Contents

1. Metric spaces

1.1. Metrics

Definition. **Metric space** is (X, d), X is set, $d : X \times X \to [0, \infty)$ is **metric** satisfying:

- $d(x, y) = 0 \Longleftrightarrow x = y$
- **Symmetry**: $d(x, y) = d(y, x)$
- **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$

Example.

- p-adic metric: for $p \in [1, \infty)$

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

- Extension of the p-adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

- Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

- Discrete metric:

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Definition. **Open ball of radius *r* around *x***:

$$B(x; r) \coloneqq \{y \in X : d(x, y) < r\}$$

Definition. **Closed ball of radius *r* around *x***:

$$D(x; r) \coloneqq \{y \in X : d(x, y) \leq r\}$$

1.2. Open and closed sets

Definition. $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

Definition. $A \subseteq X$ is **closed** if $X - A$ is open.

Note. Sets can be neither closed nor open, or both.

Example. With standard metric on $\mathbb{R}$, any singleton $\{x\} \in \mathbb{R}$ is closed and not open (same holds for $\mathbb{R}^n$).

Definition. Let X be metric space, $x \in N \subseteq X$. N is **neighbourhood** of x if

$$\exists \text{ open } V \subseteq X : x \in V \subseteq N$$

Corollary. Let $x \in X$, then $N \subseteq X$ neighbourhood of x iff $\exists \varepsilon > 0 : x \in B(x; \varepsilon) \subseteq N$.

Proposition. Open balls are open, closed balls are closed.

Lemma. Let (X, d) metric space.
- X and $\emptyset$ are both open and closed.
- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.
- Finite unions of closed sets are closed.
- Arbitrary intersections of closed sets are closed.

Example. If X has discrete metric, any $A \subseteq X$ is open and closed.

1.3. Continuity

Definition.

- **Sequence** in X is $a : \mathbb{N}_0 \to X$, written $(a_n)_{n\in\mathbb{N}}$.
- (a_n) **converges to** $\boldsymbol{a}$, $\lim_{n\to\infty} a_n = a$, if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, d(a, a_n) < \varepsilon$$

Proposition. Let X, Y metric spaces, $a \in X$, $f : X \to Y$. The following are equivalent:
- $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in X, d_X(a, x) < \delta \Longrightarrow d_Y(f(a), f(x)) < \varepsilon$.
- For every sequence (a_n) in X with $a_n \to a$, $f(a_n) \to f(a)$.
- For every open $U \subseteq Y$ with $f(a) \in U$, $f^{-1}(U)$ is a neighbourhood of a.

If f satisfies these, it is **continuous at** $\boldsymbol{a}$.

Definition. f **continuous** if continuous at every $a \in X$.

Proposition. $f : X \to Y$ continuous iff $f^{-1}(U)$ open for every open $U \subseteq Y$.

Example. Let d be discrete metric, d_2 be 2-adic metric.
- Any $f : (X, d) \to (\mathbb{R}, d_2)$ is continuous.
- $\mathrm{id} : (\mathbb{R}, d_2) \to (\mathbb{R}, d)$ is not continuous.

2. Topological spaces

2.1. Topologies

Definition. **Power set** of X: $\mathcal{P}(X) := \{A : A \subseteq X\}$.

Definition. **Topology** on set X is $\tau \subseteq \mathcal{P}(X)$ with:
- $\emptyset \in \tau$, $X \in \tau$.
- **Closure under arbitrary unions**: if $\forall i \in I, U_i \in \tau$, then

$$\bigcup_{i\in I} U_i \in \tau$$

- **Closure under finite intersections**: $U_1, U_2 \in \tau \Longrightarrow U_1 \cap U_2 \in \tau$ (this is equivalent to $U_1, ..., U_n \in \tau \Longrightarrow \bigcap_{i\in[n]} U_i \in \tau$).

(X, τ) is **topological space**. Elements of τ are **open** subsets of X. $A \subseteq X$ **closed** if $X - A$ is open.

Definition. $\tau = \mathcal{P}(X)$ is the **discrete topology** on X.

Definition. $\tau = \{\emptyset, X\}$ is the **indiscrete topology** on X.

Example.

- For metric space (M, d), let τ_d exactly contain sets which are open with respect to d. Then (M, τ_d) is a topological space. d **induces** topology τ_d.
- Let $X = \mathbb{N}_0$ and $\tau = \{\emptyset\} \cup \{U \subseteq X : X - U \text{ is finite}\}$, then (X, τ) is topological space.

Proposition. For topological space X:

- X and $\emptyset$ are closed
- Arbitrary intersections of closed sets are closed
- Finite unions of closed sets are closed

Proposition. For topological space (X, τ) and $A \subseteq X$, the **induced (subspace) topology on A**

$$\tau_A = \{A \cap U : U \in \tau\}$$

is a topology on A.

Example. Let $X = \mathbb{R}$ with standard topology induced by metric $d(x, y) = |x - y|$. Let $A = [1, 5]$. Then $[1, 3) = A \cap (0, 3)$ and $[1, 5] = A \cap (0, 6)$ are open in A.

Example. Consider $\mathbb{R}$ with standard topology τ. Then

- $\tau_{\mathbb{Z}}$ is the discrete topology on $\mathbb{Z}$.
- $\tau_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.

Proposition. Metrics d_p for $p \in [1, \infty)$ and d_∞ all induce same topology on $\mathbb{R}^n$, alled the **standard topology** on $\mathbb{R}^n$.

Definition. (X, τ) is **Hausdorff** if

$$\forall x \neq y \in X, \exists U, V \in \tau : U \cap V = \emptyset \land x \in U, y \in V$$

Lemma. Any metric space (M, d) with topology induced by d is Hausdorff.

Example. Let $|X| \geq 2$ with indiscrete topology. Then X is not Hausdorff, since $\tau = \{X, \emptyset\}$ and if $x \neq y \in X$, only open set containing x is X (same for y). But $X \cap X = X \neq \emptyset$.

Definition. **Furstenberg's topology on** $\mathbb{Z}$: define $U \subseteq \mathbb{Z}$ to be open if

$$\forall a \in U, \exists 0 \neq d \in \mathbb{Z} : a + d\mathbb{Z} := \{a + dn : n \in \mathbb{Z}\} \subseteq U$$

- Furstenberg's topology is Hausdorff.

2.2. Continuity

Definition. Let X, Y topological spaces.

- $f : X \to Y$ is **continuous** if

$$\forall V \text{ open in } Y, f^{-1}(V) \text{ open in } X$$

- f is **continuous at $a \in X$** if

$$\forall V \text{ open in } Y \text{ with } f(a) \in V, \exists U \text{ open in } X : a \in U \subseteq f^{-1}(V)$$

Lemma. $f : X \to Y$ continuous iff f continuous at every $a \in X$. (Key idea for proof: $\cup_{a \in f^{-1}(V)} U_a \subseteq f^{-1}(V) = \cup_{a \in f^{-1}(V)} \{a\} \subseteq \cup_{a \in f^{-1}(V)} U_a$)

Example. Inclusion $i : (A, \tau_A) \to (X, \tau_X)$, $A \subseteq X$, is always continuous.

Lemma. Compositions of continuous functions are continuous.

Lemma. Let $f : X \to Y$ be function between topological spaces. Then f is continuous iff

$$\forall A \text{ closed in } Y, \quad f^{-1}(A) \text{ closed in } X$$

Remark. We can use continuous functions to decide that sets are open or closed.

Definition. ***n*-sphere** is

$$S^n := \left\{ (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

Example. In the standard topology, the n-sphere is a closed subset of $\mathbb{R}^{n+1}$. (Consider the preimage of $\{1\}$ which is closed in $\mathbb{R}$).

Example.

- Can consider set of square matrices $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and give it the standard topology.
- Note

  $$\det(A) = \sum_{\sigma \in \text{ sym}(n)} \left(\text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

  is a polynomial in the entries of A so is continuous function from $M_n(\mathbb{R})$ to $\mathbb{R}$.
- $\text{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} = \det^{-1}(\mathbb{R} - \{0\})$ is open.
- $\text{SL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} = \det^{-1}(\{1\})$ is closed.
- $O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I\}$ is closed: $f_{i,j}(A) = (AA^T)_{i,j}$ is continuous and

  $$O(n) = \bigcap_{1 \leq i,j \leq n} (f_{i,j})^{-1}(\{\delta_{i,j}\})$$

- $\text{SO}(n) = O(n) \cap \text{SL}_n(\mathbb{R})$ is closed.

Definition. For X, Y topological spaces, $h : X \to Y$ is **homeomorphism** if h is bijective, continuous and h^{-1} is continuous. X and Y are **homeomorphic**, $X \cong Y$. h induces bijection between τ_X and τ_Y which commutes with unions and intersections.

Proposition. Compositions of homeomorphisms are homeomorphisms.

Example. In standard topology, $(0, 1)$ is homeomorphic to $\mathbb{R}$. (Consider $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$, $f = \tan$, $g : (0, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $g(x) = \pi\left(x - \frac{1}{2}\right)$ and $f \circ g$).

Example. $\mathbb{R}$ with standard topology τ_{st} is not homoeomorphic to $\mathbb{R}$ with the discrete topology τ_d. (Consider $h^{-1}(\{a\}) = \{h^{-1}(a)\}$, $\{a\} \in \tau_d$ but $\{h^{-1}(a)\} \notin \tau_{\text{st}}$).

Example. Let $X = \mathbb{R} \cup \{\overline{0}\}$. Define $f_0 : \mathbb{R} \to X$, $f_0(a) = a$ and $f_{\overline{0}} : \mathbb{R} \to X$, $f_{\overline{0}}(a) = a$ for $a \neq 0$, $f_{\overline{0}}(0) = \overline{0}$. Topology on X has $A \subseteq X$ open iff $f_0^{-1}(A)$ and $f_{\overline{0}}^{-1}(A)$

open. Every point in X lies in open set: for $a \notin \{0, \overline{0}\}$, $a \in (a - \frac{|a|}{2}, a + \frac{|a|}{2})$ and both pre-images of this are same open interval, for 0, set $U_0 = (-1, 0) \cup \{0\} \cup (0, 1) \subseteq X$ then $f_0^{-1}(U_0) = (-1, 1)$ and $f_{\overline{0}}^{-1}(U_0) = (-1, 0) \cup (0, 1)$ are both open. For $\overline{0}$, set $U_{\overline{0}} = (-1, 0) \cup \{\overline{0}\} \cup (0, 1) \subseteq X$, then $f_{\overline{0}}^{-1}(U_{\overline{0}}) = (-1, 1)$ and $f_0^{-1}(U_{\overline{0}}) = (-1, 0) \cup (0, 1)$ are both open. So U_0 and $U_{\overline{0}}$ both open in X. X is not Hausdorff since any open sets containing 0 and $\overline{0}$ must contain "open intervals" such as U_0 and $U_{\overline{0}}$.

Example (Furstenberg's proof of infinitude of primes). Since $a + d\mathbb{Z}$ is infinite, any nonempty finite set is not open, so any set with finite complement is not closed. For fixed d, sets $d\mathbb{Z}$, $1 + d\mathbb{Z}, ..., (d-1) + d\mathbb{Z}$ partition $\mathbb{Z}$. So the complement of each is the union of the rest, so each is open and closed. Every $n \in \mathbb{Z} - \{-1, 1\}$ is prime or product of primes, so $\mathbb{Z} - \{-1, 1\} = \bigcup_{p \text{ prime}} p\mathbb{Z}$, but finite unions of closed sets are closed, and since $\mathbb{Z} - \{-1, 1\}$ has finite complement, the union must be infinite.

3. Limits, bases and products

3.1. Limit points, interiors and closures

Definition. For topological space X, $x \in X$, $A \subseteq X$:
- **Open neighbourhood of x** is open set N, $x \in N$.
- x is **limit point** of A if every open neighbourhood N of x satisfies

$$(N - \{x\}) \cap A \neq \emptyset$$

Corollary. x is not limit point of A iff exists neighbourhood N of x with

$$A \cap N = \begin{cases} \{x\} & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

Example. Let $X = \mathbb{R}$ with standard topology.
- $0 \in X$, then $(-1/2, 1/2)$ is open neighbourhood of 0.
- If $U \subseteq X$ open, U is open neighbourhood for any $x \in U$.
- Let $A = \left\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\right\}$, then only limit point in A is 0.

Definition. Let $A \subseteq X$.
- **Interior** of A is largest open set contained in A:

$$A^\circ := \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U$$

- **Closure** of A is smallest closed set containing A:

$$\overline{A} := \bigcap_{\substack{F \text{ closed} \\ A \subseteq F}}$$

  If $\overline{A} = X$, A is **dense** in X.

Lemma.

- $\overline{X - A} = X - A^\circ$

- $\overline{A} = X - (X - A)^\circ$

Example. Let $\mathbb{Q} \subset \mathbb{R}$ with standard topology. Then $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$ (since every nonempty open set in $\mathbb{R}$ contains rational and irrational numbers).

Lemma. $\overline{A} = A \cup L$ where L is the set of limit points of A.

Theorem (Dirichlet prime number theorem). Let a, d coprime, then $a + d\mathbb{Z}$ contains infinitely many primes.

Example. Let A be set of primes in $\mathbb{Z}$ with Furstenberg topology. By above lemma, only need to find limit points in $\mathbb{Z} - A$ to find $\overline{A}$. $10\mathbb{Z}$ is an open neighbourhood of 0 for 0 inside $\mathbb{Z} - A$. For $a \notin \{-1, 0, 1\}$, $a + 10a\mathbb{Z}$ is an open neighbourhood of a. These sets have no primes so the corresponding points are not limit points of A. For ± 1, any open neighbourhood of 1 contains a set $\pm 1 + d\mathbb{Z}$ for some $d \neq 0$, but by the Dirichlet prime number theorem, this set contains at least one prime. So $\overline{A} = A \cup \{\pm 1\}$.

Lemma.

- Let $A \subseteq M$ for metric space M. If x is limit point of A then exists sequence x_n in A such that $\lim_{n \to \infty} x_n = x$.
- If $x \in M - A$ and exists sequence x_n in A with $\lim_{n \to \infty} x_n = x$ then x is limit point of A.

3.2. Bases

Definition. A **basis** for topology τ on X is collection $\mathcal{B} \subseteq \tau$ such that

$$\forall U \in \tau, \exists B \subseteq \mathcal{B} : U = \bigcup_{b \in B} b$$

(every open U is a union of sets in B).

Example.

- For metric space (M, d), $\mathcal{B} = \{B(x; r) : x \in M, r > 0\}$ is basis for the induced topology. (Since if U open, $U = \cup_{u \in U} \{u\} \subseteq \cup_{u \in U} B(u, r_u) \subseteq U$.)
- In $\mathbb{R}^n$ with standard topology, $\mathcal{B} = \{B(q; 1/m) : q \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a **countable** basis. (Find $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{r}{2}$ and $q \in \mathbb{Q}^n$ such that $q \in B\left(p; \frac{1}{m}\right)$, then $B\left(q; \frac{1}{m}\right) \subseteq B(p; r) \subseteq U$ using the triangle inequality).

Theorem. Let $f : X \to Y$ be map between topological spaces. The following are equivalent:

- f is continuous.
- If $\mathcal{B}$ is basis for topology τ on Y then $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.
- $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$.
- $\forall V \subseteq Y, \overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$.
- $f^{-1}(C)$ closed for any closed set $C \subseteq Y$.

Theorem. Let X be a set and collection $\mathcal{B} \subseteq \mathcal{P}(X)$ be such that:

- $\forall x \in X, \exists B \in \mathcal{B} : x \in B$
- If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$.

Then there is unique topology $\tau_{\mathcal{B}}$ on X for which $\mathcal{B}$ is a basis. We say $\mathcal{B}$ **generates** $\tau_{\mathcal{B}}$. We have $\tau_B = \{\cup_{i\in I} B_i : B_i \in \mathcal{B}, I \text{ indexing set}\}$.

3.3. Product topologies

Definition. **Cartesian product** of topological spaces X, Y is $X \times Y := \{(x, y) : x \in X, y \in Y\}$. We give it the **product topology** which is generated by $\mathcal{B}_{X\times Y} := \{U \times V : U \in \tau_X, V \in \tau_Y\}$.

Example.

- Let $X = Y = \mathbb{R}$, then product topology is same as standard topology on $\mathbb{R}^2$.
- Let $X = Y = S^1$, then $X \times Y = T^2 = S^1 \times S^1$ is the **2-torus**. ***n*-torus** is defined for $n \geq 3$ by

$$T^n := S^1 \times T^{n-1}$$

Definition. If $\tau_1 \subseteq \tau_2$ are topologies, then τ_1 is **smaller** than τ_2 (τ_2 is **larger** than τ_1).

Definition. For topological spaces X, Y, **projection maps** $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are

$$\pi_X(x, y) = x, \quad \pi_Y(x, y) = y$$

Proposition. For $X \times Y$ with product topology,

- π_X and π_Y are continuous.
- π_X and π_Y map open sets to open sets.
- Product topology is smallest topology for which π_X and π_Y are continuous.

Proposition. Let X, Y, Z topological spaces, then $f : Z \to X \times Y$ (with product topology on $X \times Y$) continuous iff both $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous.

Example. Let $f : X \to \mathbb{R}^n$, $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$, $f_i = \pi_i \circ f$, then f is continuous iff all f_i are continuous.

Proposition. Let X, Y nonempty topological spaces. Then $X \times Y$ with product topology is Hausdorff iff X and Y are both Hausdorff.

4. Connectedness

4.1. Clopen sets and examples

Definition. Let X topological space, then $A \subseteq X$ is **clopen** if A is open and closed.

Definition. X is **connected** if the only clopen sets in X are X and $\emptyset$.

Example.

- $\mathbb{R}$ with standard topology is connected.
- $\mathbb{Q}$ with induced topology from $\mathbb{R}$ is not connected (consider $L = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q} - L = \mathbb{Q} \cap (\sqrt{2}, \infty)$).
- The connected subsets of $\mathbb{R}$ are the intervals.

Definition. $A \subseteq \mathbb{R}$ is an interval iff $\forall x, y, z \in A, x < z < y \Longrightarrow z \in A$.

Example.

- $X = \{0, 1\}$ with discrete topology is not connected ($\{1\}$ and $\{0\}$ both open so both closed).
- $X = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$ is connected.
- $\mathbb{Z}$ with Furstenberg topology is not connected.

Theorem (continuity preserves connectedness). If $h : X \to Y$ continuous and X connected, then $h(X) \subseteq Y$ is connected.

Corollary. If $h : X \to Y$ is homeomorphism and X is connected then Y is connected.

Theorem. Let X topological space. The following are equivalent:
- X is connected.
- X cannot be written as disjoint union of two non-empty sets.
- There exists no continuous surjective function from X to a discrete space with more than one point.

Example.

- $\mathrm{GL}_n(\mathbb{R})$ is not connected (since $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R} - \{0\}$ is continuous and surjective and $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$).
- $O(n)$ is not connected.
- $(0, 1)$ is connected (since $\mathbb{R} \cong (0, 1)$ and $\mathbb{R}$ is connected).
- $X = (0, 1]$ and $Y = (0, 1)$ are not homeomorphic (if they are, then $(0, 1]$ is connected since $(0, 1)$ is).

Definition. Let $A = B \cup C$, $B \cap C = \emptyset$, then B and C are **complementary subsets** of A.

Remark. If complementary B and C open in A, then B and C clopen in A. So if $B, C \neq \emptyset$ then A not connected.

4.2. Constructing more connected sets, components, path-connectedness

Proposition. Let X topological space, $Z \subseteq X$ connected. If $Z \subseteq Y \subseteq \overline{Z}$ then Y is connected. In particular, with $Y = \overline{Z}$, the closure of a connected set is connected.

Proposition. Let $A_i \subseteq X$ connected, $i \in I$, $A_i \cap A_j \neq \emptyset$ and $\cup_{i \in I} A_i = X$. Then X is connected.

Theorem. If X and Y are connected then $X \times Y$ is connected.

Example.

- $\mathbb{R}^n$ is connected.
- $B^n = \{x \in \mathbb{R}^n : d_2(0, x) < 1\}$ (B^n is homeomorphic to $\mathbb{R}^n$).
- $D^n = \{x \in \mathbb{R}^n : d_2(0, x) \leq 1\} = \overline{B^n}$ is connected.

Example.

- $\forall n \in \mathbb{N}$, S^n is connected.
- $\forall n \in \mathbb{N}$, T^n is connected.

Definition. **Component** of topological space X is maximal connected subset of X.

Proposition. In a topological space X:
- Every $p \in X$ is in a unique component.
- If $C_1 \neq C_2$ are components, then $C_1 \cap C_2 = \emptyset$.
- X is the union of its components.
- Every component is closed in X.

Example.

- If X connected, then its only component is itself.
- If X discrete, then each singleton in τ_X is a component.
- In $\mathbb{Q}$ with induced standard topology from $\mathbb{R}$, every singleton is a component.

Definition. **Path** in topological space X is continuous function $\gamma : [0,1] \to X$. γ is said to be path from $\gamma(0)$ to $\gamma(1)$.

Definition. X is **path-connected** if for every $p, q \in X$, there is a path from p to q.

Proposition. Every path-connected topological space is connected.

Example. Let

$$Z = \left\{(x, \sin(1/x)) \in \mathbb{R}^2 : 0 < x \leq 1\right\}$$

Z is path-connected, as a path from $(x_1, \sin(1/x_1))$ to $(x_2, \sin(1/x_2))$ is given by

$$\gamma(t) = \left(x_1 + (x_2 - x_1)t, \sin\left(\frac{1}{x_1 + (x_2 - x_1)t}\right)\right)$$

So then Z is connected by the above proposition, and since the closure of a connected set is connected, $\overline{Z}$ is connected.

Every point $(0, y)$, $y \in [-1, 1]$ is a limit point of Z. Assume $\overline{Z}$ is path-connected. Then there is a path $\gamma : [0,1] \to \overline{Z}$ from $(0,0)$ to $(1, \sin(1))$. Since $(\pi_X \circ \gamma)(0) = 0$ and $(\pi_X \circ \gamma)(1) = 1$ and $\pi_X \circ \gamma$ is continuous, by the Intermediate Value Theorem, $\exists t_1 \in [0,1] : (\pi_X \circ \gamma)(t_1) = 2/\pi$. By IVT again, $\exists t_2 \in [0, t_1] : (\pi_X \circ \gamma)(t_2) = \frac{2}{2\pi}$. We obtain a strictly decreasing sequence $(t_n) \subseteq [0,1]$ where $(\pi_X \circ \gamma)(t_n) = \frac{2}{n\pi}$ which is bounded below by 0, so must converge with limit t^*.

Now $\pi_Y \circ \gamma$ is continuous, so $\lim_{n\to\infty}(\pi_Y \circ \gamma)(t_n) = (\pi_Y \circ \gamma)(t^*)$. But $(\pi_Y \circ \gamma)(t_n) = \sin\left(\frac{n\pi}{2}\right)$, and as $n \to \infty$, this oscillates between -1 and 1 and does not converge, so contradiction.

5. Compactness

Definition. Let X topological space, **cover** of X is collection $(U_i)_{i \in I}$ of subsets of X with

$$\bigcup_{i \in I} U_i = X$$

If every U_i is open, it is an **open cover**. If $J \subseteq I$, then $(U_i)_{i \in J}$ is a **subcover** of $(U_i)_{i \in I}$ if it is also a cover.

Definition. X is **compact** if every open cover of X admits a finite subcover.

Example.

- If X is finite then X is compact.
- $\mathbb{R}$ is not compact.
- If X infinite with $\tau = \{U \subseteq X : X - U \text{ is finite}\} \cup \{\emptyset\}$, then X is compact.

Proposition. Let X have topology with basis $\mathcal{B}$. Then X is compact iff every cover $(B_i)_{i \in I}$ of X, $B_i \in \mathcal{B}$, admits a finite subcover of X.

Remark. To determine compactness of $Y \subseteq X$ with induced topology, consider open covers $Y = \cup_{i \in I} (U_i \cap Y)$ for U_i open in X, which is equivalent to $Y \subseteq \cup_{i \in I} U_i$.

Example. $[0, 1]$ is compact.

Proposition. If $f : X \to Y$ continuous, X compact, then $f(X)$ is compact.

Proposition. If X compact, $A \subseteq X$ closed in X, then A is compact.

Theorem. If X is Hausdorff and $A \subseteq X$ is compact then A is closed.

Corollary. If X compact, Y is Hausdorff, $f : X \to Y$ continuous bijection, then f is homeomorphism.

Theorem. If X, Y compact, then $X \times Y$ is compact.

Definition. $S \subseteq \mathbb{R}^n$ is **bounded** if

$$\exists r \in \mathbb{R} : S \subseteq B(0; r)$$

Theorem (Heine-Borel). $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

Example.

- S^n is compact.
- T^n is compact.
- $X = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^3 = 1\}$ is not compact, since $\forall n \in \mathbb{N}$, $(n, 0, (n^2 - 1)^{1/3}) \in X$, so $X \nsubseteq B(n)$, so is unbounded, so not compact by Heine-Borel.

Corollary. Let $f : X \to \mathbb{R}$, X compact, f continuous. Then f attains its maximum and minimum.

Theorem (Bolzano-Weierstrass). An infinite subset A of a compact space X has a limit point in X.

6. Quotient spaces

Definition. Let X topological space, $\sim$ equivalence relation on X. Write $X/\sim$ for the set of equivalence classes of $\sim$: for $x \in X$,

$$[x] := \{y \in X : y \sim x\}, \quad X/\sim := \{[x] : x \in X\}$$

There is a surjective map, the **quotient map**, $\pi : X \to X/\sim$, $\pi(x) = [x]$.

Example. Let $X = \mathbb{R}^3$, define equivalence relation

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \Leftrightarrow z_1 = z_2$$

Then $\pi(a,b,c) = [(a,b,c)] = \{(x,y,z) \in \mathbb{R}^3 : z = c\}$. Elements of $\mathbb{R}^3/\sim$ are horizontal planes.

Definition. Let X topological space, $\sim$ equivalence relation on X. Then $X/\sim$ is given **quotient topology** defined by

$$U \subseteq X/\sim \text{ open} \iff \pi^{-1}(U) \text{ open in } X$$

Proposition. Quotient topology defines a topology on $X/\sim$.

Proposition. Quotient topology on $X/\sim$ is largest such that π is continuous.

Proposition. Let X topological space with equivalence relation $\sim$, Y topological space. Then $f : X/\sim \to Y$ continuous iff $f \circ \pi : X \to Y$ is continuous.

Example. In $\mathbb{R}$, let $x \sim y \iff x - y \in \mathbb{Z}$. Define $\exp : \mathbb{R} \to S^1 \subseteq \mathbb{C}$, $\exp(t) = e^{2\pi i t}$) and $\overline{\exp} : \mathbb{R}/\sim \to S^1$, $\overline{\exp}([t]) = \exp(t)$. Then

$$[s] = [t] \iff s - t = k \in \mathbb{Z} \iff \overline{\exp}(s) = e^{2\pi i k} e^{2\pi i t} = e^{2\pi i t} = \overline{\exp}(t)$$

Hence $\overline{\exp}$ is well-defined and injective, and is surjective since exp is. Also, $\overline{\exp}$ is continuous since $\exp = \overline{\exp} \circ \pi$ is. $\mathbb{R}^2$ is a metric space and so is Hausdorff, so $S^1 \subset \mathbb{R}^2$ with the induced topology is Hausdorff. Now e.g. $\pi([-10, 10]) = \mathbb{R}/\sim$, $[-10, 10]$ is compact and π continuous so $\mathbb{R}/\sim$ is compact. Since $\overline{\exp}$ is a continuous bijection, these three properties imply $\overline{\exp}$ is a homeomorphism. Hence $\mathbb{R}/\sim \cong S^1$.

Definition. Let $A \subseteq X$, define $x \sim y \iff x = y$ or $x, y \in A$. Then define $X/A := X/\sim$.

Example. $S^n \cong D^n/S^{n-1}$. Any point in D^n can be written as $t \cdot \varphi$, $t \in [0,1]$, $\varphi \in S^{n-1}$. Define

$$\begin{aligned} f : D^n \to S^n, \quad & f(t \cdot \varphi) := (\cos(\pi t), \varphi \sin(\pi t)) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1} \\ & \implies f(0 \cdot \varphi) = (1, \mathbf{0}),\ f(1/2 \cdot \varphi) = (0, \varphi),\ f(1 \cdot \varphi) = (-1, 0) \end{aligned}$$

Define $\overline{f} : D^n/S^{n-1} \to S^n$, $\overline{f}([t \cdot \varphi]) = f(t \cdot \varphi)$. If $t_1 \cdot \varphi_1 \neq t_2 \cdot \varphi_2$, then

$$\begin{aligned} [t_1 \cdot \varphi_1] = [t_2 \cdot \varphi_2] \iff t_1 \cdot \varphi_1, t_2 \cdot \varphi_2 \in S^{n-1} & \iff t_1 = t_2 = 1 \\ & \iff f(t_1 \cdot \varphi_1) = (-1, \mathbf{0}) = f(t_2 \cdot \varphi_2) \\ & \iff \overline{f}([t_1 \cdot \varphi_1]) = \overline{f}([t_2 \cdot \varphi_2]) \end{aligned}$$

f is surjective, so $\overline{f}$ is also. Now $\overline{f} \circ \pi = f$ which is continuous, so by above proposition, $\overline{f}$ is continuous. $S^n \subset \mathbb{R}^{n+1}$ is Hausdorff, $D^n \subset \mathbb{R}^n$ is closed and bounded so is compact by Heine-Borel, and so D^n/S^{n-1} is compact (since π continuous). Also, f is a continuous bijection. These imply that $\overline{f}$ is homeomorphism.

7. Topological groups

7.1. Examples

Definition. A **topological group** G is Hausdorff space which is also a group such that

$$\bullet : G \times G \to G,\ \bullet(g, h) = gh \quad \text{and} \quad i : G \to G,\ i(g) = g^{-1}$$

are continuous.

Example.

- $\mathbb{R}^n$ with addition is topological group.
- $\mathrm{GL}_n(\mathbb{R})$ with multiplication and its subgroups $O(n)$ and $\mathrm{SO}(n)$ are topological groups (each entry in AB is sum of products of entries of A and B, so matrix multiplication is continuous, matrix inversion also continuous).

Proposition.

- Any group with discrete topology is topological group.
- Any subgroup of topological group is also topological group.

Example.

- $\mathbb{C} - \{0\}$ with multiplication has topological subgroup $S^1 \subset \mathbb{C} - \{0\}$.
- Define **quaternions** as vector space $\mathbb{H} := \langle 1, i, j, k\rangle$, with topology taken from $\mathbb{R}^4$. $\mathbb{H} - \{0\}$ is a multiplicative group with S^3 a topological subgroup. For $q = a + bi + cj + dk \in \mathbb{H}$, $a, b, c, d \in \mathbb{R}$, we have $ij := k$, $jk := i$, $ki := j$, $ji := -k$, $kj := -i$, $ik := -j$. For $q \neq 0$,

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

- Note however that S^2 is not a topological group.

Definition. For topological group G, $x \in G$, define **left translation by x** as

$$L_x : G \to G, \quad L_x(g) := xg$$

Similarly, **right translation by x** is

$$R_x : G \to G, \quad R_x(g) := gx$$

Proposition. L_x has inverse $(L_x)^{-1} = L_{x^{-1}}$ and is homeomorphism. Similarly for R_x.

Notation. A specified inclusion $G \overset{x}{\hookrightarrow} G \times G$ is the map $G \to \{x\} \times G$ composed with the inclusion map $\{x\} \times G \to G \times G$. (similarly for $G \times \{x\}$).

Proposition. Let G topological group, K the component containing identity of G. Then K is normal subgroup of G.

Example. $O(n)$ is not connected, but $\mathrm{SO}(n)$ is connected and contains I_n, so is a normal subgroup of $O(n)$

7.2. Actions, orbits, orbit spaces

Definition. **Action** of group G on topological space X is map $\bullet : G \times X \to X$ such that $\forall g, h \in G$, $\forall x \in X$,

- $(hg) \bullet x = h \bullet (g \bullet x)$.
- $1 \bullet x = x$.
- $g : X \to X$ defined by $g(x) = g \bullet x$ is continous. Note: g has inverse map g^{-1} which is also continuous, so both are homeomorphisms.

Definition. **Action** of topological group G on topological space X is continuous map $\bullet : G \times X \to X$ such that $\forall g, h \in G$, $\forall x \in X$,

- $(hg) \bullet x = h \bullet (g \bullet x)$.
- $1 \bullet x = x$.

Remark. For the above definition, the condition $g(x) = g \bullet x$ being continuous isn't required since g is the comoposition of continuous maps:

$$X \overset{g}{\hookrightarrow} G \times X \xrightarrow{\bullet} X, \quad x \to (g, x) \to g \bullet x$$

Example.

- Trivial action: $(g, x) \mapsto g \bullet x = x$, so $\bullet = \pi_X$.
- Let $G = \mathrm{GL}_n(\mathbb{R})$, $X = \mathbb{R}^n$, let the action be matrix multiplication: $(A, \boldsymbol{x}) \to A \bullet \boldsymbol{x} = A\boldsymbol{x}$. This induces an action of subgroups $O(n)$ or $\mathrm{SO}(n)$ on $X = \mathbb{R}^n$.
- Let H subgroup of topological group G, **left translation action** of H on G is $\bullet : H \times G \to G$, $h \bullet g = hg$. Equivalently, $\varphi(h) = L_h$.
- Let N normal subgroup of topological group G, **conjugation action** of G on N is $\bullet : G \times N \to N$, $g \bullet n = gng^{-1}$.

Definition. Let G act on topological space X, define equivalence relation $\sim$ on X by

$$x \sim y \iff \exists g \in G : g(x) := g \bullet x = y$$

An equivalence class for this relation is an **orbit**, denoted Gx. **Orbit space**, X/G, is quotient space $X/\sim$. Action is **transitive** if X/G is a singleton.

Example.

- If G acts trivially, every orbit is singleton and $X/G = X$.
- $\mathbb{R}^n/\mathrm{GL}_n(\mathbb{R})$ contains two points and has neither discrete nor indiscrete topology.
- Action of $O(n)$ on S^{n-1} is transitive for $n \in \mathbb{N}$. Action of $\mathrm{SO}(n)$ on S^{n-1} is transitive for $n \geq 2$.

Lemma. If connected topological group G acts on topological space X, then the orbits are connected.

Theorem. Let G connected topological group act on topological space X. If X/G is connected, then X is connected.

Notation. Define specified inclusion $i_1 : M_n(\mathbb{R}) \overset{1}{\hookrightarrow} M_{n+1}(\mathbb{R})$ by $A \to \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$. So $M_n(\mathbb{R})$ can be regarded as subspace of $M_{n+1}(\mathbb{R})$.

Proposition.

- Using the inclusion $\overset{1}{\hookrightarrow}$, $\mathrm{SO}(n)$ is subgroup of $\mathrm{SO}(n+1)$.
- Viewing these as topological groups, if subgroup $\mathrm{SO}(n)$ acts on $\mathrm{SO}(n+1)$, orbit space is $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$.

Corollary. The topological group $\mathrm{SO}(n)$ is connected for $n \in \mathbb{N}$.

8. Introduction

Notation. Let $I = [0, 1]$.

Definition. **Closed n-disc** is

$$D^n := \{\boldsymbol{x} \in \mathbb{R}^n : \|x\| \leq 1\}$$

Definition. **Open n-disc** is

$$E^n := \{\boldsymbol{x} \in \mathbb{R}^n : \|x\| < 1\}$$

Definition. **n-sphere** is

$$S^n := \left\{\boldsymbol{x} \in \mathbb{R}^{n+1} : \|x\| = 1\right\}$$

Definition. **Cylinder** is $S^1 \times I$.

Definition. The **2-torus (torus)** can be defined as $\mathbb{T} := S^1 \times S^1$ or $\mathbb{T} := (I \times I)/\sim$ where

$$\forall x \in I, (x, 0) \sim (x, 1), \quad \forall y \in I, (0, y) \sim (1, y)$$

Definition. **Klein bottle** is given by $\mathbb{K} := (I \times I)/\sim$ where

$$\forall x \in I, (x, 0) \sim (x, 1), \quad \forall y \in I, (0, y) \sim (1, 1 - y)$$

Definition. **Map** is continuous $f : X \to Y$ where X, Y are topological spaces.

9. Simplicial complexes

9.1. Simplicial complexes and triangulations

Definition. Let $v_0, ..., v_n \in \mathbb{R}^N$, $n \leq N$.

- $v_0, ..., v_n$ are in **general position** if $\{v_1 - v_0, ..., v_n - v_0\}$ are linearly independent.
- **Convex hull** of $v_0, ..., v_n$ is set of all **convex linear combinations** of $v_0, ..., v_n$:

$$\langle v_0, ..., v_n \rangle := \left\{\sum_{i=0}^{n} \lambda_i v_i : \sum_{i=0}^{n} \lambda_i = 1, \forall i \in \{0, ..., n\}, \lambda_i \geq 0\right\}$$

- An **n-simplex**, $\sigma^n = \langle v_0, ..., v_n \rangle$, is convex hull of $v_0, ..., v_n$ in general position. The **vertices** $v_0, ..., v_n$ **span** σ^n and σ^n is **n-dimensional**.

Example.

- 0-simplex is a point.
- 1-simplex is a closed line segment.
- 2-simplex is closed triangle including its interior.
- 3-simplex is closed tetrahedron including its interior.

Definition. If $\sigma^n = \langle v_0, ..., v_n \rangle$ is n-simplex and $\{i_0, ..., i_r\} \subseteq \{0, ..., n\}$, then $\langle v_{i_0}, ..., v_{i_r} \rangle$ is r-simplex and $\langle v_{i_0}, ..., v_{i_r} \rangle \subseteq \sigma^n$. Any such sub-simplex is called ***r*-face** of σ^n. A **proper face** is an $(n-1)$-face. The ***i*th face** of σ^n is the $(n-1)$-simplex $\langle v_0, ..., v_{i-1}, v_{i+1}, ..., v_n \rangle$.

Definition. A **finite simplicial complex** $K \subset \mathbb{R}^N$ is finite union of simplices in $\mathbb{R}^N$ such that

- If σ^n is simplex in K and τ^r is r-face of σ^n, then τ^r is simplex in K.
- If σ_1^n and σ_2^m are simplices in K with $\sigma_1^n \cap \sigma_2^m \neq \emptyset$, then there exists $r \in \{0, ..., \min(n, m)\}$ and r-simplex τ^r in K such that τ^r is r-face of both σ_1^n and σ_2^m and $\sigma_1^n \cap \sigma_2^m = \tau^r$.

Dimension of K is maximum value of n for which there is an n-simplex in K.

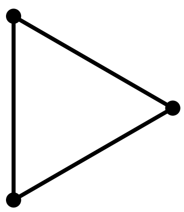
Remark. A finite simplicial complex $K \subset \mathbb{R}^N$ is a topological space when equipped with subspace topology from $\mathbb{R}^N$.

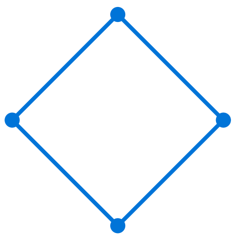
Remark. Second condition implies that two simplices can meet in at most one common face (this is important when considering quotient topologies and identifying edges with each other).

Definition. **Triangulation** of topological space X is homeomorphism $h : X \to K$ for some finite simplicial complex K. We say K **triangulates** X. X is **triangulable** if it has at least one triangulation.

Remark. If a triangulation exists, it is not unique.

Example. The black and blue figures are simplicial complexes that triangulate S^1:





9.2. Simplicial maps

Definition. A map $f : K \to L$ between finite simplicial complexes K and L is **simplicial** if

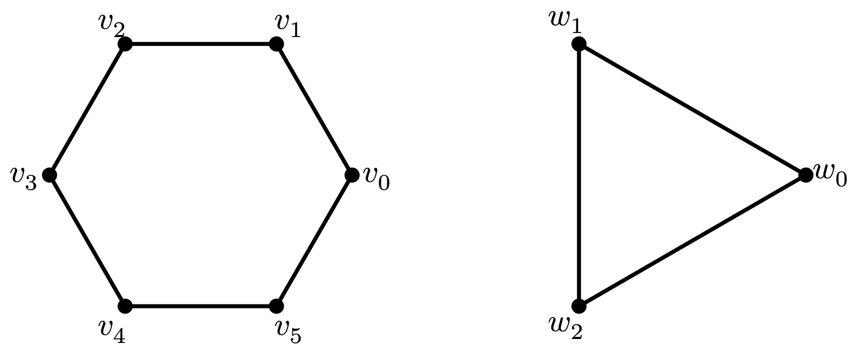
- For every vertex v of K, $f(v)$ is a vertex of L.
- If $\sigma = \langle v_0, ..., v_n \rangle$ is simplex σ in K, $f(\sigma)$ is simplex of L with vertices $f(v_0), ..., f(v_n)$, where map $f|_\sigma$ is defined as

$$f\left(\sum_{i=0}^{n} \lambda_i v_i\right) = \sum_{i=0}^{n} \lambda_i f(v_i)$$

Remark. Vertices $f(v_0), ..., f(v_n)$ of simplex $f(\sigma)$ may not be distinct, so $f(\sigma)$ may be simplex of lower dimension than σ.

Remark. For triangulations $h_X : X \to K_X$ and $h_Y : Y \to K_Y$ of topological spaces X and Y, a simplicial map $f : K_X \to K_Y$ induces a map $F : X \to Y$ by $F = h_Y^{-1} \circ f \circ h_X$.

Example. $F : S^1 \to S^1$, $F(e^{i\pi t}) = e^{2i\pi t}$ is the **2 times** map. Let $f : K_1 \to K_2$, $f(v_i) = w_{i \bmod 3}$, f is simplicial map. Then F is induced by f, where K_1 and K_2 are as below:



9.3. Barycentric subdivision and simplicial approximation

Definition. **Barycentre** of $\sigma^k = \langle v_0, ..., v_k \rangle \subset \mathbb{R}^N$ is

$$\overline{\sigma^k} = \frac{1}{k+1}(v_0 + \dots + v_k) \in \mathbb{R}^N$$

Example.

- Barycentre of 0-simplex is itself.
- Barycentre of 1-simplex is midpoint of the line.

Definition. Let $K \subset \mathbb{R}^N$ be finite simplicial complex. **First barycentric subdivision** of K is the simplicial complex $K^{(1)}$ such that:
- The vertices of $K^{(1)}$ are the barycentres $\overline{\sigma^k}$ for every simplex σ^k in K.
- The vertices $\overline{\sigma^{k_0}}, ..., \overline{\sigma^{k_m}} \in K^{(1)}$ span an m-simplex in $K^{(1)}$ if the original simplices $\sigma^{k_0}, ..., \sigma^{k_m}$ in K are (up to relabelling) strictly nested:

$$\sigma^{k_0} \prec \dots \prec \sigma^{k_m}$$

  where $\sigma^i \prec \sigma^j$ iff σ^i is i-face of σ^j with $i < j$ (thus $k_0 < \dots < k_m$).

Definition. The rth barycentric subdivision of K is defined inductively for $r > 1$ by $K^{(r)} := \left(K^{(r-1)}\right)^{(1)}$.

Remark. Let K be finite simplicial complex.
- If K is triangulation of topological space X, then so is $K^{(r)}$ for all $r \in \mathbb{N}$.
- Each simplex in $K^{(1)}$ is contained in a simplex of K.

Theorem (Simplicial approximation theorem). For each $i \in \{1, 2\}$, let $h_i : X_i \to K_i$ be triangulation of topological space X_i by finite simplicial complex K_i. Let $f : X_1 \to X_2$ be map. Then $\forall \varepsilon > 0$ there exist $n, m \in \mathbb{N}$ and a simplicial map $s : K_1^{(n)} \to K_2^{(m)}$ such that for $F := h_2 \circ f \circ h_1^{-1}$,

$$s \simeq F \quad \text{and} \quad \forall x \in K_1, \quad |F(x) - s(x)| < \varepsilon$$

10. Surfaces

10.1. Surfaces

Definition. Let S be Hausdorff, compact, connected topological space.

- S is **surface** if for all $x \in S$, there exists $U \subseteq S$ such that $x \in U$ and $U \cong E^2$ or $U \cong E^2 \cap \mathbb{R} \times \mathbb{R}_{\geq 0}$.
- **Boundary** of S, ∂S, is set of all $x \in S$ such that there is not a $U \subseteq S$ with $x \in U$ and $U \cong E^2$.
- **Interior** of S is $\text{int}(S) := S - \partial S$.
- S is **closed surface** if $\partial S = \emptyset$ (S is **locally Euclidean of dimension 2**).
- S is **surface with boundary** if $\partial S \neq \emptyset$. Surface with boundary is closed surface from which interiors of finite number of pairwise disjoint closed discs have been removed.

Definition. Let K be finite simplicial complex, $x \in K$. **Open star** of x in K, $\text{St}(x, K)$, is union of $\{x\}$ and interiors of all simplices containing x.

Example. Let K be 2d finite simplicial complex, $x \in K$.

- If there exists a 2-simplex $\sigma^2 \subseteq K$ such that $x \in \text{int}(\sigma^2)$, then $\text{St}(x, K) = \text{int}(\sigma^2) \cong E^2$.
- If there exists a 1-simplex $\sigma^1 \subseteq K$ such that $x \in \text{int}(\sigma^1)$, then

$$\text{St}(x, K) = \text{int}(\sigma^1) \cup \left\{\text{int}(\sigma^2) : \sigma^1 \text{ is face of } \sigma^2 \subseteq K, \sigma^2 \text{ is 2-simplex}\right\}$$

  Here, $\text{St}(x, K) \cong E^2$ iff there are exactly two 2-simplices meeting along σ^1.
- If $x \in K$ is vertex, then

$$\begin{aligned} \text{St}(x, K) = \{x\} &\cup \left\{\text{int}(\sigma^1) : x \text{ vertex of } \sigma^1 \subseteq K, \sigma^1 \text{ is 1-simplex}\right\} \\ &\cup \left\{\text{int}(\sigma^2) : x \text{ vertex of } \sigma^2 \subseteq K, \sigma^2 \text{ is 2-simplex}\right\} \end{aligned}$$

  Here $\text{St}(x, K) \cong E^2$ iff x is vertex of $n \geq 3$ 2-simplices, and along any of its edges containing x, each of these 2-simplices meets precisely one other 2-simplex (from the remaining $n - 1$).

Lemma. Let M be topological space triangulated by connected, finite simplicial complex K. Then M is closed surface iff

$$\forall x \in K, \quad \text{St}(x, K) \cong E^2$$

and the ways that this can happen are as listed above, with exactly two 2-simplices meeting along each 1-simplex.

Remark. If $h : M \to K$ is triangulation of topological space M and $\dim(K) \neq 2$, then M is not closed surface. It is enough to check the open star condition (in above example) at all vertices of K: if there is $x \in K$ such that $\text{St}(x, K) \ncong E^2$, then there exists vertex v of K such that $\text{St}(v, K) \ncong E^2$.

Corollary. Let X topological space, triangulated by connected finite simplicial complex K, $\dim(K) = 2$. Then X is closed surface iff for every vertex $v \in K$, $\mathrm{St}(v, K) \cong E^2$.

Definition. **Real projective plane** is closed surface arising from identifying the edges of the unit square with the following:

$$\mathbb{P} := (I \times I)/\sim, \quad (x, 0) \sim (1 - x, 1), \quad (0, y) \sim (1, 1 - y)$$

It may also be defined as quotient of S^2 by identifying diametrically opposite points:

$$\mathbb{P} = S^2/\sim, \quad \forall x \in S^2, \quad x \sim -x$$

10.2. Orientations on surfaces

Definition. An **orientation on $\mathbb{R}^2$** is choice of direction in which to travserse circles around the origin. There are exactly two choices.

Definition. **Simple closed curve** in topological space is subspace homeomorphic to circle, i.e. connected curve with no self-intersections and ends where it begins.

Definition. Surface S is **orientable** if for all $x \in \mathrm{int}(S)$, any choice of local orientation at x is preserved after translation along any simple closed curve in $\mathbf{int}(S)$ containing x. S is **non-orientable** if there exists $x \in \mathrm{int}(S)$ and simple closed curve $C \subseteq \mathrm{int}(S)$ through x such that translation along C reverses any choice of local orientation at x. Every surface is either orientable or non-orientable.

Example. $S^2, \mathbb{T}$ are orientable. Mobius band and Klein bottle are non-orientable.

Lemma. S is non-orientable iff it contains subspace homeomorphic to Mobius band.

Theorem. Let S_1, S_2 be homeomorphic surfaces. S_1 is orientable iff S_2 is orientable.

Remark. 2-simplex can be given orientation by drawing a direction around it (anticlockwise or clockwise) or by drawing direction around its boundary. A 2-simplex can be oriented in 2 ways, which can be represented by ordering of the vertices: $\langle v_0, v_1, v_2\rangle$, $\langle v_1, v_2, v_0\rangle$ and $\langle v_2, v_0, v_1\rangle$ represent same orientation, $\langle v_1, v_0, v_2\rangle$ represents different orientation.

Definition. Let K finite simplicial complex that triangulates surface S such that all 2-simplices in K are oriented.

- The orientations of two 2-simplices in K which share an edge are **compatible** if they induce opposite orientations on the shared edge.
- K is **Δ-orientable** if there exists choice of orientations on its 2-simplices such that any two 2-simplices which share an edge have compatible orientations. Such a choice, if it exists, is a **Δ-orientation** on K.

Theorem. Surface is orientable iff one (and so every) finite simplicial complex which triangulates it is Δ-orientable.

10.3. Constructions on surfaces

Definition. For surfaces S_1, S_2, their **connected sum**, $S_1 \# S_2$, is obtained by removing the interiors of one small open disc from interior of each surface, and identifying the two newly formed boundary circles. If S_1, S_2 oriented, directions around the boundary circles must be identified such that their induced orientations are opposite to each other. Then $S_1 \# S_2$ inherits an orientation which agrees (upon restriction) with those of the original surfaces S_1 and S_2.

Proposition.

- Since S_1, S_2 connected, it does not matter which two open discs are removed, the result is the same up to homeomorphism.
- $\#$ is commutative and associative.
- S^2 is the identity for $\#$ operation: $M \# S^2 \cong M$.

Definition. For $g \in \mathbb{N}_0$, **closed orientable surface of genus g (g-holed torus)** is

$$M_g = S^2 \# \underbrace{T \# \cdots \# T}_{g \text{ times}}$$

Example. The Klein bottle is given by $\mathbb{K} \cong \mathbb{P} \# \mathbb{P}$.

Definition. **Adding handle** to surface S is as follows: remove two open discs from S. Attach the ends of cylinder $S^1 \times I$ to the resulting boundary circles. If S (and cylinder) are oriented, require that the two resulting boundary circles are glued to those of the cylinder with opposite orientations, which ensures the new surface is still oriented. But if S is not orientable, this doesn't matter, as all possible results are homeomorphic.

Example.

- S^2 with handle added is homeomorphic to the torus.
- S^2 with g handles added is homeomorphic to M_g.
- M_n with handle added is homeomorphic to M_{n+1}.

Definition. **Attaching a cross cap (Mobius band)** to surface S is as follows: remove open disc from S, and identify resulting boundary circle with boundary circle of Mobius band. Attaching a cross-cap always makes the surface non-orientable.

Example. Adding cross-cap to S^2 gives real projective plane $\mathbb{P}$.

Remark. Connected sums of surfaces, surfaces with handles and surfaces with cross caps are always surfaces.

11. Homotopy and the fundamental group

11.1. Homotopy

Definition. Let X, Y topological spaces. **Homotopy** between f and g is map $H : X \times [0, 1] \to Y$ with

$$\forall x \in X, \quad H(x, 0) = f(x) \land H(x, 1) = g(x)$$

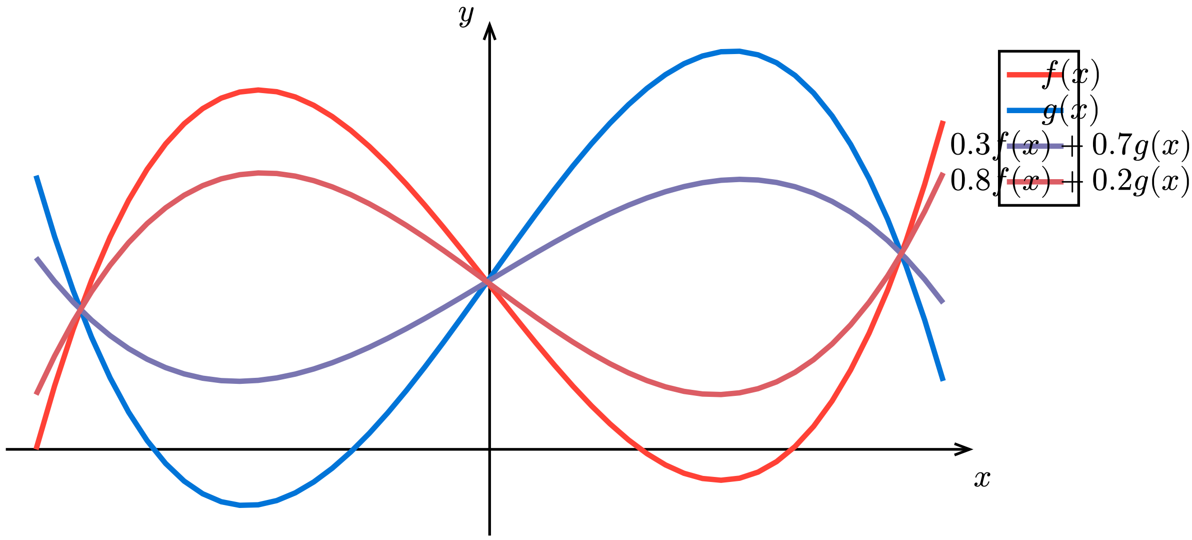
f and g are **homotopic**, $f \simeq g$, if there is a homotopy between them. We can think of homotopy as "path of maps" starting at $f : X \to Y$ and ending at $g : X \to Y$: for

$t \in [0,1]$, define $h_t : X \to Y$, $h_t(x) = H(x,t)$, which varies continuously from f at $t = 0$ to g at $t = 1$.

Example. Let $f, g : \mathbb{R} \to \mathbb{R}$ maps, then

$$H : \mathbb{R} \times [0,1] \to \mathbb{R}, \quad (x,t) \mapsto (1-t)f(x) + tg(x)$$

is homotopy between f and g.



Example. Consider $S^1 \subset \mathbb{C}$, so $S^1 = \{e^{i\pi s} : s \in [0,2)\}$. Let $a : S^1 \to S^1$ be the **antipodal map**, $a(e^{i\pi s}) = e^{-i\pi s}$. Then $a \simeq \mathrm{id}$, with homotopy given by $H : S^1 \times I \to S^1$, $H(e^{i\pi s}) = e^{i\pi(s+t)}$.

Lemma. Homotopy is equivalence relation between maps.

Definition. Map $f : X \to Y$ is **null homotopic** if it is homotopic to a constant map, i.e. to map $c : X \to Y$ with $c(x) = y_0$, $y_0 \in Y$ fixed.

Example. Identity map $\mathrm{id}_{D^2} : D^2 \to D^2$ is null homotopic: let $c : D^2 \to D^2$, $c(x) = 0$. Consider $H : D^2 \times [0,1] \to D^2$, $H(x,t) = (1-t)x$, then H is homotopy between id_{D^2} and c, since H is continuous and $H(x,0) = x = \mathrm{id}_{D^2}(x)$, $H(x,1) = 0 = c(x)$.

Definition. Map $f : X \to Y$ is **homotopy equivalence** if there exists a map $g : Y \to X$ (a **homotopy inverse**) such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. X and Y are **homotopy equivalent**, $X \simeq Y$ if there exists homotopy equivalence between them. If $X \simeq Y$, we say they have the same **homotopy type**.

Theorem. Homotopy equivalence is equivalence relation on topological spaces.

Example. Let $P = \{\boldsymbol{p}\}$ be the one point space, then $D^2 \simeq P$: let $f : D^2 \to P$, $f(x) = \boldsymbol{p}$, $g : P \to D^2$, $g(\boldsymbol{p}) = 0$. Then $f \circ g = \mathrm{id}_P \simeq \mathrm{id}_P$. Now $\forall x \in D^2$, $(g \circ f)(x) = 0$ so $g \circ f \simeq \mathrm{id}_{D^2}$ as $g \circ f$ is constant map.

Definition. Topological space X is **contractible** if it is homotopy equivalent to a one-point space.

Example. Let X topological space. The **cone on $\boldsymbol{X}$** is

$$CX = (X \times [0,1])/\sim$$

where $\sim$ identifies all points of the form $(x, 0)$ with each other, i.e. it collapses the end $X \times \{0\}$ to a single point. We have $D^n \cong CS^{n-1}$.

Proposition. For all topological spaces X, the cone CX is contractible.

Lemma. Every contractible space is path connected.

Lemma. If X and Y are homeomorphic, they are homotopy equivalent (converse does not hold).

Definition.

- It is useful to assume that every topological space X has a particular distinguished **base point** $x_0 \in X$.
- We then require that all maps and homotopies between spaces map base points to base points.
- The pair (X, x_0) is a **based space**.
- A **based map** $f : (X, x_0) \to (Y, y_0)$ is a map $X \to Y$ and satisfies $f(x_0) = y_0$.
- A **based homotopy** $H : (X, x_0) \times [0, 1] \to (Y, y_0)$ between based maps $f, g : (X, x_0) \to (Y, y_0)$ is homotopy $H : X \times [0, 1] \to Y$ with $\forall t \in [0, 1]$, $H(x_0, t) = y_0$.
- All results shown for homotopies are true for based homotopies.

11.2. The fundamental group

Remark. We consider circle S^1 as unit circle in $\mathbb{C}$ and give it base point 1.

Definition. A **loop** in based space (X, x_0) is based map

$$\lambda : (S^1, 1) \to (X, x_0)$$

Equivalently, a loop in (X, x_0) is path in X beginning and ending at x_0:

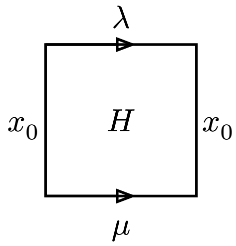
$$\lambda : [0, 1] \to (X, x_0), \quad \lambda(0) = \lambda(1) = x_0$$

Two loops $\lambda, \mu : [0, 1] \to (X, x_0)$ are homotopic if there exists based homotopy between them, i.e. if there is map

$$\begin{aligned} &H : [0,1] \times [0,1] \to X, \\ \forall s \in I, \quad &H(s, 0) = \lambda(s), \\ \forall s \in I, \quad &H(s, 1) = \mu(s), \\ \forall t \in I, \quad &H(0, t) = H(1, t) = x_0 \end{aligned}$$

This corresponds with λ being continuously deformed into μ.

Remark. It is useful to represented based homotopy H between λ and μ on the unit square. Bottom edge corresponds to λ, top edge corresponds to μ, right and left edges are mapped entirely to x_0. Horizontal line drawn across unit square represents loop in (X, x_0) and homotopy H describes path of loops from λ to μ. Vertical line describes path traced from $\lambda(s)$ to $\mu(s)$ under H.



Definition. **Homotopy class** of loop λ in (X, x_0) is

$$[\lambda] \coloneqq \{\mu : \mu \text{ loop in } (X, x_0), \mu \simeq \lambda\}$$

Fundamental group of (X, x_0) is set of homotopy classes of loops in (X, x_0):

$$\pi_1(X, x_0) \coloneqq \{[\lambda] : \lambda \text{ loop in } (X, x_0)\}$$

with group operation $*$ defined by

$$\begin{aligned} * : \pi_1(X, x_0) \times \pi_1(X, x_0) &\to p_1(X, x_0), \\ ([\lambda_1], [\lambda_2]) &\to [\lambda_1 * \lambda_2] \end{aligned}$$

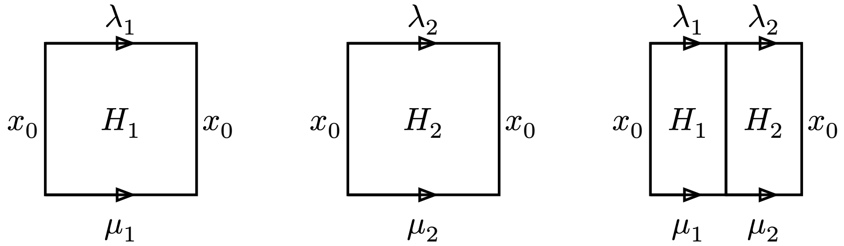
where **concatenation (product)** of λ_1 and λ_2 is the loop in (X, x_0) given by

$$\begin{aligned} \lambda_1 * \lambda_2 : [0,1] &\to X, \\ s &\mapsto \begin{cases} \lambda_1(2s) & \text{if } 0 \le s \le \frac{1}{2} \\ \lambda_2(2s-1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases} \end{aligned}$$

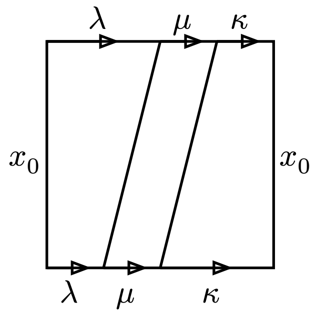
Group identity is $e : [0,1] \to X$, $e(s) = x_0$, inverse of loop λ is $\overline{\lambda} : s \mapsto \lambda(1-s)$, then $\left[\overline{\lambda}\right] = [\lambda]^{-1}$ (equivalently, define $[\lambda]^{-1} = [\lambda \circ w]$ where $w : [0,1] \to [0,1]$, $w(s) = 1 - s$).

Remark. Group operation $*$ is well defined, since if $\lambda_1 \simeq \mu_1$ by homotopy H_1, $\lambda_2 \simeq \mu_2$ by homotopy H_2, then $\lambda_1 * \lambda_2 \simeq \mu_1 * \mu_2$ by homotopy H where

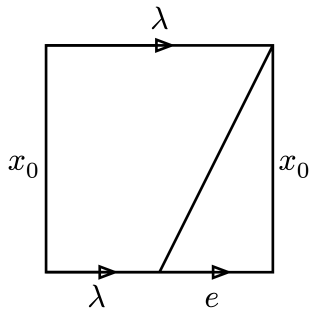
$$H(s,t) = \begin{cases} H_1(2s, t) & \text{if } 0 \le s \le \frac{1}{2} \\ H_2(2s-1, t) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

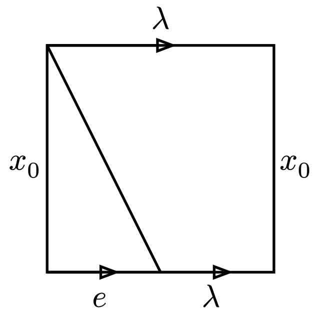


Associativity between $(\lambda * \mu) * \kappa$ and $(\lambda * (\mu * \kappa))$ is shown by homotopy diagram with $(\lambda * \mu) * \kappa$ at bottom and $\lambda * (\mu * \kappa)$ at top. At any intermediate time, path traverses λ at between 2 and 4 times speed, and correspondingly adjusts speed of κ to finish path at $t = 1$. μ is traversed at 4 times speed at all times.

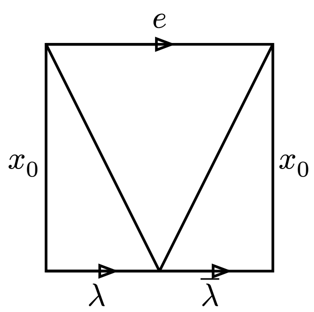


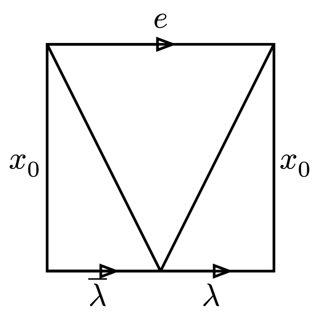
Can show identity $e(s) = x_0$ satisfies $[\lambda] * [e] = [e] * [\lambda] = [\lambda]$ with diagrams





Can show that $[\lambda] * [\overline{\lambda}] = [\overline{\lambda}] * [\lambda] = e$ by





where, for the first diagram, a horizontal path at fixed t is given by

$$s \mapsto \begin{cases} \lambda(2s) & \text{if } 0 \leq s \leq \frac{1-t}{2} \\ \lambda(1-t) & \text{if } \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \overline{\lambda}(2s-1) & \text{if } \frac{1+t}{2} \leq s \leq 1 \end{cases}$$

Definition. Let $f : (X, x_0) \to (Y, y_0)$ based map. Define a function

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad f_*([\lambda]) := [f \circ \lambda]$$

Lemma. $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is well-defined and group homomorphism.

Proof.

- Well-defined: show that $\lambda_1 \simeq \lambda_2 \implies f \circ \lambda_1 \simeq f \circ \lambda_2$ (use definition of $\lambda_1 \simeq \lambda_2$).
- Homomorphism: use that $f \circ (\lambda * \mu) = (f \circ \lambda) * (f \circ \mu)$.

□

Lemma. Let $f, g : (X, x_0) \to (Y, y_0)$ based maps, $f \simeq g$ (f and g are based homotopic), then $f_* = g_*$.

Proof. For loop λ in (X, x_0), find based homotopy between $f \circ \lambda$ and $g \circ \lambda$ in terms of based homotopy H between f and g. □

Lemma. The operation of passing from based map f to induced homomorphism f_* preserves/respects composition and the identity, i.e. if we have

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

then $(g \circ f)_* = g_* \circ f_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}$.

Proof. Straightforward, just use the definition of f_*. □

Corollary. Fundamental group is homotopy-invariant: if (X, x_0), (Y, y_0) are homotopy equivalent, then

$$\pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

Proof. Use definition of homotopy equivalent based spaces and above lemma, to show the induced homomorphisms of the homotopy equivalences are inverse to each other. □

Theorem. Let X path-connected space, $x_0, x_1 \in X$. Then

$$\pi_1(X, x_0) \cong \pi_1(X, x_1)$$

Proof.

- There is path p from x_0 to x_1.
- Let λ loop in X based at x_0.
- Define $\overline{p}(s) = p(1 - s)$, define loop λ_p in X based at x_1 by

$$\lambda_p(s) = \begin{cases} \overline{p(3s)} & \text{if } s \in [0, 1/3] \\ \lambda(3s - 1) & \text{if } s \in [1/3, 2/3] \\ p(3s - 2) & \text{if } s \in [2/3, 1] \end{cases}$$

- Claim:

$$\Phi_p : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad \Phi_p([\lambda]) = \left[\lambda_p\right]$$

  is isomorphism.
  - Well-defined: show if λ, μ loops based at x_0, $\lambda \simeq \mu \implies \lambda_p \simeq \mu_p$ by homotopy diagram (merge $\overline{p}, \lambda, p$ on bottom and $\overline{p}, \mu, p$ on top).
  - Homomorphism: show $(\lambda \times \mu)_p \simeq \lambda_p * \mu_p$ using homotopy diagram (for each $t \in [0, 1]$, we want to partially traverse p (until $s = \frac{1}{2}$) then back along $\overline{p}$, before traversing μ).
  - Isomorphism: show that $\Phi_{\overline{p}}$ defined analogously satisfies $\Phi_{\overline{p}} = (\Phi_p)^{-1}$, i.e. $(\lambda_p)_{\overline{p}} \simeq \lambda$ and $(\mu_{\overline{p}})_p \simeq \mu$ for all loops λ based at x_0, μ based at x_1. (As $t \to 1$, want to retract the spurs $p * \overline{p}$ of the loop back to x_0).

$\square$

Notation. Write $\pi_1(X)$ for fundamental group of path-connected space X (although isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ is not canonical).

Proposition. Let X contractible space, then $\pi_1(X) \cong \mathbb{1}$, the trivial group with one element.

Proof.
- Show we can omit based point in notation.
- Reason that there is only loop in one point space.
- Use definition of contractibility and above corollary.

$\square$

Definition. Topological space X is **simply connected** if path-connected and $\pi_1(X) = \mathbb{1}$ (i.e. its fundamental group is trivial).

Example.
- $\pi_1(S^1) \cong \mathbb{Z}$ where $n \in \mathbb{Z}$ corresponds to homotopy class of $\boldsymbol{n}$ **times** map $\varphi_n : S^1 \to S^1$, $\varphi_n(z) = z^n$.
- $\pi_1(S^n) \cong \mathbb{1}$ for all $n \geq 2$.
- $\pi_1(T) \cong \mathbb{Z}^2$.
- $\pi_1(\mathbb{P}) \cong \mathbb{Z}/2$.

11.3. Brouwer's fixed point theorem

Theorem. Every map $f : D^2 \to D^2$ has a fixed point:

$$\exists x \in D^2 : f(x) = x$$

Proof.
- Assume no fixed point, so every $x, f(x) \in D^2$ defines straight line L_x passing through D^2.
- Define $g(x)$ as point of intersection of boundary and L_x (the one closer to x than $f(x)$). Note $g(x) = x$ if $x \in \partial D^2$.
- Let $i : S^1 \to D^2$ be inclusion map of boundary circle to disc, then $g \circ i = \mathrm{id}_{S^1}$, and $g_* \circ i_* = (g \circ i)_* = \mathrm{id}_{\pi_1(S^1)}$.
- Obtain contradiction using Lemma 11.2.9 and Example 11.2.15.

$\square$

12. Computing the fundamental group

12.1. Finitely presented groups

Definition. **Free group on $\boldsymbol{n}$ letters** $x_1, ..., x_n$, $F^n = \langle x_1, ..., x_n \rangle$, is the group whose elements are finite words in the generators $x_1, ..., x_n$ and their formal inverses $x_1^{-1}, ..., x_n^{-1}$, where the group operation $*$ is given by concatenation of words: $x_i * x_j = x_i x_j$. Identity element is the empty word e. We assume $\forall i \in [n],\ x_i x_i^{-1} = x_i^{-1} x_i = e$.

Notation. If $k \in \mathbb{Z}$, then define

$$x_j^k = \begin{cases} e & \text{if } k = 0 \\ x_j ... x_j & \text{if } k > 0 \\ x_j^{-1} ... x_j^{-1} & \text{if } k < 0 \end{cases}$$

Note. F^n is not abelian for all $n \geq 2$ since e.g. $x_1 x_2 \neq x_2 x_1$. $\forall n \in \mathbb{N}$, F^n is infinite group.

Example.
- $F^1 = \langle x \rangle \cong \mathbb{Z}$ since every element is of the form x^k, $k \in \mathbb{Z}$. There is isomorphism $\varphi : F^1 \to \mathbb{Z}$ given by $\varphi(x) = 1$.
- $F^2 = \langle x, y \rangle \ncong \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ since $xy \neq yx$.

Definition. **Finitely presented group** is group which isomorphic to a group denoted by the **group presentation**

$$\langle x_1, ..., x_n \mid r_1, ..., r_m \rangle$$

consisting of finite words in **generators** $x_1, ..., x_n$ and their formal inverses $x_1^{-1}, ..., x_n^{-1}$, subject to **relations** $r_1, ..., r_m \in F^n = \langle x_1, ..., x_n \rangle$ (i.e. $\forall j \in [m],\ r_j = r_j^{-1} = e$), and group operation is concatenation of words.

Note.
- A finitely presented group is a quotient of a free group.
- Free groups on n letters are finitely presented (with no relations).
- Group presentations are **not unique**.

Example.
- $\langle x, y \mid xy^{-1} \rangle \cong \langle a \rangle = F^1 \cong \mathbb{Z}$ via isomorphism $\varphi : \langle x, y \mid xy^{-1} \rangle \to \langle a \rangle$ defined by $\varphi(x) = \varphi(y) = a$ since $xy^{-1} = e \iff x = y$ so can replace every y in words of $\langle x, y \mid xy^{-1} \rangle$ with x, yielding an element of $\langle x \rangle$.
- $\mathbb{Z}/2 \cong \langle x \mid x^2 \rangle$ since $x = x^{-1}$ and $\forall n \in \mathbb{N}$, $x^{-n} = x^n = e$ if n even, x if n odd.
- $\langle x, y \mid xyx^{-1}y^{-1} \rangle \cong \mathbb{Z}^2$ since $xy = yx$ so the group is abelian. Isomorphism given by $\varphi(x) = (1, 0)$, $\varphi(y) = (0, 1)$.
- $\langle x, y \mid xyx^{-1}y^{-1}, y^2 \rangle \cong \mathbb{Z} \times \mathbb{Z}/2$.

Definition. Let G_1, G_2 finitely presented groups, $G_1 \cap G_2 = \emptyset$, given by

$$G_1 = \langle x_1, ..., x_n \mid r_1, ..., r_m \rangle, \quad G_2 = \langle y_1, ..., y_k \mid s_1, ..., s_\ell \rangle$$

Free product of G_1 and G_2 is the finitely presented group

$$G_1 * G_2 := \langle x_1, ..., x_n, y_1, ..., y_m \mid r_1, ..., r_k, s_1, ..., s_\ell \rangle$$

If H is another group and there exist homomorphisms $i_1 : H \to G_1$, $i_2 : H \to G_2$, then **amalgamated product** of G_1 and G_2 with respect to H is

$$G_1 *_H G_2 := \langle x_1, ..., x_n, y_1, ..., y_k \mid r_1, ..., r_m, s_1, ..., s_\ell, i_1(h) i_2(h)^{-1}\ \forall h \in H \rangle$$

i.e. we impose the additional relations $i_1(h) = i_2(h)$ for all $h \in H$, on $G_1 * G_2$.
$G_1 *_H G_2$ is finitely presented iff H is finitely generated.

Example.
- If $H \cong \mathbb{1}$, then $G_1 *_H G_2 = G_1 * G_2$.
- If F^n, F^k are free groups, then $F^n * F^k = F^{n+k}$.
- $F^n = F^1 * \cdots * F^1 \cong \mathbb{Z} * \cdots * \mathbb{Z}$ (not this is $\ncong \mathbb{Z}^n$ for $n \geq 2$).
- If $G_1 \cong \mathbb{1}$, then

$$G_1 *_H G_2 = \langle y_1, ..., y_k \mid s_1, ..., s_\ell, i_2(h)\ \forall h \in H \rangle$$

12.2. The fundamental group of a triangulated surface

Definition. **Tree** is finite, connected graph with no cycles.

Definition. Let K be connected, finite simplicial complex of dimension ≤ 2.
- A **maximal tree** in K is tree formed out of the 0 and 1-simplices in K which cannot be extended to any larger tree (and necessarily contains all 0-simplices).
- If L is maximal tree in K, **maximal simply connected subcomplex** M of K (associated to L) is constructed as follows: let $M_0 = L$ and for each $j \in \mathbb{N}$, let M_j be subcomplex of K given by

  $$M_j = M_{j-1} \cup \left\{\text{2-simplices in } K \text{ with two edges contained in } M_{j-1}\right\}$$

  Since K is finite simplicial complex, this process must stabilise after finite number of steps. Let M be final subcomplex obtained.

Algorithm. Let K be connected, finite simplicial complex of dimension ≤ 2, let $x_0 \in K$ be 0-simplex. To compute $\pi_1(K, x_0)$:
1. Find a maximal tree L in K.
2. Extend L to maximal simply connected subcomplex M of K.
3. Assign a direction and a label to each 1-simplex in K which is not contained in M. The labels give the generators of a group presentation for $\pi_1(K, x_0)$.
4. Impose relations on the labels as follows:
   1. For 2-simplices with exactly one edge in M: if the directions of the other two edges, a and b, either both point towards or both point away from the edge in M, impose the relation $a = b$. If one points towards and the other away, then impose the relation $a = b^{-1}$.
   2. For 2-simplices with no edges in M and with labels a, b, c: (up to permutation of a, b, c) if the directions of b and c point towards a common vertex and the directions of a and c point away from a common vertex, then impose the relation $c = ab$, otherwise (in this case, the directions form a cycle), if a has direction pointing away from c and b has direction pointing towards c, impose $c = (ab)^{-1}$.
5. We have $\pi_1(K, x_0) \cong \langle \text{labels} \mid \text{relations} \rangle$.

Note. We can use step 4 to more efficiently choose labels and directions in step 3.

Definition. Each directed 1-simplex in M^c gives a **basic loop** (opposite choice of direction yields the inverse loop).

Proof. (Proof of algorithm) Let K be connected finite simplicial complex, v_0 be 0-simplex in K, L be maximal tree in K, M be maximal simply connected subcomplex in K associated to L.

- Simplices are convex, so every path in K is homotopic to one which passes through only 0- and 1-simplices (with no doubling back). In particular, every element of $\pi_1(K, v_0)$ can be represented by a loop based at v_0 which passes through only 0- and 1-simplices.
- If v is 0-simplex then $v \in L \subseteq M$, and L has no cycles, so there exists unique path from v to v_0 in L with no doubling back.
- For all 0-simplices $v \in K$, there exists unique homotopy class of paths in M from v to v_0 and this class can be represented by a unique path in L that doesn't double back on itself.
- Trees are contractible and so L is simply connected, hence M is simply connected.
- Thus, if there is another path in M from v to v_0, there is a loop in M, which must be null-homotopic. Hence, the paths must be homotopic in M.
- If $M^c = K - M \neq \emptyset$, it consists of 1- and 2-simplices (minus points on boundaries) and every 1-simplex in M^c with a choice of direction yields a homotopically non-trivial loop in (K, v_0). Each vertex of a 1-simplex in M^c can be connected back to v_0 by a unique (up to homotopy) path in L. So each directed 1-simplex in M^c gives a **basic loop** (opposite choice of direction yields the inverse loop).
- Every non-trivial loop in K is homotopic to a product of basic loops:
  - If λ is loop in (K, v_0), we have $\lambda \simeq \mu$, where μ is loop through only 0- and 1-simplices (so μ consists of a sequence of directed 1-simplices, with some in M and others not).
  - Every time μ enters/exists M (necessarily at a vertex), draw a path through L back to v_0. This shows that μ is homotopic to product of basic loops.
- Therefore, by assigning labels to directed 1-simplices in M^c, we obtain a list of generators of $\pi_1(K, v_0)$.
- Also, 2-simplices in M^c give relations between the generators of $\pi_1(K, v_0)$.
- So we have surjective homomorphism

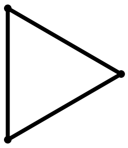
  $$\langle \text{labels of directed 1-simplices in } M^c \mid \text{2-simplex relations} \rangle \longrightarrow \pi_1(K, v_0)$$

  and this can be shown to be injective.

$\square$

Example. Consider S^1 triangulated with three 1-simplices and no 2-simplices. A maximal tree consists of two edges, the maximal connected subcomplex M is already the maximal tree. There is one 1-simplex not in M and there are no relations (since no 2-simplices). Hence $\pi_1(S^1, x_0) \cong \langle a \rangle = F^1 \cong \mathbb{Z}$.

TODO: add diagrams.



12.3. Van Kampen's theorem

Theorem (van Kampen's theorem). Let (K, v_0) be based, connected finite simplicial complex. Suppose there exists connected simplicial subcomplexes $A, B \subseteq K$ such that:

- $K = A \cup B$
- $A \cap B$ is path-connected simplicial subcomplex.
- $v_0 \in A \cap B$.

Then

$$\pi_1(K, v_0) \cong \pi_1(A, v_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$$

where the homomorphisms $(i_A)_* : \pi_1(A \cap B, v_0) \to \pi_1(A, v_0)$,
$(i_B)_* : \pi_1(A \cap B, v_0) \to \pi_1(B, v_0)$ are those induced from the respective inclusion maps $i_A : A \cap B \to A$, $i_B : A \cap B \to B$.

Proof.

- Find maximal tree $L_{A \cap B}$ in $A \cap B$. Extend to maximal trees L_A in A, L_B in B. Then $L = L_A \cup L_B$ is maximal tree in K.
- Extend $L_{A \cap B}$, L_A, L_B to maximal simply connected subcomplexes to $M_{A \cap B}$, M_A, M_B. Then $M = M_A \cup M_B$ is maximal simply connected subcomplex in K.
- By the algorithm,
  $\pi_1(K, v_0) \cong \langle$directed 1-simplices in $M^c \mid$ relations from 2-simplices in $M^c\rangle$ where

  $$\begin{aligned}\{\text{directed 1-simplices in } M^c\} &= D_A \cup D_B \\ &:= \{\text{directed 1-simplices in } A - M_A\} \\ &\quad \cup \{\text{directed 1-simplices in } B - M_B\}\end{aligned}$$

  and

  $$\begin{aligned}\{\text{2-simplices in } M^c\} &= \{\text{2-simplices in } A - M_A\} \\ &\quad \cup \{\text{2-simplices in } B - M_B\}\end{aligned}$$

- However, $D_A \cap D_B = \{$directed 1-simplices in $(A \cap B) - M_{A \cap B}\}$. So to avoid counting homotopy classes of twice, we identify corresponding generators in thre free product $\pi_1(A, v) * \pi_1(B, v_0)$, which gives the amalgamated product.

□

Example. Let W_2 be **figure 8 space** - the one-point union of two copies of S^1, i.e. two copies of S^1 joined at single point (the base point) w_0. So $W_2 = A \cup B$ where $A = B = S^1$, $A \cap B = \{p\}$, the one-point space, is path-connected. $\pi_1(\{p\}, w_0) \cong \mathbb{1}$.
So

$$
\begin{aligned}
\pi_1(W_2, w_0) &\cong \pi_1(S^1, w_0) *_{\mathbb{1}} \pi_1(S^1, w_0) \\
&= \pi_1(S^1, w_0) * \pi_1(S^1, w_0) \\
&\cong \langle x \rangle * \langle y \rangle \\
&= \langle x, y \rangle = F^2
\end{aligned}
$$

Example.

- Decompose torus as $\mathbb{T} = A \cup B$ where A is small closed disc in $\mathbb{T}$, B is closure of the remainder (i.e. remainder together with circle boundary of the disc) so $A \cap B = S^1$.
- B is homotopy equivalent to the figure-eight space so $\pi_1(B) \cong \langle x, y \rangle$. A is contractible so $\pi_1(A) \cong \mathbb{1}$, and $\pi_1(A \cap B) \cong \langle x \rangle$. But the loop going once around $A \cap B$ is homotopy equivalent to the loop going along the boundary of unit square whose quotient gives $\mathbb{T}$, which corresponds to $xyx^{-1}y^{-1}$.
- So by van Kampen's theorem,

$$
\begin{aligned}
\pi_1(\mathbb{T}) &\cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B) \\
&\cong \mathbb{1} *_{\langle z \rangle} \langle x, y \rangle \\
&= \langle x, y \mid z \rangle \\
&= \langle x, y \mid xyx^{-1}y^{-1} \rangle \cong \mathbb{Z}^2
\end{aligned}
$$

  since the image of z under the homomorphism $h_1 : \pi_1(A \cap B) \to A$ must be e.

13. Euler characteristics and the classification of closed surfaces

13.1. The Euler characteristic of a graph

Definition. Let G be finite graph with v vertices and e edges. **Euler characteristic** of G is

$$
\chi(G) := v - e
$$

Definition. **Degree** of vertex in graph of number of edge ends with that vertex as endpoint (so degree of vertex with loop is 2).

Lemma. A non-trivial finite tree contains a vertex of degree 1.

Lemma. Let G be finite tree, then $\chi(G) = 1$.

Proof. Induction of number of vertices. □

Lemma. If G is finite connected graph, then $\chi(G) \leq 1$ with equality iff G is tree.

Proof. Remove edges from cycles until G is a tree, note what happens to Euler characteristic each time. □

Remark. $\chi(G)$ is a homotopy invariant, i.e. $G_1 \simeq G_2$, then $\chi(G_1) = \chi(G_2)$.

Theorem (Euler's Theorem). Let G be finite, connected graph drawn on S^2. Then $S^2 - G$ consists of $f = 2 - \chi(G)$ **faces** (connected regions homeomorphic to open discs).

Proof.
- Assume first that $S^2 - G$ consists of f connected regions homeomorphic to open discs.
- If G has cycle, remove edge to obtain new connected graph G'. This means two of the f regions are joined into one region, and $\chi(G') = \chi(G) + 1$. Hence $\chi(G') + f' = \chi(G) + f$.
- Remove edges until resulting graph is tree T. Then $S^2 - T \cong E^2$, a single connected region. Deduce that $\chi(G) + f = 2$.
- Assumption was correct, since if T is tree then $S^2 - T$ is always homeomorphic to open disc. With reverse of above argument, every edge added to T creates cycle and divides disc into two sub-discs.

□

Corollary. Let K be finite simplicial complex which triangulates S^2, with v 0-simplices, e 1-simplices and f 2-simplices. Then

$$v - e + f = 2$$

Proof. Let $G = \{\text{0-simplices}\} \cup \{\text{1-simplices}\}$, then G is finite connected graph and $S^2 - G = \{\text{interiors of 2-simplices}\}$. □

13.2. The Euler characteristic of a surface

Definition. Let S be surface.
- Finite connected graph $G \subset S$ is **good** if every connected component of $S - G$ is homeomorphic to an open disc.
- Let $G \subset S$ be a good graph. The ***G*-Euler characteristic** of S is

$$\chi_G(S) := \chi(G) + (\text{number of components of } S - G)$$

Note.
- If $\partial S \neq \emptyset$ and $G \subset S$ is a good graph, then $\partial S \subset G$ as a subgraph.
- Not every graph in a surface is good, e.g. if G is single vertex in the torus, then $\mathbb{T} - G$ is homotopy equivalent to the figure 8 space.

Lemma. Let G, G' be good graphs on surface S, with G subgraph of G' (G' is an **enlargement** of G). Then

$$\chi_G(S) = \chi_{G'}(S)$$

Proof.
- G' can be constructed from G by sequence of one or more of the following operations, none of which change $\chi_G(S)$:
  - Add new vertex to interior of existing edge. This adds one vertex, one edge, number of components in complement does not change.

- Add new edge between existing vertices. Number of components in complement increases by 1.
- Add new edge and new vertex by attaching new edge at one to existing vertex and at other end to new vertex. Number of components in complement does not change.

$\square$

Theorem. The G-Euler characteristic of surface S does not depend on choice of good graph G.

Proof.
- Let G_1, G_2 be good graphs on S. If $\partial S \neq \emptyset$, consider ∂S as (possibly) disconnected graph with $\{\text{vertices}\} = \{\text{all vertices in } (G_1 \cup G_2) \cap \partial S\}$.
- Position G_1 such that it crosses G_2 in only isolated points in $\text{int}(S) = S - \partial S$. Add new vertices at these interior intersection points.
- Now $G_1 \cup G_2$ is common enlargment of G_1 and G_2 and a good graph, so result follows by above lemma.

$\square$

Definition. **Euler characteristic** of surface S, is $\chi(S) \coloneqq \chi_G(S)$ where G is any good graph on S.

Theorem. Euler characteristic is homeomorphism-invariant: i.e. if S_1, S_2 homeomorphic surfaces, then $\chi(S_1) = \chi(S_2)$.

Example. If surface S is triangulated by finite simplicial complex K with v 0-simplices, e 1-simplices and f 2-simplices, then

$$\chi(S) = v - e + f$$

Proposition. Euler characteristic is also homotopy-invariant:
$X \simeq Y \implies \chi(X) = \chi(Y)$.

Proof. Non-examinable. $\square$

Remark. For n-dimensional finite simplicial complex K, Euler characteristic is defined as

$$\sum_{k=0}^{n} (-1)^k (\text{number of k-simplices in } K)$$

Lemma (Union Lemma). Let $K = A \cup B$ be 2-dimensional finite simplicial complex with A, B, $A \cap B$ simplicial sub-complexes. Then

$$\chi(K) = \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

Proof. Express number of vertices, edges and faces of $A \cup B$ in terms of those of A, B and $A \cap B$. $\square$

Lemma. Let X, Y surfaces. Then

$$\chi(X\#Y) = \chi(X) + \chi(Y) - 2$$

Proof.

- For closed surface X, $\chi(X - \text{open disc}) = \chi(X) - 1$ (justify using triangulations).
- Use Union lemma with $A = X - \text{open disc}$, $B = Y - \text{open disc}$, $A \cap B = S^1$.

□

Lemma. Let S be surface, let S_M be S with cross-cap attached. Then

$$\chi(S_M) = \chi(S) - 1$$

Proof.

- $S_M = A \cup B$ where $A = (S - \text{open disc})$, B is Mobius band.
- Use Union lemma.

□

Lemma. Let S surface, let S_H be S with handle added. Then

$$\chi(S_H) = \chi(S) - 2$$

Proof.

- $S_H = A \cup B$, where $A = (S - 2 \text{ open discs})$, B is cylinder.
- $A \cap B$ is disjoint union of S^1 and S^1, $S^1 \coprod S_1$.
- Use Union lemma to show $\chi(A \cap B) = 0$.
- Use Union lemma again to deduce the result.

□

13.3. The classification of closed surfaces

Theorem (Classification Theorem for Closed Surfaces). The complete list of closed surfaces, up to homeomorphism, is

- **Orientable**: for $g \in \mathbb{N}_0$,

$$M_g \cong S^2\#\underbrace{T\#\cdots\#T}_{g \text{ times}} \cong S^2 \text{ with } g \text{ handles attached}$$

- **Non-orientable**: for $g \in \mathbb{N}$,

$$N_g \cong \underbrace{\mathbb{P}\#\cdots\#\mathbb{P}}_{g \text{ times}} \cong S^2 \text{ with } g \text{ cross caps attached}$$

Since $\chi(M_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$, a closed surface is classified up to homeomorphism (its **homeomorphism type**) by its Euler characteristic and whether it is orientable or not.

Definition. Let K be finite simplicial complex that triangulates a closed surface S, let L be sub-complex of K of dimension ≤ 1. The **thickening** of L, $\Theta(L)$, is sub-complex of $K^{(2)}$ given by the union of all 2-simplices in $K^{(2)}$ (including all their faces) which meet in L.

Proposition. Let L be 1-dimensional sub-complex of 2-dimensional finite simplicial complex K. Then

- $\Theta(L) \simeq L$.
- If L is tree, then $\Theta(L)$ is homeomorphic to closed disc D^2.
- If L is simple closed curve (i.e. homeomorphic to S^1) then $\Theta(L)$ is homeomorphic to either cylinder or Mobius band.

Lemma. Let K be finite simplicial complex that triangulates closed surface, let L be 1-dimensional sub-complex of K. Then

$$\chi(L) = \chi(\Theta(L))$$

Proof.

- Let $\partial\Theta(L)$ be boundary of $\Theta(L)$ in K, let $C = \Theta(L) - L - \partial\Theta(L)$ (note C is not simplicial complex).
- By definition, L, $\partial\Theta(L)$ and C are pairwise disjoint and $\Theta(L) = L \cup \partial\Theta(L) \cup C$.
- By Union lemma, $\chi(\Theta(L)) = \chi(L) + \chi(\partial\Theta(L)) + \chi(C)$.

□

Definition. Let K finite simplicial complex that triangulates closed surface.

- A **maximal tree** in K is tree formed out of the 0- and 1-simplices in K which cannot be extended to any larger tree (and necessarily contains all 0-simplices).
- For maximal tree L in K, **dual graph** of L, $L^* \subset K$, is defined as follows:
  - The vertices of L^* are the barycentres of the 2-simplices of K.
  - Two vertices of L^* are joined by an edge iff the corresponding two 2-simplices meet in a 1-simplex not in L.

Proposition. Let L be maximal tree in K.

- L^* is connected.
- $\Theta(L) \cup \Theta(L^*) = K^{(2)} \cong K$.
- $\Theta(L) \cap \Theta(L^*) =$ boundary of disc $\Theta(L) \cong S^1$.

Theorem. Let K finite simplicial complex that triangulates closed surface. Then $\chi(K) = 1 + \chi(L^*) \leq 2$ (where L is maximal tree in K) with equality iff $K \cong S^2$.

Proof.

- Let L be maximal tree in K with dual graph L^*. Use Union lemma and above two propositions to show $\chi(K) \leq 2$.
- Show if $\chi(K) = 2$ then L^* is tree, conclude that K is homeomorphic to union of two closed discs glued along their boundary circles, so is copy of S^2.

□

Lemma. Let S be closed surface. Then $(S + \text{handle}) \cong S\#\mathbb{T}$.

Proof.

- Up to homeomorphism, we can always arrange that a handle is attached to the interior of a region homeomorphic to a closed disc in S.
- So $S + \text{handle} \cong (S - \text{open disc}) \cup (\mathbb{T} - \text{open disc}) \cong S\#\mathbb{T}$.

□

Lemma. Let S be non-orientable closed surface. Then
$(S + \text{handle}) \cong S\#\mathbb{K} \cong S\#\mathbb{P}\#\mathbb{P} \cong (S + 2 \text{ cross caps})$.

Theorem. Up to homeomorphism, any closed surface can be obtained from a sphere by adding a finite number of handles and/or a finite number of cross caps.

Proof.

- Let S be closed surface, triangulated by a finite simpicial complex K. Let L be maximal tree in K with dual graph L^*.
- Then $\chi(S) = \chi(K) = 1 + \chi(L^*)$. If L^* is tree, then $\chi(S) = 2$ so $S \cong S^2$.
- So suppose L^* contains cycle C, i.e. a simple closed curve.
- We show we can always reduce to the case that the dual graph is a tree by performing "surgeries", and hence, that S can be constructed from S^2.
- Claim: the cycle $C \subset L^*$ is **non-separating**, i.e. $K - C$ is path-connected.
  - Suppose not. Then each connected component of $K - C$ must contain a 0-simplex in K (since $C \subset L^*$ avoids them all).
  - Clearly, 0-simplices in different components cannot be joined by a path in $K - C$. However, $L \subset K - C$ contains all 0-simplices by definition of the dual graph: contradiction.
- Since $C \cong S^1$, $\Theta(C) \cong$ cylinder or Mobius band.
- Perform surgery along the non-separating closed curve:
  - Remove interior of $\Theta(C)$ from $K^{(2)} \cong K \cong S$, giving a simplicial complex $K^{(2)} - \text{int}(\Theta(C))$ which has either two holes (if $\Theta(C) \cong$ cylinder) or one hole (if $\Theta(C) \cong$ Mobius band), with the boundary of each of these holes being a triangulated circle.
  - Glue triangulated closed discs onto each boundary circle to "cap off" the holes.
  - This gives a new finite simplicial complex K' which triangulates a closed surface S'. So

$$S \cong \begin{cases} S' + \text{handle} & \text{if } \Theta(C) \cong \text{cylinder} \\ S' + \text{cross cap} & \text{if } \Theta(C) \cong \text{Mobius band} \end{cases}$$

  - So

$$\chi(S') = \begin{cases} \chi(S) + 2 \text{ if } \Theta(C) \cong \text{cylinder} \\ \chi(S) + 1 \text{ if } \Theta(C) \cong \text{Mobius band} \end{cases}$$

- We can repeat this surgery procedure along cycles in "the" dual graph in each such new surface obtained.
- This process must terminate after a finite number of surgeries in a closed surface Z for which "the" dual graph (in every triangulation) has no cycles (i.e. the dual graph is a tree, i.e. $Z \cong S^2$), since each surgery increases the Euler characteristic and $\chi(S) \leq 2$ for all closed surfaces S.

□

Corollary. Up to homeomorphism, every closed surface S is given by precisely one of the closed surfaces

- If S is orientable: $M_g = S^2 + g$ handles $\cong S^2 \# T \# \cdots \# T$, $g \in \mathbb{N}_0$.
- If S is non-orientable: $N_g = S^2 + g$ cross caps $\cong \mathbb{P} \# \cdots \# \mathbb{P}$ where $g \in \mathbb{N}$.

Proof.

- If S orientable, then S can only be obtained by attaching handles to S^2 by above theorem. So $S^2 \cong S^2 + g$ handles $= M_g$. But $\chi(M_g) = 2 - 2g$ so homeomorphism type is determined by number of handles.
- If S non-orientable, then $S \cong S^2 + k$ handles $+ \ell$ cross caps, $k \geq 0$, $\ell \geq 1$. But attaching handle to non-orientable surface is same as attaching two cross caps, so $S^2 \cong S^2 + (2k + \ell)$ cross caps $= N_{2k+\ell}$. But $\chi(N_g) = 2 - g$ so homeomorphism type is determined by number of cross caps.

$\square$