

Complex Analysis II Course Notes

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1 The complex plane and Riemann sphere

1.1 Complex numbers

Definition 1.1.1. A **complex number** z is a number $z = x + iy$ where $x, y \in \mathbb{R}^2$ and i is the **imaginary unit**. The set of all complex numbers is written as \mathbb{C} .

Definition 1.1.2. For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, addition, subtraction and multiplication of complex numbers is defined as

$$\begin{aligned} z_1 \pm z_2 &:= (x_1 \pm x_2) + i(y_1 \pm y_2) \\ z_1 z_2 &:= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Definition 1.1.3. For a complex number $z = x + iy$, the **real part** of z , $\text{Re}(z)$, is x and the **imaginary part**, $\text{Im}(z)$, is y .

Definition 1.1.4. For a complex number $z = x + iy$, the **complex conjugate** of z , \bar{z} is defined as $\bar{z} = x - iy$.

Definition 1.1.5. Division of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$ is given by

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

This gives a multiplicative inverse for every $z = x + iy \neq 0$:

$$z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

Definition 1.1.6. The **modulus** or **absolute value** of a complex number $z = x + iy$, $|z|$, is defined as $|z| = \sqrt{x^2 + y^2}$.

Lemma 1.1.7.

1. $\forall z_1, z_2 \in \mathbb{C}, z_1 z_2 \iff z_1 = 0 \text{ or } z_2 = 0$.
2. $\forall z \in \mathbb{C}, |z| = \sqrt{z \bar{z}}$.
3. $\text{Re}(z) = \frac{z + \bar{z}}{2}$ and $\text{Im}(z) = \frac{z - \bar{z}}{2i}$.
4. $z^{-1} = \frac{\bar{z}}{|z|^2}$.

Proof. Easy. □

Definition 1.1.8. For a complex number $z = x + iy$ plotted on an Argand diagram (where z is at the point (x, y)), the **argument** of z , $\arg(z)$, is the anticlockwise angle θ from the real axis to z .

Definition 1.1.9. For a complex number z with $|z| = r$ and $\arg(z) = \theta$, z can be written in polar coordinates:

$$z = r(\cos(\theta) + i \sin(\theta)) = r e^{i\theta}$$

Definition 1.1.10. $\arg(z)$ is only defined up to multiples of 2π . The **principal value** of $\arg(z)$ is the value of $\arg(z)$ in the interval $(-\pi, \pi]$, written as $\text{Arg}(z)$.

Lemma 1.1.11.

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$.
2. $\arg(1/z) = -\arg(z) \pmod{2\pi}$.
3. $\arg(\bar{z}) = -\arg(z) \pmod{2\pi}$.

Proof. Easy. □

Lemma 1.1.12. Multiplication in \mathbb{C} can be geometrically described as a dilated rotation: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Proof.

$$\begin{aligned}
z_1 z_2 &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\
&= r_1 r_2 ((\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + i(\cos(\theta_2) \sin(\theta_1) + \sin(\theta_2) \cos(\theta_1))) \\
&= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \\
&= r_1 r_2 e^{i(\theta_1 + \theta_2)}
\end{aligned}$$
□

Remark.

- Multiplying z_1 by z_2 represents a rotation of z_1 by θ_2 anticlockwise, followed by a dilation of factor r_2 .
- Addition represents to translation.
- Complex represents reflection in the real axis.
- Taking the real part represents projection onto the real axis.
- Taking the imaginary part represents projection onto the imaginary axis.

Corollary 1.1.13.

1. $\forall z_1, z_2 \in \mathbb{C}^2, |z_1 z_2| = |z_1| |z_2|$.
2. $(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$ (de Moivre's formula).
3. $\forall z_1, z_2, |z_1 + z_2| \leq |z_1| + |z_2|$.
4. $\forall z \in \mathbb{C}, |z| \geq 0$ and $|z| = 0 \iff z = 0$.
5. $\max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\} \leq |z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$.

Proof. Easy. □

Definition 1.1.14. The **upper half** of the complex plane, \mathbb{H} , is defined as

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$$

1.2 Exponential and trigonometric functions

Definition 1.2.1. The **complex exponential function** $\exp : \mathbb{C} \rightarrow \mathbb{C}$, written as e^z , is defined as

$$e^z = \exp(z) := e^x(\cos(y) + i \sin(y))$$

Proposition 1.2.2.

1. $\forall z \in \mathbb{C}, e^z \neq 0$.
2. $\forall z_1, z_2 \in \mathbb{C}, e^{z_1+z_2} = e^{z_1} e^{z_2}$.
3. $e^z = 1 \iff \exists k \in \mathbb{Z}, z = 2\pi i k$.
4. $e^{-z} = 1 e^z$.
5. $|e^z| = e^{\operatorname{Re}(z)}$.

Proof. Easy. For 3, $\exp(z) = 1 \iff e^x \cos(y) = 1$ and $e^x \sin(y) = 0$. $e^x > 0$ so $\sin(y) = 0$ so $y = n\pi$ for some $n \in \mathbb{Z}$. So $1 = e^x \cos(n\pi) = e^x (-1)^n$ so n is even and $x = 0$. \square

Corollary 1.2.3. $\exp(2\pi i) = 1$ and $\exp(\pi i) = -1$ (Euler's formula).

Corollary 1.2.4. $\forall k \in \mathbb{Z}, \forall z \in \mathbb{C}, \exp(z + 2k\pi i) = \exp(z)$.

Definition 1.2.5. The following functions from \mathbb{C} to \mathbb{C} are defined:

$$\begin{aligned}\sin(z) &:= \frac{1}{2i} (e^{iz} - e^{-iz}) \\ \cos(z) &:= \frac{1}{2} (e^{iz} + e^{-iz}) \\ \sinh(z) &:= \frac{1}{2} (e^z - e^{-z}) = -i \sin(iz) \\ \cosh(z) &:= \frac{1}{2} (e^z + e^{-z}) = \cos(iz)\end{aligned}$$

Remark. The usual trigonometric function identities hold, e.g. $\cos(z)^2 + \sin(z)^2 = 1$.

2 Möbius Transformations

Corollary 2.0.1. Any Möbius transformation is a bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Let $T \in GL_2(\mathbb{C})$ and M_T be a Möbius transformation, then a point z is a fixed point of M_T if $M_T(z) = z$.

Lemma 2.0.2. Let $T \in GL_2(\mathbb{C})$. If $M_T : \mathbb{C} \rightarrow \mathbb{C}$ is not the identity map, then M_T has at most two fixed points in \mathbb{C} . If a Möbius transformation has three fixed points then it is the identity map.

Proof. Case 1: Suppose $M_T(\infty) = \infty$. From the definition, $M_T(z) = \frac{az+b}{cz+d}$, therefore $c = 0$. So $M_T(z) = \frac{a}{d}z + \frac{b}{d}$, with $a \neq 0, d \neq 0$ (since $\det T \neq 0$).

Such an affine linear map has at most one fixed point because:

- If $a \neq d$ then $\frac{a}{d}z + \frac{b}{d} = z \iff z = \frac{b}{d-a}$ so M_T has a unique fixed point.
- If $a = d$ then $b \neq 0$ (since we assume M_T is not the identity). So $M_T(z) = z + \frac{b}{a}$ is a translation which has no fixed points.

Case 2: Suppose $M_T(\infty) \neq \infty$. Suppose $z_0 \in \mathbb{C}$ is such that $M_T(z_0) = z_0$. We have $M_T(z_0) = z_0 \iff \frac{az_0+b}{cz_0+d} = z_0 \iff cz_0^2 + (d-a)z_0 - b = 0$. This quadratic equation has at most two roots so there are at most two fixed points of M_T . \square

Definition 2.0.3. Given four distinct points $z_0, z_1, z_2, z_3 \in \mathbb{C}$, the cross-ratio of these points denoted $(z_0, z_1; z_2, z_3)$ is defined by

$$\frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

We extend the definition to the case where one of the points is ∞ by removing all differences involving that point e.g. $(\infty, z_0; z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2}$.

Theorem 2.0.4. (Three points theorem) Let z_1, z_2, z_3 and w_1, w_2, w_3 be two sets of three ordered points in $\hat{\mathbb{C}}$. Then there exists a unique Möbius transformation f such that $f(z_i) = w_i$ for every $i \in \{1, 2, 3\}$.

Proof. Existence:

We consider the functions $F(z) = (z, w_1; w_2, w_3) = \frac{(z-w_2)(w_1-w_3)}{(z-w_3)(w_1-w_2)}$ and $G(z) = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$. These are Möbius transformations with the properties that $F(w_1) = 1$, $F(w_2) = 0$, $F(w_3) = \infty$ and similarly, $G(z_1) = 1$, $G(z_2) = 0$, $G(z_3) = \infty$. Therefore $F^{-1} \circ G$ maps each z_i to w_i .

Uniqueness:

Assume that there are two such maps, say f_1 and f_2 . Then the Möbius transformation $H = f_1^{-1} \circ f_2$ satisfies $H(z_i) = z_i$.

This shows that H has three fixed points so, by Three Point Theorem, it must be the identity. Thus $f_1 = f_2$. \square

Proposition 2.0.5. Möbius transformations preserve the cross ratio. That is, if z_0, z_1, z_2, z_3 are four distinct points in $\hat{\mathbb{C}}$ and f is a Möbius transformation, then $(f(z_0), f(z_1); f(z_2), f(z_3)) = (z_0, z_1; z_2, z_3)$.

Proof. Let $w_i = f(z_i)$ for every $i \in \{1, 2, 3\}$. Let $F(z) = (z, w_1; w_2, w_3)$ and $G(z) = (z, z_1; z_2, z_3)$. Recall $F^{-1} \circ G$ maps z_i to w_i like f does. Since there is a unique Möbius transformation with this property, we have

$$f = F^{-1} \circ G$$

and

$$F \circ f = G$$

That is, $(f(z_0), w_1; w_2, w_3) = F \circ f(z_0) = G(z_0) = (z_0, z_1; z_2, z_3)$. □

Remark. General strategy: to find Möbius transformation, find image of 3 points and use the fact that cross ratio is preserved. Plug known points into (*) and rearrange for $f(z_0)$.

2.1 The Riemann Sphere Revisited

Circles in $\hat{\mathbb{C}}$ correspond to circles in S^2 that don't pass through N (the North pole). Lines in $\hat{\mathbb{C}}$ correspond to circle in S^2 that pass through N .

Remark. Möbius transformations give all biholomorphic maps from S^2 to S^2 .

Remark. Stereographic projections are conformal.

2.2 Möbius transformations preserving the upper half plane and the unit disc

Notation: for a domain $D \subset \mathbb{C}$, let $Mob(D)$ be the set of Möbius transformations f such that $f(D) = D$.

Proposition 2.2.1. (H2H) Every Möbius transformation mapping \mathbb{H} to \mathbb{H} ($\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$) is of the form M_T with $T \in SL_2(\mathbb{R}) := \{T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \det T = 1\}$

Conversely, every such Möbius transformation maps \mathbb{H} to \mathbb{H} and hence a biholomorphism from \mathbb{H} to \mathbb{H} .

i.e. H2H: $f \in Mob(\mathbb{H}) \Leftrightarrow f = M_T$ with $T \in SL_2(\mathbb{R})$.

Remark. $T \mapsto M_T$ gives a group homomorphism $SL_2(\mathbb{R}) \rightarrow Aut(\mathbb{H})$

Proof. Any Möbius transformation $f : \mathbb{H} \rightarrow \mathbb{H}$ must map $\partial\mathbb{H}$ to $\partial\mathbb{H}$. As $\partial\mathbb{H}$ is the real line, $f : \mathbb{R} \cup \infty \rightarrow \mathbb{R} \cup \infty$. So f must map the ordered set $\{1, 0, \infty\}$ to $\{x_1, x_2, x_3\}$ for some $x_i \in \mathbb{R} \cup \infty$.

We know that the cross ratio is preserved under a Möbius transformation:

$$\begin{aligned} (f(z), x_1; x_2, x_3) &= \frac{(f(z) - x_2)(x_1 - x_3)}{(f(z) - x_3)(x_1 - x_2)} = \frac{z - 0}{1 - 0} = (z, 1; 0, \infty) \\ &\Leftrightarrow (f(z) - x_2)(x_1 - x_3) = z(f(z) - x_3)(x_1 - x_2) \\ &\Leftrightarrow f(z) = \frac{x_3(x_1 - x_2)z + x_2(x_3 - x_1)}{(x_1 - x_2)z + x_3 - x_1} \end{aligned}$$

We see that the coefficients of T are real.

If $T \in GL_2(\mathbb{R})$ and $z = x + iy$ then

$$\begin{aligned} \operatorname{Im}(M_T(z)) &= \operatorname{Im}\left(\frac{az + b}{cz + d}\right) = \operatorname{Im}\left(\frac{(az + b)(c\bar{z} + d)}{|cz + d|^2}\right) \\ &= \operatorname{Im}\left(\frac{bc\bar{z} + adz}{(cz + d)}\right) = \frac{y \det T}{|cz + d|} \end{aligned}$$

We have $z \in \mathbb{H} \Leftrightarrow y > 0$ so $M_T(z) \in H \Leftrightarrow T \in GL_2(\mathbb{R})$, $\det T > 0$. We can therefore replace T by a real matrix of determinant 1 by scaling T by a real number. \square

Proposition 2.2.2. (D2D): Every Mobius transformation from the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to \mathbb{D} is of the form $T \in SU(1, 1)$

Conversely, every such Mobius transformation maps \mathbb{D} to \mathbb{D} and hence gives a biholomorphic automorphism of \mathbb{D} .

i.e. $f \in \operatorname{Mob}(\mathbb{D}) \Leftrightarrow f = M_T$, $T \in SU(1, 1)$.

Proof. (\Rightarrow): Let $M_T : \mathbb{D} \rightarrow \mathbb{D}$ be a Mobius transformation. The Cayley map H_C maps \mathbb{H} to \mathbb{D} . We have that $f = M_C^{-1} \circ M_T \circ M_C$ is a Mobius transformation from \mathbb{H} to \mathbb{H} . By proposition 4.20, we have $f = M_S$ where $S \in SL_2(\mathbb{R})$.

Hence $C^{-1}TC = S \in SL_2(\mathbb{R})$ by Lemma 4.4.

Let $S \in M_2(\mathbb{R})$, $\det S = 1$. Then $T = CSC^{-1}$. Evaluating this shows $T \in SU(1, 1)$.

(\Leftarrow): If $T \in SU(1, 1)$, then the same calculation in reverse shows that the matrix $S = C^{-1}TC \in SL_2(\mathbb{R})$. Then $M_S : \mathbb{H} \rightarrow \mathbb{H}$ is a Mobius transformation by proposition 4.20 (H2H), and the map $M_T := M_C \circ M_S \circ M_C^{-1}$ is a Mobius transformation from \mathbb{D} to \mathbb{D} \square

Remark. $T \rightarrow M_T$ gives a group homomorphism from $SU(1, 1)$ to $\operatorname{Aut}(\mathbb{D})$.

Corollary 2.2.3. (D2D*):

1. Every Mobius transformation f from \mathbb{D} to \mathbb{D} can be written as

$$f(z) = e^{i\theta} \frac{z - z_0}{\bar{z}_0 z - 1}$$

for some angle θ and $z_0 \in \mathbb{D}$ where z_0 is the unique point in \mathbb{D} such that $f(z_0) = 0$.

2. Every Mobius transformation of the unit disc \mathbb{D} to \mathbb{D} for which $f(0) = 0$ are rotations about 0.

Proof. 1. By proposition D2D, we have

$$f(z) = \frac{az + b}{\bar{b}z + \bar{a}} = \frac{a(z + b/a)}{-\bar{a}((-\bar{b}/\bar{a})z - 1)} = -\frac{a}{\bar{a}} \frac{z - (-b/a)}{(-\bar{b}/\bar{a})z - 1}$$

So $z_0 = -\frac{b}{a}$. Since $|\frac{a}{\bar{a}}| = 1$, $-\frac{a}{\bar{a}} = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$.

$|z_0|^2 - 1 = |-\frac{b}{a}|^2 - 1 = \frac{|b|^2}{|a|^2} - 1$. Now $1 = |a|^2 - |b|^2$ so $|z_0|^2 - 1 = \frac{-1}{|a|^2} < 0$ so $|z_0|^2 < 1$ and so $|z_0| < 1$.

$$2. f(0) = 0 \Leftrightarrow e^{i\theta} \frac{0-z_0}{z_0 \cdot 0-1} = 0 \Leftrightarrow z_0 = 0 \Leftrightarrow f(z) = e^{i\theta} \frac{z-0}{0-1} = e^{-i\theta} z.$$

So f is a rotation.

□

Remark. The map $g(z) = \frac{z-z_0}{z_0 z-1}$ swaps z_0 and 0 and is an involution ($g \circ g = Id$). Also, $z \rightarrow e^{i\theta} z$ is a rotation.

So every Mobius transformation from \mathbb{D} to \mathbb{D} is given by an involution followed by a rotation.

2.3 Finding biholomorphic maps between domains

To find a biholomorphism f between domains, we build f in various stages using simpler known maps.

Example 2.3.1. Find biholomorphism from $D = \{z \in \mathbb{D} : \text{Im}(z) < 0\}$ to \mathbb{H} .

The Cayley Map M_C is a map from \mathbb{H} to \mathbb{D} , so $M_C^{-1} : \mathbb{D} \rightarrow \mathbb{H}$, $M_C^{-1}(z) = \frac{iz+i}{-z+1}$.

To find the image of D under M_C^{-1} , consider how it acts on two segments of δD :

- Under M_C^{-1} , $-1 \rightarrow 0$, $0 \rightarrow i$ and $1 \rightarrow \infty$. Therefore the line segment from -1 to 1 through 0 is mapped to the positive imaginary axis.
- Under M_C^{-1} , $-i \rightarrow 1$, so the circular arc from -1 to 1 through $-i$ is mapped to the positive real axis.

Now $-\frac{i}{2} \in D$ and $M_C^{-1}(-\frac{i}{2}) = \frac{4+3i}{5}$. The image of D under M_C^{-1} is $\Omega = \{w \in \mathbb{C} : 0 < \text{Arg}(w) < \frac{\pi}{2}\}$.

Now we find a biholomorphic map from Ω to \mathbb{H} . $g(z) = z^2$ satisfies this, as it doubles the argument of z .

So the map is $f = g \circ M_C^{-1}$, $f : D \rightarrow \mathbb{H}$.

3 Notions of convergence in complex analysis and power series

3.1 Pointwise and uniform convergence

Definition 3.1.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. A sequence of functions $\{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$ converges pointwise (on X) to f if for every $x \in X$, the limit function $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists in Y .

In other words, we have for every $x \in X$ and for every $\epsilon > 0$, for some $N \in \mathbb{N}$, for every $n > N$, $d_Y(f_n(x), f(x)) < \epsilon$. (Not that N depends on x).

Remark. For every $x \in X$, $f_n(x)$ is just a sequence of points in Y . The above definition is what we get by applying definition 2.11 (in notes) to the sequence $f_n(z)$.

Example 3.1.2. Let $f_n(z) = z^n$, $f_n : \mathbb{C} \rightarrow \mathbb{C}$. There are the following cases:

1. $z \in \mathbb{D}$. Let $\epsilon > 0$. Then $|z|^N < \epsilon$ for every $N > \frac{\log \epsilon}{\log |z|}$. So for every $n > N$ we have $f_n(z) - 0 = |z|^n < |z|^N \epsilon$, hence $\lim_{n \rightarrow \infty} f_n(z) = 0 \in \mathbb{D}$.
2. $|z| = 1$. The point z rotates around the unit circle $\partial \mathbb{D}$ by $\text{Arg}(z)$ anticlockwise every iteration. For $z \neq 1$, this sequence doesn't converge. But for $z = 1$, $\lim_{n \rightarrow \infty} f_n(z) = \lim_{n \rightarrow \infty} 1 = 1$.
3. $|z| > 1$. The value of $|z|^n$ is unbounded so doesn't converge.

The sequence f_n doesn't converge pointwise on \mathbb{C} . But it is pointwise convergent on $\mathbb{D} \cup 1$ with limit function:

$$f(z) = \begin{cases} 0 & \text{if } z \in \mathbb{D} \\ 1 & \text{if } z = 1 \end{cases} \quad (1)$$

Definition 3.1.3. Let (X, d_X) and (Y, d_Y) be two metric spaces. A sequence of functions $\{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$ converges uniformly (on X) to the limit function f if for every $\epsilon > 0$ for some $N \in \mathbb{N}$, for every $n > N$, $d_Y(f_n(x), f(x)) < \epsilon$ for every $x \in X$.

Theorem 3.1.4. Let (X, d_X) and (Y, d_Y) be two metric spaces and let $\{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$ be a sequence of functions that converges uniformly to f on X .

Then f is continuous on X .

Proof. Same as in Analysis I. □

Lemma 3.1.5. let $\{f_n\}_{n \in \mathbb{N}} : X \rightarrow \mathbb{C}$ be a sequence of functions converging pointwise to a limit function f .

1. If $|f_n(x) - f(x)| \leq s_n$ for every $x \in X$ where $\{s_n\}_{n \in \mathbb{N}}$ is some sequence in $\mathbb{R} > 0$ (independent of x) with $\lim_{n \rightarrow \infty} s_n = 0$ then f_n converge uniformly to f on X .
2. If for some sequence $x_n \in X$, $|f_n(x_n) - f(x_n)| \geq c$ for some positive constant c then f_n does not converge uniformly to f on X .

Theorem 3.1.6. (Weierstrass M-test): Let $f_n : X \rightarrow \mathbb{C}$ be a sequence of functions such that $|f_n(x)| \leq M_n$ for every $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on X to some limit function $f : X \rightarrow \mathbb{C}$.

Proof. Similar to Analysis I. □

Theorem 3.1.7. Let a sequence of functions $f_n : [a, b] \rightarrow \mathbb{R}$ converge uniformly on an interval $[a, b]$ to some function f , such that $\{f_n\}$ are all continuous. Then

$$\lim_{n \rightarrow \infty} \int_a^c f_n(x) dx = \int_a^c f(x) dx \text{ for every } c \in [a, b]$$

Definition 3.1.8. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions in a metric space X . f_n converges locally uniformly (on X) to the limit function f if for every $x \in X$, for some open set $U \subset X$ containing x , f_n converges uniformly to f on U .

Theorem 3.1.9. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of continuous functions which converges locally uniformly on X to a limit function f . Then f is continuous on X .

Proof. For every $x \in X$, f_n converges uniformly on some open set U containing x . Hence f is continuous on U by theorem 5.5 (in notes). So f is continuous at x for every $x \in X$. □

Remark. The limit of a locally uniform convergent sequence of holomorphic functions is again holomorphic.

Example 3.1.10. For every $w \in \mathbb{D}$, for some $r < 1$, $w \in B_r(0)$ and $B_r(0)$ is open. Then for every $z \in B_r(0)$, $|z|^n < r^n$ and $\lim_{n \rightarrow \infty} r^n = 0$. So by lemma 5.6 (in notes), with $s_n = r^n$, f_n converges uniformly to f in $B_r(0)$.

Remark. To prove that the limit function is continuous on all of \mathbb{D} , it is enough to prove locally uniform convergence on every ball $B_r(0)$, $0 < r < 1$, in \mathbb{D} .

Theorem 3.1.11. Let X be a metric space and let $f_n : X \rightarrow \mathbb{C}$ be a sequence of continuous functions such that for any $y \in X$, there is an open $U \subset X$ containing y and constants $M_n > 0$ with $\sum_{n=1}^{\infty} M_n < \infty$ and $|f_n(x)| \leq M_n$ for every $x \in U$. Then $\sum_{n=1}^{\infty} f_n$ converges locally uniformly to a continuous function on X .

Proof. Given $y \in X$, the hypotheses of the theorem imply that for some constants $M_n > 0$, $|f_n(y)| \leq M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$.

$$|F_k(y)| = \left| \sum_{n=1}^k f_n(y) \right| \leq \sum_{n=1}^{\infty} |f_n(y)| \leq \sum_{n=1}^k M_n$$

As $k \rightarrow \infty$, the RHS $\sum_{n=1}^k M_n$ converges so it must be bounded, and let the upper bound by L . Thus for every k , $|F_k(y)| \leq L$. So the sequence $(F_k(y))_k$ is bounded, hence it lies in some bounded, closed ball in \mathbb{C} , which is compact by Heine-Borel.

Therefore there is a subsequence $(F_{k_j}(y))_{k_j}$ that converges to $F(y)$.

Now, for $k_j > k$,

$$|F_{k_j}(y) - F_k(y)| = \left| \sum_{n=k+1}^{k_j} f_n(y) \right| \leq \sum_{n=k+1}^{k_j} |f_n(y)| \leq \sum_{n=k+1}^{k_j} M_n$$

Taking the limit as $j \rightarrow \infty$, both the LHS and RHS converge, and we get

$$|F(y) - F_k(y)| \leq \sum_{n=k+1}^{\infty} M_n$$

Now taking the limit as $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} |F(y) - F_k(y)| = 0$$

since the RHS tends to zero.

Repeating this for every y , $F_k \rightarrow F$ pointwise on X .

From the hypotheses of the theorem, we have that for every $y \in X$, for some open $U \subset X$ containing y and constants $M_n > 0$ with $\sum_{n=1}^{\infty} M_n < \infty$ and $|f_n(x)| \leq M_n$ for every $x \in U$.

Then, for every $x \in U$ and for every $L > k$,

$$|F_L(x) - F_k(x)| = \left| \sum_{n=k+1}^L f_n(x) \right| \leq \sum_{n=k+1}^L |f_n(x)| \leq \sum_{n=k+1}^L M_n$$

Taking the limit as $l \rightarrow \infty$:

$$|F(x) - F_k(x)| \leq \sum_{n=k+1}^{\infty} M_n$$

for every $x \in U$.

$\lim_{k \rightarrow \infty} \sum_{n=k+1}^{\infty} M_n = 0$. So by lemma 5.6 (in notes), $F_k \rightarrow F$ uniformly on U . \square

3.2 Complex power series

Theorem 3.2.1. A complex power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n(z - c)^n, \quad a_n, c \in \mathbb{C}^2$$

There are three cases:

1. $\sum_{n=0}^{\infty} a_n(z - c)^n$ converges only for $z = c$ ($R = 0$).
2. There exists $R > 0$ (radius of convergence) such that
 - $\sum_{n=0}^{\infty} a_n(z - c)^n$ converges absolutely for $|z - c| < R$ (We call $B_R(c)$ the disc of convergence).
 - $\sum_{n=0}^{\infty} a_n(z - c)^n$ diverges for $|z - c| > R$ (anything can happen on the circle $|z - c| = R$).
3. $\sum_{n=0}^{\infty} a_n(z - c)^n$ converges absolutely for every $z \in \mathbb{C}$ ($R = \infty$).

Remark. Radius of convergence is usually determined via ratio test or root test.

Theorem 3.2.2. A power series $\sum_{n=0}^{\infty} a_n(z - c)^n$ with radius of convergence $0 < R < \infty$ converges uniformly on every ball $B_r(c)$ with $0 < r < R$. This implies that the power series is locally uniformly convergent on its disc of convergence.

Proof. Follows via the M-test. \square

Remark. The power series do not converge uniformly in the entire disc of convergence $B_R(c)$.

Proposition 3.2.3. Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series with radius of convergence $0 < R < \infty$. Then the formal derivatives and antiderivatives

$$\sum_{n=0}^{\infty} n a_n (z-c)^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-c)^{n+1}$$

have the same radius of convergence R .

Theorem 3.2.4. Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series with radius of convergence $0 < R < \infty$ and let $f : B_R(c) \rightarrow \mathbb{C}$ be the resulting limit function. Then f is holomorphic on $B_R(c)$ with

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-c)^{n-1}$$

for $z \in B_R(c)$.

Proof. Assume $c = 0$ (the general case for c is analogous).

$$f(z) - f(w) = \sum_{n=1}^{\infty} a_n (z^n - w^n) = \sum_{n=1}^{\infty} (z-w) q_n(z)$$

where $q_n(z) = \sum_{k=0}^{n-1} w^k z^{n-1-k}$.

So for $z \neq w$, let $h(z) := \frac{f(z)-f(w)}{z-w} = \sum_{n=1}^{\infty} a_n q_n(z)$

Given $z_0 \in B_R(0)$, let $r < R$ such that $w, z_0 \in B_r(0)$. To apply the local M-test, we need constants M_n for this set $B_r(0)$ that bound the terms $a_n q_n(z)$ defining h .

For $z \in B_r(0)$,

$$|a_n q_n(z)| = |a_n \sum_{k=0}^{n-1} w^k z^{n-1-k}| \leq |a_n| \sum_{k=0}^{n-1} |w|^k |z|^{n-1-k} < |a_n| \sum_{k=0}^{n-1} r^{n-1} = n |a_n| r^{n-1}$$

So let $M_n = n |a_n| r^{n-1}$, then $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} n |a_n| r^{n-1}$ which converges by proposition 5.19 (in lecture notes).

The formal derivative $\sum_{n=1}^{\infty} n a_n r^{n-1}$ has radius of convergence R so converges absolutely on its disc of convergence $B_R(0)$. In particular, it converges at $z = R$. By the local M-test, the series defining h converges locally uniformly to a continuous function on $B_R(0)$. Hence

$$\lim_{z \rightarrow w} \frac{f(z) - f(w)}{z - w} = \lim_{h \rightarrow w} h(z) = h(w) = \sum_{n=1}^{\infty} a_n q_n(w) = \sum_{n=1}^{\infty} n a_n w^{n-1}$$

□

Corollary 3.2.5. A power series f as theorem 5.21 (in lecture notes) with positive radius of convergence R can be differentiated infinitely many times and

$$f^{(k)} := \sum_{n=k}^{\infty} k! \binom{n}{k} a_n (z-c)^{n-k}$$

for $z \in B_R(c)$

Corollary 3.2.6. A power series f as in theorem 5.21 (in lecture notes) with positive radius of convergence R has a holomorphic antiderivative $F : B_R(c) \rightarrow \mathbb{C}$, with $F'(z) = f(z)$, defined by

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-c)^{n+1}$$

4 Complex integration over contours

4.1 Definition of contour integrals

Definition 4.1.1. For a continuous function $f : [a, b] \rightarrow \mathbb{C}$, with $f(z) = u(z) + iv(z)$,

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt \in \mathbb{C}$$

Lemma 4.1.2.

1. Let f_1 and f_2 be continuous functions from $[a, b]$ to \mathbb{C} . Then $\int_a^b (f_1(t) + f_2(t))dt = \int_a^b f_1(t)dt + \int_a^b f_2(t)dt$.
2. For any complex number $c \in \mathbb{C}$ and continuous function $f : [a, b] \rightarrow \mathbb{C}$,

$$\int_a^b cf(t)dt = c \int_a^b f(t)dt$$

Definition 4.1.3. A smooth curve in \mathbb{C} is a continuously differentiable function $\gamma : [0, 1] \rightarrow \mathbb{C}$ (i.e. differentiable with continuous derivative). More generally we can consider continuously differentiable curves $\gamma : [a, b] \rightarrow \mathbb{C}$. We say that such curves are C^1 .

Remark. We write $\gamma(t) = u(t) + iv(t)$ with $u, v : [a, b] \rightarrow \mathbb{R}$. Then the derivative γ' is defined as

$$\gamma'(t) := u'(t) + iv'(t)$$

At the endpoints, we demand that the one-sided derivative exists and is continuous from the one side:

$$\gamma'(b) := \lim_{h \rightarrow 0^-} \frac{u(b+h) - u(b)}{h} + i \lim_{h \rightarrow 0^-} \frac{v(b+h) - v(b)}{h}$$

exists and

$$\lim_{t \rightarrow b^-} \gamma'(t) = \gamma'(b)$$

Definition 4.1.4. Let $U \subset \mathbb{C}$ be an open set, and $f : U \rightarrow \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \rightarrow U$ be a C^1 curve. The integral of f along the curve γ is defined as

$$\int_{\gamma} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Corollary 4.1.5. Properties of the integral along a curve:

1. $\int_{\gamma} (f_1(z) + f_2(z))dz = \int_{\gamma} f_1(z)dz + \int_{\gamma} f_2(z)dz$
2. For $c \in \mathbb{C}$, $\int_{\gamma} cf(z)dz = c \int_{\gamma} f(z)dz$

Proof. Easy □

Definition 4.1.6. Given $\gamma : [a, b] \rightarrow \mathbb{C}$, the curve $(-\gamma) : [-b, -a] \rightarrow \mathbb{C}$ is defined as

$$(-\gamma)(t) := \gamma(-t)$$

Then we have

$$\int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$$

Lemma 4.1.7. Let $U \subset \mathbb{C}$ be an open set, $f : U \rightarrow \mathbb{C}$ be continuous and $\gamma : [a, b] \rightarrow \mathbb{C}$ be a C^1 curve. If $\phi : [a', b'] \rightarrow [a, b]$ with $\phi(a') = a$ and $\phi(b') = b$ is continuously differentiable and we define $\delta : [a', b'] \rightarrow \mathbb{C}$, $\delta := \gamma \circ \phi$, then

$$\int_{\gamma} f(z)dz = \int_{\delta} f(z)dz$$

Proof.

$$\begin{aligned} \int_{\delta} f(z)dz &= \int_{a'}^{b'} f(\delta(t))\delta'(t)dt = \int_{a'}^{b'} f(\gamma(\phi(t)))(\gamma(\phi(t)))'dt \\ &= \int_{a'}^{b'} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt \end{aligned}$$

With a change of variables $s = \phi(t)$, $ds = \phi'(t)dt$:

$$\int_{a'}^{b'} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt = \int_a^b f(\gamma(s))\gamma'(s)ds = \int_{\gamma} f(z)dz$$

□

Definition 4.1.8. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve and suppose there exist $a = a_0 < a_1 < \dots < a_n = b$ such that the curves $\gamma_i : [a_{i-1}, a_i] \rightarrow \mathbb{C}$, defined by $\gamma_i(t) = \gamma(t)$ for $t \in [a_{i-1}, a_i]$ are C^1 curves. Then γ is a piecewise C^1 curve or contour.

For a contour γ above, a contour integral is defined as

$$\int_{\gamma} f(z)dz = \sum_{n=1}^n \int_{\gamma_i} f(z)dz$$

Definition 4.1.9. If $\gamma : [a, b] \rightarrow \mathbb{C}$ and $\delta : [c, d] \rightarrow \mathbb{C}$ are two contours with $\gamma(b) = \delta(c)$ the contour $\gamma \cup \delta : [a, b + d - c] \rightarrow \mathbb{C}$ is defined as

$$(\gamma \cup \delta)(t) := \begin{cases} \gamma(t) & \text{if } a \leq t \leq b \\ \delta(t) & \text{if } c \leq t \leq d \end{cases}$$

Then

$$\int_{\gamma \cup \delta} f(z)dz = \int_{\gamma} f(z)dz + \int_{\delta} f(z)dz$$

4.2 The fundamental theorem of calculus

Theorem 4.2.1. Let $U \subset \mathbb{C}$ be an open set and let $F : U \rightarrow \mathbb{C}$ be holomorphic with continuous derivative f . Then for every contour $\gamma : [a, b] \rightarrow U$,

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, if γ is closed, so $\gamma(a) = \gamma(b)$, then

$$\int_{\gamma} f(z) dz = 0$$

Proof. First consider the case where γ is a C^1 curve. Let $F = u + iv$. Then

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} F'(z) dz = \int_a^b F'(\gamma(t)) \gamma'(t) dt = \int_a^b (F(\gamma(t)))' dt \\ &= \int_a^b (u(\gamma(t)))' dt + i \int_a^b (v(\gamma(t)))' dt = [u(\gamma(t))]_a^b + i[v(\gamma(t))]_a^b \\ &= u(\gamma(b)) - u(\gamma(a)) + i(v(\gamma(b)) - v(\gamma(a))) = F(\gamma(b)) - F(\gamma(a)) \end{aligned}$$

Now extend this proof to any contour.

Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a contour, then for some $a = a_0 < a_1 < \dots < a_n = b$, the curves $\gamma_i : [a_{i-1}, a_i] \rightarrow \mathbb{C}$, $i = 1, \dots, n$, defined by $\gamma_i(t) = \gamma(t)$ for $t \in [a_{i-1}, a_i]$ are C^1 curves. Then

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} F'(z) dz = \sum_{i=1}^n \int_{\gamma_i} F'(z) dz \\ &= \sum_{i=1}^n (F(\gamma(a_i)) - F(\gamma(a_{i-1}))) = F(\gamma(a_n)) - F(\gamma(a_0)) = F(\gamma(b)) - F(\gamma(a)) \end{aligned}$$

□

Remark. Under the hypotheses on F , the integral only depends on the endpoints of the curve.

Theorem 4.2.2. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous,

$$\int_a^b f(t) dt \leq \int_a^b \max_{t \in [a, b]} f(t) dt \leq (b - a) \max_{t \in [a, b]} f(t)$$

Proof. From Analysis I. □

Definition 4.2.3. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a contour. The **length** of γ is defined as

$$L(\gamma) = \int_a^b |\gamma'(t)| dt$$

Lemma 4.2.4. (The Estimation Lemma) Let $f : U \rightarrow \mathbb{C}$ be continuous and $\gamma : [a, b] \rightarrow U$ be a contour. Then

$$\left| \int_{\gamma} f(z) dz \right| \leq L(\gamma) \sup_{\gamma} |f|$$

where $\sup_{\gamma} |f| := \sup\{|f(z)| : z \in \gamma\}$.

Proof. First prove that for a continuous function $g : [a, b] \rightarrow \mathbb{C}$,

$$\left| \int_a^b g(t) dt \right| \leq \int_a^b |g(t)| dt$$

If we write $\int_a^b g(t) dt = re^{i\theta}$ with $r \geq 0$, then

$$\begin{aligned} \left| \int_a^b g(t) dt \right| &= |re^{i\theta}| = r = \operatorname{Re} \left(e^{-i\theta} \int_a^b g(t) dt \right) \\ &= \operatorname{Re} \left(\int_a^b g(t) e^{-i\theta} dt \right) = \int_a^b \operatorname{Re}(g(t) e^{-i\theta}) dt \leq \int_a^b |e^{-i\theta} g(t)| dt = \int_a^b |g(t)| dt \end{aligned}$$

Let $g(t) = f(\gamma(t))\gamma'(t)$, then

$$\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)| dt$$

Then

$$\int_a^b |f(\gamma(t))\gamma'(t)| dt \leq \sup_{\gamma} |f| \int_a^b |\gamma'(t)| dt = L(\gamma) \sup_{\gamma} |f|$$

□

Theorem 4.2.5. (Converse to FTC) Let $f : D \rightarrow \mathbb{C}$ be continuous on a domain D . If $\int_{\gamma} f(z) dz = 0$ for every closed contour $\gamma \in D$, for some $F : D \rightarrow \mathbb{C}$, $F'(z) = f(z)$.

Proof. Let $a_0 \in D$. For every $a_0 \neq w \in D$, let $\gamma(w)$ be a contour connecting a_0 to w and is contained in D .

Since D is a domain, it is path-connected, i.e. there is a smooth path γ_w connecting a_0 to w , therefore the collection of contours contained in D and connecting a_0 and w is non-empty. Let

$$F(w) := \int_{\gamma(w)} f(z) dz$$

Let $\tilde{\gamma}(w)$ be another contour that connects a_0 to w and is contained in D . Then let $c(w) = \gamma(w) \cup (-\tilde{\gamma}(w))$ that is obtained by moving through γ then through $\tilde{\gamma}$ in the opposite direction. Since c is a closed contour in D , $\int_C f(z) dz = 0$.

Then $0 = \int_C f(z) dz = \int_{\gamma(w) \cup (-\tilde{\gamma}(w))} f(z) dz = \int_{\gamma(w)} f(z) dz + \int_{-\tilde{\gamma}(w)} f(z) dz = \int_{\gamma(w)} f(z) dz - \int_{\tilde{\gamma}(w)} f(z) dz$. Hence

$$\int_{\gamma(w)} f(z) dz = \int_{\tilde{\gamma}(w)} f(z) dz$$

Therefore F does not depend on the contour chosen to join a_0 to w .

Now we claim F is holomorphic and we claim that F is holomorphic and $\forall z \in D, F'(z) = f(z) \Rightarrow \lim_{h \rightarrow 0} \frac{F(w+h) - F(w)}{h} = f(w)$.

To evaluate $F(w+h)$ we need a contour joining a_0 to $w+h$ contained in D . For every $w \in D$, let $r > 0$ such that $B_r(w) \subset D$. This ball must exist since D is open. Then for every $h \in \mathbb{C}$ with $|h| < r$ consider the straight line δ_h that connects w to $w+h$.

A parameterisation of this line is given by

$$\delta_h : [0, 1] \rightarrow D, \quad \delta_h(t) = w + th$$

The contour $\gamma_w \cup \delta_h$ is contained in D . So

$$\begin{aligned} F(w+h) &= \int_{\gamma_w \cup \delta_h} f(z) dz = \int_{\gamma_w} f(z) dz + \int_{\delta_h} f(z) dz = F(w) + \int_{\delta_h} f(z) dz \\ \int_{\delta_h} f(z) dz &= f(w) \int_{\delta_h} dz = f(w) \int_0^1 h dt = hf(w) \end{aligned}$$

We can rewrite the previous equation as

$$F(w+h) = F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w)) dz$$

For $h \neq 0$,

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| = \frac{1}{|h|} \left| \int_{\delta_h} (f(z) - f(w)) dz \right|$$

□

4.3 First Version of Cauchy's Theorem

Definition 4.3.1. A domain D is **starlike** if for some point $a_0 \in D$, for every $b \neq a_0 \in D$, the straight line connecting a_0 and b is contained in D .

Example 4.3.2.

1. \mathbb{C} is starlike.
2. The ball $B_r(a)$ is starlike.
3. Any convex set is starlike.

Example 4.3.3.

1. \mathbb{C}^* is not starlike, because a straight line between two points could pass through 0, and $0 \notin \mathbb{C}^*$.
2. Similarly, $B_r^*(a) = B_r(a) - \{a\}$ is not starlike.

Lemma 4.3.4. Let U be an open set and let $f : U \rightarrow \mathbb{C}$ be holomorphic. Then

$$\int_{\partial \Delta} f(z) dz = 0$$

for every **triangle** Δ in U .

Remark. Here $\partial\Delta$ is the boundary of Δ , traversed anticlockwise.

Remark. Given any closed contour without a parameterisation given, we will assume that it is traversed anticlockwise.

Proof. First, split Δ into four triangles, $\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}, \Delta^{(4)}$, using the midpoints of each side. Then

$$\int_{\partial\Delta} f(z)dz = \sum_{i=1}^4 \int_{\partial\Delta^{(i)}} f(z)dz$$

Let Δ_1 be one of these four triangles which has the largest integral, then

$$\left| \int_{\partial\Delta} f(z)dz \right| \leq 4 \left| \int_{\partial\Delta_1} f(z)dz \right|$$

We then continue this procedure to produce a sequence of triangles

$$\Delta > \Delta_1 > \dots > \Delta_n > \dots$$

The length of Δ_1 , $L(\Delta_1)$ satisfies $L(\Delta_1) = \frac{1}{2}L(\Delta)$, therefore

$$L(\Delta_n) = \frac{1}{2^n} L(\Delta) \implies L(\Delta_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

Also,

$$\bigcap_{n \in \mathbb{N}} \Delta_n = \{w\}$$

is a single point in D . Now, notice that

$$\int_{\partial\Delta_n} 1dz = 0 = \int_{\partial\Delta_n} zdz$$

and that $w, f(w), f'(w)$ are constants. Then sneakily,

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} (f(z) - f(w) - (z - w)f'(w))$$

Define the auxiliary function

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} - f'(w) & \text{if } z \in D \setminus \{w\} \\ 0 & \text{if } z = w \end{cases}$$

which is continuous at $z = w$, so is continuous on D . So

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} (z - w)g(z)dz$$

Now,

$$\left| \int_{\partial\Delta} f(z)dz \right| \leq 4^n \left| \int_{\partial\Delta_n} f(z)dz \right| = 4^n \left| \int_{\partial\Delta_n} (z - w)g(z)dz \right|$$

Note

$$\sup_{z \in \partial\Delta_n} |z - w| \leq L(\partial\Delta_n)$$

so by the Estimation Lemma,

$$\begin{aligned}
\left| \int_{\partial\Delta} f(z)dz \right| &\leq 4^n L(\partial\Delta_n) \sup_{z \in \partial\Delta_n} |(z-w)g(z)| \\
&\leq 4^n L(\partial\Delta_n) \sup_{z \in \partial\Delta_n} |z-w| \sup_{z \in \partial\Delta_n} |g(z)| \\
&\leq 4^n (L(\partial\Delta_n))^2 \sup_{z \in \partial\Delta_n} |g(z)| \\
&= L(\Delta)^2 \sup_{z \in \partial\Delta_n} |g(z)|
\end{aligned}$$

As $n \rightarrow \infty$, $\sup_{z \in \partial\Delta_n} |g(z)| \rightarrow g(w) = 0$. This completes the proof. \square

Lemma 4.3.5. Let D be a starlike domain and $f : D \rightarrow \mathbb{C}$ be continuous. Then, if

$$\int_{\partial\Delta} f(z)dz = 0$$

for every $\Delta \subset D$, then for some $F : D \rightarrow \mathbb{C}$,

$$F'(z) = f(z) \quad \forall z \in D$$

Proof. Similar to the proof of converse of FTC. \square

Theorem 4.3.6. (Cauchy's Theorem for Starlike Domains - CTSD) Let D be a starlike domain and let $f : D \rightarrow \mathbb{C}$ be holomorphic. Then for every closed contour $\gamma \in D$,

$$\int_{\gamma} f(z)dz = 0$$

Proof. By Lemma Reflem:ctLem1,

$$\int_{\partial\Delta} f(z)dz = 0 \quad \forall \Delta \in D$$

By Lemma Reflem:ctLem2, f has a holomorphic antiderivative F . Then, by FTC,

$$\int_{\gamma} f(z)dz = 0 \quad \forall \text{ closed } \gamma \in D$$

\square

Remark. The same result holds if f is holomorphic on $D - S$, where S is a finite set of points and f is continuous on D . We will need this in proofs but is not used much elsewhere.

Example 4.3.7. Consider

$$\int_{|z|=\frac{1}{2}} \frac{e^z (\sin z)^2}{e^{z^2}} dz$$

Because the function in the integral is holomorphic and $|z| = \frac{1}{2}$ is a closed contour, by CTSD, this integral is equal to 0.

4.4 Cauchy's integral formula

Theorem 4.4.1. (Cauchy's integral formula - CIF) Let $B_r(a)$ be a ball in \mathbb{C} and $f : B_r(a) \rightarrow \mathbb{C}$ be holomorphic. Then for every $w \in B_r(a)$,

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz$$

where ρ is any real number with $|w-a| < \rho < r$.

Proof. Define an auxiliary function g by

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

Note that g is continuous at $z = w$, and holomorphic elsewhere. By CTSD,

$$\int_{|z-a|=\rho} g(z) dz = 0$$

Therefore

$$\int_{|z-a|=\rho} \frac{f(z)}{z-w} dz = \int_{|z-a|=\rho} \frac{f(w)}{z-w} dz$$

Now,

$$\begin{aligned} \frac{1}{z-w} &= \frac{1}{z-a+a-w} \\ &= \frac{1}{(z-a)(1-\frac{w-a}{z-a})} \\ &= \frac{1}{z-a} \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a} \right)^n \end{aligned}$$

which converges uniformly, since $|\frac{w-a}{z-a}| = |\frac{w-a}{\rho}| < 1$. So we have

$$\begin{aligned} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz &= f(w) \int_{|z-a|=\rho} \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} dz \\ &= \sum_{n=0}^{\infty} \left(f(w)(w-a)^n \int_{|z-a|=\rho} \frac{1}{(z-a)^{n+1}} dz \right) \end{aligned}$$

The inner integral is equal to 0 except when $n = 0$, when it's value is $2\pi i$. So

$$\int_{|z-a|=\rho} \frac{f(z)}{z-w} dz = f(w)(w-a)^0 \cdot 2\pi i = 2\pi i \cdot f(w)$$

□

5 Features of holomorphic functions

Theorem 5.0.1. (Cauchy-Taylor theorem) Let U be an open set and $f : U \rightarrow \mathbb{C}$ be holomorphic on U . Then for every $r > 0$ such that $B_r(a) \subset U$, f has a power series converging on $B_r(a)$ given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

is a constant for every $0 < \rho < r$. This is the **Taylor series** of f about a .

Proof. By the CIF, for every w with $|w - a| < \rho$,

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz \\ &= \frac{1}{2\pi i} \int_{|z-a|=\rho} f(z) \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} dz \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz \right) (w-a)^n \\ &= \sum_{n=0}^{\infty} c_n (w-a)^n \end{aligned}$$

□

Theorem 5.0.2. (CIF for derivatives) Let $f : B_r(a) \rightarrow \mathbb{C}$ be holomorphic. Then for every $0 < \rho < r$,

$$\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Proof. By Cauchy-Taylor, we have a convergent power series such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz$$

But we also have (corollary 5.22 in lecture notes),

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Equating these two expressions for c_n completes the proof. □

Remark. Combining theorem 7.1 (lecture notes) and theorem 7.2 (lecture notes), every holomorphic function f has power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

Remark. Cauchy-Taylor does not hold in real analysis. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

f is differentiable for $x \neq 0$. For $x = 0$,

$$f'(x) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x)}{x}$$

$$\lim_{x \rightarrow 0^-} \frac{f(x)}{x} = \lim_{x \rightarrow 0^-} \frac{0}{x} = 0$$

and

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = \lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x} = \lim_{x \rightarrow 0^+} \frac{1/x}{e^{1/x}} = \lim_{y \rightarrow \infty} \frac{y}{e^y} = 0$$

so $f'(0) = 0$, hence f is differentiable on \mathbb{R} . But if f had a Taylor series at $x = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$$

around $x = 0$.

Corollary 5.0.3. (Holomorphic functions have infinitely many derivatives) If $f : U \rightarrow \mathbb{C}$ is holomorphic on an open set U then f has derivatives of all orders and each derivative is also holomorphic.

Proof. Since U is open, $\exists B_r(a) \subset U$ around a point $z = a$. But then by Cauchy-Taylor, f has a power series. By theorem 5.21 (lecture notes), this power series is holomorphic. By corollary 5.22 (lecture notes) we can term-by-term differentiate to get a power series for $f'(z)$. By theorem 5.21 (lecture notes), $f'(z)$ is holomorphic. This can be repeated indefinitely. \square

Remark. This is a huge difference between real and complex analysis. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by defined as

$$f_n(x) = |x|x^n$$

$$f'_n(0) = \lim_{x \rightarrow 0} \frac{f_n(x) - f_n(0)}{x - 0} = \lim_{x \rightarrow 0} |x|x^{n-1} = 0$$

$f'_n(x) = (n+1)|x|x^{n-1}$ and $f^{(n)}(x) = c|x|$ which is not differentiable.

Theorem 5.0.4. (Morera's Theorem) Let $f : D \rightarrow \mathbb{C}$ be continuous on a domain D . If

$$\int_{\gamma} f(z) dz = 0 \quad \forall \text{ closed } \gamma \subset D$$

then f is holomorphic.

Proof. By the converse FTC, f has a holomorphic antiderivative $F : D \rightarrow \mathbb{C}$ such that $F'(z) = f(z) \quad \forall z \in D$. By corollary 7.6 (lecture notes), if F is holomorphic, its derivative f must be. \square

Example 5.0.5. Consider

$$\int_{|z|=3} \frac{e^z}{z^2(z-1)} dz$$

We use partial fractions:

$$\frac{1}{z^2(z-1)} = \frac{a}{z} + \frac{b}{z^2} + \frac{c}{z-1}$$

So $1 = (c+a)z^2 + (b-a)z - b$, so $b = -1, a = -1, c = 1$. Using the CIF and CIF for derivatives,

$$\begin{aligned} \int_{|z|=3} \frac{e^z}{z^2(z-1)} dz &= - \int_{|z|=3} \frac{e^z}{z} dz - \int_{|z|=3} \frac{e^z}{z^2} dz + \int_{|z|=3} \frac{e^z}{z-1} dz \\ &= -2\pi i e^0 - 2\pi i e^0 + 2\pi i e^1 \\ &= 2\pi i (e - 2) \end{aligned}$$

5.1 Liouville's theorem

Definition 5.1.1. A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is **entire** if f is holomorphic on \mathbb{C} .

Definition 5.1.2. A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is **bounded** if for some $M > 0$, $|f(z)| \leq M \forall z \in \mathbb{C}$.

Theorem 5.1.3. (Liouville's theorem) Every bounded entire function is constant.

Proof. Let f be entire and bounded. We will show that $\forall w \in \mathbb{C}$, $f(w) = f(0)$. By the CIF, for every $\rho > |w|$,

$$\begin{aligned} |f(w) - f(0)| &= \left| \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z-w} dz - \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z} dz \right| \\ &= \frac{|w|}{2\pi} \left| \int_{|z|=\rho} f(z) \frac{1}{z(z-w)} dz \right| \\ &= \end{aligned}$$

Using the Estimation lemma, boundedness of f and the reverse triangle inequality,

$$\begin{aligned} |f(w) - f(0)| &\leq \frac{|w|}{2\pi} 2\pi\rho \cdot \sup_{|z|=\rho} \frac{|f(z)|}{|z||z-w|} \\ &\leq |w|\rho \frac{M}{\rho} \sup_{|z|=\rho} \frac{1}{|z-w|} \\ &\leq |w|M \sup_{|z|=\rho} \frac{1}{||z| - |w||} \\ &= \frac{|w|M}{\rho - |w|} \\ &\rightarrow 0 \quad \text{as } \rho \rightarrow \infty \end{aligned}$$

and ρ can be arbitrarily large. □

Remark. The holomorphicity condition is essential (we can't just say that f is continuous). For example,

$$f(z) = f(x + iy) = \sin(x) + i \sin(y)$$

is continuous and bounded on \mathbb{C} but is not entire.

Theorem 5.1.4. (Fundamental theorem of Algebra) Every non-constant polynomial with complex coefficients $p(z) = a_d z^d + \cdots + a_1 z + a_0$, $a_d \neq 0$ has a complex root: for some $z_0 \in \mathbb{C}$, $P(z_0) = 0$.

Proof. By assumption, $d \geq 1$, so $|p(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$. In particular, $\exists R > 0$, $|p(z)| > 1$ if $|z| = R$. Assume the converse, that p has no roots.

Then $f(z) := \frac{1}{p(z)}$ is holomorphic on \mathbb{C} . On the set $|z| > R$, f is bounded, since $|f(z)| = \frac{1}{|p(z)|} < 1$. But $\overline{B_R}(0) = \{z \in \mathbb{C} : |z| \leq R\}$ is compact, so by theorem 2.30 (lecture notes), $|f(z)|$ attains a maximum on $\overline{B_R}(0)$. In particular, f is bounded on $\overline{B_R}(0)$. Thus f is bounded and entire, so by Liouville's theorem, f is constant, which is a contradiction. \square

Theorem 5.1.5. (Local maximum modulus principle) Let $f : B_r(a) \rightarrow \mathbb{C}$ be holomorphic. If for every $z \in B_r(a)$, $|f(z)| \leq |f(a)|$ then f is constant on $B_r(a)$.

Proof. First, we show $|f|$ is constant. Pick any $w \in B_r(a)$. We will show $|f(w)| = |f(a)|$. Let $\rho = |w - a| < r$ so that the contour $\gamma(t) = a + \rho e^{2\pi i t}$, $t \in [0, 1]$ passes through w . By the CIF,

$$\begin{aligned} |f(a)| &= \left| \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-a} dz \right| \\ &= \frac{1}{2\pi} \left| \int_{\gamma} \frac{f(z)}{z-a} dz \right| \\ &= \frac{1}{2\pi} \left| \int_0^1 \frac{f(\gamma(t))}{\gamma(t)-a} \gamma'(t) dt \right| \\ &= \frac{1}{2\pi} \left| \int_0^1 \frac{f(\gamma(t))}{\rho e^{2\pi i t}} \rho \cdot 2\pi i e^{2\pi i t} dt \right| \\ &= \left| \int_0^1 f(\gamma(t)) dt \right| \\ &\leq \int_0^1 |f(\gamma(t))| dt \\ &\leq \int_0^1 |f(a)| dt \\ &\leq |f(a)| \int_0^1 1 dt \\ &= |f(a)| \end{aligned}$$

Therefore every inequality above must be an equality. In particular,

$$\int_0^1 |f(\gamma(t))| dt = \int_0^1 |f(a)| dt$$

So by continuity, $|f(\gamma(t))| = |f(a)|$. So $|f(w)| = |f(a)|$, and w was arbitrary, so $|f(z)| = |f(a)| \forall z \in B_r(a)$.

Now, we show that f is constant. Let $|f(z)| = c \forall z \in B_r(a)$. If $c = 0$, the $|f(z)| = 0 \Rightarrow f(z) = 0 \forall z \in B_r(a)$. So assume $c \neq 0$. Now,

$$c^2 = |f(z)|^2 = f(z) \cdot \overline{f(z)}$$

So $\overline{f(z)} = \frac{c^2}{f(z)}$ is holomorphic (since f is holomorphic). Let $f = u + iv$, then $\overline{f} = u - iv = u + i(-v)$. Then by the Cauchy-Riemann equations, $u_x = v_y$ but also

$u_x = (-v)_y = -v_y$ so $u_x = v_y = 0$, and $u_y = -v_x$ but also $u_y = (-v)_x = v_x$ so $u_y = v_x = 0$. By proposition 3.3 (lecture notes), $f'(z) = u_x + iv_x = 0 + i \cdot 0 = 0$, hence f is constant. \square

Theorem 5.1.6. (Maximum modulus theorem) Let D be a domain and $f : D \rightarrow \mathbb{C}$. If

$$\exists a \in D, |f(z)| \leq |f(a)| \quad \forall z \in D$$

then f is constant on D .

Proof. Let $U_1 = \{z \in D : f(z) = f(a)\}$ and let $U_2 = \{z \in D : f(z) \neq f(a)\}$. Let $U = U_1 \cup U_2$ (TODO: check this last sentence, might not be U). U_1 is non-empty, because $a \in U_1$. We will show that U_1 is open. Pick $z \in U_1$. Then by the openness of U , there exists a ball $B_r(z)$ in U . Pick $w \in B_r(z)$. We have $|f(w)| \leq |f(a)| = |f(z)|$, so by the Local Maximum Modulus Principle, f is constant on $B_r(z)$, so $f(w) = f(z) = f(a) \implies w \in U_1 \implies B_r(z) \subset U_1$. Hence U_1 is open.

Now $U_2 = D - U_1 = f^{-1}(\mathbb{C} - \{f(a)\})$, where f^{-1} denotes the preimage. Therefore U_2 is open by theorem 2.17 (lecture notes), since $\mathbb{C} - \{f(a)\}$ is open.

So $D = U_1 \cup U_2$ where U_1 is non-empty and open and U_2 is open. By fact 7.14 (lecture notes), U_2 is empty hence $D = U_1$, so f is constant on D . \square

Example 5.1.7. Find the maximum absolute value of $f(z) = z^2 + 2z - 3$ on $\overline{B_1(0)} = \{z \in \mathbb{C} : |z| \leq 1\}$.

$\overline{B_1(0)}$ is compact so by theorem 2.30 (lecture notes), $|f(z)|$ attains a maximum on it. Also, $B_1(0)$ is a domain, so by the maximum modulus theorem, the function does not attain the maximum in $B_1(0)$. So the maximum must be obtained on the boundary. So $|z| = 1$, so let $z = e^{it}$ for some $t \in [0, 2\pi]$. Then $|f(z)|^2 = |z^2 + 2z - 3|^2 = (z^2 + 2z - 3)(\overline{z^2 + 2z - 3}) = 14 - 3e^{-2it} - 3e^{2it} - 4e^{-it} - 4e^{-it} = 14 - 6\cos(2t) - 8\cos(t) = -12\cos(t)^2 - 8\cos(t) + 20 = -12x^2 - 8x + 20$ where $x = \cos(t)$. The maximum is attained when $x = \cos(t) = -\frac{1}{3}$. This gives $|f(z)| = \frac{8}{\sqrt{3}}$.

5.2 Analytic continuation and the identity theorem

Let $f : B_r(a) \rightarrow \mathbb{C}$ be holomorphic. Then by Cauchy-Taylor, f has a convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

for $z \in B_r(a)$. Assume $f \not\equiv 0$ so at least one coefficient is non-zero. Let $m = \min\{n \geq 0 : c_n \neq 0\}$. Then

$$f(z) = (z-a)^m \sum_{n=m}^{\infty} c_n(z-a)^{n-m} = (z-a)^m \sum_{k=0}^{\infty} c_{k+m}(z-a)^k$$

where $k = n - m$. Let

$$h(z) = \sum_{k=0}^{\infty} c_{k+m}(z-a)^k$$

Then h is a convergent power series so is holomorphic by theorem 5.21 (lecture notes) and $h(a) = c_m \neq 0$. Note $f(a) = 0 \iff m > 0$.

Definition 5.2.1. (Orders of zeros) $f : B_r(a) \rightarrow \mathbb{C}$ has a **zero of order** m at a if for some holomorphic function $h : B_r(a) \rightarrow \mathbb{C}$ such that $f(z) = (z-a)^m h(z)$ and $h(a) \neq 0$.

Remark. We can show that f has a zero of order m at $z = a$ iff

$$f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$$

Example 5.2.2. Let $f(z) = z(e^z - 1)$. Let $z = a = 0$. Then

$$\begin{aligned} f(z) &= z \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 \right) \\ &= z \sum_{n=1}^{\infty} \frac{z^{n+1}}{n!} \\ &= z^2 \sum_{m=0}^{\infty} \frac{z^m}{(m+1)!} \end{aligned}$$

so f has a zero of order 2 at $a = 0$. As a check, $f'(z) = e^z - 1 + ze^z$ and $f''(z) = e^z + e^z + ze^z$ so $f(0) = f'(0) = 0$ but $f''(0) = e^0 + e^0 + 0 = 2$.

Remark. Because holomorphic functions are continuous, if $f(a) \neq 0$, we can always find $B_\rho(a)$ such that $f(z) \neq 0$ on $B_\rho(a)$.

Theorem 5.2.3. (Principle of isolated zeroes) Let $f : B_r(a) \rightarrow \mathbb{C}$ be holomorphic and $f \not\equiv 0$. Then for some $\rho > 0$,

$$f(z) \neq 0 \quad \forall z \in B_r(a) - \{a\}$$

In particular, the zeros are isolated from one another.

Proof. For $f(a) \neq 0$, we are done by continuity. If $f(a) = 0$, for some $m > 0$,

$$f(z) = (z - a)^m h(z) \quad \forall z \in B_\rho(a)$$

where $h : B_r(a) \rightarrow \mathbb{C}$ is holomorphic, $\rho > 0$ and $h(a) \neq 0$. Thus $h(z) \neq 0$ on $B_\rho(a)$ and $(z - a)^m \neq 0$ on $B_\rho(a) - \{a\}$. So $f(z) \neq 0$ on $B_\rho(a)$. \square