# 1. Metric spaces

## 1.1. Metrics

- **Metric space**: $(X, d)$, $X$ is set, $d : X \times X \to [0, \infty)$ is **metric** satisfying:
  - $d(x, y) = 0 \iff x = y$
  - **Symmetry**: $d(x, y) = d(y, x)$
  - **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- Examples of metrics:
  - $p$-adic metric:

$$d_p(x, y) = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

  - Extension of the $p$-adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

  - Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

  - Discrete metric:

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Open ball of radius $r$ around $x$**:

$$B(x; r) = \{y \in X : d(x, y) < r\}$$

- **Closed ball of radius $r$ around $x$**:

$$D(x; r) = \{y \in X : d(x, y) \leq r\}$$

## 1.2. Open and closed sets

- $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

- $A \subseteq X$ is **closed** if $X - A$ is open.
- Sets can be neither closed nor open, or both.
- Any singleton $\{x\} \in \mathbb{R}$ is closed and not open.
- Let $X$ be metric space, $x \in N \subseteq X$. $N$ is **neighbourhood** of $x$ if

$$\exists \text{ open } V \subseteq X : x \in V \subseteq N$$

- **Corollary**: let $x \in X$, then $N \subseteq X$ neighbourhood of $x$ iff $\exists \varepsilon > 0 : x \in B(x; \varepsilon) \subseteq N$.
- **Proposition**: open balls are open, closed balls are closed.
- **Lemma**: let $(X, d)$ metric space.
  - $X$ and $\emptyset$ are both open and closed.
  - Arbitrary unions of open sets are open.
  - Finite intersections of open sets are open.

- Finite unions of closed sets are closed.
- Arbitrary intersections of closed sets are closed.

### 1.3. Continuity

- **Sequence** in $X$: $a : \mathbb{N} \to X$, written $(a_n)_{n \in \mathbb{N}}$.
- $(a_n)$ converges to $a$ if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, d(a, a_n) < \varepsilon$$

- **Proposition**: let $X, Y$ metric spaces, $a \in X$, $f : X \to Y$. The following are equivalent
  - $\forall \varepsilon > 0, \exists \delta > 0 : d_X(a, x) < \delta \Longrightarrow d_Y(f(a), f(x)) < \varepsilon$.
  - For every sequence $(a_n)$ in $X$ with $a_n \to a$, $f(a_n) \to f(a)$.
  - For every open $U \subseteq Y$ with $f(a) \in U$, $f^{-1}(U)$ is a neighbourhood of $a$.

If $f$ satisfies these, it is **continuous at $a$**.

- $f$ **continuous** if continuous at every $a \in X$.
- **Proposition**: $f : X \to Y$ continuous iff $f^{-1}(U)$ open for every open $U \subseteq Y$.

## 2. Topological spaces

### 2.1. Topologies

- **Power set** of $X$: $\mathcal{P}(X) := \{A : A \subseteq X\}$.
- **Topology** on set $X$ is $\tau \subseteq \mathcal{P}(X)$ with:
  - $\emptyset \in \tau$, $X \in \tau$.
  - If $\forall i \in I, U_i \in \tau$, then

$$\bigcup_{i \in I} U_i \in \tau$$

  - $U_1, U_2 \in \tau \Longrightarrow U_1 \cap U_2 \in \tau$ (this is equivalent to $U_1, ..., U_n \in \tau \Longrightarrow \cap_{i \in [n]} U_i \in \tau$).
- $(X, \tau)$ is **topological space**. Elements of $\tau$ are **open** subsets of $X$.
- $A \subseteq X$ **closed** if $X - A$ is open.
- Let $X$ be a set. Then $\tau = \mathcal{P}(X)$ is the **discrete topology** on $X$.
- $\tau = \{\emptyset, X\}$ is the **indiscrete topology** on $X$.
- **Examples**:
  - For metric space $(M, d)$, find the open sets with respect to metric $d$. Let $\tau_d \subseteq \mathcal{P}(M)$ exactly contain these open sets. Then $(M, \tau_d)$ is a topological space. The metric $d$ **induces** the topology $\tau_d$.
  - Let $X = \mathbb{N}_0$ and $\tau = \{\emptyset\} \cup \{U \subseteq X : X - U \text{ is finite}\}$.
- **Proposition**: for topological space $X$:
  - $X$ and $\emptyset$ are closed
  - Arbitrary intersections of closed sets are closed
  - Finite unions of closed sets are closed
- **Proposition**: for topological space $(X, \tau)$ and $A \subseteq X$, the **induced (subspace) topology on $A$**

$$\tau_A = \{A \cap U : U \in \tau\}$$

is a topology on $A$.

- **Example**: let $X = \mathbb{R}$ with standard topology induced by metric $d(x, y) = |x - y|$. Let $A = [1, 5]$. Then $[1, 3) = A \cap (0, 3)$ and $[1, 5] = A \cap (0, 6)$ are open in $A$.
- **Example**: consider $\mathbb{R}$ with standard topology $\tau$. Then
  - $\tau_{\mathbb{Z}}$ is the discrete topology on $\mathbb{Z}$.
  - $\tau_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.
- **Proposition**: the metrics $d_p$ for $p \in [1, \infty)$ and $d_\infty$ all induce the same topology on $\mathbb{R}^n$.
- **Definition**: $(X, \tau)$ is **Hausdorff** if

$$\forall x \neq y \in X, \exists U, V \in \tau : U \cap V = \emptyset \wedge x \in U, y \in V$$

- **Lemma**: any metric space $(M, d)$ is Hausdorff.
- **Example**: let $|X| \geq 2$ with the indiscrete topology. Then $X$ is not Hausdorff, since $\tau = \{X, \emptyset\}$ and if $x \neq y \in X$, the only open set containing $x$ is $X$ (same for $y$). But $X \cap X = X \neq \emptyset$.
- **Furstenberg's topology on $\mathbb{Z}$**: define $U \subseteq \mathbb{Z}$ to be open if

$$\forall a \in U, \exists 0 \neq d \in \mathbb{Z} : a + d\mathbb{Z} =: \{a + dn : n \in \mathbb{Z}\} \subseteq U$$

- Furstenberg's topology is Hausdorff.

## 2.2. Continuity

- **Definition**: let $X, Y$ topological spaces.
  - $f : X \to Y$ is **continuous** if

$$\forall V \text{ open in } Y, f^{-1}(V) \text{ open in } X$$

  - $f$ is **continuous at $a \in X$** if

$$\forall V \text{ open in } Y, f(a) \in V, \exists U \text{ open in } X : a \in U \subseteq f^{-1}(V)$$

- **Lemma**: $f : X \to Y$ continuous iff $f$ continuous at every $a \in X$. (Key idea for proof: $\cup_{a \in f^{-1}(V)} U_a \subseteq f^{-1}(V) = \cup_{a \in f^{-1}(V)} \{a\} \subseteq \cup_{a \in f^{-1}(V)} U_a$)
- **Example**: inclusion $i : (A, \tau_A) \to (X, \tau_X)$, $A \subseteq X$, is always continuous.
- **Lemma**: a composition of continuous functions is continuous.
- **Lemma**: let $f : X \to Y$ be function between topological spaces. Then $f$ is continuous iff

$$\forall A \text{ closed in } Y, \quad f^{-1}(A) \text{ closed in } X$$

- **Remark**: we can use continuous functions decide that sets are open or closed.
- **Definition**: **$n$-sphere** is

$$S^n := \left\{ (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

- **Example**: in the standard topology, the $n$-sphere is a closed subset of $\mathbb{R}^{n+1}$. (Consider the preimage of $\{1\}$ which is closed in $\mathbb{R}$).
- Can consider set of square matrices $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and give it the standard topology.

- **Example**:
  - Note

$$\det(A) = \sum_{\sigma \in \text{ sym}(n)} \left( \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

  is a polynomial in the entries of $A$ so is continuous function from $M_n(\mathbb{R})$ to $\mathbb{R}$.
  - $\text{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} = \det^{-1}(\mathbb{R} - \{0\})$ is open.
  - $\text{SL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} = \det^{-1}(\{1\})$ is closed.
  - $O(n) = \left\{A \in M_n(\mathbb{R}) : AA^T = I\right\}$ is closed - consider $f_{i,j}(A) = \left(AA^T\right)_{i,j}$ then

$$O(n) = \bigcap_{1 \leq i,j \leq n} \left(f_{i,j}\right)^{-1}\left(\left\{\delta_{i,j}\right\}\right)$$

  - $\text{SO}(n) = O(n) \cap \text{SL}_n(\mathbb{R})$ is closed.
- **Definition**: for $X, Y$ topological spaces, $h : X \to Y$ is **homeomorphism** if $h$ is bijective, continuous and $h^{-1}$ is continuous. $X$ and $Y$ are **homeomorphic**. A homeomorphism gives bijection between $\tau_X$ and $\tau_Y$ which satisfies

$$h(A \cap B) = h(A) \cap h(B), \quad h(A \cup B) = h(A) \cup h(B)$$

- **Example**: in standard topology, $(0,1)$ is homeomorphic to $\mathbb{R}$. (Consider $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$, $f = \tan$, $g : (0,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $g(x) = \pi\left(x - \frac{1}{2}\right)$ and $f \circ g$).
- **Example**: $\mathbb{R}$ with standard topology $\tau_{\text{st}}$ is not homoeomorphic to $\mathbb{R}$ with the discrete topology $\tau_d$. (Consider $h^{-1}(\{a\}) = \left\{h^{-1}(a)\right\}$, $\{a\} \in \tau_{\text{st}}$ but $\left\{h^{-1}(a)\right\} \notin \tau_{\text{st}}$).
- **Example**: let $X = \mathbb{R} \cup \left\{\overline{0}\right\}$. Define $f_0 : \mathbb{R} \to X$, $f_0(a) = a$ and $f_{\overline{0}} : \mathbb{R} \to X$, $f_{\overline{0}}(a) = a$ for $a \neq 0$, $f_{\overline{0}}(0) = \overline{0}$. Topology on $X$ has $A \subseteq X$ open iff $f_0^{-1}(A)$ and $f_{\overline{0}}^{-1}(A)$ open. Every point in $X$ lies in open set: for $a \notin \left\{0, \overline{0}\right\}$, $a \in \left(a - \frac{|a|}{2}, a + \frac{|a|}{2}\right)$ and both pre-images of this are same open interval, for 0, set $U_0 = (-1,0) \cup \{0\} \cup (0,1) \subseteq X$ then $f_0^{-1}(U_0) = (-1,1)$ and $f_{\overline{0}}^{-1}(U_0) = (-1,0) \cup (0,1)$ are both open. For $\overline{0}$, set $U_{\overline{0}} = (-1,0) \cup \left\{\overline{0}\right\} \cup (0,1) \subseteq X$, then $f_{\overline{0}}^{-1}(U_{\overline{0}}) = (-1,1)$ and $f_0^{-1}(U_{\overline{0}}) = (-1,0) \cup (0,1)$ are both open. So $U_0$ and $U_{\overline{0}}$ both open in $X$. $X$ is not Hausdorff since any open sets containing 0 and $\overline{0}$ must contain "open intervals" such as $U_0$ and $U_{\overline{0}}$.
- **Example (Furstenberg's proof of infinitude of primes)**: since $a + d\mathbb{Z}$ is infinite, any nonempty finite set is not open, so any set with finite complement is not closed. For fixed $d$, sets $d\mathbb{Z}$, $1 + d\mathbb{Z}, ..., (d-1) + d\mathbb{Z}$ partition $\mathbb{Z}$. So the complement of each is the union of the rest, so each is open and closed. Every $n \in \mathbb{Z} - \{-1, 1\}$ is prime or product of primes, so $\mathbb{Z} - \{-1,1\} = \cup_{p \text{ prime}} p\mathbb{Z}$, but finite unions of closed sets are closed, and since $\mathbb{Z} - \{-1,1\}$ has finite complement, the union must be infinite.

# 3. Limits, bases and products

## 3.1. Limit points, interiors and closures

- **Definition**: for topological space $X$, $x \in X$, $A \subseteq X$:

- **Open neighbourhood of $x$** is open set $N$, $x \in N$.
- $x \in X$ is **limit point** of $A$ if every open neighbourhood $N$ of $x$ satisfies

$$(N - \{x\}) \cap A \neq \emptyset$$

- **Corollary**: $x$ is not limit point of $A$ iff exists neighbourhood $N$ of $x$ with

$$A \cap N = \begin{cases} \{x\} & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

- **Example**: let $X = \mathbb{R}$ with standard topology.
  - $0 \in X$, then $(-1/2, 1/2)$ is open neighbourhood of 0.
  - If $U \subseteq X$ open, $U$ is open neighbourhood for any $x \in U$.
  - Let $A = \left\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\right\}$, then only limit point in $A$ is 0.
- **Definition**: let $A \subseteq X$.
  - **Interior** of $A$ is largest open set contained in $A$:

$$A^\circ = \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U$$

  - **Closure** of $A$ is smallest closed set containing $A$:

$$\overline{A} = \bigcap_{\substack{F \text{ closed} \\ A \subseteq F}}$$

  If $A^\circ = X$, $A$ is **dense** in $X$.
- **Lemma**:
  - $\overline{X - A} = X - A^\circ$
  - $\overline{A} = X - (X - A)^\circ$
- **Example**:
  - Let $\mathbb{Q} \subset \mathbb{R}$ with standard topology. Then $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$ (since every nonempty open set in $\mathbb{R}$ contains rational and irrational numbers).
- **Lemma**: $\overline{A} = A \cup L$ where $L$ is the set of limit points of $A$.
- **Dirichlet prime number theorem**: let $a, d$ coprime, the set $a + d\mathbb{Z}$ contains infinitely many primes.
- **Example**: let $A$ be the set of primes in $\mathbb{Z}$ with the Furstenberg topology. By the above lemma, we only need to find the limit points in $\mathbb{Z} - A$ to find $\overline{A}$. $10\mathbb{Z}$ is an open neighbourhood of 0 for 0 inside $\mathbb{Z} - A$. For $a \notin \{-1, 0, 1\}$, $a + 10a\mathbb{Z}$ is an open neighbourhood of $a$. These sets have no primes so the corresponding points are not limit points of $A$. For $\pm 1$, any open neighbourhood of 1 contains a set $\pm 1 + d\mathbb{Z}$ for some $d \neq 0$, but by the Dirichlet prime number theorem, this set contains at least one prime. So $\overline{A} = A \cup \{\pm 1\}$.
- **Lemma**:
  - Let $A \subseteq M$ for metric space $M$. If $x$ is limit point of $A$ then exists sequence $x_n$ in $A$ such that $\lim_{n \to \infty} x_n = x$.
  - If $x \in M - A$ and exists sequence $x_n$ in $A$ with $\lim_{n \to \infty} x_n = x$ then $x$ is limit point of $A$.

## 3.2. Bases

- **Definition**: a **basis** for topology $\tau$ on $X$ is collection $\mathcal{B} \subseteq \tau$ such that

$$\forall U \in \tau, U = \bigcup_{b \in \mathcal{B}} b$$

  (every open $U$ is a union of sets in $B$).
- **Example**:
  - For metric space $(M, d)$, $\mathcal{B} = \{B(x; r) : x \in M, r > 0\}$ is basis for the induced topology. (Since if $U$ open, $U = \cup_{u \in U} \{u\} \subseteq \cup_{u \in U} B(u, r_u) \subseteq U$.)
  - In $\mathbb{R}^n$ with standard topology, $\mathcal{B} = \{B(q; 1/m) : q \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a **countable** basis. (Find $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{r}{2}$ and $q \in \mathbb{Q}^n$ such that $q \in B\left(p; \frac{1}{m}\right)$, then $B\left(q; \frac{1}{m}\right) \subseteq B(p; r) \subseteq U$ using the triangle inequality).
- **Theorem**: let $f : X \to Y$ be map between topological spaces. The following are equivalent:
  - $f$ is continuous.
  - If $\mathcal{B}$ is basis for topology $\tau$ on $Y$ then $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.
  - $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$.
  - $\forall V \subseteq Y, \overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$.
  - $f^{-1}(C)$ closed for any closed set $C \subseteq Y$.
- **Theorem**: let $X$ be a set and collection $\mathcal{B} \subseteq \mathcal{P}(X)$ be such that:
  - $\forall x \in X, \exists B \in \mathcal{B} : x \in B$
  - If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$.

  Then there is unique topology $\tau_{\mathcal{B}}$ on $X$ for which $\mathcal{B}$ is a basis. We say $\mathcal{B}$ **generates** $\tau_{\mathcal{B}}$.

## 3.3. Product topologies

- **Definition**: **Cartesian product** of topological spaces $X, Y$ is $X \times Y \coloneqq \{(x, y) : x \in X, y \in Y\}$. We give it the **product topology** which is generated by $\mathcal{B}_{X \times Y} \coloneqq \{U \times V : U \in \tau_X, V \in \tau_Y\}$.
- **Example**:
  - Let $X = Y = \mathbb{R}$, then product topology is same as standard topology on $\mathbb{R}^2$.
  - Let $X = Y = S^1$, then $X \times Y = T^2 = S^1 \times S^1$ is the 2-torus.
- **Definition**: if $\tau_1 \subseteq \tau_2$, then $\tau_1$ is **smaller** than $\tau_2$ ($\tau_2$ is **larger** than $\tau_1$).
- **Definition**: for topological spaces $X, Y$, **projection maps** $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are

$$\pi_X(x, y) = x, \quad \pi_Y(x, y) = y$$

- **Proposition**: for $X \times Y$ with product topology,
  - $\pi_X$ and $\pi_Y$ are continuous.
  - $\pi_X$ and $\pi_Y$ map open sets to open sets.
  - Product topology is smallest topology for which $\pi_X$ and $\pi_Y$ are continuous.
- **Proposition**: let $X, Y, Z$ topological spaces, then $f : Z \to X \times Y$ (with product topology on $X \times Y$) continuous iff both $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous.

- **Exercise (todo)**: prove above proposition.
- **Example**: let $f : X \to \mathbb{R}^n$, $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$, $f_i = \pi_i \circ f$, then $f$ is continuous iff all $f_i$ are continuous.
- **Proposition**: let $X, Y$ nonempty topological spaces. Then $X \times Y$ is Hausdorff iff $X$ and $Y$ are both Hausdorff.

# 4. Connectedness

## 4.1. Clopen sets and examples

- **Definition**: let $X$ topological space, then $A \subseteq X$ is **clopen** if $A$ is open and closed.
- **Definition**: $X$ is **connected** if the only clopen sets in $X$ are $X$ and $\emptyset$.
- **Example**:
  - $\mathbb{R}$ with standard topology is connected.
  - $\mathbb{Q}$ with induced topology from $\mathbb{R}$ is not connected (consider $L = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q} - L = \mathbb{Q} \cap (\sqrt{2}, \infty)$).
  - The connected subsets of $\mathbb{R}$ are the intervals.
- $A \subseteq \mathbb{R}$ is an interval iff $\forall x, y \in A, x < z < y \Longrightarrow z \in A$.
- **Example**:
  - $X = \{0, 1\}$ with discrete topology is not connected ($\{1\}$ and $\{0\}$ both open so both closed).
  - $X = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$ is connected.
  - $\mathbb{Z}$ with Furstenberg topology is not connected.
- **Theorem (continuity preserves connectedness)**: if $h : X \to Y$ continuous and $X$ connected, then $h(X) \subseteq Y$ is connected.
- **Corollary**: if $h : X \to Y$ is homeomorphism and $X$ is connected then $Y$ is connected.
- **Theorem**: let $X$ topological space. The following are equivalent:
  - $X$ is connected.
  - $X$ cannot be written as disjoint union of two non-empty sets.
  - There exists no continuous surjective function from $X$ to a discrete space with more than one point.
- **Example**:
  - $\mathrm{GL}_n(\mathbb{R})$ is not connected (since $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R} - \{0\}$ is continuous and surjective and $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$).
  - $O(n)$ is not connected.
  - $(0, 1)$ is connected (since $\mathbb{R} \cong (0, 1)$ and $\mathbb{R}$ is connected).
  - $X = (0, 1]$ and $Y = (0, 1)$ are not homeomorphic (if they are, then $(0, 1]$ is connected since $(0, 1)$ is).
- **Definition**: let $A = B \cup C$, $B \cap C = \emptyset$, then $B$ and $C$ are **complementary subsets** of $A$.
- **Remark**: if $B$ and $C$ are open in $A$, then $B$ and $C$ are clopen in $A$. So if $B, C \neq \emptyset$ then $A$ is not connected.

## 4.2. Constructing more connected sets, components, path-connectedness

- **Proposition**: let $X$ topological space, $Z \subseteq X$ connected. If $Z \subseteq Y \subseteq \overline{Z}$ then $Y$ is connected. In particular, with $Y = \overline{Z}$, the closure of a connected set is connected.
- **Proposition**: let $A_i \subseteq X$ connected, $i \in I$, $A_i \cap A_j \neq \emptyset$ and $\cup_{i \in I} A_i = X$. Then $X$ is connected.
- **Theorem**: if $X$ and $Y$ are connected then $X \times Y$ is connected.
- **Example**:
  - $\mathbb{R}^n$ is connected.
  - $B^n = \{x \in \mathbb{R}^n : d_2(0, x) < 1\}$ ($B^n$ is homeomorphic to $\mathbb{R}^n$).
  - $D^n = \{x \in \mathbb{R}^n : d_2(0, x) \leq 1\} = \overline{B^n}$ is connected.
- **Example**:
  - $\forall n \geq 1$, $S^n$ is connected.
  - $\forall n \geq 1$, $T^n := \left(S^1\right)^n$ is connected.
- **Definition**: **component** of topological space $X$ is maximal connected subset of $X$.
- **Proposition**: in a topological space $X$:
  - Every $p \in X$ is in a unique component.
  - If $C_1 \neq C_2$ are components, then $C_1 \cap C_2 = \emptyset$.
  - $X$ is the union of its components.
  - Every component is closed in $X$.
- **Example**:
  - If $X$ connected, then its only component is itself.
  - If $X$ discrete, then each singleton in $\tau_X$ is a component.
  - In $\mathbb{Q}$ with induced standard topology from $\mathbb{R}$, every singleton is a component.
- **Definition**: **path** in topological space $X$ is continuous function $\gamma : [0, 1] \to X$. $\gamma$ is said to be path from $\gamma(0)$ to $\gamma(1)$.
- **Definition**: $X$ is **path-connected** if for every $p, q \in X$, there is a path from $p$ to $q$.
- **Proposition**: every path-connected topological space is connected.
- **Example**: let

  $$Z = \left\{(x, \sin(1/x)) \in \mathbb{R}^2 : 0 < x \leq 1\right\}$$

  $Z$ is path-connected, as a path from $(x_1, \sin(1/x_1))$ to $(x_2, \sin(1/x_2))$ is given by

  $$\gamma(t) = \left(x_1 + (x_2 - x_1)t, \sin\left(\frac{1}{x_1 + (x_2 - x_1)t}\right)\right)$$

  So then $Z$ is connected by the above proposition, and since the closure of a connected set is connected, $\overline{Z}$ is connected.

  Every point $(0, y)$, $y \in [-1, 1]$ is a limit point of $Z$. Assume $\overline{Z}$ is path-connected. Then there is a path $\gamma : [0, 1] \to \overline{Z}$ from $(0, 0)$ to $(1, \sin(1))$. Since $(\pi_X \circ \gamma)(0) = 0$ and $(\pi_X \circ \gamma)(1) = 1$ and $\pi_X \circ \gamma$ is continuous, by the Intermediate Value Theorem, $\exists t_1 \in [0, 1] : (\pi_X \circ \gamma)(t_1) = 2/\pi$. By IVT again, $\exists t_2 \in [0, t_1] : (\pi_X \circ \gamma)(t_2) = \frac{2}{2\pi}$. We obtain a strictly decreasing sequence $(t_n) \subseteq [0,1]$ where $(\pi_X \circ \gamma)(t_n) = \frac{2}{n\pi}$ which is bounded below by 0, so must converge with limit $t^*$.

Now $\pi_Y \circ \gamma$ is continuous, so $\lim_{n\to\infty} (\pi_Y \circ \gamma)(t_n) = (\pi_Y \circ \gamma)(t^*)$. But $(\pi_Y \circ \gamma)(t_n) = \sin\left(\frac{n\pi}{2}\right)$, and as $n \to \infty$, this oscillates between $-1$ and 1 and does not converge, so contradiction.

# 5. Compactness

- **Definition**: let $X$ topological space, **cover** is collection $(U_i)_{i\in I}$ of subsets of $X$ with

$$\bigcup_{i\in I} U_i = X$$

  If every $U_i$ is open, it is an **open cover**. If $J \subseteq I$, then $(U_i)_{i\in J}$ is a **subcover** of $(U_i)_{i\in I}$ if it is also a cover.
- **Definition**: $X$ is **compact** if every open cover admits a finite subcover.
- **Example**:
  - If $X$ is finite then $X$ is compact.
  - $\mathbb{R}$ is not compact.
  - If $X$ infinite with $\tau = \{U \subseteq X : X - U \text{ is finite}\} \cup \emptyset$, then $X$ is compact.
- **Proposition**: let $X$ have topology with basis $\mathcal{B}$. Then $X$ is compact iff every cover $(B_i)_{i\in I}$ of $X$, $B_i \in \mathcal{B}$, admits a finite subcover of $X$.
- **Remark**: to determine compactness of $Y \subseteq X$, consider open covers $Y = \cup_{i\in I} (U_i \cap y)$ for $U_i$ open in $X$, which is equivalent to $Y \subseteq \cup_{i\in I} U_i$.
- **Example**: $[0,1]$ is compact.
- **Proposition**: if $f : X \to Y$ continuous, $X$ compact, then $f(X)$ is compact.
- **Proposition**: if $X$ compact, $A \subseteq X$ closed in $X$, then $A$ is compact.
- **Theorem**: if $X$ is Hausdorff and $A \subseteq X$ is compact then $A$ is closed.