Introduction

- **Tropical semiring or min-plus algebra**: $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ where

$$x \oplus y \coloneqq \min(x, y)$$
$$x \odot y \coloneqq x + y$$

  $\oplus$ is **tropical sum**, $\odot$ is **tropical product**.
- Tropical addition and multiplication are **commutative**:

$$x \oplus y = y \oplus x$$
$$x \odot y = y \odot x$$

- Tropical addition and multiplication are **associative**:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$
$$x \odot (y \odot z) = (x \odot y) \odot z$$

- Tropical addition and multiplication satisfy **distributivity**:

$$x \odot (y \oplus z) = x \odot y \oplus x \odot z$$

- As for normal addition and multiplication, $\odot$ has higher precedence than $\oplus$.
- Polynomials and rational functions defined over tropical semiring are piecewise linear.
- **Identity element** for addition is ∞, **identity element** for multplication is 0:

$$x \oplus \infty = x$$
$$\odot\, 0 = x$$

- The identity elements satisfy

$$x \odot \infty = \infty$$
$$x \oplus 0 = \begin{cases} 0 \text{ if } x \geq 0 \\ x \text{ if } x < 0 \end{cases}$$

- $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ is a **semiring** rather than a ring since there exists no additive inverse.
- We have

$$\forall n \in \mathbb{N}, \quad (x \oplus y)^n = x^n \oplus y^n$$

  since $(x \oplus y)^n = n \cdot \min(x, y) = \min(nx, ny) = x^n \oplus y^n$.