1. Metric spaces

1.1. Metrics

- **Definition**: **metric space** is (X, d), X is set, $d : X \times X \to [0, \infty)$ is **metric** satisfying:
  - $d(x, y) = 0 \Longleftrightarrow x = y$
  - **Symmetry**: $d(x, y) = d(y, x)$
  - **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- **Example**:
  - p-adic metric: for $p \in [1, \infty)$

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

  - Extension of the p-adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

  - Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

  - Discrete metric:

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Definition**: **open ball of radius r around x**:

$$B(x; r) := \{y \in X : d(x, y) < r\}$$

- **Definition**: **closed ball of radius r around x**:

$$D(x; r) := \{y \in X : d(x, y) \leq r\}$$

1.2. Open and closed sets

- **Definition**: $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

- **Definition**: $A \subseteq X$ is **closed** if $X - A$ is open.
- Sets can be neither closed nor open, or both.
- With standard metric on $\mathbb{R}$, any singleton $\{x\} \in \mathbb{R}$ is closed and not open (same holds for $\mathbb{R}^n$).
- **Definition**: let X be metric space, $x \in N \subseteq X$. N is **neighbourhood** of x if

$$\exists \text{ open } V \subseteq X : x \in V \subseteq N$$

- **Corollary**: let $x \in X$, then $N \subseteq X$ neighbourhood of x iff $\exists \varepsilon > 0 : x \in B(x; \varepsilon) \subseteq N$.
- **Proposition**: open balls are open, closed balls are closed.
- **Lemma**: let (X, d) metric space.
  - X and $\emptyset$ are both open and closed.

- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.
- Finite unions of closed sets are closed.
- Arbitrary intersections of closed sets are closed.
- **Example**: if X has discrete metric, any $A \subseteq X$ is open and closed.

1.3. Continuity

- **Definition**:
  - **Sequence** in X is $a : \mathbb{N}_0 \to X$, written $(a_n)_{n \in \mathbb{N}}$.
  - (a_n) **converges to a**, $\lim_{n \to \infty} a_n = a$, if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, d(a, a_n) < \varepsilon$$

- **Proposition**: let X, Y metric spaces, $a \in X$, $f : X \to Y$. The following are equivalent:
  - $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in X, d_X(a, x) < \delta \Longrightarrow d_Y(f(a), f(x)) < \varepsilon$.
  - For every sequence (a_n) in X with $a_n \to a$, $f(a_n) \to f(a)$.
  - For every open $U \subseteq Y$ with $f(a) \in U$, $f^{-1}(U)$ is a neighbourhood of a.

  If f satisfies these, it is **continuous at a**.
- **Definition**: f **continuous** if continuous at every $a \in X$.
- **Proposition**: $f : X \to Y$ continuous iff $f^{-1}(U)$ open for every open $U \subseteq Y$.
- **Example**: let d be discrete metric, d_2 be 2-adic metric.
  - Any $f : (X, d) \to (\mathbb{R}, d_2)$ is continuous.
  - $\mathrm{id} : (\mathbb{R}, d_2) \to (\mathbb{R}, d)$ is not continuous.

2. Topological spaces

2.1. Topologies

- **Definition**: **power set** of X: $\mathcal{P}(X) \coloneqq \{A : A \subseteq X\}$.
- **Definition**: **topology** on set X is $\tau \subseteq \mathcal{P}(X)$ with:
  - $\emptyset \in \tau$, $X \in \tau$.
  - **Closure under arbitrary unions**: if $\forall i \in I, U_i \in \tau$, then

$$\bigcup_{i \in I} U_i \in \tau$$

  - **Closure under finite intersections**: $U_1, U_2 \in \tau \Longrightarrow U_1 \cap U_2 \in \tau$ (this is equivalent to $U_1, ..., U_n \in \tau \Longrightarrow \bigcap_{i \in [n]} U_i \in \tau$).

  (X, τ) is **topological space**. Elements of τ are **open** subsets of X. $A \subseteq X$ **closed** if $X - A$ is open.
- **Definition**: $\tau = \mathcal{P}(X)$ is the **discrete topology** on X.
- **Definition**: $\tau = \{\emptyset, X\}$ is the **indiscrete topology** on X.
- **Example**:
  - For metric space (M, d), let τ_d exactly contain sets which are open with respect to d. Then (M, τ_d) is a topological space. d **induces** topology τ_d.

- Let $X = \mathbb{N}_0$ and $\tau = \{\emptyset\} \cup \{U \subseteq X : X - U \text{ is finite}\}$, then (X, τ) is topological space.

- **Proposition**: for topological space X:
  - X and $\emptyset$ are closed
  - Arbitrary intersections of closed sets are closed
  - Finite unions of closed sets are closed
- **Proposition**: for topological space (X, τ) and $A \subseteq X$, the **induced (subspace) topology on A**

$$\tau_A = \{A \cap U : U \in \tau\}$$

is a topology on A.
- **Example**: let $X = \mathbb{R}$ with standard topology induced by metric $d(x, y) = |x - y|$. Let $A = [1, 5]$. Then $[1, 3) = A \cap (0, 3)$ and $[1, 5] = A \cap (0, 6)$ are open in A.
- **Example**: consider $\mathbb{R}$ with standard topology τ. Then
  - $\tau_{\mathbb{Z}}$ is the discrete topology on $\mathbb{Z}$.
  - $\tau_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.
- **Proposition**: metrics d_p for $p \in [1, \infty)$ and d_∞ all induce same topology on $\mathbb{R}^n$, alled the **standard topology** on $\mathbb{R}^n$.
- **Definition**: (X, τ) is **Hausdorff** if

$$\forall x \neq y \in X, \exists U, V \in \tau : U \cap V = \emptyset \wedge x \in U, y \in V$$

- **Lemma**: any metric space (M, d) with topology induced by d is Hausdorff.
- **Example**: let $|X| \geq 2$ with indiscrete topology. Then X is not Hausdorff, since $\tau = \{X, \emptyset\}$ and if $x \neq y \in X$, only open set containing x is X (same for y). But $X \cap X = X \neq \emptyset$.
- **Definition**: **Furstenberg's topology on $\mathbb{Z}$**: define $U \subseteq \mathbb{Z}$ to be open if

$$\forall a \in U, \exists 0 \neq d \in \mathbb{Z} : a + d\mathbb{Z} := \{a + dn : n \in \mathbb{Z}\} \subseteq U$$

- Furstenberg's topology is Hausdorff.

2.2. Continuity

- **Definition**: let X, Y topological spaces.
  - $f : X \to Y$ is **continuous** if

$$\forall V \text{ open in } Y, f^{-1}(V) \text{ open in } X$$

  - f is **continuous at $a \in X$** if

$$\forall V \text{ open in } Y \text{ with } f(a) \in V, \exists U \text{ open in } X : a \in U \subseteq f^{-1}(V)$$

- **Lemma**: $f : X \to Y$ continuous iff f continuous at every $a \in X$. (Key idea for proof: $\cup_{a \in f^{-1}(V)} U_a \subseteq f^{-1}(V) = \cup_{a \in f^{-1}(V)} \{a\} \subseteq \cup_{a \in f^{-1}(V)} U_a$)
- **Example**: inclusion $i : (A, \tau_A) \to (X, \tau_X)$, $A \subseteq X$, is always continuous.
- **Lemma**: compositions of continuous functions are continuous.
- **Lemma**: let $f : X \to Y$ be function between topological spaces. Then f is continuous iff

$$\forall A \text{ closed in } Y, \quad f^{-1}(A) \text{ closed in } X$$

- **Remark**: we can use continuous functions to decide that sets are open or closed.
- **Definition**: ***n*-sphere** is

$$S^n := \left\{ (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

- **Example**: in the standard topology, the n-sphere is a closed subset of $\mathbb{R}^{n+1}$. (Consider the preimage of $\{1\}$ which is closed in $\mathbb{R}$).
- **Example**:
  - Can consider set of square matrices $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and give it the standard topology.
  - Note

$$\det(A) = \sum_{\sigma \in \text{ sym}(n)} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

    is a polynomial in the entries of A so is continuous function from $M_n(\mathbb{R})$ to $\mathbb{R}$.
  - $\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} = \det^{-1}(\mathbb{R} - \{0\})$ is open.
  - $\mathrm{SL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} = \det^{-1}(\{1\})$ is closed.
  - $O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I\}$ is closed: $f_{i,j}(A) = (AA^T)_{i,j}$ is continuous and

$$O(n) = \bigcap_{1 \leq i,j \leq n} (f_{i,j})^{-1}(\{\delta_{i,j}\})$$

  - $\mathrm{SO}(n) = O(n) \cap \mathrm{SL}_n(\mathbb{R})$ is closed.
- **Definition**: for X, Y topological spaces, $h : X \to Y$ is **homeomorphism** if h is bijective, continuous and h^{-1} is continuous. X and Y are **homeomorphic**, $X \cong Y$. h induces bijection between τ_X and τ_Y which commutes with unions and intersections.
- **Proposition**: compositions of homeomorphisms are homeomorphisms.
- **Example**: in standard topology, $(0,1)$ is homeomorphic to $\mathbb{R}$. (Consider $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$, $f = \tan$, $g : (0,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $g(x) = \pi\left(x - \frac{1}{2}\right)$ and $f \circ g$).
- **Example**: $\mathbb{R}$ with standard topology τ_{st} is not homoeomorphic to $\mathbb{R}$ with the discrete topology τ_d. (Consider $h^{-1}(\{a\}) = \{h^{-1}(a)\}$, $\{a\} \in \tau_{\text{st}}$ but $\{h^{-1}(a)\} \notin \tau_{\text{st}}$).
- **Example**: let $X = \mathbb{R} \cup \{\overline{0}\}$. Define $f_0 : \mathbb{R} \to X$, $f_0(a) = a$ and $f_{\overline{0}} : \mathbb{R} \to X$, $f_{\overline{0}}(a) = a$ for $a \neq 0$, $f_{\overline{0}}(0) = \overline{0}$. Topology on X has $A \subseteq X$ open iff $f_0^{-1}(A)$ and $f_{\overline{0}}^{-1}(A)$ open. Every point in X lies in open set: for $a \notin \{0, \overline{0}\}$, $a \in (a - \frac{|a|}{2}, a + \frac{|a|}{2})$ and both pre-images of this are same open interval, for 0, set $U_0 = (-1,0) \cup \{0\} \cup (0,1) \subseteq X$ then $f_0^{-1}(U_0) = (-1,1)$ and $f_{\overline{0}}^{-1}(U_0) = (-1,0) \cup (0,1)$ are both open. For $\overline{0}$, set $U_{\overline{0}} = (-1,0) \cup \{\overline{0}\} \cup (0,1) \subseteq X$, then $f_{\overline{0}}^{-1}(U_{\overline{0}}) = (-1,1)$ and $f_0^{-1}(U_{\overline{0}}) = (-1,0) \cup (0,1)$ are both open. So U_0 and $U_{\overline{0}}$ both open in X. X is not Hausdorff since any open sets containing 0 and $\overline{0}$ must contain "open intervals" such as U_0 and $U_{\overline{0}}$.

- **Example (Furstenberg's proof of infinitude of primes)**: since $a + d\mathbb{Z}$ is infinite, any nonempty finite set is not open, so any set with finite complement is not closed. For fixed d, sets $d\mathbb{Z}$, $1 + d\mathbb{Z}, ..., (d - 1) + d\mathbb{Z}$ partition $\mathbb{Z}$. So the complement of each is the union of the rest, so each is open and closed. Every $n \in \mathbb{Z} - \{-1, 1\}$ is prime or product of primes, so $\mathbb{Z} - \{-1, 1\} = \bigcup_{p \text{ prime}} p\mathbb{Z}$, but finite unions of closed sets are closed, and since $\mathbb{Z} - \{-1, 1\}$ has finite complement, the union must be infinite.

3. Limits, bases and products

3.1. Limit points, interiors and closures

- **Definition**: for topological space X, $x \in X$, $A \subseteq X$:
  - **Open neighbourhood of x** is open set N, $x \in N$.
  - x is **limit point** of A if every open neighbourhood N of x satisfies

$$(N - \{x\}) \cap A \neq \emptyset$$

- **Corollary**: x is not limit point of A iff exists neighbourhood N of x with

$$A \cap N = \begin{cases} \{x\} & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

- **Example**: let $X = \mathbb{R}$ with standard topology.
  - $0 \in X$, then $(-1/2, 1/2)$ is open neighbourhood of 0.
  - If $U \subseteq X$ open, U is open neighbourhood for any $x \in U$.
  - Let $A = \left\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\right\}$, then only limit point in A is 0.
- **Definition**: let $A \subseteq X$.
  - **Interior** of A is largest open set contained in A:

$$A^\circ := \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U$$

  - **Closure** of A is smallest closed set containing A:

$$\overline{A} := \bigcap_{\substack{F \text{ closed} \\ A \subseteq F}}$$

  If $\overline{A} = X$, A is **dense** in X.
- **Lemma**:
  - $\overline{X - A} = X - A^\circ$
  - $\overline{A} = X - (X - A)^\circ$
- **Example**: let $\mathbb{Q} \subset \mathbb{R}$ with standard topology. Then $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$ (since every nonempty open set in $\mathbb{R}$ contains rational and irrational numbers).
- **Lemma**: $\overline{A} = A \cup L$ where L is the set of limit points of A.
- **Dirichlet prime number theorem**: let a, d coprime, then $a + d\mathbb{Z}$ contains infinitely many primes.

- **Example**: let A be set of primes in $\mathbb{Z}$ with Furstenberg topology. By above lemma, only need to find limit points in $\mathbb{Z} - A$ to find $\overline{A}$. $10\mathbb{Z}$ is an open neighbourhood of 0 for 0 inside $\mathbb{Z} - A$. For $a \notin \{-1, 0, 1\}$, $a + 10a\mathbb{Z}$ is an open neighbourhood of a. These sets have no primes so the corresponding points are not limit points of A. For ± 1, any open neighbourhood of 1 contains a set $\pm 1 + d\mathbb{Z}$ for some $d \neq 0$, but by the Dirichlet prime number theorem, this set contains at least one prime. So $\overline{A} = A \cup \{\pm 1\}$.
- **Lemma**:
  - Let $A \subseteq M$ for metric space M. If x is limit point of A then exists sequence x_n in A such that $\lim_{n \to \infty} x_n = x$.
  - If $x \in M - A$ and exists sequence x_n in A with $\lim_{n \to \infty} x_n = x$ then x is limit point of A.

3.2. Bases

- **Definition**: a **basis** for topology τ on X is collection $\mathcal{B} \subseteq \tau$ such that

$$\forall U \in \tau, \exists B \subseteq \mathcal{B} : U = \bigcup_{b \in B} b$$

  (every open U is a union of sets in B).
- **Example**:
  - For metric space (M, d), $\mathcal{B} = \{B(x; r) : x \in M, r > 0\}$ is basis for the induced topology. (Since if U open, $U = \cup_{u \in U} \{u\} \subseteq \cup_{u \in U} B(u, r_u) \subseteq U$.)
  - In $\mathbb{R}^n$ with standard topology, $\mathcal{B} = \{B(q; 1/m) : q \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a **countable** basis. (Find $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{r}{2}$ and $q \in \mathbb{Q}^n$ such that $q \in B\left(p; \frac{1}{m}\right)$, then $B\left(q; \frac{1}{m}\right) \subseteq B(p; r) \subseteq U$ using the triangle inequality).
- **Theorem**: let $f : X \to Y$ be map between topological spaces. The following are equivalent:
  - f is continuous.
  - If $\mathcal{B}$ is basis for topology τ on Y then $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.
  - $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$.
  - $\forall V \subseteq Y, \overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$.
  - $f^{-1}(C)$ closed for any closed set $C \subseteq Y$.
- **Theorem**: let X be a set and collection $\mathcal{B} \subseteq \mathcal{P}(X)$ be such that:
  - $\forall x \in X, \exists B \in \mathcal{B} : x \in B$
  - If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$.

  Then there is unique topology $\tau_{\mathcal{B}}$ on X for which $\mathcal{B}$ is a basis. We say $\mathcal{B}$ **generates** $\tau_{\mathcal{B}}$. We have $\tau_B = \{\cup_{i \in I} B_i : B_i \in \mathcal{B}, I \text{ indexing set}\}$.

3.3. Product topologies

- **Definition**: **Cartesian product** of topological spaces X, Y is $X \times Y := \{(x, y) : x \in X, y \in Y\}$. We give it the **product topology** which is generated by $\mathcal{B}_{X \times Y} := \{U \times V : U \in \tau_X, V \in \tau_Y\}$.
- **Example**:
  - Let $X = Y = \mathbb{R}$, then product topology is same as standard topology on $\mathbb{R}^2$.

- Let $X = Y = S^1$, then $X \times Y = T^2 = S^1 \times S^1$ is the **2-torus**. ***n*-torus** is defined for $n \geq 3$ by

$$T^n := S^1 \times T^{n-1}$$

- **Definition**: if $\tau_1 \subseteq \tau_2$ are topologies, then τ_1 is **smaller** than τ_2 (τ_2 is **larger** than τ_1).
- **Definition**: for topological spaces X, Y, **projection maps** $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are

$$\pi_X(x, y) = x, \quad \pi_Y(x, y) = y$$

- **Proposition**: for $X \times Y$ with product topology,
  - π_X and π_Y are continuous.
  - π_X and π_Y map open sets to open sets.
  - Product topology is smallest topology for which π_X and π_Y are continuous.
- **Proposition**: let X, Y, Z topological spaces, then $f : Z \to X \times Y$ (with product topology on $X \times Y$) continuous iff both $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous.
- **Example**: let $f : X \to \mathbb{R}^n$, $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$, $f_i = \pi_i \circ f$, then f is continuous iff all f_i are continuous.
- **Proposition**: let X, Y nonempty topological spaces. Then $X \times Y$ with product topology is Hausdorff iff X and Y are both Hausdorff.

4. Connectedness

4.1. Clopen sets and examples

- **Definition**: let X topological space, then $A \subseteq X$ is **clopen** if A is open and closed.
- **Definition**: X is **connected** if the only clopen sets in X are X and $\emptyset$.
- **Example**:
  - $\mathbb{R}$ with standard topology is connected.
  - $\mathbb{Q}$ with induced topology from $\mathbb{R}$ is not connected (consider $L = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q} - L = \mathbb{Q} \cap (\sqrt{2}, \infty)$).
  - The connected subsets of $\mathbb{R}$ are the intervals.
- **Definition**: $A \subseteq \mathbb{R}$ is an interval iff $\forall x, y, z \in A, x < z < y \Longrightarrow z \in A$.
- **Example**:
  - $X = \{0, 1\}$ with discrete topology is not connected ($\{1\}$ and $\{0\}$ both open so both closed).
  - $X = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$ is connected.
  - $\mathbb{Z}$ with Furstenberg topology is not connected.
- **Theorem (continuity preserves connectedness)**: if $h : X \to Y$ continuous and X connected, then $h(X) \subseteq Y$ is connected.
- **Corollary**: if $h : X \to Y$ is homeomorphism and X is connected then Y is connected.
- **Theorem**: let X topological space. The following are equivalent:
  - X is connected.

    - X cannot be written as disjoint union of two non-empty sets.
    - There exists no continuous surjective function from X to a discrete space with more than one point.
- **Example**:
    - $\mathrm{GL}_n(\mathbb{R})$ is not connected (since $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R} - \{0\}$ is continuous and surjective and $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$).
    - $O(n)$ is not connected.
    - $(0, 1)$ is connected (since $\mathbb{R} \cong (0, 1)$ and $\mathbb{R}$ is connected).
    - $X = (0, 1]$ and $Y = (0, 1)$ are not homeomorphic (if they are, then $(0, 1]$ is connected since $(0, 1)$ is).
- **Definition**: let $A = B \cup C$, $B \cap C = \emptyset$, then B and C are **complementary subsets** of A.
- **Remark**: if complementary B and C open in A, then B and C clopen in A. So if $B, C \neq \emptyset$ then A not connected.

4.2. Constructing more connected sets, components, path-connectedness

- **Proposition**: let X topological space, $Z \subseteq X$ connected. If $Z \subseteq Y \subseteq \overline{Z}$ then Y is connected. In particular, with $Y = \overline{Z}$, the closure of a connected set is connected.
- **Proposition**: let $A_i \subseteq X$ connected, $i \in I$, $A_i \cap A_j \neq \emptyset$ and $\cup_{i \in I} A_i = X$. Then X is connected.
- **Theorem**: if X and Y are connected then $X \times Y$ is connected.
- **Example**:
    - $\mathbb{R}^n$ is connected.
    - $B^n = \{x \in \mathbb{R}^n : d_2(0, x) < 1\}$ (B^n is homeomorphic to $\mathbb{R}^n$).
    - $D^n = \{x \in \mathbb{R}^n : d_2(0, x) \leq 1\} = \overline{B^n}$ is connected.
- **Example**:
    - $\forall n \in \mathbb{N}$, S^n is connected.
    - $\forall n \in \mathbb{N}$, T^n is connected.
- **Definition**: **component** of topological space X is maximal connected subset of X.
- **Proposition**: in a topological space X:
    - Every $p \in X$ is in a unique component.
    - If $C_1 \neq C_2$ are components, then $C_1 \cap C_2 = \emptyset$.
    - X is the union of its components.
    - Every component is closed in X.
- **Example**:
    - If X connected, then its only component is itself.
    - If X discrete, then each singleton in τ_X is a component.
    - In $\mathbb{Q}$ with induced standard topology from $\mathbb{R}$, every singleton is a component.
- **Definition**: **path** in topological space X is continuous function $\gamma : [0, 1] \to X$. γ is said to be path from $\gamma(0)$ to $\gamma(1)$.
- **Definition**: X is **path-connected** if for every $p, q \in X$, there is a path from p to q.

- **Proposition**: every path-connected topological space is connected.
- **Example**: let

$$Z = \{(x, \sin(1/x)) \in \mathbb{R}^2 : 0 < x \leq 1\}$$

  Z is path-connected, as a path from $(x_1, \sin(1/x_1))$ to $(x_2, \sin(1/x_2))$ is given by

$$\gamma(t) = \left(x_1 + (x_2 - x_1)t, \sin\left(\frac{1}{x_1 + (x_2 - x_1)t}\right)\right)$$

  So then Z is connected by the above proposition, and since the closure of a connected set is connected, $\overline{Z}$ is connected.

  Every point $(0, y)$, $y \in [-1, 1]$ is a limit point of Z. Assume $\overline{Z}$ is path-connected. Then there is a path $\gamma : [0, 1] \to \overline{Z}$ from $(0, 0)$ to $(1, \sin(1))$. Since $(\pi_X \circ \gamma)(0) = 0$ and $(\pi_X \circ \gamma)(1) = 1$ and $\pi_X \circ \gamma$ is continuous, by the Intermediate Value Theorem, $\exists t_1 \in [0, 1] : (\pi_X \circ \gamma)(t_1) = 2/\pi$. By IVT again, $\exists t_2 \in [0, t_1] : (\pi_X \circ \gamma)(t_2) = \frac{2}{2\pi}$. We obtain a strictly decreasing sequence $(t_n) \subseteq [0, 1]$ where $(\pi_X \circ \gamma)(t_n) = \frac{2}{n\pi}$ which is bounded below by 0, so must converge with limit t^*.

  Now $\pi_Y \circ \gamma$ is continuous, so $\lim_{n\to\infty} (\pi_Y \circ \gamma)(t_n) = (\pi_Y \circ \gamma)(t^*)$. But $(\pi_Y \circ \gamma)(t_n) = \sin\left(\frac{n\pi}{2}\right)$, and as $n \to \infty$, this oscillates between -1 and 1 and does not converge, so contradiction.

5. Compactness

- **Definition**: let X topological space, **cover** of X is collection $(U_i)_{i\in I}$ of subsets of X with

$$\bigcup_{i\in I} U_i = X$$

  If every U_i is open, it is an **open cover**. If $J \subseteq I$, then $(U_i)_{i\in J}$ is a **subcover** of $(U_i)_{i\in I}$ if it is also a cover.
- **Definition**: X is **compact** if every open cover of X admits a finite subcover.
- **Example**:
  - If X is finite then X is compact.
  - $\mathbb{R}$ is not compact.
  - If X infinite with $\tau = \{U \subseteq X : X - U \text{ is finite}\} \cup \{\emptyset\}$, then X is compact.
- **Proposition**: let X have topology with basis $\mathcal{B}$. Then X is compact iff every cover $(B_i)_{i\in I}$ of X, $B_i \in \mathcal{B}$, admits a finite subcover of X.
- **Remark**: to determine compactness of $Y \subseteq X$ with induced topology, consider open covers $Y = \cup_{i\in I} (U_i \cap Y)$ for U_i open in X, which is equivalent to $Y \subseteq \cup_{i\in I} U_i$.
- **Example**: $[0, 1]$ is compact.
- **Proposition**: if $f : X \to Y$ continuous, X compact, then $f(X)$ is compact.
- **Proposition**: if X compact, $A \subseteq X$ closed in X, then A is compact.
- **Theorem**: if X is Hausdorff and $A \subseteq X$ is compact then A is closed.

- **Corollary**: if X compact, Y is Hausdorff, $f : X \to Y$ continuous bijection, then f is homeomorphism.
- **Theorem**: if X, Y compact, then $X \times Y$ is compact.
- **Definition**: $S \subseteq \mathbb{R}^n$ is **bounded** if

$$\exists r \in \mathbb{R} : S \subseteq B(0; r)$$

- **Theorem (Heine-Borel)**: $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.
- **Example**:
  - S^n is compact.
  - T^n is compact.
  - $X = \left\{\boldsymbol{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^3 = 1\right\}$ is not compact, since $\forall n \in \mathbb{N}$, $(n, 0, (n^2 - 1)^{1/3}) \in X$, so $X \not\subseteq B(n)$, so is unbounded, so not compact by Heine-Borel.
- **Corollary**: let $f : X \to \mathbb{R}$, X compact, f continuous. Then f attains its maximum and minimum.
- **Theorem (Bolzano-Weierstrass)**: an infinite subset A of a compact space X has a limit point in X.

6. Quotient spaces

- **Definition**: let X topological space, $\sim$ equivalence relation on X. Write $X/\sim$ for the set of equivalence classes of $\sim$: for $x \in X$,

$$[x] := \{y \in X : y \sim x\}, \quad X/\sim := \{[x] : x \in X\}$$

  There is a surjective map, the **quotient map**, $\pi : X \to X/\sim$, $\pi(x) = [x]$.
- **Example**: let $X = \mathbb{R}^3$, define equivalence relation

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \Leftrightarrow z_1 = z_2$$

  Then $\pi(a, b, c) = [(a, b, c)] = \left\{(x, y, z) \in \mathbb{R}^3 : z = c\right\}$. Elements of $\mathbb{R}^3/\sim$ are horizontal planes.
- **Definition**: let X topological space, $\sim$ equivalence relation on X. Then $X/\sim$ is given **quotient topology** defined by

$$U \subseteq X/\sim \text{ open} \Longleftrightarrow \pi^{-1}(U) \text{ open in } X$$

- **Proposition**: quotient topology defines a topology on $X/\sim$.
- **Proposition**: quotient topology on $X/\sim$ is largest such that π is continuous.
- **Proposition**: let X topological space with equivalence relation $\sim$, Y topological space. Then $f : X/\sim \to Y$ continuous iff $f \circ \pi : X \to Y$ is continuous.
- **Example**: in $\mathbb{R}$, let $x \sim y \Longleftrightarrow x - y \in \mathbb{Z}$. Define $\exp : \mathbb{R} \to S^1 \subseteq \mathbb{C}$, $\exp(t) = e^{2\pi i t}$) and $\overline{\exp} : \mathbb{R}/\sim \to S^1$, $\overline{\exp}([t]) = \exp(t)$. Then

$$[s] = [t] \Longleftrightarrow s - t = k \in \mathbb{Z} \Longleftrightarrow \overline{\exp}(s) = e^{2\pi i k} e^{2\pi i t} = e^{2\pi i t} = \overline{\exp}(t)$$

  Hence $\overline{\exp}$ is well-defined and injective, and is surjective since exp is. Also, $\overline{\exp}$ is continuous since $\exp = \overline{\exp} \circ \pi$ is. $\mathbb{R}^2$ is a metric space and so is Hausdorff, so $S^1 \subset \mathbb{R}^2$ with the induced topology is Hausdorff. Now e.g. $\pi([-10, 10]) = \mathbb{R}/\sim$,

$[-10, 10]$ is compact and π continuous so $\mathbb{R}/\sim$ is compact. Since $\overline{\exp}$ is a continuous bijection, these three properties imply $\overline{\exp}$ is a homeomorphism. Hence $\mathbb{R}/\sim\, \cong S^1$.

- **Definition**: let $A \subseteq X$, define $x \sim y \Longleftrightarrow x = y$ or $x, y \in A$. Then define $X/A := X/\sim$.
- **Example**: $S^n \cong D^n/S^{n-1}$. Any point in D^n can be written as $t \cdot \varphi$, $t \in [0, 1]$, $\varphi \in S^{n-1}$. Define

$$f : D^n \to S^n, \quad f(t \cdot \varphi) := (\cos(\pi t), \varphi \sin(\pi t)) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$$
$$\Longrightarrow f(0 \cdot \varphi) = (1, \mathbf{0}),\ f(1/2 \cdot \varphi) = (0, \varphi),\ f(1 \cdot \varphi) = (-1, 0)$$

Define $\overline{f} : D^n/S^{n-1} \to S^n$, $\overline{f}([t \cdot \varphi]) = f(t \cdot \varphi)$. If $t_1 \cdot \varphi_1 \neq t_2 \cdot \varphi_2$, then

$$\begin{aligned}
[t_1 \cdot \varphi_1] = [t_2 \cdot \varphi_2] \Longleftrightarrow t_1 \cdot \varphi_1, t_2 \cdot \varphi_2 \in S^{n-1} &\Longleftrightarrow t_1 = t_2 = 1 \\
&\Longleftrightarrow f(t_1 \cdot \varphi_1) = (-1, \mathbf{0}) = f(t_2 \cdot \varphi_2) \\
&\Longleftrightarrow \overline{f}([t_1 \cdot \varphi_1]) = \overline{f}([t_2 \cdot \varphi_2])
\end{aligned}$$

f is surjective, so $\overline{f}$ is also. Now $\overline{f} \circ \pi = f$ which is continuous, so by above proposition, $\overline{f}$ is continuous. $S^n \subset \mathbb{R}^{n+1}$ is Hausdorff, $D^n \subset \mathbb{R}^n$ is closed and bounded so is compact by Heine-Borel, and so D^n/S^{n-1} is compact (since π continuous). Also, f is a continuous bijection. These imply that $\overline{f}$ is homeomorphism.

7. Topological groups

7.1. Examples

- **Definition**: a **topological group** G is Hausdorff space which is also a group such that

$$\bullet : G \times G \to G,\ \bullet(g, h) = gh \quad \text{and} \quad i : G \to G,\ i(g) = g^{-1}$$

are continuous.
- **Example**:
  - $\mathbb{R}^n$ with addition is topological group.
  - $\mathrm{GL}_n(\mathbb{R})$ with multiplication and its subgroups $O(n)$ and $\mathrm{SO}(n)$ are topological groups (each entry in AB is sum of products of entries of A and B, so matrix multiplication is continuous, matrix inversion also continuous).
- **Proposition**:
  - Any group with discrete topology is topological group.
  - Any subgroup of topological group is also topological group.
- **Example**:
  - $\mathbb{C} - \{0\}$ with multiplication has topological subgroup $S^1 \subset \mathbb{C} - \{0\}$.
  - Define **quaternions** as vector space $\mathbb{H} := \langle 1, i, j, k \rangle$, with topology taken from $\mathbb{R}^4$. $\mathbb{H} - \{0\}$ is a multiplicative group with S^3 a topological subgroup. For $q = a + bi + cj + dk \in \mathbb{H}$, $a, b, c, d \in \mathbb{R}$, we have $ij := k$, $jk := i$, $ki := j$, $ji := -k$, $kj := -i$, $ik := -j$. For $q \neq 0$,

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

- Note however that S^2 is not a topological group.
- **Definition**: for topological group G, $x \in G$, define **left translation by x** as

$$L_x : G \to G, \quad L_x(g) := xg$$

Similarly, **right translation by x** is

$$R_x : G \to G, \quad R_x(g) := gx$$

- **Proposition**: L_x has inverse $(L_x)^{-1} = L_{x^{-1}}$ and is homeomorphism. Similarly for R_x.
- **Notation**: a specified inclusion $G \overset{x}{\hookrightarrow} G \times G$ is the map $G \to \{x\} \times G$ composed with the inclusion map $\{x\} \times G \to G \times G$. (similarly for $G \times \{x\}$).
- **Proposition**: let G topological group, K the component containing identity of G. Then K is normal subgroup of G.
- **Example**: $O(n)$ is not connected, but $\mathrm{SO}(n)$ is connected and contains I_n, so is a normal subgroup of $O(n)$

7.2. Actions, orbits, orbit spaces

- **Definition**: **action** of group G on topological space X is map $\bullet : G \times X \to X$ such that $\forall g, h \in G$, $\forall x \in X$,
  - $(hg) \bullet x = h \bullet (g \bullet x)$.
  - $1 \bullet x = x$.
  - $g : X \to X$ defined by $g(x) = g \bullet x$ is continous. Note: g has inverse map g^{-1} which is also continuous, so both are homeomorphisms.
- **Definition**: **action** of topological group G on topological space X is continuous map $\bullet : G \times X \to X$ such that $\forall g, h \in G$, $\forall x \in X$,
  - $(hg) \bullet x = h \bullet (g \bullet x)$.
  - $1 \bullet x = x$.
- **Remark**: for the above definition, the condition $g(x) = g \bullet x$ being continuous isn't required since g is the comoposition of continuous maps:

$$X \overset{g}{\hookrightarrow} G \times X \overset{\bullet}{\longrightarrow} X, \quad x \to (g, x) \to g \bullet x$$

- **Example**:
  - Trivial action: $(g, x) \mapsto g \bullet x = x$, so $\bullet = \pi_X$.
  - Let $G = \mathrm{GL}_n(\mathbb{R})$, $X = \mathbb{R}^n$, let the action be matrix multiplication: $(A, \boldsymbol{x}) \to A \bullet \boldsymbol{x} = A\boldsymbol{x}$. This induces an action of subgroups $O(n)$ or $\mathrm{SO}(n)$ on $X = \mathbb{R}^n$.
  - Let H subgroup of topological group G, **left translation action** of H on G is $\bullet : H \times G \to G$, $h \bullet g = hg$. Equivalently, $\varphi(h) = L_h$.
  - Let N normal subgroup of topological group G, **conjugation action** of G on N is $\bullet : G \times N \to N$, $g \bullet n = gng^{-1}$.
- **Definition**: let G act on topological space X, define equivalence relation $\sim$ on X by

$$x \sim y \Longleftrightarrow \exists g \in G : g(x) := g \bullet x = y$$

An equivalence class for this relation is an **orbit**, denoted Gx. **Orbit space**, X/G, is quotient space $X/\sim$. Action is **transitive** if X/G is a singleton.

- **Example**:
  - If G acts trivially, every orbit is singleton and $X/G = X$.
  - $\mathbb{R}^n/\mathrm{GL}_n(\mathbb{R})$ contains two points and has neither discrete nor indiscrete topology.
  - Action of $O(n)$ on S^{n-1} is transitive for $n \in \mathbb{N}$. Action of $\mathrm{SO}(n)$ on S^{n-1} is transitive for $n \geq 2$.
- **Lemma**: if connected topological group G acts on topological space X, then the orbits are connected.
- **Theorem**: let G connected topological group act on topological space X. If X/G is connected, then X is connected.
- **Notation**: define specified inclusion $i_1 : M_n(\mathbb{R}) \overset{1}{\hookrightarrow} M_{n+1}(\mathbb{R})$ by $A \to \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$. So $M_n(\mathbb{R})$ can be regarded as subspace of $M_{n+1}(\mathbb{R})$.
- **Proposition**:
  - Using the inclusion $\overset{1}{\hookrightarrow}$, $\mathrm{SO}(n)$ is subgroup of $\mathrm{SO}(n+1)$.
  - Viewing these as topological groups, if subgroup $\mathrm{SO}(n)$ acts on $\mathrm{SO}(n+1)$, orbit space is $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$.
- **Corollary**: the topological group $\mathrm{SO}(n)$ is connected for $n \in \mathbb{N}$.

8. Introduction

- **Notation**: let $I = [0, 1]$.
- **Definition**: **closed n-disc** is

$$D^n := \{\boldsymbol{x} \in \mathbb{R}^n : \|x\| \leq 1\}$$

- **Definition**: **open n-disc** is

$$E^n := \{\boldsymbol{x} \in \mathbb{R}^n : \|x\| < 1\}$$

- **Definition**: **n-sphere** is

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

- **Definition**: **cylinder** is $S^1 \times I$.
- **Definition**: the **2-torus (torus)** can be defined as $\mathbb{T} := S^1 \times S^1$ or $\mathbb{T} := (I \times I)/\sim$ where

$$\forall x \in I, (x, 0) \sim (x, 1), \quad \forall y \in I, (0, y) \sim (1, y)$$

- **Definition**: **Klein bottle** is given by $\mathbb{K} := (I \times I)/\sim$ where

$$\forall x \in I, (x, 0) \sim (x, 1), \quad \forall y \in I, (0, y) \sim (1, 1 - y)$$

- **Definition**: **map** is continuous $f : X \to Y$ where X, Y are topological spaces.

9. Simplicial complexes

9.1. Simplicial complexes and triangulations

- **Definition**: let $v_0, ..., v_n \in \mathbb{R}^N$, $n \le N$.
  - $v_0, ..., v_n$ are in **general position** if $\{v_1 - v_0, ..., v_n - v_0\}$ are linearly independent.
  - **Convex hull** of $v_0, ..., v_n$ is set of all **convex linear combinations** of $v_0, ..., v_n$:

$$\langle v_0, ..., v_n \rangle := \left\{ \sum_{i=0}^{n} \lambda_i v_i : \sum_{i=0}^{n} \lambda_i = 1, \forall i \in \{0, ..., n\}, \lambda_i \ge 0 \right\}$$

  - An $\boldsymbol{n}$**-simplex**, $\sigma^n = \langle v_0, ..., v_n \rangle$, is convex hull of $v_0, ..., v_n$ in general position. The **vertices** $v_0, ..., v_n$ **span** σ^n and σ^n is $\boldsymbol{n}$**-dimensional**.
- **Example**:
  - 0-simplex is a point.
  - 1-simplex is a closed line segment.
  - 2-simplex is closed triangle including its interior.
  - 3-simplex is closed tetrahedron including its interior.
- **Definition**: if $\sigma^n = \langle v_0, ..., v_n \rangle$ is n-simplex and $\{i_0, ..., i_r\} \subseteq \{0, ..., n\}$, then $\langle v_{i_0}, ..., v_{i_r} \rangle$ is r-simplex and $\langle v_{i_0}, ..., v_{i_r} \rangle \subseteq \sigma^n$. Any such sub-simplex is called $\boldsymbol{r}$**-face** of σ^n. A **proper face** is an $(n-1)$-face. The $\boldsymbol{i}$**th face** of σ^n is the $(n-1)$-simplex $\langle v_0, ..., v_{i-1}, v_{i+1}, ..., v_n \rangle$.
- **Definition**: a **finite simplicial complex** $K \subset \mathbb{R}^N$ is finite union of simplices in $\mathbb{R}^N$ such that
  - If σ^n is simplex in K and τ^r is r-face of σ^n, then τ^r is simplex in K.
  - If σ_1^n and σ_2^m are simplices in K with $\sigma_1^n \cap \sigma_2^m \ne \emptyset$, then there exists $r \in \{0, ..., \min(n, m)\}$ and r-simplex τ^r in K such that τ^r is r-face of both σ_1^n and σ_2^m and $\sigma_1^n \cap \sigma_2^m = \tau^r$.

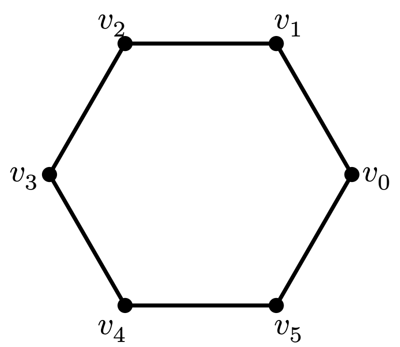
  Dimension of K is maximum value of n for which there is an n-simplex in K.
- **Remark**: a finite simplicial complex $K \subset \mathbb{R}^N$ is a topological space when equipped with subspace topology from $\mathbb{R}^N$.
- **Remark**: second condition implies that two simplices can meet in at most one common face (this is important when considering quotient topologies and identifying edges with each other).
- **Definition**: **triangulation** of topological space X is homeomorphism $h : X \to K$ for some finite simplicial complex K. We say K **triangulates** X. X is **triangulable** if it has at least one triangulation.
- **Remark**: if a triangulation exists, it is not unique.
- **Example**: the black and blue figures are simplicial complexes that triangulate S^1:

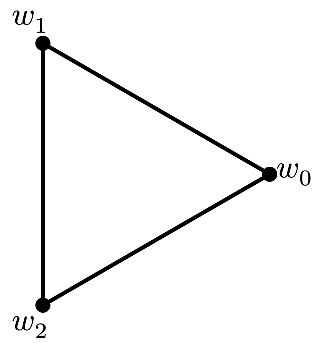
9.2. Simplicial maps

- **Definition**: a map $f : K \to L$ between finite simplicial complexes K and L is **simplicial** if
  - For every vertex v of K, $f(v)$ is a vertex of L.
  - If $\sigma = \langle v_0, ..., v_n \rangle$ is simplex σ in K, $f(\sigma)$ is simplex of L with vertices $f(v_0), ..., f(v_n)$, where map $f|_\sigma$ is defined as

$$f\left(\sum_{i=0}^{n} \lambda_i v_i\right) = \sum_{i=0}^{n} \lambda_i f(v_i)$$

- **Remark**: vertices $f(v_0), ..., f(v_n)$ of simplex $f(\sigma)$ may not be distinct, so $f(\sigma)$ may be simplex of lower dimension than σ.
- **Remark**: for triangulations $h_X : X \to K_X$ and $h_Y : Y \to K_Y$ of topological spaces X and Y, a simplicial map $f : K_X \to K_Y$ induces a map $F : X \to Y$ by $F = h_Y^{-1} \circ f \circ h_X$.
- **Example**: $F : S^1 \to S^1$, $F(e^{i\pi t}) = e^{2i\pi t}$ is the **2 times** map. Let $f : K_1 \to K_2$, $f(v_i) = w_{i \bmod 3}$, f is simplicial map. Then F is induced by f, where K_1 and K_2 are as below:





9.3. Barycentric subdivision and simplicial approximation

- **Definition**: **barycentre** of $\sigma^k = \langle v_0, ..., v_k \rangle \subset \mathbb{R}^N$ is

$$\overline{\sigma^k} = \frac{1}{k+1}(v_0 + \cdots + v_k) \in \mathbb{R}^N$$

- **Example**:
  - Barycentre of 0-simplex is itself.
  - Barycentre of 1-simplex is midpoint of the line.
- **Definition**: let $K \subset \mathbb{R}^N$ be finite simplicial complex. **First barycentric subdivision** of K is the simplicial complex $K^{(1)}$ such that:
  - The vertices of $K^{(1)}$ are the barycentres $\overline{\sigma^k}$ for every simplex σ^k in K.
  - The vertices $\overline{\sigma^{k_0}}, ..., \overline{\sigma^{k_m}} \in K^{(1)}$ span an m-simplex in $K^{(1)}$ if the original simplices $\sigma^{k_0}, ..., \sigma^{k_m}$ in K are (up to relabelling) strictly nested:

$$\sigma^{k_0} \prec \cdots \prec \sigma^{k_m}$$

  where $\sigma^i \prec \sigma^j$ iff σ^i is i-face of σ^j with $i < j$ (thus $k_0 < \cdots < k_m$).

- **Definition**: the rth barycentric subdivision of K is defined inductively for $r > 1$ by $K^{(r)} := \left(K^{(r-1)}\right)^{(1)}$.
- **Remark**: let K be finite simplicial complex.
  - If K is triangulation of topological space X, then so is $K^{(r)}$ for all $r \in \mathbb{N}$.
  - Each simplex in $K^{(1)}$ is contained in a simplex of K.
- **Simplicial approximation theorem**: for each $i \in \{1, 2\}$, let $h_i : X_i \to K_i$ be triangulation of topological space X_i by finite simplicial complex K_i. Let $f : X_1 \to X_2$ be map. Then $\forall \varepsilon > 0$ there exist $n, m \in \mathbb{N}$ and a simplicial map $s : K_1^{(n)} \to K_2^{(m)}$ such that for $F := h_2 \circ f \circ h_1^{-1}$,

$$s \simeq F \quad \text{and} \quad \forall x \in K_1, \quad |F(x) - s(x)| < \varepsilon$$

10. Surfaces

10.1. Surfaces

- **Definition**: let S be Hausdorff, compact, connected topological space.
  - S is **surface** if for all $x \in S$, there exists $U \subseteq S$ such that $x \in U$ and $U \cong E^2$ or $U \cong E^2 \cap \mathbb{R} \times \mathbb{R}_{\geq 0}$.
  - **Boundary** of S, ∂S, is set of all $x \in S$ such that there is not a $U \subseteq S$ with $x \in U$ and $U \cong E^2$.
  - **Interior** of S is $\text{int}(S) := S - \partial S$.
  - S is **closed surface** if $\partial S = \emptyset$ (S is **locally Euclidean of dimension 2**).
  - S is **surface with boundary** if $\partial S \neq \emptyset$. Surface with boundary is closed surface from which interiors of finite number of pairwise disjoint closed discs have been removed.
- **Definition**: let K be finite simplicial complex, $x \in K$. **Open star** of x in K, $\text{St}(x, K)$, is union of $\{x\}$ and interiors of all simplices containing x.
- **Example**: let K be 2d finite simplicial complex, $x \in K$.
  - If there exists a 2-simplex $\sigma^2 \subseteq K$ such that $x \in \text{int}(\sigma^2)$, then $\text{St}(x, K) = \text{int}(\sigma^2) \cong E^2$.
  - If there exists a 1-simplex $\sigma^1 \subseteq K$ such that $x \in \text{int}(\sigma^1)$, then

$$\text{St}(x, K) = \text{int}(\sigma^1) \cup \left\{\text{int}(\sigma^2) : \sigma^1 \text{ is face of } \sigma^2 \subseteq K, \sigma^2 \text{ is 2-simplex}\right\}$$

    Here, $\text{St}(x, K) \cong E^2$ iff there are exactly two 2-simplices meeting along σ^1.
  - If $x \in K$ is vertex, then

$$\begin{aligned} \text{St}(x, K) = \{x\} &\cup \left\{\text{int}(\sigma^1) : x \text{ vertex of } \sigma^1 \subseteq K, \sigma^1 \text{ is 1-simplex}\right\} \\ &\cup \left\{\text{int}(\sigma^2) : x \text{ vertex of } \sigma^2 \subseteq K, \sigma^2 \text{ is 2-simplex}\right\} \end{aligned}$$

    Here $\text{St}(x, K) \cong E^2$ iff x is vertex of $n \geq 3$ 2-simplices, and along any of its edges containing x, each of these 2-simplices meets precisely one other 2-simplex (from the remaining $n - 1$).
- **Lemma**: let M be topological space triangulated by connected, finite simplicial complex K. Then M is closed surface iff

$$\forall x \in K, \quad \mathrm{St}(x, K) \cong E^2$$

and the ways that this can happen are as listed above, with exactly two 2-simplices meeting along each 1-simplex.

- **Remark**: if $h : M \to K$ is triangulation of topological space M and $\dim(K) \neq 2$, then M is not closed surface. It is enough to check the open star condition (in above example) at all vertices of K: if there is $x \in K$ such that $\mathrm{St}(x, K) \ncong E^2$, then there exists vertex v of K such that $\mathrm{St}(v, K) \ncong E^2$.
- **Corollary**: let X topological space, triangulated by connected finite simplicial complex K, $\dim(K) = 2$. Then X is closed surface iff for every vertex $v \in K$, $\mathrm{St}(v, K) \cong E^2$.
- **Definition**: **real projective plane** is closed surface arising from identifying the edges of the unit square with the following:

$$\mathbb{P} := (I \times I) / \sim, \quad (x, 0) \sim (1 - x, 1), \quad (0, y) \sim (1, 1 - y)$$

10.2. Orientations on surfaces

- **Definition**: an **orientation on $\mathbb{R}^2$** is choice of direction in which to travserse circles around the origin. There are exactly two choices.
- **Definition**: **simple closed curve** in topological space is subspace homeomorphic to circle, i.e. connected curve with no self-intersections and ends where it begins.
- **Definition**: surface S is **orientable** if for all $x \in \mathrm{int}(S)$, any choice of local orientation at x is preserved after translation along any simple closed curve in $\mathrm{int}(S)$ containing x. S is **non-orientable** if there exists $x \in \mathrm{int}(S)$ and simple closed curve $C \subseteq \mathrm{int}(S)$ through x such that translation along C reverses any choice of local orientation at x. Every surface is either orientable or non-orientable.
- **Example**: $S^2, \mathbb{T}$ are orientable. Mobius band and Klein bottle are non-orientable.
- **Lemma**: S is non-orientable iff it contains subspace homeomorphic to Mobius band.
- **Theorem**: let S_1, S_2 be homeomorphic surfaces. S_1 is orientable iff S_2 is orientable.
- **Remark**: 2-simplex can be given orientation by drawing a direction around it (anticlockwise or clockwise) or by drawing direction around its boundary. A 2-simplex can be oriented in 2 ways, which can be represented by ordering of the vertices: $\langle v_0, v_1, v_2 \rangle$, $\langle v_1, v_2, v_0 \rangle$ and $\langle v_2, v_0, v_1 \rangle$ represent same orientation, $\langle v_1, v_0, v_2 \rangle$ represents different orientation.
- **Definition**: let K finite simplicial complex that triangulates surface S such that all 2-simplices in K are oriented.
  - The orientations of two 2-simplices in K which share an edge are **compatible** if they induce opposite orientations on the shared edge.
  - K is **Δ-orientable** if there exists choice of orientations on its 2-simplices such that any two 2-simplices which share an edge have compatible orientations. Such a choice, if it exists, is a **Δ-orientation** on K.
- **Theorem**: surface is orientable iff one (and so every) finite simplicial complex which triangulates it is Δ-orientable.

10.3. Constructions on surfaces

- **Definition**: for surfaces S_1, S_2, their **connected sum**, $S_1 \# S_2$, is obtained by removing the interiors of one small open disc from interior of each surface, and identifying the two newly formed boundary circles. If S_1, S_2 oriented, directions around the boundary circles must be identified such that their induced orientations are opposite to each other. Then $S_1 \# S_2$ inherits an orientation which agrees (upon restriction) with those of the original surfaces S_1 and S_2.
- **Proposition**:
  - Since S_1, S_2 connected, it does not matter which two open discs are removed, the result is the same up to homeomorphism.
  - $\#$ is commutative and associative.
  - S^2 is the identity for $\#$ operation: $M \# S^2 \cong M$.
- **Definition**: for $g \in \mathbb{N}_0$, **closed orientable surface of genus g (g-holed torus)** is

$$M_g = S^2 \# \underbrace{T \# \cdots \# T}_{g \text{ times}}$$

- **Example** the Klein bottle is given by $\mathbb{K} \cong \mathbb{P} \# \mathbb{P}$.
- **Definition**: **adding handle** to surface S is as follows: remove two open discs from S. Attach the ends of cylinder $S^1 \times I$ to the resulting boundary circles. If S (and cylinder) are oriented, require that the two resulting boundary circles are glued to those of the cylinder with opposite orientations, which ensures the new surface is still oriented. But if S is not orientable, this doesn't matter, as all possible results are homeomorphic.
- **Example**:
  - S^2 with handle added is homeomorphic to the torus.
  - S^2 with g handles added is homeomorphic to M_g.
  - M_n with handle added is homeomorphic to M_{n+1}.
- **Definition**: **attaching a cross cap (Mobius band)** to surface S is as follows: remove open disc from S, and identify resulting boundary circle with boundary circle of Mobius band. Attaching a cross-cap always makes the surface non-orientable.
- **Example**: adding cross-cap to S^2 gives real projective plane $\mathbb{P}$.
- **Remark**: connected sums of surfaces, surfaces with handles and surfaces with cross caps are always surfaces.

11. Homotopy and the fundamental group

11.1. Homotopy

- **Definition**: let X, Y topological spaces. **Homotopy** between f and g is map $H : X \times [0,1] \to Y$ with

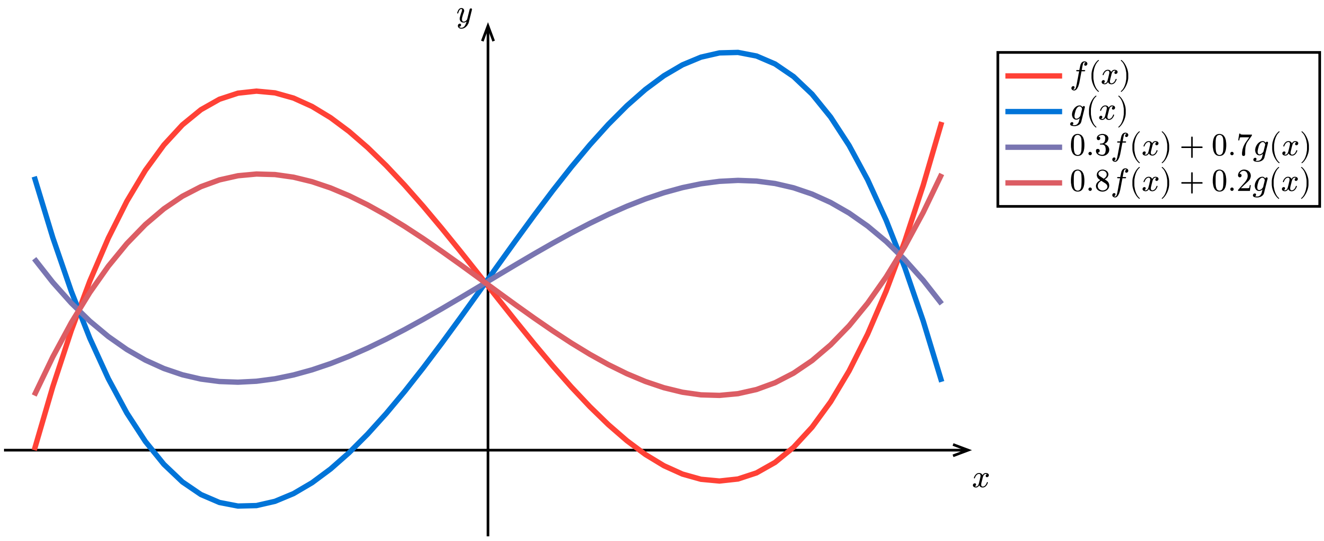
$$\forall x \in X, \quad H(x,0) = f(x) \wedge H(x,1) = g(x)$$

f and g are **homotopic**, $f \simeq g$, if there is a homotopy between them. We can think of homotopy as "path of maps" starting at $f : X \to Y$ and ending at $g : X \to Y$: for $t \in [0,1]$, define $h_t : X \to Y$, $h_t(x) = H(x,t)$, which varies continuously from f at $t = 0$ to g at $t = 1$.

- **Example**: let $f, g : \mathbb{R} \to \mathbb{R}$ maps, then

$$H : \mathbb{R} \times [0,1] \to \mathbb{R}, \quad (x,t) \mapsto (1-t)f(x) + tg(x)$$

is homotopy between f and g.



- **Example**: consider $S^1 \subset \mathbb{C}$, so $S^1 = \{e^{i\pi s} : s \in [0,2)\}$. Let $a : S^1 \to S^1$ be the **antipodal map**, $a(e^{i\pi s}) = e^{-i\pi s}$. Then $a \simeq \text{id}$, with homotopy given by $H : S^1 \times I \to S^1$, $H(e^{i\pi s}) = e^{i\pi(s+t)}$.
- **Lemma**: Homotopy is equivalence relation between maps.
- **Definition**: map $f : X \to Y$ is **null homotopic** if it is homotopic to a constant map, i.e. to map $c : X \to Y$ with $c(x) = y_0$, $y_0 \in Y$ fixed.
- **Example**: identity map $\text{id}_{D^2} : D^2 \to D^2$ is null homotopic: let $c : D^2 \to D^2$, $c(x) = 0$. Consider $H : D^2 \times [0,1] \to D^2$, $H(x,t) = (1-t)x$, then H is homotopy between id_{D^2} and c, since H is continuous and $H(x,0) = x = \text{id}_{D^2}(x)$, $H(x,1) = 0 = c(x)$.
- **Definition**: map $f : X \to Y$ is **homotopy equivalence** if there exists a map $g : Y \to X$ (a **homotopy inverse**) such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. X and Y are **homotopy equivalent**, $X \simeq Y$ if there exists homotopy equivalence between them. If $X \simeq Y$, we say they have the same **homotopy type**.
- **Theorem**: homotopy equivalence is equivalence relation on topological spaces.
- **Example**: let $P = \{\boldsymbol{p}\}$ be the one point space, then $D^2 \simeq P$: let $f : D^2 \to P$, $f(x) = \boldsymbol{p}$, $g : P \to D^2$, $g(\boldsymbol{p}) = 0$. Then $f \circ g = \text{id}_P \simeq \text{id}_P$. Now $\forall x \in D^2$, $(g \circ f)(x) = 0$ so $g \circ f \simeq \text{id}_{D^2}$ as $g \circ f$ is constant map.
- **Definition**: topological space X is **contractible** if it is homotopy equivalent to a one-point space.
- **Example**: let X topological space. The **cone on $\boldsymbol{X}$** is

$$CX = (X \times [0,1]) / \sim$$

where $\sim$ identifies all points of the form $(x, 0)$ with each other, i.e. it collapses the end $X \times \{0\}$ to a single point. We have $D^n \cong CS^{n-1}$.
- **Proposition**: for all topological spaces X, the cone CX is contractible.
- **Lemma**: every contractible space is path connected.
- **Lemma**: if X and Y are homeomorphic, they are homotopy equivalent (converse does not hold).
- **Definition**:
  - It is useful to assume that every topological space X has a particular distinguished **base point** $x_0 \in X$.
  - We then require that all maps and homotopies between spaces map base points to base points.
  - The pair (X, x_0) is a **based space**.
  - A **based map** $f : (X, x_0) \to (Y, y_0)$ is a map $X \to Y$ and satisfies $f(x_0) = y_0$.
  - A **based homotopy** $H : (X, x_0) \times [0, 1] \to (Y, y_0)$ between based maps $f, g : (X, x_0) \to (Y, y_0)$ is homotopy $H : X \times [0, 1] \to Y$ with $\forall t \in [0, 1]$, $H(x_0, t) = y_0$.
  - All results shown for homotopies are true for based homotopies.

11.2. The fundamental group

- **Remark**: we consider circle S^1 as unit circle in $\mathbb{C}$ and give it base point 1.
- **Definition**: a **loop** in based space (X, x_0) is based map

$$\lambda : (S^1, 1) \to (X, x_0)$$

Equivalently, a loop in (X, x_0) is path in X beginning and ending at x_0:

$$\lambda : [0, 1] \to (X, x_0), \quad \lambda(0) = \lambda(1) = x_0$$