Contents

1. Hidden subgroup problem

1.1. Review of Shor's algorithm

Problem 1.1 (Factoring)

Input a positive integer N.

Promise N is composite.

Task Find a non-trivial factor of N in $O(\text{poly}(n))$ time, where $n = \log N$.

Definition 1.2 An **efficient problem** is one that can be solved in polynomial time.

Remark 1.3 Clasically, the best known factoring algorithm runs in $e^{O\left(n^{1/3}(\log n)^{2/3}\right)}$. Shor's algorithm (quantum) runs in $O(n^3)$ by converting factoring into period finding:

- Given input N, choose $a < N$ which is coprime to N.
- Define $f : \mathbb{Z} \to \mathbb{Z}/N$, $f(x) = a^x \bmod N$. f is periodic with period r (the order of $a \bmod N$), i.e. $f(x + r) = f(x)$ for all $x \in \mathbb{Z}$. Finding r allows us to factor N.

1.2. Period finding

Problem 1.4 (Periodicity Determination Problem)

Input An oracle for a function $f : \mathbb{Z}/M \to \mathbb{Z}/N$.

Promise

- f is periodic with period $r < M$ (i.e. $\forall x \in \mathbb{Z}/M$, $f(x + r) = f(x)$), and
- f is injective in each period (i.e. if $0 \leq x < y < r$, then $f(x) \neq f(y)$).

Task Determine the period r.

Remark 1.5 Solving the periodicity determination problem classically requires takes time $O(\sqrt{M})$.

Definition 1.6 Let $f : \mathbb{Z}/M \to \mathbb{Z}/N$. Let H_M and H_N be quantum state spaces with orthonormal state bases $\{|i\rangle : i \in \mathbb{Z}/N\}$ and $\{|j\rangle : j \in \mathbb{Z}/M\}$. Define the unitary **quantum oracle** for f by U_f by

$$U_f|x\rangle|z\rangle = |x\rangle|z + f(x)\rangle.$$

The first register $|x\rangle$ is the **input register**, the last register $|z\rangle$ is the **output register**.

Definition 1.7 The **quantum query complexity** of an algorithm is the number of times it queries f (i.e. uses U_f).

Definition 1.8 The **quantum Fourier transform** over $\mathbb{Z}/M$ is the unitary QFT defined by its action on the computational basis:

$$\text{QFT}|x\rangle = \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} \omega^{xy}|y\rangle,$$

where $\omega = e^{2\pi i/M}$ is an M-th root of unity. Note that QFT requires only $O((\log M)^2)$ gates to implement, whereas a general $M \times M$ unitary requires $O(4^M/M)$ elementary gates.

Lemma 1.9 Let $\alpha = e^{2\pi i y/M}$. Then

$$\sum_{j=0}^{k-1} \alpha^j = \begin{cases} \frac{1-\alpha^k}{1-\alpha} = 0 & \text{if } \alpha \neq 1 \text{ i.e. } M \nmid y \\ k & \text{if } \alpha = 1 \text{ i.e. } M \mid y \end{cases}.$$

Proof (Hints). Trivial. □

Proof. The sum is a geometric series with common ratio α. □

Lemma 1.10 (Boosting success probability) If a process succeeds with probability p on one trial, then

$$\Pr(\text{at least one success in } t \text{ trials}) = 1 - (1-p)^t > 1 - \delta$$

for $t = \frac{\log(1/d)}{p}$.

Proof (Hints). Trivial. □

Proof. Trivial. □

Theorem 1.11 (Co-primality Theorem) The number of integers less than r that are coprime to r is $O(r/\log\log r)$.

Algorithm 1.12 (Quantum Period Finding) The algorithm solves the Periodicity Determination Problem: Let $f : \mathbb{Z}/M \to \mathbb{Z}/N$ be periodic with period $r < M$ and one-to-one in each period. Let $A = \frac{M}{r}$ be the number of periods. We work over the state space $H_M \otimes H_N$.

1. Construct the state $\frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle|0\rangle$ and query U_f on it.
2. Measure second register in computational basis and discard the second register.
3. Apply the quantum Fourier transform to the input state.
4. Measure the input state, yielding outcome c.
5. Compute the denominator r_0 of the simplified fraction $\frac{c}{M}$.
6. Repeat the previous steps $O(\log\log r) = O(\log\log M) = O(\log m)$ times, halting if at any iteration, $f(0) = f(r_0)$.

Theorem 1.13 (Correctness of Quantum Period Finding Algorithm) When repeated, $O(\log\log r) = O(\log\log M)$ times, the quantum period finding algorithm obtains the correct value of r with high probability.

Proof. After querying U_f, we have the state $\frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle|f(i)\rangle$. Upon measuring the second register in the computational basis, the input state collapses to $|\text{per}\rangle = \frac{1}{\sqrt{A}} \sum_{j=0}^{A-1} |x_0 + jr\rangle$, where $f(x_0) = y$ and $0 \leq x_0 < r$. Applying the quantum Fourier transform to $|\text{per}\rangle$ then gives Quantum Fourier Transform to $|\text{per}\rangle$:

$$\begin{aligned}\mathrm{QFT}|\mathrm{per}\rangle &= \frac{1}{\sqrt{M}}\sum_{y=0}^{M-1}\frac{1}{\sqrt{A}}\sum_{j=0}^{A-1}\omega^{(x_0+jr)y}|y\rangle \\ &= \frac{1}{\sqrt{MA}}\sum_{y=0}^{M-1}\omega^{x_0 y}\sum_{j=0}^{A-1}\omega^{jry}|y\rangle \\ &= \sqrt{\frac{A}{M}}\sum_{k=0}^{r-1}\omega^{x_0 kM/r}|kM/r\rangle\end{aligned}$$

Importantly, now the outcomes and probabilities are independent of x_0, so carry useful information about r. TODO add diagram showing amplitudes for this state. The outcome after the measuring the input state is $c = k_0 M/r$ for some $0 \leq k_0 < r$ (so $c/M = k_0/r$). If k_0 is coprime to r, then the denominator r_0 of the simplified fraction $\frac{c}{M}$ is equal to r. By the coprimality theorem, the probability that k_0 is coprime to r is $O(1/\log\log r)$. Checking if $f(0) = f(r_0)$ tells us if $r_0 = r$, since f is periodic and one-to-one in each period, and $r_0 \leq r$. □

1.3. Analysis of QFT part of period finding algorithm

Notation 1.14 For $R = \{0, r, ..., (A-1)r\} \subseteq \mathbb{Z}/M$ ($Ar = M$), write $|R\rangle$ for the uniform superposition of all computational basis states in R:

$$|R\rangle = \frac{1}{\sqrt{A}}\sum_{k=0}^{A-1}|kr\rangle.$$

Definition 1.15 For each $x_0 \in \mathbb{Z}/M$, define the lienar map by its action on the computational basis states:

$$\begin{aligned}U(x_0) : H_M &\to H_M, \\ |k\rangle &\mapsto |x_0 + k\rangle.\end{aligned}$$

Definition 1.16 Note that since $(\mathbb{Z}/M, +)$ is abelian, all $U(x_i)$ commute: $U(x_1)U(x_2) = U(x_1 + x_2) = U(x_2)U(x_1)$. Hence, they have a simultaneous basis of eigenvectors $\{|\chi_k\rangle : k \in \mathbb{Z}/M\}$, i.e. for all $k, x_0 \in \mathbb{Z}/M$, $U(x_0)|\chi_k\rangle = w(x_0, k)|\chi_k\rangle$, where $|w(x_0, k)| = 1$. The $|\chi_k\rangle$ are called **shift-invariant states** and form an orthonormal basis for H_M. The $|\chi_k\rangle$ are given explicitly by

$$|\chi_k\rangle = \frac{1}{\sqrt{M}}\sum_{\ell=0}^{M-1} e^{-2\pi i k\ell/M}|\ell\rangle.$$

Proposition 1.17 The explicit definition of the $|\chi_k\rangle$ indeed satisfies the property $\forall k, x_0 \in \mathbb{Z}/M$, $U(x_0)|\chi_k\rangle = w(x_0, k)|\chi_k\rangle$, and we have $w(x_0, k) = \omega^{kx_0}$, where $\omega = e^{2\pi i/M}$.

Proof (Hints). Straightforward. □

Proof. We have that

$$\begin{aligned} U(x_0)|\chi_k\rangle &= \frac{1}{\sqrt{M}} \sum_{\ell=0}^{M-1} e^{-2\pi i k\ell/M} |x_0 + \ell\rangle \\ &= \frac{1}{\sqrt{M}} \sum_{\tilde{l}=0}^{M-1} e^{-2\pi i(\tilde{l}-x_0)k/M} |\tilde{l}\rangle \\ &= e^{2\pi i k x_0/M} |\chi_k\rangle \\ &=: w(x_0,k)|\chi_k\rangle \end{aligned}$$

□

Remark 1.18 Let $U : H_M \to H_M$ be the unitary mapping the shift-invariant basis to the computational basis: $U : |\chi_k\rangle \mapsto |k\rangle$. The matrix representation of U^{-1} with respect to the computational basis has entries

$$\left(U^{-1}\right)_{jk} = \langle j|U^{-1}|k\rangle = \langle j|\chi_k\rangle = \frac{1}{\sqrt{M}} e^{-2\pi i jk/M}$$

So the matrix representation of U with respect to the same basis has entries $U_{kj} = \overline{\left(U^{-1}\right)_{jk}} = \frac{1}{\sqrt{M}} e^{2\pi i jk/M}$. Hence, we have

$$U|k\rangle = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{2\pi i jk/M} |j\rangle,$$

and so U is precisely the QFT mod M.

1.4. The hidden subgroup problem (HSP)

Problem 1.19 (Discrete Logarithm Problem (DLP))

Input $g, x \in G$ for an abelian group G.

Promise g is a generator of G.

Task Find $\log_g x$, i.e. find $L \in \mathbb{Z}/|G|$ such that $x = g^L$.

Notation 1.20 Write $[n]$ for $\{1, ..., n\}$. Write e.g. ij for the set $\{i, j\}$.

Definition 1.21 Let $\Gamma_1 = ([n], E_1)$ and $\Gamma_2 = ([n], E_2)$ be (undirected) graphs. Γ_1 and Γ_2 are **isomorphic** if there exists a permutation $\pi \in S_n$ such that for all $1 \leq i, j < n$, $ij \in E_1$ iff $\pi(i)\pi(j) \in E_2$.

Definition 1.22 Let $\Gamma = ([n], E)$ be a graph. The **automorphism group** of Γ is

$$\mathrm{Aut}(\Gamma) = \{\pi \in S_n : ij \in E \text{ iff } \pi(i)\pi(j) \in E \quad \forall i, j \in [n]\}.$$

$\mathrm{Aut}(\Gamma)$ is a subgroup of S_n, and $\pi \in \mathrm{Aut}(\Gamma)$ iff π leaves Γ invariant as a labelled graph.

Definition 1.23 The **adjacency matrix** of a graph $\Gamma = (V, E)$ is the $n \times n$ matrix M_A defined by its entries:

$$\left(M_A\right)_{ij} := \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}.$$

Problem 1.24 (Graph Isomorphism Problem)

Input Adjacency matrices M_1 and M_2 of graphs $\Gamma_1 = ([n], E_1)$ and $\Gamma_2 = ([n], E_2)$.

Task Determine whether Γ_1 and Γ_2 are isomorphic.

Remark 1.25 The best known classical algorithm for solving the graph isomorphism problem has quasi-polynomial time complexity $n^{O((\log n)^2)}$.

Problem 1.26 (Hidden Subgroup Problem (HSP)) Let G be a finite group.

Input An oracle for a function $f : G \to X$.

Promise There is a subgroup $K < G$ such that:

1. f is constant on the (left) cosets of K in G.
2. f takes a different value on each coset.

Task Determine K.

Remark 1.27

- To find K, we either find a generating set for K, or sample uniformly random elements from K.
- We want to determine K with high probability in $O(\text{poly}\log|G|)$ queries. Using $O(|G|)$ queries is easy, as we just query all values $f(g)$ and find the "level sets" (sets where f is constant).

Example 1.28 The following problems are special cases of HSP:

- The Periodicity Determination Problem: $G = \mathbb{Z}/M$, $K = \langle r \rangle = \{0, r, ..., (A-1)r\}$. The cosets are $x_0 + K = \{x_0, x_0 + r, ..., x_0 + (A-1)r\}$ for each $0 \leq x_0 < r$.
- The DLP on $(\mathbb{Z}/p)^\times$: let $f : \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1) \to (\mathbb{Z}/p)^\times$ be defined by $f(a, b) = g^a x^{-b} = g^{a-Lb}$. $G = \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$, the hidden subgroup is $K = \{\lambda(L, 1) : \lambda \in \mathbb{Z}/(p-1)\}$. (Note that if we know K, we can pick any $(c, d) = (\lambda L, \lambda) \in G$ and compute $L = \frac{c}{d}$ to find L.)
- The Graph Isomorphism Problem: $G = S_n$, hidden subgroup is $K = \text{Aut}(G)$. Let $f_\Gamma : S_n \to X$ where X is set of adjacency matrices of labelled graphs on $[n]$, defined by $f_\Gamma(\pi) = \pi(A)$. Note $|S_n| = |G| = n!$, so $\log|G| \approx n \log n$, so $O(\text{poly}\log|G|) = O(\text{poly}\, n)$.

Definition 1.29 An **irreducible representation (irrep)** of a finite abelian group G is a homomorphism $\chi : G \to \mathbb{C}^\times$.

Theorem 1.30

- Let $\chi : G \to \mathbb{C}^\times$ be an irrep. For all $g \in G$, $\chi(g)$ is a $|G|$-th root of unity.
- There are always exactly $|G|$ distinct irreps. In particular, we can label each irrep uniquely by some $g \in G$.

Theorem 1.31 (Schur's Lemma) Let χ_i and χ_j be irreps of G. Then

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij}.$$

Example 1.32 $\chi_0 : G \to \mathbb{C}^\times$, $\chi_0(g) = 1$ is the **trivial irrep**. Note that for any $\chi_i \neq \chi_0$, $\sum_{g\in G} \chi_i(g) = 0$ by Schur's lemma.

Definition 1.33 For finite abelian G, we define the **shift operators** on $H_{|G|}$ for each $k \in G$ by

$$\begin{aligned} U(k) : H_{|G|} &\to H_{|G|}, \\ |g\rangle &\mapsto |k+g\rangle. \end{aligned}$$

Note that since G is abelian, the $U(k)$ commute: $U(k)U(l) = U(l)U(k)$ for all $k, l \in G$. Hence, they have simultaneous eigenstates, which gives an orthonormal basis for $H_{|G|}$.

Proposition 1.34 For each $k \in G$, consider the state

$$|\chi_k\rangle = \frac{1}{\sqrt{|G|}} \sum_{g\in G} \overline{\chi_k(g)}|g\rangle.$$

The $|\chi_k\rangle$ are shift-invariant (invariant up to a phase under the action of all $U(g)$, $g \in G$).

Proof (Hints). Straightforward. □

Proof. Since χ_k is a homomorphism, we have $\overline{\chi_k(g)} = \chi_k(-g)$. Now

$$\begin{aligned} U(g_0)|\chi_k\rangle &= \frac{1}{\sqrt{|G|}} \sum_{g\in G} \overline{\chi_k(g)}|g_0 + g\rangle \\ &= \frac{1}{\sqrt{|G|}} \sum_{g'\in G} \overline{\chi_k(g' - g_0)}|g'\rangle \\ &= \frac{1}{\sqrt{|G|}} \sum_{g'\in G} \overline{\chi_k(g')}\chi_k(g_0)|g'\rangle \\ &= \chi_k(g_0)|\chi_k\rangle. \end{aligned}$$

□

Definition 1.35 The **quantum Fourier transform (QFT)** on $H_{|G|}$ is the unitary implementing the change of basis from the shift-invariant states $\{|\chi_g\rangle : g \in G\}$ to the computational basis $\{|g\rangle : g \in G\}$.

Note that $\mathrm{QFT}^{-1}|g\rangle = |\chi_g\rangle$. So $\left(\mathrm{QFT}^{-1}\right)_{kg} = \langle k|\chi_g\rangle = \frac{1}{\sqrt{|G|}}\overline{\chi_g(k)}$, so $\mathrm{QFT}_{kg} = \frac{1}{\sqrt{|G|}}\chi_k(g)$. So the explicit form is

$$\mathrm{QFT}|g\rangle = \frac{1}{\sqrt{|G|}} \sum_{k\in G} \chi_k(g)|k\rangle.$$

Example 1.36

- For $G = \mathbb{Z}/M$, we can check that $\chi_a(b) = e^{2\pi i ab/M}$ are irreps. So the irreps of $\mathbb{Z}/M$ are naturally labelled by $a \in \mathbb{Z}/M$ and this gives the usual $\mathrm{QFT} \bmod M$ as defined earlier.

- Similarly, for $G = \mathbb{Z}/(M_1) \times \cdots \times \mathbb{Z}/(M_r)$, $\chi_g(h) = e^{2\pi i(g_1h_1/M_1+\cdots+g_rh_r/M_r)}$ are the irreps.

Algorithm 1.37 (Quantum HSP solver for finite abelian G) The algorithm solves the HSP for finite abelian G. We work in the state space $H_{|G|} \otimes H_{|X|}$.

1. Prepare the uniform superposition state
$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle|0\rangle$$
and query U_f on it.
2. Measure the output register, then discard this register.
3. Apply QFT mod $|G|$ to the input register, then measure this register.
4. Repeat the above steps $O(\log|G|)$ times.

Theorem 1.38 (Correctness of Quantum HSP Solver) The quantum HSP solver algorithm solves the HSP for finite abelian groups with high probability.

Proof. Querying U_f on the state gives
$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle|f(g)\rangle$$
Upon measurement of the output register, we obtain a uniformly random value $f(g_0)$ from $f(G)$, and the state collapses to a **coset state**
$$|g_0 + K\rangle = \frac{1}{\sqrt{|K|}} \sum_{k \in K} |g_0 + k\rangle.$$
We have $|K\rangle = \sum_{g \in G} a_g|\chi_g\rangle$, so $|g_0 + K\rangle = U(g_0)|K\rangle = \sum_{g \in G} a_g\chi_g(g_0)|\chi_g\rangle$. So applying QFT to the input state gives $\sum_{g \in G} a_g\chi_g(g_0)|g\rangle$, so the probability of measuring outcome k is $|a_k\chi_k(g_0)|^2 = |a_k|^2$. Now
$$\begin{aligned}\mathrm{QFT}|K\rangle &= \frac{1}{\sqrt{|K|}} \sum_{k \in K} \mathrm{QFT}|k\rangle \\ &= \frac{1}{\sqrt{|G||K|}} \sum_{g \in G} \left(\sum_{k \in K} \chi_g(k) \right) |g\rangle\end{aligned}$$
Note that irreps of G restricted to K are irreps of K. The trivial irrep $\chi_0 : G \to \mathbb{C}$ remains the trivial irrep χ_0 for K. But there may be other irreps that become the trivial irrep on restriction to K. Hence
$$\sum_{k \in K} \chi_g(k) = \begin{cases} |K| & \text{if } \chi_g|_K = \chi_0|_K \\ 0 & \text{otherwise} \end{cases}$$
Hence

$$\mathrm{QFT}|K\rangle = \sqrt{\frac{|K|}{|G|}} \sum_{\substack{g\in G \\ \chi_g|_K = \chi_0|_K}} |g\rangle$$

and measuring in the computational basis on this state yields random $g \in G$ such that $\forall k \in K$, $\chi_g(k) = 1$.

If K has generators $k_1, ..., k_m$ (note that for an arbitrary group, we have $m = O(\log|G|)$), then we have a set of equations $\chi_g(k_i) = 1$ for all $i \in [m]$. We can show that if $O(\log|G|)$ such g are drawn uniformly at random, then with probability at least $2/3$, we have enough equations to determine $k_1, ..., k_m$. □

Example 1.39 Let $G = \mathbb{Z}/M_1 \times \cdots \times \mathbb{Z}/M_r$. The irreps are

$$\chi_g(h) = e^{2\pi i(g_1h_1/M_1 + \cdots + g_rh_r/M_r)}.$$

For $k \in K$, $\chi_g(k) = 1$ iff $\frac{g_1k_1}{M_1} + \cdots + \frac{g_rk_r}{M_r} = 0 \bmod 1$. This is a homogenous linear equation in k, and $O(\log|G|)$ independent such equations determine K as the nullspace.

Remark 1.40 We can implement QFT over abelian groups (and some non-abelian groups, including S_n) using circuits with $O((\log|G|)^2)$ elementary gates.

In the non-abelian case, we can still easily prepare coset states with one query to f. But the shift operators $U(g_0)$ no longer commute, so we don't have a (canonical) shift-invariant basis.

Definition 1.41 A ***d*-dimensional unitary representation** of a finite group G is a homomorphism

$$\chi : G \to U(d)$$

where $U(d)$ is the group of $d \times d$ unitary matrices.

Definition 1.42 A d-dimensional unitary representation χ of G is **irreducible** if no non-trivial subspace of $\mathbb{C}^d$ is invariant under the action of $\{\chi(g_1), ..., \chi(g_{|G|})\}$ (i.e. we cannot simultaneously block diagonalise all the $\chi(g)$ matrices by a basis change).

Definition 1.43 A set of irreps $\{\chi_1, ..., \chi_m\}$ is a **complete set of irreps** for every irrep χ of G, there exists $1 \le i \le m$ such that χ is unitarily equivalent to χ_i, i.e. for some $V \in U(d)$, $\forall g \in G, \chi(g) = V\chi_i(g)V^\dagger$.

Theorem 1.44 Let the dimensions of a complete set of irreps $\chi_1, ..., \chi_m$ be $d_1, ..., d_m$. Then $d_1^2 + \cdots + d_m^2 = |G|$.

Notation 1.45 Write $\chi_{i,jk}(g)$ for the (j,k)-th entry of the matrix $\chi_i(g)$.

Theorem 1.46 (Schur Orthogonality) Let $\chi_1, ..., \chi_m$ be a complete set of irreps for G with respective dimensions $d_1, ..., d_m$, and let $i \in [m]$, $j, k \in [d_i]$. Then

$$\sum_{g\in G} \chi_{i,jk}(g)\overline{\chi_{i',j'k'}(g)} = |G|\delta_{ii'}\delta_{jj'}\delta_{kk'}.$$

Definition 1.47 The **Fourier basis** for a group G consists of

$$|\chi_{i,jk}\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \overline{\chi_{i,jk}(g)} |g\rangle$$

for each $i \in [n]$ and $j, k \in [d_i]$. Note that by Schur orthogonality, this is an orthonormal basis.

Remark 1.48 Note that these states are not shift invariant for every $U(g_0) : |g\rangle \mapsto |g_0 g\rangle$. So measurement of the coset state $|g_0 K\rangle$ yields an output distribution that is not independent of g_0.

Definition 1.49 The **Quantum Fourier transform** over $H_{|G|}$ is the unitary mapping the Fourier basis to the computational basis:

$$\mathrm{QFT}|\chi_{i,jk}\rangle = |i, jk\rangle.$$

$|i, jk\rangle$ is a relabelling of the states $|g\rangle$ for $g \in G$ (note this is valid by Theorem 1.44).

Remark 1.50

- Measuring $\mathrm{QFT}|g_0 K\rangle$ does **not** give g_0-independent outcomes. A complete measurement in the computational basis gives an outcome i, j, k.
- However, there is an incomplete measurement which projects into the d_i^2-dimensional subspaces

  $$S_i = \mathrm{span}\{|\chi_{i,jk}\rangle : j, k \in [d_i]\}.$$

  for each $i \in [n]$. Call this measurement operator M_{rep}. Note that this distinguishes only between the irreps.
- Measuring only the representation labels of $\mathrm{QFT}|g_0 K\rangle$ gives an outcome distribution of the i values that i independent of the random shift g_0, since the χ_i are homomorphisms.
- Note this only gives partial information about K. If K is a normal subgroup, then in fact we can then determine K with $O(\log|G|)$ queries.

2. Quantum phase estimation (QPE)

Quantum phase estimation is a unifying algorithmic primitive, e.g. there is an alternative factoring algorithm based on QPE, and has many important applications in physics.

Problem 2.1 (Quantum Phase Estimation)

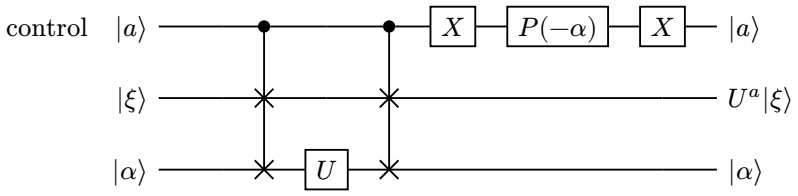
Input Unitary $U \in U(d)$ acting on $\mathbb{C}^d$; state $|v_\varphi\rangle \in \mathbb{C}^d$; level of precision $n \in \mathbb{N}$.

Promise $|v_\varphi\rangle$ is an eigenstate of U with **phase** (eigenvalue) $e^{2\pi i\varphi}$, $\varphi \in [0, 1)$ (i.e. $U|v_\varphi\rangle = e^{2\pi i\varphi}|v_\varphi\rangle$).

Task Output an estimate $\tilde{\varphi}$ of φ, accurate to n binary bits of precision.

Remark 2.2 If U is given as a circuit, we can implement the controlled-U operation, $C\text{–}U$, by controlling each elementary gate in the circuit of U.

If U is given as a black box, we need more information. Note that U is equivalent to $U' = e^{i\theta}U$ and $|\psi\rangle$ is equivalent to $e^{i\theta}|\psi\rangle$, but $C\text{–}U$ is not equivalent to $C\text{–}U'$. Given an eigenstate $|\alpha\rangle$ with known phase $e^{i\alpha}$ (so $U|\alpha\rangle = e^{i\alpha}|\alpha\rangle$), we have $U'|\alpha\rangle = e^{i(\theta+\alpha)}|\alpha\rangle$. so U and U' can be distinguished using this additional information. The following circuit implements $C\text{–}U$ (the top two lines end in state $C\text{–}U|a\rangle|\xi\rangle$):

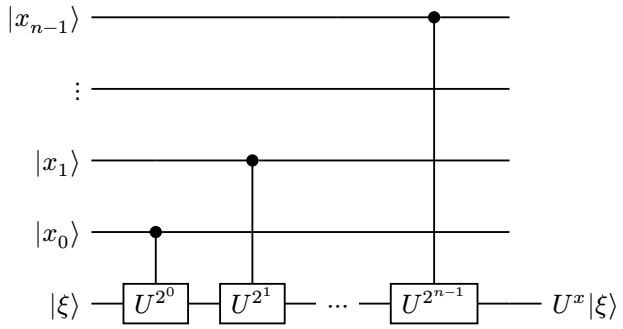


where $P(-\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$, and $\bullet\text{–}\times\text{–}\times$ denotes the controlled SWAP operation.

Definition 2.3 For a unitary U, the **generalised control** unitary $C\text{–}U$ is defined linearly by

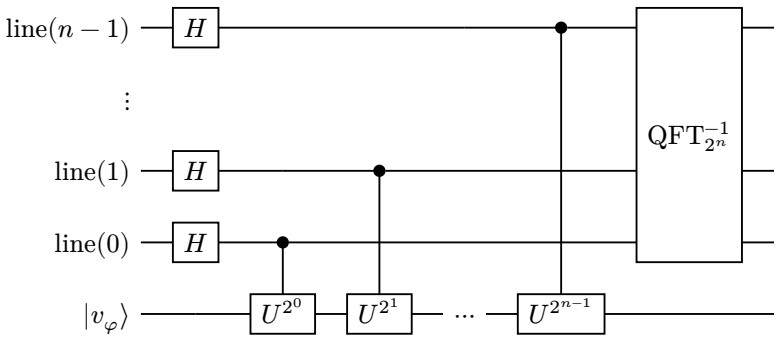
$$\forall x \in \{0,1\}^n, \quad C\text{–}U|x\rangle|\xi\rangle = |x\rangle U^x|\xi\rangle,$$

where U^x denotes U applied x times (e.g. $C\text{–}U|11\rangle|\xi\rangle = |11\rangle U^3|\xi\rangle$). Note that $C\text{–}U^k = (C\text{–}U)^k$. The following circuit implements $C\text{–}U$:



Algorithm 2.4 (Quantum Phase Estimation) Work over the space $(\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^d$, where $(\mathbb{C}^2)^{\otimes n}$ is the n-qubit register, $\mathbb{C}^d$ is the "qudit" register.

1. Apply the following circuit to $|0...0\rangle|v_\varphi\rangle$:



2. Discard the qudit register holding $|v_\varphi\rangle$, and measure the input qubits, yielding outcome $y_0...y_{n-1}$ from lines $0, ..., n-1$.
3. The estimate of φ is $\tilde{\varphi} = y/2^n = y_0/2 + \cdots + y_{n-1}/2^n$.

Remark 2.5 After $C\text{–}U^{2^{n-1}}$, the input qubits are in the state

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} e^{2\pi i \varphi x} |x\rangle.$$

If φ had an exact n-bit expansion $0.i_1 i_2 ... i_n = (i_1 ... i_n)/2^n =: \varphi_n/2^n$, then this state is precisely $\mathrm{QFT}_{2^n} |\varphi_n\rangle$, in which case, after applying QFT^{-1}, we have $|\varphi_n\rangle$, so measuring the input bits gives φ_n, and so φ, exactly.

Lemma 2.6 For all $\alpha \in \mathbb{R}$,
1. If $|\alpha| \leq \pi$, then $\left|1 - e^{i\alpha}\right| = 2|\sin(\alpha/2)| \geq \frac{2}{\pi}|\alpha|$.
2. If $\alpha \geq 0$, then $\left|1 - e^{i\alpha}\right| \leq \alpha$.

Proof (Hints). For both, think graphically. □

Proof.
1. The line $y = \frac{2}{\pi}\alpha$ lies below $2\sin(\alpha/2)$ for $0 \leq \alpha \leq \pi$).
2. On the complex unit circle, the arc length α from 1 to $e^{i\alpha}$ is at least the chord length from 1 to $e^{i\alpha}$.

□

Theorem 2.7 (Phase Estimation Theorem) Let $\tilde{\varphi}$ be the estimate of φ from the quantum phase estimation algorithm. Then
1. $\Pr(\tilde{\varphi}$ is closest n-bit approximation of $\varphi) \geq \frac{4}{\pi^2} \approx 0.4$.
2. For all $\varepsilon > 0$, $\Pr(|\tilde{\varphi} - \varphi| > \varepsilon) = O\left(\frac{1}{2^n \varepsilon}\right)$. So for any desired accuracy ε, the probability of failure decays exponentially with the number of bits of precision (lines in the circuit).

Proof (Hints). Let $\delta(y) = \varphi - y/2^n = \varphi - \tilde{\varphi}$. Show the probability of the measuring yielding outcome y is

$$p_y = \frac{1}{2^{2n}} \left| \frac{1 - e^{2^n 2\pi i \delta(y)}}{1 - e^{2\pi i \delta(y)}} \right|^2.$$

1. Find an upper bound on $\delta(a)$ where a is the closest n-bit approximation of φ.
2. Show that

$$p_y \leq \frac{1}{2^{2n}} \left(\frac{2}{4\delta(y)} \right)^2 = \frac{1}{2^{2n+2}\delta(y)^2}.$$

Let $B = \{y \in \{0,1\}^n : |\delta(y)| > \varepsilon\}$. Show that for each $y \in B$, $|\delta(y)| \leq \varepsilon + k_y/2^n$ for some $k_y \in \mathbb{N}$, and that each k_y occurs at most twice here. Conclude the upper bound using an integral.

□

Proof. Let

$$|A\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n - 1} e^{2\pi i \varphi x} |x\rangle.$$

Let $\delta(y) = \varphi - y/2^n = \varphi - \tilde{\varphi}$. Since $\mathrm{QFT}^{-1}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2\pi i x y/2^n} |y\rangle$, we have

$$\mathrm{QFT}^{-1}|A\rangle = \frac{1}{2^n} \sum_{y=0}^{2^n-1} \sum_{x=0}^{2^n-1} e^{2\pi i x \delta(y)} |y\rangle$$

so the probability of measuring outcome y is

$$p_y = \Pr\Big(\tilde{\varphi} = \frac{y}{2^n}\Big) = \frac{1}{2^{2n}} \left| \frac{1 - e^{2^n 2\pi i \delta(y)}}{1 - e^{2\pi i \delta(y)}} \right|^2.$$

1. Let $\alpha = 2^n 2\pi\delta(a)$, where a is the closest n-bit approximation of φ. Note we can imagine the possible values of $\tilde{\varphi}$ as lying on the unit circle, spaced by angle $\frac{2\pi}{2^n}$. This gives a visual intuition to the fact that $|\delta(a)| \leq \frac{1}{2^{n+1}}$. Hence $|\alpha| \leq \pi$, and so by the above lemma,

$$p_a = \Pr(\tilde{\varphi} = a) \geq \frac{1}{2^{2n}} \left(\frac{2^{n+2}\delta(a)}{2\pi\delta(a)} \right)^2 = \frac{4}{\pi^2}.$$

2. Note that $\left|1 - e^{2^n 2\pi i \delta(y)}\right| \leq 2$ by the triangle inequality. Let $B = \{y \in \{0,1\}^n : |\delta(y)| > \varepsilon\}$ denote the set of "bad" values of y. For all $y \in \{0,1\}^n$, we have $\delta(y) \in [-1, 1]$. If $|\delta(y)| \leq 1/2$, then, by the above lemma, we have $\left|1 - e^{2\pi i \delta(y)}\right| \geq 4|\delta(y)|$. If $\delta(y) > 1/2$, then $\delta(y) - 1 \in [-1/2, 1/2]$, so by the above lemma, $\left|1 - e^{2\pi i \delta(y)}\right| \geq 4|\delta(y) - 1|$ hence

$$p_y \leq \frac{1}{2^{2n}} \left(\frac{2}{4\delta(y)} \right)^2 = \frac{1}{2^{2n+2}\delta(y)^2}.$$

Let $\delta^+ = \min\{\delta(y) : y \in B, \delta(y) > 0\}$ be the smallest $\delta(y)$ such that $\delta(y) > \varepsilon$, and $\delta^- = \max\{\delta(y) : y \in B : \delta(y) < 0\}$ be the largest $\delta(y)$ such that $\delta(y) < -\varepsilon$. For all $y \in B$, we have $\delta(y) = \delta^+ + k_y/2^n$ or $\delta(y) = \delta^- - k_y/2^n$ for some $k_y \in \mathbb{N}$, so $|\delta(y)| > \varepsilon + k_y/2^n$. Note that each $k \in \mathbb{N}$, $k = k_y$ for at most 2 values of $y \in B$. Hence,

$$\begin{aligned}
\Pr(|\delta(y)| > \varepsilon) = \Pr(y \in B) &= \sum_{y \in B} p_y \\
&\leq \sum_{y \in B} \frac{1}{2^{2n+2}\big(\varepsilon + k_y/2^n\big)^2} \\
&< 2 \sum_{k=0}^{\infty} \frac{1}{2^{2n+2}} \frac{1}{(\varepsilon + k/2^n)^2} \\
&\leq \frac{1}{2^{2n+1}\varepsilon^2} + \sum_{k=1}^{\infty} \frac{1}{2^{2n+1}} \frac{1}{(\varepsilon + k/2^n)^2} \\
&= \frac{1}{2^{2n+1}\varepsilon^2} + \int_0^{\infty} \frac{1}{2^{2n+1}} \frac{1}{(\varepsilon + x/2^n)^2} \, \mathrm{d}x \\
&= \frac{1}{2^{2n+1}\varepsilon^2} + \int_{2^n\varepsilon}^{\infty} \frac{1}{2u^2} \, \mathrm{d}u = \frac{1}{2^{2n+1}\varepsilon^2} + \frac{1}{2^{n+1}\varepsilon}.
\end{aligned}$$

□

Remark 2.8 The QPE algorithm excluding the measurement is a unitary - call this unitary U_{PE}. If we apply U_{PE} to an arbitrary state $|\psi\rangle = \sum_j c_j|v_j\rangle$ where $|v_j\rangle$ are the eigenstates of U with eigenvalue $e^{2\pi i\varphi_j}$, then we have

$$U_{\text{PE}}|\psi\rangle = \sum_j c_j|\tilde{\varphi}_j\rangle|v_j\rangle$$

If every φ_j has an exact n-bit representation, then this is exact. Otherwise, we have $|\tilde{\varphi}_j\rangle = \sqrt{1-\eta}|\tilde{\varphi}_1\rangle + \sqrt{\eta}|\tilde{\varphi}_0\rangle$, where $|\tilde{\varphi}_1\rangle$ is a superposition of all n-bit strings that are correct to the first n-bits of φ, and $|\tilde{\varphi}_0\rangle$ is a superposition of strings with the first n bits not all correct.

Remark 2.9 Complexity of QPE: we use $C\text{–}U, ..., C\text{–}U^{2^{n-1}}$, so the number of uses of $C\text{–}U$ is $\approx 2^n$. So this initially looks like exponential time, but there are special cases of U where by repeated squaring, this can be implemented with $\text{poly}(n)$ gates.

If we want to estimate φ accurate to m bits of precision with probability $1-\eta$, then by the phase estimation theorem with $\varepsilon = \frac{1}{2^m}$, we need $n = O(m + \log(1/\eta))$ lines. Note this is a modest, polynomial increase in the number of lines of the circuit for an exponential reduction in η.

3. Amplitude amplification

Amplitude amplification is an extension of the key insights in Grover's algorithm (TODO: read part II notes for Grover's).

Notation 3.1 Given $|\alpha\rangle \in H_d$, write $L_{|\alpha\rangle} = \text{span}\{|\alpha\rangle\}$ for the one-dimensional subspace generated by $|\alpha\rangle$, and $L_a^\perp$ for its $(d-1)$-dimensional orthogonal complement.

Notation 3.2 Given a k-dimensional subspace $A \leq H_d$ with orthonormal basis $\{|a_1\rangle, ...,|a_k\rangle\}$, denote the projector onto the subspace A by $P_A = \sum_{i=1}^k |a_i\rangle\langle a_i|$. Note that P_A is independent of the orthonormal basis.

Notation 3.3 Given a subspace $A \leq H_d$, define the unitary $I_A = I - 2P_A$, which is the reflection in the "mirror" $A^\perp$: indeed, not that for all $|\varphi\rangle \in A$, $I_A = -|\varphi\rangle$, and for all $|\psi\rangle \in A^\perp$, $I_A|\psi\rangle = |\psi\rangle$, since $P_A|\psi\rangle = 0$.

In the case that A is one-dimensional and spanned by $|\alpha\rangle$, we have $P_A = |\alpha\rangle\langle\alpha|$, and write $I_{|\alpha\rangle} = I - 2|\alpha\rangle\langle\alpha|$.

Proposition 3.4 Let $|\alpha\rangle \in H_d$. For any unitary $U \in U(d)$, we have

$$UI_{|\alpha\rangle}U^\dagger = I_{U|\alpha\rangle}.$$

Proof (Hints). Trivial. □

Proof. $UI_{|\alpha\rangle}U^\dagger = UU^\dagger - 2U|\alpha\rangle\langle\alpha|U^\dagger = I_{U|\alpha\rangle}$. □

Problem 3.5 (Unstructured Search Problem)

Input An oracle for a function $f : \{0,1\}^n \to \{0,1\}$.
Promise There is a unique $x_0 \in \{0,1\}^n$ such that $f(x_0) = 1$.
Task Find x_0.

Remark 3.6 The unstructured search problem is closely related to the complexity class NP and to Boolean satisfiability.

Definition 3.7 For fixed $|x_0\rangle \in H_2^{\otimes n}$, the **Grover iteration operator** Q is defined as

$$Q := -H^{\otimes n} I_{|0\rangle} H^{\otimes n} I_{|x_0\rangle} = -I_{H^{\otimes n}|0\rangle} I_{|x_0\rangle}.$$

Remark 3.8 Note that for a function $f : \{0,1\}^n \to \{0,1\}$ fulfilling the promise of the Unstructured Search Problem, we can implement $I_{|x_0\rangle}$ without knowing x_0: we have $U_f|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{f(x)}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Hence, implementing Q requires only one query to f.

Theorem 3.9 (Grover) In the 2-dimensional subspace spanned by $|\psi\rangle = H^{\otimes n}|0\rangle$ and $|x_0\rangle$, the action of Q is a rotation by angle 2α, where $\sin(\alpha) = \frac{1}{\sqrt{2^n}} = \langle x_0|\psi\rangle$.

Algorithm 3.10 (Grover's Algorithm) Work in the state space $H_2^{\otimes n}$.
1. Prepare $|\psi\rangle = H^{\otimes n}|0\rangle$.
2. Apply Q^m to $|\psi\rangle$, where m is closest integer to $\frac{\arccos\left(1/\sqrt{N}\right)}{2\arcsin\left(1/\sqrt{N}\right)} = \frac{\theta}{2\alpha}$ and $\cos(\theta) = \sin(\alpha) = \langle x_0|\psi\rangle = 1/\sqrt{2^n}$. This rotates $|\psi\rangle$ to be close to $|x_0\rangle$ (within angle $\pm\alpha$ of $|x_0\rangle$).
3. Measure to get x_0 with probability $p = |\langle x_0|Q^m|\psi\rangle|^2 = 1 - \frac{1}{N}$. For large N, $\arccos\left(1/\sqrt{N}\right) \approx \frac{\pi}{2}$, and $\arcsin\left(1/\sqrt{N}\right) \approx 1/\sqrt{N}$. The number of iterations is $m = \frac{\pi}{4}\sqrt{N} = O\left(\sqrt{N}\right)$. So we need $O\left(\sqrt{N}\right)$ queries to U_f. In contrast, clasically we need $\Omega(N)$ queries to f to find x_0 with any desired constant probability. Note that $\Omega(N)$ queries are both necessary and sufficient.

Notation 3.11 Write G for the subspace of the state space H whose associated amplitudes in a given state we wish to amplify. G is called the "good" subspace. We call the subspace $G^\perp$ the "bad" subspace. Note that $H = G \oplus G^\perp$, and for any state $|\varphi\rangle \in H$, there is a unique decomposition with real, positive coefficients $|\varphi\rangle = \sin(\theta)|g\rangle + \cos(\theta)|b\rangle$, where $|g\rangle = P_G|\varphi\rangle$ and $|b\rangle = P_{G^\perp}|\varphi\rangle$.

Theorem 3.12 (Amplitude Amplification Theorem/2D-subspace Lemma) Let $G \leq H_2^{\otimes n}$ be a subspace and $|g\rangle = P_G|\psi\rangle$, $|b\rangle = P_{G^\perp}|\psi\rangle$. In the 2-dimensional subspace $\text{span}\{|\psi\rangle,|g\rangle\} = \text{span}\{|b\rangle,|g\rangle\}$, the unitary

$$Q = -I_{|\psi\rangle} I_G$$

is a rotation by angle 2θ, where $\sin(\theta) = \|P_G|\psi\rangle\|_2^2 = |\langle g|g\rangle|$ is the length of the "good" projection of $|\psi\rangle$.

Proof (Hints). Consider the matrix representation of Q in the $\text{span}\{|b\rangle,|g\rangle\}$ basis. □

Proof. By definition, we have $I_G|g\rangle = -|g\rangle$, and $I_G|b\rangle = |b\rangle$. Hence $Q|g\rangle = I_{|\psi\rangle}|g\rangle$ and $Q|b\rangle = -I_{|\psi\rangle}|b\rangle$. The matrix representation of $I_{|\psi\rangle}$ in the $\{|b\rangle,|g\rangle\}$ basis is

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 2\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}[\cos(\theta)\ \ \sin(\theta)] = \begin{bmatrix}1-2\cos(\theta)^2 & -2\sin(\theta)\cos(\theta)\\ -2\sin(\theta)\cos(\theta) & 1-2\sin(\theta)^2\end{bmatrix}$$
$$= \begin{bmatrix}-\cos(2\theta) & -\sin(2\theta)\\ -\sin(2\theta) & \cos(2\theta)\end{bmatrix}.$$

So $Q|b\rangle = \cos(2\theta)|b\rangle + \sin(2\theta)|g\rangle$, and $Q|g\rangle = -\sin(2\theta)|b\rangle + \cos(2\theta)|g\rangle$. So the matrix representation of Q in the $\{|b\rangle,|g\rangle\}$ basis is

$$\begin{bmatrix}\cos(2\theta) & -\sin(2\theta)\\ \sin(2\theta) & \cos(2\theta)\end{bmatrix}$$

which indeed is a rotation by angle 2θ. □

Corollary 3.13 We have $Q^m|\psi\rangle = \cos((2m+1)\theta)|b\rangle + \sin((2m+1)\theta)|g\rangle$.

Proof (Hints). Trivial. □

Proof. Induction on m. □

Notation 3.14 Denote by m_{opt} the $m \in \mathbb{Z}$ which maximises the probability of measuring (in the $\{|b\rangle,|g\rangle\}$ basis) an outcome in G on the state $Q^m|\psi\rangle$. Note that this probability is equal to $\sin((2m+1)\theta)^2$, which is maximised when

$$(2m+1)\theta = \pi/2 \quad \Longrightarrow m = \pi/4\theta - 1/2.$$

So m_{opt} is the nearest integer to $\pi/4\theta - 1/2$.

Example 3.15 Let $\theta = \pi/6$, then $m_{\text{opt}} = 1$ and $Q|\psi\rangle = |g\rangle$. So we obtain a good outcome with certainty on measurement.

Remark 3.16 Note that since Q is a rotation by angle 2θ, $Q^{m_{\text{opt}}}|\psi\rangle$ lies within angle $\pm\theta$ of $|g\rangle$, hence the $|g\rangle$ component of $Q^{m_{\text{opt}}}|\psi\rangle$ has amplitude $\geq \cos(\theta)^2$. TODO: insert diagram. So for small θ,

$$\Pr(\text{measuring good outcome}) \geq \cos(\theta)^2 \approx 1 - O(\theta^2).$$

Also, for small θ,

$$m_{\text{opt}} = O(1/\theta) \approx O(1/\sin(\theta)).$$

Remark 3.17 Q can be implemented (efficiently) if $I_{|\psi\rangle}$ and I_G can be implemented (efficiently). For an efficient implementation of I_G, it suffices for G to be spanned by some subset of computational basis states, and the indicator function $\mathbb{1}_G$ is efficiently computable. In this case, we have

$$U_{\mathbb{1}_G}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{\mathbb{1}_G(x)}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Since I_G is defined by its action $|g\rangle \mapsto -|g\rangle$ for $g \in G$ and $|b\rangle \mapsto |b\rangle$ for $b \in G^\perp$, the first register holds the value $I_G|x\rangle$.

For an efficient implementation of $I_{|\psi\rangle}$, we usually have $|\psi\rangle = H^{\otimes n}|0...0\rangle$, and then $I_{|\psi\rangle} = H^{\otimes n} I_{|0\rangle} H^{\otimes n}$ can be implemented with $O(n)$ gates.

Remark 3.18 In the amplitude amplification process, the relative amplitudes of basis states inside $|g\rangle$ and $|b\rangle$ won't change. So amplitude amplification boosts the overall amplitude of $|g\rangle$ at the expense of the amplitude of $|b\rangle$.

3.1. Applications of amplitude amplification

Example 3.19 We can generalise Grover search from 1 marked item to k marked items out of $N = 2^n$ items. In this case,

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \\ &= \sqrt{\frac{k}{N}} \frac{1}{\sqrt{k}} \sum_{x\in G} |x\rangle + \sqrt{\frac{N-k}{N}} \frac{1}{\sqrt{N-k}} \sum_{x\in G^\perp} |x\rangle \\ &=: \sqrt{\frac{k}{N}} |g\rangle + \sqrt{\frac{N-k}{N}} |b\rangle\end{aligned}$$

so $\sin(\theta) = \sqrt{k/N}$. For $k \ll N$, $\sin(\theta) \approx \theta$, so $m_{\text{opt}} = O(\sqrt{N/k})$ uses of Q required. E.g. $N = 4 = 2^2$ items and $k = 1$ marked item, we have $\sin(\theta) = 1/2$, so $\theta = \pi/6$, so Grover search is exact, and requires exactly one application of Q.

Example 3.20 (Quadratic speedup of general quantum algorithms) Let U be a unitary representing a quantum algorithm/circuit, with $U|0...0\rangle = \alpha|g\rangle + \beta|b\rangle$ where $|g\rangle$ is a (generally non-uniform) superposition of good/correct outcomes, and $|b\rangle$ is a (generally non-uniform) superposition of bad/incorrect outcomes. Note $|g\rangle = \sum_{x\in\{0,1\}^n} c_x|x\rangle$ is generally a non-uniform superposition. We have

$$\Pr(\text{measuring } U|0...0\rangle \text{ yields good outcome}) = |\alpha|^2.$$

Thus, we need to run U about $O(1/|\alpha|^2)$ times to succeed with high probability.

Now define $|\psi\rangle = U|0...0\rangle$ and $Q = -I_{|\psi\rangle} I_G$. We can implement Q if we have a method to verify the output of U; so in particular, we can use this method for any NP problem. This will mean we can efficiently implement the indicator function $\mathbb{1}_G$ of good labels and therefore also I_G. So by the Amplitude Amplification Theorem, Q performs a rotation of 2θ where $\sin(\theta) = |\alpha|$. So after approximately

$$m_{\text{opt}} \approx \pi/4\theta = O(1/\theta) = O(1/\sin(\theta)) = O(1/|\alpha|)$$

(for θ small) uses of Q, we get a good outcome with high probability.

Problem 3.21 (Counting Problem)
Input $f : \{0,1\}^n \to \{0,1\}$.
Task Estimate the number $k = \left|f^{-1}(\{1\})\right|$ of inputs that evaluate to 1.

Example 3.22 (Quantum Counting) This combines amplitude amplification and quantum phase estimation. Let the "good" subspace G be the subspace with basis $f^{-1}(\{1\})$. As usual, let

$$|g\rangle := \frac{1}{\sqrt{k}} \sum_{x \in f^{-1}(\{1\})} |x\rangle, \quad |b\rangle := \frac{1}{\sqrt{2^n - k}} \sum_{x \in f^{-1}(\{0\})} |x\rangle,$$

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle = \sqrt{\frac{k}{N}} \frac{1}{\sqrt{k}} \sum_{x \in f^{-1}(\{1\})} |x\rangle + \sqrt{\frac{N-k}{N}} \frac{1}{\sqrt{N-k}} \sum_{x \in f^{-1}(\{0\})} |x\rangle.$$

Recall that Q has matrix representation

$$Q = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

in the orthonormal basis $\{|b\rangle, |g\rangle\}$ where $\sin(\theta) = \|P_G|\psi\rangle\|$. The eigenvalues and eigenstates of Q are $\lambda_\pm = e^{\pm 2i\theta}$ and $|e_\pm\rangle = \frac{1}{\sqrt{2}}(|b\rangle \mp i|g\rangle)$. So we can write $|\psi\rangle = \sin(\theta)|g\rangle + \cos(\theta)|b\rangle = \frac{1}{\sqrt{2}}\big(e^{-i\theta}|e_+\rangle + e^{i\theta}|e_-\rangle\big)$. So $|\psi\rangle$ is an equally-weighted superposition of eigenstates of Q. Write $e^{\pm 2i\theta} = e^{2\pi i \varphi_\pm}$ with $\varphi_\pm \in (0,1)$. We have $\varphi_+ = \theta/\pi$ and $\varphi_- = (-2\theta + 2\pi)/2\pi = 1 - \theta/\pi$. When $k \ll N$, $\sin(\theta) = \sqrt{k/N} \approx \theta$, so using U_{PE} with m qubits of precision

$$U_{\text{PE}}|\psi\rangle = \frac{1}{\sqrt{2}}\big(e^{-i\theta}|e_+\rangle|\tilde{\varphi}_+\rangle + e^{i\theta}|e_-\rangle|\tilde{\varphi}_-\rangle\big)$$

Measuring the QPE output gives (with probability 1/2) an estimate of $\varphi_+ = \theta/\pi \approx \frac{1}{\pi}\sqrt{k/N}$ or (with probability 1/2) an estimate of $\varphi_- = 1 - \theta/\pi \approx 1 - \frac{1}{\pi}\sqrt{k/N}$. So in either case, we get an estimate of $\sqrt{k/N}$ (since we can tell when $k \ll N$ which case we are in). By the Phase Estimation Theorem, with probability at least $4/\pi^2$, QPE with m lines gives us an approximation of $\sqrt{k/N}$ to precision $O(1/2^m)$, using $O(2^m)$ C–Q operations, each of which requires one query to f. Write $\delta/\sqrt{2^n} = 1/2^m$ for some $\delta > 0$. So we can estimate $\sqrt{k}$ to precision δ, and since $\Delta(x^2) = 2x\Delta(x)$, we estimate $\sqrt{k}$ to additive error (precision) $O\big(\delta\sqrt{k}\big)$ using $O(2^m) = O\big(\sqrt{N}/\delta\big)$ queries to f.

Remark 3.23 The quantum counting algorithm is quadratically faster than the best possible classical algorithm, which is:

- Sample random x from $\{0,1\}^n$, then $\Pr(f(x) = 1) = k/N$.
- Draw m samples $x_1, ..., x_m$, then the estimate is $\tilde{k} = \ell N/m$, where $m = |\{i \in [m] : f(x_i) = 1\}|$.

We need $m = O(N/\delta^2)$ to estimate k to high precision.

4. Hamiltonian simulation

We want to use a quantum system to simulate the evolution/dynamics of another quantum system, given its Hamiltonian H. For an n-qubit system, in general this

requires $O(2^n)$ time on a classical computer. For some physically interesting classes of H, we have quantum algorithms that run in $O(\text{poly}(n))$ time.

Proposition 4.1 The Schrodinger equation which governs the time evolution of a physical state $|\psi(t)\rangle$, which is given by (assuming $\hbar = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -iH|\psi(t)\rangle,$$

has solution

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

when H is time-independent.

Definition 4.2 The **exponential** of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\exp(A) = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Note that if $[A, B] = 0$, then $\exp(A)\exp(B) = \exp(A + B)$, but generally this does not hold when $[A, B] \neq 0$.

Theorem 4.3 If H is Hermitian, then e^{-iHt} is unitary for all $t \in \mathbb{R}$.

Definition 4.4 $U(t) = e^{-iHt}$ is called the **evolution operator**. Given H and $t > 0$, we want to simulate $U(t)$ accurately.

Proposition 4.5 If $A \in \mathbb{C}^{n\times n}$ has spectrum $\{(\lambda_i, |e_i\rangle) : i \in [n]\}$, then

$$\exp(A) = \sum_{i=1}^{n} \exp(\lambda_i)|e_i\rangle.$$

Definition 4.6 The **operator norm** (**spectral norm**) of an operator $A : H \to H$ acting on the space H of states is

$$\|A\| := \max\{\|A|\psi\rangle\| : \psi \in H, \|\psi\| = 1\}.$$

Theorem 4.7 If A is diagonalisable with eigenvalues $\lambda_1, ..., \lambda_n$, then

$$\|A\| = \max\{|\lambda_1|, ..., |\lambda_n|\}.$$

Proposition 4.8 The operator norm satisfies the following properties:
1. **Submultiplicative**: $\|AB\| \leq \|A\|\|B\|$
2. **Triangle inequality**: $\|A + B\| \leq \|A\| + \|B\|$.

Definition 4.9 Let $U, \tilde{U} : H \to H$ be operators. $\tilde{U}$ **ε-approximates** U if

$$\left\|U - \tilde{U}\right\| \leq \varepsilon,$$

i.e. for all normalised states $|\psi\rangle$, $\left\|U|\psi\rangle - \tilde{U}|\psi\rangle\right\| \leq \varepsilon$.

Lemma 4.10 Let $U_1, ..., U_m, \tilde{U}_1, ..., \tilde{U}_m$ be unitaries. Suppose $\tilde{U}_i$ ε-approximates U_i for each $1 \leq i \leq m$. Then

$$\left\|U_n \cdots U_1 - \tilde{U}_n \cdots \tilde{U}_1\right\| \leq n\varepsilon$$

So the error increases at most linearly.

Definition 4.11 H is a ***k*-local Hamiltonian on *n* qubits** if we can write

$$H = \sum_{j=1}^{m} H_j$$

where each H_j acts non-trivially on at most k qubits, in which case we write $H_j = \widetilde{H}_j \otimes I$ (note these qubits need not be adjacent).

Note that $m \leq \binom{n}{k} = O\big(n^k\big)$, and we usually take k to be a constant.

Notation 4.12 Write $U_{(i)}$ for the unitary

$$I \otimes \cdots \otimes I \otimes U \otimes I \otimes \cdots \otimes I$$

where U is in the i-th position, i.e. $U_{(i)}$ is the unitary acting on the i-th qubit on n-qubits.

Example 4.13

- $H = X \otimes I \otimes I - 5Z \otimes Y \otimes I$ is 2-local on 3 qubits.
- For the **Ising model** on an $n \times n$ grid, where each qubit acts non-trivially only with its neighbours, the Hamiltonian is

$$H = J \sum_{i,j=1}^{n} \left(Z_{(i,j)}Z_{(i,j+1)} + Z_{(i,j)}Z_{(i+1,j)}\right)$$

  where $J \in \mathbb{R}$ is a coupling constant.
- For the **Heisenberg model** on a line, the Hamiltonain is

$$H = \sum_{i=1}^{n-1}\Big(J_X X_{(i)}X_{(i+1)} + J_Y Y_{(i)}Y_{(i+1)} + J_Z Z_{(i)}Z_{(i+1)}\Big),$$

  where $J_X, J_Y, J_Z \in \mathbb{R}$ are constants.

Theorem 4.14 (Solovay-Kitaev) Let U be a unitary on k-qubits, and S be a universal set of elementary gates. Then U can be ε-approximated using $O((\log(1/\varepsilon))^c)$ gates from S, where $c < 4$ is a constant.

Proof. Omitted. □

Proposition 4.15 Let $H = \sum_{j=1}^{m} H_j$ be a k-local Hamiltonian where all the local terms H_j commute. Then for all $t > 0$ and $\varepsilon > 0$, the evolution operator $U(t) = e^{-iHt}$ can be ε-approximated by a circuit with $O(m\,\mathrm{polylog}(m/\varepsilon))$ gates from any universal gate set.

Note that $m = O\big(n^k\big)$, so the time-complexity is polynomial in n.

Proof (Hints). Straightforward. □

Proof. Fix $t > 0$ and $\varepsilon > 0$. We have

$$U(t) = e^{-iHt} = e^{-i\sum_{j=1}^{m} H_j t} = \prod_{j=1}^{m} e^{-iH_j t}.$$

Each $e^{-iH_j t}$ is a unitary that acts non-trivially on at most k qubits. So we have a circuit for e^{-iHt} in terms of some set of k-qubit gates. By Solovay-Kitaev, each $e^{-iH_j t}$ can be δ-approximated by a unitary $\tilde{U}_j(t)$ circuit with $O(\text{polylog}(1/\delta))$ gates. By Lemma 4.10, we have

$$\left\| U(t) - \prod_{i=1}^{m} \tilde{U}_j(t) \right\| < m\delta.$$

So choosing $\delta = \varepsilon/m$, we obtain a circuit of size $O(m \,\text{polylog}(m/\varepsilon))$ which ε-approximates $U(t)$. □

Notation 4.16 For $N \times N$ matrices X and Y, write $X = Y + O(\varepsilon)$ to mean $X = Y + E$ where $\|E\| \leq \varepsilon$.

Lemma 4.17 (Lie-Trotter Product Formula) Let A and B be $N \times N$ matrices with $\|A\|, \|B\| \leq \delta < 1$. Then

$$e^{-iA}e^{-iB} = e^{-i(A+B)} + O(\delta^2).$$

Proof (Hints). Write $e^{-iA} = I - iA + E_A$ and show that $\|E_A\| = O(\delta^2)$, do the same for two other matrices.b □

Proof. We have

$$e^{-iA} = I - iA + \sum_{j=2}^{\infty} \frac{(-iA)^j}{j!} =: I - iA + E_A.$$

Now

$$\begin{aligned}
\|E_A\| &= \left\| (-iA)^2 \sum_{j=0}^{\infty} \frac{(-iA)^j}{(j+2)!} \right\| \\
&\leq \|(-iA)^2\| \cdot \left\| \sum_{j=0}^{\infty} \frac{(-iA)^j}{(j+2)!} \right\| && \text{by submultiplicativity} \\
&\leq \|(-iA)^2\| \cdot \sum_{j=0}^{\infty} \frac{\|(-iA)^j\|}{(j+2)!} && \text{by triangle inequality and continuity} \\
&\leq \|A\|^2 \sum_{j=0}^{\infty} \frac{\|(-iA)\|^j}{j!} && \text{by submultiplicativity} \\
&= \delta^2 e^{\delta} \leq \delta^2.
\end{aligned}$$

So $e^{-iA} = I - iA + O(\delta^2)$. By the same argument, we have

$$\begin{aligned}
e^{-iB} &= I - iB + O(\delta^2), \\
e^{-i(A+B)} &= I - i(A+B) + O(2\delta^2) = I - i(A+B) + O(\delta^2)
\end{aligned}$$

since $\|A+B\| \le \|A\| + \|B\| = 2\delta$. Hence,

$$\begin{aligned} e^{-iA}e^{-iB} &= (I - iA + O(\delta^2))(I - iB + O(\delta^2)) \\ &= I - i(A+B) + O(\delta^2) \\ &= e^{-i(A+B)} + O(\delta^2), \end{aligned}$$

since $\|AB\| \le \delta^2$ by submultiplicativity. □

Proposition 4.18 There is a $\mathrm{poly}(n, 1/\varepsilon, t)$-time quantum algorithm for simulating the evolution operators of k-local Hamiltonians.

Proof. Let $H = \sum_{j=1}^{m} H_j$ be a k-local Hamiltonian and $U(t) = e^{-iHt}$ be its evolution operator. We can assume that not all the H_j commute, otherwise we are done by Proposition 4.15. Assume $t = 1$ and each $\|H_j\| \le \delta$ with $\delta \le 1/m$, since then $\|H_1 + \cdots + H_\ell\| \le \ell\delta$, and we need the Lie-Trotter approximation to hold for all $\ell \in [m]$. We have

$$\begin{aligned} &(e^{-iH_1}e^{-iH_2}) \cdots e^{-iH_m} \\ &= \big(e^{-i(H_1+H_2)} + O(\delta^2)\big)e^{-iH_3} \cdots e^{-iH_m} && \text{by Lie-Trotter} \\ &= e^{-i(H_1+H_2)}e^{-iH_3} \cdots e^{-iH_m} + O(\delta^2) && \text{by submultiplicativity} \\ &= \big(e^{-i(H_1+H_2+H_3)} + O((2\delta)^2)\big)e^{-iH_4} \cdots e^{-iH_m} + O(\delta^2). \end{aligned}$$

since each e^{-iH_j} is unitary, so has unit norm. Repeatedly applying Lie-Trotter, we obtain

$$\begin{aligned} e^{-iH_1} \cdots e^{-iH_m} &= e^{-i(H_1+\cdots+H_m)} + O(\delta^2) + \cdots + O(((m-1)\delta)^2) \\ &= e^{-i(H_1+\cdots+H_m)} + O(m^3\lambda^2). \end{aligned}$$

Let the $O(m^3\lambda^2)$ error be $Cm^3\lambda^2$. For general $\|H_i\|$ and $t > 0$, introduce M large (to be chosen later), and define $\widetilde{H}_j = H_j t/M$. Note that $\|\widetilde{H}_j\| \le \delta t/M =: \tilde{\delta}$. Now

$$U(t) = e^{-i(H_1+\cdots+H_m)t} = \big(e^{-i(H_1+\cdots+H_m)t/M}\big)^M.$$

So we need the error in approximating $e^{-iHt/M}$ to be at most ε/M. So using the above error bound, we want $Cm^3\tilde{\delta}^2 < \varepsilon/M$, i.e. $M > Cm^3(\delta t)^2/\varepsilon$. With this choice of M, we have

$$\|e^{-iH_1 t/M} \cdots e^{-iH_m t/M} - e^{-i(H_1+\cdots+H_m)t/M}\| \le \varepsilon/M.$$

Hence by Lemma 4.10,

$$\|e^{-iH_1 t} \cdots e^{-iH_m t} - e^{-i(H_1+\cdots+H_m)t}\| \le \varepsilon.$$

The circuit is composed of Mm gates of the form $e^{-iH_j t/M}$. So the circuit size is $O(m^4(\delta t)^2/\varepsilon)$. Recall that if H is k-local, then $m \le \binom{n}{k} = O(n^k)$. So circuit size is $O(n^{4k}(\delta t)^2/\varepsilon)$ which is $\mathrm{poly}(n, t, 1/\varepsilon)$. □

Remark 4.19

- The time dependence is quadratic, but there are improved product formulae that allow the dependence of the circuit size on t to be $O(t^{1+\alpha})$ for any $\alpha > 0$.
- The ε-dependence is $\text{poly}(1/\varepsilon)$ whereas in the commuting case it was $\text{polylog}(1/\varepsilon)$. Again, there are methods that decrease this to $(1/\varepsilon)^{1/2k}$.
- We have a circuit of size $|C| = O\big(n^{4k}(\delta t)^2/\varepsilon\big)$ gates of the form $e^{-iH_j t/M}$ approximating e^{-iHt} to precision ε. If we apply Solovay-Kitaev to get this in some universal 2-qubit gate set, we need to approximate each $e^{-iH_j t/M}$ to precision $\varepsilon/|C|$. Solovay-Kitaev requires overhead $O(\log^4|C|/\varepsilon) = O\Big(\log^4\Big(\frac{n^{4k}(\delta t)^2}{\varepsilon^2}\Big)\Big)$. So final complexity is $\tilde{O}\big(n^{4k}(\delta t)^2/\varepsilon\big)$ TODO: add this to proof of theorem.

Algorithm 4.20 (Harrow-Hassidim-Lloyd (HHL)) Given $A \in \mathbb{C}^{N\times N}$, $\boldsymbol{b} \in \mathbb{C}^N$, we want to solve $A\boldsymbol{x} = \boldsymbol{b}$.

Remark 4.21 The best known classical algorithms require $O(\text{poly}(N)\cdot\log(1/\varepsilon))$ time. Just reading the inputs A or b, or writing the solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ requires $O(\text{poly}(N))$ time. Instead we focus on computing/estimating properties of the solution vector of the form $\mu = x^T M x$ (quadratic expressions defined by some Hermitian matrix M), e.g. the total weight assigned by x to a subset of indices/components. Classically, there is no better known way of doing this than computing the entire solution first. HHL can solve such tasks in $O\big(\text{polylog}(N)\cdot\frac{1}{\varepsilon}\cdot\kappa^2\big)$ time, where κ is the condition number of A, and when $\kappa = \text{polylog}(N)$, this is an exponential speedup over the best known classical algorithms.

Preliminary requirements for HHL algorithm to be applicable:

- A is Hermitian (if not, double the system size and set $\tilde{A} = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix}$, $\tilde{b} = \begin{bmatrix} 0 \\ b \end{bmatrix}$, then solution is $\tilde{x} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ where $A\boldsymbol{x} = \boldsymbol{b}$).
- A is **well-conditioned**: condition number κ is defined as $\kappa := \|A^{-1}\|\|A\|$. For A Hermitian, $\kappa = \frac{\max\{|\lambda|:\lambda \text{ eigenvalue of } A\}}{\min\{|\lambda|:\lambda \text{ eigenvalue of } A\}}$. Note $\kappa := \infty$ when A is non-invertible. κ is measure of "how invertible" A is.
- $\boldsymbol{b}$ is given as a quantum state: let $\boldsymbol{b} = (b_1, ..., b_N)$, then state is

$$|b\rangle = \frac{1}{\|b\|_2}\sum_{i=1}^{N} b_i|i\rangle.$$

  We also assume that $\boldsymbol{b}$ is normalised, (or that $\|b\|_2$ is efficiently computable), and that the state $|b\rangle$ can be efficiently prepared on a quantum computer.
- For the property $\mu = x^T M x$, assume M is Hermitian matrix, and that the corresponding measurement on $n = \log N$ qubits can be efficiently implemented.
- Assume that there is an efficient Hamiltonian simulation algorithm for A, i.e. we can implement $U(t) = e^{-iHt}$ in $O(\text{poly}(n)\cdot t)$ gates (for example, local Hamiltonians, but also a larger class, which naturally occurs in applications, is the class of locally-computable and row-sparse (every row contains at most $O(\text{poly}(n))$ non-zero entries) matrices).

The quantum algorithm will work on $n = \log N$ qubits and will never need to "write down" A, b, or $x = A^{-1}b$ as lists of numbers. It will output a state $|\hat{x}'\rangle$ that is ε-close

to $|\hat{x}\rangle$, and $|\hat{x}\rangle$ is proportional to $A^{-1}|b\rangle$ in $O\big(\text{poly}(n)\cdot\kappa^2\cdot\frac{1}{\varepsilon}\big)$. Using $|\hat{x}'\rangle$ a further $O(\text{poly}(n)\kappa^2/\varepsilon)$ times, we can estimate any $\mu = x^T M x$.

The best known classical algorithm requires $O\big(\text{poly}(N)\cdot\kappa\cdot\log\big(\frac{1}{\varepsilon}\big)\big)$ time, even with assumptions comparable to our assumptions for HHL.

Note that when ε is constant (or even $\varepsilon = O\big(\frac{1}{\text{poly}(n)}\big)$) and for well-conditioned A, we have an exponential speedup.

Algorithm 4.22 (HHL) Assume that Hamiltonian simulation and phase estimation are exact, let $A = \sum_{i=1}^N \lambda_i |v_i\rangle\langle v_i|$, assume $\lambda_{\max} = 1$, and assume that $\kappa(A)$ is known or bounded $\leq \kappa_{\max}$. This means $|\lambda_i| \in [1/\kappa_{\max}, 1]$ for each i. Work in the n-qubit Hilbert space spanned by $\{|0\rangle, ..., |N-1\rangle\}$.

Write

$$|b\rangle = \sum_{i=1}^N b_i |i\rangle = \sum_{j=1}^N \beta_j |v_j\rangle$$

The solution vector to $Ax = b$ is $|\hat{x}\rangle := A^{-1}|b\rangle = \sum_{j=1}^N \beta_j \cdot \frac{1}{\lambda_j}|v_j\rangle$ (since $A^{-1} = \sum_{i=1}^N \frac{1}{\lambda_j}|v_j\rangle\langle v_j|$). The transformation $|b\rangle \mapsto A^{-1}|b\rangle$ is linear but not unitary so cannot be directly implemented. Instead, we implemented it probabilistically using QPE, performed on $U = e^{-iA}$, which in turn is implemented by Hamiltonian simulation. At the end, we'll have a measurement step that introduces the non-unitarity. Apply U_{PE} for e^{-iA} with m lines on the state $|b\rangle|0\rangle^{\otimes m}$, which gives $\sum_i \beta_i |v_i\rangle|\lambda_i\rangle$ (assuming e^{-iA} and U_{PE} are exact and error-free). Consider the controlled rotation C–Rot acting on $n+1$ qubits: $C\text{–Rot}|\lambda\rangle|0\rangle = |\lambda\rangle(\cos(\theta)|0\rangle + \sin(\theta)|1\rangle) = |\lambda\rangle\Big(\sqrt{1 - c^2/\lambda^2}|0\rangle + \frac{c}{\lambda}|1\rangle\Big)$, with $\theta = \arcsin(c/\lambda)$ and $c \leq \min\{|\lambda_i| : i \in [N]\}$. Since $\lambda \in [1/\kappa, 1]$, $1/\lambda \in [1, \kappa]$ is larger than 1, so can choose $c = \kappa$ or $O(\kappa)$. So in C–Rot, the angle depends on the first register but not on A or b (so we're not sneaking extra info in here). C–Rot can be implemented efficiently using $O(\text{poly}(n))$ one and two-qubit gates (by e.g. Solovay-Kitaev). Assume we can implement C–Rot efficiently and exactly.

Applying C–Rot to the state $U_{\text{PE}}|b\rangle|0\rangle$, we get

$$\sum_{j=1}^N \beta_j \sqrt{1 - c^2/\lambda_j^2}|v_j\rangle|\lambda_j\rangle|0\rangle + \beta_j \frac{c}{\lambda_j}|v_j\rangle|\lambda_j\rangle|1\rangle$$

Now we measure the last qubit, and accept if outcome is 1. This is called a post-selection step. The state collapses to a state proportional to

$$\sum_{j=1}^N \frac{c}{\lambda_j}\beta_j |v_j\rangle|\lambda_j\rangle|1\rangle$$

Remark 4.23 Uses of solving linear systems:

- Numerical solutions of PDEs using discretisation leads to linear systems of size far larger than original problem description.
- Machine learning, pattern matching etc.