Contents

1. Set systems

1.1. Chains and antichains

Note 1.1 The ideas in combinatorics often occur in the proofs, so it is advisable to learn the techniques used in proofs, rather than just learning the results and not their proofs.

Definition 1.2 Let X be a set. A **set system** on X (also called a **family of subsets of X**) is a collection $\mathcal{F} \subseteq \mathbb{P}(X)$.

Notation 1.3 $X^{(r)} := \{A \subseteq X : |A| = r\}$ denotes the family of subsets of X of size r.

Remark 1.4 Usually, we take $X = [n] = \{1, ..., n\}$, so $|X^{(r)}| = \binom{n}{r}$.

Notation 1.5 For brevity, we write e.g. $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$.

Definition 1.6 We can visualise $\mathbb{P}(X)$ as a graph by joining nodes $A \in \mathbb{P}(X)$ and $B \in \mathbb{P}(X)$ if $|A \Delta B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa.

This graph is the **discrete cube** Q_n.

Diagram 1.7

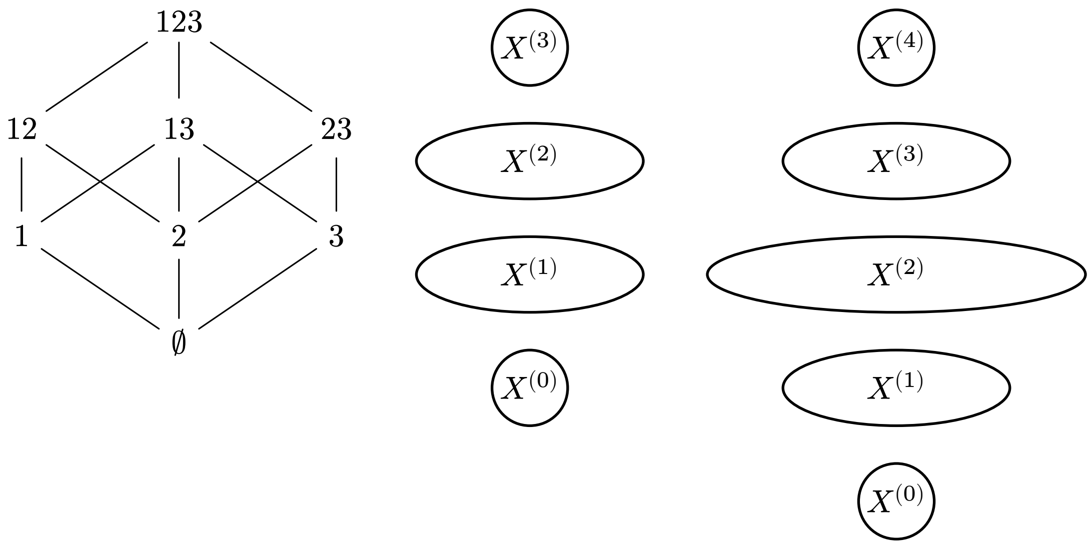


Figure 1: Q_3, Q_3, and Q_4.

Remark 1.8 Alternatively, we can view Q_n as an n-dimensional unit cube $\{0, 1\}^n$ by identifying e.g. $\{1, 3\} \subseteq [5]$ with 10100 (i.e. identify A with $\mathbb{1}_A$, the characteristic/indicator function of A).

Diagram 1.9

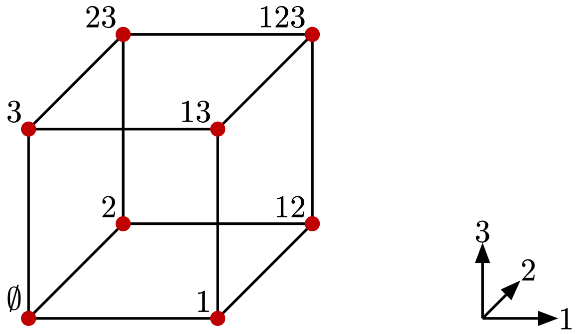


Figure 2: The cube Q_3 as the unit cube in $\mathbb{R}^3$

Definition 1.10 $\mathcal{F} \subseteq \mathbb{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{F}$, $A \subseteq B$ or $B \subseteq A$.

Example 1.11

- $\mathcal{F} = \{23, 1235, 123567\}$ is a chain.
- $\mathcal{F} = \{\emptyset, 1, 12, ..., [n]\} \subseteq \mathbb{P}([n])$ is a chain.

Definition 1.12 $\mathcal{F} \subseteq \mathbb{P}(X)$ is an **antichain** if $\forall A \neq B \in \mathcal{F}$, $A \nsubseteq B$.

Diagram 1.13

Figure 3: A chain and antichain.

Example 1.14

- $\mathcal{F} = \{23, 137\}$ is an antichain.
- $\mathcal{F} = \{1, ..., n\} \subseteq \mathbb{P}([n])$ is an antichain.
- More generally, $\mathcal{F} = X^{(r)}$ is an antichain for any r.

Proposition 1.15 A chain and an antichain can meet at most once.

Proof (Hints). Trivial. □

Proof. By definition. □

Proposition 1.16 A chain $\mathcal{F} \subseteq \mathbb{P}([n])$ can have at most $n + 1$ elements.

Proof (Hints). Trivial. □

Proof. For each $0 \leq r \leq n$, $\mathcal{F}$ can contain at most 1 r-set (set of size r). □

Theorem 1.17 (Sperner's Lemma) Let $\mathcal{F} \subseteq \mathbb{P}(X)$ be an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$, i.e. the maximum size of an antichain is achieved by the set of $X^{(\lfloor n/2 \rfloor)}$.

Proof (Hints).

- Let $r < \frac{n}{2}$.

- Let G be bipartite subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$.
- By considering an expression and upper bound for number of S-$\Gamma(S)$ edges in G for each $S \subseteq X^{(r)}$, show that there is a matching from $X^{(r)}$ to $X^{(r+1)}$.
- Reason that this induces a matching from $X^{(r)}$ to $X^{(r-1)}$ for each $r > \frac{n}{2}$.
- Reason that joining these matchings together, together with length 1 chains of subsets of $X^{(\lfloor n/2 \rfloor)}$ not included in a matching, result in a partition of $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, and conclude result from here.

$\square$

Proof. We use the idea: from "a chain meets each layer in ≤ 1 points, because a layer is an antichain", we try to decompose the cube into chains.

Diagram 1.18

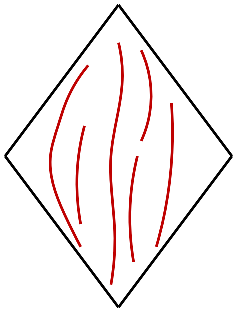


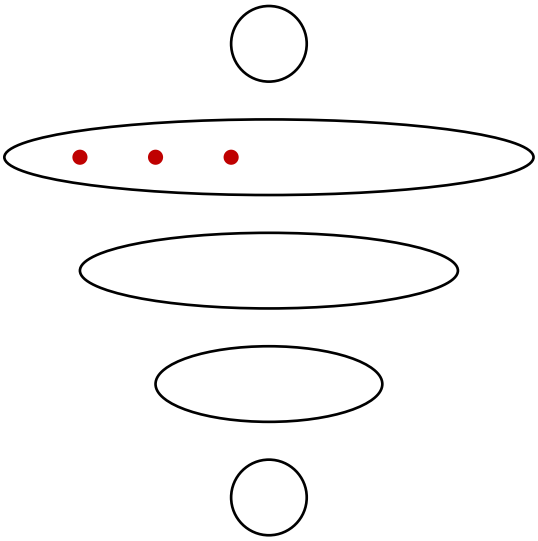
Figure 4: Decomposition of $\mathbb{P}(X)$ into chains.

In particular, we partition $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, so each subset of X appears exactly once in one chain. Then we are done (since to form an antichain, we can pick at most one element from each chain). To achieve this, it is sufficient to find:

- For each $r < \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r+1)}$ (a matching is a set of disjoint edges, one for each point in $X^{(r)}$).
- For each $r > \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r-1)}$.

Then put these matchings together to form a set of chains, each passing through $X^{(\lfloor n/2 \rfloor)}$.

Diagram 1.19



$X^{(4)}$

If a subset $X^{(\lfloor n/2 \rfloor)}$ has a chain passing through it, then this chain is unique. The subsets with no chain passing through form their own one-element chain. By taking complements, it is enough to construct the matchings just for $r < \frac{n}{2}$ (since a matching from $X^{(r)}$ to $X^{(r+1)}$ induces a matching from $X^{(n-r-1)}$ to $X^{(n-r)}$: there is a correspondence between $X^{(r)}$ and $X^{(n-r)}$ by taking complements, and taking complements reverse inclusion, so edges in the induced matching are guaranteed to exist).

Let G be the (bipartite) subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$. For any $S \subseteq X^{(r)}$, the number of S-$\Gamma(S)$ edges in G is $|S|(n-r)$ (counting from below) since there are $n-r$ ways to add an element. This number is $\leq |\Gamma(S)|\ (r+1)$ (counting from above), since $r+1$ ways to remove an element. Hence $|\Gamma(S)| \geq \frac{|S|\ (n-r)}{r+1} \geq |S|$ as $r < \frac{n}{2}$. So by Hall's theorem, since there is a matching from S to $\Gamma(S)$, there is a matching from $X^{(r)}$ to $X^{(r+1)}$. $\square$

Remark 1.20 The proof above doesn't tell us when we have equality in Sperner's Lemma.

Definition 1.21 For $\mathcal{F} \subseteq X^{(r)}$ $(1 \leq r \leq n)$, the **shadow** of $\mathcal{F}$ is the set of subsets which can be obtained by removing one element from a subset in $\mathcal{F}$:

$$\partial \mathcal{F} = \partial^- \mathcal{F} := \left\{ B \in X^{(r-1)} : B \subseteq \mathcal{F} \text{ for some } A \in \mathcal{F} \right\}.$$

Example 1.22 Let $\mathcal{F} = \{123, 124, 134, 137\} \in [7]^{(3)}$. Then $\partial \mathcal{F} = \{12, 13, 23, 14, 24, 34, 17, 37\}$.

Proposition 1.23 (Local LYM) Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq r \leq n$. Then

$$\frac{|\mathcal{F}|}{\binom{n}{r}} \leq \frac{|\partial \mathcal{F}|}{\binom{n}{r-1}}.$$

i.e. the proportion of the level occupied by $\partial \mathcal{F}$ is at least the proportion of the level occupied by $\mathcal{F}$.

Proof (Hints). Find equation and upper bound for number of $\mathcal{F}$-$\partial\mathcal{F}$ edges in Q_n. $\square$

Proof. The number of $\mathcal{F}$-$\partial\mathcal{F}$ edges in Q_n is $|\mathcal{F}|r$ (counting from above, since we can remove any of r elements from $|\mathcal{F}|$ sets) and is $\le |\partial\mathcal{F}|\ (n-r+1)$ (since adding one of the $n-r+1$ elements not in $A \in \partial\mathcal{F}$ to A may not result in a subset of $\mathcal{F}$).
Hence,

$$\frac{|\mathcal{F}|}{|\partial\mathcal{F}|} \le \frac{n-r+1}{r} = \binom{n}{r} \Big/ \binom{n}{r-1}. \qquad \square$$

Remark 1.24 For equality in Local LYM, we must have that $\forall A \in \mathcal{F}$, $\forall i \in A$, $\forall j \notin A$, we must have $(A - \{i\}) \cup \{j\} \in \mathcal{F}$, i.e. $\mathcal{F} = \emptyset$ or $X^{(r)}$ for some r.

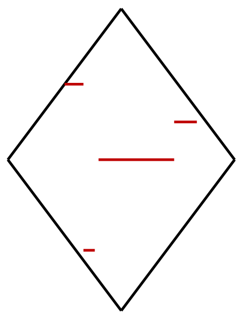
Notation 1.25 Write $\mathcal{F}_r$ for $\mathcal{F} \cap X^{(r)}$.

Theorem 1.26 (LYM Inequality) Let $\mathcal{F} \subseteq \mathbb{P}(X)$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{F} \cap X^{(r)}|}{\binom{n}{r}} \le 1,$$

i.e. the proportions of each layer occupied add to ≤ 1.

Diagram 1.27



Proof (Hints).

- Method 1: show the result for the sum $\sum_{r=k}^{n}$ by induction, starting with $k = n$. Use local LYM, and that $\partial\mathcal{F}_n$ and $\mathcal{F}_{n-1}$ are disjoint (and analogous results for lower levels).
- Method 2: let $\mathcal{C}$ be uniformly random maximal chain, find an expression for $\Pr(\mathcal{C} \text{ meets } \mathcal{F})$.
- Method 3: determine number of maximal chains in X, determine number of maximal chains passing through a fixed r-set, deduce maximal number of chains passing through $\mathcal{F}$.

$\square$

Proof. **Method 1**: "bubble down with local LYM". We trivially have that $\mathcal{F}_n/\binom{n}{n} \le 1$. $\partial\mathcal{F}_n$ and $\mathcal{F}_{n-1}$ are disjoint, as $\mathcal{F}$ is an antichain, so

$$\frac{|\partial\mathcal{F}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial\mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} \le 1.$$

So by local LYM,

$$\frac{|\mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} \leq 1.$$

Now, $\partial(\partial\mathcal{F}_n \cup \mathcal{F}_{n-1})$ and $\mathcal{F}_{n-2}$ are disjoint, as $\mathcal{F}$ is an antichain, so

$$\frac{|\partial(\partial\mathcal{F}_n \cup \mathcal{F}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

So by local LYM,

$$\frac{|\partial\mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

So

$$\frac{|\mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

Continuing inductively, we obtain the result.

Method 2: Choose uniformly at random a maximal chain $\mathcal{C}$ (i.e. $C_0 \subsetneq C_1 \subseteq \cdots \subsetneq C_n$ with $|C_r| = r$ for all r). For any r-set A, $\Pr(A \in \mathcal{C}) = 1/\binom{n}{r}$, since all r-sets are equally likely. So $\Pr(\mathcal{C} \text{ meets } \mathcal{F}_r) = |\mathcal{F}_r|/\binom{n}{r}$, since the events are disjoint. Thus, $\Pr(\mathcal{C} \text{ meets } \mathcal{F}) = \sum_{r=0}^{n} |\mathcal{F}_r|/\binom{n}{r} \leq 1$ since the events are disjoint (since $\mathcal{F}$ is an antichain).

Method 3 (same as method 2 but counting instead of using probability): The number of maximal chains is $n!$, and the number through any fixed r-set is $r!(n-r)!$, so $\sum_r |\mathcal{F}_r| r!(n-r)! \leq n!$. □

Remark 1.28 To have equality in LYM, we must have equality in each use of local LYM in proof method 1. In this case, the maximum r with $\mathcal{F}_r \neq \emptyset$ has $\mathcal{F}_r = X^{(r)}$. So equality holds iff $\mathcal{F} = X^{(r)}$ for some r. Hence equality in Sperner's Lemma holds iff $\mathcal{F} = X^{(\lfloor n/2 \rfloor)}$ or $\mathcal{F} = X^{(\lceil n/2 \rceil)}$.

1.2. Two total orders on $X^{(r)}$

Definition 1.29 Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **lexicographic (lex)** ordering if for some j, we have $a_i = b_i$ for all $i < j$, and $a_j < b_j$. "use small elements".

Example 1.30 The elements of $[4]^{(2)}$ in lexicographic order are $12, 13, 14, 23, 24, 34$. The elements of $[6]^{(3)}$ in lexicographic order are
$123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456.$

Definition 1.31 Let $A \neq B$ be r-sets, $A = a_1 ... a_r$, $B = b_1 ... b_r$ (where $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$). $A < B$ in the **colexicographic (colex)** order if for some j, we have $a_i = b_i$ for all $i > j$, and $a_j < b_j$. "avoid large elements".

Example 1.32 The elements of $[4]^{(2)}$ in colex order are $12, 13, 23, 14, 24, 34$. The elements of $[6]^{(3)}$ are
$123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 146, 246, 346, 156, 256, 356, 456$.

Remark 1.33 Lex and colex are both total orders. Note that in colex, $[n-1]^{(r)}$ is an initial segment of $[n]^{(r)}$ (this does not hold for lex). So we can view colex as an enumeration of $\mathbb{N}^{(r)}$.

Remark 1.34 $A < B$ in colex iff $A^c < B^c$ in lex with ground set order reversed.

Remark 1.35 By Local LYM, we know that $|\partial\mathcal{F}| \geq |\mathcal{F}|r/(n-r+1)$. Equality is rare (only for $\mathcal{F} = X^{(r)}$ for $0 \leq r \leq n$). What happens in between, i.e., given $|\mathcal{F}|$, how should we choose $\mathcal{F}$ to minimise $|\partial\mathcal{F}|$?

You should be able to convince yourself that if $|\mathcal{F}| = \binom{k}{r}$, then we should take $\mathcal{F} = [k]^{(r)}$. If $\binom{k}{r} < |\mathcal{F}| < \binom{k+1}{r}$, then convince yourself that we should take some $[k]^{(r)}$ plus some r-sets in $[k+1]^{(r)}$.

E.g. for $\mathcal{F} \subseteq X^{(r)}$ with $|\mathcal{F}| = \binom{8}{3} + \binom{4}{2}$, take $\mathcal{F} = [8]^{(3)} \cup \{9 \cup B : B \in [4]^{(2)}\}$.

Remark 1.36 We want to show that if $\mathcal{F} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{F}|$, then $|\partial\mathcal{C}| \leq |\partial\mathcal{F}|$. In particular, if $|\mathcal{F}| = \binom{k}{r}$ (so $\mathcal{C} = [k]^{(r)}$), then $|\partial\mathcal{F}| \geq \binom{k}{r-1}$.

1.3. Compressions

Remark 1.37 We want to transform $\mathcal{F} \subseteq X^{(r)}$ into some $\mathcal{F}' \subseteq X^{(r)}$ such that:
- $|\mathcal{F}'| = |\mathcal{F}|$,
- $|\partial\mathcal{F}'| \leq |\partial\mathcal{F}|$.

Ideally, we want a family of such "compressions" $\mathcal{F} \to \mathcal{F}' \to \ldots \to \mathcal{B}$ such that either $\mathcal{B} = \mathcal{C}$, or $\mathcal{B}$ is similar enough to $\mathcal{C}$ that we can directly check that $|\partial\mathcal{C}| \leq |\partial\mathcal{B}|$.

Definition 1.38 Let $1 \leq i < j \leq n$. The ***ij*-compression** C_{ij} is defined as:
- For $A \in X^{(r)}$,

$$C_{ij}(A) = \begin{cases} (A \cup i) - j & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}.$$

- For $\mathcal{F} \subseteq X^{(r)}$, $C_{ij}(A) = \left\{C_{ij}(A) : A \in \mathcal{F}\right\} \cup \left\{A \in \mathcal{F} : C_{ij}(A) \in \mathcal{F}\right\}$.

"replace j by i where possible". This definition is inspired by "colex prefers $i < j$ to j". Note that $C_{ij}(\mathcal{F}) \subseteq X^{(r)}$ and $\left|C_{ij}(\mathcal{F})\right| = |\mathcal{F}|$.

Definition 1.39 $\mathcal{F}$ is ***ij*-compressed** if $C_{ij}(\mathcal{F}) = \mathcal{F}$.

Example 1.40 Let $\mathcal{F} = \{123, 134, 234, 235, 146, 567\}$, then $C_{12}(\mathcal{F}) = \{123, 134, 234, 135, 146, 567\}$.

Lemma 1.41 Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq i < j \leq n$. Then $\left|\partial C_{ij}(\mathcal{F})\right| \leq |\partial\mathcal{F}|$.

Proof (Hints).
- Let $\mathcal{F}' = C_{ij}(\mathcal{F})$, $B \in \partial\mathcal{F}' - \partial\mathcal{F}$.
- Show that $i \in B$ and $j \notin B$.

- Reason that $B \cup j - i \in \partial\mathcal{F}'$.
- Show that $B \cup j - i \notin \partial\mathcal{F}'$ by contradiction.
- Conclude the result.

$\square$

Proof. Let $\mathcal{F}' = C_{ij}(\mathcal{F})$. Let $B \in \partial\mathcal{F}' - \partial\mathcal{F}$. We'll show that $i \in B$, $j \notin B$, $(B \cup j) - i \in \partial\mathcal{F} - \partial\mathcal{F}'$.

Note that $B \cup x \in \mathcal{F}'$ and $B \cup x \notin \mathcal{F}$ (since $B \notin \partial\mathcal{F}$) for some x. So $i \in B \cup x$, $j \notin B \cup x$, $(B \cup x \cup j) - i \in \mathcal{F}$. We can't have $x = i$, since otherwise $(B \cup x \cup j) - i = B \cup j$, which gives $B \in \partial\mathcal{F}$, a contradiction. So $i \in B$ and $j \notin B$. Also, $B \cup j - i \in \partial\mathcal{F}$, since $B \cup x \cup j - i \in \mathcal{F}$.

Suppose $B \cup j - i \in \partial\mathcal{F}'$: so $(B \cup j - i) \cup y \in \mathcal{F}'$ for some y. We cannot have $y = i$, since otherwise $B \cup j \in \mathcal{F}'$, so $B \cup j \in \mathcal{F}$ (as $j \in B \cup j$), contradicting $B \notin \partial\mathcal{F}$. Hence $j \in (B \cup j - i) \cup y$ and $i \notin (B \cup j - i) \cup y$. Thus, both $(B \cup j - i) \cup y$ and $B \cup y = C_{ij}((B \cup j - i) \cup y)$ belong to $\mathcal{F}$ (by definition of $\mathcal{F}'$), contradicting $B \notin \partial\mathcal{F}$. $\square$

Remark 1.42 In the above proof, we actually showed that $\partial C_{ij}(\mathcal{F}) \subseteq C_{ij}(\partial\mathcal{F})$.

Definition 1.43 $\mathcal{F} \subseteq X^{(r)}$ is **left-compressed** if $C_{ij}(\mathcal{F}) = \mathcal{F}$ for all $i < j$.

Corollary 1.44 Let $\mathcal{F} \subseteq X^{(r)}$. Then there exists a left-compressed $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{F}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{F}|$.

Proof (Hints). Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of subsets of $X^{(r)}$ with $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} i$ strictly decreasing. $\square$

Proof. Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ as follows: set $\mathcal{F}_0 = \mathcal{F}$. Having defined $\mathcal{F}_0, \ldots, \mathcal{F}_k$, if $\mathcal{F}_k$ is left-compressed the end the sequence with $\mathcal{F}_k$; if not, choose $i < j$ such that $\mathcal{F}_k$ is not ij-compressed, and set $\mathcal{F}_{k+1} = C_{ij}(\mathcal{F}_k)$.

This must terminate after a finite number of steps, e.g. since $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} i$ is strictly decreasing with k. The final term $\mathcal{B} = \mathcal{F}_k$ satisfies $|\mathcal{B}| = |\mathcal{F}|$, and $|\partial\mathcal{B}| \leq |\partial\mathcal{F}|$ by the above lemma. $\square$

Remark 1.45
- Another way of proving this is: among all $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{F}| = |\mathcal{F}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{F}|$, choose one with minimal $\sum_{A \in \mathcal{B}} \sum_{i \in A} i$.
- We can choose an order of the C_{ij} so that no C_{ij} is applied twice.
- Any initial segment of colex is left-compressed, but the converse is false, e.g. $\{123, 124, 125, 126\}$ is left-compressed.

Definition 1.46 Let $U, V \subseteq X$, $|U| = |V|$, $U \cap V = \emptyset$ and $\max U < \max V$. Define the ***UV*-compression** C_{UV} as:
- For $A \subseteq X$,

$$C_{UV}(A) = \begin{cases} (A - V) \cup U & \text{if } V \subseteq A, U \cap A = \emptyset \\ A & \text{otherwise} \end{cases}.$$

- For $\mathcal{F} \subseteq X^{(r)}$,

$$C_{UV}(\mathcal{F}) = \{C_{UV}(A) : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : C_{UV}(A) \in \mathcal{F}\}.$$

We have $C_{UV}(\mathcal{F}) \subseteq X^{(r)}$ and $|C_{UV}(\mathcal{F})| = |\mathcal{F}|$. This definition is inspired by "colex prefers 23 to 14".

Definition 1.47 $\mathcal{F}$ is ***UV*-compressed** if $C_{UV}(\mathcal{F}) = \mathcal{F}$.

Example 1.48 Let $\mathcal{F} = \{123, 124, 147, 237, 238, 149\}$, then $C_{23,14}(\mathcal{F}) = \{123, 124, 147, 237, 238, 239\}$.

Example 1.49 We can have $|\partial C_{UV}(\mathcal{F})| > |\partial \mathcal{F}|$. E.g. $\mathcal{F} = \{147, 157\}$ has $|\partial \mathcal{F}| = 5$, but $C_{23,14}(\mathcal{F}) = \{237, 157\}$ has $\left|\partial C_{23,14}(\mathcal{F})\right| = 6$.

Lemma 1.50 Let $\mathcal{F} \subseteq X^{(r)}$ be UV-compressed for all $U, V \subseteq X$ with $|U| = |V|$, $U \cap V = \emptyset$ and $\max U < \max V$. Then $\mathcal{F}$ is an initial segment of colex.

Proof (Hints). Suppose not, consider a compression for appropriate U and V. $\square$

Proof. Suppose not, then there exists $A, B \in X^{(r)}$ with $B < A$ in colex but $A \in \mathcal{F}$, $B \notin \mathcal{F}$. Let $V = A \setminus B$, $U = B \setminus A$. Then $|V| = |U|$, $U \cap V = \emptyset$, and $\max V > \max U$ (since $\max(A \Delta B) \in A$, by definition of colex). Since $\mathcal{F}$ is UV-compressed, we have $C_{UV}(A) = B \in C_{UV}(\mathcal{F}) = \mathcal{F}$, contradiction. $\square$

Lemma 1.51 Let $U, V \subseteq X$, $|U| = |V|$, $U \cap V = \emptyset$, $\max U < \max V$. For $\mathcal{F} \subseteq X^{(r)}$, suppose that

$$\forall u \in U, \exists v \in V : \quad \mathcal{F} \text{ is } (U - u, V - v)\text{-compressed.}$$

Then $|\partial C_{UV}(\mathcal{F})| \leq |\partial \mathcal{F}|$.

Proof (Hints).

- Let $\mathcal{F}' = C_{UV}(\mathcal{F})$, $B \in \partial \mathcal{F}' - \partial \mathcal{F}$.
- Show that $U \subseteq B$ and $V \cap B = \emptyset$.
- Reason that $(B - U) \cup V \in \partial \mathcal{F}$.
- Show that $(B - U) \cup V \notin \partial \mathcal{F}'$ by contradiction.

$\square$

Proof. Let $\mathcal{F}' = C_{UV}(\mathcal{F})$. For $B \in \partial \mathcal{F}' - \partial \mathcal{F}$, we will show that $U \subseteq B$, $V \cap B = \emptyset$ and $B \cup V - U \in \partial \mathcal{F} - \partial \mathcal{F}'$, then we will be done.

We have $B \cup x \in \mathcal{F}'$ for some $x \in X$, and $B \cup x \notin \mathcal{F}$. So $U \subseteq B \cup x$, $V \cap (B \cup x) = \emptyset$, and $(B \cup x \cup V) - U \in \mathcal{F}$, by definition of C_{UV}. If $x \in U$, then $\exists y \in V$ such that $\mathcal{F}$ is $(U - x, V - y)$-compressed, so from $(B \cup x \cup V) - U \in \mathcal{F}$, we have $B \cup y \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$. Thus $x \notin U$, so $U \subseteq B$ and $V \cap B = \emptyset$. Certainly $B \cup V - U \in \partial \mathcal{F}$ (since $(B \cup x \cup V) - U \in \mathcal{F}$), so we just need to show that $B \cup V - U \notin \partial \mathcal{F}'$.

Assume the opposite, i.e. $(B - U) \cup V \in \partial \mathcal{F}'$, so $(B - U) \cup V \cup w \in \mathcal{F}'$ for some $w \in X$. (This also belongs to $\mathcal{F}$, since it contains V). If $w \in U$, then since $\mathcal{F}$ is $(U - w, V - z)$-compressed for some $z \in V$, we have $B \cup z = C_{U-w,V-z}((B - U) \cup V \cup w) \in \mathcal{F}$, contradicting $B \notin \partial \mathcal{F}$. So $w \notin U$, and since $V \subseteq (B - U) \cup V \cup w$ and $U \cap ((B -$

$U) \cup V \cup w) = \emptyset$, by definition of C_{UV}, we must have that both $(B - U) \cup V \cup w$ and $B \cup w = C_{UV}((B - U) \cup V \cup w) \in \mathcal{F}$, contradicting $B \notin \partial\mathcal{F}$. □

Theorem 1.52 (Kruskal-Katona) Let $\mathcal{F} \subseteq X^{(r)}$, $1 \leq r \leq n$, let $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial\mathcal{C}| \leq |\partial\mathcal{F}|$.

In particular, if $|\mathcal{F}| = \binom{k}{r}$, then $|\partial\mathcal{F}| \geq \binom{k}{r-1}$.

Proof (Hints).

- Let $\Gamma = \{(U, V) \in \mathbb{P}(X) \times \mathbb{P}(X) : |U| = |V| > 0, U \cap V = \emptyset, \max U < \max V\} \cup \{(\emptyset, \emptyset)\}$.
- Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of UV-compressions where $(U, V) \in \Gamma$, choosing $|U| = |V| > 0$ minimal each time. Show that this (U, V) satisfies condition of above lemma.
- Reason that sequence terminates by considering $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} 2^i$.

□

Proof. Let $\Gamma = \{(U, V) \in \mathbb{P}(X) \times \mathbb{P}(X) : |U| = |V| > 0, U \cap V = \emptyset, \max U < \max V\} \cup \{(\emptyset, \emptyset)\}$. Define a sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of set systems in $X^{(r)}$ as follows: set $\mathcal{F}_0 = \mathcal{F}$. Having chosen $\mathcal{F}_0, \ldots, \mathcal{F}_k$, if $\mathcal{F}_k$ is (UV)-compressed for all $(U, V) \in \Gamma$ then stop. Otherwise, choose $(U, V) \in \Gamma$ with $|U| = |V| > 0$ minimal, such that $\mathcal{F}_k$ is not (UV)-compressed.

Note that $\forall u \in U$, $\exists v \in V$ such that $(U - u, V - v) \in \Gamma$ (namely $v = \min(V)$). So by the above lemma, $|\partial C_{UV}(\mathcal{F}_k)| \leq |\partial\mathcal{F}_k|$. Set $\mathcal{F}_{k+1} = C_{UV}(\mathcal{F}_k)$, and continue. The sequence must terminate, as $\sum_{A \in \mathcal{F}_k} \sum_{i \in A} 2^i$ is strictly decreasing with k. The final term $\mathcal{B} = \mathcal{F}_k$ satisfies $|\mathcal{B}| = |\mathcal{F}|$, $|\partial\mathcal{B}| \leq |\partial\mathcal{F}|$, and is (UV)-compressed for all $(U, V) \in \Gamma$. So $\mathcal{B} = \mathcal{C}$ by lemma before previous lemma. □

Remark 1.53

- Equivalently, if $|\mathcal{F}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \cdots + \binom{k_s}{s}$ where each $k_i > k_{i-1}$ and $s \geq 1$, then
$$|\partial\mathcal{F}| \geq \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \cdots + \binom{k_s}{s-1}.$$
- Equality in Kruskal-Katona: if $|\mathcal{F}| = \binom{k}{r}$ and $|\partial\mathcal{F}| = \binom{k}{r-1}$, then $\mathcal{F} = Y^{(r)}$ for some $Y \subseteq X$ with $|Y| = k$. However, it is not true in general that if $|\partial\mathcal{F}| = |\partial C|$, then $\mathcal{F}$ is isomorphic to $\mathcal{C}$ (i.e. there is a permutation of the ground set X sending $\mathcal{F}$ to $\mathcal{C}$).

Definition 1.54 For $\mathcal{F} \subseteq X^{(r)}$, $0 \leq r \leq n - 1$, the **upper shadow** of $\mathcal{F}$ is
$$\partial^+\mathcal{F} := \{A \cup x : A \in \mathcal{F}, x \notin A\} \subseteq X^{(r+1)}.$$

Corollary 1.55 Let $\mathcal{F} \subseteq X^{(r)}$, $0 \leq r \leq n - 1$, let $\mathcal{C}$ be the initial segment of lex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial^+\mathcal{C}| \leq |\partial^+\mathcal{F}|$.

Proof (Hints). By Kruskal-Katona. □

Proof. By Kruskal-Katona, since $A < B$ in colex iff $A^c < B^c$ in lex with ground-set (X) order reversed, and if $\mathcal{F}' = \{A^c : A \in \mathcal{F}\}$, then $|\partial^+ \mathcal{F}'| = |\partial \mathcal{F}|$. □

Remark 1.56 The fact that the shadow of an initial segment of colex on $X^{(r)}$ is an initial segment of colex on $X^{(r-1)}$ (since if $\mathcal{C} = \left\{A \in X^{(r)} : A \le a_1 ... a_r \text{ in colex}\right\}$, then $\partial \mathcal{C} = \left\{B \in X^{(r-1)} : B \le a_2 ... a_r \text{ in colex}\right\}$) gives:

Corollary 1.57 Let $\mathcal{F} \subseteq X^{(r)}$, $1 \le r \le n$, $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{F}|$. Then $|\partial^t \mathcal{C}| \le |\partial^t \mathcal{F}|$ for all $1 \le t \le r$ (where ∂^t is shadow applied t times).

Proof (Hints). Straightforward. □

Proof. If $|\partial^t \mathcal{C}| \le |\partial^t \mathcal{F}|$, then $\left|\partial^{t+1} \mathcal{C}\right| \le \left|\partial^{t+1} \mathcal{F}\right|$, since $\partial^t \mathcal{C}$ is an initial segment of colex. So we are done by induction (base case is Kruskal-Katona). □

Remark 1.58 So if $|\mathcal{F}| = \binom{k}{r}$, then $|\partial^t \mathcal{F}| \ge \binom{k}{r-t}$.

1.4. Intersecting families

Definition 1.59 A family $\mathcal{F} \in \mathbb{P}(X)$ is **intersecting** if for all $A, B \in \mathcal{F}$, $A \cap B \ne \emptyset$.

We are interested in finding intersecting families of maximum size.

Proposition 1.60 For all intersecting families $\mathcal{F} \subseteq \mathbb{P}(X)$, $|\mathcal{F}| \le 2^{n-1} = \frac{1}{2}|\mathbb{P}(X)|$.

Proof (Hints). Straightforward. □

Proof. Given any $A \subseteq X$, at most one of A and A^c can belong to $\mathcal{F}$. □

Example 1.61
- $\mathcal{F} = \{A \subseteq X : 1 \in A\}$ is intersecting, and $|\mathcal{F}| = 2^{k-1}$.
- $\mathcal{F} = \left\{A \subseteq X : |A| > \frac{n}{2}\right\}$ for n odd.

Example 1.62 Let $\mathcal{F} \subseteq X^{(r)}$:
- If $r > \frac{n}{2}$, then $\mathcal{F} = X^{(r)}$ is intersecting.
- If $r = \frac{n}{2}$, then choose one of A and A^c for all $A \in X^{(r)}$. This gives $|\mathcal{F}| = \frac{1}{2}\binom{n}{r}$.
- If $r < \frac{n}{2}$, then $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$ has size $\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$ (since the probability of a random r-set containing 1 is $\frac{r}{n}$). If $(n, r) = (8, 3)$, then $|\mathcal{F}| = \binom{7}{2} = 21$.
- Let $\mathcal{F} = \left\{A \in X^{(r)} : |A \cap \{1, 2, 3\}| \ge 2\right\}$. If $(n, r) = (8, 3)$, then $|\mathcal{F}| = 1 + \binom{3}{2}\binom{5}{1} = 16 < 21$ (since 1 set A has $|B \cap [3]| = 3$, 15 sets A have $|A \cap [3]| = 2$).

Theorem 1.63 (Erdos-Ko-Rado) Let $\mathcal{F} \subseteq X^{(r)}$ be an intersecting family, where $r < \frac{n}{2}$. Then $|\mathcal{F}| \le \binom{n-1}{r-1}$.

Proof (Hints).
- Method 1:
  - Let $\overline{\mathcal{F}} = \{A^c : A \in \mathcal{F}\}$. Show that $\partial^{n-2r}\overline{\mathcal{F}}$ and $\mathcal{F}$ are disjoint families of r-sets.
  - Assume the opposite, show that the size of the union of these two sets is greater than the size of $X^{(r)}$.
- Method 2:

- Let $c : [n] \to \mathbb{Z}/n$ be bijection, i.e. cyclic ordering of $[n]$. Show there at most r sets in $\mathcal{F}$ that are intervals (sets with r consecutive elements) under this ordering.
- Find expression for number of times an r-set in $\mathcal{F}$ is an interval all possible orderings, and find an upper bound for this using the above.

□

Proof. Proof 1 ("bubble down with Kruskal-Katona"): note that $A \cap B \neq \emptyset$ iff $A \not\subseteq B^c$. Let $\overline{\mathcal{F}} = \{A^c : A \in \mathcal{F}\} \subseteq X^{(n-r)}$. We have $\partial^{n-2r}\overline{\mathcal{F}}$ and $\mathcal{F}$ are disjoint families of r-sets (if not, then there is some $A \in \mathcal{F}$ such that $A \subseteq B^c$ for some $B \in \mathcal{F}$, but then $A \cap B = \emptyset$). Suppose $|\mathcal{F}| > \binom{n-1}{r-1}$. Then $\left|\overline{\mathcal{F}}\right| = |\mathcal{F}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$. So by Kruskal-Katona, we have $\left|\partial^{n-2r}\overline{\mathcal{F}}\right| \geq \binom{n-1}{r}$. So $|\mathcal{F}| + \left|\partial^{n-2r}\overline{\mathcal{F}}\right| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r} = \left|X^{(r)}\right|$, a contradiction, since $\mathcal{F}, \partial^{n-2r}\overline{\mathcal{F}} \subseteq X^{(r)}$.

Proof 2: pick a cyclic ordering of $[n]$, i.e. a bijection $c : [n] \to \mathbb{Z}/n$. There are at most r sets in $\mathcal{F}$ that are intervals (r consecutive elements) under this ordering: for $c_1 ... c_r \in \mathcal{F}$, for each $2 \leq i \leq r$, at most one of the two intervals $c_i ... c_{i+r-1}$ and $c_{i-r} ... c_{i-1}$ can belong to $\mathcal{F}$, since they are disjoint and $\mathcal{F}$ is intersecting (the indices of c are taken $\operatorname{mod} n$). For each r-set A, out of the $n!$ cyclic orderings, there are $n \cdot r!(n-r)!$ which map A to an interval ($r!$ orderings inside A, $(n-r)!$ orderings outside A, n choices for the start of the interval). Hence, by counting the number of times an r-set in $\mathcal{F}$ is an interval under a given ordering (over all r-sets in $\mathcal{F}$ and all cyclic orderings), we obtain $|\mathcal{F}| n r! (n-r)! \leq n! r$, i.e. $|\mathcal{F}| \leq \binom{n-1}{r-1}$. □

Remark 1.64

- The calculation at the end of proof method 1 had to give the correct answer, as the shadow calculations would all be exact if $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$ (in this case, $\mathcal{F}$ and $\partial^{n-2r}\overline{\mathcal{F}}$ partition $X^{(r)}$).
- The calculations at the end of proof method 2 had to work out, given equality for the family $\mathcal{F} = \left\{A \in X^{(r)} : 1 \in A\right\}$.
- In method 2, equivalently, we are double-counting the edges in the bipartite graph, where the vertex classes (partition sets) are $\mathcal{F}$ and all cyclic orderings, with A joined to c if A is an interval under c. This method is called **averaging** or **Katona's method**.
- Equality in Erdos-Ko-Rado holds iff $\mathcal{F} = \left\{A \in X^{(r)} : i \in A\right\}$, for some $1 \leq i \leq n$. This can be obtained from proof 1 and equality in Kruskal-Katona, or from proof 2.

2. Isoperimetric inequalities

We seek to answer questions of the form "how do we minimise the boundary of a set of given size?"

Example 2.1 In the continuous setting:

- Among all subsets of $\mathbb{R}^2$ of a given fixed area, the disc minimises the perimeter.
- Among all subsets of $\mathbb{R}^3$ of a given fixed volume, the solid sphere minimises the surface area.

- Among all subsets of S^2 of given fixed surface area, the circular cap minimises the perimeter.

Definition 2.2 For a A of vertices of a graph G, the **boundary** of A is

$$b(A) = \{x \in G : x \notin A, xy \in E \text{ for some } y \in A\}.$$

Definition 2.3 An **isoperimetric inequality** on a graph G is an inequality of the form

$$\forall A \subseteq G, \quad |b(A)| \geq f(|A|)$$

for some function $f : \mathbb{N} \to \mathbb{R}$.

Definition 2.4 The **neighbourhood** of $A \subseteq V(G)$ is $N(A) := A \cup b(A)$, i.e.

$$N(A) = \{x \in G : d(x, A) \leq 1\}.$$

Example 2.5 A good (and natural) example for A that minimises $|b(A)|$ in the discrete cube Q_n might be a ball $B(x, r) = \{y \in G : d(x, y) \leq r\}$. Let $A \subseteq \mathbb{P}(X) = V(Q_3)$, $|A| = 4$.

A good guess is that balls are best, i.e. sets of the form $B(\emptyset, r) = X^{(\leq r)} = X^{(0)} \cup \dots \cup X^{(r)}$. What if $\left|X^{(\leq r)}\right| \leq |A| \leq \left|X^{(\leq r+1)}\right|$? A good guess is take A with $X^{(\leq r)} \subsetneq A \subsetneq X^{(\leq r+1)}$. If $A = X^{(\leq r)} \cup B$, where $B \subseteq X^{(r+1)}$, then $b(A) = \left(X^{(r+1)} - B\right) \cup \partial^+ B$, so we would take B to be an initial segment of lex by Kruskal-Katona. This motivates the following definition.

Definition 2.6 The **simplicial ordering** on $\mathbb{P}(X)$ defines $x < y$ if either $|x| < |y|$, or both $|x| = |y|$ and $x < y$ in lex.

We want to show the initial segments of the simplicial ordering minimise the boundary.

Definition 2.7 For $A \subseteq \mathbb{P}(X)$ and $1 \leq i \leq n$, the ***i*-sections** of A are the families $A_-^{(i)}, A_+^{(i)} \subseteq \mathbb{P}(X \setminus i)$, given by

$$A_-^{(i)} = A_- := \{x \in A : i \notin x\},$$
$$A_+^{(i)} = A_+ := \{x - i : x \in A, i \in x\}$$

Note that $A = A_-^{(i)} \cup \{x \cup i : x \in A_+^{(i)}\}$, so we can define a family by its i-sections.

Remark 2.8 When viewing $\mathbb{P}(X)$ as the n-dimensional cube Q_n, we view the i-sections as subgraphs of the $(n-1)$-dimensional cube Q_{n-1} (which we view $\mathbb{P}(X \setminus i)$ as).

Definition 2.9 A **Hamming ball** is a family $A \subseteq \mathbb{P}(X)$ with $X^{(\leq r)} \subseteq A \subseteq X^{(\leq r+1)}$ for some r.

Definition 2.10 The ***i*-compression** of $A \subseteq \mathbb{P}(X)$ is the family $C_i(A) \subseteq \mathbb{P}(X)$ given by its i-sections:

- $(C_i(A))_-^{(i)}$ is the first $\left|A_-^{(i)}\right|$ elements of the simplicial order on $\mathbb{P}(X - i)$, and

- $(C_i(A))_+^{(i)}$ is the first $\left|A_+^{(i)}\right|$ elements of the simplicial order on $\mathbb{P}(X - i)$.

Note that $|C_i(A)| = |A|$, and $C_i(A)$ "looks more like" a Hamming ball than A does.

Definition 2.11 $A \subseteq \mathbb{P}(X)$ is ***i*-compressed** if $C_i(A) = A$.

Example 2.12 Note that a set that is i-compressed for all $i \in [n]$ is not necessarily an initial segment of simplicial, e.g. take $\{\emptyset, 1, 2, 12\}$ in Q_3. However…

Lemma 2.13 Let $B \subseteq Q_n$ be i-compressed for all $i \in [n]$ but not an initial segment of the simplicial order. Then either:
- n is odd (say $n = 2k + 1$) and

$$B = X^{\leq k} \setminus \underbrace{\{k+2, k+3, ..., 2k+1\}}_{\text{last } k\text{-set}} \cup \underbrace{\{1, 2, ..., k+1\}}_{\text{first } (k+1)\text{-set}},$$

- or n is even (say $n = 2k$), and

$$B = X^{(<k)} \cup \left\{x \in X^{(k)} : 1 \in x\right\} \setminus \underbrace{\{1, k+2, k+3, ..., 2k\}}_{\text{last } k\text{-set with } 1} \cup \underbrace{\{2, 3, ..., k+1\}}_{\text{first } k\text{-set without } 1}.$$

Proof (Hints). For $x \notin B$ and $y \in B$, show by contradiction that any $i \in [n]$ is in exactly one of x and y (it helps to visualise this), and deduce that no elements lie strictly between x and y in the simplicial ordering. □

Proof. As B is not an initial segment, there are $x < y$ in simplicial ordering with $x \notin B$ and $y \in B$. For each $i \in [n]$, assume $i \in x, y$. Since the i-section that y lives in is an initial segment of simplicial on $\mathbb{P}(X \setminus i)$ (as B is i-compressed), and $x - i < y - i$ in simplicial on $\mathbb{P}(X \setminus i)$, we have that $x - i$ lives in the same i-section, and so $x \in B$: contradiction. Similarly, $i \notin x, y$ leads to a contradiction (as then $x < y$ in simplicial on $\mathbb{P}(X \setminus i)$). So $x = y^c$.

Thus for each $y \in B$, there is at most one $x < y$ with $x \notin B$ (namely $x = y^c$), and for each $x \notin B$, there is at most one $y > x$ with $y \in B$ (namely $y = x^c$). So no sets lie between x and y in the simplicial ordering. So $B = \{z : z \leq y\} \setminus \{x\}$, with x the predecessor of y, and $x = y^c$. Hence if $n = 2k + 1$, then x is the last k-set (otherwise sizes of x and $y = x^c$ don't match), and if $n = 2k$, then x is the last k-set containing 1. □

Theorem 2.14 (Harper) Let $A \subseteq V(Q_n)$ and let C be the initial segment of the simplicial order on $\mathbb{P}(X) = V(Q_n)$, with $|C| = |A|$. Then $|N(A)| \geq |N(C)|$. So initial segments of the simplicial order minimise the boundary. In particular, if $|A| = \sum_{i=0}^{r} \binom{n}{i}$, then $|N(A)| \geq \sum_{i=0}^{r+1} \binom{n}{i}$.

Proof (Hints).
- Using induction, prove the claim that $|N(C_i(A))| \leq |N(A)|$:
  - Find expressions for $N(A)_-$ as union of two sets, similarly for $N(A)_+$, same for $N(B)_-$ and $N(B)_+$.
  - Explain why $N(B_-)$ and B_+ are nested, use this to show $\left|N(B_-) \cup B_+\right| \leq \left|N(A_-) \cup A_+\right|$.

- ‣ Do the same with the $+$ and $-$ switched.
- Define a suitable sequence of families $A_0, A_1, \ldots \in Q_n$.
- Reason that the sequence terminates by considering $\sum_{x \in A_k}$(position of x in simplicial order).
- Conclude by above lemma.

□

Proof. By induction on n. $n = 1$ is trivial. Given $n > 1$, $A \subseteq Q_n$ and $1 \leq i \leq n$, we claim that $|N(C_i(A))| \leq |N(A)|$.

Proof of claim. Write $B = C_i(A)$. We have $N(A)_- = N(A_-) \cup A_+$, and $N(A)_+ = N(A_+) \cup A_-$. Similarly, $N(B)_- = N(B_-) \cup B_+$, and $N(B)_+ = N(B_+) \cup B_-$.

Now $|B_+| = |A_+|$ by definition of B, and by the inductive hypothesis, $|N(B_-)| \leq |N(A_-)|$ (since $C_i(A_-) = B_-$). But B_+ is an initial segment of the simplicial ordering, and $N(B_-)$ is as well (since the neighbourhood of an initial segment of the simplicial ordering is also an initial segment). So B_+ and $N(B_-)$ are nested (one is contained in the other). Hence, $|N(B_-) \cup B_+| \leq |N(A_-) \cup A_+|$.

Similarly, $|B_-| = |A_-|$ by definition of B. Since B_+ and $C_i(A_+)$ are both initial segments of size $|B_+| = |A_+|$, we have $B_+ = C_i(A_+)$, hence by the inductive hypothesis, $|N(B_+)| \leq |N(A_+)|$. B_- and $N(B_+)$ are initial segments, so are nested. Hence $|N(B_+) \cup B_-| \leq |N(A_+) \cup A_-|$.

This gives $|N(B)| = |N(B)_-| + |N(B)_+| \leq |N(A)_-| + |N(A)_+| = |N(A)|$, which proves the claim.

Define a sequence $A_0, A_1, \ldots \subseteq Q_n$ as follows:
- Set $A_0 = A_1$.
- having chosen $A_0, \ldots, A_k$, if A_k is i-compressed for all $i \in [n]$, then end the sequence with A_k. If not, pick i with $C_i(A_k) \neq A_k$ and set $A_{k+1} = C_i(A_k)$, and continue.

The sequence must terminate, since $\sum_{x \in A_k}$(position of x in simplicial order) is strictly decreasing. The final family $B = A_k$ satisfies $|B| = |A|$, $|N(B)| \leq |N(A)|$, and is i-compressed for all $i \in [n]$.

So we are done by above lemma, since in each case certainly we have $|N(B)| \geq |N(C)|$. □

Remark 2.15
- If A was a Hamming ball, then we would be already done by Kruskal-Katona.
- Conversely, Harper's theorem implies Kruskal-Katona: given $B \subseteq X^{(r)}$, apply Harper's theorem to $A = X^{(\leq r-1)} \cup B$.
- We could also prove Harper's theorem using UV-compressions.
- Conversely, we can also prove Kruskal-Katona using these "codimension 1" compressions.

Definition 2.16 For $A \subseteq Q_n$ and $t \in \mathbb{N}$, the ***t*-neighbourhood** of A is

$$A_{(t)} = N^t(A) := \{x \in Q_n : d(x, A) \leq t\}.$$

Corollary 2.17 Let $A \subseteq Q_n$ with $|A| \geq \sum_{i=0}^{r} \binom{n}{i}$. Then

$$\forall t \leq n - r, \quad |N^t(A)| \geq \sum_{i=0}^{r+t} \binom{n}{i}.$$

Proof (Hints). By Harper's theorem. □

Proof. By Harper's theorem and induction on t. □

Remark 2.18 To get a feeling for the strength of the above corollary, we'll need some estimates on quantities such as $\sum_{i=0}^{r} \binom{n}{i}$. Note that $i = n/2$ maximises $\binom{n}{i}$, while $i = (1/2 - \varepsilon)n$ makes it small: we are going $\varepsilon\sqrt{n}$ standard deviations away from the mean $n/2$.

Proposition 2.19 Let $0 < \varepsilon < 1/4$. Then

$$\sum_{i=0}^{\lfloor (1/2-\varepsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2} \cdot 2^n.$$

For ε fixed and $n \to \infty$, the upper bound is an exponentially small fraction of 2^n.

Proof (Hints).

- For $1 \leq i \leq \lfloor (1/2 - \varepsilon)n \rfloor$, show that $\binom{n}{i-1} / \binom{n}{i} \leq 1 - 2\varepsilon$, use this to show that $\sum_{i=0}^{\lfloor (1/2-\varepsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{2\varepsilon} \binom{n}{\lfloor (1/2-\varepsilon)n \rfloor}$.
- TODO.

□

Proof. Let $L = \lfloor (1/2 - \varepsilon)n \rfloor$. For $1 \leq i \leq L$,

$$\binom{n}{i-1} / \binom{n}{i} = \frac{i}{n - i + 1} \leq \frac{(1/2 - \varepsilon)n}{(1/2 + \varepsilon)n} = \frac{1/2 - \varepsilon}{1/2 + \varepsilon} = 1 - \frac{2\varepsilon}{1/2 + \varepsilon} \leq 1 - 2\varepsilon.$$

Hence by induction, $\binom{n}{i} \leq (1 - 2\varepsilon)^{L-i} \binom{n}{L}$ for each $0 \leq i \leq L$, and so

$$\sum_{i=0}^{L} \binom{n}{i} \leq \frac{1}{2\varepsilon} \binom{n}{L} = \frac{1}{2\varepsilon} \binom{n}{\lfloor (1/2 - \varepsilon)n \rfloor}$$

(since this is the sum of geometric progression). The same argument tells us that

$$\begin{aligned} \binom{n}{\lfloor (1/2 - \varepsilon)n \rfloor} &\leq \binom{n}{\lfloor 1/2 - \varepsilon/2 \rfloor n} \left(1 - 2\frac{\varepsilon}{2}\right)^{\varepsilon n/2 - 1} \\ &\leq 2^n \cdot 2(1 - \varepsilon)^{\varepsilon n/2} \\ &\leq 2^n \cdot 2e^{-\varepsilon^2 n/2} \end{aligned}$$

since $1 - \varepsilon \leq e^{-\varepsilon}$ (we include -1 in the exponent due to taking floors). Then

$$\sum_{i=0}^{L} \binom{n}{i} \le \frac{1}{2\varepsilon} \cdot 2e^{-\varepsilon^2 n/2} \cdot 2^n.$$

$\square$

Theorem 2.20 Let $0 < \varepsilon < 1/4$, $A \subseteq Q_n$. If $|A|/2^n \ge 1/2$, then

$$\frac{|N^{\varepsilon n}(A)|}{2^n} \ge 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

So sets of at least half density have exponentially dense εn-neighbourhoods.

Proof (Hints).

- Enough to show that if $\varepsilon n \in \mathbb{N}$, then $|N^{\varepsilon n}(A)|/2^n \ge 1 - \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2}$.
- Give lower bound on $|A|$ which is a binomial sum, deduce lower bound on $N^{\varepsilon n}(A)$.
- Give an upper bound on $|N^{\varepsilon n}(A)^c|$ using the above proposition.

$\square$

Proof. It is enough to show that if $\varepsilon n \in \mathbb{N}$, then $|N^{\varepsilon n}(A)|/2^n \ge 1 - \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2}$. We have $|A| \ge \sum_{i=0}^{\lceil n/2-1 \rceil} \binom{n}{i}$, so by Corollary 2.17,

$$|N^{\varepsilon n}(A)| \ge \sum_{i=0}^{\lceil n/2-1+\varepsilon n \rceil} \binom{n}{i}.$$

So

$$\begin{aligned} |N^{\varepsilon n}(A)^c| &\le \sum_{i=\lceil n/2+\varepsilon n \rceil}^{n} \binom{n}{i} \\ &= \sum_{i=\lceil n/2+\varepsilon n \rceil}^{n} \binom{n}{n-i} \\ &= \sum_{i=0}^{\lceil n/2-\varepsilon n \rceil} \binom{n}{i} \\ &\le \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2} \cdot 2^n. \end{aligned}$$

by Proposition 2.19. $\square$

Remark 2.21 The same argument would give a result for "small" sets: if $|A|/2^n \ge \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}$, then $\left|N^{2\varepsilon n}(A)\right|/2^n \ge 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}$.

Definition 2.22 $f : Q_n \to \mathbb{R}$ is **Lipschitz** if for all adjacent $x, y \in Q_n$, $|f(x) - f(y)| \le 1$.

Definition 2.23 For $f : Q_n \to \mathbb{R}$, we say $M \in \mathbb{R}$ is a **Levy mean** (or **median**) of f if $|\{x \in Q_n : f(x) \le M\}| \ge 2^{n-1}$ and $|\{x \in Q_n : f(x) \ge M\}| \ge 2^{n-1}$.

Example 2.24 Let $f : Q_n \to \mathbb{R}$, $f(x) = 1$ if $1 \in x$ and $f(x) = 0$ otherwise. Then any $M \in [0, 1]$ is a Levy mean of f.

Theorem 2.25 (Concentration of Measure Phenomenon) Let $f : Q_n \to \mathbb{R}$ be Lipschitz with Levy mean M. Then for all $0 < \varepsilon < \frac{1}{4}$,

$$\frac{|\{x \in Q_n : |f(x) - M| \leq \varepsilon n\}|}{2^n} \geq 1 - \frac{4}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

So "every well-behaved function on the cube Q_n is roughly constant nearly everywhere".

Proof (Hints).

- Consider two subsets $A, B \subseteq Q_n$ of density at least $1/2$, and apply Theorem 2.20 on them.
- Use the fact that f is Lipschitz to find upper bound for the image of the εn-neighbourhood of one of A and B, similarly find a lower bound for the image of the εn-neighbourhood of the other.

□

Proof. Let $A = \{x \in Q_n : f(x) \leq M\}$. Then by definition, $|A|/2^n \geq 1/2$, so by the above theorem,

$$\frac{|N^{\varepsilon n}(A)|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

But f is Lipschitz, so $x \in N^{\varepsilon n}(A) \implies f(x) \leq M + \varepsilon n$, so $N^{\varepsilon n}(A) \subseteq \{x \in Q_n : f(x) \leq M + \varepsilon n\} =: L$. Thus,

$$\frac{|L|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

Similarly, let $U = \{x \in Q_n : f(x) \geq M - \varepsilon n\}$, then $|U|/2^n \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}$. Hence, we have

$$\begin{aligned}
\frac{|L \cap U|}{2^n} &= \frac{|L|}{2^n} + \frac{|U|}{2^n} - \frac{|L \cup U|}{2^n} \\
&\geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2} + 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2} - 1 \\
&= 1 - \frac{4}{\varepsilon} e^{-\varepsilon^2 n/2}.
\end{aligned}$$

□

Definition 2.26 The **diameter** of a graph $G = (V, E)$ is $\max\{d(x, y) : x, y \in V\}$.

Definition 2.27 Let G be a graph of diameter D. Write

$$\alpha(G, \varepsilon) = \max\left\{1 - \frac{|N^{\varepsilon D}(A)|}{|G|} : A \subseteq G, \frac{|A|}{|G|} \geq \frac{1}{2}\right\}.$$

So if $\alpha(G, \varepsilon)$ is small, then sets of at least half density have large εD-neighbourhoods.

Definition 2.28 A sequence of graphs $(G_n)_{n \in \mathbb{N}}$ is a **Levy family** if

$$\forall \varepsilon > 0, \quad \alpha(G_n, \varepsilon) \to 0 \text{ as } n \to \infty.$$

It is a **normal Levy family** if for all $\varepsilon > 0$, $\alpha(G_n, \varepsilon)$ decays exponentially with n.

Example 2.29 By the above theorem, the sequence (Q_n) is a normal Levy family (note that Q_n has diameter $n+1$). More generally, we have concentration of measure for any Levy family.

Example 2.30 Many naturally-occurring families of graphs are Levy families, e.g. $(S_n)_{n\in\mathbb{N}}$, where the permutation group S_n is made into a graph by including an edge between σ and τ if $\tau\sigma^{-1}$ is a transposition.

Example 2.31 Similarly, we can define $\alpha(X, \varepsilon)$ for any metric measure space X (of finite measure and finite diameter). E.g. the sequence of spheres $(S^n)_{n\in\mathbb{N}}$ is a Levy family. To prove this, we have:

1. An isoperimetric inequality on S^n: for $A \subseteq S^n$ and C a circular cap with $|C| = |A|$, we have $|N^\varepsilon(A)| \geq |N^\varepsilon(C)|$.
2. An estimate: a circular cap C of measure $1/2$ is the cap of angle $\pi/2$. So $N^\varepsilon(C)$ is the circular cap of angle $\pi/2 + \varepsilon$. This has measure roughly equal to $\int_\varepsilon^{\pi/2} \cos^{n-1}(t)\,\mathrm{d}t \to 0$ as $n \to \infty$.

Remark 2.32 We deduced concentration of measure from an isoperimetric inequality. Conversely, we have:

Proposition 2.33 Let G be a graph such that for any Lipschitz function $f : G \to \mathbb{R}$ with Levy mean M, we have

$$\frac{|\{x \in G : |f(x) - M| > t\}|}{|G|} \leq \alpha$$

for some given $t, \alpha \geq 0$. Then for all $A \subseteq G$ with $|A|/|G| \geq 1/2$, we have

$$\frac{|N^t(A)|}{|G|} \geq 1 - \alpha.$$

Proof (Hints). Consider an appropriate Lipschitz function with Levy mean 0. □

Proof. The function $f(x) = d(x, A)$ is Lipschitz, and has 0 as a Levy mean. So

$$\frac{|\{x \in G : d(x, A) > t\}|}{|G|} = \frac{|\{x \in G : x \notin N^t(A)\}|}{|G|} \leq \alpha.$$

□

2.1. Concentration of measure

2.2. Edge-isoperimetric inequalities

Definition 2.34 For a graph G and $A \subseteq V(G)$, the **edge-boundary** of A is

$$\partial_e A = \partial A := \{xy \in E : x \in A, y \notin A\}.$$

Definition 2.35 An **edge-isoperimetric inequality** on a graph G is an inequality of the form

$$\forall A \subseteq G, \quad |\partial A| \geq f(|A|).$$

Example 2.36 We are interested in the case $G = Q_n$. Given $|A|$, which $A \subseteq Q_n$ should we take to minimise $|\partial A|$? Let $|A| = 4$, $A \subseteq Q_3$. TODO: insert diagram. This suggests that subcubes are best. If $2^k < |A| < 2^{k+1}$, then it is natural to take $A = \mathbb{P}([k]) \cup \text{some sets in } \mathbb{P}([k+1])$. So we define:

Definition 2.37 For $x, y \in Q_n$, $x \neq y$, say $x < y$ in the **binary ordering** on Q_n if $\max(x \Delta y) \in y$. Equivalently,

$$x < y \iff \sum_{i \in x} 2^i < \sum_{i \in y} 2^i.$$

"Go up in subcubes". Effectively, we are counting the naturals up to 2^{n-1} (where an n-bit binary string corresponds to a subset of Q_n in the obvious way).

Example 2.38 The elements of Q_3 in ascending binary order are

$$\emptyset, 1, 2, 12, 3, 13, 23, 123.$$

Definition 2.39 For $A \subseteq Q_n$, $1 \leq i \leq n$, the ***i*-binary-compression** $B_i(A) \subseteq Q_n$ is defined by its i-sections:
- $(B_i(A))_-^{(i)}$ is the initial segment of binary ordering on $\mathbb{P}(X - i)$ of size $\left|A_-^{(i)}\right|$.
- $(B_i(A))_+^{(i)}$ is the initial segment of binary ordering on $\mathbb{P}(X - i)$ of size $\left|A_+^{(i)}\right|$.

So $|B_i(A)| = |A|$.

Definition 2.40 $A \subseteq Q_n$ is ***i*-binary-compressed** if $B_i(A) = A$.

Example 2.41 A set $B \subseteq Q_n$ which is i-binary-compressed for all $1 \leq i \leq n$ is not necessarily an initial segment of binary, e.g. $\{\emptyset, 1, 2, 3\} \subseteq Q_3$. However, we have:

Lemma 2.42 Let $B \subseteq Q_n$ be i-binary-compressed for all $1 \leq i \leq n$ but not an initial segment of binary. Then

$$B = \mathbb{P}([n-1]) \setminus \underbrace{\{1, 2, ..., n-1\}}_{\text{last point in binary order in } \mathbb{P}([n-1])} \cup \underbrace{\{n\}}_{\text{first point in binary order not in } \mathbb{P}([n-1])}$$

Proof (Hints). For $x \notin B$ and $y \in B$, show by contradiction that any $i \in [n]$ is in exactly one of x and y (it helps to visualise this), and deduce that no elements lie strictly between x and y in the binary ordering. $\square$

Proof. As B is not an initial segment, there are $x < y$ with $x \notin B$ and $y \in B$. For each $1 \leq i \leq n$, assume that $i \in x, y$. Since the i-section that y lives in is an initial segment of binary on $\mathbb{P}(X \setminus i)$ (as B is i-binary-compressed), and $x - i < y - i$ in binary on $\mathbb{P}(X \setminus i)$, we have that $x - i$ lives in the same i-section, and so $x \in B$: contradiction. Similarly, $i \notin x, y$ leads to a contradiction (as then $x < y$ in binary on $\mathbb{P}(X \setminus i)$). So $x = y^c$.

Thus, for each $y \in B$, there is at most one $x < y$ with $x \notin B$ (namely $x = y^c$), and for each $x \notin B$, there is at most one $y > x$ with $y \in B$ (namely $y = x^c$). So $B = \{z : z \leq y\} \setminus \{x\}$, where x is the predecessor of y and $y = x^c$. So we must have $y = \{n\}$ and $x = \{1, 2, ..., n-1\}$. □

Theorem 2.43 (Edge-isoperimetric Inequality in Q_n) Let $A \subseteq Q_n$ and let C be the initial segment of binary on Q_n with $|C| = |A|$. Then $|\partial C| \leq |\partial A|$. In particular, if $|A| = 2^k$, then $|\partial A| \geq 2^k(n-k)$.

Proof (Hints).

- By induction on n.
- Prove for each $1 \leq i \leq n$, $|\partial B_i(A)| \leq |\partial A|$, by expressing ∂A as a disjoint union of three sets (it helps to visualise this), and using that B_+ and B_- are nested (why?).
- Define a sequence $A_0, A_1, ...$ in the obvious way, show it terminates by considering a suitable function A_k.
- Use above lemma to conclude the result.

□

Proof. By induction on n. $n = 1$ is trivial. For $n > 1$ and $A \subseteq Q_n$, and $1 \leq i \leq n$, we claim that $|\partial B_i(A)| \leq |\partial A|$.

Proof of claim. Write $B = B_i(A)$. We have (remember $A_-, A_+ \subseteq Q_{n-1}$ not Q_n)

$$|\partial A| = \underbrace{|\partial A_-|}_{\text{downstairs}} + \underbrace{|\partial A_+|}_{\text{upstairs}} + \underbrace{|A_+ \Delta A_-|}_{\text{across}}$$

and similarly, $|\partial B| = |\partial B_-| + |\partial B_+| + |B_+ \Delta B_-|$. Now, $|\partial B_-| \leq |\partial A_-|$ and $|\partial B_+| \leq |\partial A_+|$ by the induction hypothesis. Also, the sets B_+ and B_- are nested/comparable (one is contained in the other), as each is an initial segment of binary on $\mathbb{P}(X - i)$. So, since $|B_-| = |A_-|$ and $|B_+| = |A_+|$ by definition, we have

$$|B_+ \Delta B_-| = |B_+| - |B_-| = |A_+| - |A_-| \leq |A_+ - A_-| \leq |A_+ \Delta A_-|.$$

if $B_- \subseteq B_+$, and similarly this holds if $B_+ \subseteq B_-$. So $|\partial B| \leq |\partial A|$. This proves the claim. □

Define a sequence $A_0, A_1, ...$ as follows: set $A_0 = A$. Having defined $A_0, ..., A_k$, if A_k is i-binary-compressed for all $1 \leq i \leq n$, then stop the sequence with A_k. Otherwise, choose i with $B_i(A_k) \neq A_k$, and set $A_{k+1} = A_k$. This must terminate, as the function $k \mapsto \sum_{x \in A_k}$ (position of x in binary) is strictly decreasing.

The final family in this sequence $B = A_k$ satisfies $|B| = |A|$, $|\partial B| \leq |\partial A|$, and B is i-binary-compressed for all $1 \leq i \leq n$. Then by Lemma 2.42, we are done, since if B is not initial segment, then clearly we have $|\partial B| \geq |\partial C|$, since in that case $C = \mathbb{P}([n-1])$.

Remark 2.44 It is vital in the proof, and of Harper's theorem, that the extremal sets, i.e. the i-sections of the compression (in dimension $n-1$) were nested.

Definition 2.45 The **isoperimetric number** of a graph G is

$$i(G) := \min\left\{\frac{|\partial A|}{|A|} : A \subseteq G, \frac{|A|}{|G|} \le \frac{1}{2}\right\}.$$

$|\partial A|/|A|$ can be thought as the average out-degree of A.

Corollary 2.46 We have $i(Q_n) = 1$.

Proof (Hints). Straightforward. □

Proof. Taking $A = \mathbb{P}(n-1)$ shows that $i(Q_n) \le 1$. To show $i(Q_n) \ge 1$, by the above theorem, we just need to show that if C is an initial segment of binary with $|C| \le 2^{n-1}$, then $|\partial C| \ge |C|$. But in this case, $C \subseteq \mathbb{P}(n-1)$, so certainly $|\partial C| \ge |C|$. □

2.3. Inequalities in the grid

Definition 2.47 For $k \ge 2$ and $n \in \mathbb{N}$, the **grid** is the graph on $[k]^n$ in which x is joined to y if

$$\exists 1 \le i \le n : |x_i - y_i| = 1 \text{ and } \forall j \ne i, \quad x_j = y_j.$$

"The distance is the ℓ_1 distance". Note that for $k = 2$, this is precisely the definition of Q_n.

Notation 2.48 For a point x in the grid on $[k]^n$, write $|x|$ for $\sum_{i=1}^n |x_i| = \|x\|_{\ell_1}$. So x is joined to y in the grid on $[k]^n$ iff $\|x - y\|_{\ell_1} = 1$.

Example 2.49 Which sets $A \subseteq [k]^n$ (of a given size) minimise $|N(A)|$? TODO: insert diagram. This suggests we "go up in levels" according to $|x| = \sum_{i=1}^n |x_i|$, so we'd take $\{x \in [k]^n : |x| \le r\}$. If

$$|\{x \in [k]^n : |x| \le r\}| < |A| < |\{x \in [k]^n : |x| \le r+1\}|,$$

then a reasonable guess is to take $A = \{x \in [k]^n : |x| \le r\}$ together with some points with x with $|x| = r + 1$. This suggests the following definition:

Definition 2.50 The **simplicial order** on the grid $[k]^n$ defines $x < y$ if either $|x| < |y|$, or $|x| = |y|$ and $x_i > y_i$, where $i = \min\{j \in [n] : x_j \ne y_j\}$.

Note that this definition agrees with the definition of simplicial order on the cube (i.e. when $k = 2$).

Example 2.51 The elements of $[3]^2$ in ascending simplicial order are

$$(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), (3,2), (2,3), (3,3).$$

The elements of $[4]^2$ in ascending simplicial order are

$$(1,1,1), (2,1,1), (1,2,1), (1,1,2), (3,1,1), (2,2,1), (2,1,2), (1,3,1), (1,2,2), (1,1,3),$$
$$(4,1,1), (3,2,1), \ldots$$

Definition 2.52 For $A \subseteq [k]^n$, $n \ge 2$, and $1 \le i \le n$, the ***i*-sections** of A are the sets

$$A_j^{(i)} = A_j := \{x \in [k]^{n-1} : (x_1, ..., x_{i-1}, j, x_{i+1}, ..., x_{n-1}) \in A\} \subseteq [k]^{n-1}.$$

for each $1 \leq j \leq k$.

Definition 2.53 The ***i*-compression** of $A \subseteq [k]^n$ is the set $C_i(A) \subseteq [k]^n$ which is defined by its i-sections: $C_i(A)_j$ is the initial segment of simplicial on $[k]^{n-1}$ of size $|A_j|$, for each $1 \leq j \leq k$.

We have $|C_i(A)| = |A|$.

Definition 2.54 $A \subseteq [k]^n$ is ***i*-compressed** if $C_i(A) = A$.

Definition 2.55 $A \subseteq [k]^n$ is a **down-set** if

$$\forall x \in A, \forall y \in [k]^n, \quad (y_i \leq x_i \ \forall 1 \leq i \leq n) \Longrightarrow y \in A.$$

Theorem 2.56 (Vertex-isoperimetric Inequality in the Grid) Let $A \subseteq [k]^n$ and C be the initial segment of simplicial on $[k]^n$ with $|C| = |A|$. Then $|N(C)| \leq |N(A)|$. In particular, if

$$|A| \geq |\{x \in [k]^n : |x| \leq r\}| \quad \Longrightarrow \quad |N(A)| \geq |\{x \in [k]^n : |x| \leq r+1\}|.$$

Proof (Hints).
- Use induction on n.
- Prove that $|N(C_i(A))| \leq |N(A)|$ by writing the i-section $N(A)_j^{(i)}$ as a union of three sets, doing the same for $N(C_i(A))_j^{(i)}$, using the fact that these three sets (for $C_i(A)$) are nested (why?).

$\square$

Proof. By induction on n. The case $n = 1$ follows since if $A \subseteq [k]^1$ and $A \neq \emptyset, [k]^1$, then $|N(A)| \geq |A| + 1 = |N(C)|$. Now given $n > 1$, and $A \subseteq [k]^n$, fix $1 \leq i \leq n$, we claim that $|N(C_i(A))| \leq |N(A)|$.

Proof of claim. Write $B = C_i(A)$. For any $1 \leq j \leq k$, we have

$$N(A)_j = \underbrace{N(A_j)}_{\text{from } x_i = j} \cup \underbrace{A_{j-1}}_{\text{from } x_i = j-1} \cup \underbrace{A_{j+1}}_{\text{from } x_i = j+1}$$

where we set $A_0 = A_{k+1} = \emptyset$. Similarly, $N(B)_j = N(B_j) \cup B_{j-1} \cup B_{j+1}$. Now, $|B_{j-1}| = |A_{j-1}|$ and $|B_{j+1}| = |A_{j+1}|$ by definition, and $|N(B_j)| \leq |N(A_j)|$ by the induction hypothesis. But the sets B_{j-1}, B_{j+1} and $N(B_j)$ are nested, as each is an initial segment of simplicial on $[k]^{n-1}$ (since neighbourhood of initial segment of simplicial is initial segment of simplicial). Hence $|N(B)_j| \leq |N(A)_j|$ for each $1 \leq j \leq k$, thus $|N(B)| \leq |N(A)|$. This proves the claim. $\square$

Among all $B \subseteq [k]^n$ with $|B| = |A|$ and $|N(B)| \leq |N(A)|$, pick one with $\sum_{x \in B}(\text{position of } x \text{ in simplicial})$ minimal. Then B is i-compressed for all $1 \leq i \leq n$. We consider the following cases:
- Case $n = 2$: what we know is precisely that B is a down-set. Let $r = \min\{|x| : x \notin B\}$ and $s = \max\{|x| : x \in B\}$. We may assume that $r \leq s$, since if $r = s + 1$, then

$B = \{x : |x| \leq r-1\}$, hence $B = C$. If $r = s$, then $\{x : |x| \leq r-1\} \subseteq B \subseteq \{x : |x| \leq r\}$, so clearly, $|N(B)| \geq |N(C)|$. We cannot have $\{x : |x| = s\} \subseteq B$ because then also $\{x : |x| = r\} \subseteq B$ (as B is a down-set). So there are y, y' with $|y| = |y'| = s$, $y \in B$, $y' \notin B$, and $y' = y \pm (e_1 - e_2)$ (where $e_1 = (1,0)$, $e_2 = (0,1)$). Similarly, we cannot have $\{x : |x| = r\} \cap B = \emptyset$, because then $\{x : |x| = s\} \cap B = \emptyset$ (since B is a down-set): contradiction. So there are x, x' with $|x| = |x'| = r$, $x \notin B$, $x' \in B$, and $x' = x \pm (e_1 - e_2)$. Now let $B' = B \cup \{x\} \setminus \{y\}$. From B we lost at least one point in the neighbourhood (namely z) and gained at most one point, so $|N(B')| \leq |N(B)|$, but this contradicts the minimality of B.

- Case $n \geq 3$: for any $1 \leq i \leq n-1$ and any $x \in B$ with $x_n > 1$ and $x_i < k$, we have $x - e_n + e_i \in B$, since B is j-compressed for any $j \neq i, n$. So, considering the n-sections of B, we have $N(B_t) \subseteq B_{t-1}$ for all $t = 2, ..., k$. Recall that $N(B)_t = N(B_t) \cup B_{t+1} \cup B_{t-1}$. So in fact, $N(B)_t = B_{t-1}$ for all $t \geq 2$. Thus
$$|N(B)| = \underbrace{|B_{k-1}|}_{\text{level } k} + \underbrace{|B_{k-2}|}_{\text{level } k-1} + \cdots + \underbrace{|B_1|}_{\text{level } 2} + \underbrace{|N(B_1)|}_{\text{level } 1} = |B| - |B_k| + |N(B_1)|.$$
Similarly, $|N(C)| = |C| - |C_k| + |N(C_1)|$. So to show $|N(C)| \leq |N(B)|$, it is enough to show that $|B_k| \leq |C_k|$ and $|B_1| \geq |C_1|$ (since B_1, C_1 and their neighbourhoods are initial segments of simplicial).

$|B_k| \leq |C_k|$: define a set $D \subseteq [k]^n$ as follows: let $D_k := B_k$, and for $t = k-1, k-2, ..., 1$, set $D_t := N(D_{t-1})$. Then $D \subseteq B$, so $|D| \leq |B|$. Also, D is an initial segment of simplicial. So in fact, $D \subseteq C$, whence $|B_k| = |D_k| \leq |C_k|$.

$|B_1| \geq |C_1|$: define a set $E \subseteq [k]^n$ as follows: set $E_1 = B_1$ and for $t = 2, 3, ..., k$, set $E_t = \{x \in [k]^{n-1} : N(\{x\}) \subseteq E_{t-1}\}$, so E_t is the biggest set whose neighbourhood is contained in E_{t-1}. Then $B \subseteq E$, so $|E| \geq |B|$. Also, E is an initial segment of simplicial. So $C \subseteq E$, whence $|B_1| = |E_1| \geq |C_1|$.

Corollary 2.57 Let $A \subseteq [k]^n$ and $|A| \geq |\{x : |x| \leq r\}|$. Then $|N^t(A)| \geq |\{x : |x| \leq r+t\}|$ for all t.

Proof. By induction, using above. □

Remark 2.58 We can check from the above corollary that, for k fixed, the sequence $\{[k]^n : n \in \mathbb{N}\}$ is a Levy family.

2.4. The edge-isoperimetric inequality in the grid

Example 2.59 Which set $A \subseteq [k]^n$ of given size should we take to minimise $|\partial A|$? In $[k]^2$, TODO: insert diagram. This suggests squares are best. However, TODO: insert diagram. So we have "phase transitions" at $|A| \approx k^2/4$ and $|A| \approx 3k^2/4$. So the extremal sets are not nested. This seems to rule out all our compression methods. And in $[k]^3$? We start with cube $[a]^3$, then square column $[a]^2 \times [k]$, then "half space" $[a] \times [k]^2$, then complement of square column, then copmlement of cube. So in $[k]^n$, up to $|A| = k^n/2$, we get $n-1$ of these "phase transitions".

Theorem 2.60 (Edge-isoperimetric Inequality in the Grid) Let $A \subseteq [k]^n$. If $|A| \leq k^n/2$, then

$$|\partial A| \geq \min\left\{d|A|^{1-1/d}k^{n/d-1} : 1 \leq d \leq n\right\}.$$

Proof (Hints). Non-examinable. □

Proof. Non-examinable. □

Remark 2.61 Note that if $A = [a]^d \times [k]^{n-d}$, then $|\partial A| = da^{d-1}k^{n-d} = d|A|^{1-1/d}k^{n/d-1}$. So Edge-isoperimetric Inequality in the Grid says that some set of the form $[a]^d \times [k]^{n-d}$ minimises the edge boundary.

Remark 2.62 Very few isoperimetric inequalities are known (even approximately), e.g. "iso in a layer": in a graph $X^{(r)}$, with x, y joined if $|x \cap y| = r - 1$. This is open. A nice special case is $r = n/2$, where it is conjectured that balls are best, i.e. sets of the form $\left\{x \in [2r]^{(r)} : |x \cap [r]| \geq t\right\}$.

3. Intersecting families

3.1. t-intersecting families

Definition 3.1 $A \subseteq \mathbb{P}(X)$ is ***t*-intersecting** if

$$\forall x, y \in A, \quad |x \cap y| \geq t.$$

Example 3.2 How large can a t-intersecting family be? For $t = 2$, we could take $\{x \subseteq X : 1, 2 \in x\}$, which has size $\frac{1}{4} \cdot 2^n$, but better is $\{x \subseteq X : |x| \geq n/2 + 1\}$.

Theorem 3.3 (Katona's t-intersecting Theorem) Let $A \subseteq \mathbb{P}(X)$ be t-intersecting, where $n \equiv t \bmod 2$. Then

$$|A| \leq \left|X^{(\geq(n+t)/2)}\right|.$$

Proof. For any $x, y \in A$, we have $|x \cap y| \geq t$, so $d(x, y^c) \geq t$. Writing $\overline{A} = \{y^c : y \in A\}$, we have $d\left(A, \overline{A}\right) \geq t$, i.e. $A_{(t-1)}$ is disjoint from $\overline{A}$. Suppose for a contradiction $|A| > \left|X^{(\geq(n+t)/2)}\right|$. Then by Harper, we have

$$\left|N^{t-1}(A)\right| \geq \left|X^{(\geq(n+t)/2-(t-1))}\right| = \left|X^{(\geq(n-t)/2+1)}\right|.$$

But $N^{t-1}(A)$ is disjoint from $\overline{A}$ which has size $> \left|X^{(\leq(n-t)/2)}\right|$, contradicting $\left|N^{t-1}(A)\right| + \left|\overline{A}\right| \leq 2^n$. □

Example 3.4 What about t-intersecting A with $A \subseteq X^{(r)}$? We might guess that the best is $A_0 = \left\{x \in X^{(r)} : [t] \subseteq x\right\}$. We could also try $A_\alpha = \left\{x \in X^{(r)} : |x \cap [t + 2\alpha]| \geq t + \alpha\right\}$ for $\alpha = 1, \ldots, r - t$.

For 2-intersecting families in $[7]^{(4)}$: $|A_0| = \binom{5}{2} = 10$, $|A_1| = 1 + \binom{4}{3}\binom{3}{1} = 13$, $|A_2| = \binom{6}{4} = 15$.

For 2-intersecting families in $[8]^{(4)}$: $|A_0| = \binom{6}{2} = 15$, $|A_1| = 1 + \binom{4}{3}\binom{4}{1} = 17$, $|A_2| = \binom{6}{4} = 15$.

For 2-intersecting families in $[9]^{(4)}$: $|A_0| = \binom{7}{2} = 21$, $|A_1| = 1 + \binom{4}{3}\binom{5}{1} = 21$, $|A_2| = \binom{6}{4} = 15$.

Note that A_0 grows quadratically, A_1 grows linearly, A_2 is constant, so A_0 is the largest of these for large n.

Theorem 3.5 Let $A \subseteq X^{(r)}$ be t-intersecting. Then, for n sufficiently large, we have $|A| \leq |A_0| = \binom{n-t}{r-t}$.

Proof. Idea: "A_0 has $t - t$ degrees of freedom".

Extending A to a maximal t-intersecting family, we must have some $x, y \in A$ with $|x \cap y| = t$ (if not, then by maximality, we have that $\forall x \in A, \forall i \in x, \forall j \notin x$, $x - i \cup j \in A$; repeating this, we have $A = X^{(r)}$: contradiction). We may assume that there exists $z \in A$ with $x \cap y \nsubseteq z$; otherwise, all $z \in A$ have the t-set $x \cap y \subseteq z$, whence $|A| \leq \binom{n-t}{r-t} = |A_0|$. So each $w \in A$ must meet $x \cup y \cup z$ in $\geq t+1$ points. Thus

$$|A| \leq \underbrace{2^{3r}}_{w \text{ on } x \cup y \cup z} \cdot \underbrace{\left(\binom{n}{r-t-1} + \binom{n}{r-t-2} + \cdots + \binom{n}{0}\right)}_{w \text{ off } x \cup y \cup z}$$

which is a polynomial in n of degree $r - t - 1$. □

Remark 3.6

- The bound we obtain for n would be $\geq (16r)^r$ (crude) or $2tr^3$ (careful).
- The theorem is often called the "second Erdos-Ko-Rado" theorem.

3.2. Modular intersections

Example 3.7 For intersecting families, we ban $|x \cap y| = 0$. What if we banned $|x \cap y| = 0 \bmod k$ for some $k \in \mathbb{N}$?

e.g. want $A \subseteq X^{(r)}$ with $|x \cap y|$ odd for all $x \neq y \in A$.

Try r odd: can achieve $|A| = \binom{\lfloor (n-1)/2 \rfloor}{(r-1)/2}$ by the diagram. What if, still for r odd, we want $|x \cap y|$ even for all $x \neq y \in A$. Can achieve $n - r + 1$ by picture, but this is only linear in n.

Similarly: for r even, if we want $|x \cap y|$ even for all $x \neq y \in A$, can achieve $|A| = \binom{\lfloor n/2 \rfloor}{r/2}$ by picture. If we want $|x \cap y|$ odd for all $x \neq y \in A$, can achieve $n - r + 1$ as above.

Seems to be that banning $|x \cap y| = r \pmod 2$ forces the family to be very small (poly in n, even a linear poly).