Contents

1. Set systems

Definition. Let X be a set. A **set system** on X (also called a **family of subsets of** X) is a collection $\mathcal{A} \subseteq \mathbb{P}(X)$.

Notation. $X^{(r)} := \{A \subseteq X : |A| = r\}$ denotes the family of subsets of X of size r.

Remark. Usually, we take $X = [n] = \{1, ..., n\}$, so $|X^{(r)}| = \binom{n}{r}$.

Notation. For brevity, we write e.g. $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$.

Definition. We can visualise $\mathbb{P}(A)$ as a graph by joining nodes $A \in \mathbb{P}(X)$ and $B \in \mathbb{P}(X)$ if $|A \Delta B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa.

This graph is the **discrete cube** Q_n.

Alternatively, we can view Q_n as an n-dimensional unit cube $\{0, 1\}^n$ by identifying e.g. $\{1, 3\} \subseteq [5]$ with 10100 (i.e. identify A with $\mathbb{1}_A$, the characteri stic/indicator function of A).

Definition. $\mathcal{A} \subseteq \mathbb{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{A}$, $A \subseteq B$ or $B \subseteq A$.

Example.
- $\mathcal{A} = \{23, 1235, 123567\}$ is a chain.
- $\mathcal{A} = \{\emptyset, 1, 12, ..., [n]\} \subseteq \mathbb{P}([n])$ is a chain.

Definition. $\mathcal{A} \subseteq \mathbb{P}(X)$ is an **antichain** if $\forall A \neq B \in \mathcal{A}$, $A \nsubseteq B$.

Example.
- $\mathcal{A} = \{23, 137\}$ is an antichain.
- $\mathcal{A} = \{1, ..., n\} \subseteq \mathbb{P}([n])$ is an antichain.
- More generally, $\mathcal{A} = X^{(r)}$ is an antichain for any r.

Proposition. A chain $\mathcal{A} \subseteq \mathbb{P}([n])$ can have at most $n + 1$ elements.

Proof. For each $0 \leq r \leq n$, $\mathcal{A}$ can contain at most 1 r-set (set of size r). □

Theorem (Sperner's Lemma). Let $\mathcal{A} \subseteq \mathbb{P}(X)$ be an antichain. Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$, i.e. the maximum size of an antichain is achieved by the set of $X^{(\lfloor n/2 \rfloor)}$.

Proof.
- We use the idea: from "a chain meets each layer in ≤ 1 points, because a layer is an antichain", we try to decompose the cube into chains.
- We decompose $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, then we are done (since a chain cannot contain a subset of a chain of size > 1).
- To achieve this, it is sufficient to find:
  - For each $r < \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r+1)}$ (a matching is a set of disjoint edges, one for each point in $X^{(r)}$).
  - For each $r > \frac{n}{2}$, a matching from $X^{(r)}$ to $X^{(r-1)}$.
- Then put these matchings together to form a set of chains, each passing through $X^{(\lfloor n/2 \rfloor)}$.
- By taking complements, it is enough to construct the matchings just for $r < \frac{n}{2}$.
- Let G be the (bipartite) subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$.

- For any $S \subseteq X^{(r)}$, the number of S-$\Gamma(S)$ edges in G is $|S|(n-r)$ (counting from below) since there are $n-r$ ways to add an element.
- This number is $\leq |\Gamma(S)|\ (r+1)$ (counting from above), since $r+1$ ways to remove an element.
- Hence $|\Gamma(S)| \leq \frac{|S|\ (n-r)}{r+1} \geq |S|$ as $r < \frac{n}{2}$.
- So by Hall's theorem, there is a matching from S to $\Gamma(S)$.

$\square$

Remark. The proof above doesn't tell us when we have equality in Sperner's Lemma.

Definition. For $\mathcal{A} \subseteq X^{(r)}$ $(1 \leq r \leq n)$, the **shadow** of $\mathcal{A}$ is

$$\partial\mathcal{A} = \partial^{-}\mathcal{A} := \left\{B \in X^{(r-1)} : B \subseteq \mathcal{A} \text{ for some } A \in \mathcal{A}\right\}.$$

Example. Let $\mathcal{A} = \{123, 124, 134, 137\} \in [7]^{(3)}$. Then $\partial\mathcal{A} = \{12, 13, 23, 14, 24, 34, 17, 37\}$.

Proposition (Local LYM). Let $\mathcal{A} \subseteq X^{(r)}$, $1 \leq r \leq n$. Then

$$\frac{|\partial\mathcal{A}|}{\binom{r}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

i.e. the proportion of the level occupied by $\partial\mathcal{A}$ is at least the proportion of the level occupied by $\mathcal{A}$.

Proof.
- The number of $\mathcal{A}$-$\partial\mathcal{A}$ edges in Q_n is $|A|r$ (counting from above) and is $\leq |\partial\mathcal{A}|\ (n-r+1)$.
- So $\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1} = \binom{n}{r-1}/\binom{n}{r}$.

$\square$

Remark. For equality in Local LYM, we must have that $\forall A \in \mathcal{A}$, $\forall i \in A$, $\forall j \in A$, we must have $A - \{i\} \cup \{j\} \in \mathcal{A}$, i.e. $\mathcal{A} = \emptyset$ or $X^{(r)}$.

Notation. Write $\mathcal{A}_r$ for $\mathcal{A} \cap X^{(r)}$.

Theorem (LYM Inequality). Let $\mathcal{A} \subseteq \mathbb{P}(X)$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

Proof.
- Method 1: "bubble down with local LYM".
  - We trivially have that $\mathcal{A}_n/\binom{n}{n} \leq 1$.
  - $\partial\mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are disjoint, as $\mathcal{A}$ is an antichain.
  - So

$$\frac{|\partial \mathcal{A}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1.$$

- So by local LYM,

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1.$$

- Now, $\partial(\partial A_n \cup A_{n-1})$ and $\mathcal{A}_{n-2}$ are disjoint, as $\mathcal{A}$ is an antichain.
- So

$$\frac{|\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- So by local LYM,

$$\frac{|\partial A_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- So

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

- Continuing inductively, we obtain the result.

$\square$

Remark. To have equality in LYM, we must have equality in each use of LYM in proof method 1. In this case, the maximum r with $\mathcal{A}_r \neq \emptyset$ has $\mathcal{A}_r = X^{(r)}$. So equality holds iff $\mathcal{A} = X^{(r)}$ for some r. Hence equality in Sperner's Lemma holds iff $\mathcal{A} = X^{(\lfloor n/2 \rfloor)}$ or $\mathcal{A} = X^{(\lceil n/2 \rceil)}$.

2. Isoperimetric inequalities

3. Intersecting families