Contents

1. Definitions and examples

1.1. Categories

Definition. A **category** $\mathcal{C}$ consists of:

1. a collection $\mathrm{ob}(\mathcal{C})$ of **objects** $A, B, C, \ldots$,
2. a collection $\mathrm{mor}(\mathcal{C})$ of **morphisms** $f, g, h, \ldots$,
3. two operations dom and cod from $\mathrm{mor}(\mathcal{C})$ to $\mathrm{ob}(\mathcal{C})$. We write $f : A \to B$ to mean f is a morphism, with domain A and codomain B.
4. an operation from $\mathrm{ob}(\mathcal{C})$ to $\mathrm{mor}(\mathcal{C})$ sending A to $1_A : A \to A$.
5. a partial binary **composition** operation $(f, g) \mapsto fg$ on $\mathrm{mor}(C)$, such that fg is defined iff $\mathrm{dom}(f) = \mathrm{cod}(g)$, and in this case $\mathrm{dom}(fg) = \mathrm{dom}(g)$ and $\mathrm{cod}(fg) = \mathrm{cod}(f)$.

and satisfies the following:

1. $f1_A = f$ and $1_A g = g$ when the composites are defined.
2. $f(gh) = (fg)h$ whenever fg and gh are defined.

Remark.

- $\mathrm{ob}(\mathcal{C})$ and $\mathrm{mor}(\mathcal{C})$ are not necessarily sets. If they are, then $\mathcal{C}$ is called a **small** category, otherwise it is called **large**.
- An equivalent definition exists without using objects (since objects A biject with identity morphisms 1_A).
- fg means first apply g, then f.

Example. **Set** = the category of all sets and functions between them. (Formally, a morphism of **Set** is a pair (f, B) where f is a set-theoretic function and B is its codomain).

Example. Algebraic categories:

- **Gp** is the category of groups and group homomorphisms.
- **Rng** is the category of rings and ring homomorphisms.
- $\mathbf{Vect}_K$ is the category of vector spaces over a field K with K-linear maps.

Example. Topological categories:

- **Top** is the category of topological spaces and continuous maps.
- **Met** is the category of metric spaces and non-expansive maps (i.e. $d(f(x), f(y)) \leq d(x, y)$).
- **Mfd** is the category of smooth manifolds and smooth (C^∞) maps.
- **TopGp** is the category of topological groups and continuous homomorphisms.
- **Htpy** is the category with same objects as Top but morphisms are homotopy classes of continuous maps.

Definition. Given a category $\mathcal{C}$ and an equivalence relation $\sim$ on $\mathrm{mor}(\mathcal{C})$ such that $f \sim g \Rightarrow (\mathrm{dom}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(f) = \mathrm{cod}(g))$, and $f \sim g \Rightarrow fh \sim gh$ when the composites fh and gh are defined, we can form a **quotient** category $\mathcal{C}/\sim$, which has the same objects as $\mathcal{C}$, but morphisms are equivalence classes of morphisms in $\mathcal{C}$ under $\sim$. $\sim$ is called a **congruence**.

Example. Relation categories:

- **Rel** is the category with the same objects as **Set** but with morphisms that are relations $R \subseteq A \times B$, with composition defined by $R \circ S = \{(a, c) \in A \times C : \exists b : (a, b) \in S \wedge (b, c) \in R\}$. If R and S are functions, then $\circ$ is the function composition operation.
- **Part** is the category with sets as objects and partial functions as morphisms. **Part** is a subcategory of **Rel**, and **Set** is a subcategory of **Part**.

Definition. For every category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{\mathrm{op}}$ has the same objects and morphisms as $\mathcal{C}$, but dom and cod are interchanged and composition is reversed. This yields a **duality principle**: if P is a true statement about categories, then so is the dual statement P^* (which is obtained by reversing arrows in P).

Definition. A **monoid** (a group but inverses not guaranteed) is a small category with one object $*$. In particular, a group is a 1-object where all morphisms are isomorphisms.

Definition. A **groupoid** is a category where every morphism is an isomorphism.

Example. The **fundamental groupoid** of a space X, $\pi_1(X)$, is the category where objects are the points of X, and morphisms $x \to y$ are homotopy classes of paths from x to y. (Note this depends only on X, whereas the fundamental group depends on X and a point $x \in X$).

Definition. A category is **discrete** if the only morphisms are identities.

Definition. A category $\mathcal{C}$ is a **preorder** if for every pair of objects (A, B), there exists at most 1 morphism $A \to B$, then $\mathrm{mor}(C)$ becomes a reflexive and transitive relation on $\mathrm{ob}(C)$ (so existence of morphism $A \to B$ corresponds to $A \preccurlyeq B$).

In particular, a poset is a small preorder where the only isomorphisms are identity morphisms.

Example. For a field K, the category $\mathbf{Mat}_K$ has natural numbers as objects, morphisms $n \to m$ are $m \times n$ matrices with entries from K, and composition is matrix multiplication.

1.2. Functors

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of mappings $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ and $F : \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$ such that $F(\mathrm{dom}(f)) = \mathrm{dom}(Ff)$, $F(\mathrm{cod}(f)) = \mathrm{cod}(Ff)$, $F(1_A) = 1_{FA}$ and $F(fg) = (Ff)(Fg)$ whenever fg is defined.

Write **Cat** for the category with objects as small categories and morphisms as functors between them.

Example. We have **forgetful** functors **Gp** $\to$ **Set**, **Rng** $\to$ **Set**, **Top** $\to$ **Set**, **Rng** $\to$ **AbGp**, **Met** $\to$ **Top**, **TopGp** $\to$ **Top**, **TopGp** $\to$ **Gp**. They "forget" the structure of the objects, and/or "forget" the conditions on the morphisms.

Example. The construction of free groups is a functor $F : \mathbf{Set} \to \mathbf{Gp}$: given a set A, FA is the group freely generated by A, such that every mapping $A \to G$, where G is a group, extends uniquely to a homomorphism $FA \to G$.

Given $f : A \to B$, define $Ff : FA \to FB$ to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow FB$. If we also have $g : B \to C$, then $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow FC$, so are equal by uniqueness.

Example. Given a set A, PA is the set of all subsets of A. Given $f : A \to B$, define $Pf : PA \to PB$ by $Pf(A') = \{f(a) : a \in A'\} \subseteq B$. So P is a functor $\mathbf{Set} \to \mathbf{Set}$.

Example. We also have a functor $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ (or $\mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$), where $P^*A = PA$ and for $f : A \to B$, $P^*f : PB \to PA$ is given by $P^*f(B') = \{a \in A : f(a) \in B'\}$. So P^* is the same construction as the power-set functor, except each subset of B is mapped by P^*f to its inverse image under f rather than its image under f.

Definition. A **contravariant** functor is a functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$, i.e. F reverses the direction of arrows. Functors which do not reverse arrow directions are called **covariant**.

Example. Given a vector space V over K, write V^* for the space of linear maps $V \to K$. Given $f : V \to W$, write $f^* : W^* \to V^*$ for the map $\theta \mapsto \theta f$. This defines a functor $(-)^* : {\mathbf{Vect}_K}^{\mathrm{op}} \to \mathbf{Vect}_K$.

Example. The mappings $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, $F \mapsto F$ define a covariant functor $\mathbf{Cat} \to \mathbf{Cat}$.

Example. A functor between monoids is a monoid homomorphism, a functor between groups is a group homomorphism, and a functor between posets is a monatone map.

Example. Given a group G, a functor $G \to \mathbf{Set}$ is given by a set A equipped with a G-action $(g, a) \mapsto g \cdot a$, i.e. a permutation representation of G.

Similarly, a functor $G \to \mathbf{Vect}_K$ is a K-linear representation of G.

Example. The fundamental group construction is a functor $\pi_1 : \mathbf{Top}^* \to \mathbf{Gp}$, where $\mathbf{Top}^*$ is the category of topological spaces with basepoints and continuous maps preserving basepoints.

1.3. Natural transformations

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha : F \to G$ is a mapping $\mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$ which assigns to each $A \in \mathrm{ob}(\mathcal{C})$ a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$, such that for any $f : A \to B$ in $\mathrm{mor}(\mathcal{C})$, the following **naturality square** commutes:

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

If $\alpha : F \to G$, $\beta : G \to H$ are natural transformation, define $\beta\alpha : F \to H$ by $(\beta\alpha)_A = \beta_A \alpha_A$. Write $[\mathcal{C}, \mathcal{D}]$ for the category with objects as functors $\mathcal{C} \to \mathcal{D}$ and morphisms as natural transformations between the functors.

Example. Given a vector space V, we have a linear map $\alpha_V : V \to V^{**}$ which sends $v \in V$ to the linear form $\theta \mapsto \theta(v)$ on V^*. These maps define a natural transformation $1_{\mathbf{Vect}_K} \to (-)^{**}$.

Example. Therre is a natural transformation $\alpha : 1_{\mathbf{Set}} \to UF$ where F is the free group functor and U is the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$ whose value at A is the inclusion $A \hookrightarrow UFA$. The naturality square is

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ UFA \arrow[r, "UFf"] & UFB \end{tikzcd}$$

Example. For any set A, we have a mapping $\alpha_A : A \to PA$ given by $\alpha_A(a) = \{a\}$. This is a natural transformation $1_{\mathbf{Set}} \to P$ since $Pf(\{a\}) = \{f(a)\}$ for any $a \in A$.

Example. Given order-preserving maps $f, g : P \to Q$ between posets, there exists a unique natural transformation $f \to g$ iff $f(p) \leq g(p)$ for all $p \in P$.

Example. Given two group homomorphisms $u, v : G \to H$, a natural transformation $u \to v$ is given by $h \in H$ such that $hu(g) = v(g)h$ for all $g \in G$, or equivalently, $u(g) = h^{-1}v(g)h$ for all $g \in G$, i.e. u and v are **conjugate** homomorphisms.

In particular, the group of natural transformations $u \to u$ is the **centraliser** of the image of u.

Example. If A and B are G-sets, considered as functors $G \to \mathbf{Set}$, a natural transformation $f : A \to B$ is a G**-equivariant** map, i.e. $f : A \to B$ such that $g \cdot f(a) = f(g \cdot a)$ for all $a \in A$, $g \in G$.

Example. The **Hurewicz homomorphism** links the homotopy and homology gruops of a space X. Elements of $\pi_n(X, x)$ are homotopy classes of basepoint-preserving maps $f : S^n \to X$. If we think of S^n as $\partial\Delta^{n+1}$, f defines a singular n-cycle on X and homotopic maps differ by an n-boundary, so we get a well-defined map $h_n : \pi_n(Xx) \to H_n(x)$. h_n is a homomorphism and a natural transformation $\pi_n \to H_n U$, where U is the forgetful functor $\mathbf{Top}^* \to \mathbf{Top}$.

1.4. Equivalence of categories

Example. **Rel** is isomorphic to $\mathbf{Rel}^{\mathrm{op}}$ in the category **Cat** via the functor $F : \mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$, $FA = A$, $FR = R^o = \{(b, a) : (a, b) \in R\}$.

Lemma. Let $\alpha : F \to G$ be a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$. Then α is an isomorphism in the functor category $[\mathcal{C}, \mathcal{D}]$ iff α_A is an isomorphism in $\mathcal{D}$ for each A.

Proof.

- $\Rightarrow$ is trivial as composition in $[\mathcal{C}, \mathcal{D}]$ is pointwise.
- $\leq$: suppose each α_A has an inverse β_A. Given $f : A \to B$ in $\mathcal{C}$, in the diagram

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

TODO finish

- We have $(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$ by naturality of α. So β is natural and an inverse for α.

$\square$

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ consists of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$.

Write $\mathcal{C} \simeq \mathcal{D}$ if there is an equivalence between $\mathcal{C}$ and $\mathcal{D}$.

Definition. P is a **categorical property** if $\mathcal{C}$ satisfies P and $\mathcal{C} \simeq \mathcal{D}$ implies that $\mathcal{D}$ satisfies P.

Example. The category **Part** of sets and partial functions is equivalent to $\mathbf{Set}^*$: define $F : \mathbf{Set}^* \to \mathbf{Part}$ by $F(A, a) = A - \{a\}$, and for $f : (A, a) \to (B, b)$, $(Ff)(x) = f(x)$ if $f(x) \neq b$ and undefined otherwise. Define $G : \mathbf{Part} \to \mathbf{Set}^*$ by $G(A) = (A \cup \{a\}, A)$, and for $f : A \dashrightarrow B$, $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ is defined, and B otherwise.

Note $FG = 1_{\mathbf{Part}}$; $GF \neq 1_{\mathbf{Set}^*}$, but there is an isomorphism $1_{\mathbf{Set}^*} \to GF$. Note also that $\mathbf{Part} \ncong \mathbf{Set}^*$.

Example. We have an equivalence $\mathbf{fdVect}_K \simeq \mathbf{fdVect}_K^{\mathrm{op}}$: both functors are $(-)^*$, and both isomorphisms are $\alpha : 1_{\mathbf{fdVect}_K} \to (-)^{**}$.

Example. $\mathbf{fdVect}_K \simeq \mathtt{Mat}_K$: define $F : \mathtt{Mat}_K \to \mathbf{fdVect}_K$, $F(n) = k^n$, $F(A : n \to p)$ is the linear map $k^n \to k^p$ represented by A (w.r.t. standard bases). To define G, choose a basis for each V, and define $G(v) = \dim(V)$, $G(f : V \to W)$ is the matrix representing f w.r.t. the chosen basis.

$GF = 1_{\mathtt{Mat}_K}$; the choice of bases yields isomorphisms $k^{\dim(V)} \to V$ for each V.

Definition. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is **faithful** if, given f and g in $\mathrm{mor}(\mathcal{C})$, if $(Ff = Fg)$, $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$, then $f = g$.

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is **full** if for every $g : FA \to FB$ in $\mathcal{D}$, there exists $f : A \to B$ in $\mathcal{C}$ with $Ff = g$.

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if, for any $B \in \mathrm{ob}(D)$, there exists an $A \in \mathrm{ob}(\mathcal{C})$ with $FA \cong B$.

Remark. If F is full and faithful, then it is essentially surjective: given $g : FA \to FB$ in $\mathcal{D}$, the unique $f : A \to B$ with $Ff = g$ is an isomorphism.

Definition. A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is a **full** subcategory if the inclusion functor $\mathcal{D} \to \mathcal{C}$ is a full functor.

Lemma. Let $F : \mathcal{C} \to \mathcal{D}$. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ iff F is full, faithful and essentially surjective.