1. The action principle

- For small $\delta s \in \mathbb{R}$, $f(s+\delta s) = f(s) + \frac{df(s)}{ds}\delta s + R(s, \delta s)$
- With $\delta f := f(s+\delta s) - f(s)$, $\delta f = \frac{df(s)}{ds}\delta s + R(s, \delta s)$, with

$$\lim_{\delta s \to 0} \frac{R(s, \delta s)}{\delta s} = 0$$

  So δf vanishes to first order in δs, so $R(s, \delta s)$ can be written as $O\left((\delta s)^2\right)$
- At the extrema of f, $\frac{df(s)}{ds} = 0$ so $\delta f = O\left((\delta s)^2\right)$
- **Functional**: map from functions to $\mathbb{R}$
- $y(t)$ **stationary** for functional S if

$$\frac{dS[y(t) + \varepsilon z(t)]}{de} \Big|_{\varepsilon=0} = 0$$

  for every smooth $z(t)$ with $z(a) = z(b) = 0$. We use the notation $\delta y(t) = \varepsilon z(t)$. $y(t)$ is called a **path**.
- **Action principle (variational principle)**: paths described by particles are stationary paths of S:

$$\delta S := S[x + \delta x] - S[x] = O\left((\delta x)^2\right)$$

  for arbitrary smooth small deformations $\delta x(t)$ around true path $x(t)$.
- **Fundamental lemma of the calculus of variations**: Let $f(x)$ be continuous in $[a, b]$ and

$$\int_a^b f(x) g(x) \,\mathrm{d}x = 0$$

  for every smooth $g(x)$ in $[a, b]$ with $g(a) = g(b) = 0$. Then $f(x) = 0$ in $[a, b]$.
- **Notation**:

$$\frac{\partial L}{\partial x} = \frac{\partial L(r,s)}{\partial r} \Big|_{(r,s)=(x(t),\dot{x}(t))}, \quad \frac{\partial L}{\partial \dot{x}} = \frac{\partial L(r,s)}{\partial s} \Big|_{(r,s)=(x(t),\dot{x}(t))}$$

- For a path $\underline{q}$ and a Lagrangian $L\left(\underline{q}, \dot{\underline{q}}\right)$, the action for the path is

$$S = \int_{t_0}^{t_1} L\left(\underline{q}(t), \dot{\underline{q}}(t)\right) \mathrm{d}t$$

- The action above satisfies

$$0 = \delta S = \int_{t_0}^{t_1} \left(\sum_{i=1}^{N} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \mathrm{d}t$$

- **Euler-Lagrange equation**:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

- The arguments in a Lagrangian, x and $\dot{x}$, are independent:

$$\frac{\partial x}{\partial \dot{x}} = \frac{\partial \dot{x}}{\partial x} = 0$$

- **Configuration space, $\mathcal{C}$**: set of all possible instantaneous confiugratinons of a physical system. (Includes positions but not velocities).
- For configuration space $\mathcal{C}$ of system $\mathcal{S}$, S has $\dim(\mathcal{C})$ **degrees of freedom**.
- **Generalised coordinates**: A set of coordinates in configuration space.
- Notation: $\underline{q}$ shows results holds for arbitrary choices of generalised coordinates.
- **Euler-Lagrange equation for configuration space $\mathcal{C}$**:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \forall i \in \{1, ..., \dim(\mathcal{C})\}$$

- For system with kinetic energy $T(\underline{q}, \underline{\dot{q}})$ and potential energy $V(\underline{q})$, the Lagrangian for the system is

$$L(\underline{q}, \underline{\dot{q}}) = T(\underline{q}, \underline{\dot{q}}) - V(\underline{q})$$

- **Ignorable coordinate q_i**: Lagrangian does not depend on q_i:

$$\frac{\partial L(q_1, ...q_N, \dot{q}_1, ...\dot{q}_N)}{\partial q_i} = 0$$

- **Generalised momentum** of coordinate q_i:

$$p_i := \frac{\partial L}{\partial \dot{q}_i}$$

- Generalised momentum of ignorable coordinate is conserved.

2. Symmetries, Noether's theorem and conservation laws

- **Transformation depending on ε**: family of smooth maps $\varphi(\varepsilon) : \mathcal{C} \to \mathcal{C}$ with $\varphi(0)$ the identity map. Can be written as

$$q_i \to {q_i}' = \phi_i(q_1, ..., q_N, \varepsilon)$$

where the ϕ_i are a set of $N = \dim(\mathcal{C})$ functions representing the transformation in the given coordinate system. Change in velocities is

$$\dot{q}_i \to \frac{d}{dt}\phi_i$$

- **Generator of φ**:

$$\frac{d\varphi(\varepsilon)}{d\varepsilon} \mid_{\varepsilon=0} = \varphi'(0)$$

- In any coordinate system,

$$q_i \to \phi_i(\underline{q}, \varepsilon) = q_i + \varepsilon a_i(\underline{q}) + O(\varepsilon^2)$$

where

$$a_i = \frac{\partial \phi_i(\underline{q}, \varepsilon)}{\partial \varepsilon} \mid_{\varepsilon=0}$$

So the generator of the transformation is a_i.

- For velocities,

$$\dot{q}_i \to \dot{q}_i + \varepsilon \dot{a}_i(q_1, ..., q_N, \dot{q}_1, ..., \dot{q}_N) + O(\varepsilon^2)$$

generated by $\dot{a}_i$.

- Equations of motion don't change when total derivative of function of coordinates and time is added to Lagrangian:

$$L \to L + \frac{dF(q_1, ..., q_N, t)}{dt}$$

doesn't change equations of motion.

- Transformation $\varphi(\varepsilon)$ is **symmetry** if for some $F(\underline{q}, t)$,

$$L \to L' = L(\phi(q_1, \varepsilon), ..., \phi(q_N, \varepsilon)) = L + \varepsilon \frac{dF(q_1, ..., q_N, t)}{dt} + O(\varepsilon^2)$$

$F(\underline{q}, t)$ defined up to a constant.

- For ignorable coordinate q_i, transformation $q_i \to q_i + c_i$ is symmetry since q_i doesn't appear in Lagrangian and $\dot{q}_i$ stays invariant. So $F = 0$ here and $a_k = \delta_{ik}$.
- **Noether's theorem**: Let a symmetric transformation be generated by $a_i(q_1, ..., q_N)$, so

$$L \to L + \varepsilon \frac{dF(q_1, ..., q_N, t)}{dt} + O(\varepsilon^2)$$

Then

$$Q := \left(\sum_{i=1}^{N} a_i \frac{\partial L}{\partial \dot{q}_i} \right) - F$$

is conserved (so $\frac{dQ}{dt} = 0$).

- Q is called **Noether charge**.
- Given Lagrangian $L(\underline{q}, \underline{\dot{q}}, t)$, **energy** is

$$E := \left(\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L$$

- Along path $\underline{q}(t)$ satisfying equations of motion,

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}$$

- So energy conserved iff Lagrangian doesn't depend explicitly on time.

3. Normal modes

- **Canonical** kinetic term: of the form $T = \frac{1}{2} \sum_{i=1}^{n} \dot{q}_i^2$.

- **Normal mode**: solution to $\underline{\ddot{q}} + A\underline{q} = 0$, associated with eigenvalue $\lambda^{(i)} > 0$ of A, of form

$$\underline{q}(t) = \underline{v}^{(i)}\left(\alpha^{(i)}\cos\left(\sqrt{\lambda^{(i)}}t\right) + \beta^{(i)}\sin\left(\sqrt{\lambda^{(i)}}t\right)\right)$$

- **Zero mode**: solution to $\underline{\ddot{q}} + A\underline{q} = 0$, associated with eigenvalue $\lambda^{(i)} = 0$ of A, of form

$$\underline{q}(t) = \underline{v}^{(i)}\left(\alpha^{(i)}t + \beta^{(i)}\right)$$

- **Instability**: solution to $\underline{\ddot{q}} + A\underline{q} = 0$, associated with eigenvalue $\lambda^{(i)} < 0$ of A, of form

$$\underline{q}(t) = \underline{v}^{(i)}\left(\alpha^{(i)}\cosh\left(\sqrt{-\lambda^{(i)}}t\right) + \beta^{(i)}\sinh\left(\sqrt{-\lambda^{(i)}}t\right)\right)$$

- When no instabilities, general solution is superposition (sum) of normal modes and zero modes.

4. Fields and the wave equation

- **Generalised Euler-Lagrange equations for fields**:

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) = 0$$

and for n fields $u^{(i)}$:

$$\frac{\partial \mathcal{L}}{\partial u^{(i)}} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x^{(i)}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t^{(i)}}\right) = 0 \quad \forall i$$

- If fields don't depend on (t, x) but on d coordinates x_i,

$$\frac{\partial \mathcal{L}}{\partial u^{(i)}} - \sum_{k=1}^{d}\frac{\partial}{\partial x_k}\left(\frac{\partial \mathcal{L}}{\partial u_k^{(i)}}\right)$$

where $u_k^{(i)} = \frac{\partial u^{(i)}}{\partial x_k}$

- **Massless scalar field Lagrangian**:

$$\mathcal{L} = \frac{1}{2}\rho u_t^2 - \frac{1}{2}\tau u_x^2$$

ρ is **density**, τ is **tension**. The field u is the **massless scalar**.

- Equation of motion for massless scalar field is

$$\rho u_{tt} - \tau u_{xx} = 0$$

which rearranges to **wave equation**:

$$u_{tt} = c^2 u_{xx}$$

where $c^2 = \tau / \rho$.

- **D'Alembert's solution to wave equation**:

$$u(x, t) = f(x - ct) + g(x + ct)$$

$f(x - ct)$ corresponds to a wave moving to the right with speed c, $g(x + ct)$ corresponds to a wave moving to the left with speed c.

- If $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = \psi(x)$ then

$$u(x, t) = \frac{1}{2}(\varphi(x - ct) + \varphi(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, \mathrm{d}s$$

- In field theory, **symmetry** is transformation

$$u \to u' = u + \varepsilon a(u)$$

such that $\delta \mathcal{L} = O(\varepsilon^2)$. $a(u)$ **generates** the transformation.

- **Note**: often, x_0 chosen to be t.
- Let $u_i = \frac{\partial u}{\partial x_i}$, **generalised momentum vector** is

$$\underline{\Pi} := \left(\frac{\partial \mathcal{L}}{\partial u_0}, ..., \frac{\partial \mathcal{L}}{\partial u_d} \right)$$

- **Noether current** associated to transformation generated by a is

$$\underline{J} = a \underline{\Pi}$$

- If $\underline{J}$ associated to symmetry,

$$\underline{\nabla} \cdot \underline{J} = \sum_{i=0}^{d} \frac{\partial J_i}{\partial x_i} = 0$$

- **(Noether) charge density**:

$$\mathcal{Q} := J_0$$

- For $d = 1$, **charge contained in interval $(\boldsymbol{a}, \boldsymbol{b})$**:

$$Q_{(a,b)} = \int_a^b \mathcal{Q} \, \mathrm{d}x$$

- For $d = 1$,

$$\frac{dQ_{(a,b)}}{dt} = J_1(a) - J_1(b)$$

- **Noehter charge** is total charge over all space. For $d = 1$:

$$Q := Q_{(-\infty,\infty)} = \int_{-\infty}^{\infty} J_0 \, \mathrm{d}x$$

- If $d = 1$ and $\lim_{x \to \pm\infty} J_1 = 0$,

$$\frac{dQ}{dt} = 0$$

- **Energy-momentum tensor**:

$$T_{ij} := \frac{\partial \mathcal{L}}{\partial u_j} \frac{\partial u}{\partial x_i} - \delta_{ij} \mathcal{L}$$

- **Energy density**:

$$\mathcal{E} := T_{00}$$

- **Conservation law for energy-momentum tensor**:

$$\sum_{j=0}^{d} \frac{\partial T_{ij}}{\partial x_j} = 0$$

- **Dirichlet boundary condition** for wave equation: $u_t(0, t) = 0$ (so $u(0, t) = 0$ as u has shift symmetry) which gives

$$u(x, t) = f(x - ct) - f(-x - ct)$$

  Here, waves reflected off boundary and turned upside down.
- **Neumann (free) boundary condition**: $u_x(0, t) = 0$ which gives

$$u(x, t) = f(x - ct) + f(-x - ct)$$

  So waves reflected off boundary and not turned upside down.
- **Junction conditions**:
  - u continuous at 0:

$$\lim_{\varepsilon \to 0^+} u(\varepsilon, t) = \lim_{\varepsilon \to 0^-} u(\varepsilon, t)$$

  - Energy conservation across junction:

$$\frac{d}{dt}\left(\lim_{\varepsilon \to 0} T(-\varepsilon, \varepsilon)\right) = \lim_{\varepsilon \to 0} (T_{tx})_{x=-\varepsilon} - \lim_{\varepsilon \to 0} (T_{tx})_{x=\varepsilon}$$

- **Ansatz for wave function with spring at junction at $x = 0$**:

$$u(x, t) = \begin{cases} \mathrm{Re}((e^{ipx} + Re^{-ipx})e^{-ipct}) & \text{if } x \leq 0 \\ \mathrm{Re}\left(Te^{ip(x-ct)}\right) & \text{if } x > 0 \end{cases}$$