1. Metric spaces

1.1. Metrics

- **Definition**: **metric space** is (X, d), X is set, $d : X \times X \to [0, \infty)$ is **metric** satisfying:
  - $d(x, y) = 0 \Longleftrightarrow x = y$
  - **Symmetry**: $d(x, y) = d(y, x)$
  - **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- **Example**:
  - p-adic metric: for $p \in [1, \infty)$

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

  - Extension of the p-adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

  - Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

  - Discrete metric:

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Definition**: **open ball of radius r around x**:

$$B(x; r) := \{y \in X : d(x, y) < r\}$$

- **Definition**: **closed ball of radius r around x**:

$$D(x; r) := \{y \in X : d(x, y) \leq r\}$$

1.2. Open and closed sets

- **Definition**: $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

- **Definition**: $A \subseteq X$ is **closed** if $X - A$ is open.
- Sets can be neither closed nor open, or both.
- With standard metric on $\mathbb{R}$, any singleton $\{x\} \in \mathbb{R}$ is closed and not open (same holds for $\mathbb{R}^n$).
- **Definition**: let X be metric space, $x \in N \subseteq X$. N is **neighbourhood** of x if

$$\exists \text{ open } V \subseteq X : x \in V \subseteq N$$

- **Corollary**: let $x \in X$, then $N \subseteq X$ neighbourhood of x iff $\exists \varepsilon > 0 : x \in B(x; \varepsilon) \subseteq N$.
- **Proposition**: open balls are open, closed balls are closed.
- **Lemma**: let (X, d) metric space.
  - X and $\emptyset$ are both open and closed.

- Arbitrary unions of open sets are open.
  - Finite intersections of open sets are open.
  - Finite unions of closed sets are closed.
  - Arbitrary intersections of closed sets are closed.
- **Example**: if X has discrete metric, any $A \subseteq X$ is open and closed.

1.3. Continuity

- **Definition**:
  - **Sequence** in X is $a : \mathbb{N}_0 \to X$, written $(a_n)_{n \in \mathbb{N}}$.
  - (a_n) **converges to a**, $\lim_{n \to \infty} a_n = a$, if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, d(a, a_n) < \varepsilon$$

- **Proposition**: let X, Y metric spaces, $a \in X$, $f : X \to Y$. The following are equivalent:
  - $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in X, d_X(a, x) < \delta \Longrightarrow d_Y(f(a), f(x)) < \varepsilon$.
  - For every sequence (a_n) in X with $a_n \to a$, $f(a_n) \to f(a)$.
  - For every open $U \subseteq Y$ with $f(a) \in U$, $f^{-1}(U)$ is a neighbourhood of a.

  If f satisfies these, it is **continuous at a**.
- **Definition**: f **continuous** if continuous at every $a \in X$.
- **Proposition**: $f : X \to Y$ continuous iff $f^{-1}(U)$ open for every open $U \subseteq Y$.
- **Example**: let d be discrete metric, d_2 be 2-adic metric.
  - Any $f : (X, d) \to (\mathbb{R}, d_2)$ is continuous.
  - $\text{id} : (\mathbb{R}, d_2) \to (\mathbb{R}, d)$ is not continuous.

2. Topological spaces

2.1. Topologies

- **Definition**: **power set** of X: $\mathcal{P}(X) := \{A : A \subseteq X\}$.
- **Definition**: **topology** on set X is $\tau \subseteq \mathcal{P}(X)$ with:
  - $\emptyset \in \tau$, $X \in \tau$.
  - **Closure under arbitrary unions**: if $\forall i \in I, U_i \in \tau$, then

$$\bigcup_{i \in I} U_i \in \tau$$

  - **Closure under finite intersections**: $U_1, U_2 \in \tau \Longrightarrow U_1 \cap U_2 \in \tau$ (this is equivalent to $U_1, ..., U_n \in \tau \Longrightarrow \bigcap_{i \in [n]} U_i \in \tau$).

  (X, τ) is **topological space**. Elements of τ are **open** subsets of X. $A \subseteq X$ **closed** if $X - A$ is open.
- **Definition**: $\tau = \mathcal{P}(X)$ is the **discrete topology** on X.
- **Definition**: $\tau = \{\emptyset, X\}$ is the **indiscrete topology** on X.
- **Example**:
  - For metric space (M, d), let τ_d exactly contain sets which are open with respect to d. Then (M, τ_d) is a topological space. d **induces** topology τ_d.

- Let $X = \mathbb{N}_0$ and $\tau = \{\emptyset\} \cup \{U \subseteq X : X - U \text{ is finite}\}$, then (X, τ) is topological space.
- **Proposition**: for topological space X:
  - X and $\emptyset$ are closed
  - Arbitrary intersections of closed sets are closed
  - Finite unions of closed sets are closed
- **Proposition**: for topological space (X, τ) and $A \subseteq X$, the **induced (subspace) topology on** $\boldsymbol{A}$

$$\tau_A = \{A \cap U : U \in \tau\}$$

  is a topology on A.
- **Example**: let $X = \mathbb{R}$ with standard topology induced by metric $d(x, y) = |x - y|$. Let $A = [1, 5]$. Then $[1, 3) = A \cap (0, 3)$ and $[1, 5] = A \cap (0, 6)$ are open in A.
- **Example**: consider $\mathbb{R}$ with standard topology τ. Then
  - $\tau_{\mathbb{Z}}$ is the discrete topology on $\mathbb{Z}$.
  - $\tau_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.
- **Proposition**: metrics d_p for $p \in [1, \infty)$ and d_∞ all induce same topology on $\mathbb{R}^n$, alled the **standard topology** on $\mathbb{R}^n$.
- **Definition**: (X, τ) is **Hausdorff** if

$$\forall x \neq y \in X, \exists U, V \in \tau : U \cap V = \emptyset \wedge x \in U, y \in V$$

- **Lemma**: any metric space (M, d) with topology induced by d is Hausdorff.
- **Example**: let $|X| \geq 2$ with indiscrete topology. Then X is not Hausdorff, since $\tau = \{X, \emptyset\}$ and if $x \neq y \in X$, only open set containing x is X (same for y). But $X \cap X = X \neq \emptyset$.
- **Definition**: **Furstenberg's topology on** $\mathbb{Z}$: define $U \subseteq \mathbb{Z}$ to be open if

$$\forall a \in U, \exists 0 \neq d \in \mathbb{Z} : a + d\mathbb{Z} := \{a + dn : n \in \mathbb{Z}\} \subseteq U$$

- Furstenberg's topology is Hausdorff.

2.2. Continuity

- **Definition**: let X, Y topological spaces.
  - $f : X \to Y$ is **continuous** if

$$\forall V \text{ open in } Y, f^{-1}(V) \text{ open in } X$$

  - f is **continuous at** $\boldsymbol{a \in X}$ if

$$\forall V \text{ open in } Y \text{ with } f(a) \in V, \exists U \text{ open in } X : a \in U \subseteq f^{-1}(V)$$

- **Lemma**: $f : X \to Y$ continuous iff f continuous at every $a \in X$. (Key idea for proof: $\cup_{a \in f^{-1}(V)} U_a \subseteq f^{-1}(V) = \cup_{a \in f^{-1}(V)} \{a\} \subseteq \cup_{a \in f^{-1}(V)} U_a$)
- **Example**: inclusion $i : (A, \tau_A) \to (X, \tau_X)$, $A \subseteq X$, is always continuous.
- **Lemma**: compositions of continuous functions are continuous.
- **Lemma**: let $f : X \to Y$ be function between topological spaces. Then f is continuous iff

$$\forall A \text{ closed in } Y, \quad f^{-1}(A) \text{ closed in } X$$

- **Remark**: we can use continuous functions to decide that sets are open or closed.
- **Definition**: ***n*-sphere** is

$$S^n := \left\{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\right\}$$

- **Example**: in the standard topology, the n-sphere is a closed subset of $\mathbb{R}^{n+1}$. (Consider the preimage of $\{1\}$ which is closed in $\mathbb{R}$).
- **Example**:
  - Can consider set of square matrices $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and give it the standard topology.
  - Note

$$\det(A) = \sum_{\sigma \in \text{ sym}(n)} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}\right)$$

    is a polynomial in the entries of A so is continuous function from $M_n(\mathbb{R})$ to $\mathbb{R}$.
  - $\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} = \det^{-1}(\mathbb{R} - \{0\})$ is open.
  - $\mathrm{SL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} = \det^{-1}(\{1\})$ is closed.
  - $O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I\}$ is closed: $f_{i,j}(A) = (AA^T)_{i,j}$ is continuous and

$$O(n) = \bigcap_{1 \leq i,j \leq n} (f_{i,j})^{-1}(\{\delta_{i,j}\})$$

  - $\mathrm{SO}(n) = O(n) \cap \mathrm{SL}_n(\mathbb{R})$ is closed.
- **Definition**: for X, Y topological spaces, $h : X \to Y$ is **homeomorphism** if h is bijective, continuous and h^{-1} is continuous. X and Y are **homeomorphic**, $X \cong Y$. h induces bijection between τ_X and τ_Y which commutes with unions and intersections.
- **Proposition**: compositions of homeomorphisms are homeomorphisms.
- **Example**: in standard topology, $(0,1)$ is homeomorphic to $\mathbb{R}$. (Consider $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$, $f = \tan$, $g : (0,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $g(x) = \pi\left(x - \frac{1}{2}\right)$ and $f \circ g$).
- **Example**: $\mathbb{R}$ with standard topology τ_{st} is not homoeomorphic to $\mathbb{R}$ with the discrete topology τ_d. (Consider $h^{-1}(\{a\}) = \{h^{-1}(a)\}$, $\{a\} \in \tau_{\text{st}}$ but $\{h^{-1}(a)\} \notin \tau_{\text{st}}$).
- **Example**: let $X = \mathbb{R} \cup \{\overline{0}\}$. Define $f_0 : \mathbb{R} \to X$, $f_0(a) = a$ and $f_{\overline{0}} : \mathbb{R} \to X$, $f_{\overline{0}}(a) = a$ for $a \neq 0$, $f_{\overline{0}}(0) = \overline{0}$. Topology on X has $A \subseteq X$ open iff $f_0^{-1}(A)$ and $f_{\overline{0}}^{-1}(A)$ open. Every point in X lies in open set: for $a \notin \{0, \overline{0}\}$, $a \in (a - \frac{|a|}{2}, a + \frac{|a|}{2})$ and both pre-images of this are same open interval, for 0, set $U_0 = (-1,0) \cup \{0\} \cup (0,1) \subseteq X$ then $f_0^{-1}(U_0) = (-1,1)$ and $f_{\overline{0}}^{-1}(U_0) = (-1,0) \cup (0,1)$ are both open. For $\overline{0}$, set $U_{\overline{0}} = (-1,0) \cup \{\overline{0}\} \cup (0,1) \subseteq X$, then $f_{\overline{0}}^{-1}(U_{\overline{0}}) = (-1,1)$ and $f_0^{-1}(U_{\overline{0}}) = (-1,0) \cup (0,1)$ are both open. So U_0 and $U_{\overline{0}}$ both open in X. X is not Hausdorff since any open sets containing 0 and $\overline{0}$ must contain "open intervals" such as U_0 and $U_{\overline{0}}$.

- **Example (Furstenberg's proof of infinitude of primes)**: since $a + d\mathbb{Z}$ is infinite, any nonempty finite set is not open, so any set with finite complement is not closed. For fixed d, sets $d\mathbb{Z}$, $1 + d\mathbb{Z}, ..., (d-1) + d\mathbb{Z}$ partition $\mathbb{Z}$. So the complement of each is the union of the rest, so each is open and closed. Every $n \in \mathbb{Z} - \{-1, 1\}$ is prime or product of primes, so $\mathbb{Z} - \{-1, 1\} = \bigcup_{p \text{ prime}} p\mathbb{Z}$, but finite unions of closed sets are closed, and since $\mathbb{Z} - \{-1, 1\}$ has finite complement, the union must be infinite.

3. Limits, bases and products

3.1. Limit points, interiors and closures

- **Definition**: for topological space X, $x \in X$, $A \subseteq X$:
  - **Open neighbourhood of x** is open set N, $x \in N$.
  - x is **limit point** of A if every open neighbourhood N of x satisfies

$$(N - \{x\}) \cap A \neq \emptyset$$

- **Corollary**: x is not limit point of A iff exists neighbourhood N of x with

$$A \cap N = \begin{cases} \{x\} & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

- **Example**: let $X = \mathbb{R}$ with standard topology.
  - $0 \in X$, then $(-1/2, 1/2)$ is open neighbourhood of 0.
  - If $U \subseteq X$ open, U is open neighbourhood for any $x \in U$.
  - Let $A = \left\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\right\}$, then only limit point in A is 0.
- **Definition**: let $A \subseteq X$.
  - **Interior** of A is largest open set contained in A:

$$A^\circ := \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U$$

  - **Closure** of A is smallest closed set containing A:

$$\overline{A} := \bigcap_{\substack{F \text{ closed} \\ A \subseteq F}}$$

  If $\overline{A} = X$, A is **dense** in X.
- **Lemma**:
  - $\overline{X - A} = X - A^\circ$
  - $\overline{A} = X - (X - A)^\circ$
- **Example**: let $\mathbb{Q} \subset \mathbb{R}$ with standard topology. Then $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$ (since every nonempty open set in $\mathbb{R}$ contains rational and irrational numbers).
- **Lemma**: $\overline{A} = A \cup L$ where L is the set of limit points of A.
- **Dirichlet prime number theorem**: let a, d coprime, then $a + d\mathbb{Z}$ contains infinitely many primes.

- **Example**: let A be set of primes in $\mathbb{Z}$ with Furstenberg topology. By above lemma, only need to find limit points in $\mathbb{Z} - A$ to find $\overline{A}$. $10\mathbb{Z}$ is an open neighbourhood of 0 for 0 inside $\mathbb{Z} - A$. For $a \notin \{-1, 0, 1\}$, $a + 10a\mathbb{Z}$ is an open neighbourhood of a. These sets have no primes so the corresponding points are not limit points of A. For ± 1, any open neighbourhood of 1 contains a set $\pm 1 + d\mathbb{Z}$ for some $d \neq 0$, but by the Dirichlet prime number theorem, this set contains at least one prime. So $\overline{A} = A \cup \{\pm 1\}$.
- **Lemma**:
  - Let $A \subseteq M$ for metric space M. If x is limit point of A then exists sequence x_n in A such that $\lim_{n \to \infty} x_n = x$.
  - If $x \in M - A$ and exists sequence x_n in A with $\lim_{n \to \infty} x_n = x$ then x is limit point of A.

3.2. Bases

- **Definition**: a **basis** for topology τ on X is collection $\mathcal{B} \subseteq \tau$ such that

$$\forall U \in \tau, \exists B \subseteq \mathcal{B} : U = \bigcup_{b \in B} b$$

  (every open U is a union of sets in B).
- **Example**:
  - For metric space (M, d), $\mathcal{B} = \{B(x; r) : x \in M, r > 0\}$ is basis for the induced topology. (Since if U open, $U = \cup_{u \in U} \{u\} \subseteq \cup_{u \in U} B(u, r_u) \subseteq U$.)
  - In $\mathbb{R}^n$ with standard topology, $\mathcal{B} = \{B(q; 1/m) : q \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a **countable** basis. (Find $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{r}{2}$ and $q \in \mathbb{Q}^n$ such that $q \in B\left(p; \frac{1}{m}\right)$, then $B\left(q; \frac{1}{m}\right) \subseteq B(p; r) \subseteq U$ using the triangle inequality).
- **Theorem**: let $f : X \to Y$ be map between topological spaces. The following are equivalent:
  - f is continuous.
  - If $\mathcal{B}$ is basis for topology τ on Y then $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.
  - $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$.
  - $\forall V \subseteq Y, \overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$.
  - $f^{-1}(C)$ closed for any closed set $C \subseteq Y$.
- **Theorem**: let X be a set and collection $\mathcal{B} \subseteq \mathcal{P}(X)$ be such that:
  - $\forall x \in X, \exists B \in \mathcal{B} : x \in B$
  - If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$.

  Then there is unique topology $\tau_{\mathcal{B}}$ on X for which $\mathcal{B}$ is a basis. We say $\mathcal{B}$ **generates** $\tau_{\mathcal{B}}$. We have $\tau_B = \{\cup_{i \in I} B_i : B_i \in \mathcal{B}, I \text{ indexing set}\}$.

3.3. Product topologies

- **Definition**: **Cartesian product** of topological spaces X, Y is $X \times Y := \{(x, y) : x \in X, y \in Y\}$. We give it the **product topology** which is generated by $\mathcal{B}_{X \times Y} := \{U \times V : U \in \tau_X, V \in \tau_Y\}$.
- **Example**:
  - Let $X = Y = \mathbb{R}$, then product topology is same as standard topology on $\mathbb{R}^2$.

- Let $X = Y = S^1$, then $X \times Y = T^2 = S^1 \times S^1$ is the **2-torus**. ***n*-torus** is defined for $n \geq 3$ by

$$T^n := S^1 \times T^{n-1}$$

- **Definition**: if $\tau_1 \subseteq \tau_2$ are topologies, then τ_1 is **smaller** than τ_2 (τ_2 is **larger** than τ_1).
- **Definition**: for topological spaces X, Y, **projection maps** $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are

$$\pi_X(x, y) = x, \quad \pi_Y(x, y) = y$$

- **Proposition**: for $X \times Y$ with product topology,
  - π_X and π_Y are continuous.
  - π_X and π_Y map open sets to open sets.
  - Product topology is smallest topology for which π_X and π_Y are continuous.
- **Proposition**: let X, Y, Z topological spaces, then $f : Z \to X \times Y$ (with product topology on $X \times Y$) continuous iff both $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous.
- **Example**: let $f : X \to \mathbb{R}^n$, $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$, $f_i = \pi_i \circ f$, then f is continuous iff all f_i are continuous.
- **Proposition**: let X, Y nonempty topological spaces. Then $X \times Y$ with product topology is Hausdorff iff X and Y are both Hausdorff.

4. Connectedness

4.1. Clopen sets and examples

- **Definition**: let X topological space, then $A \subseteq X$ is **clopen** if A is open and closed.
- **Definition**: X is **connected** if the only clopen sets in X are X and $\emptyset$.
- **Example**:
  - $\mathbb{R}$ with standard topology is connected.
  - $\mathbb{Q}$ with induced topology from $\mathbb{R}$ is not connected (consider $L = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q} - L = \mathbb{Q} \cap (\sqrt{2}, \infty)$).
  - The connected subsets of $\mathbb{R}$ are the intervals.
- **Definition**: $A \subseteq \mathbb{R}$ is an interval iff $\forall x, y, z \in A, x < z < y \Longrightarrow z \in A$.
- **Example**:
  - $X = \{0, 1\}$ with discrete topology is not connected ($\{1\}$ and $\{0\}$ both open so both closed).
  - $X = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$ is connected.
  - $\mathbb{Z}$ with Furstenberg topology is not connected.
- **Theorem (continuity preserves connectedness)**: if $h : X \to Y$ continuous and X connected, then $h(X) \subseteq Y$ is connected.
- **Corollary**: if $h : X \to Y$ is homeomorphism and X is connected then Y is connected.
- **Theorem**: let X topological space. The following are equivalent:
  - X is connected.

- X cannot be written as disjoint union of two non-empty sets.
  - There exists no continuous surjective function from X to a discrete space with more than one point.
- **Example**:
  - $\mathrm{GL}_n(\mathbb{R})$ is not connected (since $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R} - \{0\}$ is continuous and surjective and $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$).
  - $O(n)$ is not connected.
  - $(0, 1)$ is connected (since $\mathbb{R} \cong (0, 1)$ and $\mathbb{R}$ is connected).
  - $X = (0, 1]$ and $Y = (0, 1)$ are not homeomorphic (if they are, then $(0, 1]$ is connected since $(0, 1)$ is).
- **Definition**: let $A = B \cup C$, $B \cap C = \emptyset$, then B and C are **complementary subsets** of A.
- **Remark**: if complementary B and C open in A, then B and C clopen in A. So if $B, C \neq \emptyset$ then A not connected.

4.2. Constructing more connected sets, components, path-connectedness

- **Proposition**: let X topological space, $Z \subseteq X$ connected. If $Z \subseteq Y \subseteq \overline{Z}$ then Y is connected. In particular, with $Y = \overline{Z}$, the closure of a connected set is connected.
- **Proposition**: let $A_i \subseteq X$ connected, $i \in I$, $A_i \cap A_j \neq \emptyset$ and $\cup_{i \in I} A_i = X$. Then X is connected.
- **Theorem**: if X and Y are connected then $X \times Y$ is connected.
- **Example**:
  - $\mathbb{R}^n$ is connected.
  - $B^n = \{x \in \mathbb{R}^n : d_2(0, x) < 1\}$ (B^n is homeomorphic to $\mathbb{R}^n$).
  - $D^n = \{x \in \mathbb{R}^n : d_2(0, x) \leq 1\} = \overline{B^n}$ is connected.
- **Example**:
  - $\forall n \in \mathbb{N}$, S^n is connected.
  - $\forall n \in \mathbb{N}$, T^n is connected.
- **Definition**: **component** of topological space X is maximal connected subset of X.
- **Proposition**: in a topological space X:
  - Every $p \in X$ is in a unique component.
  - If $C_1 \neq C_2$ are components, then $C_1 \cap C_2 = \emptyset$.
  - X is the union of its components.
  - Every component is closed in X.
- **Example**:
  - If X connected, then its only component is itself.
  - If X discrete, then each singleton in τ_X is a component.
  - In $\mathbb{Q}$ with induced standard topology from $\mathbb{R}$, every singleton is a component.
- **Definition**: **path** in topological space X is continuous function $\gamma : [0, 1] \to X$. γ is said to be path from $\gamma(0)$ to $\gamma(1)$.
- **Definition**: X is **path-connected** if for every $p, q \in X$, there is a path from p to q.

- **Proposition**: every path-connected topological space is connected.
- **Example**: let

$$Z = \{(x, \sin(1/x)) \in \mathbb{R}^2 : 0 < x \leq 1\}$$

  Z is path-connected, as a path from $(x_1, \sin(1/x_1))$ to $(x_2, \sin(1/x_2))$ is given by

$$\gamma(t) = \left(x_1 + (x_2 - x_1)t, \sin\left(\frac{1}{x_1 + (x_2 - x_1)t}\right)\right)$$

  So then Z is connected by the above proposition, and since the closure of a connected set is connected, $\overline{Z}$ is connected.

  Every point $(0, y)$, $y \in [-1, 1]$ is a limit point of Z. Assume $\overline{Z}$ is path-connected. Then there is a path $\gamma : [0, 1] \to \overline{Z}$ from $(0, 0)$ to $(1, \sin(1))$. Since $(\pi_X \circ \gamma)(0) = 0$ and $(\pi_X \circ \gamma)(1) = 1$ and $\pi_X \circ \gamma$ is continuous, by the Intermediate Value Theorem, $\exists t_1 \in [0, 1] : (\pi_X \circ \gamma)(t_1) = 2/\pi$. By IVT again, $\exists t_2 \in [0, t_1] : (\pi_X \circ \gamma)(t_2) = \frac{2}{2\pi}$. We obtain a strictly decreasing sequence $(t_n) \subseteq [0, 1]$ where $(\pi_X \circ \gamma)(t_n) = \frac{2}{n\pi}$ which is bounded below by 0, so must converge with limit t^*.

  Now $\pi_Y \circ \gamma$ is continuous, so $\lim_{n \to \infty} (\pi_Y \circ \gamma)(t_n) = (\pi_Y \circ \gamma)(t^*)$. But $(\pi_Y \circ \gamma)(t_n) = \sin\left(\frac{n\pi}{2}\right)$, and as $n \to \infty$, this oscillates between -1 and 1 and does not converge, so contradiction.

5. Compactness

- **Definition**: let X topological space, **cover** of X is collection $(U_i)_{i \in I}$ of subsets of X with

$$\bigcup_{i \in I} U_i = X$$

  If every U_i is open, it is an **open cover**. If $J \subseteq I$, then $(U_i)_{i \in J}$ is a **subcover** of $(U_i)_{i \in I}$ if it is also a cover.
- **Definition**: X is **compact** if every open cover of X admits a finite subcover.
- **Example**:
  - If X is finite then X is compact.
  - $\mathbb{R}$ is not compact.
  - If X infinite with $\tau = \{U \subseteq X : X - U \text{ is finite}\} \cup \{\emptyset\}$, then X is compact.
- **Proposition**: let X have topology with basis $\mathcal{B}$. Then X is compact iff every cover $(B_i)_{i \in I}$ of X, $B_i \in \mathcal{B}$, admits a finite subcover of X.
- **Remark**: to determine compactness of $Y \subseteq X$ with induced topology, consider open covers $Y = \cup_{i \in I} (U_i \cap Y)$ for U_i open in X, which is equivalent to $Y \subseteq \cup_{i \in I} U_i$.
- **Example**: $[0, 1]$ is compact.
- **Proposition**: if $f : X \to Y$ continuous, X compact, then $f(X)$ is compact.
- **Proposition**: if X compact, $A \subseteq X$ closed in X, then A is compact.
- **Theorem**: if X is Hausdorff and $A \subseteq X$ is compact then A is closed.

- **Corollary**: if X compact, Y is Hausdorff, $f : X \to Y$ continuous bijection, then f is homeomorphism.
- **Theorem**: if X, Y compact, then $X \times Y$ is compact.
- **Definition**: $S \subseteq \mathbb{R}^n$ is **bounded** if

$$\exists r \in \mathbb{R} : S \subseteq B(0; r)$$

- **Theorem (Heine-Borel)**: $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.
- **Example**:
  - S^n is compact.
  - T^n is compact.
  - $X = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^3 = 1\}$ is not compact, since $\forall n \in \mathbb{N}$, $(n, 0, (n^2 - 1)^{1/3}) \in X$, so $X \nsubseteq B(n)$, so is unbounded, so not compact by Heine-Borel.
- **Corollary**: let $f : X \to \mathbb{R}$, X compact, f continuous. Then f attains its maximum and minimum.
- **Theorem (Bolzano-Weierstrass)**: an infinite subset A of a compact space X has a limit point in X.

6. Quotient spaces

- **Definition**: let X topological space, $\sim$ equivalence relation on X. Write $X/\sim$ for the set of equivalence classes of $\sim$: for $x \in X$,

$$[x] := \{y \in X : y \sim x\}, \quad X/\sim := \{[x] : x \in X\}$$

  There is a surjective map, the **quotient map**, $\pi : X \to X/\sim$, $\pi(x) = [x]$.
- **Example**: let $X = \mathbb{R}^3$, define equivalence relation

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \Leftrightarrow z_1 = z_2$$

  Then $\pi(a, b, c) = [(a, b, c)] = \{(x, y, z) \in \mathbb{R}^3 : z = c\}$. Elements of $\mathbb{R}^3/\sim$ are horizontal planes.
- **Definition**: let X topological space, $\sim$ equivalence relation on X. Then $X/\sim$ is given **quotient topology** defined by

$$U \subseteq X/\sim \text{ open} \Longleftrightarrow \pi^{-1}(U) \text{ open in } X$$

- **Proposition**: quotient topology defines a topology on $X/\sim$.
- **Proposition**: quotient topology on $X/\sim$ is largest such that π is continuous.
- **Proposition**: let X topological space with equivalence relation $\sim$, Y topological space. Then $f : X/\sim \to Y$ continuous iff $f \circ \pi : X \to Y$ is continuous.
- **Example**: in $\mathbb{R}$, let $x \sim y \Longleftrightarrow x - y \in \mathbb{Z}$. Define $\exp : \mathbb{R} \to S^1 \subseteq \mathbb{C}$, $\exp(t) = e^{2\pi i t}$) and $\overline{\exp} : \mathbb{R}/\sim \to S^1$, $\overline{\exp}([t]) = \exp(t)$. Then

$$[s] = [t] \Longleftrightarrow s - t = k \in \mathbb{Z} \Longleftrightarrow \overline{\exp}(s) = e^{2\pi i k} e^{2\pi i t} = e^{2\pi i t} = \overline{\exp}(t)$$

  Hence $\overline{\exp}$ is well-defined and injective, and is surjective since exp is. Also, $\overline{\exp}$ is continuous since $\exp = \overline{\exp} \circ \pi$ is. $\mathbb{R}^2$ is a metric space and so is Hausdorff, so $S^1 \subset \mathbb{R}^2$ with the induced topology is Hausdorff. Now e.g. $\pi([-10, 10]) = \mathbb{R}/\sim$,

$[-10, 10]$ is compact and π continuous so $\mathbb{R}/\sim$ is compact. Since $\overline{\exp}$ is a continuous bijection, these three properties imply $\overline{\exp}$ is a homeomorphism. Hence $\mathbb{R}/\sim\, \cong S^1$.

- **Definition**: let $A \subseteq X$, define $x \sim y \iff x = y$ or $x, y \in A$. Then define $X/A := X/\sim$.
- **Example**: $S^n \cong D^n/S^{n-1}$. Any point in D^n can be written as $t \cdot \varphi$, $t \in [0, 1]$, $\varphi \in S^{n-1}$. Define

$$f : D^n \to S^n, \quad f(t \cdot \varphi) := (\cos(\pi t), \varphi \sin(\pi t)) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$$
$$\Longrightarrow f(0 \cdot \varphi) = (1, \mathbf{0}),\, f(1/2 \cdot \varphi) = (0, \varphi),\, f(1 \cdot \varphi) = (-1, 0)$$

Define $\overline{f} : D^n/S^{n-1} \to S^n$, $\overline{f}([t \cdot \varphi]) = f(t \cdot \varphi)$. If $t_1 \cdot \varphi_1 \neq t_2 \cdot \varphi_2$, then

$$\begin{aligned}
[t_1 \cdot \varphi_1] = [t_2 \cdot \varphi_2] \iff t_1 \cdot \varphi_1, t_2 \cdot \varphi_2 \in S^{n-1} &\iff t_1 = t_2 = 1 \\
&\iff f(t_1 \cdot \varphi_1) = (-1, \mathbf{0}) = f(t_2 \cdot \varphi_2) \\
&\iff \overline{f}([t_1 \cdot \varphi_1]) = \overline{f}([t_2 \cdot \varphi_2])
\end{aligned}$$

f is surjective, so $\overline{f}$ is also. Now $\overline{f} \circ \pi = f$ which is continuous, so by above proposition, $\overline{f}$ is continuous. $S^n \subset \mathbb{R}^{n+1}$ is Hausdorff, $D^n \subset \mathbb{R}^n$ is closed and bounded so is compact by Heine-Borel, and so D^n/S^{n-1} is compact (since π continuous). Also, f is a continuous bijection. These imply that $\overline{f}$ is homeomorphism.

7. Topological groups

7.1. Examples

- **Definition**: a **topological group** G is Hausdorff space which is also a group such that

$$\bullet : G \times G \to G,\ \bullet(g, h) = gh \quad \text{and} \quad i : G \to G,\ i(g) = g^{-1}$$

are continuous.
- **Example**:
  - $\mathbb{R}^n$ with addition is topological group.
  - $\mathrm{GL}_n(\mathbb{R})$ with multiplication and its subgroups $O(n)$ and $\mathrm{SO}(n)$ are topological groups (each entry in AB is sum of products of entries of A and B, so matrix multiplication is continuous, matrix inversion also continuous).
- **Proposition**:
  - Any group with discrete topology is topological group.
  - Any subgroup of topological group is also topological group.
- **Example**:
  - $\mathbb{C} - \{0\}$ with multiplication has topological subgroup $S^1 \subset \mathbb{C} - \{0\}$.
  - Define **quaternions** as vector space $\mathbb{H} := \langle 1, i, j, k \rangle$, with topology taken from $\mathbb{R}^4$. $\mathbb{H} - \{0\}$ is a multiplicative group with S^3 a topological subgroup. For $q = a + bi + cj + dk \in \mathbb{H}$, $a, b, c, d \in \mathbb{R}$, we have $ij := k$, $jk := i$, $ki := j$, $ji := -k$, $kj := -i$, $ik := -j$. For $q \neq 0$,

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

- Note however that S^2 is not a topological group.
- **Definition**: for topological group G, $x \in G$, define **left translation by $\boldsymbol{x}$** as

$$L_x : G \to G, \quad L_x(g) := xg$$

Similarly, **right translation by $\boldsymbol{x}$** is

$$R_x : G \to G, \quad R_x(g) := gx$$

- **Proposition**: L_x has inverse $(L_x)^{-1} = L_{x^{-1}}$ and is homeomorphism. Similarly for R_x.
- **Notation**: a specified inclusion $G \overset{x}{\hookrightarrow} G \times G$ is the map $G \to \{x\} \times G$ composed with the inclusion map $\{x\} \times G \to G \times G$. (similarly for $G \times \{x\}$).
- **Proposition**: let G topological group, K the component containing identity of G. Then K is normal subgroup of G.
- **Example**: $O(n)$ is not connected, but $\mathrm{SO}(n)$ is connected and contains I_n, so is a normal subgroup of $O(n)$

7.2. Actions, orbits, orbit spaces

- **Definition**: **action** of group G on topological space X is map $\bullet : G \times X \to X$ such that $\forall g, h \in G$, $\forall x \in X$,
  - $(hg) \bullet x = h \bullet (g \bullet x)$.
  - $1 \bullet x = x$.
  - $g : X \to X$ defined by $g(x) = g \bullet x$ is continous. Note: g has inverse map g^{-1} which is also continuous, so both are homeomorphisms.
- **Definition**: **action** of topological group G on topological space X is continuous map $\bullet : G \times X \to X$ such that $\forall g, h \in G$, $\forall x \in X$,
  - $(hg) \bullet x = h \bullet (g \bullet x)$.
  - $1 \bullet x = x$.
- **Remark**: for the above definition, the condition $g(x) = g \bullet x$ being continuous isn't required since g is the comoposition of continuous maps:

$$X \overset{g}{\hookrightarrow} G \times X \overset{\bullet}{\longrightarrow} X, \quad x \to (g, x) \to g \bullet x$$

- **Example**:
  - Trivial action: $(g, x) \mapsto g \bullet x = x$, so $\bullet = \pi_X$.
  - Let $G = \mathrm{GL}_n(\mathbb{R})$, $X = \mathbb{R}^n$, let the action be matrix multiplication: $(A, \boldsymbol{x}) \to A \bullet \boldsymbol{x} = A\boldsymbol{x}$. This induces an action of subgroups $O(n)$ or $\mathrm{SO}(n)$ on $X = \mathbb{R}^n$.
  - Let H subgroup of topological group G, **left translation action** of H on G is $\bullet : H \times G \to G$, $h \bullet g = hg$. Equivalently, $\varphi(h) = L_h$.
  - Let N normal subgroup of topological group G, **conjugation action** of G on N is $\bullet : G \times N \to N$, $g \bullet n = gng^{-1}$.
- **Definition**: let G act on topological space X, define equivalence relation $\sim$ on X by

$$x \sim y \iff \exists g \in G : g(x) \coloneqq g \bullet x = y$$

An equivalence class for this relation is an **orbit**, denoted Gx. **Orbit space**, X/G, is quotient space $X/\sim$. Action is **transitive** if X/G is a singleton.

- **Example**:
  - If G acts trivially, every orbit is singleton and $X/G = X$.
  - $\mathbb{R}^n/\mathrm{GL}_n(\mathbb{R})$ contains two points and has neither discrete nor indiscrete topology.
  - Action of $O(n)$ on S^{n-1} is transitive for $n \in \mathbb{N}$. Action of $\mathrm{SO}(n)$ on S^{n-1} is transitive for $n \geq 2$.
- **Lemma**: if connected topological group G acts on topological space X, then the orbits are connected.
- **Theorem**: let G connected topological group act on topological space X. If X/G is connected, then X is connected.
- **Notation**: define specified inclusion $i_1 : M_n(\mathbb{R}) \overset{1}{\hookrightarrow} M_{n+1}(\mathbb{R})$ by $A \to \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$. So $M_n(\mathbb{R})$ can be regarded as subspace of $M_{n+1}(\mathbb{R})$.
- **Proposition**:
  - Using the inclusion $\overset{1}{\hookrightarrow}$, $\mathrm{SO}(n)$ is subgroup of $\mathrm{SO}(n+1)$.
  - Viewing these as topological groups, if subgroup $\mathrm{SO}(n)$ acts on $\mathrm{SO}(n+1)$, orbit space is $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$.
- **Corollary**: the topological group $\mathrm{SO}(n)$ is connected for $n \in \mathbb{N}$.