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# 1. Hidden subgroup problem

## 1.1. Review of Shor's algorithm

**Problem 1.1** (Factoring)

**Input** a positive integer  $N$ .

**Promise**  $N$  is composite.

**Task** Find a non-trivial factor of  $N$  in  $O(\text{poly}(n))$  time, where  $n = \log N$ .

**Definition 1.2** We say an **efficient problem** is one that can be solved in polynomial time.

**Remark 1.3** Classically, the best known factoring algorithm runs in  $e^{O(n^{1/3}(\log n)^{2/3})}$ . Shor's algorithm runs in  $O(n^3)$  by converting factoring into period finding:

- Given input  $N$ , choose  $a < N$  which is coprime to  $N$ .
- Define  $f : \mathbb{Z} \rightarrow \mathbb{Z}/N$ ,  $f(x) = a^x \bmod N$ .  $f$  is periodic with period  $r$  (the order of  $a \bmod N$ ), i.e.  $f(x+r) = f(x)$  for all  $x \in \mathbb{Z}$ . Finding  $r$  allows us to factor  $N$ .

## 1.2. Period finding

**Problem 1.4** (Periodicity Determination Problem)

**Input** An oracle for a function  $f : \mathbb{Z}/M \rightarrow \mathbb{Z}/N$ .

**Promise**

- $f$  is periodic with period  $r < M$  (i.e.  $\forall x \in \mathbb{Z}/M$ ,  $f(x+r) = f(x)$ ), and
- $f$  is injective in each period (i.e. if  $0 \leq x < y < r$ , then  $f(x) \neq f(y)$ ).

**Task** Determine the period  $r$ .

**Remark 1.5** Solving the periodicity determination problem classically requires takes time  $O(\sqrt{M})$ .

**Definition 1.6** Let  $f : \mathbb{Z}/M \rightarrow \mathbb{Z}/N$ . Let  $H_M$  and  $H_N$  be quantum state spaces with orthonormal state bases  $\{|i\rangle : i \in \mathbb{Z}/N\}$  and  $\{|j\rangle : j \in \mathbb{Z}/M\}$ . Define the unitary **quantum oracle** for  $f$  by  $U_f$  by

$$U_f|x\rangle|z\rangle = |x\rangle|z + f(x)\rangle.$$

The first register  $|x\rangle$  is the **input register**, the last register  $|z\rangle$  is the **output register**.

**Definition 1.7** The **query complexity** of an algorithm is the number of times it queries  $f$  (i.e. for the quantum case, the number of times it uses  $U_f$ ).

**Definition 1.8** The **quantum Fourier transform** over  $\mathbb{Z}/M$  is the unitary QFT defined by its action on the computational basis:

$$\text{QFT}|x\rangle = \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} \omega^{xy} |y\rangle,$$

where  $\omega = e^{2\pi i/M}$  is an  $M$ -th root of unity. Note that QFT requires only  $O((\log M)^2)$  gates to implement, whereas a general  $M \times M$  unitary requires  $O(4^M/M)$  elementary gates.

**Lemma 1.9** Let  $\alpha = e^{2\pi iy/M}$ . Then

$$\sum_{j=0}^{k-1} \alpha^j = \begin{cases} \frac{1-\alpha^k}{1-\alpha} = 0 & \text{if } \alpha \neq 1 \text{ i.e. } M \nmid y \\ k & \text{if } \alpha = 1 \text{ i.e. } M \mid y \end{cases}.$$

*Proof (Hints).* Trivial. □

*Proof.* The sum is a geometric series with common ratio  $\alpha$ . □

**Lemma 1.10** (Boosting success probability) If a process succeeds with probability  $p$  on one trial, then

$$\Pr(\text{at least one success in } t \text{ trials}) = 1 - (1 - p)^t > 1 - \delta$$

for  $t = O\left(\frac{\log(1/\delta)}{p}\right)$ .

*Proof (Hints).* Trivial. □

*Proof.* Trivial. □

**Theorem 1.11** (Co-primality Theorem) The number of integers less than  $r$  that are coprime to  $r$  is  $O(r/\log \log r)$ .

**Algorithm 1.12** (Quantum Period Finding) The algorithm solves the Periodicity Determination Problem: Let  $f : \mathbb{Z}/M \rightarrow \mathbb{Z}/N$  be periodic with period  $r < M$  and one-to-one in each period. Let  $A = \frac{M}{r}$  be the number of periods. We work over the state space  $H_M \otimes H_N$ .

1. Construct the state  $\frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle|0\rangle$  and query  $U_f$  on it.
2. Measure second register in computational basis and discard the second register.
3. Apply the quantum Fourier transform to the input state.
4. Measure the input state, yielding outcome  $c$ .
5. Compute the denominator  $r_0$  of the simplified fraction  $\frac{c}{M}$ .
6. Repeat the previous steps  $O(\log \log r) = O(\log \log M) = O(\log m)$  times, halting if at any iteration,  $f(0) = f(r_0)$ .

**Theorem 1.13** (Correctness of Quantum Period Finding Algorithm) When repeated,  $O(\log \log r) = O(\log \log M)$  times, the quantum period finding algorithm obtains the correct value of  $r$  with high probability.

*Proof (Hints).* Straightforward. □

*Proof.* After querying  $U_f$ , we have the state  $\frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle|f(i)\rangle$ . Upon measuring the second register in the computational basis, the input state collapses to  $|\text{per}\rangle = \frac{1}{\sqrt{A}} \sum_{j=0}^{A-1} |x_0 + jr\rangle$ , where  $f(x_0) = y$  and  $0 \leq x_0 < r$ . Applying the quantum Fourier transform to  $|\text{per}\rangle$  then gives Quantum Fourier Transform to  $|\text{per}\rangle$ :

$$\text{QFT}|\text{per}\rangle = \frac{1}{\sqrt{M}} \sum_{y=0}^{M-1} \frac{1}{\sqrt{A}} \sum_{j=0}^{A-1} \omega^{(x_0+jr)y} |y\rangle$$

$$\begin{aligned}
&= \frac{1}{\sqrt{MA}} \sum_{y=0}^{M-1} \omega^{x_0 y} \sum_{j=0}^{A-1} \omega^{jry} |y\rangle \\
&= \sqrt{\frac{A}{M}} \sum_{k=0}^{r-1} \omega^{x_0 kM/r} |kM/r\rangle
\end{aligned}$$

Importantly, now the outcomes and probabilities are independent of  $x_0$ , so carry useful information about  $r$ . The outcome after the measuring the input state is  $c = k_0 M/r$  for some  $0 \leq k_0 < r$  (so  $c/M = k_0/r$ ). If  $k_0$  is coprime to  $r$ , then the denominator  $r_0$  of the simplified fraction  $\frac{c}{M}$  is equal to  $r$ . By the coprimality theorem, the probability that  $k_0$  is coprime to  $r$  is  $O(1/\log \log r)$ . Checking if  $f(0) = f(r_0)$  tells us if  $r_0 = r$ , since  $f$  is periodic and one-to-one in each period, and  $r_0 \leq r$ .  $\square$

### 1.3. Analysis of QFT part of period finding algorithm

**Notation 1.14** For  $R = \{0, r, \dots, (A-1)r\} \subseteq \mathbb{Z}/M$  ( $Ar = M$ ), write  $|R\rangle$  for the uniform superposition of all computational basis states in  $R$ :

$$|R\rangle = \frac{1}{\sqrt{A}} \sum_{k=0}^{A-1} |kr\rangle.$$

**Definition 1.15** For each  $x_0 \in \mathbb{Z}/M$ , define the linear map by its action on the computational basis states:

$$\begin{aligned}
U(x_0) : H_M &\rightarrow H_M, \\
|k\rangle &\mapsto |x_0 + k\rangle.
\end{aligned}$$

**Definition 1.16** Note that since  $(\mathbb{Z}/M, +)$  is abelian, all  $U(x_i)$  commute:  $U(x_1)U(x_2) = U(x_1 + x_2) = U(x_2)U(x_1)$ . Hence, they have a simultaneous basis of eigenvectors  $\{|\chi_k\rangle : k \in \mathbb{Z}/M\}$ , i.e. for all  $k, x_0 \in \mathbb{Z}/M$ ,  $U(x_0)|\chi_k\rangle = w(x_0, k)|\chi_k\rangle$ , where  $|w(x_0, k)| = 1$ . The  $|\chi_k\rangle$  are called **shift-invariant states** and form an orthonormal basis for  $H_M$ . The  $|\chi_k\rangle$  are given explicitly by

$$|\chi_k\rangle = \frac{1}{\sqrt{M}} \sum_{\ell=0}^{M-1} e^{-2\pi i k \ell / M} |\ell\rangle.$$

**Proposition 1.17** The explicit definition of the  $|\chi_k\rangle$  indeed satisfies the property  $\forall k, x_0 \in \mathbb{Z}/M$ ,  $U(x_0)|\chi_k\rangle = w(x_0, k)|\chi_k\rangle$ , and we have  $w(x_0, k) = \omega^{kx_0}$ , where  $\omega = e^{2\pi i / M}$ .

*Proof (Hints).* Straightforward.  $\square$

*Proof.* We have that

$$\begin{aligned}
U(x_0)|\chi_k\rangle &= \frac{1}{\sqrt{M}} \sum_{\ell=0}^{M-1} e^{-2\pi i k \ell / M} |x_0 + \ell\rangle \\
&= \frac{1}{\sqrt{M}} \sum_{\tilde{\ell}=0}^{M-1} e^{-2\pi i (\tilde{\ell} - x_0) k / M} |\tilde{\ell}\rangle
\end{aligned}$$

$$\begin{aligned}
&= e^{2\pi i k x_0 / M} |\chi_k\rangle \\
&=: w(x_0, k) |\chi_k\rangle
\end{aligned}$$

□

**Remark 1.18** Let  $U : H_M \rightarrow H_M$  be the unitary mapping the shift-invariant basis to the computational basis:  $U : |\chi_k\rangle \mapsto |k\rangle$ . The matrix representation of  $U^{-1}$  with respect to the computational basis has entries

$$(U^{-1})_{jk} = \langle j | U^{-1} | k \rangle = \langle j | \chi_k \rangle = \frac{1}{\sqrt{M}} e^{-2\pi i j k / M}$$

So the matrix representation of  $U$  with respect to the same basis has entries  $U_{kj} = \overline{(U^{-1})_{jk}} = \frac{1}{\sqrt{M}} e^{2\pi i j k / M}$ . Hence, we have

$$U|k\rangle = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{2\pi i j k / M} |j\rangle,$$

and so  $U$  is precisely the QFT mod  $M$ .

## 1.4. The hidden subgroup problem (HSP)

**Problem 1.19** (Discrete Logarithm Problem (DLP))

**Input**  $g, x \in G$  for an abelian group  $G$ .

**Promise**  $g$  is a generator of  $G$ .

**Task** Find  $\log_g x$ , i.e. find  $L \in \mathbb{Z}/|G|$  such that  $x = g^L$ .

**Notation 1.20** Write  $[n]$  for  $\{1, \dots, n\}$ . Write e.g.  $ij$  for the set  $\{i, j\}$ .

**Definition 1.21** Let  $\Gamma_1 = ([n], E_1)$  and  $\Gamma_2 = ([n], E_2)$  be (undirected) graphs.  $\Gamma_1$  and  $\Gamma_2$  are **isomorphic** if there exists a permutation  $\pi \in S_n$  such that for all  $1 \leq i, j < n$ ,  $ij \in E_1$  iff  $\pi(i)\pi(j) \in E_2$ .

**Definition 1.22** Let  $\Gamma = ([n], E)$  be a graph. The **automorphism group** of  $\Gamma$  is

$$\text{Aut}(\Gamma) = \{\pi \in S_n : ij \in E \text{ iff } \pi(i)\pi(j) \in E \quad \forall i, j \in [n]\}.$$

$\text{Aut}(\Gamma)$  is a subgroup of  $S_n$ , and  $\pi \in \text{Aut}(\Gamma)$  iff  $\pi$  leaves  $\Gamma$  invariant as a labelled graph.

**Definition 1.23** The **adjacency matrix** of a graph  $\Gamma = (V, E)$  is the  $n \times n$  matrix  $M_A$  defined by its entries:

$$(M_A)_{ij} := \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}.$$

**Problem 1.24** (Graph Isomorphism Problem)

**Input** Adjacency matrices  $M_1$  and  $M_2$  of graphs  $\Gamma_1 = ([n], E_1)$  and  $\Gamma_2 = ([n], E_2)$ .

**Task** Determine whether  $\Gamma_1$  and  $\Gamma_2$  are isomorphic.

**Remark 1.25** The best known classical algorithm for solving the graph isomorphism problem has quasi-polynomial time complexity  $n^{O((\log n)^2)}$ .

**Problem 1.26** (Hidden Subgroup Problem (HSP)) Let  $G$  be a finite group.

**Input** An oracle for a function  $f : G \rightarrow X$ .

**Promise** There is a subgroup  $K < G$  such that:

1.  $f$  is constant on the (left) cosets of  $K$  in  $G$ .
2.  $f$  takes a different value on each coset.

**Task** Determine (a set of generators of)  $K$ .

**Remark 1.27**

- To find  $K$ , we either find a generating set for  $K$ , or sample uniformly random elements from  $K$ .
- We want to determine  $K$  with high probability in  $O(\text{poly log}|G|)$  queries. Using  $O(|G|)$  queries is easy, as we just query all values  $f(g)$  and find the “level sets” (sets where  $f$  is constant).

**Example 1.28** The following problems are special cases of HSP:

- The **Periodicity Determination Problem**:  $G = \mathbb{Z}/M$ ,  $K = \langle r \rangle = \{0, r, \dots, (A-1)r\}$ . The cosets are  $x_0 + K = \{x_0, x_0 + r, \dots, x_0 + (A-1)r\}$  for each  $0 \leq x_0 < r$ .
- The **DLP** on  $(\mathbb{Z}/p)^\times$ : let  $f : \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1) \rightarrow (\mathbb{Z}/p)^\times$  be defined by  $f(a, b) = g^a x^{-b} = g^{a-Lb}$ .  $G = \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$ , the hidden subgroup is  $K = \{\lambda(L, 1) : \lambda \in \mathbb{Z}/(p-1)\}$ . (Note that if we know  $K$ , we can pick any  $(c, d) = (\lambda L, \lambda) \in K$  and compute  $L = \frac{c}{d}$  to find  $L$ .)
- The **Graph Isomorphism Problem**:  $G = S_n$ , hidden subgroup is  $K = \text{Aut}(G)$ . Let  $f_\Gamma : S_n \rightarrow X$  where  $X$  is set of adjacency matrices of labelled graphs on  $[n]$ , defined by  $f_\Gamma(\pi) = \pi(A)$ . Note  $|S_n| = |G| = n!$ , so  $\log|G| \approx n \log n$ , so  $O(\text{poly log}|G|) = O(\text{poly } n)$ .

**Definition 1.29** An **irreducible representation (irrep)** of a finite abelian group  $G$  is a homomorphism  $\chi : G \rightarrow \mathbb{C}^\times$ .

**Theorem 1.30**

- Let  $\chi : G \rightarrow \mathbb{C}^\times$  be an irrep. For all  $g \in G$ ,  $\chi(g)$  is a  $|G|$ -th root of unity.
- There are always exactly  $|G|$  distinct irreps. In particular, we can label each irrep uniquely by some  $g \in G$ .

**Theorem 1.31** (Schur's Lemma) Let  $\chi_i$  and  $\chi_j$  be irreps of  $G$ . Then

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij}.$$

*Proof.* Omitted. □

**Example 1.32**  $\chi_0 : G \rightarrow \mathbb{C}^\times$ ,  $\chi_0(g) = 1$  is the **trivial irrep**. Note that for any  $\chi_i \neq \chi_0$ ,  $\sum_{g \in G} \chi_i(g) = 0$  by Schur's lemma.

**Definition 1.33** For finite abelian  $G$ , we define the **shift operators** on  $H_{|G|}$  for each  $k \in G$  by

$$U(k) : H_{|G|} \rightarrow H_{|G|},$$

$$|g\rangle \mapsto |k + g\rangle.$$

Note that since  $G$  is abelian, the  $U(k)$  commute:

$$U(k)U(l) = U(k + l) = U(l)U(k) \quad \forall k, l \in G.$$

Hence, they have simultaneous eigenstates, which gives an orthonormal basis for  $H_{|G|}$ .

**Proposition 1.34** For each  $k \in G$ , consider the state

$$|\chi_k\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \overline{\chi_k(g)} |g\rangle.$$

The  $|\chi_k\rangle$  are shift-invariant (invariant up to a phase under the action of all  $U(g)$ ,  $g \in G$ ).

*Proof (Hints).* Straightforward.  $\square$

*Proof.* Since  $\chi_k$  is a homomorphism, we have  $\overline{\chi_k(g)} = \chi_k(-g)$ . Now

$$\begin{aligned} U(g_0)|\chi_k\rangle &= \frac{1}{\sqrt{|G|}} \sum_{g \in G} \overline{\chi_k(g)} |g_0 + g\rangle \\ &= \frac{1}{\sqrt{|G|}} \sum_{g' \in G} \overline{\chi_k(g' - g_0)} |g'\rangle \\ &= \frac{1}{\sqrt{|G|}} \sum_{g' \in G} \overline{\chi_k(g')} \chi_k(g_0) |g'\rangle \\ &= \chi_k(g_0) |\chi_k\rangle. \end{aligned}$$

$\square$

**Definition 1.35** The **quantum Fourier transform (QFT)** on  $H_{|G|}$  is the unitary implementing the change of basis from the shift-invariant states  $\{|\chi_g\rangle : g \in G\}$  to the computational basis  $\{|g\rangle : g \in G\}$ .

Note that  $\text{QFT}^{-1}|g\rangle = |\chi_g\rangle$ . So  $(\text{QFT}^{-1})_{kg} = \langle k | \chi_g \rangle = \frac{1}{\sqrt{|G|}} \overline{\chi_g(k)}$ , so  $\text{QFT}_{kg} = \frac{1}{\sqrt{|G|}} \chi_k(g)$ . So the explicit form is

$$\text{QFT}|g\rangle = \frac{1}{\sqrt{|G|}} \sum_{k \in G} \chi_k(g) |k\rangle.$$

**Example 1.36**

- For  $G = \mathbb{Z}/M$ , we can check that  $\chi_a(b) = e^{2\pi i ab/M}$  are irreps. So the irreps of  $\mathbb{Z}/M$  are naturally labelled by  $a \in \mathbb{Z}/M$  and this gives the usual QFT mod  $M$  as defined earlier.
- Similarly, for  $G = \mathbb{Z}/(M_1) \times \cdots \times \mathbb{Z}/(M_r)$ ,  $\chi_g(h) = e^{2\pi i(g_1 h_1/M_1 + \cdots + g_r h_r/M_r)}$  are the irreps.

**Algorithm 1.37** (Quantum HSP solver for finite abelian  $G$ )

1. Working in the state space  $H_{|G|} \otimes H_{|X|}$ , prepare the uniform superposition state

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |0\rangle$$

and query  $U_f$  on it.

2. Measure the output register, then discard this register.
3. Apply QFT mod  $|G|$  to the input register, then measure this register.
4. Repeat the above steps  $O(\log|G|)$  times.

**Theorem 1.38** (Correctness of Quantum HSP Solver) The quantum HSP solver algorithm solves the HSP for finite abelian groups with high probability.

*Proof (Hints).* Use that irreps of  $G$  restricted to  $K$  are irreps of  $K$ , and that if  $O(\log|G|)$  measurements are made, then with probability at least  $2/3$ , we have enough equations to determine the generators of  $K$ .  $\square$

*Proof.* Querying  $U_f$  on the state gives

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |f(g)\rangle$$

Upon measurement of the output register, we obtain a uniformly random value  $f(g_0)$  from  $f(G)$ , and the state collapses to a **coset state**

$$|g_0 + K\rangle = \frac{1}{\sqrt{|K|}} \sum_{k \in K} |g_0 + k\rangle.$$

We have  $|K\rangle = \sum_{g \in G} a_g |\chi_g\rangle$ , so  $|g_0 + K\rangle = U(g_0)|K\rangle = \sum_{g \in G} a_g \chi_g(g_0) |\chi_g\rangle$ . So applying QFT to the input state gives  $\sum_{g \in G} a_g \chi_g(g_0) |g\rangle$ , so the probability of measuring outcome  $k$  is  $|a_k \chi_k(g_0)|^2 = |a_k|^2$ . Now

$$\begin{aligned} \text{QFT}|K\rangle &= \frac{1}{\sqrt{|K|}} \sum_{k \in K} \text{QFT}|k\rangle \\ &= \frac{1}{\sqrt{|G||K|}} \sum_{g \in G} \left( \sum_{k \in K} \chi_g(k) \right) |g\rangle \end{aligned}$$

Note that irreps of  $G$  restricted to  $K$  are irreps of  $K$ . The trivial irrep  $\chi_0 : G \rightarrow \mathbb{C}$  remains the trivial irrep  $\chi_0$  for  $K$ . But there may be other irreps that become the trivial irrep on restriction to  $K$ . Hence

$$\sum_{k \in K} \chi_g(k) = \begin{cases} |K| & \text{if } \chi_g|_K = \chi_0|_K \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\text{QFT}|K\rangle = \sqrt{\frac{|K|}{|G|}} \sum_{\substack{g \in G \\ \chi_g|_K = \chi_0|_K}} |g\rangle$$



and measuring in the computational basis on this state yields random  $g \in G$  such that  $\forall k \in K, \chi_g(k) = 1$ .

If  $K$  has generators  $k_1, \dots, k_m$  (note that for an arbitrary group, we have  $m = O(\log|G|)$ ), then we have a set of equations  $\chi_g(k_i) = 1$  for all  $i \in [m]$ . We can show that if  $O(\log|G|)$  such  $g$  are drawn uniformly at random, then with probability at least  $2/3$ , we have enough equations to determine  $k_1, \dots, k_m$ .  $\square$

**Example 1.39** Let  $G = \mathbb{Z}/M_1 \times \dots \times \mathbb{Z}/M_r$ . The irreps are

$$\chi_g(h) = e^{2\pi i(g_1 h_1/M_1 + \dots + g_r h_r/M_r)}.$$

For  $k \in K$ ,  $\chi_g(k) = 1$  iff  $\frac{g_1 k_1}{M_1} + \dots + \frac{g_r k_r}{M_r} = 0 \pmod{1}$ . This is a homogenous linear equation in  $k$ , and  $O(\log|G|)$  independent such equations determine  $K$  as the nullspace.

**Remark 1.40** We can implement QFT over abelian groups (and some non-abelian groups, including  $S_n$ ) using circuits with  $O((\log|G|)^2)$  elementary gates.

In the non-abelian case, we can still easily prepare coset states with one query to  $f$ . But the shift operators  $U(g_0)$  no longer commute, so we don't have a (canonical) shift-invariant basis.

**Definition 1.41** A  $d$ -dimensional unitary representation of a finite group  $G$  is a homomorphism

$$\chi : G \rightarrow U(d)$$

where  $U(d)$  is the group of  $d \times d$  unitary matrices.

**Definition 1.42** A  $d$ -dimensional unitary representation  $\chi$  of  $G$  is **irreducible** if no non-trivial subspace of  $\mathbb{C}^d$  is invariant under the action of  $\{\chi(g_1), \dots, \chi(g_{|G|})\}$  (i.e. we cannot simultaneously block diagonalise all the  $\chi(g)$  matrices by a basis change).

**Definition 1.43** A set of irreps  $\{\chi_1, \dots, \chi_m\}$  is a **complete set of irreps** for every irrep  $\chi$  of  $G$ , there exists  $1 \leq i \leq m$  such that  $\chi$  is unitarily equivalent to  $\chi_i$ , i.e. for some  $V \in U(d)$ ,  $\forall g \in G, \chi(g) = V\chi_i(g)V^\dagger$ .

**Theorem 1.44** Let the dimensions of a complete set of irreps  $\chi_1, \dots, \chi_m$  be  $d_1, \dots, d_m$ . Then  $d_1^2 + \dots + d_m^2 = |G|$ .

**Notation 1.45** Write  $\chi_{i,jk}(g)$  for the  $(j,k)$ -th entry of the matrix  $\chi_i(g)$ .

**Theorem 1.46** (Schur Orthogonality) Let  $\chi_1, \dots, \chi_m$  be a complete set of irreps for  $G$  with respective dimensions  $d_1, \dots, d_m$ , and let  $i \in [m]$ ,  $j, k \in [d_i]$ . Then

$$\frac{1}{|G|} \sum_{g \in G} \chi_{i,jk}(g) \overline{\chi_{i',j'k'}(g)} = \frac{1}{d_i} \delta_{ii'} \delta_{jj'} \delta_{kk'}.$$

**Definition 1.47** The **Fourier basis** for a group  $G$  consists of

$$|\chi_{i,jk}\rangle = \sqrt{\frac{d_i}{|G|}} \sum_{g \in G} \overline{\chi_{i,jk}(g)} |g\rangle$$

for each  $i \in [n]$  and  $j, k \in [d_i]$ . Note that by Schur orthogonality, this is an orthonormal basis.

**Remark 1.48** Note that these states are not shift invariant for every  $U(g_0) : |g\rangle \mapsto |g_0 g\rangle$ . So measurement of the coset state  $|g_0 K\rangle$  yields an output distribution that is not independent of  $g_0$ .

**Definition 1.49** The **Quantum Fourier transform** over  $H_{|G|}$  is the unitary mapping the Fourier basis to the computational basis:

$$\text{QFT}|\chi_{i,jk}\rangle = |i, jk\rangle.$$

$|i, jk\rangle$  is a relabelling of the states  $|g\rangle$  for  $g \in G$  (note this is valid by Theorem 1.44).

**Remark 1.50**

- Measuring  $\text{QFT}|g_0 K\rangle$  does **not** give  $g_0$ -independent outcomes. A complete measurement in the computational basis gives an outcome  $i, j, k$ .
- However, there is an incomplete measurement which projects into the  $d_i^2$ -dimensional subspaces

$$S_i = \text{span}\{|\chi_{i,jk}\rangle : j, k \in [d_i]\}.$$

for each  $i \in [n]$ . Call this measurement operator  $M_{\text{rep}}$ . Note that this distinguishes only between the irreps.

- Measuring only the representation labels of  $\text{QFT}|g_0 K\rangle$  gives an outcome distribution of the  $i$  values that is independent of the random shift  $g_0$ , since the  $\chi_i$  are homomorphisms.
- Note this only gives partial information about  $K$ . If  $K$  is a normal subgroup, then in fact we can then determine  $K$  with  $O(\log|G|)$  queries.

## 2. Quantum phase estimation (QPE)

Quantum phase estimation is a unifying algorithmic primitive, e.g. there is an alternative factoring algorithm based on QPE, and has many important applications in physics.

**Problem 2.1** (Quantum Phase Estimation)

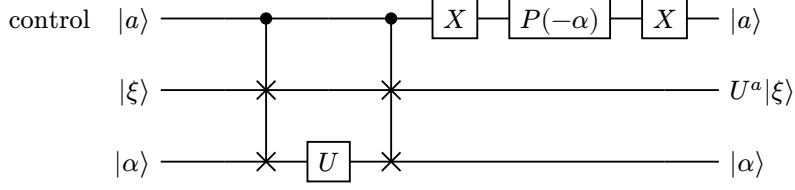
**Input** Unitary  $U \in U(d)$  acting on  $\mathbb{C}^d$ ; state  $|v_\varphi\rangle \in \mathbb{C}^d$ ; level of precision  $n \in \mathbb{N}$ .

**Promise**  $|v_\varphi\rangle$  is an eigenstate of  $U$  with **phase** (eigenvalue)  $e^{2\pi i \varphi}$ ,  $\varphi \in [0, 1)$  (i.e.  $U|v_\varphi\rangle = e^{2\pi i \varphi}|v_\varphi\rangle$ ).

**Task** Output an estimate  $\tilde{\varphi}$  of  $\varphi$ , accurate to  $n$  binary bits of precision.

**Remark 2.2** If  $U$  is given as a circuit, we can implement the controlled- $U$  operation,  $C-U$ , by controlling each elementary gate in the circuit of  $U$ .

If  $U$  is given as a black box, we need more information. Note that  $U$  is equivalent to  $U' = e^{i\theta}U$  and  $|\psi\rangle$  is equivalent to  $e^{i\theta}|\psi\rangle$ , but  $C-U$  is not equivalent to  $C-U'$ . It suffices to be given an eigenstate  $|\alpha\rangle$  with known phase  $e^{i\alpha}$  (so  $U|\alpha\rangle = e^{i\alpha}|\alpha\rangle$ ): we have  $U'|\alpha\rangle = e^{i(\theta+\alpha)}|\alpha\rangle$ . so  $U$  and  $U'$  can be distinguished using this additional information. The following circuit implements  $C-U$  (the top two lines end in state  $C-U|a\rangle|\xi\rangle$ ):

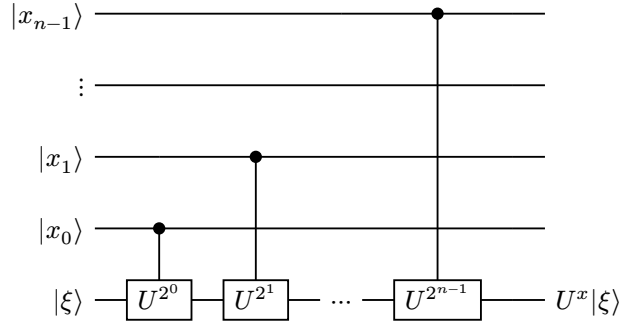


where  $P(-\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$ , and  $\bullet - \times - \times$  denotes the controlled SWAP operation. We can prove the top two lines end in the state  $C-U|a\rangle|\xi\rangle$  by checking the action of the circuit for  $|a\rangle = |0\rangle$  and  $|a\rangle = |1\rangle$ .

**Definition 2.3** For a unitary  $U$ , the **generalised control** unitary  $C-U$  is defined linearly by

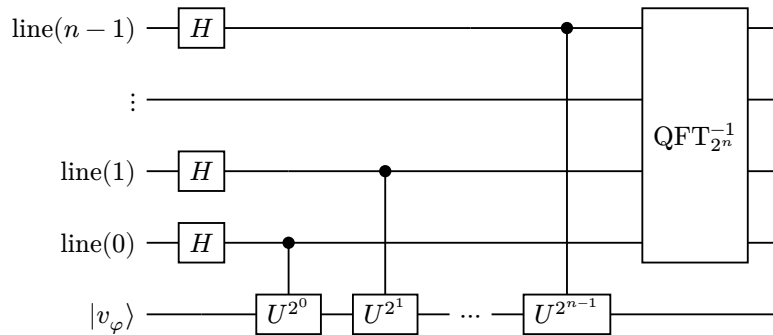
$$\forall x \in \{0, 1\}^n, \quad C-U|x\rangle|\xi\rangle = |x\rangle U^x|\xi\rangle,$$

where  $U^x$  denotes  $U$  applied  $x$  times (e.g.  $C-U|11\rangle|\xi\rangle = |11\rangle U^3|\xi\rangle$ ). Note that  $C-U^k = (C-U)^k$ . The following circuit implements  $C-U$ :



**Algorithm 2.4** (Quantum Phase Estimation) Work over the space  $(\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^d$ , where  $(\mathbb{C}^2)^{\otimes n}$  is the  $n$ -qubit register,  $\mathbb{C}^d$  is the “qudit” register.

1. Apply the following circuit to  $|0\dots 0\rangle|v_\varphi\rangle$ :



We write  $U_{\text{PE}}$  for this unitary part of the circuit.

2. Discard the qudit register holding  $|v_\varphi\rangle$ , and measure the input qubits, yielding outcome  $y_0\dots y_{n-1}$  from lines  $0, \dots, n-1$ .
3. The estimate of  $\varphi$  is  $\tilde{\varphi} = y/2^n = y_0/2 + \dots + y_{n-1}/2^n$ .

**Remark 2.5** After  $C-U^{2^{n-1}}$ , the input qubits are in the state

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} e^{2\pi i \varphi x} |x\rangle.$$

If  $\varphi$  had an exact  $n$ -bit expansion  $0.i_1 i_2 \dots i_n = (i_1 \dots i_n)/2^n =: \varphi_n/2^n$ , then this state is precisely  $\text{QFT}_{2^n} |\varphi_n\rangle$ , in which case, after applying  $\text{QFT}^{-1}$ , we have  $|\varphi_n\rangle$ , so measuring the input bits gives  $\varphi_n$ , and so  $\varphi$ , exactly.

**Lemma 2.6** For all  $\alpha \in \mathbb{R}$ ,

1. If  $|\alpha| \leq \pi$ , then  $|1 - e^{i\alpha}| = 2|\sin(\alpha/2)| \geq \frac{2}{\pi}|\alpha|$ .
2. If  $\alpha \geq 0$ , then  $|1 - e^{i\alpha}| \leq \alpha$ .

*Proof (Hints).* For both, think graphically. □

*Proof.*

1. The line  $y = \frac{2}{\pi}\alpha$  lies below  $2\sin(\alpha/2)$  for  $0 \leq \alpha \leq \pi$ .
  2. On the complex unit circle, the arc length  $\alpha$  from 1 to  $e^{i\alpha}$  is at least the chord length from 1 to  $e^{i\alpha}$ .
- 

**Theorem 2.7** (Phase Estimation Theorem) Let  $\tilde{\varphi}$  be the estimate of  $\varphi$  from the quantum phase estimation algorithm. Then

1.  $\Pr(\tilde{\varphi} \text{ is closest } n\text{-bit approximation of } \varphi) \geq \frac{4}{\pi^2} \approx 0.4$ .
2. For all  $\varepsilon > 0$ ,  $\Pr(|\tilde{\varphi} - \varphi| > \varepsilon) = O(\frac{1}{2^n \varepsilon})$ . So for any desired accuracy  $\varepsilon$ , the probability of failure decays exponentially with the number of bits of precision (lines in the circuit).

*Proof (Hints).* Let  $\delta(y) = \varphi - y/2^n + \varepsilon_y$ , where  $\varepsilon_y \in \{-1, 0, 1\}$  is such that  $\delta(y) \in [-1/2, 1/2]$ . Show the probability of the measuring yielding outcome  $y$  is

$$p_y = \frac{1}{2^{2n}} \left| \frac{1 - e^{2^n 2\pi i \delta(y)}}{1 - e^{2\pi i \delta(y)}} \right|^2.$$

1. Find an upper bound on  $\delta(a)$  where  $a$  is the closest  $n$ -bit approximation of  $\varphi$ .
2. Show that

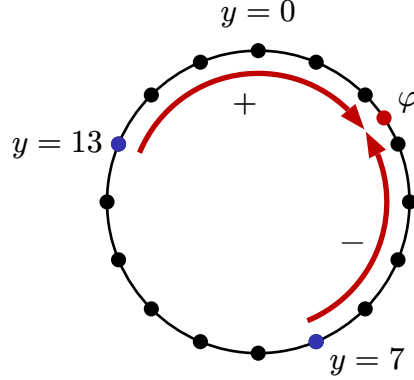
$$p_y \leq \frac{1}{2^{2n}} \left( \frac{2}{4\delta(y)} \right)^2 = \frac{1}{2^{2n+2}\delta(y)^2}.$$

Let  $B = \{y \in \{0,1\}^n : |\delta(y)| > \varepsilon\}$ . Show that for each  $y \in B$ ,  $|\delta(y)| \leq \varepsilon + k_y/2^n$  for some  $k_y \in \mathbb{N}$ , and that each  $k_y$  occurs at most twice here. Conclude the upper bound using an integral. □

*Proof.* Let

$$|A\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{2\pi i \varphi x} |x\rangle.$$

Let  $\delta(y) = \varphi - y/2^n + \varepsilon_y$ , where  $\varepsilon_y \in \{-1, 0, 1\}$  is such that  $\delta(y) \in [-1/2, 1/2]$ .  $\delta(y)$  can be thought of as the signed (positive if clockwise, negative if anticlockwise) arc distance between the points  $y/2^n$  and  $\varphi$  on a circle of circumference 1.



Example when  $n = 4$ : the 16 possible values of  $y$  are equally spaced around the circle.  $\delta(13)$  is positive,  $\delta(7)$  is negative.

Since  $\text{QFT}^{-1}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2\pi i x y / 2^n} |y\rangle$ , we have

$$\text{QFT}^{-1}|A\rangle = \frac{1}{2^n} \sum_{y=0}^{2^n-1} \sum_{x=0}^{2^n-1} e^{2\pi i x \delta(y)} |y\rangle$$

so the probability of measuring outcome  $y$  is

$$p_y = \Pr\left(\tilde{\varphi} = \frac{y}{2^n}\right) = \frac{1}{2^{2n}} \left| \frac{1 - e^{2^n 2\pi i \delta(y)}}{1 - e^{2\pi i \delta(y)}} \right|^2.$$

1. Let  $\alpha = 2^n 2\pi \delta(a)$ , where  $a$  is the closest  $n$ -bit approximation of  $\varphi$ . Note we can imagine the possible values of  $\tilde{\varphi}$  as lying on the unit circle, spaced by angle  $\frac{2\pi}{2^n}$ . This gives a visual intuition to the fact that  $|\delta(a)| \leq \frac{1}{2^{n+1}}$ . Hence  $|\alpha| \leq \pi$ , and so by the above lemma,

$$p_a = \Pr(\tilde{\varphi} = a) \geq \frac{1}{2^{2n}} \left( \frac{2^{n+2} \delta(a)}{2\pi \delta(a)} \right)^2 = \frac{4}{\pi^2}.$$

2. Note that  $|1 - e^{2^n 2\pi i \delta(y)}| \leq 2$  by the triangle inequality. Let  $B = \{y \in \{0, 1\}^n : |\delta(y)| > \varepsilon\}$  denote the set of “bad” values of  $y$ . For all  $y \in \{0, 1\}^n$ , we have  $\delta(y) \in [-1/2, 1/2]$ . So by Lemma 2.6, we have  $|1 - e^{2\pi i \delta(y)}| \geq 4|\delta(y)|$ , thus

$$p_y \leq \frac{1}{2^{2n}} \left( \frac{2}{4\delta(y)} \right)^2 = \frac{1}{2^{2n+2} \delta(y)^2}.$$

Let  $\delta^+ = \min\{\delta(y) : y \in B, \delta(y) > 0\}$  be the smallest  $\delta(y)$  such that  $\delta(y) > \varepsilon$ , and  $\delta^- = \max\{\delta(y) : y \in B, \delta(y) < 0\}$  be the largest  $\delta(y)$  such that  $\delta(y) < -\varepsilon$ . For all  $y \in B$ , we have  $\delta(y) = \delta^+ + k_y/2^n$  or  $\delta(y) = \delta^- - k_y/2^n$  for some  $k_y \in \mathbb{N}$ , so  $|\delta(y)| > \varepsilon + k_y/2^n$ . Note that each  $k \in \mathbb{N}$ ,  $k = k_y$  for at most 2 values of  $y \in B$ . Hence,

$$\begin{aligned}
\Pr(|\delta(y)| > \varepsilon) &= \Pr(y \in B) = \sum_{y \in B} p_y \\
&\leq \sum_{y \in B} \frac{1}{2^{2n+2} (\varepsilon + k_y/2^n)^2} \\
&< 2 \sum_{k=0}^{\infty} \frac{1}{2^{2n+2}} \frac{1}{(\varepsilon + k/2^n)^2} \\
&\leq \frac{1}{2^{2n+1} \varepsilon^2} + \sum_{k=1}^{\infty} \frac{1}{2^{2n+1}} \frac{1}{(\varepsilon + k/2^n)^2} \\
&= \frac{1}{2^{2n+1} \varepsilon^2} + \int_0^{\infty} \frac{1}{2^{2n+1}} \frac{1}{(\varepsilon + x/2^n)^2} dx \\
&= \frac{1}{2^{2n+1} \varepsilon^2} + \int_{2^n \varepsilon}^{\infty} \frac{1}{2u^2} du = \frac{1}{2^{2n+1} \varepsilon^2} + \frac{1}{2^{n+1} \varepsilon}.
\end{aligned}$$

□

**Remark 2.8** The QPE algorithm excluding the measurement is a unitary - call this unitary  $U_{\text{PE}}$ . If we apply  $U_{\text{PE}}$  to an arbitrary state  $|\psi\rangle = \sum_j c_j |v_j\rangle$  where  $|v_j\rangle$  are the eigenstates of  $U$  with eigenvalue  $e^{2\pi i \varphi_j}$ , then we have

$$U_{\text{PE}}|\psi\rangle = \sum_j c_j |\tilde{\varphi}_j\rangle |v_j\rangle$$

If every  $\varphi_j$  has an exact  $n$ -bit representation, then this is exact. Otherwise, we have  $|\tilde{\varphi}_j\rangle = \sqrt{1-\eta}|\tilde{\varphi}_1\rangle + \sqrt{\eta}|\tilde{\varphi}_0\rangle$ , where  $|\tilde{\varphi}_1\rangle$  is a superposition of all  $n$ -bit strings that are correct to the first  $n$ -bits of  $\varphi$ , and  $|\tilde{\varphi}_0\rangle$  is a superposition of strings with the first  $n$  bits not all correct.

**Remark 2.9** Complexity of QPE: we use  $C-U, \dots, C-U^{2^{n-1}}$ , so the number of uses of  $C-U$  is  $\approx 2^n$ . So this initially looks like exponential time, but there are special cases of  $U$  where by repeated squaring, this can be implemented with  $\text{poly}(n)$  gates.

If we want to estimate  $\varphi$  accurate to  $m$  bits of precision with probability  $1 - \eta$ , then by the phase estimation theorem with  $\varepsilon = \frac{1}{2^m}$ , we need  $n = O(m + \log(1/\eta))$  lines. Note this is a modest, polynomial increase in the number of lines of the circuit for an exponential reduction in  $\eta$ .

### 3. Amplitude amplification

Amplitude amplification is an extension of the key insights in Grover's algorithm.

**Notation 3.1** Given  $|\alpha\rangle \in H_d$ , write  $L_{|\alpha\rangle} = \text{span}\{|\alpha\rangle\}$  for the one-dimensional subspace generated by  $|\alpha\rangle$ .

**Notation 3.2** Given a subspace  $A \leq H_d$ , denote the projector onto  $A$  by  $P_A$ . Note that

$$P_A = \sum_{i=1}^k |a_i\rangle \langle a_i|$$

for any orthonormal basis  $\{|a_1\rangle, \dots, |a_k\rangle\}$  of  $A$ .

**Notation 3.3** Given a subspace  $A \leq H_d$ , define the unitary  $I_A = I - 2P_A$ , which is the reflection in the “mirror”  $A^\perp$ : indeed, note that for all  $|\varphi\rangle \in A$ ,  $I_A = -|\varphi\rangle\langle\varphi|$ , and for all  $|\psi\rangle \in A^\perp$ ,  $I_A|\psi\rangle = |\psi\rangle$ , since  $P_A|\psi\rangle = 0$ .

In the case that  $A$  is one-dimensional and spanned by  $|\alpha\rangle$ , we have  $P_A = |\alpha\rangle\langle\alpha|$ , and write  $I_{|\alpha\rangle} = I - 2|\alpha\rangle\langle\alpha|$ .

**Proposition 3.4** Let  $|\alpha\rangle \in H_d$ . For any unitary  $U \in U(d)$ , we have

$$UI_{|\alpha\rangle}U^\dagger = I_{U|\alpha\rangle}.$$

*Proof (Hints).* Trivial. □

*Proof.*  $UI_{|\alpha\rangle}U^\dagger = UU^\dagger - 2U|\alpha\rangle\langle\alpha|U^\dagger = I_{U|\alpha\rangle}$ . □

**Problem 3.5** (Unstructured Search Problem)

**Input** An oracle for a function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ .

**Promise** There is a unique  $x_0 \in \{0, 1\}^n$  such that  $f(x_0) = 1$ .

**Task** Find  $x_0$ .

**Remark 3.6** The unstructured search problem is closely related to the complexity class NP and to Boolean satisfiability.

**Definition 3.7** For fixed  $|x_0\rangle \in H_2^{\otimes n}$ , the **Grover iteration operator**  $Q$  is defined as

$$Q := -H^{\otimes n}I_{|0\rangle}H^{\otimes n}I_{|x_0\rangle} = -I_{H^{\otimes n}|0\rangle}I_{|x_0\rangle}.$$

**Remark 3.8** Note that for a function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  fulfilling the promise of the Unstructured Search Problem, we can implement  $I_{|x_0\rangle}$  without knowing  $x_0$ : we have  $U_f|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{f(x)}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = I_{|x_0\rangle}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ . Hence, implementing  $Q$  requires only one query to  $f$ .

**Theorem 3.9** (Grover) In the 2-dimensional subspace spanned by  $|\psi\rangle = H^{\otimes n}|0\rangle$  and  $|x_0\rangle$ , the action of  $Q$  is a rotation by angle  $2\alpha$ , where  $\sin(\alpha) = \frac{1}{\sqrt{2^n}} = \langle x_0|\psi\rangle$ .

**Algorithm 3.10** (Grover's Algorithm) Work in the state space  $H_2^{\otimes n}$ .

1. Prepare  $|\psi\rangle = H^{\otimes n}|0\rangle$ .
2. Apply  $Q^m$  to  $|\psi\rangle$ , where  $m$  is closest integer to  $\frac{\arccos(1/\sqrt{N})}{2\arcsin(1/\sqrt{N})} = \frac{\theta}{2\alpha}$  and  $\cos(\theta) = \sin(\alpha) = \langle x_0|\psi\rangle = 1/\sqrt{2^n}$ . This rotates  $|\psi\rangle$  to be close to  $|x_0\rangle$  (within angle  $\pm\alpha$  of  $|x_0\rangle$ ).
3. Measure to get  $x_0$  with probability  $p = |\langle x_0|Q^m|\psi\rangle|^2 = 1 - \frac{1}{N}$ . For large  $N$ ,  $\arccos(1/\sqrt{N}) \approx \frac{\pi}{2}$ , and  $\arcsin(1/\sqrt{N}) \approx 1/\sqrt{N}$ . The number of iterations is  $m = \frac{\pi}{4}\sqrt{N} = O(\sqrt{N})$ . So we need  $O(\sqrt{N})$  queries to  $U_f$ . In contrast, classically we need  $\Omega(N)$  queries to  $f$  to find  $x_0$  with any desired constant probability. Note that  $\Omega(N)$  queries are both necessary and sufficient.

**Notation 3.11** Write  $G$  for the subspace of the state space  $H$  whose associated amplitudes in a given state we wish to amplify.  $G$  is called the “good” subspace. We call the subspace  $G^\perp$  the “bad” subspace. Note that  $H = G \oplus G^\perp$ , and for any state  $|\varphi\rangle \in$

$H$ , there is a unique decomposition with real, positive coefficients  $|\varphi\rangle = \sin(\theta)|g\rangle + \cos(\theta)|b\rangle$ , where  $|g\rangle = P_G|\varphi\rangle$  and  $|b\rangle = P_{G^\perp}|\varphi\rangle$ .

**Theorem 3.12** (Amplitude Amplification Theorem/2D-subspace Lemma) Let  $G \leq H_2^{\otimes n}$  be a subspace and  $|g\rangle = P_G|\psi\rangle$ ,  $|b\rangle = P_{G^\perp}|\psi\rangle$ . In the 2-dimensional subspace  $\text{span}\{|\psi\rangle, |g\rangle\} = \text{span}\{|b\rangle, |g\rangle\}$ , the unitary

$$Q = -I_{|\psi\rangle} I_G$$

is a rotation by angle  $2\theta$ , where  $\sin(\theta) = \|P_G|\psi\rangle\|_2$  is the length of the “good” projection of  $|\psi\rangle$ .

*Proof (Hints).* Consider the matrix representation of  $Q$  in the  $\text{span}\{|b\rangle, |g\rangle\}$  basis.  $\square$

*Proof.* By definition, we have  $I_G|g\rangle = -|g\rangle$ , and  $I_G|b\rangle = |b\rangle$ . Hence  $Q|g\rangle = I_{|\psi\rangle}|g\rangle$  and  $Q|b\rangle = -I_{|\psi\rangle}|b\rangle$ . The matrix representation of  $I_{|\psi\rangle}$  in the  $\{|b\rangle, |g\rangle\}$  basis is

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \end{bmatrix} &= \begin{bmatrix} 1 - 2\cos(\theta)^2 & -2\sin(\theta)\cos(\theta) \\ -2\sin(\theta)\cos(\theta) & 1 - 2\sin(\theta)^2 \end{bmatrix} \\ &= \begin{bmatrix} -\cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}. \end{aligned}$$

So  $Q|b\rangle = \cos(2\theta)|b\rangle + \sin(2\theta)|g\rangle$ , and  $Q|g\rangle = -\sin(2\theta)|b\rangle + \cos(2\theta)|g\rangle$ . So the matrix representation of  $Q$  in the  $\{|b\rangle, |g\rangle\}$  basis is

$$\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

which indeed is a rotation by angle  $2\theta$ .  $\square$

**Corollary 3.13** We have  $Q^m|\psi\rangle = \cos((2m+1)\theta)|b\rangle + \sin((2m+1)\theta)|g\rangle$ .

*Proof (Hints).* Trivial.  $\square$

*Proof.* Induction on  $m$ .  $\square$

**Notation 3.14** Denote by  $m_{\text{opt}}$  the  $m \in \mathbb{Z}$  which maximises the probability of measuring (in the  $\{|b\rangle, |g\rangle\}$  basis) an outcome in  $G$  on the state  $Q^m|\psi\rangle$ . Note that this probability is equal to  $\sin((2m+1)\theta)^2$ , which is maximised when

$$(2m+1)\theta = \pi/2 \implies m = \pi/4\theta - 1/2.$$

So  $m_{\text{opt}}$  is the nearest integer to  $\pi/4\theta - 1/2$ .

**Example 3.15** Let  $\theta = \pi/6$ , then  $m_{\text{opt}} = 1$  and  $Q|\psi\rangle = |g\rangle$ . So we obtain a good outcome with certainty on measurement.

**Remark 3.16** Note that since  $Q$  is a rotation by angle  $2\theta$ ,  $Q^{m_{\text{opt}}}|\psi\rangle$  lies within angle  $\pm\theta$  of  $|g\rangle$ , hence the  $|g\rangle$  component of  $Q^{m_{\text{opt}}}|\psi\rangle$  has amplitude  $\geq \cos(\theta)^2$ . So for small  $\theta$ ,

$$\Pr(\text{measuring good outcome}) \geq \cos(\theta)^2 \approx 1 - O(\theta^2).$$



Also, for small  $\theta$ ,

$$m_{\text{opt}} = O(1/\theta) \approx O(1/\sin(\theta)).$$

**Remark 3.17**  $Q$  can be implemented (efficiently) if  $I_{|\psi\rangle}$  and  $I_G$  can be implemented (efficiently). For an efficient implementation of  $I_G$ , it suffices for  $G$  to be spanned by some subset of computational basis states, and the indicator function  $\mathbb{1}_G$  is efficiently computable. In this case, we have

$$U_{\mathbb{1}_G}|x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{\mathbb{1}_G(x)} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Since  $I_G$  is defined by its action  $|g\rangle \mapsto -|g\rangle$  for  $g \in G$  and  $|b\rangle \mapsto |b\rangle$  for  $b \in G^\perp$ , the first register holds the value  $I_G|x\rangle$ .

For an efficient implementation of  $I_{|\psi\rangle}$ , we usually have  $|\psi\rangle = H^{\otimes n}|0\dots 0\rangle$ , and then  $I_{|\psi\rangle} = H^{\otimes n}I_{|0\rangle}H^{\otimes n}$  can be implemented with  $O(n)$  gates.

**Remark 3.18** In the amplitude amplification process, the relative amplitudes of basis states inside  $|g\rangle$  and  $|b\rangle$  won't change. So amplitude amplification boosts the overall amplitude of  $|g\rangle$  at the expense of the amplitude of  $|b\rangle$ .

### 3.1. Applications of amplitude amplification

**Example 3.19** We can generalise Grover search from 1 marked item to  $k$  marked items out of  $N = 2^n$  items. In this case,

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \\ &= \sqrt{\frac{k}{N}} \frac{1}{\sqrt{k}} \sum_{x \in G} |x\rangle + \sqrt{\frac{N-k}{N}} \frac{1}{\sqrt{N-k}} \sum_{x \in G^\perp} |x\rangle \\ &=: \sqrt{\frac{k}{N}} |g\rangle + \sqrt{\frac{N-k}{N}} |b\rangle \end{aligned}$$

so  $\sin(\theta) = \sqrt{k/N}$ . For  $k \ll N$ ,  $\sin(\theta) \approx \theta$ , so  $m_{\text{opt}} = O(\sqrt{N/k})$  uses of  $Q$  required. E.g.  $N = 4 = 2^2$  items and  $k = 1$  marked item, we have  $\sin(\theta) = 1/2$ , so  $\theta = \pi/6$ , so Grover search is exact, and requires exactly one application of  $Q$ .

**Example 3.20** (Quadratic speedup of general quantum algorithms) Let  $U$  be a unitary representing a quantum algorithm/circuit, with  $U|0\dots 0\rangle = \alpha|g\rangle + \beta|b\rangle$  where  $|g\rangle$  is a superposition of good/correct outcomes, and  $|b\rangle$  is a superposition of bad/incorrect outcomes. Note  $|g\rangle = \sum_{x \in \{0,1\}^n} c_x |x\rangle$  is generally a non-uniform superposition. We have

$$\Pr(\text{measuring } U|0\dots 0\rangle \text{ yields good outcome}) = |\alpha|^2.$$

Thus, we need to run  $U$  about  $O(1/|\alpha|^2)$  times to succeed with high probability.

Now define  $|\psi\rangle = U|0\dots 0\rangle$  and  $Q = -I_{|\psi\rangle}I_G$ . We can implement  $Q$  if we have a method to verify the output of  $U$ ; so in particular, we can use this method for any NP problem. This will mean we can efficiently implement the indicator function  $\mathbb{1}_G$  of good labels and

therefore also  $I_G$ . So by the [Amplitude Amplification Theorem](#),  $Q$  performs a rotation of  $2\theta$  where  $\sin(\theta) = |\alpha|$ . So after approximately

$$m_{\text{opt}} \approx \pi/4\theta = O(1/\theta) = O(1/\sin(\theta)) = O(1/|\alpha|)$$

(for  $\theta$  small) uses of  $Q$ , we get a good outcome with high probability.

**Problem 3.21** (Counting Problem)

**Input**  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ .

**Task** Estimate the number  $k = |f^{-1}(\{1\})|$  of inputs that evaluate to 1.

**Example 3.22** (Quantum Counting) This combines amplitude amplification and quantum phase estimation. Let the “good” subspace  $G$  be the subspace with basis  $f^{-1}(\{1\})$ . As usual, let

$$\begin{aligned} |g\rangle &:= \frac{1}{\sqrt{k}} \sum_{x \in f^{-1}(\{1\})} |x\rangle, \quad |b\rangle := \frac{1}{\sqrt{2^n - k}} \sum_{x \in f^{-1}(\{0\})} |x\rangle, \\ |\psi\rangle &= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1\}^n} |x\rangle = \sqrt{\frac{k}{N}} \frac{1}{\sqrt{k}} \sum_{x \in f^{-1}(\{1\})} |x\rangle + \sqrt{\frac{N-k}{N}} \frac{1}{\sqrt{N-k}} \sum_{x \in f^{-1}(\{0\})} |x\rangle. \end{aligned}$$

Recall that  $Q$  has matrix representation

$$Q = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

in the orthonormal basis  $\{|b\rangle, |g\rangle\}$  where  $\sin(\theta) = \|P_G|\psi\rangle\|$ . The eigenvalues and eigenstates of  $Q$  are  $\lambda_{\pm} = e^{\pm 2i\theta}$  and  $|e_{\pm}\rangle = \frac{1}{\sqrt{2}}(|b\rangle \mp i|g\rangle)$ . So we can write  $|\psi\rangle = \sin(\theta)|g\rangle + \cos(\theta)|b\rangle = \frac{1}{\sqrt{2}}(e^{-i\theta}|e_+\rangle + e^{i\theta}|e_-\rangle)$ . So  $|\psi\rangle$  is an equally-weighted superposition of eigenstates of  $Q$ . Write  $e^{\pm 2i\theta} = e^{2\pi i\varphi_{\pm}}$  with  $\varphi_{\pm} \in (0, 1)$ . We have  $\varphi_+ = \theta/\pi$  and  $\varphi_- = (-2\theta + 2\pi)/2\pi = 1 - \theta/\pi$ . When  $k \ll N$ ,  $\sin(\theta) = \sqrt{k/N} \approx \theta$ , so using  $U_{\text{PE}}$  with  $m$  qubits of precision

$$U_{\text{PE}}|\psi\rangle = \frac{1}{\sqrt{2}}(e^{-i\theta}|e_+\rangle|\tilde{\varphi}_+\rangle + e^{i\theta}|e_-\rangle|\tilde{\varphi}_-\rangle)$$

Measuring the QPE output gives (with probability 1/2) an estimate of  $\varphi_+ = \theta/\pi \approx \frac{1}{\pi}\sqrt{k/N}$  or (with probability 1/2) an estimate of  $\varphi_- = 1 - \theta/\pi \approx 1 - \frac{1}{\pi}\sqrt{k/N}$ . So in either case, we get an estimate of  $\sqrt{k/N}$  (since we can tell when  $k \ll N$  which case we are in). By the [Phase Estimation Theorem](#), with probability at least  $4/\pi^2$ , QPE with  $m$  lines gives us an approximation of  $\sqrt{k/N}$  to precision  $O(1/2^m)$ , using  $O(2^m)$   $C$ - $Q$  operations, each of which requires one query to  $f$ . Write  $\delta/\sqrt{2^n} = 1/2^m$  for some  $\delta > 0$ . So we can estimate  $\sqrt{k}$  to precision  $\delta$ , and since  $\Delta(x^2) = 2x\Delta(x)$ , we estimate  $k$  to additive error (precision)  $O(\delta\sqrt{k})$  using  $O(2^m) = O(\sqrt{N}/\delta)$  queries to  $f$ .

**Remark 3.23** The quantum counting algorithm is quadratically faster than the best possible classical algorithm, which is:

- Sample random  $x$  from  $\{0, 1\}^n$ , then  $\Pr(f(x) = 1) = k/N$ .

- Draw  $m$  samples  $x_1, \dots, x_m$ , then the estimate is  $\tilde{k} = \frac{\ell}{m}N$ , where  $\ell = |\{i \in [m] : f(x_i) = 1\}|$ .

We need  $m = O(N/\delta^2)$  to estimate  $k$  to high precision.

## 4. Hamiltonian simulation

We want to use a quantum system to simulate the evolution/dynamics of another quantum system, given its Hamiltonian  $H$ . For an  $n$ -qubit system, in general this requires  $O(2^n)$  time on a classical computer. For some physically interesting classes of  $H$ , we have quantum algorithms that run in  $O(\text{poly}(n))$  time.

**Definition 4.1** The **exponential** of a matrix  $A \in \mathbb{C}^{n \times n}$  is defined as

$$\exp(A) = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Note that if  $[A, B] = 0$ , then  $\exp(A)\exp(B) = \exp(A+B)$ , but generally this does not hold when  $[A, B] \neq 0$ .

**Theorem 4.2** If  $H$  is Hermitian, then  $e^{-iHt}$  is unitary for all  $t \in \mathbb{R}$ .

*Proof (Hints).* Straightforward. □

*Proof.* We have  $(e^{-iHt})^* = e^{iHt}$  since  $H = H^*$ , and since  $[-iHt, iHt] = 0$ ,  $e^{-iHt}e^{iHt} = e^0 = I$ . □

**Proposition 4.3** If  $A \in \mathbb{C}^{n \times n}$  has orthonormal spectrum  $\{(\lambda_i, |e_i\rangle) : i \in [n]\}$ , then

$$\exp(A) = \sum_{i=1}^n \exp(\lambda_i) |e_i\rangle \langle e_i|.$$

*Proof (Hints).* Straightforward. □

*Proof.* By orthonormality, we have

$$A^k = \left( \sum_{i=1}^n \lambda_i |e_i\rangle \langle e_i| \right)^k = \sum_{i=1}^n \lambda_i^k |e_i\rangle \langle e_i|.$$

Hence,

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{i=1}^n \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} |e_i\rangle \langle e_i|.$$

□

**Definition 4.4** Given a Hamiltonian  $H$ , the unitary  $U(t) = e^{-iHt}$  is called the **evolution operator** of  $H$ . Given  $H$  and  $t > 0$ , we want to simulate  $U(t)$  accurately.

**Proposition 4.5** The Schrodinger equation which governs the time evolution of a physical state  $|\psi(t)\rangle$ , given by (assuming  $\hbar = 1$ )

$$\frac{d}{dt}|\psi(t)\rangle = -iH|\psi(t)\rangle,$$

has solution

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

when  $H$  is time-independent.

*Proof (Hints).* Use the spectral decomposition. □

*Proof.* Let  $H$  have orthonormal spectrum  $\{(\lambda_j, |e_j\rangle) : j \in [n]\}$ . By Proposition [4.3](#), we have

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \sum_{j=1}^n e^{-i\lambda_j t} |e_j\rangle \langle e_j | \psi(0) \rangle.$$

Hence,

$$\begin{aligned} \frac{d}{dt}|\psi(t)\rangle &= \sum_{j=1}^n -i\lambda_j t e^{-i\lambda_j t} |e_j\rangle \langle e_j | \psi(0) \rangle \\ &= \left( \sum_{j=1}^n -i\lambda_j |e_j\rangle \langle e_j| \right) \sum_{k=1}^n e^{-i\lambda_k t} |e_k\rangle \langle e_k | \psi(0) \rangle \end{aligned}$$

by orthonormality. □

**Definition 4.6** The **operator norm (spectral norm)** of an operator  $A : H \rightarrow H$  acting on the space  $H$  of states is

$$\|A\| := \max\{\|A|\psi\rangle\| : \psi \in H, \|\psi\| = 1\}.$$

**Theorem 4.7** If  $A$  is diagonalisable with eigenvalues  $\lambda_1, \dots, \lambda_n$ , then

$$\|A\| = \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

*Proof (Hints).* Straightforward. □

*Proof.* Let  $\{(\lambda_j, |e_j\rangle) : j \in [n]\}$  be the orthonormal spectrum of  $A$ . For  $\geq$ , we have

$$\|A\| \geq \max\{\|A|e_1\rangle\|, \dots, \|A|e_n\rangle\|\} = \max\{\|\lambda_1|e_1\rangle\|, \dots, \|\lambda_n|e_n\rangle\|\} = \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

For  $\leq$ , let  $|\psi\rangle \in H$  with  $\|\psi\| = 1$ . The  $|e_j\rangle$  form an orthonormal basis, so  $|\psi\rangle = \sum_{j=1}^n a_j |e_j\rangle$  for some coefficients  $a_j \in [-1, 1]$ . Let  $\lambda' = \max\{|\lambda_1|, \dots, |\lambda_n|\}$ . Then

$$\begin{aligned} \|A|\psi\rangle\|^2 &= \left\| \sum_{j=1}^n a_j \lambda_j |e_j\rangle \right\|^2 \\ &= \sum_{j=1}^n |a_j|^2 |\lambda_j|^2 \quad \text{by orthonormality} \\ &\leq \sum_{j=1}^n |a_j|^2 \lambda'^2 = \lambda'^2. \end{aligned}$$

□

**Proposition 4.8** The operator norm satisfies the following properties:

1. **Submultiplicative:**  $\|AB\| \leq \|A\| \cdot \|B\|$ .
2. **Triangle inequality:**  $\|A + B\| \leq \|A\| + \|B\|$ .

*Proof (Hints).* Straightforward. □

*Proof.*

1. Let  $|\psi_0\rangle$  achieve the maximum in  $\|AB\| = \max\{\|AB|\psi\rangle\| : |\psi\rangle \in H, \|\psi\rangle\| = 1\}$ . Let  $b = \|B|\psi_0\rangle\|$ . We have

$$\frac{1}{b}\|AB\| = \frac{1}{b}\|AB|\psi_0\rangle\| = \left\|A \frac{B|\psi_0\rangle}{b}\right\| \leq \|A\|,$$

hence  $\|AB\| \leq \|A\| \cdot b$ , but also  $b \leq \|B\|$ .

2. We have

$$\begin{aligned} \|A + B\| &= \max\{\|A|\psi\rangle + B|\psi\rangle\| : |\psi\rangle \in H, \|\psi\rangle = 1\} \\ &\leq \max\{\|A|\psi\rangle\| + \|B|\psi\rangle\| : |\psi\rangle \in H, \|\psi\rangle = 1\} \\ &\leq \max\{\|A|\psi\rangle\| : |\psi\rangle \in H, \|\psi\rangle = 1\} + \max\{\|B|\psi\rangle\| : |\psi\rangle \in H, \|\psi\rangle = 1\}. \end{aligned}$$

□

**Definition 4.9** Let  $U, \tilde{U} : H \rightarrow H$  be operators.  $\tilde{U}$   $\varepsilon$ -approximates  $U$  if

$$\|U - \tilde{U}\| \leq \varepsilon,$$

i.e. for all normalised states  $|\psi\rangle$ ,  $\|U|\psi\rangle - \tilde{U}|\psi\rangle\| \leq \varepsilon$ .

**Lemma 4.10** Let  $U_1, \dots, U_m, \tilde{U}_1, \dots, \tilde{U}_m$  be unitaries. Suppose  $\tilde{U}_i$   $\varepsilon$ -approximates  $U_i$  for each  $1 \leq i \leq m$ . Then

$$\|U_m \cdots U_1 - \tilde{U}_m \cdots \tilde{U}_1\| \leq m\varepsilon.$$

So the error increases at most linearly.

*Proof (Hints).* Straightforward. □

*Proof.* By induction on  $m$ . The case  $m = 1$  is true by assumption. Assume it is true for  $m = k$ . We have

$$\begin{aligned} &\|U_{k+1} \cdots U_1 - \tilde{U}_{k+1} \cdots \tilde{U}_1\| \\ &= \|U_{k+1}U_k \cdots U_1 - \tilde{U}_{k+1}U_k \cdots U_1 + \tilde{U}_{k+1}U_k \cdots U_1 - \tilde{U}_{k+1}\tilde{U}_k \cdots \tilde{U}_1\| \\ &\leq \|(U_{k+1} - \tilde{U}_{k+1})U_k \cdots U_1\| + \|\tilde{U}_{k+1}(U_k \cdots U_1 - \tilde{U}_k \cdots \tilde{U}_1)\| \\ &\leq \varepsilon \cdot 1 + 1 \cdot k\varepsilon = (k+1)\varepsilon \end{aligned}$$

by assumption, Proposition 4.8 and the inductive hypothesis. □

**Definition 4.11**  $H$  is a  $k$ -local Hamiltonian on  $n$  qubits if we can write

$$H = \sum_{j=1}^m H_j$$

where each  $H_j$  acts non-trivially on at most  $k$  qubits, in which case we write  $H_j = \tilde{H}_j \otimes I$  (note these qubits need not be adjacent).

Note that  $m \leq \binom{n}{k} = O(n^k)$ , and we usually take  $k$  to be a constant.

**Notation 4.12** Write  $U_{(i)}$  for the unitary

$$I \otimes \cdots \otimes I \otimes U \otimes I \otimes \cdots \otimes I$$

where  $U$  is in the  $i$ -th position, i.e.  $U_{(i)}$  is the unitary acting on the  $i$ -th qubit on  $n$ -qubits.

**Example 4.13**

- $H = X \otimes I \otimes I - 5Z \otimes Y \otimes I$  is 2-local on 3 qubits.
- For the **Ising model** on an  $n \times n$  grid, where each qubit acts non-trivially only with its neighbours, the Hamiltonian is

$$H = J \sum_{i,j=1}^n (Z_{(i,j)} Z_{(i,j+1)} + Z_{(i,j)} Z_{(i+1,j)})$$

where  $J \in \mathbb{R}$  is a coupling constant.

- For the **Heisenberg model** on a line, the Hamiltonian is

$$H = \sum_{i=1}^{n-1} (J_X X_{(i)} X_{(i+1)} + J_Y Y_{(i)} Y_{(i+1)} + J_Z Z_{(i)} Z_{(i+1)}),$$

where  $J_X, J_Y, J_Z \in \mathbb{R}$  are constants.

**Theorem 4.14** (Solovay-Kitaev) Let  $U$  be a unitary on  $k$ -qubits, and  $S$  be a universal set of elementary gates. Then  $U$  can be  $\varepsilon$ -approximated using  $O((\log(1/\varepsilon))^c)$  gates from  $S$ , where  $c < 4$  is a constant.

*Proof.* Omitted. □

**Proposition 4.15** Let  $H = \sum_{j=1}^m H_j$  be a  $k$ -local Hamiltonian where all the local terms  $H_j$  commute. Then for all  $t > 0$  and  $\varepsilon > 0$ , the evolution operator  $U(t) = e^{-iHt}$  can be  $\varepsilon$ -approximated by a circuit with  $O(m \text{ polylog}(m/\varepsilon))$  gates from any universal gate set.

Note that  $m = O(n^k)$ , so the time-complexity is polynomial in  $n$ .

*Proof (Hints).* Straightforward. □

*Proof.* Fix  $t > 0$  and  $\varepsilon > 0$ . We have

$$U(t) = e^{-iHt} = e^{-i \sum_{j=1}^m H_j t} = \prod_{j=1}^m e^{-iH_j t}.$$

Each  $e^{-iH_j t}$  is a unitary that acts non-trivially on at most  $k$  qubits. So we have a circuit for  $e^{-iHt}$  in terms of some set of  $k$ -qubit gates. By [Solovay-Kitaev](#), each  $e^{-iH_j t}$  can be  $\delta$ -approximated by a unitary  $\tilde{U}_j(t)$  circuit with  $O(\text{polylog}(1/\delta))$  gates. By Lemma [4.10](#), we have

$$\left\| U(t) - \prod_{i=1}^m \tilde{U}_j(t) \right\| < m\delta.$$

So choosing  $\delta = \varepsilon/m$ , we obtain a circuit of size  $O(m \text{polylog}(m/\varepsilon))$  which  $\varepsilon$ -approximates  $U(t)$ .  $\square$

**Notation 4.16** For  $N \times N$  matrices  $X$  and  $Y$ , write  $X = Y + O(\varepsilon)$  to mean  $X = Y + E$  where  $\|E\| \leq \varepsilon$ .

**Lemma 4.17** (Lie-Trotter Product Formula) Let  $A$  and  $B$  be  $N \times N$  matrices with  $\|A\|, \|B\| \leq \delta < 1$ . Then

$$e^{-iA}e^{-iB} = e^{-i(A+B)} + O(\delta^2).$$

*Proof (Hints).* Write  $e^{-iA} = I - iA + E_A$  and show that  $\|E_A\| = O(\delta^2)$ , do the same for two other matrices.  $\square$

*Proof.* We have

$$e^{-iA} = I - iA + \sum_{j=2}^{\infty} \frac{(-iA)^j}{j!} =: I - iA + E_A.$$

Now

$$\begin{aligned} \|E_A\| &= \left\| (-iA)^2 \sum_{j=0}^{\infty} \frac{(-iA)^j}{(j+2)!} \right\| \\ &\leq \|(-iA)^2\| \cdot \left\| \sum_{j=0}^{\infty} \frac{(-iA)^j}{(j+2)!} \right\| && \text{by submultiplicativity} \\ &\leq \|(-iA)^2\| \cdot \sum_{j=0}^{\infty} \frac{\|(-iA)^j\|}{(j+2)!} && \text{by triangle inequality and continuity} \\ &\leq \|A\|^2 \sum_{j=0}^{\infty} \frac{\|(-iA)\|^j}{j!} && \text{by submultiplicativity} \\ &= \delta^2 e^{\delta} \leq \delta^2. \end{aligned}$$

So  $e^{-iA} = I - iA + O(\delta^2)$ . By the same argument, we have

$$\begin{aligned} e^{-iB} &= I - iB + O(\delta^2), \\ e^{-i(A+B)} &= I - i(A+B) + O(4\delta^2) = I - i(A+B) + O(\delta^2) \end{aligned}$$

since  $\|A+B\| \leq \|A\| + \|B\| = 2\delta$ . Hence,

$$\begin{aligned}
e^{-iA}e^{-iB} &= (I - iA + O(\delta^2))(I - iB + O(\delta^2)) \\
&= I - i(A + B) + O(\delta^2) \\
&= e^{-i(A+B)} + O(\delta^2),
\end{aligned}$$

since  $\|AB\| \leq \delta^2$  by submultiplicativity.  $\square$

**Proposition 4.18** There is a  $\text{poly}(n, 1/\varepsilon, t)$ -time quantum algorithm for simulating the evolution operators of  $k$ -local Hamiltonians.

*Proof (Hints).*

- Let  $\tilde{H}_j = H_j t / M$  for some constant  $M$  to be determined later. Show that

$$e^{-i\tilde{H}_1} \dots e^{-i\tilde{H}_m} = e^{-i(\tilde{H}_1 + \dots + \tilde{H}_m)} + O(m^3 \tilde{\delta}^2),$$

assuming a bound on  $\|\tilde{H}_j\|$ , which you should determine.

- Using that

$$U(t) = e^{-i(H_1 + \dots + H_m)t} = (e^{-i(H_1 + \dots + H_m)t/M})^M$$

and the above, find an  $O(\dots)$  lower bound for  $M$ .  $\square$

*Proof.* Let  $H = \sum_{j=1}^m H_j$  be a  $k$ -local Hamiltonian and  $U(t) = e^{-iHt}$  be its evolution operator. If all the  $H_j$  commute, then we are done by Proposition 4.15. Otherwise, define  $\tilde{H}_j = H_j t / M$  for each  $j \in [m]$ , for some fixed constant  $M$  to be determined later. We want each  $\|\tilde{H}_j\| \leq \tilde{\delta}$  with  $\tilde{\delta} \leq 1/m$ , since then  $\|\tilde{H}_1 + \dots + \tilde{H}_\ell\| \leq \ell \tilde{\delta}$ , and we need the Lie-Trotter approximation to hold for all  $\ell \in [m]$ . We have

$$\begin{aligned}
& (e^{-i\tilde{H}_1} e^{-i\tilde{H}_2}) \dots e^{-i\tilde{H}_m} \\
&= \left( e^{-i(\tilde{H}_1 + \tilde{H}_2)} + O(\tilde{\delta}^2) \right) e^{-i\tilde{H}_3} \dots e^{-i\tilde{H}_m} \quad \text{by Lie-Trotter} \\
&= e^{-i(\tilde{H}_1 + \tilde{H}_2)} e^{-i\tilde{H}_3} \dots e^{-i\tilde{H}_m} + O(\tilde{\delta}^2) \quad \text{by submultiplicativity} \\
&= \left( e^{-i(\tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3)} + O((2\tilde{\delta})^2) \right) e^{-i\tilde{H}_4} \dots e^{-i\tilde{H}_m} + O(\tilde{\delta}^2).
\end{aligned}$$

since each  $e^{-i\tilde{H}_j}$  is unitary, so has unit norm. Repeatedly applying Lie-Trotter, we obtain

$$\begin{aligned}
e^{-i\tilde{H}_1} \dots e^{-i\tilde{H}_m} &= e^{-i(\tilde{H}_1 + \dots + \tilde{H}_m)} + O(\tilde{\delta}^2) + \dots + O((m-1)\tilde{\delta})^2 \\
&= e^{-i(\tilde{H}_1 + \dots + \tilde{H}_m)} + O(m^3 \tilde{\delta}^2).
\end{aligned}$$

Let the  $O(m^3 \tilde{\delta}^2)$  error be  $Em^3 \tilde{\delta}^2$ . Now

$$U(t) = e^{-i(H_1 + \dots + H_m)t} = (e^{-i(H_1 + \dots + H_m)t/M})^M = (e^{-i(\tilde{H}_1 + \dots + \tilde{H}_m)})^M.$$



So we need the error in approximating  $e^{-iHt/M}$  to be at most  $\varepsilon/M$ , so we require  $Em^3\tilde{\delta}^2 < \varepsilon/M$ . Let  $\delta = \tilde{\delta}M/t$  (so that  $\|H_j\| \leq \delta$  for each  $j$ ): we want  $\|H_j\| \leq \delta = \tilde{\delta}M/t \leq M/(mt)$ , i.e.  $M \geq mt\|H_j\|$  for all  $j$ . So any

$$M > \max\{Em^3(\delta t)^2/\varepsilon, mt\|H_1\|, \dots, mt\|H_m\|\} = O(m^3(\delta t)^2/\varepsilon)$$

suffices. We have

$$\|e^{-iH_1t/M} \dots e^{-iH_mt/M} - e^{-i(H_1+\dots+H_m)t/M}\| \leq \varepsilon/M.$$

and so by Lemma 4.10,

$$\|e^{-iH_1t} \dots e^{-iH_mt} - e^{-i(H_1+\dots+H_m)t}\| \leq \varepsilon.$$

The circuit is composed of  $Mm$  gates of the form  $e^{-iH_jt/M}$ , so the entire circuit consists of  $O(m^4(\delta t)^2/\varepsilon)$  of these gates. Recall that if  $H$  is  $k$ -local, then  $m \leq \binom{n}{k} = O(n^k)$ . So we have a circuit with  $C = O(n^{4k}(\delta t)^2/\varepsilon)$  gates of the form  $e^{-iH_jt/M}$  approximating  $e^{-iHt}$  to precision  $\varepsilon$ . By Solovay-Kitaev, each of these gates can be  $(\varepsilon/C)$ -approximated by  $O(\log^4(C/\varepsilon))$  gates from an elementary universal gate set. So the final complexity is  $\tilde{O}(n^{4k}(\delta t)^2/\varepsilon)$  which is  $\text{poly}(n, 1/\varepsilon, t)$ .  $\square$

#### Remark 4.19

- The time dependence is quadratic, but there are improved product formulae that allow the dependence of the circuit size on  $t$  to be  $O(t^{1+\alpha})$  for any  $\alpha > 0$ .
- The  $\varepsilon$ -dependence is  $\text{poly}(1/\varepsilon)$  whereas in the commuting case it was  $\text{polylog}(1/\varepsilon)$ . However, there are methods that decrease this to  $(1/\varepsilon)^{1/2k}$ .

## 5. The Harrow-Hassidim-Lloyd (HHL) algorithm

**Definition 5.1** The **condition number** of a square matrix  $A \in \mathbb{C}^{N \times N}$  is defined as

$$\kappa(A) := \begin{cases} \|A^{-1}\| \cdot \|A\| & \text{if } A \text{ is invertible} \\ \infty & \text{otherwise} \end{cases}.$$

We say  $A$  is **well-conditioned** if  $\kappa(A)$  is small, i.e. if  $\kappa(A) = O(\text{polylog}(N))$ .

**Remark 5.2**  $\kappa(A)$  can be thought of a measure of “how invertible”  $A$  is. As  $\kappa(A)$  increases, numerically computing  $A^{-1}$  becomes more unstable.

**Proposition 5.3** If  $A \in \mathbb{C}^{N \times N}$  is Hermitian with non-zero eigenvalues  $\lambda_1, \dots, \lambda_N$ , then

$$\kappa(A) = \frac{\max\{|\lambda_i| : i \in [N]\}}{\min\{|\lambda_i| : i \in [N]\}}.$$

*Proof (Hints).* Straightforward.  $\square$

*Proof.* We have

$$A = \sum_{i=1}^N \lambda_i |v_i\rangle\langle v_i|$$

where  $|v_1\rangle, \dots, |v_n\rangle$  are the eigenvalues of  $A$ . Hence,

$$A^{-1} = \sum_{i=1}^N \lambda_i^{-1} |v_i\rangle\langle v_i|,$$

so  $A^{-1}$  has eigenvalues  $\lambda_1^{-1}, \dots, \lambda_N^{-1}$ . So by Theorem 4.7, we have

$$\begin{aligned} \kappa(A) &= \|A^{-1}\| \cdot \|A\| = \max\{|\lambda_i^{-1}| : i \in [N]\} \cdot \max\{|\lambda_i| : i \in [N]\} \\ &= \frac{\max\{|\lambda_i| : i \in [N]\}}{\min\{|\lambda_i| : i \in [N]\}}. \end{aligned}$$

□

**Problem 5.4** (Linear System Solution Problem)

**Input** matrix  $A \in \mathbb{C}^{N \times N}$ , vector  $b \in \mathbb{C}^N$ .

**Task** find a vector  $x \in \mathbb{C}^N$  such that  $Ax = b$ .

**Remark 5.5** The best known classical algorithms for solving linear systems require  $O(\text{poly}(N) \cdot \log(1/\varepsilon))$  time. Note that even just reading the inputs  $A$  and  $b$ , or writing the solution  $x$  requires  $O(\text{poly}(N))$  time.

Instead of computing the full solution  $x$ , the HHL algorithm estimates properties of  $x$  of the form  $\mu = x^T M x$  (i.e. quadratic forms), where  $M$  is Hermitian, e.g. the total weight assigned by  $x$  to a subset of indices/components.

**Algorithm 5.6** (HHL) We are given  $|b\rangle = \frac{1}{\|b\|_2} \sum_{i=0}^{N-1} b_i |i\rangle$  and a Hermitian matrix  $A \in \mathbb{C}^{N \times N}$ . Let  $n = \log N$ .

1. Apply the unitary part  $U_{\text{PE}}$  of [Quantum Phase Estimation](#) for the unitary  $e^{-iA}$  with  $m$  bits of precision to the state  $|b\rangle|0\rangle^{\otimes m}$ .
2. Append an ancillary qubit (in state  $|0\rangle$ ) to the state, then apply a controlled rotation (acting on the last  $m$  qubits plus the ancillary qubit) to it.
3. Perform a **post-selection** step: measure the last qubit, and if the outcome is 0, reject and go back to step 1, otherwise accept.
4. Apply  $U_{\text{PE}}^{-1}$  (the inverse of phase estimation) to the resulting state.
5. Perform a measurement in the  $M$  basis on the first  $n$  qubits of the resulting state.
6. Repeat all of the above  $O(\log(\frac{1}{\eta})/\delta^2)$  times and compute the empirical mean of the measurements, where  $\eta$  controls the probability of success and  $\delta$  controls the approximation error.

**Remark 5.7**

- We can also use HHL for non-Hermitian  $A$ : double the system size and set

$$\tilde{A} = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ b \end{bmatrix}.$$

Then run HHL on  $\tilde{A}$  and  $\tilde{b}$ : if  $Ax = b$ , then  $\tilde{A}\tilde{x} = \tilde{b}$  where  $\tilde{x} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ .

- We can also use HHL for non-Hermitian  $M$ : run HHL on  $M_1 = \frac{1}{2}(M + M^\dagger)$  and  $M_2 = \frac{1}{2i}(M - M^\dagger)$  (which are Hermitian) to give estimates  $\hat{\mu}_1$  and  $\hat{\mu}_2$ , then combine to give  $\hat{\mu} := \hat{\mu}_1 + i\hat{\mu}_2$ .

**Theorem 5.8** (Chernoff-Hoeffding) Let  $X_1, \dots, X_k$  be IID RVs on  $[a, b]$  with mean  $\mu$ , and let  $\bar{X} = \frac{1}{k}(X_1 + \dots + X_k)$ . Then

$$\Pr(|\bar{X} - \mu| > \delta) \leq e^{-2k\delta^2/(b-a)^2}.$$

*Proof.* Omitted. □

**Theorem 5.9** If Hamiltonian simulation and phase estimation are exact, then the HHL algorithm computes a estimate  $\hat{\mu}$  of  $\mu = x^T M x$  to accuracy  $\delta$ , with probability at least  $1 - \eta$ .

*Proof (Sketch).* Let

$$A = \sum_{i=1}^N \lambda_i |v_i\rangle\langle v_i|$$

be the spectral decomposition of  $A$ . For simplicity, we assume that Hamiltonian simulation and phase estimation are exact; in particular, for each  $i \in [N]$ ,  $\lambda_i = \ell_i/2^n$  for some  $\ell_i \in \{0, \dots, N-1\}$ . Assume that  $\max\{|\lambda_1|, \dots, |\lambda_n|\} = 1$  and that  $\kappa(A)$  is known or bounded above by some value  $\kappa_{\max}$ . Writing  $\kappa = \kappa(A)$ , we have by Proposition 5.3 that  $|\lambda_i| \in [1/\kappa, 1]$  for each  $i \in [N]$ . Work in the  $n$ -qubit Hilbert space spanned by  $\{|1\rangle, \dots, |N\rangle\}$ . Write

$$|b\rangle = \sum_{i=1}^N b_i |i\rangle = \sum_{j=1}^N \beta_j |v_j\rangle$$

Due to the bounds on the  $|\lambda_i|$ ,  $A^{-1}$  exists and is equal to  $\sum_{i=1}^N \frac{1}{\lambda_j} |v_j\rangle\langle v_j|$ . Thus, the (unnormalised) solution to  $A|x\rangle = |b\rangle$  is

$$|x\rangle = A^{-1}|b\rangle = \sum_{j=1}^N \beta_j \cdot \frac{1}{\lambda_j} |v_j\rangle.$$

1. Applying  $U_{\text{PE}}$  on the state  $|\psi\rangle := |b\rangle|0\rangle^{\otimes m}$  gives the state

$$U_{\text{PE}}|\psi\rangle = \sum_{i=1}^N \beta_i |v_i\rangle |\ell_i\rangle$$

(assuming  $e^{-iA}$  and  $U_{\text{PE}}$  are exact and error-free). Consider the controlled rotation  $C\text{-Rot}$  acting on  $n+1$  qubits, defined by the following action: for all  $j \in [N]$ ,

$$C\text{-Rot } |\ell_j\rangle|0\rangle = |\ell_j\rangle(\cos(\theta_j)|0\rangle + \sin(\theta_j)|1\rangle) = |\ell_j\rangle \left( \sqrt{1 - \frac{c^2}{\lambda_j^2}} |0\rangle + \frac{c}{\lambda_j} |1\rangle \right),$$

where  $\theta = \arcsin(c/\lambda)$  and  $c \leq \min\{|\lambda_j| : j \in [N]\}$ , and e.g.  $C\text{-Rot } |y\rangle|0\rangle = |y\rangle|0\rangle$  for all other  $y$  (this part is not important and could be chosen to be something else which is consistent with  $C\text{-Rot}$  being unitary). By linearity, this fully determines  $C\text{-Rot}$ .

By the above bounds on the  $|\lambda_j|$ , we can choose  $c = 1/\kappa$ . So in  $C\text{-Rot}$ , the angle

depends on the first register but not on  $A$  or  $b$  (so we're not sneaking extra info in here).  $C$ -Rot can be implemented efficiently using  $O(\text{poly}(n))$  one and two-qubit gates (by e.g. [Solovay-Kitaev](#)).

2. Applying  $C$ -Rot to the state  $U_{\text{PE}}|\psi\rangle \otimes |0\rangle$  produces the state

$$\sum_{j=1}^N \beta_j \sqrt{1 - c^2/\lambda_j^2} |v_j\rangle |\ell_j\rangle |0\rangle + \beta_j \frac{c}{\lambda_j} |v_j\rangle |\ell_j\rangle |1\rangle$$

3. At the post-selection step, the probability of success (i.e. measuring outcome 1) is (Probability of successfully preparing state  $|x\rangle$  is

$$p = \left\| \sum_{j=1}^N \beta_j \frac{c}{\lambda_j} |v_j\rangle |\ell_j\rangle \right\|^2 = \sum_{j=1}^N |\beta_j c / \lambda_j|^2 \geq c^2 \sum_{j=1}^N |\beta_j|^2 = c^2 = \frac{1}{\kappa^2}$$

In this case, the post-measurement state has collapsed to (ignoring the ancillary qubit)

$$|\varphi\rangle = \frac{1}{\sqrt{p}} \sum_{j=1}^N \beta_j \frac{c}{\lambda_j} |v_j\rangle |\ell_j\rangle$$

To boost the success probability to at least  $1 - \eta$ , we can either repeat the post-selection step  $O\left(\frac{\log(1/\eta)}{p}\right) = O(\log(1/\eta)\kappa^2)$  times.

4. Applying  $U_{\text{PE}}^{-1}$  to  $|\varphi\rangle$  “uncomputes” the register holding the  $|\ell_j\rangle$ , resetting it to  $|0\rangle$ , so gives

$$|\tilde{x}\rangle = \frac{c}{\sqrt{p}} \sum_{j=1}^N \frac{\beta_j}{\lambda_j} |v_j\rangle = \frac{c}{\sqrt{p}} |x\rangle.$$

5. Performing a measurement on  $|\tilde{x}\rangle$  in the  $M$  basis on the first  $n$  qubits (i.e. measuring in the  $M \otimes I_m$  basis on the entire state, where  $I_m$  is the identity operator on  $m$  qubits) yields an eigenvalue of  $M$ .
6. We have  $\mu = \langle x | M | x \rangle = \frac{p}{c^2} \langle \tilde{x} | M \otimes I | \tilde{x} \rangle$ . Computing the empirical average of the measurements gives us an estimate of  $\tilde{\mu} = \langle \tilde{x} | M \otimes I_m | \tilde{x} \rangle$ . By the [Chernoff-Hoeffding](#) bound, to estimate  $\tilde{\mu}$  with probability at least  $1 - \eta$  to accuracy  $\delta$ , we need  $O\left(\frac{\log(1/\eta)}{\delta^2}\right)$  of the measurements in the  $M$  basis. Since  $c$  is known, we just need an estimate of  $p$  to be able to estimate  $\mu$ .

To estimate  $p$ : the post-selection step is a Bernoulli trial, with probability of outcome 1 being  $p$ . So the mean is  $0 \cdot (1 - p) + 1 \cdot p = p$ . This can also be estimated by the empirical average of the proportion of post-selection steps performed that succeed, and we again use the [Chernoff-Hoeffding](#) bound.

□

**Remark 5.10** Note that in reality [HHL](#) outputs a state  $|\tilde{x}'\rangle$  that is  $\varepsilon$ -close to  $|\tilde{x}\rangle$ , rather than  $|\tilde{x}\rangle$  itself.

**Remark 5.11** Instead of repeating the post-selection step (step 3 of [HHL](#)) until we succeed, we can use amplitude amplification to the state before we measure, i.e. to the state

$$U_{\text{HHL}}|b\rangle = \sqrt{1-p}|\text{junk}\rangle|0\rangle + \sqrt{p}|\tilde{x}\rangle|1\rangle,$$

which gives a quadratic improvement on the  $O(\kappa^2)$  measurements needed for success in the post-selection step.

Similarly, instead of estimating the probability  $p$  using the [Chernoff-Hoeffding](#) bound, we could use amplitude amplification in the form of [Quantum Counting](#).

**Remark 5.12** In [HHL](#), we want to be able to apply the transformation  $|b\rangle \mapsto A^{-1}|b\rangle = |x\rangle$ . However, this is generally non-unitary so cannot be directly implemented. Instead, we implemented it probabilistically using [Quantum Phase Estimation](#), performed on the unitary  $U = e^{-iA}$ , which in turn is implemented by Hamiltonian simulation. At the end, the measurement in  $M$  basis re-introduces the non-unitarity.

**Definition 5.13** We say a matrix  $A \in \mathbb{C}^{N \times N}$  is **row sparse** if each row of  $A$  contains  $O(\text{polylog}(N))$  non-zero entries.

More generally,  $A$  is **row  $s$ -sparse** if each row contains at most  $s$  non-zero entries.

**Definition 5.14** We say an  $s$ -sparse matrix  $A$  is **row-computable** if the entries of  $A$  can be efficiently computed in the following sense: there is a (classical)  $O(s)$ -time algorithm  $C$  which, given a row index  $1 \leq i \leq N$  and an integer  $k$ , outputs the  $k$ -th non-zero entry  $A_{ij}$  of row  $i$  and its column index  $j$ :

$$C(i, k) = (j, A_{ij}).$$

**Theorem 5.15** Under the following assumptions, for row  $s$ -sparse  $A$ , HHL runs in time  $O((\log N)\kappa^2 s^2 \cdot \frac{1}{\epsilon})$ :

- There is an efficient Hamiltonian simulation algorithm for  $A$ .
- $\|b\|_2$  is 1 (or is efficiently computable), and the state  $|b\rangle$  can be prepared exactly and efficiently.
- The unitary  $C$ -Rot can be implemented exactly and efficiently.
- Measurements in the  $M$  basis can be performed efficiently.

*Proof.* Non-examinable. □

**Remark 5.16** The condition that there is an efficient Hamiltonian simulation algorithm for  $A$  holds for local Hamiltonians and for row-computable matrices.

Since reading  $A$  and  $b$  would take  $\text{poly}(N)$  time if they were stored as their components, there needs to be a more efficient presentation of  $A$  (e.g. row-computable matrices) and  $b$  (stored as a quantum state). The only time the presentation of  $A$  features in HHL is in the Hamiltonian simulation step, so requiring this to run in  $\text{poly}(n, t)$  time is sufficient.

**Remark 5.17** Given  $s$ -sparse  $A$ , the best known classical algorithm for computing an  $\epsilon$ -approximation of  $\mu = x^T M x$  runs in  $O(Ns\sqrt{\kappa} \cdot \log(\frac{1}{\epsilon}))$  time, even with assumptions comparable to our assumptions for HHL. Classically, there is no known method of

estimating the quadratic forms  $\mu = x^T M x$  which is faster than computing  $x$  first. Thus, when  $\varepsilon$  is constant (or even  $\varepsilon = O(\frac{1}{\text{poly}(n)})$ ) and  $A$  is well-conditioned and row-sparse, the HHL runtime is an exponential speedup over the classical algorithm runtime.

## 6. Clifford computations and classical simulation of quantum computation

### 6.1. Classical simulation of quantum computation

We want to know whether there is a “key quantum effect or resource” that gives quantum computing its (potential) benefits over classical computing.

To formalise this comparison of quantum vs classical computing, we will define a precise mathematical framework of classical simulation of quantum computation.

**Problem 6.1** (Classical Simulation)

**Input**

- Description of a quantum circuit  $C$  as a list of 1- and 2- qubit gates acting on  $n$  qubit lines.
- Description of an input product state  $|\alpha_1\rangle \otimes \cdots \otimes |\alpha_n\rangle$  (note this also has a  $\text{poly}(n)$ -sized classical description).
- Description of the designated output qubit(s) line(s).

**Promise**

- $C$  has size  $N = |C| = \text{poly}(n)$ .
- We only measure one qubit, for a decision answer.

**Task** By (randomised) classical means only, perform in  $\text{poly}(n)$  time one of the following:

- **Weak simulation:** sample a bit from the output distribution of  $C|\alpha_1\rangle \cdots |\alpha_n\rangle$  with the output qubit measured in the computational basis.
- **Strong simulation:** calculate the output probabilities  $\Pr(\text{output is } 0) = p$ .

**Remark 6.2** Note that the ability to perform strong simulation implies the ability to perform weak simulation.

**Definition 6.3** If  $C$  is classical simulable (in  $\text{poly}(n)$  time), then we say there is no **quantum advantage** (up to polynomial overheads).

**Remark 6.4**

- Any quantum process performs a weak simulation of itself, i.e. the final measurement gives a sample from the output distribution.
- Strong simulability is a much stronger property.
- “Direct” strong simulation is always possible, but generally not in polynomial time: the action of successive gates is simply matrix-vector multiplication in a  $2^n$ -dimensional space, and so we can compute all amplitudes of the output state in  $O(2^n)$  time. Although this direct simulation isn’t efficient, it shows that any quantum-computable function is also classically computable.

**Theorem 6.5** If the state (including the input state) at each stage of a  $\text{poly}(n)$ -sized circuit  $C$  is a product of (single-qubit) states, then direct strong simulation can be efficiently performed for  $C$  with that input state.

*Proof (Hints).* Assume that the gate acting on a state in a given stage acts non-trivially on the first two qubits (why can we assume this?).  $\square$

*Proof.* We can assume that each gate of the circuit acts two qubits (by extending the 1-qubit gates to act trivially on another qubit). Say at a given stage of the circuit, the state is

$$|\psi\rangle = \sum_{i_1, \dots, i_n \in \{0,1\}} c_{i_1 \dots i_n} |i_1 \dots i_n\rangle,$$

and is transformed by a gate  $U$ . Suppose for simplicity that  $U$  acts on the first two qubits (the same argument works for any qubit indices pair). Let  $U_i^j$  denote the  $(i, j)$ -th entry of the matrix representation of  $U$  in the computational basis. The action of  $U$  on  $|\psi\rangle$  produces the state

$$U|\psi\rangle = \sum_{i_1, \dots, i_n \in \{0,1\}} d_{i_1 \dots i_n} |i_1 \dots i_n\rangle,$$

where  $d_{i_1 \dots i_n} = \sum_{k_1, k_2 \in \{0,1\}} U_{i_1 i_2}^{k_1 k_2} c_{k_1 k_2 i_3 \dots i_n}.$

But now since  $|\psi\rangle$  is a product state, it can be expressed as

$$|\psi\rangle = \bigotimes_{j=1}^n \left( \alpha_j^{(0)} |0\rangle + \alpha_j^{(1)} |1\rangle \right),$$

where  $c_{i_1 \dots i_n} = \prod_{j=1}^n \alpha_j^{(i_j)}$ . Hence,

$$d_{i_1 \dots i_n} = \underbrace{\sum_{k_1, k_2 \in \{0,1\}} U_{i_1 i_2}^{k_1 k_2} \alpha_1^{(k_1)} \alpha_2^{(k_2)}}_{=: \gamma_{i_1 i_2}} \cdot \prod_{j=3}^n \alpha_j^{(i_j)}.$$

The first term in the product is simply an inner product of two 4-dimensional vectors, so can be computed in  $O(1)$  time. Finally, since by assumption  $U|\psi\rangle$  is a product state, it can be expressed as

$$U|\psi\rangle = \bigotimes_{j=1}^n \left( \beta_j^{(0)} |0\rangle + \beta_j^{(1)} |1\rangle \right),$$

where  $d_{i_1 \dots i_n} = \prod_{j=1}^n \beta_j^{(i_j)}$ . We have  $\beta_j^{(k)} = \alpha_j^{(k)}$  for all  $k \in \{0,1\}$  and  $j \geq 3$  since  $U$  acts non-trivially only on the first two qubits.  $\beta_0^{(0)}, \beta_0^{(1)}, \beta_1^{(0)}, \beta_1^{(1)}$  can be determined from  $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$  in  $O(1)$  time.

So, inductively, the amplitudes of each single-qubit state which appears in each product state in the circuit can be computed, given those of the previous product state, in

constant time, so the final state's amplitudes can be computed in  $O(|C|)$  time from the initial state's amplitudes.  $\square$

**Remark 6.6** Recall that a state is a product state iff it has no entanglement. So it is sometimes claimed that entanglement is the source of quantum speedups. However, the converse is not true, i.e. entanglement is necessary but not sufficient.

## 6.2. Clifford computations

**Definition 6.7** The **1-qubit Pauli group** is the subgroup of  $U(2)$  defined as

$$\mathcal{P}_1 := \{\pm i, \pm 1\} \cdot \{I, X, Y, Z\}$$

with the usual rules  $XY = iZ$ , etc. The  **$n$ -qubit Pauli group** is the finite subgroup of  $U(2^n)$  defined as

$$\mathcal{P}_n := \mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_1 = \{P_1 \otimes \cdots \otimes P_n : P_i \in \mathcal{P}_1 \forall i\}.$$

**Definition 6.8** A **Clifford operation** on  $n$ -qubits is a unitary  $C \in U(2^n)$  which preserves the Pauli group under conjugation, i.e.

$$\forall P \in \mathcal{P}_n, \quad CPC^{-1} \in \mathcal{P}_n.$$

The **Clifford group** is the set of all Clifford operations - it is the normaliser of the subgroup  $\mathcal{P}_n$  in  $U(2^n)$ .

**Remark 6.9** Clifford groups are important in applications:

- Quantum error correction (e.g. stabiliser codes) and fault-tolerance.
- They give insights into the power of quantum vs classical computing.

### Example 6.10

- All Pauli matrices are Clifford operations.
- 1-qubit Clifford operations include  $H$  (e.g.  $HZH = X$ ) and  $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ .
- 2-qubit Clifford operations include  $C-X$  and SWAP, e.g.  $C-X_{12}Z_1C-X_{12} = Z_1Z_2$ , and  $C-X_{12}C-X_{21}C-X_{12} = \text{SWAP}_{12}$ .

**Theorem 6.11**  $C \in U(2^n)$  is Clifford iff it can be decomposed into a circuit of  $H$ ,  $S$  and  $C-X$  gates.

*Proof.* Omitted.  $\square$

**Definition 6.12** A **Clifford computation/circuit** is a circuit consisting only of Clifford gates, with a measurement in the computational basis at the end, and possibly with intermediate (for our purposes, 1-qubit) measurements in the  $Z$  basis.

We treat each intermediate measurement as an extra elementary computational step (called a “measurement gate”). Note we can apply unitary gates to the post-intermediate-measurement states.

### Example 6.13

$$|0\rangle|0\rangle \xrightarrow{H} |+\rangle|0\rangle \xrightarrow{C-X} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



is a Clifford circuit. More generally, the family of **cat**/**GHZ** circuits are Clifford circuits:

$$|0\rangle^{\otimes n} \xrightarrow{H} |+\rangle|0\rangle^{\otimes n-1} \xrightarrow{C-X_{12}} \dots \xrightarrow{C-X_{1n}} \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}).$$

**Theorem 6.14** (Gottesman-Knill) Let  $C$  be any  $M = \text{poly}(n)$ -size Clifford circuit on  $n$  qubits with no intermediate measurements, let any product state  $|\alpha_1\rangle \dots |\alpha_n\rangle$  be the input state and let the output be a measurement on the (WLOG, by applying SWAP gates) first qubit line. Then the output can always be classically strongly (and so also weakly) simulated efficiently.

*Proof (Hints).* Denote by  $p_i$  the probability of the measurement yielding outcome  $i$  for  $i = 0, 1$ . Express  $p_0 - p_1$  in bra-ket notation, the rest is straightforward.  $\square$

*Proof.* Idea: instead of evolving the input state  $|\psi_{\text{in}}\rangle$  to  $C|\psi_{\text{in}}\rangle =: |\psi_{\text{end}}\rangle$ , we “backpropagate” the final measurement to  $|\psi_{\text{in}}\rangle$ .

Noting that  $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ , write  $Z_1 = \Pi_0 - \Pi_1$ , where  $\Pi_0$  and  $\Pi_1$  are projectors onto the subspaces where first qubit is 0 and 1 respectively. Let the unitary part of  $C$  be  $C_M \dots C_1$ . Denoting by  $p_i$  the probability of the measurement yielding outcome  $i$  for  $i = 0, 1$ , we have

$$\begin{aligned} p_0 - p_1 &= \langle \psi_{\text{end}} | Z_1 | \psi_{\text{end}} \rangle = \langle \psi_{\text{in}} | C^\dagger Z_1 C | \psi_{\text{in}} \rangle \\ &= \langle \psi_{\text{in}} | C_1^\dagger \dots C_M^\dagger Z_1 C_M \dots C_1 | \psi_{\text{in}} \rangle \end{aligned}$$

$C_1^\dagger \dots C_M^\dagger Z_1 C_M \dots C_1$  is a successive conjugation of  $Z_1$  by Clifford operations (and each is a 1- or 2- qubit conjugation, hence each conjugation is a constant size computation).

By the definition of Clifford operations, we obtain

$$p_0 - p_1 = \langle \psi_{\text{in}} | \tilde{P}_1 \otimes \dots \otimes \tilde{P}_n | \psi_{\text{in}} \rangle$$

where each  $\tilde{P}_i$  is in the Pauli group. Hence, since  $|\psi_{\text{in}}\rangle = |\alpha_1\rangle \dots |\alpha_n\rangle$ , the above factorises:

$$p_0 - p_1 = \prod_{i=1}^n \langle \alpha_i | \tilde{P}_i | \alpha_i \rangle.$$

Each  $\langle \alpha_i | \tilde{P}_i | \alpha_i \rangle$  is a  $2 \times 2$  matrix computation. So computing  $p_0 - p_1$  takes  $O(n)$  time classically. We also need  $O(M)$  time for computing  $\tilde{P}_1 \otimes \dots \otimes \tilde{P}_n = C^\dagger Z_1 C$ . Since we also know  $p_0 + p_1 = 1$ , we can obtain  $p_0$  and  $p_1$ .  $\square$

**Definition 6.15** A Clifford circuit with intermediate measurements is **non-adaptive** if it does not depend on the intermediate measurement outcomes, and **adaptive** otherwise.

**Theorem 6.16** The set of Clifford gates together with the gate  $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$  are a universal gate set for quantum computation.

*Proof.* Omitted.  $\square$

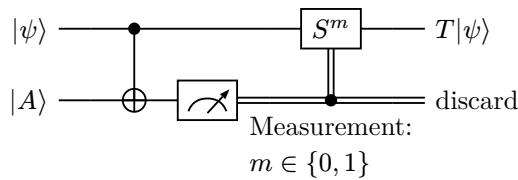
**Theorem 6.17** Let  $C$  be any  $\text{poly}(n)$ -sized Clifford circuit consisting of intermediate measurements, let  $|\psi_{\text{in}}\rangle = |\alpha_1\rangle \dots |\alpha_n\rangle$  be the input state, and say we measure on the (WLOG) first qubit.

1. If  $C$  is non-adaptive, then the output is classically strongly simulable efficiently.
2. Adaptive Clifford circuits are sufficient for fully universal quantum computation.

*Proof (Hints).* See proof sketch. □

*Proof sketch.*

1. Non-adaptive Clifford circuits are reducible to fully unitary Clifford circuits, using ancillary qubits and  $C$ - $X$  gates (then done by [Gottesman-Knill](#)).
2. Use the Bravyi-Kitaev idea of “magic states” and Theorem [6.16](#). Let  $|A\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$  denote the 1-qubit “magic state”. We’ll implement the  $T$  gate using the following “ $T$ -gadget”:



□

**Remark 6.18** Theorem [6.16](#) and Theorem [6.17](#) show we can achieve an increase in power from classically simulable to quantum universal by either:

- Allowing Clifford circuits to use non-Clifford gates, e.g.  $T$ , or
- Allowing only Clifford operations and intermediate measurements, but also the exotic resource “magic” states.

**Remark 6.19** If  $C$  is a  $\text{poly}(n)$ -sized adaptive Clifford circuit whose only inputs are computational basis states, then the output of  $C$  is classically *weakly* simulable efficiently.