Contents

1. Monochromatic sets

1.1. Ramsey's theorem

Notation. $\mathbb{N}$ denotes the set of positive integers, $[n] = \{1, ..., n\}$, and $X^{(r)} = \{A \subseteq X : |A| = r\}$. Elements of a set are written in ascending order, e.g. $\{i, j\}$ means $i < j$. Write e.g. ijk to mean the set $\{i, j, k\}$ with the ordering (unless otherwise stated) $i < j < k$.

Definition. A k**-colouring** on $A^{(r)}$ is a function $c : A^{(r)} \to [k]$.

Example.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ is even and blue if $i + j$ is odd. Then $M = 2\mathbb{N}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$ is even and blue otherwise. $M = \{4^n : n \in \mathbb{N}\}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

Theorem (Ramsey's Theorem for Pairs). Let $\mathbb{N}^{(2)}$ are 2-coloured by $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Then there exists an infinite monochromatic subset M.

Proof.
- Let $a_1 \in A_0 := \mathbb{N}$. There exists an infinite set $A_1 \subseteq A_0$ such that $c(a_1, i) = c_1$ for all $i \in A_1$.
- Let $a_2 \in A_1$. There exists infinite $A_2 \subseteq A_1$ such that $c(a_2, i) = c_2$) for all $i \in A_2$.
- Repeating this inductively gives a sequence $a_1 < a_2 < \cdots < a_k < \cdots$ and $A_1 \supseteq A_2 \supseteq \cdots$ such that $c(a_i, j) = c_i$ for all $j \in A_i$.
- One colour appears infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c_{i_k} = \cdots = c$.
- $M = \{a_{i_1}, a_{i_2}, ...\}$ is a monochromatic set.

$\square$

Remark.
- The same proof works for any $k \in \mathbb{N}$ colours.
- The proof is called a "2-pass proof".
- An alternative proof for k colours is split the k colours $1, ..., k$ into 2 colours: 1 and "2 or ... or k", and use induction.

Note. An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

Example. Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, etc. Let $\{i, j\}$ be red if $i, j \in A_k$ for some k. There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

Example. Colour $\{i < j < k\}$ red iff $i \mid (j + k)$. A monochromatic subset $M = \{2^n : n \in \mathbb{N}_0\}$ is a monochromatic set.

Theorem (Ramsey's Theorem for r-sets). Let $\mathbb{N}^{(r)}$ be finitely coloured. Then there exists a monochromatic infinite set.

Proof.

- $r = 1$: use pigeonhole principle.
- $r = 2$: Ramsey's theorem for pairs.
- For general r, use induction.
- Let $c : \mathbb{N}^r \to [k]$ be a k-colouring. Let $a_1 \in \mathbb{N}$, and consider all $r-1$ sets of $\mathbb{N} \setminus \{a_1\}$, induce colouring $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \to [k]$ via $c'(F) = c(F \cup \{a_1\})$.
- By inductive hypothesis, there exists $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that c' is constant on it (taking value c_1).
- Now pick $a_2 \in A_1$ and induce a colouring $c' : (A_1 \setminus \{a_2\})^{(r-1)} \to [k]$ such that $c'(F) = c(F \cup \{a_2\})$. By inductive hypothesis, there exists $A_2 \subseteq A_1 \setminus \{a_2\}$ such that c' is constant on it (taking value c_2).
- Repeating this gives $a_1, a_2, \ldots$ and $A_1, A_2, \ldots$ such that $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$ and $c(F \cup \{a_i\}) = c_i$ for all $F \subseteq A_{i+1}$, for $|F| = r - 1$.
- One colour must appear infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c$.
- $M = \left\{a_{i_1}, a_{i_2}, \ldots\right\}$ is a monochromatic set.

□

1.2. Applications of Ramsey's theorem

Example. In a totally ordered set, any sequence has monotonic subsequence.

Proof.

- Let (x_n) be a sequence, colour $\{i, j\}$ red if $x_i \leq x_j$ and blue otherwise.
- By Ramsey's theorem for pairs, $M = \{i_1 < i_2 < \cdots\}$ is monochromatic. If M is red, then the subsequence $x_{i_1}, x_{i_2}, \ldots$ is increasing, and is strictly decreasing otherwise.
- We can insist that (x_{i_j}) is either concave or convex: 2-colour $\mathbb{N}^{(3)}$ by colouring $\{j < k < \ell\}$ red if $\left(i, x_{i_j}\right), \left(j, x_{i_k}\right), \left(k, x_{i_\ell}\right)$ form a convex triple, and blue if they form a concave triple. Then by Ramsey's theorem for r-sets, there is an infinite convex or concave subsequence.

□

Theorem (Finite Ramsey). Let $r, m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k-coloured, we can find a monochromatic set of size (at least) m.

Proof.

- Assume not, i.e. $\forall n \in \mathbb{N}$, there exists colouring $c_n : [n]^{(r)} \to [k]$ with no monochromatic m-sets.
- There are only finitely many (k) ways to k-colour $[r]^{(r)}$, so there are infinitely many of colourings $c_r, c_{r+1}, \ldots$ that agree on $[r]^{(r)}$: $c_i \mid_{[r]^{(r)}} = d_r$ for all i in some infinite set A_1, where d_r is a k-colouring of $[r]^{(r)}$.
- Similarly, $[r+1]^{(r)}$ has only finitely many possible k-colourings. So there exists infinite $A_2 \subseteq A_1$ such that for all $i \in A_2$, $c_i \mid_{[r+1]^{(r)}} = d_{r+1}$, where d_{r+1} is a k-colouring of $[r+1]^{(r)}$.
- Continuing this process inductively, we obtain $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$. There is no monochromatic m-set for any $d_n : [n]^{(r)} \to [k]$ (because $d_n = c_i|_{[n]^{(r)}}$ for some i).
- These d_n's are nested: $d_\ell|_{[n]^{(r)}} = d_n$ for $\ell > n$.

- Finally, we colour $\mathbb{N}^{(r)}$ by the colouring $c : \mathbb{N}^{(r)} \to [k]$, $c(F) = d_n(F)$ where $n = \max(F)$ (or in fact $n \geq \max(F)$, which is well-defined by above). So c has no monochromatic m-set (since M was a monochromatic m-set, then taking $\ell = \max(M)$, d_ℓ has a monochromatic m-set), which contradicts Ramsey's Theorem for r-sets.

□

Remark.
- This proof gives no bound on $n = n(k, m)$, there are other proofs that give a bound.
- It is a proof by compactness (essentially, we proved that $\{0, 1\}^{\mathbb{N}}$ with the product topology, i.e. the topology derived from the metric $d(f, g) = \frac{1}{\min\{n \in \mathbb{N} : f(n) \neq g(n)\}}$, is sequentially compact).

Remark. Now consider a colouring $c : \mathbb{N}^{(2)} \to X$ with X potentially infinite. This does not necessarily admit an infinite monochromatic set, as we could colour each edge a different colour. Such a colouring would be injective. We can't guarantee either the colouring being constant or injective though, as $c(ij) = i$ satisfies neither.

Theorem (Canonical Ramsey). Let $c : \mathbb{N}^{(2)} \to X$ be a colouring with X an arbitrary set. Then there exists an infinite set $M \subseteq \mathbb{N}$ such that:
1. c is constant on $M^{(2)}$, or
2. c is injective on $M^{(2)}$, or
3. $c(ij) = c(kl)$ iff $i = k$ for all $i < j$ and $k < l$, $i, j, k, l \in M$, or
4. $c(ij) = c(kl)$ iff $j = l$ for all $i < j$ and $k < l$, $i, j, k, l \in M$.

Proof (Hints).
- First consider the 2-colouring c_1 of $\mathbb{N}^{(4)}$ where $ijkl$ is coloured SAME if $c(ij) = c(kl)$ and DIFF otherwise. Show that an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ (why does this exist?) coloured SAME leads to case 1.
- Assume M_1 is coloured DIFF, consider the 2-colouring of $M_1^{(4)}$, which colours $ijkl$ SAME if $c(il) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic $M_2 \subseteq M_1$ (why does this exist?) must be coloured DIFF by contradiction.
- Consider the 2-colouring of $M_2^{(4)}$ where $ijkl$ is coloured SAME if $c(ik) = c(jl)$ and DIFF otherwise. Show an infinite monochromatic set $M_3 \subseteq M_2$ (why does this exist?) must be coloured DIFF by contradiction.
- 2-colour $M_3^{(3)}$ by: ijk is coloured SAME if $c(ij) = c(jk)$ and DIFF otherwise. Show an infinite monochromatic set $M_4 \subseteq M_3$ (why does this exist) must be coloured DIFF by contradiction.
- 2-colour $M_4^{(3)}$ by the other two similar colourings to above, obtaining monochromatic $M_6 \subseteq M_5 \subseteq M_4$.
- Consider 4 combinations of these colourings on M_6, show 3 lead to one of the cases in the theorem, and the other leads to contradiction.

□

Proof.

- 2-colour $\mathbb{N}^{(4)}$ by: $ijkl$ is red if $c(ij) = c(kl)$ and blue otherwise. By Ramsey's Theorem for 4-sets, there is an infinite monochromatic set $M_1 \subseteq \mathbb{N}$ for this colouring.
- If M_1 is red, then c is constant on $M_1^{(2)}$: for all pairs $ij, i'j' \in M_1^{(2)}$, pick $m < n$ with $j, j' < m$, then $c(ij) = c(mn) = c(i'j')$.
- So assume M_1 is blue.
- Colour $M_1^{(4)}$ by giving $ijkl$ colour green if $c(il) = c(jk)$ and purple otherwise. By Ramsey's theorem for 4-sets, there exists an infinite monochromatic $M_2 \subseteq M_1$ for this colouring.
- Assume M_2 is coloured green: if $i < j < k < l < m < n \in M_2$, then $c(jk) = c(in) = c(lm)$ (consider $ijkn$ and $ilmn$): contradiction, since M_1 is blue.
- Hence M_2 is purple, i.e. for $ijkl \in M_2^{(4)}$, $c(il) \neq c(jk)$.
- Colour M_2 by: $ijkl$ is orange if $c(ik) = c(jl)$, and pink otherwise.
- By Ramsey's theorem for 4-sets, there exists infinite monochromatic $M_3 \subseteq M_2$ for this colouring.
- Assume M_3 is orange, then for $i < j < k < l < m < n \in M_3$, we have $c(jm) = c(ln)$ (consider $jlmn$) and $c(jm) = c(ik)$ (consider $ijkm$): contradiction, since $M_3 \subseteq M_1$.
- Hence M_3 is pink, i.e. for $ijkl$, $c(ik) \neq c(jl)$.
- Colour $M_3^{(3)}$ by: ijk is yellow if $c(ij) = c(jk)$ and grey otherwise. By Ramsey's theorem for 3-sets, there exists infinite monochromatic $M_4 \subseteq M_3$ for this colouring.
- Assume M_4 is yellow: then (considering $ijkl \in M_4^{(4)}$) $c(ij) = c(jk) = c(kl)$: contradiction, since $M_4 \subseteq M_1$.
- So for any $ijk \in M_4^{(3)}$, $c(ij) \neq c(jk)$.
- Finally, colour $M_4^{(3)}$ by: ijk is gold if $c(ij) = c(ik)$ and $c(ik) = c(jk)$, silver if $c(ij) = c(ik)$ and $c(ik) \neq c(jk)$, bronze if $c(ij) \neq c(ik)$ and $c(ik) = c(jk)$, and platinum if $c(ij) \neq c(ik)$ and $c(ik) \neq c(jk)$.
- By Ramsey's theorem for 3-sets, there exists monochromatic $M_5 \subseteq M_4$. M_5 cannot be gold, since then $c(ij) = c(jk)$: contradiction, since $M_5 \subseteq M_4$. If silver, then we have case 3 in the theorem. If bronze, then we have case 4 in the theorem. If platinum, then we have case 2 in the theorem.

$\square$

Remark.
- A more general result of the above theorem states: let $\mathbb{N}^{(r)}$ be arbitrarily coloured. Then we can find an infinite M and $I \subseteq [r]$ such that for all $x_1 ... x_r \in M^{(r)}$ and $y_1 ... y_r \in M^{(r)}$, $c(x_1 ... x_r) = c(y_1 ... y_r)$ iff $x_i = y_i$ for all $i \in I$.
- In canonical Ramsey, $I = \emptyset$ is case 1, $I = \{1, 2\}$ is case 2, $I = \{1\}$ is case 3 and $I = \{2\}$ is case 4.
- These 2^r colourings are called the **canonical colourings** of $\mathbb{N}^{(r)}$.

Exercise. Prove the general statement.

1.3. Van der Waerden's theorem

Remark. We want to show that for any 2-colouring of $\mathbb{N}$, we can find a monochromatic arithmetic progression of length m for any $m \in \mathbb{N}$. By compactness, this is equivalent to showing that for all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for any 2-colouring of $[n]$, there exists a monochromatic arithmetic progression of length m. (If not, there for each n, there is a colouring $c_n : [n] \to \{1, 2\}$ with no monochromatic arithmetic progression of length m. Infinitely many agree on [1], infinitely many agree on [2], and so on - we obtain a 2-colouring of $\mathbb{N}$ with no monochromatic arithmetic progression of length m).

We will prove a slightly stronger result: whenever $\mathbb{N}$ is k-coloured, there exists a monochromatic arithmetic progression, i.e. for any $k, m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k-coloured, we have a length m monochromatic progression.

Definition. Let $A_1, ..., A_k$ be length m arithmetic progressions: $A_i = \{a_i, a_i + d_i, ..., a_i + (m-1)d_i\}$. $A_1, ..., A_k$ are **focussed** at f if $a_i + md_i = f$ for all i.

Example. $\{4, 8\}$ and $\{6, 9\}$ are focussed at 12.

Definition. If length m arithmetic progressions $A_1, ..., A_k$ are focused at f and are monochromatic with each a different colour (for a given colouring), they are called **colour-focussed** at f.

Theorem. Whenever $\mathbb{N}$ is k-coloured, there exists a monochromatic arithmetic progression of length 3, i.e. for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any k-colouring of $[n]$ admits a length 3 monochromatic progression.

Proof.

- We claim that for all $r \leq k$, there exists an n such that if $[n]$ is k-coloured, then either:
  - There exists a monochromatic arithmetic progression of length 3.
  - There exist r colour-focussed arithmetic progressions of length 2.
- We prove the claim by induction on r:
  - $r = 1$: take $n = k + 1$, then by pigeonhole, some two elements of $[n]$ have the same colour, so form a length two arithmetic progression.
  - Assume true for $r - 1$ with witness n. We claim that $N = 2n(k^{2n} + 1)$ works for r.
  - Let $c : [2n(k^{2n} + 1)] \to [k]$ be a colouring. We partition $[N]$ into $k^{2n} + 1$ sets: $B_1 = \{1, ..., 2n\}$, $B_2 = \{2n + 1, ..., 4n\}$, ...
  - Assume there is no length 3 monochromatic progression for c. By inductive hypothesis, each B_i has $r - 1$ colour-focussed arithmetic progressions of length 2.
  - Since $|B_i| = 2n$, each block also contains their focus. For a set M with $|M| = 2n$, there are k^{2n} ways to k-colour M. So by pigeonhole, there are blocks B_s and B_{s+t} that have the same colouring.
  - Let $\{a_i, a_i + d_i\}$ be the $r - 1$ colour-focussed arithmetic progressions in B_s, then $\{a_i + 2nt, a_i + d_i + 2nt\}$ is the corresponding set in B_{s+t}. Let f be the focus in B_s, then $f + 2nt$ is the focus in B_{s+t}.

- Now $\{a_i, a_i + d_i + 2nt\}$, $i \in [r-1]$, are $r-1$ arithmetic progresions colour-focused at $f + 4nt$. Also, $\{f, f + 2nt\}$ is monochromatic of a different colour to the $r-1$ colours used. Hence, there are r arithmetic progressions of length 2 colour-focussed at $f + 4nt$.
- TODO finish proof.

$\square$

Remark. The idea of looking at all possible colourings of a set is called a **product argument**.

Definition. The **Van der Waerden** number $W(k, m)$ is the smallest n such that for any k-colouring of $[n]$, there exists a monochromatic arithmetic progression of length m.

Remark. The above theorem gives a tower-type upper bound $W(k, 3) \le k^{k^{(\cdot^{\cdot})^{k^{4k}}}}$.

2. Partition regular systems

3. Euclidean Ramsey theory