1. Introduction

- By Central Limit Theorem, if sample $(x_1, ..., x_n)$ with each $X_i \sim D(\mu, \sigma^2)$ (D is some distribution) then as $n \to \infty$,

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

  So distribution of sample mean always tends to normal distribution, with standard deviation $\sigma / \sqrt{n}$.
- **Unbiased estimate of standard deviation of sample mean**:

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2}$$

- **Standard error of sample mean**: estimate of standard deviation of sample mean: $s / \sqrt{n}$.
- If n too small then s is poor estimator and mean may not be normally distributed.
- If population distribution is normal and n small then sample mean is t-distributed:

$$\frac{X - \mu}{s / \sqrt{n}} \sim t_{n-1}$$

  $\frac{X-\mu}{s/\sqrt{n}}$ is **pivotal quantity** as distribution doesn't depend on parameters of X.
- **Hypothesis test** for $\underline{x}$:
  - Define **null hypothesis** which identifies distribution believed to have generated each x_i.
  - Choose **test statistic** h (function of $\underline{x}$), extreme when null is false, not extreme when null is true.
  - **Observed test statistic** is $t = h(\underline{x})$.
  - Determine how extreme t is as a realisation of $T = h(X_1, ..., X_N)$ (so need to know distribution of T).
- **One sided p-value**:

$$\mathbb{P}(T \geq t \mid H_0 \text{ true}) \quad \text{or} \quad \mathbb{P}(T \leq t \mid H_0 \text{ true})$$

- **Two sided p-value**:

$$\mathbb{P}(T \geq |t| \cup T \leq -|t| \mid H_0 \text{ true})$$

2. Monte Carlo testing

- **Monte Carlo testing**: given observed test stat $t = h(\underline{x})$ distribution $F(x \mid \theta)$ hypotheses $H_0 : \theta = \theta_0$, $H_1 : \theta > \theta_0$:
  - For $j \in \{1, ..., N\}$:
    - Simulate n observations $(z_1, ..., z_n)$ from $F(\cdot \mid \theta_0)$.
    - Compute $t_j = h(z_1, ..., z_n)$.
  - Estimate p-value by

$$P(T \geq t \mid H_0 \text{ true}) \approx \frac{1}{N} \sum_{j=1}^{N} \mathbb{I}\{t_j \geq t\}$$