Contents

1. Basic notions in quantum information theory

The field is motivated by the fact that we want to control quantum systems.

1. Can we construct and manipulate quantum systems?
2. If so, which are the scientific and technological applications?

Entanglement frontier: highly complex quantum systems, which are more complex and richer than classical systems. However, quantum systems have *decoherence*, which classical systems don't. "Quantum advantage" gives speed up over classical systems.

Quantum vs classical information theory:

- True randomness.
- Uncertainty.
- Entanglement.

Note we always work with finite-dimensional Hilbert spaces, so take $\mathbb{H} = \mathbb{C}^N$.

1.1. Qubits and basic operations

Notation 1.1 Vectors are denoted by $|\psi\rangle \in \mathbb{C}^n$, dual vectors by $\langle\psi| \in (\mathbb{C}^n)^*$, and inner products by $\langle\psi|\phi\rangle \in \mathbb{C}$. $|\psi\rangle\langle\psi| : \mathbb{C}^n \to \mathbb{C}^n$ are rank-one projectors.

Definition 1.2 Another important basis of $\mathbb{C}^2$ is $\{|+\rangle, |-\rangle\}$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Definition 1.3 For an operator $T : \mathbb{H} \to \mathbb{H}$, the **operator norm** of T is

$$\|T\| = \|T\|_{\mathbb{H}\to\mathbb{H}} := \sup_{x\in H} \frac{\|T(x)\|_{\mathbb{H}}}{\|x\|_{\mathbb{H}}}$$

Notation 1.4 Let $B(\mathbb{H})$ denote the space of bounded linear operators, i.e. T such that $\|T\| < \infty$.

Notation 1.5 Denote the dual of the operator T by T^*, i.e. the operator that satisfies $\langle y|T(x)\rangle = \langle T^*(y)|x\rangle$ for all $x, y \in \mathbb{H}$.

Definition 1.6 A **quantum measurement** is a collection of measurement operators $\{M_n\}_n \subseteq B(\mathbb{H})$ which satisfies $\sum_n M_n^* M_n = \mathbb{I}$, the identity operator.

Given $|\phi\rangle$, the probability that $|n\rangle$ occurs after this operation is $p(n) = \langle\phi|M_n^* M_n|\phi\rangle$. After performing this operation, the state of the system is $\frac{1}{\sqrt{p(n)}} M_n|\phi\rangle$. This is the **Born rule**.

Example 1.7 A measurement in the computational basis is $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$. Note M_0 and M_1 are self-adjoint. Let $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$. Then $p(i) = \langle\phi|M_i|\phi\rangle = |\alpha_i|^2$. The state after measurement is $\frac{\alpha_i}{|\alpha_i|}|i\rangle$, which is equivalent to $|i\rangle$.

Note that $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ are operationally identical: the phase does not affect the measurement probabilities.

Definition 1.8 A quantum measurement $\{M_n\}_n \subseteq B(\mathbb{H})$ is **projective measurement** if the M_n are orthogonal projections (i.e. they are self-adjoint (Hermitian) and $M_n M_m = \delta_{nm} M_n$).

Definition 1.9 An **observable** is a Hermitian operator, which we can express as its spectral decomposition

$$M = \sum_n \lambda_n M_n,$$

where $\{M_n\}_n$ is a projective measurement. The possible outcomes of the measurement correspond to its eigenvalues λ_n of the observable. Note that the expected value of the measurement is

$$\sum_n \lambda_n p(n) = \sum_n \lambda_n \langle \phi | M_n | \phi \rangle = \langle \phi | M | \phi \rangle.$$

Definition 1.10 $T : \mathbb{H} \to \mathbb{H}$ is **positive (semi-definite)** (written $T \geq 0$) if $\langle \psi | T | \psi \rangle \geq 0$ for all $|\psi\rangle \in H$.

Definition 1.11 A **POVM (positive operator valued measurement)** is a collection $\{E_n\}_n$ where each $E_n = M_n^* M_n$ for a general measurement $\{M_n\}_n$ (i.e. each E_n is positive and Hermitian, and $\sum_n E_n = \mathbb{I}$).

Note that the probability of obtaining outcome m on $|\psi\rangle$ is $p(m) = \langle \psi | E_m | \psi \rangle$. We use POVMs when we care only about the probabilities of the different measurement outcomes, and not the post-measurement states.

Conversely, given a POVM $\{E_n\}_n$, we can define a general measurement $\{\sqrt{E_n}\}_n$.

Remark 1.12 Any transformation on a normalised quantum state must map it to a normalised quantum state, and so the operation must be unitary.

Definition 1.13 The **Pauli matrice** are

$$\sigma_0 = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_X = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\sigma_Y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Pauli matrices are unitaries, and we can think of them as quantum logical gates.

Definition 1.14 The **trace** of $T : \mathbb{H} \to \mathbb{H}$ is

$$\operatorname{tr} T = \operatorname{tr} M = \sum_i M_{ii} \in \mathbb{C},$$

where M is a matrix representation of T in any basis (this is well-defined since the trace is cyclic and linear).

Proposition 1.15 For any state $|\phi\rangle$ and any operator A,

$$\operatorname{tr}(A|\phi\rangle\langle\phi|) = \langle \phi | A | \phi \rangle.$$

Proof (Hints). Straightforward. $\square$

Proof. $\operatorname{tr}(A|\phi\rangle\langle\phi|) = \sum_i \langle i|A|\phi\rangle\langle\phi|i\rangle$ for an orthonormal basis $\{|i\rangle\}$. Any basis where $|\phi\rangle = |j\rangle$ for some j instantly yields the result. Alternatively, we have

$$\operatorname{tr}(A|\phi\rangle\langle\phi|) = \sum_i \langle i|A|\phi\rangle\langle\phi|i\rangle = \sum_i \langle\phi|i\rangle\langle i|A|\phi\rangle = \langle\phi|I|A|\phi\rangle = \langle\phi|A|\phi\rangle.$$

□

Suppose we don't fully know the state of the system, but know that it is $|\phi_i\rangle$ with probability p_i. We want to be able to consider the $\sum_i p_i|\phi_i\rangle$ as a state, but this isn't normalised (except when some $p_i = 1$). To solve this issue, we assume each $|\phi_i\rangle$ to the rank-one projector $|\phi_i\rangle\langle\phi_i|$, and we describe the unknown state by $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$. This gives rise to the following definition:

Definition 1.16 A **density matrix/operator** is a linear operator $\rho \in B(\mathbb{H})$ which is:
- Hermitian,
- Positive semi-definite, and
- Satisfies $\operatorname{tr}\rho = 1$.

1.2. Postulates of quantum mechanics (Heisenberg picture)

Postulate 1.17 Given an isolated physical system, there exists a complex (separable) Hilbert space $\mathbb{H}$ associated with it, called **state space**. The physical system is described by a **state vector**, which is a normalised vector in $\mathbb{H}$.

Postulate 1.18 Given an isolated physical system, its evolution is described by a unitary. If the state of the system at time t_1 is $|\phi_1\rangle$ and at time t_2 is $|\phi_2\rangle$, then there exists a unitary U_{t_1,t_2} such that $|\phi_2\rangle = U_{t_1,t_2}|\phi_1\rangle$.

This can be generalised with the Schrodinger equation: the time evolution of a closed quantum system is given by $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)\rangle = H|\phi(t)\rangle$. The Hermitian operator H is called the **Hamiltonian** and is generally time-dependent.

Definition 1.19 Let the spectral decomposition of H be

$$H = \sum_i E_i|E_i\rangle\langle E_i|,$$

where the E_i are the **energy eigenvalues** and the $|E_i\rangle$ are the **energy eigenstates** (or **stationary states**).

The minimum energy is called the **ground state energy** and its associated eigenstate is called the **ground state**. The **(spectral) gap** of H is the (absolute) difference between the ground state energy and the next largest energy eigenvalue. When the gap is strictly positive, we say the system is **gapped**. The states $|E_i\rangle$ are called **stationary**, since they evolve as $|E_i\rangle \to \exp(-iE_it/\hbar)|E_i\rangle$.

We have $|\phi(t_2)\rangle = U(t_1,t_2)|\phi(t_1)\rangle$ where $U(t_1,t_2) = \exp(-iH(t_2-t_1)/\hbar)$ which is a unitary. In fact, any unitary U can be written in the form $U = \exp(iK)$ for some Hermitian K.

Postulate 1.20 Given a physical system with associated Hilbert space $\mathbb{H}$, quantum measurements in the system are described by a collection of measurements $\{M_n\}_n \subseteq B(\mathbb{H})$ such that $\sum_n M_n^* M_n = \mathbb{I}$, as in Definition 1.6. The index n refers to the measurement outcomes that may occur in the experiment, and given a state $|\phi\rangle$ before measurement, the probability that n occurs is

$$p(n) = \langle\phi| M_n^* M_n |\phi\rangle.$$

The state of the system after measurement is $\frac{1}{\sqrt{p(n)}} M_n |\phi\rangle$

Postulate 1.21 Given a composite physical system, its state space $\mathbb{H}$ is also composite and corresponds to the tensor product of the individual state spaces $\mathbb{H}_i$ of each component: $\mathbb{H} = \mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_N$. If the state in each system i is $|\phi_i\rangle$, then the state in the composite system is $|\phi_1\rangle \otimes \cdots \otimes |\phi_N\rangle$.

Definition 1.22 Given $|\phi\rangle \in H_1 \otimes \cdots \otimes H_N$, $|\phi\rangle$ is **entangled** if it cannot be written as a tensor product of the form $|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$. Otherwise, it is **separable** or a **product state**.

Example 1.23 The **EPR pair** (**Bell state**) $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is entangled.

1.3. Postulates of quantum mechanics (Schrodinger picture)

Postulate 1.24 Given an isolated physical system, the state of the system is completely described by its density operator, which is Hermitian, positive semi-definite and has trace one.

If we know the system is in state ρ_i with probability p_i, then the state of the system is $\sum_i p_i \rho_i$.

Pure states are of the form $\rho = |\phi\rangle\langle\phi|$, **mixed states** are of the form $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$.

Postulate 1.25 Given an isolated physical system, its evolution is described by a unitary. If the state of the system is ρ_1 at time t_1 and is ρ_2 at time t_2, then there is a unitary U depending only on t_1, t_2 such that $\rho_2 = U\rho_1 U^*$.

Postulate 1.26 The same as Postulate 1.20, except we specify that after measurement $\{M_n\}_n$, the probability of observing n is $p(n) = \mathrm{tr}(M_n^* M_n \rho)$ and the state after measurement is $\frac{1}{p(n)} M_n \rho M_n^*$.

Postulate 1.27 The same as Postulate 1.21, except that the state of the composite system is $\rho = \rho_1 \otimes \cdots \otimes \rho_n$, where ρ_i is the state of ith individual system.

Remark 1.28 The Heisenberg and Schrodinger postulates are mathematically equivalent.

1.4. States, entanglement and measurements

Theorem 1.29 (Schmidt Decomposition) Let $|\psi\rangle$ be a pure state in a bipartite system $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$, where $\mathbb{H}_A$ has dimension N_A and $\mathbb{H}_B$ has dimension $N_B \geq N_A$. Then there exist orthonormal states $\{|e_i\rangle : i \in [N_A]\} \subseteq \mathbb{H}_A$ and $\{|f_i\rangle : i \in [N_A]\} \subseteq \mathbb{H}_B$ such that

$$|\psi\rangle = \sum_{i=1}^{N_A} \lambda_i |e_i\rangle \otimes |f_i\rangle,$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i^2 = 1$.

The λ_i are unique up to re-ordering. The λ_i are called the **Schmidt coefficients** and the number of $\lambda_i > 0$ is the **Schmidt rank** of the state.

Proof. Let $|\psi\rangle = \sum_{k=1}^{N_A} \sum_{\ell=1}^{N_B} \beta_{k\ell} |\phi_k\rangle \otimes |\phi_\ell\rangle$ for orthonormal bases $\{|\phi_k\rangle : k \in [N_A]\} \subseteq \mathbb{H}_A$, $\{|\chi_\ell\rangle : \ell \in [N_B]\} \subseteq \mathbb{H}_B$. Let $(\beta_{k\ell})$ have singular value decomposition

$$U[\Sigma\ \ 0]V,$$

where U is an $N_B \times N_B$ unitary, Σ is an $N_A \times N_A$ diagonal matrix with non-negative entries, and V is an $N_A \times N_A$ unitary. So

$$\beta_{k\ell} = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} U_{ki} \Sigma_{ij} V_{j\ell} = \sum_{i=1}^{N_A} \Sigma_{ii} U_{ki} V_{i\ell}.$$

Hence,

$$|\psi\rangle = \sum_{k,\ell} \sum_i \Sigma_{ii} U_{ki} |\phi_k\rangle \otimes V_{i\ell} |\chi_\ell\rangle = \sum_i \Sigma_{ii} \underbrace{\left(\sum_k U_{ki} |\phi_k\rangle \right)}_{|e_i\rangle} \otimes \underbrace{\left(\sum_\ell V_{j\ell} |\chi_\ell\rangle \right)}_{|j_B\rangle}.$$

□

Proposition 1.30 $|\psi\rangle$ is entangled iff its Schmidt rank is > 1. Otherwise, it is separable (i.e. a product state).

Definition 1.31 Let $|\psi\rangle$ be a pure state in a bipartite system $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$, where $\mathbb{H}_A$ has dimension N_A and $\mathbb{H}_B$ has dimension $N_B \geq N_A$. $|\psi\rangle$ is **maximally entangled** if all its Schmidt coefficients are equal (to $1/\sqrt{N_A}$).

Notation 1.32 Write $S(\mathbb{H}) = \left\{\rho \in B(\mathbb{H}) : \rho = \rho^\dagger, \rho \geq 0, \operatorname{tr} p = 1\right\}$ for the set of density matrices on $\mathbb{H}$.

Definition 1.33 The **partial trace** over B, tr_B, on the bipartite system $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$ is the operator defined linearly by

$$\begin{gathered}\operatorname{tr}_B : S(\mathbb{H}_{AB}) \to S(\mathbb{H}_A), \\ |a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2| \mapsto \operatorname{tr}(|b_1\rangle\langle b_2|) \cdot |a_1\rangle\langle a_2|.\end{gathered}$$

Note that if $\rho_{AB} = \rho_A \otimes \rho_B$, then $\operatorname{tr}_B \rho_{AB} = \operatorname{tr}(\rho_B) \cdot \rho_A = \rho_A$.

Definition 1.34 Let ρ_{AB} be a density matrix in $S(\mathbb{H}_{AB})$. $\rho_A = \operatorname{tr}_B(\rho_{AB})$ is called the **reduced density matrix** or **marginal** of ρ_{AB} in A

Proposition 1.35 Let $M_A \in B(\mathbb{H}_A)$. We have

$$\operatorname{tr}(M_A \rho_A) = \operatorname{tr}((M_A \otimes \mathbb{I}_B)\rho_{AB}).$$

for all $\rho_{AB} \in S(\mathbb{H}_{AB})$, $\rho_A = \operatorname{tr}_B(\rho_{AB})$. In fact, this can be taken to be an equivalent definition of partial trace.

Remark 1.36 Let $\rho_{AB} = |\psi\rangle\langle\psi| \in S(\mathbb{H}_{AB})$ be a pure state and let r_ψ be its Schmidt rank. Then

$$\rho_A = \operatorname{tr}_B(|\psi\rangle\langle\psi|) = \sum_{k=1}^{r_\psi} p_k |u_k\rangle\langle u_k|.$$

So ρ_A is pure iff $r_\psi = 1$, i.e. iff $|\psi\rangle$ is separable.

Proposition 1.37 Let $\rho_{AB} \in B(\mathbb{H}_{AB})$ and $\rho_A = \operatorname{tr}_B(\rho_{AB})$. Then:
1. $\operatorname{tr} \rho_A = \operatorname{tr} \rho_{AB}$.
2. If $\rho_{AB} \geq 0$, then $\rho_A \geq 0$.
3. If ρ_{AB} is a density matrix then ρ_A is a density matrix.
4. We have

$$\langle \phi_i | \rho_A | \phi_i \rangle = \sum_k \langle \phi_i \otimes \psi_k | \rho_{AB} | \phi_i \otimes \psi_k \rangle,$$

   for an orthonormal bases $\{|\phi_i\rangle\}$ and $\{|\psi_k\rangle\}$.
5. If $\rho_{AB} = \sigma_A \otimes \sigma_B$ and $\operatorname{tr}(\sigma_B) = 1$, then $\sigma_A = \rho_A$.

Proof.
1. This follows from linearity of trace and the fact that $\operatorname{tr}(\rho \otimes \sigma) = \operatorname{tr}(\rho) \cdot \operatorname{tr}(\sigma)$.
2. By 1, $\langle \psi | \rho_A | \psi \rangle = \operatorname{tr}(\rho_A |\psi\rangle\langle\psi|) = \operatorname{tr}(\rho_{AB}(|\psi\rangle\langle\psi| \otimes \mathbb{I})) \geq 0$.
3. From 1 and 2, by definition.

□

Definition 1.38 Let $\rho_A \in \mathbb{S}(H_A)$ be a (pure or mixed) state. We may introduce an auxiliary space $\mathbb{H}_R$ of dimension $\operatorname{rank}(\rho_A)$ and construct a pure state $|\psi_{AR}\rangle \in \mathbb{H}_A \otimes \mathbb{H}_R$ such that $\rho_A = \operatorname{tr}_R(|\psi_{AR}\rangle\langle\psi_{AR}|)$. This is called **purification**.

Remark 1.39 Let $\left\{M_n^A\right\}_n$ be a POVM in $\mathbb{H}_A$. Then $\left\{M_n^A \otimes \mathbb{I}_B\right\}_n$ is a POVM in $\mathbb{H}_{AB}$.

Theorem 1.40 (Naimark) For every POVM $\{E_n\}_{n=1}^m \subseteq B(\mathbb{H})$, there is a state $|\psi\rangle \in \mathbb{C}^m$ and a projective measurement $\{P_n\}_{n=1}^m \subseteq B(\mathbb{H} \otimes \mathbb{C}^m)$ such that

$$\operatorname{tr}(\rho E_n) = \operatorname{tr}((\rho \otimes |\psi\rangle\langle\psi|)P_n) \quad \forall n \in [m], \forall \rho \in S(\mathbb{H}).$$

2. Quantum channels and open systems

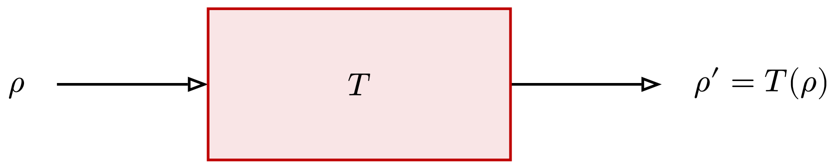
2.1. Quantum channels

Definition 2.1 A **quantum channel** is a linear map $T : S(\mathbb{H}_{\text{in}}) \to S(\mathbb{H}_{\text{out}})$ which satisfies:

- **Preserves trace**: $\operatorname{tr}(T(\rho)) = \operatorname{tr}(\rho)$ for all $\rho \in S(\mathbb{H}_{\text{in}})$.

- **Positive**: if $\rho \geq 0$, then $T(\rho) \geq 0$.
- **Completely positive**: for all ρ, σ if $\rho \otimes \sigma \geq 0$, then $(T \otimes \mathbb{I}_n)(\rho \otimes \sigma) = T(\rho) \otimes \sigma \geq 0$ (note that this implies the second condition, but the converse is false).

So quantum channels are completely positive trace-preserving (CPTP) maps. We may depict a quantum channel T as follows:



Example 2.2 Examples of quantum channels:
- Unitary evolution: $\rho \mapsto U \rho U^*$.
- Adding an ancilla: $\rho \mapsto \rho \otimes \rho_E$ (the E denotes "environment").
- Partial trace: $\rho \mapsto \mathrm{tr}_B(\rho)$ or $\rho \mapsto \mathrm{tr}_A(\rho)$.

We will see that in fact, any quantum channel is a combination of these three.

Definition 2.3 We define the **maximally entangled state** in $\left(\mathbb{C}^d\right)^{\otimes 2}$ as

$$|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |kk\rangle.$$

Definition 2.4 Recall the transposition map is defined as

$$\Theta : A \to A^T, \quad \langle i|A^T|j\rangle = \langle j|A|i\rangle.$$

We define the **partial transpose** by its action on the maximally entangled state $|\phi\rangle = \frac{1}{d}\sum_{i=1}^{d} |ii\rangle$:

$$(|\phi\rangle\langle\phi|)^{T_A} = (|\phi\rangle\langle\phi|)^{T_1} = (\Theta \otimes \mathrm{id})(|\phi\rangle\langle\phi|) = \frac{1}{d}F,$$

where $F = \sum_{i,j=1}^{n} |ij\rangle\langle ji|$ is the flip operator. Note the partial transpose is positive but not CP. Alternatively, we can define it by its action on an orthonormal basis:

$$\langle ij|X^{T_A}|k\ell\rangle = \langle kj|X|i\ell\rangle.$$

Remark 2.5 Note that the partial transpose is useful for detecting entanglement but is not physically implementable (as not CP).

Definition 2.6 Let $T : B\left(\mathbb{C}^{d\times d}\right) \to B\left(\mathbb{C}^{d'\times d'}\right)$ be a linear map. The **Choi-Jamiolkowski matrix** $C \in B\left(\mathbb{C}^{d'} \otimes \mathbb{C}^d\right)$ of T is defined as

$$C := (T \otimes \mathrm{id}_d)|\phi\rangle\langle\phi|.$$

Note that in fact, $C \in S\left(\mathbb{C}^{d'} \otimes \mathbb{C}^d\right)$ is a density matrix if T is a quantum channel.

Remark 2.7 Note that the Choi-Jamiolkowski matrix completely determines T: since $|\phi\rangle\langle\phi| = \frac{1}{d}\sum_{n,m=1}^{d} |nn\rangle\langle mm|$, we have

$$\langle ij|C|k\ell\rangle = \frac{1}{d}\sum_{m,n=1}^{d}\langle ij|(T(|n\rangle\langle m|)\otimes|n\rangle\langle m|)|k\ell\rangle$$
$$= \frac{1}{d}\sum_{m,n=1}^{d}\langle j|n\rangle\cdot\langle m|\ell\rangle\cdot\langle i|T(|n\rangle\langle m|)|k\rangle = \frac{1}{d}\langle i|T(|j\rangle\langle\ell|)|k\rangle,$$

and so we can determine any entry of any $T(\rho)$ by linearity. This state-channel duality is called the **Choi-Jamiolkowski isomorphism**, and can be expressed as

$$\operatorname{tr}(AT(B)) = d\operatorname{tr}(CA\otimes B^T)\quad\forall A\in B\left(\mathbb{C}^{d'}\right), B\in B(\mathbb{C}^d).$$

Indeed, let $\mathbb{F}|ij\rangle = |ji\rangle$ be the flip operator: note that $\mathbb{F}^{T_2} = d|\phi\rangle\langle\phi|$, then if $d = d'$,

$$d\operatorname{tr}(C(A\otimes B^T)) = d\operatorname{tr}((T\otimes\mathrm{id}_d)(|\phi\rangle\langle\phi|)(A\otimes B^T))$$
$$= \operatorname{tr}(\mathbb{F}^{T_2}(T^*(A)\otimes B^T)) = \operatorname{tr}(T^*(A)\otimes B) = \operatorname{tr}(AT(B)).$$

Definition 2.8 The **Hilbert-Schmidt inner product** of $A, B\in B(\mathbb{C}^d)$ is

$$\langle A|B\rangle_{\mathrm{HS}} := \operatorname{tr}(A^*B).$$

Theorem 2.9 (Characterisation of Quantum Channels) Let $T : B(\mathbb{C}^d)\to B\left(\mathbb{C}^{d'}\right)$ be a linear map. TFAE:

1. T is a quantum channel.
2. Let $C = (T\otimes\mathbb{I}_d)(|\phi\rangle\langle\phi|)$ be the Choi-Jamiolkowski matrix of T, then $C\geq 0$ and $\operatorname{tr}_1(C) = \frac{1}{d}\mathbb{I}_d$.
3. **Kraus decomposition**: There exists $\{A_k\}_{k=1}^{dd'}\subseteq\mathbb{C}^{d'\times d}$ with $\sum_{k=1}^{dd'}A_k^*A_k = \mathbb{I}_d$ such that

   $$T(\rho) = \sum_{k=1}^{dd'}A_k\rho A_k^*\quad\forall\rho\in S(\mathbb{C}^d).$$

   We call the number of non-trivial A_k in the Kraus decomposition the **Kraus rank** of T.
4. **Stinespring dilation**: there exists a unitary U on $\mathbb{C}^d\otimes\mathbb{C}^{dd'}$ and a state $|\psi\rangle\in\mathbb{C}^{dd'}$ such that $T(\rho) = \operatorname{tr}_2(U(\rho\otimes|\psi\rangle\langle\psi|)U^*)$ for all $\rho\in S(\mathbb{C}^d)$.

Proof (Hints).

- $1\Rightarrow 2$: straightforward.
- $4\Rightarrow 1$: use that compositions of quantum channels are quantum channels.

□

Proof.

- $1\Rightarrow 2$: $C\geq 0$ follows from the completely positive property of T and linearity. Also,

$$\operatorname{tr}_1(C) = \frac{1}{d}\sum_{n,m=1}^{d}\operatorname{tr}(T|n\rangle\langle m|)\cdot|n\rangle\langle m|$$

$$= \frac{1}{d}\sum_{n,m=1}^{d} \operatorname{tr}(|n\rangle\langle m|)\cdot|n\rangle\langle m| \quad \text{since } T \text{ preserves trace}$$

$$= \frac{1}{d}\sum_{n,m}\delta_{mn}|n\rangle\langle m| = \frac{1}{d}\sum_{n=1}^{d}|n\rangle\langle n| = \frac{1}{d}\mathbb{I}_d.$$

- $2 \Rightarrow 3$: we use that (verify this) $(A\otimes\mathbb{I})|\phi\rangle = (\mathbb{I}\otimes A^T)|\phi\rangle$ for all $A \in B(\mathbb{C}^d)$, where $|\phi\rangle$ is the maximally entangled state, and that $\forall|\psi\rangle \in \mathbb{C}^{d^2}$, there exists A such that $|\psi\rangle = (A\otimes\mathbb{I})|\phi\rangle$. Since $C \geq 0$, we can write $C = \sum_{k=1}^{dd'}|\psi_k\rangle\langle\psi_k|$ ($|\psi_k\rangle$ are not necessarily normalised). So

$$C = \sum_{k=1}^{dd'}(A_k\otimes\mathbb{I})|\phi\rangle\langle\phi|(A_k^*\otimes\mathbb{I})$$
$$= (T\otimes\mathbb{I})|\phi\rangle\langle\phi|.$$

Also,

$$\frac{1}{d}\mathbb{I} = \operatorname{tr}_1(C) = \sum_{n=1}^{d}\langle n_1|C_{12}|n_1\rangle$$
$$= \frac{1}{d}\sum_{n=1}^{d}\sum_{m=1}^{dd'}(\mathbb{I}\otimes A_m^T)(|\phi\rangle\langle\phi|)\big(\mathbb{I}\otimes\overline{A}_k\big)|n\rangle$$
$$= \sum_{n=1}^{d}\langle n|\sum_{k=1}^{dd'}(\mathbb{I}\otimes A_m^T)\frac{1}{d}\left(\sum_{k,\ell=1}^{d}|kk\rangle\langle\ell\ell|\right)\big(\mathbb{I}\otimes\overline{A}_k\big)|n\rangle$$
$$= \frac{1}{d}\sum_{n=1}^{d}\sum_{m=1}^{dd'}\sum_{k,\ell=1}^{d}\langle n|k\rangle\langle\ell|n\rangle A_m^T|k\rangle\langle\ell|\overline{A}_k$$
$$= \frac{1}{d}\sum_{n=1}^{d}\sum_{m=1}^{dd'}A_m^T|n\rangle\langle n|\overline{A}_m$$
$$= \frac{1}{d}\sum_{m=1}^{dd'}A_m^T\overline{A}_m$$

So we set $\tilde{A}_m := \overline{A}_m$.

- $3 \Rightarrow 4$: let $V = \sum_{k=1}^{dd'}A_k\otimes|k\rangle$, where $\{|k\rangle\}_{k=1}^{dd'}$ is an orthonormal basis of $\mathbb{C}^{dd'}$. V is an isometry, i.e. $V^*V = \sum_{k=1}^{dd'}A_k^*A_k = \mathbb{I}_d$. Then for all $\rho \in S\big(\mathbb{C}^{dd'}\big)$, since $(A_k\otimes|k\rangle)\rho = (A_k\rho)\otimes|k\rangle$,

$$\operatorname{tr}_2(V\rho V^*) = \operatorname{tr}_2\left(\sum_{k,\ell=1}^{dd'}(A_k\rho A_\ell^*)\otimes|k\rangle\langle\ell|\right)$$
$$= \sum_{k,\ell=1}^{dd'}(A_k\rho A_\ell^*)\operatorname{tr}(|k\rangle\langle\ell|)$$
$$= \sum_{k=1}^{dd'}A_k\rho A_k^* = T(\rho).$$

Now choose $V = U(\mathbb{I} \otimes |\psi\rangle)$ for some pure state $|\psi\rangle$ and unitary U.

- $4 \Rightarrow 1$: the maps

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi| \mapsto U(\rho \otimes |\psi\rangle\langle\psi|)U^* \mapsto \mathrm{tr}_2(U(\rho \otimes |\psi\rangle\langle\psi|)U^*)$$

are all quantum channels, and so their composition is also a quantum channel.

□

Remark 2.10

- The number k in the Kraus decomposition is called the **Kraus rank** of T, which is the same as the Choi rank (rank of the Choi-Jamiolkowski matrix). Note: this is not the same as the rank of T as a map.
- We can always express T with $r = \mathrm{rank}(C)$ Kraus operators which are orthogonal (w.r.t Hilbert-Schmidt inner product), since T is a completely positive linear map.
- Two sets of Kraus operator $\{K_j\}$ and $\{J_\ell\}$ represent the same map T iff there exists a unitary U such that $K_j = \sum_\ell U_{j\ell} J_\ell$.

2.2. Examples of quantum channels

Definition 2.11 In two dimensions, there are three kinds of errors:

1. Bit flip errors, modelled by the Pauli X: $|0\rangle \mapsto |1\rangle$, $|1\rangle \mapsto |0\rangle$.
2. Phase flip error: modelled by Pauli Z: $|0\rangle \mapsto |0\rangle$, $|1\rangle \mapsto -|1\rangle$.
3. Combination of bit and phase flip errors: modelled by Pauli Y.

A map describing the depolarising channel is

$$U_{A\to AE} : |\psi\rangle_A \mapsto \sqrt{1-p}|\psi\rangle_A \otimes |0\rangle_E + \sqrt{\frac{p}{3}}(X|\psi\rangle_A \otimes |1\rangle_E + Y|\psi\rangle_A \otimes |2\rangle_E + Z|\psi\rangle_A \otimes |3\rangle_E)$$

(the environment H_E has dimension 4). We can express this in the Kraus decomposition: let $M_a := \langle a|_E U_{A\to AE}$, $a \in \{0,1,2,3\}$, and $M_0 = \sqrt{1-p}\mathbb{I}$, $M_1 = \sqrt{p/3}X$, $M_2 = \sqrt{p/3}Y$, $M_3 = \sqrt{p/3}Z$. It is straightforward to see that

$$\sum_{a=0}^{3} M_a^\dagger M_a = \left(1 - p + \frac{p}{3} + \frac{p}{3} + \frac{p}{3}\right)\mathbb{I} = \mathbb{I}.$$

The channel is $T(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$. For arbitrary dimensions D, the depolarising channel is $\rho \mapsto (1-p)\rho + p\sigma$, where $\sigma \in S(\mathbb{C}^D)$, usually $\sigma = \mathbb{I}/d$.

Definition 2.12 The **phase damping channel** is the map

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \mapsto \begin{bmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{bmatrix}.$$

Let the environment have orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$, then the state representation is

$$|0\rangle_A \mapsto \sqrt{1-p}|0\rangle_A \otimes |0\rangle_E + \sqrt{p}|0\rangle_A \otimes |1\rangle_E$$
$$|1\rangle_A \mapsto \sqrt{1-p}|1\rangle_A \otimes |0\rangle_E + \sqrt{p}|1\rangle_A \otimes |2\rangle_E$$

The Kraus operators are $M_0 = \sqrt{1-p} \cdot \mathbb{I}$, $M_1 = \sqrt{p}|0\rangle\langle 0|$, $M_2 = \sqrt{p}|1\rangle\langle 1|$. We have $M_0^2 + M_1^2 + M_2^2 = \mathbb{I}$. The map is $T(\rho) = (1-p/2)\rho + \frac{1}{2}pZ\rho Z$.

Definition 2.13 A density matrix $\rho \in S(\mathbb{H}_A \otimes \mathbb{H}_B)$ is **separable** if it can be expressed as a convex combination

$$\rho = \sum_i p_i \rho_i^A \otimes \sigma_i^B,$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and $\rho_i^A \in S(\mathbb{H}_A)$ and $\sigma_i^B \in S(\mathbb{H}_B)$.

Definition 2.14 A quantum channel T is **entanglement breaking** if its Choi-Jamiolkowski matrix is separable. This is equivalent to the existence of a POVM $\{M_k\}$ and a set of density matrices $\{\rho_k\}$ such that $T(\rho) = \sum_k \operatorname{tr}(M_k \rho)\rho_k$.

2.3. Properties of channels

Remark 2.15 Let $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$, $d = \min\{\dim H_A, \dim H_B\}$, not necessarily normalised. The Schmidt decomposition is

$$|\psi\rangle = \sum_{j=1}^{d} \lambda_j |e_j\rangle \otimes |f_j\rangle,$$

$\lambda_j \geq 0$, $\sum_j \lambda_j^2 = \langle\psi|\psi\rangle$, $\{e_j\}$, $\{f_j\}$ orthonormal bases.

The reduced density operators of $|\psi\rangle$ are diagonal in the bases $\{|e_j\rangle\}$, $\{|f_j\rangle\}$, with eigenvalues λ_j^2. Conversely, if $\rho_A \in S(\mathbb{H}_A)$ has spectral decomposition $\rho_A = \sum_j \lambda_j |e_j\rangle\langle e_j|$, then $|\psi\rangle$ provides a purification for $\rho_A = \operatorname{tr}_B(|\psi\rangle\langle\psi|)$; the minimal dilation space we can choose, $\mathbb{H}_{\min}$ has dimension $\mathsf{rank}(\rho_A)$. If $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_{\min}$, then all other purifications of ρ_A are of the form $|\psi'\rangle = (\mathbb{I}_A \otimes V)|\psi\rangle$, with $V \in B(\mathbb{H}_{\min}, \mathbb{H}_B)$ an isometry. Hence, all purifications are related by $\mathbb{I}_A \otimes U$ with U an isometry.

Proposition 2.16 (Equivalence of Ensembles) Let $\{|\psi_j\rangle : j \in [M]\}$ and $\{|\phi_\ell\rangle : \ell \in [N]\}$ be (not necessarily normalised) ensembles. Then

$$\sum_{j=1}^{M} |\psi_j\rangle\langle\psi_j| = \sum_{\ell=1}^{N} |\phi_\ell\rangle\langle\phi_\ell|$$

iff there is an isometry $U \in \mathbb{C}^{M\times N}$ such that $|\psi_j\rangle = \sum_{\ell=1}^{N} U_{j\ell}|\phi_\ell\rangle$.

Proof (Hints).

- $\Longleftarrow$: straightforward.
- $\Longrightarrow$: explain why we can assume that $\rho = \sum_j |\psi_j\rangle\langle\psi_j|$ and $\sigma = \sum_\ell |\phi_\ell\rangle\langle\phi_\ell|$ are density matrices. Consider purifications of ρ and σ which use the same orthonormal basis in the dilation space.

□

Proof.

- $\Longleftarrow$: this is straightforward to show.

- $\Longrightarrow$: WLOG (by rescaling ρ), we can assume $\rho := \sum_j |\psi_j\rangle\langle\psi_j|$ is a density matrix. We have $\rho = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ (through purification), where $|\psi\rangle = \sum_j |\psi_j\rangle \otimes |j\rangle$. Similarly, let $|\phi\rangle = \sum_\ell |\phi_\ell\rangle \otimes |\ell\rangle$ (so we use the same orthonormal basis $\{|\ell\rangle\} = \{|j\rangle\}$). So $|\psi\rangle$ and $|\phi\rangle$ differ by a unitary (or an isometry if the dimensions are not equal), hence $|\psi\rangle = (1 \otimes U)|\phi\rangle$. Taking the scalar product with $\langle j|$, we obtain $|\psi_j\rangle = \sum_\ell U_{j\ell}|\phi_\ell\rangle$.

$\square$

Notation 2.17 Let T_1, T_2 be linear maps. Write $T_2 \geq T_1$ to mean $T_2 - T_1$ is completely positive. By the Choi-Jamiolkowski isomorphism, this is equivalent to $C_2 \geq C_1$ where C_i is the Choi matrix of T_i (i.e. $C_2 - C_1$ is positive semi-definite).

Theorem 2.18 Let $T_1, T_2 : \mathbb{C}^{d' \times d'} \to \mathbb{C}^{d \times d}$ be completely positive maps, with $T_2 \geq T_1$. Let $V_i : \mathbb{C}^d \to \mathbb{C}^{d'} \otimes \mathbb{C}^{r_i}$ be Stinespring representations for T_i (i.e. $T_i(A) = V_i^*\left(A \otimes \mathbb{I}_{r_i}\right)V_i$), then there is a contraction (i.e. $W^*W \leq \mathbb{I}$) $W : \mathbb{C}^{r_2} \to \mathbb{C}^{r_1}$ such that $V_1 = (\mathbb{I}_{d'} \otimes W)V_2$.

Moreover, if V_2 belongs to a minimal dilation, then W is unique.

Proof (Hints).

$\square$

Proof. We use the equivalence $T_2 \geq T_1 \Leftrightarrow C_2 \geq C_1$. Define the map

$$R_i = \left(\mathbb{I}_{r_i} \otimes \langle\phi|\right)(V_i \otimes \mathbb{I}_{d'}) \in B\left(\mathbb{C}^d \otimes \mathbb{C}^{d'}, \mathbb{C}^{r_i}\right)$$

Let $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$. We want to show $\||R_2|\psi\rangle\|^2 \geq \|R_1|\psi\rangle\|^2$. Indeed,

$$\begin{aligned}
\|R_2|\psi\rangle\|^2 &= \langle\psi|R_2^*R_2|\psi\rangle \\
&= \langle\psi|(V_2^* \otimes \mathbb{I}_{d'})\left(\mathbb{I}_{r_2} \otimes |\phi\rangle\right)\left(\mathbb{I}_{r_2} \otimes \langle\phi|\right)(V_2 \otimes \mathbb{I}_{d'})|\psi\rangle \\
&= \langle\psi|(T_2 \otimes \mathrm{id})(|\phi\rangle\langle\phi|)\rangle \\
&= \langle\psi|C_2|\psi\rangle \geq \langle\psi|C_1|\psi\rangle.
\end{aligned}$$

And $\langle\psi|C_1|\psi\rangle = \|R_1|\psi\rangle\|^2$ by the same argument. So there exists a contraction $W : \mathbb{C}^{r_2} \to \mathbb{C}^{r_1}$, such that $R_1 = WR_2$. So $V_1 = (\mathbb{I}_{d'} \otimes W)V_2$. If $r_2 = \mathrm{rank}(C_2)$, then R_2 is surjective, and so W is uniquely determined. $\square$

Theorem 2.19 (Radon-Nikodym) Let $\{T_i\}$ be a set of CP maps such that $\sum_i T_i = T \in B\left(\mathbb{C}^{d' \times d'}, \mathbb{C}^{d \times d}\right)$ with Stinespring representation $T(A) = V^*(A \otimes \mathbb{I}_r)V$. Then there exists a set of non-negative operators $P_i \in \mathbb{C}^{r \times r}$ such that $\sum_i P_i = \mathbb{I}_r$ and $T_i(A) = V^*(A \otimes P_i)V$.

Remark 2.20 Since $T = \sum_i T_i$, this gives $T(A) = \sum_i V^*(A \otimes P_i)V$, where $\{P_i\}$ is a POVM. This gives an identification between quantum channels of this form and POVMs.

Definition 2.21 An **instrument** is a set of CP maps $\{T_i\}$ whose sum is trace-preserving.

TODO: insert diagram.

Remark 2.22 Instruments encompass the notions of quantum channels and POVMs:

- We can assing a quantum channel $T : \rho \mapsto \sum_i T_i(\rho)$. (Measurement outcome ignored.)
- By contrast, POVMs ignore the quantum system: $p_i = \operatorname{tr}(T_i(\rho)) = \operatorname{tr}(T_i(\rho)\mathbb{I}) = \operatorname{tr}(\rho T_i^*(\mathbb{I})) =: \operatorname{tr}(\rho M_i)$: $\{M_i\}$ is a POVM.

Remark 2.23 Instruments can viewed as a special case of quantum channels by assigning to them the quantum channel

$$\rho \mapsto \sum_i T_i(\rho) \otimes |i\rangle\langle i|,$$

where $\{|i\rangle\}$ is an orthonormal basis.

Proposition 2.24 (Quantum Steering) Let $\rho \in B(\mathbb{H}_A)$ be a density operator with purification $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$. Let $\rho = \sum_i \lambda_i \rho_i$ be a convex combination. Then there is an instrument $\{T_i\}$ with each $T_i : B(\mathbb{H}_B) \to B(\mathbb{H}_B)$, such that $\lambda_i \rho_i = \operatorname{tr}_B((\mathbb{I} \otimes T_i)(|\psi\rangle\langle\psi|))$.

2.4. Description of open quantum many-body systems

Assume evolution is

$$\rho_{SE}(t) = \rho_S(t) \otimes \rho_E \overset{\mathrm{d}t}{\mapsto} \rho_{SE}(t + \mathrm{d}t) = \rho_S(t + \mathrm{d}t) \otimes \rho_E(t + \mathrm{d}t) = \rho_S(t + \mathrm{d}t) \otimes \rho_E$$

Definition 2.25 A **quantum Markov semigroup** is a 1-parameter continuous semigroup $\{T_t : t \geq 0\}$ of quantum channels (so each $T_t : S(\mathbb{H}) \to S(\mathbb{H})$).

Note that $T_0 = \mathbb{I}$ and $T_s \circ T_t = T_{t+s}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t = \mathcal{L} \circ T_t = T_t \circ \mathcal{L},$$

where $\mathcal{L}$ is the infinitesimal generator of the semigroup, called the **Liouvillian** or **Lindbladian**. This equation is called the **master equation** or **Liouville equation**. This gives

$$T_t = e^{t\mathcal{L}}.$$

2.5. Separability criteria

Notation 2.26 Let $A(\mathbb{H})$ denote the set of bounded linear Hermitian operators on $\mathbb{H}$.

Definition 2.27 The **covariance** (or **operator correlation**) of ρ between subsystems A and B is

$$\operatorname{Cor}_\rho(A : B) = \sup_{\|M_A\|, \|M_B\| \leq 1} |\operatorname{tr}(\rho M_A T_B) - \operatorname{tr}(\rho M_A) \operatorname{tr}(\rho M_B)|,$$

where $M_A \in A(H_A)$, $M_B \in A(H_B)$, and $\|\cdot\|$ is the standard operator norm.

Example 2.28 If ρ is separable, then $\mathrm{Cor}_\rho(A:B)$ measures classical correlation. If $\rho = \rho_A \otimes \rho_B$, then $\mathrm{Cor}_\rho(A:B) = 0$.

Definition 2.29 Let $|\psi\rangle = \sum_{i=1}^d \sqrt{p_i}|e_i\rangle \otimes |f_i\rangle$ be the Schmidt decomposition of $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$. Let $\rho = |\psi\rangle\langle\psi|$. The **entanglement entropy** of ρ is the Shannon entropy of the probability distribution $(p_1, ..., p_d)$:

$$S_{\mathrm{ENT}}(\rho) := -\sum_{i=1}^d p_i \log(p_i).$$

Proposition 2.30
- $S_{\mathrm{ENT}(\rho)} = 0$ iff the Schmidt rank of $|\psi\rangle$ is 1.
- The maximum value of $S_{\mathrm{ENT}}(\rho)$ is $\log(d)$, and is achieved iff $|\psi\rangle$ is maximally entangled, i.e. $\lambda_i = 1/d$ for all $i \in [d]$.

Proposition 2.31 (PPT Criterion) Let $\rho \in S(\mathbb{H}_A \otimes \mathbb{H}_B)$. If ρ^{T_A} has a negative eigenvalue, then ρ is entangled.

Proof (Hints). Prove the contrapositive. □

Proof. Assume ρ is separable, so $\rho = \sum_j p_j \rho_j^A \otimes \rho_j^B$. Then

$$\rho^{T_A} = (\Theta \otimes \mathrm{id})(\rho) = \sum_j p_j \left(\rho_j^A\right)^T \otimes \rho_j^B,$$

and so $\rho^{T_A} \geq 0$, as it is a sum of positive matrices. □

Definition 2.32 Write $S_{\mathrm{SEP}} = \{\text{separable density matrices}\}$, which is convex and compact. By the Hahn-Banach theorem, for all $\rho \notin S_{\mathrm{SEP}}$, there exists a hyperplane determined by a Hermitian operator ω such that $\mathrm{tr}(\rho\omega) < 0$ and $\mathrm{tr}(\sigma\omega) \geq 0$ for all $\sigma \in S_{\mathrm{SEP}}$. ω is called an **entanglement witness** for ρ.

By the Choi-Jamiolkowski isomorphism, ω corresponds to a map Λ via the following:

$$\omega = (\Lambda \otimes \mathrm{id}_B)(|\phi\rangle\langle\phi|).$$

Remark 2.33 The entanglement witness corresponding to the transposition map is the flip operator F.

Proposition 2.34 Let $\mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$ and let $\rho \in S(\mathbb{H}_{AB})$. Then ρ is separable iff $(\Lambda \otimes \mathrm{id}_B)(\rho) \geq 0$ for every positive map $\Lambda : B(\mathbb{H}_A) \to B(\mathbb{H}_A)$.

Proof (Hints).
- $\Longrightarrow$: straightforward.
- $\Longleftarrow$: TODO.

□

Proof. $\Longrightarrow$: let ρ be separable, so we can write $\rho = \sum_j p_j \rho_j \otimes \sigma_j$. Then for every positive $\Lambda : B(\mathbb{H}_A) \to B(\mathbb{H}_A)$,

$$(\Lambda \otimes \mathrm{id}_B)(\rho) = \sum_j \lambda_j \Lambda(\rho_j) \otimes \sigma_j \geq 0,$$

since each $\Lambda(\rho_j) \geq 0$.

$\Longleftarrow$: let ρ be entangled. We want to find a positive map $\Lambda : B(\mathbb{H}_A) \to B(\mathbb{H}_A)$ such that $(\Lambda \otimes \mathrm{id}_B)(\rho)$ has a negative eigenvalue. By Definition 2.32, ρ has an entanglement witness ω, with $\mathrm{tr}(\rho\omega) < 0$. By the Choi-Jamiolkowski isomorphism, this defines a map Λ such that

$$\omega = (\Lambda^* \otimes \mathrm{id}_B)(|\phi\rangle\langle\phi|).$$

Since $\mathrm{tr}(XY) = \mathrm{tr}(\mathbb{F}(X \otimes Y))$, and $F = d|\phi\rangle\langle\phi|$, we have for all $A \in B(\mathbb{H}_A)$, $B \in B(\mathbb{H}_B)$,

$$\begin{aligned} \mathrm{tr}(B^T \Lambda(A)) &= \mathrm{tr}(F(\Lambda(A) \otimes B^T)) \\ &= d\,\mathrm{tr}((\Lambda \otimes \mathrm{id}_B)(A \otimes B)(|\phi\rangle\langle\phi|)) \\ &= d\langle\phi|(\Lambda \otimes \mathrm{id}_B)(A \otimes B)|\phi\rangle. \end{aligned}$$

TODO: finish. $\square$

Remark 2.35

- In the above proof, we use that $\mathrm{tr}(\rho\omega) = d\langle\phi|(\Lambda \otimes \mathrm{id}_B)(\rho)|\phi\rangle < 0$ implies that $(\Lambda \otimes \mathrm{id}_B)$ has a negative eigenvalue. However, the converse is false. Hence, the positive map Λ corresponding to a witness ω in fact "detects more entanglement" than ω.
- It can be shown that Λ constructed from ω detects an entangled state ρ iff ρ is detected by a witness of the form $(\mathbb{I} \otimes \mathbb{X})\omega(\mathbb{I} \otimes X^*)$ for some $X \in B(\mathbb{H}_B)$.

Remark 2.36 Note that Proposition 2.34 is a theoretical result but is not implementable (in a lab) since Λ is only required to be positive (but not CP). However, the map

$$T(\rho) = \frac{p}{d^2}\mathbb{I}_d \otimes \mathbb{I}_d + (1-p)(\Lambda \otimes \mathrm{id}_B)(\rho)$$

is a CP map. If ρ is separable, then the minimal eigenvalue of $T(\rho)$ must exceed a certain threshold. If it doesn't exceed this threshold, then ρ is entangled.

Remark 2.37 Note that by using a change of abasis via a unitary U, we can obtain a different partial transpose $\tilde{T}_A$ from the "usual" partial transpose T_A:

$$\rho^{\tilde{T}_A} = (U \otimes \mathbb{I})((U^* \otimes \mathbb{I})\rho(U \otimes \mathbb{I}))^{T_A}(U^* \otimes \mathbb{I}) = ((UU^T) \otimes \mathbb{I})\rho^{T_A}\left((UU^T)^* \otimes \mathbb{I}\right) \neq \rho^{T_A}.$$

Note that this non-uniqueness of the partial transpose does not affect the previous criteria, as they only deal with the eigenvalues, which are invariant under basis changes. Also, we have $\rho^{\tilde{T}_A} \iff \rho^{T_A} \geq 0 \iff \rho^{T_B} \geq 0$, since ρ^{T_A} and ρ^{T_B} differ only by a global transposition.

Definition 2.38 A map $\Lambda : B(\mathbb{H}) \to B(\mathbb{H})$ is called **decomposable** if $\Lambda = \Lambda_1 + \Lambda_2 \circ \Theta$, where Λ_1 and Λ_2 are positive maps and Θ is a partial transpose. Otherwise, it is called **non-decomposable**.

Example 2.39 The entanglement witness corresponding to a decomposable map $\Lambda = \Lambda_1 + \Lambda_2 \circ \Theta$ is $\omega = Q_1 + Q_2^T$, where $Q_i = d(\Lambda_i \otimes \mathbb{I})(|\phi\rangle\langle\phi|)$ is the entanglement witness of Λ_i

Proposition 2.40 (Reduction Criterion) Let $\Lambda_{\text{red}}(A) = \text{tr}(A)\mathbb{I} - A$. Note that Λ_{red} is positive. Proposition 2.34 gives us

$$(\Lambda_{\text{red}} \otimes \mathbb{I})(\rho) \Longrightarrow \begin{cases} \rho_A \otimes \mathbb{I}_B \geq \rho_{AB} \\ \mathbb{I}_A \otimes \rho_B \geq \rho_{AB}. \end{cases}$$

The entanglement witness corresponding to Λ_{red} is $(\mathbb{I} - F)^{T_A} = 2P_-^{T_A}$, where P_- is the projector onto the anti-symmetric subspace (the space of anti-Hermitian operators). In this case, we obtain

$$\text{tr}(\rho\omega) < 0 \quad \text{iff} \quad \langle\phi|\rho|\phi\rangle \leq \frac{1}{d},$$

where $|\phi\rangle$ is the maximally entangled state.

Proof. Omitted. □

Remark 2.41 If $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, $P_-^{T_A}$ is 1-dimensional, which gives that entanglement being detected by ω is equivalent to the PPT criterion.

Proposition 2.42 Entangled states with positive partial transpose exist iff there are non-decomposable maps. Specifically, there exists a non-decomposable map $T : B(\mathbb{H}_A) \to B(\mathbb{H}_B)$ iff there exists an entangled state $\rho \in B(\mathbb{H}_A) \otimes B(\mathbb{H}_B)$ with positive partial transpose $\rho^{T_A} \geq 0$.

Proof. Omitted. □

Proposition 2.43 Let $\rho \in S(\mathbb{C}^2 \otimes \mathbb{C}^3)$ or $S(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Then ρ is separable iff $\rho^{T_A} \geq 0$.

Proof (Hints). Use the fact that every positive Λ on a Hilbert space of dimension $2 \otimes 2$ or $2 \otimes 3$ is decomposable. □

Proof. This follows from the PPT Criterion and Proposition 2.42 combined with the fact that every positive Λ on a Hilbert space of dimension $2 \otimes 2$ or $2 \otimes 3$ is decomposable. □