Contents

1. Monochromatic sets

1.1. Ramsey's theorem

Notation. $\mathbb{N}$ denotes the set of positive integers, $[n] = \{1, ..., n\}$, and $X^{(r)} = \{A \subseteq X : |A| = r\}$. Elements of a set are written in ascending order, e.g. $\{i, j\}$ means $i < j$.

Example.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ is even and blue if $i + j$ is odd. Then $M = 2\mathbb{N}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $\max\{n \in \mathbb{N} : 2^n \mid (i + j)\}$ is even and blue otherwise. $M = \{4^n : n \in \mathbb{N}\}$ is a monochromatic subset.
- Colour $\{i, j\} \in \mathbb{N}^{(2)}$ red if $i + j$ has an even number of distinct prime divisors and blue otherwise. No explicit monochromatic subset is known.

Theorem (Ramsey's Theorem for Pairs). Let $\mathbb{N}^{(2)}$ are 2-coloured by $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Then there exists an infinite monochromatic subset M.

Proof.
- Let $a_1 \in A_0 := \mathbb{N}$. There exists an infinite set $A_1 \subseteq A_0$ such that $c(a_1, i) = c_1$ for all $i \in A_1$.
- Let $a_2 \in A_1$. There exists infinite $A_2 \subseteq A_1$ such that $c(a_2, i) = c_2)$ for all $i \in A_2$.
- Repeating this inductively gives a sequence $a_1 < a_2 < \cdots < a_k < \cdots$ and $A_1 \supseteq A_2 \supseteq \cdots$ such that $c(a_i, j) = c_i$ for all $j \in A_i$.
- One colour appears infinitely many times: $c_{i_1} = c_{i_2} = \cdots = c_{i_k} = \cdots = c$.
- $M = \{a_{i_1}, a_{i_2}, ...\}$ is a monochromatic set.

□

Remark.
- The same proof works for any $k \in \mathbb{N}$ colours.
- The proof is called a "2-pass proof".
- An alternative proof for k colours is split colours $1, ...k$ into 1 and $2, ..., k$ and use induction.

Note. An infinite monochromatic set is **very** different from an arbitrarily large finite monochromatic set.

Example. Let $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, etc. Let $\{i, j\}$ be red if $i, j \in A_k$ for some k. There exist arbitrarily large monochromatic red sets but no infinite monochromatic red sets.

Example. Colour $\{i < j < k\}$ red iff $i \mid (j + k)$. A monochromatic subset $M = \{2^n : n \in \mathbb{N}_0\}$ is a monochromatic set.

Theorem (Ramsey's Theorem for r-sets). Let $\mathbb{N}^{(r)}$ be finitely coloured. Then there exists a monochromatic infinite set.

Proof.
- $r = 1$: use pigeonhole principle.
- $r = 2$: Ramsey's theorem for pairs.
- For general r, use induction.

- Let $c : \mathbb{N}^r \to [k]$ be a k-colouring. Let $a_1 \in \mathbb{N}$, and consider all $r - 1$ sets of $\mathbb{N} \setminus \{a_1\}$, induce colouring $c' : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \to [k]$ via $c'(F) = c(F \cup \{a_1\})$.
- By inductive hypothesis, there exists $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that c' is constant on it (taking value c_1).
- Now pick $a_2 \in A_1$ and induce a colouring $c' : (A_1 \setminus \{a_2\})^{(r-1)} \to [k]$ such that $c'(F) = c(F \cup \{a_2\})$. By inductive hypothesis, there exists $A_2 \subseteq A_1 \setminus \{a_2\}\}$ such that c' is constant on it (taking value c_2).
- Repeating this gives $a_1, a_2, ...$ and $A_1, A_2, ...$ such that $A_{i+1} \subseteq A_i \setminus \{a_{i+1}\}$ and $c(F \cup \{a_i\}) = c_i$ for all $F \subseteq A_{i+1}$, for $|F| = r - 1$.
- One colour must appear infinitely many times: $c_{i_1} = c_{i_2} = \dots = c$.
- $M = \left\{a_{i_1}, a_{i_2}, ...\right\}$ is a monochromatic set.

□

1.2. Applications of Ramsey's theorem

Example. In a totally ordered set, any sequence has monotonic subsequence.

Proof.

- Let the sequence be $x_1, x_2, ...$ Colour $\{i, j\}$ red if $x_i \leq x_j$ and blue otherwise.
- By Ramsey's theorem for pairs, $M = \{i_1 < i_2 < \cdots\}$ is monochromatic. If M is red, then the subsequence $x_{i_1}, x_{i_2}, ...$ is increasing, and is strictly decreasing otherwise.
- We can insist that $\left(x_{i_j}\right)$ is either concave or convex. For a triple $\left(i_{j_1}, i_{j_2}, i_{j_3}\right)$, it is convex… TODO finish.

□

Theorem (Finite Ramsey's Theorem). Let $r, m, k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k-coloured, we can find a monochromatic set of size at least m.

Proof.

- Assume not, i.e. $\forall n \in \mathbb{N}$, there exists colouring $c_n : [n]^{(r)} \to [k]$ with no monochromatic m-sets.
- There are only finitely many ways to k-colour $[r]^{(r)}$, so there are infinitely many c_n that agree on $[r]^{(r)}$ for some $n \in A_1$: $c_n \mid_{[r]^{(r)}} =$. TODO.
- $[r + 1]^{(r)}$ has only finitely many possible k-colourings.
- So there exists $A_2 \subseteq A_1$ such that $c_n \mid_{[r+1]^{(r)}} = d_{r+1}$.
- Continuing this process, we obtain $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$. There is no monochromatic m-set for any $d_n : [n]^{(r)} \to [k]$ (because $d_n = c_i|_{[n]^{(r)}}$).
- These d_n's are nested: $d_j|_{[i]^{(r)}} = d_i$ for $j > i$.
- Finally, we colour $\mathbb{N}^{(r)}$ be $c(F) = d_n(F)$ where $n = \max(F)$ (or in fact $n \geq \max(F)$, which is well-defined by above). This contradicts Ramsey's Theorem for r -sets.

□

2. Partition regular systems

3. Euclidean Ramsey theory