# Contents

# 1. Monochromatic sets

## 1.1. Ramsey's theorem

**Notation**. $\mathbb{N}$ denotes the set of positive integers, $[n] = \{1, ..., n\}$, and $X^{(r)} = \{A \subseteq X : |A| = r\}$. Elements of a set are written in ascending order, e.g. $\{i, j\}$ means $i < j$.

**Definition**. A **$k$-colouring** on $A^{(r)}$ is a function $c : A^{(r)} \to [k]$.

**Example**.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ is even and blue if $i + j$ is odd. Then $M = 2\mathbb{N}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$ is even and blue otherwise. $M = \{4^n : n \in \mathbb{N}\}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

**Theorem** (Ramsey's Theorem for Pairs). Let $\mathbb{N}^{(2)}$ are 2-coloured by $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Then there exists an infinite monochromatic subset $M$.

*Proof.*
- Let $a_1 \in A_0 := \mathbb{N}$. There exists an infinite set $A_1 \subseteq A_0$ such that $c(a_1, i) = c_1$ for all $i \in A_1$.
- Let $a_2 \in A_1$. There exists infinite $A_2 \subseteq A_1$ such that $c(a_2, i) = c_2)$ for all $i \in A_2$.
- Repeating this inductively gives a sequence $a_1 < a_2 < \cdots < a_k < \cdots$ and $A_1 \supseteq A_2 \supseteq \cdots$ such that $c(a_i, j) = c_i$ for all $j \in A_i$.
- One colour appears infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c_{i_k} = \cdots = c$.
- $M = \{a_{i_1}, a_{i_2}, ...\}$ is a monochromatic set.

$\square$

**Remark**.
- The same proof works for any $k \in \mathbb{N}$ colours.
- The proof is called a "2-pass proof".
- An alternative proof for $k$ colours is split the $k$ colours $1, ..., k$ into 2 colours: 1 and "2 or ... or $k$", and use induction.

**Note**. An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

**Example**. Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, etc. Let $\{i, j\}$ be red if $i, j \in A_k$ for some $k$. There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

**Example**. Colour $\{i < j < k\}$ red iff $i \mid (j + k)$. A monochromatic subset $M = \{2^n : n \in \mathbb{N}_0\}$ is a monochromatic set.

**Theorem** (Ramsey's Theorem for $r$-sets). Let $\mathbb{N}^{(r)}$ be finitely coloured. Then there exists a monochromatic infinite set.

*Proof.*

- $r = 1$: use pigeonhole principle.
- $r = 2$: Ramsey's theorem for pairs.
- For general $r$, use induction.
- Let $c : \mathbb{N}^r \to [k]$ be a $k$-colouring. Let $a_1 \in \mathbb{N}$, and consider all $r-1$ sets of $\mathbb{N} \setminus \{a_1\}$, induce colouring $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \to [k]$ via $c'(F) = c(F \cup \{a_1\})$.
- By inductive hypothesis, there exists $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that $c'$ is constant on it (taking value $c_1$).
- Now pick $a_2 \in A_1$ and induce a colouring $c' : (A_1 \setminus \{a_2\})^{(r-1)} \to [k]$ such that $c'(F) = c(F \cup \{a_2\})$. By inductive hypothesis, there exists $A_2 \subseteq A_1 \setminus \{a_2\}\}$ such that $c'$ is constant on it (taking value $c_2$).
- Repeating this gives $a_1, a_2, \ldots$ and $A_1, A_2, \ldots$ such that $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$ and $c(F \cup \{a_i\}) = c_i$ for all $F \subseteq A_{i+1}$, for $|F| = r - 1$.
- One colour must appear infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c$.
- $M = \left\{a_{i_1}, a_{i_2}, \ldots\right\}$ is a monochromatic set.

□

## 1.2. Applications of Ramsey's theorem

**Example**. In a totally ordered set, any sequence has monotonic subsequence.

*Proof.*

- Let $(x_n)$ be a sequence, colour $\{i, j\}$ red if $x_i \leq x_j$ and blue otherwise.
- By Ramsey's theorem for pairs, $M = \{i_1 < i_2 < \cdots\}$ is monochromatic. If $M$ is red, then the subsequence $x_{i_1}, x_{i_2}, \ldots$ is increasing, and is strictly decreasing otherwise.
- We can insist that $(x_{i_j})$ is either concave or convex: 2-colour $\mathbb{N}^{(3)}$ by colouring $\{j < k < \ell\}$ red if $\left(i, x_{i_j}\right), \left(j, x_{i_k}\right), \left(k, x_{i_\ell}\right)$ form a convex triple, and blue if they form a concave triple. Then by Ramsey's theorem for $r$-sets, there is an infinite convex or concave subsequence.

□

**Theorem** (Finite Ramsey). Let $r, m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is $k$-coloured, we can find a monochromatic set of size (at least) $m$.

*Proof.*

- Assume not, i.e. $\forall n \in \mathbb{N}$, there exists colouring $c_n : [n]^{(r)} \to [k]$ with no monochromatic $m$-sets.
- There are only finitely many ($k$) ways to $k$-colour $[r]^{(r)}$, so there are infinitely many of colourings $c_r, c_{r+1}, \ldots$ that agree on $[r]^{(r)}$: $c_i \mid_{[r]^{(r)}} = d_r$ for all $i$ in some infinite set $A_1$, where $d_r$ is a $k$-colouring of $[r]^{(r)}$.
- Similarly, $[r+1]^{(r)}$ has only finitely many possible $k$-colourings. So there exists infinite $A_2 \subseteq A_1$ such that for all $i \in A_2$, $c_i \mid_{[r+1]^{(r)}} = d_{r+1}$, where $d_{r+1}$ is a $k$-colouring of $[r+1]^{(r)}$.
- Continuing this process inductively, we obtain $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$. There is no monochromatic $m$-set for any $d_n : [n]^{(r)} \to [k]$ (because $d_n = c_i|_{[n]^{(r)}}$ for some $i$).
- These $d_n$'s are nested: $d_\ell|_{[n]^{(r)}} = d_n$ for $\ell > n$.

- Finally, we colour $\mathbb{N}^{(r)}$ by the colouring $c : \mathbb{N}^{(r)} \to [k]$, $c(F) = d_n(F)$ where $n = \max(F)$ (or in fact $n \geq \max(F)$, which is well-defined by above). So $c$ has no monochromatic $m$-set (since $M$ was a monochromatic $m$-set, then taking $\ell = \max(M)$, $d_\ell$ has a monochromatic $m$-set), which contradicts Ramsey's Theorem for $r$-sets.

□

**Remark**.
- This proof gives no bound on $n = n(k, m)$, there are other proofs that give a bound.
- It is a proof by compactness (essentially, we proved that $\{0,1\}^{\mathbb{N}}$ with the product topology, i.e. the topology derived from the metric $d(f,g) = \frac{1}{\min\{n \in \mathbb{N} : f(n) \neq g(n)\}}$, is sequentially compact).

**Remark**. Now consider a colouring $c : \mathbb{N}^{(2)} \to X$ with $X$ potentially infinite. Can $c$ be injective?
- $c(\{i,j\}) = i$ is… TODO finish

**Theorem** (Canonical Ramsey). Let $c : \mathbb{N}^{(2)} \to X$ be a colouring with $X$ an arbitrary set. Then there exists an infinite set $M$ such that:
1. $c$ is constant on $M^{(2)}$, or
2. $c$ is injective on $M^{(2)}$, or
3. $c(\{i,j\}) = c(\{k,l\})$ iff $i = k$ for all $i < j$ and $k < l$, $i, j, k, l \in M$, or
4. $c(\{i,j\}) = c(\{k,l\})$ iff $j = l$ for all $i < j$ and $k < l$, $i, j, k, l \in M$.

*Proof.*
- 2-colour $\mathbb{N}^{(4)}$ by: $ijkl$ is red if $c(ij) = c(kl)$ and blue otherwise. By Ramsey's Theorem for 4-sets, there is an infinite monochromatic set $M_1$.
- If $M_1$ is red, then $c$ is constant on $M_1^{(2)}$: if pick $m < n$ with $m > l$, then $c(ij) = c(mn) = c(kl)$.
- So assume $M_1$ is blue.
- Colour $M_1^{(4)}$ by giving $ijkl$ colour green if $c(il) = c(jk)$ and purple otherwise. By Ramsey's theorem for 4-sets, there exists an infinite monochromatic $M_2 \subseteq M_1$ for this colouring.
- Assume $M_2$ is coloured green: if $i < j < k < l < m < n \in M_2$, then $c(jk) = c(in) = c(lm)$ (consider $ijkn$ and $ilmn$): contradiction, since $M_1$ is blue.
- Hence $M_2$ is purple, i.e. for $ijkl \in M_2^{(4)}$, $c(il) \neq c(jk)$.
- Colour $M_2$ by: $ijkl$ is orange if $c(ik) = c(jl)$, and pink otherwise.
- By Ramsey's theorem for 4-sets, there exists infinite monochromatic $M_3 \subseteq M_2$ for this colouring.
- Assume $M_3$ is orange, then for $i < j < k < l < m < n \in M_3$, we have $c(jm) = c(ln)$ (consider $jlmn$) and $c(jm) = c(ik)$ (consider $ijkm$): contradiction, since $M_3 \subseteq M_1$.
- Hence $M_3$ is pink, i.e. for $ijkl$, $c(ik) \neq c(jl)$.

- Colour $M_3^{(3)}$ by: $ijk$ is yellow if $c(ij) = c(jk)$ and grey otherwise. By Ramsey's theorem for 3-sets, there exists infinite monochromatic $M_4 \subseteq M_3$ for this colouring.
- Assume $M_4$ is yellow: then (considering $ijkl \in M_4^{(4)}$) $c(ij) = c(jk) = c(kl)$: contradiction, since $M_4 \subseteq M_1$.
- So for any $ijk \in M_4^{(3)}$, $c(ij) \neq c(jk)$.
- Finally, colour $M_4^{(3)}$ by: $ijk$ is gold if $c(ij) = c(ik)$ and $c(ik) = c(jk)$, silver if $c(ij) = c(ik)$ and $c(ik) \neq c(jk)$, bronze if $c(ij) \neq c(ik)$ and $c(ik) = c(jk)$, and platinum if $c(ij) \neq c(ik)$ and $c(ik) \neq c(jk)$.
- By Ramsey's theorem for 3-sets, there exists monochromatic $M_5 \subseteq M_4$. $M_5$ cannot be gold, since then $c(ij) = c(jk)$: contradiction, since $M_5 \subseteq M_4$. If silver, then we have case 3 in the theorem. If bronze, then we have case 4 in the theorem. If platinum, then we have case 2 in the theorem.

$\square$

# 2. Partition regular systems

# 3. Euclidean Ramsey theory