# 1. The real numbers

# 1.1. Conventions on sets and functions

**Definition.** For $f : X \to Y$, **preimage** of $Z \subseteq Y$ is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

**Definition.** $f : X \to Y$ **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

**Definition.** $f : X \to Y$ **surjective** if $Y = f(X)$.

**Proposition.** Let $f : X \to Y$, $A, B \subseteq X$, then

$$f(A \cap B) \subseteq f(A) \cap f(B),$$
$$f(A \cup B) = f(A) \cup f(B),$$
$$f(X) - f(A) \subseteq f(X - A)$$

**Proposition.** Let $f : X \to Y$, $C, D \subseteq Y$, then

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D),$$
$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D),$$
$$f^{-1}(Y - C) = X - f^{-1}(C)$$

# 1.2. The real numbers

**Definition.** $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \leq a$.

**Definition.** $c \in \mathbb{R}$ is a **least upper bound (supremum)** of $E$, $c = \sup(E)$, if $c \leq a$ for every upper bound $a$.

**Definition.** $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \geq a$.

**Definition.** $c \in \mathbb{R}$ is a **greatest lower bound (infimum)**, $c = \inf(E)$, if $c \geq a$ for every lower bound $a$.

**Theorem** (Completeness axiom of the real numbers). Every $E \subseteq \mathbb{R}$ with an upper bound has a least upper bound. Every $E \subseteq \mathbb{R}$ with a lower bound has a greatest lower bound.

**Proposition** (Archimedes' principle).

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

**Remark.** Every non-empty subset of $\mathbb{N}$ has a minimum.

**Proposition.** $\mathbb{Q}$ is dense in $\mathbb{R}$:

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

# 1.3. Sequences, limits and series

**Definition.** $l \in \mathbb{R}$ is **limit** of $(x_n)$ $((x_n)$ converges to $l)$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad |x_n - l| < \varepsilon$$

A sequence **converges in $\mathbb{R}$ (is convergent)** if it has a limit $l \in \mathbb{R}$. Limit $l = \lim_{n \to \infty} x_n$ is unique.

**Definition.** $(x_n)$ **tends to infinity** if

$$\forall K > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad x_n > K$$

**Definition.** **Subsequence** of $(x_n)$ is sequence $(x_{n_j})$, $n_1 < n_2 < \cdots$.

**Definition.** **Limit inferior** of sequence $x_n$ is

$$\liminf_{n \to \infty} x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{m \geq n} x_m \right\} = \lim_{n \to \infty} \left( \inf_{m \geq n} x_m \right)$$

**Definition.** **Limit superior** of sequence $x_n$ is

$$\limsup_{n\to\infty} x_n := \inf_{n\in\mathbb{N}} \left\{ \sup_{m\geq n} x_m \right\} = \lim_{n\to\infty} \left( \sup_{m\geq n} x_m \right)$$

**Proposition.** Let $(x_n)$ bounded, $l \in \mathbb{R}$. Then $l = \limsup x_n$ iff both of the following hold:

- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < l + \varepsilon.$
- $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \geq N : x_n > l - \varepsilon.$

**Proposition.** Let $(x_n)$ bounded, $l \in \mathbb{R}$. Then $l = \liminf x_n$ iff both of the following hold:

- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > l - \varepsilon.$
- $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \geq N : x_n < l + \varepsilon.$

**Theorem** (Bolzano-Weierstrass). Every bounded sequence has a convergent subsequence.

**Proposition.** Let $(x_n)$ bounded. There exists convergent subsequence with limit $\limsup x_n$ and convergent subsequence with limit $\liminf x_n$.

**Proposition.** Let $(x_n)$ bounded, then $(x_n)$ is convergent iff $\limsup x_n = \liminf x_n$.

**Theorem** (Monotone convergence theorem for sequences).
Monotone sequence converges in $\mathbb{R}$ or tends to either $\infty$ or $-\infty$.

**Definition.** $(x_n)$ is **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad |x_n - x_m| < \varepsilon$$

**Theorem**.  Every Cauchy sequence in $\mathbb{R}$ is convergent.

# 1.4. Open and closed sets

**Definition.** $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subseteq U$$

**Proposition.** Arbitrary unions of open sets are open. Finite intersections of open sets are open.

**Definition.** $x \in \mathbb{R}$ is **point of closure (limit point)** for $E \subseteq \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists y \in E : |x - y| < \varepsilon$$

Equivalently, $x$ is point of closure of $E$ if every open interval containing $x$ contains a point of $E$.

**Definition.** **Closure** of $E$, $\overline{E}$, is set of points of closure. Note $E \subseteq \overline{E}$.

**Definition.** $F$ is **closed** if $F = \overline{F}$.

**Proposition.** $\overline{A \cup B} = \overline{A} \cup \overline{B}$. If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$.

**Proposition.** For any set $E$, $\overline{E}$ is closed, i.e. $\overline{E} = \overline{\overline{E}}$.

**Proposition.** $E \subseteq \mathbb{R}$ is closed iff $\mathbb{R} - E$ is open.

**Proposition.** Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.

**Definition.** Collection $C$ of subsets of $\mathbb{R}$ **covers** (is a **covering** of) $F \subseteq \mathbb{R}$ if $F \subseteq \cup_{S \in C} S$. If each $S$ in $C$ open, $C$ is **open covering**. If $C$ is finite, $C$ is **finite covering**.

**Definition.** Covering $C$ of $F$ **contains a finite subcover** if exists $\{S_1, ..., S_n\} \subseteq C$ with $F \subseteq \cup_{i=1}^n S_i$ (i.e. a finite subset of $C$ covers $F$).

**Definition**.  $F$ is **compact** if any open covering of $F$ contains a finite subcover.

**Example.** $\mathbb{R}$ is not compact, $[a, b]$ is compact.

**Theorem** (Heine Borel). $F$ compact iff $F$ closed and bounded.

# 1.5. Continuity, pointwise and uniform convergence of functions

**Definition.** Let $E \subseteq \mathbb{R}$. $f : E \to \mathbb{R}$ is **continuous at** $a \in E$ if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

$f$ is **continuous** if continuous at all $y \in E$.

**Definition.** $\lim_{x \to a} f(x) = l$ if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

**Proposition.** $\lim_{x \to a} f(x) = l$ iff for every sequence $(a_n)$ with $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} f(a_n) = l$.

**Proposition.** $f$ is continuous at $a \in E$ iff $\lim_{x \to a} f(x) = f(a)$ (and this limit exists).

**Definition.** $f : E \to \mathbb{R}$ is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in E, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

**Proposition**. Let $F$ closed and bounded, $f : F \to \mathbb{R}$ continuous. Then $f$ is uniformly continuous.

**Definition.** Let $f_n : E \to \mathbb{R}$ sequence of functions, $f : E \to \mathbb{R}$. $(f_n)$ **converges pointwise** to $f$ if

$$\forall \varepsilon > 0, \forall x \in E, \exists N \in \mathbb{N} : \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

$(f_n)$ **converges uniformly** to $f$ is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$$

**Theorem.** Let $f_n : E \to \mathbb{R}$ sequence of continuous functions converging uniformly to $f : E \to \mathbb{R}$. Then $f$ is continuous.

**Definition.** $P = \{x_0, ..., x_n\}$ is **partition** of $[a, b]$ if $a = x_0 < \cdots < x_n = b$.

**Definition.** $f : [a, b] \to \mathbb{R}$ is **piecewise linear** if there exists partition $P = \{x_0, ..., x_n\}$ and $m_i, c_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad f(x) = m_i x + c_i$$

$f$ is continuous on $[a, b] - P$.

**Definition.** $g : [a, b] \to \mathbb{R}$ is **step function** if there exists partition $P = \{x_0, ..., x_n\}$ and $m_i \in \mathbb{R}$ such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad g(x) = m_i$$

$g$ is continuous on $[a, b] - P$.

**Theorem.** Let $f : E \to \mathbb{R}$ continuous, $E$ closed and bounded. Then there exist continuous piecewise linear $f_n$ with $f_n \to f$ uniformly, and step functions $g_n$ with $g_n \to f$ uniformly.

**Definition.** $f : E \to \mathbb{R}$ is **Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad |f(x) - f(y)| \leq C|x - y|$$

**Definition.** $f : E \to \mathbb{R}$ is **bi-Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

# 1.6. The extended real numbers

**Definition.** **Extended reals** are $\mathbb{R} \cup \{-\infty, \infty\}$ with the order relation $-\infty < \infty$ and $\forall x \in \mathbb{R}, -\infty < x < \infty$. $\infty$ is an upper bound and $-\infty$ is a lower bound for every $x \in \mathbb{R}$, so $\sup(\mathbb{R}) = \infty$, $\inf(\mathbb{R}) = -\infty$, $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$.

- Addition: $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$. $\infty + \infty = \infty - (-\infty) = \infty$. $\infty - \infty$ is undefined.

- Multiplication: $\forall a > 0, a \cdot \infty = \infty$, $\forall a < 0, a \cdot \infty = -\infty$. Also $\infty \cdot \infty = \infty$.

- $\limsup$ and $\liminf$ are defined as

$$\limsup x_n := \inf\{\sup\{x_k : k \geq n\} : n \in \mathbb{N}\}, \quad \liminf x_n := \sup\{\inf\{x_k : k \geq n\} : n \in \mathbb{N}\}$$

**Definition.** Extended real number $l$ is **limit** of $(x_n)$ if either

- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$. Then $(x_n)$ **converges to** $l$. or
- $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$ (limit is $\infty$) or
- $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$ (limit is $-\infty$).

$(x_n)$ **converges in the extended reals** if it has a limit in the extended reals.

# 2. Further analysis of subsets of $\mathbb{R}$

# 2.1. Countability and uncountability

**Definition.** $A$ is **countable** if $A = \emptyset$, $A$ is finite or there is a bijection $\varphi : \mathbb{N} \to A$ (in which case $A$ is **countably infinite**). Otherwise $A$ is **uncountable**. **Enumeration** is bijection to $A$ from $[n]$ or $\mathbb{N}$.

**Proposition.** If there is surjection from countable set to $A$, or injection from $A$ to countable set, then $A$ is countable.

**Proposition.** Any subset of $\mathbb{N}$ is countable.

**Proposition.** $\mathbb{Q}$ is countable.

**Proposition.** If $(a_n)$ is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

**Proposition.** If $(a_{n,k})$ is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

**Definition.** $f : X \to Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.

**Proposition.** Let $f$ be monotone on $(a, b)$. Then it is discontinuous on a countable set.

**Lemma.** Set of sequences in $\{0, 1\}$, $\{(x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.

**Theorem**. $\mathbb{R}$ is uncountable.

## 2.2. The structure theorem for open sets

**Definition.** Collection $\{A_i : i \in I\}$ of sets is **(pairwise) disjoint** if $n \neq m \implies A_n \cap A_m = \emptyset$.

**Theorem** (Structure theorem for open sets). Let $U \subseteq \mathbb{R}$ open. Then exists countable collection of disjoint open intervals $\{I_n : n \in \mathbb{N}\}$ such that $U = \cup_{n \in \mathbb{N}} I_n$.

## 2.3. Accumulation points and perfect sets

**Definition.** $x \in \mathbb{R}$ is **accumulation point** of $E \subseteq \mathbb{R}$ if $x$ is point of closure of $E - \{x\}$. Equivalently, $x$ is a point of closure if

$$\forall \varepsilon > 0, \exists y \in E : y \neq x \land |x - y| < \varepsilon$$

Equivalently, there exists a sequence of distinct $y_n \in E$ with $y_n \to x$ as $n \to \infty$.

**Proposition.** Set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$.

**Proposition.** Set of accumulation points $E'$ of $E$ is closed.

**Definition.** $E \subseteq \mathbb{R}$ is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

**Proposition.** $E$ is isolated iff it has no accumulation points.

**Definition**.  Bounded set $E$ is **perfect** if it equals its set of accumulation points.

**Theorem**. Every non-empty perfect set is uncountable.

## 2.4. The middle-third Cantor set

**Proposition.** Let $\{F_n : n \in \mathbb{N}\}$ be collection of non-empty nested closed sets (so $F_{n+1} \subseteq F_n$), one of which is bounded. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$$

**Definition.** The **middle third Cantor set** is defined by:

- Define $C_0 := [0, 1]$
- Given $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$, $a_1 < b_1 < a_2 < \cdots < a_{2^n} < b_{2^n}$, with $|b_i - a_i| = 3^{-n}$, define

$$C_{n+1} := \cup_{i=1}^{2^n} \left[a_i, a_i + 3^{-(n+1)}\right] \cup \left[b_i - 3^{-(n+1)}, b_i\right]$$

  which is a union of $2^{n+1}$ disjoint intervals, with all differences in endpoints equalling $3^{-(n+1)}$.

- The **middle third Cantor set** is

$$C := \bigcap_{n \in \mathbb{N}_0} C_n$$

Observe that if $a$ is an endpoint of an interval in $C_n$, it is contained in $C$.

**Proposition.** The middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and so uncountable.

**Definition.** Let $k \in \mathbb{N} - \{1\}$, $x \in [0, 1)$. $0.a_1 a_2 ...$, $a_i \in \{0, ..., k-1\}$, is a **$k$-ary expansion** of $x$ if

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{k^i}$$

**Remark.** The $k$-ary expansion may not be unique, but there is a countable set $E \subseteq [0, 1)$ such that every $x \in [0, 1) - E$ has a unique $k$-ary expansion.

**Remark.** For every $x \in C$, the ternary $(k = 3)$ expansion of $x$ is unique and

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}$$

Moreover, every choice of sequence $(a_i)$, $a_i \in \{0, 2\}$, gives $x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i} \in C$.

**Definition.** **Cantor-Lebesgue function**, $g : [0, 1] \to [0, 1]$, is defined by

$$g(x) := \begin{cases} \sum_{i \in \mathbb{N}} \frac{a_i/2}{2^i} & \text{if } x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, a_i \in \{0, 2\} \\ \sup\{g(y) : y \in C, y \leq x\} & \text{if } x \notin C \end{cases}$$

$g$ is a surjection, monotone and continuous.

**2.5.** $G_\delta, F_\sigma$

**Definition.** $E \subseteq \mathbb{R}$ is $G_\delta$ if $E = \cap_{n \in \mathbb{N}} U_n$ with $U_n$ open.

**Definition.** $E \subseteq \mathbb{R}$ is $F_\sigma$ if $E = \cup_{n \in \mathbb{N}} F_n$ with $F_n$ closed.

**Lemma.** Set of points where $f : \mathbb{R} \to \mathbb{R}$ is continuous is $G_\delta$.

# 3. Construction of Lebesgue measure

# 3.1. Lebesgue outer measure

**Definition.** Let $I$ non-empty interval with endpoints $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$ and $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$. The **length** of $I$ is

$$\ell(I) := b - a$$

and set $\ell(\emptyset) = 0$.

**Definition.** Let $A \subseteq \mathbb{R}$. **Lebesgue outer measure** of $A$ is infimum of all sums of lengths of intervals covering $A$:

$$\mu^*(A) := \inf \left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subseteq \bigcup_{k \in \mathbb{N}} I_k, I_k \text{ intervals} \right\}$$

It satisfies **monotonicity**: $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

**Proposition.** Outer measure is **countably subadditive**:

$$\mu^* \left( \bigcup_{k \in \mathbb{N}} E_k \right) \leq \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

This implies **finite subadditivity**:

$$\mu^* \left( \bigcup_{k=1}^{n} E_k \right) \leq \sum_{k=1}^{n} \mu^*(E_k)$$

**Lemma.** We have

$$\mu^*(A) = \inf\left\{\sum_{k \in \mathbb{N}} \ell(I_k) : A \subset \bigcup_{k \in \mathbb{N}} I_k, I_k \neq \emptyset \text{ open intervals}\right\}$$

**Proposition.** Outer measure of interval is its length: $\mu^*(I) = \ell(I)$.

# 3.2. Measurable sets

**Notation.** $E^c = \mathbb{R} - E$.

**Proposition.** Let $E = (a, \infty)$. Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

**Definition.** $E \subseteq \mathbb{R}$ is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Collection of such sets is $\mathcal{F}_{\mu^*}$.

**Lemma** (Excision Property). Let $E$ Lebesgue measurable set with finite measure and $E \subseteq B$, then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

**Proposition.** If $E_1, ..., E_n$ Lebesgue measurable then $\cup_{k=1}^n E_k$ is Lebesgue measurable. If $E_1, ..., E_n$ disjoint then

$$\mu^* \left( A \cap \bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(A \cap E_k)$$

for any $A \subseteq \mathbb{R}$. In particular, for $A = \mathbb{R}$,

$$\mu^* \left( \bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(E_k)$$

**Remark.** Not every set is Lebesgue measurable.

**Definition.** Collection of subsets of $\mathbb{R}$ is an **algebra** if contains $\emptyset$ and closed under taking complements and finite unions: if $A, B \in \mathcal{A}$ then $\mathbb{R} - A, A \cup B \in \mathcal{A}$.

**Remark.** A union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if $\{A_k\}_{k \in \mathbb{N}}$ is countable collection of Lebesgue measurable sets, then let $A_1' := A_1$ and for $k > 1$, define

$$A_k' := A_k - \cup_{i=1}^{k-1} A_i$$

then $\{A_k'\}_{k \in \mathbb{N}}$ is disjoint union of Lebesgue measurable sets and $\cup_{k \in \mathbb{N}} A_k' = \cup_{k \in \mathbb{N}} A_k$.

**Proposition.** If $E$ is countable union of Lebesgue measurable sets, then $E$ is Lebesgue measurable. Also, if $\{E_k\}_{k \in \mathbb{N}}$ is countable disjoint collection of Lebesgue measurable sets then

$$\mu\left(\bigcup_{k \in \mathbb{N}} E_k\right) = \sum_{k \in \mathbb{N}} \mu(E_k)$$

## 3.3. Abstract definition of a measure

**Definition.** Let $X \subseteq \mathbb{R}$. Collection of subsets of $\mathcal{F}$ of $X$ is $\sigma$-**algebra** if

- $\emptyset \in \mathcal{F}$
- $E \in \mathcal{F} \implies E^c \in \mathcal{F}$
- If $\forall k \in \mathbb{N}, E_k \in \mathcal{F}$ then $\cup_{k \in \mathbb{N}} E_k \in \mathcal{F}$.

**Example.**

- Trivial examples are $\mathcal{F} = \{\emptyset, \mathbb{R}\}$ and $\mathcal{F} = \mathcal{P}(\mathbb{R})$.
- Countable intersections of $\sigma$-algebras are $\sigma$-algebras.

**Definition.** Let $\mathcal{F}$ $\sigma$-algebra of $X$. $\nu : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is **measure** satisfying

- $\nu(\emptyset) = 0$
- $\forall E \in \mathcal{F}, \nu(E) \geq 0$
- **Countable additivity**: if $E_1, E_2, \dots \in \mathcal{F}$ are disjoint then

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k)$$

Elements of $\mathcal{F}$ are **measurable** (as they are the only sets on which the measure $\nu$ is defined).

**Proposition.** If $\nu$ is measure then it satisfies:

- **Monotonicity**: $A \subseteq B \implies \nu(A) \leq \nu(B)$.
- **Countable subadditivity**: $\nu(\cup_{k \in \mathbb{N}} E_k) \leq \sum_{k \in \mathbb{N}} \nu(E_k)$.
- **Excision**: if $B$ has finite measure, then $A \subseteq B \implies \nu(B - A) = \nu(B) - \nu(A)$.

# 3.4. Lebesgue measure

**Lemma.** $F_{\mu^*}$ is $\sigma$-algebra and contains every interval.

**Theorem** (Carathéodory Extension). Restriction of the $\mu^*$ to $F_{\mu^*}$ is a measure.

**Theorem** (Hahn extension theorem). There exists unique measure $\mu$ defined on $\mathcal{F}_{\mu^*}$ for which $\mu(I) = \ell(I)$ for any interval $I$.

**Definition.** The measure $\mu$ of $\mu^*$ restricted to $\mathcal{F}_{\mu^*}$ is the **Lebesgue measure**. It satisfies $\mu(I) = \ell(I)$ for any interval $I$ and is translation invariant.

# 3.5. Sets of measure 0

**Proposition**. Middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.

**Proposition**. Any countable set is Lebesgue measurable and has Lebesgue measure 0.

**Proposition.** Any $E$ with $\mu^*(E) = 0$ is Lebesgue measurable and has $\mu(E) = 0$.

**Lemma.** Let $E$ Lebesgue measurable set with $\mu(E) = 0$, then $\forall E' \subseteq E$, $E'$ is Lebesgue measurable.

# 3.6. Continuity of measure

**Definition.** Countable collection $\{E_k\}_{k \in \mathbb{N}}$ is **ascending** if $\forall k \in \mathbb{N}, E_k \subseteq E_{k+1}$ and **descending** if $\forall k \in \mathbb{N}, E_{k+1} \subseteq E_k$.

**Theorem.** Every measure $m$ satisfies:

- If $\{A_k\}_{k \in \mathbb{N}}$ is ascending collection of measurable sets, then

$$m\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \lim_{k \to \infty} m(A_k)$$

- If $\{B_k\}_{k \in \mathbb{N}}$ is descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k \in \mathbb{N}} B_k\right) = \lim_{k \to \infty} m(B_k)$$

## 3.7. An approximation result for Lebesgue measure

**Definition.** **Borel $\sigma$-algebra** $\mathcal{B}(\mathbb{R})$ is smallest $\sigma$-algebra containing all intervals: for any other $\sigma$-algebra $\mathcal{F}$ containing all intervals, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

$$\mathcal{B}(\mathbb{R}) := \bigcap \{\mathcal{F} : \mathcal{F} \ \sigma\text{-algebra containing all intervals}\}$$

$E \in \mathcal{B}(\mathbb{R})$ is **Borel** or **Borel measurable**.

**Lemma.** All open subsets of $\mathbb{R}$, closed subsets of $\mathbb{R}$, $G_\delta$ sets and $F_\sigma$ sets are Borel.

**Proposition.** The following are equivalent:

- $E$ is Lebesgue measurable
- $\forall \varepsilon > 0, \exists$ open $G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
- $\forall \varepsilon > 0, \exists$ closed $F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
- $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
- $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

# 4. Measurable functions

# 4.1. Definition of a measurable function

**Proposition.** Let $f : \mathbb{R} \to \mathbb{R}$. $f$ continuous iff $\forall$ open $U \subseteq \mathbb{R}$, $f^{-1}(U) \subseteq \mathbb{R}$ is open.

**Lemma.** Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ with $E$ Lebesgue measurable. The following are equivalent:

- $\forall c \in \mathbb{R}, \{x \in E : f(x) > c\}$ is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) \geq c\}$ is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) < c\}$ is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) \leq c\}$ is Lebesgue measurable.

The same statement holds for Borel measurable sets.

**Definition.** $f : E \to \mathbb{R} \cup \{\pm\infty\}$ is **(Lebesgue) measurable** if it satisfies any of the above properties and if $E$ is Lebesgue measurable. $f$ being **Borel measurable** is defined similarly.

**Corollary.** If $f$ is Lebesgue measurable then for every $B \in \mathcal{B}(\mathbb{R})$, $f^{-1}(B)$ is measurable. In particular, if $f$ is Lebesgue measurable, preimage of any interval is measurable.

**Definition. Indicator function** on set $A$, $\mathbb{1}_A : \mathbb{R} \to \{0, 1\}$, is

$$\mathbb{1}_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

**Definition.** $\varphi : \mathbb{R} \to \mathbb{R}$ is **simple (measurable) function** if $\varphi$ is measurable function that has finite codomain.

## 4.2. Fundamental aspects of measurable functions

**Definition.** Let $E \subseteq F \subseteq \mathbb{R}$, let $f : F \to \mathbb{R}$. **Restriction** $f_E$ is function with domain $E$ and for which $\forall x \in E, f_E(x) = f(x)$.

**Definition.** Real-valued function which is increasing or decreasing is **monotone**.

**Definition**. Sequence $(f_n)$ on domain $E$ is increasing if $f_n \leq f_{n+1}$ on $E$ for all $n \in \mathbb{N}$.

**Example**.  Continuous functions are measurable.

**Definition.** For $f_1 : E \to \mathbb{R}, ..., f_n : E \to \mathbb{R}$, define

$$\max\{f_1, ..., f_n\}(x) := \max\{f_1(x), ..., f_n(x)\}$$

$\min\{f_1, ..., f_n\}$ is defined similarly.

**Proposition**. For finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain $E$, $\max\{f_1, ..., f_n\}$ and $\min\{f_1, ..., f_n\}$ are measurable.

**Definition.** For $f : E \to \mathbb{R}$, functions $|f|, f^+, f^-$ defined on $E$ are

$$|f|(x) := \max\{f(x), -f(x)\}, \quad f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}$$

**Corollary.** If $f$ measurable on $E$, so are $|f|$, $f^+$ and $f^-$.

**Proposition.** Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$. For measurable $D \subseteq E$, $f$ measurable on $E$ iff restrictions of $f$ to $D$ and $E - D$ are measurable.

**Theorem**. Let $f, g : E \to \mathbb{R}$ measurable.

- **Linearity**: $\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g$ is measurable.
- **Products**: $fg$ is measurable.

**Proposition.** Let $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ be sequence of measurable functions that converges pointwise to $f : E \to \mathbb{R} \cup \{\pm\infty\}$. Then $f$ is measurable.

**Lemma** (Simple approximation lemma). Let $f : E \to \mathbb{R}$ measurable and bounded, so $\exists M \geq 0 : \forall x \in E, |f|(x) < M$. Then $\forall \varepsilon > 0$, there exist simple measurable functions $\varphi_\varepsilon, \psi_\varepsilon : E \to \mathbb{R}$ such that

$$\forall x \in E, \quad \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \wedge 0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$$

**Theorem** (Simple approximation theorem). Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$, $E$ measurable. Then $f$ is measurable iff there exists sequence $(\varphi_n)$ of simple functions on $E$ which converge pointwise on $E$ to $f$ and satisfy

$$\forall n \in \mathbb{N}, \forall x \in E, |\varphi_n|(x) \leq |f|(x)$$

If $f$ is nonnegative, $(\varphi_n)$ can be chosen to be increasing.

**Definition.** Let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$. Then $f = g$ **almost everywhere** if $\{x \in E : f(x) \neq g(x)\}$ has measure 0.

**Proposition.** Let $f_1, f_2, f_3 : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable. If $f_1 = f_2$ almost everywhere and $f_2 = f_3$ almost everywhere then $f_1 = f_3$ almost everywhere.

**Remark.** Lebesgue measurable functions can be modified arbitrarily on a set of measure $0$ without affecting measurability.

**Proposition.** Let $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ sequence of measurable functions, $f : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable. Set of points where $(f_n)$ converges pointwise to $f$ is measurable.

**Proposition.** Let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable and finite almost everywhere on $E$.

- **Linearity**: $\forall \alpha, \beta \in \mathbb{R}$, there exists function equal to $\alpha f + \beta g$ almost everywhere on $E$ (any such function is measurable).
- **Products**: there exists function equal to $fg$ almost everywhere on $E$ (any such function is measurable).

**Definition.** Sequence of functions $(f_n)$ with domain $E$ **converge in measure** to $f$ if $(f_n)$ and $f$ are finite almost everywhere and

$$\forall \varepsilon > 0, \quad \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \to 0 \text{ as } n \to \infty$$

# 5. The Lebesgue integral

# 5.1. The integral of a simple measurable function

**Definition.** Let $\varphi$ be real-valued function taking finitely many values $\alpha_1 < \cdots < \alpha_n$, then **standard representation** of $\varphi$ is

$$\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}, \quad A_i = \varphi^{-1}(\{\alpha_i\})$$

**Lemma.** Let $\varphi = \sum_{i=1}^{m} \beta_i \mathbb{1}_{B_i}$, $B_i$ disjoint measurable collection, $\beta_i \in \mathbb{R}$, then $\varphi$ is simple measurable. If $\varphi$ takes value $0$ outside a set of finite measure then

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \beta_i \mu(B_i)$$

where $A_i$ in standard representation.

**Definition.** Let $\varphi$ be simple nonnegative measurable function or simple measurable function taking value $0$ outside set of finite measure. **Integral** of $\varphi$ with respect to $\mu$ is

$$\int \varphi = \sum_{i=1}^{n} \alpha_i \mu(A_i)$$

where $\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$ is standard representation. Here, use convention $0 \cdot \infty = 0$. For measurable $E \subseteq \mathbb{R}$, define

$$\int_E \varphi = \int \mathbb{1}_E \varphi$$

**Example.**

- Let $\varphi_2 = \mathbb{1}_{[0,2]} + \mathbb{1}_{[1,3]} = \mathbb{1}_{[0,1)\cup(2,3]} + 2\mathbb{1}_{[1,2]}$ so $\int \varphi_2 = 4$.
- Let $\varphi_3 = \mathbb{1}_{\mathbb{R}}$, then $\int \varphi_3 = 1 \cdot \infty = \infty$.
- Let $\varphi_4 = \mathbb{1}_{(0,\infty)} + (-1)\mathbb{1}_{(-\infty,0)}$. This can't be integrated.
- Let $\varphi_5 = \mathbb{1}_{(-1,0)} + (-1)\mathbb{1}_{(0,1)}$, then $\int \varphi_5 = 0$.

**Lemma.** Let $B_1, ..., B_m$ be measurable sets, $\beta_1, ..., \beta_m \in \mathbb{R} - \{0\}$. Then $\varphi = \sum_{i=1}^{m} \beta_i \mathbb{1}_{B_i}$ is simple measurable function. Also,

$$\mu\left(\bigcup_{i=1}^{m} B_i\right) < \infty \implies \sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \beta_i \mu(B_i)$$

where $A_i$ in standard representation.

**Proposition.** Let $\varphi, \psi$ be simple measurable functions:

- If $\varphi, \psi$ take value $0$ outside a set of finite measure, then $\forall \alpha, \beta \in \mathbb{R}$,

$$\int (\alpha \varphi + \beta \psi) = \alpha \int \varphi + \beta \int \psi$$

- If $\varphi, \psi$ nonnegative, then $\forall \alpha, \beta \geq 0$,

$$\int (\alpha \varphi + \beta \psi) = \alpha \int \varphi + \beta \int \psi$$

- **Monotonicity**:

$$0 \le \varphi \le \psi \implies 0 \le \int \varphi \le \int \psi$$

**Corollary.** Let $\varphi$ nonnegative simple function, then

$$\int \varphi = \sup\left\{\int \psi : 0 \leq \psi \leq \varphi, \psi \text{ simple measurable}\right\}$$

**Lemma.** Let $\varphi$ simple measurable nonnegative function. $\varphi$ takes value 0 outside a set of finite measure iff $\int \varphi < \infty$. Also, $\int \varphi = \infty$ iff there exist $\alpha > 0$, measurable $A$ with $\mu(A) = \infty$ and $\forall x \in A, \varphi(x) \geq \alpha$.

**Lemma.** Let $\{E_n\}$ be ascending collection of measurable sets, $\cup_{n \in \mathbb{N}} E_n = \mathbb{R}$. Let $\varphi$ be simple nonnegative measurable function. Then

$$\int_{E_n} \varphi \to \int \varphi \quad \text{as } n \to \infty$$

## 5.2. The integral of a nonnegative function

**Notation.** Let $\mathcal{M}^+$ denote collection of nonnegative measurable functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

**Definition.** **Support** of measurable function $f$ with domain $E$ is $\mathrm{supp}(f) := \{x \in E : f(x) \neq 0\}$.

**Definition.** Let $f \in \mathcal{M}^+$. **Integral of $f$ with respect to $\mu$ is**

$$\int f := \sup\left\{\int \varphi : 0 \le \varphi \le f, \varphi \text{ simple measurable}\right\} \in \mathbb{R} \cup \{\infty\}$$

For measurable set $E$, define

$$\int_E f := \int \mathbb{1}_E f$$

**Proposition** (Monotonicity). Let $f, g$ measurable, nonnegative. If $g \leq f$ then $\int g \leq \int f$. Let $E, F$ measurable. If $E \subseteq F$ then $\int_E f \leq \int_F f$.

**Theorem** (Monotone convergence theorem). Let $(f_n)$ be sequence in $\mathcal{M}^+$. If $(f_n)$ is increasing on measurable set $E$ and converges pointwise to $f$ on $E$ then

$$\int_E f_n \to \int_E f \quad \text{as } n \to \infty$$

**Corollary.** Restriction of integral to nonnegative functions is linear: $\forall f, g \in \mathcal{M}^+$, $\forall \alpha \geq 0$,

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

**Lemma** (Fatou's Lemma). Let $(f_n)$ be sequence in $\mathcal{M}^+$, then

$$\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n$$

**Lemma.** Let $(f_n) \subset \mathcal{M}^+$, then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

**Proposition** (Chebyshev's inequality). Let $f$ be nonnegative measurable function on $E$. Then

$$\forall \lambda > 0, \quad \mu(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f$$

**Proposition.** Let $f$ be nonnegative measurable function on $E$. Then

$$\int_E f = 0 \iff f = 0 \text{ almost everywhere on } E$$

# 5.3. Integration of measurable functions

**Notation**. Let $\mathcal{M}$ denote set of measurable functions.

**Definition.** $f \in \mathcal{M}^+$ is **integrable** if $\int f < \infty$. By Chebyshev's inequality, if $f$ is integrable, then $f$ is finite almost everywhere.

**Definition.** Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ measurable function. $f$ is **integrable** if $\int f^+$ and $\int f^-$ are finite. In this case, for any measurable set $E$, define

$$\int_E f := \int_E f^+ - \int_E f^-$$

Note that if $f$ integrable then $f^+ - f^-$ is well-defined.

**Proposition.** If $f = f_1 - f_2$, $f_1, f_2 \in \mathcal{M}^+$, $f_1, f_2$ integrable, then

$$\int f^+ - \int f^- = \int f_1 - \int f_2$$

**Definition.** $f \in \mathcal{M}$ is **integrable over** $E$ ($E$ is measurable) if $\int_E f^+$ and $\int_E f^-$ are finite (i.e. $f \cdot \mathbb{1}_E$ is integrable).

**Theorem.** $f \in \mathcal{M}$ is integrable iff $|f|$ is integrable. If $f$ integrable, then

$$\left| \int f \right| \leq \int |f|$$

**Corollary.** Let $f, g \in \mathcal{M}$, $|f| \leq |g|$. If $g$ integrable then $|f|$ is integrable, and $\int |f| \leq \int |g|$.

**Example.** sin is not integrable over $\mathbb{R}$, but is integrable over $[0, 2\pi]$, since $|f_{[0,2\pi]}| \leq \mathbb{1}_{[0,2\pi]}$.

**Theorem** (Linearity of Integration). Let $f, g \in \mathcal{M}$ integrable. Then $f + g$ is integrable and $\forall \alpha \in \mathbb{R}$, $\alpha f$ is integrable. The integral is linear:

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

**Theorem** (Dominated Convergence Theorem). Let $(f_n)$ be sequence of integrable functions. If there exists an integrable $g$ with $\forall n \in \mathbb{N}, |f_n| \leq g$, and $f_n \to f$ pointwise almost everywhere then $f$ is integrable and

$$\int f = \lim_{n \to \infty} \int f_n$$

# 5.4. Integrability: Riemann vs Lebesgue

**Proposition.** Let $f$ bounded function on bounded measurable domain $E$. Then $f$ is measurable and $\int_E |f| < \infty$ iff

$$\left\{ \int_E \varphi : \varphi \leq f, \varphi \text{ simple measurable} \right\} = \inf \left\{ \int_E \psi : f \leq \psi : \psi \text{ simple measurable} \right\}$$

(If $f$ satisfies either condition then $\int_E f$ is equal to the two above expressions).

**Definition.** Bounded function $f$ is **Lebesgue integrable** if it satisfies either of the equivalences in the above proposition.

**Definition.** Let $P = \{x_0, ..., x_n\}$ partition of $[a, b]$, $f : [a, b] \to \mathbb{R}$ bounded. **Lower and upper Darboux sums** for $f$ with respect to $P$ are

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}), \quad U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

where

$$m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

If $P \subseteq Q$ ($Q$ is a **refinement of** $P$), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

**Definition. Lower and upper Riemann integrals** of $f$ over $[a, b]$ are

$$\underline{\mathcal{I}}_a^b(f) := \sup\{L(f, P) : P \text{ partition of } [a, b]\}$$

$$\overline{\mathcal{I}}_a^b(f) := \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

**Definition.** Let $f : [a, b] \to \mathbb{R}$ bounded, then $f$ is **Riemann integrable** $(f \in \mathcal{R})$, if

$$\underline{\mathcal{I}}_a^b(f) = \overline{\mathcal{I}}_a^b(f)$$

and common value $\mathcal{I}_a^b(f) = \int_a^b f(x)\, \mathrm{d}x$ is **Riemann integral** of $f$.

**Remark.** Let $g : [a, b] \to \mathbb{R}$ step function with discontinuities at $P = \{x_0, ..., x_n\}$, so $g = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{(x_{i-1}, x_i)}$ almost everywhere. So $g$ is simple measurable and

$$L(g, P) = \sum_{i=1}^{n} \alpha_i (x_i - x_{i-1}) = U(g, P) = \int g = \mathcal{I}_a^b(g)$$

Hence for any bounded $f : [a, b] \to \mathbb{R}$,

$$\underline{\mathcal{I}}_a^b(f) = \sup\left\{\int \varphi : \varphi \leq f, \varphi \text{ step function}\right\},$$

$$\overline{\mathcal{I}}_a^b(f) = \inf\left\{\int \psi : f \leq \psi, \psi \text{ step function}\right\}$$

**Theorem.** Let $f : [a, b] \to \mathbb{R}$ bounded, $a, b \neq \pm\infty$. If $f$ Riemann integrable over $[a, b]$ then $f$ Lebesgue integrable over $[a, b]$ and the two integrals are equal.

**Theorem.** Let $f : [a, b] \to \mathbb{R}$ bounded, $a, b \neq \pm\infty$. Then $f$ is Riemann integrable on $[a, b]$ iff $f$ is continuous on $[a, b]$ except on a set of measure zero.

**Lemma.** Let $(\varphi_n)$, $(\psi_n)$ be sequences of functions, all integrable over $E$, $(\varphi_n)$ increasing on $E$, $(\psi_n)$ decreasing on $E$. Let $f : E \to \mathbb{R}$ with

$$\forall n \in \mathbb{N}, \varphi_n \leq f \leq \psi_n \text{ on } E, \quad \lim_{n \to \infty} \int_E (\psi_n - \varphi_n) = 0$$

Then $\varphi_n, \psi_n \to f$ pointwise almost everywhere on $E$, $f$ is integrable over $E$ and

$$\lim_{n \to \infty} \int_E \varphi_n = \lim_{n \to \infty} \int_E \psi_n = \int_E f$$

**Definition.** For partition $P = \{x_0, ..., x_n\}$, **gap** of $P$ is

$$\text{gap}(P) := \max\{|x_i - x_{i-1}| : i \in \{1, ..., n\}\}$$

**Lemma.** Let $f : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$ be set where $f$ is continuous. Let $(P_n)$ be sequence of partitions of $[a, b]$ with $P_{n+1} \subseteq P_n$ and $\mathrm{gap}(P_n) \to 0$ as $n \to \infty$. Let $\varphi_n, \psi_n : [a, b] \to \mathbb{R}$ step functions with

$$\varphi_n(x) := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad \psi_n(x) := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

for $P_n = \{x_0, ..., x_n\}$. Then $\forall x \in E - \cup_{n \in \mathbb{N}} P_n$,

$$\varphi_n(x), \psi_n(x) \to f(x) \quad \text{as } n \to \infty$$

**Definition.** Let $f : (a, b] \to \mathbb{R}$, $-\infty \leq a < b < \infty$, $f$ bounded and Riemann integrable on all closed bounded sub-intervals of $(a, b]$. If

$$\lim_{t \to a, t > a} \mathcal{I}_t^b(f)$$

exists then this is defined as the **improper Riemann integral** $\mathcal{I}_a^b(f)$. Similar definitions exist for $f : (a, b) \to \mathbb{R}$ and $f : [a, b) \to \mathbb{R}$.

**Note**. Improper Riemann integral may exist without function being Lebesgue integral.

**Proposition.** If $f$ is integrable, the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.

**Definition.** Let $\alpha : [a, b] \to \mathbb{R}$ monotonically increasing (and so bounded). For partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ and bounded $f : [a, b] \to \mathbb{R}$, define

$$L(f, P, \alpha) := \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})), \quad U(f, P, \alpha) := \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

where $m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}$, $M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$. Then $f$ is **integrable with respect to** $\alpha$, $f \in \mathcal{R}(\alpha)$, if

$$\inf\{U(f, P, \alpha) : P \text{ partition of } [a, b]\} = \sup\{L(f, P, \alpha) : P \text{ partition of } [a, b]\}$$

and the common value $\int_a^b f \, d\alpha$ is the **Riemann-Stieltjes integral** of $f$ with respect to $\alpha$.

**Proposition.** Let $f : (a, b) \to \mathbb{R}$, then set of points where $f$ is differentiable is measurable.

**Remark.** If $\alpha : [0, 1] \to [a, b]$ bijection, then

$$\int_0^1 f \circ \alpha \, \mathrm{d}\alpha = \int_a^b f(x) \, \mathrm{d}x$$

**Proposition.** Let $\alpha$ be monotonically increasing and differentiable with $\alpha' \in \mathcal{R}$. Then $g \in \mathcal{R}(\alpha)$ iff $g\alpha' \in \mathcal{R}$, and in that case,

$$\int_a^b g \, \mathrm{d}\alpha = \int_a^b g(x)\alpha'(x) \, \mathrm{d}x$$

**Remark.** When $g = 1$, this says $\int_a^b 1 \, \mathrm{d}\alpha = \alpha(b) - \alpha(a) = \int \alpha'(x) \, \mathrm{d}x$, similar to the fundamental theorem of calculus.

# 6. Lebesgue spaces

# 6.1. Normed linear spaces

**Definition.** Let $X$ be **complex linear space** (vector space over $\mathbb{C}$). $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ is **norm on** $X$ if

- $\forall x \in X, \|x\| = 0 \iff x = 0$.
- $\forall x \in X, \forall \lambda \in \mathbb{C}, \|\lambda x\| = |\lambda| \, \|x\|$.
- $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.

$X$ equipped with norm $\|\cdot\|$, $(X, \|\cdot\|)$, is called **complex normed linear space**.

**Example.**

- $\|x\| = \sqrt{x\overline{x}}$ is norm on $\mathbb{C}$.
- Let $C[a, b]$ denote linear space of continuous real-valued functions on $[a, b]$. Then

$$\|f\|_{\max} := \max\{|f(x)| : x \in [a, b]\}$$

is norm on $C[a, b]$.

**Proposition.** Norm induces metric on $X$: $d(x, y) = \|x - y\|$.

**Definition.** Let $(X, \|\cdot\|)$ be normed linear space.

- Sequence $(f_n)$ in $X$ is **Cauchy sequence** in $X$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad \|f_n - f_m\| < \varepsilon$$

- Sequence $(f_n)$ in $X$ **converges in** $X$, $\|f_n - f\| \to 0$ as $n \to \infty$, if

$$\exists f \in X : \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad \|f_n - f\| < \varepsilon$$

- $(X, \|\cdot\|)$ is **complete** if every Cauchy sequence converges in $X$.
- **Banach space** is complete normed linear space.

**Proposition.** Let $(X, \|\cdot\|)$ be normed linear space.

- If $(x_n)$ converges in $X$, $(x_n)$ is Cauchy sequence in $X$.
- Let $(x_n)$ be Cauchy sequence in $X$. If $(x_n)$ has convergent subsequence in $X$ then $(x_n)$ converges in $X$.

## 6.2. Lebesgue spaces $L^p$, $p \in [1, \infty)$

**Definition.** Let $p \in [1, \infty)$, $E \subseteq \mathbb{R}$.

- Linear space $L^p(E)$ is defined as

$$L^p(E) := \left\{ f : E \to \mathbb{C} : f \text{ is measurable and } \int_E |f|^p < \infty \right\} / \cong$$

where $f \cong g$ iff $f = g$ almost everywhere:

$$f \cong g \iff \exists F \subseteq E : \mu(F) = 0 \land \forall x \in E - F, f(x) = g(x)$$

- Define $\|\cdot\|_{L^p} : L^p(E) \to \mathbb{R}$ as

$$\|f\|_{L^p} := \left( \int_E |f|^p \right)^{1/p}$$

**Remark.**

- We often consider space $L^p(E)$ of real-valued measurable functions $f : E \to \mathbb{R}$ such that $\int_E |f|^p < \infty$.
- For $f : E \to \mathbb{C}$, $f = f_1 + if_2$, $f$ is measurable iff $f_1 : E \to \mathbb{R}$ and $f_2 : E \to \mathbb{R}$ are measurable. Also,

$$\int_E |f|^p < \infty \iff \left( \int_E |f_1|^p < \infty \wedge \int_E |f_2|^p < \infty \right)$$

**Example.** Let $E = \mathbb{R}$, $f(x) = \mathbb{1}_{\mathbb{R}-\mathbb{Q}}(x) + i\mathbb{1}_{\mathbb{Q}}(x)$ and $g(x) = 1$. Then $\mu(\mathbb{Q}) = 0$ so $f \cong g$.

**Proposition.** Let $(f_n), (g_n)$ sequences of measurable functions, $\forall n \in \mathbb{N}, f_n \cong g_n$, $\lim_{n \to \infty} f_n = f$ and $\lim_{n \to \infty} g_n = g$. Then $f \cong g$.

**Definition.** $p, q \in \mathbb{R}$ are **conjugate exponents** if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

**Lemma** (Young's inequality). Let $p, q$ conjugate exponents, then

$$\forall A, B \in \mathbb{R}_{\geq 0}, \quad AB \leq \frac{A^p}{p} + \frac{B^q}{q}$$

with equality iff $A^p = B^q$.

**Lemma** (Hölder's inequality).  Let $p, q$ conjugate exponents. If $f \in L^p(E)$, $g \in L^q(E)$, then

$$\int_E |fg| \leq \|f\|_{L^p} \|g\|_{L^q}$$

**Corollary** (Cauchy-Schwarz inequality for $L^2(E)$). If $f, g \in L^2(E)$, then

$$\left| \int_E f\overline{g} \right| \leq \int_E |fg| \leq \|f\|_{L^2} \|g\|_{L^2}$$

**Lemma** (Minkowski's inequality). Let $p \in [1, \infty)$. If $f, g \in L^p(E)$ then $f + g \in L^p(E)$ and

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

**Theorem.** For $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is normed linear space.

**Proposition.** Let $1 \leq p < q < \infty$. If $\mu(E) < \infty$ then $L^q(E) \subseteq L^p(E)$ and

$$\|f\|_{L^p} \leq \mu(E)^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q}$$

**Remark.**

- Convergence in $L^p$ is also called convergence in the mean of order $p$.

- This notion of convergence is different to pointwise convergence, uniform convergence and convergence in measure.

**Theorem** (Riesz-Fischer). For $p \in [1, \infty)$, $(L^p(E), \|\cdot\|_{L^p})$ is complete.

# 6.3. Lebesgue space $L^\infty$

**Definition**.

- Let $f : E \to \mathbb{C}$ measurable. $f$ is **essentially bounded** if

$$\exists M \geq 0 : |f(x)| \leq M \quad \text{almost everywhere on } E$$

- $L^\infty(E)$ is collection of equivalence classes of essentially bounded functions where $f \cong g$ iff $f = g$ almost everywhere.

- For $f \in L^\infty(E)$, define

$$\|f\|_{L^\infty} := \operatorname{ess\,sup}|f| := \inf\{M \in \mathbb{R} : \mu(\{x \in E : |f(x)| > M\}) = 0\}$$

**Proposition.**

- $0 \leq |f(x)| \leq \|f\|_{L^\infty}$ almost everywhere.
- $\|f\|_{L^\infty}$ is norm on $L^\infty(E)$.
- If $f \in L^1(E)$, $g \in L^\infty(E)$, then

$$\int_E |fg| \leq \|f\|_{L^1} \|g\|_{L^\infty}$$

**Proposition.** Let $(f_n)$ sequence of functions in $L^\infty(E)$. Then $(f_n)$ converges to $f \in L^\infty(E)$ iff there exists $G \subseteq E$ with $\mu(G) = 0$ and $(f_n)$ converges to $f$ uniformly on $E - G$.

**Theorem.** $(L^\infty(E), \|\cdot\|_{L^\infty})$ is complete.

**Remark.** If $\mu(E) < \infty$, then $L^\infty(E) \subset L^p(E)$ for $p \in [1, \infty)$ and

$$\|f\|_{L^p} \leq \mu(E)^{1/p} \|f\|_{L^\infty}$$

since

$$\|f\|_{L^p}^p = \int_E |f|^p \leq \int_E \|f\|_{L^\infty}^p \cdot \mathbb{1}_E = \|f\|_{L^\infty}^p \mu(E)$$

# 6.4. Approximation and separability

**Definition.** Let $(X, \|\cdot\|)$ be normed linear space. Let $F \subseteq G \subseteq X$. $F$ is **dense in** $G$ if

$$\forall g \in G, \forall \varepsilon > 0, \exists f \in F : \quad \|f - g\| < \varepsilon$$

**Proposition.**

- $F$ is dense in $G$ iff for every $g \in G$, there exists sequence $(f_n)$ in $F$ such that $\lim_{n \to \infty} f_n = g$ in $X$.
- For $F \subseteq G \subseteq H \subseteq X$, if $F$ dense in $G$ and $G$ dense in $H$, then $F$ dense in $H$.

**Proposition.** Let $p \in [1, \infty]$. Then subspace of simple functions in $(L^p(E), \|\cdot\|_{L^p})$ is dense in $(L^p(E), \|\cdot\|_{L^p})$.

**Definition.** $\psi : \mathbb{R} \to \mathbb{R}$ is **step function** if it can be written as

$$\psi = \sum_{k=1}^{N} \tilde{a}_k \mathbb{1}_{(a_k, b_k)}$$

where the intervals $(a_k, b_k)$ are disjoint.

**Proposition.** Let $[a, b]$ be bounded, $p \in [1, \infty)$. Then subspace of step functions on $[a, b]$ is dense in $(L^p([a, b]), \|\cdot\|_{L^p})$.

**Definition**. Normed linear space $(X, \|\cdot\|)$ is **separable** if there exists countable, dense subset $X' \subseteq X$.

**Example.** $\mathbb{R}$ is separable, since $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

**Theorem.** Let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$. Then $(L^p(E), \|\cdot\|_{L^p})$ is separable. In particular, step functions are dense in $L^p(E)$ for $p \in [1, \infty)$.

**Proposition.** Let $\varepsilon > 0$, $f \in L^p(E)$, $p \in [1, \infty)$. There exists continuous $g \in L^p(E)$ such that $\|f - g\|_{L^p} < \varepsilon$.

**Remark.** Linear space of continuous functions that vanish outside bounded set is dense in $(L^p(E), \|\cdot\|_{L^p})$ for $p \in [1, \infty)$.

**Remark.** Differentiable functions are also dense in $\left(L^p(E), \|\cdot\|_{L^p}\right)$ for $p \in [1, \infty)$.

**Remark.** Step functions and continuous functions are not dense in $(L^\infty(E), \|\cdot\|_{L^\infty})$.

**Example.** In general, $(L^\infty(E), \|\cdot\|_{L^\infty})$ is not separable. Let $[a,b]$ be bounded, $a \neq b$. Assume there is countable $\{f_n : n \in \mathbb{N}\}$ which is dense in $(L^\infty([a,b]), \|\cdot\|_{L^\infty})$. Then for every $x \in [a,b]$, can choose $g(x) \in \mathbb{N}$ such that

$$\left\|\mathbb{1}_{[a,x]} - f_{g(x)}\right\|_{L^\infty} < \frac{1}{2}$$

Also, for $x_1 \le x_2$,

$$\left\|\mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]}\right\|_{L^\infty} = \begin{cases} 1 & \text{if } a \le x_1 < x_2 \le b \\ 0 & \text{if } x_1 = x_2 \end{cases}$$

and

$$\left\| \mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]} \right\|_{L^\infty} \leq \left\| \mathbb{1}_{[a,x_1]} - f_{g(x_1)} \right\|_{L^\infty} + \left\| f_{g(x_1)} - f_{g(x_2)} \right\|_{L^\infty} + \left\| f_{g(x_2)} - \mathbb{1}_{[a,x_2]} \right\|$$

$$< 1 + \left\| f_{g(x_1)} - f_{g(x_2)} \right\|_{L^\infty}$$

If $g(x_1) = g(x_2)$ then $\left\| \mathbb{1}_{[a,x_1]} - \mathbb{1}_{[a,x_2]} \right\|_{L^\infty} = 0$ so $g : [a,b] \to \mathbb{N}$ is injective. But $\mathbb{N}$ is countable and $[a,b]$ is not countable: contradiction.

## 6.5. Riesz representation theorem for $L^p(E)$, $p \in [1, \infty)$

**Definition.** Let $X$ be linear space. $T : X \to \mathbb{R}$ is **linear functional** if

$$\forall f, g \in X, \forall a, b \in \mathbb{R}, \quad T(af + bg) = aT(f) + bT(g)$$

Any linear combination of linear functionals is linear, so set of linear functionals on linear space is also linear space.

**Definition.** Let $(X, \|\cdot\|)$ be normed linear space. $T : X \to \mathbb{R}$ is **bounded functional** if

$$\exists M \geq 0 : \forall f \in X, \quad |T(f)| \leq M\|f\|$$

**Norm** of $T$, $\|T\|_*$, is the smallest such $M$.

**Remark.** For bounded linear functional $T$ on normed linear space $(X, \|\cdot\|)$,

$$|T(f) - T(g)| \leq \|T\|_* \|f - g\|$$

This gives the following continuity property: if $f_n \to f \in X$, then $T(f_n) \to T(f)$.

**Example.** Let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, $q$ conjugate to $p$. Let $h \in L^q(E)$. Define $T : L^p(E) \to \mathbb{R}$ by

$$T(f) = \int_E h \cdot f$$

By Holder's inequality,

$$|T(f)| = \left| \int_E hf \right| \leq \int_E |hf| \leq \|h\|_{L^q} \|f\|_{L^p}$$

So $T$ is bounded linear functional.

**Remark.** We can write $\|\cdot\|_*$ as

$$\|T\|_* := \inf\{M \in \mathbb{R} : \forall f \in X, |T(f)| \leq M\|f\|\} = \sup\{|T(f)| : f \in X, \|f\| \leq 1\}$$

**Definition.** **Dual space** of $X$, $X^*$, is set of bounded linear functionals on $X$ with norm $\|\cdot\|_*$.

**Proposition.** Let $(X, \|\cdot\|)$ be normed linear space, then dual space of $X$ is linear space with norm $\|\cdot\|_*$.

**Remark.** Bounded linear functional is special case of **bounded linear transformation** between normed spaces. $T : X \to Y$ is bounded linear transformation if $T(af + bg) = aT(f) + bT(g)$ and $\exists M \geq 0 : \|T(f)\|_Y \leq M\|f\|_X$.

**Proposition.** Let $E \subseteq \mathbb{R}$ measurable, $p \in [1, \infty)$, $q$ conjugate to $p$, $h \in L^q(E)$. Define $T : L^p(E) \to \mathbb{R}$ by

$$T(f) = \int_E h f$$

Then $\|T\|_* = \|h\|_{L^q}$.

**Theorem** (Riesz representation theorem for $L^p$). Let $p \in [1, \infty)$, $q$ conjugate to $p$, $E \subseteq \mathbb{R}$ measurable. For $h \in L^q(E)$, define bounded linear functional $R_h : L^p(E) \to \mathbb{R}$ by

$$R_h(f) = \int_E hf$$

Then for every bounded linear functional $T : L_p(E) \to \mathbb{R}$, there is unique $h \in L^q(E)$ such that

$$R_h = T \quad \wedge \quad \|T\|_* = \|h\|_{L^q}$$

**Theorem.** Let $[a, b]$ be non-degenerate, bounded interval, $p \in [1, \infty)$, $q$ conjugate to $p$. If $T$ is bounded linear functional on $L^p([a, b])$ then there exists $h \in L^q([a, b])$ such that

$$T(f) = \int_a^b hf$$

# 7. Hilbert spaces

# 7.1. Inner product spaces

**Definition.** Let $H$ be complex linear space. **Inner product** on $H$ is function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ such that $\forall a, b \in \mathbb{C}, \forall x, y, z \in H$,

- **Linear in first variable**: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
- **Conjugate symmetric**: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- **Positive**: $x \neq 0 \implies \langle x, x \rangle \in (0, \infty)$
- $\langle x, x \rangle = 0 \iff x = 0$.

These imply that $\langle 0, x \rangle = 0$ and inner product is conjugate linear in second variable: $\langle z, ax + by \rangle = \overline{a}\langle z, x \rangle + \overline{b}\langle z, y \rangle$.

**Example.**

- $\mathbb{R}^n$ has inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$.
- $\mathbb{C}^n$ has inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$.
- Inner product induces metric on $H$:

$$d(x, y) = \langle x - y, x - y \rangle^{1/2}$$

**Definition**. Complex linear space $H$ with inner product $\langle \cdot, \cdot \rangle$ is called **pre-Hilbert space** or **inner product space**.

**Definition.** Let $H$ inner product space. For $x \in H$, define the norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

**Proposition.** $\|x \pm y\|^2 = \|x\|^2 \pm 2\operatorname{Re}(\langle x, y \rangle) + \|y\|^2.$

**Theorem** (Cauchy-Schwarz inequality). Let $(H, \langle \cdot, \cdot \rangle)$ be pre-Hilbert space. Then

$$\forall x, y \in H, \quad |\langle x, y \rangle| \leq \|x\| \|y\|$$

with equality iff $x$ and $y$ linearly dependent.

**Theorem** (Parallelogram Identity). A normed linear space $X$ is an inner product space with norm derived from the inner product (i.e. $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$) iff

$$\forall x, y \in X, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

**Definition.** Let $\left(X, \langle \cdot, \cdot \rangle_X\right), \left(Y, \langle \cdot, \cdot \rangle_Y\right)$ be inner product spaces.

- An inner product on $X \times Y$ is

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{X \times Y} = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y$$

- The associated norm on $X \times Y$ is

$$\|(x, y)\|_{X \times Y} = \sqrt{\langle (x, x), (y, y) \rangle_{X \times Y}} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$$

**Theorem.** Let $X$ inner product space, $x_n \to x$, $y_n \to y$ in $X$. Then $\langle x_n, y_n \rangle_X \to \langle x, y \rangle_X$.

*Proof.* Use $|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n - x, y_n \rangle + \langle x, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n - y \rangle|$ and Cauchy-Schwarz, reverse triangle inequality to show $\|y_n\| \to \|y\|$. $\square$

**Proposition.** The norm and inner product are continuous.

# 7.2. Hilbert spaces

**Definition.** Hilbert space is inner product space which is complete with respect to norm induced by inner product.

**Example.** $\mathbb{R}^n$ with standard inner product is Hilbert space.

**Example.** Define inner product on $L^2(E)$

$$\langle f, g \rangle_{L^2} := \int_E f\overline{g}$$

Induced norm is the $L^2$ norm. So by Riesz-Fischer theorem, $(L^2(E), \langle \cdot, \cdot \rangle_{L^2})$ is Hilbert space.

**Definition**. Let $H$ Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

- $x, y \in H$ are **orthogonal**, $x \perp y$ if $\langle x, y \rangle = 0$.
- $A, B \subseteq H$ are **orthogonal**, $A \perp B$ if $\forall x \in A, \forall y \in B, \quad x \perp y$.
- **Orthogonal complement** of $A \subseteq H$ is

$$A^{\perp} := \{x \in H : \forall y \in A, \quad x \perp y\}$$

**Theorem** (Pythagorean Theorem). If $x_1, ..., x_n \in H$, $x_i \perp x_j$ for $i \neq j$, then

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

*Proof.* Use linearity of inner product and orthogonal condition. □

**Theorem.** Let $H$ Hilbert space, $A \subseteq H$, then $A^{\perp}$ is closed subspace of $H$.

*Proof.*

- Subspace:
  - ‣ For $y, z \in A^\perp$, $\lambda, \mu \in \mathbb{C}$, show $\forall x \in A$, $\lambda y + \mu z \in A^\perp$.

- Closed:
  - ‣ Show if $(y_n) \subseteq A^\perp$, $y_n \to y$, then $y \in A^\perp$:
    - – Let $x \in A$, then show $|\langle x, y \rangle| \to 0$ by squeezing, triangle inequality and Cauchy-Schwarz.

$\square$

**Theorem** (Projection). Let $M$ closed subspace of Hilbert space $H$.

- For every $x \in H$, there exists unique closest point $y \in M$:

$$\forall x \in H, \exists! y \in M : \quad \|x - y\| = \min\{\|x - z\| : z \in M\}$$

  We say $y$ is "the best approximation" to $x$ in $M$.

- The point $y \in M$ closest to $x \in H$ is unique element of $M$ such that $(x - y) \perp M$.

*Proof.*

- Let $d = \inf\{\|x - z\| : z \in M\}$. Show that $\exists y \in M : \|x - y\| = d$:
  - ‣ There is sequence $(y_n) \subset M$ with $\|x - y_n\| \to d$. Show that $(y_n)$ is Cauchy:
    - $-$ $\|y_m - y_n\|^2 + \|2x - y_m - y_n\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2$ by parallelogram identity.
    - $-$ $\frac{y_m + y_n}{2} \in M$, so $\|2x - y_m - y_n\| \geq 2d$.
  - ‣ Deduce that $y_n \to y \in M$ and $\|x - y\| \to d$ by squeezing.
- Uniqueness of $y$:
  - ‣ Let $\|x - y\| = d = \|x - y'\|$.

- By parallelogram identity, $2\|x - y\|^2 + 2\|x - y'\|^2 = \|2x - y - y'\|^2 + \|y - y'\|^2$.
  - Use that $\frac{y+y'}{2} \in M$ to show $\|y - y'\| = 0$.
- To show $z = x - y \perp M$:
  - For $w \in M$, write $\langle z, w \rangle = |\langle z, w \rangle|\, \lambda$ where $\lambda = e^{i\theta}$, set $u = \lambda w$.
  - Define $f(t) = \|z + tu\|^2$, show $t = 0$ is minimum of $f$ and so $0 = f'(0)$, hence $z \in M^\perp$.
- To show uniqueness of $z$:
  - Show for $y, y' \in M$ such that $x - y \perp M$ and $x - y' \perp M$, then $\langle y - y', w \rangle = 0$ for any $w \in M$. Set $w = y - y'$ to give $y = y'$.

$\square$

**Definition.** **Direct sum** of subspaces $M$ and $N$ of linear space is

$$M \oplus N := \{y + z : y \in M, z \in N\}$$

**Corollary.** If $M$ closed subspace of Hilbert space $H$, then $H = M \oplus M^\perp$.

For all $x \in H$, $x$ can be written uniquely as $x = y + z$ where $y$ is best approximation to $x$ in $M$ and $z = x - y \perp M$.

*Proof.* By above theorem. □

**Definition.** Let $H$ Hilbert space. $\{u_\alpha\}_{\alpha \in I}$ is **orthonormal** if it is **orthogonal**: $u_\alpha \perp u_\beta$ for $\alpha \neq \beta$, and **normalised**: $\forall \alpha \in I, \|u_\alpha\| = 1$.

**Definition.** Let $X$ Banach space, $\{x_\alpha \in X : \alpha \in I\}$ be indexed set where $I$ is countable or uncountable.

- For each finite $J \subseteq I$, define **partial sum** as

$$S_J := \sum_{\alpha \in J} x_\alpha$$

- Unordered sum of $\{x_\alpha \in X : \alpha \in I\}$ **converges unconditionally** to $x \in X$, written $x = \sum_{\alpha \in I} x_\alpha$, if $\forall \varepsilon > 0$, there exists finite $J \subseteq I$ such that $\|S_K - x\| < \varepsilon$ for every finite $J \subseteq K \subseteq I$.

- Unordered sum $\sum_{\alpha \in I} x_\alpha$ is **Cauchy** if $\forall \varepsilon > 0$, there exists finite $J \subseteq I$ such that $\|S_L\| < \varepsilon$ for every finite $L \subseteq I - J$. Note that

$$\|S_L\| = \left\| \sum_{\alpha \in L \cup J} x_\alpha - \sum_{\alpha \in J} x_\alpha \right\|$$

- Unordered sum of $\{x_\alpha \in X : \alpha \in I\}$ **converges absolutely** if $\sum_{\alpha \in I} \|x_\alpha\|$ converges unconditionally in $\mathbb{R}$.

**Proposition**. Unordered sum in Banach space converges unconditionally iff it is Cauchy.

**Definition.** Let $\{c_\alpha : \alpha \in I\} \subseteq [0, \infty]$. Define

$$\sum_{\alpha \in I} c_\alpha = \sup \left\{ \sum_{\alpha \in J} c_\alpha : J \subseteq I, J \text{ finite} \right\}$$

**Proposition.** Let $\{c_\alpha : \alpha \in I\} \subseteq [0, \infty]$, $K = \{\alpha \in I : c_\alpha > 0\}$. If $\sum_{\alpha \in I} c_\alpha < \infty$, then $K$ is countable.

**Theorem** (Bessel's inequality). Let $U = \{u_\alpha : \alpha \in I\}$ orthonormal in Hilbert space $H$. Then

$$\forall x \in H, \quad \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

In particular, $\forall x \in H$, $\{\alpha \in I : \langle x, u_\alpha \rangle \neq 0\}$ is countable.

*Proof.*

- Prove for any finite $J \subseteq I$, then take supremum on LHS.
- Show that

$$\left\| x - \sum_{\alpha \in J} \langle x, u_\alpha \rangle u_\alpha \right\| = \|x\|^2 - \sum_{\alpha \in J} |\langle x, u_\alpha \rangle|^2$$

using equation 2.2 and Pythagorean theorem.

$\square$

**Theorem.** If $U = \{u_\alpha : \alpha \in I\}$ is orthonormal subset of Hilbert space $H$ then the following are equivalent:

- If $\forall \alpha \in I, \langle x, u_\alpha \rangle = 0$, then $x = 0$.
- $\forall x \in H, x = \sum_{\alpha \in I} \langle x, u_\alpha \rangle u_\alpha$ where sum converges unconditionally in $H$ and only has countably many non-zero terms.
- **Parseval's identity**:

$$\forall x \in H, \quad \|x\|^2 = \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2$$

*Proof.*

- (i) $\implies$ (ii): let $\{\alpha_j : j \in \mathbb{N}\}$ be set of indices where $\langle x, u_{\alpha_j} \rangle \neq 0$. Show the partial sums of $\sum_{j \in \mathbb{N}} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$ are Cauchy using Pythagorean theorem and so show converges.

- Set

$$y = x - \sum_{j \in \mathbb{N}} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$$

  and show $\langle y, u_\alpha \rangle = 0$.

- (ii) $\implies$ (iii): let $\varepsilon > 0$. Use definition of unconditional convergence of $x$ and Pythagorean theorem to show $\|x\|^2 - \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2 < \varepsilon$.

**Definition.** Orthonormal subset $U = \{u_\alpha : \alpha \in I\}$ of Hilbert space $H$ is **complete** if it satisfies any of the conditions in Theorem 7.2.16. An **orthonormal basis** of $H$ is a complete orthonormal subset of $H$.

**Definition.** $U$ is **maximal orthonormal set** if $\forall V \subseteq H$ such that $U \subsetneq V$, $V$ is not orthonormal.

**Lemma.** $U$ is maximal orthonormal set iff it is an orthonormal basis.

**Remark.** For orthonormal basis $\{u_\alpha : \alpha \in \mathbb{N}\}$, representation $x = \sum_{\alpha \in \mathbb{N}} c_\alpha u_\alpha$ is unique (consider $\langle x - x, u_\beta \rangle = \lim_{n \to \infty} \langle \sum_{\alpha=1}^{n} (c_\alpha - d_\alpha) u_\alpha, u_\beta \rangle$).

**Theorem.** Every Hilbert space $H$ has orthonormal basis. If $V \subseteq H$ is orthonormal set, then $H$ has orthonormal basis containing $V$.

*Proof.*

- Assume $H \neq \{0\}$. Use partial ordering $\subseteq$.
- Let $\{U_\alpha : \alpha \in I\}$ be totally ordered collection of orthonormal sets. Find upper bound of $\{U_\alpha : \alpha \in I\}$ which is orthonormal.
- Show result using Theorem 7.2.25 and Lemma 7.2.19.
- To show orthonormal sets $V$ can be extended to orthonormal bases, use same argument on family of all orthonormal subsets of $H$ containing $V$.

$\square$

**Definition.** A set $X$ is **partially ordered** if it is equipped with relation $\leq$ satisfying:

- **Reflexivity**: $\forall x \in X, x \leq x$.
- **Transitivity**: $(x \leq y \wedge y \leq z) \Longrightarrow x \leq z$.
- **Anti-symmetry**: $(x \leq y \wedge y \leq x) \Longrightarrow x = y$.

$X$ is **totally ordered** if partially ordered and $\forall x, y \in X$, either $x \leq y$ or $y \leq x$.

**Definition.** Let $X$ totally ordered set with relation $\leq$. $x \in X$ is **upper bound** for $Y \subseteq X$ if $\forall y \in Y, y \leq x$. $x \in X$ is **maximal** if $\forall y \in X, x \leq y \implies y = x$.

**Example.** Let $X$ be non-empty collection of sets. Then $\subseteq$ is partial ordering on $X$. $A \in X$ is upper bound for $X' \subseteq X$ if every set in $X'$ is subset of $A$. $M \in X$ is maximal if it is not proper subset of any set in $X$.

**Theorem** (Zorn's Lemma). A partially ordered set $X$ that has upper bounds for its totally ordered subsets has a maximal element.

**Proposition.** Hilbert space is separable iff it has countable orthonormal basis.

*Proof.*

- $\Longrightarrow$: let $U = \{u_n : n \in \mathbb{N}\}$ countable, dense in $H$. Recursively discard any $u_n$ in linear span of $u_1, ..., u_{n-1}$ to obtain linearly independent set $V = \{v_n : n \in \mathbb{N}\}$ whose linear span is dense in $H$. Applying Gram-Schmidt, set

$$w_1 = \frac{v_1}{\|v_1\|}, ..., w_{n+1} = c_{n+1}\left(v_{n+1} - \sum_{k=1}^{n} \langle w_k, v_{n+1} w_k \rangle \right)$$

where $c_n \in \mathbb{C}$ chosen so that $\|w_n\| = 1$. $\{w_n : n \in \mathbb{N}\}$ is countable orthonormal basis.

- $\Longleftarrow$: let $\{w_n : n \in \mathbb{N}\}$ be orthonormal basis, show that

$$S_m = \left\{ \sum_{k=1}^{m} c_k w_k : c_k \in \mathbb{Q} + i\mathbb{Q} \right\}$$

is countable and $\cup_{m \in \mathbb{N}} S_m$ dense in $H$. $\quad\square$

**Theorem** (Riesz Representation Theorem for Hilbert Spaces). Let $H$ Hilbert space with inner product $\langle \cdot, \cdot \rangle$, $T : H \to \mathbb{R}$ bounded linear functional. Then

$$\exists! y \in H : \forall x \in H, \quad T(x) = \langle x, y \rangle$$

Note RHS gives bounded linear functional by Cauchy-Schwarz.

*Proof.*

- Existence:
  - ‣ Show $N = \{x \in H : T(x) = 0\}$ is closed subspace of $H$, use that $H = N \oplus N^\perp$.
  - ‣ Assume $N^\perp$ contains $v$ with $\|v\| = 1$. For $x \in H$, define $u = T(x)v - T(v)x$.
  - ‣ Show that $\langle u, v \rangle = 0$, deduce a value for $y$ from this.
- Uniqueness: straightforward.

$\square$

# 8. Convergence of Fourier series

**Note.** We can view $f : [-\pi, \pi] \to \mathbb{C}$ as being $2\pi$-periodic by extending it on the real line.

**Definition.** $m$-th **partial Fourier sum** of $2\pi$-periodic integrable function $f : [-\pi, \pi] \to \mathbb{C}$ is given by

$$(S_m f)(x) = \sum_{k=-m}^{m} a_k(f) e^{ikx}$$

where

$$a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} \, \mathrm{d}y$$

are **Fourier coefficients** of $f$.

**Definition**. Let $f, g : [-\pi, \pi] \to \mathbb{C}$ be $2\pi$-periodic integrable functions. **Convolution** $f * g$ is

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(x - y)\,\mathrm{d}y$$

**Proposition.** Let $f, g, h : [-\pi, \pi] \to \mathbb{C}$ be $2\pi$-periodic integrable functions, $c \in \mathbb{C}$. Then $*$ satisfies:

- **Commutativity**: $f * g = g * f$.
- **Distributivity**: $f * (g + h) = (f * g) + (f * h)$.
- **Homogeneity**: $(cf) * g = c(f * g) = f * (cg)$.
- **Associativity**: $(f * g) * h = f * (g * h)$.

# 8.1. Pointwise convergence of Fourier series via Dirichlet kernel

**Definition.** Let $m \in \mathbb{N}_0$. The $m$**-th Dirichlet kernel** is

$$D_m(x) := \sum_{k=-m}^{m} e^{ikx}$$

**Proposition.**

- $D_m$ is trigonometric polynomial of degree $m$ with coefficients equal to 1 for $k \in [-m, m]$ and 0 otherwise.
- $D_m$ is real-valued and $2\pi$-periodic.
- $$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(x) \, \mathrm{d}x = 1$$

**Proposition.** Let $f : [-\pi, \pi] \to \mathbb{C}$ be $2\pi$-periodic integrable function. Then

$$(D_m * f)(x) = \sum_{k=-m}^{m} a_k(f)e^{ikx} = (S_m f)(x)$$

where $a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)e^{-iky} \, dy$.

**Proposition.**

$$D_m(x) = \frac{\sin\left(\left(m + \frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$

**Remark.** RHS in <u>Proposition 8.1.4</u> has removable singularity at $x = 0$, and $D_m(0) = 2m + 1$. Applying l'Hopital's rule to RHS gives

$$\lim_{x \to 0} \frac{\sin\left(\left(m + \frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} = 2m + 1$$

**Theorem** (Riemann-Lebesgue Lemma). Let $E \subseteq \mathbb{R}$ measurable, $f \in L^1(E)$. Then

$$\lim_{n \to \infty} \int_E f(x) \sin(nx) = \lim_{n \to \infty} \int_E f(x) \cos(nx) = \lim_{n \to \infty} \int_E f(x) e^{-inx} = 0$$

*Proof.*

- First consider when $f(x) = \mathbb{1}_{(a,b)}(x)$. Define $I_j = \left( \frac{2\pi j}{n}, \frac{2\pi(j+1)}{n} \right)$, so integral of $\sin(nx)$ over each $I_j$ is 0.

- Write

$$(a, b) = L \cup \bigcup_{j=1}^{N} I_j \cup R$$

so that $\text{length}(L), \text{length}(R) < \frac{2\pi}{n}$.

- Show that

$$\left| \int_E f(x) \sin(nx) \right| < \frac{4\pi}{n}$$

- Deduce the sin result for step functions.
- Use that step functions are dense in $L^1$ to show sin result for $f \in L^1(E)$ by writing $f = (f - \psi) + \psi$ and finally take $\limsup$.
- Same argument works for cos.
- Conclude exp result.

$\square$

**Theorem.** Let $f \in L^1([-\pi, \pi])$ be $2\pi$-periodic, assume $f$ differentiable at $b \in [-\pi, \pi]$. Then

$$f(b) = \lim_{m \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_m(b - y) \, \mathrm{d}y = \lim_{m \to \infty} (f * D_m)(b) = \lim_{m \to \infty} S_m f(b)$$

*Proof.*

- First assume $b = 0$. Let $0 < \varepsilon < 1$, show that $f(y)/\sin(y/2)$ is integrable on $[\varepsilon, \pi]$ and show

$$\lim_{m \to \infty} \int_{\varepsilon}^{\pi} \frac{f(y)}{\sin\left(\frac{y}{2}\right)} \sin\left(\left(m + \frac{1}{2}\right)y\right) dy = 0$$

Conclude the same for $\int_{-\pi}^{-\varepsilon}$.
- Write $f(y) = f(0) + s(y)$ and split the integral $\int_{-\pi}^{\pi}$ as such.
- Use <u>Proposition 8.1.2</u> and split integral of $s(y)$ to show

$$\lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)D_m(y)\,\mathrm{d}y = f(0) + \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} s(y)D_m(y)\,\mathrm{d}y$$

- Use differentiability at $0$ to show for $\varepsilon$ small and $y \in [-\varepsilon, \varepsilon]$, $|s(y)| \le C|y|$.
- Show that $|x|/|\sin(x)| \le 2$ for $x$ small (for $\cos(x) \ge \frac{1}{2}$) by considering $g(x) = 2\sin(x) - x$, and then that

$$0 \le \left| \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} s(y)D_m(y)\,\mathrm{d}y \right| \le \frac{4C\varepsilon}{\pi}$$

- Conclude the result for $b = 0$.

- To show for $b \in [-\pi, \pi]$, define $G(y) = f(b - y)$ and use commutativity of convolution.

$\square$

## 8.2. Uniform convergence of Cesàro mean Fourier series via Fejér kernel

**Definition.** Let $x \in \mathbb{R}$, $N \in \mathbb{N}$. **Fejér kernel** is

$$F_N(x) = \frac{1}{N} \sum_{m=0}^{N-1} D_m(x) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=-m}^{m} e^{ikx}$$

**Proposition.**

- $$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x)\, \mathrm{d}x = 1$$

- $$F_N(x) = \frac{1}{N}\left(\frac{\sin(Nx/2)}{\sin(x/2)}\right)^2$$

- Fejér kernel is non-negative, so

$$F_N(x) = |F_N(x)| \implies \int_{-\pi}^{\pi} |F_N(x)|\, \mathrm{d}x = 2\pi$$

- For $\varepsilon > 0$ and $\varepsilon < |x| < \pi$, there exists $C_\varepsilon > 0$ such that $(\sin(x/2))^{-2} \le C_\varepsilon$, hence

$$\int_\varepsilon^\pi |F_N(x)|\,\mathrm{d}x = \frac{1}{N}\int_\varepsilon^\pi \left|\frac{\sin(Nx/2)}{\sin(x/2)}\right|^2 \mathrm{d}x \leq \frac{1}{N}\int C_\varepsilon \leq \frac{\pi C_\varepsilon}{N} \to 0 \quad \text{as } N \to \infty$$

and similarly for $-\pi < x < -\varepsilon$.

**Definition.** The **$N$-th Cesàro mean** is the average of the first $N$ partial Fourier sums of $f$:

$$\frac{1}{N}\sum_{m=0}^{N-1}(S_m f)(x)$$

**Proposition.** Let $f : [-\pi, \pi] \to \mathbb{C}$ integrable, then convolution of $f$ with Fejér kernel is the Cesàro mean:

$$(f * F_N)(x) = \frac{1}{N} \sum_{m=0}^{N-1} (S_m f)(x)$$

**Theorem.** Let $f : [-\pi, \pi] \to \mathbb{C}$ continuous and $2\pi$-periodic, then

$$\forall x \in [-\pi, \pi], \quad f(x) = \lim_{N \to \infty} (f * F_N)(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} (S_m f)(x)$$

and the convergence is uniform.

*Proof.*

- Reason that $f$ is bounded: $|f| \leq B$ on $[-\pi, \pi]$.
- Let $\rho > 0$. Show that $\forall x, y \in [-\pi, \pi]$, for some $\varepsilon > 0$, $|y| < \varepsilon \implies |f(x - y) - f(x)| < \rho$.
- Show that

$$|(f * F_N)(x) - f(x)|$$

$$\leq \frac{1}{2\pi} \left( \int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) |F_N(y)| \, |f(x - y) - f(x)| \, \mathrm{d}y + \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} |F_N(y)| \, |f(x - y) - f(x)|$$

- Show that first terms of RHS tend to zero as $N \to \infty$.
- Show last term on RHS is $< \rho$.

- Conclude the result.

$\square$

**Remark.**

- By above theorem, any $2\pi$-periodic continuous function on $[-\pi, \pi]$ can be uniformly approximated by trigonometric polynomials, i.e. if $\varepsilon > 0$, then there exists trigonometric polynomial $p$ such that $\forall x \in [-\pi, \pi], |f(x) - p(x)| < \varepsilon$.

- This is analogue of Weierstrass Approximation Theorem for $2\pi$-periodic functions. Weierstrass Approximation Theorem states that for continuous function $f : [a, b] \to \mathbb{R}$ and $\varepsilon > 0$, there exists polynomial $p$ such that $\forall x \in [a, b], |f(x) - p(x)| < \varepsilon$.

- Continuous functions are dense in $L^p([a, b])$ for $p \in [1, \infty)$. Let $\varepsilon > 0$, $f \in L^p([a, b])$ and $g : [a, b] \to \mathbb{R}$ continuous such that

$\|f - g\|_{L^p} < \varepsilon$. By Weierstrass Approximation Theorem, there exists polynomial $\tilde{p}$ such that

$$\forall x \in [a, b], \quad |g(x) - \tilde{p}(x)| < \frac{\varepsilon}{(b - a)^{1/p}}$$

Hence

$$\int_a^b |g(x) - \tilde{p}(x)|^p < \varepsilon^p \quad \text{i.e.} \quad \|g - \tilde{p}\|_{L^p} < \varepsilon$$

Hence by Minkowski's inequality, $\|f - \tilde{p}\|_{L^p} < 2\varepsilon$. Hence polynomials are dense in $L^p([a, b])$ for $p \in [1, \infty)$.

- **Note**: for $p = \infty$, any continuous function in $L^{\infty}([a, b])$ can be approximated by polynomials, but continuous functions are not dense in $L^{\infty}([a, b])$.
- Similarly, trigonometric polynomials are dense in $L^p([-\pi, \pi])$ for $p \in [1, \infty)$.

## 8.3. Mean convergence of Fourier series in $L^2([-\pi, \pi])$

**Notation.** Define an inner product on $L^2([-\pi, \pi])$ by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[-\pi, \pi]} f\bar{g}$$

and denote $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. $(L^2([-\pi, \pi]), \langle \cdot, \cdot \rangle)$ is Hilbert space by Riesz-Fischer.

For $k \in \mathbb{Z}$, $x \in [-\pi, \pi]$, let $\varphi_k(x) = e^{ikx}$, then for $2\pi$-periodic integrable function $f : [-\pi, \pi] \to \mathbb{C}$,

$$a_k(f) = \langle f, \varphi_k \rangle, \quad S_N f(x) = \sum_{k=-N}^{N} \langle f, \varphi_k \rangle \varphi_k$$

**Lemma.** Let $f \in L^2([-\pi, \pi])$ be $2\pi$-periodic, define

$$\mathcal{P}_N = \left\{ \sum_{k=-n}^{n} c_k \varphi_k : c_k \in \mathbb{C}, n \leq N \right\}$$

Then:

- $\{\varphi_n : n \in \mathbb{Z}\}$ is orthonormal in $L^2([-\pi, \pi])$ with respect to $\langle \cdot, \cdot \rangle$.
- $\forall p \in \mathcal{P}_N$, $f - S_N f$ is orthogonal to $p$.
- $\forall N \geq 0$, $\forall p \in \mathcal{P}_N$,

$$\|f - S_N f\| \leq \|f - p\|$$

with equality iff $p = S_N f$.

*Proof.*
- Show $\frac{1}{2\pi} \int_{[-\pi,\pi]} \varphi_m \overline{\varphi_n} = 0 = \delta_{mn}$ (justify use of Riemann integral).
- Show that $(f - S_N f) \perp \varphi_m$ for each $|m| \leq N$ to show $(f - S_N f) \perp p$ for $p \in \mathcal{P}_N$.
- Write $f - p = f - S_N f + S_N f - \sum_{k=-N}^{N} c_k \varphi_k$, use Pythagoras.

$\square$

**Remark.** Above lemma is projection result, i.e. $S_N f$ is best approximation to $f$ in $\mathcal{P}_N$.

**Theorem.** Let $f \in L^2([-\pi, \pi])$ be $2\pi$-periodic function. Then Fourier series for $f$ converges to $f$ in $(L^2([-\pi, \pi]), \|\cdot\|)$, i.e.

$$\lim_{N \to \infty} \|S_N f - f\| = 0$$

*Proof.*

- First show if $g : [-\pi, \pi] \to \mathbb{C}$ continuous, then $\|S_N g - G\| \to 0$ as $N \to \infty$.
  - Let $\varepsilon > 0$, then for some $M$, there exists $p \in \mathcal{P}_M$ such that

  $$\forall x \in [-\pi, \pi], \quad |g(x) - p(x)| < \varepsilon$$

  - Use that $g(x) = \lim_{N \to \infty} (g * F_N)(x)$ and $g * F_{M+1} \in \mathcal{P}_M$.
  - Deduce that $\|g - p\|^2 < \varepsilon^2$.
  - Show if $M \leq N$ then $\|g - S_N g\| \leq \|g - p\| < \varepsilon$, conclude result for continuous functions.

- Let $f \in L^2([-\pi, \pi])$, $\varepsilon > 0$. Using that continuous functions are dense in $L^2([-\pi, \pi])$, there is $g : [-\pi, \pi] \to \mathbb{C}$ such that $\|f - g\| < \varepsilon$.
- Since $g$ continuous, for large enough $M$, $\|S_M g - g\| < \varepsilon$ by above.
- Use triangle inequality, the fact that $N \geq M \implies S_M g \in \mathcal{P}_N$ and projection theorem to conclude the result.

$\square$

**Lemma.** $\{\varphi_n : n \in \mathbb{Z}\}$ is orthonormal basis of $(L^2([-\pi, \pi])$ with respect to inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[-\pi,\pi]} f\overline{g}$$

*Proof.*

- Note that $\left(L^2([-\pi, \pi]), \langle \cdot, \cdot \rangle\right)$ is Hilbert space.
- Show Parseval's identity holds.
- Write $f = f - S_N f + S_N f$, use projection theorem, Pythagorean theorem and orthonormality of $\{\varphi_n : n \in \mathbb{Z}\}$ to show

$$\|f\|^2 = \|f - S_N f\|^2 + \sum_{k=-N}^{N} |\langle f, \varphi_k \rangle|^2$$

- Take limit as $N \to \infty$ to conclude result.

□