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1. Combinatorial methods

Definition 1.1 Let G be an abelian group and $A, B \subseteq G$. The **sumset** of A and B is

$$A + B := \{a + b : a \in A, b \in B\}.$$

The **difference set** of A and B is

$$A - B := \{a - b : a \in A, b \in B\}.$$

Proposition 1.2 $\max\{|A|, |B|\} \leq |A + B| \leq |A| \cdot |B|$.

Proof. Trivial. □

Example 1.3 Let $A = [n] = \{1, \dots, n\}$. Then $A + A = \{2, \dots, 2n\}$ so $|A + A| = 2|A| - 1$.

Lemma 1.4 Let $A \subseteq \mathbb{Z}$ be finite. Then $|A + A| \geq 2|A| - 1$ with equality iff A is an arithmetic progression.

Proof (Hints). Consider two sequences in $A + A$ which are strictly increasing and of the same length. □

Proof. Let $A = \{a_1, \dots, a_n\}$ with $a_i < a_{i+1}$. Then $a_1 + a_1 < a_1 + a_2 < \dots < a_1 + a_n < a_2 + a_n < \dots < a_n + a_n$. Note this is not the only choice of increasing sequence that works, in particular, so does $a_1 + a_1 < a_1 + a_2 < a_2 + a_2 < a_2 + a_3 < a_2 + a_4 < \dots < a_2 + a_n < a_3 + a_n < \dots < a_n + a_n$. So when equality holds, all these sequences must be the same. In particular, $a_2 + a_i = a_1 + a_{i+1}$ for all i . □

Lemma 1.5 If $A, B \subseteq \mathbb{Z}$, then $|A + B| \geq |A| + |B| - 1$ with equality iff A and B are arithmetic progressions with the same step.

Proof (Hints). Similar to above, consider 4 sequences in $A + B$ which are strictly increasing and of the same length. □

Example 1.6 Let $A, B \subseteq \mathbb{Z}/p$ for p prime. If $|A| + |B| \geq p + 1$, then $A + B = \mathbb{Z}/p$.

Proof (Hints). Consider $A \cap (g - B)$ for $g \in \mathbb{Z}/p$. □

Proof. Note that $g \in A + B$ iff $A \cap (g - B) \neq \emptyset$ where $(g - B = \{g\} - B)$. Let $g \in \mathbb{Z}/p$, then use inclusion-exclusion on $|A \cap (g - B)|$ to conclude result. □

Theorem 1.7 (Cauchy-Davenport) Let p be prime, $A, B \subseteq \mathbb{Z}/p$ be non-empty. Then

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

Proof (Hints).

- Assume $|A| + |B| < p + 1$, and WLOG that $1 \leq |A| \leq |B|$ and $0 \in A$ (by translation).
- Induct on $|A|$.
- Let $a \in A$, find B' such that $0 \in B'$, $a \notin B'$ and $|B'| = |B|$ (use fact that p is prime).

- Apply induction with $A \cap B'$ and $A \cup B'$, while reasoning that $(A \cap B') + (A \cup B') \subseteq A + B'$.

□

Proof. Assume $|A| + |B| < p + 1$, and WLOG that $1 \leq |A| \leq |B|$ and $0 \in A$ (by translation). We use induction on $|A|$. $|A| = 1$ is trivial. Let $|A| \geq 2$ and let $0 \neq a \in A$. Then since p is prime, $\{a, 2a, \dots, pa\} = \mathbb{Z}/p$. There exists $m \geq 0$ such that $ma \in B$ but $(m+1)a \notin B$ (why?). Let $B' = B - ma$, so $0 \in B'$, $a \notin B'$ and $|B'| = |B|$.

Now $1 \leq |A \cap B'| < |A|$ (why?) so the inductive hypothesis applies to $A \cap B'$ and $A \cup B'$. Since $(A \cap B') + (A \cup B') \subseteq A + B'$ (why?), we have $|A + B| = |A + B'| \geq |(A \cap B') + (A \cup B')| \geq |A \cap B'| + |A \cup B'| - 1 = |A| + |B| - 1$. □

Example 1.8 Cauchy-Davenport does not hold general abelian groups (e.g. \mathbb{Z}/n for n composite): for example, let $A = B = \{0, 2, 4\} \subseteq \mathbb{Z}/6$, then $A + B = \{0, 2, 4\}$ so $|A + B| = 3 < \min\{6, |A| + |B| - 1\}$.

Example 1.9 Fix a small prime p and let $V \subseteq \mathbb{F}_p^n$ be a subspace. Then $V + V = V$, so $|V + V| = |V|$. In fact, if $A \subseteq \mathbb{F}_p^n$ satisfies $|A + A| = |A|$, then A is an affine subspace (a coset of a subspace).

Proof. If $0 \in A$, then $A \subseteq A + A$, so $A = A + A$. General result follows by considering translation of A . □

Example 1.10 Let $A \subseteq \mathbb{F}_p^n$ satisfy $|A + A| \leq \frac{3}{2} |A|$. Then there exists a subspace $V \subseteq \mathbb{F}_p^n$ such that $|V| \leq \frac{3}{2} |A|$ and A is contained in a coset of V .

Proof. Exercise (sheet 1). □

Definition 1.11 Let $A, B \subseteq G$ be finite subsets of an abelian group G . The **Ruzsa distance** between A and B is

$$d(A, B) := \log \frac{|A - B|}{\sqrt{|A| \cdot |B|}}.$$

Lemma 1.12 (Ruzsa Triangle Inequality) Let $A, B, C \subseteq G$ be finite. Then

$$d(A, C) \leq d(A, B) + d(B, C).$$

Proof (Hints). Consider a certain map from $B \times (A - C)$ to $(A - B) \times (B - C)$. □

Proof. Note that $|B| |A - C| \leq |A - B| |B - C|$. Indeed, writing each $d \in A - C$ as $d = a_d - c_d$ with $a_d \in A$, $c_d \in C$, the map $\varphi : B \times (A - C) \rightarrow (A - B) \times (B - C)$, $\varphi(b, d) = (a_d - b, b - c_d)$ is injective (why?). The triangle inequality now follows from definition of Ruzsa distance. □

Definition 1.13 The **doubling constant** of finite $A \subseteq G$ is $\sigma(A) := |A + A|/|A|$.

Definition 1.14 The **difference constant** of finite $A \subseteq G$ is $\delta(A) := |A - A|/|A|$.

Remark 1.15 The Ruzsa triangle inequality shows that

$$\log \delta(A) = d(A, A) \leq d(A, -A) + d(-A, A) = 2 \log \sigma(A).$$

So $\delta(A) \leq \sigma(A)^2$, i.e. $|A - A| \leq |A + A|^2/|A|$.

Notation 1.16 Let $A \subseteq G$, $\ell, m \in \mathbb{N}_0$. Then

$$\ell A + mA := \underbrace{A + \dots + A}_{\ell \text{ times}} - \underbrace{A - \dots - A}_{m \text{ times}}$$

This is referred to as the **iterated sum and difference set**.

Theorem 1.17 (Plunnecke's Inequality) Let $A, B \subseteq G$ be finite and $|A + B| \leq K|A|$ for some $K \geq 1$. Then $\forall \ell, m \in \mathbb{N}_0$,

$$|\ell B - mB| \leq K^{\ell+m}|A|.$$

Proof (Hints).

- Let $A' \subseteq A$ minimise $|A' + B|/|A'|$ with value K' .
- Show that for every finite $C \subseteq G$, $|A' + B + C| \leq K'|A' + C|$ by induction on $|C|$ (note two sets need to be written as disjoint unions here).
- Show that $\forall m \in \mathbb{N}_0$, $|A' + mB| \leq (K')^m|A'|$ by induction.
- Use Ruzsa triangle inequality to conclude result.

□

Proof. Choose $\emptyset \neq A' \subseteq A$ which minimises $|A' + B|/|A'|$. Let the minimum value be K' . Then $|A' + B| = K'|A'|$, $K' \leq K$ and $\forall A'' \subseteq A$, $|A'' + B| \geq K'|A''|$.

We claim that for every finite $C \subseteq G$, $|A' + B + C| \leq K'|A' + C|$:

Use induction on $|C|$. $|C| = 1$ is true by definition of K' . Let claim be true for C , consider $C' = C \cup \{x\}$ for $x \notin C$. $A' + B + C' = (A' + B + C) \cup ((A' + B + x) - (D + B + x))$, where $D = \{a \in A' : a + B + x \subseteq A' + B + C\}$. By definition of K' , $|D + B| \geq K'|D|$. Hence,

$$\begin{aligned} |A' + B + C| &\leq |A' + B + C| + |A' + B + x| - |D + B + x| \\ &\leq K'|A' + C| + K'|A'| - K'|D| \\ &= K'(|A' + C| + |A'| - |D|). \end{aligned}$$

Applying this argument a second time, write $A' + C' = (A' + C) \cup ((A' + x) - (E + x))$, where $E = \{a \in A' : a + x \in A' + C\} \subseteq D$. Finally,

$$\begin{aligned} |A' + C'| &= |A' + C| + |A' + x| - |E + x| \\ &\geq |A' + C| + |A'| - |D|. \end{aligned}$$

This proves the claim.

We now show that $\forall m \in \mathbb{N}_0$, $|A' + mB| \leq (K')^m|A'|$ by induction: $m = 0$ is trivial, $m = 1$ is true by assumption. Suppose it is true for $m - 1 \geq 1$. By the claim with $C = (m - 1)B$, we have

$$|A' + mB| = |A' + B + (m - 1)B| \leq K'|A' + (m - 1)B| \leq (K')^m|A'|.$$

As in the proof of Ruzsa's triangle inequality, $\forall \ell, m \in \mathbb{N}_0$,

$$\begin{aligned} |A'| |\ell B - mB| &\leq |A' + \ell B| |A' + mB| \\ &\leq (K')^\ell |A'| (K')^m |A'| \\ &= (K')^{\ell+m} |A'|^2. \end{aligned}$$

□

Theorem 1.18 (Freiman-Ruzsa) Let $A \subseteq \mathbb{F}_p^n$ and $|A + A| \leq K|A|$. Then A is contained in a subspace $H \subseteq \mathbb{F}_p^n$ with $|H| \leq K^2 p^{K^4} |A|$.

Proof (Hints).

- Let $X \subseteq 2A - A$ be of maximal size such that all $x + A$, $x \in X$, are disjoint.
- Use [Plunnecke's Inequality](#) to obtain an upper bound on $|X||A|$.
- Show that $\forall \ell \geq 2$, $\ell A - A \subseteq (\ell - 1)X + A - A$ by induction.
- Let H be subgroup generated by A . By writing H as an infinite union, show that $H \subseteq Y + A - A$, where Y is subgroup generated by X .
- Find an upper bound for $|Y|$, conclude using [Plunnecke's Inequality](#).

□

Proof. Choose maximal $X \subseteq 2A - A$ such that the translates $x + A$ with $x \in X$ are disjoint. Such an X cannot be too large: $\forall x \in X$, $x + A \subseteq 3A - A$, so by [Plunnecke's Inequality](#), since $|3A - A| \leq K^4 |A|$,

$$|X||A| = \left| \bigcup_{x \in X} (x + A) \right| \leq |3A - A| \leq K^4 |A|.$$

Hence $|X| \leq K^4$. We next show that $2A - A \subseteq X + A - A$. Indeed, if, $y \in 2A - A$ and $y \notin X$, then by maximality of X , then $(y + A) \cap (x + A) \neq \emptyset$ for some $x \in X$. If $y \in X$, then $y \in X + A - A$. It follows from above, by induction, that $\forall \ell \geq 2$, $\ell A - A \subseteq (\ell - 1)X + A - A$:

$$\begin{aligned} \ell A - A &= A + (\ell - 1)A - A \\ &\subseteq (\ell - 2)X + 2A - A \\ &\subseteq (\ell - 2)X + X + A - A \\ &= (\ell - 1)X + A - A. \end{aligned}$$

Now, let $H \subseteq \mathbb{F}_p^n$ be the subgroup generated by A :

$$H = \bigcup_{\ell \geq 1} (\ell A - A) \subseteq Y + A - A$$

where $Y \subseteq \mathbb{F}_p^n$ is the subgroup generated by X . Every element of Y can be written as a sum of $|X|$ elements of X with coefficients in $\{0, \dots, p - 1\}$. Hence, $|Y| \leq p^{|X|} \leq p^{K^4}$. Finally, $|H| \leq |Y||A - A| \leq p^{K^4} K^2 |A|$ by [Plunnecke's Inequality](#)/[Ruzsa Triangle Inequality](#).

□

Example 1.19 Let $A = V \cup R$, where $V \subseteq \mathbb{F}_p^n$ is a subspace with $\dim(V) = d = n/K$ satisfying $K \ll d \ll n - K$, and R consists of $K - 1$ linearly independent vectors not in V . Then $|A| = |V \cup R| = |V| + |R| = p^{n/K} + K - 1 \approx p^{n/K} = |V|$.

Now $|A + A| = |(V \cup R) + (V \cup R)| = |V \cup (V + R) \cup 2R| \approx K|V| \approx K|A|$ (since $V \cup (V + R)$ gives K cosets of V). But any subspace $H \subseteq \mathbb{F}_p^n$ containing A must have size at least $p^{n/K+(K-1)} \approx |V|p^K$. Hence, the exponential dependence on K in Freiman-Ruzsa is necessary.

Theorem 1.20 (Polynomial Freiman-Ruzsa Theorem) Let $A \subseteq \mathbb{F}_p^n$ be such that $|A + A| \leq K|A|$. Then there exists a subspace $H \subseteq \mathbb{F}_p^n$ of size at most $C_1(K)|A|$ such that for some $x \in \mathbb{F}_p^n$,

$$|A \cap (x + H)| \geq \frac{|A|}{C_2(K)},$$

where $C_1(K)$ and $C_2(K)$ are polynomial in K .

Proof. Very difficult (took Green, Gowers and Tao to prove it). □

Definition 1.21 Given $A, B \subseteq G$ for an abelian group G , the **additive energy** between A and B is

$$E(A, B) := |\{(a, a', b, b') \in A \times A \times B \times B : a + b = a' + b'\}|.$$

Additive quadruples (a, a', b, b') are those such that $a + b = a' + b'$. Write $E(A)$ for $E(A, A)$.

Example 1.22 Let $V \subseteq \mathbb{F}_p^n$ be a subspace. Then $E(V) = |V|^3$. On the other hand, if $A \subseteq \mathbb{Z}/p$ is chosen at random from \mathbb{Z}/p (where each $a \in \mathbb{Z}/p$ is included with probability $\alpha > 0$), with high probability, $E(A) = \alpha^4 p^3 = \alpha|A|^3$.

Definition 1.23 For $A, B \subseteq G$, the **representation function** is $r_{A+B}(x) := |\{(a, b) \in A \times B : a + b = x\}| = |A \cap (x - B)|$.

Lemma 1.24 Let $\emptyset \neq A, B \subseteq G$ for an abelian group G . Then

$$E(A, B) \geq \frac{|A|^2 |B|^2}{|A \pm B|}.$$

Proof (Hints).

- Show that using Cauchy-Schwarz that

$$E(A, B) = \sum_{x \in G} r_{A+B}(x)^2 \geq \frac{\left(\sum_{x \in G} r_{A+B}(x)\right)^2}{|A + B|}.$$

- By using indicator functions, show that $\sum_{x \in G} r_{A+B}(x) = |A||B|$.

□

Proof. Observe that

$$\begin{aligned}
E(A, B) &= |\{(a, a', b, b') \in A^2 \times B^2 : a + b = a' + b'\}| \\
&= \left| \bigcup_{x \in G} \{(a, a', b, b') \in A^2 \times B^2 : a + b = x \text{ and } a' + b' = x\} \right| \\
&= \bigcup_{x \in G} |\{(a, a', b, b') \in A^2 \times B^2 : a + b = x \text{ and } a' + b' = x\}| \\
&= \sum_{x \in G} r_{A+B}(x)^2 \\
&= \sum_{x \in A+B} r_{A+B}(x)^2 \\
&\geq \frac{\left(\sum_{x \in A+B} r_{A+B}(x) \right)^2}{|A+B|} \quad \text{by } \text{Cauchy-Schwarz}
\end{aligned}$$

But now

$$\begin{aligned}
\sum_{x \in G} r_{A+B}(x) &= \sum_{x \in G} |A \cap (x - B)| = \sum_{x \in G} \sum_{y \in G} \mathbb{1}_A(y) \mathbb{1}_{x-B}(y) \\
&= \sum_{x \in G} \sum_{y \in G} \mathbb{1}_A(y) \mathbb{1}_B(x - y) = |A||B|.
\end{aligned}$$

Note that the same argument works for $|A - B|$. □

Corollary 1.25 If $|A + A| \leq K|A|$, then $E(A) \geq \frac{|A|^4}{|A+A|} \geq \frac{|A|^3}{K}$. So if A has small doubling constant, then it has large additive energy.

Proof (Hints). Trivial. □

Proof. Trivial. □

Example 1.26 The converse of the above lemma does not hold: e.g. let G be a (class of) abelian group(s). Then there exist constants $\theta, \eta > 0$ such that for all n large enough, there exists $A \subseteq G$ with $|A| \geq n$ satisfying $E(A) \geq \eta|A|^3$, and $|A + A| \geq \theta|A|^2$.

Definition 1.27 Given $A \subseteq G$ and $\gamma > 0$, let $P_\gamma := \{x \in G : |A \cap (x + A)| \geq \gamma|A|\}$ be the set of **γ -popular differences** of A .

Lemma 1.28 Let $A \subseteq G$ be finite such that $E(A) = \eta|A|^3$ for some $\eta > 0$. Then $\forall c > 0$, there is a subset $X \subseteq A$ with $|X| \geq \frac{\eta}{3}|A|$ such that for all (16c)-proportion of pairs $(a, b) \in X^2$, $a - b \in P_{c\eta}$.

Proof. We use a technique called “dependent random choice”. Let $U = \{x \in G : |A \cap (x + A)| \leq \frac{1}{2}\eta|A|\}$. Then

$$\begin{aligned}
\sum_{x \in U} |A \cap (x + A)|^2 &\leq \frac{1}{2}\eta|A| \sum_{x \in G} |A \cap (x + A)| \\
&= \frac{1}{2}\eta|A|^3 = \frac{1}{2}E(A).
\end{aligned}$$

For $0 \leq i \leq \lceil \log_2 \eta^{-1} \rceil$, let $Q_i = \{x \in G : |A|/2^{i+1} < |A \cap (x + A)| \leq |A|/2^i\}$ and set $\delta_i = \eta^{-1}2^{-2i}$. Then

$$\begin{aligned}
\sum_{i=0}^{\lceil \log_2 \eta^{-1} \rceil} \delta_i |Q_i| &= \sum_i \frac{|Q_i|}{\eta 2^{2i}} \\
&= \frac{1}{\eta |A|^2} \sum_i \frac{|A|^2}{2^{2i}} |Q_i| \\
&= \frac{1}{\eta |A|^2} \sum_i \frac{|A|^2}{2^{2i}} \sum_{x \notin U} \mathbb{1}_{\{|A|/2^{i+1} < |A \cap (x+A)| \leq |A|/2^i\}} \\
&\geq \frac{1}{\eta |A|^2} \sum_{x \notin U} |A \cap (x + A)|^2 \\
&\geq \frac{1}{\eta |A|^2} \cdot \frac{1}{2} E(A) = \frac{1}{2} |A|.
\end{aligned}$$

Let $S = \{(a, b) \in A^2 : a - b \notin P_{c\eta}\}$. Now

$$\begin{aligned}
\sum_i \sum_{(a,b) \in S} |(A-a) \cap (A-b) \cap Q_i| &\leq \sum_{(a,b) \in S} |(A-a) \cap (A-b)| \\
&= \sum_{(a,b) \in S} |A \cap (a-b+A)| \\
&\leq \sum_{(a,b) \in S} c\eta |A| \quad \text{by definition of } S \\
&= |S| c\eta |A| \\
&\leq c\eta |A|^3 = 2c\eta |A|^2 \cdot \frac{1}{2} |A| \\
&\leq 2c\eta |A|^2 \sum_i \delta_i |Q_i| \quad \text{by above inequality.}
\end{aligned}$$

Hence $\exists i_0$ such that

$$\sum_{(a,b) \in S} |(A-a) \cap (A-b) \cap Q_{i_0}| \leq 2c\eta |A|^2 \delta_{i_0} |Q_{i_0}|.$$

Let $Q = Q_{i_0}$, $\delta = \delta_{i_0}$, $\lambda = 2^{-i_0}$, so that

$$\sum_{(a,b) \in S} |(A-a) \cap (A-b) \cap Q| \leq 2c\eta |A|^2 \delta |Q|.$$

Given $x \in G$, let $X(x) = A \cap (x + A)$. Then

$$\mathbb{E}_{x \in Q} |X(x)| = \frac{1}{|Q|} \sum_{x \in Q} |A \cap (x + A)| \geq \frac{1}{2} \lambda |A|.$$

Define $T(x) = \{(a, b) \in X(x)^2 : a - b \in P^{c\eta}\}$. Then

$$\begin{aligned}
\mathbb{E}_{x \in Q} |T(x)| &= \mathbb{E}_{x \in Q} |\{(a, b) \in (A \cap (x + A))^2 : a - b \notin P_{c\eta}\}| \\
&= \frac{1}{|Q|} \sum_{x \in Q} |\{(a, b) \in S : x \in (A - a) \cap (A - b)\}| \\
&= \frac{1}{|Q|} \sum_{(a, b) \in S} |(A - a) \cap (A - b) \cap Q| \\
&\leq \frac{1}{|Q|} 2c\eta |A|^2 \delta |Q| = 2c\eta \delta |A|^2 = 2c\lambda^2 |A|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_{x \in Q} (|X(x)|^2 - (16c)^{-1} |T(x)|) &\geq (\mathbb{E}_{x \in Q} |X(x)|)^2 - (16c)^{-1} \mathbb{E}_{x \in Q} |T(x)| \text{ by [Cauchy-Schwarz](#)} \\
&\geq \left(\frac{\lambda}{2}\right)^2 |A|^2 - (16c)^{-1} 2c\lambda^2 |A|^2 \\
&= \left(\frac{\lambda^2}{4} - \frac{\lambda^2}{8}\right) |A|^2 = \frac{\lambda^2}{8} |A|^2.
\end{aligned}$$

So $\exists x \in Q$ such that $|X(x)|^2 \geq \frac{\lambda^2}{8} |A|^2$, so $|X| \geq \frac{\lambda}{\sqrt{8}} |A| \geq \frac{\eta}{3} |A|$ and $|T(x)| \leq 16c |X|^2$. \square

Theorem 1.29 (Balog-Szemerédi-Gowers, Schoen) Let $A \subseteq G$ be finite such that $E(A) \geq \eta |A|^3$ for some $\eta > 0$. Then there exists $A' \subseteq A$ with $|A'| \geq c_1(\eta) |A|$ such that $|A' + A'| \leq |A|/c_2(\eta)$, where $c_1(\eta)$ and $c_2(\eta)$ are both polynomial in η .

Proof. The idea is to find $A' \subseteq A$ such that $\forall a, b \in A'$, $a - b$ has many representations as $(a_1 - a_2) + (a_3 - a_4)$ with each $a_i \in A$. Apply the above lemma with $c = 2^{-7}$ to obtain $X \subseteq A$ with $|X| \geq \frac{\eta}{3} |A|$ such that for all but $\frac{1}{8}$ of pairs $(a, b) \in X^2$, $a - b \in P_{\eta/2^7}$. In particular, the bipartite graph $G = (X \sqcup X, \{(x, y) \in X \times X : x - y \in P_{\eta/2^7}\})$ has at least $\frac{7}{8} |X|^2$ edges.

Let $A' = \{x \in X : \deg_G(x) \geq \frac{3}{4} |X|\}$. Clearly $|A'| \geq |X|/8$. For any $a, b \in A'$, there are at least $|X|/2$ elements $y \in X$ such that $(a, y), (b, y) \in E(G)$ (so $a - y, b - y \in P_{\eta/2^7}$). Hence $a - b = (a - y) - (b - y)$ has at least

$$\underbrace{\frac{\eta}{6} |A|}_{\text{choices for } y} \cdot \frac{\eta}{2^7} |A| \frac{\eta}{2^7} |A| \geq \frac{\eta^3}{2^{17}} |A|^3$$

representations of the form $a_1 - a_2 - (a_3 - a_4)$ with each $a_i \in A$. It follows that $\frac{\eta^3}{2^{17}} |A|^3 |A' - A'| \leq |A|^4$, hence $|A' - A'| \leq 2^{17} \eta^{-3} |A| \leq 2^{22} \eta^{-4} |A'|$, and so $|A' + A'| \leq 2^{44} \eta^{-8} |A'|$. \square

2. Fourier-analytic techniques

In this chapter, assume that G is a *finite* abelian group.

Definition 2.1 The group \hat{G} of **characters** of G is the group of homomorphisms $\gamma : G \rightarrow \mathbb{C}^\times$. In fact, $\hat{\hat{G}}$ is isomorphic to G .

Notation 2.2 Norm and inner product notation:

- Write

$$\|f\|_q = \|f\|_{L^q(G)} = (\mathbb{E}_{x \in G} |f(x)|^q)^{1/q},$$

$$\|\hat{f}\|_q = \|\hat{f}\|_{\ell^q(\hat{G})} = \left(\sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)|^q \right)^{1/q},$$

$$\langle f, g \rangle_{L^2(G)} = \mathbb{E}_{x \in G} f(x) \overline{g(x)},$$

$$\langle f, g \rangle_{\ell^2(\hat{G})} = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma) \overline{\hat{g}(\gamma)}$$

- If Fourier support of function is restricted to $\Lambda \subseteq \hat{G}$, write $\|\hat{f}\|_{\ell^q(\Lambda)} = \left(\sum_{\gamma \in \Lambda} |\hat{f}(\gamma)|^q \right)^{1/q}$.

Notation 2.3 Asymptotic notation:

- Write $f(n) = O(g(n))$ if

$$\exists C > 0 : \forall n \in \mathbb{N}, \quad |f(n)| \leq C|g(n)|.$$

- Write $f(n) = o(g(n))$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |f(n)| \leq \varepsilon |g(n)|,$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

- Write $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$.
- If the implied constant depends on a fixed parameter, this may be indicated by a subscript, e.g. $\exp(pn^2) = O_p(\exp(n^2))$.

Theorem 2.4 (Hölder's Inequality) Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(G)$, $g \in L^q(G)$. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Theorem 2.5 (Cauchy-Schwarz Inequality) For $f, g \in L^2(G)$, we have

$$\langle f, g \rangle_{L^2(G)} \leq \|f\|_2 \|g\|_2.$$

Note this is a special case of Hölder's inequality with $p = q = 2$.

Theorem 2.6 (Young's Convolution Inequality) Let $p, q, r \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(G)$, $g \in L^q(G)$. Then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Notation 2.7 $e(y)$ denotes the function $e^{2\pi i y}$.

Example 2.8

- Let $G = \mathbb{F}_p^n$, then for any $\gamma \in \hat{G}$, we have a corresponding character $\gamma(x) = e((\gamma \cdot x)/p)$.
- If $G = \mathbb{Z}/N$, then any $\gamma \in \hat{G}$ has a corresponding character $\gamma(x) = e(\gamma x/N)$.

Notation 2.9 Given a non-empty $B \subseteq G$ and $g : B \rightarrow \mathbb{C}$, write $\mathbb{E}_{x \in B} g(x)$ for $\frac{1}{|B|} \sum_{x \in B} g(x)$. If $B = G$, we may simply write \mathbb{E} instead of $\mathbb{E}_{x \in B}$.

Lemma 2.10 For all $\gamma \in \hat{G}$,

$$\mathbb{E}_{x \in G} \gamma(x) = \begin{cases} 1 & \text{if } \gamma = 1 \\ 0 & \text{otherwise} \end{cases}.$$

and for all $x \in G$,

$$\sum_{\gamma \in \hat{G}} \gamma(x) = \begin{cases} |G| & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Proof (Hints).

- For $1 \neq \gamma \in \hat{G}$, consider $y \in G$ with $\gamma(y) \neq 1$.
- For $0 \neq x \in G$, by considering $G/\langle x \rangle$, show by contradiction that there is $\gamma \in \hat{G}$ with $\gamma(x) \neq 1$.

□

Proof. The first case for both equations is trivial. Let $1 \neq \gamma \in \hat{G}$. Then $\exists y \in G$ with $\gamma(y) \neq 1$. So

$$\begin{aligned} \gamma(y) \mathbb{E}_{z \in G} \gamma(z) &= \mathbb{E}_{z \in G} \gamma(y + z) \\ &= \mathbb{E}_{z' \in G} \gamma(z'). \end{aligned}$$

Hence $\mathbb{E}_{z \in G} \gamma(z) = 0$.

For second equation, given $0 \neq x \in G$, there exists $\gamma \in \hat{G}$ such that $\gamma(x) \neq 1$, since otherwise \hat{G} would act trivially on $\langle x \rangle$, hence would also be the dual group for $G/\langle x \rangle$, a contradiction. □

Definition 2.11 Given $f : G \rightarrow \mathbb{C}$, define the **Fourier transform** of f to be

$$\begin{aligned} \hat{f} : \hat{G} &\rightarrow \mathbb{C}, \\ \gamma &\mapsto \mathbb{E}_{x \in G} f(x) \overline{\gamma(x)}. \end{aligned}$$

Proposition 2.12 (Fourier Inversion Formula) Let $f : G \rightarrow \mathbb{C}$. Then for all $x \in G$,

$$f(x) = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma) \gamma(x).$$

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned}
\sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma) \gamma(x) &= \sum_{\gamma \in \widehat{G}} \mathbb{E}_{y \in G} f(y) \overline{\gamma(y)} \gamma(x) \\
&= \mathbb{E}_{y \in G} f(y) \sum_{\gamma \in \widehat{G}} \gamma(x - y) \\
&= f(x)
\end{aligned}$$

by [Lemma 2.10](#). □

Definition 2.13 For $A \subseteq G$, the **indicator** (or **characteristic**) function of A is

$$\begin{aligned}
\mathbb{1}_A : G &\rightarrow \{0, 1\}, \\
x &\mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}
\end{aligned}$$

Definition 2.14 $\widehat{\mathbb{1}}_A(1) = \mathbb{E}_{x \in G} \mathbb{1}_A(x) \cdot 1 = |A|/|G|$ is the **density** of A in G . This is often denoted by α .

Definition 2.15 Given $\emptyset \neq A \subseteq G$, the **characteristic measure** $\mu_A : G \rightarrow [0, |G|]$ is defined by

$$\mu_A(x) := \alpha^{-1} \mathbb{1}_A(x).$$

Note that $\mathbb{E}_{x \in G} \mu_A(x) = 1 = \widehat{\mu}_A(1)$.

Definition 2.16 The **balanced function** $f_A : G \rightarrow [-1, 1]$ of A is given by

$$f_A(x) = \mathbb{1}_A(x) - \alpha.$$

Note that $\mathbb{E}_{x \in G} f_A(x) = 0 = \widehat{f}_A(1)$.

Example 2.17 Let $V \leq \mathbb{F}_p^n$ be a subspace. Then for $t \in \widehat{\mathbb{F}}_p^n$,

$$\begin{aligned}
\widehat{\mathbb{1}}_V(t) &= \mathbb{E}_{x \in \mathbb{F}_p^n} \mathbb{1}_V(x) e(-x \cdot t/p) \\
&= \frac{|V|}{p^n} \mathbb{1}_{V^\perp}(t).
\end{aligned}$$

where $V^\perp = \{t \in \widehat{\mathbb{F}}_p^n : x \cdot t = 0 \ \forall x \in V\}$ is the **annihilator** of V . Hence, $\widehat{\mathbb{1}}_V = \mu_{V^\perp}$.

Example 2.18 Let $R \subseteq G$ be such that each $x \in G$ lies in R independently with probability $\frac{1}{2}$. Then with high probability,

$$\sup_{\gamma \neq 1} |\widehat{\mathbb{1}}_R(\gamma)| = O\left(\sqrt{\frac{\log |G|}{|G|}}\right).$$

This follows from Chernoff's inequality.

Theorem 2.19 (Chernoff's Inequality) Given complex-valued independent random variables X_1, \dots, X_n with mean 0, for all $\theta > 0$, we have

$$\Pr \left[\left| \sum_{i=1}^n X_i \right| \geq \theta \sqrt{\sum_{i=1}^n \|X_i\|_{L^\infty(\Pr)}^2} \right] \leq 4 \exp(-\theta^2/4).$$

Example 2.20 Let $Q = \{x \in \mathbb{F}_p^n : x.x = 0\}$ with $p > 2$. Then $|Q|/p^n = \frac{1}{p} + O(p^{-n/2})$ and $\sup_{t \neq 0} |\hat{1}_Q(t)| = O(p^{-n/2})$.

Lemma 2.21 (Plancherel's Identity) For all $f, g : G \rightarrow \mathbb{C}$,

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle.$$

Proof. Exercise. □

Corollary 2.22 (Parseval's Identity) For all $f, g : G \rightarrow \mathbb{C}$,

$$\|f\|_{L^2(G)}^2 = \|\hat{f}\|_{\ell^2(\hat{G})}^2.$$

Proof (Hints). Trivial from [Plancherel's Identity](#). □

Proof. By [Plancherel's Identity](#). □

Definition 2.23 Let $\rho > 0$ and $f : G \rightarrow \mathbb{C}$. The **ρ -large Fourier spectrum** of f is

$$\text{Spec}_\rho(f) := \left\{ \gamma \in \hat{G} : |\hat{f}(\gamma)| \geq \rho \|f\|_1 \right\}.$$

Example 2.24 Let $A \subseteq G$, then $\|f\|_1 = \alpha = |A|/|G|$, so

$$\text{Spec}_\rho(1_A) = \left\{ t \in \hat{\mathbb{F}}_p^n : |\hat{1}_V(t)| \geq \rho \alpha \right\}.$$

In particular, if $V \leq \mathbb{F}_p^n$ is a subspace, then by [Example 2.17](#), $\text{Spec}_\rho(1_V) = V^\perp$ for all $\rho \in (0, 1]$.

Lemma 2.25 For all $\rho > 0$,

$$|\text{Spec}_\rho(f)| \leq \rho^{-2} \frac{\|f\|_2^2}{\|f\|_1^2}$$

In particular, if $f = 1_A$ for $A \subseteq G$, then $\|f\|_1 = \alpha = |A|/|G| = \|f\|_2^2$. So $|\text{Spec}_\rho(1_A)| \leq \rho^{-2} \alpha^{-1}$.

Proof (Hints). Use [Parseval](#). □

Proof. By [Parseval](#),

$$\begin{aligned} \|f\|_2^2 &= \|\hat{f}\|_2^2 = \sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)|^2 \\ &\geq \sum_{\gamma \in \text{Spec}_\rho(f)} |\hat{f}(\gamma)|^2 \\ &\geq |\text{Spec}_\rho(f)| (\rho \|f\|_1)^2. \end{aligned}$$

□

Definition 2.26 The **convolution** of $f, g : \mathbb{G} \rightarrow \mathbb{C}$ is

$$\begin{aligned} f * g : G &\rightarrow \mathbb{C}, \\ x &\mapsto \mathbb{E}_{y \in G} f(y)g(x - y). \end{aligned}$$

Example 2.27 Given $A, B \subseteq G$,

$$\begin{aligned} (\mathbb{1}_A * \mathbb{1}_B)(x) &= \mathbb{E}_{y \in G} \mathbb{1}_A(y) \mathbb{1}_B(x - y) \\ &= \mathbb{E}_{y \in G} \mathbb{1}_A(y) \mathbb{1}_{x-B}(y) \\ &= \mathbb{E}_{y \in G} \mathbb{1}_{A \cap (x-B)}(y) \\ &= \frac{|A \cap (x - B)|}{|G|} = \frac{1}{|G|} r_{A+B}(x). \end{aligned}$$

In particular, $\text{supp}(\mathbb{1}_A * \mathbb{1}_B) = A + B$.

Lemma 2.28 Given $f, g : G \rightarrow \mathbb{C}$,

$$\forall \gamma \in \hat{G}, \quad (\widehat{f * g})(\gamma) = \hat{f}(\gamma) \hat{g}(\gamma).$$

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned} (\widehat{f * g})(\gamma) &= \mathbb{E}_{x \in G} (f * g)(x) \overline{\gamma(x)} \\ &= \mathbb{E}_{x \in G} \mathbb{E}_{y \in G} f(y)g(x - y) \overline{\gamma(x)} \\ &= \mathbb{E}_{u \in G} \mathbb{E}_{y \in G} f(y)g(u) \overline{\gamma(u + y)} \quad (u = x - y) \\ &= \mathbb{E}_{u \in G} \mathbb{E}_{y \in G} f(y)g(u) \overline{\gamma(u) \gamma(y)} \\ &= \hat{f}(\gamma) \hat{g}(\gamma). \end{aligned}$$

□

Example 2.29 $\mathbb{E}_{x+y=z+w} f(x)f(y)\overline{f(z)}\overline{f(w)} = \|\hat{f}\|_{\ell^4(\hat{G})}^4$. In particular, $\|\hat{\mathbb{1}}_A\|_{\ell^4(\hat{G})}^4 = E(A)/|G|^3$ for any $A \subseteq G$.

Theorem 2.30 (Bogolyubov's Lemma) Let $A \subseteq \mathbb{F}_p^n$ be of density α . Then there exists a subspace $V \leq \mathbb{F}_p^n$ with $\text{codim}(V) \leq 2\alpha^{-2}$, such that $V \subseteq A + A - A - A$.

Proof (Hints).

- Let $g = \mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_{-A} * \mathbb{1}_{-A}$, reason that if $g(x) > 0$ for all $x \in V$, then $V \subseteq 2A - 2A$.
- Let $S = \text{Spec}_\rho(\mathbb{1}_A)$, with ρ for now unspecified.
- Show that $g(x) = \alpha^4 + \sum_{t \in S \setminus \{0\}} |\hat{\mathbb{1}}_A(t)|^4 e(x \cdot t/p) + \sum_{t \notin S} |\hat{\mathbb{1}}_A(t)|^4 e(x \cdot t/p)$.
- Find an appropriate subspace V from S , bound $g(x)$ from below in terms of ρ , and use this to determine a suitable value for ρ .

□

Proof. Observe $2A - 2A = \text{supp}(g)$ where $g = \mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_{-A} * \mathbb{1}_{-A}$, so we want to find $V \leq \mathbb{F}_p^n$ such that $g(x) > 0$ for all $x \in V$. Let $S = \text{Spec}_\rho(\mathbb{1}_A)$ with ρ a constant to be specified later, and let $V = \langle S \rangle^\perp$. By [Lemma 2.25](#), $\text{codim}(V) = \dim \langle S \rangle \leq |S| \leq \rho^{-2} \alpha^{-1}$. Fix $x \in V$. Now

$$\begin{aligned} g(x) &= \sum_{t \in \hat{\mathbb{F}}_p^n} \hat{g}(t) e(x.t/p) \\ &= \sum_{t \in \hat{\mathbb{F}}_p^n} |\hat{\mathbb{1}}_A(t)|^4 e(x.t/p) \quad \text{by [Lemma 2.28](#)} \\ &= \alpha^4 + \sum_{t \neq 0} |\hat{\mathbb{1}}_A(t)|^4 e(x.t/p) \\ &= \alpha^4 + \sum_{t \in S \setminus \{0\}} |\hat{\mathbb{1}}_A(t)|^4 e(x.t/p) + \sum_{t \notin S} |\hat{\mathbb{1}}_A(t)|^4 e(x.t/p) \end{aligned}$$

Each term in the first sum is non-negative, since $\forall t \in S, x.t = 0$. The absolute value of the second sum is bounded above, by the triangle inequality, by

$$\begin{aligned} \sum_{t \notin S} |\hat{\mathbb{1}}_A(t)|^4 &\leq \sup_{t \notin S} |\hat{\mathbb{1}}_A(t)|^2 \sum_{t \notin S} |\hat{\mathbb{1}}_A(t)|^2 \\ &\leq \sup_{t \notin S} |\hat{\mathbb{1}}_A(t)|^2 \sum_{t \in \hat{\mathbb{F}}_p^n} |\hat{\mathbb{1}}_A(t)|^2 \\ &\leq (\rho\alpha)^2 \|\mathbb{1}_A\|_2^2 = \rho^2 \alpha^3 \end{aligned}$$

by [Example 2.24](#) and [Parseval](#). Note the second sum must be real since all other terms in the equation are. So we have $g(x) \geq \alpha^4 - \rho^2 \alpha^3$. Thus, it is sufficient that $\rho^2 \alpha^3 \leq \frac{\alpha^4}{2}$, so set $\rho = \sqrt{\alpha/2}$. Hence $g(x) > 0$ (in fact, $g(x) \geq \frac{\alpha^4}{2}$) for all $x \in V$, and $\text{codim}(V) \leq 2\alpha^{-2}$. \square

Example 2.31 The set $A = \left\{x \in \mathbb{F}_2^n : |x| \geq \frac{n}{2} + \frac{\sqrt{n}}{2}\right\}$ (where $|x|$ is number of 1s in x) has density $\geq \frac{1}{8}$ but there is no coset C of any subspace of codimension \sqrt{n} such that $C \subseteq A + A$. Hence, the $2A - 2A$ part of Bogolyubov's lemma is necessary: $2A$ is not sufficient.

Lemma 2.32 Let $A \subseteq \mathbb{F}_p^n$ have density α with $\sup_{t \neq 0} |\hat{\mathbb{1}}_A(t)| \geq \rho\alpha$ for some $\rho > 0$. Then there exists a subspace $V \leq \mathbb{F}_p^n$ with $\text{codim}(V) = 1$ and $x \in \mathbb{F}_p^n$ such that

$$|A \cap (x + V)| \geq \alpha \left(1 + \frac{\rho}{2}\right) |V|.$$

Proof (Hints).

- Let $V = \langle t \rangle^\perp$ for some suitable t (can determine later).
- Define $a_j = \frac{|A \cap (v_j + V)|}{|v_j + V|} - \alpha$ for each $j \in [p]$, where $x.v_j = j$.
- Show that $\hat{\mathbb{1}}_A(t) = \mathbb{E}_{j \in [p]} a_j e(-j/p)$.
- Show that $\mathbb{E}_{j \in [p]} a_j + |a_j| \geq \rho\alpha$.

\square

Proof. Let $t \neq 0$ be such that $|\hat{1}_A(t)| \geq \rho\alpha$ and let $V = \langle t \rangle^\perp$. Write $v_j + V = \{x \in \mathbb{F}_p^n : x \cdot t = j\}$ for $j \in [p]$ for the p distinct cosets of V . Then

$$\begin{aligned}\hat{1}_A(t) &= \hat{f}_A(t) = \mathbb{E}_{x \in \mathbb{F}_p^n} (\mathbb{1}_A(x) - \alpha) e(-x \cdot t/p) \\ &= \mathbb{E}_{j \in [p]} \mathbb{E}_{x \in v_j + V} (\mathbb{1}_A(x) - \alpha) e(-j/p) \\ &= \mathbb{E}_{j \in [p]} \left(\frac{|A \cap (v_j + V)|}{|v_j + V|} - \alpha \right) e(-j/p) \\ &=: \mathbb{E}_{j \in [p]} a_j e(-j/p).\end{aligned}$$

By the triangle inequality, $\mathbb{E}_{j \in [p]} |a_j| \geq \rho\alpha$. Note that $\mathbb{E}_{j \in [p]} a_j = 0$. So $\mathbb{E}_{j \in [p]} a_j + |a_j| \geq \rho\alpha$, so $\exists j \in [p]$ such that $a_j + |a_j| \geq \rho\alpha$, hence $a_j \geq \rho\alpha/2$. So take $x = v_j$. \square

Notation 2.33 Given $f, g, h : G \rightarrow \mathbb{C}$, write

$$T_3(f, g, h) = \mathbb{E}_{x, d \in G} f(x) g(x + d) h(x + 2d).$$

Notation 2.34 Given $A \subseteq G$, write $2 \cdot A = \{2a : a \in A\}$. Note this is not the same as $2A = A + A$.

Lemma 2.35 Let $p \geq 3$ and $A \subseteq \mathbb{F}_p^n$ be of density $\alpha > 0$, such that $\sup_{t \neq 0} |\hat{1}_A(t)| \leq \varepsilon$. Then the number of 3-APs in A differs from $\alpha^3(p^n)^2$ by at most $\varepsilon(p^n)^2$.

Proof (Hints).

- Express $T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A)$ as an inner product of functions $\mathbb{F}_p^n \rightarrow \mathbb{C}$, rewrite as inner product of functions $\hat{\mathbb{F}}_p^n \rightarrow \mathbb{C}$.
- Find upper bound of the absolute value of a sub-sum of this inner product, using triangle inequality and Cauchy-Schwarz.

\square

Proof. The number of 3-APs in A is $(p^n)^2$ multiplied by

$$\begin{aligned}T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) &= \mathbb{E}_{x, d} \mathbb{1}_A(x) \mathbb{1}_A(x + d) \mathbb{1}_A(x + 2d) \\ &= \mathbb{E}_{x, y} \mathbb{1}_A(x) \mathbb{1}_A(y) \mathbb{1}_A(2y - x) \\ &= \mathbb{E}_y \mathbb{1}_A(y) \mathbb{E}_x \mathbb{1}_A(x) \mathbb{1}_A(2y - x) \\ &= \mathbb{E}_y \mathbb{1}_A(y) (\mathbb{1}_A * \mathbb{1}_A)(2y) \\ &= \langle \mathbb{1}_{2 \cdot A}, \mathbb{1}_A * \mathbb{1}_A \rangle.\end{aligned}$$

By [Plancherel's Identity](#) and [Lemma 2.28](#), this is equal to

$$\begin{aligned}\langle \hat{\mathbb{1}}_{2 \cdot A}, \hat{\mathbb{1}}_A^2 \rangle &= \sum_{t \in \hat{\mathbb{F}}_p^n} \hat{\mathbb{1}}_{2 \cdot A}(t) \overline{\hat{\mathbb{1}}_A(t)}^2 \\ &= \alpha^3 + \sum_{t \neq 0} \hat{\mathbb{1}}_{2 \cdot A}(t) \overline{\hat{\mathbb{1}}_A(t)}^2\end{aligned}$$

But

$$\begin{aligned}
\left| \sum_{t \neq 0} \hat{\mathbb{1}}_{2 \cdot A}(t) \overline{\hat{\mathbb{1}}_A(t)}^2 \right| &\leq \sup_{t \neq 0} |\hat{\mathbb{1}}_A(t)| \sum_{t \neq 0} |\hat{\mathbb{1}}_{2 \cdot A}(t)| |\hat{\mathbb{1}}_A(t)| \\
&\leq \varepsilon \sum_{t \in \mathbb{F}_p^n} |\hat{\mathbb{1}}_{2 \cdot A}(t)| |\hat{\mathbb{1}}_A(t)| \\
&\leq \varepsilon \left(\sum_t |\hat{\mathbb{1}}_{2 \cdot A}(t)|^2 \sum_t |\hat{\mathbb{1}}_A(t)|^2 \right)^{1/2} \quad \text{by [Cauchy-Schwarz](#)} \\
&= \varepsilon \|\hat{\mathbb{1}}_{2 \cdot A}\|_2 \|\hat{\mathbb{1}}_A\|_2 \\
&= \varepsilon \cdot \alpha^2 \leq \varepsilon \quad \text{by [Parseval](#).}
\end{aligned}$$

□

Theorem 2.36 (Meshulam) Let $A \subseteq \mathbb{F}_p^n$ be a set containing no non-trivial 3-APs. Then $|A| = O(p^n / \log p^n)$, i.e. $\alpha = O(1/n)$.

Proof (Hints).

- Use similar proof as that of above lemma to show that $|T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^3| \leq \sup_{t \neq 0} |\hat{\mathbb{1}}_A(t)| \cdot \alpha$.
- Reason that provided $p^n \geq 2\alpha^{-2}$, we have $\sup_{t \neq 0} |\hat{\mathbb{1}}_A(t)| \geq \frac{\alpha^2}{2}$.
- Use this to iteratively generate $A_1, V_1, A_2, V_2, \dots$.
- Reason that each A_i contains no non-trivial 3 APs.
- Find an expression for maximum number of steps it takes for the density of the A_i to increase from $2^k \alpha$ to $2^{k+1} \alpha$ (in terms of k and α). Use this to deduce an upper bound for the maximum number steps it takes for the density to reach 1.
- Find lower bound for $\dim(V_m)$ where V_m is the final V_i in the sequence, use fact that iteration halted to deduce that $p^{\dim(V_m)} \leq 2\alpha^{-2}$.
- Reason that we can assume $\alpha \geq \sqrt{2}p^{-n/4}$, and conclude that $\alpha \leq 16n$.

□

Proof. By assumption, $T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = |A|/(p^n)^2 = \alpha/p^n$ (there are $|A|$ trivial APs). By the proof of the above lemma,

$$|T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^3| \leq \sup_{t \neq 0} |\hat{\mathbb{1}}_A(t)| \cdot \alpha.$$

So provided that $p^n \geq 2\alpha^{-2}$, we have $T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) \leq \alpha^3/2$, so $|T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^3| \geq \alpha^3/2$, hence

$$\sup_{t \neq 0} |\hat{\mathbb{1}}_A(t)| \geq \frac{\alpha^2}{2}.$$

So by [Lemma 2.32](#) with $\rho = \frac{\alpha}{2}$, there exists a subspace $V \leq \mathbb{F}_p^n$ of codimension 1 and $x \in \mathbb{F}_p^n$ such that $|A \cap (x + V)| \geq (\alpha + \alpha^2/4)|V|$.

We iterate this observation: let $A_0 = A$, $V_0 = \mathbb{F}_p^n$, $\alpha_0 = |A_0|/|V_0|$. At this i -th step, we are given a set $A_{i-1} \subseteq V_{i-1}$ of density α_{i-1} with no non-trivial 3-APs. Provided that

$p^{\dim(V_{i-1})} \geq 2\alpha_{i-1}^{-2}$, there exists a subspace $V_i \leq V_{i-1}$ of codimension 1 and $x_i \in V_{i-1}$ such that

$$|(A - x_i) \cap V_i| = |A \cap (x_i + V_i)| \geq (\alpha_{i-1} + \alpha_{i-1}^2/4)|V_i|$$

So set $A_i = (A - x_i) \cap V_i$. A_i has density $\alpha_i \geq \alpha_{i-1} + \alpha_{i-1}^2/4$, and contains no non-trivial 3-APs (since the translate $A - x_i$ contains no non-trivial 3-APs). Through this iteration, the density increases:

- from α to 2α in at most $\alpha/(\alpha^2/4) = 4\alpha^{-1}$ steps,
- from 2α to 4α in at most $(2\alpha)/((2\alpha)^2/4) = 2\alpha^{-1}$ steps.
- and so on, ...

So the density reaches 1 in at most $4\alpha^{-1}(1 + \frac{1}{2} + \frac{1}{4} + \dots) = 8\alpha^{-1}$ steps. The iteration must end with $\dim(V_i) \geq n - 8\alpha^{-1}$, at which point we must have had $p^{\dim(V_i)} < 2\alpha_{i-1}^{-2} \leq 2\alpha^{-2}$, or else we could have iterated again.

But we may assume that $\alpha \geq \sqrt{2}p^{-n/4}$ (since otherwise we would be done), so $\alpha^{-2} < \frac{1}{2}p^{n/2}$, whence $p^{n-8\alpha^{-1}} \leq p^{n/2}$, i.e. $\frac{n}{2} \leq 8\alpha^{-1}$. \square

Remark 2.37 The current largest known subset of \mathbb{F}_3^n containing no non-trivial 3-APs has size 2.2202^n .

Lemma 2.38 Let $A \subseteq [N]$ be of density $\alpha > 0$ and contain no non-trivial 3-APs, with $N > 50\alpha^{-2}$. Let p be a prime with $p \in [N/3, 2N/3]$, and write $A' = A \cap [p] \subseteq \mathbb{Z}/p$. Then one of the following holds:

1. $\sup_{t \neq 0} |\hat{1}_{A'}(t)| \geq \alpha^2/10$ (where the Fourier coefficient is computed in \mathbb{Z}/p).
2. There exists an interval $J \subseteq [N]$ of length $\geq N/3$ such that $|A \cap J| \geq \alpha(1 + \alpha/400)|J|$.

Proof (Hints).

- Show that we can assume $|A'| \geq \alpha(1 - \alpha/200)p$.

\square

Proof. TODO: fill in details in proof.

We may assume that $|A'| = |A \cap [p]| \geq \alpha(1 - \alpha/200)p$, since otherwise $|A \cap [p + 1, N]| \geq \alpha N - (\alpha(1 - \alpha/200)p) = \alpha(N - p) + \frac{\alpha^2}{200}p \geq (\alpha + \alpha^2/400)(N - p)$ since $p \geq N/3$, which implies case 2 with $J = [p + 1, N]$.

Let $A'' = A' \cap [p/3, 2p/3]$. Note that all 3-APs of the form $(x, x + d, x + 2d) \in A' \times A'' \times A''$ are in fact APs in $[N]$. If $|A' \cap [p/3]|$ or $|A' \cap [2p/3, p]|$ is at least $\frac{2}{5}|A'|$, then again we are in case 2. So we may assume that $|A''| \geq |A'|/5$. Now as in above lemmas, we have

$$\frac{\alpha''}{p} = \frac{|A''|}{p^2} = T_3(\mathbb{1}_{A'}, \mathbb{1}_{A''}, \mathbb{1}_{A''}) = \alpha'(\alpha'')^2 + \sum_t \overline{\hat{1}_{A'}(t)} \hat{1}_{A''}(t) \hat{1}_{2 \cdot A''}(t)$$

where $\alpha' = |A'|/p$ and $\alpha'' = |A''|/p$. So as before,

$$\frac{\alpha' \alpha''}{2} \leq \sup_{t \neq 0} |\mathbb{1}_{A'}(t)| \cdot \alpha''$$

provided that $\alpha''/p \leq \frac{1}{2} \alpha' (\alpha'')^2$, i.e. $2/p \leq \alpha' \alpha''$ (check this inequality indeed holds). Hence, $\sup_{t \neq 0} |\widehat{\mathbb{1}}_{A'}(t)| \geq \frac{\alpha' \alpha''}{2} \geq \frac{1}{2} \alpha (1 - \alpha/200)^2 \cdot \frac{2}{5} \geq \alpha^2/10$. TODO: constants need to change somewhere here. \square

Lemma 2.39 Let $m \in \mathbb{N}$, and let $\varphi : [m] \rightarrow \mathbb{Z}/p$ be given by $\varphi(x) = tx$ for some $t \neq 0$. For all $\varepsilon > 0$, there exists a partition of $[m]$ into progressions P_i of length $\ell_i \in [\varepsilon\sqrt{m}/2, \varepsilon\sqrt{m}]$, such that

$$\forall i, \quad \text{diam}(\varphi(P_i)) := \max_{x, y \in P_i} |\varphi(x) - \varphi(y)| \leq \varepsilon p$$

(where $|\varphi(x) - \varphi(y)|$ views $\varphi(x), \varphi(y) \in \{0, \dots, p-1\}$).

Proof. Let $u = \lfloor \sqrt{m} \rfloor$ and consider $0, t, \dots, ut$. By the pigeonhole principle, there exists $0 \leq v < w \leq u$ such that $|wt - vt| = |(w-v)t| \leq p/u$. Set $s = w - v$, so $|st| \leq p/u$. Divide $[m]$ into residue classes mod s , each of which has size at least $m/s \geq m/u$. But each residue class can be divided into APs of the form $a, a+s, \dots, a+ds$ for some $\varepsilon u/2 < d \leq \varepsilon u$. The diameter of the image of each progression under φ is $|dst| \leq dp/u \leq \varepsilon up/u = \varepsilon p$. \square

Lemma 2.40 Let $A \subseteq [N]$ be of density $\alpha > 0$, let p be prime with $p \in [N/3, 2N/3]$, and write $A' = A \cap [p] \subseteq \mathbb{Z}/p$. Suppose that $|\widehat{\mathbb{1}}_{A'}(t)| \geq \alpha^2/20$ for some $t \neq 0$. Then there exists a progression $P \subseteq [N]$ of length at least $\alpha^2 \sqrt{N}/500$ such that $|A \cap P| \geq \alpha(1 + \alpha/80)|P|$.

Proof. Let $\varepsilon = \alpha^2/40\pi$ and use above lemma to partition $[p]$ into progressions P_i of length $\geq \varepsilon \sqrt{p/2} \geq \alpha^2/40\pi \frac{\sqrt{N/3}}{2} \geq \alpha \sqrt{N}/500$, and $\text{diam}(\varphi(P_i)) \leq \varepsilon p$. Fix one x_i from each of the P_i . Then

$$\begin{aligned} \frac{\alpha^2}{20} &\leq |\widehat{f}_{A'}(t)| = \frac{1}{p} \sum_i \sum_{x \in P_i} f_{A'}(x) e(-xt/p) \\ &= \frac{1}{p} \left| \sum_i \sum_{x \in P_i} f_{A'}(x) e(-xit/p) + \sum_i \sum_{x \in P_i} f_{A'}(x) (e(-xt/p) - e(-xit/p)) \right| \\ &\leq \frac{1}{p} \sum_i \left| \sum_{x \in P_i} f_{A'}(x) \right| + \frac{1}{p} \sum_i \sum_{x \in P_i} |f_{A'}(x)| \underbrace{|e(-xt/p) - e(-xit/p)|}_{\leq 2\pi\varepsilon \text{ since } \text{diam}(\varphi(P_i)) \leq \varepsilon p} \end{aligned}$$

So

$$\sum_i \left| \sum_{x \in P_i} f_{A'}(x) \right| \geq \frac{\alpha^2}{40} p$$

Since $f_{A'}$ has mean zero,

$$\sum_i \left(\left| \sum_{x \in P_i} f_{A'}(x) \right| + \sum_{x \in P_i} f_{A'}(x) \right) \geq \frac{\alpha^2}{40} p$$

hence $\exists i$ such that

$$\left| \sum_{x \in P_i} f_{A'}(x) \right| + \sum_{x \in P_i} f_{A'}(x) \geq \frac{\alpha^2}{80} |P_i|$$

and so

$$\sum_{x \in P_i} f_{A'}(x) \geq \frac{\alpha^2}{160} |P_i|.$$

□

Definition 2.41 Let $\Gamma \subseteq \hat{G}$ and $\rho > 0$. The **Bohr set** $B(\Gamma, \rho)$ is the set

$$B(\Gamma, \rho) = \{x \in G : |\gamma(x) - 1| < \rho \ \forall \gamma \in \Gamma\}.$$

The **rank** of $B(\Gamma, \rho)$ is $|B(\Gamma, \rho)|$, and is **width** (or **radius**) is ρ .

Example 2.42 Let $G = \mathbb{F}_p^n$, then $B(\Gamma, \rho) = \langle \Gamma \rangle^\perp$ for all sufficiently small ρ . Here, the rank gives an upper bound on $\text{codim}(\langle \Gamma \rangle^\perp)$.

Lemma 2.43 Let $\Gamma \subseteq \hat{G}$ and $|\Gamma| = d$, and let $\rho > 0$. Then

$$|B(\Gamma, \rho)| \geq \left(\frac{\rho}{8} \right)^d |G|.$$

Proposition 2.44 (Bogolyubov's Lemma for Finite Abelian Groups) Let $A \subseteq G$ be of density $\alpha > 0$. Then there exists $\Gamma \subseteq \hat{G}$ with $|\Gamma| \leq 2\alpha^{-2}$ such that

$$B\left(\Gamma, \frac{1}{2}\right) \subseteq A + A - (A + A).$$

Proof. Recall that

$$(\mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_A)(x) = \sum_{\gamma \in \hat{G}} \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x)$$

Let $\Gamma = \text{Spec}_{\sqrt{\alpha/2}}(\mathbb{1}_A)$ and note that for $x \in B(\Gamma, 1/2)$ and $\gamma \in \Gamma$, $\text{Re}(\gamma(x)) > 0$. Hence, for $x \in B(\Gamma, 1/2)$,

$$\text{Re} \left(\sum_{\gamma \in \hat{G}} \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x) \right) = \text{Re} \left(\sum_{\gamma \in \Gamma} \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x) \right) + \text{Re} \left(\sum_{x \notin \Gamma} \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x) \right)$$

and

$$\begin{aligned} \left| \operatorname{Re} \left(\sum_{\gamma \notin \Gamma} |\hat{\mathbb{1}}_A(\gamma)|^4 \gamma(x) \right) \right| &\leq \sup_{\gamma \notin \Gamma} |\hat{\mathbb{1}}_A(\gamma)|^2 \sum_{\gamma \notin \Gamma} |\hat{\mathbb{1}}_A(\gamma)|^2 \\ &\leq \left(\sqrt{\frac{\alpha}{2}} \cdot \alpha \right)^2 \cdot \alpha = \frac{\alpha^4}{2} \end{aligned}$$

by Parseval. □

Theorem 2.45 (Roth) Let $A \subseteq [N]$ be a set containing no non-trivial 3-APs. Then $|A| = O(N / \log \log N)$.

Proof. □

Example 2.46 (Behrend's Example) There exists a set $A \subseteq [N]$ of size $|A| \geq \exp(-c\sqrt{\log N})N$ containing no non-trivial 3-APs.

3. Probabilistic tools

All probability spaces here will be finite.

Theorem 3.1 (Khinchine's Inequality) Let $p \in [2, \infty)$. Let X_1, \dots, X_n be independent random variables such that

$$\forall i \in [n], \quad \mathbb{P}(X_i = x_i) = \mathbb{P}(X_i = -x_i) = \frac{1}{2}$$

for some $x_1, \dots, x_n \in \mathbb{C}$. Then

$$\left\| \sum_{i=1}^n X_i \right\|_{L^p(\mathbb{P})} = O \left(p^{1/2} \left(\sum_{i=1}^n \|X_i\|_{L^2(\mathbb{P})}^2 \right)^{1/2} \right)$$

Proof (Hints).

- Explain why sufficient to prove for the case that $p = 2k$ for $k \in \mathbb{N}$.
- Explain why $\sum_{i=1}^n \|X_i\|_{L^\infty(\mathbb{P})}^2 = \sum_{i=1}^n \|X_i\|_{L^2(\mathbb{P})}^2$, and assume they are equal to 1.
- Show that $\|X\|_{L^{2k}(\mathbb{P})}^{2k} \leq 8kI(k)$, where $I(k) = \int_0^\infty t^{2k-1} \exp(-t^2/4) dt$.
- Show by induction on k that $I(k) \leq 2^{2k}(2k)^k/4k$.

□

Proof. Since L^p norms are nested, it suffices to prove in the case that $p = 2k$ for some $k \in \mathbb{N}$. Write $X = \sum_{i=1}^n X_i$, and assume the quantity $\sum_{i=1}^n \|X_i\|_{L^\infty(\mathbb{P})}^2 = \sum_{i=1}^n \|x_i\|^2 = \sum_{i=1}^n \|X_i\|_{L^2(\mathbb{P})}^2$ is equal to 1. By [Chernoff's Inequality](#), $\forall \theta > 0$,

$$\Pr(|X| \geq \theta) \leq 4 \exp(-\theta^2/4),$$

and so, since $\int_0^t P_X(s) ds = \Pr(|X| \leq t)$,

$$\begin{aligned}
\|X\|_{L^{2k}(\text{Pr})}^{2k} &= \int_0^\infty t^{2k} P_X(t) dt \\
&= \int_0^\infty 2kt^{2k-1} \text{Pr}(|X| \geq t) dt \text{ by integration by parts} \\
&\leq 8k \int_0^\infty t^{2k-1} \exp(-t^2/4) dt =: 8kI(k)
\end{aligned}$$

We will show by induction on k that $I(k) \leq 2^{2k}(2k)^k/4k$. Indeed, when $k = 1$,

$$\begin{aligned}
\int_0^\infty t \exp(-t^2/4) dt &= [-2 \exp(-t^2/4)]_0^\infty = 2 \\
&= 2^{2 \cdot 1} (2 \cdot 1)^1 / (4 \cdot 1)
\end{aligned}$$

For $k > 1$, we integrate by parts to find that

$$\begin{aligned}
I(k) &:= \int_0^\infty \underbrace{t^{2k-2}}_u \cdot \underbrace{t \exp(-t^2/4)}_{v'} dt \\
&= [t^{2k-2} \cdot (-2 \exp(-t^2/4))]_0^\infty - \int_0^\infty (2k-2)t^{2k-3} \cdot (-2 \exp(-t^2/4)) dt \\
&= 4(k-1) \int_0^\infty t^{2(k-1)-1} \exp(-t^2/4) dt \\
&= 4(k-1)I(k-1) \\
&\leq \frac{4(k-1)2^{2k-1}(2(k-1))^{k-1}}{4(k-1)} \text{ by induction hypothesis} \\
&\leq \frac{2^{2k}(2k)^k}{4k}.
\end{aligned}$$

□

Corollary 3.2 (Rudin's Inequality) Let $\Gamma \subseteq \hat{\mathbb{F}}_2^n$ be a linearly independent set and let $p \in [2, \infty)$. Then $\forall \hat{f} \in \ell^2(\Gamma)$,

$$\left\| \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \gamma \right\|_{L^p(\mathbb{F}_2^n)} = O(\sqrt{p} \cdot \|\hat{f}\|_{\ell^2(\Gamma)})$$

Proof. Exercise. □

Corollary 3.3 (Dual Rudin) Let $\Gamma \subseteq \hat{\mathbb{F}}_2^n$ be a linearly independent set and let $p \in (1, 2]$. Then $\forall f \in L^p(\mathbb{F}_2^n)$,

$$\|\hat{f}\|_{\ell^2(\Gamma)} = O\left(\sqrt{\frac{p}{p-1}} \cdot \|f\|_{L^p(\mathbb{F}_2^n)}\right).$$

Proof (Hints). Let $g(x) = \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \gamma(x)$. Show that $\|\hat{f}\|_{\ell^2(\Gamma)}^2 \leq \|f\|_{L^p(\mathbb{F}_2^n)} \|g\|_{L^q(\mathbb{F}_2^n)}$ where $1/p + 1/q = 1$, and conclude using [Rudin's Inequality](#). □

Proof. Let $f \in L^p(\mathbb{F}_2^n)$ and let $g(x) = \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \gamma(x)$. Then

$$\begin{aligned} \|\hat{f}\|_{\ell^2(\Gamma)}^2 &:= \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^2 \\ &= \langle \hat{f}, \hat{g} \rangle_{\ell^2(\Gamma)} = \langle \hat{f}, \hat{g} \rangle_{\ell^2(\hat{\mathbb{F}}_2^n)} \\ &= \langle f, g \rangle_{L^2(\mathbb{F}_2^n)} && \text{by [Plancherel's Identity](#)} \\ &\leq \|f\|_{L^p(\mathbb{F}_2^n)} \|g\|_{L^q(\mathbb{F}_2^n)} && \text{by [Hölder's Inequality](#).} \end{aligned}$$

where $1/p + 1/q = 1$. By [Rudin's Inequality](#),

$$\begin{aligned} \|g\|_{L^q(\mathbb{F}_2^n)} &= O(\sqrt{q} \cdot \|\hat{g}\|_{\ell^2(\Gamma)}) \\ &= O\left(\sqrt{\frac{p}{p-1}} \cdot \|\hat{f}\|_{\ell^2(\Gamma)}\right). \end{aligned}$$

□

Recall that given $A \subseteq \mathbb{F}_2^n$ of density $\alpha > 0$, we have $|\text{Spec}_\rho(\mathbb{1}_A)| \leq \rho^{-2} \alpha^{-1}$. This is the best possible bound as the example of a subspace A shows. However, in this case, the large spectrum is highly structured.

Theorem 3.4 (Special Case of Chang's Theorem) Let $A \subseteq \mathbb{F}_2^n$ be of density $\alpha > 0$. Then

$$\forall \rho > 0, \exists H \leq \hat{\mathbb{F}}_2^n : \dim(H) = O(\rho^{-2} \log \alpha^{-1}) \quad \text{and} \quad \text{Spec}_\rho(\mathbb{1}_A) \subseteq H.$$

Proof (Hints). Use [Dual Rudin](#) on a maximal linearly independent set in $\text{Spec}_\rho(\mathbb{1}_A)$, with $p = 1 + (\log \alpha^{-1})^{-1}$. □

Proof. Let $\Gamma \subseteq \text{Spec}_\rho(\mathbb{1}_A)$ be maximal linearly independent set. Let $H = \langle \text{Spec}_\rho(\mathbb{1}_A) \rangle$. Clearly $\dim(H) = |\Gamma|$. By [Dual Rudin](#), $\forall p \in (1, 2]$,

$$(\rho\alpha)^2 |\Gamma| \leq \sum_{\gamma \in \Gamma} |\hat{\mathbb{1}}_A(\gamma)|^2 = \|\hat{\mathbb{1}}_A\|_{\ell^2(\Gamma)}^2 = O\left(\frac{p}{p-1} \|\mathbb{1}_A\|_{L^p(\mathbb{F}_2^n)}^2\right) = O\left(\frac{p}{p-1} \alpha^{2/p}\right).$$

Hence, $|\Gamma| \leq O(\rho^{-2} \alpha^{-2} \alpha^{2/p} \frac{p}{p-1})$. Setting $p = 1 + (\log \alpha^{-1})^{-1}$, we obtain $|\Gamma| \leq O(\rho^{-2} \alpha^{-2} (\alpha^2 e^2) (\log \alpha^{-1} + 1)) = O(\rho^{-2} \log \alpha^{-1})$. □

Definition 3.5 Let G be a finite abelian group. $S \subseteq G$ is **dissociated** if, whenever $\sum_{s \in S} \varepsilon_s s = 0$ with each $\varepsilon_s \in \{-1, 0, 1\}$, then we have $\varepsilon_s = 0$ for all $s \in S$.

Example 3.6 Clearly, if $G = \mathbb{F}_2^n$, then $S \subseteq G$ is dissociated iff S is linearly independent.

Theorem 3.7 (Chang) Let G be a finite abelian group, and let $A \subseteq G$ be of density $\alpha > 0$. If $\Lambda \subseteq \text{Spec}_\rho(\mathbb{1}_A)$ is dissociated, then $|\Lambda| = O(\rho^{-2} \log \alpha^{-1})$.

Theorem 3.8 (Marcinkiewicz-Zygmund) Let $p \in [2, \infty)$ and let $X_1, \dots, X_n \in L^p(\text{Pr})$ be independent RVs with $\mathbb{E}[X_1 + \dots + X_n] = 0$. Then

$$\left\| \sum_{i=1}^n X_i \right\|_{L^p(\Pr)} = O \left(p^{1/2} \cdot \left\| \sum_{i=1}^n |X_i|^2 \right\|_{L^{p/2}(\Pr)}^{1/2} \right).$$

Proof. First assume that the distribution of the X_i is symmetric, i.e. $\Pr(X_i = a) = \Pr(X_i = -a)$ for all $a \in \mathbb{R}$ and $i \in [n]$. Partition the probability space Ω into sets $\Omega_1, \Omega_2, \dots, \Omega_M$ and write \Pr_j for the induced measure on Ω , such that all X_i are symmetric and take at most 2 values. By Khintchine's inequality, for each $j \in [M]$,

$$\begin{aligned} \left\| \sum_{i=1}^n X_i \right\|_{L^p(\Pr_j)}^p &= O \left(p^{p/2} \cdot \left(\sum_{i=1}^n \|X_i\|_{L^2(\Pr_j)}^2 \right)^{p/2} \right) \\ &= O \left(p^{p/2} \cdot \left\| \sum_{i=1}^n |X_i|^2 \right\|_{L^{p/2}(\Pr_j)}^{p/2} \right). \end{aligned}$$

Summing over all $j \in [M]$ and taking p -th roots gives the result for the symmetric case.

Now suppose the X_i are arbitrary RVs, and let Y_1, \dots, Y_n be such that $Y_i \sim X_i$ and $X_1, Y_1, \dots, X_n, Y_n$ are all independent. Applying the symmetric case to the RVs $X_i - Y_i$, we obtain

$$\begin{aligned} \left\| \sum_{i=1}^n (X_i - Y_i) \right\|_{L^p(\Pr \times \Pr)} &= O \left(p^{1/2} \cdot \left\| \sum_{i=1}^n |X_i - Y_i|^2 \right\|_{L^{p/2}(\Pr \times \Pr)}^{1/2} \right) \\ &= O \left(p^{1/2} \cdot \left\| \sum_{i=1}^n |X_i|^2 \right\|_{L^{p/2}(\Pr)}^{1/2} \right) \quad \text{TODO: check this explicitly} \end{aligned}$$

But then

$$\begin{aligned} \left\| \sum_{i=1}^n X_i \right\|_{L^p(\Pr)}^p &= \left\| \sum_{i=1}^n X_i - \mathbb{E}_Y \left[\sum_{i=1}^n Y_i \right] \right\|_{L^p(\Pr)}^p \\ &= \mathbb{E}_X \left| \sum_{i=1}^n X_i - \mathbb{E}_Y \left[\sum_{i=1}^n Y_i \right] \right|^p \\ &= \mathbb{E}_X \left| \mathbb{E}_Y \sum_{i=1}^n (X_i - Y_i) \right|^p \\ &\leq \mathbb{E}_X \mathbb{E}_Y \left| \sum_{i=1}^n (X_i - Y_i) \right|^p \quad \text{by Jensen's inequality} \\ &= \left\| \sum_{i=1}^n (X_i - Y_i) \right\|_{L^p(\Pr \times \Pr)}^p. \end{aligned}$$

□

Theorem 3.9 (Crooot-Sisask Almost Periodicity) Let G be a finite abelian group, let $\varepsilon > 0$, and $p \in [2, \infty)$. Let $A, B \subseteq G$ be such that $|A + B| \leq K|A|$, and let $f : G \rightarrow \mathbb{C}$. Then there is $b \in B$ and a set $X \subseteq B - b$ such that $|X| \geq \frac{1}{2}K^{-O(\varepsilon^{-2p})}|B|$ and

$$\|\tau_x(f * \mu_A) - f * \mu_A\|_{L^p(G)} \leq \varepsilon \|f\|_{L^p(G)} \quad \forall x \in X,$$

where $\tau_x g(y) = g(y + x)$ for all $y \in G$.

Proof. The main idea is to approximated

$$(f * \mu_A)(y) = \mathbb{E}_{x \in G} f(y - x) \mu_A(x) = \mathbb{E}_{x \in A} f(y - x)$$

by $\frac{1}{m} \sum_{i=1}^m f(y - z_i)$ where the z_i are sampled independently and uniformly from A , and m is to be chosen later. For each $y \in G$, define $Z_i(y) = \tau_{-z_i} f(y) - (f * \mu_A)(y)$.

For each $y \in G$, these are independent random variables with mean 0. So by Marcinkiewicz-Zygmund,

$$\begin{aligned} \left\| \sum_{i=1}^m Z_i(y) \right\|_{L^p(\text{Pr})}^p &= O \left(p^{p/2} \cdot \left\| \sum_{i=1}^m |Z_i(y)|^2 \right\|_{L^{p/2}(\text{Pr})}^{p/2} \right) \\ &= O \left(p^{p/2} \cdot \mathbb{E}_{(z_1, \dots, z_m) \in A^m} \left| \sum_{i=1}^m |Z_i(y)|^2 \right|^{p/2} \right). \end{aligned}$$

By Holder's inequality with $1/p' + 2/p = 1$,

$$\begin{aligned} \left| \sum_{i=1}^m |Z_i(y)|^2 \right|^{p/2} &\leq \left(\sum_{i=1}^m 1^{p'} \right)^{\frac{1}{p'} \cdot \frac{p}{2}} \cdot \left(\sum_{i=1}^m |Z_i(y)|^{2 \cdot \frac{p}{2}} \right)^{\frac{2}{p} \cdot \frac{p}{2}} \\ &= m^{p/2-1} \cdot \sum_{i=1}^m |Z_i(y)|^p. \end{aligned}$$

So

$$\left\| \sum_{i=1}^m Z_i(y) \right\|_{L^p(\text{Pr})}^p = O \left(p^{p/2} m^{p/2-1} \cdot \mathbb{E}_{(z_1, \dots, z_m) \in A^m} \sum_{i=1}^m |Z_i(y)|^p \right).$$

Summing over all $y \in G$, we have

$$\mathbb{E}_{y \in G} \left\| \sum_{i=1}^m Z_i(y) \right\|_{L^p(\text{Pr})}^p = O \left(p^{p/2} m^{p/2-1} \mathbb{E}_{(z_1, \dots, z_m) \in A^m} \sum_{i=1}^m \mathbb{E}_{y \in G} |Z_i(y)|^p \right)$$

and $(\mathbb{E}_{y \in G} |Z_i(y)|^p)^{1/p} = \|Z_i\|_{L^p(G)} = \|\tau_{-z_i} f - f * \mu_A\|_{L^p(G)} \leq \|\tau_{-z_i} f\|_{L^p(G)} + \|f * \mu_A\|_{L^p(G)} \leq \|f\|_{L^p(G)} + \|f\|_{L^p(G)} \cdot \|\mu_A\|_{L^1(G)} \leq 2\|f\|_{L^p(G)}$ by Young's convolution inequality. So we have

$$\begin{aligned}\mathbb{E}_{(z_1, \dots, z_m) \in A^m} \mathbb{E}_{y \in G} \left| \sum_{i=1}^m Z_i(y) \right|^p &= O \left(p^{p/2} m^{p/2-1} \sum_{i=1}^m (2\|f\|_{L^p(G)})^p \right) \\ &= O((4p)^{p/2} m^{p/2} \|f\|_{L^p(G)}^p).\end{aligned}$$

Choose $m = O(\varepsilon^{-2}p)$ so that the RHS is at most $(\frac{\varepsilon}{4}\|f\|_{L^p(G)})^p$, and for $(z_1, \dots, z_m) \in A^m$, define

$$M_{(z_1, \dots, z_m)} := \mathbb{E}_{y \in G} \left| \frac{1}{m} \sum_{i=1}^m \tau_{-z_i} f(y) - (f * \mu_A)(y) \right|^p.$$

Then we have

$$\mathbb{E}_{(z_1, \dots, z_m) \in A^m} M_{(z_1, \dots, z_m)} = O((4p)^{p/2} m^{p/2} \|f\|_{L^p(G)}^p) = \left(\frac{\varepsilon}{4} \|f\|_{L^p(G)} \right)^p.$$

Also define

$$L = \left\{ z \in A^m : M_z \leq \left(\frac{\varepsilon}{2} \|f\|_{L^p(G)} \right)^p \right\}.$$

By Markov's inequality, since

$$\mathbb{E}_{z \in A^m} M_z \leq \left(\frac{\varepsilon}{4} \|f\|_{L^p(G)} \right)^p = 2^{-p} \left(\frac{\varepsilon}{2} \|f\|_{L^p(G)} \right)^p,$$

we have

$$\frac{|A^m \setminus L|}{|A^m|} = \Pr \left(M_z \geq \left(\frac{\varepsilon}{2} \|f\|_{L^p(G)} \right)^p \right) \leq \Pr(M_z \geq 2^p \mathbb{E}_{z \in A^m} M_z) \leq 2^{-p},$$

hence $|L| \geq (1 - 1/2^p)|A|^m \geq \frac{1}{2}|A|^m$. Let $D = \{(b, \dots, b) : b \in B\} \subseteq B^m$. Then $L + D \subseteq (A + B)^m$, and so

$$|L + D| \leq |A + B|^m \leq K^m |A|^m \leq 2K^m |L|.$$

By [Lemma 1.24](#),

$$E(L, D) \geq \frac{|L|^2 |D|^2}{|L + D|} \geq \frac{1}{2} K^{-m} |D|^2 |L|,$$

so there are at least $|D|^2/2K^m$ pairs $(d_1, d_2) \in D^2$ such that $r_{L-L}(d_2 - d_1) > 0$. In particular, there exists $b \in B$ and $X \subseteq B - b$ such that $|X| \geq |D|/2K^m = |B|/2K^m$ and for all $x \in X$, there exists $\ell_2(x) \in L$ such that for all $i \in [m]$, $\ell_1(x)_i - \ell_2(x)_i = x$. But now for all $x \in X$, by the triangle inequality, we have,

$$\begin{aligned}
\|\tau_{-x}f * \mu_A - f * \mu_A\|_{L^p(G)} &\leq \left\| \tau_{-x}f * \mu_A - \tau_{-x} \left(\frac{1}{m} \sum_{i=1}^m \tau_{-\ell_2(x)_i} f \right) \right\|_{L^p(G)} \\
&\quad + \left\| \tau_{-x} \left(\frac{1}{m} \sum_{i=1}^m \tau_{-\ell_2(x)_i} f - f * \mu_A \right) \right\|_{L^p(G)} \\
&= \left\| f * \mu_A - \frac{1}{m} \sum_{i=1}^m \tau_{-\ell_2(x)_i} f \right\|_{L^p(G)} \\
&\quad + \left\| \frac{1}{m} \sum_{i=1}^m \tau_{-x-\ell_2(x)_i} f - f * \mu_A \right\|_{L^p(G)} \\
&\leq 2 \cdot \frac{\varepsilon}{2} \|f\|_{L^p(G)}
\end{aligned}$$

by definition of L . □

Theorem 3.10 (Bogolyubov, after Sanders) Let $A \subseteq \mathbb{F}_p^n$ have density $\alpha > 0$. There exists a subspace $V \leq \mathbb{F}_p^n$ of codimension $O((\log \alpha^{-1})^4)$ such that

$$V \subseteq (A + A) - (A + A)$$

4. Further topics

Theorem 4.1 (Ellenberg-Gijswijt) If $A \subseteq \mathbb{F}_3^n$ contains no non-trivial 3-term APs, then $|A| = o(2.756^n)$.

Notation 4.2 Let M_n denote the set of monomials in x_1, \dots, x_n whose degree in each variable is at most 2.

Notation 4.3 Let V_n denote the vector space of polynomials over \mathbb{F}_3 whose basis is M_n .

Notation 4.4 For any $0 \leq d \leq 2n$, let M_n^d denote the set of monomials in M_n of total degree at most d , and let V_n^d denote the corresponding vector space of polynomials. Write $m_d = \dim(V_n^d) = |M_n^d|$.

Lemma 4.5 Let $A \subseteq \mathbb{F}_3^n$ and $P \in V_n^d$ be a polynomial. If $P(a + b) = 0$ for all $a \neq b \in A$, then

$$|\{a \in A : P(2a) \neq 0\}| \leq 2m_{d/2}.$$

Proof. Every $P \in V_n^d$ can be written as a linear combination of monomials in M_n^d , so

$$P(x + y) = \sum_{\substack{m, m' \in M_n^d \\ \deg(mm') \leq d}} c_{m, m'} m(x) m'(y)$$

for some coefficients $c_{m, m'}$. Clearly, at least one of m, m' must have degree $\leq d/2$, whence

$$P(x+y) = \sum_{m \in M_n^{d/2}} m(x)F_m(y) + \sum_{m' \in M_n^{d/2}} m'(y)G_{m'}(x),$$

for some families of polynomials $\{F_m : m \in M_n^{d/2}\}$ and $\{G_{m'} : m' \in M_n^{d/2}\}$. Viewing $(P(x+y))_{x,y \in A}$ as an $|A| \times |A|$ matrix C , we see that C can be written as the sum of at most $2m_{d/2}$ matrices, each of which has rank 1. Thus, $\text{rank}(C) \leq 2m_{d/2}$. But by assumption, C is diagonal, and so its rank is equal to $|\{a \in A : P(a+a) \neq 0\}|$. \square

Proposition 4.6 Let $A \subseteq \mathbb{F}_3^n$ be a set containing no non-trivial 3-APs. Then $|A| \leq 3m_{2n/3}$.

Proof. Let $d \in [0, 2n]$ be an integer which we will determine later. Let W be the space of polynomials in V_n^d that vanish in $(2 \cdot A)^c$. We have $\dim(W) \geq \dim(V_n^d) - |(2 \cdot A)^c| = m_d - (3^n - |A|)$.

We claim that there exists $P \in W$ such that $|\text{supp}(P)| \geq \dim(W)$. Indeed, pick $P \in W$ with maximal support. If $|\text{supp}(P)| < \dim(W)$, then there would be a non-zero polynomial $Q \in W$ vanishing on $\text{supp}(P)$, in which case $\text{supp}(P+Q) \supsetneq \text{supp}(P)$, contradicting the maximality of $\text{supp}(P)$.

Now by assumption, $\{a + a' : a \neq a' \in A\} \cap 2 \cdot A = \emptyset$, so any polynomial that vanishes on $(2 \cdot A)^c$ also vanishes on $\{a + a' : a \neq a' \in A\}$. Thus by above lemma,

$$\begin{aligned} m_d - (3^n - |A|) &\leq \dim(W) \leq |\text{supp}(P)| = |\{x \in \mathbb{F}_3^n : P(x) \neq 0\}| \\ &= |\{a \in A : P(2a) \neq 0\}| \leq 2m_{d/2}. \end{aligned}$$

Hence, $|A| \leq 3^n - m_d + 2m_{d/2}$. But the monomials in $M_n \setminus M_n^d$ are in bijection with the ones in M_{2n-d} by $x_1^{\alpha_1} \dots x_n^{\alpha_n} \leftrightarrow x_1^{2-\alpha_1} \dots x_n^{2-\alpha_n}$, whence $3^n - m_d = m_{2n-d}$. Thus, setting $d = 4n/3$, we have

$$|A| \leq m_{2n/3} + 2m_{2n/3} = 3m_{2n/3}.$$

\square

Example 4.7 Recall from (find lemma) that given $A \subseteq G$,

$$|T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^3| \leq \sup_{\gamma \neq 1} |\hat{\mathbb{1}}_A(\gamma)|.$$

However, it is impossible to bound $T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^4$, where

$$T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = \mathbb{E}_{x,d} \mathbb{1}_A(x) \mathbb{1}_A(x+d) \mathbb{1}_A(x+2d) \mathbb{1}_A(x+3d),$$

by $\sup_{\gamma \neq 1} |\hat{\mathbb{1}}_A(\gamma)|$. Indeed, consider $Q = \{x \in \mathbb{F}_p^n : x \cdot x = 0\}$. By (find example), $|Q|/p^n = 1/p + O(p^{-n/2})$ and $\sup_{t \neq 0} |\hat{\mathbb{1}}_Q(t)| = O(p^{-n/2})$. But given a 3-AP $x, x+d, x+2d \in Q$, by the identity

$$\forall x, d, \quad x^2 - 3(x+d)^2 + 3(x+2d)^2 - (x+3d)^2 = 0,$$

$x+3d$ automatically lies in Q , so $T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = (1/p)^3 + O(p^{-n/2})$.

Definition 4.8 Given $f : G \rightarrow \mathbb{C}$, define its U^2 -norm by

$$\|f\|_{U^2(G)}^4 = \mathbb{E}_{x,a,b \in G} f(x) \overline{f(x+a)} \overline{f(x+b)} f(x+a+b)$$

By (find example), we have $\|f\|_{U^2(G)} = \|\hat{f}\|_{\ell^4(\widehat{G})}$, so it is indeed a norm.

Lemma 4.9 Let $f_1, f_2, f_3 : G \rightarrow \mathbb{C}$. Then

$$|T_3(f_1, f_2, f_3)| \leq \min_{i \in [3]} \left(\|f_i\|_{U^2(G)} \cdot \prod_{j \neq i} \|f_j\|_{L^\infty(G)} \right).$$

Note that

$$\sup_{\gamma \in \widehat{G}} |\hat{f}(\gamma)|^4 \leq \sum_{\gamma \in \widehat{G}} |\hat{f}(\gamma)|^4 \leq \sup_{\gamma \in \widehat{G}} |\hat{f}(\gamma)|^2 \sum_{\gamma \in \widehat{G}} |\hat{f}(\gamma)|^2$$

and so by Parseval,

$$\|\hat{f}\|_{\ell^\infty(\widehat{G})} = \|f\|_{U^2(G)}^4 = \|\hat{f}\|_{\ell^\infty(\widehat{G})}^4 \leq \|\hat{f}\|_{\ell^\infty(\widehat{G})}^2 \cdot \|f\|_{L^2(G)}^2.$$

Moreover, if $f = f_A = \mathbb{1}_A - \alpha$, then

$$T_3(f, f, f) = T_3(\mathbb{1}_A - \alpha, \mathbb{1}_A - \alpha, \mathbb{1}_A - \alpha) = T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^3.$$

We may therefore reformulate the first step in the proof of [Meshulam](#) as follows: if $p^n \geq 2\alpha^{-2}$, then by (find lemma), $\frac{\alpha^3}{2} \leq \left| \frac{\alpha}{p^n} - \alpha^3 \right| = |T_3(f_A, f_A, f_A)| \leq \|f_A\|_{U^2(\mathbb{F}_p^n)}$. It remains to show that if $\|f_A\|_{U^2(\mathbb{F}_p^n)}$ is non-trivial, then there exists a subspace $V \leq \mathbb{F}_p^n$ of bounded codimension on which A has increased density.

Theorem 4.10 (U^2 Inverse Theorem) Let $f : \mathbb{F}_p^n \rightarrow \mathbb{C}$ satisfy $\|f\|_{L^\infty(\mathbb{F}_p^n)} \leq 1$ and $\|f\|_{U^2(\mathbb{F}_p^n)} \geq \delta$ for some $\delta > 0$. Then there exists $b \in \mathbb{F}_p^n$ such that

$$\left| \mathbb{E}_{x \in \mathbb{F}_p^n} f(x) e(-x \cdot b/p) \right| \geq \delta^2.$$

In other words, $\langle f, \varphi \rangle \geq \delta^2$ for $\varphi(x) = e(-x \cdot b/p)$. We say “ f correlates with a linear phase function”.

Proof. We have seen that $\|f\|_{U^2(\mathbb{F}_p^n)} \leq \|\hat{f}\|_{\ell^\infty(\widehat{\mathbb{F}_p^n})} \|f\|_{L^2(\mathbb{F}_p^n)} \leq \|\hat{f}\|_{\ell^\infty(\widehat{\mathbb{F}_p^n})}$. So

$$\delta^2 \leq \|\hat{f}\|_{\ell^\infty(\widehat{\mathbb{F}_p^n})} = \sup_{t \in \widehat{\mathbb{F}_p^n}} \left| \mathbb{E}_{x \in \mathbb{F}_p^n} f(x) e(-x \cdot t/p) \right|.$$

□

Definition 4.11 Given $f : G \rightarrow \mathbb{C}$, the U^3 norm of f is defined by

$$\begin{aligned}
\|f\|_{U^3(G)}^8 &= \mathbb{E}_{x,a,b,c \in G} f(x) \overline{f(x+a)} \overline{f(x+b)} \overline{f(x+c)} \\
&\quad f(x+a+b) f(x+b+c) f(x+a+c) \overline{f(x+a+b+c)} \\
&= \mathbb{E}_{x,h_1,h_2,h_3 \in G} \prod_{\varepsilon \in \{0,1\}^3} \mathcal{C}^{|\varepsilon|} f(x + \varepsilon \cdot h),
\end{aligned}$$

where $\mathcal{C}g(x) = \overline{g(x)}$ and $|\varepsilon| = |\{i \in [3] : \varepsilon_i = 1\}|$ is the number of 1's in ε .

TODO: insert diagram of cube with vertices $x, x+a, \dots, x+a+b+c$.

Remark 4.12 It is easy to verify that $\|f\|_{U^3(G)}^8 = \mathbb{E}_{c \in G} \|\Delta_c f\|_{U^2(G)}^4$ where $\Delta_c g(x) = g(x) \overline{g(x+c)}$.

Definition 4.13 Given eight functions $f_\varepsilon : G \rightarrow \mathbb{C}$ for $\varepsilon \in \{0,1\}^3$, define their **U^3 inner product** by

$$\langle (f_\varepsilon)_{\varepsilon \in \{0,1\}^3} \rangle_{U^3(G)} := \mathbb{E}_{x,h_1,h_2,h_3 \in G} \prod_{\varepsilon \in \{0,1\}^3} \mathcal{C}^{|\varepsilon|} f_\varepsilon(x + \varepsilon \cdot h)$$

Observe that $\langle f, f, f, f, f, f, f, f \rangle_{U^3(G)} = \|f\|_{U^3(G)}^8$.

Lemma 4.14 (Gowers-Cauchy-Schwarz Inequality) Given $f_\varepsilon : G \rightarrow \mathbb{C}$ for $\varepsilon \in \{0,1\}^3$,

$$\left| \langle (f_\varepsilon)_{\varepsilon \in \{0,1\}^3} \rangle_{U^3(G)} \right| \leq \prod_{\varepsilon \in \{0,1\}^3} \|f_\varepsilon\|_{U^3(G)}.$$

Proof. Exercise (helpful to do for U^2 first). □

Remark 4.15 Setting $f_\varepsilon = f$ for $\varepsilon \in \{0,1\}^2 \times \{0\}$ and $f_\varepsilon = 1$ otherwise, it follows that

$$\|f\|_{U^2(G)}^4 \leq \|f\|_{U^3(G)}^4 \quad \text{hence} \quad \|f\|_{U^2(G)} \leq \|f\|_{U^3(G)}.$$

Proposition 4.16 Let $f_1, f_2, f_3, f_4 : \mathbb{F}_5^n \rightarrow \mathbb{C}$. Then

$$|T_4(f_1, f_2, f_3, f_4)| \leq \min_{i \in [4]} \|f_i\|_{U^3(G)} \cdot \prod_{j \neq i} \|f_j\|_{L^\infty(\mathbb{F}_5^n)}.$$

Proof. Assume $f_i = f$ for all i and that $\|f\|_{L^\infty(\mathbb{F}_5^n)} \leq 1$ (we can remove these assumptions). Reparameterising (by subtracting $a+b+c+d$), we have

$$T_4(f, f, f, f) = \mathbb{E}_{a,b,c,d \in \mathbb{F}_5^n} f(3a+2b+c) f(2a+b-d) f(a-c-2d) f(-b-2c-3d)$$

Now

$$\begin{aligned}
|T_4(f, f, f, f)|^8 &\leq \left(\mathbb{E}_{a,b,c} |\mathbb{E}_d f(2a+b-d) f(a-c-2d) f(-b-2c-3d)|^2 \right)^4 \text{ by C-S} \\
&\quad \left(\mathbb{E}_{d,d'} \mathbb{E}_{a,b} f(2a+b-d) \overline{f(2a+b-d')} \right)^4 \\
&= \cdot \mathbb{E}_c f(a-c-2d) \overline{f(a-c-2d')} f(-b-2c-3d) \overline{f(-b-2c-3d')} \\
&\leq \mathbb{E}_{d,d'} \mathbb{E}_{a,b} \left| \mathbb{E}_c f(a-c-2d) \overline{f(a-c-2d')} f(-b-2c-3d) \overline{f(-b-2c-3d')} \right|^2 \\
&\quad \left(\mathbb{E}_{c,c',d,d'} \mathbb{E}_a f(a-c-2d) \overline{f(a-c'-2d)} f(a-c-2d') \overline{f(a-c'-2d')} \right)^2 \\
&= \cdot \mathbb{E}_b f(-b-2c-3d) \overline{f(-b-2c'-3d)} f(-b-2c-3d') \overline{f(-b-2c'-3d')} \\
&\leq \mathbb{E}_{c,c',d,d',a} \left| \mathbb{E}_b f(-b-2c-3d) \overline{f(-b-2c'-3d)} f(-b-2c-3d') \overline{f(-b-2c'-3d')} \right|^2 \\
&= \mathbb{E}_{b,b',c,c',d,d'} f(-b-2c-3d) \overline{f(-b'-2c-3d)} f(-b-2c'-3d) \overline{f(-b'-2c'-3d)} \\
&\quad \overline{f(-b-2c-3d')} f(-b'-2c-3d') f(-b-2c'-3d') \overline{f(-b'-2c'-3d')}
\end{aligned}$$

where all the inequalities are by Cauchy-Schwarz. \square

Example 4.17 Let M be an $\mathbb{F}_5^{n \times n}$ be a symmetric matrix. Then $f(x) = e(x^T M x / 5)$ satisfies $\|f\|_{U^3} = 1$.

Theorem 4.18 (U^3 Inverse Theorem) Let $f : \mathbb{F}_5^n \rightarrow \mathbb{C}$ satisfy $\|f\|_{L^\infty(\mathbb{F}_5^n)} \leq 1$ and $\|f\|_{U^3(\mathbb{F}_5^n)} \geq \delta$ for some $\delta > 0$. Then there exists a symmetric matrix $M \in \mathbb{F}_5^{n \times n}$ and $b \in \mathbb{F}_5^n$ such that

$$|\mathbb{E}_x f(x) e(x^T M x + b^T x / p)| \geq c(\delta),$$

where $c(\delta)$ is a polynomial in δ . In other words, $|\langle f, \varphi \rangle| \geq c(\delta)$ for $\varphi(x) = e(x^T M x + b^T x / p)$, and we say “ f correlates with a quadratic phase function”.

Proof sketch. We have $\|f\|_{U^3}^8 = \mathbb{E}_h \|\Delta_h f\|_{U^2}^4$ where $\Delta_h f(x) = f(x) \overline{f(x+h)}$.

1. Weak linearity: if $\|f\|_{U^3}^8 \geq \delta^8$, then for at least a $\delta^8/2$ -proportion of $h \in \mathbb{F}_5^n$, $\delta^8/2 \leq \|\Delta_h f\|_{U^2}^4 \leq \|\widehat{\Delta_h f}\|_{\ell^\infty}^2$. So for each such $h \in \mathbb{F}_5^n$, there exists t_h such that $|\widehat{\Delta_h f}(t_h)|^2 \geq \delta^8/2$. We have

Proposition 4.19 Let $f : \mathbb{F}_5^n \rightarrow \mathbb{C}$ satisfy $\|f\|_{L^\infty(\mathbb{F}_5^n)} \leq 1$ and $\|f\|_{U^3(\mathbb{F}_5^n)} \geq \delta$ for some $\delta > 0$. Suppose $|\mathbb{F}_5^n| = \Omega_\delta(1)$. Then there exists $S \subseteq \mathbb{F}_5^n$ with $|S| = \Omega_\delta(|\mathbb{F}_5^n|)$ and a function $\varphi : S \rightarrow \widehat{\mathbb{F}_5^n}$ such that:

- $|\widehat{\Delta_h f}(\varphi(h))| = \Omega_\delta(1)$, and
- There are at least $\Omega_\delta(|\mathbb{F}_5^n|^3)$ quadruples $(s_1, s_2, s_3, s_4) \in S^4$ such that $s_1 + s_2 = s_3 + s_4$ and $\varphi(s_1) + \varphi(s_2) = \varphi(s_3) + \varphi(s_4)$.

2. Strong linearity. If S and φ are as above, then there is a linear function $\psi : \mathbb{F}_5^n \rightarrow \widehat{\mathbb{F}_5^n}$ which coincides with φ for many elements of S . We have

Proposition 4.20 Let S and φ be as given by above proposition. Then there exists a $M \in \mathbb{F}_5^{n \times n}$ and $b \in \mathbb{F}_5^n$ such that $\psi : \mathbb{F}_5^n \rightarrow \widehat{\mathbb{F}_5^n}$, $\psi(x) = Mx + b$ satisfies $\psi(x) = \varphi(x)$ for $\Omega_\delta(|\mathbb{F}_5^n|)$ elements $x \in S$.

Proof. Consider the graph of φ : $\Gamma = \{(h, \varphi(h)) : h \in S\} \subseteq \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$. By above proposition, Γ has $\Omega_\delta(|\mathbb{F}_5^n|)$ additive quadruples. By Balog-Szemerédi-Gowers, there exists $\Gamma' \subseteq \Gamma$ with $|\Gamma'| = \Omega_\delta(|\Gamma|) = \Omega_\delta(|\mathbb{F}_5^n|)$ and $|\Gamma' + \Gamma'| = O_\delta(|\Gamma'|)$. Define $S' \subseteq S$ by $\Gamma' = \{(h, \varphi(h)) : h \in S'\}$. Note that $|S'| = \Omega_\delta(|\mathbb{F}_5^n|)$. By Freiman-Ruzsa applied to $\Gamma' \subseteq \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$, there exists a subspace $H \leq \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$ with $|H| = O_\delta(|\Gamma'|) = O_\delta(|\mathbb{F}_5^n|)$ such that $\Gamma' \subseteq H$.

Denote by $\pi : \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$ the projection onto the first n coordinates. By construction, $\pi(H) \supseteq S'$. Moreover, since $|S'| = \Omega_\delta(|\mathbb{F}_5^n|)$, we have

$$|\ker(\pi|_H)| = \frac{|H|}{|\text{im}(\pi|_H)|} \leq \frac{O_\delta(|\mathbb{F}_5^n|)}{|S'|} = O_\delta(1).$$

We may thus partition H into $O_\delta(1)$ cosets of some subspace H^* such that $\pi|_H$ is injective on each coset. By averaging, there exists a coset $x + H^*$ such that $|\Gamma' \cap (x + H^*)| = \Omega_\delta(|\Gamma'|) = \Omega_\delta(|\mathbb{F}_5^n|)$.

Set $\Gamma'' = \Gamma' \cap (x + H^*)$ and define S'' accordingly. Now $\pi|_{x+H^*}$ is injective and surjective onto $V := \text{im}(\pi|_{x+H^*})$. This means there is an affine-linear map $\psi : V \rightarrow \widehat{\mathbb{F}}_5^n$ such that $(h, \psi(h)) \in \Gamma'$ for all $h \in S''$. \square

3. Symmetry argument.

4. Integration step.

Theorem 4.21 (Szemerédi's Theorem for 4-APs) Let $A \subseteq \mathbb{F}_5^n$ be a set containing no non-trivial 4-APs. Then $|A| = O(5^n)$.

Proof. Idea: by above proposition with $f = f_A = \mathbb{1}_A - \alpha$,

$$T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^4 = T_4(f_A, f_A, f_A, f_A) + 14 \text{ other terms},$$

in which between one and three of the inputs are equal to f_A . These are controlled by $\|f_A\|_{U^2(\mathbb{F}_5^n)} \leq \|f_A\|_{U^3(\mathbb{F}_5^n)}$, whence

$$|T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^4| \leq 15\|f_A\|_{U^3(\mathbb{F}_5^n)}$$

So if A contains no non-trivial 4-APs and $5^n > 2\alpha^{-3}$, then $\|f_A\|_{U^3(\mathbb{F}_5^n)} \geq \frac{\alpha^4}{30}$. What can we say about functions with large U^3 norm? \square