Contents

1. Non-classical logic

1.1. Intuitionistic logic

Idea: a statement is true if there is a proof of it. A proof of $\varphi \Rightarrow \psi$ is a "procedure" that can convert a proof of φ to a proof of ψ. A proof of $\neg\varphi$ is a proof that there is no proof of φ.

In particular, $\neg\neg\varphi$ is not always the same as φ.

Fact. The Law of Excluded Middle (LEM) ($\varphi \vee \neg\varphi$) is not (generally) intuitionistically valid.

Moreover, the Axiom of Choice is incompatible with intuitionistic set theory.

In intuitionistic logic, $\exists$ means an explicit element can be found.

Why bother with intuitionistic logic?
- Intuitionistic mathematics is more general, as we assume less (no LEM or AC).
- Several notions that are conflated in classical mathematics are genuinely different constructively.
- Intuitionistic proofs have a computable content that may be absent in classical proofs.
- Intuitionistic logic is the internal logic of an arbitrary topos.

We will inductively define a provability relation by enforcing rules that implement the BHK-interpretation.

Definition. A set is **inhabited** if there is a proof that it is non-empty.

Axiom (Choice - Intutionistic Version). Any family of inhabited sets admits a choice function.

Theorem (Diaconescu). The Law of Excluded Middle can be intutionistically deduced from the Axiom of Choice.

Proof (Hints).
- Proof should use Axioms of Separation, Extensionality and Choice.
- For proposition φ, consider $A = \{x \in \{0,1\} : \varphi \vee (x = 0)\}$ and $B = \{x \in \{0,1\} : \varphi \vee (x = 1)\}$.
- Show that we have a proof of $f(A) = 0 \vee f(A) = 1$, similarly for $f(B)$.
- Consider the possibilities that arise from above, show that they lead to either a proof of φ or a proof of $\neg\varphi$.

□

Proof.
- Let φ be a proposition. By the Axiom of Separation, the following are sets:

$$A = \{x \in \{0,1\} : \varphi \vee (x = 0)\},$$
$$B = \{x \in \{0,1\} : \varphi \vee (x = 1)\}.$$

- Since $0 \in A$ and $1 \in B$, we have a proof that $\{A, B\}$ is a family of inhabited sets, thus admits a choice function $f : \{A, B\} \to A \cup B$ by the Axiom of Choice.
- f satisfies $f(A) \in A$ and $f(B) \in B$ by definition.
- So we have $f(A) = 0$ or φ is true, and $f(B) = 1$ or φ is true. Also, $f(A), f(B) \in \{0, 1\}$.
- Now $f(A) \in \{0, 1\}$ means we have a proof of $f(A) = 0 \vee f(A) = 1$ and similarly for $f(B)$.
- There are the following possibilities:
  1. We have a proof that $f(A) = 1$, so $\varphi \vee (1 = 0)$ has a proof, so we must have a proof of φ.
  2. We have a proof that $f(B) = 0$, so $\varphi \vee (0 = 1)$ has a proof, so we must have a proof of φ.
  3. We have a proof that $f(A) = 0 \wedge f(B) = 1$, in which case we can prove $\neg\varphi$: assume there is a proof of φ, we can prove that $A = B$ (by the Axiom of Extensionality), in which case $0 = f(A) = f(B) = 1$: contradiction.
- So we can always specify a proof of φ or a proof of $\neg\varphi$.

$\square$

Notation. We write $\Gamma \vdash \varphi$ to mean that φ is a consequence of the formulae in the set Γ. Γ is called the **set of hypotheses or open assumptions**.

Notation. Notation for assumptions and deduction.

Definition. The rules of the **intuitionistic propositional calculus (IPC)** are:
- conjunction introduction,
- conjunction elimination,
- disjunction introduction,
- disjunction elimination,
- implication introduction,
- implication elimination,
- assumption,
- weakening,
- construction,
- and for any A,

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}.$$

as defined below.

Definition. The **conjunction introduction ($\wedge$-I)** rule:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}.$$

Definition. The **conjunction elimination ($\wedge$-E)** rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \lor B}, \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \lor B}.$$

Definition. The **disjunction introduction (∨-I)** rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \lor B}, \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \lor B}.$$

Definition. The **disjunction elimination (proof by cases) (∨-E)** rule:

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C \quad \Gamma \vdash A \lor B}{\Gamma \vdash C}.$$

Definition. The **implication/arrow introduction (→-I)** rule (note the similarity to the deduction theorem):

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}.$$

Definition. The **implication/arrow elimination (→-E)** rule (note the similarity to modus ponens):

$$\frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}.$$

Definition. The **assumption (Ax)** rule: for any A,

$$\frac{}{\Gamma, A \vdash A}$$

Definition. The **weakening** rule:

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}.$$

Definition. The **construction** rule:

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}.$$

Remark. We obtain classical propositional logic (CPC) from IPC by adding either:

- $\Gamma \vdash A \lor \neg A$:

$$\frac{}{\Gamma \vdash A \lor \neg A},$$

  or

- If $\Gamma, \neg A \vdash \perp$, then $\Gamma \vdash A$:

$$\frac{\Gamma, \neg A \vdash \perp}{\Gamma \vdash A}.$$

Notation. see scan

Definition. We obtain **intuitionistic first-order logic (IQC)** by adding the following rules to IPC for quantification:

- existental inclusion,
- existential elimination,
- universal inclusion,
- universal elimination

as defined below.

Definition. The **existential inclusion (∃-I)** rule: for any term t,

$$\frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x.\varphi(x)}.$$

Definition. The **existential elimination (∃-I)** rule:

$$\frac{\Gamma \vdash \exists x.\varphi \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi},$$

where x is not free in Γ or ψ.

Definition. The **universal inclusion (∀-I)** rule:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x.\varphi},$$

where x is not free in Γ.

Definition. The **universal exclusion (∀-E)** rule:

$$\frac{\Gamma \vdash \forall x.\varphi(x)}{\Gamma \vdash \varphi[t/x]},$$

where t is a term.

Definition. We define the notion of **discharging/closing** open assumptions, which informally means that we remove them as open assumptions, and append them to the consequence by adding implications. We enclose discharged assumptions in square brackets [] to indicate this, and add discharged assumptions in parentheses to the right of the proof. For example, →-I is written as

$$\begin{array}{c} \Gamma, [A] \\ \vdots \\ \dfrac{B}{\Gamma \vdash A \to B}(A) \end{array}$$

Example. A natural deduction proof that $A \wedge B \to B \wedge A$ is given below:

$$\dfrac{\dfrac{\frac{[A\wedge B]}{A} \quad \frac{[A\wedge B]}{B}}{B\wedge A}}{A\wedge B\to B\wedge A}(A \wedge B)$$

Example. A natural deduction proof of $\varphi \to (\psi \to \varphi)$ is given below (note clearly we must use $\to$-I):

$$\cfrac{\cfrac{[\varphi] \quad [\psi]}{\psi \to \varphi}}{\varphi \to (\psi \to \varphi)}$$

Example. A natural deduction proof of $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ (note clearly we must use $\to$-I):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\varphi \to (\psi \to \chi)] \quad [\varphi \to \psi] \quad [\varphi]}{\psi \to \chi \qquad \psi}}{\chi}}{\varphi \to \chi}}{(\varphi \to \psi) \to (\varphi \to \chi)}}{(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))}$$

Notation. If Γ is a set of propositions, φ is a proposition and $L \in \{\mathrm{IPC}, \mathrm{IQC}, \mathrm{CPC}, \mathrm{CQC}\}$, write $\Gamma \underset{L}{\vdash} \varphi$ if there is a proof of φ from Γ in the logic L.

Lemma. If $\Gamma \underset{\mathrm{IPC}}{\vdash} \varphi$, then $\Gamma, \psi \underset{\mathrm{IPC}}{\vdash} \varphi$ for any proposition ψ. If p is a primitive proposition (doesn't contain any logical connectives or quantifiers) and ψ is any proposition, then $\Gamma[\psi/p] \underset{\mathrm{IPC}}{\vdash} \varphi[\psi/p]$.

Proof. Induction on number of lines of proof (exercise). □

1.2. The simply typed λ-calculus

Definition. The set Π of **simple types** is generated by the grammar

$$\Pi \coloneqq U \mid \Pi \to \Pi$$

where U is a countable set of **type variables (primitive types)** together with an infinite set of V of **variables**. So Π consists of U and is closed under $\to$: for any $a, b \in \Pi$, $a \to b \in \Pi$.

Definition. The set Λ_Π of **simply typed λ-terms** is defined by the grammar

$$\Lambda_\Pi \coloneqq V \mid \lambda V : \pi \, . \, \Lambda_\Pi \mid \Lambda_\Pi \, \Lambda_\Pi$$

In the term $\lambda x : \tau . M$, x is a variable, τ is type and M is a λ-term. Forming terms of this form is called **λ-abstraction**. Forming terms of the form $\Lambda_\Pi \Lambda_\Pi$ is called **λ-application**.

Definition. A **context** is a set of pairs $\Gamma = \{x_1 : \tau_1, ..., x_n : \tau_n\}$ where the x_i are distinct variables and each τ_i is a type. So a context is an assignment of a type to each variable in a given set. Write C for the set of all possible contexts. Given a context $\Gamma \in C$, write $\Gamma, x : \tau$ for the context $\Gamma \cup \{x : \tau\}$ (if x does not appear in Γ).

The **domain** of Γ is the set of variables $\{x_1, ..., x_n\}$ that occur in it, and its **range**, $|\Gamma|$, is the set of types $\{\tau_1, ..., \tau_n\}$ that it manifests.

Definition. Recursively define the **typability relation** $\mid\vdash \subseteq C \times \Lambda_\Pi \times \Pi$ via:

1. For every context Γ, variable x not occurring in Γ and type τ, we have $\Gamma, x : \tau \mid\vdash x : \tau$.
2. For a context Γ, variable x not occurring in Γ, types $\sigma, \tau \in \Pi$, and λ-term. If $\Gamma, x : \sigma \mid\vdash M : \tau$, then $\Gamma \mid\vdash (\lambda x : \sigma.M) : (\sigma \to t)$.
3. Let Γ be a context, $\sigma, \tau \in \Pi$ be types, and $M, N \in \Lambda_\Pi$ be terms. If $\Gamma \mid\vdash M : (\sigma \to t)$ and $\Gamma \mid\vdash N : \sigma$, then $\Gamma \mid\vdash (MN) : \tau$.

Notation. We will refer to the λ-calculus of Λ_Π with this typability relation as $\lambda(\to)$.

Definition. A variable x occurring in a λ-abstraction $\lambda x : \sigma.M$ is **bound** and is **free** otherwise. A term with no free variables is called **closed**.

Definition. Terms M and N are **α-equivalent** if they differ only in the names of teh bound variables.

Definition. If M and N are λ-terms and x is a variable, then we define the **substitution of N for x in M** by the following rules:

- $x[x := N] = N$.
- $y[x := N] = N$.
- $(PQ)[x := N] = P[x := N]Q[x := N]$ for λ-terms P, Q.
- $(\lambda y : \sigma.P)[x := N] = \lambda y : \sigma.(P[x := N])$ for $x \neq y$ and y not free in N.

Definition. The **β-reduction** relation is the smallest relation $\underset{\beta}{\longrightarrow}$ on Λ_Π closed under the following rules:

- $(\lambda x : \sigma.P)Q \underset{\beta}{\longrightarrow} P[x := Q]$. The term being reduced is called a **β-redex**, and the result is called its **β-contraction**.
- If $P \underset{\beta}{\longrightarrow} P'$, then for all variables x and types $\sigma \in \Pi$, we have $\lambda x : \sigma.P \underset{\beta}{\longrightarrow} \lambda x : \sigma.P'$.
- If $P \underset{\beta}{\longrightarrow} P'$ and Z is a λ-term, then $PZ \underset{\beta}{\longrightarrow} P'Z$ and $Zp \underset{\beta}{\longrightarrow} ZP'$.

Definition. We define **β-equivalence**, $\underset{\beta}{\equiv}$, as the smallest equivalence relation containing $\underset{\beta}{\longrightarrow}$.

Example. We have $(\lambda x : \mathbb{Z}.(\lambda y : \tau.x))2 \underset{\beta}{\longrightarrow} (\lambda y : \tau.2)$.

Lemma (Free Variables Lemma). Let $\Gamma \mid\vdash M : \sigma$. Then

- If $\Gamma \subseteq \Gamma'$, then $\Gamma' \mid\vdash M : \sigma$.
- The free variables of M occur in Γ.

- There is a context $\Gamma^* \subseteq \Gamma$ comprising exactly the free variables in M, with $\Gamma^* \mid\vdash M : \sigma$.