0.1. Integration and measure

- Dirichlet's function: $f : [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

1. The real numbers

- $a \in \mathbb{R}$ is an **upper bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \leq a$.
- $c \in \mathbb{R}$ is a **least upper bound (supremum)** if $c \leq a$ for every upper bound a.
- $a \in \mathbb{R}$ is an **lower bound** of $E \subseteq \mathbb{R}$ if $\forall x \in E, x \geq a$.
- $c \in \mathbb{R}$ is a **greatest lower bound (supremum)** if $c \geq a$ for every upper bound a.
- **Completeness axiom of the real numbers**: every subset E with an upper bound has a least upper bound. Every subset E with a lower bound has a greatest lower bound.
- **Archimedes' principle**:

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

- Every non-empty subset of $\mathbb{N}$ has a minimum.
- **The rationals are dense in the reals**:

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

1.1. Conventions on sets and functions

- For $f : X \to Y$, **preiamge** of $Z \subseteq Y$ is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

- $f : X \to Y$ **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

- $f : X \to Y$ **surjective** if $Y = f(X)$.
- **Limit inferior** of sequence x_n:

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \left(\inf_{m\geq n} x_m \right) = \sup_{n\geq 0} \inf_{m\geq n} x_m$$

- **Limit superior** of sequence x_n:

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \left(\sup_{m\geq n} x_m \right) = \inf_{n\geq 0} \sup_{m\geq n} x_m$$

1.2. Open and closed sets

- $U \subseteq \mathbb{R}$ is **open** if

$$\forall x \in U, \exists \varepsilon : (x - \varepsilon, x + \varepsilon) \subseteq U$$

- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.
- $x \in \mathbb{R}$ is **point of closure (limit point)** for $E \subseteq \mathbb{R}$ if

$$\forall \delta > 0, \exists y \in E : |x - y| < \delta$$

Equivalently, x is point of closure if every open interval containing x contains another point of E.

- **Closure** of E, $\overline{E}$, is set of points of closure.
- F is **closed** if $F = \overline{F}$.
- If $A \subset B \subseteq \mathbb{R}$ then $\overline{A} \subset \overline{B}$.
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- For any set E, $\overline{E}$ is closed.
- Let $E \subseteq \mathbb{R}$. The following are equivalent:
  - E is closed.
  - $\mathbb{R} - E$ is open.
- Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.
- **Definition**: collection C of subsets of $\mathbb{R}$ **covers** (is a **covering** of) $F \subseteq \mathbb{R}$ if $F \subseteq \cup_{S \in C} S$. If each S in C open, G is **open covering**. If C is finite, C is **finite cover**.
- Covering C of F **contains a finite subcover** if exists $\{S_1, ..., S_n\} \subseteq C$ with $F \subseteq \cup_{i=1}^{n} S_i$ (i.e. a finite subset of C covers F). F is **compact** if any open covering U contains a finite subcover.
- **Example**: $\mathbb{R}$ is not compact, $[a, b]$ is compact.
- **Heine-Borel theorem**: if $F \subset \mathbb{R}$ closed and bounded then any open covering of F has finite subcovering (so F is compact). If F compact then F closed and bounded.

1.3. The extended real numbers

- **Definition**: **extended reals** are $\mathbb{R} \cup \{-\infty, \infty\}$ with the order relation $-\infty < \infty$ and $\forall x \in \mathbb{R}, -\infty < x < \infty$. ∞ is an upper bound and $-\infty$ is a lower bound for every $x \in \mathbb{R}$, so $\sup(\mathbb{R}) = \infty$, $\inf(\mathbb{R}) = -\infty$.
  - Addition: $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$. $\infty + \infty = \infty - (-\infty) = \infty$. $\infty - \infty$ is undefined.
  - Multiplication: $\forall a \in \mathbb{R}_{>0}, a \cdot \infty = \infty$, $\forall a \in \mathbb{R}_{<0}, a \cdot = -\infty$. $\infty \cdot \infty = \infty$ and $0 \cdot \infty = \infty$.
  - $\limsup$ and $\liminf$ are defined as

$$\limsup x_n \coloneqq \inf_{n \in \mathbb{N}} \left\{ \sup_{k \geq n} x_k \right\}, \quad \liminf x_n \coloneqq \sup_{n \in \mathbb{N}} \left\{ \inf_{k \geq n} x_k \right\}$$

- **Definition**: extended real number l is **limit** of (x_n) if either
  - $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$. Then (x_n) **converges to l**. or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$ (limit is ∞) or
  - $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$ (limit is $-\infty$).

(x_n) **converges in the extended reals** if it has a limit in the extended reals.

2. Further analysis of subsets of $\mathbb{R}$

TODO: up to here, check that all notes are made from these topics

2.1. Countability and uncountability

- A is **countable** if $A = \emptyset$, A is finite or there is a bijection $\varphi : \mathbb{N} \to A$ (in which case A is **countably infinite**). Otherwise A is **uncountable**. φ is called an **enumeration**.
- If surjection from $\mathbb{N}$ to A, or injection from A to $\mathbb{N}$, then A is countable.
- Examples of countable sets:
  - $\mathbb{N}$ ($\varphi(n) = n$)
  - $2\mathbb{N}$ ($\varphi(n) = 2n$)
- $\mathbb{Q}$ is countable.
- **Exercise (todo)**: show that $\mathbb{N}^k$ is countable for any $k \in \mathbb{N}$.
- **Exercise (todo)**: show that if a_n is a nonnegative sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- **Exercise (todo)**: show that if $a_{n,k}$ is a nonnegative sequence and $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection then

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

- $f : X \to Y$ is **monotone** if $x \geq y \Rightarrow f(x) \geq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.
- Let f be monotone on (a, b). Then it is discountinuous on a countable set.
- Set of sequences in $\{0, 1\}$, $\{((x_n))_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$ is uncountable.
- **Theorem**: $\mathbb{R}$ is uncountable.

2.2. The structure theorem for open sets

- Collection $\{A_i : i \in I\}$ of sets is **(pairwise) disjoint** if $n \neq m \Longrightarrow A_n \cap A_m = \emptyset$.
- **Structure theorem for open sets**: let $U \subseteq \mathbb{R}$ open. Then exists countable collection of disjoint open intervals $\{I_n : n \in \mathbb{N}\}$ such that $U = \cup_{n \in \mathbb{N}} I_n$.

2.3. Accumulation points and perfect sets

- $x \in \mathbb{R}$ is **accumulation point** of $E \subseteq \mathbb{R}$ if x is point of closure of $E - \{x\}$. Equivalently, x is a point of closure if

$$\forall \delta > 0, \exists y \in E : y \neq x \land |x - y| < \delta$$

  Equivalently, there exists a sequence of distinct $y_n \in E$ with $y_n \to x$ as $n \to \infty$.
- **Exercise**: set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$.
- $E \subseteq \mathbb{R}$ is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

- **Proposition**: set of accumulation points E' of E is closed.
- Bounded set E is **perfect** if it equals its set of accumulation points.

- **Exercise (todo)**: what is the set of accumulation points of an isolated set?
- Every non-empty perfect set is uncountable.

2.4. The middle-third Cantor set

- **Middle third Cantor set**:
  - Define $C_0 := [0, 1]$
  - Given $C_n = \cup_{i=1}^{2^n} [a_i, b_i]$, $a_i < b_1 < a_2 < \cdots$, with $|b_i - a_i| = 3^{-n}$, define

    $$C_{n+1} := \cup_{i=1}^{2^n} \left[a_i, a_i + 3^{-(n+1)}\right] \cup \left[b_i - 3^{-(n+1)}, b_i\right]$$

    which is a union of 2^{n+1} disjoint intervals, with difference in endpoints equalling $3^{-(n+1)}$.
  - The **middle third Cantor set** is

    $$C := \bigcup_{n \in \mathbb{N}} C_n$$

    Observe that if a is an endpoint of an interval in C_n, it is contained in C.
- **Proposition**: the middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and uncountable.

2.5. G_s, F_σ

- Set E is $\boldsymbol{G_\delta}$ if $E = \cap_{n \in \mathbb{N}} U_n$ with U_n open.
- Set E is $\boldsymbol{F_\sigma}$ if $E = \cup_{n \in \mathbb{N}} F_n$ with F_n closed.
- **Lemma**: set of points where $f : \mathbb{R} \to \mathbb{R}$ is continuous is G_δ.

3. Construction of Lebesgue measure

3.1. Lebesgue outer measure

- **Definition**: let I non-empty interval with endpoints $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$ and $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$. The **length** of I is

  $$\ell(I) := b - a$$

  and set $\ell(\emptyset) = 0$.
- **Example**: if $I = (-\infty, b] = (-\infty, a] \cup [a, b]$ then $\ell(I) = \infty = \ell(-\infty, a]) + \ell([a, b])$
- **Definition**: let $A \subseteq \mathbb{R}$. **Lebesgue outer measure** of A is infimum of all sums of lengths of intervals covering A:

  $$\mu^*(A) := \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k, I_k \text{ intervals}\right\}$$

  It satisfies **monotonicity**: $A \subseteq B \Longrightarrow \mu^*(A) \leq \mu^*(B)$.
- **Proposition**: outer measure is **countably subadditive**: if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets then

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$$

- **Lemma**: we have

$$\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k, I_k \neq \emptyset \text{ open intervals}\right\}$$

- Lebesgue outer measure of interval is its length: $\mu^*(I) = \ell(I)$.

3.2. Measurable sets

- **Notation**: $E^c = \mathbb{R} - E$.
- **Proposition**: let $E = (a, \infty)$. Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

- **Definition**: $E \subseteq \mathbb{R}$ is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

  Collection of such sets is $\mathcal{F}_{\mu^*}$.
- **Lemma (excision property)**: let E Lebesgue measurable set with finite measure and $E \subseteq B$, then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

- **Remark**: not every set is Lebesgue measurable.
- **Definition**: collection of subsets of $\mathbb{R}$ is an **algebra** if contains $\emptyset$ and closed under taking complements and finite unions: if $A, B \in \mathcal{A}$ then $\mathbb{R} - A, A \cup B \in \mathcal{A}$.
- **Remark**: if a union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if $\{A_k\}_{k=1}^{\infty}$ is countable collection of Lebesgue measurable sets, then let $A_{1'} = A_1$ and for $k > 1$, define

$$A_{k'} = A_k - \cup_{i=1}^{k-1} A_i$$

  then $\{A_{k'}\}_{k=1}^{\infty}$ is disjoint union of Lebesgue measurable sets.
- **Proposition**: if $E_1, ..., E_n$ Lebesgue measurable then $\cup_{k=1}^{n} E_k$ is Lebesgue measurable. If $E_1, ..., E_n$ disjoint then

$$\mu^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

  for any $A \subseteq \mathbb{R}$. In particular, for $A = \mathbb{R}$,

$$\mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k)$$

- **Proposition**: if E is countable union of Lebesgue measurable sets, then E is Lebesgue measurable. Also, if $\{E_k\}_{k \in \mathbb{N}}$ is countable disjoint collection of Lebesgue measurable sets then

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu^*(E_k)$$

3.3. Abstract definition of a measure

- **Definition**: let $X \subseteq \mathbb{R}$. Collection of subsets of $\mathcal{F}$ of X is **σ-algebra** if
  - $\emptyset \in F$
  - $E \in F \Longrightarrow E^c \in F$
  - $E_1, ..., E_n \in F \Longrightarrow \cup_{k=1}^{\infty} E_k \in \mathcal{F}$.
- **Example**:
  - Trivial examples are $\mathcal{F} = \{\emptyset, \mathbb{R}\}$ and $\mathcal{F} = \mathcal{P}(\mathbb{R})$.
  - Arbitrary intersections of σ-algebras are σ-algebras.
- **Definition**: let $\mathcal{F}$ σ-algebra of X. $\nu : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is **measure** satisfying
  - $\nu(\emptyset) = 0$
  - $\forall E \in \mathcal{F}, \nu(E) \geq 0$
  - **Countable additivity**: if $E_1, E_2, ... \in \mathcal{F}$ are disjoint then

    $$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k)$$

  Elements of $\mathcal{F}$ are **measurable** (as they are the only sets on which the measure ν is defined).
- **Proposition**: if ν is measure then it satisfies:
  - **Monotonicity**: $A \subseteq B \Longrightarrow \nu(A) \leq \nu(B)$.
  - **Countable subadditivity**: $\nu(\cup_{k \in \mathbb{N}} E_k) \leq \sum_{k \in \mathbb{N}} \nu(E_k)$.
  - **Excision**: if A has finite measure, then $A \subseteq B \Longrightarrow m(B - A) = m(B) - m(A)$.

3.4. Lebesgue measure

- **Lemma**: the Lebesgue measurable sets form a σ-algebra and contain every interval.
- **Theorem (Caratheodory extension)**: the restriction of the outer measure μ^* to the σ-algebra of Lebesgue measurable sets is a measure.
- **Definition**: the measure μ of μ^* restricted to $\mathcal{F}_{\mu^*}$ is the **Lebesgue measure**. It satisfies $\mu(I) = \ell(I)$ for any interval I and is translation invariant.
- **Hahn extension theorem**: there exists unique measure μ defined on $\mathcal{F}_{\mu^*}$ for which $\mu(I) = \ell(I)$ for any interval I.

3.5. Sets of measure 0

- **Exercise (todo)**: middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.
- **Exercise (todo)**: any countable set is Lebesgue measurable and has Lebesgue measure 0.
- **Exercise (todo)**: any E with $\mu^*(E) = 0$ is Lebesgue measurable and has $\mu(E) = 0$.

- **Lemma**: let E Lebesgue measurable set with $\mu(E) = 0$, then $\forall E' \subseteq E$, E' is Lebesgue measurable.

3.6. An approximation result for Lebesgue measure

- **Definition**: **Borel σ-algebra** $\mathcal{B}(\mathbb{R})$ is smallest σ-algebra containing all intervals: for any other σ-algebra $\mathcal{F}$ containing all intervals, $\mathcal{B}(\mathbb{R}) \subset \mathcal{F}$.

$$\mathcal{B}(\mathbb{R}) = \bigcap \{\mathcal{F} : \mathcal{F}\ \sigma\text{ -algebra containing all intervals}\}$$

  $E \in \mathcal{B}(\mathbb{R})$ is **Borel** or **Borel measurable**.
- Every open subset of $\mathbb{R}$, every closed subset of $\mathbb{R}$, every G_δ set, every F_σ set is Borel.
- **Proposition**: the following are equivalent:
  - E is Lebesgue measurable
  - $\forall \varepsilon > 0, \exists$ open $G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
  - $\forall \varepsilon > 0, \exists$ closed $F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
  - $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
  - $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

4. Measurable functions

4.1. Definition of a measurable function

- **Lemma**: let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ with E Lebesgue measurable. The following are equivalent:
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) > c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) \geq c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) < c\}$ is Lebesgue measurable.
  - $\forall c \in \mathbb{R}, \{x \in E : f(x) \leq c\}$ is Lebesgue measurable.
- **Definition**: $f : E \to \mathbb{R}$ is **(Lebesgue) measurable** if it satisfies any one of the above properties.