- **Q**: What is the field $\mathbb{C}\{\{t\}\}$ (field of complex Puiseux series)?
- **A**: Field of complex Puiseux series, $\mathbb{C}\{\{t\}\}$, is defined as

$$\mathbb{C}\{\{t\}\} := \bigcup_{n\in\mathbb{N}} \mathbb{C}((t^{1/n})) = \left\{ \sum_{m=-\infty}^{\infty} c_m t^{m/n} : n \in \mathbb{N}, c_i \in \mathbb{C} \right\}$$

- **Q**: Let $K = \mathbb{C}\{\{t\}\}$. What is the tropicalisation $\mathrm{trop}(V(w - x^2y + xy^2)) \subseteq \mathbb{R}^3$ where $w - x^2y + xy^2 \in K[w, x, y]$ where V is vanishing set (where polynomial is zero). (Draw a picture with an explanation). How is it different for $w - x^\alpha + x^\beta$, $x^\alpha = x_1^{\alpha_1} ... x_m^{\alpha_m}$, similarly for β?
- **A**: Let $f(w, x, y) = w - x^2y + xy^2 \in K[w, x, y]$, then $\mathrm{trop}(f) = w \oplus x^2 \odot y \oplus x \odot y^2 = \min(w, 2x + y, x + 2y)$ (since the coefficient of each monomial is a constant, so valuation val is equal to 0, using the natural "exponent" valuation on $\mathbb{C}\{\{t\}\}$). Now the tropical hypersurface is

$$\begin{aligned}\mathrm{trop}(V(f)) &= V(\mathrm{trop}(f)) \\ &= \big\{(w, x, y) \in \mathbb{R}^3 : \mathrm{trop}(f)(w, x, y) \text{ attains the minimum twice}\big\} \\ &= \big\{(w, x, y) \in \mathbb{R}^3 : w = y + 2x \leq x + 2y \\ &\quad \text{or } w = x + 2y \leq 2x + y \text{ or } 2x + y = x + 2y \leq w\big\}\end{aligned}$$

  Let

$$g(w, x_1, ..., x_m) = w - \underline{x}^{\underline{\alpha}} + \underline{x}^{\underline{\beta}} = w - x_1^{\alpha_1} \cdots x_m^{\alpha_m} + x_1^{\beta_1} \cdots x_m^{\beta_m}$$

  Then

$$\begin{aligned}\mathrm{trop}(g)(w, x_1, ..., x_m) &= \min(w, \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_mx_m, \beta_1x_1 + \cdots + \beta_mx_m) \\ &= \min\big(w, \underline{\alpha} \cdot \underline{x}, \underline{\beta} \cdot \underline{x}\big)\end{aligned}$$

  Hence

$$\begin{aligned}\mathrm{trop}(V(f)) &= V(\mathrm{trop}(f)) \\ &= \big\{(w, x_1, ..., x_m) \in \mathbb{R}^3 : w = \underline{\alpha} \cdot \underline{x} \leq \underline{\beta} \cdot \underline{x} \\ &\quad \text{or } w = \underline{\beta} \cdot \underline{x} \leq \underline{\alpha} \cdot \underline{x} \text{ or } \underline{\alpha} \cdot \underline{x} = \underline{\beta} \cdot \underline{x} \leq w\big\}\end{aligned}$$

Questions I have:

- Does it matter what valuation you use, for example, if you use the trivial valuation? E.g. for Puiseux series, if the trivial valuation $\mathrm{val}(c(t)) = 0$ was used instead of the exponent valuation, this would result in a different tropicalisation?
- Not sure how to sketch the tropicalisation in Q2, I thought it had to be 3D, since f has three parameters.

- Are there any major differences between the min-plus and max-plus algebras, why has the book chosen min-plus? Do they lead to different results?