Contents

1. Background 2
1.1. Notes on second quantisation 2
2. The transverse-field Ising model 3
3. Quantum Ising model 4
3.1. Brilloin-Wigner perturbation theory 4

Themes:
- quantum matter
  - topological order (TO)
- quantum computing
  - quantum error correction (QEC)
  - topological quantum computing

Methods:
- mostly operator algebra (Pauli operators, Fermion operators)
- **some** field theory (second quantisation, path integrals)
- just a **little** band theory

1. Background

1.1. Notes on second quantisation

We can define an action of S_n on an n qudit state (a representation of the n-qudit Hilbert space by S_n) linearly by

$$\sigma|i_1...i_n\rangle = |i_{\sigma(1)}...i_{\sigma(n)}\rangle.$$

Definition 1.1 A **boson** is a quantum state $|\psi\rangle$ that is invariant under the action of S_n (symmetric under permutations), i.e.

$$\forall \sigma \in S_n, \quad \sigma|\psi\rangle = |\psi\rangle.$$

Definition 1.2 A **fermion** is a quantum state $|\varphi\rangle$ that is anti-symmetric under permutations, i.e. invariant under even permutations and is negated under odd permutations:

$$\begin{aligned} \forall \sigma \in A_n, &\quad \sigma|\varphi\rangle = |\varphi\rangle \\ \forall \tau \in S_n \setminus A_n, &\quad \tau|\varphi\rangle = -|\varphi\rangle \end{aligned}$$

Definition 1.3 The symmetrisation of a state $|\chi\rangle$ is

$$S_\pm|\chi\rangle = \frac{1}{|S_n|} \sum_{\sigma \in S_n} (\pm 1)^{\mathrm{sgn}(\sigma)} \sigma|\chi\rangle$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ. S_+ results in a boson, S_- results in a fermion.

Notation 1.4 **Second quantisation** is a compact way of expressing bosons and fermions:

$$|n_1, ..., n_d\rangle_\pm = S_\pm|i_1...i_n\rangle$$

where n_j denotes the number of single qudit states that are in state $|j\rangle$, in any basis state of $|n_1, ..., n_d\rangle_\pm$. The number of qudits is $n = \sum_{j=1}^d n_j$.

The states $|n_1, ..., n_d\rangle_\pm$ are called **occupation (number) states**.

Proposition 1.5 Occupation states satisfy:
1. $\langle n_1, ..., n_d | m_1, ..., m_d \rangle = \delta_{n_1 m_1} \cdots \delta_{n_d m_d}$.
2. $\sum_{n_1+\cdots+n_d=n} |n_1, ..., n_d\rangle\langle n_1, ..., n_d| = I$.

Definition 1.6 For a fixed number of qudits n, the space of all occupated number states is called **Fock space**.

Define the creation and annihilation operators

$$\hat{a}_j^\dagger |..., n_j, ...\rangle_\pm = \sqrt{n_j + 1} |..., n_j + 1, ...\rangle_\pm$$
$$\hat{a}_j |..., n_j + 1, ...\rangle_\pm = \sqrt{n_j + 1} |..., n_j, ...\rangle_\pm$$

This gives

$$\left[\hat{a}_i, \hat{a}_j\right] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \quad \left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij} \quad \text{for bosons}$$
$$\left\{\hat{a}_i, \hat{a}_j\right\} = \left\{\hat{a}_i^\dagger, \hat{a}_j^\dagger\right\} = 0, \quad \left\{\hat{a}_i, \hat{a}_j^\dagger\right\} = \delta_{ij} \quad \text{for bosons}$$

A corollary of $\left\{\hat{a}_j^\dagger, \hat{a}_j^\dagger\right\} = 2\hat{a}_j^\dagger\hat{a}_j^\dagger = 0$ is the Pauli principle that no single particle state can be occupied by more than one fermion.

Definition 1.7 The **occupation number operator** is $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$. Note that $\hat{n}_j |..., n_j, ...\rangle = n_j |..., n_j, ...\rangle$.

Example 1.8 The total particle number operator is

$$\hat{n} = \sum_j \hat{n}_j$$

For a single-qudit operator $\hat{T} = \sum_{i,j} t_{ij} |i\rangle\langle j|$, we have

$$\hat{T} = \sum_{ij} t_{ij} \hat{a}_i^\dagger \hat{a}_j$$

(since $|i\rangle\langle j||k\rangle = \hat{a}_i^\dagger \hat{a}_j |k\rangle$)

Noting that $|\varphi\rangle = \sum_i \langle i|\varphi\rangle |i\rangle$, we define

$$\hat{a}_\varphi^\dagger = \sum_i \langle i|\lambda\rangle \hat{a}_i^\dagger$$

(note this is analogous to a basis transformation)

2. The transverse-field Ising model

Notation 2.1 When working with N qubits (an N-site system), write X_j, Y_j, Z_j for the Pauli X, Y, Z on site j, e.g.

$$X_j = \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes X \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I},$$

where X is in the j-th position.

3. Quantum Ising model

Definition 3.1 The **classical Ising model** describes the energy of a system $\{z_j : j \in [N]\}$ as

$$E(\{z_j : j \in [N]\}) = -J \sum$$

TODO: familiarise with classical Ising model

Quantum ising model: $H = -J \sum_{i,j,nn} Z_i Z_j - h \sum_j Z_j$, $J > 0$. nn denotes nearest neighbours. We have $H|\{z_j\}\rangle = E(\{z_j\})|\{z_j\}\rangle$, $Z_i|\{z_j\}\rangle = z_i|\{z_j\}\rangle$ where $z_i \in \{-1, 1\}$.

Transverse field Ising model: $H = -J \sum_{i,j:nn} Z_i Z_j - h \sum_j X_j$, $J > 0$ (feromagn), $h > 0$. It has a $\mathbb{Z}_2$ symmetry: $P = \prod_j X_j$, $HP = PH$, $P^2 = I$.

$P|\{z_j\}\rangle = |\{-z_j\}\rangle$ (spin flip).

If $J = 0$: ground state is $|\mathrm{GS}\rangle = \otimes_{j=1}^{N} |+\rangle_j =: |\underline{X}\rangle$. Denote $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$.

If $h = 0$: ground states are $|\Uparrow\rangle = \bigotimes_{j=1}^{N} |0\rangle_j$, $|\Downarrow\rangle = \bigotimes_{j=1}^{N} |1\rangle_j$, or any linear combination of these.

We have $P|\underline{X}\rangle = \underline{X}$, and $\langle \underline{X} | Z_j | \underline{X} \rangle = 0$, since $Z_j|+\rangle_j = |-\rangle_j$. So order param (z_j) is 0, can think of as paramagnet.

Also, $P|\Uparrow\rangle = |\Downarrow\rangle$, and $\langle \Uparrow | Z_j | \Uparrow \rangle \neq 0$, so order param (z_j) is not 0, so can think of as feromagnet.

Since $[H, P] = 0$, so there exists a basis $|\psi_{E,P}\rangle$ such that $H|\psi_{E,P}\rangle = E_P|\psi_{E,P}\rangle$, and $P|\psi_{E,P}\rangle = p|\psi_{E,P}\rangle$, where $p \in \{-1, 1\}$.

The ground states are $|\mathrm{GS}_\pm\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle + |\Downarrow\rangle)$. We have $P|\mathrm{GS}_\pm\rangle = \pm|\mathrm{GS}_\pm\rangle$, and $\langle \mathrm{GS}_\pm | Z_j | \mathrm{GS}_\pm \rangle = 0$.

Now consider $H = H_0 + \delta H$, where $H_0 = -J \sum_{i,j:nn} Z_i Z_j$, and $\delta H = -h \sum_j X_j$, where $|h| \ll J$. δH is the perturbation, with coupling h.

3.1. Brilloin-Wigner perturbation theory

Write the eigenstates of H_0 as $H_0|n\rangle = E_n|n\rangle$, and $H|\tilde{n}\rangle = E_{\tilde{n}}|\tilde{n}\rangle$. Write $P = \sum_{n \in S} |n\rangle\langle n|$ and $Q = P^\perp = I - P = \sum_{n \in S^\perp} |n\rangle\langle n|$. Denote perturbed ground state energies by $E_{\widetilde{m}}$. Let $|\widetilde{m}^{(n)}\rangle$ denote unnormalised perturbed ground-space eigenstates, i.e. $H|\widetilde{m}^{(n)}\rangle = E_{\widetilde{m}}|\widetilde{m}^{(n)}\rangle$, and $|\psi_{\widetilde{m}}\rangle := P|\widetilde{m}^{(n)}\rangle$ is normalised.

We have $(H_0 + \delta H)|\widetilde{m}^{(n)}\rangle = E_{\widetilde{m}}|\widetilde{m}^{(n)}\rangle$, so $(E_{\widetilde{m}} - H_0)|\widetilde{m}^{(n)}\rangle = \delta H|\widetilde{m}^{(n)}\rangle$. So

$$(E_{\widetilde{m}} - E_n)\langle n | \widetilde{m}^{(n)} \rangle = \langle n | \delta H | \widetilde{m}^{(n)} \rangle$$

. If $|n\rangle \in S^\perp$, then $|n\rangle\langle n|\widetilde{m}^{(n)}\rangle = \frac{|n\rangle\langle n|}{E_{\widetilde{m}}-E_n}\delta H|\widetilde{m}^{(n)}\rangle$ and so $\sum_{|n\rangle\in S^\perp}|n\rangle\langle n|\widetilde{m}^{(n)}\rangle = \sum_{|n\rangle\in S^\perp}\frac{|n\rangle\langle n|}{E_{\widetilde{m}}-E_n}\delta H|\widetilde{m}^{(n)}\rangle$. We rewrite this as $Q|\widetilde{m}^{(n)}\rangle = G\delta H|\widetilde{m}^{(n)}\rangle$. So $|\widetilde{m}^{(n)}\rangle = |\psi_{\widetilde{m}}\rangle + G\delta H|\widetilde{m}^{(n)}\rangle$, and so we have

$$|\widetilde{m}^{(n)}\rangle = (I - G\delta H)^{-1}|\psi_{\widetilde{m}}\rangle$$

Now for $|n\rangle \in S$, we have $(E_{\widetilde{m}} - E_0)\langle n|\widetilde{m}^{(n)}\rangle = \langle n|\underbrace{\delta H(I-G\delta H)^{-1}}_{=:A^{(\widetilde{m})}}|\psi_{\widetilde{m}}\rangle = \sum_{n'\in S}\underbrace{\langle n|A^{(\widetilde{m})}|n'\rangle}_{H^{\text{eff}}_{nn'}}\underbrace{\langle n'|\widetilde{m}^{(n)}\rangle}_{\delta_{n'}}$. $H^{\text{eff}}_{nn'}$) is a $d_G \times d_G$ "effective" Hamiltonian.

Now

$$\begin{aligned}\delta E_\pm &= \langle \mathrm{GS}_\pm|\delta H(\mathbb{I}-G\delta H)^{-1}|\mathrm{GS}_\pm\rangle \\ &= \underbrace{\langle \mathrm{GS}_\pm|\delta H|\mathrm{GS}_\pm\rangle}_{=0} + \langle \mathrm{GS}_\pm|\delta HG\delta H|\mathrm{GS}_\pm\rangle + \langle \mathrm{GS}_\pm|\delta HG\delta G\delta H|\mathrm{GS}_\pm\rangle + \cdots\end{aligned}$$

and $G\delta H|\mathrm{GS}_\pm\rangle = -Gh\sum_j X_j|\mathrm{GS}_\pm\rangle \approx \frac{h}{J}\sum_j X_j|\mathrm{GS}_\pm\rangle$. So $\langle \mathrm{GS}_\pm|\delta HG\delta H|\mathrm{GS}_\pm\rangle \approx -N\frac{h^2}{J}\langle \mathrm{GS}_\pm|\mathrm{GS}_\pm\rangle$. Note this is independent of p.

Also,

$$\begin{aligned}\langle \mathrm{GS}_\pm|\delta H(G\delta H)^{m-1}|\mathrm{GS}_\pm\rangle &\to \frac{\delta\varepsilon^{(m)}}{2}\langle \mathrm{GS}_\pm|\prod_{j=1}^N X_j|\mathrm{GS}_\pm\rangle, \quad m \geq N \\ &= \pm\frac{\delta\varepsilon^{(m)}}{2}\langle \mathrm{GS}_\pm|\mathrm{GS}_\pm\rangle\end{aligned}$$