Contents

0.1. Measurements

von Neumann measurements: $\sum_i P_i = \mathbb{I}$, $P_i P_j = \delta_{ij} P_i$. Then when measuring ρ_A, it collapses to $\frac{1}{\operatorname{tr}(P_i \rho_A)} P_i \rho_A P_i$. If we measure system C on the state $U_{AC}(|0\rangle\langle 0| \otimes \rho_A) U_{AC}^\dagger$ gives $\operatorname{tr}_C\Big(\Big(P_i^{(C)} \otimes \mathbb{I}\Big) U_{AC}(|0\rangle\langle 0| \otimes \rho_A) U_{AC}^\dagger \Big(P_i^{(c)} \otimes \mathbb{I}\Big)\Big)$

Let $A_0 = \sqrt{\mathbb{I} - \mathrm{d}t \sum_i L_i^\dagger L_i}$, $\{L_i\}$ are Limdblod operators, $A_i = \sqrt{\mathrm{d}t} L_i$. This gives

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = i[H, \rho] + \sum_i L_i \rho L_i^\dagger - \frac{1}{2} \sum_i \Big(L_i^\dagger L_i \rho + \rho L_i^\dagger L_i\Big).$$

Ky-Fan principle for Hermitian matrices: $\lambda_1 = \max_{P_1} \operatorname{tr}(P_1 \rho) = \max_{|\psi\rangle} \langle \psi | \rho | \psi \rangle$, $\lambda_1 + \lambda_2 = \max_{P_2} \operatorname{tr}(P_2 \rho)$, $\lambda_1 + \lambda_2 + \lambda_3 = \max_{P_3} \operatorname{tr}(P_3 \rho)$. P_i are projectors.

Theorem 0.1 (Quantum Steering) Let $|\psi\rangle$ be a pure state in $\mathbb{H} = \mathbb{H}_A \otimes \mathbb{H}_B$ and let $\rho_B = \operatorname{tr}_A(|\psi\rangle\langle\psi|)$. A POVM measurement on system A can produce the ensemble $\{(p_i, \rho_i) : i \in [M]\}$ at system B iff $\rho_B = \sum_{i=1}^M p_i \rho_i$.

Remark 0.2 The Quantum Steering theorem is also known as the Hughston, Jozsa, Wootters theorem.

Definition 0.3 An **entanglement monotone** is a function on the set of quantum states in $\mathbb{H}_A \otimes \mathbb{H}_B$ which does not increase, on average, under local transformations on $\mathbb{H}_A$ and $\mathbb{H}_B$. In particular, it is invariant under local unitary operations.

Theorem 0.4 (Vidal) A function of a bipartite pure state is an entanglement monotone iff it is a concave unitarily invariant function of its local density matrix.

Example 0.5 Let $\mathbb{H} = \mathbb{H}_A \otimes \mathbb{H}_B$ with $n = \min\{\dim \mathbb{H}_A, \dim \mathbb{H}_B\}$. A family of entanglement monotones on $\mathbb{H}$ is given by

$$\mu_m(|\psi\rangle) = -\sum_{i=1}^m \lambda_i,$$

for each $m \in [n]$, where $\lambda_1, \dots, \lambda_n$ are the Schmidt coefficients of $|\psi\rangle$ in decreasing order.

Definition 0.6 Let $x, y \in \mathbb{R}^n$, and let $x^{(i)}$ denote the i-th largest element of x. We say x **weakly majorises** y, written $y \underset{w}{\prec} x$, if

$$\sum_{i=1}^m y^{(i)} \leq \sum_{i=1}^m x^{(i)} \quad \forall m \in [n].$$

x **majorises** y, $x \prec y$, if it weakly majorises y and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

Theorem 0.7 The probabilistic transformation $|\psi\rangle \mapsto \{(p_i, |\psi_i\rangle) : i \in [M]\}$ can be accomplished using LOCC iff

$$\lambda(|\psi\rangle) \prec \sum_{i=1}^M p_i \lambda(|\psi_i\rangle),$$

where $\lambda(|\varphi\rangle)$ denotes the vector of Schmidt coefficients of $|\varphi\rangle$.

Theorem 0.8 (Bennett) Given an asymptotic number N of copies of a bipartite pure state $|\psi\rangle_{AB}$ with local density operator ρ, there exists a local transformation that transforms $N \cdot S(\rho)$ Bell states with fidelity tending to 1. Conversely, $N \cdot S(\rho)$ Bell states can be diluted into N copies of the original state with fidelity tending to 1.

Definition 0.9 The **entanglement of formation** of a mixed state is the minimal number of EPR pairs needed to construct the state:

$$E_f(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle),$$

where $E(|\psi_i\rangle)$ is the von-Neumann entropy of the local density operator of $|\psi_i\rangle$, and the minimum is taken over all ensembles $\{(p_i, |\psi_i\rangle)\}$ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

Note that ρ is separable iff $E_f(\rho) = 0$.

Definition 0.10 For $n \in \mathbb{N}$, the **entanglement cost** of ρ is $E_f(\rho^{\otimes n})$.

Theorem 0.11 Let ρ be a bipartite pure state. The **negativity** of ρ is twice the sum of the absolute values of values of all negative eigenvalues of ρ^{T_B}, where T_B is the partial transpose with respect to system B. The negativity is an entanglement monotone.

Definition 0.12 The **entanglement of distillation** is the maximal fraction of EPR pairs that ca be distilled out of a large number of copies of the state.

Definition 0.13 A **ground state** of a Hamiltonian H is an eigenstate of H corresponding to the smallest eigenvalue E_0 of H.

An **excited state** of H is an eigenstate corresponding to a non-minimal eigenvalue of H. The smallest energy of an excited state is called the **mass gap** E_1 of the Hamiltonian.

When the number of qubits $N \to \infty$, H is **gapped** if there is δ independent of N such that $E_1 - E_0 > \delta$.

Definition 0.14 A **symmetry** of a Hamiltonian H is a unitary operator U such that $[H, U] = 0$, i.e. $UHU^* = H$.

Proposition 0.15 If H has a symmetry U and $|\psi_{\text{GS}}\rangle$ is the unique ground state of H, then $|\psi_{\text{GS}}\rangle$ is invariant under U, i.e. $U|\psi_{\text{GS}}\rangle = |\psi_{\text{GS}}\rangle$ (up to a phase).

Theorem 0.16 Fundamental theorem of MPS: $|\psi(A)\rangle = |\psi(B)\rangle$ iff $\exists \varphi, X$ such that $B^i = e^{i\varphi} X A^i X^{-1}$.

Smith normal form: if matrix M has integer entries, can write $M = U \Sigma V^T$, where $\det(U), \det(V) = \pm 1$, U, V have integer entries, Σ is diagonal with entries

1. Entanglement theory

Theorem 1.1 (Schmidt Decomposition) Let $|\psi\rangle$ be a pure state in a bipartite system $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$, where $\mathbb{H}_A$ has dimension N_A and $\mathbb{H}_B$ has dimension $N_B \geq N_A$. Then there exist orthonormal states $\{|e_i\rangle : i \in [N_A]\} \subseteq \mathbb{H}_A$ and $\{|f_i\rangle : i \in [N_A]\} \subseteq \mathbb{H}_B$ such that

$$|\psi\rangle = \sum_{i=1}^{N_A} \lambda_i |e_i\rangle \otimes |f_i\rangle,$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i^2 = 1$.

The λ_i are unique up to re-ordering. The λ_i are called the **entanglement spectrum**, **Schmidt coefficients**, **Schmidt weights** or **Schmidt numbers** of $|\psi\rangle$ and the number of $\lambda_i > 0$ is the **Schmidt rank** of the state.

Proof (Hints). Use the singular value decomposition of the matrix of amplitudes of $|\psi\rangle$. □

Proof. Let $|\psi\rangle = \sum_{k=1}^{N_A} \sum_{\ell=1}^{N_B} \beta_{k\ell} |\varphi_k\rangle \otimes |\varphi_\ell\rangle$ for orthonormal bases $\{|\varphi_k\rangle : k \in [N_A]\} \subseteq \mathbb{H}_A$, $\{|\chi_\ell\rangle : \ell \in [N_B]\} \subseteq \mathbb{H}_B$. Let $(\beta_{k\ell})$ have singular value decomposition

$$U[\Sigma \;\; 0]V,$$

where U is an $N_A \times N_A$ unitary, Σ is an $N_A \times N_A$ diagonal matrix with non-negative entries, and V is an $N_B \times N_B$ unitary. So

$$\beta_{k\ell} = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} U_{ki} \Sigma_{ij} V_{j\ell} = \sum_{i=1}^{N_A} \Sigma_{ii} U_{ki} V_{i\ell}.$$

Hence,

$$|\psi\rangle = \sum_{k,\ell} \sum_i \Sigma_{ii} U_{ki} |\varphi_k\rangle \otimes V_{i\ell} |\chi_\ell\rangle = \sum_i \Sigma_{ii} \underbrace{\left(\sum_k U_{ki} |\varphi_k\rangle \right)}_{|e_i\rangle} \otimes \underbrace{\left(\sum_\ell V_{j\ell} |\chi_\ell\rangle \right)}_{|j_B\rangle}.$$

□

Proposition 1.2 The squared Schmidt coefficients of $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$ are the eigenvalues of the reduced density operators $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \mathrm{tr}_A(|\psi\rangle\langle\psi|)$.

Proof (Hints). Straightforward. □

Proof. We have

$$|\psi\rangle\langle\psi| = \sum_{i,j} \lambda_i \lambda_j^* |e_i\rangle\langle e_j| \otimes |f_i\rangle\langle f_j|.$$

Since $\mathrm{tr}\big(|f_i\rangle\langle f_j|\big) = \langle f_j | f_i\rangle = \delta_{ij}$, the result for ρ_A follows. The result for ρ_B follows similarly. □

Definition 1.3 An **entanglement monotone** is a non-negative function on the set of quantum states in $\mathbb{H}_A \otimes \mathbb{H}_B$ which does not increase, on average, under local transformations on $\mathbb{H}_A$ and $\mathbb{H}_B$. In particular, it is invariant under local unitary operations.

More specifically, an **entanglement monotone** μ is a function from the set $S(\mathbb{H}_A \otimes \mathbb{H}_B)$ of quantum states in $\mathbb{H}_A \otimes \mathbb{H}_B$ to $\mathbb{R}$ which satisfies:

- **Non-negativity**: $\mu(\rho) \geq 0$ for all $\rho \in S(\mathbb{H}_A \otimes \mathbb{H}_B)$.

- $\mu(\rho) = 0$ if ρ is separable.
- **Monotonicity under LOCC**: TODO.

Entanglement monotones quantify the amount of entanglement in a quantum state.

Theorem 1.4 (Vidal) A function of a bipartite pure state is an entanglement monotone iff it is a concave unitarily invariant function of its local density matrix.

Example 1.5 Let $\mathbb{H} = \mathbb{H}_A \otimes \mathbb{H}_B$ with $n = \min\{\dim \mathbb{H}_A, \dim \mathbb{H}_B\}$. A family of entanglement monotones on $\mathbb{H}$ is given by

$$\mu_m(|\psi\rangle) = -\sum_{i=1}^{m} \lambda_i,$$

for each $m \in [n]$, where $\lambda_1, ..., \lambda_n$ are the Schmidt coefficients of $|\psi\rangle$ in decreasing order.

Definition 1.6 Let $\rho \in S(\mathbb{H})$ be a quantum state with spectral decomposition $\rho = \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i|$. The **von-Neumann entropy** of ρ is

$$S(\rho) = -\sum_{i=1}^{n} \lambda_i \log(\lambda_i) = -\operatorname{tr}(\rho \log(\rho)).$$

The von-Neumann entropy is a measure of how mixed the state ρ is: it is non-negative and is zero iff ρ is a pure state.

Definition 1.7 Let $|\psi\rangle \in S(\mathbb{H}_A \otimes \mathbb{H}_B)$ be a bipartite pure state. The **entanglement entropy** of $|\psi\rangle$ is the von-Neumann entropy of either of its reduced density operators. So the entanglement entropy is

$$S(\rho_A) = S(\rho_B) = -\sum_{i=1}^{n} \lambda_i^2 \log(\lambda_i^2),$$

where $\rho_A = \operatorname{tr}_B(|\psi\rangle\langle\psi|)$, $\rho_B = \operatorname{tr}_A(|\psi\rangle\langle\psi|)$ and $\lambda_1, ..., \lambda_n$ are the Schmidt coefficients of $|\psi\rangle$.

Definition 1.8 A **completely positive (CP)** map is a linear map $T : B(\mathbb{H}) \to B(\mathbb{H})$ such that for all $n \in \mathbb{N}$, $T \otimes \mathbb{I}_n$ is positive (i.e. if $A \geq 0$, then $(T \otimes \mathrm{id}_n)(A) \geq 0$).

CP maps can be expressed in their **Kraus decomposition** as

$$T(\rho) = \sum_k A_k \rho A_k^\dagger,$$

where the $\{A_k\}$ are **Kraus operators**.

Definition 1.9 A **completely positive trace preserving (CPTP)** map is a CP map T such that $\operatorname{tr}(T(A)) = \operatorname{tr}(A)$ for all $A \in B(\mathbb{H})$. In particular, CPTP maps map density operators to density operators, and describe the most general evolution of a quantum system.

If A has Kraus decomposition $T(\rho) = \sum_k A_k \rho A_k^\dagger$, then the Kraus operators satisfy $\sum_k A_k^\dagger A_k = \mathbb{I}$.

Definition 1.10 A matrix $U \in \mathbb{C}^{m \times n}$ is called an **isometry** if $U^\dagger U = \mathbb{I}_n$.

Remark 1.11 The Kraus decomposition of a CPTP map is not unique: given a set of Kraus operators $\{A_k : k \in [K]\}$, we can define an equivalent set of Kraus operators $\{B_\ell : \ell \in [L]\}$ for the same map by $B_\ell = \sum_{k=1}^K U_{\ell k} A_k$, where U is an isometry. Moreover, two sets of Kraus operators are equivalent if and only if they are related by such an isometry.

Definition 1.12 Given a set of **Lindblad operators** $\{L_i : i \in [M]\}$ (which are arbitrary square matrices), define the Kraus operators

$$
\begin{aligned}
A_0 &= \sqrt{I - \mathrm{d}t \sum_{i=1}^M L_i^\dagger L_i}, \\
A_i &= \sqrt{\mathrm{d}t} L_i, \quad i \in [M].
\end{aligned}
$$

The CP map T defined by these Kraus operators satisfies $T(\rho) = \rho + O(\mathrm{d}t)$, which gives

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \sum_i L_i \rho L_i^\dagger - \frac{1}{2}\left(\sum_i L_i^\dagger L_i \rho + \rho \sum_i L_i^\dagger L_i\right) \\
&= \sum_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\left\{L_i^\dagger L_i, \rho\right\}\right)
\end{aligned}
$$

Given that the system evolves according to a Hamiltonian H, we obtain the **Lindblad equation**

$$
\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H, \rho] + \sum_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\left\{L_i^\dagger L_i, \rho\right\}\right).
$$

Remark 1.13 Physically, evolution according to the Lindblad equation corresponds to when we the system of interest to an ancilla through an infinitesimal interaction / evolution with a Hamiltonian which couple both systems, then take the trace over the ancilla. This only makes sense when the ancillary system cannot interact with the system of interest anymore at later times.

Remark 1.14 CPTP maps and the Lindblad equation are the two ways of describing the evolution of a quantum system: the Lindblad equation is the continuous version of a CPTP map.

2. Tensor networks

Definition 2.1 A **rank-r** tensor of dimension $d_1 \times d_2 \times ... \times d_r$ is an element of $\mathbb{C}^{d_1 \times \cdots \times d_r}$.

In tensor network notation (TNN), a rank-r tensor is represented by a box with r legs, with each leg corresponding to an index.

Example 2.2
- A scalar is a rank-0 tensor.

- A vector is a rank-1 tensor.
- A matrix is a rank-2 tensor.

Definition 2.3 The **tensor product** $A \otimes B$ of a rank-r tensor A and a rank-s tensor B is given by

$$(A \otimes B)_{i_1,\dots,i_r,j_1,\dots,j_s} = A_{i_1,\dots,i_r} \cdot B_{j_1,\dots,j_s}.$$

In TNN, the tensor product is represented by placing two tensors next to each other without joining them.

Definition 2.4 Let A be a tensor of dimension $d_1 \times d_2 \times \dots \times d_r$ and suppose the k-th and ℓ-th indices have the same dimension $d = d_k = d_\ell$. The **partial trace** $\mathrm{tr}_{k,\ell}\, A$ of A over the k-th and ℓ-th indices is given by jointly summing over those indices:

$$\left(\mathrm{tr}_{k,\ell}\, A\right)_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_{\ell-1},i_{\ell+1},\dots,i_r} = \sum_{j=1}^{d} A_{i_1,\dots,i_{k-1},j,i_{k+1},\dots,i_{\ell-1},j,i_{\ell+1},\dots,i_r}.$$

In TNN, the partial trace is represented by joining the legs corresponding to the indices being traced out.

Definition 2.5 A **tensor contraction** is a tensor product followed by a partial trace.

In TNN, a tensor contraction is represented by joining the legs corresponding to the indices being summed over.

Example 2.6 Vector inner products, matrix-vector multiplication, matrix multiplication, and the trace of a matrix are all examples of tensor contractions.

Remark 2.7 It is easy to see that the matrix trace is cyclic by writing it in tensor network notation, and "sliding" one of the matrices around.

Definition 2.8 Using the fact that the vector spaces $\mathbb{C}^{a_1 \times \dots \times a_r}$ and $\mathbb{C}^{b_1 \times \dots \times b_s}$ are isomorphic iff $a_1 \times \cdots \times a_r = b_1 \times \cdots \times b_s$, we can **group** or **split** indices to respectively increase or decrease the rank of a tensor.

As a concrete example, if T is a rank $r+s$ tensor, we can group its first r indices together and its last s indices together to form a matrix:

$$T_{I,J} = T_{i_1,\dots,i_r,j_1,\dots,j_s},$$

where the group indices have been defined as

$$\begin{aligned} I &= i_1 + d_1 i_2 + \dots + d_1 d_2 \dots d_{r-1} i_r, \\ J &= j_1 + d_{r+1} j_2 + \dots + d_{r+1} d_{r+2} \dots d_{r+s-1} j_s. \end{aligned}$$

Such a partioning of the indices into two sets is called a **bisection** of the tensor.

Example 2.9 For a general contraction of two tensors, we can group the indices involved in the contraction, and the indices not involved in the contraction, to simplify this contraction to a matrix multiplication.

Example 2.10 The singular value decomposition (SVD) of a matrix T indexed by I and J is given by

$$T_{I,J} = \sum_k U_{I,k} S_{k,k} V^\dagger_{k,J}.$$

By grouping and splitting, we obtain a higher-dimensional version of the SVD:

$$T_{i_1,\ldots,i_r,j_1,\ldots,j_s} = \sum_k U_{i_1,\ldots,i_r,k} S_{k,k} V^\dagger_{k,j_1,\ldots,j_s}.$$

Remark 2.11 The rank of a tensor given in a tensor network diagram is the number of unmatched legs in the diagram. The value of the tensor is independent of the order in which the constituent tensors are contracted.

Definition 2.12 An MPS $|\psi(A)\rangle$ is in **left-canonical form** if

$$\sum_{j=0}^{d-1} A_j^\dagger A_j = \mathbb{I}_D$$

Definition 2.13 Let $|\psi(A)\rangle$ be an MPS in left-canonical form. $|\psi(A)\rangle$ is **injective** if ....