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# 1. Introduction

## 1.1. Cubic equations over $\mathbb{C}$

- For a polynomial equation, a **solution by radicals** is a formula for solutions using only addition, subtraction, multiplication, division and radicals  $\sqrt[m]{\cdot}$  for  $m \in \mathbb{N}$ .
- For general cubic equation  $x^3 + a_2x^2 + a_1x + a_0 = 0$ :
  - Tschirnhaus transformation** is substitution  $t = x + \frac{a_2}{3}$ , giving

$$t^3 + pt + q = 0, \quad p := \frac{-a_2^2 + 3a_1}{3}, \quad q := \frac{2a_2^3 - 9a_1a_2 + 27a_0}{27}$$

This is a **reduced** (or **depressed**) cubic equation.

- When  $t = u + v$ ,  $t^3 - (3uv)t - (u^3 + v^3) = 0$  which is in the reduced cubic form with  $p = -3uv$ ,  $q = -(u^3 + v^3)$ .
- We have

$$(y - u^3)(y - v^3) = y^2 - (u^3 + v^3)y + u^3v^3 = y^2 + qy - \frac{p^3}{27} = 0$$

$$\text{so } u^3, v^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

- So a solution to  $t^3 + pt + q = 0$  is

$$t = u + v = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

The other solutions are  $\omega u + \omega^2 v$  and  $\omega^2 u + \omega v$  where  $\omega = e^{2\pi i/3}$  is the 3rd root of unity. This is because  $u$  and  $v$  each have three solutions independently to  $u^3, v^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ , but also  $uv = -\frac{p}{3}$ .

**Remark.** The above method doesn't work for fields of characteristic 2 or 3 since the formulas involve division by 2 or 3 (which is dividing by zero in these respective fields).

## 1.2. Quartic equations over $\mathbb{C}$

- For general quartic equation  $x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ :
  - Substitution  $t = x + \frac{a_3}{4}$  gives **reduced** quartic equation

$$t^4 + pt^2 + qt + r = 0$$

- We then manipulate the polynomial so that it is the sum or difference of two squares and use  $a^2 + b^2 = (a + ib)(a - ib)$  or  $a^2 - b^2 = (a + b)(a - b)$ :

$$(t^2 + w)^2 + (p - 2w)t^2 + qt + (r - w^2) = 0$$

- $(p - 2w)t^2 + qt + (r - w^2) = 0$  is a square iff its discriminant is zero:

$$q^2 - 4(p - 2w)(r - w^2) = 0 \iff w^3 - \frac{1}{2}pw^2 - rw + \frac{1}{8}(4pr - q^2) = 0$$

- This **cubic resolvent** is solvable by radicals. Taking any of the solutions and substituting for  $w$  gives a sum or difference of two squares in  $t$ . The quadratic factors can then be solved.

## 2. Fields and polynomials

### 2.1. Basic properties of fields

**Definition.** Ring  $R$  is **field** if every element of  $R - \{0\}$  has multiplicative inverse and  $1 \neq 0 \in R$ .

**Lemma.** Every field is integral domain.

**Definition.** **Field homomorphism** is ring homomorphism  $\varphi : K \rightarrow L$  between fields:

- $\varphi(a + b) = \varphi(a) + \varphi(b)$
- $\varphi(ab) = \varphi(a)\varphi(b)$
- $\varphi(1) = 1$

These imply  $\varphi(0) = 0$ ,  $\varphi(-a) = -\varphi(a)$ ,  $\varphi(a^{-1}) = \varphi(a)^{-1}$ .

**Lemma.** Let  $\varphi : K \rightarrow L$  field homomorphism.

- $\text{im}(\varphi) = \{\varphi(a) : a \in K\}$  is field.
- $\ker(\varphi) = \{a \in K : \varphi(a) = 0\} = \{0\}$ , i.e.  $\varphi$  is injective.

**Definition.** **Subfield**  $K$  of field  $L$  is subring of  $L$  where  $K$  is field.  $L$  is **field extension** of  $K$ .

- The above lemma shows image of  $\varphi : K \rightarrow L$  is subfield of  $L$ .

**Lemma.** Intersections of subfields are subfields.

**Definition.** **Prime subfield** of  $L$  is intersection of all subfields of  $L$ .

**Definition.** **Characteristic**  $\text{char}(K)$  of field  $K$  is

$$\text{char}(K) := \min\{n \in \mathbb{N} : \chi(n) = 0\}$$

(or 0 if this does not exist) where  $\chi : \mathbb{Z} \rightarrow K$ ,  $\chi(m) = 1 + \dots + 1$  ( $m$  times).

**Example.**  $\text{char}(\mathbb{Q}) = \text{char}(\mathbb{R}) = \text{char}(\mathbb{C}) = 0$ ,  $\text{char}(\mathbb{F}_p) = p$  for  $p$  prime.

**Lemma.** For any field  $K$ ,  $\text{char}(K)$  is either 0 or prime.

**Theorem.**

- If  $\text{char}(K) = 0$  then prime subfield of  $K$  is  $\cong \mathbb{Q}$ .
- If  $\text{char}(K) = p > 0$  then prime subfield of  $K$  is  $\cong \mathbb{F}_p$ .

**Corollary.**

- If  $\mathbb{Q}$  is subfield of  $K$  then  $\text{char}(K) = 0$ .
- If  $\mathbb{F}_p$  is subfield of  $K$  for prime  $p$  then  $\text{char}(K) = p$ .

**Remark.** Let  $\text{char}(K) = p$ , then  $p \mid \binom{p}{i}$  so  $(a + b)^p = a^p + b^p$  in  $K$ . Also in  $K[x]$  for  $p > 2$  prime,  $x^p - 1 = (x - 1)^p$ .

**Theorem** (Fermat's little theorem).  $\forall a \in \mathbb{F}_p, a^p = a$ .

### 2.2. Polynomials over fields

**Definition.** **Degree** of  $f(x) = a_0 + a_1x + \dots + a_nx_n$ ,  $a_n \neq 0$  is  $\deg(f(x)) = n$ .

- Degree of zero polynomial is  $\deg(0) = -\infty$ .
- $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$ .

- $\deg(f(x) + g(x)) \leq \max\{\deg(f(x)), \deg(g(x))\}$  with equality if  $\deg(f(x)) \neq \deg(g(x))$ .
- Only invertible elements in  $K[x]$  are non-zero constants  $f(x) = a_0 \neq 0$ .
- Similarities between  $\mathbb{Z}$  and  $K[x]$  for field  $K$ :
  - $K[x]$  is integral domain.
  - There is a division algorithm for  $K[x]$ : for  $f(x), g(x) \in K[x]$ ,  $\exists! q(x), r(x) \in K[x]$  with  $\deg(r(x)) < \deg(g(x))$  such that

$$f(x) = q(x)g(x) + r(x)$$

- Every  $f(x), g(x) \in K[x]$  have greatest common divisor  $\gcd(f(x), g(x))$  unique up to multiplication by non-zero constants. By Euclidean algorithm for polynomials,

$$\exists a(x), b(x) \in K[x] : a(x)f(x) + b(x)g(x) = \gcd(f(x), g(x))$$

- Can construct field from  $K[x]$ : **field of fractions** of  $K[x]$  is

$$K(x) := \text{Frac}(K[x]) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in K[x], g(x) \neq 0 \right\}$$

where  $f_1(x)/g_1(x) = f_2(x)/g_2(x) \iff f_1(x)g_2(x) = f_2(x)g_1(x)$ . (We can construct the field of fractions for any integral domain).

- $K[x]$  is PID and so UFD.

**Definition.** For field  $K$ ,  $f(x) \in K[x]$  **irreducible in  $K[x]$**  (or  $f(x)$  is **irreducible over  $K$** ) if

- $\deg(f(x)) \geq 1$  and
- $f(x) = g(x)h(x) \implies g(x)$  or  $h(x)$  is constant

### 2.3. Tests for irreducibility

- If  $f(x)$  has linear factor in  $K[x]$ , it has root in  $K[x]$ .

**Proposition** (Rational root test). If  $f(x) = a_0 + \dots + a_n x^n \in \mathbb{Z}[x]$  has rational root  $\frac{b}{c} \in \mathbb{Q}$  with  $\gcd(b, c) = 1$  then  $b \mid a_0$  and  $c \mid a_n$ . **Note:** this can't be used to show  $f$  is irreducible for  $\deg(f(x)) \geq 4$ .

**Theorem** (Gauss's lemma). Let  $f(x) \in \mathbb{Z}[x]$ ,  $f(x) = g(x)h(x)$ ,  $g(x), h(x) \in \mathbb{Q}[x]$ . Then  $\exists r \in \mathbb{Q} : rg(x), r^{-1}h(x) \in \mathbb{Z}[x]$ . i.e. if  $f(x)$  can be factored in  $\mathbb{Q}[x]$  it can be factored in  $\mathbb{Z}[x]$ .

**Example.** Let  $f(x) = x^4 - 3x^3 + 1 \in \mathbb{Q}[x]$ . Using the rational root test,  $f(\pm 1) \neq 0$  so no linear factors in  $\mathbb{Q}[x]$ . Checking quadratic factors, let

$$f(x) = (ax^2 + bx + c)(rx^2 + sx + t), \quad a, b, c, r, s, t \in \mathbb{Z} \text{ by Gauss's lemma}$$

So  $1 = ar \implies a = r = \pm 1$ .  $1 = ct \implies c = t = \pm 1$ .  $-3 = b + s$  and  $0 = c(b + s)$ : contradiction. So  $f(x)$  irreducible in  $\mathbb{Q}[x]$ .

**Example.** Let  $f(x) = x^4 - 3x^2 + 1 \in \mathbb{Q}[x]$ . The rational root test shows there are no linear factors. Checking quadratic factors, let

$$f(x) = (ax^2 + bx + c)(rx^2 + sx + t), \quad a, b, c, r, s, t \in \mathbb{Z} \text{ by Gauss's lemma}$$

As before,  $a = r = \pm 1$ ,  $c = t = \pm 1$ .  $0 = b + s \Rightarrow b = -s$ ,  $-3 = at + bs + cr = -b^2 \pm 2$ .  $b = 1$  works. So  $f(x) = (x^2 - x - 1)(x^2 + x - 1)$ .

**Proposition.** Let  $f(x) = a_0 + \dots + a_n x^n \in \mathbb{Z}[x]$ . If exists prime  $p \nmid a_n$  such that  $\bar{f}(x)$  is irreducible in  $\mathbb{F}_p[x]$ , then  $f(x)$  irreducible in  $\mathbb{Q}[x]$ .

**Example.** Let  $f(x) = 8x^3 + 14x - 9$ . Reducing mod 7,  $\bar{f}(x) = x^3 - 2 \in \mathbb{F}_7[x]$ . No roots exist for this, so  $f(x)$  irreducible in  $\mathbb{Q}[x]$ . For polynomials, no  $p$  is suitable, e.g.  $f(x) = x^4 + 1$ .

- Gauss's lemma works with any UFD  $R$  instead of  $\mathbb{Z}$  and field of fractions  $\text{Frac}(R)$  instead of  $\mathbb{Q}$ : e.g. let  $F$  field,  $R = F[t]$ ,  $K = F(t)$ , then  $f(x) \in R[x]$  irreducible in  $K[x]$  iff  $f(x)$  has no proper factors in  $R[x]$ .

**Proposition** (Eisenstein's criterion). Let  $f(x) = a_0 + \dots + a_n x^n \in \mathbb{Z}[x]$ , prime  $p \in \mathbb{Z}$  such that  $p \mid a_0, \dots, p \mid a_{n-1}$ ,  $p \nmid a_n$ ,  $p^2 \nmid a_0$ . Then  $f(x)$  irreducible in  $\mathbb{Q}[x]$ .

**Example.** Let  $f(x) = x^3 - 3x + 1$ . Consider  $f(x - 1) = x^3 - 3x^2 + 3$ . Then by Eisenstein's criterion with  $p = 3$ ,  $f(x - 1)$  irreducible in  $\mathbb{Q}[x]$  so  $f(x)$  is as well, since factoring  $f(x - 1)$  is equivalent to factoring  $f(x)$ .

**Example.**  $p$ -th cyclotomic polynomial is

$$f(x) = \frac{x^p - 1}{x - 1} = 1 + \dots + x^{p-1}$$

Now

$$f(x + 1) = \frac{(1 + x)^p - 1}{1 + x - 1} = x^{p-1} + px^{p-2} + \dots + \binom{p}{p-2}x + p$$

so can apply Eisenstein with  $p = p$ .

**Proposition** (Generalised Eisenstein's criterion). Let  $R$  be integral domain,  $K = \text{Frac}(R)$ ,

$$f(x) = a_0 + \dots + a_n x^n \in R[x]$$

If there is irreducible  $p \in R$  with

$$p \mid a_0, \dots, p \mid a_{n-1}, p \nmid a_n, p^2 \nmid a_0$$

then  $f(x)$  is irreducible in  $K[x]$ .

## 3. Field extensions

### 3.1. Definitions and examples

**Definition.** Field extension  $L/K$  is field  $L$  containing subfield  $K$ . Can specify homomorphism  $\iota : K \rightarrow L$  (which is injective).

**Example.**

- $\mathbb{C}/\mathbb{R}$ ,  $\mathbb{C}/\mathbb{Q}$ ,  $\mathbb{R}/\mathbb{Q}$ .
- $L = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$  is field extension of  $\mathbb{Q}$ .  $\mathbb{Q}(\theta)$  is field extension of  $\mathbb{Q}$  where  $\theta$  is root of  $f(x) \in \mathbb{Q}[x]$ .

- $L = \mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$  is smallest subfield of  $\mathbb{R}$  containing  $\mathbb{Q}$  and  $\sqrt[3]{2}$ .
- $K(t)$  is field extension of  $K$ .

**Definition.** Let  $L/K$  field extension,  $S \subseteq L$ . Then  **$K$  with  $S$  adjoined**,  $K(S)$ , is minimal subfield of  $L$  containing  $K$  and  $S$ . If  $|S| = 1$ ,  $L/K$  is a **simple extension**.

**Example.**  $\mathbb{Q}(\sqrt{2}, \sqrt{7}) = \{a + b\sqrt{2} + c\sqrt{7} + d\sqrt{14} : a, b, c, d \in \mathbb{Q}\}$  is  $\mathbb{Q}$  with  $S = \{\sqrt{2}, \sqrt{7}\}$ .

**Example.**  $\mathbb{R}/\mathbb{Q}$  is not simple extension.

**Definition.** **Tower** is chain of field extensions, e.g.  $K \subset M \subset L$ .

### 3.2. Algebraic elements and minimal polynomials

**Definition.** Let  $L/K$  field extension,  $\theta \in L$ . Then  $\theta$  is **algebraic over  $K$**  if

$$\exists 0 \neq f(x) \in K[x] : f(\theta) = 0$$

Otherwise,  $\theta$  is **transcendental over  $K$** .

**Example.** For  $n \geq 1$ ,  $\theta = e^{2\pi i/n}$  is algebraic over  $\mathbb{Q}$  (root of  $x^n - 1$ ).

**Example.**  $t \in K(t)$  is transcendental over  $K$ .

**Lemma.** The algebraic elements in  $K(t)/K$  are precisely  $K$ .

**Lemma.** Let  $L/K$  field extension,  $\theta \in L$ . Define  $I_K(\theta) := \{f(x) \in K[x] : f(\theta) = 0\}$ . Then  $I_K(\theta)$  is ideal in  $K[x]$  and

- If  $\theta$  transcendental over  $K$ ,  $I_K(\theta) = \{0\}$
- If  $\theta$  algebraic over  $K$ , then exists unique monic irreducible polynomial  $m(x) \in K[x]$  such that  $I_K(\theta) = \langle m(x) \rangle$ .

**Definition.** For  $\theta \in L$  algebraic over  $K$ , **minimal polynomial of  $\theta$  over  $K$**  is the unique monic polynomial  $m(x) \in K[x]$  such that  $I_K(\theta) = \langle m(x) \rangle$ . The **degree** of  $\theta$  over  $K$  is  $\deg(m(x))$ .

**Remark.** If  $f(x) \in K[x]$  irreducible over  $K$ , monic and  $f(\theta) = 0$  then  $f(x) = m(x)$ .

**Example.**

- Any  $\theta \in K$  has minimal polynomial  $x - \theta$  over  $K$ .
- $i \in \mathbb{C}$  has minimal polynomial  $x^2 + 1$  over  $\mathbb{R}$ .
- $\sqrt{2}$  has minimal polynomial  $x^2 - 2$  over  $\mathbb{Q}$ .  $\sqrt[3]{2}$  has minimal polynomial  $x^3 - 2$  over  $\mathbb{Q}$ .

### 3.3. Constructing field extensions

**Lemma.** Let  $K$  field,  $f(x) \in K[x]$  non-zero. Then

$$f(x) \text{ irreducible over } K \iff K[x]/\langle f(x) \rangle \text{ is a field}$$

**Definition.** Let  $L_1/K, L_2/K$  field extensions,  $\varphi : L_1 \rightarrow L_2$  field homomorphism.  $\varphi$  is  **$K$ -homomorphism** if  $\forall a \in K, \varphi(a) = a$  ( $\varphi$  fixes elements of  $K$ ).

- If  $\varphi$  is isomorphism then it is  **$K$ -isomorphism**.
- If  $L_1 = L_2$  and  $\varphi$  is bijective then  $\varphi$  is  **$K$ -automorphism**.

**Theorem.** Let  $m(x) \in K[x]$  irreducible, monic,  $K_m := K[x]/\langle m(x) \rangle$ . Then

- $K_m/K$  is field extension.
- Let  $\theta = \pi(x)$  where  $\pi : K[x] \rightarrow K_m$  is canonical projection, then  $\theta$  has minimal polynomial  $m(x)$  and  $K_m \cong K(\theta)$ .

**Proposition.** Let  $L/K$  field extension,  $\tau \in L$  with  $m(\tau) = 0$  and  $K_L(\tau)$  be minimal subfield of  $L$  containing  $K$  and  $\tau$ . Then exists unique  $K$ -isomorphism  $\varphi : K_m \rightarrow K_L(\tau)$  such that  $\varphi(\theta) = \tau$ .

**Example.**

- Complex conjugation  $\mathbb{C} \rightarrow \mathbb{C}$  is  $\mathbb{R}$ -automorphism.
- Let  $K$  field,  $\text{char}(K) \neq 2$ ,  $\sqrt{2} \notin K$ , so  $x^2 - 2$  is minimal polynomial of  $\sqrt{2}$  over  $K$ , then  $K(\sqrt{2}) \cong K[x]/\langle x^2 - 2 \rangle$  is field extension of  $K$  and  $a + b\sqrt{2} \mapsto a - b\sqrt{2}$  is  $K$ -automorphism.

**Proposition.** Let  $\theta$  transcendental over  $K$ , then exists unique  $K$ -isomorphism  $\varphi : K(t) \rightarrow K(\theta)$  such that  $\varphi(t) = \theta$ :

$$\varphi\left(\frac{f(t)}{g(t)}\right) = \varphi\left(\frac{f(\theta)}{g(\theta)}\right)$$

### 3.4. Explicit examples of simple extensions

- Let  $r \in K^\times$  non-square in  $K$ ,  $\text{char}(K) \neq 2$ , then  $x^2 - r$  irreducible in  $K[x]$ . E.g. for  $K = \mathbb{Q}(t)$ ,  $x^2 - t \in K[x]$  is irreducible. Then  $K(\sqrt{t}) = \mathbb{Q}(\sqrt{t}) \cong K[x]/\langle x^2 - t \rangle$ .
- Define  $\mathbb{F}_9 = \mathbb{F}_3[x]/\langle x^2 - 2 \rangle \cong \mathbb{F}_3(\theta) = \{a + b\theta : a, b \in \mathbb{F}_3\}$  for  $\theta$  a root of  $x^2 - 2$ .

**Proposition.** Let  $K(\theta)/K$  where  $\theta$  has minimal polynomial  $m(x) \in K[x]$  of degree  $n$ . Then

$$K[x]/\langle m(x) \rangle \cong K(\theta) = \{c_0 + c_1\theta + \dots + c_{n-1}\theta^{n-1} : c_i \in K\}$$

and its elements are written uniquely:  $K(\theta)$  is vector space over  $K$  of dimension  $n$  with basis  $\{1, \theta, \dots, \theta^{n-1}\}$ .

**Example.**  $\mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\} \cong \mathbb{Q}[x]/\langle x^3 - 2 \rangle$ .  $\mathbb{Q}(\omega\sqrt[3]{2})$  and  $\mathbb{Q}(\omega^2\sqrt[3]{2})$  where  $\omega = e^{2\pi i/3}$  are isomorphic to  $\mathbb{Q}(\sqrt[3]{2})$  as  $\omega\sqrt[3]{2}, \omega\sqrt[3]{4}$  have same minimal polynomial.

### 3.5. Degrees of field extensions

**Definition.** Degree of field extension  $L/K$  is

$$[L : K] := \dim_L(F)$$

**Example.**

- When  $\theta$  algebraic over  $K$  of degree  $n$ ,  $[K(\theta) : K] = n$ .
- Let  $\theta$  transcendental over  $K$ , then  $[K(\theta) : K] = \infty$ , so  $[K(t) : K] = \infty$ ,  $[\mathbb{Q}(\pi) : \mathbb{Q}]$ ,  $[\mathbb{R} : \mathbb{Q}] = \infty$ .

**Definition.**  $L/K$  is **algebraic extension** if every element in  $L$  is algebraic over  $K$ .

**Proposition.** Let  $[L : K] < \infty$ , then  $L/K$  is algebraic extension and  $L = K(\alpha_1, \dots, \alpha_n)$  for some  $\alpha_1, \dots, \alpha_n \in L$ .

**Theorem** (Tower law). Let  $K \subseteq M \subseteq L$  tower of field extensions. Then

- $[L : K] < \infty \iff [L : M] < \infty \wedge [M : K] < \infty$ .
- $[L : K] = [L : M][M : K]$ .

**Example.**

- $K = \mathbb{Q} \subset M = \mathbb{Q}(\sqrt{2}) \subset L = \mathbb{Q}(\sqrt{2}, \sqrt{7})$ .  $M/K$  has basis  $\{1, \sqrt{2}\}$  so  $[M : K] = 2$ . Let  $\sqrt{7} \in \mathbb{Q}(\sqrt{2})$ , then  $\sqrt{7} = c + d\sqrt{2}$ ,  $c, d \in \mathbb{Q}$  so  $7 = (c^2 + 2d^2) + 2cd\sqrt{2}$  so  $7 = c^2 + 2d^2$ ,  $0 = 2cd$  so  $d^2 = \frac{7}{2}$  or  $c^2 = 7$ , which are both contradictions. So  $[L : K] = 4$  with basis  $\{1, \sqrt{2}, \sqrt{7}, \sqrt{14}\}$ .
- Let  $K = \mathbb{Q} \subset M = \mathbb{Q}(i) \subset \mathbb{Q}(i, \sqrt{2})$ . We know  $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ , and  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ ,  $[\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q}] = 2$  (since  $i \notin \mathbb{R}$ ) so  $[\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q}(\sqrt{2})] = 2$ .
- Let  $K = \mathbb{Q} \subset M = \mathbb{Q}(\sqrt{2}) \subset L = \mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$ . Then  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ ,  $[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}] = 3$  so  $2 \mid [L : K]$  and  $3 \mid [L : K]$  so  $6 \mid [L : K]$  so  $[L : K] \geq 6$ . But  $[L : M] \leq 3$  and  $[M : K] \leq 2$  so  $[L : K] \leq 6$  hence  $[L : K] = 6$ .
- More generally, we have  $[K(\alpha, \beta) : K] \leq [K(\alpha) : K][K(\beta) : K]$ .

**Example.**

- Let  $\theta = \sqrt[3]{4} + 1$ .  $\mathbb{Q}(\theta) = \mathbb{Q}(\sqrt[3]{4})$  so minimal polynomial over  $\mathbb{Q}$ ,  $m$ , has  $\deg(m) = 3$ .  $(\theta - 1)^3 = 4$  so minimal polynomial is  $x^3 - 3x^2 + 3x - 5$ .
- Let  $\theta = \sqrt{2} + \sqrt{3}$ .  $\mathbb{Q}(\sqrt{2}, \theta) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$  which has degree 2 over  $\mathbb{Q}(\sqrt{2})$  so minimal polynomial of  $\theta$  over  $\mathbb{Q}(\sqrt{2})$  has degree 2,  $\theta - \sqrt{2} = \sqrt{3}$  so minimal polynomial is  $x^2 - 2\sqrt{2}x - 1$ .
- Let  $\theta = \sqrt{2} + \sqrt{3}$ .  $\mathbb{Q} \subset \mathbb{Q}(\theta) \subset \mathbb{Q}(\sqrt{2}, \sqrt{7})$  so  $[\mathbb{Q}(\theta) : \mathbb{Q}] \mid [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$  so  $[\mathbb{Q}(\theta) : \mathbb{Q}] \in \{1, 2, 4\}$ . Can't be 1 as  $\theta \notin \mathbb{Q}$ . If it was 2 then  $1, \theta, \theta^2$  are linearly dependent over  $\mathbb{Q}$  which leads to a contradiction. So degree of minimal polynomial of  $\theta$  over  $\mathbb{Q}$  is 4.  $\theta^2 = 5 + 2\sqrt{6} \Rightarrow (\theta^2 - 5)^2 = 24$  so minimal polynomial is  $x^4 - 10x^2 + 1$ .

## 4. Galois extensions

### 4.1. Splitting fields

**Definition.** For field  $K$ ,  $0 \neq f(x) \in K[x]$ ,  $L/K$  is **splitting field** of  $f(x)$  over  $K$  if

- $\exists c \in K^\times, \theta_1, \dots, \theta_n \in L : f(x) = c(x - \theta_1) \cdots (x - \theta_n)$  ( $f(x)$  **splits over**  $L$ ).
- $L = K(\theta_1, \dots, \theta_n)$ .

**Example.**

- $\mathbb{C}$  is splitting field of  $x^2 + 1$  over  $\mathbb{R}$ , since  $x^2 + 1 = (x + i)(x - i)$  and  $\mathbb{C} = \mathbb{R}(i, -i) = \mathbb{R}(i)$ .
- $\mathbb{C}$  is not splitting field of  $x^2 + 1$  over  $\mathbb{Q}$  as  $\mathbb{C} \neq \mathbb{Q}(i, -i)$ .
- $\mathbb{Q}$  is splitting field of  $x^2 - 36$  over  $\mathbb{Q}$ .
- $\mathbb{C}$  is splitting of  $x^4 + 1$  over  $\mathbb{R}$ .
- $\mathbb{Q}(i, \sqrt{2})$  is splitting field of  $x^4 - x^2 - 2 = (x^2 + 1)(x^2 - 2) = (x + i)(x - i)(x + \sqrt{2})(x - \sqrt{2})$  over  $\mathbb{Q}$ .
- $\mathbb{F}_2(\theta)$  where  $\theta^3 + \theta + 1 = 0$  is splitting field of  $x^3 + x + 1$  over  $\mathbb{F}_2$ .



- Consider splitting field of  $x^3 - 2$  over  $\mathbb{Q}$ . Let  $\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2$  then  $\mathbb{Q}(\sqrt[3]{2}, \omega)$  is splitting field since it must contain  $\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}$ .

**Theorem.** Let  $0 \neq f(x) \in K[x]$ ,  $\deg(f) = n$ . Then there exists a splitting field  $L$  of  $f(x)$  over  $K$  with

$$[L : K] \leq n!$$

**Notation.** For field homomorphism  $\varphi : K \rightarrow K'$  and  $f(x) = a_0 + \dots + a_n x^n \in K[x]$ , write

$$\varphi_*(f(x)) := \varphi(a_0) + \dots + \varphi(a_n)x^n \in K'[x]$$

**Lemma.** Let  $\sigma : K \rightarrow K'$  isomorphism and  $K(\theta)/K$ ,  $\theta$  has minimal polynomial  $m(x) \in K[x]$ ,  $\theta'$  be root of  $\sigma_*(m(x))$ . Then there exists unique  $K$ -isomorphism  $\tau : K(\theta) \rightarrow K'(\theta')$  such that  $\tau(\theta) = \theta'$ .

**Theorem.** For field isomorphism  $\sigma : K \rightarrow K'$  and  $0 \neq f(x) \in K[x]$ , let  $L$  be splitting field of  $f(x)$  over  $K$ ,  $L'$  be splitting field of  $\sigma_*(f(x))$  over  $K'$ . Then there exists a field isomorphism  $\tau : L \rightarrow L'$  such that  $\forall a \in K, \tau(a) = \sigma(a)$ .

**Corollary.** Setting  $K = K'$  and  $\sigma = \text{id}$  implies that splitting fields are unique.

## 4.2. Normal extensions

**Definition.**  $L/K$  is **normal** if: for all  $f(x) \in K[x]$ , if  $f$  is irreducible and has a root in  $L$  then all its roots are in  $L$ . In particular,  $f(x)$  splits completely as product of linear factors in  $L[x]$ . So the minimal polynomial of  $\theta \in L$  over  $K$  has all its roots in  $L$  and can be written as product of linear factors in  $L[x]$ .

**Example.**

- If  $[L : K] = 1$  then  $L/K$  is normal.
- If  $[L : K] = 2$  then  $L/K$  is normal: let  $\theta \in L$  have minimal polynomial  $m(x) \in K[x]$ , then  $K \subseteq K(\theta) \subseteq L$  so  $\deg(m(x)) = [K(\theta) : K] \in \{1, 2\}$ :
  - If  $\deg(m(x)) = 1$  then  $m(x)$  is already linear.
  - If  $\deg(m(x)) = 2$  then  $m(x) = (x - \theta)m_1(x)$ ,  $m_1(x) \in L[x]$  is linear so  $m(x)$  splits completely in  $L[x]$ .
- If  $[L : K] = 3$  then  $L/K$  is not necessarily normal. Let  $\theta$  be root of  $x^3 - 2 \in \mathbb{Q}[x]$ . Other two roots are  $\omega\theta, \omega^2\theta$  where  $\omega = e^{2\pi i/3}$ . If  $\omega\theta \in \mathbb{Q}(\theta)$  then  $\omega = \frac{\omega\theta}{\theta} \in L$  so  $\mathbb{Q} \subset \mathbb{Q}(\omega) \subset \mathbb{Q}(\theta)$  but  $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$  which doesn't divide  $[\mathbb{Q}(\theta) : \mathbb{Q}] = 3$ .
- Let  $\theta \in \mathbb{C}$  be root of irreducible  $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$ . Let  $\theta = u + v$ , then  $(u + v)^3 - 3uv(u + v) - (u^3 + v^3) \equiv 0$  implies  $uv = 1 = u^3v^3$ ,  $u^3 + v^3 = 1$ . So  $(y - u^3)(y - v^3) = y^2 - y + 1$  has roots  $u^3$  and  $v^3$ . So the three roots of  $f$  are

$$\theta_1 = u + v = e^{\pi i/9} + e^{-\pi i/9} = 2 \cos(\pi/9)$$

$$\theta_2 = \omega u + \omega^2 v = e^{7\pi i/9} + e^{-7\pi i/9} = 2 \cos(7\pi/9)$$

$$\theta_3 = \omega^2 u + \omega v = e^{13\pi i/9} + e^{-13\pi i/9} = 2 \cos(13\pi/9)$$

Furthermore, for each  $i, j$ ,  $\theta_i \in \mathbb{Q}(\theta_j)$ , e.g.

$$\theta_2 = 2 \cos\left(\pi - \frac{2\pi}{9}\right) = -2 \cos\left(\frac{2\pi}{9}\right) = -2 \left(2 \cos\left(\frac{\pi}{9}\right)^2 - 1\right) = 2 - \theta_1^2$$

Also  $\theta_1 + \theta_2 + \theta_3 = 0$  so  $\theta_3 \in \mathbb{Q}(\theta_1)$ . So  $\mathbb{Q}(\theta_1)$  contains all roots of  $f(x)$ .

**Theorem** (normality criterion).  $L/K$  is finite and normal iff  $L$  is splitting field for some  $0 \neq f(x) \in K[x]$  over  $K$ .

**Example.**

- $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})/\mathbb{Q}$  is normal as it is the splitting field of  $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)(x^2 - 7) \in \mathbb{Q}[x]$ .
- $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$  is not normal but  $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$  is normal as it is the splitting field of  $x^3 - 2 \in \mathbb{Q}$ .
- $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$  is not normal but  $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$  is normal.
- Let  $\theta$  root of  $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$ . Then  $\mathbb{Q}(\theta)/\mathbb{Q}$  is normal as is splitting field of  $f(x)$  over  $\mathbb{Q}$ .
- $\mathbb{F}_2(\theta)/\mathbb{F}_2$  where  $\theta^3 + \theta^2 + 1 = 0$  is normal, as  $\mathbb{F}_2(\theta)$  contains all roots of  $x^3 + x^2 + 1$ .
- $\mathbb{F}_p(\theta)/\mathbb{F}_p(t)$  where  $\theta^p = t$  is normal as it is the splitting field of  $x^p - t = x^p - \theta^p = (x - \theta)^p$  so  $f(x)$  splits into linear factors in  $L[x]$ .

**Definition.** Field  $N$  is **normal closure** of  $L/K$  if  $K \subseteq L \subseteq N$ ,  $N/K$  is normal, and if  $K \subseteq L \subseteq N' \subseteq N$  with  $N'/K$  normal then  $N = N'$ .

**Theorem.** Every finite extension  $L/K$  has normal closure, unique up to a  $K$ -isomorphism.

**Definition.**  $\text{Aut}(L/K)$  is group of  $K$ -automorphisms of  $L/K$  with composition as the group operation.

**Example.**

- $\text{Aut}(\mathbb{C}/\mathbb{R})$  contains at least two elements: complex conjugation:  $\sigma(a + bi) = a - bi$  and the identity map  $\text{id} = \sigma^2$ . If  $\tau \in \text{Aut}(\mathbb{C}/\mathbb{R})$  then  $\tau(a + bi) = a + b\tau(i)$ . But  $\tau(i)^2 = \tau(i^2) = \tau(-1) = -1$  hence  $\tau(i) = \pm i$ . So there are only two choices for  $\tau$ . So  $\text{Aut}(\mathbb{C}/\mathbb{R}) = \{\text{id}, \sigma\}$ .
- Let  $f(x) = x^2 + px + q \in \mathbb{Q}[x]$  irreducible with distinct roots  $\theta, \theta'$ . Then  $\text{Aut}(\mathbb{Q}(\theta)/\mathbb{Q}) = \{\text{id}, \sigma\} \cong \mathbb{Z}/2$  where  $\sigma(a + b\theta) = a + b\theta'$ .
- Let  $\theta$  root of  $x^3 - 2$ , let  $\sigma \in \text{Aut}(\mathbb{Q}(\theta)/\mathbb{Q})$ . Now  $\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2$  so  $\sigma(\theta) \in \{\theta, \omega\theta, \omega^2\theta\}$  but  $\omega\theta, \omega^2\theta \notin \mathbb{Q}(\theta)$  so  $\sigma(\theta) = \theta \implies \sigma = \text{id}$ .
- Let  $\theta^p = t$ ,  $\sigma \in \text{Aut}(\mathbb{F}_p(\theta)/\mathbb{F}_p(t))$ . Then

$$\sigma(\theta)^p = \sigma(\theta^p) = \sigma(t) = t = \theta^p$$

$$\text{so } (\sigma(\theta) - \theta)^p = \sigma(\theta)^p - \theta^p = 0 \implies \sigma(\theta) = \theta \implies \sigma = \text{id}.$$

- Let  $\sigma \in \text{Aut}(\mathbb{R}/\mathbb{Q})$ . Then  $\alpha \leq \beta \in \mathbb{R} \implies \beta - \alpha = \gamma^2$ ,  $\gamma \in \mathbb{R}$ , so  $\sigma(\beta) - \sigma(\alpha) = \sigma(\gamma)^2 \geq 0$  so  $\sigma(\alpha) \leq \sigma(\beta)$ . Given  $\alpha \in \mathbb{R}$ , there exist sequences  $(r_n), (s_n) \subset \mathbb{Q}$  with  $r_n \leq \alpha \leq s_n$  and  $r_n \rightarrow \alpha$ ,  $s_n \rightarrow \alpha$  as  $n \rightarrow \infty$ . Hence  $r_n = \sigma(r_n) \leq \sigma(\alpha) \leq \sigma(s_n) = s_n$  so  $\sigma(\alpha) = \alpha$  by squeezing. Hence  $\text{Aut}(\mathbb{R}/\mathbb{Q}) = \{\text{id}\}$ .

**Theorem.** Let  $L = K(\theta)$ ,  $\theta$  root of irreducible  $f(x) \in K[x]$ ,  $\deg(f) = n$ . Then  $|\text{Aut}(L/K)| \leq n$ , with equality iff  $f(x)$  has  $n$  distinct roots in  $L$ .

**Theorem.** Let  $L/K$  be finite extension. Then  $|\text{Aut}(L/K)| \leq [L : K]$ , with equality iff  $L/K$  is normal and minimal polynomial of every  $\theta \in L$  over  $K$  has no repeated roots (in a splitting field).

### 4.3. Separable extensions

**Definition.** Let  $L/K$  finite extension.

- $\theta \in L$  is **separable over  $K$**  if its minimal polynomial over  $K$  has no repeated roots (in its splitting field).
- $L/K$  is **separable** if every  $\theta \in L$  is separable over  $K$ .

**Example.** Let  $K = \mathbb{F}_p(t)$ , then  $f(x) = x^p - t \in K[x]$  is irreducible by Eisenstein's criterion with  $p = t$ , and  $f(x) = x^p - \theta^p = (x - \theta)^p$  so  $\theta$  is root of multiplicity  $p \geq 2$ . So  $\mathbb{F}_p(\theta)/\mathbb{F}_p(t)$  is normal but not separable.

**Definition.** Let  $f(x) = \sum_{i=0}^n a_i x^i \in K[x]$ . **Formal derivative** of  $f(x)$  is

$$Df(x) = D(f) := \sum_{i=1}^n i a_i x^{i-1} \in K[x]$$

- Formal derivative satisfies:

$$D(f + g) = D(f) + D(g), \quad D(fg) = f \cdot D(g) + D(f) \cdot g, \quad \forall a \in K, D(a) = 0$$

Also  $\deg(D(f)) < \deg(f)$ . But if  $\text{char}(K) = p$ , then  $D(x^p) = px^{p-1} = 0$  so it is not always true that  $\deg(D(f)) = \deg(f) - 1$ .

**Theorem** (sufficient conditions for separability). Finite extension  $L/K$  is separable if any of the following hold:

- $\text{char}(K) = 0$ ,
- $\text{char}(K) = p$  and  $K = \{b^p : b \in K\}$  for prime  $p$ ,
- $\text{char}(K) = p$  and  $p \nmid [L : K]$

**Definition.**  $K$  is **perfect field** if either of first two of above properties hold.

**Remark.** All finite extensions of any perfect extension (e.g.  $\mathbb{Q}, \mathbb{F}_p$ ) are separable (recall Fermat's little theorem:  $\forall a \in \mathbb{F}_p, a = a^p$ ). So to find a non-separable extension  $L/K$ , we need  $\text{char}(K) = p > 0$ ,  $K$  infinite and  $p \mid [L : K]$ . For example,  $L = \mathbb{F}_p(\theta)$ ,  $K = \mathbb{F}_p(t)$  where  $\theta^p = t$ .

**Theorem.** Let  $\alpha_1, \dots, \alpha_n$  algebraic over  $K$ , then  $K(\alpha_1, \dots, \alpha_n)/K$  is separable iff every  $\alpha_i$  is separable over  $K$ .

**Remark.** For tower  $K \subseteq M \subseteq L$ ,  $L/K$  is separable iff  $L/M$  and  $M/K$  are separable. However, the same statement for normality does not hold.

**Theorem** (Theorem of the Primitive Element). Let  $L/K$  finite and separable. Then  $L/K$  is simple, i.e.  $\exists \alpha \in L : L = K(\alpha)$ .

### 4.4. The fundamental theorem of Galois theory

**Definition.** Finite extension  $L/K$  is **Galois extension** if it is normal and separable. Equivalently,  $|\text{Aut}(L/K)| = [L : K]$ . When  $L/K$  is Galois, the **Galois group** is  $\text{Gal}(L/K) := \text{Aut}(L/K)$ .

**Definition.** Let  $\mathcal{F} := \{\text{intermediate fields of } L/K\}$  and  $\mathcal{G} := \{\text{subgroups of } \text{Gal}(L/K)\}$ . Define the map  $\Gamma : \mathcal{F} \rightarrow \mathcal{G}$ ,  $\Gamma(M) = \text{Gal}(L/M)$ .

**Definition.** Let  $L$  field,  $G$  a group of automorphisms of  $L$ . **Fixed field**  $L^G$  of  $G$  is set of elements in  $L$  which are invariant under all automorphisms in  $G$ :

$$L^G := \{\alpha \in L : \forall \sigma \in G, \sigma(\alpha) = \alpha\}$$

**Theorem.** If  $G$  is finite group of automorphisms of  $L$  then  $L^G$  is subfield of  $L$  and  $[L : L^G] = |G|$ .

**Corollary.** If  $L/K$  is Galois then

- $L^{\text{Gal}(L/K)} = K$ .
- If  $L^G = K$  for some group  $G$  of  $K$ -automorphisms of  $L$ , then  $G = \text{Gal}(L/K)$ .

**Remark.** If  $L/K$  is Galois and  $\alpha \in L$  but  $\alpha \notin K$ , then there exists an automorphism  $\sigma \in \text{Gal}(L/K)$  such that  $\sigma(\alpha) \neq \alpha$ .

**Definition.** For  $H$  subgroup of  $\text{Gal}(L/K)$ , set  $L^H := \{\alpha \in L : \forall \sigma \in H, \sigma(\alpha) = \alpha\}$ , then  $K \subseteq L^H \subseteq L$ . Define  $\Phi : \mathcal{G} \rightarrow \mathcal{F}$ ,  $\Phi(H) = L^H$ .

- $\Gamma$  and  $\Phi$  are inclusion-reversing:  $M_1 \subseteq M_2 \implies \Gamma(M_2) \subseteq \Gamma(M_1)$ , and  $H_1 \subseteq H_2 \implies \Phi(H_2) \subseteq \Phi(H_1)$ .

**Theorem** (Fundamental theorem of Galois theory - Theorem A). For finite Galois extension  $L/K$ ,

- $\Gamma : \mathcal{F} \rightarrow \mathcal{G}$  and  $\Phi : \mathcal{G} \rightarrow \mathcal{F}$  are mutually inverse bijections (the **Galois correspondence**).
- For  $M \in \mathcal{F}$ ,  $L/M$  is Galois and  $|\text{Gal}(L/M)| = [L : M]$ .
- For  $H \in \mathcal{G}$ ,  $L/L^H$  is Galois and  $\text{Gal}(L/L^H) = H$ .

**Remark.**  $\text{Gal}(L/K)$  acts on  $\mathcal{F}$ : given  $\sigma \in \text{Gal}(L/K)$  and  $K \subseteq M \subseteq L$ , consider  $\sigma(M) = \{\sigma(\alpha) : \alpha \in M\}$  which is a subfield of  $L$  and contains  $K$ , since  $\sigma$  fixes elements of  $K$ . Given another automorphism  $\tau : L \rightarrow L$ ,

$$\begin{aligned} \tau \in \text{Gal}(L/\sigma(M)) &\iff \forall \alpha \in M, \tau(\sigma(\alpha)) = \sigma(\alpha) \\ &\iff \forall \alpha \in M, \sigma^{-1}(\tau(\sigma(\alpha))) = \alpha \\ &\iff \sigma^{-1}\tau\sigma \in \text{Gal}(L/M) \\ &\iff \tau \in \sigma \text{Gal}(L/M)\sigma^{-1} \end{aligned}$$

Hence  $\sigma \text{Gal}(L/M)\sigma^{-1}$  and  $\text{Gal}(L/M)$  are conjugate subgroups of  $\text{Gal}(L/K)$ . Now

$$[M : K] = \frac{[L : K]}{[L : M]} = \frac{|\text{Gal}(L/K)|}{|\text{Gal}(L/M)|}$$

**Theorem** (Fundamental theorem of Galois theory - Theorem B). For finite Galois extension  $L/K$ ,  $G = \text{Gal}(L/K)$  and  $K \subseteq M \subseteq L$ . Then the following are equivalent:

- $M/K$  is Galois.
- $\forall \sigma \in G, \sigma(M) = M$ .
- $H = \text{Gal}(L/M)$  is normal subgroup of  $G = \text{Gal}(L/K)$ .

When these conditions hold, we have  $\text{Gal}(M/K) \cong G/H$ .

**Example.** Let  $L/K$  be Galois,  $[L : K] = p$  prime.

- By the tower law, any  $K \subseteq M \subseteq L$  has  $[L : M] \in \{1, p\}$ ,  $[M : K] \in \{p, 1\}$ , so  $M = L$  or  $K$ . In both cases,  $M/K$  is normal.
- $|\text{Gal}(L/K)| = [L : K] = p$  so  $\text{Gal}(L/M) \cong \mathbb{Z}/p$ , so the only subgroups are  $\text{Gal}(L/K)$  and  $\{\text{id}\}$ . In both cases,  $H$  is normal subgroup of  $\text{Gal}(L/K)$ .

## 4.5. Computations with Galois groups

**Example** (quadratic extension).  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$  is normal (since degree is 2) and separable (since characteristic is zero). Any element of  $\varphi \in G = \text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$  is determined by the image of  $\sqrt{2}$ . But  $\varphi(\sqrt{2})^2 = \varphi(2) = 2$  so  $\varphi(\sqrt{2}) = \pm\sqrt{2}$ . This gives two automorphisms  $\text{id}(\sqrt{2}) = \sqrt{2}$  and  $\sigma(\sqrt{2}) = -\sqrt{2}$ . So  $G = \{\text{id}, \sigma\} = \langle \sigma \rangle \cong \mathbb{Z}/2$ . Subgroup  $\{\text{id}\}$  corresponds to  $\mathbb{Q}(\sqrt{2})$ ,  $G$  corresponds to  $\mathbb{Q}$ .

**Example** (biquadratic extension).  $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$  over  $\mathbb{Q}$  is normal (as splitting field of  $(x^2 - 2)(x^2 - 3)$  over  $\mathbb{Q}$ ) and separable (as  $\text{char}(\mathbb{Q}) = 0$ ), so is Galois extension. Let  $\sigma$  be given as before.

- Suppose  $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$ , then  $\sigma(\sqrt{3})^2 = \sigma(3) = 3$ , so  $\sigma(\sqrt{3}) = \pm\sqrt{3}$ .
- If  $\sigma(\sqrt{3}) = \sqrt{3}$ , then  $\sqrt{3} \in \mathbb{Q}(\sqrt{2})^{\{\text{id}, \sigma\}} = \mathbb{Q}$ : contradiction.
- If  $\sigma(\sqrt{3}) = -\sqrt{3}$ , then  $\sigma(\sqrt{2})\sigma(\sqrt{3}) = \sigma(\sqrt{6}) = (-\sqrt{2})(-\sqrt{3}) = \sqrt{6}$ , so  $\sqrt{6} \in \mathbb{Q}(\sqrt{2})^{\{\text{id}, \sigma\}} = \mathbb{Q}$ : contradiction.
- So  $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ , hence  $[L : \mathbb{Q}] = [L : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4$ .
- Now  $G = \text{Gal}(L/\mathbb{Q})$  has order  $[L : \mathbb{Q}] = 4$ , so  $G \cong \mathbb{Z}/4$  or  $\mathbb{Z}/2 \times \mathbb{Z}/2$ .
- For  $\varphi \in G$ ,  $\varphi(\sqrt{2})^2 = 2 \implies \varphi(\sqrt{2}) = \pm\sqrt{2}$ ,  $\varphi(\sqrt{3})^2 = 3 \implies \varphi(\sqrt{3}) = \pm\sqrt{3}$ . So there are four choices, corresponding to choices of  $\pm$  signs.
- Define  $\sigma, \tau$  by  $\sigma(\sqrt{2}) = -\sqrt{2}$ ,  $\sigma(\sqrt{3}) = \sqrt{3}$ ,  $\tau(\sqrt{2}) = \sqrt{2}$ ,  $\tau(\sqrt{3}) = -\sqrt{3}$ . Now  $\sigma^2 = \tau^2 = \text{id}$ ,  $\sigma\tau(\sqrt{2}) = -\sqrt{2}$ ,  $\sigma\tau(\sqrt{3}) = -\sqrt{3}$  and  $\sigma\tau = \tau\sigma$ .
- So  $G = \langle \sigma, \tau : \sigma^2 = \tau^2 = \text{id}, \sigma\tau = \tau\sigma \rangle = \langle \sigma \rangle \times \langle \tau \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ .
- $G$  has proper subgroups  $H_1 = \langle \sigma \rangle$ ,  $H_2 = \langle \tau \rangle$ ,  $H_3 = \langle \sigma\tau \rangle$ .
- So the intermediate fields are  $L^{H_1}, L^{H_2}, L^{H_3}$ .
- $\sigma(\sqrt{3}) = \sqrt{3} \implies \sqrt{3} \in L^{H_1}$  so  $\mathbb{Q}(\sqrt{3}) \subseteq L^{H_1}$ , but  $[L : \mathbb{Q}(\sqrt{3})] = 2 = |H_1| = [L : L^{H_1}]$ . Hence  $L^{H_1} = \mathbb{Q}(\sqrt{3})$ . Similarly  $L^{H_2} = \mathbb{Q}(\sqrt{2})$ .
- $\sigma\tau(\sqrt{6}) = \sqrt{6} \implies \sqrt{6} \in L^{H_3}$ , so  $L^{H_3} = \mathbb{Q}(\sqrt{6})$ .

**Remark.** It is not generally true that  $[K(\sqrt{a}, \sqrt{b}) : K] = 4$ , e.g.  $\mathbb{Q}(\sqrt{2}, \sqrt{8}) = \mathbb{Q}(\sqrt{2})$ .

**Remark.** Can generalise above example to arbitrary  $K(\sqrt{a}, \sqrt{b})/K$  where  $\text{char}(K) \neq 2$ , and  $a, b \in K$ ,  $a, b, ab \notin (K^\times)^2$  where  $(K^\times)^2$  is set of squares of  $K^\times$ .

**Example** (degree 8 extension).

- Consider  $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$  over  $\mathbb{Q}$ .  $L$  is splitting field of  $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ , so is normal, and  $\text{char}(\mathbb{Q}) = 0$ , so is separable, so is Galois.
- Let  $M = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . By above,  $\text{Gal}(M/\mathbb{Q}) = \langle \sigma \rangle \times \langle \tau \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ .
- Suppose  $\sqrt{5} \in M$ . Then  $\sigma(\sqrt{5})^2 = \tau(\sqrt{5})^2 = 5$ , so  $\sigma(\sqrt{5}) = \pm\sqrt{5}$ ,  $\tau(\sqrt{5}) = \pm\sqrt{5}$ .
- If  $\sigma(\sqrt{5}) = \sqrt{5}$ , then  $\sqrt{5} \in M^{(\sigma)} = \mathbb{Q}(\sqrt{3})$ .
  - If  $\tau(\sqrt{5}) = \sqrt{5}$ ,  $\sqrt{5} \in M^{(\sigma, \tau)} = \mathbb{Q}$ : contradiction.
  - If  $\tau(\sqrt{5}) = -\sqrt{5}$ , then since  $\sqrt{15} \in M^{(\sigma)}$ ,  $\tau(\sqrt{15}) = \sqrt{15}$ , so  $\sqrt{15} \in M^{(\sigma, \tau)} = \mathbb{Q}$ : contradiction.

- If  $\sigma(\sqrt{5}) = -\sqrt{5}$ , then  $\sigma(\sqrt{10}) = \sigma(\sqrt{2})\sigma(\sqrt{5}) = (-\sqrt{2})(-\sqrt{5}) = \sqrt{10}$ , so  $\sqrt{10} \in M^{(\sigma)} = \mathbb{Q}(\sqrt{3})$ .
  - If  $\tau(\sqrt{5}) = \sqrt{5}$ ,  $\tau(\sqrt{10}) = \sqrt{10} \in M^{(\sigma, \tau)} = \mathbb{Q}$ : contradiction.
  - If  $\tau(\sqrt{5}) = -\sqrt{5}$ ,  $\tau(\sqrt{30}) = \tau(\sqrt{5})\tau(\sqrt{3})\tau(\sqrt{2}) = \sqrt{30} \in M^{(\sigma, \tau)} = \mathbb{Q}$ : contradiction.
- So  $\sqrt{5} \notin M$ , so  $[L : \mathbb{Q}] = [L : M][M : \mathbb{Q}] = 8$ . The 8 elements in  $\text{Gal}(L/\mathbb{Q})$  are determined by choices of  $\sqrt{a} \mapsto \pm\sqrt{a}$  where  $a \in \{2, 3, 5\}$ .
- $\text{Gal}(L/\mathbb{Q}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$  where  $\sigma_1(\sqrt{2}) = -\sqrt{2}$ ,  $\sigma_2(\sqrt{3}) = -\sqrt{3}$ ,  $\sigma_3(\sqrt{5}) = -\sqrt{5}$  and the  $\sigma_i$  fix all other square roots.
- More generally, write  $\sigma(\sqrt{5}) = (-1)^j \sqrt{5}$ ,  $\tau(\sqrt{5}) = (-1)^k \sqrt{5}$ ,  $j, k \in \{0, 1\}$ . Define  $m = 2^j 3^k$ , then  $\sigma(\sqrt{m}) = (-1)^j \sqrt{m} \Rightarrow \sigma(\sqrt{5m}) = \sqrt{5m}$  and  $\tau(\sqrt{m}) = (-1)^k \sqrt{m} \Rightarrow \tau(\sqrt{5m}) = \sqrt{5m}$ , so  $\sqrt{5m} \in M^{(\sigma, \tau)} = \mathbb{Q}$ : contradiction.

**Example** (cubic extension and its normal closure).

- Let  $L = \mathbb{Q}(\theta)$ ,  $\theta^3 - 2 = 0$ .  $L/\mathbb{Q}$  isn't Galois since not normal. Take the normal closure  $N = \mathbb{Q}(\theta, \omega) = \mathbb{Q}(\theta, \sqrt{-3})$ .
- Let  $M = \mathbb{Q}(\omega)$  so  $[M : \mathbb{Q}] = 2$ ,  $[L : \mathbb{Q}] = 3$  and  $[N : \mathbb{Q}] = 6$ . Let  $G = \text{Gal}(N/\mathbb{Q})$ .
- Since  $|G| = [N : \mathbb{Q}] = 6$ ,  $G \cong \mathbb{Z}/6$  or  $G \cong D_3 \cong S_3$ .
- $G$  contains  $\text{Gal}(N/L)$ . Since  $N = L(\omega)$ ,

$$\text{Gal}(N/L) = \{\text{id}, \tau\} = \langle \tau \rangle \cong \mathbb{Z}/2$$

where  $\tau(\sqrt{-3}) = -\sqrt{-3}$  (i.e.  $\tau(\omega) = \omega^2$ ) and  $\tau(\theta) = \theta$  as  $\theta \in L$ .

- $G$  contains  $H = \text{Gal}(N/M)$ .  $N = M(\theta)$ ,  $|H| = [N : M] = 3$  so  $\text{Gal}(N/M)$  is cyclic so

$$H = \{\text{id}, \sigma, \sigma^2\} = \langle \sigma \rangle \cong \mathbb{Z}/3$$

where  $\sigma(\theta) = \omega\theta$ , also  $\sigma(\omega) = \omega$  as  $\omega \in M$  and  $\sigma^2(\theta) = \omega^2\theta$ , so  $H$  permutes the three roots of  $x^3 - 2$ .

- $\tau \notin H$  so  $H = \{\text{id}, \sigma, \sigma^2\}$  and  $\tau H = \{\tau, \tau\sigma, \tau\sigma^2\}$  are disjoint cosets. So  $G = H \cup \tau H = \langle \tau, \sigma \rangle$  so  $|G| = 6$ .  $\tau^2 = \sigma^3 = \text{id}$  and  $\sigma\tau = \tau\sigma^2$ . So  $G \cong S_3 \cong D_3$ .
- $G$  has one subgroup of order 3,  $H = \langle \sigma \rangle$ . Fixed field is  $N^H = M$ .  $H$  is only proper normal subgroup of  $G$ . Correspondingly,  $M$  is only normal extension of  $\mathbb{Q}$  in  $N$ .
- There are 3 order 2 subgroups:  $\langle \tau \rangle$ ,  $\langle \tau\sigma \rangle$ ,  $\langle \tau\sigma^2 \rangle$ .  $N^{(\tau)} = \mathbb{Q}(\theta) = L$ ,  $N^{(\tau\sigma)} = \mathbb{Q}(\omega\theta) = \sigma(L)$ ,  $N^{(\tau\sigma^2)} = \mathbb{Q}(\omega^2\theta) = \sigma^2(L)$ .

**Example.** Show  $\sqrt[3]{3} \notin \mathbb{Q}(\sqrt[3]{2})$ .

- Assume  $\sqrt[3]{3} \in \mathbb{Q}(\sqrt[3]{2})$ . Then  $\sqrt[3]{3} \in N = \mathbb{Q}(\omega, \sqrt[3]{2})$ , the normal closure.
- As above, let  $\sigma \in \text{Gal}(N/\mathbb{Q})$ ,  $\sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2}$  and  $N^{(\sigma)} = \mathbb{Q}(\omega)$ . Also,

$$\sigma(\sqrt[3]{3})^3 = \sigma(3) = 3 \Rightarrow \sigma(\sqrt[3]{3}) \in \{\sqrt[3]{3}, \omega\sqrt[3]{3}, \omega^2\sqrt[3]{3}\}$$

- If  $\sigma(\sqrt[3]{3}) = \sqrt[3]{3}$ , then  $\sqrt[3]{3} \in N^{(\sigma)} = \mathbb{Q}(\omega)$ , so  $\mathbb{Q}(\sqrt[3]{3}) \subseteq \mathbb{Q}(\omega)$ : contradiction.
- If  $\sigma(\sqrt[3]{3}) = \omega\sqrt[3]{3}$ , then  $\sigma(\sqrt[3]{3}/\sqrt[3]{2}) = \sqrt[3]{3}/\sqrt[3]{2}$  hence  $\sqrt[3]{3/2} \in N^{(\sigma)} = \mathbb{Q}(\omega)$ , so  $\mathbb{Q}(\sqrt[3]{3/2}) = \mathbb{Q}(\sqrt[3]{12}) \subseteq \mathbb{Q}(\omega)$ : contradiction.
- If  $\sigma(\sqrt[3]{3}) = \omega^2\sqrt[3]{3}$ ,  $\mathbb{Q}(\sqrt[3]{3/4}) = \mathbb{Q}(\sqrt[3]{6}) \subseteq \mathbb{Q}(\omega)$ : contradiction.

**Remark.** In the above example,  $N = \mathbb{Q}(\theta_1, \theta_2, \theta_3) = \mathbb{Q}(\sqrt[3]{2}, \omega)$  where  $\theta_i$  are the roots of  $x^3 - 2$ . Plotting this roots on Argand diagram gives the symmetry group  $S_3 \cong D_3$  of

an equilateral triangle.  $\tau$  reflects the  $\theta_i$  (complex conjugation),  $\sigma$  rotates the roots (but **doesn't** rotate all of  $N$ , as it fixes  $\mathbb{Q}$ ). For  $g \in G$ ,  $g(\theta_j) = \theta_{\pi(j)}$  where  $\pi$  is permutation of  $\{1, 2, 3\}$ . So there is a group homomorphism  $\varphi : G \rightarrow S_3$ ,  $\varphi(g) = \pi$ .  $\ker(\varphi) = \{\text{id}\}$ , so  $\varphi$  is injective and also surjective, since  $|G| = |S_3| = 6$ , so  $\varphi$  is isomorphism.

**Definition.** For  $f(x) \in K[x]$ ,  $\deg(f) = n \geq 1$ , with  $n$  distinct roots, the **Galois group** of  $f(x)$ ,  $G_f$ , is Galois group of splitting field of  $f(x)$  over  $K$  (provided it is separable).

**Remark.** Elements of  $G_f$  permute roots of  $f$ , so  $G_f$  is subgroup of  $S_n$ . If  $f(x)$  irreducible over  $K$ , then  $G_f$  is **transitive** subgroup, i.e. given 2 roots  $\alpha, \beta$  of  $f$ , there is a  $g \in G_f$  with  $g(\alpha) = \beta$ . This gives a general pattern

polynomial  $\longrightarrow$  field extension  $\longrightarrow$  permutation group

**Example.** Consider  $\mathbb{Q} \subset L = \mathbb{Q}(\theta) \subset N = \mathbb{Q}(\theta, i)$  where  $\theta = \sqrt[4]{2}$ .  $N$  is normal closure of  $\mathbb{Q}(\theta)$ ,  $[N : \mathbb{Q}] = 8$  so  $|\text{Gal}(N/\mathbb{Q})| = 8$ .

- Define  $\sigma(\theta) = i\theta$ ,  $\sigma(i) = i$ ,  $\tau(\theta) = \theta$ ,  $\tau(i) = -i$ . Then  $\tau^2 = \sigma^4 = \text{id}$ . We have

	id	$\sigma$	$\sigma^2$	$\sigma^3$	$\tau$	$\tau\sigma$	$\tau\sigma^2$	$\tau\sigma^3$
$\theta$	$\theta$	$i\theta$	$-\theta$	$-i\theta$	$\theta$	$-i\theta$	$-\theta$	$i\theta$
$i$	$i$	$i$	$i$	$i$	$-i$	$-i$	$-i$	$-i$

so  $G = \text{Gal}(N/\mathbb{Q}) = \langle \sigma, \tau : \sigma^4 = \tau^2 = \text{id}, \sigma\tau = \tau\sigma^3 \rangle \cong D_4$ .

- Order 2 subgroups are  $\langle \tau \rangle$ ,  $\langle \tau\sigma \rangle$ ,  $\langle \tau\sigma^2 \rangle$ ,  $\langle \tau\sigma^3 \rangle$ ,  $\langle \sigma^2 \rangle$ .
- Order 4 subgroups are  $\langle \sigma^2, \tau \rangle \cong (\mathbb{Z}/2)^2$ ,  $\langle \sigma \rangle \cong \mathbb{Z}/4$ ,  $\langle \sigma^2, \tau\sigma \rangle \cong (\mathbb{Z}/2)^2$ .
- Respectively, intermediate field extensions of degree 4 are  $\mathbb{Q}(\sqrt[4]{2})$ ,  $\mathbb{Q}(i\sqrt[4]{2})$ ,  $\mathbb{Q}(\sqrt{2}, i)$ ,  $\mathbb{Q}((1-i)\sqrt[4]{2})$ ,  $\mathbb{Q}((1+i)\sqrt[4]{2})$ .
- Respectively, intermediate field extensions of degree 2 are  $\mathbb{Q}(\sqrt{2})$ ,  $\mathbb{Q}(i)$ ,  $\mathbb{Q}(i\sqrt{2})$ .

## 5. Cyclotomic field extensions

### 5.1. Roots of unity

**Definition.** If  $L/K$  is Galois,  $\text{Gal}(L/K) \cong \mathbb{Z}/n$ , then  $L$  is **cyclic extension** of  $K$  of degree  $n$ .

**Definition.**  $\zeta \in K^*$  is  **$n$ -th primitive root of unity** if  $\zeta^n = 1$  and  $\forall 0 < m < n$ ,  $\zeta^m \neq 1$ , i.e. order of  $\zeta$  in  $K^*$  is  $n$ .

**Example.**

- $\zeta$  is primitive 1-st root of unity iff  $\zeta = 1$ .
- $-1$  is primitive 2-nd root of unity iff  $\text{char}(K) \neq 2$ .
- If  $\text{char}(K) = p$  prime, then  $K$  contains no  $p$ -th primitive roots of unity (since  $\zeta^p = 1 \iff (\zeta - 1)^p = 0 \iff \zeta = 1$ ).
- If  $K = \mathbb{C}$ ,  $\exp(2\pi i/n)$  is  $n$ -th primitive root of unity.

**Proposition.** Let  $\zeta \in K^*$  primitive  $n$ -th root of unity, let  $d = \gcd(m, n)$ . Then  $\zeta^m$  is primitive  $(n/d)$ -th root of unity.

**Corollary.** Let  $\zeta \in K^*$  primitive  $n$ -th root of unity.

- $\zeta^m = 1 \iff m \equiv 0 \pmod n$ .
- $\zeta^m$  is primitive  $n$ -th root of unity iff  $\gcd(m, n) = 1$ .

**Definition.** Let  $\mu(K)$  denote subgroup of all roots of unity in  $K^*$ .

**Theorem.** Let  $K$  field,  $H$  finite subgroup of  $K^*$ , then  $H$  is cyclic.

**Corollary.** Let  $K$  field,  $n \in \mathbb{N}$  be largest such that  $K$  contains primitive  $n$ -th root of unity  $\zeta$ . Then  $\mu(K)$  is cyclic subgroup in  $K^*$  generated by  $\zeta$ .

## 5.2. $n$ -th cyclotomic field extensions

**Notation.** Let  $\zeta_n = \exp(2\pi i/n) \in \mathbb{C}$ .

**Definition.**  $\mathbb{Q}(\zeta_n)/\mathbb{Q}$  is  **$n$ -th cyclotomic field extension**.

**Proposition.**  $\mathbb{Q}(\zeta_n)/\mathbb{Q}$  is Galois.

**Definition.**  $\Phi_n(x) := \prod_{a \in A} (x - \zeta_n^a)$  where  $A = \{a \in \mathbb{N} : 0 < a < n, \gcd(a, n) = 1\}$ .

**Proposition.**  $\Phi_n(x) \in \mathbb{Q}[x]$  is irreducible and so is minimal polynomial of a primitive  $n$ -th root of unity over  $\mathbb{Q}$ . In particular,  $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$ , where  $\varphi(n) = |(\mathbb{Z}/n)^\times|$  is Euler function.

**Proposition.** Properties of  $\varphi$  function:

- For prime  $p$ ,  $\varphi(p) = p - 1$ .
- For prime  $p$ ,  $\varphi(p^k) = p^k - p^{k-1}$ .
- If  $\gcd(n, m) = 1$ , then  $\varphi(nm) = \varphi(n)\varphi(m)$ .
- If  $n = \prod_{i=1}^r p_i^{k_i}$  is prime factorisation of  $n$ , then

$$\varphi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right)$$

**Proposition.**  $\forall n \in \mathbb{N}, x^n - 1 = \prod_{n_1|n} \Phi_{n_1}(x)$ .

**Example.**

- $\Phi_1(x) = x - 1$ .
- $\Phi_1(x)\Phi_2(x) = x^2 - 1 \implies \Phi_2(x) = x + 1$ .
- $\Phi_1(x)\Phi_3(x) = x^3 - 1 \implies \Phi_3(x) = x^2 + x + 1$ .

**Proposition.**

- For  $p$  prime,  $\Phi_p(x) = x^{p-1} + \dots + x + 1$ .
- For  $p$  prime,  $\Phi_{p^k}(x) = \Phi_p(x^{p^{k-1}})$ .
- For every  $n \in \mathbb{N}$ ,  $\Phi_n(x)$  has integer coefficients.

## 5.3. Galois properties of cyclotomic extensions

**Theorem.**  $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^\times$ .

**Corollary.**  $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$  is abelian so every subgroup is normal, so any subfield of  $\mathbb{Q}(\zeta_n)$  is Galois over  $\mathbb{Q}$ .

**Corollary.** For  $p$  prime,  $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p-1)$ . In particular, for  $d \mid (p-1)$ ,  $\mathbb{Q}(\zeta_p)$  contains exactly one subfield of degree  $d$  and there are no other subfields.



**Remark.** For  $d = 2$  in above corollary,  $\mathbb{Q}(\zeta_p)$  contains unique quadratic subfield  $\mathbb{Q}(\sqrt{D_p})$ .  $D_p = p$  if  $p \equiv 1 \pmod{4}$  and  $D_p = -p$  if  $p \equiv 3 \pmod{4}$ .

**Example.**  $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$  not always cyclic, e.g.  $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ .

**Proposition.**

- If  $n$  odd,  $\mu(\mathbb{Q}(\zeta_n))$  is cyclic of order  $2n$  and is generated by  $-\zeta_n$ .
- If  $n$  even,  $\mu(\mathbb{Q}(\zeta_n))$  is of order  $n$  and is generated by  $\zeta_n$ .
- If  $\gcd(m, n) = 1$ , then  $\mathbb{Q}(\zeta_m, \zeta_n) = \mathbb{Q}(\zeta_{mn})$ .
- $\forall m, n \in \mathbb{N}$ ,  $\mathbb{Q}(\zeta_m, \zeta_n) = \mathbb{Q}(\zeta_{\text{lcm}(m, n)})$

## 5.4. Special properties of $\mathbb{Q}(\zeta_p)$ , where $p > 2$ is prime

**Example.**  $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong (\mathbb{Z}/5)^\times$  has generator  $\tau : \zeta_5 \mapsto \zeta_5^2$ .  $\mathbb{Q}$ -basis  $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$  is not invariant under action of  $\tau$  or any power of  $\tau$  (since  $\tau(\zeta_5^2) = \zeta_5^4$ ) but  $\{\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$  is invariant. The same holds for general  $p > 2$  prime. For  $\alpha_i \in \mathbb{Q}$ ,  $\alpha_1 \zeta_p + \dots + \alpha_{p-1} \zeta_p^{p-1} \in \mathbb{Q}$  iff  $\alpha_1 = \dots = \alpha_{p-1}$ .

**Example.** If  $x \in \mathbb{Q}(\zeta_p)$ ,  $[\mathbb{Q}(x) : \mathbb{Q}] = |\{\sigma(x) : \sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})\}|$ . In particular, if  $\tau$  is generator of  $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$  and  $x = \alpha_1 \zeta_p + \dots + \alpha_{p-1} \zeta_p^{p-1}$  then set of all conjugates of  $x$  is equal to (note not all elements are distinct)

$$\{\tau^a(x) : a \in [p-1]\} = \left\{ \sum_{i=1}^{p-1} \alpha_i \zeta_p^{ai} : a \in [p-1] \right\}$$

**Example.** Let  $x = \zeta_5 + \zeta_5^4$ ,  $\tau : \zeta_5 \mapsto \zeta_5^2$  is a generator of  $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ .  $\tau(x) = \zeta_5^2 + \zeta_5^3 \neq x$  but  $\tau^2(x) = x$ , so  $[\mathbb{Q}(x) : \mathbb{Q}] = 2$ , i.e.  $\mathbb{Q}(\zeta_5 + \zeta_5^4)$  is unique quadratic subfield in  $\mathbb{Q}(\zeta_5)$ .

**Definition.** Let  $x \in \mathbb{Q}(\zeta_p)$ , let minimal polynomial of  $x$  over  $\mathbb{Q}$  be  $m(t) = (t - x^{(1)}) \dots (t - x^{(d)})$ . **Conjugates** of  $x$  over  $\mathbb{Q}$  are  $x^{(1)} = x, \dots, x^{(d)}$ .

**Example.** Minimal polynomial of  $\zeta_5 + \zeta_5^4 = 2 \cos(2\pi/5)$  over  $\mathbb{Q}$  is  $m(x) = (x - \zeta_5 - \zeta_5^4)(x - \zeta_5^2 - \zeta_5^3) = x^2 + x - 1$ , with roots  $(-1 \pm \sqrt{5})/2$ . So  $\cos(2\pi/5) = (-1 + \sqrt{5})/4$ , and unique quadratic subfield of  $\mathbb{Q}(\zeta_5)$  over  $\mathbb{Q}$  is  $\mathbb{Q}(\sqrt{5})$ .

**Example.** Let  $\tau \in G$  be generator of  $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ , i.e.  $\tau(\zeta_p) = \zeta_p^a$ ,  $a \pmod{p}$  generates  $(\mathbb{Z}/p)^\times$ . Let

$$\Theta_p = \zeta_p - \tau(\zeta_p) + \tau^2(\zeta_p) - \dots + \tau^{p-3}(\zeta_p) - \tau^{p-2}(\zeta_p)$$

$\Theta_p$  behaves like  $\sqrt{D_p}$ :  $\tau(\Theta_p) = -\Theta_p$ ,  $\tau^2(\Theta_p) = \Theta_p$ . So  $\Theta_p \in \mathbb{Q}(\zeta_p)^{\langle \tau^2 \rangle}$ . Also,  $\tau(\Theta_p^2) = \Theta_p^2$  so  $\Theta_p^2 \in \mathbb{Q}(\zeta_p)^{\langle \tau \rangle} = \mathbb{Q}$ . In fact,  $\Theta_p^2 = D_p$ . Therefore

$$\Theta_p^2 = A + B(\zeta_p + \dots + \zeta_p^{p-1}) = A - B$$

So when computing  $\Theta_p^2$ , only need to consider coefficients for 1 and  $\zeta_p$ .

## 6. Cyclic field extensions

### 6.1. Cyclic extensions of degree 2

**Definition.**  $L/K$  is **cyclic of degree 2** if it is Galois and  $\text{Gal}(L/K) \cong \mathbb{Z}/2$ .

**Example.** Let  $L/K$  cyclic of degree 2, so  $\text{Gal}(L/K) = \{e, \tau\}$ ,  $\tau^2 = e$ . Let  $\theta \in L - K$ , then  $\tau(\theta) \neq \theta$  (as otherwise  $\theta \in L^{\langle \tau \rangle} = K$ ). Let  $\theta_1 = \tau(\theta) - \theta$ , so  $\tau(\theta_1) = \tau^2(\theta) - \tau(\theta) = -\theta_1$ . If  $\text{char}(K) \neq 2$ , then  $\theta_1 \neq -\theta_1$  and so  $\theta_1 \notin K$ ,  $L = K(\theta_1)$ .  $\theta_1$  is “better” than  $\theta$ , since  $\tau(\theta_1) = -\theta_1$ . Now if  $a = \theta_1^2$ , then  $\tau(a) = a$ , so  $L = K(\sqrt{a})$ .

**Theorem.** If  $\text{char}(K) \neq 2$  and  $L/K$  is cyclic quadratic extension, then

$$\exists a \in K^\times - K^{\times 2} : L = K(\sqrt{a})$$

**Definition.**  $a_1, \dots, a_n$  are **independent modulo  $K^{\times 2}$**  (independent modulo squares) if

$$a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n} \in K^{\times 2} \iff \text{all } \varepsilon_i \text{ are even}$$

**Proposition.** If  $\text{char}(K) \neq 2$ :

- $K(\sqrt{a_1}) = K(\sqrt{a_2}) \iff a_1 \equiv a_2 \pmod{K^{\times 2}}$ , i.e.  $a_1 = a_2 \cdot b^2$ ,  $b \in K^\times$ .
- If  $a_1, \dots, a_n \in K^\times$  are independent modulo  $K^{\times 2}$  then  $K(\sqrt{a_1}, \dots, \sqrt{a_n})$  has degree  $2^n$  over  $K$  with Galois group  $\cong (\mathbb{Z}/2)^n$ .
- If  $L/K$  Galois with Galois group  $(\mathbb{Z}/2)^n$ , then

$$\exists a_1, \dots, a_n \in K^\times : L = K(\sqrt{a_1}, \dots, \sqrt{a_n})$$

**Remark.** Let  $\text{char}(K) = 2$ , then  $\forall a \in K^\times$ ,  $L = K(\sqrt{a})$  is normal but not separable (since minimal polynomial of e.g.  $\sqrt{a}$  is  $x^2 - a = (x + \sqrt{a})(x - \sqrt{a}) = (x - \sqrt{a})^2$  so has repeated roots).

## 6.2. Cyclic extensions of degree $n$ (the Kummer theory)

**Definition.**  $L/K$  is **cyclic of degree  $n$**  if it is Galois and  $\text{Gal}(L/K)$  is cyclic of order  $n$ .

**Theorem.** If  $K$  contains primitive  $n$ -th root of unity and for all divisors  $d > 1$  of  $n$ ,  $a \in K^\times$  is not  $d$ -th power in  $K$ , then  $L = K(\sqrt[n]{a})$  is cyclic extension of  $K$  of degree  $n$ . In particular,  $x^n - a \in K[x]$  is irreducible.

**Proposition.** If  $\zeta_p \in K$ ,  $a \in K^\times - K^{\times p}$ , then  $K(\sqrt[p]{a})/K$  is cyclic of degree  $p$ . In particular,  $x^p - a \in K[x]$  is irreducible.

**Theorem.** Let  $K$  contain  $n$ -th primitive root of unity,  $L/K$  is cyclic extension of degree  $n$ . Then

$$\exists a \in K^\times : L = K(\sqrt[n]{a})$$

Such an  $a$  is given by  $\theta_{b_0}^n$  for some  $b_0 \in L$ , where

$$\theta_b = b + \zeta_n^{-1} \sigma(b) + \dots + \zeta_n^{-(n-1)} \sigma^{n-1}(b)$$

is **Lagrange resolvent** for  $b$ , i.e.  $L = K(\theta_b)$ .

**Lemma** (Artin's lemma). There exists  $b_0 \in L$  such that  $\theta_{b_0} \neq 0$ .

## 7. Finite fields

## 7.1. Existence and uniqueness

**Lemma.** Let  $K$  finite field, then  $K$  is field extension of  $\mathbb{F}_p$  for some prime  $p$  and  $|K| = p^n$  where  $n = [K : \mathbb{F}_p]$ .

**Theorem.** Let  $p$  prime. Then  $\forall n \in \mathbb{N}$ , there is field  $K$  with  $|K| = p^n$ .

**Theorem.** Let  $K$  finite field with  $|K| = q = p^n$ . Then

- $\forall \alpha \in K, \alpha^q = \alpha$ .
- $x^q - x = \prod_{\alpha \in K} (x - \alpha)$
- $K$  is splitting field of  $x^q - x$  over  $\mathbb{F}_p$ .

**Corollary.** If  $K_1, K_2$  finite fields,  $|K_1| = |K_2|$ , then  $K_1 \cong K_2$ .

**Definition.** Let  $q = p^n$ , then  $\mathbb{F}_q$  is the unique (up to isomorphism) field containing  $q$  elements.

**Definition.** For  $q = p^n$ , the **Frobenius automorphism** is

$$\sigma : \mathbb{F}_q \rightarrow \mathbb{F}_q, \quad \sigma(\alpha) = \alpha^p$$

which is an  $\mathbb{F}_p$ -automorphism by Fermat's little theorem.

**Theorem.** Let  $q = p^n$ ,  $p$  prime.

- $\mathbb{F}_q/\mathbb{F}_p$  is Galois of degree  $n$ .
- Frobenius automorphism generates  $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$  and there is group isomorphism

$$\text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \leftrightarrow \mathbb{Z}/n, \quad \sigma \leftrightarrow 1 \bmod n$$

## 7.2. Counting irreducible polynomials over finite fields

**Notation.** Let  $\text{Irr}_{\mathbb{F}_p}(m)$  denote set of all irreducible polynomials in  $\mathbb{F}_p[x]$  of degree  $m$ . Let  $N_p(m) = |\text{Irr}_{\mathbb{F}_p}(m)|$ .

**Theorem.** Let  $q = p^m$ , then  $mN_p(m) = |\{\alpha \in \mathbb{F}_q : \mathbb{F}_p(\alpha) = \mathbb{F}_q\}|$ .

**Remark.** To use above theorem, note that  $\mathbb{F}_p(\alpha) \neq \mathbb{F}_{p^m}$  iff  $\alpha$  belongs to proper subfield of  $\mathbb{F}_{p^m}$ .

**Example.** We construct  $L = \mathbb{F}_{3^{16}}$  by finding irreducible polynomial of degree 16 in  $\mathbb{F}_3[x]$ .

- $\mathbb{F}_9 = \mathbb{F}_3(\theta)$  where  $\theta^2 + 1 = 0$ ,  $\mathbb{F}_9 = \{a + b\theta : a, b \in \mathbb{F}_3\}$ .  $K := \mathbb{F}_9$  contains primitive 8-th root of unity since  $\mathbb{F}_9^\times \cong \mathbb{Z}/8$ .
- $L/K$  is cyclic extension of degree 8, so by Kummer theory there exists  $\alpha \in K$  such that  $L = K(\sqrt[8]{\alpha})$ .  $\alpha$  must be element that is not square or fourth power in  $\mathbb{F}_9$  and has order exactly 8.
- $\alpha = \theta$  doesn't work since  $\theta^2 = -1 \implies \theta^4 = 1$ .  $\alpha = 1 + \theta$  works since

$$(1 + \theta)^2 = \theta^2 + \theta + 1 = -\theta, \quad (1 + \theta)^4 = \theta^2 = -1, \quad (1 + \theta)^8 = 1$$

so  $\alpha = 1 + \theta$  has order 8 in  $\mathbb{F}_9$ .

- So  $L = K(\sqrt[8]{\alpha}) = \mathbb{F}_9(\sqrt[8]{1 + \theta}) = \mathbb{F}_3(\theta, \sqrt[8]{1 + \theta}) = \mathbb{F}_3(\eta)$  where  $\eta^8 = 1 + \theta$ . Now  $[L : \mathbb{F}_3] = 16$  by tower law, so  $L = \mathbb{F}_{3^{16}}$  by uniqueness of finite fields.

- $\eta^8 = 1 + \theta \implies (\eta^8 - 1)^2 = \theta^2 = -1 \implies \eta^{16} + \eta^8 + 2 = 0$  so  
 $f(x) = x^{16} + x^8 + 2 \in \mathbb{F}_3[x]$  is irreducible.

## 8. Galois groups of polynomials

### 8.1. Symmetric functions

**Definition.** Define action of  $S_n$  on  $L = k(x_1, \dots, x_n)$  by  $\tau : x_j \mapsto x_{\pi(j)}$  where  $\pi \in S_n$ , which gives  $k$ -automorphism

$$\tau : L \rightarrow L, \quad \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} \mapsto \frac{f(x_{\pi(1)}, \dots, x_{\pi(n)})}{g(x_{\pi(1)}, \dots, x_{\pi(n)})}$$

The **symmetric functions** in  $L$  are elements of fixed field  $L^{S_n}$ .

**Definition.** The **elementary symmetric polynomials**  $e_r \in L$  for  $r \in [n]$  are

$$\begin{aligned} e_1 &= \sum_{1 \leq i \leq n} x_i \\ e_2 &= \sum_{1 \leq i < j \leq n} x_i x_j \\ &\vdots \\ e_r &= \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdots x_{i_r} \\ &\vdots \\ e_n &= x_1 \cdots x_n \end{aligned}$$

Define  $K = k(e_1, \dots, e_n)$ .

**Theorem.**  $K = L^{S_n}$  and  $L/K$  is Galois with  $\text{Gal}(L/K) \cong S_n$ .

*Proof.*

- Note that  $f(x) = (x - x_1) \cdots (x - x_n) = x^n - e_1 x^{n-1} + \dots + (-1)^n e_n$ .
- Show  $L$  splitting field of  $f(x)$  over  $K$  and  $[L : K] \leq n!$ .
- Show  $[L : K] \geq n!$ .

□

**Remark.** Every finite group  $G$  is subgroup of  $S_n$  for some  $n$ , so there is always Galois extension with Galois group  $G$ : let  $L = k(x_1, \dots, x_n)$ , let  $G \subseteq S_n$  act on  $L$  as above, then  $\text{Gal}(L/L^G) = G$ .

**Definition.** For  $f(x) \in K[x]$ , **Galois group** of  $f(x)$ ,  $G_f$ , is Galois group of splitting field of  $f(x)$  over  $K$  (provided this extension is separable). If  $\deg(f) = n$ ,  $G_f$  acts by permuting roots  $\theta_1, \dots, \theta_n$  of  $f$ , so is subgroup of  $S_n$ . There can be non-trivial relationships between roots, so  $G_f$  may be proper subgroup.

**Corollary.** Any symmetric polynomial in  $k[x_1, \dots, x_n]$  can be expressed as polynomial in elementary symmetric polynomials, i.e.

$$k[x_1, \dots, x_n]^{S_n} = k[e_1, \dots, e_n]$$

where LHS is set of symmetric polynomials, RHS is set of polynomials in elementary symmetric polynomials.

**Example.**

- When  $n = 2$ ,  $x_1^2 + x_2^2 = e_1^2 - 2e_2$  and  $x_1^3 + x_2^3 = e_1^3 - 3e_1e_2$ .
- When  $n = 3$ ,  $x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_2x_3^2 + x_3^2x_1 + x_3x_1^2 = e_1e_2 - 3e_3$ .

**Definition. Lexicographic ordering of monomials,  $>_{\text{lex}}$  (or  $\succ_L$ ), is**

$$x_1^{a_1} \cdots x_n^{a_n} >_{\text{lex}} x_1^{b_1} \cdots x_n^{b_n}$$

iff  $\exists 0 \leq j \leq n-1$  such that  $a_1 = b_1, \dots, a_j = b_j$  and  $a_{j+1} > b_{j+1}$ .

**Example.**  $x_1^2x_2^3x_3 >_{\text{lex}} x_1^2x_2^2x_3^4$ .

**Definition. Leading term** of  $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$  is largest monomial  $cx_1^{a_1} \cdots x_n^{a_n}$  with  $c \neq 0$ ,  $a_i \neq 0$  for some  $i$ , appearing in  $f$  with respect to lexicographic ordering.

**Note.** If  $f$  is symmetric, then  $a_1 \geq \dots \geq a_n$ .

**Algorithm.** Given  $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]^{S_n}$ , express  $f$  as polynomial in elementary symmetric polynomials:

1. Find leading term  $cx_1^{a_1} \cdots x_n^{a_n}$  of  $f$ , compute

$$f_1 = f - ce_1^{a_1-a_2} \cdots e_{n-1}^{a_{n-1}-a_n} e_n^{a_n}$$

Note leading term of  $ce_1^{a_1-a_2} \cdots e_{n-1}^{a_{n-1}-a_n} e_n^{a_n}$  is also  $cx_1^{a_1} \cdots x_n^{a_n}$  so leading term of  $f_1$  is strictly smaller than leading term of  $f$ . Also,  $f_1$  is symmetric.

2. If  $f_1 \neq 0$ , apply step 1 to get  $f_2, f_3, \dots$ . Since leading term of  $f_1, f_2, \dots$  is strictly decreasing, eventually  $f_i = 0$ .

**Example.** Express  $f(x_1, x_2) = x_1^3 + x_2^3$  in elementary symmetric polynomials:

- Leading term of  $f$  is  $x_1^3 = x_1^3x_2^0$ , so

$$f_1 = f - e_1^{3-0}e_2^0 = -3x_1^2x_2 - 3x_1x_2^2$$

- Leading term of  $f_1$  is  $-3x_1^2x_2$ , so

$$f_2 = f_1 - (-3)e_1^{2-1}e_2^1 = -3x_1^2x_2 - 3x_1x_2^2 + 3(x_1 + x_2)x_1x_2 = 0$$

- So  $f_1 = f_2 + (-3)e_1^{2-1}e_2^1 = -3e_1e_2$  and  $f = e_1^3 + f_1 = e_1^3 - 3e_1e_2$ .

**Example.**

- Let  $\theta_1 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3)$ ,  $\theta_2 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3)$ , where  $\omega = \zeta_3$ .
- Let  $\sigma = (1 \ 2 \ 3) \in S_3$ , then  $\sigma(\theta_1) = \omega^2 \theta_1$ ,  $\sigma(\theta_2) = \omega \theta_2$ , hence

$$\sigma(\theta_1^3 + \theta_2^3) = \omega^6 \theta_1^3 + \omega^3 \theta_2^3 = \theta_1^3 + \theta_2^3$$

- Let  $\tau = (2 \ 3) \in S_3$ , then  $\tau(\theta_1) = \theta_2$ ,  $\tau(\theta_2) = \theta_1$  so  $\tau(\theta_1^3 + \theta_2^3) = \theta_1^3 + \theta_2^3$ .
- Since  $S_3 = \langle \sigma, \tau \rangle$ ,  $f(x_1, x_2, x_3) = 27(\theta_1^3 + \theta_2^3) \in \mathbb{Q}[x_1, x_2, x_3]^{S_3}$ . Applying the algorithm:

- $f_1 = f - 2e_1^3 = 9(x_1^2x_2 + \dots)$ .
- $f_2 = f_1 - (-9)e_1e_2 = 27x_1x_2x_3$ .
- $f_3 = f_2 - 27e_3 = 0$ .

- So  $f = 2e_1^3 - 9e_1e_2 + 27e_3$ .
- By a similar process,  $9\theta_1\theta_2 = e_1^2 - 3e_2$ .

## 8.2. Galois theory for cubic polynomials

**Example** (Solving quadratic). Let  $\text{char}(k) \neq 2$ . General quadratic polynomial can be written as

$$f(x) = x^2 - e_1x + e_2 = (x - x_1)(x - x_2) \in K[x]$$

where  $e_1 = x_1 + x_2, e_2 = x_1x_2 \in K = k(e_1, e_2)$ . Let  $L = k(x_1, x_2) = K(x_1)$ , then  $L/K$  is Galois and  $\text{Gal}(L/K) = \{\text{id}, \sigma\} \cong S_2 \cong \mathbb{Z}/2$  where  $\sigma(x_1) = x_2, \sigma(x_2) = x_1$ . Since  $L/K$  cyclic and  $\zeta_2 = -1 \in K$ , by [Theorem 6.2.4](#), Lagrange resolvent of  $x_1$  is

$$\theta = \theta_{x_1} = x_1 + \zeta_2^{-1}\sigma(x_1) = x_1 - x_2$$

So  $\sigma(\theta) = -\theta$  and  $\theta^2 = e_1^2 - 4e_2$ .  $\Delta = \theta^2$  is **discriminant** of  $f(x)$ . So we have  $x_1 = (e_1 + \sqrt{\Delta})/2, x_2 = (e_1 - \sqrt{\Delta})/2$ . If  $f(x)$  is irreducible, it has distinct roots, and so Galois group  $G_f \cong \mathbb{Z}/2$ .

**Example** (Solving cubic).

- Let  $\text{char}(k) \neq 2, 3$ , let

$$f(x) = x^3 - e_1x^2 + e_2x - e_3 = (x - x_1)(x - x_2)(x - x_3) \in K[x]$$

where  $e_1 = x_1 + x_2 + x_3, e_2 = x_1x_2 + x_1x_3 + x_2x_3, e_3 = x_1x_2x_3 \in K = k(e_1, e_2, e_3) \subset L = K(x_1, x_2, x_3)$ .

- By [Theorem 8.1.3](#),  $\text{Gal}(L/K) = S_3$  with normal subgroup  $A_3 \cong \mathbb{Z}/3$ . We have tower  $K \subset M = L^{A_3} \subset L$ . So  $\text{Gal}(L/M) \cong A_3 \cong \mathbb{Z}/2, \text{Gal}(M/K) \cong S_3/A_3 \cong \mathbb{Z}/2$ .
- Assume  $k$  contains primitive 3rd root of unity  $\omega$ , so  $\omega^2$  is also primitive 3rd root of unity. Define

$$\theta_1 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3), \quad t_1 = \theta_1^3,$$

$$\theta_2 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3), \quad t_2 = \theta_2^3$$

then  $t_1, t_2 \in M$  and  $L = M(\theta_1) = M(\theta_2)$ . By [Example 8.1.14](#),  $27(\theta_1^3 + \theta_2^3) = 2e_1^3 - 9e_1e_2 + 27e_3, 9\theta_1\theta_2 = e_1^2 - 3e_2$ , so  $t_1, t_2$  are roots of **quadratic resolvent** of  $f(x)$ :

$$(t - t_1)(t - t_2) = t^2 - \left( \frac{2e_1^3 - 9e_1e_2 + 27e_3}{27} \right)t + \left( \frac{e_1^2 - 3e_2}{9} \right)^3$$

- To find roots  $x_1, x_2, x_3$  of  $f$ :
  - Solve quadratic resolvent to find  $t_1, t_2$ .
  - Choose  $\theta_1 = \sqrt[3]{t_1}$ , find  $\theta_2$  from  $9\theta_1\theta_2 = e_1^2 - 3e_2$ .
  - Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 = e_1 \\ x_1 + \omega x_2 + \omega^2 x_3 = 3\theta_1 \\ x_1 + \omega^2 x_2 + \omega x_3 = 3\theta_2 \end{cases} \implies \begin{cases} x_1 = e_1/3 + \theta_1 + \theta_2 \\ x_2 = e_1/3 + \omega^2 \theta_1 + \omega \theta_2 \\ x_3 = e_1/3 + \omega \theta_1 + \omega^2 \theta_2 \end{cases}$$

**Remark.** To solve general cubic  $f(x) = x^3 + ax^2 + bx + c$ , first perform shift:

$$f(x - a/3) = x^3 + px + q$$

then quadratic resolvent is (*memorise*)

$$t^2 + qt - \frac{p^3}{27}$$

with roots  $t_1 = \theta_1^3, t_2 = \theta_2^3$ , choose  $\theta_1, \theta_2$  such that  $\theta_1 \theta_2 = -\frac{p}{3}$ , then roots of  $f(x - a/3)$  are  $x_1 = \theta_1 + \theta_2, x_2 = \omega^2 \theta_1 + \omega \theta_2, \omega \theta_1 + \omega^2 \theta_2$ .

**Example** (Galois groups of cubic polynomials). Let  $\text{char}(K) \neq 2, 3$ ,  $f(x) = x^3 + ax^2 + bx + c \in K[x]$ , let  $L$  be splitting field for  $f(x)$  over  $K$ , then  $G_f = \text{Gal}(L/K)$ . Let  $\alpha_1, \alpha_2, \alpha_3$  be roots of  $f(x)$  in  $L$ .

- If  $\alpha_1, \alpha_2, \alpha_3 \in K$ , then  $L = K, G_f = \{\text{id}\}$ .
- If  $f(x) = (x - \alpha_j)g(x)$  where  $\alpha_j \in K, g(x) \in K[x]$  irreducible quadratic, then  $[L : K] = 2, G_f \cong \mathbb{Z}/2$ .
- If  $f(x)$  irreducible in  $K[x]$ , then  $K \subset K(\alpha_1) \subseteq K(\alpha_1, \alpha_2, \alpha_3) = L$ , then either  $[L : K(\alpha_1)] = 1$ , so  $[L : K] = 3$  and  $G_f \cong A_3 \cong \mathbb{Z}/3$ , or  $[L : K(\alpha_1)] = 2$ , so  $[L : K] = 6$  and  $G_f \cong S_3$ .

**Definition. Discriminant** of  $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$  is  $\Delta = \delta^2$  where

$$\delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$$

Note  $\Delta \neq 0$  if  $f$  has distinct roots.

**Note.** If  $G_f \cong A_3$ , then  $G_f = \langle \tau \rangle$  where  $\tau : \alpha_1 \mapsto \alpha_2, \alpha_2 \mapsto \alpha_3, \alpha_3 \mapsto \alpha_1$ , then  $\tau(\delta) = \delta$  so  $\delta \in L^{G_f} = K$  and  $\Delta \in K^{\times 2}$ . But if  $G_f \cong S_3$ , then if  $\tau \in A_3, \tau(\delta) = \delta$  and if  $\tau \in S_3 - A_3$ , then  $\tau(\delta) = -\delta$  so  $\delta \notin K$  but  $\Delta \in K$ .

**Theorem.** Let  $f(x) \in K[x]$  irreducible,  $\deg(f) = 3$ . Then

- $G_f \cong A_3 \iff \Delta \in K^{\times 2}$ ,
- $G_f \cong S_3 \iff \Delta \in K^\times - K^{\times 2}$ .

**Theorem.** Let  $f(x) = x^3 + ax^2 + bx + c \in K[x]$ , then

$$\Delta = 18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^2$$

For reduced cubic  $f(x) = x^3 + px + q$ , (*memorise*)

$$\Delta = -4p^3 - 27q^2$$

**Note.** The reduced form of  $f(x)$  has same discriminant as  $f(x)$ .

### 8.3. Galois theory for quartic polynomials

**Example.** Let  $\text{char}(k) \neq 2, 3$ ,  $K = k(e_1, e_2, e_3, e_4) \subseteq L = k(x_1, x_2, x_3, x_4)$ , so  $L$  is splitting field over  $K$  of  $f(x) = x^4 - e_1x^3 + e_2x^2 - e_3x + x_4$  and  $\text{Gal}(L/K) \cong S_4$ .

**Remark.**  $S_4$  can be visualised as symmetries of regular tetrahedron with vertices labelled  $\{1, 2, 3, 4\}$ . Consider three pairs of opposite edges

$$P_1 = \{(1, 2), (3, 4)\}, \quad P_2 = \{(1, 3), (2, 4)\}, \quad P_3 = \{(1, 4), (2, 3)\}$$

Any permutation in  $S_4$  of the four vertices permutes  $P_1, P_2, P_3$ , which gives map  $\pi : S_4 \rightarrow S_3$ .

- $\pi$  is surjective group homomorphism.
- $\pi$  has kernel  $\ker(\pi) = \{\text{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ .
- $A_4 \subset S_4$  is subgroup of even permutations (orientation-preserving symmetries). Restriction of  $\pi$  to  $A_4$  gives another surjective homomorphism  $A_4 \rightarrow A_3$  (and  $\pi^{-1}(A_3) = A_4$ ) also with kernel  $V_4$ .
- $V_4$  is kernel so is normal subgroup of  $S_4$  and of  $A_4$ . Note that  $V_4$  is only subgroup of  $A_4$  isomorphic to  $\mathbb{Z}/2 \times \mathbb{Z}/2$ , but there are four subgroups of  $S_4$ , isomorphic to  $\mathbb{Z}/2 \times \mathbb{Z}/2$ , with  $V_4$  the only normal one.
- This gives increasing sequence of subgroups in  $S_4$

$$\{\text{id}\} \subset \mathbb{Z}/2 \subset V_4 \subset A_4 \subset S_4$$

and  $V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ ,  $A_4/V_4 \cong A_3 \cong \mathbb{Z}/3$ ,  $S_4/A_4 \cong \mathbb{Z}/2$ .

- Each  $G_i$  in this sequence is normal subgroup of  $G_{i+1}$  and  $G_{i+1}/G_i$  is cyclic, meaning that  $S_4$  is **solvable (soluble) group**.
- We have tower

$$K = L^{S_4} \subset L^{V_4} \subset L = L^{\{e\}}$$

By fundamental theorem,  $\text{Gal}(L/L^{V_4}) = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ , so  $L/L^{V_4}$  appears as bi-quadratic extension.

- $V_4$  is normal in  $S_4$  so by fundamental theorem,  $\text{Gal}(L^{V_4}/K) \cong S_4/V_4 \cong S_3$  by first isomorphism theorem. Hence  $L^{V_4}$  appears as splitting field of a cubic polynomial over  $K$ .

**Example** (Solving quartic equations). Define

$$\begin{aligned} \theta_1 &= \frac{1}{2}(x_1 + x_2 - x_3 - x_4), \\ \theta_2 &= \frac{1}{2}(x_1 - x_2 + x_3 - x_4), \\ \theta_3 &= \frac{1}{2}(x_1 - x_2 - x_3 + x_4) \end{aligned}$$

Then  $\forall j \in [3], \forall \sigma \in V_4$ ,  $\sigma(\theta_j) = \pm\theta_j$ . The  $\theta_j$  arise from Lagrange resolvents for the three quadratic subextensions of  $L^{V_4}$  in  $L$ . They behave like  $\sqrt{2}, \sqrt{3}, \sqrt{6}$  in  $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Each  $t_i = \theta_i^2$  is fixed by  $V_4$  and are permuted by  $S_4/V_4 \cong S_3$ . They are roots of **cubic resolvent** of  $f(x)$ :



$$(t - t_1)(t - t_2)(t - t_3) = t^3 + s_1 t^2 + s_2 t + s_3$$

which has coefficients in  $(L^{V_4})^{S_3} = L^{S_4} = K$ . To find roots  $x_1, x_2, x_3, x_4$  of  $f(x)$ :

- Solve cubic resolvent to find  $t_1, t_2, t_3$ .
- Set  $\theta_j = \pm\sqrt{t_j}$  where signs are chosen so that  $\theta_1\theta_2\theta_3 = (e_1^3 - 4e_1e_2 + 8e_3)/8$ .
- Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = e_1 \\ x_1 + x_2 - x_3 + x_4 = 2\theta_1 \\ x_1 - x_2 + x_3 - x_4 = 2\theta_2 \\ x_1 - x_2 - x_3 + x_4 = 2\theta_3 \end{cases} \implies \begin{cases} x_1 = e_1/4 + (\theta_1 + \theta_2 + \theta_3)/2 \\ x_2 = e_1/4 + (\theta_1 - \theta_2 - \theta_3)/4 \\ x_3 = e_1/4 + (-\theta_1 + \theta_2 - \theta_3)/2 \\ x_4 = e_1/4 + (-\theta_1 - \theta_2 + \theta_3)/2 \end{cases}$$

**Remark.** In practice, perform shift to kill  $x^3$  coefficient to obtain **reduced quartic**:

$$f(x - a/4) = x^4 + px^2 + qx + r$$

- Cubic resolvent is (*memorise*)

$$t^3 + 2pt^2 + (p^2 - 4r)t - q^2$$

- Choose  $\theta_1, \theta_2, \theta_3$  such that (*memorise*)

$$\theta_1\theta_2\theta_3 = -q$$

- Roots of  $f(x - a/4)$  are (*memorise*)

$$x_1 = \frac{1}{2}(\theta_1 + \theta_2 + \theta_3),$$

$$x_2 = \frac{1}{2}(\theta_1 - \theta_2 - \theta_3),$$

$$x_3 = \frac{1}{2}(-\theta_1 + \theta_3 - \theta_3),$$

$$x_4 = \frac{1}{2}(-\theta_1 - \theta_2 + \theta_3)$$

- Recover roots of  $f(x)$  by subtracting  $a/4$ .

**Example.** Find all complex roots of  $f(x) = x^4 + 6x^3 + 18x^2 + 30x + 25$ .

- Eliminate  $x^3$  term:

$$f(x - 6/4) = x^4 + \frac{9}{2}x^2 + 3x + \frac{85}{16}$$

- $p = 9/2, q = 3, r = 85/16$ , so cubic resolvent is

$$t^3 + 2pt^2 + (p^2 - 4r)t - q^2 = t^3 + 9t^2 - t - 9 = (t - 1)(t + 1)(t + 9)$$

So roots are  $t_1 = 1, t_2 = -1, t_3 = -9$ . Set  $\theta_1 = \sqrt{t_1} = 1, \theta_2 = \sqrt{t_2} = i, \theta_3 = \pm\sqrt{t_3} = \pm 3i$  so that  $\theta_1\theta_2\theta_3 = -q = -3$ , i.e.  $\theta_3 = 3i$ .

- So roots of  $f(x - 3/2)$  are

$$\begin{aligned}
x_1 &= \frac{1}{2}(\theta_1 + \theta_2 + \theta_3) = \frac{1}{2}(1 + 4i), \\
x_2 &= \frac{1}{2}(\theta_1 - \theta_2 - \theta_3) = \frac{1}{2}(1 - 4i), \\
x_3 &= \frac{1}{2}(-\theta_1 + \theta_3 - \theta_3) = \frac{1}{2}(-1 - 2i), \\
x_4 &= \frac{1}{2}(-\theta_1 - \theta_2 + \theta_3) = \frac{1}{2}(-1 + 2i)
\end{aligned}$$

- So roots of  $f(x)$  are  $-1 \pm 2i, -2 \pm i$ .

**Example** (Galois groups of quartic polynomials).

- Let  $\text{char}(K) \neq 2, 3$ ,  $f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$ . Galois group is  $G_f = \text{Gal}(L/K)$  where  $L$  is splitting field for  $f(x)$  over  $K$ , and  $G_f$  is subgroup of  $S_4$ .
- Assume that  $f(x)$  irreducible in  $K[x]$ . It can be shown there are five possible isomorphism classes of Galois groups:  $S_4, A_4, V_4, \mathbb{Z}/4$  or  $D_4$ .
- Let  $R(t) \in K[t]$  be cubic resolvent of  $f(x)$  with roots  $t_1 = \theta_1^2, t_2 = \theta_2^2, t_3 = \theta_3^2$ . Let  $M$  be splitting field of  $R(t)$  over  $K$ , so

$$K \subset K(t_1, t_2, t_3) \subset M \subset L = M(\theta_1, \theta_2, \theta_3)$$

**Theorem.** Let  $f(x) \in K[x]$  irreducible and have irreducible cubic resolvent  $R(t) \in K[t]$  with roots  $t_1 = \theta_1^2, t_2 = \theta_2^2, t_3 = \theta_3^2$ . Let  $L$  be splitting field of  $f(x)$  over  $K$  (so  $G_f = \text{Gal}(L/K)$ ) and let  $M$  be splitting field of  $R(t)$  over  $K$  (so  $G_R = \text{Gal}(M/K)$ ).

- If  $\Delta_R \in K^{\times^2}$  (i.e.  $G_R \cong A_3$  and  $[M : K] = 3$ ), then  $G_f \cong A_4$ .
- If  $\Delta_R \in K^\times - K^{\times^2}$  (i.e.  $G_R \cong S_3$  and  $[M : K] = 6$ ), then  $G_f \cong S_4$ .

*Proof.*

- Sufficient to prove  $[L : M] = 4$  since then  $[L : K] = 12$  or  $24$  by Tower Law.
- Show  $M$  does not contain  $\theta_1, \theta_2$  or  $\theta_3$ .
  - Suppose it does, so WLOG  $\theta_1 \in M$ .  $\text{Gal}(M/K) \cong A_3$  or  $S_3$ , so must be order 3 element  $\sigma \in \text{Gal}(M/K)$ .  $\sigma(\theta_1)$  and  $\sigma^2(\theta_1)$  are the other two roots  $\theta_2$  and  $\theta_3$  since  $R(t)$  is irreducible and  $\theta_1, \theta_2, \theta_3 \in M$ . But this implies  $M = L$  so  $[L : K] = 3$  or  $6$ , but  $4 \mid [L : K]$  since  $L$  contains roots of irreducible quartic.
- $M(\theta_1)/M$  is degree 2. Assume  $\theta_2 \in M(\theta_1)$ .  $\text{Gal}(M(\theta_1)/M) = \{\text{id}, \tau\}$  for some  $\tau : \theta_1 \mapsto -\theta_1$ . Also  $\theta_2^2 \in M$  so  $\tau(\theta_2) = \pm\theta_2$ .
  - If  $\tau(\theta_2) = \theta_2$ , then  $\theta_2 \in M$ : contradiction.
  - If  $\tau(\theta_2) = -\theta_2$ , then  $\tau(\theta_1\theta_2) = (-\theta_1)(-\theta_2) = \theta_1\theta_2$  hence  $\theta_1\theta_2 \in M$ . But  $\theta_1\theta_2\theta_3 \in K$  and  $\theta_1\theta_2 \neq 0$  since  $R(t)$  irreducible. But then  $\theta_3 \in M$ : contradiction.
- Hence  $[M(\theta_1, \theta_2) : M] \geq 4$ , and  $\theta_1\theta_2\theta_3 \in M$  so  $L = M(\theta_1, \theta_2)$  and  $[L : M] = 4$ .

□