1. Introduction

Definition. **Epimorphism** is surjective homomorphism.

Definition. **Embedding** or **monomorphism** is injective homomorphism.

1.1. Cubic equations over $\mathbb{C}$

- For a polynomial equation, a **solution by radicals** is a formula for solutions using only addition, subtraction, multiplication, division and radicals $\sqrt[m]{\cdot}$ for $m \in \mathbb{N}$.
- For general cubic equation $x^3 + a_2x^2 + a_1x + a_0 = 0$:
  - **Tschirnhaus transformation** is substitution $t = x + \frac{a_2}{3}$, giving

$$t^3 + pt + q = 0, \quad p := \frac{-a_2^2 + 3a_1}{3}, \quad q := \frac{2a_2^3 - 9a_1a_2 + 27a_0}{27}$$

  This is a **reduced** (or **depressed**) cubic equation.

- When $t = u + v$, $t^3 - (3uv)t - (u^3 + v^3) = 0$ which is in the reduced cubic form with $p = -3uv, q = -(u^3 + v^3)$.
- We have

$$(y - u^3)(y - v^3) = y^2 - (u^3 + v^3)y + u^3v^3 = y^2 + qy - \frac{p^3}{27} = 0$$

so $u^3, v^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$.
- So a solution to $t^3 + pt + q = 0$ is

$$t = u + v = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

The other solutions are $\omega u + \omega^2 v$ and $\omega^2 u + \omega v$ where $\omega = e^{2\pi i/3}$ is the 3rd root of unity. This is because u and v each have three solutions independently to $u^3, v^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$, but also $uv = -\frac{p}{3}$.

Remark. The above method doesn't work for fields of characteristic 2 or 3 since the formulas involve division by 2 or 3 (which is dividing by zero in these respective fields).

1.2. Quartic equations over $\mathbb{C}$

- For general quartic equation $x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$:
  - Substitution $t = x + \frac{a_3}{4}$ gives **reduced** quartic equation

$$t^4 + pt^2 + qt + r = 0$$

  - We then manipulate the polynomial so that it is the sum or difference of two squares and use $a^2 + b^2 = (a + ib)(a - ib)$ or $a^2 - b^2 = (a + b)(a - b)$:

$$\left(t^2 + w\right)^2 + (p - 2w)t^2 + qt + \left(r - w^2\right) = 0$$

  - $(p - 2w)t^2 + qt + \left(r - w^2\right) = 0$ is a square iff its discriminant is zero:

$$q^2 - 4(p-2w)\big(r-w^2\big) = 0 \Longleftrightarrow w^3 - \frac{1}{2}pw^2 - rw + \frac{1}{8}\big(4pr - q^2\big) = 0$$

- This **cubic resolvent** is solvable by radicals. Taking any of the solutions and substituting for w gives a sum or difference of two squares in t. The quadratic factors can then be solved.

2. Fields and polynomials

2.1. Basic properties of fields

Definition. Ring R is **field** if every element of $R - \{0\}$ has mutliplicative inverse and $1 \neq 0 \in R$.

Lemma. Every field is integral domain.

Definition. **Field homomorphism** is ring homomorphism $\varphi : K \to L$ between fields:

- $\varphi(a+b) = \varphi(a) + \varphi(b)$
- $\varphi(ab) = \varphi(a)\varphi(b)$
- $\varphi(1) = 1$

These imply $\varphi(0) = 0$, $\varphi(-a) = -\varphi(a)$, $\varphi\left(a^{-1}\right) = \varphi(a)^{-1}$.

Lemma. Let $\varphi : K \to L$ field homomorphism.

- $\mathrm{im}(\varphi) = \{\varphi(a) : a \in K\}$ is field.
- $\ker(\varphi) = \{a \in K : \varphi(a) = 0\} = \{0\}$, i.e. φ is injective.

Definition. **Subfield** K of field L is subring of L where K is field. L is **field extension** of K.

- The above lemma shows image of $\varphi : K \to L$ is subfield of L.

Lemma. Intersections of subfields are subfields.

Definition. **Prime subfield** of L is intersection of all subfields of L.

Definition. **Characteristic** $\mathrm{char}(K)$ of field K is

$$\mathrm{char}(K) := \min\{n \in \mathbb{N} : \chi(n) = 0\}$$

(or 0 if this does not exist) where $\chi : \mathbb{Z} \to K$, $\chi(m) = 1 + \cdots + 1$ (m times).

Example. $\mathrm{char}(\mathbb{Q}) = \mathrm{char}(\mathbb{R}) = \mathrm{char}(\mathbb{C}) = 0$, $\mathrm{char}(\mathbb{F}_p) = p$ for p prime.

Lemma. For any field K, $\mathrm{char}(K)$ is either 0 or prime.

Theorem.

- If $\mathrm{char}(K) = 0$ then prime subfield of K is $\cong \mathbb{Q}$.
- If $\mathrm{char}(K) = p > 0$ then prime subfield of K is $\cong \mathbb{F}_p$.

Corollary.

- If $\mathbb{Q}$ is subfield of K then $\mathrm{char}(K) = 0$.
- If $\mathbb{F}_p$ is subfield of K for prime p then $\mathrm{char}(K) = p$.

Remark. Let $\operatorname{char}(K) = p$, then $p \mid \binom{p}{i}$ so $(a+b)^p = a^p + b^p$ in K. Also in $K[x]$ for p prime, $x^p - 1 = (x-1)^p$.

Theorem (Fermat's little theorem). $\forall a \in \mathbb{F}_p, a^p = a$.

2.2. Polynomials over fields

Definition. **Degree** of $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_n \neq 0$, is $\deg(f(x)) = n$.

- Degree of zero polynomial is $\deg(0) = -\infty$.
- $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.
- $\deg(f(x) + g(x)) \leq \max\{\deg(f(x)), \deg(g(x))\}$ with equality if $\deg(f(x)) \neq \deg(g(x))$.
- Only invertible elements in $K[x]$ are non-zero constants $f(x) = a_0 \neq 0$.
- Similarities between $\mathbb{Z}$ and $K[x]$ for field K:
  - $K[x]$ is integral domain.

- There is a division algorithm for $K[x]$: for $f(x), g(x) \in K[x]$, $\exists! q(x), r(x) \in K[x]$ with $\deg(r(x)) < \deg(g(x))$ such that

$$f(x) = q(x)g(x) + r(x)$$

- Every $f(x), g(x) \in K[x]$ have greatest common divisor $\gcd(f(x), g(x))$ unique up to multiplication by non-zero constants. By Euclidean algorithm for polynomials,

$$\exists a(x), b(x) \in K[x] : a(x)f(x) + b(x)g(x) = \gcd(f(x), g(x))$$

- Can construct field from $K[x]$: **field of fractions** of $K[x]$ is

$$K(x) := \mathrm{Frac}(K[x]) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in K[x], g(x) \neq 0 \right\}$$

where $f_1(x)/g_1(x) = f_2(x)/g_2(x) \iff f_1(x)g_2(x) = f_2(x)g_1(x)$. (We can construct the field of fractions for any integral domain).

- $K[x]$ is PID and so UFD.

Definition. For field K, $f(x) \in K[x]$ **irreducible in** $K[x]$ (or $f(x)$ is **irreducible over** K) if

- $\deg(f(x)) \geq 1$ and
- $f(x) = g(x)h(x) \implies g(x)$ or $h(x)$ is constant

2.3. Tests for irreducibility

Proposition. If $f(x)$ has linear factor in $K[x]$, it has root in $K[x]$.

Proposition (Rational root test). If $f(x) = a_0 + \cdots + a_n x^n \in \mathbb{Z}[x]$ has rational root $\frac{b}{c} \in \mathbb{Q}$ with $\gcd(b, c) = 1$ then $b \mid a_0$ and $c \mid a_n$.
Note: this can't be used to show f is irreducible for $\deg(f(x)) \geq 4$.

Theorem (Gauss's lemma). Let $f(x) \in \mathbb{Z}[x]$, $f(x) = g(x)h(x)$, $g(x), h(x) \in \mathbb{Q}[x]$. Then $\exists r \in \mathbb{Q} : rg(x), r^{-1}h(x) \in \mathbb{Z}[x]$. i.e. if $f(x)$ can be factored in $\mathbb{Q}[x]$ it can be factored in $\mathbb{Z}[x]$.

Example. Let $f(x) = x^4 - 3x^3 + 1 \in \mathbb{Q}[x]$. Using the rational root test, $f(\pm 1) \neq 0$ so no linear factors in $\mathbb{Q}[x]$. Checking quadratic factors, let

$$f(x) = (ax^2 + bx + c)(rx^2 + sx + t), \quad a, b, c, r, s, t \in \mathbb{Z} \text{ by Gauss's lemma}$$

So $1 = ar \Rightarrow a = r = \pm 1$. $1 = ct \Rightarrow c = t = \pm 1$. $-3 = b + s$ and $0 = c(b + s)$: contradiction. So $f(x)$ irreducible in $\mathbb{Q}[x]$.

Example. Let $f(x) = x^4 - 3x^2 + 1 \in \mathbb{Q}[x]$. The rational root test shows there are no linear factors. Checking quadratic factors, let

$$f(x) = (ax^2 + bx + c)(rx^2 + sx + t), \quad a, b, c, r, s, t \in \mathbb{Z} \text{ by Gauss's lemma}$$

As before, $a = r = \pm 1$, $c = t = \pm 1$. $0 = b + s \Rightarrow b = -s$, $-3 = at + bs + cr = -b^2 \pm 2$. $b = 1$ works. So $f(x) = (x^2 - x - 1)(x^2 + x - 1)$.

Proposition. Let $f(x) = a_0 + \cdots + a_n x^n \in \mathbb{Z}[x]$. If exists prime $p \nmid a_n$ such that $\overline{f}(x)$ is irreducible in $\mathbb{F}_p[x]$, then $f(x)$ irreducible in $\mathbb{Q}[x]$.

Example. Let $f(x) = 8x^3 + 14x - 9$. Reducing mod 7, $\overline{f}(x) = x^3 - 2 \in \mathbb{F}_7[x]$. No roots exist for this, so $f(x)$ irreducible in $\mathbb{Q}[x]$. For some polynomials, no p is suitable, e.g. $f(x) = x^4 + 1$.

- Gauss's lemma works with any UFD R instead of $\mathbb{Z}$ and field of fractions $\mathrm{Frac}(R)$ instead of $\mathbb{Q}$: e.g. let F field, $R = F[t]$, $K = F(t)$, then $f(x) \in R[x]$ irreducible in $K[x]$ if $f(x)$ is irreducible in $R[x]$.

Proposition (Eisenstein's criterion). Let $f(x) = a_0 + \cdots + a_n x^n \in \mathbb{Z}[x]$, prime $p \in \mathbb{Z}$ such that $p \mid a_0, ..., p \mid a_{n-1}$, $p \nmid a_n$, $p^2 \nmid a_0$. Then $f(x)$ irreducible in $\mathbb{Q}[x]$.

Example. Let $f(x) = x^3 - 3x + 1$. Consider $f(x-1) = x^3 - 3x^2 + 3$. Then by Eisenstein's criterion with $p = 3$, $f(x-1)$ irreducible in $\mathbb{Q}[x]$ so $f(x)$ is as well, since factoring $f(x-1)$ is equivalent to factoring $f(x)$.

Example. p**-th cyclotomic polynomial** is

$$f(x) = \frac{x^p - 1}{x - 1} = 1 + \cdots + x^{p-1}$$

Now

$$f(x+1) = \frac{(1+x)^p - 1}{1 + x - 1} = x^{p-1} + px^{p-2} + \cdots + \binom{p}{p-2} x + p$$

so can apply Eisenstein with $p = p$.

Proposition (Generalised Eisenstein's criterion). Let R be integral domain, $K = \mathrm{Frac}(R)$,

$$f(x) = a_0 + \cdots + a_n x^n \in R[x]$$

If there is irreducible $p \in R$ with

$$p \mid a_0, ..., p \mid a_{n-1}, p \nmid a_n, p^2 \nmid a_0$$

then $f(x)$ is irreducible in $K[x]$.

3. Field extensions

3.1. Definitions and examples

Definition. **Field extension** L/K is field L containing subfield K. Can specify homomorphism $\iota : K \to L$ (which is injective).

Example.

- $\mathbb{C}/\mathbb{R}$, $\mathbb{C}/\mathbb{Q}$, $\mathbb{R}/\mathbb{Q}$.
- $L = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is field extension of $\mathbb{Q}$. $\mathbb{Q}(\theta)$ is field extension of $\mathbb{Q}$ where θ is root of $f(x) \in \mathbb{Q}[x]$.
- $L = \mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$ is smallest subfield of $\mathbb{R}$ containing $\mathbb{Q}$ and $\sqrt[3]{2}$.
- $K(t)$ is field extension of K.

Definition. Let L/K field extension, $S \subseteq L$. Then K **with** S **adjoined**, $K(S)$, is minimal subfield of L containing K and S. If $|S| = 1$, L/K is a **simple extension**.

Example. $\mathbb{Q}(\sqrt{2}, \sqrt{7}) = \{a + b\sqrt{2} + c\sqrt{7} + d\sqrt{14} : a, b, c, d, \in \mathbb{Q}\}$ is $\mathbb{Q}$ with $S = \{\sqrt{2}, \sqrt{7}\}$.

Example. $\mathbb{R}/\mathbb{Q}$ is not simple extension.

Definition. **Tower** is chain of field extensions, e.g. $K \subset M \subset L$.

3.2. Algebraic elements and minimal polynomials

Definition. Let L/K field extension, $\theta \in L$. Then θ is **algebraic over** K if

$$\exists 0 \neq f(x) \in K[x] : f(\theta) = 0$$

Otherwise, θ is **transcendental over** K.

Example. For $n \geq 1$, $\theta = e^{2\pi i/n}$ is algebraic over $\mathbb{Q}$ (root of $x^n - 1$).

Example. $t \in K(t)$ is transcendental over K.

Lemma. The algebraic elements in $K(t)/K$ are precisely K.

Lemma. Let L/K field extension, $\theta \in L$. Define $I_K(\theta) := \{f(x) \in K[x] : f(\theta) = 0\}$. Then $I_K(\theta)$ is ideal in $K[x]$ and

- If θ transcendental over K, $I_K(\theta) = \{0\}$
- If θ algebraic over K, then exists unique monic irreducible polynomial $m(x) \in K[x]$ such that $I_K(\theta) = \langle m(x) \rangle$.

Definition. For $\theta \in L$ algebraic over K, **minimal polynomial of θ over K** is the unique monic polynomial $m(x) \in K[x]$ such that $I_K(\theta) = \langle m(x) \rangle$. The **degree** of θ over K is $\deg(m(x))$.

Remark. If $f(x) \in K[x]$ irreducible over K, monic and $f(\theta) = 0$ then $f(x) = m(x)$.

Example.

- Any $\theta \in K$ has minimal polynomial $x - \theta$ over K.
- $i \in \mathbb{C}$ has minimal polynomial $x^2 + 1$ over $\mathbb{R}$.
- $\sqrt{2}$ has minimal polynomial $x^2 - 2$ over $\mathbb{Q}$. $\sqrt[3]{2}$ has minimal polynomial $x^3 - 2$ over $\mathbb{Q}$.

3.3. Constructing field extensions

Lemma. Let K field, $f(x) \in K[x]$ non-zero. Then

$$f(x) \text{ irreducible over } K \iff K[x]/\langle f(x)\rangle \text{ is a field}$$

Definition. Let L_1/K, L_2/K field extensions, $\varphi : L_1 \to L_2$ field homomorphism. φ is K**-homomorphism** if $\forall a \in K, \varphi(a) = a$ (φ fixes elements of K).

- If φ is isomorphism then it is K**-isomorphism**.
- If $L_1 = L_2$ and φ is bijective then φ is K**-automorphism**.

Theorem. Let $m(x) \in K[x]$ irreducible, monic, $K_m := K[x]/\langle m(x)\rangle$. Then

- K_m/K is field extension.
- Let $\theta = \pi(x)$ where $\pi : K[x] \to K_m$ is canonical projection, then θ has minimal polynomial $m(x)$ and $K_m \cong K(\theta)$.

Proposition (Universal property of simple extension). Let L/K field extension, $\tau \in L$ with $m(\tau) = 0$ and $K_L(\tau)$ be minimal subfield of L containing K and τ. Then exists unique K-isomorphism $\varphi : K_m \to K_L(\tau)$ such that $\varphi(\theta) = \tau$.

Example.

- Complex conjugation $\mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-automorphism.
- Let K field, $\mathrm{char}(K) \neq 2$, $\sqrt{2} \notin K$, so $x^2 - 2$ is minimal polynomial of $\sqrt{2}$ over K, then $K(\sqrt{2}) \cong K[x]/\langle x^2 - 2 \rangle$ is field extension of K and $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is K-automorphism.

Proposition. Let θ transcendental over K, then exists unique K-isomorphism $\varphi : K(t) \to K(\theta)$ such that $\varphi(t) = \theta$:

$$\varphi\left(\frac{f(t)}{g(t)}\right) = \varphi\left(\frac{f(\theta)}{g(\theta)}\right)$$

3.4. Explicit examples of simple extensions

Example.

- Let $r \in K^\times$ non-square in K, $\mathrm{char}(K) \neq 2$, then $x^2 - r$ irreducible in $K[x]$. E.g. for $K = \mathbb{Q}(t)$, $x^2 - t \in K[x]$ is irreducible. Then $K(\sqrt{t}) = \mathbb{Q}(\sqrt{t}) \cong K[x]/\langle x^2 - t\rangle$.
- Define $\mathbb{F}_9 = \mathbb{F}_3[x]/\langle x^2 - 2\rangle \cong \mathbb{F}_3(\theta) = \{a + b\theta : a, b \in \mathbb{F}_3\}$ for θ a root of $x^2 - 2$.

Proposition. Let $K(\theta)/K$ where θ has minimal polynomial $m(x) \in K[x]$ of degree n. Then

$$K[x]/\langle m(x)\rangle \cong K(\theta) = \{c_0 + c_1\theta + \cdots + c_{n-1}\theta^{n-1} : c_i \in K\}$$

and its elements are written uniquely: $K(\theta)$ is vector space over K of dimension n with basis $\{1, \theta, ..., \theta^{n-1}\}$.

Example. $\mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\} \cong \mathbb{Q}[x]/\langle x^3 - 2\rangle$. $\mathbb{Q}(\omega\sqrt[3]{2})$ and $\mathbb{Q}(w^2\sqrt[3]{2})$ where $\omega = e^{2\pi i/3}$ are isomorphic to $\mathbb{Q}(\sqrt[3]{2})$ as $\omega\sqrt[3]{2}$, $\omega\sqrt[3]{4}$ have same minimal polynomial.

3.5. Degrees of field extensions

Definition. **Degree** of field extension L/K is

$$[L : K] := \dim_K(L)$$

Example.

- When θ algebraic over K of degree n, $[K(\theta) : K] = n$.
- Let θ transcendental over K, then $[K(\theta) : K] = \infty$, so $[K(t) : K] = \infty$, $[\mathbb{Q}(\pi) : \mathbb{Q}]$, $[\mathbb{R} : \mathbb{Q}] = \infty$.

Definition. L/K is **algebraic extension** if every element in L is algebraic over K.

Proposition. Let $[L : K] < \infty$, then L/K is algebraic extension and $L = K(\alpha_1, ..., \alpha_n)$ for some $\alpha_1, ..., \alpha_n \in L$. The converse also holds.

Theorem (Tower law). Let $K \subseteq M \subseteq L$ tower of field extensions. Then

- $[L : K] < \infty \iff [L : M] < \infty \wedge [M : K] < \infty$.
- $[L : K] = [L : M][M : K]$.

Example.

- $K = \mathbb{Q} \subset M = \mathbb{Q}(\sqrt{2}) \subset L = \mathbb{Q}(\sqrt{2}, \sqrt{7})$. M/K has basis $\{1, \sqrt{2}\}$ so $[M : K] = 2$. Let $\sqrt{7} \in \mathbb{Q}(\sqrt{2})$, then $\sqrt{7} = c + d\sqrt{2}$, $c, d \in \mathbb{Q}$ so $7 = \left(c^2 + 2d^2\right) + 2cd\sqrt{2}$ so $7 = c^2 + 2d^2$, $0 = 2cd$ so $d^2 = \frac{7}{2}$ or $c^2 = 7$, which are both contradictions. So $[L : K] = 4$ with basis $\{1, \sqrt{2}, \sqrt{7}, \sqrt{14}\}$.
- Let $K = \mathbb{Q} \subset M = \mathbb{Q}(i) \subset \mathbb{Q}(i, \sqrt{2})$. We know $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, $[\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q}] = 4$ (since $i \notin \mathbb{R}$) so $[\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q}(\sqrt{2})] = 2$.
- Let $K = \mathbb{Q} \subset M = \mathbb{Q}(\sqrt{2}) \subset L = \mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$. Then $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, $[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}] = 3$ so $2 \mid [L : K]$ and $3 \mid [L : K]$ so $6 \mid [L :$

$K]$ so $[L : K] \geq 6$. But $[L : M] \leq 3$ and $[M : K] \leq 2$ so $[L : K] \leq 6$ hence $[L : K] = 6$.

- More generally, we have $[K(\alpha, \beta) : K] \leq [K(\alpha) : K][K(\beta) : K]$.

Example.

- Let $\theta = \sqrt[3]{4} + 1$. $\mathbb{Q}(\theta) = \mathbb{Q}(\sqrt[3]{4})$ so minimal polynomial over $\mathbb{Q}$, m, has $\deg(m) = 3$. $(\theta - 1)^3 = 4$ so minimal polynomial is $x^3 - 3x^2 + 3x - 5$.
- Let $\theta = \sqrt{2} + \sqrt{3}$. $\mathbb{Q}(\sqrt{2}, \theta) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ which has degree 2 over $\mathbb{Q}(\sqrt{2})$ so minimal polynomial of θ over $\mathbb{Q}(\sqrt{2})$ has degree 2, $\theta - \sqrt{2} = \sqrt{3}$ so minimal polynomial is $x^2 - 2\sqrt{2}x - 1$.
- Let $\theta = \sqrt{2} + \sqrt{3}$. $\mathbb{Q} \subset \mathbb{Q}(\theta) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$ so $[\mathbb{Q}(\theta) : \mathbb{Q}] \mid [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$ so $[\mathbb{Q}(\theta) : \mathbb{Q}] \in \{1, 2, 4\}$. Can't be 1 as $\theta \notin \mathbb{Q}$. If it was 2 then $1, \theta, \theta^2$ are linearly dependent over $\mathbb{Q}$ which leads to a contradiction. So degree of minimal polynomial

of θ over $\mathbb{Q}$ is 4. $\theta^2 = 5 + 2\sqrt{6} \Rightarrow \left(\theta^2 - 5\right)^2 = 24$ so minimal polynomial is $x^4 - 10x^2 + 1$.

4. Galois extensions

4.1. Splitting fields

Definition. For field K, $0 \neq f(x) \in K[x]$, L/K is **splitting field** of $f(x)$ over K if

- $\exists c \in K^{\times}, \theta_1, ..., \theta_n \in L : f(x) = c(x - \theta_1) \cdots (x - \theta_n)$ ($f(x)$ **splits over** L).
- $L = K(\theta_1, ..., \theta_n)$.

Example.

- $\mathbb{C}$ is splitting field of x^2+1 over $\mathbb{R}$, since $x^2+1=(x+i)(x-i)$ and $\mathbb{C}=\mathbb{R}(i,-i)=\mathbb{R}(i)$.
- $\mathbb{C}$ is not splitting field of x^2+1 over $\mathbb{Q}$ as $\mathbb{C}\neq\mathbb{Q}(i,-i)$.
- $\mathbb{Q}$ is splitting field of x^2-36 over $\mathbb{Q}$.
- $\mathbb{C}$ is splitting of x^4+1 over $\mathbb{R}$.
- $\mathbb{Q}(i,\sqrt{2})$ is splitting field of $x^4-x^2-2=(x^2+1)(x^2-2)=(x+i)(x-i)(x+\sqrt{2})(x-\sqrt{2})$ over $\mathbb{Q}$.
- $\mathbb{F}_2(\theta)$ where $\theta^3+\theta+1=0$ is splitting field of x^3+x+1 over $\mathbb{F}_2$.

- Consider splitting field of $x^3 - 2$ over $\mathbb{Q}$. Let $\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2$ then $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is splitting field since it must contain $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$, $\omega^2\sqrt[3]{2}$.

Theorem. Let $0 \neq f(x) \in K[x]$, $\deg(f) = n$. Then there exists a splitting field L of $f(x)$ over K with

$$[L : K] \leq n!$$

Notation. For field homomorphism $\varphi : K \to K'$ and $f(x) = a_0 + \cdots + a_n x^n \in K[x]$, write

$$\varphi_*(f(x)) := \varphi(a_0) + \cdots + \varphi(a_n)x^n \in K'[x]$$

Lemma. Let $\sigma : K \to K'$ isomorphism and $K(\theta)/K$, θ has minimal polynomial $m(x) \in K[x]$, θ' be root of $\sigma_*(m(x))$. Then there exists unique K-isomorphism $\tau : K(\theta) \to K'(\theta')$ such that $\tau(\theta) = \theta'$.

Theorem. For field isomorphism $\sigma : K \to K'$ and $0 \neq f(x) \in K[x]$, let L be splitting field of $f(x)$ over K, L' be splitting field of $\sigma_*(f(x))$ over K'. Then there exists a field isomorphism $\tau : L \to L'$ such that $\forall a \in K, \tau(a) = \sigma(a)$.

Corollary. Setting $K = K'$ and $\sigma = \mathrm{id}$ implies that splitting fields are unique.

4.2. Normal extensions

Definition. L/K is **normal** if: for all $f(x) \in K[x]$, if f is irreducible and has a root in L then all its roots are in L. In particular, $f(x)$ splits completely as product of linear factors in $L[x]$. So the minimal polynomial of $\theta \in L$ over K has all its roots in L and can be written as product of linear factors in $L[x]$.

Example.

- If $[L : K] = 1$ then L/K is normal.
- If $[L : K] = 2$ then L/K is normal: let $\theta \in L$ have minimal polynomial $m(x) \in K[x]$, then $K \subseteq K(\theta) \subseteq L$ so $\deg(m(x)) = [K(\theta) : K] \in \{1, 2\}$:
  - If $\deg(m(x)) = 1$ then $m(x)$ is already linear.
  - If $\deg(m(x)) = 2$ then $m(x) = (x - \theta)m_1(x)$, $m_1(x) \in L[x]$ is linear so $m(x)$ splits completely in $L[x]$.
- If $[L : K] = 3$ then L/K is not necessarily normal. Let θ be root of $x^3 - 2 \in \mathbb{Q}[x]$. Other two roots are $\omega\theta$, $\omega^2\theta$ where $\omega = e^{2\pi i/3}$. If

$\omega\theta \in \mathbb{Q}(\theta)$ then $\omega = \frac{\omega\theta}{\theta} \in L$ so $\mathbb{Q} \subset \mathbb{Q}(\omega) \subset \mathbb{Q}(\theta)$ but $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ which doesn't divide $[\mathbb{Q}(\theta) : \mathbb{Q}] = 3$.

- Let $\theta \in \mathbb{C}$ be root of irreducible $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$. Let $\theta = u + v$, then $(u+v)^3 - 3uv(u+v) - (u^3 + v^3) \equiv 0$ implies $uv = 1 = u^3v^3$, $u^3 + v^3 = 1$. So $(y - u^3)(y - v^3) = y^2 - y + 1$ has roots u^3 and v^3. So the three roots of f are

$$\theta_1 = u + v = e^{\pi i/9} + e^{-\pi i/9} = 2\cos(\pi/9)$$

$$\theta_2 = \omega u + \omega^2 v = e^{7\pi i/9} + e^{-7\pi i/9} = 2\cos(7\pi/9)$$

$$\theta_3 = \omega^2 u + \omega v = e^{13\pi i/9} + e^{-13\pi i/9} = 2\cos(13\pi/9)$$

Furthermore, for each i, j, $\theta_i \in \mathbb{Q}(\theta_j)$, e.g.

$$\theta_2 = 2\cos\left(\pi - \frac{2\pi}{9}\right) = -2\cos\left(\frac{2\pi}{9}\right) = -2\left(2\cos\left(\frac{\pi}{9}\right)^2 - 1\right) = 2 - \theta_1^2$$

Also $\theta_1 + \theta_2 + \theta_3 = 0$ so $\theta_3 \in \mathbb{Q}(\theta_1)$. So $\mathbb{Q}(\theta_1)$ contains all roots of $f(x)$.

Theorem (normality criterion). L/K is finite and normal iff L is splitting field for some $0 \neq f(x) \in K[x]$ over K.

Example.

- $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})/\mathbb{Q}$ is normal as it is the splitting field of $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)(x^2 - 7) \in \mathbb{Q}[x]$.
- $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal but $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ is normal as it is the splitting field of $x^3 - 2 \in \mathbb{Q}$.
- $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not normal but $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$ is normal.
- Let θ root of $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$. Then $\mathbb{Q}(\theta)/\mathbb{Q}$ is normal as is splitting field of $f(x)$ over $\mathbb{Q}$.
- $\mathbb{F}_2(\theta)/\mathbb{F}_2$ where $\theta^3 + \theta^2 + 1 = 0$ is normal, as $\mathbb{F}_2(\theta)$ contains all roots of $x^3 + x^2 + 1$.

- $\mathbb{F}_p(\theta)/\mathbb{F}_p(t)$ where $\theta^p = t$ is normal as it is the splitting field of $x^p - t = x^p - \theta^p = (x - \theta)^p$ so $f(x)$ splits into linear factors in $L[x]$.

Definition. Field N is **normal closure** of L/K if $K \subseteq L \subseteq N$, N/K is normal, and if $K \subseteq L \subseteq N' \subseteq N$ with N'/K normal then $N = N'$.

Theorem. Every finite extension L/K has normal closure, unique up to a K-isomorphism.

Definition. $\mathrm{Aut}(L/K)$ is group of K-automorphisms of L/K with composition as the group operation.

Example.

- $\mathrm{Aut}(\mathbb{C}/\mathbb{R})$ contains at least two elements: complex conjugation: $\sigma(a+bi) = a - bi$ and the identity map $\mathrm{id} = \sigma^2$. If $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{R})$ then $\tau(a+bi) = a + b\tau(i)$. But $\tau(i)^2 = \tau(i^2) = \tau(-1) = -1$ hence $\tau(i) = \pm i$. So there are only two choices for τ. So $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \sigma\}$.
- Let $f(x) = x^2 + px + q \in \mathbb{Q}[x]$ irreducible with distinct roots θ, θ'. Then $\mathrm{Aut}(\mathbb{Q}(\theta)/\mathbb{Q}) = \{\mathrm{id}, \sigma\} \cong \mathbb{Z}/2$ where $\sigma(a + b\theta) = a + b\theta'$.
- Let θ root of $x^3 - 2$, let $\sigma \in \mathrm{Aut}(\mathbb{Q}(\theta)/\mathbb{Q})$. Now $\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2$ so $\sigma(\theta) \in \{\theta, \omega\theta, \omega^2\theta\}$ but $\omega\theta, \omega^2\theta \notin \mathbb{Q}(\theta)$ so $\sigma(\theta) = \theta \implies \sigma = \mathrm{id}$.

- Let $\theta^p = t$, $\sigma \in \operatorname{Aut}\big(\mathbb{F}_p(\theta)/\mathbb{F}_p(t)\big)$. Then

$$\sigma(\theta)^p = \sigma(\theta^p) = \sigma(t) = t = \theta^p$$

so $(\sigma(\theta) - \theta))^p = \sigma(\theta)^p - \theta^p = 0 \implies \sigma(\theta) = \theta \implies \sigma = \mathrm{id}$.
- Let $\sigma \in \operatorname{Aut}(\mathbb{R}/\mathbb{Q})$. Then $\alpha \leq \beta \in \mathbb{R} \implies \beta - \alpha = \gamma^2$, $\gamma \in \mathbb{R}$, so $\sigma(\beta) - \sigma(a) = \sigma(\gamma)^2 \geq 0$ so $\sigma(\alpha) \leq \sigma(\beta)$. Given $\alpha \in \mathbb{R}$, there exist sequences $(r_n), (s_n) \subset \mathbb{Q}$ with $r_n \leq \alpha \leq s_n$ and $r_n \to \alpha$, $s_n \to \alpha$ as $n \to \infty$. Hence $r_n = \sigma(r_n) \leq \sigma(\alpha) \leq \sigma(s_n) = s_n$ so $\sigma(\alpha) = \alpha$ by squeezing. Hence $\operatorname{Aut}(\mathbb{R}/\mathbb{Q}) = \{\mathrm{id}\}$.

Theorem. Let $L = K(\theta)$, θ root of irreducible $f(x) \in K[x]$, $\deg(f) = n$. Then $|\mathrm{Aut}(L/K)| \leq n$, with equality iff $f(x)$ has n distinct roots in L.

Theorem. Let L/K be finite extension. Then $|\mathrm{Aut}(L/K)| \leq [L : K]$, with equality iff L/K is normal and minimal polynomial of every $\theta \in L$ over K has no repeated roots (in a splitting field).

4.3. Separable extensions

Definition. Let L/K finite extension.

- $\theta \in L$ is **separable over** K if its minimal polynomial over K has no repeated roots (in its splitting field).
- L/K is **separable** if every $\theta \in L$ is separable over K.

Example. Let $K = \mathbb{F}_p(t)$, then $f(x) = x^p - t \in K[x]$ is irreducible by Eisenstein's criterion with $p = t$, and $f(x) = x^p - \theta^p = (x - \theta)^p$ so θ is root of multiplicity $p \geq 2$. So $\mathbb{F}_p(\theta)/\mathbb{F}_p(t)$ is normal but not separable.

Definition. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in K[x]$. **Formal derivative** of $f(x)$ is

$$Df(x) = D(f) := \sum_{i=1}^{n} i a_i x^{i-1} \in K[x]$$

Note. Formal derivative satisfies:

$$D(f+g) = D(f) + D(g), \quad D(fg) = f \cdot D(g) + D(f) \cdot g, \quad \forall a \in K, D(a) = 0$$

Also $\deg(D(f)) < \deg(f)$. But if $\mathrm{char}(K) = p$, then $D(x^p) = px^{p-1} = 0$ so it is not always true that $\deg(D(f)) = \deg(f) - 1$.

Note. If $f(x)$ has a repeated root α, then $Df(\alpha) = 0$.

Theorem (sufficient conditions for separability). Finite extension L/K is separable if any of the following hold:

- $\mathrm{char}(K) = 0$,
- $\mathrm{char}(K) = p$ and $K = \{b^p : b \in K\} = K^p$ for prime p,
- $\mathrm{char}(K) = p$ and $p \nmid [L : K]$

Definition. K is **perfect field** if either of first two of above properties hold.

Remark. All finite extensions of any perfect extension (e.g. $\mathbb{Q}, \mathbb{F}_p$) are separable (recall Fermat's little theorem: $\forall a \in \mathbb{F}_p, a = a^p$). So to find a non-separable extension L/K, we need $\mathrm{char}(K) = p > 0$, K infinite and $p \mid [L : K]$. For example, $L = \mathbb{F}_p(\theta)$, $K = \mathbb{F}_p(t)$ where $\theta^p = t$.

Theorem. Let $\alpha_1, ..., \alpha_n$ algebraic over K, then $K(\alpha_1, ..., \alpha_n)/K$ is separable iff every α_i is separable over K.

Remark. For tower $K \subseteq M \subseteq L$, L/K is separable iff L/M and M/K are separable. However, the same statement for normality does not hold.

Theorem (Theorem of the Primitive Element). Let L/K finite and separable. Then L/K is simple, i.e. $\exists \alpha \in L : L = K(\alpha)$.

4.4. The fundamental theorem of Galois theory

Definition. Finite extension L/K is **Galois extension** if it is normal and separable. Equivalently, $|\mathrm{Aut}(L/K)| = [L : K]$. When L/K is Galois, the **Galois group** is $\mathrm{Gal}(L/K) := \mathrm{Aut}(L/K)$.

Definition. Let $\mathcal{F} := \{\text{intermediate fields of } L/K\}$ and $\mathcal{G} := \{\text{subgroups of } \mathrm{Gal}(L/K)\}$. Define the map $\Gamma : \mathcal{F} \to \mathcal{G}$, $\Gamma(M) = \mathrm{Gal}(L/M)$.

Definition. Let L field, G a group of automorphisms of L. **Fixed field** L^G of G is set of elements in L which are invariant under all automorphisms in G:

$$L^G := \{\alpha \in L : \forall \alpha \in G, \sigma(\alpha) = \alpha\}$$

Theorem. If G is finite group of automorphisms of L then L^G is subfield of L and $\left[L : L^G\right] = |G|$.

Corollary. If L/K is Galois then

- $L^{\mathrm{Gal}(L/K)} = K$.
- If $L^G = K$ for some group G of K-automorphisms of L, then $G = \mathrm{Gal}(L/K)$.

Note. Let $\sigma \in \mathrm{Gal}(L/K)$. If $\alpha \in L$ has minimal polynomial $f(x) \in K[x]$ over K, then $f(\alpha) = 0$, and

$$f(\sigma(\alpha)) = \sigma(f(\alpha))$$

by properties of field homomorphisms. Hence $\sigma(\alpha)$ is also a root of $f(x)$ for any $\sigma \in \mathrm{Gal}(L/K)$, i.e. σ permutes the roots of $f(x)$.

Remark. If L/K is Galois and $\alpha \in L$ but $\alpha \notin K$, then there exists an automorphism $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\alpha) \neq \alpha$.

Definition. For H subgroup of $\mathrm{Gal}(L/K)$, set $L^H := \{\alpha \in L : \forall \sigma \in H, \sigma(\alpha) = \alpha\}$, then $K \subseteq L^H \subseteq L$. Define $\Phi : \mathcal{G} \to \mathcal{F}$, $\Phi(H) = L^H$.

- Γ and Φ are inclusion-reversing: $M_1 \subseteq M_2 \implies \Gamma(M_2) \subseteq \Gamma(M_1)$, and $H_1 \subseteq H_2 \implies \Phi(H_2) \subseteq \Phi(H_1)$.

Theorem (Fundamental theorem of Galois theory - Theorem A).
For finite Galois extension L/K,

- $\Gamma : \mathcal{F} \to \mathcal{G}$ and $\Phi : \mathcal{G} \to \mathcal{F}$ are mutually inverse bijections (the **Galois correspondence**).
- For $M \in \mathcal{F}$, L/M is Galois and $|\mathrm{Gal}(L/M)| = [L : M]$.
- For $H \in \mathcal{G}$, L/L^H is Galois and $\mathrm{Gal}\big(L/L^H\big) = H$.

Remark. $\mathrm{Gal}(L/K)$ acts on $\mathcal{F}$: given $\sigma \in \mathrm{Gal}(L/K)$ and $K \subseteq M \subseteq L$, consider $\sigma(M) = \{\sigma(\alpha) : \alpha \in M\}$ which is a subfield of L and contains K, since σ fixes elements of K. Given another automorphism $\tau : L \to L$,

$$\begin{aligned}
\tau \in \mathrm{Gal}(L/\sigma(M)) &\iff \forall \alpha \in M, \tau(\sigma(\alpha)) = \sigma(\alpha) \\
&\iff \forall \alpha \in M, \sigma^{-1}(\tau(\sigma(\alpha))) = \alpha \\
&\iff \sigma^{-1}\tau\sigma \in \mathrm{Gal}(L/M) \\
&\iff \tau \in \sigma \ \mathrm{Gal}(L/M)\sigma^{-1}
\end{aligned}$$

Hence $\sigma \operatorname{Gal}(L/M)\sigma^{-1}$ and $\operatorname{Gal}(L/M)$ are conjugate subgroups of $\operatorname{Gal}(L/K)$. Now

$$[M:K] = \frac{[L:K]}{[L:M]} = \frac{|\operatorname{Gal}(L/K)|}{|\operatorname{Gal}(L/M)|}$$

Theorem (Fundamental theorem of Galois theory - Theorem B).
Let L/K be finite Galois extension, $G = \mathrm{Gal}(L/K)$ and $K \subseteq M \subseteq L$.
Then the following are equivalent:

- M/K is Galois.
- $\forall \sigma \in G, \quad \sigma(M) = M$.
- $H = \mathrm{Gal}(L/M)$ is normal subgroup of $G = \mathrm{Gal}(L/K)$.

When these conditions hold, we have $\mathrm{Gal}(M/K) \cong G/H$.

Example. Let L/K be Galois, $[L : K] = p$ prime.

- By the tower law, any $K \subseteq M \subseteq L$ has $[L : M] \in \{1, p\}$, $[M : K] \in \{p, 1\}$, so $M = L$ or K. In both cases, M/K is normal.
- $|\mathrm{Gal}(L/K)| = [L : K] = p$ so $\mathrm{Gal}(L/M) \cong \mathbb{Z}/p$, so the only subgroups are $\mathrm{Gal}(L/K)$ and $\{\mathrm{id}\}$. In both cases, H is normal subgroup of $\mathrm{Gal}(L/K)$.

4.5. Computations with Galois groups

Example (quadratic extension). $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal (since degree is 2) and separable (since characteristic is zero). Any element of $\varphi \in G = \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ is determined by the image of $\sqrt{2}$. But $\varphi(\sqrt{2})^2 = \varphi(2) = 2$ so $\varphi(\sqrt{2}) = \pm\sqrt{2}$. This gives two automorphisms $\mathrm{id}(\sqrt{2}) = \sqrt{2}$ and $\sigma(\sqrt{2}) = -\sqrt{2}$. So $G = \{\mathrm{id}, \sigma\} = \langle\sigma\rangle \cong \mathbb{Z}/2$. Subgroup $\{\mathrm{id}\}$ corresponds to $\mathbb{Q}(\sqrt{2})$, G corresponds to $\mathbb{Q}$.

Example (biquadratic extension). $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$ is normal (as splitting field of $(x^2-2)(x^2-3)$ over $\mathbb{Q}$) and separable (as $\mathrm{char}(\mathbb{Q}) = 0$), so is Galois extension. Let σ be given as before.

- Suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$, then $\sigma(\sqrt{3})^2 = \sigma(3) = 3$, so $\sigma(\sqrt{3}) = \pm\sqrt{3}$.
- If $\sigma(\sqrt{3}) = \sqrt{3}$, then $\sqrt{3} \in \mathbb{Q}(\sqrt{2})^{\{\mathrm{id},\sigma\}} = \mathbb{Q}$: contradiction.
- If $\sigma(\sqrt{3}) = -\sqrt{3}$, then $\sigma(\sqrt{2})\sigma(\sqrt{3}) = \sigma(\sqrt{6}) = (-\sqrt{2})(-\sqrt{3}) = \sqrt{6}$, so $\sqrt{6} \in \mathbb{Q}(\sqrt{2})^{\{\mathrm{id},\sigma\}} = \mathbb{Q}$: contradiction.
- So $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, hence $[L : \mathbb{Q}] = [L : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4$.
- Now $G = \mathrm{Gal}(L/\mathbb{Q})$ has order $[L : \mathbb{Q}] = 4$, so $G \cong \mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$.

- For $\varphi \in G$, $\varphi(\sqrt{2})^2 = 2 \implies \varphi(\sqrt{2}) = \pm\sqrt{2}$, $\varphi(\sqrt{3})^2 = 3 \implies \varphi(\sqrt{3}) = \pm\sqrt{3}$. So there are four choices, corresponding to choices of $\pm$ signs.
- Define σ, τ by $\sigma(\sqrt{2}) = -\sqrt{2}$, $\sigma(\sqrt{3}) = \sqrt{3}$, $\tau(\sqrt{2}) = \sqrt{2}$, $\tau(\sqrt{3}) = -\sqrt{3}$. Now $\sigma^2 = \tau^2 = \mathrm{id}$, $\sigma\tau(\sqrt{2}) = -\sqrt{2}$, $\sigma\tau(\sqrt{3}) = -\sqrt{3}$ and $\sigma\tau = \tau\sigma$.
- So $G = \langle \sigma, \tau : \sigma^2 = \tau^2 = \mathrm{id}, \sigma\tau = \tau\sigma \rangle = \langle \sigma \rangle \times \langle \tau \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- G has proper subgroups $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau \rangle$, $H_3 = \langle \sigma\tau \rangle$.
- So the intermediate fields are $L^{H_1}, L^{H_2}, L^{H_3}$.

- $\sigma(\sqrt{3}) = \sqrt{3} \implies \sqrt{3} \in L^{H_1}$ so $\mathbb{Q}(\sqrt{3}) \subseteq L^{H_1}$, but $[L : \mathbb{Q}(\sqrt{3})] = 2 = |H_1| = [L : L^{H_1}]$. Hence $L^{H_1} = \mathbb{Q}(\sqrt{3})$. Similarly $L^{H_2} = \mathbb{Q}(\sqrt{2})$.
- $\sigma\tau(\sqrt{6}) = \sqrt{6} \implies \sqrt{6} \in L^{H_3}$, so $L^{H_3} = \mathbb{Q}(\sqrt{6})$.

Remark. It is not generally true that $[K(\sqrt{a}, \sqrt{b}) : K] = 4$, e.g. $\mathbb{Q}(\sqrt{2}, \sqrt{8}) = \mathbb{Q}(\sqrt{2})$.

Remark. Can generalise above example to arbitrary $K(\sqrt{a}, \sqrt{b})/K$ where $\mathrm{char}(K) \neq 2$, and $a, b \in K$, $a, b, ab \notin (K^\times)^2$ where $(K^\times)^2$ is set of squares of $K^\times$.

Example (degree 8 extension).

- Consider $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. L is splitting field of $(x^2 - 2)(x^2 - 3)(x^2 - 5)$, so is normal, and $\mathrm{char}(\mathbb{Q}) = 0$, so is separable, so is Galois.
- Let $M = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. By above, $\mathrm{Gal}(M/Q) = \langle \sigma \rangle \times \langle \tau \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- Suppose $\sqrt{5} \in M$. Then $\sigma(\sqrt{5})^2 = \tau(\sqrt{5})^2 = 5$, so $\sigma(\sqrt{5}) = \pm\sqrt{5}$, $\tau(\sqrt{5}) = \pm\sqrt{5}$.
- If $\sigma(\sqrt{5}) = \sqrt{5}$, then $\sqrt{5} \in M^{\langle \sigma \rangle} = \mathbb{Q}(\sqrt{3})$.
  - If $\tau(\sqrt{5}) = \sqrt{5}$, $\sqrt{5} \in M^{\langle \sigma, \tau \rangle} = \mathbb{Q}$: contradiction.

- If $\tau(\sqrt{5}) = -\sqrt{5}$, then since $\sqrt{15} \in M^{\langle\sigma\rangle}$, $\tau(\sqrt{15}) = \sqrt{15}$, so $\sqrt{15} \in M^{\langle\sigma,\tau\rangle} = \mathbb{Q}$: contradiction.

- If $\sigma(\sqrt{5}) = -\sqrt{5}$, then $\sigma(\sqrt{10}) = \sigma(\sqrt{2})\sigma(\sqrt{5}) = (-\sqrt{2})(-\sqrt{5}) = \sqrt{10}$, so $\sqrt{10} \in M^{\langle\sigma\rangle} = \mathbb{Q}(\sqrt{3})$.
  - If $\tau(\sqrt{5}) = \sqrt{5}$, $\tau(\sqrt{10}) = \sqrt{10} \in M^{\langle\sigma,\tau\rangle} = \mathbb{Q}$: contradiction.
  - If $\tau(\sqrt{5}) = -\sqrt{5}$, $\tau(\sqrt{30}) = \tau(\sqrt{5})\tau(\sqrt{3})\tau(\sqrt{2}) = \sqrt{30} \in M^{\langle\sigma,\tau\rangle} = \mathbb{Q}$: contradiction.
- So $\sqrt{5} \notin M$, so $[L : \mathbb{Q}] = [L : M][M : \mathbb{Q}] = 8$. The 8 elements in $\mathrm{Gal}(L/\mathbb{Q})$ are determined by choices of $\sqrt{a} \mapsto \pm\sqrt{a}$ where $a \in \{2, 3, 5\}$.

- $\mathrm{Gal}(L/\mathbb{Q}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ where $\sigma_1(\sqrt{2}) = -\sqrt{2}$, $\sigma_2(\sqrt{3}) = -\sqrt{3}$, $\sigma_1(\sqrt{5}) = -\sqrt{5}$ and the σ_i fix all other square roots.
- More generally, write $\sigma(\sqrt{5}) = (-1)^j\sqrt{5}$, $\tau(\sqrt{5}) = (-1)^k\sqrt{5}$, $j, k \in \{0, 1\}$. Define $m = 2^j 3^k$, then $\sigma(\sqrt{m}) = (-1)^j\sqrt{m} \Rightarrow \sigma(\sqrt{5m}) = \sqrt{5m}$ and $\tau(\sqrt{m}) = (-1)^k\sqrt{m} \Rightarrow \tau(\sqrt{5m}) = \sqrt{5m}$, so $\sqrt{5m} \in M^{\langle\sigma,\tau\rangle} = \mathbb{Q}$: contradiction.

Example (cubic extension and its normal closure).

- Let $L = \mathbb{Q}(\theta)$, $\theta^3 - 2 = 0$. $L/\mathbb{Q}$ isn't Galois since not normal. Take the normal closure $N = \mathbb{Q}(\theta, \omega) = \mathbb{Q}(\theta, \sqrt{-3})$.
- Let $M = \mathbb{Q}(\omega)$ so $[M : \mathbb{Q}] = 2$, $[L : \mathbb{Q}] = 3$ and $[N : \mathbb{Q}] = 6$. Let $G = \mathrm{Gal}(N/\mathbb{Q})$.
- Since $|G| = [N : \mathbb{Q}] = 6$, $G \cong \mathbb{Z}/6$ or $G \cong D_3 \cong S_3$.
- G contains $\mathrm{Gal}(N/L)$. Since $N = L(\omega)$,

$$\mathrm{Gal}(N/L) = \{\mathrm{id}, \tau\} = \langle \tau \rangle \cong \mathbb{Z}/2$$

  where $\tau(\sqrt{-3}) = -\sqrt{-3}$ (i.e. $\tau(w) = \omega^2$) and $\tau(\theta) = \theta$ as $\theta \in L$.

- G contains $H = \mathrm{Gal}(N/M)$. $N = M(\theta)$, $|H| = [N : M] = 3$ so $\mathrm{Gal}(N/M)$ is cyclic so

$$H = \{\mathrm{id}, \sigma, \sigma^2\} = \langle\sigma\rangle \cong \mathbb{Z}/3$$

where $\sigma(\theta) = \omega\theta$, also $\sigma(\omega) = \omega$ as $\omega \in M$ and $\sigma^2(\theta) = \omega^2\theta$, so H permutes the three roots of $x^3 - 2$.
- $\tau \notin H$ so $H = \{\mathrm{id}, \sigma, \sigma^2\}$ and $\tau H = \{\tau, \tau\sigma, \tau\sigma^2\}$ are disjoint cosets. So $G = H \cup \tau H = \langle \mathrm{\tau}, \mathrm{\sigma} \rangle$ so $|G| = 6$. $\tau^2 = \sigma^3 = \mathrm{id}$ and $\sigma\tau = \tau\sigma^2$. So $G \cong S_3 \cong D_3$.

- G has one subgroup of order 3, $H = \langle\sigma\rangle$. Fixed field is $N^H = M$. H is only proper normal subgroup of G. Correspondingly, M is only normal extension of Q in N.
- There are 3 order 2 subgroups: $\langle\tau\rangle$, $\langle\tau\sigma\rangle$, $\langle\tau\sigma^2\rangle$. $N^{\langle\tau\rangle} = \mathbb{Q}(\theta) = L$, $N^{\langle\tau\sigma\rangle} = \mathbb{Q}(\omega\theta) = \sigma(L)$, $N^{\langle\tau\sigma^2\rangle} = \mathbb{Q}\big(\omega^2\theta\big) = \sigma^2(L)$.

Example. Show $\sqrt[3]{3} \notin \mathbb{Q}(\sqrt[3]{2})$.

- Assume $\sqrt[3]{3} \in \mathbb{Q}(\sqrt[3]{2})$. Then $\sqrt[3]{3} \in N = \mathbb{Q}(\omega, \sqrt[3]{2})$, the normal closure.
- As above, let $\sigma \in \mathrm{Gal}(N/\mathbb{Q})$, $\sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2}$ and $N^{\langle\sigma\rangle} = \mathbb{Q}(\omega)$. Also,

$$\sigma(\sqrt[3]{3})^3 = \sigma(3) = 3 \Longrightarrow \sigma(\sqrt[3]{3}) \in \{\sqrt[3]{3}, \omega\sqrt[3]{3}, \omega^2\sqrt[3]{3}\}$$

- If $\sigma(\sqrt[3]{3}) = \sqrt[3]{3}$, then $\sqrt[3]{3} \in N^{\langle\sigma\rangle} = \mathbb{Q}(\omega)$, so $\mathbb{Q}(\sqrt[3]{3}) \subseteq \mathbb{Q}(\omega)$: contradiction.
- If $\sigma(\sqrt[3]{3}) = \omega\sqrt[3]{3}$, then $\sigma(\sqrt[3]{3}/\sqrt[3]{2}) = \sqrt[3]{3}/\sqrt[3]{2}$ hence $\sqrt[3]{3/2} \in N^{\langle\sigma\rangle} = \mathbb{Q}(\omega)$, so $\mathbb{Q}(\sqrt[3]{3/2}) = \mathbb{Q}(\sqrt[3]{12}) \subseteq \mathbb{Q}(\omega)$: contradiction.

- If $\sigma(\sqrt[3]{3}) = \omega^2\sqrt[3]{3}$, $\mathbb{Q}(\sqrt[3]{3/4}) = \mathbb{Q}(\sqrt[3]{6}) \subseteq \mathbb{Q}(\omega)$: contradiction.

Remark. In the above example, $N = \mathbb{Q}(\theta_1, \theta_2, \theta_3) = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where θ_i are the roots of $x^3 - 2$. Plotting these roots on Argand diagram gives the symmetry group $S_3 \cong D_3$ of an equilateral triangle. τ reflects the θ_i (complex conjugation), σ rotates the roots (but **doesn't** rotate all of N, as it fixes $\mathbb{Q}$). For $g \in G$, $g\left(\theta_j\right) = \theta_{\pi(j)}$ where π is permutation of $\{1, 2, 3\}$. So there is a group homomorphism $\varphi : G \to S_3$, $\varphi(g) = \pi$. $\ker(\varphi) = \{\mathrm{id}\}$, so φ is injective and also surjective, since $|G| = |S_3| = 6$, so φ is isomorphism.

Definition. For $f(x) \in K[x]$, $\deg(f) = n \geq 1$, with n distinct roots, the **Galois group** of $f(x)$, G_f, is Galois group of splitting field of $f(x)$ over K (provided it is separable).

Remark. Elements of G_f permute roots of f, so G_f is subgroup of S_n. If $f(x)$ irreducible over K, then G_f is **transitive** subgroup, i.e. given 2 roots α, β of f, there is a $g \in G_f$ with $g(\alpha) = \beta$. This gives a general pattern

$$\text{polynomial} \longrightarrow \text{field extension} \longrightarrow \text{permutation group}$$

Example. Consider $\mathbb{Q} \subset L = \mathbb{Q}(\theta) \subset N = \mathbb{Q}(\theta, i)$ where $\theta = \sqrt[4]{2}$. N is normal closure of $\mathbb{Q}(\theta)$, $[N : \mathbb{Q}] = 8$ so $|\mathrm{Gal}(N/\mathbb{Q})| = 8$.

- Define $\sigma(\theta) = i\theta$, $\sigma(i) = i$, $\tau(\theta) = \theta$, $\tau(i) = -i$. Then $\tau^2 = \sigma^4 = \mathrm{id}$. We have

	id	σ	σ^2	σ^3	τ	$\tau\sigma$	$\tau\sigma^2$	$\tau\sigma^3$
θ	θ	$i\theta$	$-\theta$	$-i\theta$	θ	$-i\theta$	$-\theta$	$i\theta$
i	i	i	i	i	$-i$	$-i$	$-i$	$-i$

  so $G = \mathrm{Gal}(N/\mathbb{Q}) = \langle \sigma, \tau : \sigma^4 = \tau^2 = \mathrm{id}, \sigma\tau = \tau\sigma^3 \rangle \cong D_4$.
- Order 2 subgroups are $\langle \tau \rangle$, $\langle \tau\sigma \rangle$, $\langle \tau\sigma^2 \rangle$, $\langle \tau\sigma^3 \rangle$, $\langle \sigma^2 \rangle$.

- Order 4 subgroups are $\langle \sigma^2, \tau \rangle \cong (\mathbb{Z}/2)^2$, $\langle \sigma \rangle \cong \mathbb{Z}/4$, $\langle \sigma^2, \tau\sigma \rangle \cong (\mathbb{Z}/2)^2$.
- Respectively, intermediate field extensions of degree 4 are $\mathbb{Q}(\sqrt[4]{2})$, $\mathbb{Q}(i\sqrt[4]{2})$, $\mathbb{Q}(\sqrt{2}, i)$, $\mathbb{Q}((1-i)\sqrt[4]{2})$, $\mathbb{Q}((1+i)\sqrt[4]{2})$.
- Respectively, intermediate field extensions of degree 2 are $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i)$, $\mathbb{Q}(i\sqrt{2})$.

5. Cyclotomic field extensions

5.1. Roots of unity

Definition. $\zeta \in K^*$ is n**-th primitive root of unity** if $\zeta^n = 1$ and $\forall 0 < m < n$, $\zeta^m \neq 1$, i.e. order of ζ in K^* is n.

Example.

- ζ is primitive 1-st root of unity iff $\zeta = 1$.
- -1 is primitive 2-nd root of unity iff $\mathrm{char}(K) \neq 2$.
- If $\mathrm{char}(K) = p$ prime, then K contains no p-th primitive roots of unity (since $\zeta^p = 1 \iff (\zeta - 1)^p = 0 \iff \zeta = 1$).
- If $K = \mathbb{C}$, $\exp(2\pi i/n)$ is n-th primitive root of unity.

Proposition. Let $\zeta \in K^*$ primitive n-th root of unity, let $d = \gcd(m, n)$. Then ζ^m is primitive (n/d)-th root of unity.

Corollary. Let $\zeta \in K^*$ primitive n-th root of unity.

- $\zeta^m = 1 \iff m \equiv 0 \mod n$.
- ζ^m is primitive n-th root of unity iff $\gcd(m, n) = 1$.

Definition. Let $\mu(K)$ denote subgroup of all roots of unity in K^*.

Theorem. Let K field, H finite subgroup of K^*, then H is cyclic.

Remark. This implies that any finite field $\mathbb{F}_q$ can be written $\mathbb{F}_q = \mathbb{F}_{p^n} = \mathbb{F}_p(\alpha)$ where α is generator of $\mathbb{F}_q^\times$.

Corollary. Let K field, $n \in \mathbb{N}$ be largest such that K contains primitive n-th root of unity ζ. Then $\mu(K)$ is cyclic subgroup in K^* generated by ζ.

5.2. n-th cyclotomic field extensions

Notation. Let $\zeta_n = \exp(2\pi i/n) \in \mathbb{C}$.

Definition. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is n**-th cyclotomic field extension**.

Proposition. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois.

Definition. $\Phi_n(x) := \prod_{a\in A}(x-\zeta_n^a)$ where $A = \{a \in \mathbb{N} : 0 < a < n, \gcd(a,n) = 1\}$.

Proposition. $\Phi_n(x) \in \mathbb{Q}[x]$ is irreducible and so is minimal polynomial of a primitive n-th root of unity over $\mathbb{Q}$. In particular, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$, where $\varphi(n) = |(\mathbb{Z}/n)^\times|$ is Euler function.

Proposition. Properties of φ function:

- For prime p, $\varphi(p) = p - 1$.
- For prime p, $\varphi\left(p^k\right) = p^k - p^{k-1}$.
- If $\gcd(n, m) = 1$, then $\varphi(nm) = \varphi(n)\varphi(m)$.
- If $n = \prod_{i=1}^{r} p_i^{k_i}$ is prime factorisation of n, then

$$\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$$

Proposition. $\forall n \in \mathbb{N}, x^n - 1 = \prod_{n_1 | n} \Phi_{n_1}(x)$.

Example.

- $\Phi_1(x) = x - 1$.
- $\Phi_1(x)\Phi_2(x) = x^2 - 1 \Longrightarrow \Phi_2(x) = x + 1$.
- $\Phi_1(x)\Phi_3(x) = x^3 - 1 \Longrightarrow \Phi_3(x) = x^2 + x + 1$.

Proposition.

- For p prime, $\Phi_p(x) = x^{p-1} + \cdots + x + 1$.
- For p prime, $\Phi_{p^k}(x) = \Phi_p(x^{p^{k-1}})$.
- For every $n \in \mathbb{N}$, $\Phi_n(x)$ has integer coefficients.

5.3. Galois properties of cyclotomic extensions

Theorem. $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^\times$.

Remark. To compute $(\mathbb{Z}/n)^\times$, note that for m, n coprime, $(\mathbb{Z}/mn)^\times \cong (\mathbb{Z}/m)^\times \times (\mathbb{Z}/n)^\times$ and

- If $p \neq 2$ prime, then $(\mathbb{Z}/p^r)^\times$ is cyclic of order $\varphi(p^r)$.
- $(\mathbb{Z}/4)^\times \cong \mathbb{Z}/2$ and for $r \geq 3$, $(\mathbb{Z}/2^r)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{r-2}$.

Corollary. $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is abelian so every subgroup is normal, so any subfield of $\mathbb{Q}(\zeta_n)$ is Galois over $\mathbb{Q}$.

Corollary. For p prime, $G = \mathrm{Gal}\big(\mathbb{Q}(\zeta_p)/\mathbb{Q}\big) \cong (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p-1)$. In particular, for $d \mid (p-1)$, $\mathbb{Q}(\zeta_p)$ contains exactly one subfield of degree d and there are no other subfields.

Remark. For $d = 2$ in above corollary, $\mathbb{Q}(\zeta_p)$ contains unique quadratic subfield $\mathbb{Q}(\sqrt{D_p})$, where $D_p = (-1)^{(p-1)/2}p$

Example. $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ not always cyclic, e.g. $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

Proposition.

- If n odd, $\mu(\mathbb{Q}(\zeta_n))$ is cyclic of order $2n$ and is generated by $-\zeta_n$.
- If n even, $\mu(\mathbb{Q}(\zeta_n))$ is cyclic of order n and is generated by ζ_n.
- If $\gcd(m, n) = d$, then $\mathbb{Q}(\zeta_m, \zeta_n) = \mathbb{Q}\Big(\zeta_{mn/d}\Big)$.

5.4. Special properties of $\mathbb{Q}(\zeta_p)$, where $p > 2$ is prime

Example. $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong (\mathbb{Z}/5)^\times$ has generator $\tau : \zeta_5 \mapsto \zeta_5^2$. $\mathbb{Q}$-basis $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ is not invariant under action of τ or any power of τ (since $\tau\big(\zeta_5^2\big) = \zeta_5^4$) but $\{\zeta, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$ is invariant. The same holds for general $p > 2$ prime. For $\alpha_i \in \mathbb{Q}$, $\alpha_1\zeta_p + \cdots + \alpha_{p-1}\zeta_p^{p-1} \in \mathbb{Q}$ iff $\alpha_1 = \cdots = \alpha_{p-1}$.

Example. If $x \in \mathbb{Q}(\zeta_p)$, $[\mathbb{Q}(x) : \mathbb{Q}] = |\{\sigma(x) : \sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})\}|$ In particular, if τ is generator of $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $x = \alpha_1\zeta_p + \cdots + \alpha_{p-1}\zeta_p^{p-1}$ then set of all conjugates of x is equal to (note not all elements are distinct)

$$\{\tau^a(x) : a \in [p-1]\} = \left\{\sum_{i=1}^{p-1} \alpha_i \zeta_p^{ai} : a \in [p-1]\right\}$$

Example. Let $x = \zeta_5 + \zeta_5^4$, $\tau : \zeta_5 \mapsto \zeta_5^2$ is a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. $\tau(x) = \zeta_5^2 + \zeta_5^3 \neq x$ but $\tau^2(x) = x$, so $[\mathbb{Q}(x) : \mathbb{Q}] = 2$, i.e. $\mathbb{Q}\big(\zeta_5 + \zeta_5^4\big)$ is unique quadratic subfield in $\mathbb{Q}(\zeta_5)$.

Definition. Let $x \in \mathbb{Q}(\zeta_p)$, let minimal polynomial of x over $\mathbb{Q}$ be $m(t) = \left(t - x^{(1)}\right) \cdots \left(t - x^{(d)}\right)$. **Conjugates** of x over $\mathbb{Q}$ are $x^{(1)} = x, \ldots, x^{(d)}$.

Example. Minimal polynomial of $\zeta_5 + \zeta_5^4 = 2\cos(2\pi/5)$ over $\mathbb{Q}$ is $m(x) = (x - \zeta_5 - \zeta_5^4)(x - \zeta_5^2 - \zeta_5^3) = x^2 + x - 1$, with roots $\left(-1 \pm \sqrt{5}\right)/2$. So $\cos(2\pi/5) = \left(-1 + \sqrt{5}\right)/4$, and unique quadratic subfield of $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{5})$.

Example. Let $\tau \in G$ be generator of $G = \mathrm{Gal}\big(\mathbb{Q}(\zeta_p)/\mathbb{Q}\big)$, i.e. $\tau\big(\zeta_p\big) = \zeta_p^a$, $a \bmod p$ generates $(\mathbb{Z}/p)^\times$. Let

$$\Theta_p = \zeta_p - \tau\big(\zeta_p\big) + \tau^2\big(\zeta_p\big) - \cdots + \tau^{p-3}\big(\zeta_p\big) - \tau^{p-2}\big(\zeta_p\big)$$

Θ_p behaves like $\sqrt{D_p}$: $\tau(\Theta_p) = -\Theta_p$, $\tau^2(\Theta_p) = \Theta_p$. So $\Theta_p \in \mathbb{Q}(\zeta_p)^{\langle\tau^2\rangle}$. Also, $\tau(\Theta_p^2) = \Theta_p^2$ so $\Theta_p^2 \in \mathbb{Q}(\zeta_p)^{\langle\tau\rangle} = \mathbb{Q}$. In fact, $\Theta_p^2 = D_p$. Therefore

$$\Theta_p^2 = A + B\big(\zeta_p + \cdots + \zeta_p^{p-1}\big) = A - B$$

So when computing Θ_p^2, only need to consider coefficients for 1 and ζ_p.

6. Cyclic field extensions

6.1. Cyclic extensions of degree 2

Example. Let L/K cyclic of degree 2, so $\mathrm{Gal}(L/K) = \{e, \tau\}$, $\tau^2 = e$. Let $\theta \in L - K$, then $\tau(\theta) \neq \theta$ (as otherwise $\theta \in L^{\langle \tau \rangle} = K$). Let $\theta_1 = \tau(\theta) - \theta$, so $\tau(\theta_1) = \tau^2(\theta) - \tau(\theta) = -\theta_1$. If $\mathrm{char}(K) \neq 2$, then $\theta_1 \neq -\theta_1$ and so $\theta_1 \notin K$, $L = K(\theta_1)$. θ_1 is "better" than θ, since $\tau(\theta_1) = -\theta_1$. Now if $a = \theta_1^2$, then $\tau(a) = a$, so $L = K(\sqrt{a})$.

Theorem. If $\mathrm{char}(K) \neq 2$ and L/K is cyclic quadratic extension, then

$$\exists a \in K^\times - K^{\times^2} : \quad L = K(\sqrt{a})$$

Definition. $a_1, ..., a_n$ are **independent modulo** $K^{\times^2}$ **(independent modulo squares)** if

$$a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \in K^{\times^2} \Longleftrightarrow \text{all } \varepsilon_i \text{ are even}$$

Proposition. If $\mathrm{char}(K) \neq 2$:

- $K(\sqrt{a_1}) = K(\sqrt{a_2}) \iff a_1 \equiv a_2 \bmod {K^{\times}}^2$, i.e. $a_1 = a_2 \cdot b^2$, $b \in K^{\times}$.
- If $a_1, ..., a_n \in K^{\times}$ are independent modulo ${K^{\times}}^2$ then $K(\sqrt{a_1}, ..., \sqrt{a_n})$ has degree 2^n over K with Galois group $\cong (\mathbb{Z}/2)^n$.
- If L/K Galois with Galois group $(\mathbb{Z}/2)^n$, then

$$\exists a_1, ..., a_n \in K^{\times} : \quad L = K(\sqrt{a_1}, ..., \sqrt{a_n})$$

Remark. Let $\operatorname{char}(K) = 2$, then $\forall a \in K^\times$, $L = K(\sqrt{a})$ is normal but not separable (since minimal polynomial of e.g. $\sqrt{a}$ is $x^2 - a = (x+\sqrt{a})(x-\sqrt{a}) = (x-\sqrt{a})^2$ so has repeated roots).

6.2. Cyclic extensions of degree n (the Kummer theory)

Definition. L/K is **cyclic of degree** n if it is Galois and $\mathrm{Gal}(L/K)$ is cyclic of order n.

Theorem. If K contains primitive n-th root of unity and for all divisors $d > 1$ of n, $a \in K^\times$ is not d-th power in K, then $L = K(\sqrt[n]{a})$ is cyclic extension of K of degree n. In particular, $x^n - a \in K[x]$ is irreducible.

Proposition. If $\zeta_p \in K$, $a \in K^\times - K^{\times^p}$, then $K(\sqrt[p]{a})/K$ is cyclic of degree p. In particular, $x^p - a \in K[x]$ is irreducible.

Theorem. Let K contain primitive n-th root of unity ζ_n, L/K is cyclic extension of degree n, $\mathrm{Gal}(L/K) = \langle \sigma \rangle$. Then

$$\exists a \in K^\times : L = K(\sqrt[n]{a})$$

Such an a is given by θ_b^n for some $b \in L$, where

$$\theta_b = b + \zeta_n^{-1}\sigma(b) + \cdots + \zeta_n^{-(n-1)}\sigma^{n-1}(b)$$

is **Lagrange resolvent** for b, i.e. $L = K(\theta_b)$.

Lemma (Artin's lemma). There exists $b \in L$ such that $\theta_b \neq 0$.

7. Finite fields

7.1. Existence and uniqueness

Lemma. Let K finite field, then K is field extension of $\mathbb{F}_p$ for some prime p and $|K| = p^n$ where $n = [K : \mathbb{F}_p]$.

Theorem. Let p prime. Then $\forall n \in \mathbb{N}$, there is field K with $|K| = p^n$.

Theorem. Let K finite field with $|K| = q = p^n$. Then

- $\forall \alpha \in K, \alpha^q = \alpha$.
- $x^q - x = \prod_{\alpha \in K}(x - \alpha)$
- K is splitting field of $x^q - x$ over $\mathbb{F}_p$.

Corollary. If K_1, K_2 finite fields, $|K_1| = |K_2|$, then $K_1 \cong K_2$.

Definition. Let $q = p^n$, then $\mathbb{F}_q$ is the unique (up to isomorphism) field containing q elements.

Definition. For $q = p^n$, the **Frobenius automorphism** is

$$\sigma : \mathbb{F}_q \to \mathbb{F}_q, \quad \sigma(\alpha) = \alpha^p$$

which is an $\mathbb{F}_p$-automorphism by Fermat's little theorem.

Theorem. Let $q = p^n$, p prime.

- $\mathbb{F}_q/\mathbb{F}_p$ is Galois of degree n.
- Frobenius automorphism generates $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and there is group isomorphism

$$\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \leftrightarrow \mathbb{Z}/n, \quad \sigma \longleftrightarrow 1 \bmod n$$

7.2. Counting irreducible polynomials over finite fields

Notation. Let $\mathrm{Irr}_{\mathbb{F}_p}(m)$ denote set of all irreducible polynomials in $\mathbb{F}_p[x]$ of degree m. Let $N_p(m) = |\mathrm{Irr}_{\mathbb{F}_p}(m)|$.

Theorem. Let $q = p^m$, then $mN_p(m) = |\{\alpha \in \mathbb{F}_q : \mathbb{F}_p(\alpha) = \mathbb{F}_q\}|$.

Remark. To use above theorem, note that $\mathbb{F}_p(\alpha) \neq \mathbb{F}_{p^m}$ iff α belongs to proper subfield of $\mathbb{F}_{p^m}$.

Example.

- If m is prime, then $\mathbb{F}_{p^m}$ has only one proper subfield $\mathbb{F}_p$, so $mN_p(m) = |\mathbb{F}_{p^m}| - |\mathbb{F}_p| = p^m - p$.
- The proper subfields of $\mathbb{F}_{p^4}$ are $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$, but $\mathbb{F}_p \subset \mathbb{F}_{p^2}$, so $4N_p(4) = |\mathbb{F}_{p^4}| - |\mathbb{F}_{p^2}|$.
- $\mathbb{F}_p(\alpha) \neq \mathbb{F}_{p^6}$ iff $\alpha \in \mathbb{F}_{p^3} \cup \mathbb{F}_{p^2}$. Since $\mathbb{F}_{p^3} \cap \mathbb{F}_{p^2} = \mathbb{F}_p$, we have $6N_p(6) = |\mathbb{F}_{p^6}| - |\mathbb{F}_{p^3}| - |\mathbb{F}_{p^2}| + |\mathbb{F}_p| = p^6 - p^3 - p^2 + p$.

Proposition. We have

$$p^n = \sum_{m \mid n} m N_p(m)$$

which we can use recursively to compute any $N_p(m)$.

Example. We construct $L = \mathbb{F}_{3^{16}}$ by finding irreducible polynomial of degree 16 in $\mathbb{F}_3[x]$.

- $\mathbb{F}_9 = \mathbb{F}_3(\theta)$ where $\theta^2 + 1 = 0$, $\mathbb{F}_9 = \{a + b\theta : a, b \in \mathbb{F}_3\}$. $K := \mathbb{F}_9$ contains primitive 8-th root of unity since $\mathbb{F}_9^\times \cong \mathbb{Z}/8$.
- L/K is cyclic extension of degree 8, so by Kummer theory there exists $\alpha \in K$ such that $L = K(\sqrt[8]{\alpha})$. α must be element that is not square or fourth power in $\mathbb{F}_9$, so we can look for elements that have order 8.
- $\alpha = \theta$ doesn't work since $\theta^2 = -1 \implies \theta^4 = 1$. $\alpha = 1 + \theta$ works since

$$(1+\theta)^2 = \theta^2 + \theta + 1 = -\theta, \quad (1+\theta)^4 = \theta^2 = -1, (1+\theta)^8 = 1$$

so $\alpha = 1 + \theta$ has order 8 in $\mathbb{F}_9$.

- So $L = K(\sqrt[8]{a}) = \mathbb{F}_9(\sqrt[8]{1+\theta}) = \mathbb{F}_3(\theta, \sqrt[8]{1+\theta}) = \mathbb{F}_3(\eta)$ where $\eta^8 = 1 + \theta$. Now $[L : \mathbb{F}_3] = 16$ by tower law, so $L = \mathbb{F}_{3^{16}}$ by uniqueness of finite fields.
- $\eta^8 = 1 + \theta \implies (\eta^8 - 1)^2 = \theta^2 = -1 \implies \eta^{16} + \eta^8 + 2 = 0$ so $f(x) = x^{16} + x^8 + 2 \in \mathbb{F}_3[x]$ is irreducible.

8. Galois groups of polynomials

8.1. Symmetric functions

Definition. Define action of S_n on $L = k(x_1, ..., x_n)$ by $\tau : x_j \mapsto x_{\pi(j)}$ where $\pi \in S_n$, which gives k-automorphism

$$\tau : L \to L, \quad \frac{f(x_1, ..., x_n)}{g(x_1, ..., x_n)} \mapsto \frac{f(x_{\pi(1)}, ..., x_{\pi(n)})}{g(x_{\pi(1)}, ..., x_{\pi(n)})}$$

The **symmetric functions** in L are elements of fixed field L^{S_n}.

Definition. The **elementary symmetric polynomials** $e_r \in L$ for $r \in [n]$ are

$$\begin{aligned}
e_1 &= \sum_{1 \le i \le n} x_i \\
e_2 &= \sum_{1 \le i < j \le n} x_i x_j \\
&\vdots \\
e_r &= \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r} \\
&\vdots \\
e_n &= x_1 \cdots x_n
\end{aligned}$$

Define $K = k(e_1, ..., e_n)$.

Theorem. $K = L^{S_n}$ and L/K is Galois with $\mathrm{Gal}(L/K) \cong S_n$.

Proof.

- Note that $f(x) = (x - x_1) \cdots (x - x_n) = x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n$.
- Show L splitting field of $f(x)$ over K and $[L : K] \leq n!$.
- Show $[L : K] \geq n!$.

$\square$

Remark. Every finite group G is subgroup of S_n for some n, so there is always Galois extension with Galois group G: let $L = k(x_1, ...x_n)$, let $G \subseteq S_n$ act on L as above, then $\mathrm{Gal}\left(L/L^G\right) = G$.

Definition. For $f(x) \in K[x]$, **Galois group** of $f(x)$, G_f, is Galois group of splitting field of $f(x)$ over K (provided this extension is separable). If $\deg(f) = n$, G_f acts by permuting roots $\theta_1, ..., \theta_n$ of f, so is subgroup of S_n. There can be non-trivial relationships between roots, so G_f may be proper subgroup.

Corollary. Any symmetric polynomial in $k[x_1, ..., x_n]$ can be expressed as polynomial in elementary symmetric polynomials, i.e.

$$k[x_1, ..., x_n]^{S_n} = k[e_1, ..., e_n]$$

where LHS is set of symmetric polynomials, RHS is set of polynomials in elementary symmetric polynomials.

Example.

- When $n = 2$, $x_1^2 + x_2^2 = e_1^2 - 2e_2$ and $x_1^3 + x_2^3 = e_1^3 - 3e_1 e_2$.
- When $n = 3$, $x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + x_2 x_3^2 + x_3^2 x_1 + x_3 x_1^2 = e_1 e_2 - 3e_3$.

Definition. Lexicographic ordering of monomials, $>_{\text{lex}}$ (or $\succ_L$), is

$$x_1^{a_1} \cdots x_n^{a_n} >_{\text{lex}} x_1^{b_1} \cdots x_n^{b_n}$$

iff $\exists 0 \leq j \leq n-1$ such that $a_1 = b_1, \ldots, a_j = b_j$ and $a_{j+1} > b_{j+1}$.

Example. $x_1^2 x_2^3 x_3 >_{\text{lex}} x_1^2 x_2^2 x_3^4$.

Definition. **Leading term** of $f(x_1, ..., x_n) \in k[x_1, ..., x_n]$ is largest monomial $cx_1^{a_1} \cdots x_n^{a_n}$ with $c \neq 0$, $a_i \neq 0$ for some i, appearing in f with respect to lexicographic ordering.

Note. If f is symmetric, then $a_1 \geq \cdots \geq a_n$.

Algorithm. Given $f(x_1, ..., x_n) \in k[x_1, ..., x_n]^{S_n}$, express f as polynomial in elementary symmetric polynomials:

1. Find leading term $cx_1^{a_1} \cdots x_n^{a_n}$ of f, compute

$$f_1 = f - ce_1^{a_1 - a_2} \cdots e_{n-1}^{a_{n-1} - a_n} e_n^{a_n}$$

   Note leading term of $ce_1^{a_1 - a_2} \cdots e_{n-1}^{a_{n-1} - a_n} e_n^{a_n}$ is also $cx_1^{a_1} \cdots x_n^{a_n}$ so leading term of f_1 is strictly smaller than leading term of f. Also, f_1 is symmetric.

2. If $f_1 \neq 0$, apply step 1 to get f_2, f_3, ... Since leading term of $f_1, f_2, ...$ is strictly decreasing, eventually $f_i = 0$.

Example. Express $f(x_1, x_2) = x_1^3 + x_2^3$ in elementary symmetric polynomials:

- Leading term of f is $x_1^3 = x_1^3 x_2^0$, so

$$f_1 = f - e_1^{3-0} e_2^0 = -3x_1^2 x_2 - 3x_1 x_2^2$$

- Leading term of f_1 is $-3x_1^2 x_2$, so

$$f_2 = f_1 - (-3)e_1^{2-1} e_2^1 = -3x_1^2 x_2 - 3x_1 x_2^2 + 3(x_1 + x_2)x_1 x_2 = 0$$

- So $f_1 = f_2 + (-3)e_1^{2-1} e_2^1 = -3e_1 e_2$ and $f = e_1^3 + f_1 = e_1^3 - 3e_1 e_2$.

Example.

- Let $\theta_1 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3)$, $\theta_2 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3)$, where $\omega = \zeta_3$.
- Let $\sigma = (1\ 2\ 3) \in S_3$, then $\sigma(\theta_1) = \omega^2\theta_1$, $\sigma(\theta_2) = \omega\theta_2$, hence

$$\sigma(\theta_1^3 + \theta_2^3) = \omega^6\theta_1^3 + \omega^3\theta_2^3 = \theta_1^3 + \theta_2^3$$

- Let $\tau = (2\ 3) \in S_3$, then $\tau(\theta_1) = \theta_2$, $\tau(\theta_2) = \theta_1$ so $\tau(\theta_1^3 + \theta_2^3) = \theta_1^3 + \theta_2^3$.
- Since $S_3 = \langle \sigma, \tau \rangle$, $f(x_1, x_2, x_3) = 27(\theta_1^3 + \theta_2^3) \in \mathbb{Q}[x_1, x_2, x_3]^{S_3}$. Applying the algorithm:
  - $f_1 = f - 2e_1^3 = 9(x_1^2 x_2 + \cdots)$.
  - $f_2 = f_1 - (-9)e_1 e_2 = 27x_1 x_2 x_3$.

- $f_3 = f_2 - 27e_3 = 0$.
- So $f = 2e_1^3 - 9e_1e_2 + 27e_3$.

- By a similar process, $9\theta_1\theta_2 = e_1^2 - 3e_2$.

8.2. Galois theory for cubic polynomials

Example (Solving quadratic). Let $\mathrm{char}(k) \neq 2$. General quadratic polynomial can be written as

$$f(x) = x^2 - e_1 x + e_2 = (x - x_1)(x - x_2) \in K[x]$$

where $e_1 = x_1 + x_2, e_2 = x_1 x_2 \in K = k(e_1, e_2)$. Let $L = k(x_1, x_2) = K(x_1)$, then L/K is Galois and $\mathrm{Gal}(L/K) = \{\mathrm{id}, \sigma\} \cong S_2 \cong \mathbb{Z}/2$ where $\sigma(x_1) = x_2$, $\sigma(x_2) = x_1$. Since L/K cyclic and $\zeta_2 = -1 \in K$, by Theorem 6.2.4, Lagrange resolvent of x_1 is

$$\theta = \theta_{x_1} = x_1 + \zeta_2^{-1} \sigma(x_1) = x_1 - x_2$$

So $\sigma(\theta) = -\theta$ and $\theta^2 = e_1^2 - 4e_2$. $\Delta = \theta^2$ is **discriminant** of $f(x)$. So we have $x_1 = (e_1 + \sqrt{\Delta})/2$, $x_2 = (e_1 - \sqrt{\Delta})/2$. If $f(x)$ is irreducible, it has distinct roots, and so Galois group $G_f \cong \mathbb{Z}/2$.

Example (Solving cubic).

- Let $\mathrm{char}(k) \neq 2, 3$, let

$$f(x) = x^3 - e_1 x^2 + e_2 x - e_3 = (x - x_1)(x - x_2)(x - x_3) \in K[x]$$

  where $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $e_3 = x_1x_2x_3 \in K = k(e_1, e_2, e_3) \subset L = K(x_1, x_2, x_3)$.
- By Theorem 8.1.3, $\mathrm{Gal}(L/K) = S_3$ with normal subgroup $A_3 \cong \mathbb{Z}/3$. We have tower $K \subset M = L^{A_3} \subset L$. So $\mathrm{Gal}(L/M) \cong A_3 \cong \mathbb{Z}/3$, $\mathrm{Gal}(M/K) \cong S_3/A_3 \cong \mathbb{Z}/2$.
- Assume k contains primitive 3rd root of unity ω, so w^2 is also primitive 3rd root of unity. Define

$$\theta_1 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3), \quad t_1 = \theta_1^3,$$

$$\theta_2 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3), \quad t_2 = \theta_2^3$$

then $t_1, t_2 \in M$ and $L = M(\theta_1) = M(\theta_2)$. By Example 8.1.14, $27(\theta_1^3 + \theta_2^3) = 2e_1^3 - 9e_1e_2 + 27e_3$, $9\theta_1\theta_2 = e_1^2 - 3e_2$, so t_1, t_2 are roots of **quadratic resolvent** of $f(x)$:

$$(t - t_1)(t - t_2) = t^2 - \left(\frac{2e_1^3 - 9e_1e_2 + 27e_3}{27}\right)t + \left(\frac{e_1^2 - 3e_2}{9}\right)^3$$

- To find roots x_1, x_2, x_3 of f:
  - Solve quadratic resolvent to find t_1, t_2.
  - Choose $\theta_1 = \sqrt[3]{t_1}$, find θ_2 from $9\theta_1\theta_2 = e_1^2 - 3e_2$.
  - Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 = e_1 \\ x_1 + \omega x_2 + \omega^2 x_3 = 3\theta_1 \\ x_1 + \omega^2 x_2 + \omega x_3 = 3\theta_2 \end{cases} \implies \begin{cases} x_1 = e_1/3 + \theta_1 + \theta_2 \\ x_2 = e_1/3 + \omega^2\theta_1 + \omega\theta_2 \\ x_3 = e_1/3 + \omega\theta_1 + \omega^2\theta_2 \end{cases}$$

Remark. To solve general cubic $f(x) = x^3 + ax^2 + bx + c$, first perform shift:

$$f(x - a/3) = x^3 + px + q$$

then quadratic resolvent is (*memorise*)

$$t^2 + qt - \frac{p^3}{27}$$

with roots $t_1 = \theta_1^3$, $t_2 = \theta_2^3$, choose θ_1, θ_2 such that $\theta_1\theta_2 = -\frac{p}{3}$, then roots of $f(x - a/3)$ are $x_1 = \theta_1 + \theta_2$, $x_2 = \omega^2\theta_1 + \omega\theta_2$, $\omega\theta_1 + \omega^2\theta_2$.

Example (Galois groups of cubic polynomials). Let $\mathrm{char}(K) \neq 2, 3$, $f(x) = x^3 + ax^2 + bx + c \in K[x]$, let L be splitting field for $f(x)$ over K, then $G_f = \mathrm{Gal}(L/K)$. Let $\alpha_1, \alpha_2, \alpha_3$ be roots of $f(x)$ in L.

- If $\alpha_1, \alpha_2, \alpha_3 \in K$, then $L = K$, $G_f = \{\mathrm{id}\}$.
- If $f(x) = \big(x - \alpha_j\big)g(x)$ where $\alpha_j \in K$, $g(x) \in K[x]$ irreducible quadratic, then $[L : K] = 2$, $G_f \cong \mathbb{Z}/2$.
- If $f(x)$ irreducible in $K[x]$, then $K \subset K(\alpha_1) \subseteq K(\alpha_1, \alpha_2, \alpha_3) = L$, then either $[L : K(\alpha_1)] = 1$, so $[L : K] = 3$ and $G_f \cong A_3 \cong \mathbb{Z}/3$, or $[L : K(\alpha_1)] = 2$, so $[L : K] = 6$ and $G_f \cong S_3$.

Definition. **Discriminant** of $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ is $\Delta = \delta^2$ where

$$\delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$$

Note $\Delta \neq 0$ if f has distinct roots.

Note. If $G_f \cong A_3$, then $G_f = \langle \tau \rangle$ where $\tau : \alpha_1 \mapsto \alpha_2$, $\alpha_2 \mapsto \alpha_3$, $\alpha_3 \mapsto \alpha_1$, then $\tau(\delta) = \delta$ so $\delta \in L^{G_f} = K$ and $\Delta \in K^{\times^2}$. But if $G_f \cong S_3$, then if $\tau \in A_3$, $\tau(\delta) = \delta$ and if $\tau \in S_3 - A_3$, then $\tau(\delta) = -\delta$ so $\delta \notin K$ but $\Delta \in K$.

Theorem. Let $f(x) \in K[x]$ irreducible, $\deg(f) = 3$. Then

- $G_f \cong A_3 \iff \Delta \in {K^{\times}}^2$,
- $G_f \cong S_3 \iff \Delta \in K^{\times} - {K^{\times}}^2$.

Theorem. Let $f(x) = x^3 + ax^2 + bx + c \in K[x]$, then

$$\Delta = 18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^2$$

For reduced cubic $f(x) = x^3 + px + q$, (*memorise*)

$$\Delta = -4p^3 - 27q^2$$

Note. The reduced form of $f(x)$ has same discriminant as $f(x)$.

8.3. Galois theory for quartic polynomials

Example. Let $\mathrm{char}(k) \neq 2, 3$, $K = k(e_1, e_2, e_3, e_4) \subseteq L = k(x_1, x_2, x_3, x_4)$, so L is splitting field over K of $f(x) = x^4 - e_1 x^3 + e_2 x^2 - e_3 x + x_4$ and $\mathrm{Gal}(L/K) \cong S_4$.

Remark. S_4 can be visualised as symmetries of regular tetrahedron with vertices labelled $\{1, 2, 3, 4\}$. Consider three pairs of opposite edges

$$P_1 = \{(1, 2), (3, 4)\}, \quad P_2 = \{(1, 3), (2, 4)\}, \quad P_3 = \{(1, 4), (2, 3)\}$$

Any permutation in S_4 of the four vertices permutes P_1, P_2, P_3, which gives map $\pi : S_4 \to S_3$.

- π is surjective group homomorphism.
- π has kernel $\ker(\pi) = \{\mathrm{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

- $A_4 \subset S_4$ is subgroup of even permutations (orientation-preserving symmetries). Restriction of π to A_4 gives another surjective homomorphism $A_4 \to A_3$ (and $\pi^{-1}(A_3) = A_4$) also with kernel V_4.
- V_4 is kernel so is normal subgroup of S_4 and of A_4. Note that V_4 is only subgroup of A_4 isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, but there are four subgroups of S_4, isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, with V_4 the only normal one.
- This gives increasing sequence of subgroups in S_4

$$\{\mathrm{id}\} \subset \mathbb{Z}/2 \subset V_4 \subset A_4 \subset S_4$$

and $V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, $A_4/V_4 \cong A_3 \cong \mathbb{Z}/3$, $S_4/A_4 \cong \mathbb{Z}/2$.

- Each G_i in this sequence is normal subgroup of G_{i+1} and G_{i+1}/G_i is cyclic, meaning that S_4 is **solvable (soluble) group**.
- We have tower

$$K = L^{S_4} \subset L^{V_4} \subset L = L^{\{e\}}$$

By fundamental theorem, $\mathrm{Gal}\big(L/L^{V_4}\big) = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, so L/L^{V_4} appears as biquadratic extension.
- V_4 is normal in S_4 so by fundamental theorem, $\mathrm{Gal}\big(L^{V_4}/K\big) \cong S_4/V_4 \cong S_3$ by first isomorphism theorem. Hence L^{V_4} appears as splitting field of a cubic polynomial over K.

Example (Solving quartic equations). Define

$$\theta_1 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4),$$

$$\theta_2 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4),$$

$$\theta_3 = \frac{1}{2}(x_1 - x_2 - x_3 + x_4)$$

Then $\forall j \in [3], \forall \sigma \in V_4$, $\sigma\left(\theta_j\right) = \pm\theta_j$. The θ_j arise from Lagrange resolvents for the three quadratic subextensions of L^{V_4} in L. They behave like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Each $t_i = \theta_i^2$ is fixed by V_4

and are permuted by $S_4/V_4 \cong S_3$. They are roots of **cubic resolvent** of $f(x)$:

$$(t - t_1)(t - t_2)(t - t_3) = t^3 + s_1 t^2 + s_2 t + s_3$$

which has coefficients in $(L^{V_4})^{S_3} = L^{S_4} = K$. To find roots x_1, x_2, x_3, x_4 of $f(x)$:

- Solve cubic resolvent to find t_1, t_2, t_3.
- Set $\theta_j = \pm\sqrt{t_j}$ where signs are chosen so that $\theta_1\theta_2\theta_3 = (e_1^3 - 4e_1e_2 + 8e_3)/8$.
- Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = e_1 \\ x_1 + x_2 - x_3 + x_4 = 2\theta_1 \\ x_1 - x_2 + x_3 - x_4 = 2\theta_2 \\ x_1 - x_2 - x_3 + x_4 = 2\theta_3 \end{cases} \Longrightarrow \begin{cases} x_1 = e_1/4 + (\theta_1 + \theta_2 + \theta_3)/2 \\ x_2 = e_1/4 + (\theta_1 - \theta_2 - \theta_3)/4 \\ x_3 = e_1/4 + (-\theta_1 + \theta_2 - \theta_3)/2 \\ x_4 = e_1/4 + (-\theta_1 - \theta_2 + \theta_3)/2 \end{cases}$$

Remark. In practice, perform shift to kill x^3 coefficient to obtain **reduced quartic**:

$$f(x - a/4) = x^4 + px^2 + qx + r$$

- Cubic resolvent is *(memorise)*

$$t^3 + 2pt^2 + (p^2 - 4r)t - q^2$$

- Choose $\theta_1, \theta_2, \theta_3$ such that *(memorise)*

$$\theta_1 \theta_2 \theta_3 = -q$$

- Roots of $f(x - a/4)$ are *(memorise)*

$$\begin{aligned}
x_1 &= \frac{1}{2}(\theta_1 + \theta_2 + \theta_3), \\
x_2 &= \frac{1}{2}(\theta_1 - \theta_2 - \theta_3), \\
x_3 &= \frac{1}{2}(-\theta_1 + \theta_2 - \theta_3), \\
x_4 &= \frac{1}{2}(-\theta_1 - \theta_2 + \theta_3)
\end{aligned}$$

- Recover roots of $f(x)$ by subtracting $a/4$.

Example. Find all complex roots of $f(x) = x^4 + 6x^3 + 18x^2 + 30x + 25$.

- Eliminate x^3 term:

$$f(x - 6/4) = x^4 + \frac{9}{2}x^2 + 3x + \frac{85}{16}$$

- $p = 9/2$, $q = 3$, $r = 85/16$, so cubic resolvent is

$$t^3 + 2pt^2 + (p^2 - 4r)t - q^2 = t^3 + 9t^2 - t - 9 = (t-1)(t+1)(t+9)$$

  So roots are $t_1 = 1$, $t_2 = -1$, $t_3 = -9$. Set $\theta_1 = \sqrt{t_1} = 1$, $\theta_2 = \sqrt{t_2} = i$, $\theta_3 = \pm\sqrt{t_3} = \pm 3i$ so that $\theta_1\theta_2\theta_3 = -q = -3$, i.e. $\theta_3 = 3i$.
- So roots of $f(x - 3/2)$ are

$$
\begin{aligned}
x_1 &= \frac{1}{2}(\theta_1 + \theta_2 + \theta_3) = \frac{1}{2}(1 + 4i), \\
x_2 &= \frac{1}{2}(\theta_1 - \theta_2 - \theta_3) = \frac{1}{2}(1 - 4i), \\
x_3 &= \frac{1}{2}(-\theta_1 + \theta_3 - \theta_3) = \frac{1}{2}(-1 - 2i), \\
x_4 &= \frac{1}{2}(-\theta_1 - \theta_2 + \theta_3) = \frac{1}{2}(-1 + 2i)
\end{aligned}
$$

- So roots of $f(x)$ are $-1 \pm 2i$, $-2 \pm i$.

Example (Galois groups of quartic polynomials).

- Let $\mathrm{char}(K) \neq 2, 3$, $f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$. Galois group is $G_f = \mathrm{Gal}(L/K)$ where L is splitting field for $f(x)$ over K, and G_f is subgroup of S_4.
- Assume that $f(x)$ irreducible in $K[x]$. It can be shown there are five possible isomorphism classes of Galois groups: $S_4, A_4, V_4, \mathbb{Z}/4$ or D_4.
- Let $R(t) \in K[t]$ be cubic resolvent of $f(x)$ with roots $t_1 = \theta_1^2$, $t_2 = \theta_2^2$, $t_3 = \theta_3^2$. Let M be splitting field of $R(t)$ over K, so

$$K \subset K(t_1, t_2, t_3) \subset M \subset L = M(\theta_1, \theta_2, \theta_3)$$

Theorem. Let $f(x) \in K[x]$ irreducible and have irreducible cubic resolvent $R(t) \in K[t]$ with roots $t_1 = \theta_1^2$, $t_2 = \theta_2^2$, $t_3 = \theta_2^3$. Let L be splitting field of $f(x)$ over K (so $G_f = \mathrm{Gal}(L/K)$) and let M be splitting field of $R(t)$ over K (so $G_R = \mathrm{Gal}(M/K)$).

- If $\Delta_R \in K^{\times^2}$ (i.e. $G_R \cong A_3$ and $[M : K] = 3$), then $G_f \cong A_4$.
- If $\Delta_R \in K^\times - K^{\times^2}$ (i.e. $G_R \cong S_3$ and $[M : K] = 6$), then $G_f \cong S_4$.

Proof.

- Sufficient to prove $[L:M]=4$ since then $[L:K]=12$ or 24 by Tower Law.
- Show M does not contain θ_1, θ_2 or θ_3.
  - Suppose it does, so WLOG $\theta_1 \in M$. $\mathrm{Gal}(M/K) \cong A_3$ or S_3, so must be order 3 element $\sigma \in \mathrm{Gal}(M/K)$. $\sigma(\theta_1)$ and $\sigma^2(\theta_1)$ are the other two roots θ_2 and θ_3 since $R(t)$ is irreducible and $\theta_1, \theta_2, \theta_3 \in M$. But this implies $M = L$ so $[L:K]=3$ or 6, but $4 \mid [L:K]$ since L contains roots of irreducible quartic.
- $M(\theta_1)/M$ is degree 2. Assume $\theta_2 \in M(\theta_1)$. $\mathrm{Gal}(M(\theta_1)/M) = \{\mathrm{id}, \tau\}$ for some $\tau : \theta_1 \mapsto -\theta_1$. Also $\theta_2^2 \in M$ so $\tau(\theta_2) = \pm\theta_2$.

- If $\tau(\theta_2) = \theta_2$, then $\theta_2 \in M$: contradiction.
- If $\tau(\theta_2) = -\theta_2$, then $\tau(\theta_1\theta_2) = (-\theta_1)(-\theta_2) = \theta_1\theta_2$ hence $\theta_1\theta_2 \in M$. But $\theta_1\theta_2\theta_3 \in K$ and $\theta_1\theta_2 \neq 0$ since $R(t)$ irreducible. But then $\theta_3 \in M$: contradiction.

- Hence $[M(\theta_1, \theta_2) : M] \geq 4$, and $\theta_1\theta_2\theta_3 \in M$ so $L = M(\theta_1, \theta_2)$ and $[L : M] = 4$.

$\square$

Example.

- If $f(x) \in K[x]$ but cubic resolvent $R(t) \in K[t]$ is reducible, it is possible that all roots $t_1 = \theta_1^2$, $t_2 = \theta_2^2$, $t_3 = \theta_3^2$ are in K. Then $M = K$ and $L = K(\theta_1, \theta_2, \theta_3)$. Since $\theta_1\theta_2\theta_3 \in K$, L/K is obtained by adjoining only two square roots to K. Since $f(x)$ irreducible of degree 4, we have $[L : K] \geq 4$, hence only option is biquadratic extension $G_f = \mathrm{Gal}(L/K) = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- If only one root t_1, t_2, t_3 is in K (say it is t_1):
  - M is splitting field of irreducible quadratic over K. Hence $M = K(\sqrt{d})$ for some $d \in K^\times - K^{\times^2}$ and $\mathrm{Gal}(M/K) = \{\mathrm{id}, \varphi\} \cong \mathbb{Z}/2$ where $\varphi(\sqrt{d}) = -\sqrt{d}$.

- We have

$$K \subset M = K(\sqrt{d}) = K(\alpha, \overline{\alpha}) \subset L = M(\sqrt{\alpha}, \sqrt{\overline{\alpha}})$$

where α and $\overline{\alpha} = \varphi(\alpha)$ are conjugate elements in $M^\times - M^{\times^2}$ (corresponding to t_2 and t_3).
- In this case, L/K is normal extension, since if $\alpha, \overline{\alpha}$ are roots of $x^2 + ax + b \in K[x]$, then $\pm\sqrt{\alpha}, \pm\sqrt{\overline{\alpha}}$ are roots of $x^4 + ax^2 + b \in K[x]$. So L is splitting field of $x^4 + ax^2 + b$ over K. For above tower of fields, we have Galois groups

$$\{\mathrm{id}\} \subset \mathrm{Gal}(L/M) = H \subset \mathrm{Gal}(L/K) = G$$

and $G/H \cong \mathrm{Gal}(M/K) = \{\mathrm{id}, \varphi\} \cong \mathbb{Z}/2$.

Theorem. Let $M = K(\sqrt{d})$, $d \notin {K^\times}^2$, $\mathrm{Gal}(M/K) = \{\mathrm{id}, \varphi\}$. Let α, $\overline{\alpha} = \varphi(\alpha) \in M^\times - {M^\times}^2$, and let $L = M(\sqrt{\alpha}, \sqrt{\overline{\alpha}})$, $G = \mathrm{Gal}(L/K)$.

- If $\alpha\overline{\alpha} \in {K^\times}^2$, then $[L:K] = 4$ and $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- If $\alpha\overline{\alpha} \in {M^\times}^2 - {K^\times}^2$ then $[L:K] = 4$ and $G \cong \mathbb{Z}/4$.
- If $\alpha\overline{\alpha} \notin {M^\times}^2$, then $[L:K] = 8$ and $G \cong D_4$.

Note. In the case that $C := \alpha\overline{\alpha} \notin M^{\times^2}$ and so $G \cong D_4$:

- We have $\mathrm{Gal}(M/K) = \{\mathrm{id}, \varphi\}$, $\varphi : \sqrt{d} \mapsto -\sqrt{d}$.
- There are two lifts of φ to L:

$$\tau : (\sqrt{d}, \sqrt{C}, \sqrt{\alpha}) \mapsto (-\sqrt{d}, \sqrt{C}, \sqrt{\overline{\alpha}}),$$

$$\sigma : (\sqrt{d}, \sqrt{C}, \sqrt{\alpha}) \mapsto (-\sqrt{d}, -\sqrt{C}, \sqrt{\overline{\alpha}})$$

  (so $\tau(\sqrt{\overline{\alpha}}) = \sqrt{\alpha}$, $\sigma(\sqrt{\overline{\alpha}}) = -\sqrt{\alpha}$)
- Then $G = \mathrm{Gal}(L/K) = \langle \tau, \sigma \mid \tau^2 = \sigma^4 = e, \tau\sigma = \sigma^3\tau \rangle$.

8.4. A criterion for solvability by radicals

Note. Assume all fields in this section have characteristic 0.

Definition. L/K is **radical extension** if there is tower of field extensions

$$K = K_0 \subset \cdots \subset K_m = L$$

where for each $1 \le i \le m$, $K_i = K_{i-1}(\sqrt[n_i]{\alpha_i})$ with $\alpha_i \in K_{i-1}$ and $n_i \in \mathbb{N}$.

Example. Let $\alpha = \sqrt[3]{2 + \sqrt[5]{3 - \sqrt{7}}}$. We have

$$K_0 = \mathbb{Q} \subset K_1 = \mathbb{Q}(\sqrt{7}) \subset K_2 = K_1\left(\sqrt[5]{3 - \sqrt{7}}\right) \subset K_3 = K_2(\alpha)$$

Definition. $f(x) \in K[x]$ is **solvable in radicals** over K if there is a radical extension L of K containing at least one root of $f(x)$.

Lemma. If $f(x)$ irreducible and solvable in radicals, then all its roots belong to the radical field extension L.

Definition. A finite group G is **solvable (soluble)** if there exists decreasing sequence of subgroups

$$G = G_0 \supset \cdots \supset G_m = \{\mathrm{id}\}$$

where for each $1 \leq i \leq m$, G_i is normal subgroup of G_{i-1} and G_{i-1}/G_i is cyclic.

Lemma (Properties of solvable groups).

- Every subgroup of finite solvable group is solvable.
- Abelian groups are solvable.
- S_n is solvable iff $n \leq 4$.
- Let G finite group with normal subgroup H. Then G is solvable iff both H and G/H are solvable.

Theorem (Galois' Theorem: Criterion for solvability in radicals). Let $f(x) \in K[x]$ irreducible. Then $f(x)$ is solvable in radicals over K iff Galois group G_f is solvable.

8.5. Polynomials not solvable by radicals

Lemma. A_n is generated by 3-cycles $(i\; j\; k)$.

Proof.

- $A_1 = A_2 = \{\mathrm{id}\}$.
- For $n \geq 3$, any element in A_n is product of even number of transpositions.
- Combine pairs of transpositions as follows:
  - $(ij)(ij) = \mathrm{id}$.
  - $(ij)(ik) = (ikj)$.
  - $(ij)(kl) = (ik)(jk)(jk)(kl) = (ijk)(jkl)$.

$\square$

Theorem. For $n \geq 5$, A_n and S_n are not solvable.

Proof.

- Assume A_n solvable, so there is decreasing sequence of subgroups

$$A_n = G_0 \supset \cdots \supset G_m = \{\text{id}\}$$

with G_i normal in G_{i-1}, G_{i-1}/G_i cyclic and so abelian. So we have canonical projection homomorphism $\pi : A_n \to Q = A_n/G_1$, Q is abelian and non-trivial.

- Let $g = (i_1 i_2 i_3) \in A_n$. There are $i_4, i_5 \in [n]$ (since $n \geq 5$) such that i_1, i_2, i_3, i_4, i_5 distinct. Let $g_1 = (i_1 i_2 i_4)$, $g_2 = (i_1 i_3 i_5)$, then $g_1 g_2 g_1^{-1} g_2^{-1} = g$.
- Since Q abelian, $\pi(g) = \pi(g_1)\pi(g_2)\pi(g_1)^{-1}\pi(g_2)^{-1} = \text{id}$.

- So π sends 3-cycles to id, and A_n is generated by 3-cycles, so $\pi(A_n) = \{\mathrm{id}\}$ which is the trivial group: contradiction.

$\square$

Theorem. Let $f(x) \in \mathbb{Q}[x]$ irreducible polynomial of degree 5 with exactly 3 real roots. Then $f(x)$ has Galois group $G_f \cong S_5$ (and so $f(x)$ is not solvable by radicals over $\mathbb{Q}$).