1 Hamiltonian Formalism

Definition 1.0.1. The classical **state** of a system at a given instant in time is a **complete** set of data that fully specifies the future evolution of the system.

Remark. **Any** set of data that fully fixes future evolution is valid.

Definition 1.0.2. The **phase (or state) space** is the set of all possible states for a system at a given time.

Example 1.0.3. A free particle moving in $\mathbb{R}$. The phase space is $\mathbb{R}^2$ ($\mathbb{R}$ for position, $\mathbb{R}$ for velocity).

Definition 1.0.4. The **Hamiltonian formalism** studies dynamics in a phase space, parameterised by $\underline{q}(t)$ and $\underline{p}(t)$, where $p_i = \frac{\partial L}{\partial \dot{\underline{q_i}}}$, the momentum.

Example 1.0.5. A particle moving in $\mathbb{R}$, with $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2$.
Then $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ so $\dot{x}(x, p_x) = \frac{p_x}{m}$.
In the Hamltonian formalism, $L(x, p_x) = \frac{p_x^2}{2m}$.

Example 1.0.6. A particle moving in $\mathbb{R}^2$ (in polar coordinates).
$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. So $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$.
So $\dot{r}(r, \theta, p_r, p_\theta) = \frac{p_r}{m}$, $\dot{\theta}(r, \theta, p_r, p_\theta) = \frac{p_\theta}{mr^2}$.
$L(r, \theta, \dot{r}, \dot{\theta}) = L(r, \theta, p_r, p_\theta) = \frac{1}{2}(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2})$.

Definition 1.0.7. Given two functions $f(\underline{q}, \underline{p}, t)$ and $g(\underline{q}, \underline{p}, t)$ in phase space their **Poisson bracket** is:

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

where n is the dimension of the configuration space.

Remark. In the Hamiltonian formalism, $\frac{\partial q_i}{\partial p_j} = \frac{\partial p_j}{\partial q_i} = 0$.
Similarly, $\frac{\partial q_i}{\partial q_j} = \frac{\partial p_i}{\partial p_j} = \delta_{i,j}$

Example 1.0.8. Let $f = q_i$, $g = q_j$. $\{q_i, q_j\} = 0$, and $\{p_i, p_j\} = 0$. $\{q_i, p_j\} = \sum_{k=1}^{n} \delta_{i,j}\delta_{j,k} = \delta_{i,j}$.

Definition 1.0.9. Let $\mathbb{F}$ be the set functions from a phase space P to $\mathbb{R}$

Definition 1.0.10. The Hamiltonian flow $\Phi_f^{(s)}$, with $(s) \in \mathbb{R}$, $f \in F$ operator maps $\mathbb{F}$ to $\mathbb{F}$ and is defined as

$$\Phi_f^{(s)}(g) := e^{s\{\cdot, f\}}g := g + s\{g, f\} + \frac{s^2}{2}\{\{g, f\}, f\} + \cdots$$

Remark. The transformation generated by f has generator $a_i = \{q_i, f\}$ where $q_i \to q_i + \epsilon a_i$.
Infinitesimally, $\Phi_f^{(s)}(g) := g + \epsilon\{g, f\} + O(\epsilon^2)$

TODO: properties on poisson bracket

Example 1.0.11. (Rotation in $\mathbb{R}^2$ in Cartesian coordinates) As a guess, choose $f = q_1\dot{q}_2 - \dot{q}_1 q_2$, the angular momentum.

$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(q_1, q_2)$ so $p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1$ and $p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2 \Rightarrow f = q_1 p_2 - q_2 p_1$.

Then $q_1 \to q_1 + \epsilon\{q_1, f\} + O(\epsilon^2) = q_1 + \epsilon\{q_1, q_1 p_2 - q_2 p_1\} = q_1 + \epsilon\{q_1, q_1 p_2\} - \epsilon\{q_1, q_2 p_1\} = q_1 + \epsilon\{q_1, q_1\}p_2 + \epsilon\{q_1, p_2\}q_1 - \epsilon\{q_1, q_2\}p_1 - \epsilon\{q_1, p_1\}q_2 = q_1 - \epsilon q_2$

Similarly, $q_2 \to q_2 + \epsilon q_1$ so $(q_1, q_2) \to (q_1, q_2) + \epsilon((0, -1), (1, 0))(q_1, q_2)$ TODO make into matrices and column vectors.

Definition 1.0.12. The **Hamiltonian** is the energy expressed in Hamiltonian coordinates:

$$H = \sum_{i=1}^{n} \dot{q_i(\underline{q}, \underline{p})} p_i - L(\underline{q}, \underline{\dot{q}}(\underline{q}, \underline{p}))$$

Example 1.0.13. (Harmonic oscillator in one dimension) Let $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \Rightarrow p = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$.

$H = \dot{x}p - L = \frac{p^2}{m} - (\frac{1}{2}\frac{p^2}{m} - \frac{1}{2}kx^2) = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2$

Theorem 1.0.14. The time evolution of the phase space coordinates $\underline{q}, \underline{p}$ is generated by Hamiltonian flow Φ_H:

$$q_i(t + a) = \Phi_H^{(a)} q_i(t), p_i(t + a) = \Phi_H^{(a)} p_i(t)$$

Infinitesimally, $q_i(t) + \epsilon\dot{q}_i(t) + O(\epsilon^2) = q_i(t + \epsilon) = q_i(t) + \epsilon\{q_i, H\} + O(\epsilon^2) \Leftrightarrow \dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}$ and similarly, $\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$.

These equations are called **Hamilton's equations**.

Proof. $\frac{\partial H}{\partial q_i}$. TODO: complete this proof, finish rest of notes from lecture. □

Corollary 1.0.15. The time evolution of any function $f(\underline{q}, \underline{p})$ in phase space is generated by Φ_H:

$$\frac{df}{dt} = \{f, H\}$$

If $f(\underline{q}, \underline{p}, t)$ depends explicitly on time then

$$\frac{df}{dt} = \{f, h\} + \frac{\partial f}{\partial t}$$

Proof. $\frac{df}{dt} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) + \frac{\partial f}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t}$. □