Contents

1. Set systems

1.1. Chains and antichains

Note 1.1 The ideas in combinatorics often occur in the proofs, so it is advisable to learn the techniques used in proofs, rather than just learning the results and not their proofs.

Definition 1.2 Let X be a set. A **set system** on X (also called a **family of subsets of X**) is a collection $\mathcal{F} \subseteq \mathbb{P}(X)$.

Notation 1.3 $X^{(r)} := \{A \subseteq X : |A| = r\}$ denotes the family of subsets of X of size r.

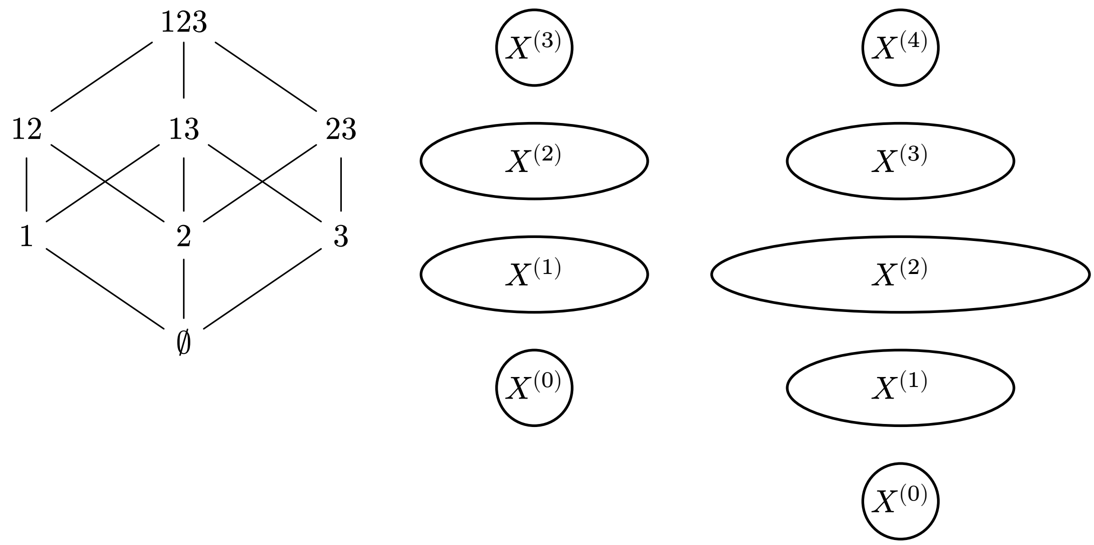
Remark 1.4 Usually, we take $X = [n] = \{1, ..., n\}$, so $|X^{(r)}| = \binom{n}{r}$.

Notation 1.5 For brevity, we write e.g. $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$.

Definition 1.6 We can visualise $\mathbb{P}(X)$ as a graph by joining nodes $A \in \mathbb{P}(X)$ and $B \in \mathbb{P}(X)$ if $|A \Delta B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa.

This graph is the **discrete cube** Q_n.

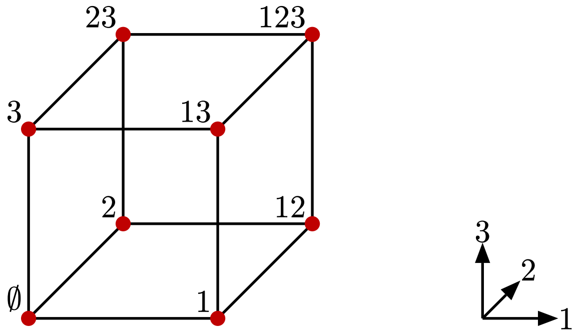
Diagram 1.7



Q_3, Q_3, and Q_4.

Remark 1.8 Alternatively, we can view Q_n as an n-dimensional unit cube $\{0, 1\}^n$ by identifying e.g. $\{1, 3\} \subseteq [5]$ with 10100 (i.e. identify A with $\mathbb{1}_A$, the characteristic/indicator function of A).

Diagram 1.9



The cube Q_3 as the unit cube in $\mathbb{R}^3$

Definition 1.10 $\mathcal{F} \subseteq \mathbb{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{F}$, $A \subseteq B$ or $B \subseteq A$.

Example 1.11

- $\mathcal{F} = \{23, 1235, 123567\}$ is a chain.
- $\mathcal{F} = \{\emptyset, 1, 12, ..., [n]\} \subseteq \mathbb{P}([n])$ is a chain.

Definition 1.12 $\mathcal{F} \subseteq \mathbb{P}(X)$ is an **antichain** if $\forall A \neq B \in \mathcal{F}$, $A \nsubseteq B$.

Diagram 1.13

A chain and antichain.

Example 1.14

- $\mathcal{F} = \{23, 137\}$ is an antichain.
- $\mathcal{F} = \{1, ..., n\} \subseteq \mathbb{P}([n])$ is an antichain.
- More generally, $\mathcal{F} = X^{(r)}$ is an antichain for any r.

Proposition 1.15 A chain and an antichain can meet at most once.

Proof (Hints). Trivial. $\square$

Proof. By definition. $\square$

Proposition 1.16 A chain $\mathcal{F} \subseteq \mathbb{P}([n])$ can have at most $n + 1$ elements.

Proof (Hints). Trivial. $\square$

Proof. For each $0 \leq r \leq n$, $\mathcal{F}$ can contain at most 1 r-set (set of size r). $\square$

Theorem 1.17 (Sperner's Lemma) Let $\mathcal{F} \subseteq \mathbb{P}(X)$ be an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$, i.e. the maximum size of an antichain is achieved by the set of $X^{(\lfloor n/2 \rfloor)}$.

Proof (Hints).

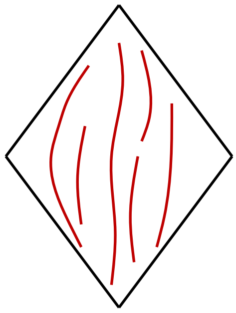
- Let $r < \frac{n}{2}$.

- Let G be bipartite subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$.
- By considering an expression and upper bound for number of S-$\Gamma(S)$ edges in G for each $S \subseteq X^{(r)}$, show that there is a matching from $X^{(r)}$ to $X^{(r+1)}$.
- Reason that this induces a matching from $X^{(r)}$ to $X^{(r-1)}$ for each $r > \frac{n}{2}$.
- Reason that joining these matchings together, (together with, if n is even, length 1 chains of subsets of $X^{(\lfloor n/2 \rfloor)}$ not included in a matching), result in a partition of $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, and conclude result from here.

□

Proof. We use the idea: from "a chain meets each layer in ≤ 1 points, because a layer is an antichain", we try to decompose the cube into chains.

Diagram 1.18



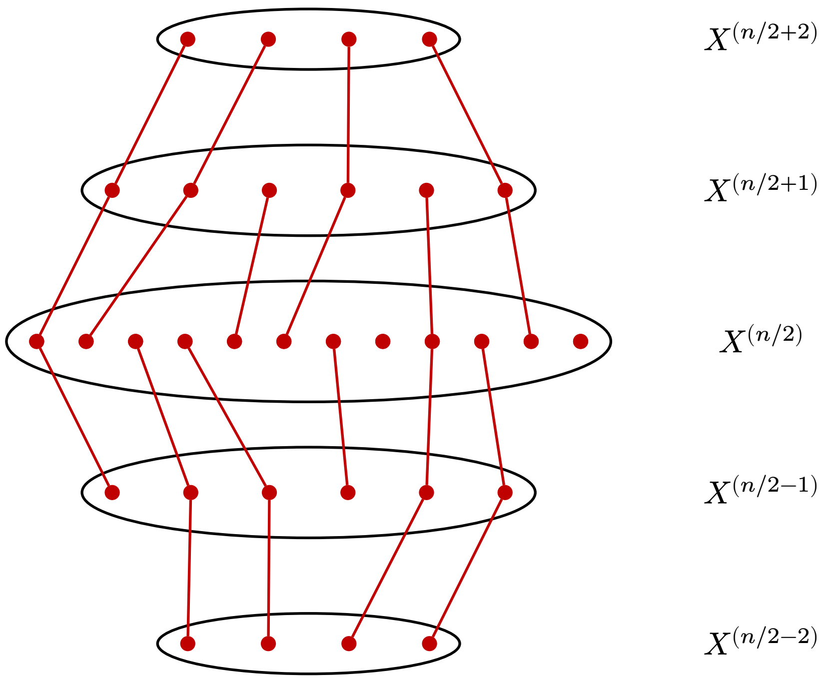
Decomposition of $\mathbb{P}(X)$ into chains.

In particular, we partition $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, so each subset of X appears exactly once in one chain. Then we are done (since to form an antichain, we can pick at most one element from each chain). To achieve this, it is sufficient to find:

- For each $r < \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r+1)}$ (a matching is a set of disjoint edges, one for each point in $X^{(r)}$).
- For each $r > \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r-1)}$.

Then put these matchings together to form a set of chains, each passing through $X^{(\lfloor n/2 \rfloor)}$.

Diagram 1.19

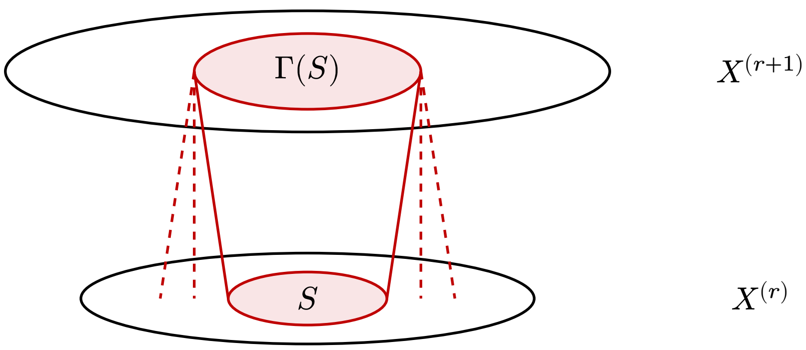


Example of joining matchings in the 5 middle layers, for n even.

If a subset $X^{(\lfloor n/2 \rfloor)}$ has a chain passing through it, then this chain is unique. The subsets with no chain passing through form their own one-element chain (these only exist if n is even, in which case they are the subsets of $X^{(n/2)}$). By taking complements, it is enough to construct the matchings just for $r < \frac{n}{2}$ (since a matching from $X^{(r)}$ to $X^{(r+1)}$ induces a matching from $X^{(n-r-1)}$ to $X^{(n-r)}$: there is a correspondence between $X^{(r)}$ and $X^{(n-r)}$ by taking complements, and taking complements reverse inclusion, so edges in the induced matching are guaranteed to exist).

Let G be the (bipartite) subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$. For any $S \subseteq X^{(r)}$, the number of S-$\Gamma(S)$ edges in G is $|S|(n-r)$ (counting from below) since there are $n-r$ ways to add an element. This number is $\leq |\Gamma(S)|\ (r+1)$ (counting from above), since $r+1$ ways to remove an element.

Diagram 1.20



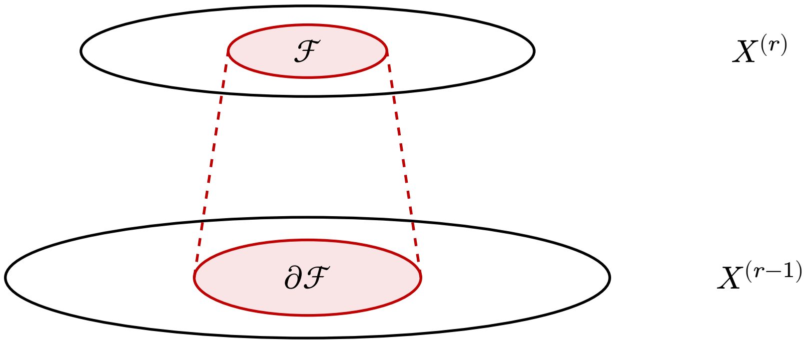
Hence $|\Gamma(S)| \geq \frac{|S|\ (n-r)}{r+1} \geq |S|$ as $r < \frac{n}{2}$. So by Hall's theorem, since there is a matching from S to $\Gamma(S)$, there is a matching from $X^{(r)}$ to $X^{(r+1)}$. $\square$

Remark 1.21 The proof above doesn't tell us when we have equality in Sperner's Lemma.

Definition 1.22 For $\mathcal{F} \subseteq X^{(r)}$ $(1 \leq r \leq n)$, the **shadow** of $\mathcal{F}$ is the set of subsets which can be obtained by removing one element from a subset in $\mathcal{F}$:

$$\partial\mathcal{F} = \partial^{-}\mathcal{F} := \left\{B \in X^{(r-1)} : B \subseteq \mathcal{F} \text{ for some } A \in \mathcal{F}\right\}.$$

Diagram 1.23



A family $\mathcal{F} \subseteq X^{(r)}$ and its shadow.

Example 1.24 Let $\mathcal{F} = \{123, 124, 134, 137\} \in [7]^{(3)}$. Then $\partial\mathcal{F} = \{12, 13, 23, 14, 24, 34, 17, 37\}$.

Proposition 1.25 (Local LYM) Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq r \leq n$. Then

$$\frac{|\mathcal{F}|}{\left|X^{(r)}\right|} \leq \frac{|\partial\mathcal{F}|}{\left|X^{(r-1)}\right|}, \quad \text{i.e.} \quad \frac{|\mathcal{F}|}{\binom{n}{r}} \leq \frac{|\partial\mathcal{F}|}{\binom{n}{r-1}}.$$

i.e. the proportion of the level occupied by $\partial\mathcal{F}$ is at least the proportion of the level occupied by $\mathcal{F}$.

Proof (Hints). Find equation and upper bound for number of $\mathcal{F}$-$\partial\mathcal{F}$ edges in Q_n. □

Proof. The number of $\mathcal{F}$-$\partial\mathcal{F}$ edges in Q_n is $|\mathcal{F}|r$ (counting from above, since we can remove any of r elements from $|\mathcal{F}|$ sets) and is $\leq |\partial\mathcal{F}|\ (n-r+1)$ (since adding one of the $n-r+1$ elements not in $A \in \partial\mathcal{F}$ to A may not result in a subset of $\mathcal{F}$). Hence,

$$\frac{|\mathcal{F}|}{|\partial\mathcal{F}|} \leq \frac{n-r+1}{r} = \binom{n}{r} \Big/ \binom{n}{r-1}.$$ □

Remark 1.26 For equality in Local LYM, we must have that $\forall A \in \mathcal{F}$, $\forall i \in A$, $\forall j \notin A$, we must have $(A - \{i\}) \cup \{j\} \in \mathcal{F}$, i.e. $\mathcal{F} = \emptyset$ or $X^{(r)}$ for some r.

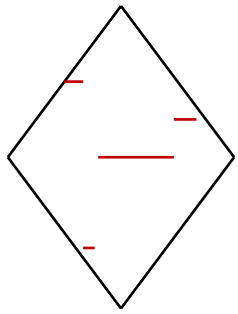
Notation 1.27 Write $\mathcal{F}_r$ for $\mathcal{F} \cap X^{(r)}$.

Theorem 1.28 (LYM Inequality) Let $\mathcal{F} \subseteq \mathbb{P}(X)$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{\left|\mathcal{F} \cap X^{(r)}\right|}{\left|X^{(r)}\right|} = \sum_{r=0}^{n} \frac{\left|\mathcal{F} \cap X^{(r)}\right|}{\binom{n}{r}} \leq 1,$$

i.e. the proportions of each layer occupied add to ≤ 1.

Diagram 1.29



Proof (Hints).

- Method 1: show the result for the sum $\sum_{r=n-1}^{n}$, by using Local LYM, and the fact that $\partial\mathcal{F}_n$ and $\mathcal{F}_{n-1}$ are disjoint. Continue inductively.
- Method 2: let $\mathcal{C}$ be uniformly random maximal chain, find an expression for $\Pr(\mathcal{C}$ meets $\mathcal{F})$.
- Method 3: determine number of maximal chains in X, determine number of maximal chains passing through a fixed r-set, deduce maximal number of chains passing through $\mathcal{F}$.

□

Proof. **Method 1**: "bubble down with local LYM". We trivially have that $\mathcal{F}_n/\binom{n}{n} \le 1$. $\partial\mathcal{F}_n$ and $\mathcal{F}_{n-1}$ are disjoint, as $\mathcal{F}$ is an antichain, so

$$\frac{|\partial\mathcal{F}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial\mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} \le 1.$$

So by local LYM,

$$\frac{|\mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} \le 1.$$

Now, $\partial(\partial\mathcal{F}_n \cup \mathcal{F}_{n-1})$ and $\mathcal{F}_{n-2}$ are disjoint, as $\mathcal{F}$ is an antichain, so

$$\frac{|\partial(\partial\mathcal{F}_n \cup \mathcal{F}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \le 1.$$

So by local LYM,

$$\frac{|\partial\mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \le 1.$$

So

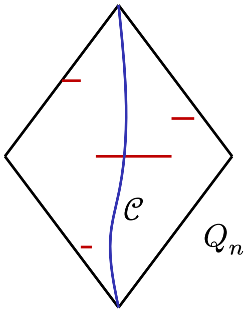
$$\frac{|\mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \le 1.$$

Continuing inductively, we obtain the result.

Method 2: Choose uniformly at random a maximal chain $\mathcal{C}$ (i.e. $C_0 \subsetneq C_1 \subseteq \cdots \subsetneq C_n$ with $|C_r| = r$ for all r). For any r-set A, $\Pr(A \in \mathcal{C}) = 1/\binom{n}{r}$, since all r-sets are equally likely. So $\Pr(\mathcal{C}$ meets $\mathcal{F}_r) = |\mathcal{F}_r|/\binom{n}{r}$, since the events are disjoint. Thus,

$\Pr(\mathcal{C} \text{ meets } \mathcal{F}) = \sum_{r=0}^{n} |\mathcal{F}_r| / \binom{n}{r} \leq 1$ since the events are disjoint (since $\mathcal{F}$ is an antichain).

Diagram 1.30



A random maximal chain $\mathcal{C}$.

Method 3 (same as method 2 but counting instead of using probability): The number of maximal chains is $n!$, and the number through any fixed r-set is $r!(n-r)!$, so $\sum_r |\mathcal{F}_r| r!(n-r)! \leq n!$. □

Remark 1.31 To have equality in LYM, we must have equality in each use of local LYM in proof method 1. In this case, the maximum r with $\mathcal{F}_r \neq \emptyset$ has $\mathcal{F}_r = X^{(r)}$. So equality holds iff $\mathcal{F} = X^{(r)}$ for some r. Hence equality in Sperner's Lemma holds iff $\mathcal{F} = X^{(\lfloor n/2 \rfloor)}$ or $\mathcal{F} = X^{(\lceil n/2 \rceil)}$.

1.2. Two total orders on $X^{(r)}$

Definition 1.32 Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **lexicographic (lex)** ordering if for some j, we have $a_i = b_i$ for all $i < j$, and $a_j < b_j$. "use small elements".

Example 1.33 The elements of $[4]^{(2)}$ in lexicographic order are $12, 13, 14, 23, 24, 34$. The elements of $[6]^{(3)}$ in lexicographic order are

$$123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456.$$

Definition 1.34 Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **colexicographic (colex)** order if for some j, we have $a_i = b_i$ for all $i > j$, and $a_j < b_j$. "avoid large elements".

Example 1.35 The elements of $[4]^{(2)}$ in colex order are $12, 13, 23, 14, 24, 34$. The elements of $[6]^{(3)}$ are $123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 146, 246, 346, 156, 256,$

Remark 1.36 Lex and colex are both total orders. Note that in colex, $[n-1]^{(r)}$ is an initial segment of $[n]^{(r)}$ (this does not hold for lex). So we can view colex as an enumeration of $\mathbb{N}^{(r)}$.

Remark 1.37 $A < B$ in colex iff $A^c < B^c$ in lex with ground set order reversed.

Remark 1.38 By Local LYM, we know that $|\partial \mathcal{F}| \geq |\mathcal{F}| r/(n-r+1)$. Equality is rare (only for $\mathcal{F} = X^{(r)}$ for $0 \leq r \leq n$). What happens in between, i.e., given $|\mathcal{F}|$, how should we choose $\mathcal{F}$ to minimise $|\partial \mathcal{F}|$?

You should be able to convince yourself that if $|\mathcal{F}| = \binom{k}{r}$, then we should take $\mathcal{F} = [k]^{(r)}$. If $\binom{k}{r} < |\mathcal{F}| < \binom{k+1}{r}$, then convince yourself that we should take some $[k]^{(r)}$ plus some r-sets in $[k+1]^{(r)}$.

E.g. for $\mathcal{F} \subseteq X^{(r)}$ with $|\mathcal{F}| = \binom{8}{3} + \binom{4}{2}$, take $\mathcal{F} = [8]^{(3)} \cup \{9 \cup B : B \in [4]^{(2)}\}$.

Remark 1.39 We want to show that if $\mathcal{F} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{F}|$, then $|\partial \mathcal{C}| \leq |\partial \mathcal{F}|$. In particular, if $|\mathcal{F}| = \binom{k}{r}$ (so $\mathcal{C} = [k]^{(r)}$), then $|\partial \mathcal{F}| \geq \binom{k}{r-1}$.

1.3. Compressions

Remark 1.40 We want to transform $\mathcal{F} \subseteq X^{(r)}$ into some $\mathcal{F}' \subseteq X^{(r)}$ such that:
- $|\mathcal{F}'| = |\mathcal{F}|$,
- $|\partial \mathcal{F}'| \leq |\partial \mathcal{F}|$.

Ideally, we want a family of such "compressions" $\mathcal{F} \to \mathcal{F}' \to ... \to \mathcal{B}$ such that either $\mathcal{B} = \mathcal{C}$, or $\mathcal{B}$ is similar enough to $\mathcal{C}$ that we can directly check that $|\partial \mathcal{C}| \leq |\partial \mathcal{B}|$.

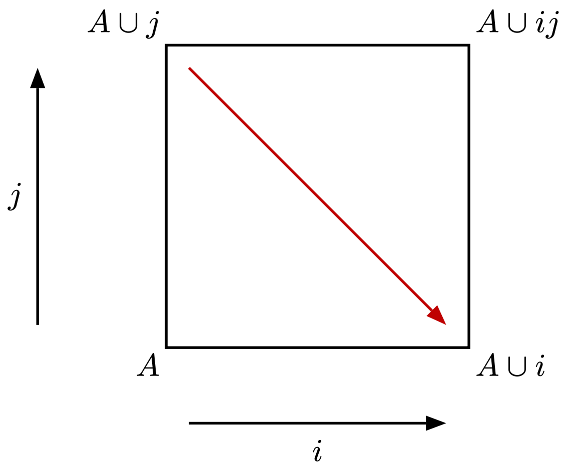
Definition 1.41 Let $1 \leq i < j \leq n$. The ***ij*-compression** C_{ij} is defined as:
- For $A \in X^{(r)}$,

$$C_{ij}(A) = \begin{cases} (A \cup i) - j & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}.$$

- For $\mathcal{F} \subseteq X^{(r)}$, $C_{ij}(\mathcal{F}) = \{C_{ij}(A) : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : C_{ij}(A) \in \mathcal{F}\}$.

"replace j by i where possible". This definition is inspired by "colex prefers $i < j$ to j". Note that $C_{ij}(\mathcal{F}) \subseteq X^{(r)}$ and $|C_{ij}(\mathcal{F})| = |\mathcal{F}|$.

Diagram 1.42



Applying an ij-compression to $A \in X^{(r)}$.

Definition 1.43 $\mathcal{F}$ is ***ij*-compressed** if $C_{ij}(\mathcal{F}) = \mathcal{F}$.

Example 1.44 Let $\mathcal{F} = \{123, 134, 234, 235, 146, 567\}$, then $C_{12}(\mathcal{F}) = \{123, 134, 234, 135, 146, 567\}$.

Lemma 1.45 Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq i < j \leq n$. Then $|\partial C_{ij}(\mathcal{F})| \leq |\partial \mathcal{F}|$.

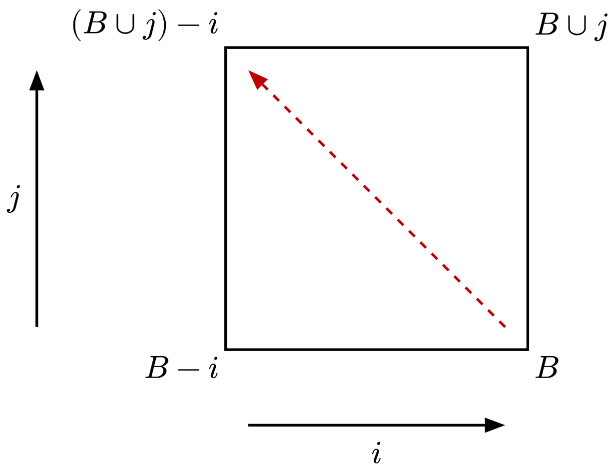
Proof (Hints).

- Let $\mathcal{F}' = C_{ij}(\mathcal{F})$, $B \in \partial\mathcal{F}' - \partial\mathcal{F}$.
- Show that $i \in B$ and $j \notin B$.
- Reason that $B \cup j - i \in \partial\mathcal{F}$.
- Show that $B \cup j - i \notin \partial\mathcal{F}'$ by contradiction (helpful to see Definition 1.41).
- Conclude the result.

$\square$

Proof. Let $\mathcal{F}' = C_{ij}(\mathcal{F})$. Let $B \in \partial\mathcal{F}' - \partial\mathcal{F}$. We'll show that $i \in B$, $j \notin B$, $(B \cup j) - i \in \partial\mathcal{F} - \partial\mathcal{F}'$.

Diagram 1.46



Note that $B \cup x \in \mathcal{F}'$ and $B \cup x \notin \mathcal{F}$ (since $B \notin \partial\mathcal{F}$) for some x. So $i \in B \cup x$, $j \notin B \cup x$, $(B \cup x \cup j) - i \in \mathcal{F}$. We can't have $x = i$, since otherwise $(B \cup x \cup j) - i = B \cup j$, which gives $B \in \partial\mathcal{F}$, a contradiction. So $i \in B$ and $j \notin B$. Also, $B \cup j - i \in \partial\mathcal{F}$, since $B \cup x \cup j - i \in \mathcal{F}$.

Suppose $B \cup j - i \in \partial\mathcal{F}'$: so $(B \cup j - i) \cup y \in \mathcal{F}'$ for some y. We cannot have $y = i$, since otherwise $B \cup j \in \mathcal{F}'$, so $B \cup j \in \mathcal{F}$ (as $j \in B \cup j$), contradicting $B \notin \partial\mathcal{F}$. Hence $j \in (B \cup j - i) \cup y$ and $i \notin (B \cup j - i) \cup y$. Thus, both $(B \cup j - i) \cup y$ and $B \cup y = C_{ij}((B \cup j - i) \cup y)$ belong to $\mathcal{F}$ (by definition of $\mathcal{F}'$ - both lie in the second set in the union in Definition 1.41), contradicting $B \notin \partial\mathcal{F}$. $\square$

Remark 1.47 In the above proof, we actually showed that $\partial C_{ij}(\mathcal{F}) \subseteq C_{ij}(\partial\mathcal{F})$.

Definition 1.48 $\mathcal{F} \subseteq X^{(r)}$ is **left-compressed** if $C_{ij}(\mathcal{F}) = \mathcal{F}$ for all $i < j$.

Corollary 1.49 Let $\mathcal{F} \subseteq X^{(r)}$. Then there exists a left-compressed $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{F}|$ and $|\partial\mathcal{B}| \le |\partial\mathcal{F}|$.

Proof (Hints). Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \dots$ of subsets of $X^{(r)}$ with $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} i$ strictly decreasing. $\square$

Proof. Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \dots$ as follows: set $\mathcal{F}_0 = \mathcal{F}$. Having defined $\mathcal{F}_0, \dots, \mathcal{F}_k$, if $\mathcal{F}_k$ is left-compressed the end the sequence with $\mathcal{F}_k$; if not, choose $i < j$ such that $\mathcal{F}_k$ is not ij-compressed, and set $\mathcal{F}_{k+1} = C_{ij}(\mathcal{F}_k)$.

This must terminate after a finite number of steps, e.g. since $\sum_{A\in\mathcal{F}_k}\sum_{i\in A} i$ is strictly decreasing with k. The final term $\mathcal{B}=\mathcal{F}_k$ satisfies $|\mathcal{B}|=|\mathcal{F}|$, and $|\partial\mathcal{B}|\leq|\partial\mathcal{F}|$ by the above lemma. $\square$

Remark 1.50
- Another way of proving this is: among all $\mathcal{B}\subseteq X^{(r)}$ with $|\mathcal{F}|=|\mathcal{F}|$ and $|\partial\mathcal{B}|\leq|\partial\mathcal{F}|$, choose one with minimal $\sum_{A\in\mathcal{B}}\sum_{i\in A} i$.
- We can choose an order of the C_{ij} so that no C_{ij} is applied twice.
- Any initial segment of colex is left-compressed, but the converse is false, e.g. $\{123,124,125,126\}$ is left-compressed.

Definition 1.51 Let $U,V\subseteq X$, $|U|=|V|$, $U\cap V=\emptyset$ and $\max U<\max V$. Define the ***UV*-compression** C_{UV} as:
- For $A\subseteq X$,

$$C_{UV}(A)=\begin{cases}(A-V)\cup U & \text{if } V\subseteq A, U\cap A=\emptyset\\ A & \text{otherwise}\end{cases}.$$

- For $\mathcal{F}\subseteq X^{(r)}$,

$$C_{UV}(\mathcal{F})=\{C_{UV}(A):A\in\mathcal{F}\}\cup\{A\in\mathcal{F}:C_{UV}(A)\in\mathcal{F}\}.$$

We have $C_{UV}(\mathcal{F})\subseteq X^{(r)}$ and $|C_{UV}(\mathcal{F})|=|\mathcal{F}|$. This definition is inspired by "colex prefers 23 to 14".

Definition 1.52 $\mathcal{F}$ is ***UV*-compressed** if $C_{UV}(\mathcal{F})=\mathcal{F}$.

Example 1.53 Let $\mathcal{F}=\{123,124,147,237,238,149\}$, then $C_{23,14}(\mathcal{F})=\{123,124,147,237,238,239\}$.

Example 1.54 We can have $|\partial C_{UV}(\mathcal{F})|>|\partial\mathcal{F}|$. E.g. $\mathcal{F}=\{147,157\}$ has $|\partial\mathcal{F}|=5$, but $C_{23,14}(\mathcal{F})=\{237,157\}$ has $\left|\partial C_{23,14}(\mathcal{F})\right|=6$.

Lemma 1.55 Let $\mathcal{F}\subseteq X^{(r)}$ be UV-compressed for all $U,V\subseteq X$ with $|U|=|V|$, $U\cap V=\emptyset$ and $\max U<\max V$. Then $\mathcal{F}$ is an initial segment of colex.

Proof (Hints). Suppose not, consider a compression for appropriate U and V. $\square$

Proof. Suppose not, then there exists $A,B\in X^{(r)}$ with $B<A$ in colex but $A\in\mathcal{F}$, $B\notin\mathcal{F}$. Let $V=A\setminus B$, $U=B\setminus A$. Then $|V|=|U|$, $U\cap V=\emptyset$, and $\max V>\max U$ (since $\max(A\Delta B)\in A$, by definition of colex). Since $\mathcal{F}$ is UV-compressed, we have $C_{UV}(A)=B\in C_{UV}(\mathcal{F})=\mathcal{F}$, contradiction. $\square$

Lemma 1.56 Let $U,V\subseteq X$, $|U|=|V|$, $U\cap V=\emptyset$, $\max U<\max V$. For $\mathcal{F}\subseteq X^{(r)}$, suppose that

$$\forall u\in U,\exists v\in V:\quad \mathcal{F}\text{ is }(U-u,V-v)\text{-compressed.}$$

Then $|\partial C_{UV}(\mathcal{F})|\leq|\partial\mathcal{F}|$.

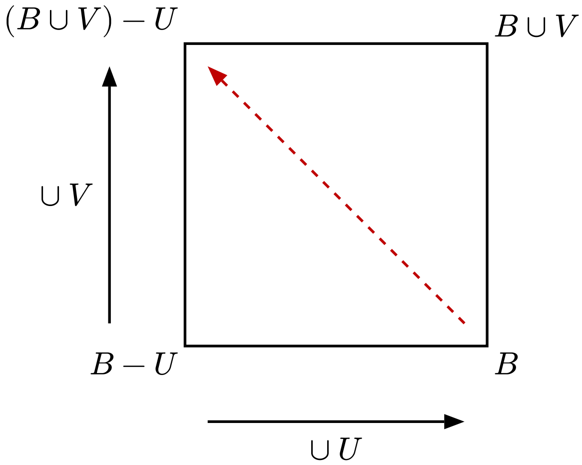
Proof (Hints).
- Let $\mathcal{F}'=C_{UV}(\mathcal{F})$, $B\in\partial\mathcal{F}'-\partial\mathcal{F}$.
- Show that $U\subseteq B$ and $V\cap B=\emptyset$.

- Reason that $(B - U) \cup V \in \partial \mathcal{F}$.
- Show that $(B - U) \cup V \notin \partial \mathcal{F}'$ by contradiction.

$\square$

Proof. Let $\mathcal{F}' = C_{UV}(\mathcal{F})$. For $B \in \partial \mathcal{F}' - \partial \mathcal{F}$, we will show that $U \subseteq B$, $V \cap B = \emptyset$ and $B \cup V - U \in \partial \mathcal{F} - \partial \mathcal{F}'$, then we will be done.

Diagram 1.57



We have $B \cup x \in \mathcal{F}'$ for some $x \in X$, and $B \cup x \notin \mathcal{F}$. So $U \subseteq B \cup x$, $V \cap (B \cup x) = \emptyset$, and $(B \cup x \cup V) - U \in \mathcal{F}$, by definition of C_{UV}. If $x \in U$, then $\exists y \in V$ such that $\mathcal{F}$ is $(U - x, V - y)$-compressed, so from $(B \cup x \cup V) - U \in \mathcal{F}$, we have $B \cup y \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$. Thus $x \notin U$, so $U \subseteq B$ and $V \cap B = \emptyset$. Certainly $B \cup V - U \in \partial \mathcal{F}$ (since $(B \cup x \cup V) - U \in \mathcal{F}$), so we just need to show that $B \cup V - U \notin \partial \mathcal{F}'$.

Assume the opposite, i.e. $(B - U) \cup V \in \partial \mathcal{F}'$, so $(B - U) \cup V \cup w \in \mathcal{F}'$ for some $w \in X$. (This also belongs to $\mathcal{F}$, since it contains V). If $w \in U$, then since $\mathcal{F}$ is $(U - w, V - z)$-compressed for some $z \in V$, we have $B \cup z = C_{U-w,V-z}((B - U) \cup V \cup w) \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$. So $w \notin U$, and since $V \subseteq (B - U) \cup V \cup w$ and $U \cap ((B - U) \cup V \cup w) = \emptyset$, by definition of C_{UV}, we must have that both $(B - U) \cup V \cup w$ and $B \cup w = C_{UV}((B - U) \cup V \cup w) \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$. $\square$

Theorem 1.58 (Kruskal-Katona) Let $\mathcal{F} \subseteq X^{(r)}$, $1 \le r \le n$, let $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial \mathcal{C}| \le |\partial \mathcal{F}|$.

In particular, if $|\mathcal{F}| = \binom{k}{r}$, then $|\partial \mathcal{F}| \ge \binom{k}{r-1}$.

Proof (Hints).

- Let $\Gamma = \{(U, V) \in \mathbb{P}(X) \times \mathbb{P}(X) : |U| = |V| > 0, U \cap V = \emptyset, \max U < \max V\} \cup \{(\emptyset, \emptyset)\}$.
- Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of UV-compressions where $(U, V) \in \Gamma$, choosing $|U| = |V| > 0$ minimal each time. Show that this (U, V) satisfies condition of above lemma.
- Reason that sequence terminates by considering $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} 2^i$.

$\square$

Proof. Let $\Gamma = \{(U,V) \in \mathbb{P}(X) \times \mathbb{P}(X) : |U| = |V| > 0, U \cap V = \emptyset, \max U < \max V\} \cup \{(\emptyset, \emptyset)\}$. Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of set systems in $X^{(r)}$ as follows: set $\mathcal{F}_0 = \mathcal{F}$. Having chosen $\mathcal{F}_0, \ldots, \mathcal{F}_k$, if $\mathcal{F}_k$ is (UV)-compressed for all $(U,V) \in \Gamma$ then stop. Otherwise, choose $(U,V) \in \Gamma$ with $|U| = |V| > 0$ minimal, such that $\mathcal{F}_k$ is not (UV)-compressed.

Note that $\forall u \in U$, $\exists v \in V$ such that $(U - u, V - v) \in \Gamma$ (namely $v = \min(V)$). So by the above lemma, $|\partial C_{UV}(\mathcal{F}_k)| \leq |\partial \mathcal{F}_k|$. Set $\mathcal{F}_{k+1} = C_{UV}(\mathcal{F}_k)$, and continue. The sequence must terminate, as $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} 2^i$ is strictly decreasing with k. The final term $\mathcal{B} = \mathcal{F}_k$ satisfies $|\mathcal{B}| = |\mathcal{F}|$, $|\partial \mathcal{B}| \leq |\partial \mathcal{F}|$, and is (UV)-compressed for all $(U,V) \in \Gamma$. So $\mathcal{B} = \mathcal{C}$ by lemma before previous lemma. □

Remark 1.59

- Equivalently, if $|\mathcal{F}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \cdots + \binom{k_s}{s}$ where each $k_i > k_{i-1}$ and $s \geq 1$, then

$$|\partial \mathcal{F}| \geq \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \cdots + \binom{k_s}{s-1}.$$

- Equality in Kruskal-Katona: if $|\mathcal{F}| = \binom{k}{r}$ and $|\partial \mathcal{F}| = \binom{k}{r-1}$, then $\mathcal{F} = Y^{(r)}$ for some $Y \subseteq X$ with $|Y| = k$. However, it is not true in general that if $|\partial \mathcal{F}| = |\partial C|$, then $\mathcal{F}$ is isomorphic to $\mathcal{C}$ (i.e. there is a permutation of the ground set X sending $\mathcal{F}$ to $\mathcal{C}$).

Definition 1.60 For $\mathcal{F} \subseteq X^{(r)}$, $0 \leq r \leq n-1$, the **upper shadow** of $\mathcal{F}$ is

$$\partial^+ \mathcal{F} := \{A \cup x : A \in \mathcal{F}, x \notin A\} \subseteq X^{(r+1)}.$$

Corollary 1.61 Let $\mathcal{F} \subseteq X^{(r)}$, $0 \leq r \leq n-1$, let $\mathcal{C}$ be the initial segment of lex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial^+ \mathcal{C}| \leq |\partial^+ \mathcal{F}|$.

Proof (Hints). By Kruskal-Katona. □

Proof. By Kruskal-Katona, since $A < B$ in colex iff $A^c < B^c$ in lex with ground-set (X) order reversed, and if $\mathcal{F}' = \{A^c : A \in \mathcal{F}\}$, then $|\partial^+ \mathcal{F}'| = |\partial \mathcal{F}|$. □

Remark 1.62 The fact that the shadow of an initial segment of colex on $X^{(r)}$ is an initial segment of colex on $X^{(r-1)}$ (since if $\mathcal{C} = \left\{A \in X^{(r)} : A \leq a_1 \ldots a_r \text{ in colex}\right\}$, then $\partial \mathcal{C} = \left\{B \in X^{(r-1)} : B \leq a_2 \ldots a_r \text{ in colex}\right\}$) gives:

Corollary 1.63 Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq r \leq n$, $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial^t \mathcal{C}| \leq |\partial^t \mathcal{F}|$ for all $1 \leq t \leq r$ (where ∂^t is shadow applied t times).

Proof (Hints). Straightforward. □

Proof. If $|\partial^t \mathcal{C}| \leq |\partial^t \mathcal{F}|$, then $\left|\partial^{t+1} \mathcal{C}\right| \leq \left|\partial^{t+1} \mathcal{F}\right|$, since $\partial^t \mathcal{C}$ is an initial segment of colex. So we are done by induction (base case is Kruskal-Katona). □

Remark 1.64 So if $|\mathcal{F}| = \binom{k}{r}$, then $|\partial^t \mathcal{F}| \geq \binom{k}{r-t}$.

1.4. Intersecting families

Definition 1.65 A family $\mathcal{F} \subseteq \mathbb{P}(X)$ is **intersecting** if for all $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

We are interested in finding intersecting families of maximum size.

Proposition 1.66 For all intersecting families $\mathcal{F} \subseteq \mathbb{P}(X)$, $|\mathcal{F}| \leq 2^{n-1} = \frac{1}{2}|\mathbb{P}(X)|$.

Proof (Hints). Straightforward. □

Proof. Given any $A \subseteq X$, at most one of A and A^c can belong to $\mathcal{F}$. □

Example 1.67

- $\mathcal{F} = \{A \subseteq X : 1 \in A\}$ is intersecting, and $|\mathcal{F}| = 2^{n-1}$.
- $\mathcal{F} = \{A \subseteq X : |A| > \frac{n}{2}\}$ for n odd.

Example 1.68 Let $\mathcal{F} \subseteq X^{(r)}$:

- If $r > \frac{n}{2}$, then $\mathcal{F} = X^{(r)}$ is intersecting.
- If $r = \frac{n}{2}$, then choose one of A and A^c for all $A \in X^{(r)}$. This gives $|\mathcal{F}| = \frac{1}{2}\binom{n}{r}$.
- If $r < \frac{n}{2}$, then $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$ has size $\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$ (since the probability of a random r-set containing 1 is $\frac{r}{n}$). If $(n, r) = (8, 3)$, then $|\mathcal{F}| = \binom{7}{2} = 21$.
- Let $\mathcal{F} = \left\{A \in X^{(r)} : |A \cap \{1, 2, 3\}| \geq 2\right\}$. If $(n, r) = (8, 3)$, then $|\mathcal{F}| = 1 + \binom{3}{2}\binom{5}{1} = 16 < 21$ (since 1 set A has $|A \cap [3]| = 3$, 15 sets A have $|A \cap [3]| = 2$).

Theorem 1.69 (Erdos-Ko-Rado) Let $\mathcal{F} \subseteq X^{(r)}$ be an intersecting family, where $r < \frac{n}{2}$. Then $|\mathcal{F}| \leq \binom{n-1}{r-1}$.

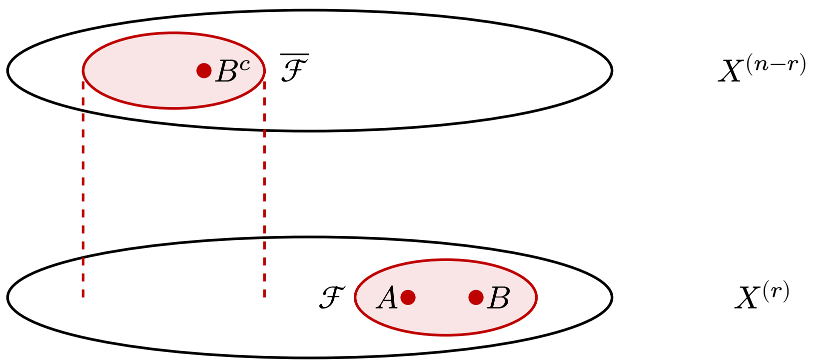
Proof (Hints).

- Method 1:
  - Let $\overline{\mathcal{F}} = \{A^c : A \in \mathcal{F}\}$. Show that $\partial^{n-2r}\overline{\mathcal{F}}$ and $\mathcal{F}$ are disjoint families of r-sets.
  - Assume the opposite, show that the size of the union of these two sets is greater than the size of $X^{(r)}$.
- Method 2:
  - Let $c : [n] \to \mathbb{Z}/n$ be bijection, i.e. cyclic ordering of $[n]$. Show there at most r sets in $\mathcal{F}$ that are intervals (sets with r consecutive elements) under this ordering.
  - Find expression for number of times an r-set in $\mathcal{F}$ is an interval all possible orderings, and find an upper bound for this using the above.

□

Proof. Proof 1 ("bubble down with Kruskal-Katona"): note that $A \cap B \neq \emptyset$ iff $A \not\subseteq B^c$.

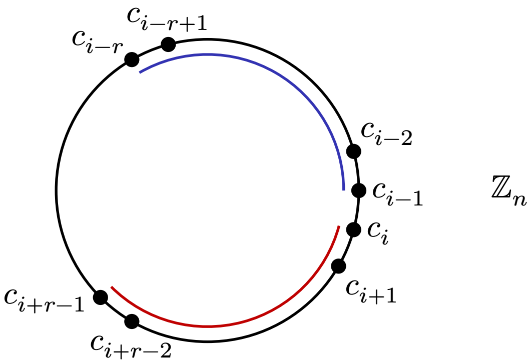
Diagram 1.70



Let $\overline{\mathcal{F}} = \{A^c : A \in \mathcal{F}\} \subseteq X^{(n-r)}$. We have $\partial^{n-2r}\overline{\mathcal{F}}$ and $\mathcal{F}$ are disjoint families of r-sets (if not, then there is some $A \in \mathcal{F}$ such that $A \subseteq B^c$ for some $B \in \mathcal{F}$, but then $A \cap B = \emptyset$). Suppose $|\mathcal{F}| > \binom{n-1}{r-1}$. Then $\left|\overline{\mathcal{F}}\right| = |\mathcal{F}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$. So by Kruskal-Katona, we have $\left|\partial^{n-2r}\overline{\mathcal{F}}\right| \geq \binom{n-1}{r}$. So $|\mathcal{F}| + \left|\partial^{n-2r}\overline{\mathcal{F}}\right| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r} = \left|X^{(r)}\right|$, a contradiction, since $\mathcal{F}, \partial^{n-2r}\overline{\mathcal{F}} \subseteq X^{(r)}$.

Proof 2: pick a cyclic ordering of $[n]$, i.e. a bijection $c : [n] \to \mathbb{Z}/n$. There are at most r sets in $\mathcal{F}$ that are intervals (r consecutive elements) under this ordering: for $c_1 ... c_r \in \mathcal{F}$, for each $2 \leq i \leq r$, at most one of the two intervals $c_i ... c_{i+r-1}$ and $c_{i-r} ... c_{i-1}$ can belong to $\mathcal{F}$, since they are disjoint and $\mathcal{F}$ is intersecting (the indices of c are taken $\bmod n$).

Diagram 1.71



For each r-set A, out of the $n!$ cyclic orderings, there are $n \cdot r!(n-r)!$ which map A to an interval ($r!$ orderings inside A, $(n-r)!$ orderings outside A, n choices for the start of the interval). Hence, by counting the number of times an r-set in $\mathcal{F}$ is an interval under a given ordering (over all r-sets in $\mathcal{F}$ and all cyclic orderings), we obtain $|\mathcal{F}|nr!(n-r)! \leq n!r$, i.e. $|\mathcal{F}| \leq \binom{n-1}{r-1}$. □

Remark 1.72

- The calculation at the end of proof method 1 had to give the correct answer, as the shadow calculations would all be exact if $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$ (in this case, $\mathcal{F}$ and $\partial^{n-2r}\overline{\mathcal{F}}$ partition $X^{(r)}$).
- The calculations at the end of proof method 2 had to work out, given equality for the family $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$.
- In method 2, equivalently, we are double-counting the edges in the bipartite graph, where the vertex classes (partition sets) are $\mathcal{F}$ and all cyclic orderings, with A joined to c if A is an interval under c. This method is called **averaging** or **Katona's method**.
- Equality in Erdos-Ko-Rado holds iff $\mathcal{F} = \left\{A \in X^{(r)} : i \in A\right\}$, for some $1 \leq i \leq n$. This can be obtained from proof 1 and equality in Kruskal-Katona, or from proof 2.

2. Isoperimetric inequalities

We seek to answer questions of the form "how do we minimise the boundary of a set of given size?"

Example 2.1 In the continuous setting:

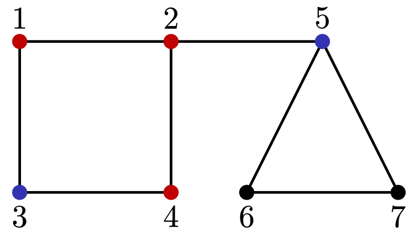
- Among all subsets of $\mathbb{R}^2$ of a given fixed area, the disc minimises the perimeter.

- Among all subsets of $\mathbb{R}^3$ of a given fixed volume, the solid sphere minimises the surface area.
- Among all subsets of S^2 of given fixed surface area, the circular cap minimises the perimeter.

Definition 2.2 For a A of vertices of a graph G, the **boundary** of A is

$$b(A) = \{x \in G : x \notin A, xy \in E \text{ for some } y \in A\}.$$

Diagram 2.3



$A = \{1, 2, 4\}$ (in red) has boundary $\{3, 5\}$ (in blue).

Definition 2.4 An **isoperimetric inequality** on a graph G is an inequality of the form

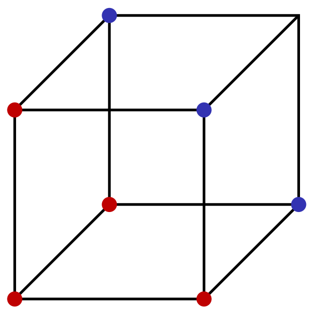
$$\forall A \subseteq G, \quad |b(A)| \geq f(|A|)$$

for some function $f : \mathbb{N} \to \mathbb{R}$.

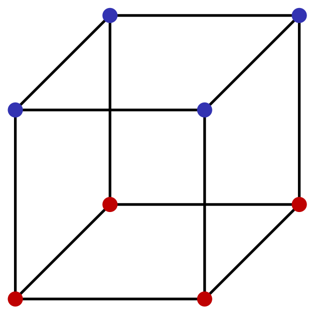
Definition 2.5 The **neighbourhood** of $A \subseteq V(G)$ is $N(A) := A \cup b(A)$, i.e.

$$N(A) = \{x \in G : d(x, A) \leq 1\}.$$

Diagram 2.6 Let $A \subseteq \mathbb{P}(X) = V(Q_3)$, $|A| = 4$.



$|b(A)| = 3$



$|b(A)| = 4$

Example 2.7 A good (and natural) example for A that minimises $|b(A)|$ in the discrete cube Q_n might be a ball $B(x, r) = \{y \in G : d(x, y) \leq r\}$.

A good guess is that balls are best, i.e. sets of the form $B(\emptyset, r) = X^{(\leq r)} = X^{(0)} \cup \cdots \cup X^{(r)}$. What if $\left|X^{(\leq r)}\right| \leq |A| \leq \left|X^{(\leq r+1)}\right|$? A good guess is take A with $X^{(\leq r)} \subsetneq A \subsetneq X^{(\leq r+1)}$. If $A = X^{(\leq r)} \cup B$, where $B \subseteq X^{(r+1)}$, then $b(A) = \left(X^{(r+1)} - B\right) \cup \partial^+ B$, so we would take B to be an initial segment of lex by Kruskal-Katona. This motivates the following definition.

Definition 2.8 The **simplicial ordering** on $\mathbb{P}(X)$ defines $x < y$ if either $|x| < |y|$, or both $|x| = |y|$ and $x < y$ in lex.

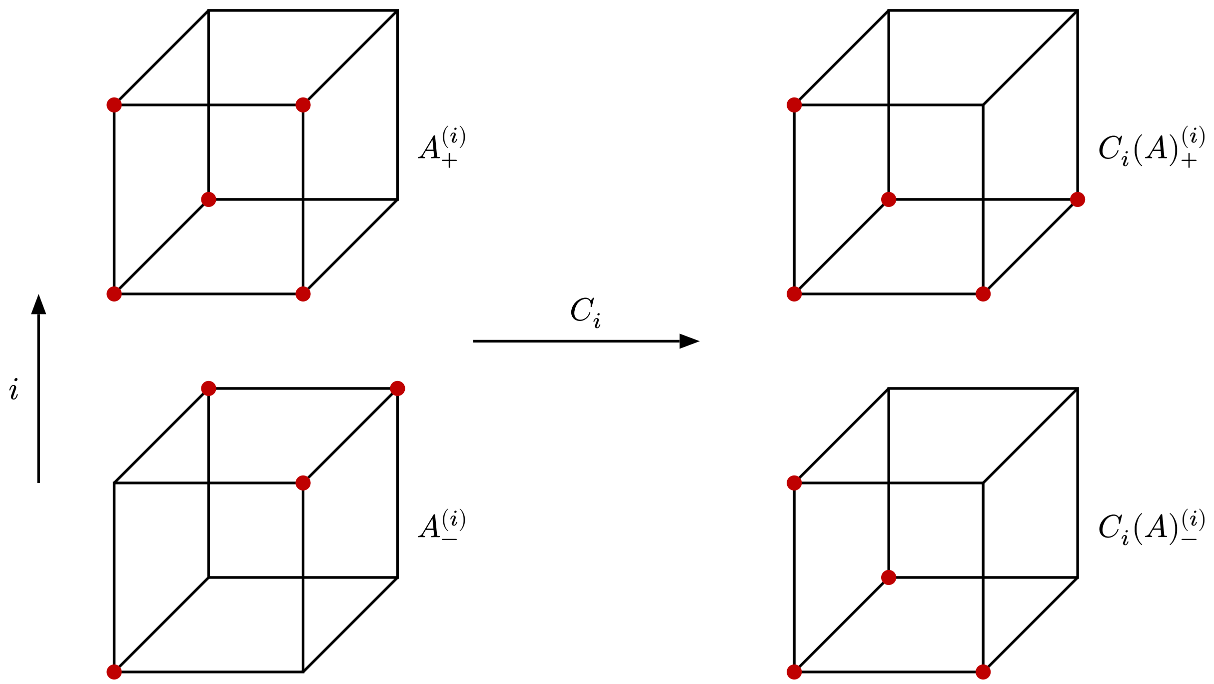
We want to show the initial segments of the simplicial ordering minimise the boundary.

Definition 2.9 For $A \subseteq \mathbb{P}(X)$ and $1 \leq i \leq n$, the ***i*-sections** of A are the families $A_-^{(i)}, A_+^{(i)} \subseteq \mathbb{P}(X \setminus i)$, given by

$$\begin{aligned} A_-^{(i)} &= A_- := \{x \in A : i \notin x\}, \\ A_+^{(i)} &= A_+ := \{x - i : x \in A, i \in x\} \end{aligned}$$

Note that $A = A_-^{(i)} \cup \{x \cup i : x \in A_+^{(i)}\}$, so we can define a family by its i-sections.

Diagram 2.10



i-compression of A

Remark 2.11 When viewing $\mathbb{P}(X)$ as the n-dimensional cube Q_n, we view the i-sections as subgraphs of the $(n-1)$-dimensional cube Q_{n-1} (which we view $\mathbb{P}(X \setminus i)$ as).

Definition 2.12 A **Hamming ball** is a family $A \subseteq \mathbb{P}(X)$ with $X^{(\leq r)} \subseteq A \subseteq X^{(\leq r+1)}$ for some r.

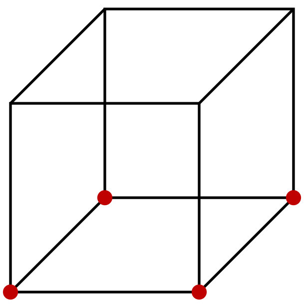
Definition 2.13 The ***i*-compression** of $A \subseteq \mathbb{P}(X)$ is the family $C_i(A) \subseteq \mathbb{P}(X)$ given by its i-sections:

- $(C_i(A))_-^{(i)}$ is the first $\left|A_-^{(i)}\right|$ elements of the simplicial order on $\mathbb{P}(X - i)$, and
- $(C_i(A))_+^{(i)}$ is the first $\left|A_+^{(i)}\right|$ elements of the simplicial order on $\mathbb{P}(X - i)$.

Note that $|C_i(A)| = |A|$, and $C_i(A)$ "looks more like" a Hamming ball than A does.

Definition 2.14 $A \subseteq \mathbb{P}(X)$ is ***i*-compressed** if $C_i(A) = A$.

Example 2.15 Note that a set that is i-compressed for all $i \in [n]$ is not necessarily an initial segment of simplicial, e.g. take $\{\emptyset, 1, 2, 12\}$ in Q_3.



However...

Lemma 2.16 Let $B \subseteq Q_n$ be i-compressed for all $i \in [n]$ but not an initial segment of the simplicial order. Then either:

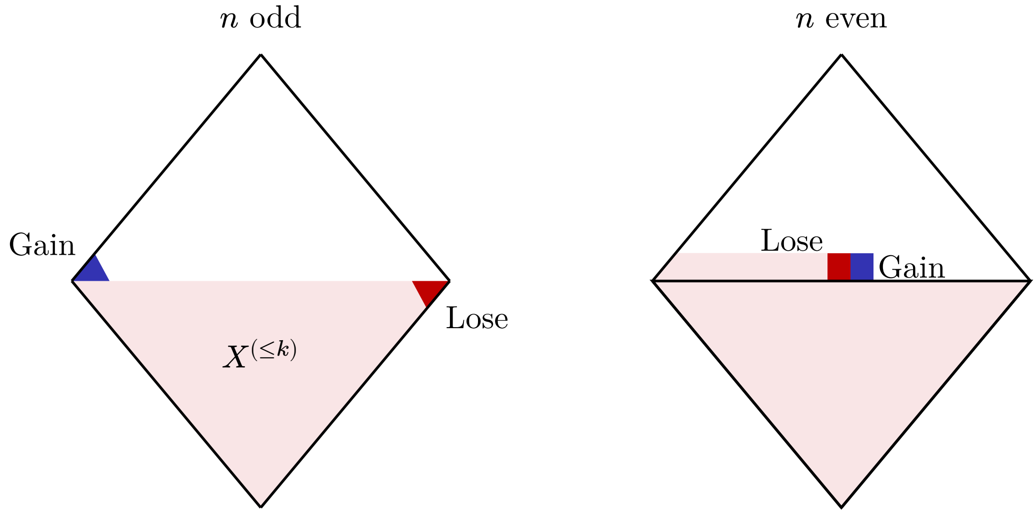
- n is odd (say $n = 2k+1$) and

$$B = X^{(\leq k)} \setminus \underbrace{\{k+2, k+3, \dots, 2k+1\}}_{\text{last } k\text{-set}} \cup \underbrace{\{1, 2, \dots, k+1\}}_{\text{first } (k+1)\text{-set}},$$

- or n is even (say $n = 2k$), and

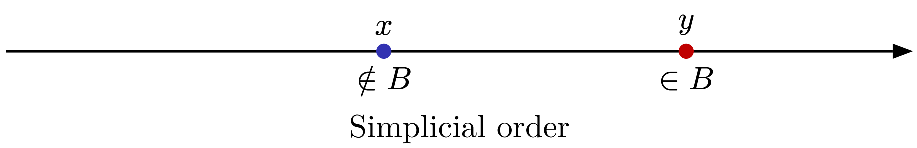
$$B = X^{(<k)} \cup \left\{x \in X^{(k)} : 1 \in x\right\} \setminus \underbrace{\{1, k+2, k+3, \dots, 2k\}}_{\text{last } k\text{-set with } 1} \cup \underbrace{\{2, 3, \dots, k+1\}}_{\text{first } k\text{-set without } 1}.$$

Diagram 2.17



Proof (Hints). For $x \notin B$ and $y \in B$, show by contradiction that any $i \in [n]$ is in exactly one of x and y (it helps to visualise this), and reason that no elements lie strictly between x and y in the simplicial ordering. □

Proof. As B is not an initial segment, there are $x < y$ in simplicial ordering with $x \notin B$ and $y \in B$.



For each $i \in [n]$, assume $i \in x, y$. Since the i-section that y lives in is an initial segment of simplicial on $\mathbb{P}(X \setminus i)$ (as B is i-compressed), and $x - i < y - i$ in simplicial on $\mathbb{P}(X \setminus i)$, we have that $x - i$ lives in the same i-section, and so $x \in B$: contradiction. Similarly, $i \notin x, y$ leads to a contradiction (as then $x < y$ in simplicial on $\mathbb{P}(X \setminus i)$). So $x = y^c$.

Thus for each $y \in B$, there is at most one $x < y$ with $x \notin B$ (namely $x = y^c$), and for each $x \notin B$, there is at most one $y > x$ with $y \in B$ (namely $y = x^c$). So no sets lie between x and y in the simplicial ordering. So $B = \{z : z \leq y\} \setminus \{x\}$, with x the predecessor of y, and $x = y^c$.

Simplicial order

Hence if $n = 2k+1$, then x is the last k-set (otherwise sizes of x and $y = x^c$ don't match), and if $n = 2k$, then x is the last k-set containing 1. □

Theorem 2.18 (Harper) Let $A \subseteq V(Q_n)$ and let C be the initial segment of the simplicial order on $\mathbb{P}(X) = V(Q_n)$, with $|C| = |A|$. Then $|N(A)| \geq |N(C)|$. So initial segments of the simplicial order minimise the boundary. In particular, if $|A| = \sum_{i=0}^{r} \binom{n}{i}$, then $|N(A)| \geq \sum_{i=0}^{r+1} \binom{n}{i}$.

Proof (Hints).

- By induction on n.
- Prove the claim that $|N(C_i(A))| \leq |N(A)|$:
  - Find expressions for $N(A)_-$ as union of two sets, similarly for $N(A)_+$, same for $N(B)_-$ and $N(B)_+$, where $B = C_i(A)$.
  - Explain why $N(B_-)$ and B_+ are nested, use this to show $|N(B_-) \cup B_+| \leq |N(A_-) \cup A_+|$.
  - Do the same with the $+$ and $-$ switched.
- Define a suitable sequence of families $A_0, A_1, \ldots \in Q_n$.
- Reason that the sequence terminates by considering $\sum_{x \in A_k}$(position of x in simplicial order).
- Conclude by above lemma.

□

Proof. By induction on n. $n = 1$ is trivial. Given $n > 1$, $A \subseteq Q_n$ and $1 \leq i \leq n$, we claim that $|N(C_i(A))| \leq |N(A)|$.

Proof of claim. Write $B = C_i(A)$. We have $N(A)_- = N(A_-) \cup A_+$, and $N(A)_+ = N(A_+) \cup A_-$. Similarly, $N(B)_- = N(B_-) \cup B_+$, and $N(B)_+ = N(B_+) \cup B_-$.

Now $|B_+| = |A_+|$ by definition of B, and by the inductive hypothesis, $|N(B_-)| \leq |N(A_-)|$ (since B_- is an initial segment of simplicial of the same size as A_-). But B_+ is an initial segment of the simplicial ordering, and $N(B_-)$ is as well (since the neighbourhood of an initial segment of the simplicial ordering is also an initial segment).

So B_+ and $N(B_-)$ are nested (one is contained in the other). Hence, $|N(B_-) \cup B_+| \leq |N(A_-) \cup A_+|$.

Similarly, $|B_-| = |A_-|$ by definition of B. Again, by the inductive hypothesis, $|N(B_+)| \leq |N(A_+)|$. B_- and $N(B_+)$ are initial segments, so are nested. Hence $|N(B_+) \cup B_-| \leq |N(A_+) \cup A_-|$.

This gives $|N(B)| = |N(B)_-| + |N(B)_+| \leq |N(A)_-| + |N(A)_+| = |N(A)|$, which proves the claim.

Define a sequence $A_0, A_1, \ldots \subseteq Q_n$ as follows:

- Set $A_0 = A_1$.
- having chosen $A_0, \ldots, A_k$, if A_k is i-compressed for all $i \in [n]$, then end the sequence with A_k. If not, pick i with $C_i(A_k) \neq A_k$ and set $A_{k+1} = C_i(A_k)$, and continue.

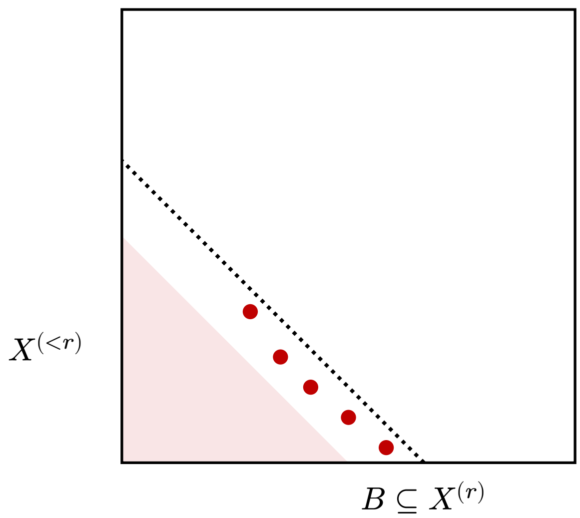
The sequence must terminate, since $\sum_{x \in A_k}$(position of x in simplicial order) is strictly decreasing. The final family $B = A_k$ satisfies $|B| = |A|$, $|N(B)| \leq |N(A)|$, and is i-compressed for all $i \in [n]$.

So we are done by above lemma, since in each case certainly we have $|N(B)| \geq |N(C)|$. □

Remark 2.19

- If A was a Hamming ball, then we would be already done by Kruskal-Katona.
- Conversely, Harper implies Kruskal-Katona: given $B \subseteq X^{(r)}$, apply Harper to $A = X^{(\leq r-1)} \cup B$.
- We could also prove Harper using UV-compressions.
- Conversely, we can also prove Kruskal-Katona using these "codimension 1" compressions.

Diagram 2.20

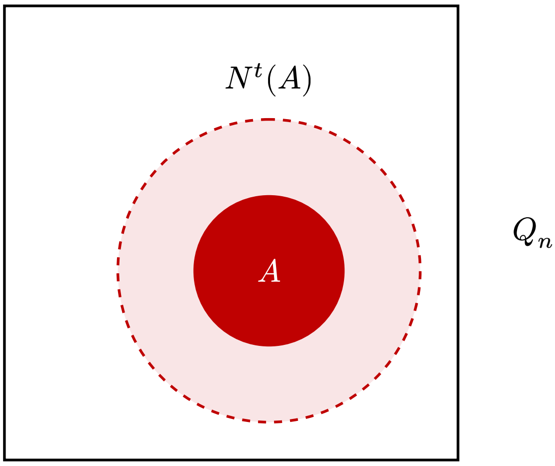


Harper implies Kruskal-Katona.

Definition 2.21 For $A \subseteq Q_n$ and $t \in \mathbb{N}$, the ***t*-neighbourhood** of A is

$$A_{(t)} = N^t(A) := \{x \in Q_n : d(x, A) \leq t\}.$$

Diagram 2.22



t-neighbourhood of A.

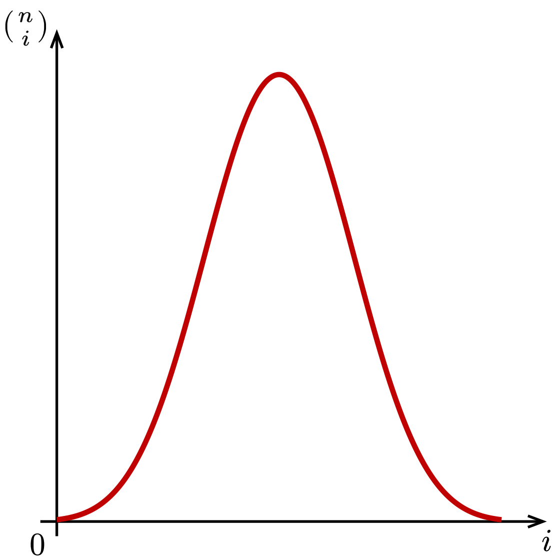
Corollary 2.23 Let $A \subseteq Q_n$ with $|A| \geq \sum_{i=0}^{r} \binom{n}{i}$. Then

$$\forall t \leq n - r, \quad |N^t(A)| \geq \sum_{i=0}^{r+t} \binom{n}{i}.$$

Proof (Hints). By Harper's theorem. □

Proof. By Harper's theorem and induction on t. □

Remark 2.24 To get a feeling for the strength of the above corollary, we'll need some estimates on quantities such as $\sum_{i=0}^{r} \binom{n}{i}$. Note that $i = n/2$ maximises $\binom{n}{i}$, while $i = (1/2 - \varepsilon)n$ makes it small: we are going $\varepsilon\sqrt{n}$ standard deviations away from the mean $n/2$.



Proposition 2.25 Let $0 < \varepsilon < 1/4$. Then

$$\sum_{i=0}^{\lfloor (1/2-\varepsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2} \cdot 2^n.$$

For ε fixed and $n \to \infty$, the upper bound is an exponentially small fraction of 2^n.

Proof (Hints).
- Let $L = \lfloor (1/2 - \varepsilon)n \rfloor$ and $M = \lfloor (1/2 - \varepsilon/2)n \rfloor$.
- For $1 \leq i \leq L$, show that $\binom{n}{i-1}/\binom{n}{i} \leq 1 - 2\varepsilon$, use this to show that

$$\sum_{i=0}^{L} \binom{n}{i} \leq \frac{1}{2\varepsilon} \binom{n}{L}.$$

- Use the same argument to show that

$$\binom{n}{L} \leq \binom{n}{M} (1 - \varepsilon)^{M-L}.$$

- Use that $1 - \varepsilon \leq e^{-\varepsilon}$ to conclude the result.

□

Proof. Let $L = \lfloor (1/2 - \varepsilon)n \rfloor$. For $1 \leq i \leq L$,

$$\binom{n}{i-1} / \binom{n}{i} = \frac{i}{n - i + 1} \leq \frac{(1/2 - \varepsilon)n}{(1/2 + \varepsilon)n} = \frac{1/2 - \varepsilon}{1/2 + \varepsilon} = 1 - \frac{2\varepsilon}{1/2 + \varepsilon} \leq 1 - 2\varepsilon.$$

Hence by induction, $\binom{n}{i} \leq (1 - 2\varepsilon)^{L-i} \binom{n}{L}$ for each $0 \leq i \leq L$, and so

$$\sum_{i=0}^{L} \binom{n}{i} \leq \frac{1}{2\varepsilon} \binom{n}{L}$$

(since this is the sum of geometric progression). Let $M = \lfloor (1/2 - \varepsilon/2)n \rfloor$. It is easy to show that $M - L > \varepsilon n/2 - 1$. By the same argument as above, $\binom{n}{i} \leq (1 - 2\frac{\varepsilon}{2})^{M-i} \binom{n}{M}$ for each $0 \leq i \leq M$. In particular,

$$\begin{aligned}
\binom{n}{L} &\leq \binom{n}{M} \left(1 - 2\frac{\varepsilon}{2}\right)^{M-L} \\
\binom{n}{L} &\leq \binom{n}{M} (1 - \varepsilon)^{\varepsilon n/2 - 1} \\
&\leq 2^n \cdot 2(1 - \varepsilon)^{\varepsilon n/2} \\
&\leq 2^n \cdot 2e^{-\varepsilon^2 n/2}
\end{aligned}$$

since $1 - \varepsilon \leq e^{-\varepsilon}$. Combining with the previous upper bound, we obtain

$$\sum_{i=0}^{L} \binom{n}{i} \leq \frac{1}{2\varepsilon} \cdot 2e^{-\varepsilon^2 n/2} \cdot 2^n.$$

□

Theorem 2.26 Let $0 < \varepsilon < 1/4$, $A \subseteq Q_n$. If $|A|/2^n \geq 1/2$, then

$$\frac{|N^{\varepsilon n}(A)|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

So sets of at least half density have exponentially dense εn-neighbourhoods.

Proof (Hints).
- Enough to show that if $\varepsilon n \in \mathbb{N}$, then $|N^{\varepsilon n}(A)|/2^n \geq 1 - \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2}$.

- Give lower bound on $|A|$ which is a binomial sum, deduce lower bound on $N^{\varepsilon n}(A)$.
- Give an upper bound on $|N^{\varepsilon n}(A)^c|$ using the above proposition.

$\square$

Proof. It is enough to show that if $\varepsilon n \in \mathbb{N}$, then $|N^{\varepsilon n}(A)|/2^n \geq 1 - \frac{1}{\varepsilon}e^{-\varepsilon^2 n/2}$. We have $|A| \geq \sum_{i=0}^{\lceil n/2-1 \rceil} \binom{n}{i}$, so by Corollary 2.23,

$$|N^{\varepsilon n}(A)| \geq \sum_{i=0}^{\lceil n/2-1+\varepsilon n \rceil} \binom{n}{i}.$$

So

$$\begin{aligned} |N^{\varepsilon n}(A)^c| &\leq \sum_{i=\lceil n/2+\varepsilon n \rceil}^{n} \binom{n}{i} \\ &= \sum_{i=\lceil n/2+\varepsilon n \rceil}^{n} \binom{n}{n-i} \\ &= \sum_{i=0}^{\lfloor n/2-\varepsilon n \rfloor} \binom{n}{i} \\ &\leq \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2} \cdot 2^n. \end{aligned}$$

by Proposition 2.25. $\square$

Remark 2.27 The same argument would give a result for "small" sets: if $|A|/2^n \geq \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$, then $\left|N^{2\varepsilon n}(A)\right|/2^n \geq 1 - \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$.

2.1. Concentration of measure

Definition 2.28 $f : Q_n \to \mathbb{R}$ is **Lipschitz** if for all adjacent $x, y \in Q_n$, $|f(x) - f(y)| \leq 1$.

Definition 2.29 For $f : Q_n \to \mathbb{R}$, we say $M \in \mathbb{R}$ is a **Levy mean** (or **median**) of f if $|\{x \in Q_n : f(x) \leq M\}| \geq 2^{n-1}$ and $|\{x \in Q_n : f(x) \geq M\}| \geq 2^{n-1}$.

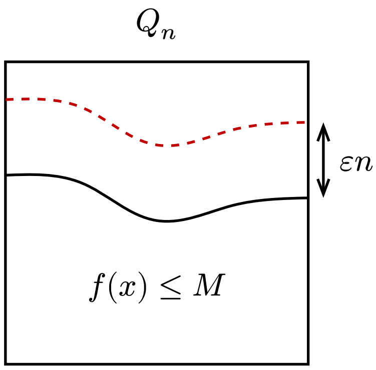
Example 2.30 Let $f : Q_n \to \mathbb{R}$, $f(x) = 1$ if $1 \in x$ and $f(x) = 0$ otherwise. Then any $M \in [0, 1]$ is a Levy mean of f.

Theorem 2.31 (Concentration of Measure Phenomenon) Let $f : Q_n \to \mathbb{R}$ be Lipschitz with Levy mean M. Then for all $0 < \varepsilon < \frac{1}{4}$,

$$\frac{|\{x \in Q_n : |f(x) - M| \leq \varepsilon n\}|}{2^n} \geq 1 - \frac{4}{\varepsilon}e^{-\varepsilon^2 n/2}.$$

So "every well-behaved function on the cube Q_n is roughly constant nearly everywhere".

Diagram 2.32



Proof (Hints).

- Consider two subsets $A, B \subseteq Q_n$ of density at least $1/2$, and apply Theorem 2.26 on them.
- Use the fact that f is Lipschitz to find a set that contains $N^{\varepsilon n}(A)$ (and similarly for B).

□

Proof. Let $A = \{x \in Q_n : f(x) \leq M\}$. Then by definition, $|A|/2^n \geq 1/2$, so by the above theorem,

$$\frac{|N^{\varepsilon n}(A)|}{2^n} \geq 1 - \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}.$$

But f is Lipschitz, so $x \in N^{\varepsilon n}(A) \implies f(x) \leq M + \varepsilon n$, so $N^{\varepsilon n}(A) \subseteq \{x \in Q_n : f(x) \leq M + \varepsilon n\} =: L$. Thus,

$$\frac{|L|}{2^n} \geq 1 - \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}.$$

Similarly, let $U = \{x \in Q_n : f(x) \geq M - \varepsilon n\}$, then $|U|/2^n \geq 1 - \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2}$. Hence, we have

$$\begin{aligned}
\frac{|L \cap U|}{2^n} &= \frac{|L|}{2^n} + \frac{|U|}{2^n} - \frac{|L \cup U|}{2^n} \\
&\geq 1 - \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2} + 1 - \frac{2}{\varepsilon}e^{-\varepsilon^2 n/2} - 1 \\
&= 1 - \frac{4}{\varepsilon}e^{-\varepsilon^2 n/2}.
\end{aligned}$$

□

Definition 2.33 The **diameter** of a graph $G = (V, E)$ is $\max\{d(x, y) : x, y \in V\}$.

Definition 2.34 Let G be a graph of diameter D. Write

$$\alpha(G, \varepsilon) = \max\left\{1 - \frac{|N^{\varepsilon D}(A)|}{|G|} : A \subseteq G, \frac{|A|}{|G|} \geq \frac{1}{2}\right\}.$$

So if $\alpha(G, \varepsilon)$ is small, then sets of at least half density have large εD-neighbourhoods.

Definition 2.35 A sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is a **Levy family** if

$$\forall \varepsilon > 0, \quad \alpha(G_n, \varepsilon) \to 0 \text{ as } n \to \infty.$$

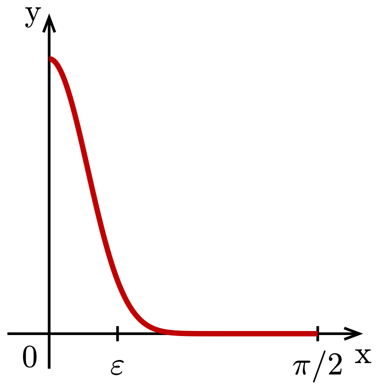
It is a **normal Levy family** if for all $\varepsilon > 0$, $\alpha(G_n, \varepsilon)$ decays exponentially with n.

Example 2.36 By Theorem 2.26, the sequence (Q_n) is a normal Levy family (note that Q_n has diameter $n+1$). More generally, we have concentration of measure for any Levy family.

Example 2.37 Many naturally-occurring families of graphs are Levy families, e.g. $(S_n)_{n\in\mathbb{N}}$, where the permutation group S_n is made into a graph by including an edge between σ and τ if $\tau\sigma^{-1}$ is a transposition.

Example 2.38 Similarly, we can define $\alpha(X, \varepsilon)$ for any metric measure space X (of finite measure and finite diameter). E.g. the sequence of spheres $(S^n)_{n\in\mathbb{N}}$ is a Levy family. To prove this, we have:

1. An isoperimetric inequality on S^n: for $A \subseteq S^n$ and C a circular cap with $|C| = |A|$, we have $|N^\varepsilon(A)| \geq |N^\varepsilon(C)|$.
2. An estimate: a circular cap C of measure $1/2$ is the cap of angle $\pi/2$. So $N^\varepsilon(C)$ is the circular cap of angle $\pi/2 + \varepsilon$. This has measure roughly equal to $\int_\varepsilon^{\pi/2} \cos^{n-1}(t)\,\mathrm{d}t \to 0$ as $n \to \infty$.



Remark 2.39 We deduced concentration of measure from an isoperimetric inequality. Conversely, we have:

Proposition 2.40 Let G be a graph such that for any Lipschitz function $f : G \to \mathbb{R}$ with Levy mean M, we have

$$\frac{|\{x \in G : |f(x) - M| > t\}|}{|G|} \leq \alpha$$

for some given $t, \alpha \geq 0$. Then for all $A \subseteq G$ with $|A|/|G| \geq 1/2$, we have

$$\frac{|N^t(A)|}{|G|} \geq 1 - \alpha.$$

Proof (Hints). Consider an appropriate Lipschitz function with Levy mean 0. □

Proof. The function $f(x) = d(x, A)$ is Lipschitz, and has 0 as a Levy mean. So

$$\frac{|\{x \in G : d(x, A) > t\}|}{|G|} = \frac{|\{x \in G : x \notin N^t(A)\}|}{|G|} \leq \alpha.$$

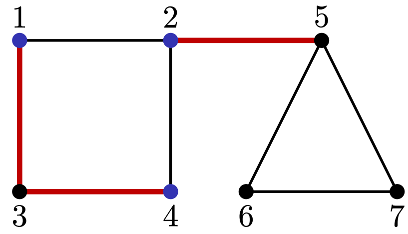
□

2.2. Edge-isoperimetric inequalities

Definition 2.41 For a graph G and $A \subseteq V(G)$, the **edge-boundary** of A is

$$\partial_e A = \partial A := \{xy \in E : x \in A, y \notin A\}.$$

Diagram 2.42

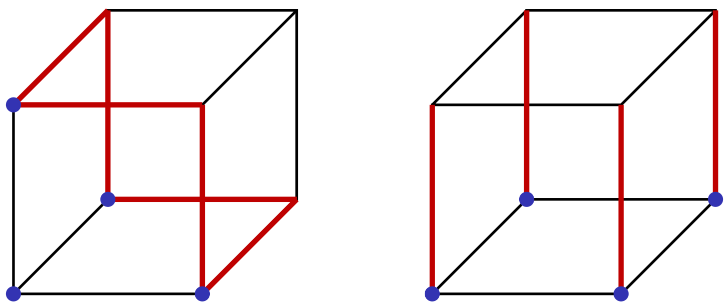


$A = \{1, 2, 4\}$ (in blue) has edge-boundary $\{13, 34, 25\}$ (in red).

Definition 2.43 An **edge-isoperimetric inequality** on a graph G is an inequality of the form

$$\forall A \subseteq G, \quad |\partial A| \geq f(|A|).$$

Example 2.44 We are interested in the case $G = Q_n$. Given $|A|$, which $A \subseteq Q_n$ should we take to minimise $|\partial A|$? Let $|A| = 4$, $A \subseteq Q_3$.



$|\partial A| = 6$ $\qquad$ $|\partial A| = 4$

The diagram suggests that subcubes are best. If $2^k < |A| < 2^{k+1}$, then it is natural to take $A = \mathbb{P}([k]) \cup$ some sets in $\mathbb{P}([k+1])$. If $A \subseteq Q_4$ has size $|A| > 2^3$, then it is natural to take all of the bottom layer and $|A| - 2^3$ elements in the top layer. Then the size of the edge boundary is the number of edges from the bottom layer to the top layer (i.e. $2^3 - (|A| - 2^3) = 2^4 - |A|$) plus the number of edges in the top layer. So now we want to minimise the number of edges in the top layer.

Definition 2.45 For $x, y \in Q_n$, $x \neq y$, say $x < y$ in the **binary ordering** on Q_n if $\max(x \Delta y) \in y$. Equivalently,

$$x < y \iff \sum_{i \in x} 2^i < \sum_{i \in y} 2^i.$$

"Go up in subcubes". Effectively, we are counting the naturals up to $2^n - 1$ (where an n-bit binary string corresponds to a subset of Q_n in the obvious way).

Example 2.46 The elements of Q_3 in ascending binary order are

$$\emptyset, 1, 2, 12, 3, 13, 23, 123.$$

Definition 2.47 For $A \subseteq Q_n$, $1 \leq i \leq n$, the ***i*-binary-compression** $B_i(A) \subseteq Q_n$ is defined by its i-sections:
- $(B_i(A))^{(i)}_-$ is the initial segment of binary ordering on $\mathbb{P}(X - i)$ of size $\left|A^{(i)}_-\right|$.
- $(B_i(A))^{(i)}_+$ is the initial segment of binary ordering on $\mathbb{P}(X - i)$ of size $\left|A^{(i)}_+\right|$.

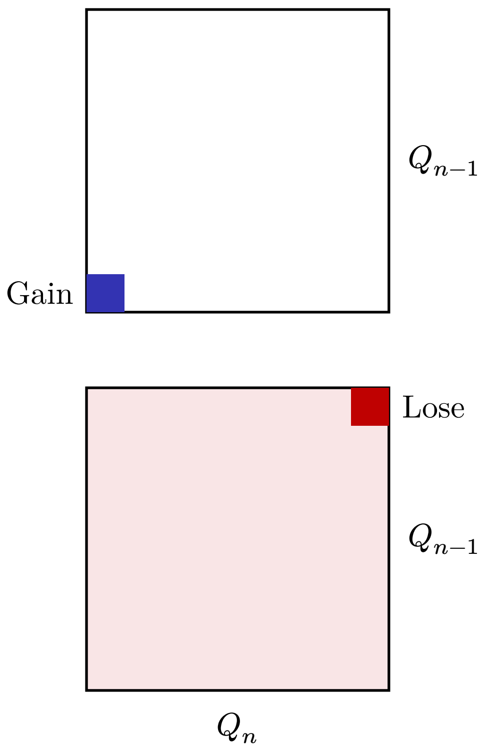
So $|B_i(A)| = |A|$.

Definition 2.48 $A \subseteq Q_n$ is ***i*-binary-compressed** if $B_i(A) = A$.

Example 2.49 A set $B \subseteq Q_n$ which is i-binary-compressed for all $1 \leq i \leq n$ is not necessarily an initial segment of binary, e.g. $\{\emptyset, 1, 2, 3\} \subseteq Q_3$. However, we have:

Lemma 2.50 Let $B \subseteq Q_n$ be i-binary-compressed for all $1 \leq i \leq n$ but not an initial segment of binary. Then

$$B = \underbrace{\mathbb{P}([n-1])}_{\text{downstairs}} \setminus \underbrace{\{1, 2, \ldots, n-1\}}_{\text{last point in binary order in } \mathbb{P}([n-1])} \cup \underbrace{\{n\}}_{\text{first point in binary order not in } \mathbb{P}([n-1])}$$



Proof (Hints). For $x \notin B$ and $y \in B$, show by contradiction that any $i \in [n]$ is in exactly one of x and y (it helps to visualise this), and deduce that no elements lie strictly between x and y in the binary ordering. $\square$

Proof. As B is not an initial segment, there are $x < y$ with $x \notin B$ and $y \in B$. For each $1 \leq i \leq n$, assume that $i \in x, y$. Since the i-section that y lives in is an initial segment of binary on $\mathbb{P}(X \setminus i)$ (as B is i-binary-compressed), and $x - i < y - i$ in binary on $\mathbb{P}(X \setminus i)$, we have that $x - i$ lives in the same i-section, and so $x \in B$: contradiction. Similarly, $i \notin x, y$ leads to a contradiction (as then $x < y$ in binary on $\mathbb{P}(X \setminus i)$). So $x = y^c$.

Thus, for each $y \in B$, there is at most one $x < y$ with $x \notin B$ (namely $x = y^c$), and for each $x \notin B$, there is at most one $y > x$ with $y \in B$ (namely $y = x^c$). So $B = \{z : z \leq y\} \setminus \{x\}$, where x is the predecessor of y and $y = x^c$. So we must have $y = \{n\}$ and $x = \{1, 2, ..., n-1\}$. □

Theorem 2.51 (Edge-isoperimetric Inequality in Q_n) Let $A \subseteq Q_n$ and let C be the initial segment of binary on Q_n with $|C| = |A|$. Then $|\partial C| \leq |\partial A|$. In particular, if $|A| = 2^k$, then $|\partial A| \geq 2^k(n-k)$.

Remark 2.52 This theorem is sometimes called the Theorem of Harper, Lindsey, Bernstein and Hart.

Proof (Hints).
- By induction on n.
- Prove for each $1 \leq i \leq n$, $|\partial B_i(A)| \leq |\partial A|$, by expressing ∂A as a disjoint union of three sets (it helps to visualise this), and using that B_+ and B_- are nested (why?).
- Define a sequence $A_0, A_1, ...$ in the obvious way, show it terminates by considering a suitable function A_k.
- Use above lemma to conclude the result.

□

Proof. By induction on n. $n = 1$ is trivial. For $n > 1$ and $A \subseteq Q_n$, and $1 \leq i \leq n$, we claim that $|\partial B_i(A)| \leq |\partial A|$.

Proof of claim. Write $B = B_i(A)$. We have (remember $A_-, A_+ \subseteq Q_{n-1}$ not Q_n)

$$|\partial A| = \underbrace{|\partial A_-|}_{\text{downstairs}} + \underbrace{|\partial A_+|}_{\text{upstairs}} + \underbrace{|A_+ \Delta A_-|}_{\text{across}}$$

and similarly, $|\partial B| = |\partial B_-| + |\partial B_+| + |B_+ \Delta B_-|$. Now, $|\partial B_-| \leq |\partial A_-|$ and $|\partial B_+| \leq |\partial A_+|$ by the induction hypothesis. Also, the sets B_+ and B_- are nested/comparable (one is contained in the other), as each is an initial segment of binary on $\mathbb{P}(X - i)$. So, since $|B_-| = |A_-|$ and $|B_+| = |A_+|$ by definition, we have

$$|B_+ \Delta B_-| = |B_+| - |B_-| = |A_+| - |A_-| \leq |A_+ - A_-| \leq |A_+ \Delta A_-|.$$

if $B_- \subseteq B_+$, and similarly this holds if $B_+ \subseteq B_-$. So $|\partial B| \leq |\partial A|$. This proves the claim. □

Define a sequence $A_0, A_1, ...$ as follows: set $A_0 = A$. Having defined $A_0, ..., A_k$, if A_k is i-binary-compressed for all $1 \leq i \leq n$, then stop the sequence with A_k. Otherwise, choose i with $B_i(A_k) \neq A_k$, and set $A_{k+1} = A_k$. This must terminate, as the function $k \mapsto \sum_{x \in A_k}$(position of x in binary) is strictly decreasing.

The final family in this sequence $B = A_k$ satisfies $|B| = |A|$, $|\partial B| \leq |\partial A|$, and B is i-binary-compressed for all $1 \leq i \leq n$. Then by Lemma 2.50, we are done, since if B is not initial segment, then clearly we have $|\partial B| \geq |\partial C|$, since in that case $C = \mathbb{P}([n-1])$.

Remark 2.53 It is vital in the proof, and of Harper's theorem, that the extremal sets, i.e. the i-sections of the compression (in dimension $n-1$) were nested.

Definition 2.54 The **isoperimetric number** of a graph G is

$$i(G) := \min\left\{ \frac{|\partial A|}{|A|} : A \subseteq G, \frac{|A|}{|G|} \leq \frac{1}{2} \right\}.$$

$|\partial A|/|A|$ can be thought as the average out-degree of A.

Corollary 2.55 We have $i(Q_n) = 1$.

Proof (Hints). Show $\leq$ and $\geq$. □

Proof. Taking $A = \mathbb{P}(n-1)$ shows that $i(Q_n) \leq 1$. To show $i(Q_n) \geq 1$, by the above theorem, we just need to show that if C is an initial segment of binary with $|C| \leq 2^{n-1}$, then $|\partial C| \geq |C|$. But in this case, $C \subseteq \mathbb{P}(n-1)$, so certainly $|\partial C| \geq |C|$. □

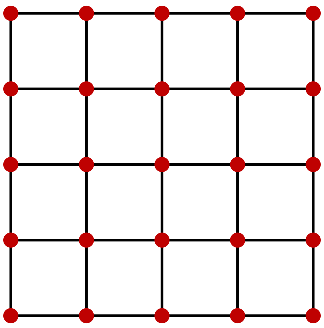
2.3. Inequalities in the grid

Definition 2.56 For $k \geq 2$ and $n \in \mathbb{N}$, the **grid** is the graph on $[k]^n$ in which x is joined to y if

$$\exists 1 \leq i \leq n : |x_i - y_i| = 1 \text{ and } \forall j \neq i, \quad x_j = y_j.$$

"The distance is the ℓ_1 distance". Note that for $k = 2$, this is precisely the definition of Q_n.

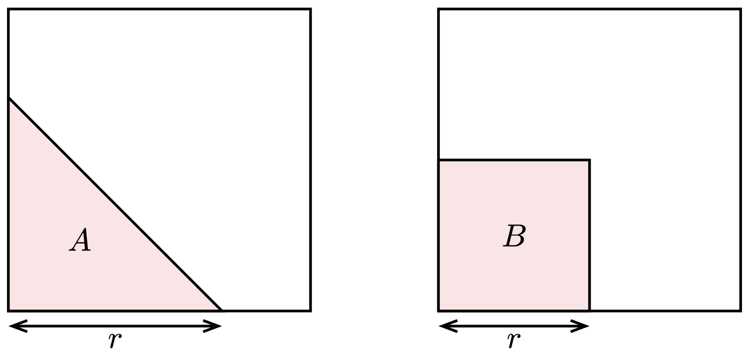
Diagram 2.57



The grid $[5]^2$

Notation 2.58 For a point x in the grid on $[k]^n$, write $|x|$ for $\sum_{i=1}^n |x_i| = \|x\|_{\ell_1}$. So x is joined to y in the grid on $[k]^n$ iff $\|x-y\|_{\ell_1} = 1$.

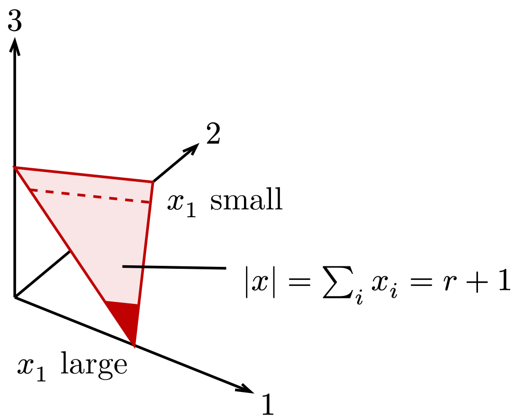
Example 2.59 Which sets $A \subseteq [k]^n$ (of a given size) minimise $|N(A)|$?



In the diagram for $[k]^2$, $|b(A)| \approx r \approx \sqrt{2|A|}$ and $|b(B)| = 2r = 2\sqrt{|B|}$ suggests we "go up in levels" according to $|x| = \sum_{i=1}^{n} |x_i|$, so we'd take $\{x \in [k]^n : |x| \leq r\}$. If

$$|\{x \in [k]^n : |x| \leq r\}| < |A| < |\{x \in [k]^n : |x| \leq r+1\}|,$$

then a reasonable guess is to take $A = \{x \in [k]^n : |x| \leq r\}$ together with some points with x with $|x| = r+1$. As suggested in the diagram below, we should take points while obeying the motto "keep x_1 large":



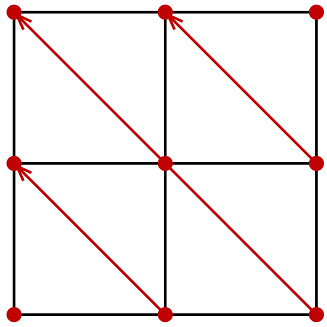
This suggests the following definition:

Definition 2.60 The **simplicial order** on the grid $[k]^n$ defines $x < y$ if either $|x| < |y|$, or $|x| = |y|$ and $x_i > y_i$, where $i = \min\{j \in [n] : x_j \neq y_j\}$.

Note that this definition agrees with the definition of simplicial order on the cube (i.e. when $k = 2$).

Example 2.61 The elements of $[3]^2$ in ascending simplicial order are

$$(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), (3,2), (2,3), (3,3).$$



The elements of $[4]^3$ in ascending simplicial order are

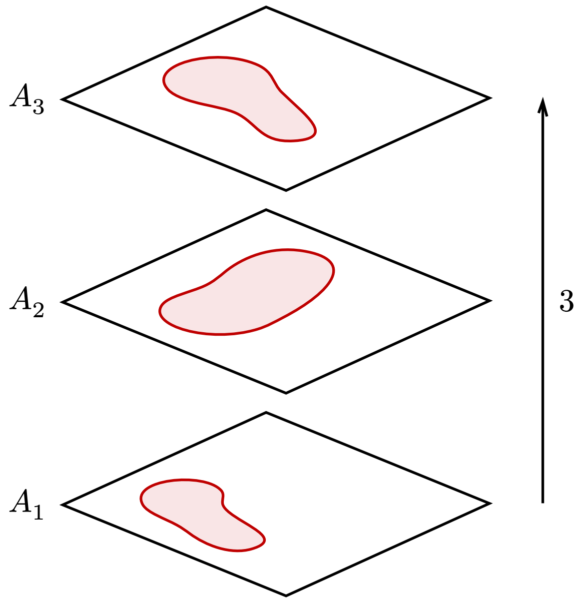
$$(1,1,1),(2,1,1),(1,2,1),(1,1,2),(3,1,1),(2,2,1),(2,1,2),(1,3,1),(1,2,2),(1,1,3),$$
$$(4,1,1),(3,2,1),\ldots$$

Definition 2.62 For $A \subseteq [k]^n$, $n \geq 2$, and $1 \leq i \leq n$, the ***i*-sections** of A are the sets

$$A_j^{(i)} = A_j := \{x \in [k]^{n-1} : (x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_{n-1}) \in A\} \subseteq [k]^{n-1}.$$

for each $1 \leq j \leq k$.

Diagram 2.63



The 3-sections of $A \subseteq [k]^3$

Definition 2.64 The ***i*-compression** of $A \subseteq [k]^n$ is the set $C_i(A) \subseteq [k]^n$ which is defined by its i-sections: $C_i(A)_j$ is the initial segment of simplicial on $[k]^{n-1}$ of size $|A_j|$, for each $1 \leq j \leq k$.

We have $|C_i(A)| = |A|$.

Definition 2.65 $A \subseteq [k]^n$ is ***i*-compressed** if $C_i(A) = A$.

Theorem 2.66 (Vertex-isoperimetric Inequality in the Grid) Let $A \subseteq [k]^n$ and C be the initial segment of simplicial on $[k]^n$ with $|C| = |A|$. Then $|N(C)| \leq |N(A)|$. In particular,

$$|A| \geq |\{x \in [k]^n : |x| \leq r\}| \implies |N(A)| \geq |\{x \in [k]^n : |x| \leq r+1\}|.$$

Proof (Hints).

- Use induction on n.
- Prove that $|N(C_i(A))| \leq |N(A)|$ by writing the i-section $N(A)_j^{(i)}$ as a union of three sets, doing the same for $N(C_i(A))_j^{(i)}$, and using the fact that these three sets (for $C_i(A)$) are nested (why?).

- Let $B \subseteq [k]^n$, $|B| = |A|$ and $|N(B)| \leq |N(A)|$, and let B be i-compressed for all $1 \leq i \leq n$ (find an expression to minimise which will imply B has this property).
- Case $n = 2$:
  - Let $r = \min\{|x| : x \notin B\}$, $s = \max\{|x| : x \in B\}$.
  - Explain why $r \leq s + 1$ and that we can assume $r \leq s$.
  - Show that if $r = s$, then $|N(C)| \leq |N(B)|$.
  - Explain why $\{x \in [k]^n : |x| = s\} \subseteq B$ implies $\{x \in [k]^n : |x| = r\} \subseteq B$, reason that this would be a contradiction. Deduce that there exist $y \in B$, $y' \notin B$ such that $|y| = |y'| = s$ and $y' = y \pm (1, -1)$.
  - By a similar argument, show that there exist $x \notin B$, $x' \in B$ with $|x| = |x'| = r$ and $x' = x \pm (1, -1)$.
  - Consider B' which is obtained from B by adding an appropriate element and removing an appropriate element.
  - Reason that $|N(B')| \leq |N(B)|$, contradicting the minimality of B (it helps to draw a diagram).
- Case $n \geq 3$:
  - For $1 \leq i \leq n-1$ and $x \in B$ with $x_n > 1$ and $x_i < k$ explain why $x - e_n + e_i \in B$.
  - Considering the n-sections of B, explain why $N(B)_j \subseteq B_{j-1}$ and hence that $N(B)_j = B_{j-1}$.
  - Use this to show that $|N(B)| = |B| - |B_k| + |N(B_1)|$, and similarly for C.
  - Show that $|B_k| \leq |C_k|$, by defining a set $D \subseteq [k]^n$ by its n-sections: $D_k := B_k$, and $D_{j-1} = N(D_j)$ for all j, and showing that $D \subseteq C$.
  - Show that $|B_1| \geq |C_1|$, by defining a set $E \subseteq [k]^n$ by its i-sections: $E_1 := B_1$, $E_j = \{x \in [k]^{n-1} : N(\{x\}) \subseteq E_{j-1}\}$, and showing that $C \subseteq E$.
  - Conclude that $|N(B_1)| \geq |N(C_1)|$.

□

Proof. By induction on n. The case $n = 1$ follows since if $A \subseteq [k]^1$ and $A \neq \emptyset, [k]^1$, then $|N(A)| \geq |A| + 1 = |N(C)|$. Now given $n > 1$, and $A \subseteq [k]^n$, fix $1 \leq i \leq n$, we claim that $|N(C_i(A))| \leq |N(A)|$.

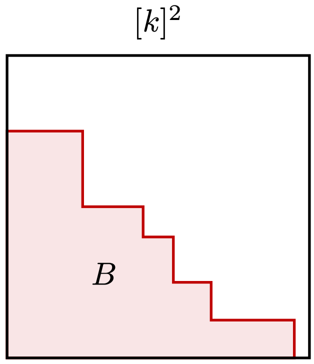
Proof of claim. Write $B = C_i(A)$. For any $1 \leq j \leq k$, we have

$$N(A)_j = \underbrace{N(A_j)}_{\text{from } x_i = j} \cup \underbrace{A_{j-1}}_{\text{from } x_i = j-1} \cup \underbrace{A_{j+1}}_{\text{from } x_i = j+1}$$

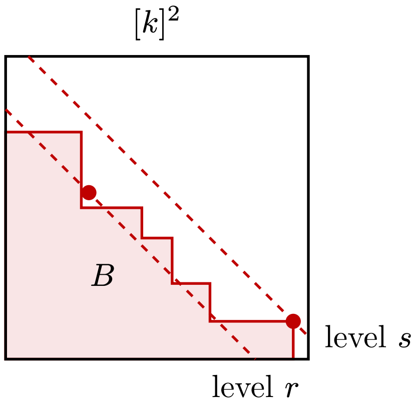
where we set $A_0 = A_{k+1} = \emptyset$. Similarly, $N(B)_j = N(B_j) \cup B_{j-1} \cup B_{j+1}$. Now, $|B_{j-1}| = |A_{j-1}|$ and $|B_{j+1}| = |A_{j+1}|$ by definition, and $|N(B_j)| \leq |N(A_j)|$ by the induction hypothesis. But the sets B_{j-1}, B_{j+1} and $N(B_j)$ are nested, as each is an initial segment of simplicial on $[k]^{n-1}$ (since neighbourhood of initial segment of simplicial is initial segment of simplicial). Hence $|N(B)_j| \leq |N(A)_j|$ for each $1 \leq j \leq k$, thus $|N(B)| \leq |N(A)|$. This proves the claim. □

Among all $B \subseteq [k]^n$ with $|B| = |A|$ and $|N(B)| \leq |N(A)|$, pick one with $\sum_{x \in B}(\text{position of } x \text{ in simplicial})$ minimal. Then B is i-compressed for all $1 \leq i \leq n$. We consider the following cases:

- Case $n = 2$: what we know is precisely that B is a down-set ($A \subseteq [k]^n$ is a **down-set** if $\forall x \in A, \forall y \in [k]^n, (y_i \leq x_i \ \forall 1 \leq i \leq n) \implies y \in A$.)



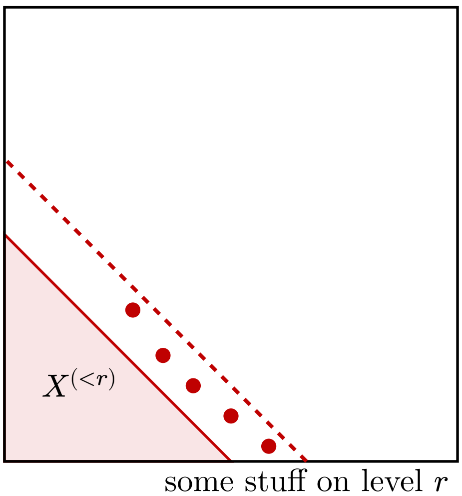
Let $r = \min\{|x| : x \notin B\}$ and $s = \max\{|x| : x \in B\}$. We have that $r \leq s+1$ by definition. We may assume that $r \leq s$, since if $r = s+1$, then $B = \{x : |x| \leq r-1\}$ which is an initial segment of simplicial, hence $B = C$.



If $r = s$, then

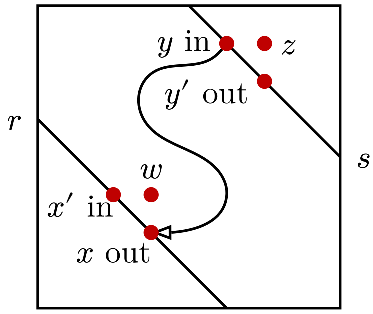
$$\{x \in [k]^n : |x| \leq r-1\} \subseteq B \subseteq \{x \in [k]^n : |x| \leq r\},$$

so clearly $|N(C)| \leq |N(B)|$.



Case when $r = s$

We cannot have $\{x \in [k]^n : |x| = s\} \subseteq B$ because then also $\{x \in [k]^n : |x| = r\} \subseteq B$ (as B is a down-set).



Case when $r < s$

So there are y, y' with $|y| = |y'| = s$, $y \in B$, $y' \notin B$, and $y' = y \pm (e_1 - e_2)$ (where $e_1 = (1, 0)$, $e_2 = (0, 1)$ are the standard basis vectors). Similarly, we cannot have $\{x \in [k]^n : |x| = r\} \cap B = \emptyset$, because then $\{x \in [k]^n : |x| = s\} \cap B = \emptyset$ (since B is a down-set): contradiction. So there are x, x' with $|x| = |x'| = r$, $x \notin B$, $x' \in B$, and $x' = x \pm (e_1 - e_2)$. Now let $B' = B \cup \{x\} \setminus \{y\}$. From B we lost at least one point in the neighbourhood (namely the unique point z which is joined to both y and y') and gained at most one point (the only point we might gain is the unique point w which is joined to both x and x'), so $|N(B')| \leq |N(B)|$, but this contradicts the minimality of B.

- Case $n \geq 3$: for any $1 \leq i \leq n - 1$ and any $x \in B$ with $x_n > 1$ and $x_i < k$, we have $x - e_n + e_i \in B$, since $x - e_n + e_i < x$ in simplicial and B is j-compressed for any $j \neq i, n$. So, considering the n-sections of B, we have $N(B_j) \subseteq B_{j-1}$ for all $j = 2, \ldots, k$. Recall that $N(B)_j = N(B_j) \cup B_{j+1} \cup B_{j-1}$. So in fact, $N(B)_j = B_{j-1}$ for all $j \geq 2$. Thus

$$|N(B)| = \underbrace{|B_{k-1}|}_{\text{level } k} + \underbrace{|B_{k-2}|}_{\text{level } k-1} + \cdots + \underbrace{|B_1|}_{\text{level } 2} + \underbrace{|N(B_1)|}_{\text{level } 1} = |B| - |B_k| + |N(B_1)|.$$

  Similarly, $|N(C)| = |C| - |C_k| + |N(C_1)|$. So to show $|N(C)| \leq |N(B)|$, it is enough to show that $|B_k| \leq |C_k|$ and $|B_1| \geq |C_1|$ (since B_1, C_1 and their neighbourhoods are initial segments of simplicial).

  $|B_k| \leq |C_k|$: define a set $D \subseteq [k]^n$ by its n-sections as follows: let $D_k := B_k$, and for $j = k - 1, k - 2, \ldots, 1$, set $D_j := N(D_{j+1})$. Then $D \subseteq B$, so $|D| \leq |B|$. Also, D is an initial segment of simplicial, since B_k is an initial segment of simplicial, and so all n-sections of D are as well. So in fact, $D \subseteq C$, whence $|B_k| = |D_k| \leq |C_k|$.

  $|B_1| \geq |C_1|$: define a set $E \subseteq [k]^n$ as follows: set $E_1 = B_1$ and for $j = 2, 3, \ldots, k$, set $E_j = \left\{x \in [k]^{n-1} : N(\{x\}) \subseteq E_{j-1}\right\}$, so E_j is the biggest set whose neighbourhood is contained in E_{j-1}. Then $B \subseteq E$, so $|E| \geq |B|$. Also, E is an initial segment of simplicial. So $C \subseteq E$, whence $|B_1| = |E_1| \geq |C_1|$.

Corollary 2.67 Let $A \subseteq [k]^n$ and $|A| \geq |\{x \in [k]^n : |x| \leq r\}|$. Then $\left|N^j(A)\right| \geq |\{x \in [k]^n : |x| \leq r + j\}|$ for all j.

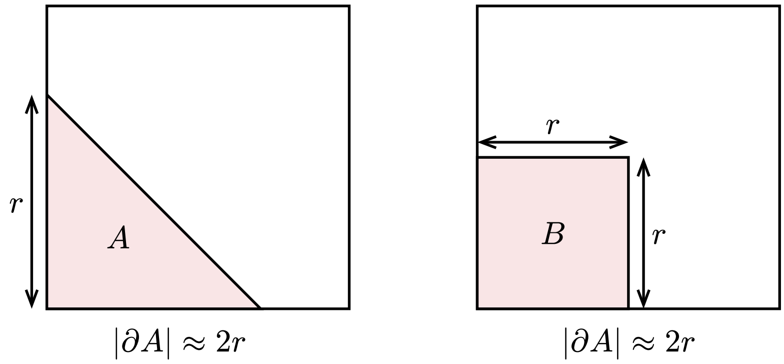
Proof (Hints). Trivial by above. $\square$

Proof. By induction, using above. $\square$

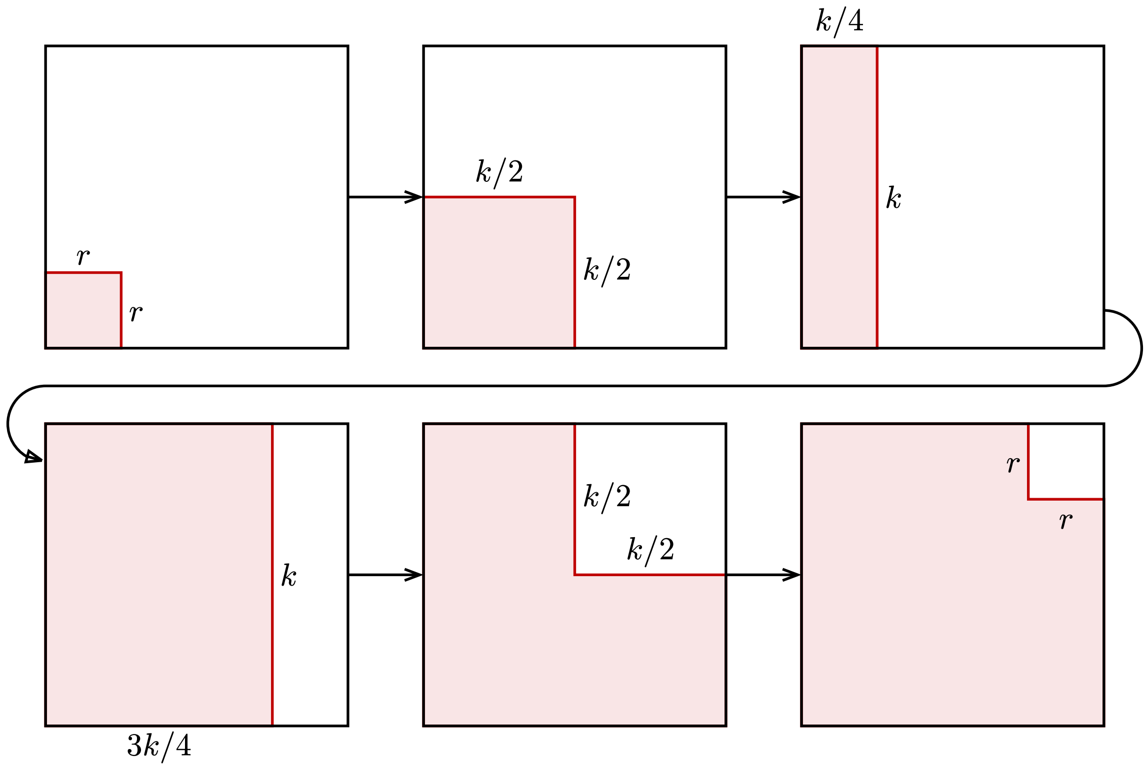
Remark 2.68 We can check from the above corollary that, for k fixed, the sequence of grids $\{[k]^n : n \in \mathbb{N}\}$ is a Levy family.

2.4. The edge-isoperimetric inequality in the grid

Example 2.69 Which set $A \subseteq [k]^n$ of given size should we take to minimise $|\partial A|$?



The diagram above for $[k]^2$ suggests squares are best. However, the diagram below shows we have "phase transitions" at $|A| \approx k^2/4$ and $|A| \approx 3k^2/4$. So the extremal sets are not nested.



This seems to rule out all our compression methods. In $[k]^3$:

- Start with $[a]^3$,
- then the square columns $[a]^2 \times [k]$,
- then the "half spaces" $[a] \times [k]^2$,
- then the complements of the square columns,
- then the complements of the cubes.

Generalising, in $[k]^n$, up to $|A| = k^n/2$, we get $n-1$ of these "phase transitions".

Theorem 2.70 (Edge-isoperimetric Inequality in the Grid) Let $A \subseteq [k]^n$. If $|A| \leq k^n/2$, then

$$|\partial A| \geq \min\big\{d|A|^{1-1/d}k^{n/d-1} : 1 \leq d \leq n\big\}.$$

Proof (Hints). Non-examinable. □

Proof. Non-examinable. □

Remark 2.71 Note that if $A = [a]^d \times [k]^{n-d}$, then

$$|\partial A| = da^{d-1}k^{n-d} = d|A|^{1-1/d}k^{n/d-1}.$$

So the Edge-isoperimetric Inequality in the Grid says that some set of the form $[a]^d \times [k]^{n-d}$ minimises the edge boundary.

Remark 2.72 Very few isoperimetric inequalities are known (even approximately), e.g. "iso in a layer": in a graph $X^{(r)}$, with x, y joined if $|x \cap y| = r - 1$. This is open. A nice special case is $r = n/2$, where it is conjectured that balls are best, i.e. sets of the form $\big\{x \in [2r]^{(r)} : |x \cap [r]| \geq t\big\}$.

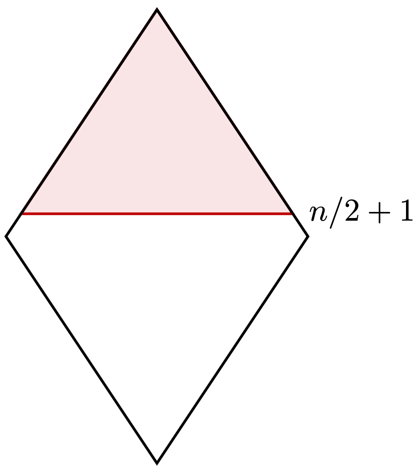
3. Intersecting families

3.1. t-intersecting families

Definition 3.1 $A \subseteq \mathbb{P}(X)$ is ***t*-intersecting** if

$$\forall x, y \in A, \quad |x \cap y| \geq t.$$

Example 3.2 How large can a t-intersecting family be? For $t = 2$, we could take $\{x \subseteq X : 1, 2 \in x\}$, which has size $\frac{1}{4} \cdot 2^n$, but better is $\{x \subseteq X : |x| \geq n/2 + 1\}$, which has size $\approx \frac{1}{2} \cdot 2^n$.



Theorem 3.3 (Katona's t-intersecting Theorem) Let $A \subseteq \mathbb{P}(X)$ be t-intersecting, where $n \equiv t \bmod 2$. Then

$$|A| \leq \big|X^{(\geq (n+t)/2)}\big|.$$

(The condition $n \equiv t \bmod 2$ is not necessary but simplifies the proof.)

Proof (Hints).

- Show that $N^{t-1}(A)$ is disjoint from $\overline{A} := \{y^c : y \in A\}$.
- Assuming the negation of the theorem statement, show that

$$\left|N^{t-1}(A)\right| \geq \left|X^{(\geq (n-t)/2+1)}\right|,$$

  and derive a contradiction (find a strict lower bound for the size of $\overline{A}$).

$\square$

Proof. For any $x, y \in A$, we have $|x \cap y| \geq t$, so $d(x, y^c) \geq t$. Writing $\overline{A} = \{y^c : y \in A\}$, we have $d(A, \overline{A}) \geq t$, i.e. $N^{t-1}(A)$ is disjoint from $\overline{A}$. Suppose for a contradiction $|A| > \left|X^{(\geq (n+t)/2)}\right|$. Then by Harper, we have

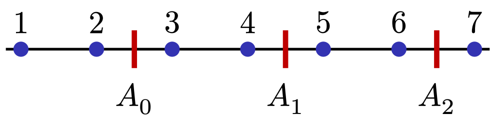
$$\left|N^{t-1}(A)\right| \geq \left|X^{(\geq (n+t)/2-(t-1))}\right| = \left|X^{(\geq (n-t)/2+1)}\right|.$$

But $N^{t-1}(A)$ is disjoint from $\overline{A}$ which has size $\left|\overline{A}\right| = |A| > \left|X^{(\leq (n-t)/2)}\right|$, contradicting $\left|N^{t-1}(A)\right| + \left|\overline{A}\right| \leq 2^n$. $\square$

Example 3.4 What about t-intersecting A with $A \subseteq X^{(r)}$? We might guess that the best is $A_0 = \left\{x \in X^{(r)} : [t] \subseteq x\right\}$. We could also try $A_\alpha = \left\{x \in X^{(r)} : |x \cap [t+2\alpha]| \geq t + \alpha\right\}$ for $\alpha = 1, \ldots, r-t$.

- For 2-intersecting families in $[7]^{(4)}$: $|A_0| = \binom{5}{2} = 10$, $|A_1| = 1 + \binom{4}{3}\binom{3}{1} = 13$, $|A_2| = \binom{6}{4} = 15$.
- For 2-intersecting families in $[8]^{(4)}$: $|A_0| = \binom{6}{2} = 15$, $|A_1| = 1 + \binom{4}{3}\binom{4}{1} = 17$, $|A_2| = \binom{6}{4} = 15$.
- For 2-intersecting families in $[9]^{(4)}$: $|A_0| = \binom{7}{2} = 21$, $|A_1| = 1 + \binom{4}{3}\binom{5}{1} = 21$, $|A_2| = \binom{6}{4} = 15$.

Note that A_0 grows quadratically, A_1 grows linearly, A_2 is constant, so A_0 is the largest of these for large n.



Theorem 3.5 (Second Erdos-Ko-Rado Theorem) Let $X = [n]$ and let $A \subseteq X^{(r)}$ be t-intersecting. Then, for n sufficiently large, we have $|A| \leq |A_0| = \binom{n-t}{r-t}$.

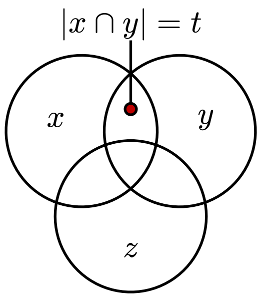
Proof (Hints).

- Show by contradiction that a maximal t-intersecting family $A' \supseteq A$ has $x, y \in A'$ with $|x \cap y| = t$.
- Explain why we can assume that there exists $z \in A'$ with $x \cap y \not\subseteq z$, and hence each $w \in A'$ meets $x \cup y \cup z$ in $\geq t + 1$ points.
- Show that $|A'|$ is bounded above by a polynomial in n of degree $r - t - 1$.

$\square$

Proof. Idea: "A_0 has $r - t$ degrees of freedom".

Extend A to a maximal t-intersecting family A', trivially $|A| \leq |A'|$. There exist $x, y \in A'$ with $|x \cap y| = t$ (if not, then by maximality, we have that $\forall x \in A', \forall i \in x, \forall j \notin x$, $x \setminus i \cup j \in A'$; repeating this, we have $A' = X^{(r)}$: contradiction). We may assume that there exists $z \in A'$ with $x \cap y \nsubseteq z$; otherwise, all $z \in A'$ contain the t-set $x \cap y \subseteq z$, whence $|A'| \leq \binom{n-t}{r-t} = |A_0|$. So each $w \in A'$ must meet $x \cup y \cup z$ in $\geq t+1$ points.



Thus

$$|A'| \leq \underbrace{2^{3r}}_{w \text{ on } x \cup y \cup z} \cdot \underbrace{\left(\binom{n}{r-t-1} + \binom{n}{r-t-2} + \cdots + \binom{n}{0}\right)}_{w \text{ off } x \cup y \cup z}$$

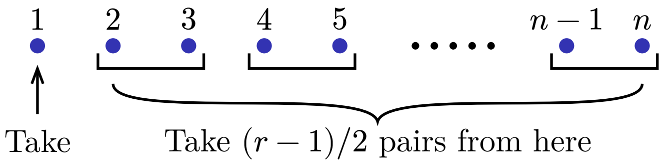
which is a polynomial in n of degree $r-t-1$, so is eventually smaller than $|A_0| = \binom{n-t}{r-t}$, a polynomial in n of degree $r-t$. $\square$

Remark 3.6 The bound we obtain for n in the Second Erdos-Ko-Rado Theorem would be $\geq (16r)^r$ (crude) or $2tr^3$ (careful).

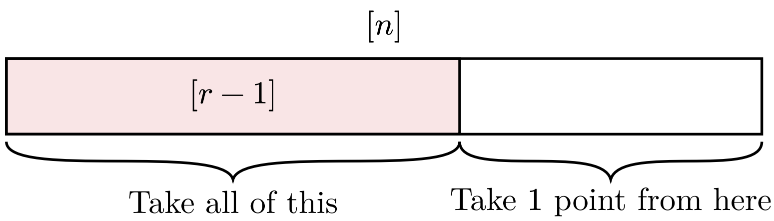
3.2. Modular intersections

Example 3.7 For intersecting families, we ban $|x \cap y| = 0$. What if we banned $|x \cap y| = 0 \bmod k$ for some $k \in \mathbb{N}$?

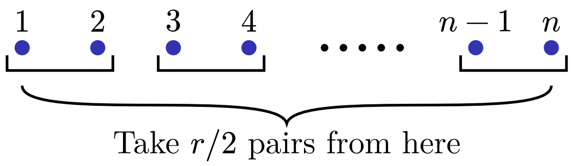
For example, for $k = 2$, we want $A \subseteq X^{(r)}$ with $|x \cap y|$ odd for all $x \neq y \in A$. When r is odd, we can achieve $|A| = \binom{\lfloor (n-1)/2 \rfloor}{(r-1)/2}$ by the diagram below.



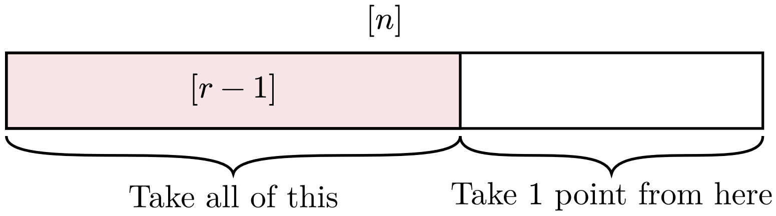
For r odd, if we want $|x \cap y|$ even for all $x \neq y \in A$, we can achieve $n - r + 1$ by the diagram below, but this is only linear in n.



Example 3.8 Similarly, for r even, if we want $|x \cap y|$ even for all $x \neq y \in A$, we can achieve $|A| = \binom{\lfloor n/2 \rfloor}{r/2}$ by the diagram below.



If we want $|x \cap y|$ odd for all $x \neq y \in A$, can achieve $n - r + 1$ by the diagram below.



It seems to be that banning $|x \cap y| = r (\bmod 2)$ forces the family to be very small (polynomial in n; in fact, a linear polynomial). Remarkably, we cannot beat linear.

Proposition 3.9 Let r be odd and $A \subseteq X^{(r)}$. If $|x \cap y|$ is even for all $x \neq y \in A$, then $|A| \leq n$.

Proof (Hints). Identify each $x \in \mathbb{P}(X)$ with a point $\overline{x}$ in an appropriate vector space, and by considering dot products, show that $\{\overline{x} : x \in A\}$ is linearly independent. □

Proof. Idea: find $|A|$ linearly independent vectors in a vector space of dimension n, namely Q_n.

View $\mathbb{P}(X)$ as $\mathbb{F}_2^n$, the n-dimensional vector space over $\mathbb{F}_2$, by identifying each $x \in \mathbb{P}(X)$ with $\overline{x}$, its characteristic sequence (where we count from the left, so $\{1, 3, 4\} \leftrightarrow$ 1011000...0). Then we have $\overline{x}.\overline{x} \neq 0$ for all $x \in A$ (as r is odd). Also, $\overline{x}.\overline{y} = 0$ for all $x \neq y \in A$, as $|x \cap y|$ is even. Hence $\{\overline{x} : x \in A\}$ is linearly independent (if $\sum_i \lambda_i \overline{x_i}$, dot with $\overline{x_j}$ to get $\lambda_j = 0$). So $|A| \leq n$. □

Corollary 3.10 Hence also, if $A \subseteq X^{(r)}$ with r even with $|x \cap y|$ odd for all $x \neq y \in A$, then $|A| \leq n + 1$.

Proof (Hints). Use the above proposition. □

Proof. Just add $n + 1$ to each $x \in A$ and apply above proposition. □

This $\bmod 2$ behaviour generalises: namely, allowing s values for $|x \cap y| \bmod p$ implies that $|A|$ is bounded above by a polynomial of degree s.

Theorem 3.11 (Frankl-Wilson) Let p be prime and $A \subseteq X^{(r)}$. Suppose that for all $x \neq y \in A$, we have $|x \cap y| \equiv \lambda_i \bmod p$ for some i, where $s \leq r$ and $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}$ with no $\lambda_i \equiv r \bmod p$. Then $|A| \leq \binom{n}{s}$.

Proof (Hints).

- For each $i \leq j$, let $M(i, j)$ be the $\binom{n}{i} \times \binom{n}{j}$ matrix with rows indexed by $X^{(i)}$, columns indexed by $X^{(j)}$, with

$$M(i, j)_{xy} = \begin{cases} 1 & \text{if } x \subseteq y \\ 0 & \text{otherwise} \end{cases}, \quad x \in X^{(i)}, y \in X^{(j)}.$$

- Let V be the vector space over $\mathbb{R}$ spanned by the rows of $M(s,r)$.
- By finding an expression for each of its entries, show that $M(i,s)M(s,r) = \binom{r-i}{s-i}M(i,r)$.
- Let $M(i) = M(i,r)^T M(i,r)$. Explain why each row of each $M(i)$ is in V.
- Let $M = \sum_{i=0}^{s} a_i M(i)$, where the a_i are chosen so that $M_{xy} = (|x \cap y| - \lambda_1) \cdots (|x \cap y| - \lambda_s)$ (explain why each $a_i \in \mathbb{Z}$).
- Consider the submatrix of M spanned by the rows and columns corresponding to the elements of A.

□

Proof. Idea: try to find $|A|$ linearly independent points in a vector space of dimension $\binom{n}{s}$, by somehow "applying" the polynomial $(t-\lambda_1)\cdots(t-\lambda_s)$ to $|x \cap y|$.

For each $i \le j$, let $M(i,j)$ be the $\binom{n}{i} \times \binom{n}{j}$ matrix with rows indexed by $X^{(i)}$, columns indexed by $X^{(j)}$, with

$$M(i,j)_{xy} = \begin{cases} 1 & \text{if } x \subseteq y \\ 0 & \text{otherwise} \end{cases}, \quad x \in X^{(i)}, y \in X^{(j)}.$$

Let V be the vector space over $\mathbb{R}$ spanned by the rows of $M(s,r)$, so $\dim(V) \le \binom{n}{s}$. For $i \le s$, consider the matrix $M(i,s)M(s,r)$. Each row of this matrix belongs to V, as we have left-multiplied $M(s,r)$ by a matrix. For $x \in X^{(i)}$, $y \in X^{(r)}$,

$$(M(i,s)M(s,r))_{xy} = \text{number of } s\text{-sets } z \text{ with } x \subseteq z \text{ and } z \subseteq y = \begin{cases} 0 & \text{if } x \not\subseteq y \\ \binom{r-i}{s-i} & \text{if } x \subseteq y. \end{cases}$$

So $M(i,s)M(s,r) = \binom{r-i}{s-i}M(i,r)$. So all rows of $M(i,r)$ belong to V. Let $M(i) = M(i,r)^T M(i,r)$. Again, each row of this matrix is in V, since we have left-multiplied $M(i,r)$ by a matrix. For $x, y \in X^{(r)}$, we have

$$M(i)_{xy} = \text{number of } i\text{-sets } z \text{ with } z \subseteq x \text{ and } z \subseteq y = \binom{|x \cap y|}{i}.$$

Write the integer polynomial $(t-\lambda_1)\cdots(t-\lambda_s)$ as $\sum_{i=0}^{s} a_i \binom{t}{i}$ with all $a_i \in \mathbb{Z}$. This is possible since $t(t-1)\cdots(t-i+1) = i!\binom{t}{i}$. Let $M = \sum_{i=0}^{s} a_i M(i)$. Note each row of each $M(i)$ is in V. Then for all $x, y \in X^{(r)}$,

$$M_{xy} = \sum_{i=0}^{s} a_i \binom{|x \cap y|}{i} = (|x \cap y| - \lambda_1) \cdots (|x \cap y| - \lambda_s).$$

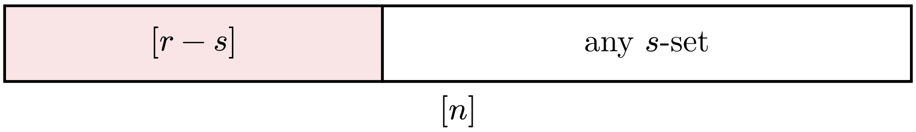
So the submatrix of M spanned by the rows and columns corresponding to the elements of A is

$$\begin{bmatrix} \not\equiv 0 \bmod p & & 0 \\ & \ddots & \\ 0 & & \not\equiv 0 \bmod p \end{bmatrix}$$

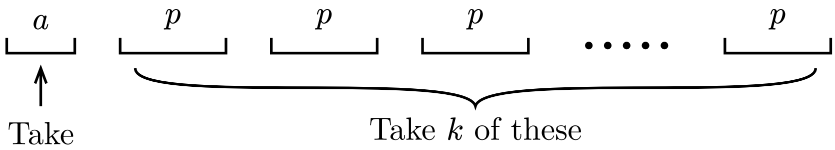
Hence the rows of M corresponding to A are linearly independent over $\mathbb{F}_p$, so also over $\mathbb{Z}$, so also over $\mathbb{Q}$, so also over $\mathbb{R}$. So $|A| \le \dim(V) = \binom{n}{s}$. $\square$

Remark 3.12

- The bound in Frankl-Wilson is a *polynomial* in n, even as r varies!
- The bound is essentially the best possible: we can achieve $|A| = \binom{n-r+s}{s} \approx \binom{n}{s}$ for large n, as shown in the diagram below.



- The condition $\lambda_i \not\equiv r \bmod p$ for all i is necessary: indeed, if $n = a + \lambda p$, $0 \le a \le p - 1$, then can have $A \subseteq X^{a+kp}$ with $|A| = \binom{\lambda}{k}$ and all $|x \cap y| \equiv a \bmod p$, but $\binom{\lambda}{k}$ is not a polynomial in n (as we can choose any k).



Remark 3.13 We do need p prime. Grolmusz constructed, for each n, a value of $r \equiv 0 \bmod 6$ and a family $A \subseteq [n]^{(r)}$ such that $\forall x \ne y \in A$, we have $|x \cap y| \not\equiv 0 \bmod 6$ and $|A| > n^{c \log n / \log\log n}$, which is not a polynomial in n.

Corollary 3.14 Let $A \subseteq [n]^{(r)}$ with $|x \cap y| \not\equiv r \bmod p$ for all $x \ne y \in A$, where $p < r$ is prime. Then $|A| \le \binom{n}{p-1}$.

Proof (Hints). Trivial by Frankl-Wilson. $\square$

Proof. We are allowed $p - 1$ values of $|x \cap y| \bmod p$, so done by Frankl-Wilson. $\square$

Two $(n/2)$-sets in $[n]$ typically meet in $\approx n/4$ points, but having the exact equality $|x \cap y| = n/4$ is very unlikely. But remarkably:

Corollary 3.15 Let p be prime, and $A \subseteq [4p]^{(2p)}$ with $|x \cap y| \ne p$ for all $x \ne y \in A$. Then $|A| \le 2\binom{4p}{p-1}$.

Proof (Hints). Remove all complements from A and use Corollary 3.14. $\square$

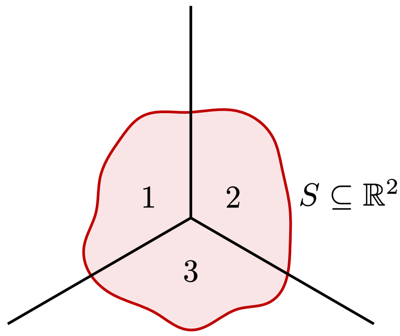
Proof. By halving $|A|$ if necessary, we may assume that no $x, x^c \in A$ (for any $x \in [4p]^{(2p)}$). Then for $x \ne y \in A$, $|x \cap y| \ne 0, p, 2p$, so $|x \cap y| \not\equiv 0 \bmod p$. So $|A| \le \binom{4p}{p-1}$ by Corollary 3.14. $\square$

Remark 3.16 $|x \cap y| \ne p$ for all $x \ne y \in A$ is a weak constraint, yet $2\binom{4p}{p-1}$ is a *tiny* (exponentially small) fraction of $\binom{4p}{2p}$. Indeed, $\binom{n}{n/2} \approx c \cdot 2^n/\sqrt{n}$, for some constant c, whereas $\binom{n}{n/4} \le 4e^{-n/32} 2^n$ by Proposition 2.25.

3.3. Borsuk's conjecture

Let $S \subseteq \mathbb{R}^n$ be bounded. How few pieces can we break S into, such that each piece has smaller diameter than that of S?

Diagram 3.17



A partition of S into three pieces

The example of a regular n-simplex in $\mathbb{R}^n$ ($n+1$ points, all at distance 1) shows that we may need $n+1$ pieces.

Conjecture 3.18 (Borsuk) $n+1$ pieces is always sufficient.

Borsuk is true when $n = 1$ (trivial), $n = 2$ (doable), $n = 3$ (hard), and also when S is a smooth convex body in $\mathbb{R}^n$ (e.g. sphere), or a symmetric ($x \in S \Rightarrow -x \in S$) convex body in $\mathbb{R}^n$ (e.g. octohedron).

However, in general, Borsuk is massively false:

Theorem 3.19 (Kahn, Kalai) For all $n \in \mathbb{N}$, there exists a bounded $S \subseteq \mathbb{R}^n$ such that to break S into pieces of smaller diameter, we need at least $C^{\sqrt{n}}$ pieces for some constant $C > 1$.

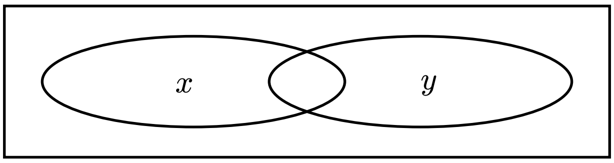
Remark 3.20 Our proof will show Borsuk is false for $n \geq 2000$.

Proof (Hints).

- Prove for all n of the form, $\binom{4p}{2}$ for p prime.
- For $x, y \in [n]^{(r)}$, find an expression for $\|x - y\|^2$ in terms of $|x \cap y|$.
- Identify $[n]$ with the edge set of an appropriate graph, and for each $x \in [4p]^{(2p)}$, let G_x be the complete bipartite subgraph with vertex classes x and x^c.
- Show that the number of edges in $G_x \cap G_y$ is $\left|G_x \cap G_y\right| = |x \cap y|^2 + (2p - |x \cap y|)^2$ and give the value of $|x \cap y|$ which minimises this.
- Let $S \subseteq [n]^{(4p^2)}$ be an appropriate set of size $|S| = \frac{1}{2}\binom{4p}{2p}$. Using Corollary 3.15, show that any subset $S' \subseteq S$ of smaller diameter than S has size at most $2\binom{4p}{p-1}$.
- Use Proposition 2.25 and the fact that $\binom{n}{n/2} \approx c \cdot 2^n / \sqrt{n}$ to conclude the result. $\square$

Proof. We will prove it for all n of the form $\binom{4p}{2}$ where p is prime. Then we are done for all $n \in \mathbb{N}$ (with a different constant C), e.g. because there exists prime p with $n/2 \leq p \leq n$. We'll find $S \subseteq Q_n \subseteq \mathbb{R}^n$. In fact, $S \subseteq [n]^{(r)}$ for some r. (These are genuine ideas). Since $S \subseteq [n]^{(r)}$, we have $\forall x, y \in S$,

$$\begin{aligned}\|x - y\|^2 &= \text{number of coordinates where } x, y \text{ differ} \\ &= |x \Delta y| = |x \setminus y| + |y \setminus x| = 2(r - |x \cap y|).\end{aligned}$$



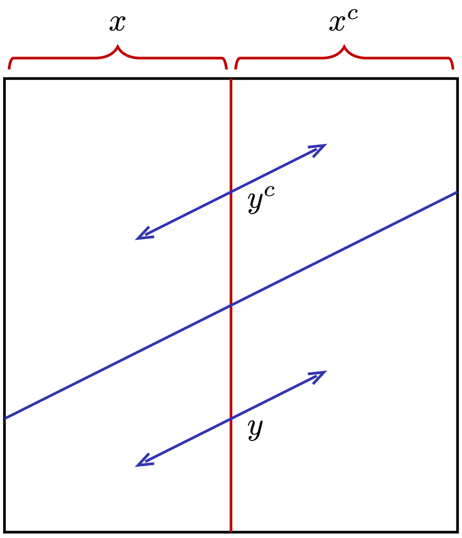
$[n]$

The diameter of S is $\max\{\|x-y\| : x, y \in S\}$, so we seek S with $\min\{|x \cap y| : x, y \in S\} = k$, where every subset $S' \subseteq S$ with $\min\{|x \cap y| : x, y \in S'\} > k$ is very small (so need many pieces).

Identify $[n]$ with the edge set of the complete graph K_{4p} on $4p$ points. For each $x \in [4p]^{(2p)}$, let G_x be the complete bipartite graph with vertex classes x, x^c. Let $S = \{G_x : x \in [4p]^{(2p)}\}$. So $S \subseteq [n]^{(4p^2)}$, and $|S| = \frac{1}{2}\binom{4p}{2p}$ (since $G_x = G_{x^c}$). Now, the number of edges in $G_x \cap G_y$ is

$$\begin{aligned} |G_x \cap G_y| &= |x \cap y| \cdot |x^c \cap y^c| + |x^c \cap y| \cdot |x \cap y^c| \\ &= |x \cap y|^2 + |x^c \cap y|^2 \\ &= d^2 + (2p-d)^2, \quad \text{where } d = |x \cap y|, \end{aligned}$$

which is minimised when $d = |x \cap y| = p$.



$[4p]$

Now let $S' \subseteq S$ have smaller diameter than that of S: $S' = \{G_x : x \in A\}$, where $A \subsetneq [4p]^{2p}$. So we must have that $\forall x \neq y \in A$, $|x \cap y| \neq p$ (as otherwise diameter of S' is equal to diameter of S). Thus $|A| \le 2\binom{4p}{p-1}$ by Corollary 3.15.

So by Proposition 2.25, the number of pieces needed is at least

$$\begin{aligned} \frac{\frac{1}{2}\binom{4p}{2p}}{2\binom{4p}{p-1}} &\ge \frac{c \cdot 2^{4p}/\sqrt{p}}{e^{-p/8}2^{4p}} && \text{for some } c \\ &\ge (c')^p && \text{for some } c' \\ &\ge (c'')^{\sqrt{n}} && \text{for some } c''. \end{aligned}$$

□