LOGIC AND COMP

By

$$\dfrac{\begin{array}{cc}[A] & [B] \\ \vdots & \vdots \\ X & Y\end{array}}{C}\,(A, B)$$

we mean "if we can prove X assuming A and Y assuming B, then we can infer C by "discharging/closing" the open assumptions A and B."

In particular, the $(\to\text{-I})$ rule can be written as

$$\dfrac{\begin{array}{c}\Gamma\,[A] \\ \vdots \\ B\end{array}}{\Gamma \vdash A \to B}\,(A)$$

Example. A natural deduction proof of $A \wedge B \to B \wedge A$.

End goal is $\dfrac{\vdots}{A\wedge B \to B \wedge A}$:

$$\dfrac{\dfrac{\dfrac{[A\wedge B]}{A} \quad \dfrac{[A \wedge B]}{B}}{B \wedge A}\,(A\wedge B)}{A \wedge B \to B \wedge A}$$

Example. Prove the Hilbert-style axioms $\phi \to (\psi \to \phi)$ and $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

Clearly need to use $\to$-introduction for both.

$$\dfrac{\dfrac{[\phi] \quad [\psi]}{\psi \to \phi}\,(\psi)}{\phi \to (\psi \to \phi)}\,(\phi)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi \to (\psi \to \chi)]^{(3)}}{\phi \to \chi} \quad \dfrac{[\phi \to \psi]^{(2)} \quad [\phi]^{(1)}}{\psi}\,(\to\text{-E})}{\chi}\,(\to \chi)}{\phi \to \chi}\,(\to\text{-I},\ \phi)}{(\phi \to \psi) \to (\phi \to \chi)}\,(\to\text{-I},\ \phi \to \psi)}{(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))}\,(\to\text{-I},\ \phi \to (\psi \to \chi))$$

If Γ is a set of propositions in the language and ϕ is a proposition, we write

$$\left.\begin{array}{l} \Gamma \vdash_{IPC} \phi \\[1em] \Gamma \vdash_{IQC} \phi \\[1em] \Gamma \vdash_{CPC} \phi \\[1em] \Gamma \vdash_{CQC} \phi \end{array}\right\} \text{ if there is a proof of } \phi \text{ from } \Gamma \text{ in the respective logic.}$$

Lemma 1.1.4 If $\Gamma \vdash_{IPC} \phi$, then $\Gamma, \psi \vdash_{IPC} \phi$ for any proposition ψ. If p is a primitive proposition and ψ is any proposition, then $\Gamma[p := \psi]$ ~~, then~~ $\vdash_{IPC} \phi[p := \psi]$ ← notation for $\phi[\psi/p]$

Proof. Induction on the size of the proofs (exercise).

1.2 The Simply Typed λ-Calculus

We ~~assume given~~ for now a set Π, of simple types generated by a grammar,

$$\Pi := U \mid \Pi \to \Pi$$ ~~where U is a countable set~~

(so Π consists of U and any $a \to b \in \Pi$ where $a, b \in \Pi$), where U is a countable set of type variables, as well as an infinite set V of variables.