0.1. Integration and measure

- Dirichlet's function: $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

1. The real numbers

- $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \leq a$.
- $c \in \mathbb{R}$ is a **least upper bound (supremum)** if $c \leq a$ for every upper bound a.
- $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \geq a$.
- $c \in \mathbb{R}$ is a **greatest lower bound (supremum)** if $c \geq a$ for every upper bound a.
- **Completeness axiom of the real numbers**: every subset E with an upper bound has a least upper bound. Every subset E with a lower bound has a greatest lower bound.
- **Archimedes' principle**:

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

- Every non-empty subset of $\mathbb{N}$ has a minimum.
- **The rationals are dense in the reals**:

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

1.1. Conventions on sets and functions

- For $f : X \to Y$, **preiamge** of $Z \subseteq Y$ is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

- $f : X \to Y$ **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

- $f : X \to Y$ **surjective** if $Y = f(X)$.
- **Limit inferior** of sequence x_n:

$$\liminf_{n \to \infty} x_n := \lim_{n \to \infty} \left(\inf_{m \geq n} x_m \right) = \sup_{n \geq 0} \inf_{m \geq n} x_m$$

- **Limit superior** of sequence x_n:

$$\limsup_{n \to \infty} x_n := \lim_{n \to \infty} \left(\sup_{m \geq n} x_m \right) = \inf_{n \geq 0} \sup_{m \geq n} x_m$$

- **Definition**: $f : E \to \mathbb{R}$ is **bi-Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

1.2. Open and closed sets

- $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon : (x - \varepsilon, x + \varepsilon) \subseteq U$$

- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.
- $x \in \mathbb{R}$ is **point of closure (limit point)** for $E \subseteq \mathbb{R}$ if

$$\forall \delta > 0, \exists y \in E : |x - y| < \delta$$

  Equivalently, x is point of closure if every open interval containing x contains another point of E.
- **Closure** of E, $\overline{E}$, is set of points of closure.
- F is **closed** if $F = \overline{F}$.
- If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$.
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- For any set E, $\overline{E}$ is closed.
- Let $E \subseteq \mathbb{R}$. The following are equivalent:
  - E is closed.
  - $\mathbb{R} - E$ is open.
- Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.
- **Definition**: collection C of subsets of $\mathbb{R}$ **covers** (is a **covering** of) $F \subseteq \mathbb{R}$ if $F \subseteq \cup_{S \in C} S$. If each S in C open, G is **open covering**. If C is finite, C is **finite cover**.
- Covering C of F **contains a finite subcover** if exists $\{S_1, ..., S_n\} \subseteq C$ with $F \subseteq \cup_{i=1}^n S_i$ (i.e. a finite subset of C covers F). F is **compact** if any open covering U contains a finite subcover.
- **Example**: $\mathbb{R}$ is not compact, $[a, b]$ is compact.
- **Heine-Borel theorem**: if $F \subset \mathbb{R}$ closed and bounded then any open covering of F has finite subcovering (so F is compact). If F compact then F closed and bounded.

1.3. The extended real numbers

- **Definition**: **extended reals** are $\mathbb{R} \cup \{-\infty, \infty\}$ with the order relation $-\infty < \infty$ and $\forall x \in \mathbb{R}, -\infty < x < \infty$. ∞ is an upper bound and $-\infty$ is a lower bound for every $x \in \mathbb{R}$, so $\sup(\mathbb{R}) = \infty$, $\inf(\mathbb{R}) = -\infty$.
  - Addition: $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$. $\infty + \infty = \infty - (-\infty) = \infty$. $\infty - \infty$ is undefined.
  - Multiplication: $\forall a \in \mathbb{R}_{>0}, a \cdot \infty = \infty$, $\forall a \in \mathbb{R}_{<0}, a \cdot = -\infty$. $\infty \cdot \infty = \infty$ and $0 \cdot \infty = \infty$.
  - $\limsup$ and $\liminf$ are defined as

$$\limsup x_n := \inf_{n \in \mathbb{N}} \left\{ \sup_{k \geq n} x_k \right\}, \quad \liminf x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{k \geq n} x_k \right\}$$

- **Definition**: extended real number l is **limit** of (x_n) if either
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$. Then (x_n) **converges to l**. or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$ (limit is ∞) or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$ (limit is $-\infty$).

(x_n) **converges in the extended reals** if it has a limit in the extended reals.

2. Further analysis of subsets of $\mathbb{R}$

TODO: up to here, check that all notes are made from these topics

2.1. Countability and uncountability

- A is **countable** if $A = \emptyset$, A is finite or there is a bijection $\varphi : \mathbb{N} \to A$ (in which case A is **countably infinite**). Otherwise A is **uncountable**. φ is called an **enumeration**.
- If surjection from $\mathbb{N}$ to A, or injection from A to $\mathbb{N}$, then A is countable.
- Examples of countable sets:
  - $\mathbb{N}$ ($\varphi(n) = n$)
  - $2\mathbb{N}$ ($\varphi(n) = 2n$)
- $\mathbb{Q}$ is countable.
- **Exercise (todo)**: show that $\mathbb{N}^k$ is countable for any $k \in \mathbb{N}$.
- **Exercise (todo)**: show that if a_n is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Exercise (todo)**: show that if $a_{n,k}$ is a nonnegative sequence and $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- $f : X \to Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.
- Let f be monotone on (a, b). Then it is discountinuous on a countable set.
- Set of sequences in $\{0, 1\}$, $\{((x_n))_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.
- **Theorem**: $\mathbb{R}$ is uncountable.

2.2. The structure theorem for open sets

- Collection $\{A_i : i \in I\}$ of sets is **(pairwise) disjoint** if $n \neq m \Longrightarrow A_n \cap A_m = \emptyset$.
- **Structure theorem for open sets**: let $U \subseteq \mathbb{R}$ open. Then exists countable collection of disjoint open intervals $\{I_n : n \in \mathbb{N}\}$ such that $U = \cup_{n \in \mathbb{N}} I_n$.

2.3. Accumulation points and perfect sets

- $x \in \mathbb{R}$ is **accumulation point** of $E \subseteq \mathbb{R}$ if x is point of closure of $E - \{x\}$. Equivalently, x is a point of closure if

$$\forall \delta > 0, \exists y \in E : y \neq x \land |x - y| < \delta$$

  Equivalently, there exists a sequence of distinct $y_n \in E$ with $y_n \to x$ as $n \to \infty$.
- **Exercise**: set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$.
- $E \subseteq \mathbb{R}$ is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

- **Proposition**: set of accumulation points E' of E is closed.
- Bounded set E is **perfect** if it equals its set of accumulation points.
- **Exercise (todo)**: what is the set of accumulation points of an isolated set?
- Every non-empty perfect set is uncountable.

2.4. The middle-third Cantor set

- **Proposition**: let $\{F_n : n \in N\}$ be collection of non-empty nested closed sets, one of which is bounded, so $F_{n+1} \subseteq F_n$. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$$

- **Middle third Cantor set**:
  - Define $C_0 := [0, 1]$
  - Given $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$, $a_i < b_1 < a_2 < \cdots$, with $|b_i - a_i| = 3^{-n}$, define

$$C_{n+1} := \cup_{i=1}^{2^n} \left[a_i, a_i + 3^{-(n+1)}\right] \cup \left[b_i - 3^{-(n+1)}, b_i\right]$$

    which is a union of 2^{n+1} disjoint intervals, with difference in endpoints equalling $3^{-(n+1)}$.
  - The **middle third Cantor set** is

$$C := \bigcup_{n \in \mathbb{N}} C_n$$

    Observe that if a is an endpoint of an interval in C_n, it is contained in C.
- **Proposition**: the middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and uncountable.
- **Definition**: let $k \in \mathbb{N} - \{1\}$, $x \in [0, 1)$. $0.a_1 a_2 \ldots$, $a_i \in \{0, \ldots, k - 1\}$, is a ***k*-ary expansion** of x if

$$x = \sum_{i=1}^{\infty} \frac{a_i}{k^i}$$

- **Remark**: the k-ary expansion may not be unique, but there is a countable set $E \subseteq [0, 1)$ such that every $x \in [0, 1) - E$ has a unique k-ary expansion.
- **Remark**: for every $x \in C$, the ternary ($k = 3$) expansion of x is unique and

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}$$

  Moreover, every choice of sequence (a_i), $a_i \in \{0, 2\}$, gives $x \in \sum_{i=1}^{\infty} \frac{a_i}{3^i} \in C$.
- **Example**: define $g : [0, 1] \to [0, 1]$ by

$$g(x) := \begin{cases} \sum_{i=1}^{\infty} \frac{a_i/2}{2^i} & \text{if } x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, a_i \in \{0, 2\} \\ \sup_{x \in C, x \leq y} f(x) & \text{if } x \notin C \end{cases}$$

  g is a surjection, monotone and continuous.

2.5. G_δ, F_σ

- Set E is $\boldsymbol{G_\delta}$ if $E = \cap_{n\in\mathbb{N}} U_n$ with U_n open.
- Set E is $\boldsymbol{F_\sigma}$ if $E = \cup_{n\in\mathbb{N}} F_n$ with F_n closed.
- **Lemma**: set of points where $f : \mathbb{R} \to \mathbb{R}$ is continuous is G_δ.

3. Construction of Lebesgue measure

3.1. Lebesgue outer measure

- **Definition**: let I non-empty interval with endpoints $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$ and $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$. The **length** of I is

$$\ell(I) := b - a$$

  and set $\ell(\emptyset) = 0$.
- **Example**: if $I = (-\infty, b] = (-\infty, a] \cup [a, b]$ then $\ell(I) = \infty = \ell(-\infty, a]) + \ell([a, b])$
- **Definition**: let $A \subseteq \mathbb{R}$. **Lebesgue outer measure** of A is infimum of all sums of lengths of intervals covering A:

$$\mu^*(A) := \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k, I_k \text{ intervals}\right\}$$

  It satisfies **monotonicity**: $A \subseteq B \Longrightarrow \mu^*(A) \leq \mu^*(B)$.
- **Proposition**: outer measure is **countably subadditive**: if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets then

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$$

- **Lemma**: we have

$$\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k, I_k \neq \emptyset \text{ open intervals}\right\}$$

- Lebesgue outer measure of interval is its length: $\mu^*(I) = \ell(I)$.

3.2. Measurable sets

- **Notation**: $E^c = \mathbb{R} - E$.
- **Proposition**: let $E = (a, \infty)$. Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

- **Definition**: $E \subseteq \mathbb{R}$ is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

  Collection of such sets is $\mathcal{F}_{\mu^*}$.
- **Lemma (excision property)**: let E Lebesgue measurable set with finite measure and $E \subseteq B$, then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

- **Remark**: not every set is Lebesgue measurable.

- **Definition**: collection of subsets of $\mathbb{R}$ is an **algebra** if contains $\emptyset$ and closed under taking complements and finite unions: if $A, B \in \mathcal{A}$ then $\mathbb{R} - A, A \cup B \in \mathcal{A}$.
- **Remark**: if a union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if $\{A_k\}_{k=1}^{\infty}$ is countable collection of Lebesgue measurable sets, then let $A_{1'} = A_1$ and for $k > 1$, define

$$A_{k'} = A_k - \cup_{i=1}^{k-1} A_i$$

then $\{A_{k'}\}_{k=1}^{\infty}$ is disjoint union of Lebesgue measurable sets.
- **Proposition**: if $E_1, ..., E_n$ Lebesgue measurable then $\cup_{k=1}^{n} E_k$ is Lebesgue measurable. If $E_1, ..., E_n$ disjoint then

$$\mu^* \left(A \cap \bigcup_{k=1}^{n} E_k \right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

for any $A \subseteq \mathbb{R}$. In particular, for $A = \mathbb{R}$,

$$\mu^* \left(\bigcup_{k=1}^{n} E_k \right) = \sum_{k=1}^{n} \mu^*(E_k)$$

- **Proposition**: if E is countable union of Lebesgue measurable sets, then E is Lebesgue measurable. Also, if $\{E_k\}_{k \in \mathbb{N}}$ is countable disjoint collection of Lebesgue measurable sets then

$$\mu^* \left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu^*(E_k)$$

3.3. Abstract definition of a measure

- **Definition**: let $X \subseteq \mathbb{R}$. Collection of subsets of $\mathcal{F}$ of X is **σ-algebra** if
  - $\emptyset \in F$
  - $E \in F \Longrightarrow E^c \in F$
  - $E_1, ..., E_n \in F \Longrightarrow \cup_{k=1}^{\infty} E_k \in \mathcal{F}$.
- **Example**:
  - Trivial examples are $\mathcal{F} = \{\emptyset, \mathbb{R}\}$ and $\mathcal{F} = \mathcal{P}(\mathbb{R})$.
  - Arbitrary intersections of σ-algebras are σ-algebras.
- **Definition**: let $\mathcal{F}$ σ-algebra of X. $\nu : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is **measure** satisfying
  - $\nu(\emptyset) = 0$
  - $\forall E \in \mathcal{F}, \nu(E) \geq 0$
  - **Countable additivity**: if $E_1, E_2, ... \in \mathcal{F}$ are disjoint then

$$\nu \left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \nu(E_k)$$

  Elements of $\mathcal{F}$ are **measurable** (as they are the only sets on which the measure ν is defined).

- **Proposition**: if ν is measure then it satisfies:
  - **Monotonicity**: $A \subseteq B \Longrightarrow \nu(A) \leq \nu(B)$.
  - **Countable subadditivity**: $\nu(\cup_{k\in\mathbb{N}} E_k) \leq \sum_{k\in\mathbb{N}} \nu(E_k)$.
  - **Excision**: if A has finite measure, then $A \subseteq B \Longrightarrow m(B - A) = m(B) - m(A)$.

3.4. Lebesgue measure

- **Lemma**: the Lebesgue measurable sets form a σ-algebra and contain every interval.
- **Theorem (Caratheodory extension)**: the restriction of the outer measure μ^* to the σ-algebra of Lebesgue measurable sets is a measure.
- **Definition**: the measure μ of μ^* restricted to $\mathcal{F}_{\mu^*}$ is the **Lebesgue measure**. It satisfies $\mu(I) = \ell(I)$ for any interval I and is translation invariant.
- **Hahn extension theorem**: there exists unique measure μ defined on $\mathcal{F}_{\mu^*}$ for which $\mu(I) = \ell(I)$ for any interval I.

3.5. Sets of measure 0

- **Exercise (todo)**: middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.
- **Exercise (todo)**: any countable set is Lebesgue measurable and has Lebesgue measure 0.
- **Exercise (todo)**: any E with $\mu^*(E) = 0$ is Lebesgue measurable and has $\mu(E) = 0$.
- **Lemma**: let E Lebesgue measurable set with $\mu(E) = 0$, then $\forall E' \subseteq E$, E' is Lebesgue measurable.

3.6. Continuity of measure

- **Definition**: countable collection $\{E_k\}_{k=1}^{\infty}$ is **ascending** if $\forall k \in \mathbb{N}, E_k \subseteq E_{k+1}$ and **descending** if $\forall k \in \mathbb{N}, E_{k+1} \subseteq E_k$.
- **Theorem**: every measure m satisfies:
  - If $\{A_k\}_{k=1}^{\infty}$ is ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$$

  - If $\{B_k\}_{k=1}^{\infty}$ is descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$$

3.7. An approximation result for Lebesgue measure

- **Definition**: **Borel σ-algebra** $\mathcal{B}(\mathbb{R})$ is smallest σ-algebra containing all intervals: for any other σ-algebra $\mathcal{F}$ containing all intervals, $\mathcal{B}(\mathbb{R}) \subset \mathcal{F}$.

$$\mathcal{B}(\mathbb{R}) = \bigcap \{\mathcal{F} : \mathcal{F}\ \sigma\text{ -algebra containing all intervals}\}$$

  $E \in \mathcal{B}(\mathbb{R})$ is **Borel** or **Borel measurable**.

- Every open subset of $\mathbb{R}$, every closed subset of $\mathbb{R}$, every G_δ set, every F_σ set is Borel.
- **Proposition**: the following are equivalent:
  - E is Lebesgue measurable
  - $\forall \varepsilon > 0, \exists$ open $G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
  - $\forall \varepsilon > 0, \exists$ closed $F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
  - $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
  - $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

4. Measurable functions

4.1. Definition of a measurable function

- **Lemma**: let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ with E Lebesgue measurable. The following are equivalent:
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) > c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) \geq c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) < c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) \leq c\}$ is Lebesgue measurable.
- **Definition**: $f : E \to \mathbb{R}$ is **(Lebesgue) measurable** if it satisfies any one of the above properties and if E is Lebesgue measurable.
- **Proposition**: let $f : \mathbb{R} \to \mathbb{R}$. f continuous iff $\forall$ open $U \subseteq, f^{-1}(U) \subseteq \mathbb{R}$ is open.
- **Definition**: **indicator function** on set A, $\mathbb{1}_A : \mathbb{R} \to \{0, 1\}$ is

$$\mathbb{1}_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

- **Definition**: $\varphi : \mathbb{R} \to \mathbb{R}$ is **simple (measurable) function** if φ is measurable function that has finite codomain.
- **Definition**: sequence of functions (f_n) with domain E **converge in measure** to f if

$$\forall \varepsilon > 0, \quad \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \to 0 \text{ as } n \to \infty$$

4.2. Fundamental aspects of measurable functions

- **Definition**: let $E \subseteq F \subseteq \mathbb{R}$, let $f : F \to \mathbb{R}$. **Restriction** f_E is function with domain E and for which $\forall x \in E, f_E(x) = f(x)$.
- **Definition**: real-valued function which is increasing or decreasing is **monotone**.
- **Definition**: sequence (f_n) on domain E is increasing if $f_n \leq f_{n+1}$ on E for all $n \in \mathbb{N}$.
- **Example**: continuous functions are measurable.
- **Definition**: for $f_1 : E \to \mathbb{R}, ..., f_n : E \to \mathbb{R}$, $\max\{f_1, ..., f_n\} : E \to \mathbb{R}$ is

$$\max\{f_1, ..., f_n\}(x) = \max\{f_1(x), ..., f_n(x)\}$$

  $\min\{f_1, ..., f_n\}$ is defined similarly.
- **Proposition**: for finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E, $\max\{f_1, ..., f_n\}$ and $\min\{f_1, ..., f_n\}$ are measurable.

- **Definition**: for $f : E \to \mathbb{R}$, functions $|f|, f^+, f^-$ defined on E are

$$|f|(x) := \max\{f(x), -f(x)\}, \quad f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}$$

- **Corollary**: if f measurable on E, so are $|f|$, f^+ and f^-.
- **Proposition**: let $f : E \to \mathbb{R} \cup \{\pm\infty\}$. For measurable $D \subseteq E$, f measurable on E iff restrictions of f to D and $E - D$ are measurable.
- **Theorem**: let f, g real-valued measurable functions with domain E.
  - **Linearity**: $\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g$ is measurable.
  - **Products**: fg is measurable.
- **Proposition**: let (f_n) be sequence of measurable functions on E that converges pointwise to f on E. Then f is measurable.
- **Simple approximation lemma**: let $f : E \to \mathbb{R}$ measurable and bounded, so $\exists M \geq 0 : \forall x \in E, |f|(x) < M$. Then $\forall \varepsilon > 0$, there exist simple measurable functions $\varphi_\varepsilon, \psi_\varepsilon : E \to \mathbb{R}$ such that

$$\forall x \in E, \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \wedge 0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$$

- **Definition**: let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$. Then $f = g$ **almost everywhere** if $\{x \in E : f(x) \neq g(x)\}$ has measure 0.
- **Proposition**: let $f_1, f_2, f_3 : E \to \mathbb{R} \cup \{\pm\infty\}$ measurable. If $f_1 = f_2$ almost everywhere and $f_2 = f_3$ almost everywhere then $f_1 = f_3$ almost everywhere.
- Let $f, g : E \to \mathbb{R} \cup \{\pm\infty\}$ finite almost everywhere on E. Let D_f and D_g be sets for which f and g are finite. Then $f + g$ is finite and well-defined on $D_f \cap D_g$ and complement of $D_f \cap D_g$ has measure 0.
- **Remark**: Lebesgue measurable functions can be modified arbitrarily on a set of measure 0 without affecting measurability.
- **Simple approximation theorem**: let $f : E \to \mathbb{R} \cup \{\pm\infty\}$, E measurable. Then f is measurable iff there exists sequence (φ_n) of simple functions on E which converge pointwise on E to f and satisfy

$$\forall n \in \mathbb{N}, \forall x \in E, |\varphi_n(x)| \leq |f|(x)$$

  If f is nonnegative, (φ_n) can be chosen to be increasing.

5. The Lebesgue integral

5.1. The integral of a simple measurable function

- **Definition**: let φ be real-valued function taking finitely many values $\alpha_1 < \cdots < \alpha_n$, then **standard representation** of φ is

$$\varphi = \sum_{i=1}^{n} \alpha \mathbb{1}_{A_i}, \quad A_i = \varphi^{-1}(\{\alpha_i\})$$

- **Lemma**: let $\varphi = \sum_{i=1}^{m} \beta_i \mathbb{1}_{B_i}$, B_i disjoint mesauble collection, $\beta_i \in \mathbb{R}$, then φ is simple measurable. If φ takes values 0 outside a finite set then

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \beta_i \mu(B_i)$$

where A_i in standard representation.

- **Definition**: let φ be simple nonnegative measurable function. **Integral** of φ with respect to μ is

$$\int \varphi = \sum_{i=1}^{n} \alpha_i \mu(A_i)$$

where $\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$ is the standard representation. Here we use the convention $0 \cdot \infty = 0$.

- **Example**:
  - Let $\varphi_2 = \mathbb{1}_{[0,2]} + \mathbb{1}_{[1,3]} = \mathbb{1}_{[0,1)\cup(2,3]} + 2\mathbb{1}_{[1,2]}$ so $\int \varphi_2 = 4$.
  - Let $\varphi_3 = \mathbb{1}_{\mathbb{R}}$, then $\int \varphi_3 = 1 \cdot \infty = \infty$.
  - Let $\varphi_4 = \mathbb{1}_{(0,\infty)} + (-1)\mathbb{1}_{(-\infty,0)}$. This can't be integrated.
  - Let $\varphi_5 = \mathbb{1}_{(-1,0)} + (-1)\mathbb{1}_{(0,1)}$.
- **Lemma**: let $B_1, ..., B_m$ be collection of measurable sets, $\beta_1, ..., \beta_m \in \mathbb{R} - \{0\}$. Then $\varphi = \sum_{i=1}^{m} \beta_i \mathbb{1}_{B_i}$ is simple measurable function. If measurable of $\cup_{i=1}^{m} B_i$ is finite, then

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \beta_i \mu(B_i)$$

where A_i in standard representation.

- **Proposition (linearity and monotonicity of integration for simple funtions)**: let φ, ψ be simple measurable functions:
  - If φ, ψ take value 0 outside a set of finite measure, then $\forall \alpha, \beta \in \mathbb{R}$,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

  -

$$0 \leq \varphi \leq \psi \Longrightarrow 0 \leq \int \varphi \leq \int \psi$$

- **Corollary**: let φ nonnegative simple function, then

$$\int \varphi = \sup\left\{ \int \psi : 0 \leq \psi \leq \varphi, \psi \text{ simple measurable} \right\}$$

- **Lemma**: let φ simple measurable nonnegative function. φ takes value 0 outside a set of finite measure iff $\int \varphi < \infty$. Also, $\int \varphi = \infty$ iff there exist $\alpha > 0$, measurable A with $\mu(A) = \infty$ with $\varphi(x) \geq \alpha$ on A.
- **Lemma**: let $\{E_n\}$ be ascending collection of measurable sets, $\cup_{n=1}^{\infty} E_n = \mathbb{R}$. Let φ be simple nonnegative measurable function. Then

$$\int_{E_n} \varphi \to \int \varphi \quad \text{as } n \to \infty$$

5.2. The integral of a nonnegative function

- **Notation**: let $\mathcal{M}^+$ denote collection of nonnegative measurable functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

- **Definition**: **support** of measurable function f with domain E is $\{x \in E : f(x) \neq 0\}$.
- **Definition**: let $f \in \mathcal{M}^+$. **Integral of f with respect to μ** is

$$\int f := \sup\left\{\int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple measurable}\right\} \in \mathbb{R} \cup \{\infty\}$$

  For measurable set E, define

$$\int_E f := \int \mathbb{1}_E f$$

- **Proposition**: let f, g measurable. If $g \leq f$ then $\int g \leq \int f$. Let E, F measurable. If $E \subseteq F$ then $\int_E f \leq \int_F f$.
- **Monotone convergence theorem**: let (f_n) be sequence in $\mathcal{M}^+$. If (f_n) is increasing on measurable set E and converges pointwise to f on E then

$$\int_E f_n \to \int_E f \quad \text{as } n \to \infty$$

- **Corollary**: restriction of integral to nonnegative functions is linear: $\forall f, g \in \mathcal{M}^+$, $\forall \alpha \geq 0$,

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

- **Fatou's lemma**: let (f_n) be sequence in $\mathcal{M}^+$, then

$$\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n$$

- **Lemma**: let $(f_n) \subset \mathcal{M}^+$, then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

- **Proposition (Chebyshev's inequality)**: let f be nonnegative measurable function on E. Then

$$\forall \lambda > 0, \quad \mu(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f$$

- **Proposition**: let f be nonnegative measurable function on E. Then

$$\int_E f = 0 \iff f = 0 \text{ almost everywhere on } E$$

5.3. Integration of measurable functions

- **Notation**: let $\mathcal{M}$ denote set of measurable functions.
- **Definition**: $f \in \mathcal{M}^+$ is **integrable** if $\int f < \infty$.

- **Definition**: let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ measurable function. f is **integrable** if $\int f^+$ and $\int f^-$ are finite. In this case, for any measurable set E, define

$$\int_E f := \int_E f^+ - \int_E f^-$$

If f integrable then $f^+ - f^-$ is well-defined.
- **Definition**: $f \in \mathcal{M}$ is **integrable over $\boldsymbol{E}$** (E is measurable) if $\int_E f^+$ and $\int_E f^-$ are finite (i.e. $f \cdot \mathbb{1}_E$ is integrable).
- **Theorem**: $f \in \mathcal{M}$ is integrable iff $|f|$ is integrable. If f integrable, then

$$\left|\int f\right| \leq \int |f|$$

- **Corollary**: let $f, g \in \mathcal{M}$, $|f| \leq |g|$. If g integrable then $|f|$ is integrable, and $\int |f| \leq \int |g|$.
- **Theorem (Linearity of Integration)**: let f, g integrable. Then $f + g$ is integrable and $\forall \alpha \in \mathbb{R}$, αf is integrable. The integral is linear:

$$\int (f + g) = \int f + \int g$$

$$\int \alpha f = \alpha \int f$$

- **Dominated Convergence Theorem**: let (f_n) be sequence of integrable functions. If there exists an integrable g with $\forall n \in \mathbb{N}, |f_n| \leq g$, and $f_n \to f$ pointwise almost everywhere then f is integrable and

$$\int f = \lim_{n \to \infty} \int f_n$$

- **Example**: sin is not integrable over $\mathbb{R}$, but is integrable over $[0, 2\pi]$, since $|f_{[0,2\pi]}| \leq \mathbb{1}_{[0,2\pi]}$.

5.4. Integrability: Riemann vs Lebesgue

- **Proposition**: let f bounded function on bounded domain. Then f is measurable and $\int |f| < \infty$ iff

$$\sup\left\{\int \varphi : \varphi \leq f, \varphi \text{ simple measurable}\right\} = \inf\left\{\int \psi : f \leq \psi : \psi \text{ simple measurable}\right\}$$

- **Definition**: bounded function f is **Lebesgue integrable** if it satisfies either of the equivalences in the above proposition.
- **Definition**: let $P = \{x_0 < \cdots < x_n\}$ be partition of $[a, b]$, let $f : [a, b] \to \mathbb{R}$ bounded. **Lower and upper Darboux sums** for f with respect to P are

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}), \quad U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

where

$$m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

If $P \subseteq Q$ (Q is a **refinement of $\boldsymbol{P}$**), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

- **Definition**: **lower and upper Riemann integrals** of f over $[a, b]$ are

$$\underline{\mathcal{I}}_a^b(f) := \sup\{L(f, P) : P \text{ partition of } [a, b]\}$$

$$\overline{\mathcal{I}}_a^b(f) := \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

- **Definition**: let $f : [a, b] \to \mathbb{R}$ bounded, then f is **Riemann integrable** ($f \in \mathcal{R}$), if

$$\underline{\mathcal{I}}_a^b(f) = \overline{\mathcal{I}}_a^b(f)$$

  and the common value $\mathcal{I}_a^b$ is the **Riemann integral** of f.
- **Theorem**: let $f : [a, b] \to \mathbb{R}$ bounded. If f is Riemann integrable over $[a, b]$ then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal:

$$\mathcal{I}_a^b(f) = \int_{[a,b]} f$$

- **Theorem**: let $f : [a, b] \to \mathbb{R}$ bounded. Then f is Riemann integrable on $[a, b]$ iff f is continuous on $[a, b]$ except on a set of measure zero.
- **Lemma**: let (φ_n), (ψ_n) be sequences of functions, all integrable over E, (φ_n) increasing on E, (ψ_n) decreasing on E. Let $f : E \to \mathbb{R}$ with

$$\forall n \in \mathbb{N}, \varphi_n \leq f \leq \psi_n \text{ on } E, \quad \lim_{n\to\infty} \int_E (\psi_n - \varphi_n) = 0$$

  Then $\varphi_n \to f$ pointwise almost everywhere on E, $\psi_n \to f$ pointwise almost everywhere on E, f is integrable over E and

$$\lim_{n\to\infty} \int_E \varphi_n = \lim_{n\to\infty} \int_E \psi_n = \int_E f$$

- **Proposition**: let $f : [a, b] \to \mathbb{R}$ bounded. If f is Riemann integrable then it is continuous on a set E whose complement $[a, b] - E$ has measure 0.