1} Z = E_i(Z - E^{(i)} Z)$, and so by Jensen's inequality

$$\begin{aligned}
\Delta_i^2 &= (E_i(Z - E^{(i)} Z))^2 = \mathbb{E}[Z - E^{(i)} Z \mid X_{1:i}]^2 \\
&\leq \mathbb{E}[(Z - E^{(i)} Z)^2 \mid X_{1:i}] = E_i((Z - E^{(i)} Z)^2).
\end{aligned}$$

Hence, by the [Law of Total Expectation](#),

$$\begin{aligned}
\text{Var}(Z) &= \sum_{i=1}^n \mathbb{E}[\Delta_i^2] \leq \sum_{i=1}^n \mathbb{E}[E_i((Z - E^{(i)} Z)^2)] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[(Z - E^{(i)} Z)^2 \mid X_{1:i}]] = \sum_{i=1}^n \mathbb{E}[(Z - E^{(i)} Z)^2].
\end{aligned}$$

Finally, we have $\mathbb{E}[E^{(i)}(Z - E^{(i)} Z)^2] = \mathbb{E}[\text{Var}(Z \mid X^{(i)})] = \mathbb{E}[\text{Var}^{(i)}(Z)]$, which gives the equality in the theorem statement. \square

Theorem 2.4 (Efron-Stein Inequality) Let X_1, \dots, X_n be independent and f be square integrable. Let $Z = f(X_1, \dots, X_n)$. Then

$$\text{Var}(Z) \leq \mathbb{E}\left[\sum_{i=1}^n (Z - E^{(i)} Z)^2\right] =: \nu.$$

Moreover, if X'_1, \dots, X'_n are IID copies of X_1, \dots, X_n , and $Z'_i = f(X_{1:(i-1)}, X'_i, X_{(i+1):n})$, then

$$\nu = \frac{1}{2} \mathbb{E}\left[\sum_{i=1}^n (Z - Z'_i)^2\right] = \mathbb{E}\left[\sum_{i=1}^n (Z - Z'_i)_+^2\right] = \mathbb{E}\left[\sum_{i=1}^n (Z - Z'_i)_-^2\right],$$

where $X_+ = \max\{0, X\}$ and $X_- = \max\{-X, 0\}$. Moreover,

$$\nu = \sum_{i=1}^n \inf_{Z_i} \mathbb{E}[(Z - Z_i)^2],$$

where the infimum is over all $X^{(i)}$ -measurable and square-integrable RVs Z_i .

Proof (Hints).

- First part is straightforward.
- For second part, show that $\text{Var}^{(i)}(Z) = \frac{1}{2} \text{Var}^{(i)}(Z - Z'_i)$.
- For last part, use that $\text{Var}(X) = \inf_a \mathbb{E}[(X - a)^2]$.

\square

Proof. The first part follows instantly from the [Efron-Stein Inequality](#) by linearity of expectation. Now $\text{Var}(X) = \frac{1}{2} \text{Var}(X - Y)$, if X and Y are IID. Conditional on $X^{(i)}$, Z and Z'_i are independent. Hence, since $\mathbb{E}[Z] = \mathbb{E}[Z'_i]$,

$$\text{Var}^{(i)}(Z) = \frac{1}{2} \text{Var}^{(i)}(Z - Z'_i) = \frac{1}{2} \mathbb{E}^{(i)}[(Z - Z'_i)^2].$$

Thus we have

$$\nu = \frac{1}{2} \sum_{i=1}^n \mathbb{E}[(Z - Z'_i)^2].$$

The equality with \cdot_+ and \cdot_- follows since $Z - Z'_i$ is a symmetric RV. Finally, recall that $\text{Var}(X) = \inf_a \mathbb{E}[(X - a)^2]$, with equality if $a = \mathbb{E}[X]$. So $\text{Var}^{(i)}(Z) = \inf_{Z'_i} \mathbb{E}^{(i)}[(Z - Z'_i)^2]$, with equality if $Z'_i = \mathbb{E}^{(i)} Z$. Taking expectations and summing completes the proof. \square

2.2. Functions with bounded differences

Definition 2.5 $f : A^n \rightarrow \mathbb{R}$ has the **bounded differences (b.d.)** property if

$$\sup_{(x, x'_i) \in A^{n+1}} |f(x_{1:(i-1)}, x_i, x_{(i+1):n}) - f(x_{1:(i-1)}, x'_i, x_{(i+1):n})| \leq c_i \quad \forall i \in [n].$$

So changing one of the coordinates changes the value of the function at most by a constant.

Corollary 2.6 Let X_1, \dots, X_n be independent and $Z = f(X_{1:n})$ have bounded differences with constants c_i . Then $\text{Var}(Z) \leq \frac{1}{4} \sum_{i=1}^n c_i^2$.

Proof (Hints). Consider the random variable

$$Z_i = \frac{1}{2} \left(\sup_{x_i \in A} f(X_{1:(i-1)}, x_i, X_{(i+1):n}) + \inf_{x_i \in A} f(X_{1:(i-1)}, x_i, X_{(i+1):n}) \right).$$

\square

Proof. Define

$$Z_i = \frac{1}{2} \left(\sup_{x_i \in A} f(X_{1:(i-1)}, x_i, X_{(i+1):n}) + \inf_{x_i \in A} f(X_{1:(i-1)}, x_i, X_{(i+1):n}) \right)$$

Z_i is a function of $X^{(i)}$. We have $|Z - Z_i| \leq c_i/2$. By the final part of the [Efron-Stein Inequality](#), we have $\text{Var}(Z) \leq \sum_{i=1}^n \mathbb{E}[(Z - Z_i)^2] \leq \frac{1}{4} \sum_{i=1}^n c_i^2$. \square

Example 2.7 (Bin packing) Given $x_1, \dots, x_n \in [0, 1]$, what is the minimum number k of bins B_j into which $\sum_{x \in B_j} x \leq 1$ for each $j = 1, \dots, k$?

Suppose X_1, \dots, X_n be independent and let $Z = f(X_{1:n})$ be the minimum number of bins. Note that changing any one x_i changes f by at most 1, so f has bounded differences with constants $c_i = 1$. So by the [Efron-Stein Inequality](#), $\text{Var}(Z) \leq \frac{1}{4}n$.

Note that this bound is tight, e.g. when $X_i \sim \text{Bern}(1/2)$, $Z \sim B(n, 1/2)$, which has variance $n \cdot \frac{1}{2} \cdot \frac{1}{2}$.

Example 2.8 (Longest common sub-sequence) Let $X_{1:n}$ and $Y_{1:n}$ be independent sequences of coin flips. Let

$$Z = f(X_{1:n}, Y_{1:n}) = \max \left\{ k : \exists i_1 < \dots < i_k, j_1 < \dots < j_k \text{ s.t. } X_{i_\ell} = Y_{j_\ell} \forall \ell \in [k] \right\}$$

Note that changing any one coin flip changes Z by at most 1, so f has bounded differences with constants $c_i = 1$, so by the [Efron-Stein Inequality](#), $\text{Var}(Z) \leq n/2 = \Theta(n)$. Since it is known that $\mathbb{E}[Z] = \Theta(n)$, the deviations from the mean are small compared to the mean.

Example 2.9 (Chromatic numbers of graphs) Let G be an **Erdos-Renyi random graph** with n vertices, i.e. each $\{i, j\} \in E(G)$ with probability p (independently). The **chromatic number** $\chi(G)$ of G is the smallest number of colors on the vertices such that there are no two adjacent vertices with the same colour. For $i < j$, let $X_{ij} = \mathbb{1}_{\{\{i,j\} \in E\}}$. We have

$$\chi(G) = f\left(\{X_{ij}\}_{1 \leq i < j \leq n}\right),$$

for some (complicated) function f . Since adding or removing an edge changes $\chi(G)$ by at most 1, f has bounded differences with constants $c_{ij} = 1$. By [Efron-Stein Inequality](#), $\text{Var}(Z) \leq \binom{n}{2}/4 = \Theta(n^2)$. It is known that $\mathbb{E}[\chi(G)] \approx n/\log n$, so the bound on the variance is not useful when applying [Chebyshev's Inequality](#). However:

Now for each $1 \leq i \leq n-1$, let $Y^{(i)}$ be a random vector taking values in $\{0, 1\}^i$ where $Y_j^{(i)} = \mathbb{1}_{\{\{i+1, j\} \in E\}}$ for each $1 \leq j \leq i$. The Y_i are independent. Also, note that $\{Y^{(i)}\}_{i=1}^{n-1}$ determines the graph. Hence, $\chi(G) = g(Y^{(1)}, \dots, Y^{(n-1)})$ for some (complicated) function g . g has bounded differences with constants 1 (e.g. by considering giving vertex $i+1$ a new colour). Then by [Efron-Stein Inequality](#), $\text{Var}(\chi(G)) \leq (n-1)/4$, which is a tighter bound. This yields a useful application of [Chebyshev's Inequality](#), which shows that $\chi(G)$ is close to its mean value.

3. Poincaré inequalities

Let X_1, \dots, X_n be real-valued random variables, and let $Z = f(X_1, \dots, X_n)$. A Poincaré inequality is of the form $\text{Var}(Z) \lesssim \mathbb{E}[\|\nabla f(X)\|^2]$. So we have a local property (smoothness) which gives a global property (bound on the variance).

Definition 3.1 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is **separately convex** if it is convex if all of its individual arguments.

Theorem 3.2 (Convex Poincaré Inequality) Let $X_{1:n}$ be independent RVs supported on $[0, 1]$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be separately convex with partial derivatives that exist. Let $Z = f(X_{1:n})$. Then

$$\text{Var}(Z) \leq \mathbb{E}[\|\nabla f(X_{1:n})\|^2],$$

where $\|\cdot\| = \|\cdot\|_2$ is the Euclidean norm.

Proof (Hints).

- Let $Z_i = \inf_{x'_i} f(X_{1:(i-1)}, x'_i, X_{(i+1):n})$. Let X'_i be the value for which the infimum is achieved (why is it achieved?).
- Use that $|Z - Z_i|^2 \leq |X_i - X'_i|^2 \cdot \left(\frac{\partial f}{\partial x_i}(X)\right)^2$ (since X'_i is a minimiser).

□

Proof. Let $Z_i = \inf_{x'_i} f(X_{1:(i-1)}, x'_i, X_{(i+1):n})$. Let X'_i be the value for which the infimum is achieved (since f is continuous and the domain $[0, 1]^n$ we consider is compact). Denote $\bar{X}^{(i)} = (X_{1:(i-1)}, X'_i, X_{(i+1):n})$. Note that since f is separately convex and X'_i is a minimiser (so $f(X'_i) \leq f(X)$),

$$|Z - Z_i|^2 = |f(X_{1:n}) - f(\bar{X}^{(i)})|^2 \leq |X_i - X'_i|^2 \cdot \left(\frac{\partial f}{\partial x_i}(X_{1:n})\right)^2.$$

By the [Efron-Stein Inequality](#),

$$\begin{aligned} \text{Var}(Z) &\leq \sum_{i=1}^n \mathbb{E}[(Z - Z_i)^2] \\ &\leq \sum_{i=1}^n \mathbb{E}\left[(X_i - X'_i)^2 \left(\frac{\partial f}{\partial x_i}(X_{1:n})\right)^2\right] \\ &\leq \sum_{i=1}^n \mathbb{E}\left[\left(\frac{\partial f}{\partial x_i}(X_{1:n})\right)^2\right] = \mathbb{E}[\|\nabla f(X_{1:n})\|^2]. \end{aligned}$$

□

Example 3.3 Let $X \in \mathbb{R}^{n \times d}$ be a random matrix with $X_{i,j} \in [-1, 1]$ independent. The spectral norm (or ℓ_2 -operator norm) of X is its largest singular value:

$$\sigma_1(X) = \sup\{\|Xu\| : u \in \mathbb{R}^d, \|u\| = 1\} = \sup_{u \in \mathbb{R}^n, \|u\|=1} \sup_{v \in \mathbb{R}^d, \|v\|=1} \langle u, Xv \rangle.$$

σ_1 is convex (and so separately convex) since it is a supremum of linear functions. Since it is a norm, we have $\sigma_1(A + B) \leq \sigma_1(A) + \sigma_1(B)$ and $\sigma_1(A - B) \geq |\sigma_1(A) - \sigma_1(B)|$. Fix A . Since X ranges over a compact set, the supremum is achieved: let u, v achieve the supremum. Then

$$\begin{aligned} \sigma_1(A) = \langle v, Xu \rangle &\leq \|v\| \cdot \|Xu\| \quad \text{by Cauchy-Schwarz} \\ &\leq \|v\| \cdot \|u\| \left(\sum_{i,j} X_{i,j}^2\right)^{1/2} = \left(\sum_{i,j} X_{i,j}^2\right)^{1/2} = \|X\|_F. \end{aligned}$$

Now if X, X' are independent, $d(X, X') = \|X - X'\|_F \geq \sigma_1(X - X') \geq |\sigma_1(X) - \sigma_1(X')|$ where d is the Euclidean distance between vectorised X and X' (i.e. Frobenius norm). So σ_1 is a 1-Lipschitz function, and note that an L -Lipschitz function satisfies $\|\nabla f\| \leq L$. So by the [Convex Poincaré Inequality](#), $\text{Var}(\sigma_1(X)) \leq 4$ (the RHS is 4, not

1, since X_{ij} take values in $[-1, 1]$ instead of $[0, 1]$). Note that this is independent of the dimension of X !

Theorem 3.4 (Gaussian Poincaré Inequality) Let $X_{1:n}$ be IID and standard Gaussian (i.e. each $X_i \sim N(0, 1)$). Then for any continuously differentiable $f \in C^1(\mathbb{R}^n)$,

$$\text{Var}(f(X_{1:n})) \leq \mathbb{E}[\|\nabla f(X_{1:n})\|^2].$$

Proof (Hints).

- Show, using the [Efron-Stein Inequality](#), that it is sufficient to prove the result for $n = 1$.
- You may assume that $f \in C^2(\mathbb{R})$ (f is twice continuously differentiable) and has compact support.
- Using the definition of conditional variance, show that $\text{Var}^{(i)}(f(S_n)) = \frac{1}{4} \left(f\left(S_n - \frac{\varepsilon_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}\right) - f\left(S_n - \frac{\varepsilon_i}{\sqrt{n}} - \frac{1}{\sqrt{n}}\right) \right)^2$, where $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i$ and ε_i are IID Rademacher random variables (taking values in $\{-1, 1\}$ with equal probability).
- Use Taylor's theorem to find an upper bound for

$$\left| f\left(S_n - \frac{\varepsilon_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}\right) - f\left(S_n - \frac{\varepsilon_i}{\sqrt{n}} - \frac{1}{\sqrt{n}}\right) \right|$$

- Use [Efron-Stein Inequality](#) for $f(S_n)$ and the central limit theorem to conclude the result.

□

Proof. Assume the result holds for the $n = 1$ case, i.e. $\text{Var}(f(X)) \leq \mathbb{E}[f'(X)^2]$ for $X \sim N(0, 1)$. Then by the [Efron-Stein Inequality](#) and [Law of Total Expectation](#),

$$\begin{aligned} \text{Var}(Z) &\leq \mathbb{E} \left[\sum_{i=1}^n \text{Var}^{(i)}(f(X_{1:n})) \right] \\ &\leq \mathbb{E} \left[\sum_{i=1}^n \mathbb{E} \left[\left(\frac{\partial f}{\partial x_i}(X_{1:n}) \right)^2 \mid X^{(i)} \right] \right] \\ &= \mathbb{E} \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(X_{1:n}) \right)^2 \right] = \mathbb{E}[\|\nabla f(X_{1:n})\|^2]. \end{aligned}$$

So it suffices to prove the result for $n = 1$: WLOG, assume $\mathbb{E}[\|\nabla f(X)\|^2] < \infty$. Let ε_i be IID Rademacher random variables (taking values in $\{-1, 1\}$ with equal probability). Consider $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i$. It suffices to prove the case when $f \in C^2(\mathbb{R})$ (f is twice continuously differentiable) and has compact support. So f' and f'' are bounded. By the [Efron-Stein Inequality](#),

$$\text{Var}(f(S_n)) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Var}^{(i)}(S_n) \right].$$

Note $\text{Var}^{(i)}$ here is conditional on $\varepsilon^{(i)}$. We have $S_n = S_n - \varepsilon_i/\sqrt{n} \pm 1/\sqrt{n}$ with equal probabilities. Note that $S_n - \varepsilon_i/\sqrt{n}$ is a function of $\varepsilon^{(i)}$. We have

$$\mathbb{E}^{(i)}[f(S_n)] = \frac{1}{2}f(S_n - \varepsilon_i/\sqrt{n} + 1/\sqrt{n}) + \frac{1}{2}f(S_n - \varepsilon_i/\sqrt{n} - 1/\sqrt{n})$$

and so

$$\begin{aligned} \text{Var}^{(i)}(f(S_n)) &= \frac{1}{2} \left(f(S_n - \varepsilon_i/\sqrt{n} + 1/\sqrt{n}) - \left(\frac{1}{2}f(S_n - \varepsilon_i/\sqrt{n} + 1/\sqrt{n}) + \frac{1}{2}f(S_n - \varepsilon_i/\sqrt{n} - 1/\sqrt{n}) \right) \right)^2 \\ &\quad + \frac{1}{2} \left(f(S_n - \varepsilon_i/\sqrt{n} - 1/\sqrt{n}) - \left(\frac{1}{2}f(S_n - \varepsilon_i/\sqrt{n} + 1/\sqrt{n}) + \frac{1}{2}f(S_n - \varepsilon_i/\sqrt{n} - 1/\sqrt{n}) \right) \right)^2 \\ &= \frac{1}{4} (f(S_n - \varepsilon_i/\sqrt{n} + 1/\sqrt{n}) - f(S_n - \varepsilon_i/\sqrt{n} - 1/\sqrt{n}))^2 \end{aligned}$$

Let K be an upper bound for $|f''|$. Then

$$\begin{aligned} &|f(S_n + (1 - \varepsilon_i)/\sqrt{n}) - f(S_n - (1 + \varepsilon_i)/\sqrt{n})| \\ &= \left| f(S_n) + \frac{1 - \varepsilon_i}{\sqrt{n}} f'(S_n - \varepsilon_i/\sqrt{n}) + \frac{(1 - \varepsilon_i)^2}{2n} f''(S_n - \varepsilon_i/\sqrt{n} + \xi_{i,m}) \right. \\ &\quad \left. - f(S_n) + \frac{1 + \varepsilon_i}{\sqrt{n}} f'(S_n - \varepsilon_i/\sqrt{n}) - \frac{(1 + \varepsilon_i)^2}{2n} f''(S_n - \varepsilon_i/\sqrt{n} + \xi_{i,m}^{(2)}) \right| \\ &\leq \left| \frac{2}{\sqrt{n}} f'(S_n) \right| + 2K/n. \end{aligned}$$

Thus, $\text{Var}^{(i)}(f(S_n)) \leq (|f'(S_n)/\sqrt{n}| + K/n)^2$. Hence,

$$\text{Var}(f(S_n)) \leq \mathbb{E} \left[\sum_{i=1}^n (|f'(S_n)/\sqrt{n}| + K/n)^2 \right] = \mathbb{E}[f'(S_n)^2] + 2 \frac{K}{\sqrt{n}} \mathbb{E}[|f'(S_n)|] + \frac{K^2}{n}$$

As $n \rightarrow \infty$, $\text{Var}(f(S_n)) \rightarrow \text{Var}(X)$, $X \sim N(0, 1)$ by the central limit theorem. Also, $\mathbb{E}[f'(S_n)^2] \rightarrow \mathbb{E}[f'(X)^2]$ by the central limit theorem. So in the limit, $\text{Var}(f(X)) \leq \mathbb{E}[f'(X)^2]$. \square

Remark 3.5 The above proof uses a **tensorisation** argument. Tensorisation roughly means decomposing a high-dimensional function into a sum of lower-dimensional functions. E.g. the formula $\text{Var}(\sum_i X_i) = \sum_i \text{Var}(X_i)$ uses the tensorisation property of variance. Also, the [Efron-Stein Inequality](#)

$$\text{Var}(Z) \leq \sum_{i=1}^n \mathbb{E}[\text{Var}^{(i)}(Z)].$$

can be thought of as an example of the tensorisation of variance.

Remark 3.6 If f is L -Lipschitz, i.e. $|f(x) - f(y)| \leq L \cdot \|x - y\|$, then $\|\nabla f\| \leq L$. The [Gaussian Poincaré Inequality](#) holds for L -Lipschitz functions (with L^2 on the RHS).

Example 3.7 Recall from earlier that the operator norm σ_1 is 1-Lipschitz. If $X \in \mathbb{R}^{n \times d}$ with each $X_{ij} \sim N(0, 1)$ IID, then by the Gaussian Poincaré Inequality, $\text{Var}(\sigma_1(X)) \leq 1$, which is a good bound, given that it is known that $\mathbb{E}[\sigma_1(X)] = O(\sqrt{n} + \sqrt{d})$.

Example 3.8 Let $X_1, \dots, X_n \sim N(0, 1)$ be independent. Let $Z = f(X) = \max_i X_i$. We have $\nabla f = (0, \dots, 1, \dots, 0)$ where 1 is at the index of the maximum. Hence, by the Gaussian Poincaré Inequality, $\text{Var}(Z) \leq 1$, which is a good bound, given it is known that $\mathbb{E}[Z_n] \approx \log n$.

3.1. Poincaré constant

Definition 3.9 Let X be an RV taking values in \mathbb{R}^d . We say X satisfies the Poincaré inequality with constant C if

$$\text{Var}(f(X)) \leq C \cdot \mathbb{E}[\|\nabla f(X)\|^2] \quad \forall f \in C^1(\mathbb{R}^d).$$

The smallest such constant $C_P(X)$ is the **Poincaré constant** of X :

$$C_P(X) = \sup_{f \in C^1(\mathbb{R}^d)} \frac{\text{Var}(f(X))}{\mathbb{E}[\|\nabla f(X)\|^2]}.$$

Proposition 3.10 The Poincaré constant satisfies the following properties:

1. $C_P(aX + b) = a^2 C_P(X)$ for constants $a \in \mathbb{R}, b \in \mathbb{R}^d$.
2. For any unit vector $\theta \in \mathbb{R}^d$, $\text{Var}(\langle X, \theta \rangle) \leq C_P(X)$. In particular, $\text{Var}(X_i) \leq C_P(X)$ for all i .
3. If X_1, \dots, X_n are independent, then

$$C_P(X_{1:n}) = \max_i C_P(X_i).$$

4. If $C_P(X) < \infty$, then X has connected support.

Proof. Exercise. □

Remark 3.11 The constant $1/C_P(X)$ is called the **spectral gap**.

Definition 3.12 We say $\{X_n\}_{n \in \mathbb{N}}$ is a **(time homogenous) Markov chain** on a finite state space S (which WLOG we can take to be $[d]$) if

$$\mathbb{P}(X_{n+1} = j \mid X_{1:n} = i_{1:n}) = \mathbb{P}(X_{n+1} = j \mid X_n = i_n)$$

for all n and $i_1, \dots, i_n, j \in S$, i.e. if X_{n+1} is conditionally independent of $X_{1:(n-1)}$ given X_n for all n .

Definition 3.13 The **transition matrix** $P \in \mathbb{R}^{d \times d}$ of the Markov chain is defined by

$$P_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i),$$

and its **discrete generator** is $\Lambda := P - I$.

Definition 3.14 Let P be the transition matrix of a Markov chain. A row vector $\pi \in \mathbb{R}^d$ (which represents a distribution on $[d]$) on state space S is called **stationary** if $\pi_j = \sum_i \pi_i P_{ij}$ for all j (i.e. $\pi P = \pi$).

Definition 3.15 Given a Markov chain with stationary distribution $\pi \in \mathbb{R}^d$ and $f, g \in \mathbb{R}^d$, the **Dirichlet form** is defined as

$$\mathcal{E}(f, g) := -\langle f, \Lambda g \rangle_\pi,$$

where $\langle x, y \rangle_\pi = \sum_{i=1}^d x_i y_i \pi_i$.

Proposition 3.16 Let $P \in \mathbb{R}^{d \times d}$ be a reversible transition matrix with stationary distribution $\pi \in \mathbb{R}^d$. Assume the **reversibility** condition:

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i, j \in [d].$$

Let $f \in \mathbb{R}^d$. Then

$$\mathcal{E}(f, f) = \frac{1}{2} \mathbb{E}_{X_{n+1}, X_n \sim \pi} [(f(X_{n+1}) - f(X_n))^2],$$

which is the **discrete gradient** (we may view f as a function $i \mapsto f_i$).

Proof (Hints). Use that $\sum_j P_{ij} = 1$ for all i to split the sum $\sum_i f_i^2 \pi_i$ into two sums. \square

Proof. Since $\sum_j P_{ij} = 1$ for all i , we have

$$\begin{aligned} \mathcal{E}(f, f) &= \langle f, (I - P)f \rangle_\pi = \sum_i f_i^2 \pi_i - \sum_i f_i \pi_i \sum_j P_{ij} f_j \\ &= \frac{1}{2} \left(\sum_{i,j} f_i^2 \pi_i P_{ij} + \sum_{i,j} f_j^2 \pi_j P_{ji} - 2 \sum_{i,j} \pi_i P_{ij} f_i f_j \right) \\ &= \frac{1}{2} \sum_{i,j} \pi_i P_{ij} (f_i - f_j)^2 \\ &= \frac{1}{2} \sum_{i,j} \mathbb{P}(X_{n+1} = j \mid X_n = i) \mathbb{P}(X_n = i) (f_i - f_j)^2 \\ &= \frac{1}{2} \sum_{i,j} \mathbb{P}(X_{n+1} = j, X_n = i) (f(i) - f(j))^2 \\ &= \frac{1}{2} \mathbb{E} [(f(X_{n+1}) - f(X_n))^2]. \end{aligned}$$

\square

Remark 3.17 If the transition matrix P is reversible, then $\Lambda = P - I$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_\pi$ (i.e. $\langle \Lambda f, g \rangle_\pi = \langle f, \Lambda g \rangle_\pi$), so has real eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$. By Proposition 3.16, we have $\langle f, -\Lambda f \rangle_\pi \geq 0$, so $-\Lambda$ is positive semi-definite, and so all $\lambda_i \leq 0$. Since $\sum_j \Lambda_{ij} = 0$ for all i , we have $\lambda_1 = 0$, corresponding to eigenvector $f_1 = (1, \dots, 1)$.

Now $\lambda_2 = \sup_{f: \langle f, f_1 \rangle_\pi = 0} \frac{\langle f, \Lambda f \rangle_\pi}{\langle f, f \rangle_\pi}$, so

$$\mathcal{E}(f, f) = -\langle f, \Lambda f \rangle_\pi \geq -\lambda_2 \langle f, f \rangle_\pi = -\lambda_2 \mathbb{E}_\pi [f(X_1)^2] = -\lambda_2 \text{Var}_\pi(f) = (\lambda_1 - \lambda_2) \text{Var}_\pi(f)$$

for all $f \in \mathbb{R}^d$ such that $\mathbb{E}_\pi[f(X_1)] = \langle f, f_1 \rangle_\pi = 0$. There is equality if $f = f_2$, the eigenvector corresponding to λ_2 .

The best constant, c , in the inequality $\text{Var}_\pi(f) \leq c \cdot \mathcal{E}(f, f)$ is $c = \frac{1}{\lambda_1 - \lambda_2}$, the spectral gap.

4. The entropy method

4.1. Entropy, chain rules and Han's inequality

In the following section, let A be a discrete (countable) alphabet and let X be an RV on A .

Definition 4.1 The **Shannon entropy** of X with PMF P is

$$H(X) = \mathbb{E}[-\log P(X)] = - \sum_{x \in A} \mathbb{P}(X = x) \log \mathbb{P}(X = x),$$

where we use the convention $0 \log 0 = 0$.

Example 4.2 The entropy of $X \sim \text{Bern}(p)$ is $H(X) = -p \log p - (1-p) \log(1-p)$.

Remark 4.3 Note that for $x_1^n \in A^n$, $P^n(x_1^n) = e^{-n \frac{1}{n} \sum_{i=1}^n -\log P(x_i)}$ (P^n is the product distribution). So $P^n(X_1^n) = e^{-n \frac{1}{n} \sum_{i=1}^n -\log P(X_i)} \approx e^{-n H(X_i)}$ for IID X_i , by the **Weak Law of Large Numbers**.

Proposition 4.4 Properties of Shannon entropy:

- H is non-negative.
- $H(\cdot)$ is concave as a functional of P .
- If $|A| < \infty$, then $H(X) \leq \log |A|$ with equality if $X \sim \text{Unif}(A)$.

Proof. Exercise. □

Definition 4.5 For PMFs Q, P on A , Q is **absolutely continuous** with respect to P , written $Q \ll P$, if $P(x) = 0 \Rightarrow Q(x) = 0$ for all $x \in A$.

Definition 4.6 Let Q, P be PMFs on A such that $Q \ll P$ (which means if $P(x) = 0$, then $Q(x) = 0$). The **relative entropy** between Q and P is

$$D(Q \parallel P) = \mathbb{E}_Q \left[\log \frac{Q(X)}{P(X)} \right] = \sum_{x \in A} Q(x) \log \frac{Q(x)}{P(x)}$$

if $Q \ll P$, and $D(Q \parallel P) = \infty$ otherwise. We use the convention that $0 \log \frac{0}{0} = 0$.

Proposition 4.7 Properties of relative entropy:

- $D(Q \parallel P) \geq 0$.
- $D(Q \parallel P)$ is convex in both arguments.
- If $X \sim P$ where P is the uniform distribution on A , and $Y \sim Q$, then $D(Q \parallel P) = H(X) - H(Y)$.

Proof. Exercise. □

Definition 4.8 The **conditional entropy** of X given Y is

$$\begin{aligned}
H(X | Y) &= \mathbb{E}[-\log P_{X|Y}(X | Y)] = - \sum_{x,y} P(x,y) \log P(x | y) \\
&= \sum_y \mathbb{P}(Y = y) H(X | Y = y)
\end{aligned}$$

Theorem 4.9 (Chain Rule for Entropy) We have

$$H(X_{1:n}) = \mathbb{E}[-\log P(X_{1:n})] = \sum_{i=1}^n H(X_i | X_{1:(i-1)}).$$

Proof (Hints). Straightforward. □

Proof. Since

$$\mathbb{P}(X_{1:n} = x_{1:n}) = \mathbb{P}(X_1 = x_1) \mathbb{P}(X_2 = x_2 | X_1 = x_1) \cdots \mathbb{P}(X_n = x_n | X_{1:(n-1)} = x_{1:(n-1)}),$$

we have

$$\begin{aligned}
H(X_{1:n}) &= \mathbb{E}[-\log P(X_{1:n})] = \mathbb{E}\left[\sum_{i=1}^n -\log P(X_i | X_{1:(i-1)})\right] \\
&= \sum_{i=1}^n \mathbb{E}[-\log P(X_i | X_{1:(i-1)})] \\
&= \sum_{i=1}^n H(X_i | X_{1:(i-1)}).
\end{aligned}$$

□

Proposition 4.10 (Conditioning Reduces Entropy) $H(X | Y) \leq H(X)$.

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned}
H(X) - H(X | Y) &= \mathbb{E}\left[\log \frac{1}{P(X)} + \log P(X | Y)\right] \\
&= \mathbb{E}\left[\log \frac{P(X | Y)P(Y)}{P(X)P(Y)}\right] = D(P_{X,Y} \| P_X P_Y) \geq 0.
\end{aligned}$$

□

Definition 4.11 Similarly to the definition of relative entropy, the **conditional relative entropy** of X and Y given Z is

$$D(X \| Y | Z) = \sum_z \mathbb{P}(Z = z) D(X | Z = z \| Y | Z = z).$$

We also write e.g.

$$D(Q_{Y|X} \| P_Y | Q_X) = \sum_x \mathbb{P}(X = x) D(Q_{Y|X=x} \| P_Y).$$

Proposition 4.12 (Chain Rule for Relative Entropy) Let P, Q be PMFs on A^n . Let $X_{1:n} \sim Q$. Then

$$\begin{aligned} D(Q_{X_{1:n}} \parallel P_{X_{1:n}}) &= \sum_{i=1}^n \mathbb{E}_{Q_{X_{1:(i-1)}}} \left[D(Q_{X_i \mid X_{1:(i-1)}} \parallel P_{X_i \mid X_{1:(i-1)}}) \right] \\ &=: \sum_{i=1}^n D(Q_{X_i \mid X_{1:(i-1)}} \parallel P_{X_i \mid X_{1:(i-1)}} \mid Q_{X_{1:(i-1)}}) \end{aligned}$$

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned} D(Q_{X_{1:n}} \parallel P_{X_{1:n}}) &= \mathbb{E}_Q \left[\log \frac{Q(X_{1:n})}{P(X_{1:n})} \right] \\ &= \mathbb{E}_Q \left[\sum_{i=1}^n \log \frac{Q_{X_i \mid X_{1:(i-1)}}(X_i \mid X_{1:(i-1)})}{P_{X_i \mid X_{1:(i-1)}}(X_i \mid X_{1:(i-1)})} \right] \\ &= \sum_{i=1}^n \mathbb{E}_{Q_{X_{1:(i-1)}}} \left[D(Q_{X_i \mid X_{1:(i-1)}} \parallel P_{X_i \mid X_{1:(i-1)}}) \right] \end{aligned}$$

□

Remark 4.13 The Chain Rule for Relative Entropy is similar to the chain rule for variance:

$$\text{Var}(Z) = \sum_{i=1}^n \mathbb{E}[\Delta_i^2],$$

$\Delta_i = \mathbb{E}[Z \mid X_{1:i}] - \mathbb{E}[Z \mid X_{1:(i-1)}]$, which led to the Efron-Stein Inequality.

Lemma 4.14 (Conditioning Reduces Conditional Entropy) $H(X \mid Y, Z) \leq H(X \mid Y)$.

Proof (Hints). Straightforward. □

Proof. $H(X \mid Y, Z) = \sum_z \mathbb{P}(Z = z) H(X \mid Y, Z = z) \leq \sum_z \mathbb{P}(Z = z) H(X \mid Z = z) = H(X \mid Z)$ by Conditioning Reduces Entropy. □

Theorem 4.15 (Han's Inequality) Let $X_{1:n}$ be discrete RVs. Then

$$H(X_{1:n}) \leq \frac{1}{n-1} \sum_{i=1}^n H(X^{(i)}).$$

Proof (Hints). Show that $H(X_{1:n}) \leq H(X^{(i)}) + H(X_i \mid X_{1:(i-1)})$. □

Proof. By the Chain Rule for Entropy and Conditioning Reduces Entropy,

$$\begin{aligned} H(X_{1:n}) &= H(X^{(i)}) + H(X_i \mid X^{(i)}) \\ &\leq H(X^{(i)}) + H(X_i \mid X_{1:(i-1)}) \end{aligned}$$

Summing over i , we obtain $nH(X_{1:n}) \leq \sum_{i=1}^n H(X^{(i)}) + H(X_{1:n})$ by the chain rule. □

Corollary 4.16 (Loomis-Whitney Inequality) The Loomis-Whitney inequality states that for finite $A \subseteq \mathbb{Z}^n$,

$$|A| \leq \prod_{i=1}^n |A^{(i)}|^{1/(n-1)}$$

Proof (Hints). Straightforward. □

Proof. Let $X_{1:n}$ be uniform on A . Then $\log|A| = H(X_{1:n})$. By Han's Inequality,

$$H(X_{1:n}) \leq \frac{1}{n-1} \sum_{i=1}^n H(X^{(i)}) \leq \frac{1}{n-1} \sum_{i=1}^n \log|A^{(i)}|$$

□

Lemma 4.17 Let Q, P be PMFs on a discrete set $A \times B \times C$. Then

$$D(Q_{Y|X,Z} \parallel P_Y \mid Q_{X,Z}) \geq D(Q_{Y|X} \parallel P_Y \mid Q_X)$$

Proof (Hints). Use convexity of relative entropy. □

Proof. By convexity of relative entropy,

$$\begin{aligned} D(Q_{Y|X,Z} \parallel P_Y \mid Q_{X,Z}) &= \sum_{x,z} Q_{X,Z}(x,z) D(Q_{Y|X=x,Z=z} \parallel P_Y) \\ &= \sum_x Q(x) \sum_z Q(z|x) D(Q_{Y|X=x,Z=z} \parallel P_Y) \\ &\geq \sum_x Q(x) D\left(\sum_z Q(z|x) Q_{Y|X=x,Z=z} \parallel P_Y\right) \\ &= \sum_x Q(x) D(Q_{Y|X=x} \parallel P_Y) \\ &= D(Q_{Y|X} \parallel P_Y \mid Q_X). \end{aligned}$$

□

Theorem 4.18 (Han's Inequality for Relative Entropy) Suppose Q, P are PMFs on A^n , and assume that $P = P_1 \otimes \cdots \otimes P_n$. Then

$$D(Q \parallel P) = D(Q_{X_{1:n}} \parallel P_{X_{1:n}}) \geq \frac{1}{n-1} \sum_{i=1}^n D(Q_{X^{(i)}} \parallel P_{X^{(i)}})$$

Equivalently,

$$D(Q \parallel P) \leq \sum_{i=1}^n D(Q_{X_i|X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}})$$

(this is tensorisation of $D(\cdot \parallel \cdot)$).

Remark 4.19 Taking P to be uniform in Han's Inequality for Relative Entropy gives Han's Inequality for Shannon entropy.

Proof (Hints). Explain why $D(Q \parallel P) = D(Q_{X^{(i)}} \parallel P_{X^{(i)}}) + D(Q_{X_i | X^{(i)}} \parallel P_{X_i | Q_{X^{(i)}}})$, then use Lemma 4.17. \square

Proof. By the Chain Rule for Relative Entropy and Lemma 4.17,

$$\begin{aligned} D(Q \parallel P) &= D(Q_{X^{(i)}} \parallel P_{X^{(i)}}) + D(Q_{X_i | X^{(i)}} \parallel P_{X_i | Q_{X^{(i)}}}) \\ &= D(Q_{X^{(i)}} \parallel P_{X^{(i)}}) + D(Q_{X_i | X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}}) \quad \text{since } P \text{ is a product distribution} \\ &\geq D(Q_{X^{(i)}} \parallel P_{X^{(i)}}) + D(Q_{X_i | X_{1:(i-1)}} \parallel P_{X_i} \mid Q_{X_{1:(i-1)}}) \end{aligned}$$

Summing over i , we obtain $nD(Q \parallel P) \geq \sum_{i=1}^n D(Q_{X^{(i)}} \parallel P_{X^{(i)}}) + D(Q \parallel P)$ by the Chain Rule for Relative Entropy, hence

$$\begin{aligned} D(Q \parallel P) &\geq \frac{1}{n-1} \sum_{i=1}^n D(Q_{X^{(i)}} \parallel P_{X^{(i)}}) \\ &= \frac{1}{n-1} \sum_{i=1}^n (D(Q \parallel P) - D(Q_{X_i | X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}})) \\ \Leftrightarrow \frac{n}{n-1} D(Q \parallel P) - D(Q \parallel P) &\leq \frac{1}{n-1} \sum_{i=1}^n D(Q_{X_i | X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}}) \end{aligned}$$

\square

Definition 4.20 There is another notion of entropy. Let $Z \geq 0$ almost surely. Let $\varphi(x) = x \log x$ for $x > 0$ and $\varphi(0) = 0$. The **entropy** of Z is defined as

$$\text{Ent}(Z) = \mathbb{E}[\varphi(Z)] - \varphi(\mathbb{E}[Z]),$$

Note the similarity to the definition $\text{Var}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$. Also, since φ is convex, $\text{Ent}(Z)$ is non-negative by Jensen's inequality.

Proposition 4.21 Let $X \sim P$, where $Q \ll P$ are PMFs on a countable alphabet A . Let $Z = \frac{Q(X)}{P(X)}$. Then

$$\text{Ent}(Z) = D(Q \parallel P).$$

Proof (Hints). Straightforward. \square

Proof. We have

$$\begin{aligned} \text{Ent}(Z) &= \mathbb{E}_P \left[\frac{Q(X)}{P(X)} \log \frac{Q(X)}{P(X)} \right] - \left(\mathbb{E}_P \frac{Q(X)}{P(X)} \right) \log \mathbb{E}_P \left[\frac{Q(X)}{P(X)} \right] \\ &= D(Q \parallel P) - 1 \log 1 = D(Q \parallel P). \end{aligned}$$

\square

Remark 4.22 In general, when Z is the Radon-Nikodym derivative $\frac{dQ}{dP}(X)$ and $X \sim P$, then $\text{Ent}(Z) = D(Q \parallel P)$.

Theorem 4.23 (Tensorisation of Entropy) Let X_1, \dots, X_n be independent RVs taking values in a countable set A , and let $f : A^n \rightarrow \mathbb{R}_{\geq 0}$. Let $Z = f(X_{1:n}) = f(X)$. Then

$$\text{Ent}(Z) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}^{(i)}(Z) \right],$$

where

$$\begin{aligned} \text{Ent}^{(i)}(Z) &= E^{(i)}[Z \log Z] - E^{(i)}[Z] \log E^{(i)}[Z] \\ &= \mathbb{E}[Z \log Z \mid X^{(i)}] - \mathbb{E}[Z \mid X^{(i)}] \log \mathbb{E}[Z \mid X^{(i)}]. \end{aligned}$$

Remark 4.24 Tensorisation of Entropy is analogous to the Efron-Stein Inequality.

Proof (Hints).

- Let $X \sim P = P_1 \otimes \cdots \otimes P_n$. Let $Q(x) = f(x)P(x)$.
- Show that $\text{Ent}(aZ) = a \text{Ent}(Z)$, and so can assume WLOG that $\mathbb{E}[Z] = 1$, so Q is PMF.
- Use Han's Inequality for Relative Entropy on Q and P .
- Show that

$$Q_{X_i \mid X^{(i)}}(x_i \mid x^{(i)}) = \frac{P(x_i)f(x)}{\mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}]}.$$

- Show that $Q^{(i)}(x^{(i)}) = P^{(i)}(x^{(i)})\mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}]$, and so that $\mathbb{E}[D(Q_{X_i \mid X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}})] = \mathbb{E}_P[\text{Ent}^{(i)}(f(X))]$.

□

Proof. Let $X \sim P = P_1 \otimes \cdots \otimes P_n$. Let $Q(x) = f(x)P(x)$. Since

$$\text{Ent}(aZ) = a\mathbb{E}[Z \log Z] + a\mathbb{E}[Z \log a] - a\mathbb{E}[Z] \log \mathbb{E}[Z] - a\mathbb{E}[Z] \log a = a \text{Ent}(Z),$$

we may assume WLOG that $\mathbb{E}[Z] = 1$, and so Q is a valid PMF. By Han's Inequality for Relative Entropy,

$$D(Q \parallel P) \leq \sum_{i=1}^n \mathbb{E}[D(Q_{X_i \mid X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}})]$$

Now

$$\begin{aligned} Q_{X_i \mid X^{(i)}}(x_i \mid x^{(i)}) &= \frac{Q_X(x)}{Q_{X^{(i)}}(x^{(i)})} = \frac{P(x)f(x)}{\sum_{x'_i \in A} Q(x_{1:(i-1)}, x'_i, x_{(i+1):n})} \\ &= \frac{P_i(x_i)P^{(i)}(x^{(i)})f(x)}{\sum_{x'_i \in A} P_i(x'_i)P^{(i)}(x^{(i)})f(x^{(i)}, x'_i)} \\ &= \frac{P_i(x_i)f(x)}{\mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}]} \end{aligned}$$

(write $f(x^{(i)}, x'_i) = f(x_{1:(i-1)}, x'_i, x_{(i+1):n})$). By definition,

$$\mathbb{E}[D(Q_{X_i \mid X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}})]$$

$$= \sum_{x^{(i)} \in A^{n-1}} Q^{(i)}(x^{(i)}) \sum_{x_i \in A} \frac{P_i(x_i) f(x)}{\mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}]} \log \frac{f(x)}{\mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}]}$$

But $Q^{(i)}(x^{(i)}) = P^{(i)}(x^{(i)}) \mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}]$. So,

$$\begin{aligned} & \mathbb{E}[D(Q_{X_i \mid X^{(i)}} \parallel P_{X_i} \mid Q_{X^{(i)}})] \\ &= \sum_{x^{(i)} \in A^{n-1}} P^{(i)}(x^{(i)}) \left(\sum_{x_i \in A} P_i(x_i) f(x) \log f(x) - \mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}] \log \mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}] \right) \\ &= \sum_{x^{(i)} \in A^{n-1}} P^{(i)}(x^{(i)}) (\mathbb{E}[f(X) \log f(X) \mid X^{(i)} = x^{(i)}] - \mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}] \log \mathbb{E}[f(X) \mid X^{(i)} = x^{(i)}]) \\ &= \mathbb{E}_P[\text{Ent}^{(i)}(f(X))] \end{aligned}$$

So $\text{Ent}(f(X)) = D(Q \parallel P) \leq \sum_{i=1}^n \mathbb{E}[\text{Ent}^{(i)}(f(X))]$. \square

4.2. Herbst's argument

Theorem 4.25 (Herbst's Argument) Let Z be a real-valued RV such that $\mathbb{E}[e^{\lambda Z}] < \infty$ for all $\lambda > 0$. Suppose there exists $\nu > 0$ such that for all $\lambda > 0$,

$$\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} \leq \lambda^2 \frac{\nu}{2}.$$

Then

$$\psi_{Z - \mathbb{E}[Z]}(\lambda) = \log \mathbb{E}[e^{\lambda(Z - \mathbb{E}[Z])}] \leq \lambda^2 \frac{\nu}{2} \quad \forall \lambda > 0.$$

Proof (Hints).

- Show that $\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} = \lambda^2 G'(\lambda)$, where $G(\lambda) = \frac{1}{\lambda} \psi_{Z - \mathbb{E}[Z]}(\lambda)$.
- Given an upper bound for $\int_0^\lambda G'(t) dt$ (explain using a Taylor expansion of $\psi_{Z - \mathbb{E}[Z]}$ why this integral is valid).

\square

Proof. Write $\psi = \psi_{Z - \mathbb{E}[Z]}$. We have

$$\begin{aligned} \text{Ent}(e^{\lambda Z}) &= \lambda \mathbb{E}[e^{\lambda Z} \cdot Z] - \mathbb{E}[e^{\lambda Z}] \log \mathbb{E}[e^{\lambda Z}] \\ &= \mathbb{E}[e^{\lambda Z}] \left(\lambda \mathbb{E} \left[\frac{Z e^{\lambda Z}}{\mathbb{E}[e^{\lambda Z}]} \right] - \log \mathbb{E}[e^{\lambda Z}] \right) \end{aligned}$$

But

$$\psi'(\lambda) = (\psi_Z(\lambda) - \lambda \mathbb{E}[Z])' = \mathbb{E} \left[\frac{Z e^{\lambda Z}}{\mathbb{E}[e^{\lambda Z}]} \right] - \mathbb{E}[Z].$$

So by the above expression for Ent ,

$$\begin{aligned}\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} &= [\lambda\psi'(\lambda) + \lambda\mathbb{E}[Z] - \lambda\mathbb{E}[Z] - \psi(\lambda)] \\ &= \lambda^2 \left(\frac{1}{\lambda}\psi'(\lambda) - \frac{1}{\lambda^2}\psi(\lambda) \right) = \lambda^2 G'(\lambda)\end{aligned}$$

where $G(\lambda) = \frac{1}{\lambda}\psi(\lambda)$. Also, by assumption,

$$\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} \leq \lambda^2 \frac{\nu}{2}$$

By Taylor's theorem, $G(\lambda) = \frac{1}{\lambda}(\psi(0) + \lambda\psi'(0) + O(\lambda^2)) = \frac{1}{\lambda}O(\lambda^2) = O(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. Hence, we may integrate $G'(\theta)$ from 0 to λ :

$$\begin{aligned}G(\lambda) &= \int_0^\lambda G'(\theta) d\theta \leq \int_0^\lambda \frac{\nu}{2} d\theta \quad \text{since } \theta^2 G'(\theta) \leq \theta^2 \frac{\nu}{2} \\ &= \lambda \frac{\nu}{2}\end{aligned}$$

So $\psi(\lambda) \leq \lambda^2 \frac{\nu}{2}$. □

Theorem 4.26 (Bounded Differences Inequality) Let $X = (X_1, \dots, X_n)$, where the X_i are independent. Let f have bounded differences with constants c_i . Let $Z = f(X)$. Then for all $t > 0$,

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t), \mathbb{P}(Z - \mathbb{E}[Z] \leq -t) \leq e^{-2t^2 / \sum_{i=1}^n c_i^2} = e^{-t^2 / 2\nu},$$

where $\nu = \frac{1}{4} \sum_{i=1}^n c_i^2$.

Proof (Hints).

- Use [Hoeffding's Lemma](#) and an equality from the proof of [Herbst's Argument](#) to show that $\frac{\text{Ent}^{(i)}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z} | X^{(i)}]} = \lambda\psi'_i(\lambda) - \psi_i(\lambda) \leq \frac{1}{8}\lambda^2 c_i^2$ (you should use an integral somewhere), where $\psi_i(\lambda) = \log \mathbb{E}[e^{\lambda(Z - \mathbb{E}[Z])} | X^{(i)}]$.
- Use [Tensorisation of Entropy](#) and [Herbst's Argument](#) to show that $Z - \mathbb{E}[Z]$ has sub-Gaussian right tail with parameter ν .
- Why does the result also hold for $-f$?

□

Proof. The first step is tensorisation of entropy: by [Tensorisation of Entropy](#), we have

$$\text{Ent}(e^{\lambda Z}) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}^{(i)}(e^{\lambda Z}) \right]$$

Write $f_{X^{(i)}}(x_i) = f(X_{1:(i-1)}, x_i, X_{(i+1):n})$. Conditional on $X^{(i)}$, $f_{X^{(i)}}$ takes values on an interval of length $\leq c_i$ by the bounded differences property.

The second step is to apply [Hoeffding's Lemma](#). Let $\psi_i(\lambda) = \log \mathbb{E}[e^{\lambda(Z - \mathbb{E}[Z])} | X^{(i)}]$. As in the proof of [Herbst's Argument](#), we have

$$\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} = \lambda \psi'_{Z-\mathbb{E}[Z]}(\lambda) - \psi_{Z-\mathbb{E}[Z]}(\lambda).$$

Note that this holds for the random variable $Z \mid X^{(i)} = x^{(i)}$, for any value of $x^{(i)}$. By [Hoeffding's Lemma](#), we have $\psi''_i(\lambda) \leq c_i^2/4$, and so

$$\begin{aligned} \frac{\text{Ent}^{(i)}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z} \mid X^{(i)}]} &= \lambda \psi'_i(\lambda) - \psi_i(\lambda) = \int_0^\lambda \theta \psi''_i(\theta) d\theta \\ &\leq \int_0^\lambda \theta \frac{c_i^2}{4} d\theta \\ &= \frac{1}{8} \lambda^2 c_i^2 \end{aligned}$$

The third step is using [Herbst's Argument](#): we have

$$\begin{aligned} \text{Ent}(e^{\lambda Z}) &\leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}^{(i)}(e^{\lambda Z}) \right] \leq \mathbb{E} \left[\sum_{i=1}^n \frac{1}{8} \lambda^2 c_i^2 \mathbb{E}[e^{\lambda Z} \mid X^{(i)}] \right] \\ &= \frac{1}{2} \lambda^2 \cdot \frac{1}{4} \sum_{i=1}^n c_i^2 \mathbb{E}[e^{\lambda Z}] \end{aligned}$$

by [Law of Total Expectation](#). By [Herbst's Argument](#), we have

$$\psi_{Z-\mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2 \nu}{2} \quad \forall \lambda > 0,$$

and so the [Chernoff Bound](#) gives $\mathbb{P}(Z - \mathbb{E}[Z]) \leq e^{-t^2/2\nu}$. Now noting that $-f$ also has bounded differences with the same constants, we obtain the left-tail bound. \square

4.3. Log-Sobolev inequalities on the hypercube

Notation 4.27 Let X_1, \dots, X_n be IID and uniform on $\{-1, 1\}$, so $X = X_{1:n}$ is uniform on the hypercube $\{-1, 1\}^n$. Let $Z = f(X)$. By [Efron-Stein Inequality](#), $\text{Var}(Z) \leq \frac{1}{2} \mathbb{E} \left[\sum_{i=1}^n (Z - Z'_i)^2 \right] =: \nu$, where $Z'_i = f(X_{1:(i-1)}, X'_i, X_{(i+1):n})$ and X'_i is an independent copy of X_i . Define $\mathcal{E}(f)$ as

$$\begin{aligned} \nu &= \frac{1}{4} \mathbb{E} \left[\sum_{i=1}^n (f(X) - f(\bar{X}^{(i)}))^2 \right] \\ &= \frac{1}{2} \mathbb{E} \left[\sum_{i=1}^n (f(X) - f(\bar{X}^{(i)}))_+^2 \right] =: \mathcal{E}(f), \end{aligned}$$

where $\bar{X}^{(i)} = (X_{1:(i-1)}, -X_i, X_{(i+1):n})$. $\frac{1}{2} (f(X) - f(\bar{X}^{(i)}))$ looks like a discrete partial derivative in the i -th direction. So $\mathcal{E}(f)$ is a discrete analogue of $\mathbb{E}[\|\nabla f(X)\|^2]$.

Theorem 4.28 (Log-Sobolev Inequality for Bernoullis) Let X be uniformly distributed on $\{-1, 1\}^n$ and $f : \{-1, 1\}^n \rightarrow \mathbb{R}$. Then

$$\text{Ent}(f^2(X)) \leq 2 \cdot \mathcal{E}(f).$$

Proof (Hints).

- Use [Tensorisation of Entropy](#) to show that it is enough to prove the result for $n = 1$.
- Based on the one-dimensional inequality that needs to be shown, construct a suitable function $h(a, b)$. Let $g(a) = h(a, b)$ for fixed b . Show that $g(b) = 0$, $g'(b) = 0$, and $g''(a) \leq 0$ for all $a \geq b$.

□

Proof. Let $Z = f(X)$. By [Tensorisation of Entropy](#),

$$\text{Ent}(Z^2) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}^{(i)}(Z^2) \right]$$

If the result was true for $n = 1$, then we would have $\text{Ent}^{(i)}(Z^2) \leq \frac{1}{2} \left(f(X) - f(\overline{X}^{(i)}) \right)^2$ (since when $X^{(i)}$ is fixed, we may think of Z^2 as being a function of X_i , and this function is $f(X)^2$ or $f(\overline{X}^{(i)})^2$ with equal probability) and so $\text{Ent}(Z^2) \leq 2\mathcal{E}(f)$. So it suffices to prove the $n = 1$ case. Let $f(1) = a$, $f(-1) = b$. In the $n = 1$ case, the inequality we want to show is

$$\frac{1}{2}a^2 \log(a^2) + \frac{1}{2}b^2 \log(b^2) - \frac{1}{2}(a^2 + b^2) \log\left(\frac{a^2 + b^2}{2}\right) \leq \frac{1}{2}(b - a)^2.$$

We may assume $a, b \geq 0$, since $\frac{(b-a)^2}{2} \geq \frac{(|b|-|a|)^2}{2}$. Also, by symmetry, WLOG we assume $a \geq b$. For fixed $b \geq 0$, define

$$h(a) = \frac{1}{2}a^2 \log(a^2) + \frac{1}{2}b^2 \log(b^2) - \frac{1}{2}(a^2 + b^2) \log\left(\frac{a^2 + b^2}{2}\right) - \frac{1}{2}(b - a)^2.$$

Since $h(b) = 0$, it is enough to show that $h'(b) = 0$ and $h''(a) \leq 0$ (so h is concave). We have

$$h'(a) = a \log \frac{2a^2}{a^2 + b^2} - (a - b)$$

Hence, $h'(b) = 0$. Also,

$$h''(a) = 1 + \log \frac{2a^2}{a^2 + b^2} - \frac{2a^2}{a^2 + b^2} \leq 0,$$

since $\log x \leq x - 1$. □

Remark 4.29 [Log-Sobolev Inequality for Bernoullis](#) is stronger than [Efron-Stein Inequality](#). Also, the constant 2 on the RHS is tight.

Theorem 4.30 (Gaussian Log-Sobolev Inequality) Let $X = (X_1, \dots, X_n)$ be a vector of n independent RVs with each $X_i \sim N(0, 1)$, let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable. Then

$$\text{Ent}(f^2(X)) \leq 2 \cdot \mathbb{E}[\|\nabla f(X)\|^2].$$

Proof. Exercise (use tensorisation and the central limit theorem). \square

Definition 4.31 $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **L -Lipschitz** if

$$|f(x) - f(y)| \leq L \cdot \|x - y\| \quad \forall x, y \in \mathbb{R}^n.$$

An L -Lipschitz function f satisfies $\|\nabla f(x)\| \leq L$ for all $x \in \mathbb{R}^n$.

Theorem 4.32 (Gaussian Concentration Inequality) Let $X = (X_1, \dots, X_n)$ be a vector of n independent RVs with each $X_i \sim N(0, 1)$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz and $Z = f(X)$. Then $Z - \mathbb{E}[Z] \in \mathcal{G}(L^2)$, i.e. for all $\lambda \in \mathbb{R}$,

$$\psi_{Z - \mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2 L^2}{2},$$

and so for all $t > 0$,

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/2L^2}, \quad \text{and} \quad \mathbb{P}(Z - \mathbb{E}[Z] \leq -t) \leq e^{-t^2/2L^2}.$$

Note that these bounds are independent of the dimension n .

Proof (Hints).

- Explain why we can assume f is continuously differentiable (think sequences).
- Use that $\|\nabla f(X)\| \leq L$ and the **Gaussian Log-Sobolev Inequality** on $e^{\lambda f/2}$ to obtain an upper bound that is a suitable assumption for **Herbst's Argument**.

\square

Proof. WLOG, we can assume f is continuously differentiable (otherwise, we can approximate f with a sequence of continuously differentiable functions which converge to f). Note that $\|\nabla f(X)\| \leq L$. By the **Gaussian Log-Sobolev Inequality** for $e^{\lambda f/2}$, we have

$$\begin{aligned} \text{Ent}(e^{\lambda f(X)}) &\leq 2 \cdot \mathbb{E} \left[\left\| \nabla e^{\lambda f(X)/2} \right\|^2 \right] \\ &= 2 \cdot \mathbb{E} \left[\left\| \frac{\lambda}{2} \nabla(f(X)) \cdot e^{\lambda f(X)/2} \right\|^2 \right] \\ &= \frac{\lambda^2}{2} \mathbb{E} [e^{\lambda f(X)} \|\nabla f(X)\|^2] \\ &\leq \frac{\lambda^2 L^2}{2} \mathbb{E} [e^{\lambda f(X)}] \end{aligned}$$

So by **Herbst's Argument**,

$$\psi_{Z - \mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2 L^2}{2},$$

and the **Chernoff Bound** gives the right tail bound. The left tail bound follows from the fact that $-f$ is also L -Lipschitz. \square

Theorem 4.33 (Concentration on the Hypercube) Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ and let $X = (X_1, \dots, X_n)$ be uniform on $\{-1, 1\}^n$. Let $Z = f(X)$ and assume

$$\max_{x \in \{-1,1\}^n} \sum_{i=1}^n (f(x) - f(\bar{x}^{(i)}))_+^2 \leq \nu$$

for some $\nu > 0$. Then for all $t > 0$,

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/\nu},$$

i.e. Z has a sub-Gaussian right tail with variance parameter $\nu/2$.

Proof (Hints).

- Explain why $\frac{e^{z/2} - e^{y/2}}{(z-y)/2} \leq e^{z/2}$ for $z > y$.
- Use the [Log-Sobolev Inequality for Bernoullis](#) on an appropriate function to obtain an upper bound that is a suitable assumption for [Herbst's Argument](#).

□

Proof. We use the [Log-Sobolev Inequality for Bernoullis](#) for the function $e^{\lambda f/2}$: for $\lambda > 0$, we have

$$\begin{aligned} \text{Ent}(e^{\lambda f(X)}) &\leq \frac{1}{2} \mathbb{E} \left[\sum_{i=1}^n \left(e^{\lambda f(X)/2} - e^{\lambda f(\bar{X}^{(i)})/2} \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{i=1}^n \left(e^{\lambda f(X)/2} - e^{\lambda f(\bar{X}^{(i)})/2} \right)_+^2 \right] \end{aligned}$$

Since for $z > y$, $\frac{e^{z/2} - e^{y/2}}{(z-y)/2} \leq e^{z/2}$ (by convexity of \exp),

$$\begin{aligned} \text{Ent}(e^{\lambda f(X)}) &\leq \mathbb{E} \left[\sum_{i=1}^n \frac{\lambda^2}{2^2} \left(f(X) - f(\bar{X}^{(i)}) \right)_+^2 \cdot e^{\lambda f(X)} \right] \\ &\leq \frac{\nu \lambda^2}{4} \mathbb{E}[e^{\lambda f(X)}]. \end{aligned}$$

By [Herbst's Argument](#), we thus have $\psi_{Z - \mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2 \nu/2}{2}$ for all $\lambda > 0$, and the [Chernoff Bound](#) gives $\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/\nu}$. □

Remark 4.34

- If the same condition for the negative part $(\cdot)_-$ holds, then we get the analogous left tail bound.
- If $\max_{x \in \{-1,1\}^n} \sum_{i=1}^n (f(x) - f(\bar{x}^{(i)}))^2 \leq \nu$, then $Z - \mathbb{E}[Z] \in \mathcal{G}(\nu/2)$. In fact, more careful analysis shows that $Z - \mathbb{E}[Z] \in \mathcal{G}(\nu/4)$.
- If f has bounded differences with constants c_i where $\sum_{i=1}^n c_i^2 \leq \nu$, then f also satisfies

$$\max_{x \in \{-1,1\}^n} \sum_{i=1}^n (f(x) - f(\bar{x}^{(i)}))^2 \leq \nu$$

so $Z - \mathbb{E}[Z] \in \mathcal{G}(\nu/4)$. [Bounded Differences Inequality](#) also gives $Z - \mathbb{E}[Z] \in \mathcal{G}(\nu/4)$ under stronger assumptions. So we are able to prove a result that is as strong as [Bounded Differences Inequality](#) but under a weaker assumption.

- The [Efron-Stein Inequality](#) gives

$$\text{Var}(Z) \leq \mathbb{E} \left[\sum_{i=1}^n (Z - Z'_i)_+^2 \right] = \frac{1}{2} \mathbb{E} \left[\sum_{i=1}^n (Z - \bar{Z}^{(i)})^2 \right] \leq \nu/2$$

if $\mathbb{E} \left[\sum_{i=1}^n (Z - \bar{Z}^{(i)})^2 \right] \leq \nu$. Note that this is a weaker result, but makes a weaker assumption than [Concentration on the Hypercube](#).

4.4. The modified log-Sobolev inequality (MLSI)

Lemma 4.35 (Variational Principle for Entropy) For any non-negative random variable Y ,

$$\text{Ent}(Y) = \inf_{u>0} \mathbb{E}[Y(\log Y - \log u) - (Y - u)]$$

and the infimum is achieved at $u = \mathbb{E}[Y]$.

Proof (Hints). Use the inequality $\log x \leq x - 1$ and show that the difference is non-positive for all $u > 0$. \square

Proof. We have

$$\begin{aligned} \text{Ent}(Y) - \mathbb{E}[Y \log Y + Y \log u - (Y - u)] &= \mathbb{E} \left[Y \log \frac{u}{\mathbb{E}[Y]} + Y - u \right] \\ &\leq \frac{\mathbb{E}[Y]}{\mathbb{E}[Y]} u - \mathbb{E}[Y] + \mathbb{E}[Y] - u = 0 \end{aligned}$$

since $\log x \leq x - 1$. For $u = \mathbb{E}[Y]$,

$$\mathbb{E}[Y \log Y] - \mathbb{E}[Y \log u + Y - u] = \text{Ent}(Y).$$

\square

Remark 4.36 This is an entropy analogue of $\text{Var}(Y) = \inf_{a \in \mathbb{R}} \mathbb{E}[(Y - a)^2]$. In fact, for any convex function φ , we can prove that the infimum

$$\inf_{u>0} \mathbb{E}[\varphi(Y) - \varphi(u) - \varphi'(u)(Y - u)]$$

is achieved when $u = \mathbb{E}[Y]$. The [Variational Principle for Entropy](#) is a special case for $\varphi(x) = x \log x$.

Theorem 4.37 (Modified Log-Sobolev Inequality) Let X_1, \dots, X_n be independent RVs taking values on A . Let $f : A^n \rightarrow \mathbb{R}$ and $Z = f(X)$. Let $f_i : A^{n-1} \rightarrow \mathbb{R}$ be an arbitrary function and $Z_i = f_i(X^{(i)})$ for each $i \in [n]$. Then

$$\text{Ent}(e^{\lambda Z}) \leq \sum_{i=1}^n \mathbb{E}[e^{\lambda Z} \varphi(-\lambda(Z - Z_i))] \quad \forall \lambda > 0,$$

where $\varphi(x) = e^x - x - 1$.

For $\lambda > 0$ and $Z \geq Z_i$, we may use the inequality $\varphi(-x) \leq x^2/2$ for $x \geq 0$ to give a simpler upper bound:

$$\text{Ent}(e^{\lambda Z}) \leq \frac{\lambda^2}{2} \sum_{i=1}^n \mathbb{E}[e^{\lambda Z} (Z - Z_i)^2].$$

Proof (Hints). Use [Tensorisation of Entropy](#) and the [Variational Principle for Entropy](#), with $u = Y_i = e^{\lambda Z_i}$ (conditional on $X^{(i)}$). \square

Proof. Let $Y = e^{\lambda Z}$ and $Y_i = e^{\lambda Z_i}$. By [Tensorisation of Entropy](#),

$$\text{Ent}(Y) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}^{(i)}(Y) \right]$$

We will bound each of the n terms on the RHS. Conditional on $X^{(i)}$, take $u = Y_i$ (note that $u > 0$). By the [Variational Principle for Entropy](#),

$$\begin{aligned} \text{Ent}^{(i)}(Y) &\leq \mathbb{E} \left[Y \log \frac{Y}{Y_i} - (Y - Y_i) \mid X^{(i)} \right] \\ &= \mathbb{E} [e^{\lambda Z} \lambda (Z - Z_i) - (e^{\lambda Z} - e^{\lambda Z_i}) \mid X^{(i)}] \\ &= \mathbb{E} [e^{\lambda Z} (\lambda (Z - Z_i) + e^{-\lambda (Z - Z_i)} - 1) \mid X^{(i)}] \\ &= \mathbb{E} [e^{\lambda Z} \varphi(-\lambda (Z - Z_i)) \mid X^{(i)}]. \end{aligned}$$

The result follows by summing and taking expectations. \square

Theorem 4.38 (Relaxed Bounded Differences) Let $Z = f(X_1, \dots, X_n)$ for independent RVs X_1, \dots, X_n which take values on A and $f : A^n \rightarrow \mathbb{R}$. Let

$$Z_i = \inf_{x'_i} f(X_{1:(i-1)}, x'_i, X_{(i+1):n}).$$

Suppose that

$$\sum_{i=1}^n (Z - Z_i)^2 \leq \nu$$

almost surely for some $\nu > 0$. Then for all $t > 0$,

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/2\nu}.$$

Proof (Hints). By the [Modified Log-Sobolev Inequality](#). \square

Proof. By the [Modified Log-Sobolev Inequality](#),

$$\text{Ent}(e^{\lambda Z}) \leq \frac{\lambda^2}{2} \mathbb{E} \left[e^{\lambda Z} \sum_{i=1}^n (Z - Z_i)^2 \right] \leq \frac{\lambda^2 \nu}{2} \mathbb{E}[e^{\lambda Z}]$$

The result follows by [Herbst's Argument](#) and the [Chernoff Bound](#). \square

Remark 4.39 If $Z_i = \sup_{x'_i} f(X_{1:(i-1)}, x'_i, X_{(i+1):n})$ and $\sum_{i=1}^n (Z - Z_i)^2 \leq \nu$, then we also obtain a left tail bound. If this condition holds for the supremum and the infimum, then $Z - \mathbb{E}[Z] \in \mathcal{G}(\nu)$.

4.5. Concentration of convex Lipschitz functions

Let $f : [0, 1]^n \rightarrow \mathbb{R}$ be separately convex and 1-Lipschitz. The [Convex Poincaré Inequality](#) says that $\text{Var}(f(X)) \leq \mathbb{E}[\|\nabla f(X)\|^2] \leq 1$.

Theorem 4.40 Let $f : [0, 1]^n \rightarrow \mathbb{R}$ be separately convex and 1-Lipschitz. Let $Z = f(X_1, \dots, X_n)$ where X_1, \dots, X_n are independent and are supported on $[0, 1]$. Then for all $t > 0$,

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/2},$$

so $Z - \mathbb{E}[Z]$ has a sub-Gaussian right tail.

Proof (Hints).

- You may assume the partial derivatives of f exist.
- Find an appropriate upper bound for $\sum_{i=1}^n \left(f(X) - f(X'_{(i)})\right)^2$, where $X'_{(i)} = (X_{1:(i-1)}, X'_i, X_{(i+1):n})$ and X'_i is the value for which the infimum is achieved (why is the infimum achieved?).
- Conclude using [Relaxed Bounded Differences](#).

□

Proof. We may assume the partial derivatives of f exist (by approximating f as a sequence of differentiable functions if necessary). By [Relaxed Bounded Differences](#), it is enough to show that $\sum_{i=1}^n (Z - Z_i)^2 \leq 1$, where $Z_i = \inf_{x'_i} f(X_{1:(i-1)}, x'_i, X_{(i+1):n})$. We have

$$\sum_{i=1}^n (Z - Z_i)^2 = \sum_{i=1}^n \left(f(X) - f(X'_{(i)})\right)^2,$$

where $X'_{(i)} = (X_{1:(i-1)}, X'_i, X_{(i+1):n})$ and X'_i is the value for which the infimum is achieved. (The infimum is achieved since f is continuous and $[0, 1]$ is compact) By convexity and the fact that X'_i is a minimiser (so $f(X'_{(i)}) \leq f(X)$),

$$\begin{aligned} \sum_{i=1}^n \left(f(X) - f(X'_{(i)})\right)^2 &\leq \sum_{i=1}^n (X_i - X'_i)^2 \left(\frac{\partial}{\partial x_i} f(X)\right)^2 \\ &\leq \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} f(X)\right)^2 \\ &= \|\nabla f(X)\|^2 \leq 1 \end{aligned}$$

since f is 1-Lipschitz. □

Remark 4.41 The proof wouldn't work for a left-tail bound, since $-f$ is concave not convex. The entropy method does not seem to give a left tail.

Remark 4.42 The naive bound using just the Lipschitz-ness of f would give $\sum_{i=1}^n (Z - Z_i)^2 \leq n$, so convexity gives a big improvement.

5. The transport method

Definition 5.1 Let Ω be a countable set and \mathcal{A} be a collection of subsets of Ω which is a σ -algebra. A **probability space** is (Ω, \mathcal{A}, P) , where P is a probability measure.

Definition 5.2 A **real-valued RV** Z is a map $\Omega \rightarrow \mathbb{R}$. We define

$$\mathbb{P}(Z \in A) = \sum_{\omega \in \Omega: Z(\omega) \in A} P(\omega)$$

for $A \subseteq \mathbb{R}$. We define $\mathbb{E}[Z] = \sum_{\omega \in \Omega} P(\omega)Z(\omega)$. If $Q \ll P$, write $\mathbb{E}_Q[Z] = \sum_{\omega \in \Omega} Q(\omega)Z(\omega)$.

Theorem 5.3 (Variational Representation for log-MGF and Relative Entropy) Let (Ω, \mathcal{A}, P) be a countable probability space and Z be a random variable with $\mathbb{E}[|Z|] < \infty$. Then

$$\log \mathbb{E}[e^Z] = \log \mathbb{E}_P[e^Z] = \sup_{Q \ll P} (\mathbb{E}_Q[Z] - D(Q \parallel P))$$

where the supremum is taken over all probability measures Q that are absolutely continuous with respect to P such that $\mathbb{E}_Q[|Z|] < \infty$.

Conversely, fix $Q \ll P$. Then

$$D(Q \parallel P) = \sup_Z (\mathbb{E}_Q[Z] - \log \mathbb{E}_P[e^Z]),$$

where the supremum is over all RVs Z such that $\mathbb{E}_P[|Z|], \mathbb{E}_Q[|Z|] < \infty$.

Proof (Hints).

- For first statement, define

$$Q^*(\omega) = \frac{e^{Z(\omega)} P(\omega)}{\mathbb{E}_P[e^Z]}$$

and show that $D(Q \parallel P) + \log \mathbb{E}_P[e^Z] - \mathbb{E}_Q[Z] = D(Q \parallel Q^*)$.

- For second statement, show that $D(Q \parallel P) \geq \mathbb{E}_Q[Z] - \log \mathbb{E}[e^Z]$ for any $Q \ll P$ and Z , with equality if $Z(\omega) = \log \frac{Q(\omega)}{P(\omega)}$.

□

Proof. Define

$$Q^*(\omega) = \frac{e^{Z(\omega)} P(\omega)}{\mathbb{E}_P[e^Z]}.$$

Note that Q^* is a valid PMF. For any $Q \ll P$ such that $\mathbb{E}_Q[|Z|] < \infty$, we have

$$\begin{aligned} 0 &\leq D(Q \parallel Q^*) \\ &= \mathbb{E}_{Y \sim Q} \left[\log \frac{Q(Y)}{Q^*(Y)} \right] \\ &= \mathbb{E}_{Y \sim Q} \left[\log \left(\frac{Q(Y)}{P(Y)} \frac{P(Y)}{Q^*(Y)} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{Y \sim Q} \left[\log \frac{Q(Y)}{P(Y)} \right] + \mathbb{E}_Q \left[\log \frac{P(Y) \mathbb{E}_{Z \sim P}[e^Z]}{P(Y) e^Z} \right] \\
&= D(Q \parallel P) + \log \mathbb{E}_P[e^Z] - \mathbb{E}_Q[Z]
\end{aligned}$$

Hence $\log \mathbb{E}[e^Z] \geq \mathbb{E}_Q Z - D(Q \parallel P)$, with equality iff $Q = Q^*$. The result follows since $Q^* \ll P$. For the second statement, note that $D(Q \parallel P) \geq \mathbb{E}_Q[Z] - \log \mathbb{E}[e^Z]$, for any $Q \ll P$ and Z . There is equality if $Z(\omega) = \log \frac{Q(\omega)}{P(\omega)}$. (Note that $\mathbb{E}_Q[|Z|] = \mathbb{E}_Q[|\log \frac{Q}{P}|] < \infty$ since $D(Q \parallel P) < \infty$ and the negative part of $x \log x$ is finitely bounded.) Note it can be shown that the result holds when $D(Q \parallel P) = \infty$ and when $\mathbb{E}_P[e^Z] = \infty$. \square

Corollary 5.4 For all $\lambda \in \mathbb{R}$, we have

$$\log \mathbb{E}_P[e^{\lambda(Z - \mathbb{E}_P[Z])}] = \sup_{Q \ll P} (\lambda(\mathbb{E}_Q Z - \mathbb{E}_P Z) - D(Q \parallel P))$$

Theorem 5.5 (Marton's Argument) Let P be a PMF and $Z \sim P$. If there exists $\nu > 0$ such that

$$\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \sqrt{2\nu D(Q \parallel P)}$$

for all PMFs Q such that $Q \ll P$, then

$$\psi_{Z - \mathbb{E}[Z]}(\lambda) = \log \mathbb{E}_P[e^{\lambda(Z - \mathbb{E}_P[Z])}] \leq \frac{\lambda^2 \nu}{2} \quad \forall \lambda > 0,$$

(and so also $\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/2\nu}$ by the [Chernoff Bound](#)). Conversely, if there exists $\nu > 0$ such that $\psi_{Z - \mathbb{E}[Z]}(\lambda) = \log \mathbb{E}_P[e^{\lambda(Z - \mathbb{E}_P[Z])}] \leq \frac{\lambda^2 \nu}{2}$ for all $\lambda > 0$, then

$$\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \sqrt{2\nu D(Q \parallel P)}$$

for all $Q \ll P$.

Proof (Hints).

- Show that $\log \mathbb{E}_P[e^{\lambda(Z - \mathbb{E}[Z])}] \leq \sup_{t \geq 0} (\lambda \sqrt{2\nu t} - t)$.
- For converse, may assume that $\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \geq 0$ (why?). The proof is similar to the first proof.

\square

Proof. By the [Variational Representation for log-MGF and Relative Entropy](#),

$$\begin{aligned}
\log \mathbb{E}_P[e^{\lambda(Z - \mathbb{E}[Z])}] &= \sup_{Q \ll P} (\lambda(\mathbb{E}_Q[Z] - \mathbb{E}_P[Z]) - D(Q \parallel P)) \\
&\leq \sup_{Q \ll P} (\lambda \sqrt{2\nu D(Q \parallel P)} - D(Q \parallel P)) \\
&\leq \sup_{t \geq 0} (\lambda \sqrt{2\nu t} - t).
\end{aligned}$$

Let $f(t) = \lambda \sqrt{2\nu t} - t$. Then $f'(t) = 0$ iff $t = \frac{\lambda^2 \nu}{2}$, and so the $\sup_{t \geq 0} f(t) = \frac{\lambda^2 \nu}{2}$.

For the converse, we may assume that $\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \geq 0$, since otherwise we are trivially done. By [Variational Representation for log-MGF and Relative Entropy](#), for all $\lambda > 0$,

$$D(Q \parallel P) \geq \lambda(\mathbb{E}_Q[Z] - \mathbb{E}_P[Z]) - \log \mathbb{E}_P e^{\lambda(Z - \mathbb{E}_P[Z])} \geq \lambda(\mathbb{E}_Q[Z] - \mathbb{E}_P[Z]) - \frac{\lambda^2 \nu}{2}$$

Taking the supremum over $\lambda > 0$, we obtain

$$D(Q \parallel P) \geq \sup_{\lambda > 0} \left(\lambda(\mathbb{E}_Q[Z] - \mathbb{E}_P[Z]) - \frac{\lambda^2 \nu}{2} \right)$$

Differentiating the RHS, we see that it is maximised when $\lambda = \frac{1}{\nu}(\mathbb{E}_Q[Z] - \mathbb{E}_P[Z])$, and so

$$D(Q \parallel P) \geq \frac{(\mathbb{E}_Q[Z] - \mathbb{E}_P[Z])^2}{2\nu}.$$

□

5.1. Concentration via Marton's argument

Definition 5.6 Let P, Q be distributions on A . Let $X \sim P$ and $Y \sim Q$. A **coupling** π is a joint distribution on (X, Y) such that X has marginal P (w.r.t π) and Y has marginal Q (w.r.t. π). Write $\Pi(P, Q)$ for the set of all couplings.

Example 5.7 $P \otimes Q$ is the independent coupling.

Lemma 5.8 $f : A^n \rightarrow \mathbb{R}$ such that $f(y) - f(x) \leq \sum_{i=1}^n c_i d(x_i, y_i)$ for some constants c_i and distance $d(\cdot, \cdot)$. Let $X \sim P_1 \otimes \cdots \otimes P_n =: P$, $Z = f(X)$. Let $C > 0$ be such that

$$\inf_{\pi \in \Pi(P, Q)} \sum_{i=1}^n \mathbb{E}_\pi [d(X_i, Y_i)]^2 \leq 2CD(Q \parallel P).$$

for all $Q \ll P$. Then

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/2\nu},$$

where $\nu = C \sum_{i=1}^n c_i^2$.

Proof (Hints). Let $Q \ll P$ and $Y \sim Q$. Show that

$$\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \left(\sum_{i=1}^n c_i^2 \right)^{1/2} \left(\sum_{i=1}^n \mathbb{E}_\pi [d(X_i, Y_i)]^2 \right)^{1/2},$$

and conclude the result using [Marton's Argument](#). □

Proof. Let $Q \ll P$ and $Y \sim Q$. Then

$$\begin{aligned} \mathbb{E}_Q[Z] - \mathbb{E}_P[Z] &= \mathbb{E}[f(Y)] - \mathbb{E}[f(X)] \\ &= \mathbb{E}_\pi[f(Y) - f(X)] \quad \text{for all } \pi \in \Pi(P, Q) \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}_\pi \left[\sum_{i=1}^n c_i d(X_i, Y_i) \right] \\
&= \sum_{i=1}^n c_i \mathbb{E}_\pi [d(X_i, Y_i)] \\
&\leq \left(\sum_{i=1}^n c_i^2 \right)^{1/2} \left(\sum_{i=1}^n \mathbb{E}_\pi [d(X_i, Y_i)]^2 \right)^{1/2} \quad \text{by Cauchy-Schwarz}
\end{aligned}$$

So

$$\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \left(\sum_{i=1}^n c_i^2 \right)^{1/2} \left(\inf_{\pi \in \Pi(P, Q)} \sum_{i=1}^n \mathbb{E}_\pi [d(X_i, Y_i)]^2 \right)^{1/2}$$

Since

$$\inf_{\pi \in \Pi(P, Q)} \sum_{i=1}^n \mathbb{E}_\pi [d(X_i, Y_i)]^2 \leq 2CD(Q \parallel P)$$

we have $\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \sqrt{2\nu D(Q \parallel P)}$, where $\nu = C \sum_{i=1}^n c_i^2$. The result follows by [Marton's Argument](#). \square

Definition 5.9 Let $X \sim P$ and $Y \sim Q$. The **transportation cost** from Q to P w.r.t a distance $d(\cdot, \cdot)$ is

$$\inf_{\pi \in \Pi(P, Q)} \mathbb{E}_\pi [d(X, Y)].$$

Definition 5.10 Let P and Q be distributions on the same space (Ω, \mathcal{A}) . The **total variation distance** between P and Q is

$$d_{\text{TV}}(P, Q) := \sup_{A \in \mathcal{A}} |P(A) - Q(A)|.$$

Proposition 5.11 Let $A^* = \{\omega \in \Omega : P(\omega) \geq Q(\omega)\}$. We have the alternative expressions

$$\begin{aligned}
d_{\text{TV}}(P, Q) &= \frac{1}{2} \sum_{\omega \in \Omega} |P(\omega) - Q(\omega)| = \sum_{\omega \in \Omega} (P(\omega) - Q(\omega))_+ \\
&= P(A^*) - Q(A^*) = 1 - \sum_{\omega \in \Omega} \min\{P(\omega), Q(\omega)\}.
\end{aligned}$$

Proof (Hints).

- For second equality, consider the $+$ and $-$ parts.
- For the first equality, show \leq by splitting sum over A and A^c for $A \in \mathcal{A}$, show \geq by considering $A^* = \{\omega : P(\omega) \geq Q(\omega)\}$.
- For the third equality, show the fourth expression is equal to the third.

\square

Proof. For the first inequality: for any $A \in \mathcal{A}$, by the triangle inequality,

$$\begin{aligned}
\sum_{\omega \in \Omega} |P(\omega) - Q(\omega)| &= \sum_{\omega \in A} |P(\omega) - Q(\omega)| + \sum_{\omega \in A^c} |P(\omega) - Q(\omega)| \\
&\geq P(A) - Q(A) + Q(A^c) - P(A^c) = 2(P(A) - Q(A))
\end{aligned}$$

and similarly $\sum_{\omega \in \Omega} |P(\omega) - Q(\omega)| \geq 2(Q(A) - P(A))$. Conversely,

$$\begin{aligned}
d_{\text{TV}}(P, Q) &\geq P(A^*) - Q(A^*) \\
&= \sum_{\omega \in \Omega} (P(\omega) - Q(\omega))_+ = \frac{1}{2} \sum_{\omega \in \Omega} |P(\omega) - Q(\omega)|,
\end{aligned}$$

since $\sum_{\omega \in \Omega} (P(\omega) - Q(\omega))_+ = \sum_{\omega \in \Omega} (P(\omega) - Q(\omega))_-$. For the third inequality,

$$\begin{aligned}
1 - \sum_{\omega \in \Omega} \min\{P(\omega), Q(\omega)\} &= \sum_{\omega \in \Omega} P(\omega) - \min\{P(\omega), Q(\omega)\} \\
&= \sum_{\omega \in \Omega} (P(\omega) - Q(\omega))_+
\end{aligned}$$

□

Lemma 5.12 Let P and Q be distributions on the same space. Then if $X \sim P$ and $Y \sim Q$,

$$\inf_{\pi \in \Pi(P, Q)} \mathbb{P}_\pi(X \neq Y) = d_{\text{TV}}(P, Q) \in [0, 1].$$

Proof (Hints). Show that LHS \geq RHS by taking a supremum and infimum and using that $|\mathbb{1}_{\{x \in A\}} - \mathbb{1}_{\{y \in A\}}| \leq \mathbb{1}_{\{x \neq y\}}$, then consider

$$\pi(\omega_1, \omega_2) = \begin{cases} \min\{P(\omega), Q(\omega)\} & \text{if } \omega_1 = \omega_2 = \omega \\ \frac{1}{d_{\text{TV}}(P, Q)} (P(\omega_1) - Q(\omega_1))(Q(\omega_2) - P(\omega_2)) & \text{if } (\omega_1, \omega_2) \in A^* \times (A^*)^c \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof. Let $\pi \in \Pi(P, Q)$ and $A \in \mathcal{A}$. Since $|\mathbb{I}_{\{X \in A\}} - \mathbb{I}_{\{Y \in A\}}| \leq \mathbb{I}_{\{X \neq Y\}}$ We have

$$\begin{aligned}
|P(A) - Q(A)| &= |\mathbb{E}_\pi[\mathbb{I}_{\{X \in A\}} - \mathbb{I}_{\{Y \in A\}}]| \\
&\leq \mathbb{E}_\pi[|\mathbb{I}_{\{X \in A\}} - \mathbb{I}_{\{Y \in A\}}|] \\
&\leq \mathbb{E}[\mathbb{I}_{\{X \neq Y\}}] \quad \text{pointwise} \\
&= \mathbb{P}(X \neq Y).
\end{aligned}$$

Taking the supremum over all $A \in \mathcal{A}$ and the infimum over all couplings gives $d_{\text{TV}}(P, Q) \leq \inf_{\pi \in \Pi(P, Q)} \mathbb{P}(X \neq Y)$. We will construct π such that $\mathbb{P}(X \neq Y) = d_{\text{TV}}(P, Q)$. Intuitively, we want to place as much mass as possible on the “diagonal”, i.e. make $\pi(\omega, \omega)$ as large as possible.

For $(\omega_1, \omega_2) \in \Omega \times \Omega$, let

$$\pi(\omega_1, \omega_2) = \begin{cases} \min\{P(\omega), Q(\omega)\} & \text{if } \omega_1 = \omega_2 = \omega \\ \frac{1}{d_{\text{TV}}(P, Q)}(P(\omega_1) - Q(\omega_1))(Q(\omega_2) - P(\omega_2)) & \text{if } (\omega_1, \omega_2) \in A^* \times (A^*)^c \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\mathbb{P}_\pi(X = Y) = \sum_{\omega \in \Omega} \pi(\omega, \omega) = \sum_{\omega \in \Omega} \min\{P(\omega), Q(\omega)\}$, and so by Proposition 5.11, $\mathbb{P}_\pi(X \neq Y) = 1 - \sum_{\omega \in \Omega} \min\{P(\omega), Q(\omega)\} = d_{\text{TV}}(P, Q)$. Also, π is indeed a valid coupling:

$$\begin{aligned} \sum_{\omega_1 \in \Omega} \pi(\omega_1, \omega_2) &= \sum_{\omega_1 \in A^*} (P(\omega_1) - Q(\omega_1)) \frac{Q(\omega_2) - P(\omega_2)}{d_{\text{TV}}(P, Q)} \mathbb{I}_{\{\omega_2 \in (A^*)^c\}} + \min\{P(\omega_2), Q(\omega_2)\} \\ &= Q(\omega_2), \end{aligned}$$

and similarly $\sum_{\omega_2 \in \Omega} \pi(\omega_1, \omega_2) = P(\omega_1)$. □

Definition 5.13 The minimising coupling

$$\pi(\omega_1, \omega_2) = \begin{cases} \min\{P(\omega), Q(\omega)\} & \text{if } \omega_1 = \omega_2 = \omega \\ \frac{1}{d_{\text{TV}}(P, Q)}(P(\omega_1) - Q(\omega_1))(Q(\omega_2) - P(\omega_2)) & \text{if } (\omega_1, \omega_2) \in A^* \times (A^*)^c \\ 0 & \text{otherwise.} \end{cases}$$

in the proof of Lemma 5.12 is called the **optimal total variation coupling**.

Lemma 5.14 (Pinsker's Inequality) Let P and Q be PMFs such that $Q \ll P$. Then

$$d_{\text{TV}}(P, Q)^2 \leq \frac{1}{2} D(Q \parallel P).$$

Proof (Hints). Let $Y(\omega) = \frac{Q(\omega)}{P(\omega)}$ and $Z = \mathbb{I}_{\{Y \geq 1\}}$. Use [Hoeffding's Lemma](#) and [Marton's Argument](#). □

Proof. Let $Y(\omega) = \frac{Q(\omega)}{P(\omega)}$. Let $Z = \mathbb{I}_{\{Y \geq 1\}}$. By [Hoeffding's Lemma](#),

$$\psi_{Z - \mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2}{8}.$$

But then by [Marton's Argument](#),

$$\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \sqrt{2 \cdot \frac{1}{4} \cdot D(Q \parallel P)},$$

i.e. $d_{\text{TV}}(P, Q) = Q(A) - P(A) \leq \sqrt{\frac{1}{2} \cdot D(Q \parallel P)}$, where $A = \{\omega \in \Omega : Q(\omega) \geq P(\omega)\}$, by Proposition 5.11. □

Theorem 5.15 (Marton's Transport Cost Inequality) Let $P = P_1 \otimes \cdots \otimes P_n$ and $Q \ll P$. Let $X \sim P$ and $Y \sim Q$. Then

$$\inf_{\pi \in \Pi(P, Q)} \sum_{i=1}^n \mathbb{E}_\pi \left[\mathbb{I}_{\{X_i \neq Y_i\}} \right]^2 = \inf_{\pi \in \Pi(P, Q)} \sum_{i=1}^n \mathbb{P}_\pi(X_i \neq Y_i)^2 \leq \frac{1}{2} D(Q \parallel P).$$

Proof. We use induction on n . The $n = 1$ case follows from Lemma 5.12 and Pinsker's Inequality. Assume that for every $n \leq k$, there exists a coupling π_n on $(X_{1:n}, Y_{1:n})$ such that $\sum_{i=1}^n \mathbb{P}(X_i \neq Y_i)^2 \leq \frac{1}{2}D(Q \parallel P)$. We will extend it to a coupling π_{k+1} on $(X_{1:(k+1)}, Y_{1:(k+1)})$. Write

$$\sum_{i=1}^{k+1} \mathbb{P}(X_i \neq Y_i)^2 = \sum_{i=1}^k \mathbb{P}(X_i \neq Y_i)^2 + \mathbb{P}(X_{k+1} \neq Y_{k+1})^2$$

For fixed $y_{1:k}$, let $\pi_{y_{1:k}} \in \Pi(P_{X_{k+1}}, Q_{Y_{k+1} | Y_{1:k}=y_{1:k}})$ be the optimal total variation coupling of X_{k+1} and $Y_{k+1} | Y_{1:k} = y_{1:k}$. Define

$$\begin{aligned} \pi_{k+1}(x_{1:(k+1)}, y_{1:(k+1)}) &:= \pi_k(x_{1:k}, y_{1:k}) \cdot \pi_{y_{1:k}}(x_{k+1}, y_{k+1}) \\ &= \mathbb{P}(X_{1:k} = x_{1:k}, Y_{1:k} = y_{1:k}) \mathbb{P}(X_{k+1} = x_{k+1}) \mathbb{P}(Y_{k+1} = y_{k+1} | X_{k+1} = x_{k+1}) \end{aligned}$$

This new coupling has two properties:

1. Given $(X_{1:k}, Y_{1:k})$, the distribution of (X_{k+1}, Y_{k+1}) depends only on $Y_{1:k}$, i.e. $X_{1:k} - Y_{1:k} - (X_{k+1}, Y_{k+1})$ form a Markov chain.
2. Also, X_{k+1} is independent of $(X_{1:k}, Y_{1:k})$.

These properties imply that $(X_{k+1}, Y_{k+1}) | X_{1:k} = x_{1:k}, Y_{1:k} = y_{1:k} \sim \pi_{y_{1:k}}$. Hence,

$$\begin{aligned} \mathbb{P}(X_{k+1} \neq Y_{k+1} | X_{1:k} = x_{1:k}, Y_{1:k} = y_{1:k}) &= d_{\text{TV}}(P_{X_{k+1}}, Q_{Y_{k+1} | Y_{1:k}=y_{1:k}}) \\ &\leq \sqrt{\frac{1}{2}D(Q_{Y_{k+1} | Y_{1:k}=y_{1:k}} \parallel P_{X_{k+1}})} \end{aligned}$$

by the $n = 1$ result. Taking expectation over π_k on the LHS gives

$$\begin{aligned} \mathbb{P}(X_{k+1} \neq Y_{k+1}) &= \mathbb{E}_{\pi_k}[\mathbb{P}(X_{k+1} \neq Y_{k+1} | X_{1:k}, Y_{1:k})] \\ &\leq \mathbb{E}_{Q_{Y_{1:k}}} \left[\sqrt{\frac{1}{2}D(Q_{Y_{k+1} | Y_{1:k}} \parallel P_{X_{k+1}})} \right] \end{aligned}$$

Squaring and using Jensen's inequality gives

$$\begin{aligned} \mathbb{P}(X_{k+1} \neq Y_{k+1})^2 &\leq \frac{1}{2} \mathbb{E}_{Q_{Y_{1:k}}} [D(Q_{Y_{k+1} | Y_{1:k}} \parallel P_{X_{k+1}})] \\ &= \frac{1}{2} D(Q_{Y_{k+1} | Y_{1:k}} \parallel P_{X_{k+1}} | Q_{Y_{1:k}}) \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} \sum_{i=1}^{k+1} \mathbb{P}(X_i \neq Y_i)^2 &\leq \frac{1}{2} (D(Q_{Y_{1:k}} \parallel P_{X_{1:k}}) + D(Q_{Y_{k+1} | Y_{1:k}} \parallel P_{X_{k+1}} | Q_{Y_{1:k}})) \\ &= \frac{1}{2} D(Q_{Y_{1:(k+1)}} \parallel P_{X_{1:(k+1)}}) \end{aligned}$$

by the Chain Rule for Relative Entropy. □

Remark 5.16 We can recover the [Bounded Differences Inequality](#) from [Marton's Transport Cost Inequality](#): the conditions of Lemma [5.8](#) are satisfied with $C = \frac{1}{4}$, since f having bounded differences with constant c_i implies

$$f(y) - f(x) \leq \sum_{i=1}^n c_i d(x_i, y_i),$$

where $d(x_i, y_i) = \mathbb{I}_{\{x_i \neq y_i\}}$. This gives the concentration bound.

5.2. Talagrand's inequality

Definition 5.17 Marton's divergence is

$$d_2^2(Q, P) = \mathbb{E}_{X \sim P} \left[\left(1 - \frac{Q(X)}{P(X)} \right)_+^2 \right] = \sum_{\omega: P(\omega) > 0} \frac{(P(\omega) - Q(\omega))_+^2}{P(\omega)}.$$

Lemma 5.18 Let P and Q be distributions on the same space (Ω, \mathcal{A}) . Then

$$\inf_{\pi \in \Pi(P, Q)} \mathbb{E}_{(X, Y) \sim \pi} [\mathbb{P}(X \neq Y \mid X)^2] = d_2^2(Q, P).$$

Proof (Hints).

- For \geq , explain why $\mathbb{P}(X = Y \mid X = x) \leq \min\{1, Q(x)/P(x)\}$, then take expectation.
- Showing equality, by showing that the optimal total variation coupling minimises the LHS, is left as an exercise.

□

Proof. We have

$$\mathbb{P}(X = Y \mid X = x) = \frac{\mathbb{P}(X = x, Y = x)}{\mathbb{P}(X = x)} \leq \min\left\{1, \frac{Q(x)}{P(x)}\right\}.$$

So for any coupling π ,

$$\mathbb{E}_\pi [\mathbb{P}(X \neq Y \mid X)^2] \geq \mathbb{E}_P \left[\left(1 - \min\left\{1, \frac{Q(X)}{P(X)}\right\} \right)^2 \right] = \mathbb{E}_P \left[\left(1 - \frac{Q(X)}{P(X)} \right)_+^2 \right] = d_2^2(Q, P).$$

Showing equality is left as an exercise.

□

Lemma 5.19 (Pinsker's Inequality for Marton Divergence) Let P, Q be distributions on the same space (Ω, \mathcal{A}) with $Q \ll P$. Then

$$d_2^2(Q, P) \leq 2D(Q \parallel P).$$

Proof (Hints).

- Let $h(t) = (1 - t) \log(1 - t) + t$ for $t \leq 1$, expression $D(Q \parallel P)$ using h (as an expectation w.r.t P).
- Show that $h(t) \geq 0$ and by considering derivatives, show that $h(t) \geq t^2/2$ for all $t \in [0, 1]$.

□

Proof. Let $h(t) = (1-t)\log(1-t) + t$ for $t \leq 1$ and $q(X) = \frac{Q(X)}{P(X)}$. Then

$$D(Q \parallel P) = \mathbb{E}_{X \sim P}[h(1 - q(X))].$$

We have $h(t) = -(1-t)\log(1 + \frac{t}{1-t}) + t \geq -t + t \geq 0$ since $\log x \leq x - 1$. Also, $h(t) \geq t^2/2$ for $t \in [0, 1]$: indeed, $h(0) = 0^2/2$, and $h'(t) = -1 - \log(1-t) + 1 = -\log(1-t)$, thus

$$\frac{d}{dt} \left(h(t) - \frac{t^2}{2} \right) = -\log(1-t) - t \geq (1-t) + 1 - t = 0.$$

So we have

$$\begin{aligned} D(Q \parallel P) &= \mathbb{E}[h(1 - q(X))] \geq \mathbb{E}[h((1 - q(X))_+)] \\ &\geq \mathbb{E} \left[\frac{(1 - q(X))_+^2}{2} \right] = \frac{1}{2} d_2^2(Q, P). \end{aligned}$$

where first inequality is since $h \geq 0$ and $h(0) = 0$. \square

Theorem 5.20 (Marton's Conditional Transport Cost Inequality) Let $X = (X_1, \dots, X_n)$, $X \sim P = P_1 \otimes \dots \otimes P_n$, and let $Q \ll P$. Then

$$\inf_{\pi \in \Pi(P, Q)} \sum_{i=1}^n \mathbb{E}_{\pi} [\mathbb{P}(X_i \neq Y_i \mid X)^2] \leq 2D(Q \parallel P).$$

Proof. We use induction on n . The $n = 1$ case follows by Lemma 5.18 and Pinsker's Inequality for Marton Divergence. Now assume that for every $n \leq k$, there exists a $\pi_n \in \Pi(P, Q)$ such that $\sum_{i=1}^n \mathbb{E}_{\pi_n} [\mathbb{P}(X_i \neq Y_i \mid X)^2] \leq 2D(Q_{X_{1:n}} \parallel P_{X_{1:n}})$. We will find a coupling π_{k+1} (extended from π_k) such that

$$\begin{aligned} \sum_{i=1}^k \mathbb{E}_{\pi_{k+1}} [\mathbb{P}(X_i \neq Y_i \mid X_{1:(k+1)})^2] + \mathbb{E}_{\pi_{k+1}} [\mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{1:(k+1)})^2] &= \sum_{i=1}^{k+1} \mathbb{E}_{\pi_{k+1}} [\mathbb{P}(X_i \neq Y_i \mid X_{1:(k+1)})^2] \\ &\leq D(Q_{Y_{1:(k+1)}} \parallel P_{X_{1:(k+1)}}) \end{aligned}$$

For fixed $y_{1:k}$, let $\pi_{y_{1:k}}$ be the optimal total variation coupling of X_{k+1} and $Y_{k+1} \mid Y_{1:k} = y_{1:k}$. Let

$$\begin{aligned} \pi_{k+1}(x_{1:(k+1)}, y_{1:(k+1)}) &= \pi_k(x_{1:k}, y_{1:k}) \cdot \pi_{y_{1:k}}(x_{k+1}, y_{k+1}) \\ &= \mathbb{P}(X_{1:k} = x_{1:k}, Y_{1:k} = y_{1:k}) \cdot \mathbb{P}(X_{k+1} = x_{k+1}) \cdot \mathbb{P}(Y_{k+1} = y_{k+1} \mid X_{k+1} = x_{k+1}), \end{aligned}$$

where the probabilities in the last line are w.r.t. the new coupling π_{k+1} . This coupling has two properties:

- $X_{1:k} - Y_{1:k} - (X_{k+1}, Y_{k+1})$ form a Markov chain, i.e. given $(X_{1:k}, Y_{1:k})$, the distribution of (X_{k+1}, Y_{k+1}) only depends on $Y_{1:k}$.
- X_{k+1} is independent of $(X_{1:k}, Y_{1:k})$.

These properties imply that given $X_{1:k} = x_{1:k}, Y_{1:k} = y_{1:k}$, we have $(X_{k+1}, Y_{k+1}) \sim \pi_{y_{1:k}}$. By the induction hypothesis,

$$\begin{aligned} \sum_{i=1}^k \mathbb{E}_{\pi_{k+1}} \left[\mathbb{P}(X_i \neq Y_i \mid X_{1:(k+1)})^2 \right] &= \sum_{i=1}^k \mathbb{E}_{\pi_{k+1}} \left[\mathbb{P}(X_i \neq Y_i \mid X_{1:k})^2 \right] \text{ by second property} \\ &= \sum_{i=1}^k \mathbb{E}_{\pi_k} \left[\mathbb{P}(X_i \neq Y_i \mid X_{1:k})^2 \right] \\ &\leq 2D(Q_{Y_{1:k}} \parallel P_{X_{1:k}}). \end{aligned}$$

We want to show

$$\mathbb{E} \left[\mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{1:(k+1)})^2 \right] \leq 2D(Q_{Y_{k+1} \mid Y_{1:k}} \parallel P_{X_{k+1} \mid Q_{Y_{1:k}}})$$

From the $n = 1$ case (and since the optimal total variation coupling $\pi_{y_{1:k}}$ is a minimiser in Lemma 5.18), we know that

$$\mathbb{E}_{\pi_{y_{1:k}}} \left[\mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{k+1}, Y_{1:k} = y_{1:k})^2 \right] \leq 2D(Q_{Y_{k+1} \mid Y_{1:k} = y_{1:k}} \parallel P_{X_{k+1}}).$$

By the two properties of π_{k+1} ,

$$\mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{k+1}, Y_{1:k} = y_{1:k}) = \mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{1:(k+1)}, Y_{1:k} = y_{1:k})$$

Taking $\mathbb{E}_{Y_{1:k}}(\cdot)$ in the above, we obtain

$$\begin{aligned} \mathbb{E} \left[\mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{1:(k+1)}, Y_{1:k})^2 \right] &= \mathbb{E} \left[\mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{k+1}, Y_{k+1})^2 \right] \\ &\leq 2D(Q_{Y_{k+1} \mid Y_{1:k}} \parallel P_{X_{k+1} \mid Q_{Y_{1:k}}}) \end{aligned}$$

The LHS is equal to

$$\begin{aligned} &\mathbb{E} \left[\mathbb{E} \left[\mathbb{I}_{\{X_{k+1} \neq Y_{k+1}\}} \mid X_{1:(k+1)}, Y_{1:k} \right]^2 \mid X_{1:(k+1)} \right] \\ &\geq \mathbb{E} \left[\mathbb{E} \left[\mathbb{E} \left[\mathbb{I}_{\{X_{k+1} \neq Y_{k+1}\}} \mid X_{1:(k+1)}, Y_{1:k} \right] \mid X_{1:(k+1)} \right]^2 \right] \text{ by Jensen} \\ &= \mathbb{E} \left[\mathbb{E} \left[\mathbb{I}_{\{X_{k+1} \neq Y_{k+1}\}} \mid X_{1:(k+1)} \right]^2 \right] \text{ by tower property} \\ &= \mathbb{E} \mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{1:(k+1)})^2 \end{aligned}$$

So

$$\begin{aligned} &\sum_{i=1}^k \mathbb{E} \mathbb{P}(X_i \neq Y_i \mid X_{1:(k+1)})^2 + \mathbb{E} \mathbb{P}(X_{k+1} \neq Y_{k+1} \mid X_{1:k})^2 \\ &\leq 2D(Q_{Y_{1:k}} \parallel P_{X_{1:k}}) + 2D(Q_{Y_{k+1} \mid Y_{1:k}} \parallel P_{X_{k+1} \mid Q_{Y_{1:k}}}) \\ &= 2D(Q \parallel P) \end{aligned}$$

by the Chain Rule for Relative Entropy. □

Definition 5.21 $f : A^n \rightarrow \mathbb{R}$ satisfies the **one-sided bounded differences** property if

$$f(y) - f(x) \leq \sum_{i=1}^n \mathbb{I}_{\{x_i \neq y_i\}} c_i(x) \quad \forall x, y \in A^n,$$

where $c_i : A^n \rightarrow \mathbb{R}_{\geq 0}$.

Remark 5.22 We can't apply results for bounded differences on functions with this property, since it is a weaker property.

Remark 5.23 By **Relaxed Bounded Differences**, if $\sum_{i=1}^n (Z_i - Z)^2 \leq \nu$, where $Z_i = \sup_{x_i} f(X_{1:(i-1)}, x_i, X_{(i+1):n})$, then $\mathbb{P}(Z - \mathbb{E}[Z] \leq -t) \leq e^{-t^2/2\nu}$. Under one-sided bounded differences,

$$0 \leq \sum_{i=1}^n (Z_i - Z)^2 \leq \sum_{i=1}^n c_i(X)^2 \leq \sup_{x \in A^n} \sum_{i=1}^n c_i(x)^2 =: \nu_\infty,$$

so we obtain the left-tail bound $\mathbb{P}(Z - \mathbb{E}[Z] \leq -t) \leq e^{-t^2/2\nu_\infty}$. But now if $Z_i = \inf_{x_i} f(X_{1:(i-1)}, x_i, X_{(i+1):n})$, with infimum achieved at $(X')^{(i)} = (X_{1:(i-1)}, x'_i, X_{(i+1):n})$, then

$$0 \leq \sum_{i=1}^n (Z - Z_i)^2 \leq \sum_{i=1}^n c_i((X')^{(i)})^2.$$

We generally can't say that this is $\leq \sup_{x \in A^n} \sum_{i=1}^n c_i(x)^2$, so can't immediately deduce a right tail bound.

However, the transport method gives us a right-tail bound with a better parameter $\nu = \mathbb{E}[\sum_{i=1}^n c_i(X)^2] \leq \nu_\infty$.

Theorem 5.24 (Talagrand's One-sided Bounded Differences Inequality) Let $X = (X_1, \dots, X_n) \sim P_1 \otimes \dots \otimes P_n$, X_i independent. Let $f : A^n \rightarrow \mathbb{R}$ be a function with one-sided bounded differences with associated functions c_i . Let $Z = f(X)$ and let $\nu = \mathbb{E}[\sum_{i=1}^n c_i(X)^2]$. Then

$$\psi_{Z - \mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2 \nu}{2} \quad \forall \lambda > 0$$

which implies that

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq e^{-t^2/2\nu} \quad \forall t > 0.$$

Proof (Hints).

- For $Q \ll P$ and $\pi \in \Pi(P, Q)$, show that, using **Law of Total Expectation**,

$$\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \sum_{i=1}^n \mathbb{E}_\pi[c_i(X) \mathbb{P}(X_i \neq Y_i \mid X)],$$

where $\mathbb{P}(X_i \neq Y_i \mid X) = \mathbb{E}_\pi[\mathbb{I}_{\{X_i \neq Y_i\}} \mid X]$.

- Apply Cauchy-Schwarz twice.

- Conclude using [Marton's Conditional Transport Cost Inequality](#) and [Marton's Argument](#).

□

Proof. Let $Q \ll P$. Then for all $\pi \in \Pi(P, Q)$,

$$\begin{aligned}
\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] &= \mathbb{E}_\pi[f(Y) - f(X)] \\
&\leq \mathbb{E}_\pi \left[\sum_{i=1}^n c_i(X) \mathbb{I}_{\{X_i \neq Y_i\}} \right] \quad \text{by assumption} \\
&= \sum_{i=1}^n \mathbb{E}_\pi \mathbb{E}_\pi [\mathbb{I}_{\{X_i \neq Y_i\}} c_i(X) \mid X] \quad \text{by [Law of Total Expectation](#)} \\
&= \sum_{i=1}^n \mathbb{E}_\pi [c_i(X) \mathbb{P}(X_i \neq Y_i \mid X)] \\
&\leq \sum_{i=1}^n (\mathbb{E}_\pi [c_i(X)^2])^{1/2} (\mathbb{E}_\pi [\mathbb{P}(X_i \neq Y_i \mid X)^2])^{1/2} \quad \text{by Cauchy-Schwarz} \\
&\leq \left(\sum_{i=1}^n \mathbb{E}_\pi [c_i(X)^2] \right)^{1/2} \left(\sum_{i=1}^n \mathbb{E} [\mathbb{P}(X_i \neq Y_i \mid X)^2] \right)^{1/2} \quad \text{by Cauchy-Schwarz}
\end{aligned}$$

where we write $\mathbb{P}(X_i \neq Y_i \mid X) = \mathbb{E}_\pi [\mathbb{I}_{\{X_i \neq Y_i\}} \mid X]$. By [Marton's Conditional Transport Cost Inequality](#),

$$\inf_{\pi \in \Pi(P, Q)} \sum_{i=1}^n \mathbb{E} [\mathbb{P}(X_i \neq Y_i \mid X)^2] \leq 2D(Q \parallel P).$$

which implies that

$$\mathbb{E}_Q[Z] - \mathbb{E}_P[Z] \leq \sqrt{\nu \cdot 2 \cdot D(Q \parallel P)}$$

and so by [Marton's Argument](#), $\psi_{Z - \mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2 \nu}{2}$ for all $\lambda > 0$, which gives the right tail bound by the [Chernoff Bound](#). □

6. Log-concave random variables

Definition 6.1 A continuous random variable $X \in \mathbb{R}^n$ with density function ρ is **log-concave** if $\log \rho$ is concave, i.e. if

$$\rho(\lambda x + (1 - \lambda)y) \geq \rho(x)^\lambda \rho(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Definition 6.2 A **convex body** is a non-empty, convex, compact set. The **diameter** of a convex body K is $\text{Diam}(K) = \sup_{x, y \in K} \|x - y\|_2$.

Example 6.3 The Gaussian

$$\frac{1}{(2\pi)^n \det(\Sigma)^{1/2}} e^{-(x \Sigma^{-1} x)/2},$$

the exponential $\alpha e^{-\|x\|}$ and the uniform distribution on convex bodies are log-concave distributions.

Theorem 6.4 (Log-concave Poincaré inequality) Let X be log-concave, supported on a convex body $K \subseteq \mathbb{R}^n$. Then X satisfies the Poincaré inequality with Poincaré constant

$$C_P(X) \leq \text{Diam}(K)^2 \cdot C_n,$$

for some absolute C_n depending only on n ; that is,

$$\text{Var}(f(X)) \leq \text{Diam}(K)^2 \cdot C_n \cdot \mathbb{E}[\|\nabla f(X)\|^2],$$

for all $f \in C^1(\mathbb{R}^n)$.

Proof. WLOG $\mathbb{E}[f(X)] = 0$. We have

$$\text{Var}(f(X)) = \frac{1}{2} \text{Var}(f(X) - f(Y)) = \frac{1}{2} \mathbb{E}[(f(X) - f(Y))^2],$$

where Y is an independent copy of X . Hence,

$$\begin{aligned} \text{Var}(f(X)) &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y) - f(x)|^2 \rho(x) \rho(y) \, dx \, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_{[0,1]} \nabla f(ty + (1-t)x) \cdot (y-x) \, dt \right|^2 \rho(x) \rho(y) \, dx \, dy \\ &\leq \frac{\text{Diam}(K)^2}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{[0,1]} \|\nabla f(ty + (1-t)x)\|^2 \, dt \rho(x) \rho(y) \, dx \, dy \quad \text{by Cauchy-Schwarz} \\ &= \frac{\text{Diam}(K)^2}{2} \int_{[0,1]} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(ty + (1-t)x)\|^2 \rho(x) \rho(y) \, dx \, dy \, dt \end{aligned}$$

First consider the case when $t \approx \frac{1}{2}$. We use the bound $\min\{\rho(x), \rho(y)\} \leq \rho(ty + (1-t)x)$ (due to concavity), which implies

$$\begin{aligned} \rho(x) \rho(y) &\leq \rho(ty + (1-t)x) \max\{\rho(x), \rho(y)\} \\ &\leq \rho(ty + (1-t)x) (\rho(x) + \rho(y)). \end{aligned}$$

So

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(ty + (1-t)x)\|^2 \rho(x) \rho(y) \, dx \, dy \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(ty + (1-t)x)\|^2 \rho(ty + (1-t)x) (\rho(x) + \rho(y)) \, dx \, dy \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(u)\|^2 \rho(u) \rho(x) \frac{du \, dx}{t^n} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(u)\|^2 \rho(u) \rho(y) \frac{du}{(1-t)^n} \, dy \\ &= \left(\frac{1}{t^n} + \frac{1}{(1-t)^n} \right) \mathbb{E}[\|\nabla f(X)\|^2]. \end{aligned}$$

using the substitutions $ty + (1-t)x = u$ (so $t^n dy = du$), $ty + (1-t)x = v$ (so $(1-t)^n dx = dv$).

In the case $t \gg 1/2$ or $t \ll 1/2$, then

$$\rho(x)\rho(y) \leq \rho(ty + (1-t)x) \cdot \rho((1-t)y + tx)$$

hence

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(ty + (1-t)x)\|^2 \rho(x)\rho(y) dx dy \\ & \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(ty + (1-t)x)\|^2 \rho(ty + (1-t)x) \rho((1-t)y + tx) dy dx \\ & = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\nabla f(u)\|^2 \rho(u)\rho(v) \frac{du dv}{|t^2 - (1-t)^2|^n} \\ & = \frac{1}{|t^2 - (1-t)^2|^n} \mathbb{E}[\|\nabla f(X)\|^2] \end{aligned}$$

since the map $(x, y) \mapsto (tx + (1-t)y, (1-t)x + ty)$ is represented by the matrix $\begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix}$ which has determinant $|t^2 - (1-t)^2|$. So $dx dy = \frac{du dv}{|t^2 - (1-t)^2|^n}$.

Combining these, we obtain

$$\begin{aligned} \text{Var}(f(X)) & \leq \frac{\text{Diam}(K)^2}{2} \mathbb{E}[\|\nabla f(X)\|^2] \int_{[0,1]} \min\left\{\frac{1}{t^n} + \frac{1}{(1-t)^n}, \frac{1}{|t^2 - (1-t)^2|^n}\right\} dt \\ & \leq \frac{\text{Diam}(K)^2}{2} C_n \mathbb{E}[\|\nabla f(X)\|^2]. \end{aligned}$$

□

Remark 6.5 Let $X \sim \text{Unif}(A)$, $A \subseteq \mathbb{R}^n$. The Poincaré constant $C_p(X)$ measures the **conductance** of A , which is large if A has a bottleneck.

6.1. One-dimensional log-concave random variables

Definition 6.6 Let X be an RV on \mathbb{R} with density function f . The **differential entropy** of X is

$$h(X) = - \int_{-\infty}^{\infty} f(x) \log f(x) dx = \mathbb{E}[-\log f(X)].$$

Definition 6.7 Let X, Y be an RVs on \mathbb{R} with density functions f, g . The **differential relative entropy** of X and Y is

$$D(f \parallel g) = D(X \parallel Y) = \int_{-\infty}^{\infty} f(x) \log \frac{f(x)}{g(x)} dx = \mathbb{E}\left[\log \frac{f(X)}{g(X)}\right] \geq 0.$$

Lemma 6.8 Let Y be an RV with density f on \mathbb{R} with variance $\text{Var}(Y) = \sigma^2$. Let $Z \sim N(\mathbb{E}[Y], \sigma^2)$. Then

$$h(Y) \leq h(Z) = \frac{1}{2} \log(2\pi e \sigma^2).$$

In other words, normally distributed random variables maximise differential entropy.

Proof (Hints).

- Explain why we can assume $\mathbb{E}[Y] = 0$ WLOG.
- Use non-negativity of differential relative entropy.

□

Proof. WLOG, $\mathbb{E}[Y] = 0$ (since entropy is invariant under constant shifts). Let $\varphi_{\sigma^2}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$. We have

$$\begin{aligned} 0 \leq D(f \parallel \varphi_{\sigma^2}) &= \int_{-\infty}^{\infty} f(x) \log f(x) dx + \frac{1}{2} \log(2\pi\sigma^2) + \int_{-\infty}^{\infty} \frac{x^2}{2\sigma^2} f(x) dx \\ &= -h(Y) + \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2\sigma^2} \mathbb{E}[Y^2] \\ &= -h(Y) + \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2} = \frac{1}{2} \log(2\pi e \sigma^2). \end{aligned}$$

It is straightforward to show that $h(Z) = \frac{1}{2} \log(2\pi e \sigma^2)$.

□

Definition 6.9 A random variable X is **isotropic** if $\mathbb{E}[X] = 0$ and $\text{Var}(X) = 1$.

Lemma 6.10 Let X be log-concave and isotropic, with density function ρ on \mathbb{R} . Then

$$\rho(0) \geq \frac{1}{\sqrt{2\pi e}}.$$

Proof (Hints). Write $\rho(0) = e^{(\log(\rho(\int_{-\infty}^{\infty} \rho(x)x dx)))}$ and use log-concavity.

□

Proof. We have

$$\begin{aligned} \rho(0) &= \rho\left(\int_{-\infty}^{\infty} \rho(x)x dx\right) = e^{\log \rho(\int_{-\infty}^{\infty} \rho(x)x dx)} \geq e^{\int_{-\infty}^{\infty} \rho(x) \log \rho(x) dx} \\ &= e^{-h(\rho)} \geq \frac{1}{\sqrt{2\pi e}}, \end{aligned}$$

where the first inequality is by log-concavity (we use that $\int_{-\infty}^{\infty} \rho(x) dx = 1$), and the second is by Lemma 6.8.

□

Remark 6.11 It can be shown that for log-concave ρ , $\max_x \rho(x) \leq c$ for some absolute constant c . So the above lemma says that $\rho(0)$ and $\max_x \rho(x)$ are comparable.

Proposition 6.12 Let X be log-concave, isotropic, with density function ρ on \mathbb{R} . Then for all $x \geq 3/\rho(0)$,

$$\rho(x) \leq \rho(0) e^{-\frac{\rho(0)}{3} \log(2)x} \leq e^{-x \log(2)/(3\sqrt{2\pi e})}$$

Proof (Hints).

- Let x_m denote the mode of X (why is this unique?). Can assume WLOG that $x_m > 0$. WLOG, $x_m > 0$. Let $x_0 = \frac{2}{\rho(0)} + x_m$. Why is $x_0 \geq x_m$?
- By writing 1 as an integral, show that $x_m \leq 1/\rho(0)$ (justify using log-concavity).
- Use the same idea to show that $\rho(x_0) \leq \rho(0)/2$.
- For $x \geq 3/\rho(0)$, write $x_0 = \frac{x_0}{x} \cdot x + (1 - \frac{x_0}{x}) \cdot 0$ (why is this a valid convex combination?). Use log-concavity and combine the above inequalities to obtain the result.

□

Proof. Write x_m for the mode of X (this is unique since X is log-concave). WLOG, $x_m > 0$ (the proof is similar if $x_m < 0$). Define $x_0 := \frac{2}{\rho(0)} + x_m$. We have $x_0 \geq x_m$ by Lemma 6.10. First note that

$$1 = \int_{-\infty}^{\infty} \rho(x) dx \geq \int_0^{x_m} \rho(x) dx \geq x_m \rho(0)$$

by log-concavity. Hence, $x_m \leq 1/\rho(0)$. Also,

$$1 = \int_{-\infty}^{\infty} \rho(x) dx \geq \int_{x_m}^{x_0} \rho(x) dx \geq \rho(x_0)(x_0 - x_m) = \rho(x_0) \frac{2}{\rho(0)}$$

where the last inequality is because ρ has one mode (unimodal). Hence, $\rho(x_0) \leq \rho(0)/2$. So we have $x \geq \frac{3}{\rho(0)} \geq \frac{2}{\rho(0)} + x_m = x_0$, so we write $x_0 = \frac{x_0}{x} \cdot x + (1 - \frac{x_0}{x}) \cdot 0$. By log-concavity,

$$\rho(x_0) \geq \rho(x)^{x_0/x} \cdot \rho(0)^{1-x_0/x}.$$

Exponentiating both sides by x/x_0 , we get

$$\begin{aligned} \rho(x) &\leq \frac{\rho(x_0)^{x/x_0}}{\rho(0)^{x/x_0-1}} = \rho(0) \left(\frac{\rho(x_0)}{\rho(0)} \right)^{x/x_0} \leq \rho(0) \left(\frac{1}{2} \right)^{x/x_0} \leq \rho(0) 2^{-\rho(0)x/3} \\ &= \rho(0) e^{-\rho(0) \log(2)x/3}. \end{aligned}$$

The final inequality is by Lemma 6.10. □

Remark 6.13 If ρ is log-concave and isotropic then so is $x \mapsto \rho(-x)$, so we can obtain a left tail bound as well.