# Complex Analysis II Course Notes

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February 9, 2023

# Contents

# 1 The complex plane and Riemann sphere

## 1.1 Complex numbers

**Definition 1.1.1.** A **complex number** $z$ is a number $z = x + iy$ where $x, y \in \mathbb{R}^2$ and $i$ is the **imaginary unit**. The set of all complex numbers is written as $\mathbb{C}$.

**Definition 1.1.2.** For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, addition, subtraction and multiplication of complex numbers is defined as

$$\begin{aligned} z_1 \pm z_2 &:= (x_1 \pm x_2) + i(y_1 \pm y_2) \\ z_1 z_2 &:= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

**Definition 1.1.3.** For a complex number $z = x + iy$, the **real part** of $z$, $\mathrm{Re}(z)$, is $x$ and the **imaginary part**, $\mathrm{Im}(z)$, is $y$.

**Definition 1.1.4.** For a complex number $z = x + iy$, the **complex conjugate** of $z$, $\overline{z}$ is defined as $\overline{z} = x - iy$.

**Definition 1.1.5.** Division of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$ is given by

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

This gives a multiplicative inverse for every $z = x + iy \neq 0$:

$$z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y_2} - i\frac{y}{x^2 + y^2}$$

**Definition 1.1.6.** The **modulus** or **absolute value** of a complex number $z = x + iy$, $|z|$, is defined as $|z| = \sqrt{x^2 + y^2}$.

**Lemma 1.1.7.**

1. $\forall z_1, z_2 \in \mathbb{C}^2, z_1 z_2 \iff z_1 = 0$ or $z_2 = 0$.
2. $\forall z \in \mathbb{C}, |z| = \sqrt{z\overline{z}}$.
3. $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.
4. $z^{-1} = \frac{\overline{z}}{|z|^2}$.

*Proof.* Easy. □

**Definition 1.1.8.** For a complex number $z = x + iy$ plotted on an Argand diagram (where $z$ is at the point $(x, y)$), the **argument** of $z$, $\arg(z)$, is the anticlockwise angle $\theta$ from the real axis to $z$.

**Definition 1.1.9.** For a complex number $z$ with $|z| = r$ and $\arg(z) = \theta$, $z$ can be written in polar coordinates:

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

**Definition 1.1.10.** $\arg(z)$ is only defined up to multiples of $2\pi$. The **principal value** of $\arg(z)$ is the value of $\arg(z)$ in the interval $(-\pi, \pi]$, written as $\mathrm{Arg}(z)$.

**Lemma 1.1.11.**

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$.
2. $\arg(1/z) = -\arg(z) \pmod{2\pi}$.
3. $\arg(\overline{z}) = -\arg(z) \pmod{2\pi}$.

*Proof.* Easy. □

**Lemma 1.1.12.** Multiplication in $\mathbb{C}$ can be geometrically described as a dilated rotation: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

*Proof.*

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2 ((\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\cos(\theta_2)\sin(\theta_1) + \sin(\theta_2)\cos(\theta_1))) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

□

**Remark.**

- Multiplying $z_1$ by $z_2$ represents a rotation of $z_1$ by $\theta_2$ anticlockwise, followed by a dilation of factor $r_2$.
- Addition represents to translation.
- Complex represents reflection in the real axis.
- Taking the real part represents projection onto the real axis.
- Taking the imaginary part represents projection onto the imaginary axis.

**Corollary 1.1.13.**

1. $\forall z_1, z_2 \in \mathbb{C}^2, |z_1 z_2| = |z_1||z_2|$.
2. $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$ (de Moivre's formula).
3. $\forall z_1, z_2, |z_1 + z_2| \leq |z_1| + |z_2|$.
4. $\forall z \in \mathbb{C}, |z| \geq 0$ and $|z| = 0 \iff z = 0$.
5. $\max\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\} \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.

*Proof.* Easy. □

**Definition 1.1.14.** The **upper half** of the complex plane, $\mathbb{H}$, is defined as

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

## 1.2 Exponential and trigonometric functions

**Definition 1.2.1.** The **complex exponential function** $\exp : \mathbb{C} \to \mathbb{C}$, written as $e^z$, is defined as

$$e^z = \exp(z) := e^x(\cos(y) + i\sin(y))$$

**Proposition 1.2.2.**

1. $\forall z \in \mathbb{C}, e^z \neq 0$.
2. $\forall z_1, z_2 \in \mathbb{C}^2, e^{z_1+z_2} = e^{z_1}e^{z_2}$.
3. $e^z = 1 \iff \exists k \in \mathbb{Z}, z = 2\pi i k$.
4. $e^{-z} = 1\ e^z$.
5. $|e^z| = e^{\mathrm{Re}(z)}$.

*Proof.* Easy. For 3, $\exp(z) = 1 \iff e^x\cos(y) = 1$ and $e^x\sin(y) = 0$. $e^x > 0$ so $\sin(y) = 0$ so $y = n\pi$ for some $n \in \mathbb{Z}$. So $1 = e^x\cos(n\pi) = e^x(-1)^n$ so $n$ is even and $x = 0$. $\square$

**Corollary 1.2.3.** $\exp(2\pi i) = 1$ and $\exp(\pi i) = -1$ (Euler's formula).

**Corollary 1.2.4.** $\forall k \in \mathbb{Z}, \forall z \in \mathbb{C}, \exp(z + 2k\pi i) = \exp(z)$.

**Definition 1.2.5.** The following functions from $\mathbb{C}$ to $\mathbb{C}$ are defined:

$$\begin{aligned}
\sin(z) &:= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
\cos(z) &:= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
\sinh(z) &:= \frac{1}{2}\left(e^{z} - e^{-z}\right) = -i\sin(iz) \\
\cosh(z) &:= \frac{1}{2}\left(e^{z} + e^{-z}\right) = \cos(iz)
\end{aligned}$$

**Remark.** The usual trigonometric function identities hold, e.g. $\cos(z)^2 + \sin(z)^2 = 1$.

## 1.3 Logarithms and complex powers

**Definition 1.3.1.** We write the **set of non-zero complex numbers** as

$$\mathbb{C}^* = \mathbb{C} - \{0\}$$

**Lemma 1.3.2.** For every $w \in \mathbb{C}^*$, $e^z = w$ has a solution for $z$. Let $w = |w|e^{i\theta}$, $\theta = \mathrm{Arg}(w)$. Then every solution for $z$ is given by

$$z = \log(|w|) + i(\theta + 2\pi k), \quad k \in \mathbb{Z}$$

*Proof.* By Proposition 1.2.2 (part 2) and Corollary 1.2.3,

$$w = |w|e^{i\theta} = e^{\log(|w|)}e^{i\theta} = e^{\log(|w|)}e^{i(\theta+2\pi k)} = e^{\log(|w|)+i(\theta+2\pi k)} = e^z$$

Let $z = x + iy$, then

$$e^z = e^x e^{iy} = w = |w|e^{i\theta} \implies |e^z| = e^x = |w| \implies x = \log(|w|)$$

Hence $e^{iy} = e^{i\theta}$ so $e^{i(y-\theta)} = 1$ so $y - \theta = 2\pi k$ for some $k \in \mathbb{Z}$ by Proposition 1.2.2 (part 3). $\square$

**Definition 1.3.3.** Let $\theta_1 < \theta_2$ with $\theta_2 - \theta_1 = 2\pi$. Let arg be the argument function with values in $(\theta_1, \theta_2]$. Then

$$\log(z) := \log(|z|) + i\arg(z)$$

is called a **branch of logarithm**. It has a jump discontinuity on the ray $R_{\theta_1} = R_{\theta_2}$. This ray is called a **branch cut**.

**Definition 1.3.4.** Choosing $\theta_1 = -\pi$ and $\theta_2 = \pi$, so that arg = Arg, gives the **principal branch of logarithm**

$$\mathrm{Log}(z) := \log(|z|) + i\mathrm{Arg}(z)$$

which has a jump discontinuity on the ray $\mathbb{R}_{\leq 0}$ (the non-positive real axis).

**Remark.** The prinical branch, Log, matches the definition of log for real numbers, so is the branch that should be used, unless otherwise stated.

**Lemma 1.3.5.**

1. $\forall z \in \mathbb{C}^*, e^{\log(z)} = z$.
2. Generally, $\log(zw) \neq \log(z) + \log(w)$.
3. Generally, $\log(e^z) \neq z$.

**Definition 1.3.6.** For a fixed $w \in \mathbb{C}^*$, we can choose any branch of log to define a **complex power function** by

$$z^w = \exp(w\log(z))$$

**Remark.** The complex power function depends on the branch of log we choose.

## 1.4 The Riemann sphere and extended complex plane

**Definition 1.4.1.** The **unit sphere** $S^2$ is defined as

$$S^2 := \{(x, y, s) \in \mathbb{R}^3 : x^2 + y^2 + s^2 = 1\}$$

**Definition 1.4.2.** We define $N = (0, 0, 1) \in S^2$ to be the **north pole**. For every point $v \in S^2 - \{N\}$, there is a unique straight line $L_{N,v}$ passing through $N$ and $v$. This is not parallel to the $(x, y)$-plane as $v \neq N$, hence $L_{N,v}$ intersects the $(x, y)$-plane at a unique point $(x, y, 0)$ which corresponds to the point $x + iy \in \mathbb{C}$.

**Definition 1.4.3.** The **stereographic projection** map $P : S^2 - \{N\} \to \mathbb{C}$ is defined as

$$P(v) = x + iy$$

where $x+iy$ is the complex number corresponding to the point $(x, y, 0)$ where the line passing through $N$ and $v$, $L_{N,v}$ intersects the $(x, y)$-plane.

The equation $L_{N,v}$ is

$$\gamma(t) = N + ((x, y, s) - N)t = (0, 0, 1) + (x, y, s - 1)t$$

which intersects the $(x, y)$-plane at $t = 1/(1 - s)$. So

$$P(x, y, s) = \frac{x}{1 - s} + i\frac{y}{1 - s}$$

**Definition 1.4.4.** The **inverse stereographic projection**, the inverse of $P$, is

$$P^{-1}(z) = \frac{1}{1+|z|^2}(2\text{Re}(z), 2\text{Im}(z), |z|^2 - 1)$$

**Remark.** $P$ is a bijection as it has an inverse.

**Definition 1.4.5.** The **extended complex plane** is defined as

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

**Remark.** $N$ corresponds to $\infty \in \hat{\mathbb{C}}$ under the stereographic projection. So $\hat{\mathbb{C}}$ can be thought of as the entire sphere $S^2$.

**Remark.** The south pole, $(0, 0, -1)$ could also be used to define a different, valid projection.

**Proposition 1.4.6.** The following are correspondences between $S^2$ and $\hat{\mathbb{C}}$:

$$\begin{aligned}
N &\longleftrightarrow \infty \\
S &\longleftrightarrow 0 \\
\text{equator} &\longleftrightarrow \text{unit circle: } \{z \in \mathbb{C} : |z| = 1\} \\
\text{open Southern hemisphere} &\longleftrightarrow \text{unit disc: } \{z \in \mathbb{C} : |z| < 1\} \\
\text{open Northern hemisphere} &\longleftrightarrow \hat{\mathbb{C}} - \overline{B_1}(0) = \hat{\mathbb{C}} - \{z \in \mathbb{C} : |z| \leq 1\}
\end{aligned}$$

# 2 Metric Spaces

## 2.1 Metric spaces

**Definition 2.1.1.** A **metric space** is a set $X$ together with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies, for every $x, y, z \in X^3$,

1. **(D1) Positivity**: $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.
2. **(D2) Symmetry**: $d(x, y) = d(y, x)$.
3. **(D3) Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

$d$ is called a **metric**. A metric space is written as $(X, d)$.

**Example 2.1.2.** Let $X = \mathbb{C}$ and $d(x, y) = |x - y|$. Then $(X, d)$ is a metric space.

**Example 2.1.3.** Let $X = \mathbb{C}^n$ and

$$d(\underline{x}, \underline{y}) = ||x - y||_2 = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

Then $(X, d)$ is a metric space.

**Example 2.1.4.** Let $V$ be a finite dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$, then

$$d(\underline{x}, \underline{y}) = ||x - y|| = \sqrt{\langle x - y, x - y \rangle}$$

is a metric.

**Definition 2.1.5.** Let $V$ be a real or complex vector space. A function $|| \cdot || : V \to \mathbb{R}_{\geq 0}$ is called a **norm** if it satisfies, for every $v, w \in V^2$ and $\lambda \in \mathbb{C}$ or $\mathbb{R}$:

1. **(N1) Positivity**: $||v|| \geq 0$ and $||v|| = 0 \iff v = 0$.
2. **(N2) Linearity in scalar multiplication**: $||\lambda v|| = |\lambda| ||v||$.
3. **(N3) Triangle inequality**: $||v + w|| \leq ||v|| + ||w||$.

**Definition 2.1.6.** Property N3 implies the **reverse triangle inequality**:

$$||v - w|| \geq | \; ||v|| - ||w|| \; |$$

**Definition 2.1.7.** A vector space equipped with a norm is called a **normed vector space**.

**Remark.** A normed vector space together with $d(v, w) = ||v - w||$ is always a metric space.

**Example 2.1.8.** For every $p \geq 1$, the $l_p$**-norm** is defined on vectors in $\mathbb{C}^n$ by

$$||\underline{x}||_p := \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

The **Taxicab norm** is $l_p$ norm when $p = 1$.

**Example 2.1.9.** The $l_\infty$**-norm** (or **sup-norm**) is defined as

$$||\underline{x}||_\infty := \max_{i=1,\dots,n} |x_i|$$

**Example 2.1.10.** The **Riemannian metric** (or **chordal metric**) is defined as

$$d(z, w) := ||f(z) - f(w)||_2$$

where $z, w \in \hat{\mathbb{C}}$ and $f : \hat{\mathbb{C}} \to S^2$ is the inverse stereographic projection.

**Definition 2.1.11.** Let $X$ be a non-empty finite set. The **discrete metric** on $X$ is defined as

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

$(X, d)$ is called a **discrete metric space**.

**Example 2.1.12.** Let $X = C([a, b])$ be the space of continuous functions on $[a, b]$. Then

$$||f|| := \max_{x \in [a,b]} |f(x)|$$

defines a norm, so is a metric.

**Example 2.1.13.** Let $(X, d)$ be a metric space. Every non-empty subset $Y \subset X$ also forms a metric space with $(Y, d)$. The metric restricted to $Y$ is called the **subspace metric**.

## 2.2 Open and closed sets

**Definition 2.2.1.** Let $(X, d)$ be a metric space, $x \in X$ and $r > 0$ be a real number. The **open ball** $B_r(x)$ **of radius** $r$ **centred at** $x$ is defined as

$$B_r(x) := \{y \in X : d(x, y) < r\}$$

**Definition 2.2.2.** Let $(X, d)$ be a metric space, $x \in X$ and $r > 0$ be a real number. The **closed ball** $B_r(x)$ **of radius** $r$ **centred at** $x$ is defined as

$$\overline{B_r}(x) := \{y \in X : d(x, y) \leq r\}$$

**Example 2.2.3.** Let $X = \mathbb{C}$ and $d(z, w) = |z - w|$. Then $B_1(0) = \mathbb{D} = \{z : |z| < 1\}$, the unit disc.

**Example 2.2.4.** Let $X = \mathbb{R}^2$. For $l_2$-norm, the unit ball $B_1(\underline{0})$ is the inside of the unit circle centred at the origin.

For the $l_\infty$-norm, $B_1(\underline{0})$ is the inside of the square with vertices $(1, 1), (-1, 1), (-1, -1), (1, -1)$, since $\max |x|, |y| < 1$ in this ball.

For the $l_1$-norm, in $B_1(0)$, we have $|x| + |y| < 1$ so in the 1st quadrant, $y < 1 - x$, in the 2nd quadrant, $y < 1+$, in the 3rd quadrant, $y > -1-x$, and in the 4th quadrant, $y > -1+x$. So the unit ball is the inside of the diamond with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$.

**Definition 2.2.5.** Let $(X, d)$ be a metric space. $U \subset X$ is called **open** (in $X$) if for every $x \in U$, for some $\epsilon > 0$, $B_\epsilon(x) \subset U$.

**Definition 2.2.6.** Let $(X, d)$ be a metric space. $U \subset X$ is called **closed** (in $X$) its complement in $X$, $X - U$, is open.

**Definition 2.2.7.** Sets in a metric space that are both open and closed are called **clopen**.

**Example 2.2.8.** $\emptyset$ and $X$ are clopen in any metric space.

**Lemma 2.2.9.** In a metric space, the open ball $B_r(x)$ is open.

*Proof.* Let $y \in B_r(x)$ and let $s := d(x, y) < r$. Let $\epsilon = r - s > 0$. Then for every $z \in B_\epsilon(y)$,

$$d(x, z) \leq d(x, y) + d(y, z) < s + \epsilon = r$$

So $z \in B_r(x)$. □

**Example 2.2.10.** $\mathbb{H}, \mathbb{D}, \mathbb{C}^*, \mathbb{C} - \mathbb{R}_{\leq 0}$ are all open. The 1st quadrant $\Omega_1 := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$ is open, since for every $z \in \Omega_1$, let $r = \min(\mathrm{Re}(z), \mathrm{Im}(z)) > 0$, then $B_r(z) \subset \Omega_1$.

**Example 2.2.11.** Let $(X, d)$ be a discrete metric space, so

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

For every $x \in X$ and $r > 0$,

$$B_r(x) = \begin{cases} \{x\} & \text{if } r \leq 1 \\ X & \text{if } r > 1 \end{cases}$$

So by Lemma 2.2.9, every singleton $\{x\}$ is an open set with respect to the discrete metric. But also, $X - \{x\}$ is open, since for every $y \in X - \{x\}$ and $r < 1$, $y \neq x$ so $B_r(y) = \{y\} \subset X - \{x\}$.

So all balls are clopen with respect to the discrete metric.

**Example 2.2.12.** $[0, 1)$ is neither open nor closed in $\mathbb{R}$ with respect to the standard metric $|\cdot|$, since $0 \in [0, 1)$ doesn't have a ball in $[0, 1)$ containing it, but $1 \in \mathbb{R} - [0, 1) = (-\infty, 0) \cup [1, \infty)$ also doesn't have a ball in $\mathbb{R} - [0, 1)$ containing it.

**Lemma 2.2.13.** Let $(X, d)$ be a metric space.

1. Arbitrary unions of open sets are open, i.e. for every (finite or infinite) collection of open sets $U_i \subseteq X$, the union
$$\bigcup_i U_i$$
is open.

2. Finite intersections of open sets are open, i.e. for every finite collection of open sets $U_i \subseteq X$, the intersection
$$\bigcap_{i=1}^n U_i$$
is open.

*Proof.*

1. Let $x \in \bigcup_i U_i$. Then for some $j$, $x \in U_j$. $U_j$ is open so for some $\epsilon > 0$, $B_\epsilon(x) \subseteq U_j \subseteq \bigcup_i U_i$.

2. Let $x \in \bigcap_{i=1}^n U_i$. For every $i$, $U_i$ is open so for some $r_i > 0$, $B_{r_i}(x) \subseteq U_i$. Let $\epsilon = \min\{r_1, \ldots, r_n\} > 0$, so $B_\epsilon(x) \subseteq B_{r_i}(x)$ for every $i$. Then
$$B_\epsilon(x) \subseteq \bigcap_{i=1}^n B_{r_i}(x) \subseteq \bigcap_{i=1}^n U_i$$

□

**Corollary 2.2.14.** Let $(X, d)$ be a metric space. Then

1. Finite unions of closed sets are closed.
2. Arbitrary intersections of closed sets are closed.

*Proof.* Use De Morgan's laws and Lemma 2.2.13. □

**Remark.** Infinite intersections of open sets are not always open, and infinite unions of closed sets are not always closed. For example,

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{i}, 1 - \frac{1}{i}\right] = (0, 1)$$

is an infinite union of closed sets but $(0, 1)$ is open in $\mathbb{R}$.

# 3 Mobius Transformations

**Corollary 3.0.1.** Any Mobius transformation is a bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Let $T \in GL_2(\mathbb{C})$ and $M_T$ be a Mobius transformation, then a point $z$ is a fixed point of $M_T$ if $M_T(z) = z$.

**Lemma 3.0.2.** Let $T \in GL_2(\mathbb{C})$. If $M_T : \mathbb{C} \to \mathbb{C}$ is not the identity map, then $M_T$ has at most two fixed points in $\mathbb{C}$. If a Mobius transformation has three fixed points then it is the identity map.

*Proof.* Case 1: Suppose $M_T(\infty) = \infty$. From the definition, $M_T(z) = \frac{az+b}{cz+d}$, therefore $c = 0$. So $M_T(z) = \frac{a}{d}z + \frac{b}{d}$, with $a \neq 0, d \neq 0$ (since $\det T \neq 0$).

Such an affine linear map has at most one fixed point because:

- If $a \neq d$ then $\frac{a}{d}z + \frac{b}{d} = z \iff z = \frac{b}{d-a}$ so $M_T$ has a unique fixed point.
- If $a = d$ then $b \neq 0$ (since we assume $M_T$ is not the identity). So $M_T(z) = z + \frac{b}{a}$ is a translation which has no fixed points.

Case 2: Suppose $M_T(\infty) \neq \infty$. Suppose $z_0 \in \mathbb{C}$ is such that $M_T(z_0) = z_0$. We have $M_T(z_0) = z_0 \iff \frac{az_0+b}{cz_0+d} = z_0 \iff cz_0^2 + (d-a)z_0 - b = 0$. This quadratic equation has at most two roots so there are at most two fixed points of $M_T$. $\square$

**Definition 3.0.3.** Given four distinct points $z_0, z_1, z_2, z_3 \in \mathbb{C}$, the cross-ratio of these points denoted $(z_0, z_1; z_2, z_3)$ is defined by

$$\frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

We extend the definition to the case where one of the points is $\infty$ by removing all differences involving that point e.g. $(\infty, z_0; z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2}$.

**Theorem 3.0.4.** (Three points theorem) Let $z_1, z_2, z_3$ and $w_1, w_2, w_3$ be two sets of three ordered points in $\hat{\mathbb{C}}$. Then there exists a unique Mobius transformation $f$ such that $f(z_i) = w_i$ for every $i \in \{1, 2, 3\}$.

*Proof.* Existence:

We consider the functions $F(z) = (z, w_1; w_2, w_3) = \frac{(z-w_2)(w_1-w_3)}{(z-z_3)(w_1-w_2)}$ and $G(z) = \frac{(z-z_2)(z-z_3)}{(z-z_3)(z_1-z_2)}$. These are Mobius transformations with the properties that $F(w_1) = 1$, $F(w_2) = 0$, $F(w_3) = \infty$ and similarly, $G(z_1) = 1$, $G(z_2) = 0$, $G(z_3) = \infty$. Therefore $F^{-1} \circ G$ maps each $z_i$ to $w_i$.

Uniqueness:

Assume that there are two such maps, say $f_1$ and $f_2$. Then the Mobius transformation $H = f_1^{-1} \circ f_2$ satisfies $H(z_i) = z_i$.

This shows that $H$ has three fixed points so, by Three Point Theorem, it must be the identity. Thus $f_1 = f_2$. $\square$

**Proposition 3.0.5.** Mobius transformations preserve the cross ratio. That is, if $z_0, z_1, z_2, z_3$ are four distinct points in $\hat{\mathbb{C}}$ and $f$ is a Mobius transformation, then $(f(z_0), f(z_1); f(z_2), f(z_3)) = (z_0, z_1; z_2, z_3)$.

*Proof.* Let $w_i = f(z_i)$ for every $i \in \{1,2,3\}$. Let $F(z) = (z, w_1; w_2, w_3)$ and $G(z) = (z, z_1; z_2, z_3)$. Recall $F^{-1} \circ G$ maps $z_i$ to $w_i$ like $f$ does. Since there is a unique Mobius transformation with this property, we have

$$f = F^{-1} \circ G$$

and

$$F \circ f = G$$

That is, $(f(z_0), w_1; w_2, w_3) = F \circ f(z_0) = G(z_0) = (z_0, z_1; z_2, z_3)$. □

**Remark.** General strategy: to find Mobius transformation, find image of 3 points and use the fact that cross ratio is preserved. Plug known points into (*) and rearrange for $f(z_0)$.

## 3.1 The Riemann Sphere Revisited

Circles in $\hat{\mathbb{C}}$ correspond to circles in $S^2$ that don't pass through $N$ (the North pole).
Lines in $\hat{\mathbb{C}}$ correspond to circle in $S^2$ that pass through $N$.

**Remark.** Mobius transformations give all biholomorphic maps from $S^2$ to $S^2$.

**Remark.** Stereographic projections are conformal.

## 3.2 Mobius transformations preserving the upper half plane and the unit disc

Notation: for a domain $D \subset \mathbb{C}$, let $Mob(D)$ be the set of Mobius transformations $f$ such that $f(D) = D$.

**Proposition 3.2.1.** (H2H) Every Mobius transformation mapping $\mathbb{H}$ to $\mathbb{H}$ ($\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$) is of the form $M_T$ with $T \in SL_2(\mathbb{R}) := \{T = () : a, b, c, d \in \mathbb{R}, \det T = 1\}$

Conversely, every such Mobius transformation maps $\mathbb{H}$ to $\mathbb{H}$ and hence a biholomorphism from $\mathbb{H}$ to $\mathbb{H}$.

i.e. H2H: $f \in \text{Mob}(\mathbb{H}) \Leftrightarrow f = M_T$ with $T \in SL_2(\mathbb{R})$.

**Remark.** $T \to M_T$ gives a group homomorphism $SL_2(\mathbb{R}) \to Aut(\mathbb{H})$

*Proof.* Any Mobius transformation $f : \mathbb{H} \to \mathbb{H}$ must map $\partial\mathbb{H}$ to $\partial\mathbb{H}$. As $\partial\mathbb{H}$ is the real line, $f : \mathbb{R} \cup \infty \to \mathbb{R} \cup \infty$. So $f$ must map the ordered set $\{1, 0, \infty\}$ to $\{x_1, x_2, x_3\}$ for some $x_i \in \mathbb{R} \cup \infty$.

We know that the cross ratio is preserved under a Mobius transformation:

$$(f(z), x_1; x_2, x_3) = \frac{(f(z) - x_2)(x_1 - x_3)}{(f(z) - x_3)(x_1 - x_2)} = \frac{z - 0}{1 - 0} = (z, 1; 0, \infty)$$

$$\Leftrightarrow (f(z) - x_2)(x_1 - x_3) = z(f(z) - x_3)(x_1 - x_2)$$

$$\Leftrightarrow f(z) = \frac{x_3(x_1 - x_2)z + x_2(x_3 - x_1)}{(x_1 - x_2)z + x_3 - x_1}$$

We see that the coefficients of $T$ are real.

If $T \in GL_2(\mathbb{R})$ and $z = x + iy$ then

$$Im(M_T(z)) = Im(\frac{az + b}{cz + d}) = Im(\frac{(az + b)(c\bar{z} + d)}{|cz + d|^2})$$

$$= Im(\frac{bc\bar{z} + adz}{(cz+d)}) = \frac{y \det T}{|cz+d|}$$

We have $z \in \mathbb{H} \Leftrightarrow y > 0$ so $M_T(z) \in H \Leftrightarrow T \in GL_2(\mathbb{R})$, $\det T > 0$. We can therefore replace $T$ by a real matrix of determinant 1 by scaling $T$ by a real number. □

**Proposition 3.2.2.** (D2D): Every Mobius transformation from the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to $\mathbb{D}$ is of the form $T \in SU(1,1)$

Conversely, every such Mobius transformation maps $\mathbb{D}$ to $\mathbb{D}$ and hence gives a h biholopmorhpic automorphism of $\mathbb{D}$.

i.e. $f \in Mob(\mathbb{D}) \Leftrightarrow f = M_T$, $T \in SU(1,1)$.

*Proof.* ($\Rightarrow$): Let $M_T : \mathbb{D} \to \mathbb{D}$ be a Mobius transformation. The Cayley map $H_C$ maps $\mathbb{H}$ to $\mathbb{D}$. We have that $f = M_C^{-1} \circ M_T \circ M_C$ is a Mobius transformation from $\mathbb{H}$ to $\mathbb{H}$. By proposition 4.20, we have $f = M_S$ where $S \in SL_2(\mathbb{R})$.

Hence $C^{-1}TC = S \in SL_2(\mathbb{R})$ by Lemma 4.4.

Let $S \in \mathbb{M}_2(\mathbb{R})$, $\det S = 1$. Then $T = CSC^{-1}$. Evalutating this shows $T \in SU(1,1)$.

($\Leftarrow$): If $T \in SU(1,1)$, then the same calculation in reverse shows that the matrix $S = C^{-1}TC \in SL_2(\mathbb{R})$. Then $M_S : \mathbb{H} \to \mathbb{H}$ is a Mobius transformation by proposition 4.20 (H2H), and the map $M_T := M_C \circ M_S \circ M_C^{-1}$ is a Mobius transformation from $\mathbb{D}$ to $\mathbb{D}$ □

**Remark.** $T \to M_T$ gives a group homomorphism from $SU(1,1)$ to $Aut(\mathbb{D})$.

**Corollary 3.2.3.** (D2D*):

1. Every Mobius transformation $f$ from $\mathbb{D}$ to $\mathbb{D}$ can be written as

$$f(z) = e^{i\theta} \frac{z - z_0}{\bar{z}_0 z - 1}$$

for some angle $\theta$ and $z_0 \in \mathbb{D}$ where $z_0$ is the unique point in $\mathbb{D}$ such that $f(z_0) = 0$.

2. Every Mobius transformation of the unit disc $\mathbb{D}$ to $\mathbb{D}$ for which $f(0) = 0$ are rotations about 0.

*Proof.* 1. By proposition D2D, we have

$$f(z) = \frac{az + b}{\bar{b}z + \bar{a}} = \frac{a(z + b/a)}{-\bar{a}((-\bar{b}/\bar{a})z - 1)} = -\frac{a}{\bar{a}} \frac{z - (-b/a)}{(-\bar{b}/\bar{a})z - 1}$$

So $z_0 = -\frac{b}{a}$. Since $| - \frac{a}{\hat{a}} = 1$, $-\frac{a}{\hat{a}} = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$.

$|z_0|^2 - 1 = | - \frac{b}{a}|^2 - 1 = \frac{|b|^2}{|a|^2} - 1$. Now $1 = |a|^2 - |b|^2$ so $|z_0|^2 - 1 = \frac{-1}{|a|^2} < 0$ so $|z_0|^2 < 1$ and so $|z_0| < 1$.

2. $f(0) = 0 \Leftrightarrow e^{i\theta} \frac{0 - z_0}{\bar{z}_0 \cdot 0 - 1} = 0 \Leftrightarrow z_0 = 0 \Leftrightarrow f(z) = e^{i\theta} \frac{z - 0}{0 - 1} = e^{-i\theta} z$.

So $f$ is a rotation.

□

**Remark.** The map $g(z) = \frac{z - z_0}{\bar{z}_0 z - 1}$ swaps $z_0$ and 0 and is an involution ($g \circ g = Id$). Also, $z \to e^{i\theta} z$ is a rotation.

So every Mobius transformation from $\mathbb{D}$ to $\mathbb{D}$ is given by an involution followed by a rotation.

## 3.3 Finding biholomorphic maps between domains

To find a biholomorphism $f$ between domains, we build $f$ in various stages using simpler known maps.

**Example 3.3.1.** Find biholomorphism from $D = \{z \in \mathbb{D} : Im(z) < 0\}$ to $\mathbb{H}$.

The Cayley Map $M_C$ is a map from $\mathbb{H}$ to $\mathbb{D}$, so $M_C^{-1} : \mathbb{D} \to \mathbb{H}$, $M_C^{-1}(z) = \frac{iz+i}{-z+1}$.

To find the image of $D$ under $M_C^{-1}$, consider how it acts on two segments of $\delta D$:

- Under $M_C^{-1}$, $-1 \to 0$, $0 \to i$ and $1 \to \infty$. Therefore the line segment from $-1$ to $1$ through $0$ is mapped to the positive imaginary axis.
- Under $M_C^{-1}$, $-i \to 1$, so the circular arc from $-1$ to $1$ through $-i$ is mapped to the positive real axis.

Now $-\frac{i}{2} \in D$ and $M_C^{-1}(-\frac{i}{2}) = \frac{4+3i}{5}$. The image of $D$ under $M_C^{-1}$ is $\Omega = \{w \in \mathbb{C} : 0 < Arg(w) < \frac{\pi}{2}\}$.

Now we find a biholomorphic map from $\Omega$ to $\mathbb{H}$. $g(z) = z^2$ satisfies this, as it doubles the argument of $z$.

So the map is $f = g \circ M_C^{-1}$, $f : D \to \mathbb{H}$.

# 4 Notions of convergence in complex analysis and power series

## 4.1 Pointwise and uniform convergence

**Definition 4.1.1.** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. A sequence of functions $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ converges pointwise (on $X$) to $f$ if for every $x \in X$, the limit function $f(x) := \lim_{n\to\infty} f_n(x)$ exists in $Y$.

In other words, we have for every $x \in X$ and for every $\epsilon > 0$, for some $N \in \mathbb{N}$, for every $n > N$, $d_Y(f_n(x), f_n(x)) < \epsilon$. (Not that $N$ depends on $x$).

**Remark.** For every $x \in X$, $f_n(x)$ is just a sequence of points in $Y$. The above definition is what we get by applying definition 2.11 (in notes) to the sequence $f_n(z)$.

**Example 4.1.2.** Let $f_n(z) = z^n$, $f_n : \mathbb{C} \to \mathbb{C}$. There are the following cases:

1. $z \in \mathbb{D}$. Let $\epsilon > 0$. Then $|z|^N < \epsilon$ for every $N > \frac{\log \epsilon}{\log |z|}$. So for every $n > N$ we have $f_n(z) - 0 = |z|^n < |z|^N \epsilon$, hence $\lim_{n\to\infty} f_n(z) = 0 \in \mathbb{D}$.

2. $|z| = 1$. The point $z$ rotates around the unit circle $\delta\mathbb{D}$ by $Arg(z)$ anticlockwise every iteration. For $z \neq 1$, this sequence doesn't converge. But for $z = 1$, $\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} 1 = 1$.

3. $|z| > 1$. The value of $|z|^n$ is unbounded so doesn't converge.

The sequence $f_n$ doesn't converge pointwise on $\mathbb{C}$. But it is pointwise convergent on $\mathbb{D} \cup 1$ with limit function:

$$f(z) = \begin{cases} 0 & \text{if } z \in \mathbb{D} \\ 1 & \text{if } z = 1 \end{cases} \tag{1}$$

**Definition 4.1.3.** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. A sequence of functions $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ converges uniformly (on $X$) to the limit function $f$ if for every $\epsilon > 0$ for some $N \in \mathbb{N}$, for every $n > N$, $d_Y(f_n(x), f(x)) < \epsilon$ for every $x \in X$.

**Theorem 4.1.4.** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces and let $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ be a sequence of functions that converges uniformly to $f$ on $X$.

Then $f$ is continuous on $X$.

*Proof.* Same as in Analysis I. □

**Lemma 4.1.5.** let $\{f_n\}_{n\in\mathbb{N}} : X \to \mathbb{C}$ be a sequence of functions converging pointwise to a limit function $f$.

1. If $|f_n(x) - f(x)| \leq s_n$ for every $x \in X$ where $\{s_n\}_{n\in\mathbb{N}}$ is some sequence in $\mathbb{R} > 0$ (independent of $x$) with $\lim_{n\to\infty} s_n = 0$ then $f_n$ converge uniformly to $f$ on $X$.

2. If for some sequence $x_n \in X$, $|f_n(x_n) - f(x_n)| \geq c$ for some positive constant $c$ then $f_n$ does not converge uniformly to $f$ on $X$.

**Theorem 4.1.6.** (Weierstrass M-test): Let $f_n : X \to \mathbb{C}$ be a sequence of fucntions such that $|f_n(x)| \leq M_n$ for every $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on $X$ to some limit function $f : X \to \mathbb{C}$.

*Proof.* Similar to Analysis I. □

**Theorem 4.1.7.** Let a sequence of functions $f_n : [a, b] \to \mathbb{R}$ converge uniformly on an interval $[a, b]$ to some function $f$, such that $\{f_n\}$ are all continuous. Then

$$\lim_{n\to\infty} \int_a^c f_n(x)dx = \int_a^c f(x)dx \text{ for every } c \in [a, b]$$

**Definition 4.1.8.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in a metric space $X$. $f_n$ converges locally uniformly (on $X$) to the limit function $f$ if for every $x \in X$, for some open set $U \subset X$ containing $x$, $f_n$ converges uniformly to $f$ on $U$.

**Theorem 4.1.9.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions which converges locally uniformly on $X$ to a limit function $f$. Then $f$ is continuous on $X$.

*Proof.* For every $x \in X$, $f_n$ converges uniformly on some open set $U$ containing $x$. Hence $f$ is continuous on $U$ by theorem 5.5 (in notes). So $f$ is continuous at $x$ for every $x \in X$. □

**Remark.** The limit of a locally uniform convergent sequence of holomorphic functions is again holomorphic.

**Example 4.1.10.** For every $w \in \mathbb{D}$, for some $r < 1$, $w \in B_r(0)$ and $B_r(0)$ is open. Then for every $z \in B_r(0)$, $|z|^n < r^n$ and $\lim_{n\to\infty} r_n = 0$. So by lemma 5.6 (in notes), with $s_n = r^n$, $f_n$ converges uniformly to $f$ in $B_r(0)$.

**Remark.** To prove that the limit function is conitnuous on all of $\mathbb{D}$, it is enough to prove locally uniform convergence on every ball $B_r(0)$, $0 < r < 1$, in $\mathbb{D}$.

**Theorem 4.1.11.** Let $X$ be a metric space and let $f_n : X \to \mathbb{C}$ be a sequence of continuous functions such that for any $y \in X$, there is an open $U \subset X$ containing $y$ and constants $M_n > 0$ with $\sum_{n=1}^{\infty} M_n < \infty$ and $|f_n(x)| \leq M_n$ for every $x \in U$. Then $\sum_{n=1}^{\infty} f_n$ converges locally uniformly to a continuous function on $X$.

*Proof.* Given $y \in X$, the hypotheses of the theorem imply that for some constants $M_n > 0$, $|f_n(y)| \leq M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$.

$$|F_k(y)| = |\sum_{n=1}^{k} f_n(y)| \leq \sum_{n=1}^{\infty} |f_n(y)| \leq \sum_{n=1}^{k} M_n$$

As $k \to \infty$, the RHS $\sum_{n=1}^{k} M_n$ converges so it must be bounded, and let the upper bound by $L$. Thus for every $k$, $|F_k(y)| \leq L$. So the sequence $(F_k(y))_k$ is bounded, hence it lies in some boundedd, closed ball in $\mathbb{C}$, which is compact by Heine-Borel.

Therefore there is a subsequence $(F_{k_j}(y))_{k_j}$ that converges to $F(y)$.

Now, for $k_j > k$,

$$|F_{k_j}(y) - F_k(y)| = |\sum_{n=k+1}^{k_j} f_n(y)| \leq \sum_{n=k+1}^{k_j} |f_n(y)| \leq \sum_{n=k+1}^{k_j} M_n$$

Taking the limit as $j \to \infty$, both the LHS and RHS converge, and we get

$$|F(y) - F_k(y)| \leq \sum_{n=k+1}^{\infty} M_n$$

Now taking the limit as $k \to \infty$, we get

$$\lim_{k\to\infty} |F(y) - F_k(y)| = 0$$

since the RHS tends to zero.

Repeating this for every $y$, $F_k \to F$ pointwise on $X$.

From the hypotheses of the theorem, we have that for every $y \in X$, for some open $U \subset X$ containing $y$ and constants $M_n > 0$ with $\sum_{n=1}^{\infty} < \infty$ and $|f_n(x)| \leq M_n$ for every $x \in U$.

Then, for every $x \in U$ and for every $L > k$,

$$|F_L(x) - F_k(x)| = |\sum_{n=k+1}^{L} f_n(x)| = \sum_{n=k+1}^{L} |f_n(x)| \leq \sum_{n=k+1}^{L} M_n$$

Taking the limit as $l \to \infty$:

$$|F(x) - F_k(x)| \leq \sum_{n=k+1}^{\infty} M_n$$

for every $x \in U$.

$\lim_{k\to\infty} \sum_{n=k+1}^{\infty} M_n = 0$. So by lemma 5.6 (in notes), $F_k \to F$ uniformly on $U$. $\square$

## 4.2 Complex power series

**Theorem 4.2.1.** A complex power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z-c)^n, \quad a_n, c \in \mathbb{C}^2$$

There are three cases:

1. $\sum_{n=0}^{\infty} a_n (z-c)^n$ converges only for $z = c$ ($R = 0$).
2. There exists $R > 0$ (radius of convergence) such that
   - $\sum_{n=0}^{\infty} a_n (z-c)^n$ converges absolutely for $|z - c| < R$ (We call $B_R(c)$ the disc of convergence).
   - $\sum_{n=0}^{\infty} a_n (z-c)^n$ diverges for $|z - c| > R$ (anything can happen on the circle $|z - c| = R$).
3. $\sum_{n=0}^{\infty} a_n (z-c)^n$ converges absolutely for every $z \in \mathbb{C}$ ($R = \infty$).

**Remark.** Radius of convergence is usually determined via ratio test or root test.

**Theorem 4.2.2.** A power series $\sum_{n=0}^{\infty} a_n (z-c)^n$ with radius of convergence $0 < R < \infty$ converges uniformly on every ball $B_r(c)$ with $0 < r < R$. This implies that the power series is locally uniformly convergent on its disc of convergence.

*Proof.* Follows via the M-test. $\square$

**Remark.** The power series do not converge uniformly in the entire disc of conergence $B_R(c)$.

**Proposition 4.2.3.** Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series with radius of convergence $0 < R < \infty$. Then the formal derivatives and antiderivatives

$$\sum_{n=0}^{\infty} n a_n (z-c)^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$$

have the same radius of convergence $R$.

**Theorem 4.2.4.** Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series with radius of convergence $0 < R < \infty$ and let $f : B_R(c) \to \mathbb{C}$ be the resulting limit function. Then $f$ is holomorphic on $B_R(c)$ with

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-c)^{n-1}$$

for $z \in B_R(c)$.

*Proof.* Assume $c = 0$ (the general case for $c$ is analogous).

$$f(z) - f(w) = \sum_{n=1}^{\infty} a_n(z^n - w^n) = \sum_{n=1}^{\infty} (z-w) q_n(z)$$

where $q_n(z) = \sum_{k=0}^{n-1} w^k z^{n-1-k}$.

So for $z \neq w$, let $h(z) := \frac{f(z)-f(w)}{z-w} = \sum_{n=1}^{\infty} a_n q_n(z)$

Given $z_0 \in B_R(0)$, let $r < R$ such that $w, z_0 \in B_r(0)$. To apply the local M-test, we need constants $M_n$ for this set $B_r(0)$ that bound the terms $a_n q_n(z)$ defining $h$.

For $z \in B_r(0)$,

$$|a_n q_n(z)| = |a_n \sum_{k=0}^{n-1} w^k z^{n-1-k}| \leq |a_n| \sum_{k=0}^{n-1} |w|^k |z|^{n-1-k} < |a_n| \sum_{k=0}^{n-1} r^{n-1} = n|a_n| r^{n-1}$$

So let $M_n = n|a_n|r^{n-1}$, then $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} n|a_n|r^{n-1}$ which converges by proposition 5.19 (in lecture notes).

The formal derivative $\sum_{n=1}^{\infty} n a_n r^{n-1}$ has radius of convergence $R$ so converges absolutely on its disc of convergence $B_R(0)$. In particular, it converges at $z = R$. By the local M-test, the series defining $h$ converges locally uniformly to a continuous function on $B_R(0)$. Hence

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = \lim_{h \to w} h(z) = h(w) = \sum_{n=1}^{\infty} a_n q_n(w) = \sum_{n=1}^{\infty} n a_n w^{n-1}$$

□

**Corollary 4.2.5.** A power series $f$ as theorem 5.21 (in lecture notes) with positive radius of convergence $R$ can be differentiated infinitely many times and

$$f^{(k)} := \sum_{n=k}^{\infty} k! \binom{n}{k} a_n (z-c)^{n-k}$$

for $z \in B_R(c)$

**Corollary 4.2.6.** A power series $f$ as in theorem 5.21 (in lecture notes) with positive radius of convergence $R$ has a holomorphic antiderivative $F : B_R(c) \to \mathbb{C}$, with $F'(z) = f(z)$, defined by

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$$

# 5 Complex integration over contours

## 5.1 Definition of contour integrals

**Definition 5.1.1.** For a continuous function $f : [a, b] \to \mathbb{C}$, with $f(z) = u(z) + iv(z)$,

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt \in \mathbb{C}$$

**Lemma 5.1.2.**

1. Let $f_1$ and $f_2$ be continuous functions from $[a, b]$ to $\mathbb{C}$. Then $\int_a^b (f_1(t) + f_2(t))dt = \int_a^b f_1(t)dt + \int_a^b f_2(t)dt$.
2. For any complex number $c \in \mathbb{C}$ and continuous function $f : [a, b] \to \mathbb{C}$,

$$\int_a^b cf(t)dt = c \int_a^b f(t)dt$$

**Definition 5.1.3.** A smooth curve in $\mathbb{C}$ is a continuously differentiable function $\gamma : [0, 1] \to \mathbb{C}$ (i.e. differentiable with continuous derivative). More generally we can consider continuously differentiable curves $\gamma : [a, b] \to \mathbb{C}$. We say that such curves are $C^1$.

**Remark.** We write $\gamma(t) = u(t) + iv(t)$ with $u, v : [a, b] \to \mathbb{R}$. Then the derivative $\gamma'$ is defined as

$$\gamma'(t) := u'(t) + iv'(t)$$

At the endpoints, we demand that the one-sided derivative exists and is continuous from the one side:

$$\gamma'(b) := \lim_{h \to 0^-} \frac{u(b + h) - u(b)}{h} + i \lim_{h \to 0^-} \frac{v(b + h) - v(b)}{h}$$

exists and

$$\lim_{t \to b^-} \gamma'(t) = \gamma'(b)$$

**Definition 5.1.4.** Let $U \subset \mathbb{C}$ be an open set, and $f : U \to \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \to U$ be a $C^1$ curve. The integral of $f$ along the curve $\gamma$ is defined as

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

**Corollary 5.1.5.** Properties of the integral along a curve:

1. $\int_\gamma (f_1(z) + f_2(z))dz = \int_\gamma f_1(z)dz + \int_\gamma f_2(z)dz$
2. For $c \in \mathbb{C}$, $\int_\gamma cf(z)dz = c \int_\gamma f(z)dz$

*Proof.* Easy $\square$

**Definition 5.1.6.** Given $\gamma : [a,b] \to \mathbb{C}$, the curve $(-\gamma) : [-b,-a] \to \mathbb{C}$ is defined as

$$(-\gamma)(t) := \gamma(-t)$$

Then we have

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$$

**Lemma 5.1.7.** Let $U \subset \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be continuous and $\gamma : [a,b] \to \mathbb{C}$ be a $C^1$ curve. If $\phi : [a', b'] \to [a,b]$ with $\phi(a') = a$ and $\phi(b') = b$ is continuously differentiable and we define $\delta : [a', b'] \to \mathbb{C}$, $\delta := \gamma \circ \phi$, then

$$\int_{\gamma} f(z)dz = \int_{\delta} f(z)dz$$

*Proof.*

$$\int_{\delta} f(z)dz = \int_{a'}^{b'} f(\delta(t))\delta'(t)dt = \int_{a'}^{b'} f(\gamma(\phi(t)))(\gamma(\phi(t)))'dt$$

$$= \int_{a'}^{b'} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt$$

With a change of variables $s = \phi(t)$, $ds = \phi'(t)dt$:

$$\int_{a'}^{b'} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt = \int_{a}^{b} f(\gamma(s))\gamma'(s)ds = \int_{\gamma} f(z)dz$$

□

**Definition 5.1.8.** Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and suppose there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that the curves $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$, defined by $\gamma_i(t) = \gamma(t)$ for $t \in [a_{i-1}, a_i]$ are $C^1$ curves. Then $\gamma$ is a piecewise $C^1$ curve or contour.

For a contour $\gamma$ above, a contour integral is defined as

$$\int_{\gamma} f(z)dz = \sum_{n=1}^{n} \int_{\gamma_i} f(z)dz$$

**Definition 5.1.9.** If $\gamma : [a,b] \to \mathbb{C}$ and $\delta : [c,d] \to \mathbb{C}$ are two contours with $\gamma(b) = \delta(c)$ the contour $\gamma \cup \delta : [a, b+d-c] \to \mathbb{C}$ is defined as

$$(\gamma \cup \delta)(t) := \begin{cases} \gamma(t) & \text{if } a \le t \le b \\ \delta(t) & \text{if } c \le t \le d \end{cases}$$

Then

$$\int_{\gamma \cup \delta} f(z)dz = \int_{\gamma} f(z)dz + \int_{\delta} f(z)dz$$

## 5.2 The fundamental theorem of calculus

**Theorem 5.2.1.** Let $U \in \mathbb{C}$ be an open set and let $F : U \to \mathbb{C}$ be holomorphic with continuous derivative $f$. Then for every contour $\gamma : [a, b] \to U$,

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, if $\gamma$ is closed, so $\gamma(a) = \gamma(b)$, then

$$\int_\gamma f(z)dz = 0$$

*Proof.* First consider the case where $\gamma$ is a $C^1$ curve. Let $F = u + iv$. Then

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F(\gamma(t)))'dt$$

$$= \int_a^b (u(\gamma(t)))'dt + i\int_a^b (v(\gamma(t)))'dt = [u(\gamma(t))]_a^b + i[v(\gamma(t))]_a^b$$

$$= u(\gamma(b)) - u(\gamma(b)) + i(v(\gamma(b)) - v(\gamma(b))) = F(\gamma(b)) - F(\gamma(a))$$

Now extend this proof to any contour.

Let $\gamma : [a, b] \to \mathbb{C}$ be a contour, then for some $a = a_0 < a_1 < \cdots < a_n = b$, the curves $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$, $i = 1, \ldots, n$, defined by $\gamma_i(t) = \gamma(t)$ for $t \in [a_{i-1}, a_i]$ are $C^1$ curves. Then

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = \sum_{i=1}^n \int_{\gamma_i} F'(z)dz$$

$$= \sum_{i=1}^n (F(\gamma(a_i)) - F(\gamma(a_{i-1}))) = F(\gamma(a_n)) - F(\gamma(a_0)) = F(\gamma(b)) - F(\gamma(a))$$

□

**Remark.** Under the hypotheses on $F$, the integral only depends on the endpoints of the curve.

**Theorem 5.2.2.** If $f : [a, b] \to \mathbb{R}$ is continuous,

$$\int_a^b f(t)dt \leq \int_a^b \max_{t\in[a,b]} f(t)dt \leq (b - a) \max_{t\in[a,b]}$$

*Proof.* From Analysis I. □

**Definition 5.2.3.** Let $\gamma : [a, b] \to \mathbb{C}$ be a contour. The **length** of $\gamma$ is defined as

$$L(\gamma) = \int_a^b |\gamma'(t)|dt$$

**Lemma 5.2.4.** (**The Estimation Lemma**) Let $f : U \to \mathbb{C}$ be continuous and $\gamma : [a, b] \to U$ be a contour. Then

$$\left|\int_\gamma f(z)dz\right| \leq L(\gamma) \sup_\gamma |f|$$

where $\sup_\gamma |f| := \sup\{|f(z)| : z \in \gamma\}$.

*Proof.* First prove that for a continuous function $g : [a, b] \to \mathbb{C}$,

$$\left| \int_a^b g(t)dt \right| \leq \int_a^b |g(t)|dt$$

If we write $\int_a^b g(t)dt = re^{i\theta}$ with $r \geq 0$, then

$$\left| \int_a^b g(t)dt \right| = |re^{i\theta}| = r = \text{Re}\left( e^{-i\theta} \int_a^b g(t)dt \right)$$

$$= \text{Re}\left( \int_a^b g(t)e^{-i\theta}dt \right) = \int_a^b \text{Re}(g(t)e^{-i\theta})dt \leq \int_a^b \left| e^{-i\theta}g(t) \right| dt = \int_a^b |g(t)|dt$$

Let $g(t) = f(\gamma(t))\gamma'(t)$, then

$$\left| \int_\gamma g(z)dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| \leq \int_a^b \left| f(\gamma(t))\gamma'(t) \right| dt$$

Then

$$\int_a^b \left| f(\gamma(t))\gamma'(t) \right| dt \leq \sup_\gamma |f| \int_a^b |\gamma'(t)|dt = L(\gamma) \sup_\gamma |f|$$

□

**Theorem 5.2.5.** (Converse to FTC) Let $f : D \to \mathbb{C}$ be continuous on a domain $D$. If $\int_\gamma f(z)dz = 0$ for every closed contour $\gamma \in D$, for some $F : D \to \mathbb{C}$, $F'(z) = f(z)$.

*Proof.* Let $a_0 \in D$. For every $a_0 \neq w \in D$, let $\gamma(w)$ be a contour connecting $a_0$ to $w$ and is contained in $D$.

Since $D$ is a domain, it is path-connected, i.e. there is a smooth path $\gamma_w$ connecting $a_0$ to $w$, therefore the collection of contours contained in $D$ and connecting $a_0$ and $W$ is non-empty. Let

$$F(w) := \int_{\gamma(w)} f(z)dz$$

Let $\tilde{\gamma}(w)$ be another contour that connects $a_0$ to $w$ and is contained in $D$. Then let $c(w) = \gamma(w) \cup (-\tilde{\gamma}(w))$ that is obtained by moving through $\gamma$ then through $\tilde{\gamma}$ in the opposite direction. Since $c$ is a closed contour in $D$, $\int_C f(z)dz = 0$.

Then $0 = \int_C f(z)dz = \int_{\gamma(w)\cup(-\tilde{\gamma}(w))} f(z)dz = \int_{\gamma(w)} f(z)dz + \int_{-\tilde{\gamma}(w)} f(z)dz = \int_{\gamma(w)} f(z)dz - \int_{\tilde{\gamma}(w)} f(z)dz$. Hence

$$\int_{\gamma(w)} f(z)dz = \int_{\tilde{\gamma}(w)} f(z)dz$$

Therefore $F$ does not depend on the contour chosen to join $a_0$ to $w$.

Now we claim $F$ is holomorphic and we claim that $F$ is holomorphic and $\forall z \in D, F'(z) = f(z) \Rightarrow \lim_{h\to 0} \frac{F(w+h)-F(w)}{h} = f(w)$.

To evaluate $F(w + h)$ we need a contour joining $a_0$ to $w + h$ contained in $D$. For every $w \in D$, let $r > 0$ such that $B_r(w) \subset D$. This ball must exist since $D$ is open. Then for every $h \in \mathbb{C}$ with $|h| < r$ consider the striaght line $\delta_h$ that connects $w$ to $w + h$.

A parameterisation of this line is given by

$$\delta_h : [0, 1] \to D, \quad \delta_h(t) = w + th$$

The contour $\gamma_w \cup \delta_h$ is contained in $D$. So

$$F(w+h) = \int_{\gamma_w \cup \delta_h} f(z)dz = \int_{\gamma_w} f(z)dz + \int_{\delta_h} f(z)dz = F(w) + \int_{\delta_h} f(z)dz$$

$$\int_{\delta_h} f(w)dz = f(w)\int_{\delta_h} dz = f(w)\int_0^1 hdt = hf(w)$$

We can rewrite the previous equation as

$$F(w+h) = F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w))dz$$

For $h \neq 0$,

$$\left|\frac{F(w+h) - F(w)}{h} - f(w)\right| = \frac{1}{|h|}\left|\int_{\delta_h} (f(z) - f(w))dz\right|$$

□

## 5.3 First Version of Cauchy's Theorem

**Definition 5.3.1.** A domain $D$ is **starlike** if for some point $a_0 \in D$, for every $b \neq a_0 \in D$, the straight line connecting $a_0$ and $b$ is contained in $D$.

**Example 5.3.2.**

1. $\mathbb{C}$ is starlike.
2. The ball $B_r(a)$ is starlike.
3. Any convex set is starlike.

**Example 5.3.3.**

1. $\mathbb{C}^\star$ is not starlike, because a straight line between two points could pass through 0, and $0 \notin \mathbb{C}^\star$.
2. Similarly, $B_r^\star(a) = B_r(a) - \{a\}$ is not starlike.

**Lemma 5.3.4.** Let $U$ be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Then

$$\int_{\partial\Delta} f(z)dz = 0$$

for every **triangle** $\Delta$ in $U$.

**Remark.** Here $\partial\Delta$ is the boundary of $\Delta$, traversed anticlockwise.

**Remark.** Given any closed contour without a parameterisation given, we will assume that it is traversed anticlockwise.

*Proof.* First, split $\Delta$ into four triangles, $\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}, \Delta^{(4)}$, using the midpoints of each side. Then

$$\int_{\partial\Delta} f(z)dz = \sum_{i=1}^{4} \int_{\partial\Delta^{(i)}} f(z)dz$$

Let $\Delta_1$ be one of these four triangles which has the largest integral, then

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq 4\left|\int_{\partial\Delta_1} f(z)dz\right|$$

We then continue this procedure to produce a sequence of triangles

$$\Delta > \Delta_1 > \cdots > \Delta_n > \cdots$$

The length of $\Delta_1$, $L(\Delta_1)$ satisfies $L(\Delta_1) = \frac{1}{2}L(\Delta)$, therefore

$$L(\Delta_n) = \frac{1}{2}nL(\Delta) \Longrightarrow L(\Delta_n) \to \text{ as } n \to \infty$$

Also,

$$\bigcap_{n\in\mathbb{N}} \Delta_n = \{w\}$$

is a single point in $D$. Now, notice that

$$\int_{\partial\Delta_n} 1dz = 0 = \int_{\partial\Delta_n} zdz$$

and that $w, f(w), f'(w)$ are constants. Then sneakily,

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} (f(z) - f(w) - (z-w)f'(w))$$

Define the auxiliary function

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} - f'(w) & \text{if } z \in D \setminus \{w\} \\ 0 & \text{if } z = w \end{cases}$$

which is continuous at $z = w$, so is continuous on $D$. So

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} (z-w)g(z)dz$$

Now,

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq 4^n\left|\int_{\partial\Delta_n} f(z)dz\right| = 4^n\left|\int_{\partial\Delta_n} (z-w)g(z)dz\right|$$

Note

$$\sup_{z\in\partial\Delta_n} |z-w| \leq L(\partial\Delta_n)$$

so by the Estimation Lemma,

$$\begin{aligned}\left|\int_{\partial\Delta} f(z)dz\right| &\leq 4^n L(\partial\Delta_n) \sup_{z\in\partial\Delta_n} |(z-w)g(z)| \\ &\leq 4^n L(\partial\Delta_n) \sup_{z\in\partial\Delta_n} |(z-w)| \sup_{z\in\partial\Delta_n} |g(z)| \\ &\leq 4^n (L(\partial\Delta_n))^2 \sup_{z\in\partial\Delta_n} |g(z)| \\ &= L(\Delta)^2 \sup_{z\in\partial\Delta_n} |g(z)|\end{aligned}$$

As $n \to \infty$, $\sup_{z\in\partial\Delta_n} |g(z)| \to g(w) = 0$. This completes the proof. $\square$

**Lemma 5.3.5.** Let $D$ be a starlike domain and $f : D \to \mathbb{C}$ be continuous. Then, if

$$\int_{\partial\Delta} f(z)dz = 0$$

for every $\Delta \subset D$, then for some $F : D \to \mathbb{C}$,

$$F'(z) = f(z) \quad \forall z \in D$$

*Proof.* Similar to the proof of converse of FTC. □

**Theorem 5.3.6.** (**Cauchy's Theorem for Starlike Domains - CTSD**) Let $D$ be a starlike domain and let $f : D \to \mathbb{C}$ be holomorphic. Then for every closed contour $\gamma \in D$,

$$\int_{\gamma} f(z)dz = 0$$

*Proof.* By Lemma Reflem:ctLem1,

$$\int_{\partial\Delta} f(z)dz = 0 \quad \forall \Delta \in D$$

By Lemma Reflem:ctLem2, $f$ has a holomorphic antiderivative $F$. Then, by FTC,

$$\int_{\gamma} f(z)dz = 0 \quad \forall \text{ closed } \gamma \in D$$

□

**Remark.** The same result holds if $f$ is holomorphic on $D - S$, where $S$ is a finite set of points and $f$ is continuous on $D$. We will need this in proofs but is not used much elsewhere.

**Example 5.3.7.** Consider

$$\int_{|z|=\frac{1}{2}} \frac{e^z (\sin z)^2}{e^{z^2}} dz$$

Because the function in the integral is holomorphic and $|z| = \frac{1}{2}$ is a closed contour, by CTSD, this integral is equal to 0.

## 5.4 Cauchy's integral formula

**Theorem 5.4.1.** (**Cauchy's integral formula - CIF**) Let $B_r(a)$ be a ball in $\mathbb{C}$ and $f : B_r(a) \to \mathbb{C}$ be holomorphic. Then for every $w \in B_r(a)$,

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz$$

where $\rho$ is any real number with $|w - a| < \rho < r$.

*Proof.* Define an auxiliary function $g$ by

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

Note that $g$ is continuous at $z = w$, and holomorphic elsewhere. By CTSD,

$$\int_{|z-a|=\rho} g(z)dz = 0$$

Therefore

$$\int_{|z-a|=\rho} \frac{f(z)}{z-w}dz = \int_{|z-a|=\rho} \frac{f(w)}{z-w}dz$$

Now,

$$\begin{aligned}
\frac{1}{z-w} &= \frac{1}{z-a+a-w} \\
&= \frac{1}{(z-a)(1-\frac{w-a}{z-a})} \\
&= \frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n
\end{aligned}$$

which converges uniformly, since $|\frac{w-a}{z-a}| = |\frac{w-a}{\rho} < 1|$. So we have

$$\begin{aligned}
\int_{|z-a|=\rho} \frac{f(z)}{z-w}dz &= f(w)\int_{|z-a|=\rho}\sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}dz \\
&= \sum_{n=0}^{\infty}\left(f(w)(w-a)^n\int_{|z-a|=\rho}\frac{1}{(z-a)^{n+1}}dz\right)
\end{aligned}$$

The inner integral is equal to 0 except when $n = 0$, when it's value is $2\pi i$. So

$$\int_{|z-a|=\rho} \frac{f(z)}{z-w}dz = f(w)(w-a)^0 \cdot 2\pi i = 2\pi i \cdot f(w)$$

$\square$

# 6 Features of holomorphic functions

**Theorem 6.0.1.** (**Cauchy-Taylor theorem**) Let $U$ be an open set and $f : U \to \mathbb{C}$ be holomorphic on $U$. Then for every $r > 0$ such that $B_r(a) \subset U$, $f$ has a power series converging on $B_r(a)$ given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

is a constant for every $0 < \rho < r$. This is the **Taylor series** of $f$ about $a$.

*Proof.* By the CIF, for every $w$ with $|w-a| < \rho$,

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz \\ &= \frac{1}{2\pi i} \int_{|z-a|=\rho} f(z) \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} dz \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz \right) (w-a)^n \\ &= \sum_{n=0}^{\infty} c_n (w-a)^n \end{aligned}$$

□

**Theorem 6.0.2.** (**CIF for derivatives**) Let $f : B_r(a) \to \mathbb{C}$ be holomorphic. Then for every $0 < \rho < r$,

$$\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

*Proof.* By Cauchy-Taylor, we have a convergent power series such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

But we also have (corollary 5.22 in lecture notes),

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Equating these two expressions for $c_n$ completes the proof. □

**Remark.** Combining theorem 7.1 (lecture notes) and theorem 7.2 (lecture notes), every holomorphic function $f$ has power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

**Remark.** Cauchy-Taylor does not hold in real analysis. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$f$ is differentiable for $x \neq 0$. For $x = 0$,

$$f'(x) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}$$

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{n\to 0^-} \frac{0}{x} = 0$$

and

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{e^{-1/x}}{x} = \lim_{x\to 0^+} \frac{1/x}{e^{1/x}} = \lim_{y\to\infty} \frac{y}{e^y} = 0$$

so $f'(0) = 0$, hence $f$ is differentiable on $\mathbb{R}$. But if $f$ had a Taylor series at $x = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = 0$$

around $x = 0$.

**Corollary 6.0.3.** (Holomorphic functions have infinitely many derivatives) If $f : U \to \mathbb{C}$ is holomorphic on an open set $U$ then $f$ has derivatives of all orders and each derivative is also holomorphic.

*Proof.* Since $U$ is open, $\exists B_r(a) \subset U$ around a point $z = a$. But then by Cauchy-Taylor, $f$ has a power series. By theorem 5.21 (lecture notes), this power series is holomorphic. By corollary 5.22 (lecture notes) we can term-by-term differentiate to get a power series for $f'(z)$. By theorem 5.21 (lecture notes), $f'(z)$ is holomorphic. This can be repeated indefinitely. □

**Remark.** This is a huge difference between real and complex analysis. Let $f : \mathbb{R} \to \mathbb{R}$ by defined as

$$f_n(x) = |x|x^n$$

$$f_n'(0) = \lim_{x\to 0} \frac{f_n(x) - f_n(0)}{x - 0} = \lim_{x\to 0} |x|x^{n-1} = 0$$

$f_n'(x) = (n+1)|x|x^{n-1}$ and $f^{(n)}(x) = c|x|$ which is not differentiable.

**Theorem 6.0.4.** (**Morera's Theorem**) Let $f : D \to \mathbb{C}$ be continuous on a domain $D$. If

$$\int_\gamma f(z)dz = 0 \quad \forall \text{ closed } \gamma \subset D$$

then $f$ is holomorphic.

*Proof.* By the converse FTC, $f$ has a holomorphic antiderivative $F : D \to \mathbb{C}$ such that $F'(z) = f(z) \quad \forall z \in D$. By corollary 7.6 (lecture notes), if $F$ is holomorphic, its derivative $f$ must be. □

**Example 6.0.5.** Consider

$$\int_{|z|=3} \frac{e^z}{z^2(z-1)} dz$$

We use partial fractions:

$$\frac{1}{z^2(z-1)} = \frac{a}{z} + \frac{b}{z^2} + \frac{c}{z-1}$$

So $1 = (c+a)z^2 + (b-a)z - b$, so $b = -1, a - -1, c = 1$. Using the CIF and CIF for derivatives,

$$\begin{aligned}
\int_{|z|=3} \frac{e^z}{z^2(z-1)} dz &= -\int_{|z|=3} \frac{e^z}{z} dz - \int_{|z|=3} \frac{e^z}{z^2} dz + \int_{|z|=3} \frac{e^z}{z-1} dz \\
&= -2\pi i e^0 - 2\pi i e^0 + 2\pi i e^1 \\
&= 2\pi i(e-2)
\end{aligned}$$

## 6.1 Liouville's theorem

**Definition 6.1.1.** A function $f : \mathbb{C} \to \mathbb{C}$ is **entire** if $f$ is holomorphic on $\mathbb{C}$.

**Definition 6.1.2.** A function $f : \mathbb{C} \to \mathbb{C}$ is **bounded** if for some $M > 0$, $|f(z)| \leq M \; \forall z \in \mathbb{C}$.

**Theorem 6.1.3.** (**Liouville's theorem**) Every bounded entire function is constant.

*Proof.* Let $f$ be entire and bounded. We will show that $\forall w \in \mathbb{C},\ f(w) = f(0)$. By the CIF, for every $\rho > |w|$,

$$\begin{aligned}
|f(w) - f(0)| &= \left| \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z-w} dz - \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z} dz \right| \\
&= \frac{|w|}{2\pi} \left| \int_{|z|=\rho} f(z) \frac{1}{z(z-w)} dz \right| \\
&=
\end{aligned}$$

Using the Estimation lemma, boundedness of $f$ and the reverse triangle inequality,

$$\begin{aligned}
|f(w) - f(0)| &\leq \frac{|w|}{2\pi} 2\pi\rho \cdot \sup_{|z|=\rho} \frac{|f(z)|}{|z||z-w|} \\
&\leq |w|\rho \frac{M}{\rho} \sup_{|z|=\rho} \frac{1}{|z-w|} \\
&\leq |w| M \sup_{|z|=\rho} \frac{1}{||z| - |w||} \\
&= \frac{|w| M}{\rho - |w|} \\
&\to 0 \quad \text{as } \rho \to \infty
\end{aligned}$$

and $\rho$ can be arbitrarily large. □

**Remark.** The holomorphicity condition is essential (we can't just say that $f$ is continuous). For example,

$$f(z) = f(x+iy) = \sin(x) + i\sin(y)$$

is continuous and bounded on $\mathbb{C}$ but is not entire.

**Theorem 6.1.4.** (**Fundamental theorem of Algebra**) Every non-constant polynomial with complex coefficients $p(z) = a_d z^d + \cdots + a_1 z + a_0$, $a_d \neq 0$ has a complex root: for some $z_0 \in \mathbb{C}, P(z_0) = 0$.

*Proof.* By assumption, $d \geq 1$, so $|p(z)| \to \infty$ as $|z| \to \infty$. In particular, $\exists R > 0, |p(z)| > 1$ if $|z| = R$. Assume the converse, that $p$ has no roots.

Then $f(z) := \frac{1}{p(z)}$ is holomorphic on $\mathbb{C}$. On the set $|z| > R$, $f$ is bounded, since $|f(z)| = \frac{1}{|p(z)|} < 1$. But $\overline{B_R}(0) = \{z \in \mathbb{C} : |z| \leq R\}$ is compact, so by theorem 2.30 (lecture notes), $|f(z)|$ attains a maximum on $\overline{B_R}(0)$. In particular, $f$ is bounded on $\overline{B_R}(0)$. Thus $f$ is bounded and entire, so by Liouville's theorem, $f$ is constant, which is a contradiction. $\square$

**Theorem 6.1.5.** (**Local maximum modulus principle**) Let $f : B_r(a) \to \mathbb{C}$ be holomorphic. If for every $z \in B_r(a)$, $|f(z)| \leq |f(a)|$ then $f$ is constant on $B_r(a)$.

*Proof.* First, we show $|f|$ is constant. Pick any $w \in B_r(a)$. We will show $|f(w)| = |f(a)|$. Let $\rho = |w - a| < r$ so that the contour $\gamma(t) = a + \rho e^{2\pi i t}, t \in [0,1]$ passes through $w$. By the CIF,

$$\begin{aligned}
|f(a)| &= \left| \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-a} dz \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{f(z)}{z-a} dz \right| \\
&= \frac{1}{2\pi} \left| \int_0^1 \frac{f(\gamma(t))}{\gamma(t)-a} \gamma'(t) dt \right| \\
&= \frac{1}{2\pi} \left| \int_0^1 \frac{f(\gamma(t))}{\rho e^{2\pi i t}} \rho \cdot 2\pi i e^{2\pi i t} dt \right| \\
&= \left| \int_0^1 f(\gamma(t)) dt \right| \\
&\leq \int_0^1 |f(\gamma(t))| dt \\
&\leq \int_0^1 |f(a)| dt \\
&\leq |f(a)| \int_0^1 1 dt \\
&= |f(a)|
\end{aligned}$$

Therefore every inequality above must be an equality. In particular,

$$\int_0^1 |f(\gamma(t))| dt = \int_0^1 |f(a)| dt$$

So by continuity, $|f(\gamma(t))| = |f(a)|$. So $|f(w)| = |f(a)|$, and $w$ was arbitrary, so $|f(z)| = |f(a)|\ \forall z \in B_r(a)$.

Now, we show that $f$ is constant. Let $|f(z)| = c\ \forall z \in B_r(a)$. If $c = 0$, the $|f(z)| = 0 \Rightarrow f(z) = 0 \forall z \in B_r(a)$. So assume $c \neq 0$. Now,

$$c^2 = |f(z)|^2 = f(z) \cdot \overline{f(z)}$$

So $\overline{f(z)} = \frac{c^2}{f(z)}$ is holomorphic (since $f$ is holomorphic). Let $f = u + iv$, then $\overline{f} = u - iv = u + i(-v)$. Then by the Cauchy-Riemann equations, $u_x = v_y$ but also $u_x = (-v)_y = -v_y$ so $u_x = v_y = 0$, and $u_y = -v_x$ but also $u_y = (-v)_x = v_x$ so $u_y = v_x = 0$. By proposition 3.3 (lecture notes), $f'(z) = u_x + iv_x = 0 + i \cdot 0 = 0$, hence $f$ is constant. $\square$

**Theorem 6.1.6.** (**Maximum modulus theorem**) Let $D$ be a domain and $f : D \to \mathbb{C}$. If

$$\exists a \in D, |f(z)| \leq |f(a)| \quad \forall z \in D$$

then $f$ is constant on $D$.

*Proof.* Let $U_1 = \{z \in D : f(z) = f(a)\}$ and let $U_2 = \{z \in D : f(z) \neq f(a)\}$. Let $U = U_1 \cup U_2$ (TODO: check this last sentence, might not be $U$). $U_1$ is non-empty, because $a \in U_1$. We will show that $U_1$ is open. Pick $z \in U_1$. Then by the openness of $U$, there exists a ball $B_r(z)$ in $U$. Pick $W \in B_r(z)$. We have $|f(w)| \leq |f(a)| = |f(z)|$, so by the Local Maximum Modulus Principle, $f$ is constant on $B_r(z)$, so $f(w) = f(z) = f(a) \implies w \in U_1 \implies B_r(z) \subset U_1$. Hence $U_1$ is open.

Now $U_2 = D - U_1 = f^{-1}(\mathbb{C} - \{f(a)\})$, where $f^{-1}$ denotes the preimage. Therefore $U_2$ is open by theorem 2.17 (lecture notes), since $\mathbb{C} - \{f(a)\}$ is open.

So $D = U_1 \cup U_2$ where $U_1$ is non-empty and open and $U_2$ is open. By fact 7.14 (lecture notes), $U_2$ is empty hence $D = U_1$, so $f$ is constant on $D$. $\square$

**Example 6.1.7.** Find the maximum absolute value of $f(z) = z^2 + 2z - 3$ on $\overline{B_1}(0) = \{z \in \mathbb{C} : |z| \leq 1\}$.

$\overline{B_1}(0)$ is compact so by theorem 2.30 (lecture notes), $|f(z)|$ attains a maximum on it. Also, $B_1(0)$ is a domain, so by the maximum modulus theorem, the function does not attain the maximum in $B_1(0)$. So the maximum must be obtained on the boundary. So $|z| = 1$, so let $z = e^{it}$ for some $t \in [0, 2\pi]$. Then $|f(z)|^2 = |z^2 + 2z - 3|^2 = (z^2 + 2z - 3)\overline{(z^2 + 2z - 3)} = 14 - 3e^{-2it} - 3e^{2it} - 4^{it} - 4e^{-it} = 14 - 6\cos(2t) - 8\cos(t) = -12\cos(t)^2 - 8\cos(t) + 20 = -12x^2 - 8x + 20$ where $x = \cos(t)$. The maximum is attained when $x = \cos(t) = -\frac{1}{3}$. This gives $|f(z)| = \frac{8}{\sqrt{3}}$.

## 6.2 Analytic continuation and the identity theorem

Let $f : B_r(a) \to \mathbb{C}$ be holomorphic. Then by Cauchy-Taylor, $f$ has a convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

for $z \in B_r(a)$. Assume $f \not\equiv 0$ so at least one coefficient is non-zero. Let $m = \min\{n \geq 0 : c_n \neq 0\}$. Then

$$f(z) = (z-a)^m \sum_{n=m}^{\infty} c_n (z-a)^{n-m} = (z-a)^m \sum_{k=0}^{\infty} c_{k+m} (z-a)^k$$

where $k = n - m$. Let

$$h(z) = \sum_{k=0}^{\infty} c_{k+m} (z-a)^k$$

Then $h$ is a convergent power series so is holomorphic by theorem 5.21 (lecture notes) and $h(a) = c_m \neq 0$. Note $f(a) = 0 \iff m > 0$.

**Definition 6.2.1.** (**Orders of zeros**) $f : B_r(a) \to \mathbb{C}$ has a **zero of order** $m$ at $a$ if for some holomorphic function $h : B_r(a) \to \mathbb{C}$ such that $f(z) = (z-a)^m h(z)$ and $h(a) \neq 0$.

**Remark.** We can show that $f$ has a zero of order $m$ at $z = a$ iff

$$f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$$

**Example 6.2.2.** Let $f(z) = z(e^z - 1)$. Let $z = a = 0$. Then

$$\begin{aligned} f(z) &= z\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right) \\ &= z\sum_{n=1}^{\infty} \frac{z^{n+1}}{n!} \\ &= z^2 \sum_{m=0}^{\infty} \frac{z^m}{(m+1)!} \end{aligned}$$

so $f$ has a zero of order 2 at $a = 0$. As a check, $f'(z) = e^z - 1 + ze^z$ and $f''(z) = e^z + e^z + ze^z$ so $f(0) = f'(0) = 0$ but $f''(0) = e^0 + e^0 + 0 = 2$.

**Remark.** Because holomorphic functions are continuous, if $f(a) \neq 0$, we can always find $B_\rho(a)$ such that $f(z) \neq 0$ on $B_\rho(a)$.

**Theorem 6.2.3.** (**Principle of isolated zeroes**) Let $f : B_r(a) \to \mathbb{C}$ be holomorphic and $f \not\equiv 0$. Then for some $\rho > 0$,

$$f(z) \neq 0 \quad \forall z \in B_r(a) - \{a\}$$

In particular, the zeros are isolated from one another.

*Proof.* For $f(a) \neq 0$, we are done by continuity. If $f(a) = 0$, for some $m > 0$,

$$f(z) = (z-a)^m h(z) \quad \forall z \in B_\rho(a)$$

where $h : B_r(a) \to \mathbb{C}$ is holomorphic, $\rho > 0$ and $h(a) \neq 0$. Thus $h(z) \neq 0$ on $B_\rho(a)$ and $(z-a)^m \neq 0$ on $B_\rho(a) - \{a\}$. So $f(z) \neq 0$ on $B_\rho(a)$. □

**Theorem 6.2.4.** (**Uniqueness of analytic continuation**) Let $D' \subset D$ be non-empty domains and $f : D' \to \mathbb{C}$ be holomorphic. Then there exists **at most one** holomorphic $g : D \to \mathbb{C}$ such that

$$g(z) = f(z) \quad \forall z \in D'$$

If $g$ exists, it is called the **analytic continuation** of $f$ to $D$.

*Proof.* Let $g_1, g_2 : D' \to \mathbb{C}$ be analytic continuations. Let $h(z) = g_1(z) - g_2(z)$. We will show that $h(z) = 0$ on $D$. Note $h(z) = 0 \; \forall z \in D'$. Let

$$D_0 = \{w \in D : \exists r > 0, h(z) = 0 \text{ on } B_r(w) D_1 = \{w \in D : \exists n \geq 0, h^{(n)}(w) \neq 0\}$$

We will show $D_0$ is non-empty and open, $D_1$ is open and $D$ is the disjoint union $D = D_0 \cup D_1$, so $D_0 \cap D_1 = \emptyset$.

First we show $D_0$ is non-empty and open. Since $D' \subset D_0$, $D_0$ is non-empty. We want to show that $\forall w \in D_0, \exists r > 0, B_r(w) \subset D_0$. By the definition of $D_0$, for some $r > 0$, $h(z) = 0$ on $B_r(w)$. Pick $z \in B_r(w)$. $B_r(w)$ is open, so $\exists B_\rho(z)$ inside $B_r(w)$, on which $h(z) = 0$. Thus $z \in D_0$. Thus $D_0$ is open.

Now we show $D_1$ is open.

$$\begin{aligned} D_1 &= \bigcup_{n=0}^{\infty} \{w \in D : h^{(n)}(w) \neq 0\} \\ &= \bigcup_{n=0}^{\infty} \left(h^{(n)}\right)^{-1}(\mathbb{C} - \{0\}) \end{aligned}$$

By Lemma 2.8 (lecture notes) and Theorem 2.17 (lecture notes), $D_1$ is open.

Now we show $D = D_0 \cup D_1$. Pick $w \in D$. If $w \notin D_1$, then $h^{(n)}(w) = 0\ \forall n \geq 0$. But by Cauchy-Taylor (lecture notes), $h$ has a Taylor series about $z = w$ with coefficients $h^{(n)}(w)/n! = 0$. Thus $h = 0$ around $w$. So $w \in D_0$.

If $w \in D_1$, it must have a non-zero Taylor series expansion (at least coefficient $c_n$ is non-zero). By the Principle of Isolated Zeroes (lecture notes), for some $B_\rho(w)$, $h(z) \neq 0$ on $B_\rho(w) - \{w\}$. So $w \notin D_0$. This completes the proof. □

**Corollary 6.2.5.** Let $f, g$ be holomorphic on a domain $D$. If $f = g$ on some $B_r(w) \subseteq D$ then $f = g$ on $D$.

**Definition 6.2.6.** Given a set $S \subset \mathbb{C}$, a point in $S$ is called

- **isolated** in $S$ if $\exists \epsilon > 0$, $B_\epsilon(w) \cap S = \{w\}$.
- **non-isolated** in $S$ if $\forall \epsilon > 0$, $\exists z \in S$, $z \in B_\epsilon(w)$, with $\omega \neq z$.

**Theorem 6.2.7.** (**Identity theorem**) Let $f, g : D \to \mathbb{C}$ be holomorphic on a domain $D$. If $S = \{z \in D : f(z) = g(z)\}$ contains a non-isolated point, then

$$f(z) = g(z) \quad \text{on } D$$

*Proof.* Let $w \in S$ be non-isolated and let $h(z) = f(z) - g(z)$. Then $h(w) = 0$. By the Principle of Isolated Zeroes (lecture notes), for some $\rho$, $\forall z \in B_\rho(w) - \{w\}$, $h(z) \neq 0$. But this contradicts $w$ being non-isolated. So $h(z) = 0$ on $B_\rho(w)$. By Corollary 6.2.5, $h(z) = 0$ on $D$. □

**Example 6.2.8.** Let $f : \mathbb{C} \to \mathbb{C}$ and assume $\forall n \in \mathbb{N}, f(1/n) = \sin(1/n)$. Then $f(z) = \sin(z)$ on $\mathbb{C}$.

By continuity (lemma 2.29 (lecture notes)),

$$f(0) = f\left(\lim_{n\to\infty} \frac{1}{n}\right) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0)$$

Since $z = 0$ is a non-isolated point in $S = \{z \in \mathbb{C} : f(z) = \sin(z)\}$, $f(z) = \sin(z)$ on $\mathbb{C}$.

**Example 6.2.9.** (Problems class) Evaluate

$$I = \int_{|z|=1} \frac{1}{(z-a)(z-b)} dz$$

in the cases:

$$|a| > 1, |b| > 1; \quad |a| < 1 < |b|; \quad |a| < 1, |b| < 1$$

- $|a| > 1, |b| > 1$: we can find a ball $B_r(0)$ with $1 < r < \min\{|a|, |b|\}$, on which the integrand is holomorphic, so by CTSD, $I = 0$.
- $|a| < 1 < |b|$: Let $f(z) = 1/(z - b)$, then

$$I = \int_{|z|=1} \frac{f(z)}{z-a} dz = 2\pi i \frac{1}{a-b}$$

by Cauchy's integral formula.

- $|a| < 1, |b| < 1$: if $a = b$ then
$$\frac{1}{(z-a)(z-b)} = \frac{1}{(z-a)^2} \implies I = 2\pi i \cdot 0 = 0$$
by CIF for derivatives. If $a \neq b$, then use partial fractions.

**Example 6.2.10.** (Problems class) Evaluate
$$I = \int_{|z|=2} \frac{\sin(z)^2}{z^2} dz$$
Let $f(z) = \sin(z)^2, a = 0, \rho = 2, n = 1$. Then by CIF for derivatives,
$$I = \frac{2\pi i}{1} \left[\sin(z)^2\right]' \Big|_{z=0} = 2\pi i [2\sin(z)\cos(z)] \Big|_{z=0} = 0$$
**Example 6.2.11.** (Problems class) Let $f(z) = 1/(1-z)^2$. Find a Taylor series of $f$.

An antiderivative of $f$ is $F(z) = 1/(1-z)$. $F$ has Taylor series
$$F(z) = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$
So by theorem 5.21 (lecture notes), $f$ has Taylor series
$$f(z) = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{m=0}^{\infty} (m+1) z^m \quad (|z| < 1)$$
**Example 6.2.12.** (Problems class) Using the Taylor series expansion of $f(z) = e^z$ about $z = 0$, prove that $|e^z - 1| \leq e^{|z|} - 1 \leq |z| e^{|z|}$.

Using the triangle inequality,
$$\begin{aligned}
|e^z - 1| &= \left| \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 \right| \\
&= \left| \sum_{n=1}^{\infty} \frac{z^n}{n!} \right| \\
&\leq \sum_{n=1}^{\infty} \frac{|z|^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{|z|^n}{n!} - 1 \\
&= e^{|z|} - 1
\end{aligned}$$
Also,
$$\begin{aligned}
e^{|z|} - 1 &= \sum_{n=1}^{\infty} \frac{|z|^n}{n!} \\
&= |z| \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{n!} \\
&= |z| \sum_{m=0}^{\infty} \frac{|z|^m}{(m+1)!} \\
&\leq |z| \sum_{m=0}^{\infty} \frac{|z|^m}{m!} \\
&= |z| e^{|z|}
\end{aligned}$$

**Example 6.2.13.** Find the Taylor series about $z_0 \in \mathbb{C}$ of $f(z) = e^z$.

$\forall n \in \mathbb{N}, f^{(n)}(z) = e^z$. By Corollary 5.22 (lecture notes),

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!}(z - z_0)^n$$

**Example 6.2.14.** Calculate the Taylor series of $f(z) = \mathrm{Log}(1 + z)$ about $z = 0$.

$$f'(z) = \frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n$$

Then by corollary 5.23 (lecture notes),

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} + c \quad (|z| < 1)$$

for some $c$. $\mathrm{Log}(1 + 0) = 0$, so $c = 0$.

## 6.3 Harmonic functions and the Dirichlet problem

**Definition 6.3.1.** A **harmonic function** is a real valued function $u : D \to \mathbb{R}$ on a domain $D \subset \mathbb{C}$ that has continuous second-order partial derivatives which satisfy the **Laplace equation**:

$$u_{xx} + u_{yy} = 0$$

**Proposition 6.3.2.** Let $f = u + iv : D \to \mathbb{C}$ be holomorphic. Then $u$ and $v$ are harmonic.

*Proof.* By proposition 3.3 (lecture notes) the first-order partial derivatives exist and $f' = u_x - iu_y = v_y + iv_x$. By corollary 7.6 (lecture notes), $f'$ is holomorphic and so continuous, hence $u_x, u_y, v_y, v_x$ are continuous as well. By the same argument with $f'$, the second-order partial derivatives exist and are continuous. By proposition 3.3 (lecture notes), $u_x, u_y, v_x, v_y$ satisfy the Cauchy-Riemann equations:

$$u_x = v_y \implies u_{xx} = v_{yx}$$
$$u_y = -v_x \implies u_{yy} = -v_{xy}$$

By the Schwartz-Clairault theorem, $v_{yx} = v_{xy}$, hence $u_{xx} + u_{yy} = v_{yx} - v_{yx} = 0$. ☐

**Example 6.3.3.** Let $f(z) = e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y))$. So $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$. $u_x(x, y) = e^x \cos(y)$, $u_y = -e^x \sin(y)$, $u_{xx} = e^x \cos(y)$ and $u_{yy} = -e^x \cos(y)$.

**Example 6.3.4.** Let $f(x + iy) = x^2 + y^2$. So $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. $u_x = 2x \implies u_{xx} = 2$ but $u_y = 2y \implies u_{yy} = 2$ so $u_{xx} + u_{yy} = 4$. Note $f(z) = |z|^2$. So $f$ is not holomorphic.

**Example 6.3.5.** Let $u(x, y) = x^2 - y^2 + 3x$, so $u_x = 2x + 3 \implies u_{xx} = 2$, $u_y = -2y \implies u_{yy} = -2$ so $u$ is harmonic.

**Proposition 6.3.6.** Let $f : D \to \mathbb{C}$ be holomorphic on a starlike domain $D$. Then for some $F : D \to \mathbb{C}$, $F$ is holomorphic and $F' = f$.

*Proof.* By Cauchy's Starlike theorem (lecture notes),

$$\int_\gamma f(z)dz = 0 \quad \forall \text{ closed } \gamma$$

By the converse FTC (lecture notes), there exists a holomorphic antiderivative of $f$, $F$. $\square$

**Theorem 6.3.7.** (**The existence of a harmonic conjugate**) If $D$ is a starlike domain and $u : D \to \mathbb{R}$ is harmonic, then for some harmonic function $v : D \to \mathbb{R}$,

$$f = u + iv$$

is holomorphic on $D$. $v$ is called the **harmonic conjugate** of $u$ and is unique up to a real additive constant.

*Proof.* If such an $f$ exists, then $f' = u_x - iu_y$ would also be holomorphic. We will first construct $f'$ and then construct $f$.

Let $g(x, y) = u_x + i(-u_y)$. We want to show $g$ is holomorphic. Using theorem 3.5 (lecture notes), we will show $g$ satisfies the Cauchy-Riemann equations.

$$(u_x)_x = u_{xx}, \quad (-u_y)_y = -u_{yy} = u_{xx}$$

as $u$ is harmonic. Similarly,

$$(u_x)_y = u_{xy}, \quad -(-u_y)_x = u_{yx}$$

by Schwarz-Clairault. So by theorem 3.5 (lecture notes), $g$ is holomorphic.

Now we will construct $f$. By proposition 7.28 (lecture notes), $g$ has a holomorphic antiderivative, $F = U + iV$, where $F'(z) = g(z)$ on $D$. $F$ satisfies the Cauchy-Riemann equations and $F' = U_x - U_y i = g = u_x - u_y i$. Hence

$$U_x = u_x, \quad U_y = u_y \implies (U - u)_x = 0 = (U - u)_y$$

Therefore $U = u + c$ for some constant $c$. So let $f = F - c = (U + iV) - c = u + iV$, and let $v = V$. By proposition 7.26 (lecture notes), $v$ is harmonic.

Finally, we show $v$ is unique (TODO). $\square$

**Example 6.3.8.** We have seen $u(x, y) = x^2 - y^2 + 3x$ is harmonic. Construct its harmonic conjugate $v$.

We want a holomorphic $f = u + iv$. By the first Cauchy-Riemann equation, $v_y = u_x = 2x + 3$. So $v(x, y) = 2xy + 3y + g(x)$ for some function $g$. By the second Cauchy-Riemann equation,

$$2y + g'(x) = v_x = -u_y = 2y$$

hence $g'(x) = 0 \implies g(x) = c$ for a constant $c \in \mathbb{R}$. So $v(x, y) = 2xy + 2y + c$. Then $f(x + iy) = x^2 - y^2 + 3x + i(2xy + 3y + c)$. Note that $(x + iy)^2 = x^2 - y^2 + 2xyi$, so $f(z) = z^2 + 3z + ic$.

**Definition 6.3.9.** The **Dirichlet boundary problem** states: Let $D \subseteq \mathbb{C}$ be a domain with closure $\bar{D}$ and boundary $\delta D$. Let $g : \delta D \to \mathbb{R}$ be continuous. Find a continuous function $\mu : \bar{D} \to \mathbb{R}$ such that $\mu$ is harmonic on $D$ and matches $g$ on $\delta D$.

**Example 6.3.10.** Let

$$D = \{x + iy \in \mathbb{C} : 2 < y < 5\}, \quad g(x, y) = \begin{cases} 4 & \text{if } y = 2 \\ 13 & \text{if } y = 5 \end{cases}$$

So $\bar{D} = \{x + iy \in \mathbb{C} : 2 \leq y \leq 5\}$ and $\delta D = \{x + iy \in \mathbb{C} : y = 2 \text{ or } y = 5\}$. We want a harmonic $\mu$ such that $\mu = g$ on $\delta D$.

Note that $g$ doesn't depend on $x$, so it is possible that $\mu = 0$. If this was true, then $\mu_{xx} = 0$ and $\mu$ is harmonic so $\mu_{yy} = -\mu_{xx} = 0$, so

$$\mu(x, y) = ay + b$$

This guess for this solution is called an **ansatz**. Check $\mu(x, 2) = 4 = 2a + b$ and $\mu(x, 5) = 13 = 5a + b$, which gives $a = 3$ and $b = -2$. Thus $\mu(x, y) = 3y - 2$ is a solution to the Dirichlet boundary problem.

**Proposition 6.3.11.** Let $f : D \to \mathbb{C}$, $f = u + iv$ be holormorphic on $D$ and $\mu$ is harmonic on $f(D)$. Then

$$\tilde{\mu} := \mu \circ f = \mu(u, v)$$

is harmonic on $D$.

**Example 6.3.12.** From Example 6.3.10, $\mu(x, y) = 3y - 2$ was a solution to the Dirichlet boundary problem on $D_1 = \{x + iy : 2 < y < 5\}$ with

$$g(x, y) = \begin{cases} 4 & \text{if } y = 2 \\ 13 & \text{if } y = 5 \end{cases}$$

Let $D_2$ be $D_1$ rotated anticlockwise by $\pi/4$ about the origin. Solve the Dirichlet boundary problem on $D_2$ when

$$g(x, y) = \begin{cases} 4 & \text{if } y = x + 2\sqrt{2} \\ 13 & \text{if } y = x + 5\sqrt{2} \end{cases}$$

Let $f(z) = e^{-\pi i/4}z$. From Example 6.3.10, we know a solution on $f(D_2) = D_1$. So by proposition 7.31 (lecture notes), $\tilde{\mu} = \mu \circ f$ is a solution on $D_2$. Note that

$$f(x, y) = e^{-\pi i/4}(x + iy) = \frac{1}{\sqrt{2}}(x + y) + i\frac{1}{\sqrt{2}}(y - x)$$

Thus $\tilde{\mu} = \mu(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(y - x)) = \frac{3}{\sqrt{2}}(y - x) - 2$.

# 7 General form of the Cauchy-Taylor theorem and Cauchy's integral formula

## 7.1 Winding number and simply connected sets

**Definition 7.1.1.** Let $\gamma : [a, b] \to \mathbb{C}$ be a contour of the form

$$\gamma(t) = \omega + r(t)e^{i\theta(t)}$$

where $\omega \in \mathbb{C}$, $\theta(t) : [a, b] \to \mathbb{C}$, $r(t) : [a, b] \to \mathbb{R}^+$ are continuous, piecewise-$C^1$. The **winding number** of $\gamma$ about $\omega$ is defined as

$$I(\gamma, \omega) = \frac{\theta(b) - \theta(a)}{2\pi}$$

**Example 7.1.2.** Let $\gamma_1(t) = e^{2\pi i t}$ for $t \in [0, 1]$. Then $r(t) = 1$, $\omega = 0$, $\theta(t) = 2\pi t$. So

$$I(\gamma_1, 0) = \frac{2\pi \cdot 1 - 2\pi \cdot 0}{2\pi} = 1$$

Let $\gamma_2(t) = e^{2\pi i t}$ for $t \in [0, 2]$. Then

$$I(\gamma_2, 0) = \frac{2\pi \cdot 2 - 2\pi \cdot 0}{2\pi} = 2$$

**Remark.** $w \notin \gamma$ since $r(t) > 0$, and $I(\gamma, \omega) \in \mathbb{Z}$ when $\gamma$ is closed, since if $\gamma$ is closed, then

$$\begin{aligned} &\gamma(a) = \gamma(b) = \omega + r(a)e^{i\theta(a)} = \omega + r(b)e^{i\theta(b)} \\ \iff &r(b) = r(a) \text{ and } \theta(a) = \theta(b) + 2\pi n \quad (n \in \mathbb{Z}) \end{aligned}$$

Thus

$$I(\gamma, \omega) = \frac{\theta(b) - \theta(a)}{2\pi} \in \mathbb{Z}$$

**Theorem 7.1.3.** Let $\gamma : [a, b] \to \mathbb{C}$ be a contour. Then for every $\omega \in \mathbb{C}$ with $\omega \notin \gamma$, for some continuous, piecewise $C^1$, $\theta : [a, b] \to \mathbb{R}$ and $r : [a, b] \to \mathbb{R}^+$,

$$\gamma(t) = \omega + r(t)e^{i\theta(t)}$$

*Proof.* Omitted. □

**Lemma 7.1.4.** Let $\gamma : [a, b] \to \mathbb{C}$ be a closed contour and $\omega \notin \gamma$. Then

$$I(\gamma, \omega) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \omega} dz$$

*Proof.* By theorem 8.2 (lecture notes),

$$\gamma(t) = \omega + r(t)e^{i\theta(t)}$$

By definition 6.4 (lecture notes),

$$\begin{aligned} \int_\gamma \frac{1}{z - w} dz &= \int_a^b \frac{1}{\gamma(t) - \omega} \gamma'(t) dt \\ &= \int_a^b \frac{1}{r(t)e^{i\theta(t)}} \left( r'(t)e^{i\theta(t)} + r(t) \cdot i\theta'(t)e^{i\theta(t)} \right) dt \\ &= \int_a^b \left( \frac{r'(t)}{r(t)} + i\theta'(t) \right) dt \\ &= [\log(r(t)) + i\theta(t)]_a^b = i(\theta(b) - \theta(a)) = 2\pi i \cdot I(\gamma, \omega) \end{aligned}$$

□

**Proposition 7.1.5.** Let $D$ be a starlike domain. Then for every closed contour $\gamma$ and every $\omega \notin D$,

$$I(\gamma, \omega) = 0$$

*Proof.* By Lemma 7.1.4,

$$I(\gamma, \omega) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \omega} dz$$

and $1/(z - w)$ is holomorphic on $D$. So by CTSD, $I(\gamma, \omega) = 0$. □

**Definition 7.1.6.** Let $U$ be an open set. A closed contour $\gamma$ in $U$ is **homologous to zero** if $I(\gamma, \omega) = 0$ for every $\omega \notin U$.

**Definition 7.1.7.** An open set $U$ is called **simply connected** if every closed contour in $U$ is homologous to zero.

**Example 7.1.8.** By proposition 8.4 (lecture notes), starlike domains are simply connected.

**Example 7.1.9.** Let $A = \{z \in \mathbb{C} : \alpha < |z| < \beta\}$. Let $\gamma(t) = \rho e^{2\pi i t}$ for $t \in [0, 1]$ and $\alpha < \rho < \beta$. Pick $\omega = 0$ then $\omega \notin A$ but

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz = 1$$

by CIF. Thus $A$ is not simply connected.

**Definition 7.1.10.** A **cycle** $\Gamma$ defined on an open set $U$ is a finite collection of closed contours in $U$. We write

$$\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

**Definition 7.1.11.** Let $\Gamma$ be a cycle. $w \in \mathbb{C}$ is **not on** $\Gamma$ ($\omega \notin \gamma_i$) if $w \notin \gamma_i$ for every $i$. The **winding number of** $\Gamma$ **around** $\omega$ is defined as

$$I(\Gamma, \omega) = \sum_{i=1}^{n} I(\gamma_i, \omega)$$

and we define

$$\int_\Gamma f(z) dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z) dz$$

$\Gamma$ is called **homologous to zero in** $U$ if $I(\Gamma, \omega) = 0$ for every $\omega \notin U$.

**Example 7.1.12.** Let $A_{\alpha,\beta}(\omega) = \{z \in \mathbb{C} : \alpha < |z - \omega| < \beta \text{ for } 0 \leq \alpha < \beta \leq \infty\}$. Let $\Gamma$ be a cycle in $A_{\alpha,\beta}(\omega)$.

Let $\gamma_1(t) = \omega + \rho_1 e^{-2\pi i t}$ for $t \in [0, 1]$, $\gamma_2(t) = \omega + \rho_2 e^{2\pi i t}$ for $t \in [0, 1]$ where $\alpha < \rho_1 < \rho_2 < \beta$. Define

$$\Gamma = \gamma_1 + \gamma_2$$

We claim that $\Gamma$ is homologous to zero in $A_{\alpha,\beta}(\omega)$. Let $\omega' \in A_{\alpha,\beta}(\omega)$. Consider $a \notin A_{\alpha,\beta}(\omega)$. Then

- If $|\omega - a| > \beta$ then

$$\begin{aligned} I(\Gamma, a) &= I(\gamma_1, a) + I(\gamma_2, a) \\ &= \frac{1}{2\pi i} \int_{\gamma_1} \frac{1}{z - a} dz + \frac{1}{2\pi i} \int_{\gamma_2} \frac{1}{z - a} dz \end{aligned}$$

But then $1/(z - a)$ is holomorphic. So by Cauchy's theorem both integrals vanish.

- If $|\omega - a| < \alpha$ then by CIF,

$$\int_{\gamma_1} \frac{1}{z-a} dz = -1, \quad \int_{\gamma_2} \frac{1}{z-a} dz = 1$$

so $I(\Gamma, a) = 0$.

## 7.2 General form of the Cauchy-Taylor theorem and CIF

**Definition 7.2.1.** A closed curve $\gamma : [a, b] \to \mathbb{C}$ is called **simple** if for all $t_1 < t_2$

$$\gamma(t_1) = \gamma(t_2) \Longrightarrow t_1 = a \text{ and } t_2 = b$$

i.e. it cannot cross itself or go back on itself.

**Theorem 7.2.2.** (**Jordan curve theorem**) Let $\gamma \subset \mathbb{C}$ be a simple closed curve. Then its complement $\mathbb{C} - \gamma$ is a disjoint union of two domains, exactly one of which is bounded.

*Proof.* Beyond the scope of this course, so omitted. □

**Definition 7.2.3.** The bounded domain is called the **interior** of $\gamma$ and we write $D_\gamma^{\text{int}}$. We say $w \in D_\gamma^{\text{int}}$ **lies inside** $\gamma$.

**Definition 7.2.4.** The other, non-bounded, domain is called the **exterior** of $\gamma$ and we write $D_\gamma^{\text{ext}}$. We say $w \in D_\gamma^{\text{ext}}$ **lies outside** $\gamma$.

**Remark.** $\mathbb{C} = D_\gamma^{\text{int}} \cup \gamma \cup D_\gamma^{\text{ext}}$ as a disjoint union.

**Remark.** Given a simple closed **contour**, it is always possible to place an orientation on $\gamma$ such that

$$\forall w \in \mathbb{C} - \gamma, \quad I(\gamma, w) = \begin{cases} 1 & \text{if } w \in D_\gamma^{\text{int}} \\ 0 & \text{if } w \in D_\gamma^{\text{ext}} \end{cases}$$

We call $\gamma$ **positively oriented** if this equation holds.

**Definition 7.2.5.** $f$ is called **holomorphic on** $D_\gamma^{\textbf{int}} \cup \gamma$ (and **holomorphic on and inside** $\gamma$) if for some domain $D$ containing $D_\gamma^{\text{int}} \cup \gamma$ on which $f$ is holomorphic.

**Remark.** For a simple closed curve, $\gamma$ is homologous to zero in $D$, since if $w \notin D$ then $w \in D_\gamma^{\text{ext}}$ so $I(\gamma, w) = 0$.

**Theorem 7.2.6.** Let $f : D \to \mathbb{C}$ be a holomorphic function on a domain $D$. Then for every cycle $\Gamma \in D$ such that $\Gamma$ is homologous to zero in $D$, for every $\omega \in D - \Gamma$, we have the **General form of the Cauchy-Taylor theorem**

$$\int_\Gamma f(z) dz = 0$$

and the **General form of the Cauchy integral formula**

$$\int_\Gamma \frac{f(z)}{z-w} dz = 2\pi i \cdot I(\Gamma, \omega) \cdot f(\omega)$$

**Example 7.2.7.** By definition, if $D$ is simply connected, every cycle in $D$ is homologous to zero, so for every closed contour $\Gamma$,

$$\int_\Gamma f(z) dz = 0$$

**Example 7.2.8.** Let $D = B_r(a)$ and let $\gamma$ be $|z-a| = \rho$. Then for every $\omega \in B_r(a)$ with $|\omega - a| < \rho$,

$$I(\gamma, \omega) = 1$$

so

$$\int_\gamma \frac{f(z)}{z-\omega} dz = 2\pi i \cdot f(\omega)$$

**Theorem 7.2.9.** Let $\gamma$ be a simple closed contour, positively oriented, and let $f$ be holomorphic on $D_\gamma^{\text{int}} \cup \gamma$. Then we have **Cauchy's theorem for simple closed curves**

$$\int_\gamma f(z) dz = 0$$

and **Cauchy's integral formula for simple closed curves**: if $w \in D_\gamma^{\text{int}}$ then

$$\int_\gamma \frac{f(z)}{z-w} dz = 2\pi i \cdot f(w)$$

**Remark.** From here onwards, this will be the version of these two theorems which we use most often.

**Example 7.2.10.** Let $\gamma$ be the square with vertices at $1+i, 1-i, -1+i, -1-i$. Consider

$$\int_\gamma \frac{\cos(z)}{z(z^2+2)} dz$$

$f(z) = \cos(z)/(z^2+2)$ is holomorphic on $D_\gamma^{\text{int}} \cup \gamma$ so by CIF for simple closed curves,

$$\int_\gamma \frac{f(z)}{z} dz = 2\pi i \cdot f(0) = \pi i$$

# 8 Holomorphic functions on punctured domains

## 8.1 Laurent series

**Definition 8.1.1.** A **Laurent series** is an infinite series of the form

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

where $c_n \in \mathbb{C}$ and $a \in \mathbb{C}$. $a$ is called the **centre**. The sum

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

is called the **analytic part** and the sum

$$\sum_{n=-\infty}^{-1} c_n(z-a)^n$$

is called the **principal part**.

**Definition 8.1.2.** We say a Laurent series **converges at** $z \in \mathbb{C}$ iff the principal part and analytic part independently converge at $z$.

**Remark.** A Laurent series is a Taylor series if the principal part is 0, otherwise, it is not defined at $z = a$.

**Definition 8.1.3.** For every $0 \le r < R \le \infty$, and $a \in \mathbb{C}$, the **annulus** of centre $a$, interior radius $r$ and external radius $R$ is defined as

$$A_{r,R}(a) = \{z \in \mathbb{C} : r < |z-a| < R\}$$

**Proposition 8.1.4.** Given a Laurent series with a non-zero principal part, then either

- The Laurent series converges nowhere **or**
- For some $r, R$, the Laurent series converges absolutely on the annulus $A_{r,R}(a)$ and does not converge for $|z-a| > R$ or $|z-a| < r$. We call $A_{r,R}(a)$ the **annulus of convergence**.

*Proof.* By definition, the Laurent series converges iff

$$F_1(z) = \sum_{n=0}^{\infty} c_n(z-a)^n, \quad F_2(z) = \sum_{n=-\infty}^{-1} c_n(z-a)^n$$

both converge. By Theorem 5.15 (lecture notes), either $F_1$ converges at $z = a$ (but then the Laurent series converges nowhere as the principal is not defined at $z = a$) or for some $0 < R \le \infty$, $F_1$ converges absolutely for $|z-a| < R$.

Now define $w = 1/(z-a)$. Then

$$F_2(z) = \sum_{n=-\infty}^{-1} c_n(z-a)^n = \sum_{n=-\infty}^{-1} c_n w^{-n} = \sum_{m=1}^{\infty} c_{-m} w^m =: \tilde{F}(w)$$

Either $\tilde{F}$ converges only at $w = 0$ (so the Laurent series converges nowhere) or for some $0 < R' \le \infty$ such that $\tilde{F}$ converges when $|w| < R$. Let

$$r = \begin{cases} 1/R' & \text{if } R' \neq \infty \\ 0 & \text{if } R' = \infty \end{cases}$$

Then $F_2$ converges when $|w| < R' \iff |z-a| >$. TODO: check lecture notes. $\square$