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1. Probability basics

TODO: weak and strong laws of large numbers, Markov chains, Cesaro lemma, Markov's inequality, ... probably others.

2. Entropy

2.1. Introduction

Notation 2.1 Write $x_1^n := (x_1, \dots, x_n) \in \{0, 1\}^n$ for an length n bit string.

Notation 2.2 We use P to denote a probability mass function. Write P_1^n for the joint probability mass function of a sequence of n random variables $X_1^n = (X_1, \dots, X_n)$.

Definition 2.3 A random variable X has a **Bernoulli distribution**, $X \sim \text{Bern}(p)$, if for some fixed $p \in (0, 1)$,

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

i.e. the probability mass function (PMF) of X is $P : \{0, 1\} \rightarrow \mathbb{R}$, $P(0) = 1 - p$, $P(1) = p$.

Notation 2.4 Throughout, we take \log to be the base-2 logarithm, \log_2 .

Definition 2.5 The **binary entropy function** $h : (0, 1) \rightarrow [0, 1]$ is defined as

$$h(p) := -p \log p - (1 - p) \log(1 - p)$$

Example 2.6 Let $x_1^n \in \{0, 1\}^n$ be an n bit string which is the realisation of binary random variables (RVs) $X_1^n = (X_1, \dots, X_n)$, where the X_i are independent and identically distributed (IID), with common distribution $X_i \sim \text{Bern}(p)$. Let $k = |\{i \in [n] : x_i = 1\}|$ be the number of ones in x_1^n . We have

$$\mathbb{P}(X_1^n = x_1^n) := P^n(x_1^n) = \prod_{i=1}^n P(x_i) = p^k (1 - p)^{n-k}.$$

Now by the law of large numbers, the probability of ones in a random x_1^n is $k/n \approx p$ with high probability for large n . Hence,

$$P^n(x_1^n) \approx p^{np} (1 - p)^{n(1-p)} = 2^{-nh(p)}.$$

Note that this reveals an amazing fact: this approximation is independent of x_1^n , so any message we are likely to encounter has roughly the same probability $\approx 2^{-nh(p)}$ of occurring.

Remark 2.7 By the above example, we can split the set of all possible n -bit messages, $\{0, 1\}^n$, into two parts: the set B_n of **typical** messages which are approximately uniformly distributed with probability $\approx 2^{-nh(p)}$ each, and the non-typical messages that occur with negligible probability. Since all but a very small amount of the probability is concentrated in B_n , we have $|B_n| \approx 2^{nh(p)}$.

Remark 2.8 Suppose an encoder and decoder both already know B_n and agree on an ordering of its elements: $B_n = \{x_1^n(1), \dots, x_1^n(b)\}$, where $b = |B_n|$. Then instead of transmitting the actual message, the encoder can transmit its index $j \in [b]$, which can be described with

$$\lceil \log b \rceil = \lceil \log |B_n| \rceil \approx nh(p)$$

bits.

Remark 2.9

- The closer p is to $\frac{1}{2}$ (intuitively, the more random the messages are), the larger the entropy $h(p)$, and the larger the number of typical strings $|B_n|$.
- Assuming we ignore non-typical strings, which have vanishingly small probability for large n , the “compression rate” of the above method is $h(p)$, since we encode n bit strings using $nh(p)$ strings. $h(p) < 1$ unless the message is uniformly distributed over all of $\{0, 1\}^n$.
- So the closer p is to 0 or 1 (intuitively, the less random the messages are), the smaller the entropy $h(p)$, so the greater the compression rate we can achieve.

2.2. Asymptotic equipartition property

Notation 2.10 We denote a finite alphabet by $A = \{a_1, \dots, a_m\}$.

Notation 2.11 If X_1, \dots, X_n are IID RVs with values in A , with common distribution described by a PMF $P : A \rightarrow [0, 1]$ (i.e. $P(x) = \mathbb{P}(X_i = x)$ for all $x \in A$), then write $X \sim P$, and we say “ X has distribution P on A ”.

Notation 2.12 For $i \leq j$, write X_i^j for the block of random variables (X_i, \dots, X_j) , and similarly write x_i^j for the length $j - i + 1$ string $(x_i, \dots, x_j) \in A^{i-j+1}$.

Notation 2.13 For IID RVs X_1, \dots, X_n with each $X_i \sim P$, denote their joint PMF by $P^n : A^n \rightarrow [0, 1]$:

$$P^n(x_1^n) = \mathbb{P}(X_1^n = x_1^n) = \prod_{i=1}^n \mathbb{P}(X_i = x_i) = \prod_{i=1}^n P(x_i),$$

and we say that “the RVs X_1^n have the product distribution P^n ”.

Definition 2.14 A sequence of RVs $(Y_n)_{n \in \mathbb{N}}$ **converges in probability** to an RV Y if $\forall \varepsilon > 0$,

$$\mathbb{P}(|Y_n - Y| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Definition 2.15 Let $X \sim P$ be a discrete RV on a countable alphabet A . The **entropy** of X is

$$H(X) = H(P) := - \sum_{x \in A} P(x) \log P(x) = \mathbb{E}[-\log P(X)].$$

Remark 2.16

- We use the convention $0 \log 0 = 0$ (this is natural due to continuity: $x \log x \rightarrow 0$ as $x \downarrow 0$, and also can be derived measure-theoretically).

- Entropy is technically a functional the probability distribution P and not of X , but we use the notation $H(X)$ as well as $H(P)$.
- $H(X)$ only depends on the probabilities $P(x)$, not on the values $x \in A$. Hence for any bijective $f : A \rightarrow A$, we have $H(f(X)) = H(X)$.
- All summands of $H(X)$ are non-negative, so the sum always exists and is in $[0, \infty]$, even if A is countable infinite.
- $H(X) = 0$ iff all summands are 0, i.e. if $P(x) \in \{0, 1\}$ for all $x \in A$, i.e. X is **deterministic** (constant, so equal to a fixed $x_0 \in A$ with probability 1).

Theorem 2.17 Let $X = \{X_n : n \in \mathbb{N}\}$ be IID RVs with common distribution P on a finite alphabet A . Then

$$-\frac{1}{n} \log P^n(X_1^n) \rightarrow H(X_1) \quad \text{in probability as } n \rightarrow \infty$$

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned} P^n(X_1^n) &= \prod_{i=1}^n P(X_i) \\ \Rightarrow \frac{1}{n} \log P^n(X_1^n) &= \frac{1}{n} \sum_{i=1}^n \log P(X_i) \rightarrow \mathbb{E}[-\log P(X_1)] \quad \text{in probability} \end{aligned}$$

by the weak law of large numbers (WLLN) for the IID RVs $Y_i = -\log P(X_i)$. □

Corollary 2.18 (Asymptotic Equipartition Property (AEP)) Let $\{X_n : n \in \mathbb{N}\}$ be IID RVs on a finite alphabet A with common distribution P and common entropy $H = H(X_i)$. Then

- (\Rightarrow): for all $\varepsilon > 0$, the set of **typical strings** $B_n^*(\varepsilon) \subseteq A^n$ defined by

$$B_n^*(\varepsilon) := \{x_1^n \in A^n : 2^{-n(H+\varepsilon)} \leq P^n(x_1^n) \leq 2^{-n(H-\varepsilon)}\}$$

satisfies

$$|B_n^*(\varepsilon)| \leq 2^{n(H+\varepsilon)} \quad \forall n \in \mathbb{N}, \quad \text{and}$$

$$P^n(B_n^*(\varepsilon)) = \mathbb{P}(X_1^n \in B_n^*(\varepsilon)) \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

- (\Leftarrow): for any sequence $(B_n)_{n \in \mathbb{N}}$ of subsets of A^n , if $P(X_1^n \in B_n) \rightarrow 1$ as $n \rightarrow \infty$, then $\forall \varepsilon > 0$,

$$|B_n| \geq (1 - \varepsilon) 2^{n(H-\varepsilon)} \quad \text{eventually}$$

$$\text{i.e. } \exists N \in \mathbb{N} : \forall n \geq N, \quad |B_n| \geq (1 - \varepsilon) 2^{n(H-\varepsilon)}.$$

Proof (Hints).

- (\Rightarrow): straightforward.
- (\Leftarrow): show that $P^n(B_n \cap B_n^*(\varepsilon)) \rightarrow 1$ as $n \rightarrow \infty$.

□

Proof.

- (\Rightarrow):
 ▶ Let $\varepsilon > 0$. By [Theorem 2.17](#), we have

$$\mathbb{P}(X_1^n \notin B_n^*(\varepsilon)) = \mathbb{P}\left(\left|-\frac{1}{n} \log P^n(X_1^n) - H\right| > \varepsilon\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

- ▶ By definition of $B_n^*(\varepsilon)$,

$$1 \geq P^n(B_n^*(\varepsilon)) = \sum_{x_1^n \in B_n^*(\varepsilon)} P^n(x_1^n) \geq |B_n^*(\varepsilon)| 2^{-n(H+\varepsilon)}.$$

- (\Leftarrow):
 ▶ We have $P^n(B_n \cap B_n^*(\varepsilon)) = P^n(B_n) + P^n(B_n^*(\varepsilon)) - P^n(B_n \cup B_n^*(\varepsilon)) \geq P^n(B_n) + P^n(B_n^*(\varepsilon)) - 1$, so $P^n(B_n \cap B_n^*(\varepsilon)) \rightarrow 1$.
 ▶ So $P^n(B_n \cap B_n^*(\varepsilon)) \geq 1 - \varepsilon$ eventually, and so

$$\begin{aligned} 1 - \varepsilon \leq P^n(B_n \cap B_n^*(\varepsilon)) &= \sum_{x_1^n \in B_n \cap B_n^*(\varepsilon)} P^n(x_1^n) \\ &\leq |B_n \cap B_n^*(\varepsilon)| 2^{-n(H-\varepsilon)} \leq |B_n| 2^{-n(H-\varepsilon)}. \end{aligned}$$

□

Remark 2.19

- The \Rightarrow part of AEP states that a specific object (in this case, the $B_n^*(\varepsilon)$) can achieve a certain performance, while the \Leftarrow part states that no other object of this type can significantly perform better. This is common type of result in information theory.
- [Theorem 2.17](#) gives a mathematical interpretation of entropy: the probability of a random string X_1^n generally decays exponentially with n ($P^n(X_1^n) \approx 2^{-nH}$ with high probability for large n). The AEP gives a more “operational interpretation”: the smallest set of strings that can carry almost all the probability of P^n has size $\approx 2^{nH}$.
- The AEP tells us that higher entropy means more typical strings, and so the possible values of X_1^n are more unpredictable. So we consider “high entropy” RVs to be “more random” and “less predictable”.

2.3. Fixed-rate lossless data compression

Definition 2.20 A **memoryless source** $X = \{X_n : n \in \mathbb{N}\}$ is a sequence of IID RVs with a common PMF P on the same alphabet A .

Definition 2.21 A **fixed-rate lossless compression code** for a source X consists of a sequence of **codebooks** $\{B_n : n \in \mathbb{N}\}$, where each $B_n \subseteq A^n$ is a set of source strings of length n .

Assume the encoder and decoder share the codebooks, each of which is sorted. To send x_1^n , an encoder checks with $x_1^n \in B_n$; if so, they send the index of x_1^n in B_n , along with a flag bit 1, which requires $1 + \lceil \log |B_n| \rceil$ bits. Otherwise, they send x_1^n

uncompressed, along with a flag bit 0 to indicate an “error”, which requires $1 + \lceil \log|A| \rceil = 1 + \lceil n \log|A| \rceil$ bits.

Definition 2.22 For each $n \in \mathbb{N}$, the **rate** of a fixed-rate code $\{B_n : n \in \mathbb{N}\}$ for a source X is

$$R_n := \frac{1}{n}(1 + \lceil \log|B_n| \rceil) \approx \frac{1}{n} \log|B_n| \quad \text{bits/symbol.}$$

Definition 2.23 For each $n \in \mathbb{N}$, the **error probability** of a fixed-rate code $\{B_n : n \in \mathbb{N}\}$ for a source X is

$$P_e^{(n)} := \mathbb{P}(X_1^n \notin B_n).$$

Theorem 2.24 (Fixed-rate Coding Theorem) Let $X = \{X_n : n \in \mathbb{N}\}$ be a memoryless source with distribution P and entropy $H = H(X_i)$.

- (\Rightarrow): $\forall \varepsilon > 0$, there is a fixed-rate code $\{B_n^*(\varepsilon) : n \in \mathbb{N}\}$ with vanishing error probability ($P_e^{(n)} \rightarrow 0$ as $n \rightarrow \infty$) and with rate

$$R_n \leq H + \varepsilon + \frac{2}{n} \quad \forall n \in \mathbb{N}.$$

- (\Leftarrow): let $\{B_n : n \in \mathbb{N}\}$ be a fixed-rate with vanishing error probability. Then $\forall \varepsilon > 0$, its rate R_n satisfies

$$R_n > H - \varepsilon \quad \text{eventually.}$$

Proof (Hints). (\Rightarrow): straightforward. (\Leftarrow): straightforward. □

Proof.

- (\Rightarrow):
 - Let $B_n^*(\varepsilon)$ be the sets of typical strings defined in AEP (**Asymptotic Equipartition Property (AEP)**). Then $P_e^{(n)} = 1 - \mathbb{P}(X_1^n \in B_n^*) \rightarrow 0$ as $n \rightarrow \infty$ by AEP.
 - Also by AEP, $R_n = \frac{1}{n}(1 + \lceil \log|B_n^*| \rceil) \leq \frac{1}{n} \log|B_n^*| + \frac{2}{n} \leq H + \varepsilon + \frac{2}{n}$.
- (\Leftarrow):
 - WLOG let $0 < \varepsilon < 1/2$. By AEP,

$$R_n \geq \frac{1}{n} \log|B_n^*| + \frac{1}{n} \geq \frac{1}{n} \log(1 - \varepsilon) + H - \varepsilon + \frac{1}{n} = H - \varepsilon + \frac{1}{n} \log(2(1 - \varepsilon)) > H - \varepsilon$$

eventually. □

3. Relative entropy

Definition 3.1 Suppose $x_1^n \in A^n$ are observations generated by IID RVs X_1^n and we want to decide whether $X_1^n \sim P^n$ or Q^n , for two distinct candidate PMFs P, Q on A .

A **hypothesis test** is described by a **decision region** $B_n \subseteq A^n$ such that

- If $x_1^n \in B_n$, then we declare that $X_1^n \sim P^n$.
- Otherwise, if $x_1^n \notin B_n$, then we declare that $X_1^n \sim Q^n$.

Definition 3.2 The associated **error probabilities** for a hypothesis test are

$$\begin{aligned} e_1^{(n)} &= e_1^{(n)}(B_n) := \mathbb{P}(\text{declare } P \mid \text{data} \sim Q) = Q^n(B_n) \\ e_2^{(n)} &= e_2^{(n)}(B_n) := \mathbb{P}(\text{declare } Q \mid \text{data} \sim P) = P^n(B_n^c). \end{aligned}$$

Definition 3.3 The **relative entropy** between PMFs P and Q on the same countable alphabet A is

$$D(P \parallel Q) := \sum_{x \in A} P(x) \log \frac{P(x)}{Q(x)} = \mathbb{E} \left[\log \frac{P(X)}{Q(X)} \right], \quad \text{where } X \sim P.$$

Remark 3.4

- We use the convention that $0 \log \frac{0}{0} = 0$ (this can be avoided by defining relative entropy measure-theoretically).
- $D(P \parallel Q)$ always exists and $D(P \parallel Q) \geq 0$ with equality iff $P = Q$.
- Relative entropy is not symmetric: $D(P \parallel Q) \neq D(Q \parallel P)$ in general, and does not satisfy the triangle inequality.
- Despite this, it is reasonable and natural to think of $D(P \parallel Q)$ as a statistical “distance” between P and Q .

Remark 3.5 Let $X \sim P$. We have, by WLLN,

$$\begin{aligned} \frac{1}{n} \log \left(\frac{P^n(X_1^n)}{Q^n(X_1^n)} \right) &= \frac{1}{n} \log \prod_{i=1}^n \frac{P(X_i)}{Q(X_i)} \\ &= \frac{1}{n} \sum_{i=1}^n \log \frac{P(X_i)}{Q(X_i)} \\ &\longrightarrow D(P \parallel Q) \text{ in probability as } n \rightarrow \infty. \end{aligned}$$

So for large n , $\frac{P^n(X_1^n)}{Q^n(X_1^n)} \approx 2^{nD(P \parallel Q)}$ with high probability. Hence, the random string X_1^n is exponentially more likely under its true distribution P than under Q .

3.1. Asymptotically optimal hypothesis testing

Theorem 3.6 (Stein's Lemma) Let P, Q be PMFs on a finite alphabet A , with $D = D(P \parallel Q) \in (0, \infty)$. Let $X = \{X_n : n \in \mathbb{N}\}$ be a memoryless source on A , with either each $X_i \sim P$ or each $X_i \sim Q$.

- (\Rightarrow): for all $\varepsilon > 0$, there is a hypothesis test with decision regions $\{B_n^*(\varepsilon) : n \in \mathbb{N}\}$ such that

$$\forall n \in \mathbb{N}, \quad e_1^{(n)}(B_n^*(\varepsilon)) \leq 2^{-n(D-\varepsilon)}$$

and $e_2^{(n)} \rightarrow 0$ as $n \rightarrow \infty$.

- (\Leftarrow): for any hypothesis test with decision regions $\{B_n : n \in \mathbb{N}\}$ such that $e_2^{(n)}(B_n) \rightarrow 0$ as $n \rightarrow \infty$, we have $\forall \varepsilon > 0$,

$$e_1^{(n)}(B_n) \geq 2^{-n(D+\varepsilon+\frac{1}{n})} \quad \text{eventually.}$$

Proof (Hints).

- (\Rightarrow):
 - Let $B_n^*(\varepsilon) = \left\{x_1^n \in A^n : 2^{n(D-\varepsilon)} \leq \frac{P^n(x_1^n)}{Q^n(x_1^n)} \leq 2^{n(D+\varepsilon)}\right\}$. The rest is straightforward (use above remark).
- (\Leftarrow):
 - Show that $P^n(B_n^*(\varepsilon) \cap B_n) \rightarrow 1$ as $n \rightarrow \infty$, use that $\frac{1}{2} = 2^{-n(1/n)}$.

□

Proof.

- (\Rightarrow):
 - Let $B_n^*(\varepsilon) = \left\{x_1^n \in A^n : 2^{n(D-\varepsilon)} \leq \frac{P^n(x_1^n)}{Q^n(x_1^n)} \leq 2^{n(D+\varepsilon)}\right\}$.
 - Then the convergence in probability of $\frac{1}{n} \sum_{i=1}^n \log \frac{P(X_i)}{Q(X_i)}$ is equivalent to $\mathbb{P}(X_1^n \notin B_n^*) = P^n(B_n^*(\varepsilon)) = e_2^{(n)} \rightarrow 0$ as $n \rightarrow \infty$, when $X_1^n \sim P^n$.
 - Also, $1 \geq P^n(B_n^*) = \sum_{x_1^n \in B_n^*(\varepsilon)} Q^n(x_1^n) \frac{P^n(x_1^n)}{Q^n(x_1^n)} \geq 2^{n(D-\varepsilon)} \sum_{x_1^n \in B_n^*(\varepsilon)} Q^n(x_1^n) = 2^{n(D-\varepsilon)} Q^n(B_n^*(\varepsilon))$.
- (\Leftarrow):
 - We have $e_2^{(n)}(B_n^*(\varepsilon)) = P^n(B_n^*(\varepsilon)) \rightarrow 0$ as $n \rightarrow \infty$. Suppose $e_2^{(n)}(B_n) = P^n(B_n^c) \rightarrow 0$. Then $P^n(B_n \cap B_n^*(\varepsilon)) \rightarrow 1$. So eventually,

$$\begin{aligned}
 \frac{1}{2} \leq P^n(B_n \cap B_n^*(\varepsilon)) &= \sum_{x_1^n \in B_n \cap B_n^*(\varepsilon)} P^n(x_1^n) \frac{Q^n(x_1^n)}{Q^n(x_1^n)} \\
 &\leq 2^{n(D+\varepsilon)} \sum_{x_1^n \in B_n} Q^n(x_1^n) \\
 &= 2^{n(D+\varepsilon)} Q^n(B_n) = 2^{n(D+\varepsilon)} e_1^{(n)}(B_n)
 \end{aligned}$$

□

Remark 3.7

- The decision regions B_n^* are asymptotically optimal in that, among all tests that have $e_2^{(n)} \rightarrow 0$, they achieve the asymptotically smallest possible $e_1^{(n)} \approx 2^{-nD}$. However, they are not the most optimal decision regions for finite n . For finite regions, the optimal regions are given by the Neyman-Pearson Lemma.
- Assuming $D \neq 0$ is a trivial assumption, as otherwise $P = Q$ on A , so any test would give the correct answer.
- Assuming $D < \infty$ is a reasonable assumption, as otherwise there is some $a \in A$ such that $P(a) > 0$ but $Q(a) = 0$. In that case, we check whether any such a appear in x_1^n or not.
- In Stein's Lemma, we assume one error vanishes at possibly an arbitrarily slow rate, while the other decays exponentially. This is a natural asymmetry in many applications, e.g. in diagnosing disease.
- Stein's Lemma shows why the relative entropy is a natural measure of "distance" between two distributions, as large D means a smaller error probability (one vanishes exponentially at rate D), so easier to tell apart the distributions from the data.

3.2. Relative entropy and optimal hypothesis testing

Theorem 3.8 (Neyman-Pearson Lemma) For a hypothesis test between P and Q based on n data samples, the **likelihood ratio decision regions**

$$B_{\text{NP}} = \left\{ x_1^n \in A^n : \frac{P^n(x_1^n)}{Q^n(x_1^n)} \geq T \right\}, \quad \text{for some threshold } T > 0,$$

are optimal in that, for any decision region $B_n \subseteq A^n$, if $e_1^{(n)}(B_n) \leq e_1^{(n)}(B_{\text{NP}})$, then $e_2^{(n)}(B_n) \geq e_2^{(n)}(B_{\text{NP}})$, and vice versa.

Proof (Hints). Consider the inequality

$$(P^n(x_1^n) - TQ^n(x_1^n))(\mathbb{1}_{B_{\text{NP}}}(x_1^n) - \mathbb{1}_{B_n}(x_1^n)) \geq 0$$

(justify why this holds). □

Proof.

- Consider the obvious inequality

$$(P^n(x_1^n) - TQ^n(x_1^n))(\mathbb{1}_{B_{\text{NP}}}(x_1^n) - \mathbb{1}_{B_n}(x_1^n)) \geq 0$$

- Then, summing over all x_1^n ,

$$\begin{aligned} 0 &\leq P^n(B_{\text{NP}}) - P^n(B_n) - TQ^n(B_{\text{NP}}) + TQ^n(B_n) \\ &= 1 - e_2^{(n)}(B_{\text{NP}}) - (1 - e_2^{(n)}(B_n)) - T(e_1^{(n)}(B_{\text{NP}}) - e_1^{(n)}(B_n)) \\ &\implies e_2^{(n)}(B_n) - e_2^{(n)}(B_{\text{NP}}) \geq T(e_1^{(n)}(B_{\text{NP}}) - e_1^{(n)}(B_n)) \end{aligned}$$

□

Remark 3.9 Neyman-Pearson says that if any decision region has an error as small as that of B_{NP} , then its other error must be larger than that of B_{NP} .

Notation 3.10 Let \hat{P}_n denote the empirical distribution (or **type**) induced by x_1^n on A^n (the frequency with which $a \in A$ occurs in x_1^n):

$$\forall a \in A, \quad \hat{P}_n(a) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x_i=a\}}$$

Proposition 3.11 The Neyman-Pearson decision region B_{NP} can be expressed in information-theoretic form as

$$B_{\text{NP}} = \left\{ x_1^n \in A^n : D(\hat{P}_n \parallel Q) \geq D(\hat{P}_n \parallel P) + T' \right\}$$

where $T' = \frac{1}{n} \log T$.

Proof (Hints). Rewrite the expression $\frac{1}{n} \log \frac{P^n(x_1^n)}{Q^n(x_1^n)}$. □

Proof. We have

$$\begin{aligned}
\frac{1}{n} \log \frac{P^n(x_1^n)}{Q^n(x_1^n)} &= \frac{1}{n} \log \left(\prod_{i=1}^n \frac{P(x_i)}{Q(x_i)} \right) \\
&= \frac{1}{n} \sum_{i=1}^n \log \frac{P(x_i)}{Q(x_i)} \\
&= \frac{1}{n} \sum_{i=1}^n \sum_{a \in A} \mathbb{1}_{\{x_i=a\}} \log \frac{P(a)}{Q(a)} \\
&= \sum_{a \in A} \left(\frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x_i=a\}} \right) \log \frac{P(a)}{Q(a)} \\
&= \sum_{a \in A} \hat{P}_n(a) \log \left(\frac{P(a)}{Q(a)} \cdot \frac{\hat{P}_n(a)}{\hat{P}_n(a)} \right) \\
&= D(\hat{P}_n \parallel Q) - D(\hat{P}_n \parallel P).
\end{aligned}$$

□

Theorem 3.12 (Jensen's Inequality) Let I be an interval, $f : I \rightarrow \mathbb{R}$ be convex and X be an RV with values in I . Then

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]).$$

Moreover, if f is strictly convex, then equality holds iff X is almost surely constant.

Proof. Omitted. □

Theorem 3.13 (Log-sum Inequality) Let $a_1, \dots, a_n, b_1, \dots, b_n$ be non-negative constants. Then

$$\sum_{i=1}^n a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^n a_i \right) \log \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}$$

with equality iff $\frac{a_i}{b_i} = c$ for all i , for some constant c . We use the convention that $0 \log 0 = 0 \log \frac{0}{0} = 0$.

Remark 3.14 This also holds for countably many a_i and b_i .

Proof (Hints). Use Jensen's inequality with X the RV such that $\mathbb{P}\left(X = \frac{a_i}{b_i}\right) = \frac{b_i}{\sum_{j=1}^n b_j}$ for all $i \in [n]$, and a suitable f . □

Proof.

- Define

$$f(x) = \begin{cases} x \log x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

f is strictly convex.

- Let $A = \sum_i a_i$, $B = \sum_i b_i$. Let X be the RV with $\mathbb{P}\left(X = \frac{a_i}{b_i}\right) = \frac{b_i}{B}$ for all $i \in [n]$.
- Then $\mathbb{E}[f(X)] = \sum_i \frac{b_i}{B} \frac{a_i}{b_i} \log \frac{a_i}{b_i} = \frac{1}{B} \sum_i a_i \log \frac{a_i}{b_i}$.

- $f(\mathbb{E}[X]) = \mathbb{E}[X] \log \mathbb{E}[X] = \sum_i \frac{a_i}{b_i} \frac{b_i}{B} \log \sum_i \frac{a_i}{b_i} \frac{b_i}{B} = \frac{A}{B} \log \frac{A}{B}$.
- So by Jensen's inequality, $\frac{A}{B} \log \frac{A}{B} \leq \frac{1}{B} \sum_i a_i \log \frac{a_i}{b_i}$.

□

Proposition 3.15

1. If P and Q are PMFs on the same finite alphabet A , then

$$D(P \parallel Q) \geq 0$$

with equality iff $P = Q$.

2. If $X \sim P$ on a finite alphabet A , then

$$0 \leq H(X) \leq \log|A|$$

with equality to 0 iff X is a constant, and equality to $\log|A|$ iff X is uniformly distributed on A .

Remark 3.16 This also holds for countably infinite A .

Proof (Hints).

1. Straightforward.
2. For $\leq \log|A|$, consider $D(P \parallel Q)$ where Q is the uniform distribution on A . ≥ 0 is straightforward.

□

Proof.

- ▶ By the log-sum inequality,

$$D(P \parallel Q) = \sum_{x \in A} P(x) \log \frac{P(x)}{Q(x)} \geq \left(\sum_{x \in A} P(x) \right) \log \frac{\sum_{x \in A} P(x)}{\sum_{x \in A} Q(x)} = 0$$

with equality if $\frac{P(x)}{Q(x)}$ is the same constant for all $x \in A$, i.e. $P = Q$.

- ▶ Let Q be the uniform distribution on A , so $H(Q) = \sum_{x \in A} \frac{1}{|A|} \log \frac{1}{1/|A|} = \log|A|$.
- ▶ Now $0 \leq D(P \parallel Q) = \sum_{x \in A} P(x) \log \frac{P(x)}{1/|A|} = \log|A| - H(X)$ with equality iff $P = Q$, i.e. P is uniform.
- ▶ Each term in $-H(X)$ is ≤ 0 , with equality iff each $P(x) \log P(x)$ is 0, i.e. $P(x) = 0$ or 1.

□

Remark 3.17 If $X = \{X_n : n \in \mathbb{N}\}$ is a memoryless source with PMF P on A , then we have shown that it can be at best compressed to $\approx H(P)$ bits/symbol. This means that we can always achieve non-trivial compression, i.e. a description using $\approx H(P) < \log|A|$ bits/symbol, unless the source X is completely random (i.e. IID and uniformly distribute), in which case we cannot do better than simply describing each x_1^n uncompressed using $\frac{\lceil \log|A|^n \rceil}{n} \approx \log|A|$ bits/symbol.

4. Properties of entropy and relative entropy

4.1. Joint entropy and conditional entropy

Definition 4.1 Let X_1^n be an arbitrary finite collection of discrete RVs on corresponding alphabets A_1, \dots, A_n . Note we can think of X_1^n itself a discrete RV on alphabet $A_1 \times \dots \times A_n$. Let X_1^n have PMF P_n , then the **joint entropy** of X_1^n is

$$H(X_1^n) = H(P_n) = H(X_1, \dots, X_n) := \mathbb{E}[-\log P_n(X_1^n)] = - \sum_{x_1^n \in A^n} P_n(x_1^n) \log P_n(x_1^n).$$

Example 4.2 Note that if X and Y are independent, then $P_{X,Y}(x, y) = P_X(x)P_Y(y)$, so

$$H(X, Y) = \mathbb{E}[-\log P_{X,Y}(X, Y)] = \mathbb{E}[-\log P_X(X) - \log P_Y(Y)] = H(X) + H(Y).$$

Example 4.3 Let X and Y have joint PMF given by

$X \backslash Y$	1	2	3	
0	1/10	1/5	1/4	11/20
1	1/5	1/20	1/5	9/20
	3/10	1/4	9/20	

Note that X and Y are not independent. We have

$$\begin{aligned} H(X) &= -\frac{3}{10} \log \frac{3}{10} - \frac{1}{4} \log \frac{1}{4} - \frac{9}{20} \log \frac{9}{20} \approx 1.539, \\ H(Y) &= -\frac{11}{20} \log \frac{11}{20} - \frac{9}{20} \log \frac{9}{20} \approx 0.993, \\ H(X, Y) &= -\frac{1}{10} \log \frac{1}{10} - \dots - \frac{1}{5} \log \frac{1}{5} \approx 2.441 < H(X) + H(Y). \end{aligned}$$

In general, if X and Y are not independent, then $P_{XY}(x, y) = P_X(x)P_{Y|X}(y | x)$, so

$$H(X, Y) = \mathbb{E}[-\log P_{XY}(x, y)] = \mathbb{E}[-\log P_X(x)] + \mathbb{E}[-\log P_{Y|X}(y | x)].$$

Definition 4.4 Let X and Y be discrete random variables with joint PMF $P_{X,Y}$, then the **conditional entropy** of Y given X is

$$H(Y | X) = \mathbb{E}[-\log P_{Y|X}(Y | X)] = - \sum_{x,y} P_{X,Y}(x, y) \log P_{Y|X}(y | x)$$

Note 4.5 $P_{Y|X}$ is a function of $(x, y) \in X$, and so for the expected value we multiply the log by the probability that $X = x$ and $Y = y$.

Proposition 4.6 For discrete RVs X and Y , we have

$$H(Y | X) = H(X, Y) - H(X).$$

Proof (Hints). Straightforward. □

Proof. Note that $P_{Y|X}(y|x) = \mathbb{P}(Y=y|X=x) = \frac{\mathbb{P}(Y=y, X=x)}{\mathbb{P}(X=x)} = P_{X,Y}(x,y)P_X(x)$.
Hence

$$\begin{aligned} H(X,Y) &= \mathbb{E}[-\log P_{X,Y}(X,Y)] \\ &= \mathbb{E}[-\log P_X(X) - \log P_{Y|X}(Y|X)] \\ &= \mathbb{E}[-\log P_X(X)] + \mathbb{E}[-\log P_{Y|X}(Y|X)]. \end{aligned}$$

□

4.2. Properties of entropy, joint entropy and conditional entropy

Proposition 4.7 (Chain Rule for Entropy) Let X_1^n be a collection of discrete RVs. Then

$$H(X_1^n) = \sum_{i=1}^n H(X_i | X_1^{i-1}).$$

In particular, if the X_1^n are independent, then

$$H(X_1^n) = \sum_{i=1}^n H(X_i).$$

Proof (Hints). By induction. □

Proof. We can write

$$\begin{aligned} P_{X_1^n}(x_1^n) &= P_{X_1}(x_1)P_{X_2|X_1}(x_2|x_1)\cdots P_{X_n|X_1,\dots,x_{n-1}}(x_n|x_1,\dots,x_{n-1}) \\ &= \prod_{i=1}^n P_{X_i|X_1^{i-1}}(x_i|x_1^{i-1}). \end{aligned}$$

Then the result follows by inductively using the above proposition. □

Proposition 4.8 (Conditioning Reduces Entropy) For discrete RVs X and Y ,

$$H(Y|X) \leq H(Y)$$

with equality iff X and Y are independent.

Proof (Hints). Express $H(Y) - H(Y|X)$ as a relative entropy. □

Proof. We have

$$\begin{aligned}
H(Y) - H(Y | X) &= \mathbb{E}[-\log P_Y(Y)] - \mathbb{E}[-\log P_{Y|X}(Y | X)] \\
&= \mathbb{E} \left[\log \frac{P_{Y|X}(Y | X)}{P_Y(Y)} \right] \\
&= \mathbb{E} \left[\log \frac{P_{Y|X}(Y | X) P_X(X)}{P_Y(Y) P_X(X)} \right] \\
&= \mathbb{E} \left[\log \frac{P_{X,Y}(X, Y)}{P_X(X) P_Y(Y)} \right] \\
&= D(P_{X,Y} \| P_X P_Y).
\end{aligned}$$

This is non-negative iff $P_{X,Y} = P_X P_Y$, i.e. X and Y are independent. \square

Definition 4.9 Discrete RVs X and Z are **conditionally independent given Y** if:

- $P_{X,Z|Y}(x, z | y) = P_{X|Y}(x | y) P_{Z|Y}(z | y)$,
- or equivalently, $P_{X|Z,Y}(x | z, y) = P_{X|Y}(x | y)$,
- or equivalently, $P_{Z|X,Y}(z | x, y) = P_{Z|Y}(z | y)$.

We denote this by writing $X - Y - Z$ and we say that X, Y, Z form a Markov chain. Note that $X - Y - Z$ is equivalent to $Z - Y - X$, but not to $X - Z - Y$.

Note 4.10 For any function g on Y , we have $X - Y - g(Y)$.

Corollary 4.11 $H(X_1^n) \leq \sum_{i=1}^n H(X_i)$ with equality iff all X_1^n are independent.

Proof. Straightforward. \square

Proof. $H(X_1^n) = \sum_{i=1}^n H(X_i | X_1^{i-1}) \leq \sum_{i=1}^n H(X_i)$ by the chain rule and conditioning reducing entropy. \square

Remark 4.12 We can write

$$\begin{aligned}
H(Y | X) &= - \sum_{x,y} (P_{X,Y}(x, y)) \log P_{Y|X}(y | x) \\
&= \sum_x P_X(x) \left(- \sum_y P_{Y|X}(y | x) \log P_{Y|X}(y | x) \right) \\
&=: \sum_x P_X(x) H(Y | X = x)
\end{aligned}$$

Note $H(Y | X = x)$ is **not** a conditional entropy, and in particular, we do not always have $H(Y | X = x) \leq H(Y)$. Since $0 \leq H(Y | X = x) \leq \log |A_Y|$, we have $0 \leq H(Y | X) \leq \log |A_Y|$ with equality to 0 iff Y is a function of X (i.e. $H(Y | X = x) = 0$ for all x).

Proposition 4.13 (Data Processing Inequality for Entropy) Let X be discrete RV on alphabet A and f be function on A . Then

1. $H(f(X)|X) = 0$.
2. $H(f(X)) \leq H(X)$ with equality iff f is injective.

Proof (Hints). Use that $x \mapsto (x, f(x))$ is injective and the chain rule. \square

Proof. We have already shown the “if” direction of 2. We have $H(X) = H(X, f(X)) = H(f(X)|X) + H(X)$, since $x \mapsto (x, f(x))$ is injective. Also, $H(X) = H(X, f(X)) = H(X | f(X)) + H(f(X)) \geq H(f(X))$. So $H(X) \geq H(f(X))$ with equality iff $H(X | f(X)) = 0$, i.e. X is a deterministic function of $f(X)$, i.e. f is invertible. \square

Proposition 4.14 (Properties of Conditional Entropy) For discrete RVs X, Y, Z :

- Chain rule: $H(X, Z | Y) = H(X | Y) + H(Z | X, Y)$.
- Subadditivity: $H(X, Z | Y) \leq H(X | Y) + H(Z | Y)$ with equality iff X and Z are conditionally independent given Y .
- Conditioning reduces entropy: $H(X | Y, Z) \leq H(X | Y)$ with equality iff X and Z are conditionally independent given Y .

Proof. Exercise. \square

Theorem 4.15 (Fano's Inequality) Let X and Y be RVs on respective alphabets A and B . Suppose we are interested in the RV X but only are allowed to observe the possibly correlated RV Y . Consider the estimate $\hat{X} = f(Y)$, with probability of error $P_e := \mathbb{P}(\hat{X} \neq X)$. Then

$$H(X | Y) \leq h(P_e) + P_e \log(|A| - 1),$$

where h is the binary entropy function.

Proof (Hints). Consider an “error” Bernoulli RV E which depends on X and Y . Use the chain rule in two directions on $H(X, E | Y)$. Merge these and split up into the cases when $E = 0$ and $E = 1$ (using) \square

Proof. Let E be the binary RV taking value 1 when there is an error (i.e. $\hat{X} \neq X$), and taking value 0 otherwise. So $E \sim \text{Bern}(P_e)$ and $H(E) = h(P_e)$. Then

$$H(X, E | Y) = H(X | Y) + H(E | X, Y) = H(X | Y)$$

since E is function of (X, Y) . Using the chain rule in the other direction,

$$H(X, E | Y) = H(E | Y) + H(X | E, Y) \leq H(E) + H(X | E, Y).$$

Now

$$\begin{aligned} H(X | Y) - h(P_e) &\leq H(X | E, Y) \\ &= P_e H(X | E = 1, Y) + (1 - P_e) H(X | E = 0, Y) \end{aligned}$$

When $E = 0$, given Y , we can determine $X = f(Y)$ as a function of Y , so $H(X | E = 0, Y) = 0$. When $E = 1$, given Y , we know X doesn't take value $f(Y)$, so there are $|A| - 1$ possible values that it takes, so $H(X | E = 1, Y) \leq \log(|A| - 1)$. \square

4.3. Properties of relative entropy

Theorem 4.16 (Data Processing Inequality for Relative Entropy) Let $X \sim P_X$ and $X' \sim Q_X$ be RVs on the same alphabet A , and $f : A \rightarrow B$ be an arbitrary function. Let $P_{f(X)}$ and $Q_{f(X)}$ be the PMFs of $f(X)$ and $f(X')$ respectively. Then

$$D(P_{f(X)} \parallel Q_{f(X)}) \leq D(P_X \parallel Q_X).$$

Proof (Hints). Use that $P_{f(X)}(y) = \sum_{x \in f^{-1}(\{y\})} P_X(x)$. □

Proof. For each $y \in B$, let $A_y = \{x \in A : f(x) = y\} = f^{-1}(\{y\})$. Then

$$\begin{aligned} D(P_{f(X)} \parallel Q_{f(X)}) &= \sum_{y \in B} P_{f(X)}(y) \log \frac{P_{f(X)}(y)}{Q_{f(X)}(y)} \\ &= \sum_{y \in B} \left(\sum_{x \in A_y} P_X(x) \right) \log \frac{\sum_{x \in A_y} P_X(x)}{\sum_{x \in A_y} Q_X(x)} \\ &\leq \sum_{y \in B} \sum_{x \in A_y} P_X(x) \log \frac{P_X(x)}{Q_X(x)} \quad \text{by log-sum inequality} \\ &= \sum_{x \in A} P_X(x) \log \frac{P_X(x)}{Q_X(x)} = D(P_X \parallel Q_X). \end{aligned}$$

□

Remark 4.17 The data processing inequality for relative entropy shows that we cannot make two distributions more “distinguishable” by first “processing” the data (by applying f).

Definition 4.18 The **total variation distance** between PMFs P and Q on the same alphabet A is

$$\|P - Q\|_{\text{TV}} = \sum_{x \in A} |P(x) - Q(x)|.$$

Remark 4.19 Let $B = \{x \in A : P(x) > Q(x)\}$, then

$$\begin{aligned} \|P - Q\|_{\text{TV}} &= \sum_{x \in A} |P(x) - Q(x)| \\ &= \sum_{x \in B} (P(x) - Q(x)) + \sum_{x \in B^c} (Q(x) - P(x)) \\ &= P(B) - Q(B) + Q(B^c) - P(B^c) \\ &= P(B) - Q(B) + (1 - Q(B)) + (1 - P(B)) \\ &= 2(P(B) - Q(B)). \end{aligned}$$

Notation 4.20 Write

$$D_e(P \parallel Q) = (\ln 2) D(P \parallel Q) = \sum_{x \in A} P(x) \log_e \frac{P(x)}{Q(x)}$$

and more generally, write

$$D_c(P \parallel Q) = (\log_c 2)P(D \parallel Q) = \sum_{x \in A} P(x) \log_c \frac{P(x)}{Q(x)}.$$

Theorem 4.21 (Pinsker's Inequality) Let P and Q be PMFs on the same alphabet A . Then

$$\|P - Q\|_{\text{TV}}^2 \leq (2 \ln 2)D(P \parallel Q) = 2D_e(P \parallel Q).$$

Proof (Hints).

- First prove for case that P and Q are PMFs of $\text{Bern}(p)$ and $\text{Bern}(q)$ (explain why we can assume $q \leq p$ WLOG), by defining $\Delta(p, q) = 2D_e(P \parallel Q) - \|P - Q\|_{\text{TV}}^2$, and showing that $\frac{\partial \Delta(p, q)}{\partial q} \leq 0$.
- Then show for general PMFs by using data processing, where $f = \mathbb{1}_B$ for $B = \{x \in A : P(x) > Q(x)\}$.

□

Proof. First, assume that P and Q are the PMFs of the distributions $\text{Bern}(p)$ and $\text{Bern}(q)$ for some $0 \leq q \leq p \leq 1$ ($q \leq p$ WLOG since we can simultaneously interchange both P with $1 - P$ and Q with $1 - Q$ if necessary). Let

$$\Delta(p, q) = (2 \ln 2)D(P \parallel Q) - \|P - Q\|_{\text{TV}}^2 = 2p \ln \frac{p}{q} + 2(1 - p) \ln \frac{1 - p}{1 - q} - (2(p - q))^2.$$

Since $\Delta(p, p) = 0$ for all p , it suffices to show that $\frac{\partial \Delta(p, q)}{\partial q} \leq 0$. Indeed,

$$\frac{\partial \Delta(p, q)}{\partial q} = -2\frac{p}{q} + 2\frac{1 - p}{1 - q} + 8(p - q) = 2(q - p) \left(\frac{1}{q(1 - q)} - 4 \right) \leq 0$$

since $q(1 - q) \leq \frac{1}{4}$ for all $q \in [0, 1]$.

Now, assume P and Q are general PMFs and let $B = \{x \in A : P(x) > Q(x)\}$ and $f = \mathbb{1}_B$. Define the RVs $X \sim P$ and $X' \sim Q$, and let P_f and Q_f be the respective PMFs of the RVs $f(X)$ and $f(X')$. Note that $f(X) \sim \text{Bern}(p)$, $f(X') \sim \text{Bern}(q)$ where $p = P(B)$ and $q = Q(B)$. Then

$$\begin{aligned} 2D_e(P \parallel Q) &\geq 2D_e(P_f \parallel Q_f) && \text{by data-processing} \\ &\geq \|P_f - Q_f\|_{\text{TV}}^2 && \text{by above} \\ &= (2(p - q))^2 \\ &= (2(P(B) - Q(B)))^2 \\ &= \|P - Q\|_{\text{TV}}^2. \end{aligned}$$

□

Theorem 4.22 (Convexity of Relative Entropy) The relative entropy $D(P \parallel Q)$ is jointly convex in P, Q : for all PMFs P, P', Q, Q' on the same alphabet and for all $0 < \lambda < 1$,

$$D(\lambda P + (1 - \lambda)P' \parallel \lambda Q + (1 - \lambda)Q') \leq \lambda D(P \parallel Q) + (1 - \lambda)D(P' \parallel Q').$$

Proof. Exercise. □

Corollary 4.23 (Concavity of Entropy) The entropy of $H(P)$ is a concave function on all PMFs P on a finite alphabet.

Proof (Hints). Use convexity of relative entropy of P and a suitable distribution. □

Proof. Let P be a PMF on finite alphabet A and U be the uniform PMF on A . Then by convexity of relative entropy, $D(P \parallel U) = \sum_{x \in A} p(x) \log \frac{P(x)}{1/|A|} = \log m - H(P)$ is convex in P , so $H(P)$ is concave in P . □

5. Poisson approximation

5.1. Poisson approximation via entropy

Theorem 5.1 Let X_1, \dots, X_n be IID RVs with each $X_i \sim \text{Bern}(\lambda/n)$, let $S_n = X_1 + \dots + X_n$. Then $P_{S_n} \rightarrow \text{Pois}(\lambda)$ in distribution as $n \rightarrow \infty$, i.e. $\forall k \in \mathbb{N}$,

$$\mathbb{P}(S_n = k) \rightarrow e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{as } n \rightarrow \infty$$

Remark 5.2 Using information theory, we can derive stronger and more general statements than the one above.

Theorem 5.3 Let X_1, \dots, X_n be (not necessarily independent) RVs with each $X_i \sim \text{Bern}(p_i)$. Let $S_n = \sum_{i=1}^n X_i$ and $\lambda = \sum_{i=1}^n p_i = \mathbb{E}[S_n]$. Then

$$D_e(P_{S_n} \parallel \text{Pois}(\lambda)) \leq \sum_{i=1}^n p_i^2 + \left(\sum_{i=1}^n H_e(X_i) - H_e(X_1^n) \right).$$

Proof (Hints).

- Let $Z_i = \text{Pois}(p_i)$ for each $i \in [n]$ be independent Poisson RVs so that $T_n = \sum_{i=1}^n Z_i \sim \text{Pois}(\lambda)$.
- Use data processing inequality for relative entropy, and prove the fact that $D_e(\text{Bern}(p) \parallel \text{Pois}(p)) \leq p^2$ for all $p \in [0, 1]$ (use that $1 - p \leq e^{-p}$).

□

Proof. Let $Z_i = \text{Pois}(p_i)$ for each $i \in [n]$ be independent Poisson RVs so that $T_n = \sum_{i=1}^n Z_i \sim \text{Pois}(\lambda)$. Then

$$\begin{aligned}
D_e(P_{S_n} \parallel \text{Pois}(\lambda)) &= D_e(P_{S_n} \parallel P_{T_n}) \\
&\leq D_e(P_{X_1^n} \parallel P_{Z_1^n}) \quad \text{by data-processing with } f(x_1^n) = x_1 + \dots + x_n \\
&= \mathbb{E} \left[\ln \frac{P_{X_1^n}(X_1^n)}{P_{Z_1^n}(X_1^n)} \right] \\
&= \mathbb{E} \left[\ln \left(\frac{P_{X_1^n}(x_1^n)}{\prod_{i=1}^n P_{Z_1^n}(x_i)} \cdot \frac{\prod_{i=1}^n P_{X_i}(x_i)}{\prod_{i=1}^n P_{X_i}(x_i)} \right) \right] \\
&= \mathbb{E} \left[\ln \left(\prod_{i=1}^n \frac{P_{X_i}(x_i)}{P_{Z_i}(x_i)} \right) \right] + \sum_{x_1^n \in A^n} P_{X_1^n}(x_1^n) \ln \frac{1}{\prod_{i=1}^n P_{X_i}(x_i)} - H_e(X_1^n) \\
&= \sum_{i=1}^n D_e(P_{X_i} \parallel P_{Z_i}) + \sum_{i=1}^n H_e(X_i) - H_e(X_1^n)
\end{aligned}$$

since for given $x_1 \in A$, $\sum_{x_2^n \in A^n} P_{X_1^n}(x_1^n) = P_{X_1}(x_1)$ (and similarly for each x_j , $j = 2, \dots, n$). Now note that $D_e(P_{X_i} \parallel P_{Z_i}) = D_e(\text{Bern}(p_i) \parallel \text{Pois}(p_i))$, and for all $p \in (0, 1)$,

$$\begin{aligned}
D_e(\text{Bern}(p) \parallel \text{Pois}(p)) &= (1-p) \ln \frac{1-p}{e^{-p}} + p \ln \frac{p}{pe^{-p}} \\
&= (1-p) \ln(1-p) + (1-p)p + p^2 \\
&\leq (1-p) \ln(e^{-p}) + p \\
&= p^2
\end{aligned}$$

since $1-p \leq e^{-p}$ for all $p \in [0, 1]$. Similarly, if $p = 0$ or 1 , then $D_e(\text{Bern}(p) \parallel \text{Pois}(p)) = 0 \leq p^2$. □

Corollary 5.4 Let X_1, \dots, X_n be independent, with each $X_i \sim \text{Bern}(p_i)$. Then

$$D_e(P_{S_n} \parallel \text{Pois}(\lambda)) \leq \sum_{i=1}^n p_i^2$$

Corollary 5.5 Theorem 5.1 follows directly from Theorem 5.3.

Proof. Let P_λ be the PMF of the $\text{Pois}(\lambda)$ distribution. Then by Pinsker's inequality,

$$\|P_{S_n} - P_\lambda\|_{\text{TV}}^2 \leq 2D_e(P_{S_n} \parallel \text{Pois}(\lambda)) \leq 2 \sum_{i=1}^n \frac{\lambda^2}{n^2} = 2 \frac{\lambda^2}{n}.$$

So for each $k \in \mathbb{N}$, $|P_{S_n}(k) - P_\lambda(k)| \leq \|P_{S_n} - P_\lambda\|_{\text{TV}} \leq \sqrt{\frac{2}{n}} \lambda \rightarrow 0$ as $n \rightarrow \infty$. □

Remark 5.6 Theorem 5.3 is stronger than Theorem 5.1 in that it holds for all n rather than being asymptotic. It also provides an easily computable bound on the difference between P_{S_n} and $\text{Pois}(\lambda)$, and does not assume the p_i are equal, or that the RVs X_1, \dots, X_n are independent.

Remark 5.7 It is known that for independent X_1, \dots, X_n , $P_{S_n} \rightarrow \text{Pois}(\lambda)$ iff $\sum_{i=1}^n p_i^2 \rightarrow 0$. So the bound in **Theorem 5.3** is the best possible.

5.2. What is the Poisson distribution?

Lemma 5.8 (Binomial Maximum Entropy) Let $B_n(\lambda)$ be set of distributions on \mathbb{N}_0 that arise from sums $\sum_{i=1}^n X_i$ where $X_i \sim \text{Bern}(p_i)$ are independent and $\sum_{i=1}^n p_i = \lambda$. For all $n \geq \lambda$,

$$H_e(\text{Bin}(n, \lambda/n)) = \sup\{H_e(P) : P \in B_n(\lambda)\}$$

Proof. Exercise. □

Theorem 5.9 (Poisson Maximum Entropy) We have

$$\begin{aligned} & H_e(\text{Pois}(\lambda)) \\ &= \sup \left\{ H_e(S_n) : S_n = \sum_{i=1}^n X_i, X_i \sim \text{Bern}(p_i) \text{ independent} \wedge \sum_{i=1}^n p_i = \lambda, n \geq 1 \right\} \\ &= \sup_{n \in \mathbb{N}} \sup \{ H_{e(P)} : P \in B_n(\lambda) \}. \end{aligned}$$

Proof. Let $H^* = \sup_{n \in \mathbb{N}} \sup \{ H_e(P) : P \in B_n(\lambda) \}$. Note that $B_n(\lambda) \subseteq B_{n+1}(\lambda)$, hence $H^* = \lim_{n \rightarrow \infty} \sup \{ H_{e(P)} : P \in B_n(\lambda) \} = \lim_{n \rightarrow \infty} H_e(\text{Bin}(n, \lambda/n))$.

Let P_n and Q be respective PMFs of $\text{Bin}(n, \lambda/n)$ and $\text{Pois}(\lambda)$. Using that $k! \leq k^k \leq e^{k^2}$, we have

$$\begin{aligned} H_e(Q) &= \sum_{k=0}^{\infty} Q(k) \ln \frac{k!}{e^{-\lambda} \lambda^k} \\ &\leq \sum_{k=0}^{\infty} Q(k) (\lambda - k \ln \lambda + k^2) \\ &= \lambda^2 + 2\lambda - \lambda \ln \lambda < \infty \end{aligned}$$

since $\mathbb{E}[X] = \lambda$ and $\mathbb{E}[X^2] = \lambda + \lambda^2$ for $X \sim \text{Pois}(\lambda)$. So $H_e(Q)$ is finite. The convergence is left as an exercise. □

6. Mutual information

Definition 6.1 The **mutual information** between discrete RVs X and Y is

$$I(X; Y) = H(X) - H(X|Y).$$

The **conditional mutual information** between X and Y given a discrete RV Z is

$$\begin{aligned} I(X; Y | Z) &= H(X | Z) - H(X | Y, Z) \\ &= H(X | Z) + H(Y | Z) - H(X, Y | Z) \\ &= H(Y | Z) - H(Y | X, Z). \end{aligned}$$

Proposition 6.2 Let X and Y be discrete RVs with marginal PMFs P_X and P_Y respectively, and joint PMF $P_{X,Y}$, then the mutual information can be expressed as:

$$\begin{aligned}
I(X; Y) &= H(X) + H(Y) - H(X, Y) \\
&= H(Y) - H(Y | X) \\
&= D(P_{X,Y} \parallel P_X P_Y).
\end{aligned}$$

Proof (Hints). Straightforward. □

Proof. The first two lines are by the chain rule. For the third, we have

$$\begin{aligned}
H(X) + H(Y) - H(X, Y) &= \mathbb{E}[-\log P_X(X)] + \mathbb{E}[-\log P_Y(Y)] - \mathbb{E}[-\log P_{X,Y}(X, Y)] \\
&= \mathbb{E} \left[\log \left(\frac{P_{X,Y}(X, Y)}{P_X(X) P_Y(Y)} \right) \right] \\
&= D(P_{X,Y} \parallel P_X P_Y).
\end{aligned}$$

□

Remark 6.3

- $I(X; Y)$ is symmetric in X and Y .
- The sum of the information contain in X and Y separately minus the information contained in the pair indeed is the amount of mutual information shared by both.
- Considering [Stein's Lemma](#), we can consider $I(X; Y)$ as a measure of how well data generated from $P_{X,Y}$ can be distinguished from independent pairs (X', Y') generated by the product distribution $P_X P_Y$, so is a measure of how far X and Y are from being independent.

Proposition 6.4

- $0 \leq I(X; Y) \leq H(X)$ with equality to 0 iff X and Y are independent.
- Similarly, $I(X; Z | Y) \geq 0$ with equality iff $X - Y - Z$, i.e. X and Z are conditionally independent given Y .

Proof. First is by [Proposition 6.2](#) and non-negativity of conditional entropy, second is an exercise. □

Proposition 6.5 (Chain Rule for Mutual Information) For all discrete RVs X_1, \dots, X_n, Y ,

$$I(X_1^n; Y) = \sum_{i=1}^n I(X_i; Y | X_1^{i-1}).$$

Proof (Hints). Straightforward. □

Proof. By the chain rule for entropy,

$$\begin{aligned}
I(X_1^n; Y) &= H(X_1^n) - H(X_1^n | Y) \\
&= \sum_{i=1}^n H(X_i | X_1^{i-1}) - \sum_{i=1}^n H(X_i | X_1^{i-1}, Y) \\
&= \sum_{i=1}^n (H(X_i | X_1^{i-1}) - H(X_i | X_1^{i-1}, Y)) \\
&= \sum_{i=1}^n I(X_i; Y | X_1^{i-1}).
\end{aligned}$$

□

Theorem 6.6 (Data Processing Inequalities for Mutual Information) If $X - Y - Z$ (so X and Z are conditionally independent given Y), then

$$I(X; Z), I(X; Y | Z) \leq I(X; Y).$$

Proof (Hints). Use chain rule for mutual information twice on the same expression. □

Proof. By the chain rule, we have

$$\begin{aligned}
I(X; Y, Z) &= I(X; Y) + I(X; Z | Y) \\
&= I(X; Z) + I(X; Y | Z).
\end{aligned}$$

Now $I(X; Z | Y) = 0$ by conditional independence, so $I(X; Y) = I(X; Z) + I(X; Y | Z)$. □

Example 6.7 We always have $X - Y - f(Y)$, hence $I(X; f(Y)) \leq I(X; Y)$, so applying a function to Y cannot make X and Y “less independent”.

6.1. Synergy and redundancy

Note 6.8 $I(X; Y_1, Y_2)$ can be greater than, equal to, or less than $I(X; Y_1) + I(X; Y_2)$.

Definition 6.9 The **synergy** of Y_1, Y_2 about X is

$$\begin{aligned}
S(X; Y_1, Y_2) &= I(X; Y_1, Y_2) - (I(X; Y_1) + I(X; Y_2)) \\
&= I(X; Y_2 | Y_1) - I(X; Y_2).
\end{aligned}$$

So the synergy can be < 0 , > 0 or $= 0$.

Definition 6.10 If $S(X; Y_1, Y_2)$ is:

- negative, then Y_1 and Y_2 contain **redundant** information about X ;
- zero, then Y_1 and Y_2 are **orthogonal**;
- positive, then Y_1 and Y_2 are **synergistic**. Intuitively, knowing Y_1 already makes the information in Y_2 more valuable (in that it gives more information about X).

Theorem 6.11 Let RVs Y_1, Y_2 be conditionally independent given X , each with distribution $P_{Y|X}$, and RVs Z_1, Z_2 be distributed according to $Q_{Z|Y}(\cdot | Y_1), Q_{Z|Y}(\cdot | Y_2)$ respectively. Let RV Y have distribution $P_{Y|X}$, and W_1, W_2 be conditionally independent given Y , distributed according to $Q_{Z|Y}(\cdot | Y)$.

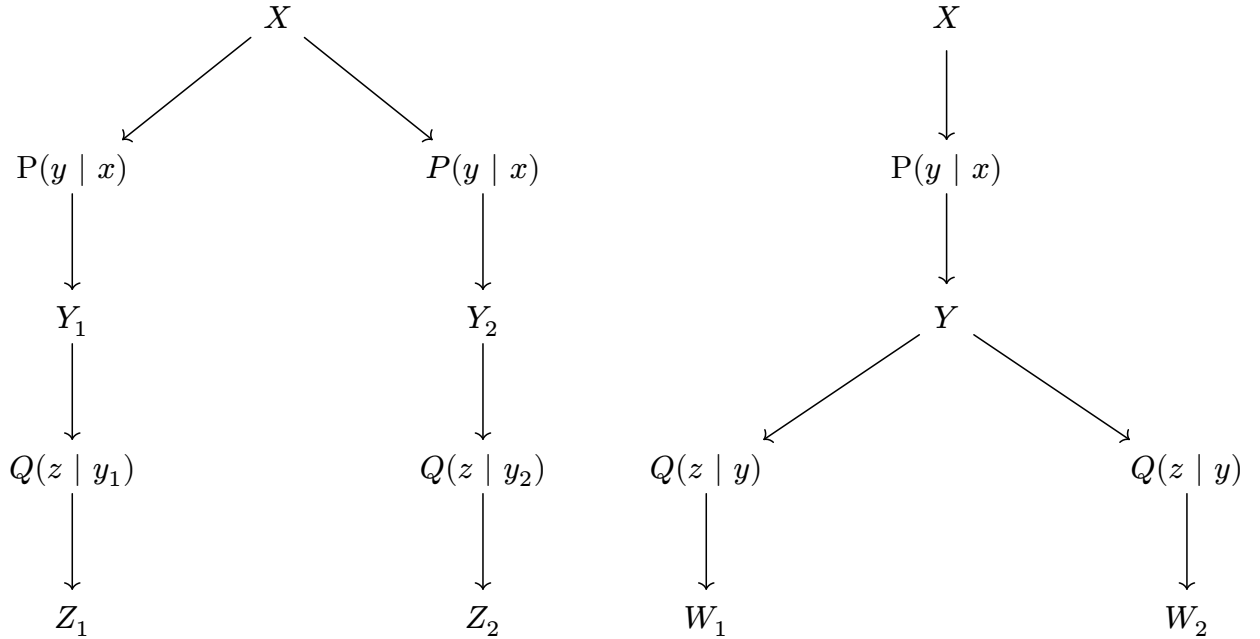
If $S(X; W_1, W_2) > 0$, then $I(X; W_1, W_2) > I(X; Z_1, Z_2)$, for independent Z_1 and Z_2 , i.e. correlated observations are better than independent ones.

Proof (Hints). Use data processing for mutual information. \square

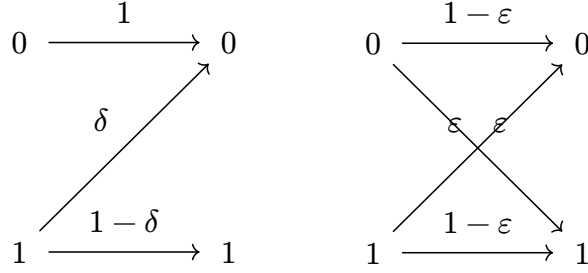
Proof. As in **Definition 6.9**, we have $I(X; W_2 | W_1) > I(X; W_2)$. $I(X; W_2) = I(X; Z_2)$ since (X, W_2) has the same joint distribution as (X, Z_2) . By the data processing inequality, we have $I(X; Z_2 | Z_1) = I(Z_2; X | Z_1) \leq I(Z_2; X) = I(X; Z_2)$, since Z_1 and Z_2 are conditionally independent given X . Hence $I(X; W_2 | W_1) > I(X; Z_2 | Z_1)$, so $I(X; W_2 | W_1) + I(X; W_1) > I(X; Z_2 | Z_1) + I(X; Z_1)$, and the result follows by the chain rule. \square

Example 6.12 Given two equally noisy channels of a signal X , we want to decide whether it is better (gives more information about X) for the channels to be independent (this corresponds with choosing the Y_1, Y_2, Z_1, Z_2) or correlated (this corresponds with choosing the Y, W_1, W_2).

The natural assumption that the conditionally independent observations Z_1, Z_2 would be “better” than W_1, W_2 (i.e. $I(X; Z_1, Z_2) \geq I(X; W_1, W_2)$) is **false**. We can show diagrammatically as



Example 6.13 For example, let $P_{Y|X}$ be the Z -channel: if $X = 0$, then $Y = 0$ with probability 1, and if $X = 1$, then $Y \sim \text{Bern}(1 - \delta)$ for some $\delta \in (0, 1)$. Let $Q_{Z|Y}$ be a binary symmetric channel: given Y taking values in $0, 1$, $Z = Y$ with probability $1 - \varepsilon$, and $Z = 1 - Y$ with probability ε for some $\varepsilon \in (0, 1)$. We can represent this as



If $X \sim \text{Bern}(1/2)$, $\delta = 0.85$ and $\varepsilon = 0.1$, then $I(X; W_1, W_2) \approx 0.047 > I(X; Z_1, Z_2) \approx 0.039$. So the correlated observations W_1, W_2 are better than the independent observations Z_1, Z_2 .

7. Entropy and additive combinatorics

7.1. Simple sumset entropy bounds

Definition 7.1 For $A, B \subseteq \mathbb{Z}$ the **sumset** of A and B is

$$A + B := \{a + b : a \in A, b \in B\}.$$

Definition 7.2 For $A, B \subseteq \mathbb{Z}$ the **difference set** of A and B is

$$A - B := \{a - b : a \in A, b \in B\}.$$

Proposition 7.3 Let $A, B \subseteq \mathbb{Z}$ be finite. Then

$$\max\{|A|, |B|\} \leq |A + B| \leq |A||B|.$$

Proof (Hints). Trivial. □

Proof. Trivial. □

Proposition 7.4 (Ruzsa Triangle Inequality) Let $A, B, C \subseteq \mathbb{Z}$ be finite. Then

$$|A - C| \cdot |B| \leq (|A - B||B - C|).$$

Proof (Hints). Show that an appropriate function is injective. □

Proof. Fix a presentation $y = a_y - c_y$ (where $a_y \in A, c_y \in C$) for each $y \in A - C$. Let

$$\begin{aligned} f : B \times (A - C) &\rightarrow (A - B) \times (B - C) \\ (b, y) &\mapsto (a_y - b, b - c_y). \end{aligned}$$

If $f(b, y) = f(b', y')$, then $a_{y'} - b' = a_y - b$ and $b' - c_{y'} = b - c_y$. So $a_y - a_{y'} = b - b' = c_y - c_{y'}$. So $y = a_y - c_y = a_{y'} - c_{y'} = y'$. Hence $a_y = a_{y'}$, and so $b = b'$. So f is injective, so $|B \times (A - C)| \leq |(A - B) \times (B - C)|$. □

Remark 7.5 If X_1^n is a large collection of IID RVs with common PMF P on alphabet A , then the AEP tells us that we can concentrate on the 2^{nH} typical strings. $2^{nH} = (2^H)^n$ is typically much smaller than all $|A|^n = (2^{\log|A|})^n$ strings. We can think of $(2^H)^n$ as the effective support size of P^n , and can of 2^H as the effective support size of a single RV with entropy H .

Remark 7.6 We can use the above interpretation to obtain useful conjectures about bounds for the entropy of discrete RVs, from corresponding results on bounds on sumsets. We start with a sumset bound, then replace subsets of \mathbb{Z} by independent RVs on \mathbb{Z} , and replace $\log|A|$ of each set A by the entropy of the corresponding RV.

Proposition 7.7 Let X and Y are independent RVs on alphabet \mathbb{Z} , then

$$\max\{H(X), H(Y)\} \leq H(X + Y) \leq H(X) + H(Y).$$

Proof (Hints).

- For lower bound, show that $H(X) \leq H(X + Y)$ using data processing and similarly for $H(Y)$. The upper bound should follow directly from this calculation. □

Proof. For the lower bound,

$$\begin{aligned} H(X) + H(Y) &= H(X, Y) && \text{by Chain Rule for Entropy} \\ &= H(Y, X + Y) && \text{by Data Processing} \\ &= H(X + Y) + H(Y \mid X + Y) && \text{by Chain Rule for Entropy} \\ &\leq H(X + Y) + H(Y) && \text{by Conditioning Reduces Entropy.} \end{aligned}$$

Note we have equality for data processing, since $(x, y) \mapsto (x, x + y)$ is injective. Hence $H(X + Y) \geq H(X)$, and the same argument shows that $H(X + Y) \geq H(Y)$.

For the upper bound, we have $H(X) + H(Y) = H(X + Y) + H(Y \mid X + Y) \geq H(X + Y)$ by non-negativity of conditional entropy. □

Lemma 7.8 Let X, Y, Z be independent RVs on alphabet \mathbb{Z} . Then

$$H(X - Z) + H(Y) \leq H(X - Y, Y - Z).$$

Proof (Hints).

- Show that $I(X; X - Z) \leq I(X; (X - Y, Y - Z))$.
- Rewrite both sides of the above inequality in terms of entropies, using **Data Processing**. □

Proof. Since $X - Z = (X - Y) + (Y - Z)$, X and $X - Z$ are conditionally independent given $(X - Y, Y - Z)$ by **Note 4.10**. Thus by **Data Processing** for mutual information, we have $I(X; (X - Y, Y - Z)) \geq I(X; X - Z)$. Now

$$\begin{aligned} I(X; X - Z) &= H(X - Z) - H(X - Z \mid X) \\ &= H(X - Z) - H(Z \mid X) = H(X - Z) - H(Z) \end{aligned}$$

by **Data Processing** (since, given $X = x$, $x - z \mapsto z$ is injective), and independence of X and Z . Also,

$$\begin{aligned}
I(X; (X - Y, Y - Z)) &= H(X - Y, Y - Z) + H(X) - H(X, X - Y, Y - Z) \\
&= H(X - Y, Y - Z) + H(X) - H(X, Y, Z) \\
&= H(X - Y, Y - Z) - H(Y) - H(Z)
\end{aligned}$$

by **Data Processing** (since $(x, x - y, y - z) \mapsto (x, y, z)$ is injective), and independence of X, Y and Z . \square

Theorem 7.9 (Ruzsa Triangle Inequality for Entropy) Let X, Y, Z be independent RVs on alphabet \mathbb{Z} . Then

$$H(X - Z) + H(Y) \leq H(X - Y) + H(Y - Z).$$

Proof (Hints). By above lemma. \square

Proof. By the above lemma, we have

$$\begin{aligned}
H(X - Z) + H(Y) &\leq H(X - Y, Y - Z) \\
&= H(X - Y) + H(Y - Z \mid X - Y) \quad \text{by Chain Rule for Entropy} \\
&\leq H(X - Y) + H(Y - Z).
\end{aligned}$$

by **Conditioning Reduces Entropy**. \square

7.2. The doubling-difference inequality for entropy

Definition 7.10 For IID RVs X_1, X_2 on alphabet \mathbb{Z} , the **entropy-increase** due to addition (Δ^+) or subtraction (Δ^-) is

$$\begin{aligned}
\Delta^+ &:= H(X_1 + X_2) - H(X_1), \\
\Delta^- &:= H(X_1 - X_2) - H(X_1).
\end{aligned}$$

Proposition 7.11 For IID X_1, X_2 on \mathbb{Z} , we have

$$\begin{aligned}
\Delta^+ &= I(X_1 + X_2; X_2), \\
\Delta^- &= I(X_1 - X_2; X_2).
\end{aligned}$$

Proof (Hints). Straightforward. \square

Proof. We have

$$\begin{aligned}
I(X_1 + X_2; X_2) &= H(X_1 + X_2) + H(X_2) - H(X_1 + X_2, X_2) \\
&= H(X_1 + X_2) + H(X_2) - H(X_1, X_2) \\
&= H(X_1 + X_2) + H(X_2) - H(X_1) - H(X_2)
\end{aligned}$$

by **Data Processing** (since $(x_1 + x_2, x_2) \mapsto (x_1, x_2)$ is injective) and **Chain Rule for Entropy**. The proof is identical for Δ^- . \square

Lemma 7.12 Let X, Y, Z be independent RVs on alphabet \mathbb{Z} . Then

$$H(X + Y + Z) + H(Y) \leq H(X + Y) + H(Y + Z).$$

Proof (Hints).

- Show that $I(X; X + Y + Z) \leq I(X + Y; X)$.
- Rewrite both sides in terms of entropies.

□

Proof. Since $X - (X + Y, Z) - (X + Y + Z)$ form a Markov chain by [Note 4.10](#), we have, by [Data Processing](#) and [Chain Rule](#) for mutual information,

$$\begin{aligned} I(X; X + Y + Z) &\leq I(X + Y, Z; X) = I(X + Y; X) + I(Z; X \mid X + Y). \\ &= I(X + Y; X) \end{aligned}$$

since Z is (conditionally) independent of X given $X + Y$. Now

$$\begin{aligned} I(X + Y; X) &= H(X + Y) + H(X) - H(X + Y, X) \\ &= H(X + Y) + H(X) - H(Y, X) \\ &= H(X + Y) + H(X) - H(Y) - H(X) \\ &= H(X + Y) - H(Y) \end{aligned}$$

since $(y, x) \mapsto (x + y, x)$ is injective and X and Y are independent. Also,

$$\begin{aligned} I(X + Y + Z; X) &= H(X + Y + Z) + H(X + Y + Z \mid X) \\ &= H(X + Y + Z) - H(Y + Z \mid X) \\ &= H(X + Y + Z) - H(Y + Z) \end{aligned}$$

since, given $X = x$, $x + y + z \mapsto y + z$ is injective, and X and $Y + Z$ are independent.

□

Theorem 7.13 (Doubling-difference Inequality) Let X_1 and X_2 be IID RVs on \mathbb{Z} . Then

$$\frac{1}{2} \leq \frac{\Delta^+}{\Delta^-} \leq 2.$$

Proof (Hints).

- For lower bound, use [Ruzsa Triangle Inequality](#) for appropriate RVs.
- For upper bound,

□

Proof. For the lower bound, let $X, -Y, Z$ be IID with the same distribution as X_1 . Then by the [Ruzsa Triangle Inequality](#),

$$H(X_1 - X_2) + H(X_1) \leq H(X_1 + X_2) + H(X_1 + X_2).$$

So $2(H(X_1 + X_2) - H(X_1)) \geq H(X_1 - X_2) - H(X_1)$.

For the upper bound, let $X, -Y, Z$ be IID with the same distribution as X_1 . Then by the above lemma and [Proposition 7.7](#),

$$H(X_1 + X_2) + H(X_1) \leq H(X_1 - X_2) + H(X_1 - X_2)$$

so $H(X_1 + X_2) - H(X_1) \leq 2(H(X_1 - X_2) - H(X_1))$.

□

8. Entropy rate

Definition 8.1 For an arbitrary source $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$, the **entropy rate** $H(\mathbf{X})$ of \mathbf{X} is the limit of the average number of bits per symbol:

$$H(\mathbf{X}) = \lim_{n \rightarrow \infty} \frac{1}{n} H(X_1^n)$$

whenever the limit exists.

Example 8.2 If \mathbf{X} is memoryless (so a sequence of IID RVs) with common entropy $H = H(X_i)$, then the entropy rate is

$$H(\mathbf{X}) = \lim_{n \rightarrow \infty} \frac{1}{n} H(X_1^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n H(X_i) = H.$$

Example 8.3 Let $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ be an irreducible, aperiodic Markov chain on a finite alphabet A with transition matrix Q , where

$$Q_{ab} = \mathbb{P}(X_{n+1} = b \mid X_n = a), \quad \forall a, b \in A$$

Let $X_1 \sim P_{X_1}$ be the initial distribution and π be the unique stationary distribution ($\mathbb{P}(X_n = x) \rightarrow \pi(x)$ as $n \rightarrow \infty$). \mathbf{X} has a unique invariant distribution π to which it converges:

$$\forall x \in A, \quad \mathbb{P}(X_n = x) \rightarrow \pi(x) \quad \text{as } n \rightarrow \infty$$

and hence also

$$\mathbb{P}(X_{n-1} = x, X_n = y) = \mathbb{P}(X_n = x)Q_{xy} \rightarrow \pi(x)Q_{xy}.$$

Then by the **Chain Rule for Entropy** and conditional independence,

$$\begin{aligned} H(X_1^n) &= \sum_{i=1}^n H(X_i \mid X_1^{i-1}) \\ &= H(X_1) + \sum_{i=2}^n H(X_i \mid X_{i-1}) \\ &= H(X_1) - H(X_{n+1} \mid X_n) + \sum_{i=1}^n H(X_{i+1} \mid X_i). \end{aligned}$$

By the convergence theorem for Markov chains, we have $P_{X_n} \rightarrow \pi$ as $n \rightarrow \infty$.

$H(X \mid Y)$ is a continuous function of the joint distribution $P_{X,Y}$, so $H(X_n \mid X_{n-1}) \rightarrow H(\overline{X}_1 \mid \overline{X}_0)$ as $n \rightarrow \infty$, where $\overline{X}_0 \sim \pi$ and $\mathbb{P}(\overline{X}_1 = b \mid \overline{X}_1 = a) = Q_{ab}$. We have

$$\frac{1}{n} H(X_1^n) = \frac{1}{n} (H(X_1) - H(X_{n+1} \mid X_n)) + \frac{1}{n} \sum_{i=1}^n H(X_{i+1} \mid X_i)$$

The first term tends to 0 since the numerator is bounded, and the summands in the second term tend to $H(\overline{X}_1 \mid \overline{X}_0)$. So the entropy rate exists and is equal to $H(\mathbf{X}) = H(\overline{X}_1 \mid \overline{X}_0)$.

Definition 8.4 A source \mathbf{X} is **stationary** if for any block length $n \in \mathbb{N}$, the distribution of X_{k+1}^{k+n} is independent of k .

Remark 8.5 If $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ is one-sided stationary process, then by Kolmogorov's extension theorem, \mathbf{X} admits a unique two-sided extension to $\mathbf{X} = \{X_n : n \in \mathbb{Z}\}$.

Theorem 8.6 If $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ is a stationary process on finite alphabet A , then its entropy rate exists and is equal to

$$H(\mathbf{X}) = \lim_{n \rightarrow \infty} H(X_n | X_1^{n-1}).$$

Proof (Hints). Show that the sequence $\{H(X_n | X_1^{n-1}) : n \in \mathbb{N}\}$ is non-increasing and use the Cèsaro Lemma. \square

Proof. The sequence $\{H(X_n | X_1^{n-1}) : n \in \mathbb{N}\}$ is non-negative by non-negativity of conditional entropy, and is non-increasing, since

$$\begin{aligned} H(X_{n+1} | X_1^n) &\leq H(X_{n+1} | X_2^n) && \text{by Conditioning Reduces Entropy} \\ &= H(X_2^{n+1}) - H(X_2^n) && \text{by Chain Rule for Entropy} \\ &= H(X_1^n) - H(X_1^{n-1}) && \text{by stationarity} \\ &= H(X_{n-1} | X_1^{n-2}) && \text{by Chain Rule for Entropy.} \end{aligned}$$

Hence the limit $\lim_{n \rightarrow \infty} H(X_n | X_1^{n-1})$ exists, and so by the Cèsaro Lemma, the averages converge to the same limit. But by the **Chain Rule for Entropy**, the averages are

$$\frac{1}{n} \sum_{i=1}^n H(X_i | X_1^{i-1}) = \frac{1}{n} H(X_1^n).$$

\square

Theorem 8.7 For a stationary process $\mathbf{X} = \{X_n : n \in \mathbb{Z}\}$ on a finite alphabet A ,

$$H(\mathbf{X}) = H(X_0 | X_{-n}^{-1}) = H(X_0 | X_{-\infty}^{-1}).$$

Proof (Hints). Non-examinable. \square

Proof. By Martingale convergence, we have that

$$P(x_0 | X_{-n}^{-1}) \rightarrow P(x_0 | X_{-\infty}^{-1}) \quad \text{almost surely as } n \rightarrow \infty,$$

where $P(\cdot | x_{-n}^{-1})$ is the conditional distribution of X_0 given $X_{-n}^{-1} = x_{-n}^{-1}$, and $P(\cdot | x_{-\infty}^{-1})$ is the conditional distribution of X_0 given $X_{-\infty}^{-1} = x_{-\infty}^{-1}$. Now, we can take expectations to obtain that, by the bounded convergence theorem (since $p \mapsto p \log p$ is continuous and bounded for $p \in [0, 1]$),

$$\begin{aligned}
H(X_0 | X_{-n}^{-1}) &= \mathbb{E} \left[- \sum_{x_0 \in A} P(x_0 | X_{-n}^{-1}) \log P(x_0 | X_{-n}^{-1}) \right] \\
&\rightarrow \mathbb{E} \left[- \sum_{x_0 \in A} P(x_0 | X_{-\infty}^{-1}) \log P(x_0 | X_{-\infty}^{-1}) \right] \\
&=: H(X_0 | X_{-\infty}^{-1}) \quad \text{almost surely as } n \rightarrow \infty.
\end{aligned}$$

Finally, $H(X_0 | X_{-n}^{-1}) = H(X_{n+1} | X_1^n)$ by stationarity, so we are done by **Theorem 8.6**. □

Definition 8.8 Let $\mathbf{X} = \{X_n : n \in \mathbb{Z}\}$ be a stationary source on finite alphabet A , and define the (left) **shift** operator $T : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ on sequences $A^{\mathbb{Z}}$ by

$$(Tx)_n = x_{n+1} \quad \forall n \in \mathbb{Z}.$$

\mathbf{X} is **ergodic** if all shift invariant events are trivial, i.e. for any measurable $B \subseteq A^{\mathbb{Z}}$, we have

$$T^{-1}B = B \implies \mathbb{P}(X_{-\infty}^{\infty} \in B) = 0 \text{ or } 1.$$

Intuitively, an ergodic process is one which satisfies the general form of the strong law of large numbers.

It turns out that ergodicity is equivalent to the validity of the following:

Theorem 8.9 (Birkhoff's Ergodic Theorem) Let $\mathbf{X} = \{X_n : n \in \mathbb{Z}\}$ be a stationary ergodic source on alphabet A . Then for any measurable function $f : A^{\mathbb{Z}} \rightarrow \mathbb{R}$ such that

$$\mathbb{E}[|f(X_{-\infty}^{\infty})|] < \infty,$$

we have

$$\frac{1}{n} \sum_{i=1}^n f(T^i X_{-\infty}^{\infty}) \rightarrow \mathbb{E}[f(X_{-\infty}^{\infty})] \quad \text{almost surely as } n \rightarrow \infty$$

Proof (Hints). Beyond the scope of this course. □

Proof. Omitted. □

Remark 8.10 The strong law of large numbers follows instantly from Birkhoff by setting $f(x_{-\infty}^{\infty}) = x_1$.

Example 8.11 Every IID source is ergodic.

Theorem 8.12 (Shannon-McMillan-Breiman) Let $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ be a stationary ergodic source on alphabet A with entropy rate $H = H(\mathbf{X})$, then

$$-\frac{1}{n} \log P_n(X_1^n) \rightarrow H \quad \text{almost surely as } n \rightarrow \infty$$

where P_n is the PMF of X_1^n .

Proof (Hints). Non-examinable. □

Proof. Idea: by **Chain Rule for Entropy**, we have

$$-\frac{1}{n} \log P_n(X_1^n) = -\frac{1}{n} \log \prod_{i=1}^n P(X_i | X_1^{i-1}) = \frac{1}{n} \sum_{i=1}^n [-\log P(X_i | X_1^{i-1})]$$

but we cannot directly apply the ergodic theorem to this, since $-\log P(X_i | X_1^{i-1})$ is not of the form $f(T^i x_\infty^\infty)$. Instead, note that by **Birkhoff's Ergodic Theorem** and **Theorem 8.7**,

$$\begin{aligned} -\frac{1}{n} \log P(X_1^n | X_{-\infty}^0) &= \frac{1}{n} \sum_{i=1}^n [-\log P(X_i | X_{-\infty}^{i-1})] \\ &\rightarrow \mathbb{E}[-\log P(X_0 | X_{-\infty}^{-1})] \\ &=: H(X_0 | X_{-\infty}^{-1}) = H \text{ almost surely as } n \rightarrow \infty. \end{aligned}$$

Also, by **Birkhoff's Ergodic Theorem**, for each fixed $k \geq 1$,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n (-\log P(X_i | X_{i-k}^{i-1})) &\rightarrow \mathbb{E}[-\log P(X_0 | X_{-k}^{-1})] \\ &=: H(X_0 | X_{-k}^{-1}) \text{ almost surely as } n \rightarrow \infty. \end{aligned}$$

We have

$$\begin{aligned} &\mathbb{P}\left(-\frac{1}{n} \log P(X_1^n | X_{-\infty}^0) - \left(-\frac{1}{n} \log P_n(X_1^n)\right) > \varepsilon\right) = \mathbb{P}\left(\frac{1}{n} \log \frac{P_n(X_1^n)}{P(X_1^n | X_{-\infty}^0)} > \varepsilon\right) \\ &= \mathbb{P}\left(\frac{P_n(X_1^n)}{P(X_1^n | X_{-\infty}^0)} > 2^{n\varepsilon}\right) \\ &\leq 2^{-n\varepsilon} \mathbb{E}\left[\frac{P_n(X_1^n)}{P(X_1^n | X_{-\infty}^0)}\right] \text{ by markov's inequality} \\ &\leq 2^{-n\varepsilon} \mathbb{E}\left[\mathbb{E}\left[\frac{P_n(X_1^n)}{P(X_1^n | X_{-\infty}^0)} \mid X_{-\infty}^0\right]\right] \\ &= 2^{-n\varepsilon} \mathbb{E}\left[\sum_{\substack{x_1^n \\ P(x_1^n | X_{-\infty}^0) > 0}} P(x_1^n | X_{-\infty}^0) \frac{P_n(x_1^n)}{P(x_1^n | X_{-\infty}^0)}\right] \\ &\leq 2^{-n\varepsilon} \end{aligned}$$

which is summable, so by Borel-Cantelli,

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log P(X_1^n | X_{-\infty}^0) \leq \liminf_{n \rightarrow \infty} -\frac{1}{n} \log P_n(X_1^n) \text{ almost surely.}$$

For each fixed k , consider the sequence of PMFs $Q_n^{(k)}(x_1^n) = P_k(x_1^k) \prod_{i=k+1}^n P(x_i | X_{i-k}^{i-1})$ for $x_1^n \in A^n$. Then

$$\begin{aligned}
& -\frac{1}{n} \log Q_n^{(k)}(X_1^n) - \left[-\frac{1}{n} \sum_{i=1}^n \log P(x_i \mid x_{i-k}^{i-1}) \right] \\
& = -\frac{1}{n} \left[\log P_k(x_1^k) - \sum_{i=1}^k \log P(X_i \mid X_{i-k}^{i-1}) \right] \\
& \rightarrow 0 \text{ almost surely as } n \rightarrow \infty
\end{aligned}$$

So suffices to show that $\limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_n(X_1^n) \leq \limsup_{n \rightarrow \infty} -\frac{1}{n} \log Q_n^{(k)}(X_1^n)$ almost surely. So again, let $\varepsilon > 0$ be arbitrary, then

$$\begin{aligned}
& \mathbb{P} \left(-\frac{1}{n} \log P_n(X_1^n) - \left(-\frac{1}{n} \log Q_n^{(k)}(X_1^n) \right) > \varepsilon \right) \\
& = \mathbb{P} \left(\frac{Q_n^{(k)}(X_1^n)}{P_n(X_1^n)} > 2^{n\varepsilon} \right) \leq 2^{-n\varepsilon} \mathbb{E} \left[\frac{Q_n^{(k)}(X_1^n)}{P_n(X_1^n)} \right] \text{ by Markov's inequality} \\
& \leq 2^{-n\varepsilon} \sum_{x_1^n \in A^n} P_n(x_1^n) \frac{Q_n^{(k)}(x_1^n)}{P_n(x_1^n)} = 2^{-n\varepsilon}
\end{aligned}$$

which is summable, so by Borel-Cantelli and the fact that $\varepsilon > 0$ was arbitrary, we have

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_n(X_1^n) \leq \limsup_{n \rightarrow \infty} -\frac{1}{n} \sum_{i=1}^n \log P(X_i \mid X_{i-k}^{i-1}).$$

□

9. Types and large deviations

9.1. The method of types

Definition 9.1 Let A be a finite alphabet and $x_1^n \in A^n$. The **type** of x_1^n is its empirical distribution $\hat{P}_n = \hat{P}_{x_1^n}$:

$$\hat{P}_n(a) = \hat{P}_{x_1^n}(a) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x_i=a\}}.$$

Notation 9.2 For a finite alphabet $A = \{a_1, \dots, a_m\}$, let \mathcal{P} denote the set of all PMFs on A :

$$\mathcal{P} = \left\{ P \in [0, 1]^m : \sum_{a \in A} P(a) = 1 \right\}.$$

Note that \mathcal{P} is an m -simplex.

Notation 9.3 We write \mathcal{P}_n for the set of all **n -types**:

$$\mathcal{P}_n = \{ P \in \mathcal{P} : nP(a) \in \mathbb{Z} \forall a \in A \}.$$

Note that \mathcal{P}_n is finite.

Proposition 9.4 We have $|\mathcal{P}_n| \leq (n+1)^m$.

Proof (Hints). Straightforward. □

Proof. Each $P \in \mathcal{P}_n$ is of the form $(k_1/n, \dots, k_m/n)$. There are at most $(n+1)$ choices $(0, \dots, n)$ for each k_i . □

Proposition 9.5 Let $x_1^n \in A^n$ have type \hat{P}_n . Then for any PMF Q ,

$$Q^n(x_1^n) = 2^{-n(H(\hat{P}_n) + D(\hat{P}_n \parallel Q))}.$$

In particular, if $Q = \hat{P}_n$, then $Q^n(x_1^n) = 2^{-nH(Q)}$.

Proof (Hints). Rewrite $\log Q^n(x_1^n)$. □

Proof. We have

$$\begin{aligned} \log Q^n(x_1^n) &= \sum_{i=1}^n \log Q(x_i) \\ &= \sum_{i=1}^n \sum_{a \in A} \mathbb{1}_{\{x_i=a\}} \log Q(a) \\ &= n \sum_{a \in A} \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x_i=a\}} \log Q(a) \\ &= n \sum_{a \in A} \hat{P}_n(a) \log Q(a) = - \sum_{a \in A} \hat{P}_n(a) \log \left(\frac{\hat{P}_n(a)}{Q(a)} \frac{1}{\hat{P}_n(a)} \right) \\ &= -n \left(\sum_{a \in A} \hat{P}_n(a) \log \frac{\hat{P}_n(a)}{Q(a)} + \sum_{a \in A} \hat{P}_n(a) \log \frac{1}{\hat{P}_n(a)} \right) \\ &= -n(D(\hat{P}_n \parallel Q) + H(\hat{P}_n)) \end{aligned}$$

□

Definition 9.6 Given a n -type P , its **type class** is

$$T(P) := \{x_1^n \in A^n : \hat{P}_{x_1^n} = P\}.$$

Note that $A^n = \coprod_{P \in \mathcal{P}_n} T(P)$: since A^n has size $|A|^n$ exponential in n , and the union is over $|\mathcal{P}_n| \leq (n+1)^m$ (polynomial in n) elements, at least one type class must contain exponentially many strings.

$T(P)$ consists of all possible arrangements of $nP(a_1)$ a_1 's, ..., $nP(a_m)$ a_m 's, so

$$|T(P)| = \frac{n!}{\prod_{j=1}^m (nP(a_j))!}.$$

Lemma 9.7 Let $P \in \mathcal{P}_n$. Then

$$P^n(T(P)) = \max\{P^n(T(Q)) : Q \in \mathcal{P}_n\}.$$

i.e. the most likely type class under P^n is $T(P)$.

Proof (Hints).

- For $Q \in \mathcal{P}_n$, find an expression for $P^n(x_1^n)$ which should be independent of x_1^n , for each case $x_1^n \in T(P)$ and $x_1^n \in T(Q)$.
- Show that $\frac{P^n(T(P))}{P^n(T(Q))} \geq 1$, using the fact that $k!/\ell! \geq \ell^{k-\ell}$ (why?).

□

Proof. Let $Q \in \mathcal{P}_n$ be arbitrary. Then

$$\begin{aligned} \frac{P^n(T(P))}{P^n(T(Q))} &= \frac{|T(P)| \cdot \prod_{i=1}^m P(a_i)^{nP(a_i)}}{|T(Q)| \cdot \prod_{i=1}^m P(a_i)^{nQ(a_i)}} \\ &= \frac{n!}{\prod_{i=1}^m (nP(a_i))!} \cdot \frac{\prod_{i=1}^m (nQ(a_i))!}{n!} \cdot \prod_{i=1}^m P(a_i)^{n(P(a_i)-Q(a_i))} \\ &= \prod_{i=1}^m P(a_i)^{n(P(a_i)-Q(a_i))} \cdot \prod_{i=1}^m \frac{(nQ(a_i))!}{(nP(a_i))!}. \end{aligned}$$

Now since $k!/\ell! \geq \ell^{k-\ell}$ (to show this, consider $k \geq \ell$ and $k < \ell$ cases separately), this is

$$\begin{aligned} &\geq \prod_{i=1}^m P(a_i)^{n(P(a_i)-Q(a_i))} \cdot \prod_{i=1}^m (nP(a_i))^{n(Q(a_i)-P(a_i))} \\ &= \prod_{i=1}^m n^{n(Q(a_i)-P(a_i))} \\ &= n^{n \sum_{i=1}^m (Q(a_i)-P(a_i))} = 1 \end{aligned}$$

since probabilities sum to 1.

□

Proposition 9.8 Let $|A| = m$. For any n -type $P \in \mathcal{P}_n$,

$$(n+1)^{-m} 2^{nH(P)} \leq |T(P)| \leq 2^{H(P)}.$$

Proof (Hints). Straightforward.

□

Proof. By [Proposition 9.5](#), we have $1 \geq P^n(T(P)) = |T(P)| 2^{-nH(P)}$. For the lower bound,

$$\begin{aligned} 1 &= \sum_{x_1^n \in A^n} P^n(x_1^n) \\ &= \sum_{Q \in \mathcal{P}_n} P^n(T(Q)) \\ &\leq |\mathcal{P}_n| P^n(T(P)) && \text{by [Lemma 9.7](#)} \\ &\leq (n+1)^m |T(P)| 2^{-nH(P)}. \end{aligned}$$

□

Corollary 9.9 For any n -type $P \in \mathcal{P}_n$ and any PMF Q on A ,

$$(n+1)^{-m} 2^{-nD(P \parallel Q)} \leq Q^n(T(P)) \leq 2^{-nD(P \parallel Q)}.$$

Proof (Hints). Straightforward. □

Proof. Let $x_1^n \in T(P)$ be arbitrary. Then by [Proposition 9.5](#),

$$Q^n(T(P)) = |T(P)| Q^n(x_1^n) = |T(P)| 2^{-n(H(P) + D(P \parallel Q))}.$$

So we are done by [Proposition 9.8](#). □

9.2. Sanov's theorem

Theorem 9.10 (Sanov) Let X_1^n be IID with common PMF Q which has full support on alphabet A (i.e. $Q(a) > 0$ for all $a \in A$) with $|A| = m$. Let \hat{P}_n be the empirical distribution of X_1^n . For all $E \subseteq \mathcal{P}$,

$$\mathbb{P}(\hat{P}_n \in E) \leq (n+1)^m 2^{-nD^*}.$$

where $D^* = \inf\{D(P \parallel Q) : P \in E\}$. Also, if $E = \overline{\text{int}(E)}$ is equal to the closure of its interior, then

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\hat{P}_n \in E) = D^* = D(P^* \parallel Q),$$

where $P^* \in E$.

Proof (Hints).

- For the inequality, use that $\mathbb{P}(\hat{P}_n \in E) = \mathbb{P}(\hat{P}_n \in E \cap \mathcal{P}_n) = \sum_{P \in E \cap \mathcal{P}_n} Q^n(T(P))$. Explain why D^* is finite.
- For the equality, use the above inequality, and explain why there is a sequence $\{P_n : n \in \mathbb{N}\}$ with each $P_n \in \mathcal{P}_n$ and $P_n \rightarrow P^*$ where $D(P^* \parallel Q) = D^*$ (why does P^* exist?)

□

Proof. Since Q has full support, for any $P \in \mathcal{P}$, we have $D(P \parallel Q) \leq -\sum_{a \in A} \log Q(a) < \infty$, so D^* is finite. For the upper bound,

$$\begin{aligned} \mathbb{P}(\hat{P}_n \in E) &= \mathbb{P}(\hat{P}_n \in E \cap \mathcal{P}_n) \\ &= \sum_{P \in E \cap \mathcal{P}_n} \mathbb{P}(\hat{P}_n = P) \\ &= \sum_{P \in E \cap \mathcal{P}_n} \mathbb{P}(X_1^n \in T(P)) \\ &= \sum_{P \in E \cap \mathcal{P}_n} Q^n(T(P)) \\ &\leq |E \cap \mathcal{P}_n| \max\{Q^n(T(P)) : P \in E \cap \mathcal{P}_n\} \\ &\leq |E \cap \mathcal{P}_n| \max\{2^{-nD(P \parallel Q)} : P \in E \cap \mathcal{P}_n\} \quad \text{by [Corollary 9.9](#)} \\ &= |E \cap \mathcal{P}_n| \cdot 2^{-n \min\{D(P \parallel Q) : P \in E \cap \mathcal{P}_n\}} \\ &\leq (n+1)^m \cdot 2^{-nD^*}. \end{aligned}$$

So $\liminf_{n \rightarrow \infty} -\frac{1}{n} \log Q^n(\hat{P}_n \in E) \geq D^*$.

For the lower bound, since E is compact and $D(P \parallel Q)$ is continuous in P , the infimum D^* is attained by some P^* . (Note that since \mathcal{P} itself is compact, there is always a minimising P^* but this is not necessarily in E). Also, note that $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ is dense in \mathcal{P} , so we can find a sequence $\{P_n : n \in \mathbb{N}\} \subseteq E$ such that each $P_n \in \mathcal{P}_n$ and $P_n \rightarrow P^*$ (as a vector). Now for each $n \in \mathbb{N}$,

$$\mathbb{P}(\hat{P}_n \in E) \geq \mathbb{P}(\hat{P}_n = P_n) = Q^n(T(P_n)) \geq (n+1)^{-m} 2^{-nD(P_n \parallel Q)}$$

by [Corollary 9.9](#). We have $D(P_n \parallel Q) \rightarrow D(P^* \parallel Q)$ as $n \rightarrow \infty$ since $D(P \parallel Q)$ is continuous in P . So $\limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\hat{P}_n \in E) \leq D(P^* \parallel Q) = D^*$. \square

Definition 9.11 For a random variable Y , the **log-moment generating function** of Y is $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\Lambda(\lambda) := \ln \mathbb{E}[e^{\lambda Y}].$$

Notation 9.12 Write $\Lambda^*(x) = \sup\{\lambda x - \Lambda(\lambda) : \lambda > 0\}$.

Proposition 9.13 (Chernoff Bound) Let X_1^n be IID RVs and $f : A \rightarrow \mathbb{R}$ have mean $\mu = \mathbb{E}[f(X_1)]$. Denote the empirical averages by $S_n := \frac{1}{n} \sum_{i=1}^n f(X_i)$. Then for all $\varepsilon > 0$,

$$\mathbb{P}(S_n \geq \mu + \varepsilon) \leq e^{-n\Lambda^*(\mu+\varepsilon)},$$

where Λ is the log-moment generating function of the $f(X_i)$.

Proof (Hints). Use Markov's inequality. \square

Proof. By Markov's inequality, for all $\lambda > 0$,

$$\mathbb{P}(S_n \geq \mu + \varepsilon) = \mathbb{P}(e^{n\lambda S_n} \geq e^{n\lambda(\mu+\varepsilon)}) \leq e^{-n\lambda(\mu+\varepsilon)} \mathbb{E}[e^{\lambda n S_n}].$$

Now since the X_i are independent,

$$\mathbb{E}[e^{\lambda n S_n}] = \mathbb{E}[e^{\lambda \sum_{i=1}^n f(X_i)}] = \mathbb{E}\left[\prod_{i=1}^n e^{\lambda f(X_i)}\right] = \prod_{i=1}^n \mathbb{E}[e^{\lambda f(X_i)}] = e^{n\Lambda(\lambda)}.$$

Hence,

$$\mathbb{P}(S_n \geq \mu + \varepsilon) \leq e^{-n\lambda(\mu+\varepsilon)} e^{n\Lambda(\lambda)} = e^{-n(\lambda(\mu+\varepsilon) - \Lambda(\lambda))},$$

and this holds for all $\lambda > 0$, so taking the supremum over λ gives the result. \square

Example 9.14 Let X_1^n be IID with common PMF Q on finite alphabet A , let $f : A \rightarrow \mathbb{R}$ with mean $\mu = \mathbb{E}_{X \sim Q}[f(X)]$. Denote the empirical averages by $S_n := \frac{1}{n} \sum_{i=1}^n f(X_i)$. Let $\varepsilon > 0$. By WLLN, $\mathbb{P}(S_n \geq \mu + \varepsilon) \rightarrow 0$ as $n \rightarrow \infty$. We want to estimate how small this probability is as a function of n . Typically, the way we bound $\mathbb{P}(S_n \geq \mu + \varepsilon)$ is by the [Chernoff Bound](#). Alternatively, we have

$$S_n = \frac{1}{n} \sum_{i=1}^n f(X_i) = \frac{1}{n} \sum_{i=1}^n \sum_{a \in A} \mathbb{1}_{\{X_i=a\}} f(a) = \sum_{a \in A} \hat{P}_n(a) f(a) = \mathbb{E}_{X \sim \hat{P}_n} [f(X)].$$

Let B be the event $B = \{S_n \geq \mu + \varepsilon\}$, then we can write B as $\{\hat{P}_n \in E\}$ where $E = \{P \in \mathcal{P} : \mathbb{E}_{X \sim P} [f(X)] \geq \mu + \varepsilon\}$. But **Sanov** says that $\mathbb{P}(S_n \geq \mu + \varepsilon) = \mathbb{P}(\hat{P}_n \in E) \leq (n+1)^m e^{-n D_e(P^* \parallel Q)}$ and in fact it tells us that $D_e(P^* \parallel Q) = \inf\{D_e(P \parallel Q) : P \in E\}$ is asymptotically the “correct” exponent.

Proposition 9.15 Let X_1^n be IID RVs with common PMF Q on alphabet A and $f : A \rightarrow \mathbb{R}$ have mean $\mu = \mathbb{E}[f(X_1)]$. Let P^* be the minimiser in **Sanov** for the event $E = \{P \in \mathcal{P} : \mathbb{E}_{X \sim P} [f(X)] \geq \mu + \varepsilon\}$. Then

$$\forall \varepsilon > 0, \quad \Lambda^*(\mu + \varepsilon) = D_e(P^* \parallel Q),$$

where Λ is the log-moment generating function of the X_i .

Proof (Hints).

- \leq : show that $S_n = \mathbb{E}_{X \sim \hat{P}_n} [f(X)]$, then use the **Chernoff Bound** and **Sanov**.
- \geq : for each $\lambda \geq 0$, define a PMF on A by

$$P_\lambda(a) = \frac{e^{\lambda f(a)}}{\mathbb{E}[e^{\lambda f(X_1)}]} Q(a).$$

- Show that $\Lambda'(\lambda) = \mathbb{E}_{Y \sim P_\lambda} [f(Y)]$ and $\Lambda''(\lambda) \geq 0$.
- Deduce that there exists $\lambda^* > 0$ such that $\Lambda'(\lambda^*) = \mu + \varepsilon$, then use the definition of P^* to conclude the result.

□

Proof. (\leq): Let $\varepsilon > 0$. We have

$$S_n = \frac{1}{n} \sum_{i=1}^n f(X_i) = \frac{1}{n} \sum_{i=1}^n \sum_{a \in A} \mathbb{1}_{\{X_i=a\}} f(a) = \sum_{a \in A} \hat{P}_n(a) f(a) = \mathbb{E}_{X \sim \hat{P}_n} [f(X)].$$

So we have $\mathbb{P}(\hat{P}_n \in E) = \mathbb{P}(S_n \geq \mu + \varepsilon)$, hence

$$\begin{aligned} \Lambda^*(\mu + \varepsilon) &\leq \liminf_{n \rightarrow \infty} -\frac{1}{n} \mathbb{P}(S_n \geq \mu + \varepsilon) \quad \text{by the \textbf{Chernoff Bound}} \\ &\leq \lim_{n \rightarrow \infty} -\frac{1}{n} \ln \mathbb{P}(\hat{P}_n \in E) \\ &= D_e(P^* \parallel Q) \quad \text{by \textbf{Sanov}.} \end{aligned}$$

(\geq): For each $\lambda \geq 0$, define the PMF P_λ on A by

$$P_\lambda(a) = \frac{e^{\lambda f(a)}}{\mathbb{E}[e^{\lambda f(X_1)}]} Q(a).$$

Then

$$\Lambda'(\lambda) = \frac{\mathbb{E}[f(X_1) e^{\lambda f(X_1)}]}{\mathbb{E}[e^{\lambda f(X_1)}]} = \frac{1}{\mathbb{E}[e^{\lambda f(X_1)}]} \sum_{a \in A} Q(a) f(a) e^{\lambda f(a)} = \mathbb{E}_{Y \sim P_\lambda} [f(Y)]$$

and also, a straightforward calculation shows that

$$\Lambda''(\lambda) = \text{Var}_{Y \sim P_\lambda}(f(Y)) \geq 0.$$

Hence, $\Lambda'(\lambda)$ is increasing from $\Lambda'(0) = \mu$ to $\lim_{\lambda \rightarrow \infty} \Lambda'(\lambda) =: f^*$, so there exists $\lambda^* > 0$ such that $\Lambda'(\lambda^*) = \mu + \varepsilon$, hence $\mathbb{E}_{Y \in P_{\lambda^*}}[f(Y)] = \mu + \varepsilon$, so $P_{\lambda^*} \in E$. Thus,

$$\begin{aligned} D_e(P^* \parallel Q) &\leq D_e(P_{\lambda^*} \parallel Q) \\ &= \mathbb{E}_{Y \sim P_{\lambda^*}} \left[\log \frac{P_{\lambda^*}(Y)}{Q(Y)} \right] \\ &= \mathbb{E}_{Y \sim P_{\lambda^*}} \left[\log \frac{e^{\lambda^* f(Y)}}{\mathbb{E}[e^{\lambda^* f(X_1)}]} \right] \\ &= \lambda^* \mathbb{E}_{Y \sim P_{\lambda^*}}[f(Y)] - \Lambda(\lambda^*) \\ &= \lambda^*(\mu + \varepsilon) - \Lambda(\lambda^*) \leq \Lambda^*(\mu + \varepsilon). \end{aligned}$$

□

Corollary 9.16 Let X_1^n be IID RVs with common PMF Q on alphabet A . The minimiser P^* in **Sanov** for the event $E = \{P \in \mathcal{P} : \mathbb{E}_{X \sim P}[f(X)] \geq \mu + \varepsilon\}$ is unique and is given by

$$P^*(a) = P_{\lambda^*}(a) = \frac{e^{\lambda^* f(a)}}{\mathbb{E}[e^{\lambda^* f(X_1)}]} Q(a).$$

where $\lambda^* > 0$ satisfies $\mathbb{E}_{Y \sim P_{\lambda^*}}[f(Y)] = \mu + \varepsilon$.

Proof (Hints). Existence: by above proposition. Uniqueness: use a property of $D(P \parallel Q)$ and the fact that E is non-empty, convex and closed. □

Proof. $D(P \parallel Q)$ is strictly convex in P for fixed Q and E is non-empty, convex and closed, so the minimising P^* is unique. The existence is by the proof of the above proposition. □

Theorem 9.17 (Pythagorean Identity) Let $E \subseteq \mathcal{P}$ be closed and convex. Let $Q \notin E$ have full support on A , and let P^* achieve the minimum in Sanov's theorem. Then

$$\forall P \in E, \quad D(P \parallel Q) \geq D(P \parallel P^*) + D(P^* \parallel Q).$$

Proof (Hints).

- For $P \in E$, define $\bar{P}_\lambda = \lambda P + (1 - \lambda)P^*$ for $\lambda \in [0, 1]$. Show that $D(\bar{P}_\lambda \parallel Q) \geq D(\bar{P}_0 \parallel Q)$ for all $\lambda \in [0, 1]$.
- Use the derivative of $D_e(\bar{P}_\lambda \parallel Q)$ at $\lambda = 0$ to obtain the result.

□

Proof. Let $P \in E$. Define the mixture $\bar{P}_\lambda = \lambda P + (1 - \lambda)P^*$ for $0 \leq \lambda \leq 1$. Since E is convex, $\bar{P}_\lambda \in E$ for all $\lambda \in [0, 1]$, and by definition of P^* , $D(\bar{P}_\lambda \parallel Q) \geq D(P^* \parallel Q) = D(\bar{P}_0 \parallel Q)$ for all $\lambda \in [0, 1]$. So we have

$$\begin{aligned}
0 &\leq \left. \frac{d}{d\lambda} D_e(\bar{P}_\lambda \parallel Q) \right|_{\lambda=0} \\
&= \left. \frac{d}{d\lambda} \sum_{a \in A} \bar{P}_\lambda(a) \ln \frac{\bar{P}_\lambda(a)}{Q(a)} \right|_{\lambda=0} \\
&= \sum_{a \in A} (P(a) - P^*(a)) \ln \frac{\bar{P}_\lambda(a)}{Q(a)} \Big|_{\lambda=0} + \sum_{a \in A} (P(a) - P^*(a)) \\
&= \sum_{a \in A} P(a) \ln \frac{P^*(a)P(a)}{Q(a)P(a)} - \sum_{a \in A} P^*(a) \ln \frac{P^*(a)}{Q(a)} \\
&= D_e(P \parallel Q) - D_e(P \parallel P^*) - D_e(P^* \parallel Q).
\end{aligned}$$

□

Remark 9.18

- The **Pythagorean Identity** is an L^2 -style bound: the minimiser P^* can be viewed as the “orthogonal projection” of Q onto E .
- The **Pythagorean Identity** provides a quantitative version of the uniqueness statement in **Corollary 9.16**: if $D(P \parallel Q) = D(P^* \parallel Q)$, then $P = P^*$; additionally, if $D(P \parallel Q) \leq D(P^* \parallel Q) + \delta$ (i.e. $D(P \parallel Q)$ is close to $D(P^* \parallel Q)$), then $D(P \parallel P^*) \leq \delta$ (i.e. P is close to P^*).

9.3. The Gibbs conditioning principle

Lemma 9.19 Let $\{Z_n : n \in \mathbb{N}\}$ be a bounded sequence of RVs which converges to $z \in \mathbb{R}$ in probability. Then

$$\mathbb{E}[Z_n] \rightarrow c \quad \text{as } n \rightarrow \infty.$$

Proof (Hints). Use Jensen’s inequality, then split the expectation into two terms, one bounded above by ε , the other $\rightarrow 0$, to show that $|\mathbb{E}[Z_n] - c| \rightarrow 0$. □

Proof. Let $\varepsilon > 0$. Since the Z_n are bounded, we have $|Z_n| \leq M$ for all $n \in \mathbb{N}$, for some constant M . By **Jensen's Inequality**,

$$|\mathbb{E}[Z_n] - z| \leq \mathbb{E}[|Z_n - z|] = \mathbb{E}[|Z_n - z| \cdot \mathbb{1}_{\{|Z_n - z| \leq \varepsilon\}}] + \mathbb{E}[|Z_n - z| \cdot \mathbb{1}_{\{|Z_n - z| > \varepsilon\}}].$$

The first term is bounded above by ε . The second term is bounded above by

$$(M + |z|) \cdot \mathbb{E}[\mathbb{1}_{\{|Z_n - z| > \varepsilon\}}] = (M + |z|) \cdot \mathbb{P}(|Z_n - z| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, $\limsup_{n \rightarrow \infty} |\mathbb{E}[Z_n] - c| \leq \varepsilon$, and $\varepsilon > 0$ was arbitrary. □

Theorem 9.20 (Gibbs' Conditioning Principle) Let X_1^n be IID with common PMF Q which has full support on A . Let \hat{P}_n be the empirical distribution of X_1^n . If $E \subseteq \mathcal{P}$ is closed, convex, has non-empty interior, and $Q \notin E$, then

$$\forall a \in A, \quad \mathbb{E}[\hat{P}_n(a) \mid \hat{P}_n \in E] = \mathbb{P}(X_1 = a \mid \hat{P}_n \in E) \rightarrow P^*(a) \quad \text{as } n \rightarrow \infty.$$

Proof (Hints).

- Showing the equality is straightforward.
- Define $B(Q, \delta) := \{P \in \mathcal{P} : D(P \parallel Q) \leq D(P^* \parallel Q) + \delta\}$, $C = B(Q, 2\delta) \cap E$ and $D = E \setminus C$.
- Show that $\mathbb{P}(\hat{P}_n \in D \mid \hat{P}_n \in E) \leq (n+1)^{2m} 2^{-n\delta}$.
- Use the **Pythagorean Identity** and **Pinsker's Inequality** to show that $\mathbb{P}(|\hat{P}_n(a) - P^*(a)| > \varepsilon \mid \hat{P}_n \in E) \rightarrow 0$.

□

Proof. The conditional distribution of each X_i given $\hat{P}_n \in E$ is the same, so

$$\mathbb{E}[\hat{P}_n(a) \mid \hat{P}_n \in E] = \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X_i = a \mid \hat{P}_n \in E) = \mathbb{P}(X_1 = a \mid \hat{P}_n \in E).$$

Define the relative entropy neighbourhoods

$$B(Q, \delta) := \{P \in \mathcal{P} : D(P \parallel Q) \leq D(P^* \parallel Q) + \delta\},$$

and write $C = B(Q, 2\delta) \cap E$ and $D = E \setminus C$.

Then

$$\mathbb{P}(\hat{P}_n \in D \mid \hat{P}_n \in E) = \frac{\mathbb{P}(\hat{P}_n \in D)}{\mathbb{P}(\hat{P}_n \in E)}.$$

By **Sanov**,

$$\mathbb{P}(\hat{P}_n \in D) \leq (n+1)^m 2^{-n \inf\{D(P \parallel Q) : P \in D\}} \leq (n+1)^m 2^{-n(D(P^* \parallel Q) + 2\delta)}$$

and for the denominator, since $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is dense in \mathcal{P} , \mathcal{P}_n eventually intersects every open set in \mathcal{P} , so eventually $B(Q, \delta) \cap E \cap \mathcal{P}_n$ is non-empty (since E has non-empty interior). So we can eventually find $P_n \in \mathcal{P}_n \cap E \cap B(Q, \delta)$. By

Proposition 9.8,

$$\begin{aligned} \mathbb{P}(\hat{P}_n \in E) &\geq \mathbb{P}(\hat{P}_n \in B(Q, \delta) \cap E) \\ &\geq \mathbb{P}(\hat{P}_n = P_n) = Q^n(T(P_n)) \\ &\geq (n+1)^{-m} 2^{-nD(P_n \parallel Q)} \\ &\geq (n+1)^{-m} 2^{-n(D(P^* \parallel Q) + \delta)}, \end{aligned}$$

since $P_n \in B(Q, \delta)$. Combining these, we obtain

$$\mathbb{P}(\hat{P}_n \in D \mid \hat{P}_n \in E) \leq (n+1)^{2m} 2^{-n\delta} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For $P \in C$, by the **Pythagorean Identity**,

$$D(P^* \parallel Q) \geq D(P \parallel Q) \geq D(P \parallel P^*) + D(P^* \parallel Q),$$

thus $D(P \parallel P^*) \leq 2\delta$. So

$$\mathbb{P}(D(\hat{P}_n \| P^*) \leq 2\delta \mid \hat{P}_n \in E) \geq \mathbb{P}(\hat{P}_n \in C \mid \hat{P}_n \in E) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Hence by **Pinsker's Inequality**, since $\delta > 0$ was arbitrary,

$$\mathbb{P}(\|\hat{P}_n - P^*\|_{\text{TV}} > \varepsilon \mid \hat{P}_n \in E) \rightarrow 0 \text{ as } n \rightarrow \infty$$

for all $\varepsilon > 0$. Thus also, $\mathbb{P}(|\hat{P}_n(a) - P^*(a)| > \varepsilon \mid \hat{P}_n \in E) \rightarrow 0$. So, conditional on $\hat{P}_n \in E$, $\hat{P}_n \rightarrow P^*$ in probability as $n \rightarrow \infty$. Therefore, since $(\hat{P}_n(a))$ is a bounded sequence, we also have $\mathbb{E}[\hat{P}_n(a) \mid \hat{P}_n \in E] \rightarrow P^*(a)$ as $n \rightarrow \infty$ by **Lemma 9.19**. \square

Example 9.21 Suppose a fair die is rolled 1000 times, and the observed average of the rolls is at least 5. What proportion of the rolls was a 6?

Let X_1^{1000} be IID RVs with uniform distribution Q on $A = \{1, 2, 3, 4, 5, 6\}$. Let $f(x) = x$, $\mu = \mathbb{E}[X_1^{1000}] = 3.5$, let $E = \{P \in \mathcal{P} : \mathbb{E}_{X \sim P}[X] \geq 5\}$. By **Corollary 9.16**, the minimiser P^* is unique and is given by

$$P^*(a) = \frac{e^{\lambda^* a}}{\sum_{k=1}^6 e^{\lambda^* k}}, \quad \forall a \in A,$$

where $\lambda^* > 0$ is such that $\mathbb{E}_{Y \sim P_{\lambda^*}}[Y] = 5$. We can directly compute $\lambda^* \approx 0.63$ and so

$$P^* \approx (0.021, 0.039, 0.14, 0.25, 0.48)$$

So we expect that about 48% of the rolls were 6.

9.4. Error probability in fixed-rate data compression

Theorem 9.22 Let $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ be a memoryless source with entropy $H = H(X_1)$ and with PMF Q which has full support on finite alphabet A . For any rate R with $H < R < \log|A|$,

- \Rightarrow : There is a fixed-rate code $\{B_n^* : n \in \mathbb{N}\}$ with asymptotic rate no more than R bits/symbol:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} (1 + \lceil \log|B_n^*| \rceil) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log|B_n^*| \leq R,$$

and with probability of error $P_e^{(n)}$ that decays to zero exponentially fast:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_e^{(n)} \leq -D^*,$$

where

$$D^* = \inf\{D(P \| Q) : H(P) \geq R\}.$$

- \Leftarrow : for any fixed-rate code $\{B_n : n \in \mathbb{N}\}$ with asymptotic rate no more than R bits/symbol:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} (1 + \lceil \log|B_n| \rceil) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log|B_n| \leq R,$$

then its probability of error $P_e^{(n)}$ cannot decay faster than exponentially with exponent D^* :

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log P_e^{(n)} \geq -D^*.$$

Proof (Hints).

- \Rightarrow : let B_n^* be the codebook which is a union over an appropriate set of type classes.
- \Leftarrow : explain why there is $\delta > 0$ such that $\inf\{D(P \parallel Q) : H(P) \geq R + \delta\} \leq D^* + \varepsilon$.
- Explain why, for all n large enough, there is $P_n \in \mathcal{P}_n$ such that $H(P_n) \geq R + \delta/2$ and $D(P_n \parallel Q) \leq D^* + 2\varepsilon$.
- Show that $|B_n|/|T(P_n)| \rightarrow 0$ as $n \rightarrow \infty$, and hence that $P_e^{(n)} \geq \frac{1}{2}(n+1)^{-m}2^{-n(D^*+2\varepsilon)}$ eventually.

□

Proof. \Rightarrow : define the codebook

$$B_n^* = \bigcup_{\substack{P \in \mathcal{P}_n \\ H(P) < R}} T(P).$$

Then by [Proposition 9.4](#) and [Proposition 9.8](#),

$$|B_n^*| = \sum_{\substack{P \in \mathcal{P}_n \\ H(P) < R}} |T(P)| \leq \sum_{\substack{P \in \mathcal{P}_n \\ H(P) < R}} 2^{nH(P)} \leq (n+1)^m 2^{nR},$$

and so $\limsup_{n \rightarrow \infty} \frac{1}{n} \log |B_n^*| \leq R$. For the probability of error,

$$P_e^{(n)} = \mathbb{P}(X_1^n \notin B_n^*) = Q^n \left(\bigcup_{\substack{P \in \mathcal{P}_n \\ H(P) \geq R}} T(P) \right) \leq \sum_{\substack{P \in \mathcal{P}_n \\ H(P) \geq R}} Q^n(T(P)) \leq (n+1)^m 2^{-nD^*}.$$

\Leftarrow : let $\varepsilon > 0$ be arbitrary. By continuity, there is a $\delta > 0$ such that

$$\inf\{D(P \parallel Q) : H(P) \geq R + \delta\} \leq D^* + \varepsilon.$$

Since the n -types $\{P_n : n \in \mathbb{N}\}$ are dense in \mathcal{P} , for all n large enough, we can find $P_n \in \mathcal{P}_n$ such that $H(P_n) \geq R + \delta/2$ and $D(P_n \parallel Q) \leq D^* + 2\varepsilon$. Also, by assumption, there is a sequence (r_n) such that $\frac{1}{n} \log |B_n| \leq R + r_n$ and $r_n \rightarrow 0$. Now

$$\begin{aligned} \frac{|B_n|}{|T(P_n)|} &\leq \frac{2^{n(R+r_n)}}{(n+1)^{-m} 2^{nH(P_n)}} = (n+1)^m 2^{n(R-H(P_n)+r_n)} \\ &\leq (n+1)^m 2^{n(r_n-\delta/2)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

So $|B_n|/|T(P_n)| \leq 1/2$ eventually. Then, for an arbitrary string $x_1^n \in T(P_n)$, we have

$$\begin{aligned}
P_e^{(n)} &= \mathbb{P}(X_1^n \in B_n^c) \geq \mathbb{P}(X_1^n \in T(P_n) \cap B_n^c) \\
&= |T(P_n) \cap B_n^c| Q^n(x_1^n) = \frac{|T(P_n) \cap B_n^c|}{|T(P_n)|} Q^n(T(P_n)) \\
&\geq \left(1 - \frac{|T(P_n) \cap B_n|}{|T(P_n)|}\right) (n+1)^{-m} 2^{-nD(P_n \parallel Q)} \\
&\geq \left(1 - \frac{|B_n|}{|T(P_n)|}\right) (n+1)^{-m} 2^{-nD(P_n \parallel Q)} \\
&\geq \frac{1}{2} (n+1)^{-m} 2^{-n(D^*+2\varepsilon)} \quad \text{eventually}
\end{aligned}$$

Thus,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log P_e^{(n)} \geq -(D^* + 2\varepsilon),$$

and since $\varepsilon > 0$ was arbitrary, we are done. \square

Remark 9.23

- **Theorem 9.22** gives the rate at which the error probabilities $P_e^{(n)}$ of the codes in the **Fixed-rate Coding Theorem** decay.
- Note that the code B_n^* is **universal**: it achieves the optimal error probability at rate R *simultaneously* for all memoryless sources with entropy $H < R$.
- The **Fixed-rate Coding Theorem** says that $P_e^{(n)}$ cannot tend to zero if $R < H$. In fact, it is possible to show a “strong converse” of the **Fixed-rate Coding Theorem**, which says that in this case, $P_e^{(n)} \rightarrow 1$ exponentially fast.

10. Variable-rate lossless data compression

Notation 10.1 Let $\{0,1\}^*$ denote the set of all binary strings of finite length.

Definition 10.2 A **variable-rate lossless compression code** of block length n on a finite alphabet A is an injective map $C_n : A^n \rightarrow \{0,1\}^*$ which maps source strings to **codewords**. C_n is also known as the **encoder**.

Each C_n has an associated **length function** $L_n : A^n \rightarrow \mathbb{N}$, defined as

$$L_n(x_1^n) = \text{length of } C_n(x_1^n).$$

Definition 10.3 A code C_n is **prefix-free** if for all $x_1^n \neq y_1^n \in \{0,1\}^n$, the codeword $C_n(x_1^n)$ is not a prefix (an initial segment) of $C_n(y_1^n)$.

Example 10.4

x	$C(x)$	x	$C(x)$	x	$C(x)$	x	$C(x)$
a	00	a	0	a	0	a	0
b	01	b	10	b	00	b	1
c	10	c	110	c	110	c	00
d	11	d	111	d	111	d	11

The first two codes are prefix-free, the last two are not.

Remark 10.5 An advantage of prefix-free codes is that once a full codeword is received, it is guaranteed to be that codeword and not the start of another.

Theorem 10.6 (Kraft's Inequality)

- (\Rightarrow): for any length function $L_n : A^n \rightarrow \mathbb{N}$ satisfying **Kraft's inequality**:

$$\sum_{x_1^n \in A^n} 2^{-L_n(x_1^n)} \leq 1,$$

there is a prefix-free code C_n on A^n with length function L_n .

- (\Leftarrow): the length function of any prefix-free code satisfies Kraft's inequality.

Proof (Hints). For both directions, consider the complete binary tree of depth $\max\{L_n(x_1^n) : x_1^n \in A^n\}$. □

Proof. \Leftarrow : let C_n be a prefix-free code with length function L_n . Let $L^* = \max\{L_n(x_1^n) : x_1^n \in A^n\}$ and consider the complete binary tree of depth L^* . If we mark all the codewords on the tree, then the prefix-free property implies that no codeword is a descendant of any other codeword. Each codeword $C_n(x_1^n)$ has $2^{L^* - L_n(x_1^n)}$ descendants (possibly including itself) at depth L^* . The prefix-free property also implies that these descendants are disjoint for different codewords. Since the total number of leaves at depth L^* is 2^{L^*} , we have

$$2^{L^*} \geq \sum_{x_1^n \in A^n} 2^{L^* - L_n(x_1^n)}.$$

\Rightarrow : given a length function L_n satisfying Kraft's inequality, consider the complete binary tree of depth $L^* = \max\{L_n(x_1^n) : x_1^n \in A^n\}$. Then, ordering the $x_1^n \in A^n$ in the order of increasing $L_n(x_1^n)$, assign to each x_1^n (in order) any available node (i.e. any node that is not a prefix or descendant of any codewords already assigned) at depth $L_n(x_1^n)$. Kraft's inequality guarantees that there will always be such a node. □

Remark 10.7 **Kraft's Inequality** informally says “not all codelengths for prefix-free codes can be short”.

10.1. The codes-distributions correspondence

Theorem 10.8 (Codes-distributions Correspondence)

- \Rightarrow : for any PMF Q_n on A^n , there is a prefix-free code C_n^* with length function L_n^* such that

$$\forall x_1^n \in A^n, \quad L_n^*(x_1^n) < -\log Q_n(x_1^n) + 1$$

- \Leftarrow : for any prefix-free code C_n with length function L_n , there is a PMF Q_n on A^n such that

$$\forall x_1^n \in A^n, \quad -\log Q_n(x_1^n) \leq L_n(x_1^n).$$

Proof (Hints).

- \Rightarrow : straightforward.

- \Leftarrow : consider **Kraft's Inequality** to define a suitable Q_n .

□

Proof. \Rightarrow : Let $L_n^*(x_1^n) = \lceil -\log Q_n(x_1^n) \rceil < -\log Q_n(x_1^n) + 1$. L_n^* satisfies Kraft's inequality:

$$\sum_{x_1^n \in A^n} 2^{-L_n^*(x_1^n)} = \sum_{x_1^n \in A^n} 2^{-\lceil -\log Q_n(x_1^n) \rceil} \leq \sum_{x_1^n \in A^n} 2^{\log Q_n(x_1^n)} = \sum_{x_1^n \in A^n} Q_n(x_1^n) = 1.$$

So we are done by the first part of **Kraft's Inequality**.

\Leftarrow : define the PMF Q_n on A^n by

$$Q_n(x_1^n) = \frac{2^{-L_n(x_1^n)}}{\sum_{y_1^n \in A^n} 2^{-L_n(y_1^n)}}.$$

Then

$$-\log Q_n(x_1^n) = L_n(x_1^n) + \log \left(\sum_{y_1^n \in A^n} 2^{-L_n(y_1^n)} \right) \leq L_n(x_1^n).$$

since L_n satisfies Kraft's inequality (i.e. $\sum_{y_1^n \in A^n} 2^{-L_n(y_1^n)} \leq 1$).

□

Remark 10.9

- **Codes-distributions Correspondence** says that the performance of any prefix-free can be dominated by a code with length function $L_n(x_1^n) \approx -\log Q_n(x_1^n)$ for some PMF Q_n on A^n , and that for any distribution Q_n such a code exists. So finding a good code is equivalent to finding a good distribution. This assumes nothing about the distribution of the source X_1^n or the block length n .

Theorem 10.10 Let X_1^n have PMF P_n on A^n .

\Rightarrow : there is a prefix-free code C_n^* with length function L_n^* that achieves an expected description length of

$$\mathbb{E}[L_n^*(X_1^n)] < H(X_1^n) + 1.$$

\Leftarrow : for any prefix-free code C_n with length function L_n on A^n ,

$$\mathbb{E}[L_n(X_1^n)] \geq H(X_1^n).$$

Proof (Hints). Straightforward.

□

Proof. \Rightarrow : let C_n^* be the code with length function $L_n^*(x_1^n) = \lceil -\log P_n(x_1^n) \rceil$ as in the **Codes-distributions Correspondence**. Then

$$\mathbb{E}[L_n^*(X_1^n)] < \mathbb{E}[-\log P_n(X_1^n) + 1] = H(X_1^n) + 1.$$

\Leftarrow : let Q_n be as in the **Codes-distributions Correspondence**. Then

$$\begin{aligned}
\mathbb{E}[L_n(X_1^n)] &\geq \mathbb{E}[-\log Q_n(X_1^n)] \\
&= \mathbb{E}\left[\log\left(\frac{1}{P_n(X_1^n)} \cdot \frac{P_n(X_1^n)}{Q_n(X_1^n)}\right)\right] \\
&= \mathbb{E}[-\log P_n(X_1^n)] + \mathbb{E}\left[\log \frac{P_n(X_1^n)}{Q_n(X_1^n)}\right] \\
&= H(X_1^n) + D(P_n \parallel Q_n) \geq H(X_1^n).
\end{aligned}$$

□

Corollary 10.11 Let $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ be a stationary source with entropy rate $H = H(\mathbf{X})$. Then H is the best asymptotically achievable compression rate among all variable-rate prefix-free codes:

$$\lim_{n \rightarrow \infty} \inf_{(C_n, L_n) \text{ prefix-free}} \frac{1}{n} \mathbb{E}[L_n(X_1^n)] = H.$$

Proof (Hints). Straightforward. □

Proof. By [Theorem 10.10](#),

$$\frac{1}{n} H(X_1^n) \leq \inf_{(C_n, L_n) \text{ prefix-free}} \frac{1}{n} \mathbb{E}[L_n(X_1^n)] < \frac{1}{n} (H(X_1^n) + 1).$$

□

10.2. Shannon codes and their properties

Definition 10.12 A **Shannon code** for a distribution Q_n on A^n is a code with length function

$$L_n(x_1^n) := \lceil -\log Q_n(x_1^n) \rceil.$$

Note this is the code used in the proof of the [Codes-distributions Correspondence](#).

Remark 10.13

- Shannon codes do not always achieve the optimal (minimal) expected description length $\mathbb{E}[L_n(X_1^n)]$, which is achieved instead by the Huffman code. However, the difference between the expected description lengths of these codes is less than one bit by [Theorem 10.10](#).
- Shannon codes give shorter descriptions to likely messages and longer descriptions to unlikely messages.

Definition 10.14 We call the $L_n(x_1^n) = -\log Q_n(x_1^n)$ for $x_1^n \in A^n$ the **ideal Shannon codelengths**.

Theorem 10.15 (Competitive Optimality of Shannon Codes) Let P_n be a distribution on A^n and $X_1^n \sim P_n$. For any other PMF Q_n on A^n ,

$$\mathbb{P}(-\log Q_n(X_1^n) \leq -\log P_n(X_1^n) - K) \leq 2^{-K}.$$

Proof (Hints). Use Markov's inequality. □

Proof. By Markov's inequality, we have

$$\begin{aligned}
\mathbb{P}(-\log Q_n(X_1^n) \leq -\log P_n(X_1^n) - K) &= \mathbb{P}\left(\frac{Q_n(X_1^n)}{P_n(X_1^n)} \geq 2^K\right) \\
&\leq 2^{-K} \mathbb{E}\left[\frac{Q_n(X_1^n)}{P_n(X_1^n)}\right] \\
&= 2^{-K} \sum_{x_1^n \in A^n} P_n(x_1^n) \cdot \frac{Q_n(x_1^n)}{P_n(x_1^n)} \\
&= 2^{-K}.
\end{aligned}$$

□

11. Universal data compression

12. Redundancy and the price of universality

12.1. Redundancy

Definition 12.1 Suppose $x_1^n \in A^n$ is generated by a memoryless source with PMF P on a finite alphabet A , with $|A| = m$. The target compression is $-\log P^n(x_1^n)$ bits. So if instead we use a code with respect to an arbitrary PMF Q_n on A^n , the **redundancy** is $\rho_n(x_1^n; P, Q_n) = -\log Q_n(x_1^n) - (-\log P^n(x_1^n)) = \log \frac{P^n(x_1^n)}{Q_n(x_1^n)}$.

The **minimax maximal redundancy** is

$$\rho_n^* = \min_{Q_n} \max_P \max_{x_1^n} \log \frac{P^n(x_1^n)}{Q_n(x_1^n)}$$

and the **minimax average redundancy** is

$$\bar{\rho}_n = \min_{Q_n} \max_P \mathbb{E}_{P^n} \left[\log \frac{P^n(x_1^n)}{Q_n(x_1^n)} \right] = \sup_P D(P^n \parallel Q_n)$$

We will see that

$$\frac{m-1}{2} \log n - C' \leq \bar{\rho}_n \leq \rho_n^* \leq \frac{m-1}{2} \log n + C$$

for some constants C, C' .

12.2. Shtarkov's upper bound

Theorem 12.2 (Normalised Maximum Likelihood Code) Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be an arbitrary parametric family of distributions on a finite alphabet B . Then

$$\rho^*(\Theta) := \min_Q \max_{\theta} \max_x \log \frac{P_\theta(x)}{Q(x)} = \log Z$$

where

$$Z = \sum_{x \in B} \max_{\theta \in \Theta} P_{\theta}(x).$$

Proof. Let $Q^*(x) = \frac{1}{Z} \max_{\theta \in \Theta} P_{\theta}(x)$. We have

$$\begin{aligned} \rho^*(\Theta) &\leq \max_{\theta} \max_x \log \frac{P_{\theta}(x)}{Q^*(x)} \\ &= \max_x \log \frac{\max_{\theta} P_{\theta}(x)}{Q^*(x)} = \log Z. \end{aligned}$$

For the lower bound, note that for every Q , $Q(x) \leq Q^*(x)$ for at least one x , say x^* . Therefore,

$$\rho^*(\theta) \geq \min_Q \max_{\theta} \log \frac{P_{\theta}(x^*)}{Q^*(x^*)} = \min_Q \log Z = \log Z.$$

□

Now consider the $(m-1)$ -dimensional parametric family $\mathcal{P} = \{P_{\theta}^n : \theta \in \Theta\}$, where $\Theta = \{\theta \in [0, 1]^{m-1} : \sum_{i=1}^{m-1} \theta_i \leq 1\}$ and writing $A = \{a_1, \dots, a_m\}$,

$$P_{\theta}(a_i) = \begin{cases} \theta_i & \text{if } i \neq m \\ 1 - \sum_{j=1}^{m-1} \theta_j & \text{if } i = m \end{cases}$$

Then applying the previous theorem to this parametric family consisting of all memoryless sources on A gives

$$\rho_n^* = \log \left(\sum_{x_1^n} \max_{\theta} P_{\theta}^n(x_1^n) \right) = \log \left(\sum_{x_1^n} \hat{P}_{x_1^n}^n(x_1^n) \right)$$

evaluating this gives:

Definition 12.3 The **Gamma function** is defined as

$$\Gamma(z) := \int_0^{\infty} x^{z-1} e^{-x} dx.$$

Theorem 12.4 (Shtarkov) The minimax maximal redundancy over the class of all memoryless sources on A satisfies, for all $n \in \mathbb{N}$,

$$\rho_n^* \leq \frac{m-1}{2} \log \left(\frac{n}{2} \right) + \log \frac{\Gamma(1/2)}{\Gamma(m/2)} + \frac{C'}{\sqrt{n}}$$

for a constant C depending only on m .

Theorem 12.5 (von Neumann minimax) TODO.

Theorem 12.6 (Redundancy-capacity Theorem) Let $\mathcal{P} = \{P_{\theta} : \theta \in \Theta\}$ be a “nice” parametric family of distributions on a finite alphabet B . Then

$$\bar{\rho}(\Theta) := \min_Q \max_{\theta \in \Theta} D(P_{\theta} \parallel Q) = \max_{\pi} I(\underline{\theta}; X),$$

where the max is over all probability distributions π on Θ , $\underline{\theta} \sim \pi$ and $X \mid \underline{\theta} = \theta \sim P_\theta$ (so the pair of RVs $(\underline{\theta}, X)$ has joint distribution $\pi(\theta)P_\theta(x)$).

Proof. Omitted (non-examinable). □

Theorem 12.7 (Rissanen) Let $\{Q_n : n \in \mathbb{N}\}$ be an arbitrary sequence of distributions on A^n , where $|A| = m$. Then for all $\varepsilon > 0$, there exists a constant C' and a subset $\Theta_0 \subseteq \Theta$ of volume less than ε such that for all $\theta \notin \Theta_0$,

$$D(P_\theta^n \parallel Q_n) \geq \frac{m-1}{2} \log n - C'' \quad \text{eventually.}$$

for some constant C'' . In particular, $\bar{\rho}_n \geq \frac{m-1}{2} \log n - C'''$ eventually.