Contents

1. Monochromatic sets

1.1. Ramsey's theorem

Notation 1.1 $\mathbb{N}$ denotes the set of positive integers, $[n] = \{1, ..., n\}$, and $X^{(r)} = \{A \subseteq X : |A| = r\}$. Elements of a set are written in ascending order, e.g. $\{i, j\}$ means $i < j$. Write e.g. ijk to mean the set $\{i, j, k\}$ with the ordering (unless otherwise stated) $i < j < k$.

Definition 1.2 A ***k*-colouring** on $A^{(r)}$ is a function $c : A^{(r)} \to [k]$.

Example 1.3
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ is even and blue if $i + j$ is odd. Then $M = 2\mathbb{N}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$ is even and blue otherwise. $M = \{4^n : n \in \mathbb{N}\}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

Theorem 1.4 (Ramsey's Theorem for Pairs) Let $\mathbb{N}^{(2)}$ are 2-coloured by $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Then there exists an infinite monochromatic subset M.

Proof.
- Let $a_1 \in A_0 := \mathbb{N}$. There exists an infinite set $A_1 \subseteq A_0$ such that $c(a_1, i) = c_1$ for all $i \in A_1$.
- Let $a_2 \in A_1$. There exists infinite $A_2 \subseteq A_1$ such that $c(a_2, i) = c_2)$ for all $i \in A_2$.
- Repeating this inductively gives a sequence $a_1 < a_2 < \cdots < a_k < \cdots$ and $A_1 \supseteq A_2 \supseteq \cdots$ such that $c(a_i, j) = c_i$ for all $j \in A_i$.
- One colour appears infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c_{i_k} = \cdots = c$.
- $M = \{a_{i_1}, a_{i_2}, ...\}$ is a monochromatic set.

$\square$

Remark 1.5
- The same proof works for any $k \in \mathbb{N}$ colours.
- The proof is called a "2-pass proof".
- An alternative proof for k colours is split the k colours $1, ..., k$ into 2 colours: 1 and "2 or ... or k", and use induction.

Note 1.6 An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

Example 1.7 Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, etc. Let $\{i, j\}$ be red if $i, j \in A_k$ for some k. There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

Example 1.8 Colour $\{i < j < k\}$ red iff $i \mid (j + k)$. A monochromatic subset $M = \{2^n : n \in \mathbb{N}_0\}$ is a monochromatic set.

Theorem 1.9 (Ramsey's Theorem for r-sets) Let $\mathbb{N}^{(r)}$ be finitely coloured. Then there exists a monochromatic infinite set.

Proof.

- $r = 1$: use pigeonhole principle.
- $r = 2$: Ramsey's theorem for pairs.
- For general r, use induction.
- Let $c : \mathbb{N}^r \to [k]$ be a k-colouring. Let $a_1 \in \mathbb{N}$, and consider all $r-1$ sets of $\mathbb{N} \setminus \{a_1\}$, induce colouring $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \to [k]$ via $c'(F) = c(F \cup \{a_1\})$.
- By inductive hypothesis, there exists $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that c' is constant on it (taking value c_1).
- Now pick $a_2 \in A_1$ and induce a colouring $c' : (A_1 \setminus \{a_2\})^{(r-1)} \to [k]$ such that $c'(F) = c(F \cup \{a_2\})$. By inductive hypothesis, there exists $A_2 \subseteq A_1 \setminus \{a_2\}$ such that c' is constant on it (taking value c_2).
- Repeating this gives $a_1, a_2, \dots$ and $A_1, A_2, \dots$ such that $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$ and $c(F \cup \{a_i\}) = c_i$ for all $F \subseteq A_{i+1}$, for $|F| = r - 1$.
- One colour must appear infinitely many times: $c_{i_1} = c_{i_2} = \dots = c$.
- $M = \left\{a_{i_1}, a_{i_2}, \dots\right\}$ is a monochromatic set.

□

1.2. Applications of Ramsey's theorem

Example 1.10 In a totally ordered set, any sequence has monotonic subsequence.

Proof.

- Let (x_n) be a sequence, colour $\{i, j\}$ red if $x_i \le x_j$ and blue otherwise.
- By Ramsey's theorem for pairs, $M = \{i_1 < i_2 < \dots\}$ is monochromatic. If M is red, then the subsequence $x_{i_1}, x_{i_2}, \dots$ is increasing, and is strictly decreasing otherwise.
- We can insist that (x_{i_j}) is either concave or convex: 2-colour $\mathbb{N}^{(3)}$ by colouring $\{j < k < \ell\}$ red if $\left(i, x_{i_j}\right), \left(j, x_{i_k}\right), \left(k, x_{i_\ell}\right)$ form a convex triple, and blue if they form a concave triple. Then by Ramsey's theorem for r-sets, there is an infinite convex or concave subsequence.

□

Theorem 1.11 (Finite Ramsey) Let $r, m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k-coloured, we can find a monochromatic set of size (at least) m.

Proof.

- Assume not, i.e. $\forall n \in \mathbb{N}$, there exists colouring $c_n : [n]^{(r)} \to [k]$ with no monochromatic m-sets.
- There are only finitely many (k) ways to k-colour $[r]^{(r)}$, so there are infinitely many of colourings $c_r, c_{r+1}, \dots$ that agree on $[r]^{(r)}$: $c_i \mid_{[r]^{(r)}} = d_r$ for all i in some infinite set A_1, where d_r is a k-colouring of $[r]^{(r)}$.
- Similarly, $[r+1]^{(r)}$ has only finitely many possible k-colourings. So there exists infinite $A_2 \subseteq A_1$ such that for all $i \in A_2$, $c_i \mid_{[r+1]^{(r)}} = d_{r+1}$, where d_{r+1} is a k-colouring of $[r+1]^{(r)}$.
- Continuing this process inductively, we obtain $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n$. There is no monochromatic m-set for any $d_n : [n]^{(r)} \to [k]$ (because $d_n = c_i|_{[n]^{(r)}}$ for some i).
- These d_n's are nested: $d_\ell|_{[n]^{(r)}} = d_n$ for $\ell > n$.

- Finally, we colour $\mathbb{N}^{(r)}$ by the colouring $c : \mathbb{N}^{(r)} \to [k]$, $c(F) = d_n(F)$ where $n = \max(F)$ (or in fact $n \geq \max(F)$, which is well-defined by above). So c has no monochromatic m-set (since M was a monochromatic m-set, then taking $\ell = \max(M)$, d_ℓ has a monochromatic m-set), which contradicts Ramsey's Theorem for r-sets.

□

Remark 1.12

- This proof gives no bound on $n = n(k, m)$, there are other proofs that give a bound.
- It is a proof by compactness (essentially, we proved that $\{0, 1\}^{\mathbb{N}}$ with the product topology, i.e. the topology derived from the metric $d(f, g) = \frac{1}{\min\{n \in \mathbb{N} : f(n) \neq g(n)\}}$, is sequentially compact).

Remark 1.13 Now consider a colouring $c : \mathbb{N}^{(2)} \to X$ with X potentially infinite. This does not necessarily admit an infinite monochromatic set, as we could colour each edge a different colour. Such a colouring would be injective. We can't guarantee either the colouring being constant or injective though, as $c(ij) = i$ satisfies neither.

Theorem 1.14 (Canonical Ramsey) Let $c : \mathbb{N}^{(2)} \to X$ be a colouring with X an arbitrary set. Then there exists an infinite set $M \subseteq \mathbb{N}$ such that:

1. c is constant on $M^{(2)}$, or
2. c is injective on $M^{(2)}$, or
3. $c(ij) = c(kl)$ iff $i = k$ for all $i < j$ and $k < l$, $i, j, k, l \in M$, or
4. $c(ij) = c(kl)$ iff $j = l$ for all $i < j$ and $k < l$, $i, j, k, l \in M$.

Proof (Hints).

- First consider the 2-colouring c_1 of $\mathbb{N}^{(4)}$ where $ijkl$ is coloured SAME if $c(ij) = c(kl)$ and DIFF otherwise. Show that an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ (why does this exist?) coloured SAME leads to case 1.
- Assume M_1 is coloured DIFF, consider the 2-colouring of $M_1^{(4)}$, which colours $ijkl$ SAME if $c(il) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic $M_2 \subseteq M_1$ (why does this exist?) must be coloured DIFF by contradiction.
- Consider the 2-colouring of $M_2^{(4)}$ where $ijkl$ is coloured SAME if $c(ik) = c(jl)$ and DIFF otherwise. Show an infinite monochromatic set $M_3 \subseteq M_2$ (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour $M_3^{(3)}$ by: ijk is coloured SAME if $c(ij) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic set $M_4 \subseteq M_3$ (why does this exist) must be coloured DIFF by contradiction.
- 2-colour $M_4^{(3)}$ by the other two similar colourings to above, obtaining monochromatic $M_6 \subseteq M_5 \subseteq M_4$.
- Consider 4 combinations of these colourings on M_6, show 3 lead to one of the cases in the theorem, and the other leads to contradiction.

□

Proof.

- 2-colour $\mathbb{N}^{(4)}$ by: $ijkl$ is red if $c(ij) = c(kl)$ and blue otherwise. By Ramsey's Theorem for 4-sets, there is an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ for this colouring.
- If M_1 is red, then c is constant on $M_1^{(2)}$: for all pairs $ij, i'j' \in M_1^{(2)}$, pick $m < n$ with $j, j' < m$, then $c(ij) = c(mn) = c(i'j')$.
- So assume M_1 is blue.
- Colour $M_1^{(4)}$ by giving $ijkl$ colour green if $c(il) = c(jk)$ and purple otherwise. By Ramsey's theorem for 4-sets, there exists an infinite monochromatic $M_2 \subseteq M_1$ for this colouring.
- Assume M_2 is coloured green: if $i < j < k < l < m < n \in M_2$, then $c(jk) = c(in) = c(lm)$ (consider $ijkn$ and $ilmn$): contradiction, since M_1 is blue.
- Hence M_2 is purple, i.e. for $ijkl \in M_2^{(4)}$, $c(il) \neq c(jk)$.
- Colour M_2 by: $ijkl$ is orange if $c(ik) = c(jl)$, and pink otherwise.
- By Ramsey's theorem for 4-sets, there exists infinite monochromatic $M_3 \subseteq M_2$ for this colouring.
- Assume M_3 is orange, then for $i < j < k < l < m < n \in M_3$, we have $c(jm) = c(ln)$ (consider $jlmn$) and $c(jm) = c(ik)$ (consider $ijkm$): contradiction, since $M_3 \subseteq M_1$.
- Hence M_3 is pink, i.e. for $ijkl$, $c(ik) \neq c(jl)$.
- Colour $M_3^{(3)}$ by: ijk is yellow if $c(ij) = c(jk)$ and grey otherwise. By Ramsey's theorem for 3-sets, there exists infinite monochromatic $M_4 \subseteq M_3$ for this colouring.
- Assume M_4 is yellow: then (considering $ijkl \in M_4^{(4)}$) $c(ij) = c(jk) = c(kl)$: contradiction, since $M_4 \subseteq M_1$.
- So for any $ijk \in M_4^{(3)}$, $c(ij) \neq c(jk)$.
- Finally, colour $M_4^{(3)}$ by: ijk is gold if $c(ij) = c(ik)$ and $c(ik) = c(jk)$, silver if $c(ij) = c(ik)$ and $c(ik) \neq c(jk)$, bronze if $c(ij) \neq c(ik)$ and $c(ik) = c(jk)$, and platinum if $c(ij) \neq c(ik)$ and $c(ik) \neq c(jk)$.
- By Ramsey's theorem for 3-sets, there exists monochromatic $M_5 \subseteq M_4$. M_5 cannot be gold, since then $c(ij) = c(jk)$: contradiction, since $M_5 \subseteq M_4$. If silver, then we have case 3 in the theorem. If bronze, then we have case 4 in the theorem. If platinum, then we have case 2 in the theorem.

$\square$

Remark 1.15

- A more general result of the above theorem states: let $\mathbb{N}^{(r)}$ be arbitrarily coloured. Then we can find an infinite M and $I \subseteq [r]$ such that for all $x_1...x_r \in M^{(r)}$ and $y_1...y_r \in M^{(r)}$, $c(x_1...x_r) = c(y_1...y_r)$ iff $x_i = y_i$ for all $i \in I$.
- In canonical Ramsey, $I = \emptyset$ is case 1, $I = \{1, 2\}$ is case 2, $I = \{1\}$ is case 3 and $I = \{2\}$ is case 4.
- These 2^r colourings are called the **canonical colourings** of $\mathbb{N}^{(r)}$.

Exercise 1.16 Prove the general statement.

1.3. Van der Waerden's theorem

Remark 1.17 We want to show that for any 2-colouring of $\mathbb{N}$, we can find a monochromatic arithmetic progression of length m for any $m \in \mathbb{N}$. By compactness, this is equivalent to showing that for all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any 2-colouring of $[n]$, there exists a monochromatic arithmetic progression of length m. (If not, then for each $n \in \mathbb{N}$, there is a colouring $c_n : [n] \to \{1, 2\}$ with no monochromatic arithmetic progression of length m. Infinitely many of these colourings agree on $[1]$, infinitely many of those agreeing in $[1]$ agree on $[2]$, and so on - we obtain a 2-colouring of $\mathbb{N}$ with no monochromatic arithmetic progression of length m).

We will prove a slightly stronger result: whenever $\mathbb{N}$ is k-coloured, there exists a length m monochromatic arithmetic progression, i.e. for any $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k-coloured, we have a length m monochromatic progression.

Definition 1.18 Let $A_1, ..., A_k$ be length m arithmetic progressions: $A_i = \{a_i, a_i + d_i, ..., a_i + (m-1)d_i\}$. $A_1, ..., A_k$ are **focussed** at f if $a_i + md_i = f$ for all i.

Example 1.19 $\{4, 8\}$ and $\{6, 9\}$ are focussed at 12.

Definition 1.20 If length m arithmetic progressions $A_1, ..., A_k$ are focused at f and are monochromatic, each with a different colour (for a given colouring), they are called **colour-focussed** at f.

Remark 1.21 We use the idea that if $A_1, ..., A_k$ are colour-focussed at f (for a k-colouring) and of length $m - 1$, then some $A_i \cup \{f\}$ is a length m monochromatic arithmetic progression.

Theorem 1.22 Whenever $\mathbb{N}$ is k-coloured, there exists a monochromatic arithmetic progression of length 3, i.e. for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any k-colouring of $[n]$ admits a length 3 monochromatic progression.

Proof (Hints).

- Prove by induction the claim: $\forall r \leq k$, $\exists n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there exists a monochromatic arithmetic progression of length 3, or r colour-focussed arithmetic progressions of length 2.
  - $r = 1$ case is straightforward.
  - Let claim be true for $r - 1$ with witness n, let $N = 2n(k^{2n} + 1)$.
  - Partition N into blocks of equal size, show that two of these blocks must have the same colouring.
  - Using the inductive hypothesis, merge the $r - 1$ colour-focussed arithmetic progressions from these two blocks into a new set of $r - 1$ colour-focussed arithmetic progressions.
  - Find another length 2 monochromatic arithmetic progression, reason that this is of different colour.
- Reason that this claim implies the result.

□

Proof.

- We claim that for all $r \leq k$, there exists an $n \in \mathbb{N}$ such that if $[n]$ is k-coloured, then either:
  - There exists a monochromatic arithmetic progression of length 3.
  - There exist r colour-focussed arithmetic progressions of length 2.
- This claim implies the result by the above remark.
- We prove the claim by induction on r:
  - $r = 1$: take $n = k + 1$, then by pigeonhole, some two elements of $[n]$ have the same colour, so form a length two arithmetic progression.
  - Assume true for $r - 1$ with witness n. We claim that $N = 2n(k^{2n} + 1)$ works for r.
  - Let $c : [2n(k^{2n} + 1)] \to [k]$ be a colouring. We partition $[N]$ into $k^{2n} + 1$ blocks of size $2n$: $B_i = \{2n(i - 1) + 1, ..., 2ni\}$ for $i = 1, ..., k^{2n} + 1$.
  - Assume there is no length 3 monochromatic progression for c. By inductive hypothesis, each block B_i has $r - 1$ colour-focussed arithmetic progressions of length 2.
  - Since $|B_i| = 2n$, each block also contains their focus. For a set M with $|M| = 2n$, there are k^{2n} ways to k-colour M. So by pigeonhole, there are blocks B_s and B_{s+t} that have the same colouring.
  - Let $\{a_i, a_i + d_i\}$ be the $r - 1$ arithmetic progressions in B_s colour-focussed at f, then $\{a_i + 2nt, a_i + d_i + 2nt\}$ is the corresponding set of arithmetic progressions in B_{s+t}, each colour-focussed at $f + 2nt$.
  - Now $\{a_i, a_i + d_i + 2nt\}$, $i \in [r - 1]$, are $r - 1$ arithmetic progresions colour-focused at $f + 4nt$. Also, $\{f, f + 2nt\}$ is monochromatic of a different colour to the $r - 1$ colours used (since there is no length 3 monochromatic progression for c). Hence, there are r arithmetic progressions of length 2 colour-focussed at $f + 4nt$.

□

Remark 1.23 The idea of looking at all possible colourings of a set is called a **product argument**.

Definition 1.24 The **Van der Waerden** number $W(k, m)$ is the smallest $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there exists a monochromatic arithmetic progression in $[n]$ of length m.

Remark 1.25 The above theorem gives a **tower-type** upper bound $W(k, 3) \leq k^{k^{(\cdot\cdot)^{k^{4k}}}}$.

Theorem 1.26 (Van der Waerden's Theorem) For all $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there is a length m monochromatic arithmetic progression.

Proof (Hints).

- Use induction on m.

- Given induction hypothesis on $m - 1$, prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, we have either a monochromatic length m arithmetic progression, or r colour-focussed arithmetic progressions of length $m - 1$. Reason that this claim implies the result.
- Use induction on r. Give an explicit n for $r = 1$.
- Let n be the witness for $r - 1$, let $N = W(k^{2n}, m - 1) \cdot 2n$. Assume a k-colouring of $[N]$, $c : [N] \to [k]$, has no arithmetic progressions of length m.
- Partition $[N]$ into the obvious choice of $W(k^{2n}, m - 1)$ blocks B_i, each of length $2n$.
- Colour the indices $1 \leq i \leq W(k^{2n}, m - 1)$ of the blocks by

$$c'(i) = (c(2n(i-1)+1), c(2n(i-1)+2)...., c(2ni))$$

- Reason that we can find monochromatic arithmetic progression $s, s+t, ..., s + (m-2)t$ of length $m - 1$ (w.r.t c'), and that this corresponds to sequence of blocks $B_s, B_{s+t}, ..., B_{s+(m-2)t}$, each identically coloured.
- Reason that B_s contains $r - 1$ colour-focussed length $m - 1$ arithmetic progressions A_i together with their focus f.
- Let A_i' be the same arithmetic progression but with common difference $2nt$ larger than that of A_i. Show the A_i' are colour-focussed at some focus in terms of f.
- Find another length $m - 1$ arithmetic progression, show this must be monochromatic and of different colour to all A_i'. Show it also has same focus as all A_i'.

□

Proof.
- By induction on m. $m = 1$ is trivial, $m = 2$ is by pigeonhole principle. $m = 3$ is the statement of the previous theorem.
- Assume true for $m - 1$ and all $k \in \mathbb{N}$.
- For fixed k, we prove the claim: for all $r \leq k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, either:
  - There is a monochromatic arithmetic progression of length m, or
  - There are r colour-focussed arithmetic progressions of length $m - 1$.
- We will then be done (by considering the focus).
- To prove the claim, we use induction on r.
- $r = 1$ is the claim of the first inductive hypothesis: take $n = W(k, m - 1)$.
- Assume the claim holds for $r - 1$ with witness n, and assume there is no monochromatic arithmetic progression of length m. We will show that $N = W(k^{2n}, m - 1)2n$ is sufficient for r.
- Partition $[N]$ into $W(k^{2n}, m - 1)$ blocks of length $2n$: $B_i = \{2n(i-1)+1, ..., 2ni\}$ for $i = 1, ..., W(k^{2n}, m - 1)$.
- Each block has k^{2n} possible colourings. Colour the blocks as

$$c'(i) = (c(2n(i-1)+1), c(2n(i-1)+2)...., c(2ni))$$

By definition of W, there exists a monochromatic arithmetic progression of length $m-1$ (w.r.t. to c'): $\{\alpha, \alpha+t, ..., \alpha+(m-2)t\}$. The repsective blocks $B_\alpha, ..., B_{\alpha+(m-2)t}$ are identically coloured.

- B_α has length $2n$, so by induction B_α contains $r-1$ colour-focussed arithmetic progressions of length $m-1$, together with their focus (as length of block is $2n$).
- Let $A_1, ..., A_{r-1}$, $A_i = \{a_i, a_i+d_i, ..., a_i+(m-2)d_i\}$, be colour-focussed at f.
- Let $A_i' = \{a_i, a_i+(d_i+2nt), ..., a_i+(m-2)(d_i+2nt)\}$ for $i = 1, ..., r-1$. The A_i' are monochromatic as the blocks are identically coloured and the A_i are monochromatic. Also, A_i and A_i' have the same colouring, and the A_i are colour-focussed, hence the A_i' have pairwise distinct colours.
- The A_i are focussed at f and the colour of f of different than the colour of all A_i. $f = a_i + (m-1)d_i$ for all i.
- Now $\{f, f+2nt, f+4nt, ..., f+2n(m-2)t\}$ is an arithmetic progression of length $m-1$, is monochromatic and of a different colour to all the A_i'.
- It is enough to show that $a_i + (m-1)(d_i+2nt) = f + 2n(m-1)t$ for all i, but this is equivalent to $a_i + (m-1)d_i = f$, which is true as all A_i were focussed at f.

□

Corollary 1.27 For any k-colouring of $\mathbb{N}$, there exists a colour class containing arbitrarily long arithmetic progressions.

Remark 1.28 We can't guarantee infinitely long arithmetic progressions, e.g.

- 2-colour $\mathbb{N}$ by 1 red, $2, 3$ blue, $4, 5, 6$ red, etc.
- The set of infinite arithmetic progressions in $\mathbb{N}$ is countable (since described by two integers: the start term and step). Enumerate them by $(A_k)_{k \in \mathbb{N}}$. Pick $x_1 < y_1 \in A_1$, colour x_1 red and y_1 blue. Then pick $x_2, y_2 \in A_2$ with $y_1 < x_2 < y_2$, colour x_2 red, y_2 blue. Continue inductively.

Theorem 1.29 (Strengthened Van der Waerden) Let $m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that for any k-colouring of $[n]$, there exists a monochromatic length m arithmetic progression whose common difference is the same colour (i.e. there exists $a, a+d, ..., a+(m-1), d$ all of the same colour).

Proof (Hints).

- Use induction on k.
- If n is the witness for $k-1$ colours, show that $N = W(k, n(m-1)+1)$ is a witness for k colours, by considering n different multiples of the step of a suitable arithmetic progression.

□

Proof.

- Fix $m \in \mathbb{N}$. We use induction on k. $k = 1$ case is trivial.
- Let n be witness for $k-1$ colours.
- We will show that $N = W(k, n(m-1)+1)$ is suitable for k colours.
- If $[N]$ is k-coloured, there exists a monochromatic (say red) arithemtic progression of length $n(m-1)+1$: $a, a+d, ..., a+n(m-1)d$.

- If rd is red for any $1 \leq r \leq n$, then we are done (consider $a, a + rd, ..., a + (m - 1)rd$).
- If not, then $\{d, 2d, ..., nd\}$ is $k - 1$-coloured, which induces a $k - 1$ colouring on $[n]$. Therefore, there exists a monochromatic arithmetic progression $b, b + s, ..., b + (m - 1)s$ (with s the same colour) by induction, which translates to $db, db + ds, ..., db + d(m - 1)s$ and ds being monochromatic.

$\square$

Remark 1.30 The case $m = 2$ of strengthened Van der Waerden is **Schur's theorem**: for any k-colouring of $\mathbb{N}$, there are monochromatic x, y, z such that $x + y = z$. This can be proved directly from Ramsey's theorem for pairs: let $c : \mathbb{N} \to [k]$ be a k -colouring, then induce $c' : \mathbb{N}^{(2)} \to [k]$ by $c'(ij) = c(j - i)$. By Ramsey, there exist $i < j < k$ such that $c'(ij) = c'(ik) = c'(jk)$, i.e. $c(j - i) = c(k - i) = c(k - j)$. So take $x = j - i$, $z = k - i$, $y = k - j$.

1.4. The Hales-Jewett theorem

Definition 1.31 Let X be finite set. We say X^n consists of **words of length n on alphabet X**.

Definition 1.32 Let X be finite. A **(combinatorial) line** in X^n is a set $L \subseteq X^n$ of the form

$$L = \left\{(x_1, ..., x_n) \in X^n : \forall i \notin I, x_i = a_i \text{ and } \forall i, j \in I, x_i = x_j\right\}$$

for some non-empty set $I \subseteq [n]$ and $a_i \in X$ (for each $i \notin I$). I is the set of **active coordinates** for L.

Note that a combinatorial line is invariant under permutations of X.

Example 1.33 Let $X = [3]$. Some lines in X^2 are:
- $I = \{1\}$: $\{(1, 1), (2, 1), (3, 1)\}$ (with $a_2 = 1$), $\{(1, 2), (2, 2), (3, 2)\}$ (with $a_2 = 2$), $\{(1, 3), (2, 3), (3, 3)\}$ (with $a_2 = 3$).
- $I = \{2\}$: $\{(1, 1), (1, 2), (1, 3)\}$ (with $a_1 = 1$), $\{(2, 1), (2, 2), (2, 3)\}$ (with $a_1 = 2$), $\{(3, 1), (3, 2), (3, 3)\}$ (with $a_1 = 3$).
- $I = \{1, 2\}$: $\{(1, 1), (2, 2), (3, 3)\}$.

Note that $\{(1, 3), (2, 2), (3, 1)\}$ is **not** a combinatorial line.

Example 1.34 Some sets of lines in $[3]^3$ are:
- $I = \{1\}$: $\{(1, 2, 3), (2, 2, 3), (3, 2, 3)\}$ (with $a_2 = 2, a_3 = 3$).
- $I = \{1, 3\}$: $\{(1, 3, 1), (2, 3, 2), (3, 3, 3)\}$ (with $a_2 = 3$).

Definition 1.35 In a line L, write L^- and L^+ for the smallest and largest points in L (with respect to the ordering on $[m]^n$ where $x \leq y$ if $x_i \leq y_i$ for all i).

Definition 1.36 Lines $L_1, ..., L_k$ are **focussed** at f if $L_i^+ = f$ for all $i \in [k]$. They are **colour-focussed** if they are focussed and $L_i \setminus \left\{L_i^+\right\}$ is monochromatic for all $i \in [k]$, with each $L_i \setminus \left\{L_i^+\right\}$ a different colour.

Theorem 1.37 (Hales-Jewett) Let $m, k \in \mathbb{N}$ (we use alphabet $X = [m]$), then there exists $n \in \mathbb{N}$ such that for any k-colouring of $[m]^n$, there exists a monochromatic combinatorial line.

Notation 1.38 Denote the smallest such n by $\mathrm{HJ}(m, k)$.

Proof (Hints).

- Induction on m. Prove by induction the claim that for all $1 \le r \le k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[m]^n$, we have either a monochromatic line, or r colour-focussed lines (reason that this claim implies the result).
- State why claim holds for $r = 1$.
- Let n be witness for $r - 1$, $n' = \mathrm{HJ}(m - 1, k^{m^n})$. Want to show that $n + n'$ is witness for r.
- Write $[m]^{n+n'} = [m]^n \times [m]^{n'}$.
- For a colouring $c : [m]^{n+n'} \to [k]$, induce a suitable colouring $c' : [m]^{n'} \to [k]^{m^n}$ and consider what the definition of n' implies. Use this to induce a colouring $c'' : [m]^n \to [k]$.
- Using the inductive hypothesis and the previous point, construct $r - 1$ lines in $[m]^{n+n'}$ which are colour-focussed. Find another line in $[m]^{n+n'}$ (which should have first n coordinates constant) of different colour which has the same focus point.

$\square$

Proof. By induction on m. The case $m = 1$ is trivial as $|[m]^n| = 1$. Assume that $\mathrm{HJ}(m - 1, k')$ exists for all $k' \in \mathbb{N}$. We claim that for all $1 \le r \le k$, there exists $n \in \mathbb{N}$ such that for any k-colouring of $[m]^n$, we have either:

- a monochromatic line, or
- r colour-focussed lines.

We can then take $r = k$ and consider the focus.

We prove the claim by induction on r. For $r = 1$, $n = \mathrm{HJ}(m - 1, k)$ suffices. Let n be a witness for $r - 1$. Let $n' = \mathrm{HJ}(m - 1, k^{m^n})$. We will show $N = n + n'$ is a witness for r. Let $c : [m]^N \to [k]$ be a k-colouring with no monochromatic lines. Writing $[m]^N = [m]^n \times [m]^{n'}$, colour $[m]^{n'}$ by $c' : [m]^{n'} \to [k]^{m^n}$, $c'(b) = (c(a_1, b), ..., c(a_{m^n}, b))$ (where $[m]^n = \{a_1, ..., a_{m^n}\}$). By the inductive hypothesis, there exists a line L in $[m]^{n'}$ with active coordinates I such that

$$\forall a \in [m]^n, \forall b, b' \in L \setminus \{L^+\}, \quad c(a, b) = c(a, b').$$

But now this induces a (well-defined) colouring $c'' : [m]^n \to [k]$, $c''(a) = c(a, b)$ for any $b \in L \setminus \{L^+\}$. By definition of n, there exist $r - 1$ lines $L_1, ..., L_{r-1}$ colour-focussed (w.r.t c'') at f, with active coordinates $I_1, ..., I_{r-1}$.

Finally, consider the $r - 1$ lines L'_i, $1 \le i \le r - 1$ in $[m]^N$ that start at (L_i^-, L^-) with active coordinates $I_i \cup I$, and the line L' in $[m]^N$ that starts at (f, L^-) with active coordinates I. By the construction of c'', the colour of each point in L'_i is determined by the first n coordinates which form a point lying in L_i. Hence, since the L_i are colour-focussed, the L'_i are colour-focussed. As for L', the first n coordinates are constant (always equal to f), and so again by the construction of c'', the colour of each point in L' is equal to $c''(f)$, which is a different colour to each colour of the L'_i. Hence all $L'_1, ..., L'_{r-1}, L'$ colour-focussed at (f, L^+), so we are done. □

Corollary 1.39 Hales-Jewett implies Van der Waerden's theorem.

Proof (Hints). For a colouring $c : \mathbb{N} \to [k]$, consider the induced colouring $c'(x_1, ..., x_n) = c(x_1 + \cdots + x_n)$ of $[m]^n$. □

Proof. Let c be a k-colouring of $\mathbb{N}$. For sufficiently large n (i.e. $n \geq \mathrm{HJ}(m, k)$), induce a k-colouring c' of $[m]^n$ by $c'(x_1, ..., x_n) = c(x_1 + \cdots + x_n)$. By Hales-Jewett, a monochromatic (with respect to c') combinatorial line L exists. This gives a monochromatic (with respect to c) length m arithmetic progression in $\mathbb{N}$. The step is equal to the number of active coordinates. The first term in the arithmetic progression corresponds to the point in L with all active coordinates equal to 1, the last term corresponds to the point in L with all active coordinates equal to m. □

Exercise 1.40 Show that the m-in-a-row noughts and crosses game cannot be a draw in sufficiently high dimensions, and that the first player can always win.

Definition 1.41 A ***d*-dimensional subspace** (or ***d*-point parameter set**) $S \subseteq X^n$ is a set such that there exist pairwise disjoint $I_1, ..., I_d \subseteq [n]$ and $a_i \in X$ for all $i \in [n] - (I_1 \cup \cdots \cup I_d)$, such that

$$S = \big\{x \in X^n : x_i = a_i \quad \forall i \in [n] - (I_1 \cup \cdots \cup I_d), \\ \text{and } x_i = x_j \quad \forall i, j \in I_k \text{ for some } k \in [d]\big\}.$$

Example 1.42 Two 2-dimensional subspaces in X^3 are $\{(x, y, 2) : x, y \in X\}$ ($I_1 = \{1\}, I_2 = \{2\}$) and $\{(x, x, y) : x, y \in X\}$ ($I_1 = \{1, 2\}, I_2 = \{3\}$).

Theorem 1.43 (Extended Hales-Jewett) For all $m, k, d \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any colouring of $[m]^n$, there exists a monochromatic d-dimensional subspace.

Proof (Hints). Use Hales-Jewett on m^d and k. □

Proof. We can view $X^{dn'}$ as $\left(X^d\right)^{n'}$. A line in $\left(X^d\right)^{n'}$ (on alphabet $Y = X^d$) corresponds to a d-dimensional subspace in $X^{dn'}$ (on alphabet X). (Each inactive coordinate in the line corresponds to d adjacent inactive coordinates in the subspace, and each active coordinate in the line corresponds to d adjacent active coordinates in the subspace). Hence, we can take $n = d \cdot \mathrm{HJ}\left(m^d, k\right)$. □

Definition 1.44 Let $S \subseteq \mathbb{N}^d$ be finite. A **homothetic copy** of S is a set of the form $a + \lambda S$ where $a \in \mathbb{N}^d$ and $\lambda \in \mathbb{N}$ ($l \neq 0$).

Theorem 1.45 (Gallai) Let $S \subseteq \mathbb{N}^d$ be finite. For every k-colouring of $\mathbb{N}^d$, there exists a monochromatic homothetic copy of S.

Proof (Hints). Let $S = \{S_1, ..., S_m\}$, consider colouring $c' : [m]^n \to [k]$ (for suitable n) given by $c'(x_1, ..., x_n) = c(S_{x_1}, ..., S_{x_m})$. □

Proof. Let $S = \{S_1, ..., S_m\}$. Let $c : \mathbb{N}^d \to [k]$ be a k-colouring. For n large enough (i.e. $n \geq \mathrm{HJ}(m, k)$), colour $[m]^n$ by $c'(x_1, ..., x_n) = c\left(S_{x_1} + \cdots + S_{x_m}\right)$. By Hales-Jewett, there exists a monochromatic line (with respect to c') in $[m]^n$ with active coordinates I. So $c\left(\sum_{i \notin I} S_i + |I| S_j\right)$ is the same colour for all $j \in [m]$. So we are done, as $\sum_{i \notin I} S_i + |I| S$ is a homothetic copy of S. □

Remark 1.46
- Gallai's theorem can also be proven with a focussing + product colouring argument.
- For $S = \{(x, y) \in \mathbb{N}^2 : x, y \in \{1, 2\}\}$, Gallai's theorem proves the existence of a monochromatic square whereas extended Hales-Jewett only guarantees a monochromatic rectangle.

2. Partition regular systems

2.1. Rado's theorem

Strengthened Van der Waerden says that the system $x_1 + x_2 = y_1, x_1 + 2x_2 = y_2, ..., x_1 + mx_2 = y_m$ has a monochromatic solution in $x_1, x_2, y_1, ..., y_m$. We want to find when a general system of equations is partition regular.

Definition 2.1 Let $A \in \mathbb{Q}^{m \times n}$ be a $m \times n$ matrix. A is **partition regular (PR)** if for any finite colouring of $\mathbb{N}$, there exists a monochromatic $\boldsymbol{x} \in \mathbb{N}^n$ such that $A\boldsymbol{x} = \mathbf{0}$.

Example 2.2
- Schur's theorem says that $x + y = z$ has a monochromatic solution for any finite colouring of $\mathbb{N}$, and so that $(1, 1, -1)$ is PR.
- Strengthened Van der Waerden states that

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

  is PR.
- $(a, b, -(a + b))$ is PR for any a, b (a monochromatic solution is $x = y = z$).
- $(2, -1)$ is not PR: colour $\mathbb{N}$ by n is red if $\max\{m \in \mathbb{N} : 2^m \mid n\}$ is even, and blue otherwise. Then if $2x = y$, x and y must have different colours.

Definition 2.3 A rational matrix A with columns $\boldsymbol{c}_1, ..., \boldsymbol{c}_n \in \mathbb{Q}^m$ has the **column property (CP)** if there exists a partition $B_1 \sqcup \cdots \sqcup B_r$ of $[n]$ such that:
1. $\sum_{i \in B_1} \boldsymbol{c}_i = \mathbf{0}$.
2. For all $s \in \{2, ..., r\}$, $\sum_{i \in B_s} \boldsymbol{c}_i \in \mathrm{span}\left\{\boldsymbol{c}_j : j \in B_1 \sqcup \cdots \sqcup B_{s-1}\right\}$ (note we can take the linear span over $\mathbb{R}$ or over $\mathbb{Q}$ here, as if a rational vector is a real linear combination of rational vectors, then it is also a rational linear combination of them).

Example 2.4

- $(1, 1, -1)$ has CP, with $B_1 = \{1, 3\}$, $B_2 = \{2\}$.
- The matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{bmatrix}$$

  from Strengthened Van der Waerden has CP, with $B_1 = \{1, 3, ..., n\}$ and $B_2 = \{2\}$.
- $(3, 4, -7)$ has CP with $B_1 = \{1, 2, 3\}$.
- $(\lambda, -1)$ has CP iff $\lambda = 1$.
- $(3, 4, -6)$ doesn't have CP.

Example 2.5

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & a \\ 4 & -4 & b \end{bmatrix}$$

has CP iff $(a, b) = (6, 12)$.

Remark 2.6 $\boldsymbol{x} = (a_1, ..., a_n)$ is PR iff $\lambda \boldsymbol{x}$ is PR (for any $\lambda \in \mathbb{Q}^\times$), so we can assume that each $a_i \in \mathbb{Z}$. Also, $\boldsymbol{x}$ has CP iff there exists $\emptyset \neq I \subseteq [n]$ such that $\sum_{i \in I} a_i = 0$. We may also assume WLOG each $a_i \neq 0$. We will first show that if $\boldsymbol{x}$ is PR, then it has CP. Even in the $1 \times n$ matrix case of Rado's theorem, neither direction is easy.

Notation 2.7 For p prime and $x = (a_k ... a_0)_p \in \mathbb{N}$, write $e(x)$ for the rightmost non-zero digit in the base-p expansion of x, i.e. $e(x) = a_{t(x)}$, where $t(x) = \min\{i : a_i \neq 0\}$.

Proposition 2.8 Let $a_1, ..., a_n \in \mathbb{Q}^*$. If $(a_1, ..., a_n)$ is PR, then it has CP.

Proof (Hints). For p large enough (determine later a bound for p), colour $\mathbb{N}$ by giving x colour $e(x)$, and consider $\min\{t(x_1), ..., t(x_n)\}$. □

Proof. Let p be a large prime ($p > \sum_{i=1}^n |a_i|$). Define a $(p-1)$-colouring of $\mathbb{N}$ giving x colour $e(x)$. By assumption, there are $x_1, ..., x_n$ of the same colour d such that $\sum_{i=1}^n a_i x_i = 0$. Let $t = \min\{t(x_1), ..., t(x_n)\}$, and let $I = \{i \in [n] : t(x_i) = t\}$ (note I is non-empty). So when summing $\sum_{i=1}^n a_i x_i = 0$ and considering the last digit in the base p expansion, we have $\sum_{i=1}^n a_i x_i = 0 \bmod p^{t+1}$ and so obtain $\sum_{i \in I} a_i d = 0 \bmod p$, so $\sum_{i \in I} a_i = 0$ (since p is prime and was chosen large enough). □

Remark 2.9 There is no other known proof of this proposition.

Lemma 2.10 Let $\lambda \in \mathbb{Q}$. Then $(1, \lambda, -1)$ is partition regular, i.e. for any finite colouring of $\mathbb{N}$, there exists monochromatic $(x, y, z) \in \mathbb{N}^3$ such that $x + \lambda y = z$.

Proof (Hints).

- Reason that we can assume $\lambda > 0$. Write $\lambda = r/s$, $r, s \in \mathbb{N}$.
- Use induction on number of colours k: given n such that any $(k-1)$-colouring of $[n]$ admits monochromatic solution, show that $N = W(k, nr + 1)ns$ works for k colours, by considering the definition of W and isd for eacah $i \in [n]$.

□

Proof. The case $\lambda = 0$ is trivial, and if $\lambda < 0$, we may rewrite the equation as $z - \lambda y = x$, so we may assume that $\lambda > 0$, so let $\lambda = \frac{r}{s}$ for $r, s \in \mathbb{N}$. In fact, we show that for any k-colouring of $[n]$ (for some n depending on k), there is a monochromatic solution.

We seek a monochromatic solution to $x + \frac{r}{s}y = z$ for some finite colouring $c : \mathbb{N} \to [k]$. We use induction on the number of colours k. For $k = 1$, $n = \max\{s, r + 1\}$ is sufficient, with monochromatic solution $(1, s, r + 1)$. Assume n is a witness for $k - 1$ colours. We will show $N = nsW(k, nr + 1)$ is suitable for k colours. By definition of W, given a k-colouring of $[N]$, there is a monochromatic AP inside $[W(k, nr + 1)] \subseteq [N]$ of length $nr + 1$: $a, a + d, ..., a + nrd$, coloured red.

Consider isd for each $i \in [n]$. Note that $isd \leq nsW(k, nr + 1)$ so each isd does indeed have a colour. If some isd is also red, then $(a, isd, a + ird)$ is a monochromatic solution. If no isd is red, then $\{sd, ..., nsd\}$ is $(k - 1)$-coloured, so by the inductive hypothesis, there exists $i, j, k \in [n]$ such that $\{isd, jsd, ksd\}$ is monochromatic and $isd + \lambda jsd = ksd$, so (isd, jsd, ksd) is a monochromatic solution. □

Remark 2.11
- Note the similarity to the proof of Strengthened Van der Waerden.
- The case $\lambda = 1$ is Schur's theorem, which can be proven directly by Ramsey's theorem; however, there is no known proof using Ramsey's theorem for general $\lambda \in \mathbb{Q}$.

Theorem 2.12 (Rado's Theorem for Single Equations) Let $a_1, ..., a_n \in \mathbb{Q} \setminus \{0\}$. $(a_1, ..., a_n)$ is PR iff it has CP.

Proof (Hints). For $\Longleftarrow$: for the obvious choice of $I \subseteq [n]$, fix $i_0 \in I$, and define $\boldsymbol{x} \in \mathbb{N}^n$ componentwise:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}.$$

Show that $\boldsymbol{x}$ is a solution to $\sum_{i=1}^{n} a_i x_i = 0$. □

Proof. $\Longrightarrow$ is by Proposition 2.8. For $\Longleftarrow$: we have that $\sum_{i \in I} a_i = 0$ for some $\emptyset \neq I \subseteq [n]$. Given a colouring $c : \mathbb{N} \to [k]$, we need to show that there are monochromatic $x_1, ..., x_n$ such that $\sum_{i=1}^{n} a_i x_i = 0$.

Fix $i_0 \in I$. We construct the following vector $\boldsymbol{x} \in \mathbb{N}^n$ by defining its components:

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I \setminus \{i_0\} \end{cases}$$

for some fixed suitable x, y, z. We need x, y, z to be monochromatic and

$$\begin{aligned} a_{i_0}x + \sum_{i\notin I} a_i y + \sum_{i\in I\setminus\{i_0\}} a_i z &= 0 \\ \iff a_{i_0}x - z a_{i_0} + \sum_{i\notin I} a_i y &= 0 \\ \iff x + \frac{\sum_{i\notin I} a_i}{a_{i_0}} y - z &= 0 \end{aligned}$$

and this holds, since x, y, z exist by the above lemma. □

Conjecture 2.13 (Rado's Boundedness Conjecture) Let A be an $m \times n$ matrix that is not PR (so there exists a "bad" colouring, i.e. a k-colouring with no monochromatic solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ for some $k \in \mathbb{N}$). Is k bounded (for given m, n)?

This is known for 1×3 matrices: 24 colours suffice.

Proposition 2.14 Let $A \in \mathbb{Q}^{m\times n}$. If A is PR, then it has CP.

Proof (Hints).

- Let $\boldsymbol{x} \in \mathbb{N}^n$ be the monochromatic solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. For fixed prime p, partition $[n]$ into $B_1, ..., B_r$ by grouping $i, j \in [n]$ by $t(x_i), t(x_j)$ (and preserving the ordering).
- Reason that the same partition exists for infinitely many p.
- Considering $\sum_{i=1}^n x_i \boldsymbol{c}_i = \boldsymbol{0} \bmod p$ for infinitely many p, show that $\sum_{i\in B_1} \boldsymbol{c}_i = 0$, and

$$p^t \sum_{i\in B_k} \boldsymbol{c}_i + \sum_{i\in B_1,...,B_{k-1}} x_i d^{-1} \boldsymbol{c}_i \equiv \boldsymbol{0} \bmod p^{t+1}.$$

- By taking the dot product with $\boldsymbol{u} \in \mathbb{N}^m$ for appropriate u, show by contradiction that $\sum_{i\in B_k} \boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{c}_i : i \in B_1, ..., B_{k-1}\}$.

□

Proof. Let $\boldsymbol{c}_1, ..., \boldsymbol{c}_n \in \mathbb{Q}^m$ be the columns of A. For fixed prime p, colour $\mathbb{N}$ as before by $c(x) = e(x)$. By assumption, there exists a monochromatic $\boldsymbol{x} \in \mathbb{N}^n$ such that $\sum_{i=1}^n x_i \boldsymbol{c}_i = \boldsymbol{0}$. We partition the columns (by partitioning $[n] = B_1 \sqcup \cdots \sqcup B_r$) as follows:

- $i, j \in B_k$ iff $t(x_i) = t(x_j)$.
- $i \in B_k$, $j \in B_\ell$ for $k < \ell$ iff $t(x_i) < t(x_j)$.

We do this for infinitely many primes p. Since there are finitely many partitions of $[n]$, for infinitely many p, we will have the same blocks $B_1, ..., B_r$.

Consider $\sum_{i=1}^n x_i \boldsymbol{c}_i = \boldsymbol{0}$ performed in base p. Each $i \in [n]$ has the same colour $d = e(x_i) \in [1, p-1]$. So $\sum_{i\in B_1} d\boldsymbol{c}_i = 0 \bmod p$ (by collecting the rightmost terms in base p), hence $\sum_{i\in B_1} c_i = 0 \bmod p$. But this holds for infinitely many p, hence

$$\sum_{i\in B_1} \boldsymbol{c}_i = 0.$$

Now $\sum_{i\in B_k} p^t d\boldsymbol{c}_i + \sum_{i\in B_1,...,B_{k-1}} x_i\boldsymbol{c}_i = \boldsymbol{0} \bmod p^{t+1}$ for some t. So

$$p^t \sum_{i\in B_k} \boldsymbol{c}_i + \sum_{i\in B_1,...,B_{k-1}} x_i d^{-1}\boldsymbol{c}_i \equiv \boldsymbol{0} \bmod p^{t+1}.$$

We claim that $\sum_{i\in B_k} \boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{c}_i : i \in B_1, ..., B_{k-1}\}$. Suppose not, then there exists $\boldsymbol{u} \in \mathbb{N}^m$ such that $\boldsymbol{u}.\boldsymbol{c}_i = 0$ for all $i \in B_1, ..., B_{k-1}$, but $\boldsymbol{u}.\left(\sum_{i\in B_k} \boldsymbol{c}_i\right) \neq 0$. Then dotting with $\boldsymbol{u}$, we obtain $p^t\boldsymbol{u}.\left(\sum_{i\in B_k} \boldsymbol{c}_i\right) \equiv 0 \bmod p^{t+1}$, so $\boldsymbol{u}.\sum_{i\in B_k} \boldsymbol{c}_i \equiv 0 \bmod p$. But this holds for infinitely many p, so $\boldsymbol{u}.\sum_{i\in B_k} \boldsymbol{c}_i = 0$: contradiction. □

Definition 2.15 For $m, p, c \in \mathbb{N}$, an $\boldsymbol{(m, p, c)}$**-set** $S \subseteq \mathbb{N}$ with generators $x_1, ..., x_m \in \mathbb{N}$ is of the form

$$S = \left\{\sum_{i=1}^{m} \lambda_i x_i : \exists j \in [m] : \lambda_j = c, \lambda_i = 0\ \forall i < j, \text{and } \lambda_k \in [-p, p]\ \forall k > j\right\}$$

where $[-p, p] = \{-p, -(p-1), ..., p\}$. So S consists of

$$\begin{aligned} cx_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_m x_m, &\quad \lambda_i \in [-p, p], \\ cx_2 + \lambda_3 x_3 + \cdots + \lambda_m x_m, &\quad \lambda_i \in [-p, p], \\ \vdots\quad & \\ cx_m. & \end{aligned}$$

These are the **rows** of S. We can think of S as a "progression of progressions".

Example 2.16

- A $(2, p, 1)$-set with generators x_1, x_2 is of the form $\{x_1 - px_2, x_1 - (p-1)x_2, ..., x_1 + px_2, x_2\}$, so is an AP of length $2p + 1$ together with its step.
- A $(2, p, 3)$-set with generators x_1, x_2 is of the form $\{3x_1 - px_2, 3x_1 - (p-1)x_2, ..., 3x_1, ..., 3x_1 + px_2, 3x_2\}$, so is an AP of length $2p + 1$, whose middle term is divisible by 3, together with three times its step.

Theorem 2.17 Let $m, p, c \in \mathbb{N}$. For any finite colouring of $\mathbb{N}$, there exists a monochromatic (m, p, c)-set.

Proof (Hints).

- Reason that an (m', p, c)-set contains an (m, p, c)-set for $m' \geq m$. With $M = k(m-1) + 1$, reason that if we can find an (M, p, c)-set with each row monochromatic, then we can find an monochromatic (m, p, c)-set.
- Let $A_1 = \{c, 2c, ..., \lfloor n/c \rfloor c\}$, reason that A_1 contains a set of the form $R_1 = \{cx_1 - n_1 d_1, cx_1 - (n_1 - 1)d_1, ..., cx_1 + n_1 d_1\}$ for some large n_1.
- Let $B_1 = \left\{d_1, 2d_1, ..., \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1\right\}$. We have $cx_1 + \lambda_1 b_1 + \cdots + \lambda_{M-1} b_{M-1} \in R_1$, explain why these are monochromatic.
- Inside B_1, define

$$A_2 = \left\{cd_1, 2cd_1, ..., \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1\right\}.$$

and apply the argument as before, where the divisor in the $\lfloor\cdot\rfloor$ expression in the new B_2 is $(M-2)p$.
- Argue that after a certain number of steps, we have formed an (M, p, c)-set with each row monochromatic.

$\square$

Proof. Let $c : \mathbb{N} \to [k]$ be the colouring of $\mathbb{N}$ with k colours. Note that an (m', p, c)-set with $m' \geq m$ contains an (m, p, c)-set (by taking any m rows, and setting some suitable λ_i to 0). Let $M = k(m-1)+1$. It is enough to find a (M, p, c)-set such that each row is monochromatic.

Let n be large (large enough to apply the argument that follows). Let $A_1 = \{c, 2c, ..., \lfloor n/c \rfloor c\}$. By Van der Waerden, A_1 contains a monochromatic AP R_1 of length $2n_1 + 1$ where n_1 is large enough:

$$R_1 = \{cx_1 - n_1 d_1, cx_1 - (n_1 - 1)d_1, ..., cx_1 + n_1 d_1\}.$$

has colour k_1. Now we restrict our attention to

$$B_1 = \left\{d_1, 2d_1, ..., \left\lfloor \frac{n_1}{(M-1)p} \right\rfloor d_1\right\}.$$

Observe that

$$cx_1 + \lambda_1 b_1 + \cdots + \lambda_{M-1} b_{M-1} \in R_1$$

for all $\lambda_i \in [-p, p]$ and $b_i \in B_1$, so all these sums have colour k_1. Inside B_1, look at

$$A_2 = \left\{cd_1, 2cd_1, ..., \left\lfloor \frac{n_1}{(M-1)pc} \right\rfloor cd_1\right\}.$$

By Van der Waerden, A_2 contains a monochromatic AP R_2 of length $2n_2 + 1$ with colour k_2:

$$R_2 = \{cx_2 - n_2 d_2, cx_2 - (n_2 - 1)d_2, ..., cx_2 + n_2 d_2\}.$$

Note that $x_2 \subseteq B_1$. Now we restrict our attention to

$$B_2 = \left\{d_2, 2d_2, ..., \left\lfloor \frac{n_2}{(M-2)p} \right\rfloor d_2\right\}.$$

Again, note that for all $\lambda_i \in [-p, p]$ and $b_i \in B_2$, we have

$$cx_2 + \lambda_1 b_1 + \cdots + \lambda_{M-2} b_{M-2} \in R_2$$

so has colour k_2.

We iterate this process M times, and obtain M generators $x_1, ..., x_M$ such that each row of the (M, p, c)-set generated by $x_1, ..., x_M$ is monochromatic. But now $M = k(m-1)+1$, so m of the rows have the same colour. $\square$

Remark 2.18 Being extremely precise in this proofs (such as considering $\lfloor\cdot\rfloor$) is much less important than the ideas in the proof. (Won't be penalised in the exam for small details like this).

Corollary 2.19 (Folkman's Theorem) Let $m \in \mathbb{N}$ be fixed. For every finite colouring of $\mathbb{N}$, there exists $x_1, ..., x_m \in \mathbb{N}$ such that

$$\mathrm{FS}(x_1, ..., x_m) := \left\{ \sum_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}$$

is monochromatic.

Proof (Hints). A specific case of Theorem 2.17. ☐

Proof. By the $(m, 1, 1)$ case of Theorem 2.17. ☐

Remark 2.20

- The case $n = 2$ of Folkman's theorem is Schur's theorem.
- For a colouring $c : \mathbb{N} \to [k]$, we induce a colouring $c' : \mathbb{N} \to [k]$ by $c'(n) = c(2^n)$. Then by Folkman's theorem for c', there exists $x_1, ..., x_m$ such that

$$\mathrm{FP}(x_1, ..., x_m) = \left\{ \prod_{i \in I} x_i : \emptyset \neq I \subseteq [m] \right\}.$$

- It is not known whether the same result holds for $\mathrm{FS}(x_1, ..., x_m) \cup \mathrm{FP}(x_1, ..., x_m)$. However, it does not hold for infinite sets $\{x_n : n \in \mathbb{N}\}$, and does hold for colourings of $\mathbb{Q}$.

Proposition 2.21 Let A have CP. Then there exist $m, p, c \in \mathbb{N}$ such that every (m, p, c)-set contains a solution $\boldsymbol{y}$ to $A\boldsymbol{y} = \boldsymbol{0}$, i.e. all y_i belong to the (m, p, c)-set.

Proof. Let $\boldsymbol{c}_1, ..., \boldsymbol{c}_n$ be the columns of A. By assumption, there is a partition $B_1 \sqcup \cdots \sqcup B_r$ of $[n]$ such that $\forall k \in [r]$,

$$\begin{aligned}
&\sum_{i \in B_k} \boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{c}_i :\in B_1 \cup \cdots \cup B_{k-1}\} \\
\implies &\sum_{i \in B_k} \boldsymbol{c}_i = \sum_{i \in B_1 \cup \cdots \cup B_{k-1}} q_{ik} \boldsymbol{c}_i \quad \text{for some } q_{ik} \in \mathbb{Q} \\
\implies &\sum_{i=1}^{n} d_{ik} \boldsymbol{c}_i = \boldsymbol{0}
\end{aligned}$$

where

$$d_{ik} = \begin{cases} 0 & \text{if } i \notin B_1 \cup \cdots \cup B_{k-1} \\ 1 & \text{if } i \in B_k \\ -q_{ik} & \text{if } i \in B_1 \cup \cdots \cup B_{k-1} \end{cases}.$$

Take $m = r$. Let $x_1, ..., x_r \in \mathbb{N}$, and let $y_i = \sum_{k=1}^{r} d_{ik} x_k$ for each $i \in [n]$. Now $\boldsymbol{y} = (y_1, ..., y_n)$ is a solution to $A\boldsymbol{y} = \boldsymbol{0}$: we have

$$\begin{aligned}\sum_{i=1}^{n} y_i \boldsymbol{c}_i &= \sum_{i=1}^{n} \sum_{k=1}^{r} d_{ik} x_k \boldsymbol{c}_i \\ &= \sum_{k=1}^{r} x_k \sum_{i=1}^{n} d_{ik} \boldsymbol{c}_i = \mathbf{0}.\end{aligned}$$

Let c be the LCD of all the q_{ik}. Now $cy_i = \sum_{k=1}^{n} cd_{ik} x_k$ is an integral linear combination of the x_k, and $c\boldsymbol{y}$ is a solution since $\boldsymbol{y}$ is. Let p be c times maximum of the absolute values of the numerators of the q_{ik}. By definition of the d_{ik}, $c\boldsymbol{y}$ is in the (m, p, c)-set generated by $x_1, ..., x_r$. □

Theorem 2.22 (Rado) $A \in \mathbb{Q}^{m \times n}$ is PR iff it has CP.

Proof. $\Longrightarrow$ is by Proposition 2.14. For $\Longleftarrow$, let $c' : \mathbb{N} \to [k]$ be a finite colouring of $\mathbb{N}$. Also, by the above proposition, since A has CP, there exists $m, p, c \in \mathbb{N}$ such that $A\boldsymbol{x} = \mathbf{0}$ has a solution $\boldsymbol{x}$ in any (m, p, c)-set by the above theorem. By Theorem 2.17, there is a monochromatic (m, p, c)-set with respect to c'. This gives a monochromatic solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$. □

Remark 2.23 From the proof of Rado's Theorem, we obtain that if A is PR for the "$\mathrm{mod}\, p$" colourings, then it is PR for *any* colouring. There is no proof of this fact that is more direct than using Rado's theorem.

Theorem 2.24 (Consistency) Let A and B be rational PR matrices. Then the matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

is PR.

Proof (Hints). Rado's Theorem. □

Proof. This is a trivial check of the CP given the CP of A and B, then we are done by Rado's Theorem. □

Remark 2.25 The Consistency Theorem says that if we can find monochromatic solutions $\boldsymbol{x}$ and $\boldsymbol{x}'$ to $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{y} = \mathbf{0}$, then we can find monochromatic solutions $\boldsymbol{x}'$ and $\boldsymbol{y}'$, of the same colour, to $A\boldsymbol{x}' = \mathbf{0}$ and $B\boldsymbol{y}' = \mathbf{0}$.

Theorem 2.26 For any finite colouring of $\mathbb{N}$, some colour class contains solutions to *all* PR equations.

Proof (Hints). Use the Consistency Theorem. □

Proof. For a given k-colouring of $\mathbb{N}$, let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_k$ be the colour classes. Assume the contrary, so for each $1 \leq i \leq k$, there exists a PR matrix A_i such that $A_i\boldsymbol{x} = \mathbf{0}$ has no monochromatic solution of the same colour as C_i. But then by inductively applying the consistency theorem, the matrix

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

has a monochromatic solution of the same colour as some C_j. But then $C_j \boldsymbol{x} = \boldsymbol{0}$ has a solution $\boldsymbol{x}$ of the same colour as C_j: contradiction. □

2.2. Ultrafilters

Definition 2.27 A **filter** on $\mathbb{N}$ is a non-empty collection $\mathcal{F}$ of subsets of $\mathbb{N}$ such that:

- $\emptyset \notin \mathcal{F}$,
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$, i.e. $\mathcal{F}$ is an **up-set**.
- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$, i.e. $\mathcal{F}$ is closed under finite intersections.

A filter is a notion of "large" subsets of $\mathbb{N}$.

Example 2.28

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$ is a filter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$ is a filter.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$ is a filter, called the **cofinite filter**.
- $\mathcal{F}_4 = \{A \subseteq \mathbb{N} : A \text{ infinite}\}$ is not a filter, since it contains $2\mathbb{N}$ and $2\mathbb{N}+1$ but not $\emptyset = (2\mathbb{N}) \cap (2\mathbb{N}+1)$.
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$ is a filter.

Definition 2.29 An **ultrafilter** is a maximal filter.

Definition 2.30 For $x \in \mathbb{N}$, the **principal ultrafilter at $\boldsymbol{x}$** is

$$\mathcal{U}_x := \{A \subseteq \mathbb{N} : x \in A\}.$$

Proposition 2.31 The principal ultrafilter at x is indeed an ultrafilter.

Proof (Hints). Straightforward. □

Proof. If $B \notin \mathcal{U}_x$, then $x \in B^c$ so $B^c \in \mathcal{U}_x$, but $B^c \cap B = \emptyset$, so $\mathcal{U}_x \cup \{B\}$ is not a filter. □

Example 2.32

- $\mathcal{F}_1 = \{A \subseteq \mathbb{N} : 1 \in A\}$ is an ultrafilter.
- $\mathcal{F}_2 = \{A \subseteq \mathbb{N} : 1, 2 \in A\}$ is not an ultrafilter as $\mathcal{F}_1$ extends it.
- $\mathcal{F}_3 = \{A \subseteq \mathbb{N} : A^c \text{ finite}\}$ is not an ultra filter, as $\mathcal{F}_5$ extends it.
- $\mathcal{F}_5 = \{A \subseteq \mathbb{N} : 2\mathbb{N} \setminus A \text{ finite}\}$ is not an ultrafilter, as $\{A \subseteq \mathbb{N} : 4\mathbb{N} \setminus A \text{ finite}\}$ extends it.

Proposition 2.33 A filter $\mathcal{F}$ is an ultrafilter iff for all $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Proof (Hints). $\Longleftarrow$: straightforward. $\Longrightarrow$: show if $A \notin \mathcal{F}$, then $\exists B \in \mathcal{F}$ such that $A \cap B = \emptyset$. □

Proof. $\Longleftarrow$: since $A \cap A^c = \emptyset \notin \mathcal{F}$.

$\Longrightarrow$: let $\mathcal{F}$ is an ultrafilter. We cannot have $A, A^c \in \mathcal{F}$ as $A \cap A^c = \emptyset \notin \mathcal{F}$. Suppose there is $A \subseteq \mathbb{N}$ such that $A, A^c \notin \mathcal{F}$. By maximality of $\mathcal{F}$, since $A \notin \mathcal{F}$, then $\exists B \in \mathcal{F}$ such that $A \cap B = \emptyset$ (suppose not, then $\mathcal{F}' = \{S \subseteq \mathbb{N} : S \supseteq A \cap B \text{ for some } B \in \mathcal{F}\}$ extends $\mathcal{F}$). Similarly, $\exists C \in \mathcal{F}$ such that $A^c \cap C = \emptyset$. So we have $C \subseteq A$, so $B \cap C = \emptyset \notin \mathcal{F}$: contradiction (or also $C \subseteq A \Longrightarrow A \in \mathcal{F}$: contradiction). $\square$

Corollary 2.34 Let $\mathcal{U}$ be an ultrafilter and $A = B \cup C \in \mathcal{U}$. Then $B \in U$ or $C \in U$.

Proof (Hints). Straightforward. $\square$

Proof. If not, then $B^c, C^c \in \mathcal{U}$ by Proposition 2.33, hence $B^c \cap C^c = (B \cup C)^c = A^c \in \mathcal{U}$: contradiction. $\square$

Proposition 2.35 Every filter is contained in an ultrafilter.

Proof (Hints). Use Zorn's Lemma. $\square$

Proof. Let $\mathcal{F}_0$ be a filter. By Zorn's Lemma, it is enough to show that every non-empty chain of filters has an upper bound. Let $\{\mathcal{F}_i : i \in I\}$ be a chain of filters in the poset of filters containing $\mathcal{F}_0$, partially ordered by inclusion, and set $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$.

- $\emptyset \notin \mathcal{F}$ since $\emptyset \notin \mathcal{F}_i$ for each $i \in I$.
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $A \in \mathcal{F}_i$ for some $i \in I$, so $B \in \mathcal{F}_i$, so $B \in \mathcal{F}$.
- Let $A, B \in \mathcal{F}$, so $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ for some i, j. WLOG, $\mathcal{F}_i \subseteq \mathcal{F}_j$, so $A \cap B \in \mathcal{F}_j$, so $A \cap B \in \mathcal{F}$.

$\mathcal{F}$ is an upper bound for the chain, so we are done. $\square$

Proposition 2.36 Let $\mathcal{U}$ be an ultrafilter. Then $\mathcal{U}$ is non-principal iff $\mathcal{U}$ extends the cofinite filter $\mathcal{F}_C$.

Proof (Hints). $\Longleftarrow$: straightforward. $\Longrightarrow$: use Corollary 2.34. $\square$

Proof. $\Longleftarrow$: if $\mathcal{U} = \mathcal{U}_x$ is principal, then we have $\{x\} \in \mathcal{U}$ so $\{x\}^c \notin \mathcal{U}$ by Proposition 2.33, but also $\{x\}^c \in \mathcal{F}_C$: contradiction.

$\Longrightarrow$: let $A \in \mathcal{F}_C$, so $A^c = \{a_1, ..., a_k\}$ is finite. Assume $A \notin \mathcal{U}$, then $A^c \in \mathcal{U}$, so by Corollary 2.34, some $a_i \in \mathcal{U}$. But then by definition of a filter, each set containing a_i is in $\mathcal{U}$, so $\mathcal{U}$ is principal: contradiction. $\square$

Notation 2.37 Let $\beta\mathbb{N}$ denote the set of all ultrafilters on $\mathbb{N}$.

Definition 2.38 Define a topology on $\beta\mathbb{N}$ by its base (basis), which consists of

$$C_A := \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$$

for each $A \subseteq \mathbb{N}$. The sets above indeed form a base: we have $\bigcup_{A \subseteq \mathbb{N}} C_A = \beta\mathbb{N}$, and $C_A \cap C_B = C_{A \cap B}$, since $A \cap B \in \mathcal{U}$ iff $A, B \in \mathcal{U}$. The open sets are of the form $\bigcup_{i \in I} C_{A_i}$ and the closed sets are of the form $\bigcap_{i \in I} C_{A_i}$.

Remark 2.39 $\beta\mathbb{N} \setminus C_A = C_{A^c}$, since $A \notin \mathcal{U}$ iff $A^c \in \mathcal{U}$. We can view $\mathbb{N}$ as being embedded in $\beta\mathbb{N}$ by identifying $n \in \mathbb{N}$ with $\tilde{n} := \mathcal{U}_n$, the principal ultrafilter at n. Each point in $\mathbb{N}$ under this correspondence is isolated in $\beta\mathbb{N}$, since $C_{\{n\}} = \{\tilde{n}\}$ is an open neighbourhood of $\tilde{n}$. Also, $\mathbb{N}$ is dense in $\beta\mathbb{N}$, since for every $n \in A$, $\tilde{n} \in C_A$, so every non-empty open set in $\beta\mathbb{N}$ intersects $\mathbb{N}$.

Theorem 2.40 $\beta\mathbb{N}$ is a compact Hausdorff topological space.

Proof. Hausdorff: let $\mathcal{U} \neq \mathcal{V}$ be ultrafilters, so there is $A \in \mathcal{U}$ such that $A \notin \mathcal{V}$. But then $A^c \in \mathcal{V}$, so $\mathcal{U} \in C_A$, $\mathcal{V} \in C_{A^c}$, and $C_A \cap C_{A^c}$ is open.

Compact: it is compact iff every open admits a finite subcover iff a collection of open sets such that no finite subcollection covers $\beta\mathbb{N}$, they don't cover $\beta\mathbb{N}$ iff for every collection of closed sets such that they have finite intersection property ($(F_i)_{i\in I}$, $\cap_{i\in J} F_i \neq \emptyset$ for all J finite), then their intersection is non-empty. We can assume each F_i is a basis set, i.e. $F_i = C_{A_i}$ for some $A_i \in \mathbb{N}$. Suppose $\left\{C_{A_i} : i \in I\right\}$ have teh finite intersection property. First, $C_{A_{i_1}} \cap \cdots \cap C_{A_{i_k}} = C_{A_{i_1} \cap \cdots \cap A_{i_k}} \neq \emptyset$, hence $\bigcap_{j=1}^k A_{i_j} \neq \emptyset$. So let $\mathcal{F} = \left\{A : A \supseteq A_{i_1} \cap \cdots \cap A_{i_k} \text{ for some } A_{i_1}, ..., A_{i_n}\right\}$. We have $\emptyset \notin \mathcal{F}$, if $B \supseteq A \in \mathcal{F}$ then $B \in \mathcal{F}$, and if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. Hence $\mathcal{F}$ is a filter. $\mathcal{F}$ extends to an ultrafilter $\mathcal{U}$. Note that $(\forall i, A_i \in \mathcal{U}) \iff \left(\mathcal{U} \in C_{A_i} \forall i\right)$. So $U \in \cap C_{A_i}$, so $\cap C_{A_i} \neq \emptyset$. □

Remark 2.41

- $\beta\mathbb{N}$ can be viewed as a subset of $\{0,1\}^{\mathbb{P}(\mathbb{N})}$ (so each ultrafilter is viewed as a function $\mathbb{P}(\mathbb{N}) \to \{0,1\}$). The topology on $\beta\mathbb{N}$ is the restriction of the product topology on $\{0,1\}^{\mathbb{P}(\mathbb{N})}$. Also, $\beta\mathbb{N}$ is a closed subset of $\{0,1\}^{\mathbb{P}(\mathbb{N})}$, so is compact by Tychonov's theorem (TODO: look up statement of this theorem).
- $\beta\mathbb{N}$ is the largest compact Hausdorff topological space in which (the embedding of) $\mathbb{N}$ is dense. In other words, if X is compact and Hausdorff, and $f : \mathbb{N} \to X$, there exists a unique continuous $\tilde{f} : \beta\mathbb{N} \to X$ extending f. TODO: insert diagram.
- $\beta\mathbb{N}$ is called the **Stone-Čech compactification** of $\mathbb{N}$.

Definition 2.42 Let p be a statement and $\mathcal{U}$ be an ultrafilter. $\forall_{\mathcal{U}} x\, p(x)$ to mean $\{x \in \mathbb{N} : p(x)\} \in \mathcal{U}$ and say $p(x)$ "for most x" or "for $\mathcal{U}$-most x".

Example 2.43

- For $\mathcal{U} = \tilde{n}$, we have $\forall_{\mathcal{U}} x\, p(x)$ iff $p(n)$.
- For non-principal $\mathcal{U}$, we have $\forall_{\mathcal{U}} x\, (x > 4)$ (if not, then $\{1,2,3\} = \{x \in \mathbb{N} : x > 4\}^c \in \mathcal{U}$, so $\{i\} \in \mathcal{U}$ for some $i = 1, 2, 3$, so $\mathcal{U}$ is principal: contradiction).

Proposition 2.44 Let $\mathcal{U}$ be an ultrafilter and p, q be statements. Then

1. $\forall_{\mathcal{U}} x\, (p(x) \wedge q(x))$ iff $(\forall_{\mathcal{U}} x\, p(x)) \wedge (\forall_{\mathcal{U}} x\, q(x))$.
2. $\forall_{\mathcal{U}} x\, (p(x) \vee q(x))$ iff $(\forall_{\mathcal{U}} x\, p(x)) \vee (\forall_{\mathcal{U}} x\, q(x))$.
3. $\neg(\forall_{\mathcal{U}} x\, p(x))$ iff $\forall_{\mathcal{U}} x\, (\neg p(x))$.

Proof. Let $A = \{x \in \mathbb{N} : p(x)\}$ and $B = \{x \in \mathbb{N} : q(x)\}$. We have

1. $A \cap B \in \mathcal{U}$ iff $A \in \mathcal{U}$ and $B \in \mathcal{U}$ by definition.
2. $A \cup B \in \mathcal{U}$ iff $A \in \mathcal{U}$ and $B \in \mathcal{U}$ by (find result).
3. $A \notin \mathcal{U}$ iff $A^c \in \mathcal{U}$ by (find result).

□

Note 2.45 $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, p(x,y)$ is not necessarily the same as $\forall_{\mathcal{V}} y \forall_{\mathcal{U}} x \, p(x,y)$, even when $\mathcal{U} = \mathcal{V}$. For example, let $\mathcal{U}$ be non-principal, and $p(x,y) = (x < y)$. Then $\forall_{\mathcal{U}} x (\forall_{\mathcal{U}} y \, (x < y))$ is true, as every x satisfies $\forall_{\mathcal{U}} y \, (x < y)$. But $\forall_{\mathcal{U}} y \forall_{\mathcal{U}} x \, (x < y)$ is false, as no y has $\forall_{\mathcal{U}} x \, (x < y)$. So **don't swap quantifiers!**.

Definition 2.46 Given ultrafilters $\mathcal{U}, \mathcal{V}$, define their sum to be

$$\mathcal{U} + \mathcal{V} := \{A \subseteq \mathbb{N} : \forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, (x + y \in A)\}.$$

Example 2.47 We have $\widetilde{m} + \tilde{n} = \widetilde{m+n}$.

Proposition 2.48 For any ultrafilters $\mathcal{U}$ and $\mathcal{V}$, $\mathcal{U} + \mathcal{V}$ is an ultrafilter.

Proof. We have $\emptyset \notin \mathcal{U} + \mathcal{V}$. If $A \in \mathcal{U} + \mathcal{V}$ and $A \subseteq B$, then $B \in \mathcal{U} + \mathcal{V}$. If $A, B \in \mathcal{U} + \mathcal{V}$, then $(\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, (x + y \in A)) \wedge (\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, (x + y \in B))$, so by above proposition, we have $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y (x + y \in A \wedge x + y \in B)$, i.e. $\forall_{\mathcal{U}} x \forall \mathcal{V} y \, (x + y \in A \cap B)$, i.e. $A \cap B \in \mathcal{U} + \mathcal{V}$. Hence $\mathcal{U} + \mathcal{V}$ is a filter.

Suppose that $A \notin \mathcal{U} + \mathcal{V}$, i.e. $\neg(\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, (x + y \in A))$. Then by above proposition twice, we have $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, \neg(x + y \in A)$. So $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, (x + y \in A^c)$, i.e. $A^c \in \mathcal{U} + \mathcal{V}$. ◻

Proposition 2.49 Ultrafilter addition is associative.

Proof. Let $A \subseteq \mathcal{U} + (\mathcal{V} + \mathcal{W})$, so $\forall_{\mathcal{U}} x \forall_{\mathcal{V}+\mathcal{W}} \, (x + y \in A)$. So $B := \{y : x + y \in A\} \in \mathcal{V} + \mathcal{W}$, i.e. $\forall_{\mathcal{V}} y_1 \forall_{\mathcal{W}} y_2 \, (y_1 + y_2 \in B)$. So we have $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y_1 \forall_{\mathcal{W}} y_2 \, (x + y_1 + y_2 \in A)$. So

$$\mathcal{U} + (\mathcal{V} + \mathcal{W}) = \{A \subseteq \mathbb{N} : \forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \forall_{\mathcal{W}} z \, (x + y + z \in A)\} = (\mathcal{U} + \mathcal{V}) + \mathcal{W}.$$

◻

Proposition 2.50 Ultrafilter addition is left-continuous: for fixed $\mathcal{V}$, $\mathcal{U} \mapsto \mathcal{U} + \mathcal{V}$ is continuous.

Proof. For $A \subseteq \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{U} + \mathcal{V} \in C_A &\iff A \in \mathcal{U} + \mathcal{V} \\
&\iff \forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \, (x + y \in A) \\
&\iff B := \{x \in \mathbb{N} : \forall_{\mathcal{V}} y \, (x + y \in A)\} \in \mathcal{U} \\
&\iff \mathcal{U} \in C_B
\end{aligned}$$

hence the preimage of C_A, which is C_B, is open. ◻

Proposition 2.51 (Idempotent Lemma) There exists an idempotent ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ (i.e. $\mathcal{U} = \mathcal{U} + \mathcal{U}$).

Proof. For $M \subseteq \beta\mathbb{N}$, define $M + M := \{x + y : x, y \in M\}$. We seek a non-empty, compact $M \subseteq \beta\mathbb{N}$ which is minimal such that $M + M \subseteq M$, and hope to show that M is a singleton.

Such an M exists ($\beta\mathbb{N}$ is one such), so the set of all such M is non-empty. By Zorn's Lemma, it suffices to show that if $\{M_i : i \in I\}$ is a chain of such sets, then $M =$

$\bigcap_{i \in I} M_i$ (an upper bound with respect to the partial ordering $\supseteq$) is another such set. This M will be compact as an intersection of closed sets, since $\beta\mathbb{N}$ is compact and Hausdorff, so any subspace is closed iff it is compact. Also, $M + M \subseteq M$: for $x, y \in M$, we have $x, y \in M_i$ so $x + y \in M_i + M_i \subseteq M_i$ for all $i \in I$, so $x + y \in M$. Finally, M is non-empty: $\{M_i : i \in I\}$ have the finite intersection property, as they are a chain, and are closed, so their intersection is non-empty.

So by Zorn's lemma, there exists such a minimal M. Given $x \in M$, we have $M + x = M$, since $M + x \neq \emptyset$, $M + x$ is compact (as the continuous image of a compact set) and $(M + x) + (M + x) = (M + x + M) + x \subseteq (M + M + M) + x \subseteq M + x$, so by minimality of M, $M + x = M$.

In particular, there exists $y \in M$ such that $y + x = x$. Let $T = \{y \in M : y + x = x\}$. We claim that $T = M$, and since $T \subseteq M$, it is enough to show that T is compact, non-empty and $T + T \subseteq T$, by minimality of M. Indeed, $y \in T$, so $T \neq \emptyset$, T is the pre-image of a singleton which is compact, hence closed, so T is closed, so compact. Finally, for $y, z \in T$, we have $y + x = x = z + x$ so $y + z + x = y + x = x$, so $y + z \in T$, so $T + T \subseteq T$.

Hence, $y + x = x$ for all $y \in M$, hence $x + x = M$. In fact, $M = \{x\}$. □

Remark 2.52 The finite subgroup problem asks whether we can find a non-trivial subgroup of $\beta\mathbb{N}$ (e.g. find $\mathcal{U}$ with $\mathcal{U} + \mathcal{U} \neq \mathcal{U}$ but $\mathcal{U} + \mathcal{U} + \mathcal{U} = \mathcal{U}$). This was recently proven to be negative.

Remark 2.53 An open problem is whether an ultrafilter can "absorb" another ultrafilter, i.e. whether there exist $\mathcal{U} \neq \mathcal{V}$ such that $\mathcal{U} + \mathcal{U} = \mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U} = \mathcal{V} + \mathcal{V} = \mathcal{V}$.

Theorem 2.54 (Hindman) For any finite colouring of $\mathbb{N}$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$\mathrm{FS}(\{x_n : n \in \mathbb{N}\}) = \left\{ \sum_{i \in I} x_i : I \subseteq \mathbb{N} \text{ finite}, I \neq \emptyset \right\}.$$

Proof. Let $\mathcal{U}$ be an idempotent ultrafilter, and partition $\mathbb{N}$ into its colour classes: $\mathbb{N} = A_1 \sqcup \cdots \sqcup A_k$. Since $\emptyset \notin \mathcal{U}$ by definition, we have $A_1 \cup \cdots \cup A_k \in \mathbb{N} \in \mathcal{U}$ by Proposition 2.33. So by Corollary 2.34, $A \coloneqq A_i \in \mathcal{U}$ for some $i \in [k]$. We have $\forall_{\mathcal{U}} y \,(y \in A)$ by definition. Thus:
1. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y \,(y \in A)$.
2. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y \,(x \in A)$.
3. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y \,(x + y \in A)$ since $A \in \mathcal{U} + \mathcal{U} = \mathcal{U}$.

Proposition 2.44 then gives that $\forall_{\mathcal{U}} x \forall_{\mathcal{V}} y \,(\mathrm{FS}(x, y) \subseteq A)$. Fix $x_1 \in A$ such that $\forall_{\mathcal{U}} y \,(\mathrm{FS}(x_1, y) \subseteq A)$.

Now assume we have found $x_1, ..., x_n$ such that $\forall_{\mathcal{U}} y \,(\mathrm{FS}(x_1, ..., x_n, y) \subseteq A)$, i.e. $B \coloneqq \{y \in \mathbb{N} : \mathrm{FS}(x_1, ..., x_n, y) \subseteq A\} \in \mathcal{U} = \mathcal{U} + \mathcal{U}$, i.e. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y \,(x + y \in B)$ by definition. We have:

1. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (\mathrm{FS}(x_1, ..., x_n, y) \subseteq A)$.
2. $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (\mathrm{FS}(x_1, ..., x_n, x) \subseteq A)$.
3. For each $z \in \mathrm{FS}(x_1, ..., x_n, y)$, we have $\forall_{\mathcal{U}} y\ (z + y \in A)$, so $\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (z + x + y \in A)$.

Proposition 2.44 then gives that

$$\forall_{\mathcal{U}} x \forall_{\mathcal{U}} y\ (\mathrm{FS}(x_1, ..., x_n, x, y) \subseteq A).$$

The result follows by induction. □

3. Euclidean Ramsey theory

If we 2-colour $\mathbb{R}^2$, there are 2 points of distance at most 1 of the same colour (consider equilateral triangel).

If we 3-colour $\mathbb{R}^3$, there are 2 points of distance at most 1 of the same colour (consider regular tetrahedron)

If we k-colour $\mathbb{R}^k$, then by considering the regular simplex with $k+1$ vertices such that any 2 points have distance 1 between them, 2 points have the same colour.

Definition 3.1 X' is an **isometric copy** of X if there exists a bijection $\varphi : X \to X'$ which prserves distances:

$$\forall x, y \in X, \quad d(x, y) = d(\varphi(x), \varphi(y)).$$

Definition 3.2 A finite set $X \subseteq \mathbb{R}^m$ is **(Euclidean) Ramsey** if for all $k \in \mathbb{N}$, there exists a finite set $S \subseteq \mathbb{R}^n$ (n could be very large) such that for any k-colouring of S, there exists a monochromatic isometric copy of X.

Example 3.3

- $\{0, 1\}$ is Ramsey, by the above simplex argument.
- The equilateral triangle of side length 1 is Ramsey, by considering the $2k$-dimensional unit simplex.
- Any $\{0, a\}$ is Ramsey.
- By the same argument, any regular simplex is Ramsey.

Remark 3.4

- If X is infinite, then (exercise) we can construct a 2-colouring of $\mathbb{R}^n$ with no monochromatic isometric copy of X.
- Above, we took S to be in $\mathbb{R}^k$ for k colours. Can we do better? We can't do it for $\{0, 1\}$ in $\mathbb{R}$: consider the colouring $x \mapsto \lfloor x \rfloor \bmod 2$. For $\{0, 1\}$ with 3 colours, can do this in $\mathbb{R}^2$: look at diagram. Actually this shows $\chi(\mathbb{R}^2) \geq 4$. Can show $\chi(\mathbb{R}^2) \leq 7$ by hexagonal argument. We know $\chi(\mathbb{R}^2) \geq 5$. In general, $1.2^n \leq \chi(\mathbb{R}^n) \leq 3^n$. The upper bound easily follows from a hexagonal colouring.

Proposition 3.5 X is Euclidean Ramsey iff $\forall k \in \mathbb{N}$, $\exists n \in \mathbb{N}$ such that for any k-colouring of $\mathbb{R}^n$, there exists a monochromatic isometric copy of X.

Proof. If X is Euclidean Ramsey then take S finite in $\mathbb{R}^n$ (for k colours).

$\Longleftarrow$: we use a compactness proof. Suppose not, therefore for any finite $S \subseteq \mathbb{R}^n$, there is a bad k-colouring (i.e. no monochromatic isometric copy of X). The space of all k-colourings is $[k]^{(\mathbb{R}^n)}$, which is compact by Tychonov (TODO: add this statement). Consider the set $C_{X'}$ of colourings under which X' is not monochromatic. $C_{X'}$ is closed. Look at $\{C_{X'} : X' \text{ isometric copy of } X\}$. It has the finite intersection property, because any finite S has a bad k-colouring. Therefore, $\bigcap C_{X'} \neq \emptyset$, so there exists a k-colouring of $\mathbb{R}^n$ with no monochromatic isometric copy of X in S. □

Lemma 3.6 If $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are both Ramsey, then $X \times Y \subseteq \mathbb{R}^{n+m}$ is also Ramsey.

Proof. Let c be a colouring of $S \times T$, where S is k-Ramsey for X and T is $k^{|S|}$-Ramsey for Y. $k^{|S|}$-colour T as follows: $c'(t) = \left(c(s_1, t), ..., c\left(s_{|S|}, t\right)\right)$. By choice of T, there is a monochromatic (with respect to c') isometric copy Y' of Y. So $c(s, y) = c(s, y')$ for all $y, y' \in Y$ and $s \in S$. Now k-colour S by $c''(s) = c(s, y)$ for any $y \in Y$ (note this is well-defined). By choice of S, there is a monochromatic (with respect to c'') isometric copy X' of X, so $X' \times Y'$ is monochromatic with respect to c.

TODO: convince yourself that this is a very standard product argument. □

Remark 3.7 Since any $\{0, a\}$ and $\{0, b\}$ are Ramsey, any rectangle is Ramsey, so any right-angle triangle is Ramsey (since it is embedded in a rectangle). Similarly, any cuboid is Ramsey, and so any acute triangle (which is embedded in a cuboid) is Ramsey.

Remark 3.8 In general, to prove sets are Ramsey, we will first embed them in "nicer" sets (with useful symmetry groups) and show instead that those sets are Ramsey. We will show:

- any triangle is Ramsey
- any regular n-gon is Ramsey
- any Platonic solid is Ramsey