Conc ineq.

Proof of Hoeffding:

$$\psi_Y'(\lambda) = \frac{\mathbb{E}[Y e^{\lambda Y}]}{\mathbb{E}[e^{\lambda Y}]} = \mathbb{E}\left[Y \frac{e^{\lambda Y}}{\mathbb{E}[e^{\lambda Y}]}\right] = \mathbb{E}_\lambda[Y]$$

where $\mathbb{E}_\lambda$ denotes the expectation w.r.t. P_λ.

If P_λ is discrete, then $P_\lambda(y) = \dfrac{e^{\lambda y}}{\sum_{y'} p(y') e^{\lambda y'}}$

If P_λ continuous with density f

$$f_\lambda(y) = \frac{e^{\lambda y} f(y)}{\int f(y') e^{\lambda y'} dy'}$$

$$\psi_Y''(\lambda) = \frac{\mathbb{E}[Y^2 e^{\lambda Y}] \cdot \mathbb{E}[e^{\lambda Y}] - \mathbb{E}[Y e^{\lambda Y}]^2}{\mathbb{E}[e^{\lambda Y}]^2}$$

$$= \mathbb{E}\left[Y^2 \frac{e^{\lambda Y}}{\mathbb{E}[e^{\lambda Y}]}\right] - \mathbb{E}[Y e^{\lambda Y}]^2$$

$$= \mathbb{E}_\lambda Y^2 - \mathbb{E}_\lambda[Y]^2 = \mathrm{Var}_\lambda(Y)$$

Now $Y \in [a,b) \Rightarrow \left|Y - \frac{b-a}{2}\right|^2 \leq \frac{(b-a)^2}{4}$

$$\mathrm{Var}_\lambda(Y) = \mathrm{Var}_\lambda\left(Y - \frac{(b-a)}{2}\right) \leq \mathbb{E}_\lambda\left[Y - \frac{(b-a)}{2}\right]^2 \leq \frac{(b-a)^2}{4}$$

Now $\psi_Y(\lambda) = \underbrace{\psi_Y(0)}_{0} + \underbrace{\psi_Y'(0)}_{0}\lambda + \psi_Y''(\xi)\frac{\lambda^2}{2}$

$$\leq \frac{\lambda^2 (b-a)^2/4}{2} \quad \text{hence } Y \in G\left(\frac{(b-a)^2}{4}\right) \quad \square$$