hw2_KleisleMurphy

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1 HW2 - Bayesian Inference in the Poisson Generalized Linear Model

STATS271/371: Applied Bayesian Statistics

Stanford University. Winter, 2021.

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Due: 11:59pm Friday, April 16, 2021 via GradeScope

In this 2nd homework, we will perform Bayesian Inference in the Poisson generalized linear model.

References: - Chapter 16 of BDA3 contains background material on generalized linear models. - Chapter 7.1 of BDA3 introduces notation for model evaluation based on predictive log likelihoods. - The data we use comes from Uzzell & Chichilnisky, 2004. If you're interested, see `README.txt` file in the `/data_RGCs` directory or the Pillow tutorial for details.

Remark: While some programming languages may incorporate packages that fit Poisson GLMs using one line of code, deriving some of the calculations yourself is an important part of this assignment. Therefore, calls to specialized GLM libraries such as pyglmnet are **prohibited**. Of course, standard libraries such as Numpy are still allowed (and encouraged!). calls to numerical optimizers (such as scipy.optimize.minimize) are fair game.

1.1 The Poisson GLM

The Poisson distribution is a common model for count data with a single parameter $\lambda \in \mathbb{R}_+$. Its pmf is,

$$\Pr(y \mid \lambda) = \frac{1}{y!} e^{-\lambda} \lambda^y, \tag{1}$$

for $y \in \mathbb{N}$. Its mean and variance are both equal to λ.

Suppose we have count observations $y_n \in \mathbb{N}$ along with covariates $x_n \in \mathbb{R}^P$. We construct a Poisson GLM by modeling the expected value as,

$$\mathbb{E}[y_n \mid x_n] = f(w^\top x_n), \tag{2}$$

with $w \in \mathbb{R}^P$ and $f : \mathbb{R} \to \mathbb{R}_+$ is the mean function. The *canonical mean function* is $f(a) = e^a$; equivalently, the canonical *link* function is the logarithm.

We assume a Gaussian prior on the weights w:

$$w \sim \mathcal{N}(0, \sigma^2 I),$$

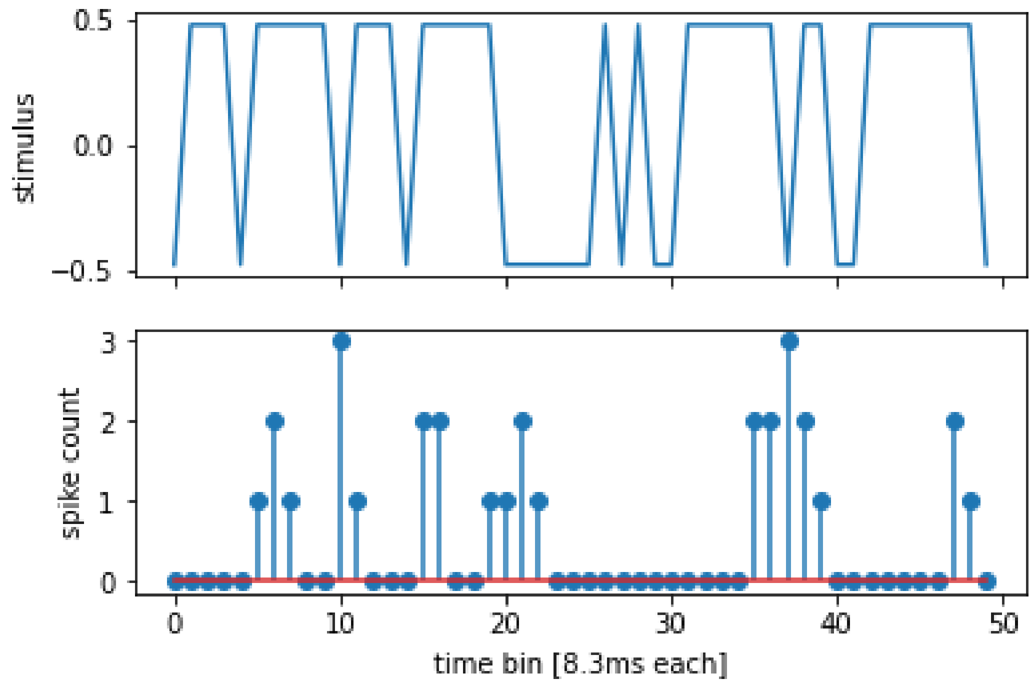
where $\sigma^2 I$ is the covariance matrix.

1.2 Load the data

The data consists of spike counts from a retinal neuron responding to a flickering light. The spike counts are measured in 8.3ms bins and they range from 0 to 3 spikes/bin. The stimulus is binary, either .48 if the light is on or -.48 if it's off. The goal of this assignment is to model how the neural spike counts relate to recent light exposure over the past 25 time bins (approximately 200ms).

We've provided some code to load the data in Python and plot it. Feel free to convert this to R if that is your preference.

[1]:
```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.linalg import hankel
from scipy.stats import multivariate_normal, norm, poisson

df = pd.read_csv("hw2.csv")

# Convert the training data to arrays
y_train = np.array(df["y_train"])
stim_train = np.array(df["stim_train"])
N_train = len(y_train)

# Convert the test data to arrays
y_test = np.array(df["y_test"])
stim_test = np.array(df["stim_test"])
N_test = len(y_test)

# Plot the stimulus and spike counts
fig, axs = plt.subplots(2, 1, sharex=True)
axs[0].plot(stim_train[:50])
axs[0].set_ylabel("stimulus")
axs[1].stem(y_train[:50], use_line_collection=True)
axs[1].set_ylabel('spike count')
_ = axs[1].set_xlabel('time bin [8.3ms each]')
```



[2]:
```
# a quick look at the dataframe
df
```

[2]:
```
     Unnamed: 0  y_train  stim_train  y_test  stim_test
0             1        0       -0.48       0      -0.48
1             2        0        0.48       0      -0.48
2             3        0        0.48       0       0.48
3             4        0        0.48       0       0.48
4             5        0       -0.48       0       0.48
..          …        …          …       …         …
995         996        2        0.48       2      -0.48
996         997        2       -0.48       0       0.48
997         998        2       -0.48       0       0.48
998         999        0        0.48       1      -0.48
999        1000        0        0.48       0       0.48

[1000 rows x 5 columns]
```

1.3 Problem 1: Construct the design matrix

Let $y_n \in \mathbb{N}$ denote the spike count in the n-th time bin and $s_n \in \mathbb{R}$ denote the corresponding stimulus at that bin.

Construct the *design matrix* for the training data $X \in \mathbb{R}^{N_{\text{train}} \times P}$ with rows

$$x_n = (1, s_n, s_{n-1}, \ldots, s_{n-L+1}) \tag{3}$$

where $L = 25$ denotes the number of stimulus bins to include in the covariates. (Thus the number of total covariates is $P = L + 1$.)

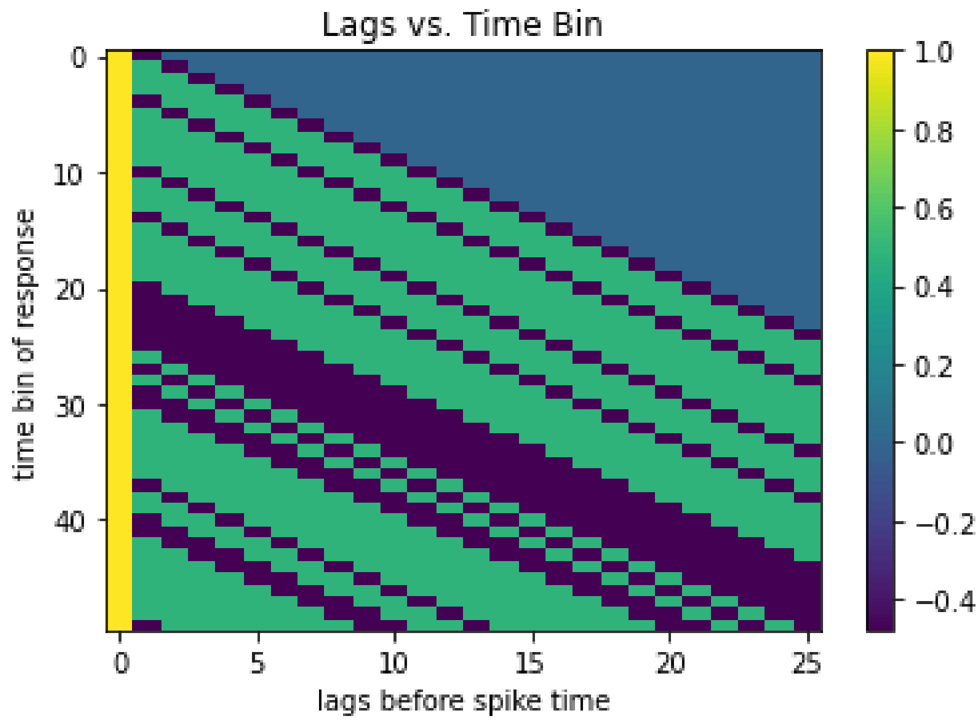
Visualize the first 50 rows of the matrix with, e.g., `imshow`. Don't forget your labels and colorbar.

Note: Pad the stimulus with zeros so that $s_i = 0$ for $i \leq 0$.

[3]:
```python
def construct_design(x, padlength=25):
    """
    Constructs a design matrix, as instructed by the problem. Uses section 3b␣
 ↪of
    attached tutorial, so most recent is on the RHS, least recent on the LHS.

    Args:
        x : list, np.array, pd.Series
            An array-like of stimuli to pad
        padlength : int
            pad size
    Returns : np.array[len(x), padlength + 1]
        The padded array
    """
    x_padded = np.hstack((np.zeros(padlength-1), x))
    x_design = hankel(x_padded[:-padlength+1], x[-padlength:])[:, ::-1]
    x_design = np.hstack([np.ones(len(x)).reshape(-1, 1), x_design])
    return x_design

y_train = df['y_train'].values
x_train = construct_design(df['stim_train'].values, padlength=25)
x_train
```

[3]:
```
array([[ 1.  , -0.48,  0.  , …,  0.  ,  0.  ,  0.  ],
       [ 1.  ,  0.48, -0.48, …,  0.  ,  0.  ,  0.  ],
       [ 1.  ,  0.48,  0.48, …,  0.  ,  0.  ,  0.  ],
       …,
       [ 1.  , -0.48, -0.48, …, -0.48,  0.48,  0.48],
       [ 1.  ,  0.48, -0.48, …, -0.48, -0.48,  0.48],
       [ 1.  ,  0.48,  0.48, …, -0.48, -0.48, -0.48]])
```

[4]:
```python
plt.clf()
plt.imshow(x_train[:50], aspect='auto', interpolation='nearest')
plt.xlabel('lags before spike time')
plt.ylabel('time bin of response')
plt.title('Lags vs. Time Bin')
plt.colorbar()
plt.show()
```



1.4 Problem 2a [Math]: Derive the log joint probability

Derive the log joint probability,

$$\begin{aligned}\mathcal{L}(w) &\triangleq \log p(\{y_n\}_{n=1}^N, w \mid \{x_n\}_{n=1}^N, \sigma^2) &(4)\\ &= \dots &(5)\end{aligned}$$

First, it may be useful to derive certain properties of the Poisson GLM, and build up to the log joint probability. For a single observation i, we have

$$\begin{aligned}p(y_i; \lambda) &= \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}\\ &= \frac{1}{y!}\exp(y_i \log(\lambda) - \lambda).\end{aligned}$$

Under this rearrangment, it is clear from a GLM perspective that

- $b(y) = \frac{1}{y!}$
- $T(y) = y$
- $\eta_i = \log(\lambda) \implies \lambda = \exp(\eta_i)$
- $\alpha(\eta_i) = \lambda = \exp(\eta_i)$

Per usual, we will model $\eta_i = w^T x_i$, which we will substitute in in short order.

Now consider the joint likelihood. We have:

$$
\begin{aligned}
L(w; x, y) &= \left(\prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{y_i}}{y_i!} \right) \pi(w) \\
&= \left(\prod_{i=1}^{n} \frac{e^{\eta_i y_i - e_i^{\eta}}}{y_i!} \right) \pi(w) \\
\ell(w; x, y) &= \left(\sum_{i=1}^{n} \eta_i y_i - e_i^{\eta} - log(y_i!) \right) + \log(\pi(w)) \\
&= \left(\sum_{i=1}^{n} w^T x_i y_i - e^{w^T x_i} - log(y_i!) \right) - \frac{1}{2\sigma^2} w^T w - \frac{P}{2} \log(2\pi) - \frac{1}{2} \log |\sigma^2 I|
\end{aligned}
$$

1.5 Problem 2b [Code]: Implement the log probability function

Write a function that computes the log joint probability and evaluate it on the training set with $w = (0, \ldots, 0) \in \mathbb{R}^P$ and $\sigma^2 = 1$. **Print your result.**

[5]:
```
def compute_log_prob(y, x, w, w_prior=np.zeros(26), sigma2=1):
    """
    Computes log density.
    Args:
        y : np.array[n, ]
            The target data
        x : np.array[n, p]
            the feature matrix
        w : np.array[p, ]
            GLM coefficients (with intercept)
    """
    prior_lik = multivariate_normal.logpdf(w, w_prior, sigma2 * np.eye(len(w)))
    glm_lik = np.sum(poisson.logpmf(y, np.exp(x.dot(w))))
    return prior_lik + glm_lik

compute_log_prob(y=y_train, x=x_train, w=np.zeros(26), w_prior=np.zeros(26),␣
↪sigma2=1)
```

[5]: -1125.7464724378729

1.6 Problem 3a [Math]: Derive the gradient

Derive the gradient of the log joint probability

$$\nabla_w \mathcal{L}(w) = \ldots \tag{6}$$

Show your work.

Next, we compute the gradient, using the chain rule early/often:

$$
\begin{aligned}
\nabla_w \ell(w; x, y) &= \nabla_w \left(\left(\sum_{i=1}^{n} w^T x_i y_i - e^{w^T x_i} - log(y_i!) \right) - \frac{1}{2\sigma^2} w^T w - \frac{P}{2}\log(2\pi) - \frac{1}{2}\log|\sigma^2 I| \right) \\
&= \left(\sum_{i=1}^{n} x_i y_i - e^{w^T x_i} x_i) \right) - w/\sigma^2 \\
&= \left(\sum_{i=1}^{n} (y_i - e^{w^T x_i}) x_i) \right) - w/\sigma^2.
\end{aligned}
$$

1.7 Problem 3b [Code]: Implement the gradient

Write a function to compute the gradient wrt w of the log probability for given values of w and evaluate it on the training set at $w = (0, \ldots, 0) \in \mathbb{R}^P$ and $\sigma^2 = 1$. **Print your result.**

Note: While this is not required in this homework, it may be helpful to do numerical checks for gradient and Hessian calculations using finite differences. See *e.g.* Section 4.2 of https://cilvr.cs.nyu.edu/diglib/lsml/bottou-sgd-tricks-2012.pdf.

[6]:
```
def compute_gradient(y, x, w, sigma2=1):
    eta = x.dot(w)
    grad = np.vstack([(y[i] - np.exp(eta[i])) * x[i, ]
                      for i in range(len(y))]).sum(axis=0) - w/sigma2
#     yhat = np.exp(x.dot(w))
#     grad = (y - yhat).T.dot(x) - w
    return grad
compute_gradient(y=y_train, x=x_train, w=np.zeros(x_train.shape[1]))
```

[6]:
```
array([-603.  ,    7.2 ,   18.24,   44.64,  119.04,  145.44,   64.32,
        -51.84,  -59.04,  -44.64,  -44.16,  -47.52,  -54.24,  -33.12,
        -27.84,  -50.4 ,  -48.  ,  -34.08,  -50.88,  -38.88,  -22.08,
        -37.44,  -29.76,  -11.04,    2.4 ,    1.92])
```

1.8 Problem 4a [Math]: Derive the Hessian

Derive the Hessian of the log joint probability

$$\nabla^2_w \mathcal{L}(w) = \ldots \tag{7}$$

Show your work.

Here, we tackle the Hessian. We begin by picking up where we left off at the gradient (just one more chain rule):

$$\nabla_w^2 \ell(w; x, y) = \nabla_w \left[\left(\sum_{i=1}^{n} (y_i - e^{w^T x_i}) x_i \right) - w/\sigma^2 \right]$$
$$= -\left(\sum_{i=1}^{n} e^{w^T x_i} x_i x_i^T \right) - \sigma^2 I$$
$$= -X\lambda X - \sigma^2 I,$$

where λ is a $n \times n$ diagonal matrix s.t.

$$\lambda_i = \exp(w^T x_i),$$

via our familiar link function.

1.9 Problem 4b [Code]: Implement the Hessian

Write a function to compute the Hessian of the log probability for given values of w and σ^2 and evaluate it on the training set at $w = (0, \ldots, 0) \in \mathbb{R}^P$ and $\sigma^2 = 1$.

Visualize the Hessian with, e.g., `imshow`. Don't forget labels and a colorbar.

We plot both the Hessian, and the inverse Hessian. Note here that when we say "lag index", we're describing the index of the features that are lags. Of course, the 0th lag index is simply the intercept, to avoid confusion here.

[7]:
```
def compute_hessian(w, x, sigma2=1):
    """"""
    V = np.diag(np.exp(x.dot(w)))
    H = -(x.T.dot(V).dot(x) + np.eye(x.shape[1])/sigma2)
    return H

hess_p4 = compute_hessian(w=np.zeros(x_train.shape[1]), x=x_train)
pd.DataFrame(hess_p4)
```

[7]:

	0	1	2	3	4	5	6 \
0	-1001.00	-13.4400	-12.9600	-12.4800	-12.9600	-13.4400	-12.9600
1	-13.44	-231.4000	-2.9952	-11.9808	-0.2304	-18.8928	1.1520
2	-12.96	-2.9952	-231.1696	-2.7648	-12.2112	-0.4608	-18.6624
3	-12.48	-11.9808	-2.7648	-230.9392	-2.9952	-12.4416	-0.2304
4	-12.96	-0.2304	-12.2112	-2.9952	-230.7088	-2.7648	-12.6720
5	-13.44	-18.8928	-0.4608	-12.4416	-2.7648	-230.4784	-2.9952
6	-12.96	1.1520	-18.6624	-0.2304	-12.6720	-2.9952	-230.2480
7	-12.48	1.3824	1.3824	-18.4320	-0.4608	-12.9024	-2.7648
8	-12.00	-2.9952	1.6128	1.6128	-18.6624	-0.6912	-12.6720
9	-11.52	-23.0400	-2.7648	1.8432	1.3824	-18.8928	-0.4608
10	-11.04	7.6032	-22.8096	-2.5344	1.6128	1.1520	-18.6624
11	-11.52	-0.9216	7.3728	-23.0400	-2.3040	1.8432	0.9216
12	-12.00	-7.1424	-1.1520	7.1424	-22.8096	-2.0736	1.6128

13	-12.48	3.6864	-7.3728	-1.3824	7.3728	-22.5792	-2.3040
14	-12.00	14.0544	3.9168	-7.1424	-1.6128	7.1424	-22.3488
15	-12.48	19.3536	13.8240	3.6864	-6.9120	-1.3824	6.9120
16	-12.96	10.3680	19.1232	13.5936	3.9168	-6.6816	-1.6128
17	-13.44	4.6080	10.1376	18.8928	13.8240	4.1472	-6.9120
18	-12.96	23.2704	4.8384	10.3680	18.6624	13.5936	4.3776
19	-12.48	4.1472	23.5008	5.0688	10.1376	18.4320	13.8240
20	-12.96	11.2896	3.9168	23.2704	5.2992	10.3680	18.2016
21	-13.44	-3.6864	11.0592	3.6864	23.5008	5.5296	10.1376
22	-13.92	-4.3776	-3.9168	10.8288	3.9168	23.7312	5.2992
23	-14.40	-4.1472	-4.6080	-4.1472	11.0592	4.1472	23.5008
24	-14.88	0.6912	-4.3776	-4.8384	-3.9168	11.2896	3.9168
25	-15.36	-6.4512	0.4608	-4.6080	-4.6080	-3.6864	11.0592

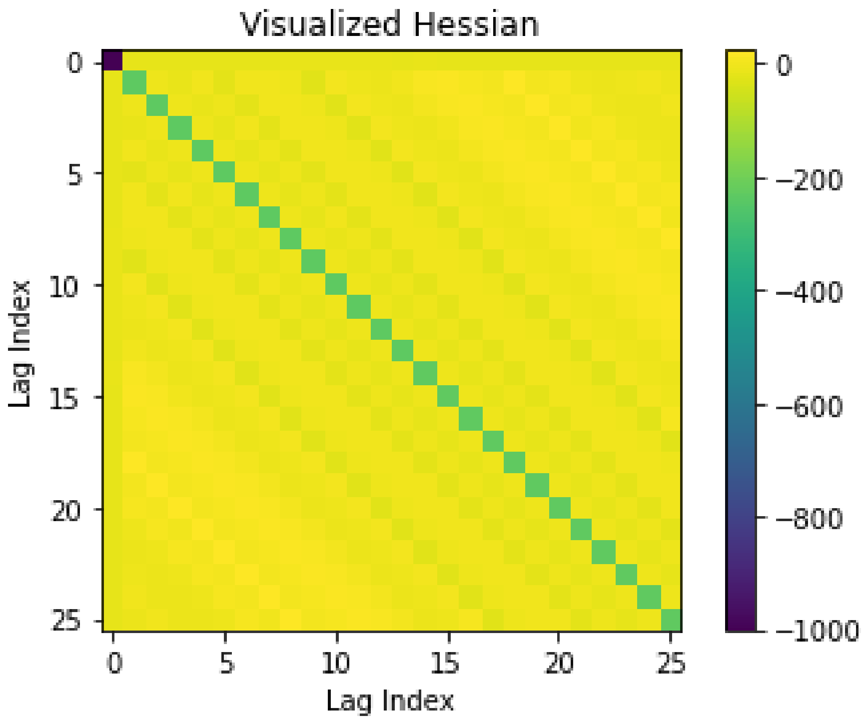
	7	8	9	…	16	17	18	19 \
0	-12.4800	-12.0000	-11.5200	…	-12.9600	-13.4400	-12.9600	-12.4800
1	1.3824	-2.9952	-23.0400	…	10.3680	4.6080	23.2704	4.1472
2	1.3824	1.6128	-2.7648	…	19.1232	10.1376	4.8384	23.5008
3	-18.4320	1.6128	1.8432	…	13.5936	18.8928	10.3680	5.0688
4	-0.4608	-18.6624	1.3824	…	3.9168	13.8240	18.6624	10.1376
5	-12.9024	-0.6912	-18.8928	…	-6.6816	4.1472	13.5936	18.4320
6	-2.7648	-12.6720	-0.4608	…	-1.6128	-6.9120	4.3776	13.8240
7	-230.0176	-2.5344	-12.4416	…	6.6816	-1.8432	-6.6816	4.6080
8	-2.5344	-229.7872	-2.3040	…	-22.8096	6.4512	-1.6128	-6.4512
9	-12.4416	-2.3040	-229.5568	…	-2.5344	-23.0400	6.6816	-1.3824
10	-0.2304	-12.2112	-2.0736	…	1.1520	-2.7648	-22.8096	6.9120
11	-18.8928	-0.4608	-12.4416	…	0.6912	1.3824	-2.9952	-23.0400
12	0.6912	-19.1232	-0.6912	…	-18.6624	0.9216	1.1520	-3.2256
13	1.3824	0.4608	-19.3536	…	-0.6912	-18.4320	0.6912	0.9216
14	-2.0736	1.6128	0.6912	…	-12.6720	-0.9216	-18.2016	0.9216
15	-22.5792	-2.3040	1.3824	…	-2.0736	-12.4416	-1.1520	-18.4320
16	6.6816	-22.8096	-2.5344	…	-227.9440	-1.8432	-12.6720	-1.3824
17	-1.8432	6.4512	-23.0400	…	-1.8432	-227.7136	-2.0736	-12.9024
18	-6.6816	-1.6128	6.6816	…	-12.6720	-2.0736	-227.4832	-1.8432
19	4.6080	-6.4512	-1.3824	…	-1.3824	-12.9024	-1.8432	-227.2528
20	13.5936	4.3776	-6.6816	…	-18.2016	-1.1520	-13.1328	-2.0736
21	17.9712	13.3632	4.1472	…	1.3824	-17.9712	-1.3824	-13.3632
22	9.9072	17.7408	13.1328	…	1.1520	1.6128	-18.2016	-1.6128
23	5.0688	9.6768	17.5104	…	-2.7648	1.3824	1.3824	-18.4320
24	23.2704	4.8384	9.4464	…	-22.3488	-2.5344	1.1520	1.1520
25	3.6864	23.0400	4.6080	…	7.8336	-22.1184	-2.7648	0.9216

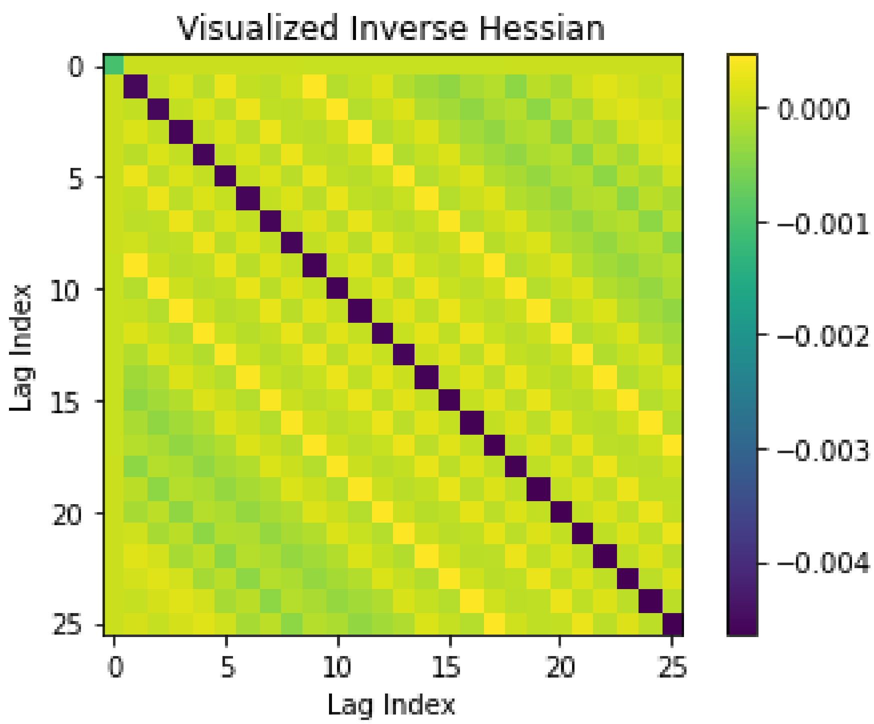
	20	21	22	23	24	25
0	-12.9600	-13.4400	-13.9200	-14.4000	-14.8800	-15.3600
1	11.2896	-3.6864	-4.3776	-4.1472	0.6912	-6.4512
2	3.9168	11.0592	-3.9168	-4.6080	-4.3776	0.4608
3	23.2704	3.6864	10.8288	-4.1472	-4.8384	-4.6080

```
4     5.2992   23.5008    3.9168   11.0592   -3.9168   -4.6080
5    10.3680    5.5296   23.7312    4.1472   11.2896   -3.6864
6    18.2016   10.1376    5.2992   23.5008    3.9168   11.0592
7    13.5936   17.9712    9.9072    5.0688   23.2704    3.6864
8     4.3776   13.3632   17.7408    9.6768    4.8384   23.0400
9    -6.6816    4.1472   13.1328   17.5104    9.4464    4.6080
10   -1.6128   -6.9120    3.9168   12.9024   17.2800    9.2160
11    7.1424   -1.3824   -6.6816    4.1472   13.1328   17.5104
12  -22.8096    7.3728   -1.1520   -6.4512    4.3776   13.3632
13   -2.9952  -22.5792    7.6032   -0.9216   -6.2208    4.6080
14    0.6912   -3.2256  -22.8096    7.3728   -1.1520   -6.4512
15    1.1520    0.9216   -2.9952  -22.5792    7.6032   -0.9216
16  -18.2016    1.3824    1.1520   -2.7648  -22.3488    7.8336
17   -1.1520  -17.9712    1.6128    1.3824   -2.5344  -22.1184
18  -13.1328   -1.3824  -18.2016    1.3824    1.1520   -2.7648
19   -2.0736  -13.3632   -1.6128  -18.4320    1.1520    0.9216
20 -227.0224   -1.8432  -13.1328   -1.3824  -18.2016    1.3824
21   -1.8432 -226.7920   -1.6128  -12.9024   -1.1520  -17.9712
22  -13.1328   -1.6128 -226.5616   -1.3824  -12.6720   -0.9216
23   -1.3824  -12.9024   -1.3824 -226.3312   -1.1520  -12.4416
24  -18.2016   -1.1520  -12.6720   -1.1520 -226.1008   -0.9216
25    1.3824  -17.9712   -0.9216  -12.4416   -0.9216 -225.8704

[26 rows x 26 columns]
```

[8]:

```
plt.clf()
plt.imshow(hess_p4)
plt.xlabel('Lag Index')
plt.ylabel('Lag Index')
plt.title('Visualized Hessian')
plt.colorbar()
plt.show()


plt.clf()
plt.imshow(np.linalg.inv(hess_p4))
plt.xlabel('Lag Index')
plt.ylabel('Lag Index')
plt.title('Visualized Inverse Hessian')
plt.colorbar()
plt.show()
```


Visualized Hessian
Lag Index
Lag Index
0
5
10
15
20
25
0
−200
−400
−600
−800
−1000


Visualized Inverse Hessian
Lag Index
Lag Index
0
5
10
15
20
25
0.000
−0.001
−0.002
−0.003
−0.004

1.10 Problem 5

1.11 Problem 5a: Optimize to find the posterior mode

Optimize the log joint probability to find the posterior mode. You may use built-in optimization libraries (e.g. `scipy.optimize.minimize`).

1.12 Problem 5b: Approximate the covariance at the mode

Solve for $\Sigma_{\mathsf{MAP}} = -[\nabla^2(\mathcal{L}(w_{\mathsf{MAP}})]^{-1}$. Plot the covariance matrix (e.g. with `imshow`). Don't forget to add a colorbar and labels.

[9]:
```
def newton(y,
           x,
           w_init=None,
           gamma=1e-1,
           tol=1e-5,
           maxiter=1e5,
           gradient=compute_gradient,
           hessian=compute_hessian):
    """
    Ya cowboy

    Args:
        y : np.array[n]
            An array of targets
        x : np.array[n, p]
            The design matrix
        w_init : np.array(p) or None
            An optional array of initial weights
        gamma : float
            Learning rate
        tol : float
            When the l2 norm between w_{t+1} and w_t dips below this value,
            the model is said to have converged
        maxiter : int
            Maximum iterations to peform before stopping, with a warning
        gradient : callable
            A function to calculate the gradient w.r.t. y, x, w
        hessian : callable
            A function to calculate the Hessian w.r.t. y, x, w
    Returns : tuple[np.array[p], np.array[n_iter, p]]
        A tuple containing:
            - the weights at last iteration
            - a matrix of weights over all iterations
    """
    w_init = np.zeros(x.shape[1]) if w_init is None else w_init
```

```python
    eps = 1e10 # l2 norm of difference between iterations
    wold = w_init # starting value for w
    n_iter = 1 # counter
    w_log = [wold] # for tracking
    eps_log = [] # for tracking

    while eps > tol and n_iter <= maxiter:
        grad = gradient(y=y, x=x, w=wold)
        H = hessian(w=wold, x=x_train)
        wnew = wold - gamma * (np.linalg.inv(H)).dot(grad)
        eps = np.linalg.norm(wnew - wold, ord=2)
        # logging
        eps_log.append(eps)
        w_log.append(wnew)
        # advancing
        wold = wnew
        n_iter += 1
    w_log = np.vstack(w_log)

    if n_iter >= maxiter:
        warnings.warn('Solver failed to converge')
    return wnew, w_log

w_map, w_map_log = newton(
    y_train,
    x_train,
    gamma=1e-1,
    tol=1e-6,
    maxiter=1e5)
cov_wmap = - np.linalg.inv(compute_hessian(w_map, x_train))
pd.DataFrame(cov_wmap)
```

[9]:

	0	1	2	3	4	5	6 \
0	0.010385	-0.000234	-0.000613	-0.001231	-0.005843	-0.012001	-0.002162
1	-0.000234	0.011771	-0.000392	-0.000201	-0.000359	-0.000673	0.000501
2	-0.000613	-0.000392	0.011855	0.000070	-0.000452	0.000212	-0.000867
3	-0.001231	-0.000201	0.000070	0.012623	-0.000784	-0.000429	0.000617
4	-0.005843	-0.000359	-0.000452	-0.000784	0.020632	-0.000918	-0.000821
5	-0.012001	-0.000673	0.000212	-0.000429	-0.000918	0.032763	-0.000956
6	-0.002162	0.000501	-0.000867	0.000617	-0.000821	-0.000956	0.014528
7	0.001329	-0.000886	0.000420	-0.000999	0.000821	-0.000650	-0.001060
8	0.000981	0.000209	-0.000381	-0.000087	-0.000658	0.000641	-0.000252
9	0.001090	-0.002132	-0.000173	-0.000329	-0.000154	-0.000833	0.000437
10	0.001169	0.000150	-0.001522	-0.000223	0.000192	-0.000141	-0.001765
11	0.000250	0.000284	0.000486	-0.000625	0.000045	0.000651	0.000375
12	0.000949	-0.000159	-0.000803	0.001083	-0.000555	0.000079	0.001057
13	0.000735	0.000381	-0.000194	-0.000374	0.000313	-0.000946	0.000083

14	0.000197	0.000115	0.000505	-0.000292	0.000433	0.000320	-0.001547
15	0.000378	0.000575	0.000285	0.000426	-0.000133	0.000636	-0.000224
16	0.000070	0.000981	0.000429	0.001122	-0.000037	-0.000065	0.001734
17	0.000229	-0.000246	0.001144	0.000534	0.000891	0.000038	-0.000795
18	0.000351	0.000966	-0.000373	-0.000026	0.000682	0.000820	-0.000115
19	0.000137	0.000954	0.000910	0.000488	-0.000413	0.000640	0.000623
20	-0.000602	0.001048	0.001115	0.001032	0.000907	-0.000203	0.000871
21	-0.000201	-0.000422	0.000853	0.000801	0.001474	0.000714	-0.000707
22	-0.000603	-0.000590	-0.000510	0.000571	0.000006	0.001841	0.001169
23	0.000259	0.000059	-0.000464	-0.001019	-0.000015	-0.001027	0.001896
24	0.000084	0.000307	-0.000581	-0.001220	0.000011	0.000103	-0.001430
25	0.000037	0.000534	-0.000134	-0.000502	-0.001007	-0.000300	0.000322

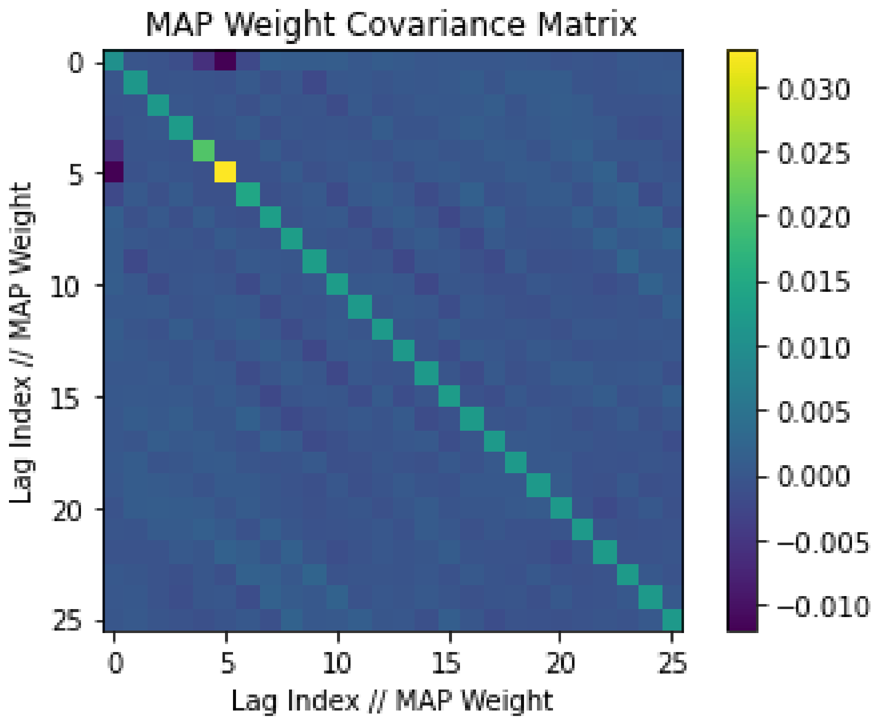
	7	8	9	…	16	17	18	19 \
0	0.001329	0.000981	0.001090	…	0.000070	0.000229	0.000351	0.000137
1	-0.000886	0.000209	-0.002132	…	0.000981	-0.000246	0.000966	0.000954
2	0.000420	-0.000381	-0.000173	…	0.000429	0.001144	-0.000373	0.000910
3	-0.000999	-0.000087	-0.000329	…	0.001122	0.000534	-0.000026	0.000488
4	0.000821	-0.000658	-0.000154	…	-0.000037	0.000891	0.000682	-0.000413
5	-0.000650	0.000641	-0.000833	…	-0.000065	0.000038	0.000820	0.000640
6	-0.001060	-0.000252	0.000437	…	0.001734	-0.000795	-0.000115	0.000623
7	0.013055	-0.000515	0.000222	…	-0.000135	0.001296	-0.000623	-0.000017
8	-0.000515	0.012931	-0.000060	…	-0.001996	0.000505	-0.000402	-0.000506
9	0.000222	-0.000060	0.012953	…	-0.000824	-0.001708	0.000522	-0.001025
10	0.000484	-0.000521	0.000076	…	-0.000162	-0.001115	-0.001048	0.000199
11	-0.001644	-0.000228	-0.000549	…	0.000401	-0.000224	-0.000888	-0.001065
12	0.000198	-0.001583	-0.000113	…	-0.000339	-0.000422	-0.000018	-0.000357
13	0.001098	-0.000003	-0.002257	…	0.000636	0.000135	-0.000525	0.000535
14	-0.000263	0.000858	0.000013	…	-0.001526	0.000571	-0.000837	-0.000287
15	-0.002284	-0.000304	0.000169	…	-0.000696	-0.001330	0.000360	-0.000788
16	-0.000135	-0.001996	-0.000824	…	0.012438	-0.000146	-0.000482	0.000313
17	0.001296	0.000505	-0.001708	…	-0.000146	0.012092	0.000569	0.000364
18	-0.000623	-0.000402	0.000522	…	-0.000482	0.000569	0.012147	0.000355
19	-0.000017	-0.000506	-0.001025	…	0.000313	0.000364	0.000355	0.011972
20	0.000003	-0.000435	-0.000816	…	0.000036	0.000036	0.000559	0.000858
21	0.001346	-0.000033	-0.000680	…	-0.000818	-0.000020	-0.000741	-0.000795
22	-0.000390	0.001788	-0.000123	…	-0.000366	-0.000624	-0.000517	-0.000908
23	0.001563	0.000395	0.002491	…	0.000501	-0.000306	-0.000674	0.000380
24	0.001822	0.000884	0.000401	…	-0.000880	-0.000331	-0.000283	-0.000987
25	-0.001095	0.001989	0.000262	…	-0.000066	-0.001417	-0.000506	-0.000802

	20	21	22	23	24	25
0	-0.000602	-0.000201	-0.000603	0.000259	0.000084	0.000037
1	0.001048	-0.000422	-0.000590	0.000059	0.000307	0.000534
2	0.001115	0.000853	-0.000510	-0.000464	-0.000581	-0.000134
3	0.001032	0.000801	0.000571	-0.001019	-0.001220	-0.000502
4	0.000907	0.001474	0.000006	-0.000015	0.000011	-0.001007

```
5  -0.000203  0.000714  0.001841 -0.001027  0.000103 -0.000300
6   0.000871 -0.000707  0.001169  0.001896 -0.001430  0.000322
7   0.000003  0.001346 -0.000390  0.001563  0.001822 -0.001095
8  -0.000435 -0.000033  0.001788  0.000395  0.000884  0.001989
9  -0.000816 -0.000680 -0.000123  0.002491  0.000401  0.000262
10 -0.000406 -0.000569 -0.001327 -0.000462  0.002025  0.000477
11 -0.000076 -0.000432 -0.000604 -0.000281 -0.000535  0.001294
12 -0.000884  0.000189  0.000079 -0.000272  0.000166 -0.000105
13  0.000004 -0.000736 -0.000360 -0.000293 -0.000219  0.000077
14  0.000816  0.000827 -0.000101  0.000190  0.000524 -0.001061
15 -0.000413  0.000130  0.000960 -0.000745 -0.000119  0.001360
16  0.000036 -0.000818 -0.000366  0.000501 -0.000880 -0.000066
17  0.000036 -0.000020 -0.000624 -0.000306 -0.000331 -0.001417
18  0.000559 -0.000741 -0.000517 -0.000674 -0.000283 -0.000506
19  0.000858 -0.000795 -0.000908  0.000380 -0.000987 -0.000802
20  0.012087 -0.000374 -0.001865 -0.000707  0.000373 -0.000831
21 -0.000374  0.012123 -0.000423 -0.000880 -0.000768 -0.000523
22 -0.001865 -0.000423  0.012345 -0.000566 -0.000484 -0.000132
23 -0.000707 -0.000880 -0.000566  0.012361  0.000286 -0.000542
24  0.000373 -0.000768 -0.000484  0.000286  0.012225  0.000463
25 -0.000831 -0.000523 -0.000132 -0.000542  0.000463  0.012244

[26 rows x 26 columns]
```

Again, bear in mind the 0th index is just for the intercept.

[10]:
```
plt.clf()
plt.imshow(cov_wmap)
plt.xlabel('Lag Index // MAP Weight')
plt.ylabel('Lag Index // MAP Weight')
plt.title('MAP Weight Covariance Matrix')
plt.colorbar()
plt.show()
```



1.13 Problem 6: Plot the posterior of the weights

Plot the posterior mean of the weights for features $s_n, \ldots, s_{n-L+1}$ (i.e. not including the bias term). Also plot 95% credible intervals around the mean by using two standard deviations of the marginal distribution of the weights. Note the diagonal of Σ_{MAP} gives the marginal variance of the posterior.

First, I present the posterior weights as is. Second, I plot a density for each of them. Third, I plot them w.r.t. their lag index, to see which of the previous time windows may be most important.

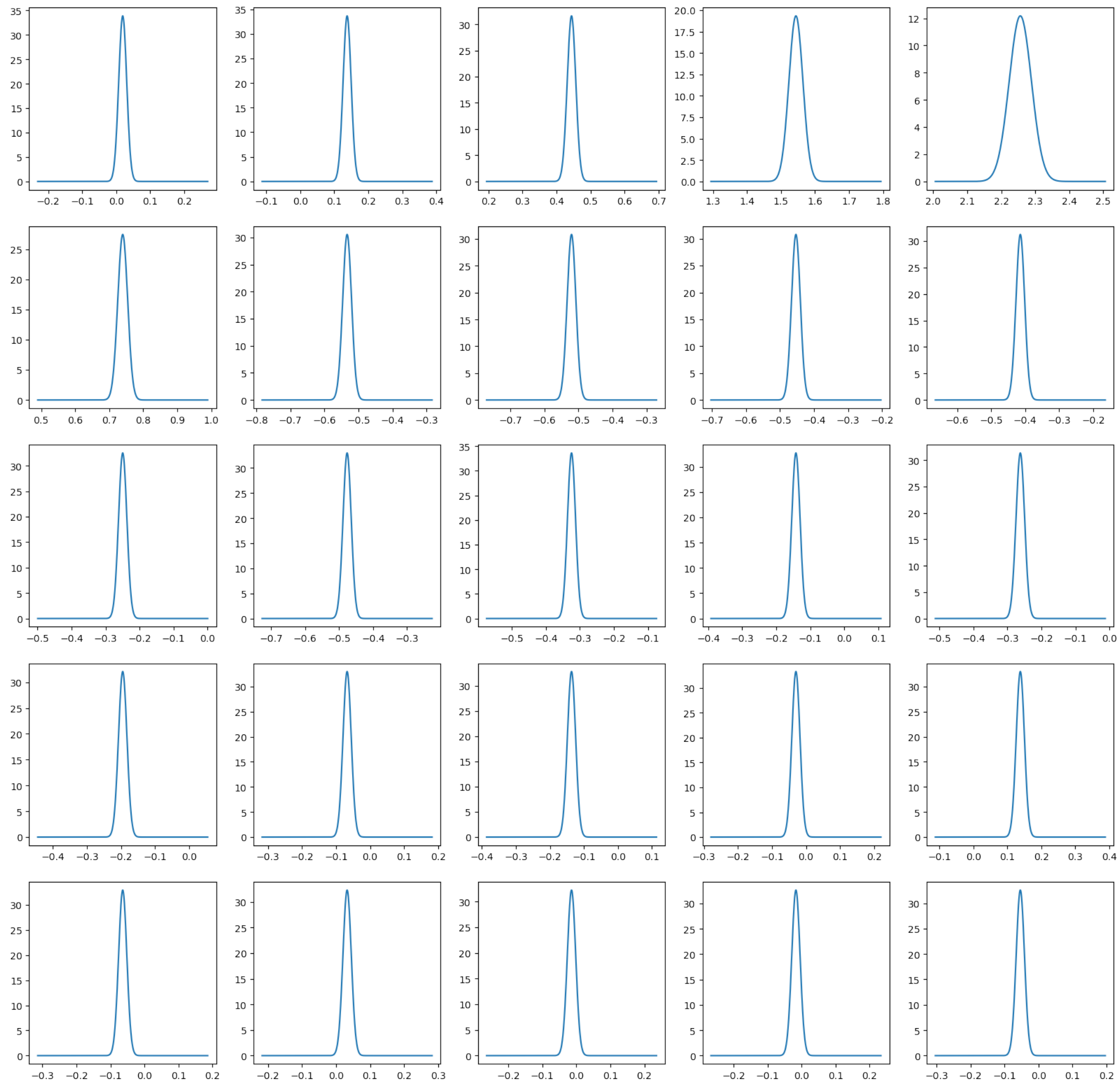
[11]:
```
w_map
```

[11]:
```
array([-1.98387467,  0.01916078,  0.13772364,  0.44424144,  1.54387934,
        2.25596082,  0.73980371, -0.53345703, -0.52034956, -0.45324512,
       -0.41421336, -0.24945054, -0.4764015 , -0.32471314, -0.14218603,
       -0.26219321, -0.19492622, -0.06752119, -0.13640114, -0.02965932,
        0.13819359, -0.06414585,  0.03259843, -0.01427306, -0.01652936,
       -0.0532132 ])
```

[12]:
```
plt.rcParams['figure.figsize'] = [20, 20]
plt.rcParams['figure.dpi'] = 100
plt.clf()
fig, axs = plt.subplots(5, 5)
row, col = 0, 0
```

```python
for i in range(1, 26):
    x_ = np.zeros(len(w_map)); x_[i] = 1 # for marginalizing
    marginal_cov = x_.T.dot(cov_wmap).dot(x_) # marginal covariance
    x_sup = np.linspace(w_map[i] - .25, w_map[i] + .25, 10000)
    posterior_density = norm.pdf(x_sup, w_map[i], marginal_cov)
    axs[row, col].plot(x_sup, posterior_density)
    col += 1
    if col > 4:
        row += 1
        col = 0
plt.figure(figsize=(1000, 1000))
plt.show()
```

```
<Figure size 2000x2000 with 0 Axes>
```

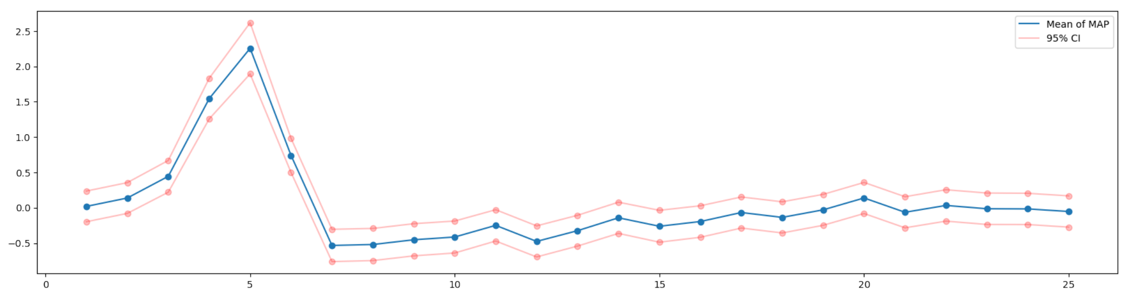


<Figure size 100000x100000 with 0 Axes>

[13]:

```python
plt.rcParams['figure.figsize'] = [20, 5]
plt.rcParams['figure.dpi'] = 100
plt.clf()
upr, lwr = [], []
for i in range(1, 26):
    x_ = np.zeros(len(w_map)); x_[i] = 1 # for marginalizing
    marginal_cov = x_.T.dot(cov_wmap).dot(x_) # marginal covariance
    upr.append(w_map[i] + np.sqrt(marginal_cov) * 2)
    lwr.append(w_map[i] - np.sqrt(marginal_cov) * 2)
plt.plot(range(1, 26), w_map[1:], label = 'Mean of MAP')
plt.scatter(range(1, 26), w_map[1:])
plt.plot(range(1, 26), lwr, color='red', alpha=.25, label = '95% CI')
plt.scatter(range(1, 26), lwr, color='red', alpha=.25)
plt.plot(range(1, 26), upr, color='red', alpha=.25)
plt.scatter(range(1, 26), upr, color='red', alpha=.25)
plt.legend()
plt.plot()
```

[13]: []



1.14 Problem 7 [Short Answer]: Interpret your results

Here, the neurons are cells from the retina, which are responsive to light. The stimulus at time bin n is either -0.5 or +0.5 depending on whether a light was off or on, respectively, at that time. What do these weights tell you about the relationship between the stimulus and the spike counts?

Judging by the apparent spike in between $\sim n-2$ and $\sim n-6$, it seems that these lagged times – in particular, the $n-5$ time index, is particularly important here. A positive stimulus for this lag will result in a *larger* value for $w^t x_i$ (as w_5 is positive), which in turn will result in a larger $\lambda = \exp(w^t x_i)$ and a greater mean and variance underlying the predicted Poisson. In contrast, a negative stimulus for this lag will result in a *smaller* value for $w^t x_i$, and thus a smaller mean and variance underlying the predicted value. In this way, we see how a positive stimulus may affect the modeled spike count.

1.15 Problem 8: Approximate the posterior predictive distribution of the rates

Draw many samples $w^{(s)}$ from the Laplace approximation of the posterior $p(w \mid \{x_n, y_n\})$. Use those samples to approximate the posterior predictive distribution on the **test** dataset,

$$p(y_{n'} = k \mid x_{n'}, \{x_n, y_n\}_{n=1}^N) = \int p(y_{n'} \mid w, x_{n'})\, p(w \mid \{x_n, y_n\}_{n=1}^N)\, \mathrm{d}w \tag{8}$$

$$\approx \frac{1}{S} \sum_{s=1}^{S} p(y_{n'} = k \mid w^{(s)}, x_{n'}) \tag{9}$$

where

$$w^{(s)} \sim p(w \mid \{x_n, y_n\}_{n=1}^N) \tag{10}$$

$$\approx \mathcal{N}(w \mid w_{\mathsf{MAP}}, \Sigma_{\mathsf{MAP}}) \tag{11}$$

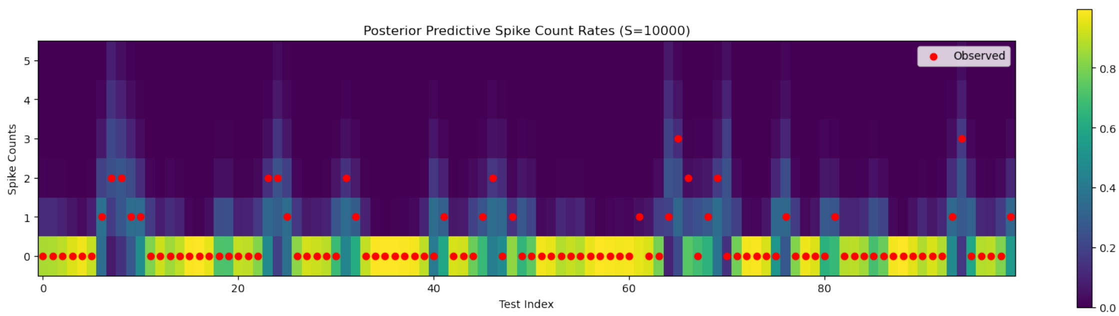
Visualize the posterior predictive distribution as an $K \times N_{\mathsf{test}}$ array where row corresponds to possible spike counts $k \in \{0, \dots, K\}$. You can set $K = 5$ for this problem. **Only show the first 100 columns (time bins), otherwise it's hard to see changes in the rate.**

Overlay the actual spike counts for the test dataset.

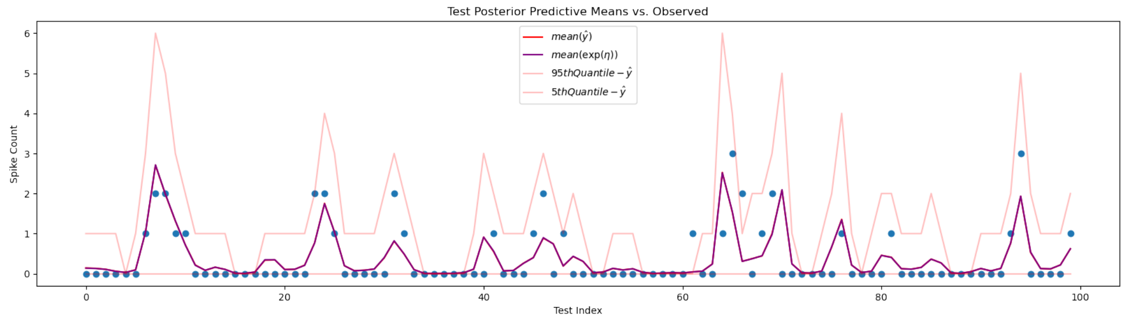
[14]:
```python
N=100 # number of test examples to show
K=5
S=10000

# retrieve the test data
y_test = df['y_test'].values
x_test = construct_design(df['stim_test'].values, padlength=25)

w_samp = multivariate_normal(w_map, cov_wmap).rvs(S, random_state=2020).T
lambda_hat = np.exp(x_test.dot(w_samp))
lambda_hat.shape
yhat = np.vstack([
    poisson(mu=lambda_hat[i, :]).rvs(S) for i in range(lambda_hat.shape[0])
])

yhat
```

[14]:
```
array([[2, 0, 1, …, 1, 0, 0],
       [0, 1, 0, …, 0, 0, 0],
       [0, 0, 0, …, 0, 0, 0],
       …,
       [0, 2, 0, …, 0, 0, 1],
       [0, 0, 1, …, 2, 0, 0],
       [1, 0, 0, …, 0, 0, 0]])
```

[15]:
```
x_test_plt, yhat_plt, lambda_hat_plt = x_test[0:N, :], yhat[0:N, :],␣
↪lambda_hat[0:N, :]
kmat = np.zeros((N, K+1))
for i in range(N):
    for k in range(K+1):
        kmat[i, k] = np.sum(yhat_plt[i, :] == k)/S
plt.rcParams['figure.figsize'] = [20, 5]
plt.rcParams['figure.dpi'] = 100
plt.clf()
plt.imshow(kmat.T, origin='lower', aspect=4)
plt.colorbar()
plt.scatter(range(N), y_test[0:N], color = 'red', label='Observed')
plt.xlabel('Test Index')
plt.title('Posterior Predictive Spike Count Rates (S=10000)')
plt.ylabel('Spike Counts')
plt.legend()
plt.show()
```



[16]:
```
plt.clf()
plt.rcParams['figure.figsize'] = [20, 5]
plt.rcParams['figure.dpi'] = 100
plt.scatter(range(N), y_test[0:N])
plt.plot(range(N), yhat_plt.mean(axis=1), label='$mean(\hat{y})$', color='red')
plt.plot(range(N), lambda_hat_plt.mean(axis=1), label='$mean(\exp(\eta))$',␣
↪color='purple')
plt.plot(range(N), [np.quantile(yhat_plt[i, :], .95) for i in range(N)],
         label='$95th Quantile - \hat{y}$', color='red', alpha=.25)
plt.plot(range(N), [np.quantile(yhat_plt[i, :], .05) for i in range(N)],
         label='$5th Quantile - \hat{y}$', color='red', alpha=.25)
plt.legend()
plt.xlabel('Test Index')
plt.ylabel('Spike Count')
plt.title('Test Posterior Predictive Means vs. Observed')
plt.show()
```



1.16 Problem 9: Compute the log predictive density

Simulate from the posterior distribution to compute a Monte Carlo approximation to what the book calls the *log pointwise predictive density* (Eq. 7.4).

$$\sum_{n'=1}^{N_{\text{test}}} \log p(y_{n'} \mid x_{n'}, \{x_n, y_n\}_{n=1}^{N}) = \sum_{n'=1}^{N_{\text{test}}} \log \int p(y_{n'} \mid w, x_{n'})\, p(w \mid \{x_n, y_n\}_{n=1}^{N})\, \mathrm{d}w \tag{12}$$

$$\approx \sum_{n'=1}^{N_{\text{test}}} \log \frac{1}{S} \sum_{s=1}^{S} p(y_{n'} = k \mid w^{(s)}, x_{n'}) \tag{13}$$

where

$$w^{(s)} \sim p(w \mid \{x_n, y_n\}_{n=1}^{N}) \tag{14}$$

$$\approx \mathcal{N}(w \mid w_{\mathsf{MAP}}, \Sigma_{\mathsf{MAP}}) \tag{15}$$

Use $S = 1000$ Monte Carlo samples and **print your result.**

Note: The book recommends a more fully Bayesian approach in which they compute the log pointwise predictive density for one data point at a time, using the remainder to compute the posterior distribution on the weights. For simplicity, we will stick with a single training and test split, as given in the dataset above.

[17]:
```
S = 1000
w_samp = multivariate_normal(w_map, cov_wmap).rvs(S, random_state=2020).T
lambda_hats = x_test.dot(w_samp)
result = 0
for i in range(lambda_hats.shape[0]):
    y_ = np.ones(lambda_hats.shape[1]) * y_test[i]
    p_y = poisson.pmf(y_, np.exp(lambda_hats[i, ]))
    avg_p_y = np.mean(p_y)
    result += np.log(avg_p_y)
result
```

[17]: -533.2948841430846

```
[ ]:
```

```
[ ]:
```