1. Metric spaces

1.1. Metrics

- **Metric space**: (X, d), X is set, $d : X \times X \to [0, \infty)$ is **metric** satisfying:
  - $d(x, y) = 0 \iff x = y$
  - **Symmetry**: $d(x, y) = d(y, x)$
  - **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$
- Examples of metrics:
  - p-adic metric:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

  - Extension of the p-adic metric:

$$d_\infty(x, y) = \max\{|x_i - y_i| : i \in [n]\}$$

  - Metric of $C([a, b])$:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

  - Discrete metric:

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

- **Open ball of radius r around x**:

$$B(x; r) = \{y \in X : d(x, y) < r\}$$

- **Closed ball of radius r around x**:

$$D(x; r) = \{y \in X : d(x, y) \leq r\}$$

1.2. Open and closed sets

- $U \subseteq X$ is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : B(x; \varepsilon) \subset U$$

- $A \subseteq X$ is **closed** if $X - A$ is open.
- Sets can be neither closed nor open, or both.