

1. Metric spaces

1.1. Metrics

- **Metric space:** (X, d) , X is set, $d : X \times X \rightarrow [0, \infty)$ is **metric** satisfying:
 - $d(x, y) = 0 \iff x = y$
 - **Symmetry:** $d(x, y) = d(y, x)$
 - **Triangle inequality:** $d(x, y) \leq d(x, z) + d(z, y)$
- Examples of metrics:
 - p -adic metric:

$$d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}$$

- Extension of the p -adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

- Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

- Discrete metric:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

- **Open ball of radius r around x :**

$$B(x; r) = \{y \in X : d(x, y) < r\}$$

- **Closed ball of radius r around x :**

$$D(x; r) = \{y \in X : d(x, y) \leq r\}$$

1.2. Open and closed sets

- $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

- $A \subseteq X$ is **closed** if $X - A$ is open.
- Sets can be neither closed nor open, or both.
- Any singleton $\{x\} \in \mathbb{R}$ is closed and not open.
- Let X be metric space, $x \in N \subseteq X$. N is **neighbourhood** of x if

$$\exists \text{ open } V \subseteq X : x \in V \subseteq N$$

- **Corollary:** let $x \in X$, then $N \subseteq X$ neighbourhood of x iff $\exists \varepsilon > 0 : x \in B(x; \varepsilon) \subseteq N$.
- **Proposition:** open balls are open, closed balls are closed.
- **Lemma:** let (X, d) metric space.
 - X and \emptyset are both open and closed.
 - Arbitrary unions of open sets are open.
 - Finite intersections of open sets are open.

- Finite unions of closed sets are closed.
- Arbitrary intersections of closed sets are closed.

1.3. Continuity

- **Sequence** in X : $a : \mathbb{N} \rightarrow X$, written $(a_n)_{n \in \mathbb{N}}$.
- (a_n) converges to a if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, d(a, a_n) < \varepsilon$$

- **Proposition:** let X, Y metric spaces, $a \in X$, $f : X \rightarrow Y$. The following are equivalent
 - $\forall \varepsilon > 0, \exists \delta > 0 : d_X(a, x) < \delta \implies d_Y(f(a), f(x)) < \varepsilon$.
 - For every sequence (a_n) in X with $a_n \rightarrow a$, $f(a_n) \rightarrow f(a)$.
 - For every open $U \subseteq Y$ with $f(a) \in U$, $f^{-1}(U)$ is a neighbourhood of a .

If f satisfies these, it is **continuous at a** .

- f **continuous** if continuous at every $a \in X$.
- **Proposition:** $f : X \rightarrow Y$ continuous iff $f^{-1}(U)$ open for every open $U \subseteq Y$.

2. Topological spaces

2.1. Topologies

- **Power set** of X : $\mathcal{P}(X) := \{A : A \subseteq X\}$.
- **Topology** on set X is $\tau \subseteq \mathcal{P}(X)$ with:
 - $\emptyset \in \tau, X \in \tau$.
 - If $\forall i \in I, U_i \in \tau$, then

$$\bigcup_{i \in I} U_i \in \tau$$

- $U_1, U_2 \in \tau \implies U_1 \cap U_2 \in \tau$ (this is equivalent to $U_1, \dots, U_n \in \tau \implies \bigcap_{i \in [n]} U_i \in \tau$).
- (X, τ) is **topological space**. Elements of τ are **open** subsets of X .
- $A \subseteq X$ **closed** if $X - A$ is open.
- Let X be a set. Then $\tau = \mathcal{P}(X)$ is the **discrete topology** on X .
- $\tau = \{\emptyset, X\}$ is the **indiscrete topology** on X .
- **Examples:**
 - For metric space (M, d) , find the open sets with respect to metric d . Let $\tau_d \subseteq \mathcal{P}(M)$ exactly contain these open sets. Then (M, τ_d) is a topological space. The metric d **induces** the topology τ_d .
 - Let $X = \mathbb{N}_0$ and $\tau = \{\emptyset\} \cup \{U \subseteq X : X - U \text{ is finite}\}$.
- **Proposition:** for topological space X :
 - X and \emptyset are closed
 - Arbitrary intersections of closed sets are closed
 - Finite unions of closed sets are closed
- **Proposition:** for topological space (X, τ) and $A \subseteq X$, the **induced (subspace) topology on A**

$$\tau_A = \{A \cap U : U \in \tau\}$$

is a topology on A .

- **Example:** let $X = \mathbb{R}$ with standard topology induced by metric $d(x, y) = |x - y|$. Let $A = [1, 5]$. Then $[1, 3) = A \cap (0, 3)$ and $[1, 5] = A \cap (0, 6)$ are open in A .
- **Example:** consider \mathbb{R} with standard topology τ . Then
 - $\tau_{\mathbb{Z}}$ is the discrete topology on \mathbb{Z} .
 - $\tau_{\mathbb{Q}}$ is not the discrete topology on \mathbb{Q} .
- **Proposition:** the metrics d_p for $p \in [1, \infty)$ and d_{∞} all induce the same topology on \mathbb{R}^n .
- **Definition:** (X, τ) is **Hausdorff** if

$$\forall x \neq y \in X, \exists U, V \in \tau : U \cap V = \emptyset \wedge x \in U, y \in V$$

- **Lemma:** any metric space (M, d) is Hausdorff.
- **Example:** let $|X| \geq 2$ with the indiscrete topology. Then X is not Hausdorff, since $\tau = \{X, \emptyset\}$ and if $x \neq y \in X$, the only open set containing x is X (same for y). But $X \cap X = X \neq \emptyset$.
- **Furstenberg's topology on \mathbb{Z} :** define $U \subseteq \mathbb{Z}$ to be open if

$$\forall a \in U, \exists 0 \neq d \in \mathbb{Z} : a + d\mathbb{Z} =: \{a + dn : n \in \mathbb{Z}\} \subseteq U$$

- Furstenberg's topology is Hausdorff.

2.2. Continuity

- **Definition:** let X, Y topological spaces.
 - $f : X \rightarrow Y$ is **continuous** if

$$\forall V \text{ open in } Y, f^{-1}(V) \text{ open in } X$$

- f is **continuous at $a \in X$** if

$$\forall V \text{ open in } Y, f(a) \in V, \exists U \text{ open in } X : a \in U \subseteq f^{-1}(V)$$

- **Lemma:** $f : X \rightarrow Y$ continuous iff f continuous at every $a \in X$. (Key idea for proof: $\cup_{a \in f^{-1}(V)} U_a \subseteq f^{-1}(V) = \cup_{a \in f^{-1}(V)} \{a\} \subseteq \cup_{a \in f^{-1}(V)} U_a$)
- **Example:** inclusion $i : (A, \tau_A) \rightarrow (X, \tau_X)$, $A \subseteq X$, is always continuous.
- **Lemma:** a composition of continuous functions is continuous.
- **Lemma:** let $f : X \rightarrow Y$ be function between topological spaces. Then f is continuous iff

$$\forall A \text{ closed in } Y, f^{-1}(A) \text{ closed in } X$$

- **Remark:** we can use continuous functions decide that sets are open or closed.
- **Definition:** **n -sphere** is

$$S^n := \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 = 1 \right\}$$

- **Example:** in the standard topology, the n -sphere is a closed subset of \mathbb{R}^{n+1} . (Consider the preimage of $\{1\}$ which is closed in \mathbb{R}).
- Can consider set of square matrices $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and give it the standard topology.

- **Example:**

- Note

$$\det(A) = \sum_{\sigma \in \text{sym}(n)} \left(\text{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)} \right)$$

is a polynomial in the entries of A so is continuous function from $M_n(\mathbb{R})$ to \mathbb{R} .

- $\text{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} = \det^{-1}(\mathbb{R} - \{0\})$ is open.
- $\text{SL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} = \det^{-1}(\{1\})$ is closed.
- $O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I\}$ is closed - consider $f_{i,j}(A) = (AA^T)_{i,j}$ then

$$O(n) = \bigcap_{1 \leq i, j \leq n} (f_{i,j})^{-1}(\{\delta_{i,j}\})$$

- $\text{SO}(n) = O(n) \cap \text{SL}_n(\mathbb{R})$ is closed.

- **Definition:** for X, Y topological spaces, $h : X \rightarrow Y$ is **homeomorphism** if h is bijective, continuous and h^{-1} is continuous. X and Y are **homeomorphic**. A homeomorphism gives bijection between τ_X and τ_Y which satisfies

$$h(A \cap B) = h(A) \cap h(B), \quad h(A \cup B) = h(A) \cup h(B)$$

- **Example:** in standard topology, $(0, 1)$ is homeomorphic to \mathbb{R} . (Consider $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow (-\infty, \infty)$, $f = \tan$, $g : (0, 1) \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$, $g(x) = \pi(x - \frac{1}{2})$ and $f \circ g$).
- **Example:** \mathbb{R} with standard topology τ_{st} is not homeomorphic to \mathbb{R} with the discrete topology τ_d . (Consider $h^{-1}(\{a\}) = \{h^{-1}(a)\}$, $\{a\} \in \tau_{\text{st}}$ but $\{h^{-1}(a)\} \notin \tau_{\text{st}}$).
- **Example:** let $X = \mathbb{R} \cup \{\bar{0}\}$. Define $f_0 : \mathbb{R} \rightarrow X$, $f_0(a) = a$ and $f_{\bar{0}} : \mathbb{R} \rightarrow X$, $f_{\bar{0}}(a) = a$ for $a \neq 0$, $f_{\bar{0}}(0) = \bar{0}$. Topology on X has $A \subseteq X$ open iff $f_0^{-1}(A)$ and $f_{\bar{0}}^{-1}(A)$ open. Every point in X lies in open set: for $a \notin \{0, \bar{0}\}$, $a \in (a - \frac{|a|}{2}, a + \frac{|a|}{2})$ and both pre-images of this are same open interval, for 0, set $U_0 = (-1, 0) \cup \{0\} \cup (0, 1) \subseteq X$ then $f_0^{-1}(U_0) = (-1, 1)$ and $f_{\bar{0}}^{-1}(U_0) = (-1, 0) \cup (0, 1)$ are both open. For $\bar{0}$, set $U_{\bar{0}} = (-1, 0) \cup \{\bar{0}\} \cup (0, 1) \subseteq X$, then $f_0^{-1}(U_{\bar{0}}) = (-1, 1)$ and $f_{\bar{0}}^{-1}(U_{\bar{0}}) = (-1, 0) \cup (0, 1)$ are both open. So U_0 and $U_{\bar{0}}$ both open in X . X is not Hausdorff since any open sets containing 0 and $\bar{0}$ must contain “open intervals” such as U_0 and $U_{\bar{0}}$.
- **Example (Furstenberg’s proof of infinitude of primes):** since $a + d\mathbb{Z}$ is infinite, any nonempty finite set is not open, so any set with finite complement is not closed. For fixed d , sets $d\mathbb{Z}, 1 + d\mathbb{Z}, \dots, (d-1) + d\mathbb{Z}$ partition \mathbb{Z} . So the complement of each is the union of the rest, so each is open and closed. Every $n \in \mathbb{Z} - \{-1, 1\}$ is prime or product of primes, so $\mathbb{Z} - \{-1, 1\} = \bigcup_{p \text{ prime}} p\mathbb{Z}$, but finite unions of closed sets are closed, and since $\mathbb{Z} - \{-1, 1\}$ has finite complement, the union must be infinite.

3. Limits, bases and products

3.1. Limit points, interiors and closures

- **Definition:** for topological space X , $x \in X$, $A \subseteq X$:

- **Open neighbourhood of x** is open set N , $x \in N$.
- $x \in X$ is **limit point** of A if every open neighbourhood N of x satisfies

$$(N - \{x\}) \cap A \neq \emptyset$$

- **Corollary:** x is not limit point of A iff exists neighbourhood N of x with

$$A \cap N = \begin{cases} \{x\} & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

- **Example:** let $X = \mathbb{R}$ with standard topology.
 - $0 \in X$, then $(-1/2, 1/2)$ is open neighbourhood of 0.
 - If $U \subseteq X$ open, U is open neighbourhood for any $x \in U$.
 - Let $A = \{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\}$, then only limit point in A is 0.
- **Definition:** let $A \subseteq X$.
 - **Interior** of A is largest open set contained in A :

$$A^\circ = \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U$$

- **Closure** of A is smallest closed set containing A :

$$\overline{A} = \bigcap_{\substack{F \text{ closed} \\ A \subseteq F}} F$$

If $A^\circ = X$, A is **dense** in X .

- **Lemma:**
 - $\overline{X - A} = X - A^\circ$
 - $\overline{A} = X - (X - A)^\circ$
- **Example:**
 - Let $\mathbb{Q} \subset \mathbb{R}$ with standard topology. Then $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$ (since every nonempty open set in \mathbb{R} contains rational and irrational numbers).
- **Lemma:** $\overline{A} = A \cup L$ where L is the set of limit points of A .
- **Dirichlet prime number theorem:** let a, d coprime, the set $a + d\mathbb{Z}$ contains infinitely many primes.
- **Example:** let A be the set of primes in \mathbb{Z} with the Furstenberg topology. By the above lemma, we only need to find the limit points in $\mathbb{Z} - A$ to find \overline{A} . $10\mathbb{Z}$ is an open neighbourhood of 0 for 0 inside $\mathbb{Z} - A$. For $a \notin \{-1, 0, 1\}$, $a + 10a\mathbb{Z}$ is an open neighbourhood of a . These sets have no primes so the corresponding points are not limit points of A . For ± 1 , any open neighbourhood of 1 contains a set $\pm 1 + d\mathbb{Z}$ for some $d \neq 0$, but by the Dirichlet prime number theorem, this set contains at least one prime. So $\overline{A} = A \cup \{\pm 1\}$.
- **Lemma:**
 - Let $A \subseteq M$ for metric space M . If x is limit point of A then exists sequence x_n in A such that $\lim_{n \rightarrow \infty} x_n = x$.
 - If $x \in M - A$ and exists sequence x_n in A with $\lim_{n \rightarrow \infty} x_n = x$ then x is limit point of A .

3.2. Bases

- **Definition:** a **basis** for topology τ on X is collection $\mathcal{B} \subseteq \tau$ such that

$$\forall U \in \tau, U = \bigcup_{b \in \mathcal{B}} b$$

(every open U is a union of sets in \mathcal{B}).

- **Example:**
 - For metric space (M, d) , $\mathcal{B} = \{B(x; r) : x \in M, r > 0\}$ is basis for the induced topology. (Since if U open, $U = \bigcup_{u \in U} \{u\} \subseteq \bigcup_{u \in U} B(u, r_u) \subseteq U$.)
 - In \mathbb{R}^n with standard topology, $\mathcal{B} = \{B(q; 1/m) : q \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a **countable** basis. (Find $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{r}{2}$ and $q \in \mathbb{Q}^n$ such that $q \in B(p; \frac{1}{m})$, then $B(q; \frac{1}{m}) \subseteq B(p; r) \subseteq U$ using the triangle inequality).
- **Theorem:** let $f : X \rightarrow Y$ be map between topological spaces. The following are equivalent:
 - f is continuous.
 - If \mathcal{B} is basis for topology τ on Y then $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.
 - $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$.
 - $\forall V \subseteq Y, \overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$.
 - $f^{-1}(C)$ closed for any closed set $C \subseteq Y$.
- **Theorem:** let X be a set and collection $\mathcal{B} \subseteq \mathcal{P}(X)$ be such that:
 - $\forall x \in X, \exists B \in \mathcal{B} : x \in B$
 - If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$.

Then there is unique topology $\tau_{\mathcal{B}}$ on X for which \mathcal{B} is a basis. We say \mathcal{B} **generates** $\tau_{\mathcal{B}}$.

3.3. Product topologies

- **Definition: Cartesian product** of topological spaces X, Y is $X \times Y := \{(x, y) : x \in X, y \in Y\}$. We give it the **product topology** which is generated by $\mathcal{B}_{X \times Y} := \{U \times V : U \in \tau_X, V \in \tau_Y\}$.
- **Example:**
 - Let $X = Y = \mathbb{R}$, then product topology is same as standard topology on \mathbb{R}^2 .
 - Let $X = Y = S^1$, then $X \times Y = T^2 = S^1 \times S^1$ is the 2-torus.
- **Definition:** if $\tau_1 \subseteq \tau_2$, then τ_1 is **smaller** than τ_2 (τ_2 is **larger** than τ_1).
- **Definition:** for topological spaces X, Y , **projection maps** $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are

$$\pi_X(x, y) = x, \quad \pi_Y(x, y) = y$$

- **Proposition:** for $X \times Y$ with product topology,
 - π_X and π_Y are continuous.
 - π_X and π_Y map open sets to open sets.
 - Product topology is smallest topology for which π_X and π_Y are continuous.
- **Proposition:** let X, Y, Z topological spaces, then $f : Z \rightarrow X \times Y$ (with product topology on $X \times Y$) continuous iff both $\pi_X \circ f : Z \rightarrow X$ and $\pi_Y \circ f : Z \rightarrow Y$ are continuous.

- **Exercise (todo):** prove above proposition.
- **Example:** let $f : X \rightarrow \mathbb{R}^n$, $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $\pi_i(x) = x_i$, $f_i = \pi_i \circ f$, then f is continuous iff all f_i are continuous.
- **Proposition:** let X, Y nonempty topological spaces. Then $X \times Y$ is Hausdorff iff X and Y are both Hausdorff.

4. Connectedness

4.1. Clopen sets and examples

- **Definition:** let X topological space, then $A \subseteq X$ is **clopen** if A is open and closed.
- **Definition:** X is **connected** if the only clopen sets in X are X and \emptyset .
- **Example:**
 - \mathbb{R} with standard topology is connected.
 - \mathbb{Q} with induced topology from \mathbb{R} is not connected (consider $L = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q} - L = \mathbb{Q} \cap (\sqrt{2}, \infty)$).
 - The connected subsets of \mathbb{R} are the intervals.
- $A \subseteq \mathbb{R}$ is an interval iff $\forall x, y \in A, x < z < y \implies z \in A$.
- **Example:**
 - $X = \{0, 1\}$ with discrete topology is not connected ($\{1\}$ and $\{0\}$ both open so both closed).
 - $X = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$ is connected.
 - \mathbb{Z} with Furstenberg topology is not connected.
- **Theorem (continuity preserves connectedness):** if $h : X \rightarrow Y$ continuous and X connected, then $h(X) \subseteq Y$ is connected.
- **Corollary:** if $h : X \rightarrow Y$ is homeomorphism and X is connected then Y is connected.
- **Theorem:** let X topological space. The following are equivalent:
 - X is connected.
 - X cannot be written as disjoint union of two non-empty sets.
 - There exists no continuous surjective function from X to a discrete space with more than one point.
- **Example:**
 - $\text{GL}_n(\mathbb{R})$ is not connected (since $\det : \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R} - \{0\}$ is continuous and surjective and $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$).
 - $O(n)$ is not connected.
 - $(0, 1)$ is connected (since $\mathbb{R} \cong (0, 1)$ and \mathbb{R} is connected).
 - $X = (0, 1]$ and $Y = (0, 1)$ are not homeomorphic (if they are, then $(0, 1]$ is connected since $(0, 1)$ is).
- **Definition:** let $A = B \cup C$, $B \cap C = \emptyset$, then B and C are **complementary subsets** of A .
- **Remark:** if B and C are open in A , then B and C are clopen in A . So if $B, C \neq \emptyset$ then A is not connected.

4.2. Constructing more connected sets, components, path-connectedness

Proposition: let X topological space, $Z \subseteq X$ connected. If $Z \subseteq Y \subseteq \overline{Z}$ then Y is connected. In particular, with $Y = \overline{Z}$, the closure of a connected set is connected.