kleisle_murphy_h5w_poisson_matrix_factorization (1)

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1 Assignment 5: Poisson Matrix Factorization

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2 Background

Poisson matrix factorization (PMF) is a mixed membership model like LDA, and it has close ties to non-negative factorization of count matrices. Let $\mathbf{X} \in \mathbb{N}^{N \times M}$ denote a count matrix with entries $x_{n,m}$. We model each entry as a Poisson random variable,

$$x_{n,m} \sim \mathrm{Po}\Big(\boldsymbol{\theta}_n^\top \boldsymbol{\eta}_m\Big) = \mathrm{Po}\Big(\sum_{k=1}^{K} \theta_{n,k}\eta_{m,k}\Big), \tag{1}$$

where $\boldsymbol{\theta}_n \in \mathbb{R}_+^K$ and $\boldsymbol{\eta}_n \in \mathbb{R}_+^K$ are *non-negative* feature vectors for row n and column m, respectively.

PMF has been used for recommender systems, aka collaborative filtering. In a recommender system, the rows correspond to users, the columns to items, and the entries $x_{n,m}$ to how much user n liked item m (on a scale of $0, 1, 2, \ldots$ stars, for example). The K feature dimensions capture different aspects of items that users may weight in their ratings.

Note that the Poisson rate must be non-negative. It is sufficient to ensure $\boldsymbol{\theta}_n$ and $\boldsymbol{\eta}_m$ are non-negative. To that end, PMF uses gamma priors,

$$\theta_{n,k} \sim \mathrm{Ga}(\alpha_\theta, \beta_\theta) \tag{2}$$

$$\eta_{m,k} \sim \mathrm{Ga}(\alpha_\eta, \beta_\eta), \tag{3}$$

where $\alpha_\star$ and $\beta_\star$ are hyperparameters. When $\alpha_\star < 1$, the gamma distribution has a sharp peak at zero and the prior induces sparsity in the feature vectors.

2.1 Latent variable formulation

PMF can be rewritten in terms of a latent variable model. Note that,

$$x_{n,m} \sim \mathrm{Po}\Big(\sum_{k=1}^{K} \theta_{n,k}\eta_{m,k}\Big) \iff x_{n,m} = \sum_{k=1}^{K} z_{n,m,k} \tag{4}$$

$$z_{n,m,k} \sim \mathrm{Po}(\theta_{nk}\eta_{mk}) \quad \text{independently.} \tag{5}$$

From this perspective, a user's rating of an item is a sum of ratings along each feature dimension, and each feature rating is an independent Poisson random variable.

The joint distribution is,

$$p(\mathbf{X}, \mathbf{Z}, \mathbf{\Theta}, \mathbf{H}) = \left[\prod_{n=1}^{N}\prod_{m=1}^{M} \mathbb{I}\Big[x_{n,m} = \sum_{k=1}^{K} z_{n,m,k}\Big] \prod_{k=1}^{K} \mathrm{Po}(z_{n,m,k} \mid \theta_{n,k}\eta_{m,k})\right] \times \left[\prod_{n=1}^{N}\prod_{k=1}^{K} \mathrm{Ga}(\theta_{n,k} \mid \alpha_\theta, \beta_\theta)\right] \times \left[\prod_{m=1}^{M}\prod_{k=1}^{K} \mathrm{G} \tag{6}\right.$$

where $\mathbf{Z} \in \mathbb{N}^{N \times M \times K}$ denotes the *tensor* of feature ratings, $\mathbf{\Theta} \in \mathbb{R}_+^{N \times K}$ is a matrix with rows $\boldsymbol{\theta}_n$, and $\mathbf{H} \in \mathbb{R}_+^{M \times K}$ is a matrix with rows $\boldsymbol{\eta}_m$.

[ ]:
```
import torch
from torch.distributions import Distribution, Gamma, Poisson, Multinomial
from torch.distributions.kl import kl_divergence

from tqdm.auto import trange

import matplotlib.pyplot as plt
from matplotlib.cm import Blues
import seaborn as sns
sns.set_context("notebook")
```

3 Problem 1: Conditional distributions [math]

Since this model is constructed from conjugate exponential family distributions, the conditionals are available in closed form. We will let $\mathbf{z}_{n,m} = (z_{n,m,1}, \ldots, z_{n,m,K})$.

3.1 Problem 1a: Derive the conditional for $\mathbf{z}_{n,m}$

Find the conditional density $p(\mathbf{z}_{n,m} \mid x_{n,m}, \boldsymbol{\theta}_n, \boldsymbol{\eta}_m)$.

First, we have,

$$
\begin{aligned}
p(z_{n,m} \mid x_{n,m}, \theta_n, \eta_m) &\propto \mathbb{1}_{(x_{n,m}=\sum_{k=1}^{K} z_{n,m,k})} \prod_{k=1}^{K} Pois(z_{n,m,k} \mid \theta_{n,k}\eta_{m,k}) && (7)\\
&\propto \mathbb{1}_{(x_{n,m}=\sum_{k=1}^{K} z_{n,m,k})} \prod_{k=1}^{K} \frac{(\theta_{n,k}\eta_{m,k})^{z_{n,m,k}}}{z_{n,m,k}!} && (8)\\
&\sim Multinom(z_{n,m} \mid x_{n,m}, \pi_{n,m}), && (9)
\end{aligned}
$$

where we have

$$
\pi_{n,m} = \left\langle \frac{\theta_{n,1}\eta_{m,1}}{\theta_n^\top \eta_m}, \ldots, \frac{\theta_{n,K}\eta_{m,K}}{\theta_n^\top \eta_m} \right\rangle,
$$

which is seen from logits of

$$
\langle \theta_{n,1}\eta_{m,1}, \ldots, \theta_{n,K}\eta_{m,K} \rangle
$$

3.2 Problem 1b: Derive the conditional for $\theta_{n,k}$

Find the conditional density $p(\theta_{n,k} \mid \mathbf{Z}, \mathbf{H})$.

$$
\begin{aligned}
p(\theta_{n,k} \mid Z, H) &\propto \left[\prod_{m=1}^{M} Pois(z_{n,m,k} \mid \theta_{n,k}\eta_{m,k})\right] Gamma(\theta_{n,k} \mid \alpha_\theta, \beta_\theta) && (10)\\
&\propto \left[\prod_{m=1}^{M} (\theta_{n,k}\eta_{m,k})^{z_{n,m,k}} e^{-\theta_{n,k}\eta_{m,k}}\right] Gamma(\theta_{n,k} \mid \alpha_\theta, \beta_\theta) && (11)\\
&\propto \left[\prod_{m=1}^{M} (\theta_{n,k})^{z_{n,m,k}}\right] e^{-\theta_{n,k}\sum_m \eta_{m,k}} \cdot Gamma(\theta_{n,k} \mid \alpha_\theta, \beta_\theta) && (12)\\
&\sim Gamma\Big(\alpha_\theta + \sum_{m=1}^{M} z_{n,m,k},\ \beta_\theta + \sum_{m=1}^{M} \eta_{m,k}\Big), && (13)
\end{aligned}
$$

\end{align}

3.3 Problem 1c: Derive the conditional for $\eta_{m,k}$

Find the conditional density $p(\eta_{m,k} \mid \mathbf{Z},\ )$.

The proof for $\eta_{n,k}$ is identical, and we have

$$
p(\eta_{n,k} \mid Z, \boldsymbol{\Theta}) \sim Gamma\Big(\alpha_\eta + \sum_{n=1}^{N} z_{n,m,k},\ \beta_\eta + \sum_{n=1}^{N} \theta_{n,k}\Big). \qquad (14)
$$

---

4 Problem 2: Coordinate ascent variational inference [math]

We will perform inference in this model using a mean-field variational posterior which factorizes according to:

$$q(\mathbf{Z}, \mathbf{H}, \mathbf{\Theta}) = q(\mathbf{Z})q(\mathbf{H})q(\mathbf{\Theta}) = \left[\prod_{n=1}^{N}\prod_{m=1}^{M} q(\mathbf{z}_{n,m})\right]\left[\prod_{n=1}^{N}\prod_{k=1}^{K} q(\theta_{n,k})\right]\left[\prod_{m=1}^{M}\prod_{k=1}^{K} q(\eta_{m,k})\right]$$

The optimal mean field factors will have the same forms as the conditional distributions above.

4.1 Problem 2a: Derive the CAVI update for $q(\mathbf{z}_{n,m})$

Show that, fixing $q(\mathbf{H})$ and $q(\mathbf{\Theta})$, the optimal $q(\mathbf{z}_{n,m})$ is given by:

$$q(\mathbf{z}_{n,m}; \boldsymbol{\lambda}^{(z)}_{n,m}) = \mathrm{Mult}(\mathbf{z}_{n,m}; x_{n,m}, \boldsymbol{\lambda}^{(z)}_{n,m}) \tag{15}$$

$$\log \lambda^{(z)}_{n,m,k} = \mathbb{E}_q[\log \theta_{n,k} + \log \eta_{m,k}] + c \tag{16}$$

Begin by taking expectation w.r.t. all approximation parameters *but for* $\lambda^{(z)}_{n,m}$ (as set forth on 9.25)

$$\begin{aligned}
\log q(z_{n,m}; \lambda^{(z)}_{n,m}) &= E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})} \log p(z_{n,m}|\theta, \eta, X, \ldots) && (17)\\
&= E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})} \log Multinom(z_{n,m}|\theta, \eta, X, \ldots) + c && (18)\\
&= E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})} \log \left[\binom{x_{n,m}}{z_{n,m}} \prod_{k=1}^{K} \left(\frac{\theta_{n,k}\eta_{m,k}}{\theta_n^T \eta_m}\right)^{z_{n,m,k}}\right] + c && (19)\\
&= E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})} \sum_{k=1}^{K} z_{n,m,k}\left[\log\left(\frac{\theta_{n,k}\eta_{m,k}}{\theta_n^T \eta_m}\right) - \log(z_{n,m,k}!)\right] + c && (20)\\
&= E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})} \sum_{k=1}^{K} [z_{n,m,k} \log \theta_{n,k}\eta_{m,k} - \log(z_{n,m,k}!)] + c' && (21)\\
&= \sum_{k=1}^{K} E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})}\left[z_{n,m,k}[\log \theta_{n,k} + \log \eta_{m,k}] - \log(z_{n,m,k}!)\right] + c' && (22)\\
&= \sum_{k=1}^{K} z_{n,m,k} \underbrace{E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})}[\log \theta_{n,k} + \log \eta_{m,k}]}_{\lambda^{(z)}_{n,m,k}} - \log(z_{n,m,k}!) + c' && (23)
\end{aligned}$$

Hence, we see the kernel of a logged multinomial (as the $z_{n,m,k}$ pulls out) with:

- counts $z_{n,m,k}$
- and logits (up to additive constant):

  $$E_{Q(H)Q(\Theta)Q(Z_{\sim n,\sim m})}[\log \theta_{n,k} + \log \eta_{m,k}] = \lambda^{(z)}_{n,m,k},$$

  So with awful notation

  $$q(z_{n,m}; \lambda^{(z)}_{n,m}) \sim MN(x_{n,m}; logit = \langle \lambda^{(z)}_{n,m,1}, \ldots, \lambda^{(z)}_{n,m,K}\rangle),$$

  as desired.

4.2 Problem 2b: Derive the CAVI update for $q(\theta_{n,k})$

Show that, fixing $q(\mathbf{Z})$ and $q(\mathbf{H})$, the optimal $q(\theta_{n,k})$ is given by:

$$q(\theta_{n,k}; \lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}) = \mathrm{Ga}(\theta_{n,k}; \lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}) \tag{24}$$

$$\lambda^{(\theta)}_{n,k,1} = \alpha_\theta + \sum_{m=1}^{M} \mathbb{E}_q[z_{n,m,k}] \tag{25}$$

$$\lambda^{(\theta)}_{n,k,2} = \beta_\theta + \sum_{m=1}^{M} \mathbb{E}_q[\eta_{m,k}] \tag{26}$$

We follow the same procedure as above, i.e. that set up in 9.25. We have, where Θ^C describes all variational Θ parameters that are not $\lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}$:

$$
\begin{aligned}
\log q(\theta_{n,k}; \lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}) &= E_{Q(H)Q(\Theta^C)Q(Z)} \log p(\theta_{n,k}|\theta, \eta, X, \ldots) && (27)\\
&= E_{Q(H)Q(\Theta^C)Q(Z)} \log Gamma(\theta_{n,k}|\alpha_\theta + \sum_m z_{n,m,k}, \beta_\theta + \sum_m \eta_{m,k}) + c && (28)\\
&= E_{Q(H)Q(\Theta^C)Q(Z)} \left[\left(\alpha_\theta + \sum_m z_{n,m,k} \right) \log \theta_{n,k} - \left(\beta_\theta + \sum_m \eta_{m,k} \right) \theta_{n,k} \right] + c' && (29)\\
&= E_{Q(H)Q(\Theta^C)Q(Z)} [\alpha_\theta \log \theta_{n,k} - \beta_\theta \theta_{n,k}] && (30)\\
&\qquad + E_{Q(H)Q(\Theta^C)Q(Z)} \left[\sum_m z_{n,m,k} \log \theta_{n,k} - \sum_m \eta_{m,k} \theta_{n,k} \right] + c' && (31)\\
&= \log \theta_{n,k} \cdot E_{Q(H)Q(\Theta^C)Q(Z)}[\alpha_\theta] - \theta_{n,k} \cdot E_{Q(H)Q(\Theta^C)Q(Z)}[\beta_\theta] && (32)\\
&\qquad + \log \theta_{n,k} \cdot E_{Q(H)Q(\Theta^C)Q(Z)} \left[\sum_m z_{n,m,k} \right] && (33)\\
&\qquad - \theta_{n,k} \cdot E_{Q(H)Q(\Theta^C)Q(Z)} \left[\sum_m \eta_{m,k} \theta_{n,k} \right] + c' && (34)\\
&= \alpha_\theta \log \theta_{n,k} - \beta_\theta \theta_{n,k} && (35)\\
&\qquad + \log \theta_{n,k} \cdot E_{Q(H)Q(\Theta^C)Q(Z)} \left[\sum_m z_{n,m,k} \right] && (36)\\
&\qquad - \theta_{n,k} \cdot E_{Q(H)Q(\Theta^C)Q(Z)} \left[\sum_m \eta_{m,k} \theta_{n,k} \right] + c' && (37)\\
&= \left(\alpha_\theta + E_{Q(H)Q(\Theta^C)Q(Z)} \left[\sum_m z_{n,m,k} \right] \right) \log \theta_{n,k} && (38)\\
&\qquad - \left(\beta_\theta + E_{Q(H)Q(\Theta^C)Q(Z)} \left[\sum_m \eta_{m,k} \theta_{n,k} \right] \right) \theta_{n,k} && (39)
\end{aligned}
$$

where as above, the $\theta_{n,k}$ terms pull out of the expectation since it is not included in Θ^C. This amounts to the log-kernel of a Gamma distribution, where

$$\lambda^{(\theta)}_{n,k,1} = \alpha_\theta + E_{Q(H)Q(\Theta^C)Q(Z)} \sum_m z_{n,m,k}$$

and

$$\lambda^{(\theta)}_{n,k,2} = \beta_\theta + E_{Q(H)Q(\Theta^C)Q(Z)} \sum_m \eta_{m,k},$$

giving

$$q(\theta_{n,k}; \lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}) \sim Gamma(\lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}).$$

We follow the same procedure as above, i.e. that set up in 9.25. We have, where Θ describes all variational Θ parameters that are not $\lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}$:

$$
\begin{aligned}
\log q(\eta_{m,k}; \lambda^{(\eta)}_{n,k,1}, \lambda^{(\eta)}_{n,k,2}) &= E_{Q(H^C)Q(\Theta)Q(Z)} \log p(\eta_{m,k}|\theta, \eta, X, \ldots) && (40)\\
&= E_{Q(H^C)Q(\Theta)Q(Z)} \log Gamma(\eta_{m,k}|\alpha_\eta + \sum_n z_{n,m,k}, \beta_\eta + \sum_n \theta_{n,k}) + c && (41)\\
&= E_{Q(H^C)Q(\Theta)Q(Z)} \left[\left(\alpha_\eta + \sum_n z_{n,m,k} \right) \log \eta_{m,k} - \left(\beta_\eta + \sum_n \theta_{n,k} \right) \eta_{m,k} \right] + c' && (42)\\
&= E_{Q(H^C)Q(\Theta)Q(Z)} [\alpha_\eta \log \eta_{m,k} - \beta_\eta \eta_{m,k}] && (43)\\
&\quad + E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n z_{n,m,k} \log \eta_{m,k} - \sum_n \theta_{n,k} \eta_{m,k} \right] + c' && (44)\\
&= \log \eta_{m,k} \cdot E_{Q(H^C)Q(\Theta)Q(Z)}[\alpha_\eta] - \eta_{m,k} \cdot E_{Q(H^C)Q(\Theta)Q(Z)}[\beta_\eta] && (45)\\
&\quad + \log \eta_{m,k} \cdot E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n z_{n,m,k} \right] && (46)\\
&\quad - \eta_{m,k} \cdot E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n \theta_{n,k} \eta_{m,k} \right] + c' && (47)\\
&= \alpha_\eta \log \eta_{m,k} - \beta_\eta \eta_{m,k} && (48)\\
&\quad + \log \eta_{m,k} \cdot E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n z_{n,m,k} \right] && (49)\\
&\quad - \eta_{m,k} \cdot E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n \theta_{n,k} \eta_{m,k} \right] + c' && (50)\\
&= \left(\alpha_\eta + E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n z_{n,m,k} \right] \right) \log \eta_{m,k} && (51)\\
&\quad - \left(\beta_\eta + E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n \theta_{n,k} \eta_{m,k} \right] \right) \eta_{m,k} && (52)
\end{aligned}
$$

where as above, the $\theta_{n,k}$ terms pull out of the expectation since it is not included in H^C. This amounts to the log-kernel of a Gamma distribution, where

$$\lambda^{(\eta)}_{m,k,1} = \alpha_\eta + E_{Q(H^C)Q(\Theta)Q(Z)} \sum_m z_{n,m,k}$$

and

$$\lambda^{(\eta)}_{m,k,2} = \beta_\eta + E_{Q(H^C)Q(\Theta)Q(Z)} \sum_n \eta_{m,k},$$

giving

$$q(\theta_{n,k}; \lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}) \sim Gamma(\lambda^{(\eta)}_{m,k,1}, \lambda^{(\eta)}_{m,k,2}).$$

4.3 Problem 2c: Derive the CAVI update for $q(\eta_{m,k})$

Show that, fixing $q(\mathbf{Z})$ and $q(\boldsymbol{\Theta})$, the optimal $q(\eta_{m,k})$ is given by:

$$q(\eta_{m,k}; \lambda^{(\eta)}_{m,k,1}, \lambda^{(\eta)}_{m,k,2}) = \mathrm{Ga}(\eta_{m,k}; \lambda^{(\eta)}_{m,k,1}, \lambda^{(\eta)}_{m,k,2}) \tag{53}$$

$$\lambda^{(\eta)}_{m,k,1} = \alpha_\eta + \sum_{n=1}^{N} \mathbb{E}_q[z_{n,m,k}] \tag{54}$$

$$\lambda^{(\eta)}_{m,k,2} = \beta_\eta + \sum_{n=1}^{N} \mathbb{E}_q[\theta_{n,k}] \tag{55}$$

This proof follows from identical algebraic steps as those above, albeit with respect to the η approximation:

$$
\begin{aligned}
\log q(\theta_{n,k}; \lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}) &= E_{Q(H^C)Q(\Theta)Q(Z)} \log p(\eta_{m,k}|\theta, \eta, X, \ldots) && (56)\\
&= E_{Q(H^C)Q(\Theta)Q(Z)} \log Gamma(\eta_{m,k}|\alpha_\eta + \sum_n z_{n,m,k}, \beta_\eta + \sum_n \theta_{n,k}) + c && (57)\\
&= E_{Q(H^C)Q(\Theta)Q(Z)} \left[\left(\alpha_\eta + \sum_n z_{n,m,k} \right) \log \eta_{m,k} - \left(\beta_\eta + \sum_n \theta_{n,k} \right) \eta_{m,k} \right] + c' && (58)\\
&= E_{Q(H^C)Q(\Theta)Q(Z)} [\alpha_\eta \log \eta_{m,k} - \beta_\eta \eta_{m,k}] && (59)\\
&\qquad + E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n z_{n,m,k} \log \eta_{m,k} - \sum_n \theta_{n,k} \eta_{m,k} \right] + c' && (60)\\
&= E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n z_{n,m,k} \log \eta_{m,k} - \sum_n \theta_{n,k} \eta_{m,k} \right] + c'' && (61)\\
&= \left(E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n z_{n,m,k} \right] \right) \log \eta_{m,k} && (62)\\
&\qquad - \left(E_{Q(H^C)Q(\Theta)Q(Z)} \left[\sum_n \theta_{n,k} \right] \right) \eta_{m,k} + c'', && (63)
\end{aligned}
$$

where

$$\lambda^{(\eta)}_{m,k,1} = E_{Q(H^C)Q(\Theta)Q(Z)} \sum_n z_{n,m,k}$$

and

$$\lambda^{(\eta)}_{m,k,2} = E_{Q(H^C)Q(\Theta)Q(Z)} \sum_n \theta_{n,k},$$

giving

$$q(\theta_{n,k}; \lambda^{(\eta)}_{m,k,1}, \lambda^{(\eta)}_{m,k,2}) \sim Gamma(\lambda^{(\eta)}_{m,k,1}, \lambda^{(\eta)}_{m,k,2}).$$

4.4 Problem 2d: Find the expected sufficient statistics

To update the variational factors, we need the expectations $\mathbb{E}_q[z_{n,m,k}]$, $\mathbb{E}_q[\log\theta_{n,k} + \log\eta_{m,k}]$, $\mathbb{E}_q[\theta_{n,k}]$, and $\mathbb{E}_q[\eta_{m,k}]$. Assume that each factor follows the forms derived above. That is, assume $q(\mathbf{z}_{n,m})$ is Multinomial with parameters $\lambda^{(z)}_{n,m}$ while $q(\theta_{n,k})$ and $q(\eta_{mk})$ are Gamma with parameters $\left(\lambda^{(\theta)}_{n,k,1}, \lambda^{(\theta)}_{n,k,2}\right)$ and $\left(\lambda^{(\eta)}_{m,k,1}, \lambda^{(\eta)}_{m,k,2}\right)$. Derive what each of these expectations are in closed form.

Here, we assume that the variational distributions are "true." Hence, we plug-and-chug with expectations of those assumed distributions.

Since `E[Multinomial]` is simply `trials * probs`, we have for $\sigma = softmax$

$$E_q(z_{n,m,k}) = x_{n,m}\sigma(\langle\lambda^{(z)}_{n,m,1}, \ldots, \lambda^{(z)}_{n,m,K}\rangle)_k.$$

We then plug in for the gammas, taking a standard ratio of parameters:

$$E_q[\theta_{n,k}] = \frac{\lambda^{(\theta)}_{n,k,1}}{\lambda^{(\theta)}_{n,k,2}} =$$

and

$$E_q[\eta_{n,k}] = \frac{\lambda^{(\eta)}_{m,k,1}}{\lambda^{(\eta)}_{m,k,2}},$$

where here we have access to the λ through whatever is most recent in the CAVI updating scheme.

Lastly, the above two expectations give:

$$\begin{aligned}
E_q[\log\theta_{n,k} + \log\eta_{m,k}] &= E_q[\log\theta_{n,k}] + E_q[\log\eta_{m,k}] && (64)\\
&= \varphi(E_q\sum_m z_{n,m,k}) - \log(E_q\sum_{m,k}\eta_{m,k}) + \varphi(E_q\sum_n z_{n,m,k}) - \log(E_q\sum_{n,k}\theta_{n,k}) && (65)\\
&= \varphi(\sum_m E_q z_{n,m,k}) - \log(\sum_{m,k}E_q\eta_{m,k}) + \varphi(E_q\sum_n z_{n,m,k}) - \log(\sum_{n,k}E_q\theta_{n,k}) && (66)
\end{aligned}$$

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5 Problem 3: Implement Coordinate Ascent Variational Inference [code]

First we'll give some helper functions and objects. Because PyTorch doesn't offer support for batched multinomial distributions in which the total counts differ (e.g. each $\mathbf{z}_{n,m}$ follows a multinomial distribution in which the total count is $x_{n,m}$), we have defined a `BatchedMultinomial` distribution for your convenience. This distribution doesn't support sampling, but will return the mean of each Multinomial variable in its batch. This is exactly what is needed for the CAVI updates.

```
[ ]: def gamma_expected_log(gamma_distbn):
         """Helper function to compute the expectation of log(X) where X follows a
         gamma distribution.
         """
         return torch.digamma(gamma_distbn.concentration) - torch.log(gamma_distbn.
      ↪rate)

     class BatchedMultinomial(Multinomial):
         """
         Creates a Multinomial distribution parameterized by `total_count` and
         either `probs` or `logits` (but not both). The innermost dimension of
         `probs` indexes over categories. All other dimensions index over batches.

         The `probs` argument must be non-negative, finite and have a non-zero sum,
         and it will be normalized to sum to 1 along the last dimension. `probs` will
         return this normalized value. The `logits` argument will be interpreted as
         unnormalized log probabilities and can therefore be any real number. It will
         likewise be normalized so that the resulting probabilities sum to 1 along
         the last dimension. `logits` will return this normalized value.

         Args:
             total_count (Tensor): number of trials
             probs (Tensor): event probabilities
                 Has shape total_count.shape + (num_categories,)
             logits (Tensor): event log probabilities (unnormalized)
                 Has shape total_count.shape + (num_categories,)

         Note: this text is mostly from the PyTorch documentation for the
             Multinomial distribution
         """
        def __init__(self, total_count, probs=None, logits=None, validate_args=None):
            super().__init__(probs=probs, logits=logits, validate_args=validate_args)
             self.total_count = total_count

         @property
         def mean(self):
             return self.total_count[..., None] * self.probs
```

5.1 Problem 3a: Implement a CAVI update step

Using the update equations derived in Problem 2, complete the `cavi_step` function below.

Hint: Given a `Distribution` named `d`, `d.mean` returns the mean of that distribution.

```
[ ]: def cavi_step(X, q_z, q_theta, q_eta, alpha_theta, beta_theta, alpha_eta,␣
      ↪beta_eta):
         """One step of CAVI.
```

```python
    Args:
        X: torch.tensor of shape (N, M)
        q_z: variational posterior over z, BatchedMultinomial distribution
        q_theta: variational posterior over theta, Gamma distribution
        q_eta: variational posterior over eta, Gamma distribution

    Returns:
       (q_z, q_theta, q_eta): Updated distributions after performing CAVI updates
    """
    ###
    # Your code here
    N, M = X.shape
    ### compute the expectations
    expected_log_theta = gamma_expected_log(q_theta)
    expected_log_eta = gamma_expected_log(q_eta)

    # Update q_z
    q_z = BatchedMultinomial(
        # data (N, M)
        X,
        # logits or pi (N, M, K)
        logits = (
            # (N, K) --> (N, 1, K)
            expected_log_theta.unsqueeze(1)
            # (M, K) --> (1, M ,K)
            + expected_log_eta.unsqueeze(0)
        )
    )


    # Update the per-user posterior q_theta
    q_theta = Gamma(
        alpha_theta + q_z.mean.sum(axis=1), # sum over M: (N, M, K) --> (N, K)
        # q_eta: (N, K) --> (K, )
        beta_theta + q_eta.mean.sum(axis=0)
    )

    # Update the per-item posterior q_eta
    q_eta = Gamma(
        alpha_eta + q_z.mean.sum(axis=0), # sum over N: (N, M, K) --> (M, K)
        beta_eta + q_theta.mean.sum(axis=0)
    )

    return q_z, q_theta, q_eta
```

5.2 Problem 3b: ELBO Calculation [math]

Recall that the evidence lower bound is defined as:

$$\mathcal{L}(q) = \mathbb{E}_q \left[\log p(\mathbf{X}, \mathbf{Z}, \mathbf{H}, \boldsymbol{\Theta}) - \log q(\mathbf{Z}, \mathbf{H}, \boldsymbol{\Theta})\right]$$

Assume that $q(\mathbf{Z})$ has support contained in $\{\mathbf{Z} : \sum_{k=1}^{K} z_{n,m,k} = x_{n,m} \text{ for all } n, m\}$. Show that we can rewrite $\mathcal{L}(q)$ as:

$$\mathcal{L}(q) = \mathbb{E}_q[\log p(\mathbf{Z} \mid \boldsymbol{\Theta}, \mathbf{H}) - \log q(\mathbf{Z})] - \mathrm{KL}(q(\boldsymbol{\Theta})||p(\boldsymbol{\Theta})) - \mathrm{KL}(q(\mathbf{H})||p(\mathbf{H}))$$

Next, use that $q(\mathbf{z}_{n,m}; \boldsymbol{\lambda}^{(z)}_{n,m}) = \mathrm{Mult}(\mathbf{z}_{n,m}; x_{n,m}, \boldsymbol{\lambda}^{(z)}_{n,m})$ and by plug in the densities of the Poisson and Multinomial distributions to show that we have:

$$\mathbb{E}_q[\log p(\mathbf{Z} \mid \boldsymbol{\Theta}, \mathbf{H}) - \log q(\mathbf{Z})] = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E}_q \left[\sum_{k=1}^{K} -\theta_{n,k}\eta_{m,k} + z_{n,m,k} \log(\theta_{n,k}\eta_{m,k}) - z_{n,m,k} \log(\lambda^{(z)}_{n,m,k}) \right] - \log(x_{n,m}!)$$

Explain why we have:

$$\mathbb{E}_q \left[-\theta_{n,k}\eta_{m,k} + z_{n,m,k} \log(\theta_{n,k}\eta_{m,k}) - z_{n,m,k} \log(\lambda^{(z)}_{n,m,k}) \right] =$$

$$-\mathbb{E}_q \left[\theta_{n,k}\right] \mathbb{E}_q \left[\eta_{m,k}\right] + \mathbb{E}_q \left[z_{n,m,k}\right] \left(\mathbb{E}_q \left[\log(\theta_{n,k})\right] + \mathbb{E}_q \left[\log(\eta_{m,k})\right] - \log(\lambda^{(z)}_{n,m,k}) \right)$$

Then, it follows that

$$\begin{gathered}
\mathbb{E}_q \left[-\theta_{n,k}\eta_{m,k} + z_{n,m,k} \log(\theta_{n,k}\eta_{m,k}) - z_{n,m,k} \log(\lambda^{(z)}_{n,m,k}) \right] \\
= \\
-E_q[\theta_{n,k}\eta m, k] + E_q[z_{n,m,k}(\log \theta_{n,k} + \log \eta_{m,k})] - E_q[\log(\lambda^{(z)}_{n,m,k})] \\
= \\
-E_q[\theta_{n,k}]E_q[\eta m, k] + E_q[[z_{n,m,k}]E_q[(\log \theta_{n,k} + \log \eta_{m,k})] - E_q[\log(\lambda^{(z)}_{n,m,k})] \\
= \\
-E_q[\theta_{n,k}]E_q[\eta_{m,k}] + E_q[[z_{n,m,k}]E_q[(\log \theta_{n,k} + \log \eta_{m,k})] - \log(\lambda^{(z)}_{n,m,k}).
\end{gathered}$$

The first step is justified by linearity of expectations and log of a product; the second step is justified by the assumption that $Z \propto \Theta \propto H$, so expectations factor; the third is by the fact that $E_q[\log \lambda]$ is effectively w.r.t. a constant. Hence, we get the desired result.

5.3 Problem 3c: Implement the ELBO [code]

Using our expression above, write a function which evaluates the evidence lower bound.

Hints: - Use the `kl_divergence` function imported above to compute the KL divergence between two `Distributions` in the same family. - Recall that for integers n, $\Gamma(n + 1) = n!$ where Γ is the Gamma function. $\log \Gamma$ is implemented in PyTorch as `torch.lgamma`.

[ ]:
```python
def expected_poisson_logpdf(x, expected_rate, expected_log_rate):
    """Helper function to compute the expected logpdf under a Poisson.
    """
    return -torch.special.gammaln(x + 1) + x * expected_log_rate - expected_rate

def elbo(X, q_z, q_theta, q_eta, p_theta, p_eta):
    """Compute the evidence lower bound.

    Args:
        X: torch.tensor of shape (N, M)
        q_z: variational posterior over z, BatchedMultinomial distribution
        q_theta: variational posterior over theta, Gamma distribution
        q_eta: variational posterior over eta, Gamma distribution
        p_theta: prior over theta, Gamma distribution
        p_eta: prior over eta, Gamma distribution

    Returns:
        elbo: torch.tensor of shape []
    """
    ### blocked by chunks of the above function, idk ###

    ### E[theta] * E[eta] ###
     # (N, K) --> (N, 1, K) // (M, K) --> (1, M ,K) ==> (N, M, K)
    expected_theta_eta = q_theta.mean.unsqueeze(1) * q_eta.mean.unsqueeze(0)

    ### E[z] ###
    expected_z = q_z.mean # (N, M, K)

    ### E[log(theta)] + E[log(eta)] ###
    expected_log_theta = gamma_expected_log(q_theta) # E[log(theta)]: (N, K)
    expected_log_eta = gamma_expected_log(q_eta) # E[log(eta)]: (M, K)
    # (N, K) --> (N, 1, K) // (M, K) --> (1, M ,K) ==> (N, M, K)
    expected_log_theta_eta = expected_log_theta.unsqueeze(1) + expected_log_eta.
 ↪unsqueeze(0)

    ### log(lambda) ###
    log_lambda_z = q_z.logits # (N, M, K)

    ### checks ###
    # assert expected_rate.shape == expected_log_rate.shape
    # assert expected_rate.shape == torch.Size([N, M, K])

    elbo = (
        #
        (
            expected_theta_eta
            + expected_z * (
```

```
                expected_log_theta_eta + log_lambda_z
            )
        ).sum()
        - torch.lgamma(X + 1).sum()
        #
        - kl_divergence(q_eta, p_eta).sum()
        #
        - kl_divergence(q_theta, p_theta).sum()
    )
    return elbo / torch.sum(X)
```

5.4 Implement CAVI loop [given]

Using your functions defined above, complete the function `cavi` below. `cavi` loops for some number of iterations, updating each of the variational factors in sequence and evaluating the ELBO at each step.

[ ]:
```
from torch.distributions import Uniform

def cavi(data,
         num_factors=10,
         num_iters=100,
         tol=1e-5,
         alpha_theta=0.1,
         beta_theta=1.0,
         alpha_eta=0.1,
         beta_eta=1.0,
         seed=0
        ):
    """Run coordinate ascent VI for Poisson matrix factorization.

    Args:

    Returns:
        elbos, (q_z, q_theta, q_eta):
    """
    data = data.float()
    N, M = data.shape
    K = num_factors      # short hand

    # Initialize the variational posteriors.
    q_eta = Gamma(Uniform(0.5 * alpha_eta, 1.5 * alpha_eta).sample((M, K)),
                  Uniform(0.5 * beta_eta, 1.5 * beta_eta).sample((M, K)))
    q_theta = Gamma(Uniform(0.5 * alpha_theta, 1.5 * alpha_theta).sample((N, K)),
                    Uniform(0.5 * beta_theta, 1.5 * beta_theta).sample((N, K)))
    q_z = BatchedMultinomial(data, logits=torch.zeros((N, M, K)))
```

```python
    p_theta = Gamma(alpha_theta, beta_theta)
    p_eta = Gamma(alpha_eta, beta_eta)

    # Run CAVI
    elbos = [elbo(data, q_z, q_theta, q_eta, p_theta, p_eta)]
    for itr in trange(num_iters):
        q_z, q_theta, q_eta = cavi_step(data, q_z, q_theta, q_eta,
                                        alpha_theta, beta_theta,
                                        alpha_eta, beta_eta)

        elbos.append(elbo(data, q_z, q_theta, q_eta, p_theta, p_eta))
    return torch.tensor(elbos), (q_z, q_theta, q_eta)
```

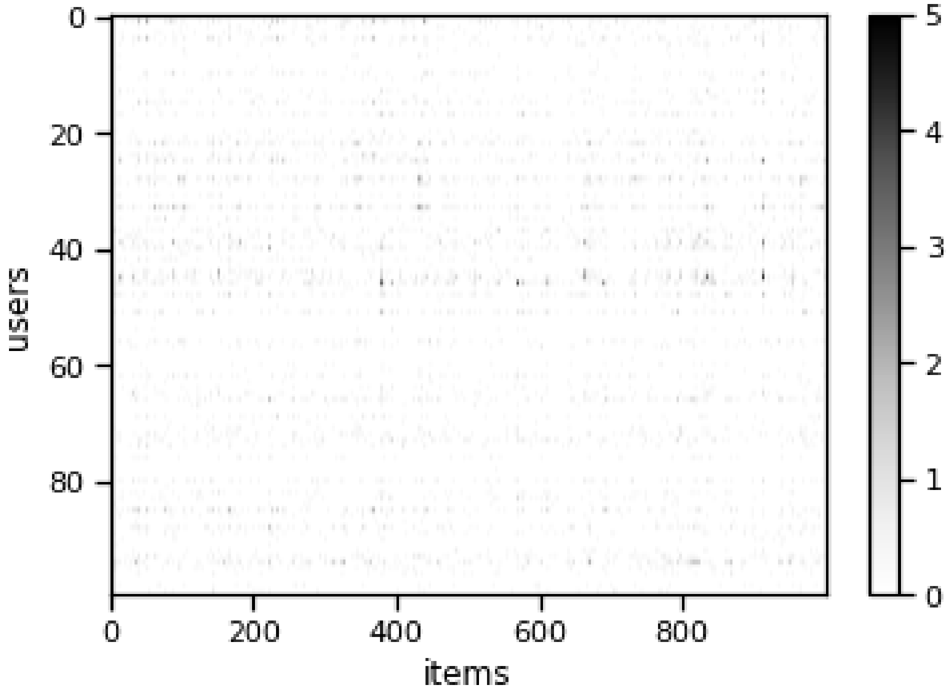
6 Test your implementation on a toy dataset

To check your implementation is working properly, we will fit a mean-field variational posterior using data sampled from the true model.

[ ]:
```python
def test_toy_datset():

    # Constants
    N = 100    # num "users"
    M = 1000 # num "items"
    K = 5      # number of latent factors

    # Hyperparameters
    alpha = 0.1  # sparse gamma prior with mean alpha/beta
    beta = 1.0

    # Sample data from the model
    torch.manual_seed(305)
    theta = Gamma(alpha, beta).sample(sample_shape=(N, K))
    eta = Gamma(alpha, beta).sample(sample_shape=(M, K))
    data = Poisson(theta @ eta.T).sample()

    print(data.shape)
    # Plot the data matrix
    plt.imshow(data, aspect="auto", vmax=5, cmap="Greys")
    plt.xlabel("items")
    plt.ylabel("users")
    plt.colorbar()
    plt.show()
```

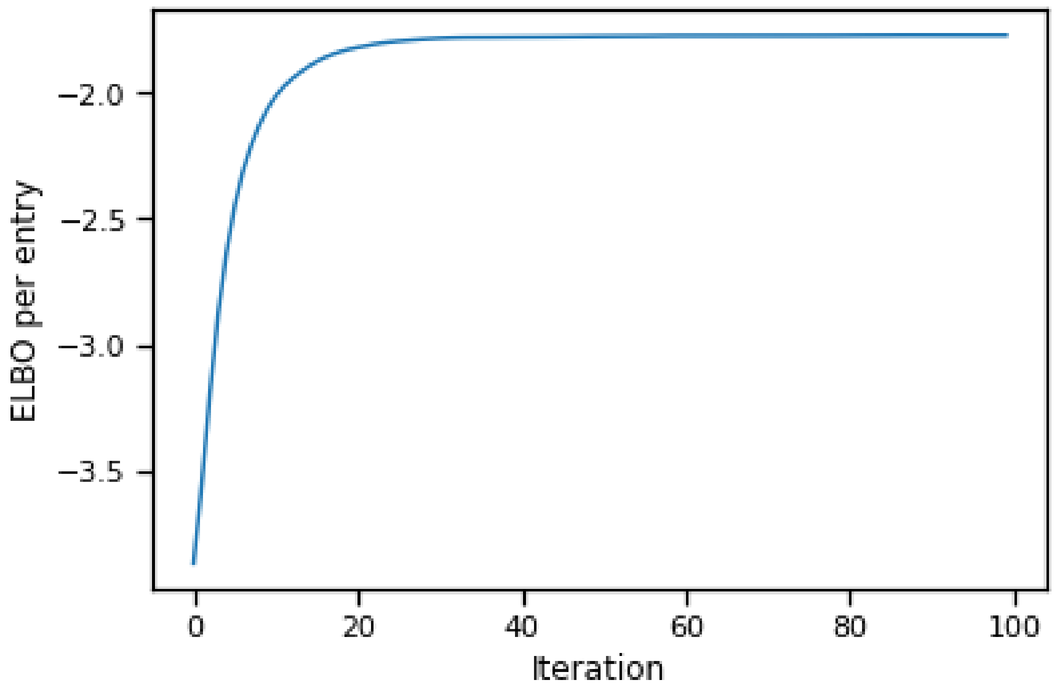
```
    print("Max data:  ", data.max())
    print("num zeros: ", torch.sum(data == 0))
    print("Data shape: ", data.shape)

    ### CAVI
    elbos, (q_z, q_theta, q_eta) = cavi(data)

    ### plot ELBOS
    plt.plot(elbos[1:])
    plt.xlabel("Iteration")
    plt.ylabel("ELBO per entry")
    plt.show()

    ### diagnostics
    true_rates = theta @ eta.T
    inf_rates = q_theta.mean @ q_eta.mean.T

    # Plot the data matrix
    plt.figure(figsize=(12, 6))
    plt.subplot(1, 2, 1)
    plt.imshow(true_rates, aspect="auto", vmax=3, cmap="Greys")
    plt.xlabel("items")
    plt.ylabel("users")
    plt.title("true rates")
    plt.colorbar()

    plt.subplot(1, 2, 2)
    plt.imshow(inf_rates, aspect="auto", vmax=3, cmap="Greys")
    plt.xlabel("items")
    plt.ylabel("users")
    plt.title("inferred rates")
    plt.colorbar()
    plt.show()

test_toy_datset()
```

```
torch.Size([100, 1000])
```

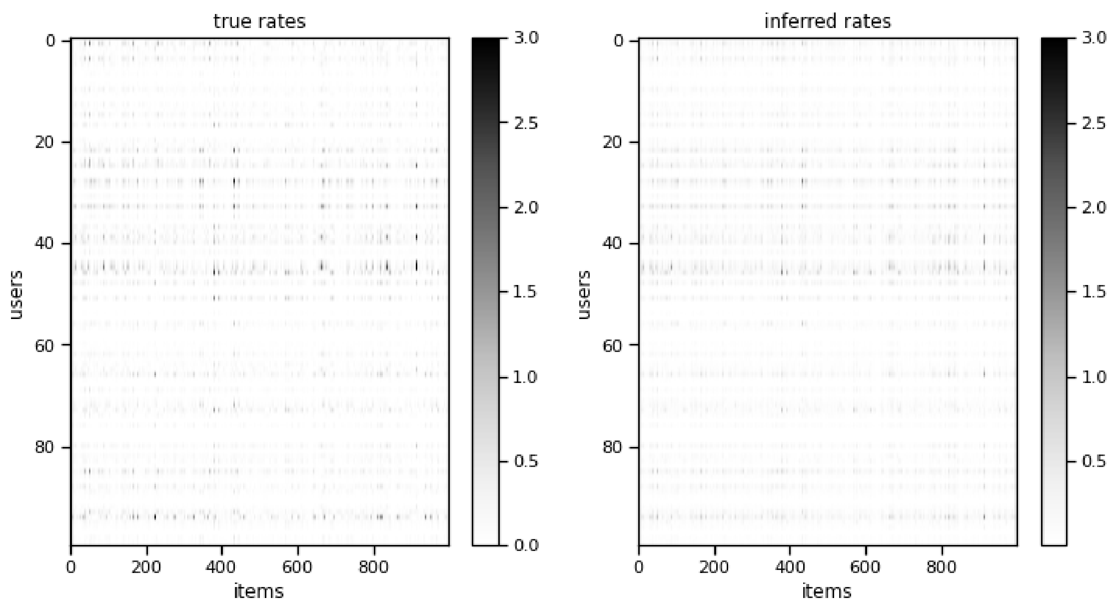



```
Max data:   tensor(14.)
num zeros:  tensor(95568)
Data shape:  torch.Size([100, 1000])
```

```
  0%|          | 0/100 [00:00<?, ?it/s]
```



ELBO per entry
−2.0
−2.5
−3.0
−3.5
0
20
40
60
80
100
Iteration



true rates
inferred rates
users
items
0
20
40
60
80
200
400
600
800
3.0
2.5
2.0
1.5
1.0
0.5
0.0

7 Problem 4: Run your code on a downsampled LastFM dataset

Next, we will use data gathered from Last.FM users to fit a PMF model. We use a downsampled version of the Last.FM-360K users dataset. This dataset records how many times each user played an artist's songs. We downsample the data to include only the 2000 most popular artists, as measured by how many users listened to the artist at least once, and the 1000 most prolific users, as measured by how many artists they have listened to.

In the code below , we use `lfm` to represent the data matrix X in the model. That is, `lfm[n, d]` denotes how many times the `n`-th user played a song by the `d`-th artist.

[ ]:
```
!wget -nc https://raw.githubusercontent.com/slinderman/stats305c/main/
↪assignments/hw5/subsampled_last_fm.csv
```

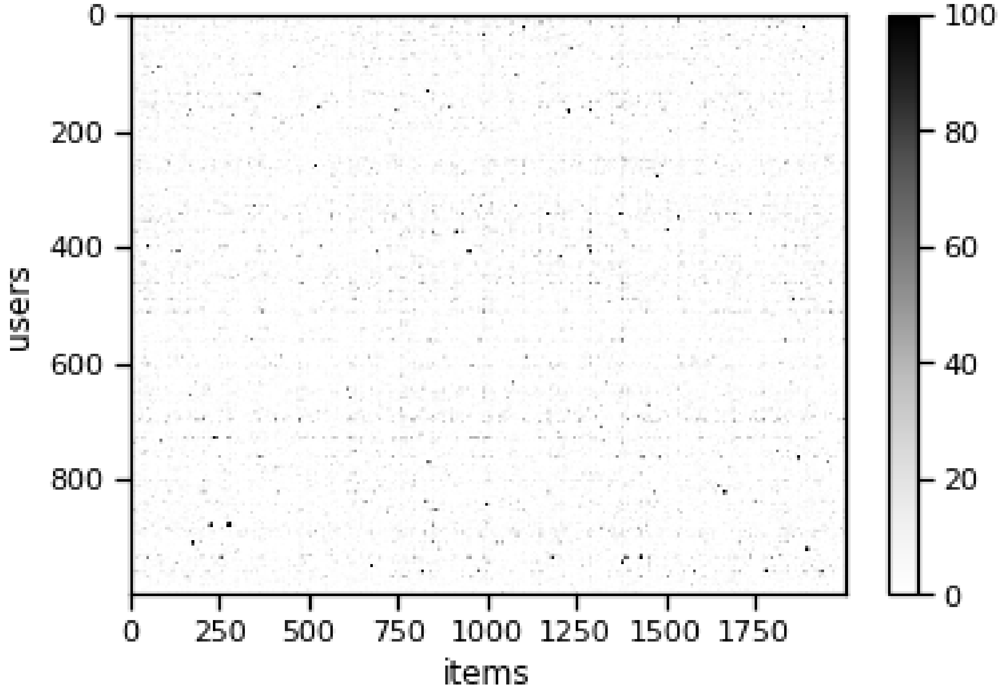
```
--2022-05-02 04:42:50--  https://raw.githubusercontent.com/slinderman/stats305c/
main/assignments/hw5/subsampled_last_fm.csv
Resolving raw.githubusercontent.com (raw.githubusercontent.com)…
185.199.108.133, 185.199.109.133, 185.199.110.133, …
Connecting to raw.githubusercontent.com
(raw.githubusercontent.com)|185.199.108.133|:443… connected.
HTTP request sent, awaiting response… 200 OK
Length: 1267973 (1.2M) [text/plain]
Saving to: 'subsampled_last_fm.csv'

subsampled_last_fm. 100%[===================>]   1.21M  --.-KB/s    in 0.05s

2022-05-02 04:42:50 (23.0 MB/s) - 'subsampled_last_fm.csv' saved
[1267973/1267973]
```

[ ]:
```
import pandas as pd

lfm_df = pd.read_csv('subsampled_last_fm.csv')
lfm = lfm_df.pivot_table(index='UserID', columns='ItemID', aggfunc=sum)\
    .fillna(0).astype(int).to_numpy()
lfm = torch.tensor(lfm, dtype=torch.int)
print(lfm.shape)
```

```
torch.Size([999, 2000])
```

[ ]:
```
plt.imshow(lfm, aspect="auto", vmax=100, cmap="Greys")
plt.xlabel("items")
plt.ylabel("users")
plt.colorbar()
```

[ ]: <matplotlib.colorbar.Colorbar at 0x7f36cc52ebd0>



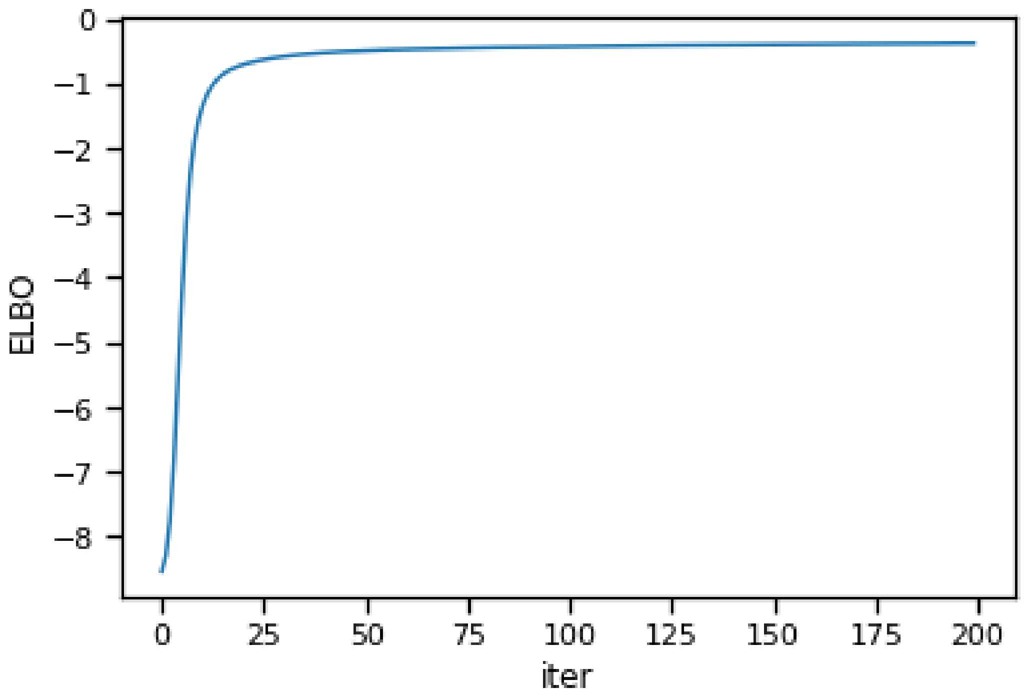
Using the code below, run coordinate ascent variational inference on this dataset. Our implementation takes around 10-15 minutes to finish, and achieves an ELBO of −1590.

[ ]:
```
elbos, (q_z, q_theta, q_eta) = cavi(lfm,
     num_factors=40,
     num_iters=200,
     alpha_theta=1.,
     beta_theta=0.5,
     alpha_eta=1.,
     beta_eta=0.5)
```

```
  0%|          | 0/200 [00:00<?, ?it/s]
```

[ ]:
```
print(elbos[-1])
plt.plot(elbos[1:])
plt.xlabel("iter")
plt.ylabel("ELBO")
plt.show()
```

```
tensor(-0.3790)
```



7.1 Investigate "genres"

The columns of **H** correspond to weights on artists. Intuitively, each of the K columns should put weight on subsets of artists that are often played together. We might think of these columns as reflecting different "genres" of music. The code below the top 10 artists for a few of these columns.

[ ]:
```
# Find the 10 most used genres
genre_loading = q_theta.mean.sum(0)
genre_order = torch.argsort(genre_loading, descending=True)

# Print the top 10 artists for each of the top 10 genres
for genre in genre_order[:10]:
    print("genre ", genre)
    artist_idx = torch.argsort(q_eta.mean[:, genre],
                               descending=True)[:10].numpy()
    subset = lfm_df[lfm_df['ItemID'].isin(artist_idx)]
    print(subset[['ItemID', 'Artist']].drop_duplicates())
    print("")
```

```
genre  tensor(1)
      ItemID          Artist
1       1647  tegan and sara
10        15  sufjan stevens
```

24	38	cat power
44	28	elliott smith
154	1306	modest mouse
155	1578	ryan adams
157	910	bright eyes
469	1959	fleet foxes
1019	815	sam cooke
1289	1933	jay-z

genre tensor(7)

	ItemID	Artist
49	1286	radiohead
204	627	david bowie
206	795	the cure
320	1658	devo
394	476	the smiths
475	12	morrissey
687	1877	new order
711	692	sonic youth
739	813	guided by voices
757	469	the magnetic fields

genre tensor(25)

	ItemID	Artist
60	990	pink floyd
61	758	metallica
105	773	led zeppelin
142	1591	iron maiden
165	806	judas priest
211	763	ac/dc
1365	614	black sabbath
1369	1301	megadeth
1375	651	motörhead
1677	1768	kiss
20257	763	acdc

genre tensor(14)

	ItemID	Artist
55	44	jimi hendrix
56	1377	the beatles
60	990	pink floyd
69	857	bob dylan
73	1210	the doors
105	773	led zeppelin
204	627	david bowie
551	988	pearl jam
563	1100	the clash
2920	1855	guns n' roses

```
2934     1855   guns n roses
18201    1377        beatles

genre  tensor(29)
       ItemID                       Artist
336      442                     ghostface
817     1150           a tribe called quest
1419    1500                        common
1431     964                     the roots
2297    1639                           nas
2312    1213                       mos def
2693    1598                       j dilla
3613    1834                        madlib
3835      99                   wu-tang clan
4726     442              ghostface killah
10000     99                   wu tang clan
15054   1319                  blessthefall
24609    964  the roots featuring d'angelo

genre  tensor(2)
       ItemID              Artist
13       676                stars
38      1605             coldplay
49      1286            radiohead
83      1196                 muse
109      981          jason mraz
167        2  death cab for cutie
235      822                  m83
282     1066           bloc party
1080    1785              incubus
1852    1279          snow patrol
29203    676            the stars

genre  tensor(32)
       ItemID             Artist
56      1377         the beatles
67        26               queen
106      730        eric clapton
116       63     frédéric chopin
183      911             madonna
280     1705                abba
324     1370  the rolling stones
330       13       elvis presley
565     1254                  u2
876      832         mushroomhead
18201   1377             beatles

genre  tensor(28)
```

```
      ItemID          Artist
170       37       daft punk
218     1007    depeche mode
222      574         blondie
865     1494       the knife
951     1930       fever ray
973     1379 crystal castles
992      657 yeah yeah yeahs
1303    1400        ladytron
1443    1313 franz ferdinand
1447    1947         justice

genre  tensor(5)
       ItemID                        Artist
59        645                   miles davis
64       1672                   johnny cash
69        857                     bob dylan
277      1043               various artists
293      1533                     tom waits
297       163  nick cave and the bad seeds
324      1370            the rolling stones
330        13                 elvis presley
343       755                 leonard cohen
443       163    nick cave & the bad seeds
913       848             bruce springsteen
18256    1043                          v.a.

genre  tensor(0)
       ItemID          Artist
328      1819      bad brains
452      1791      have heart
454       820    comeback kid
788      1692         ramones
821       277          rancid
860      1735            nofx
885      1281     against me!
894       142    bad religion
1364     1170      black flag
11243    1769  sick of it all
```

7.2 Problem 4a

Inspect the data either using the csv file or the pandas dataframe and choose a user who has listened to artists you recognize. If you are not familiar with any of the artists, use the listener with UserID 349, who mostly listens to hip-hop artists. For the particular user n you choose, find the 10 artists who are predicted to have the most plays by sorting the vector of mean song counts predicted by the model, i.e. the n^{th} row of $\mathbb{E}_q[\mathbf{\Theta}\mathbf{H}^\top]$. Are these artists you would expect the user would enjoy? Are there any artists that the user has not listened to?

Hint: Use `torch.argsort(..., descending=True)` to return the indices of the largest elements of a vector in descending order.

[ ]:
```
lfm_df.query("Artist == 'bruce springsteen'").sort_values("Count",␣
↪ascending=False).head(10)
```

[ ]:

	UserID	ItemID	Artist	Count
34674	854	848	bruce springsteen	1370
4563	107	848	bruce springsteen	324
2475	57	848	bruce springsteen	304
30416	738	848	bruce springsteen	291
14488	346	848	bruce springsteen	289
26947	654	848	bruce springsteen	264
18180	436	848	bruce springsteen	248
9074	216	848	bruce springsteen	222
23380	563	848	bruce springsteen	219
3956	94	848	bruce springsteen	211

First, the user's actual top plays:

[ ]:
```
###
# Your code here.
##

USER_DEMO = 854

lfm_df.query(f"UserID == {USER_DEMO}").sort_values("Count", ascending=False).
↪head(10)
```

[ ]:

	UserID	ItemID	Artist	Count
34673	854	857	bob dylan	3047
34674	854	848	bruce springsteen	1370
34675	854	1100	the clash	1028
34676	854	1533	tom waits	803
34677	854	836	håkan hellström	699
34678	854	476	the smiths	621
34679	854	479	jens lekman	603
34680	854	795	the cure	576
34681	854	12	morrissey	496
34682	854	1377	the beatles	444

Then the predicted top plays:

[ ]:
```
loadings = q_theta.mean[USER_DEMO, :] @ (q_eta.mean.T)
user_t10 = torch.argsort(loadings, descending=True)[0:10].numpy()
disp_df = pd.DataFrame({
```

```
    "ItemID": user_t10,
    "user_ranking": np.arange(1, 11)
}).merge(
    lfm_df[lfm_df['ItemID'].\
           isin(user_t10)].\
           groupby("Artist", as_index=False).\
           head(1)[["Artist", "ItemID"]].\
           reset_index(drop=True),
    on=["ItemID"]
)
disp_df
```

[ ]:

```
    ItemID  user_ranking                      Artist
0      857             1                   bob dylan
1      755             2               leonard cohen
2     1533             3                   tom waits
3      848             4           bruce springsteen
4       13             5               elvis presley
5     1370             6          the rolling stones
6      163             7  nick cave and the bad seeds
7      163             7    nick cave & the bad seeds
8     1043             8             various artists
9     1043             8                        v.a.
10    1672             9                 johnny cash
11     645            10                 miles davis
```

Answer:

I chose the most prolific Bruce Springsteen user in the dataset (`UserID=854`). For the most part, I expected this user to listen to middle-aged/Dad rock, i.e. Springsteen, Tom Petty, the Who, the Eagles, the Rolling Stones, that kind of thing. Further, given the apparent naming discrepancies – for instance, `nick cave & the bad seeds` and `nick cave and the bad seeds` are treated as separate artists – I expected some duplicate `bruce springsteen & the e-street band` or `bruce springsteen and the e-street band` entries.

As it turned out, results were generally in the middle-aged/Dad rock genre: Springsteen and the Rolling stones were in the top 10. More singer-songwriter-y types such as Bob Dylan, Johnny Cash, and Tom Waits were also included – while this was a tad surprising (I'd have bet more on the bands named above), it's not entirely non-sensical, given some of Springsteen's later songs. Additionally, I was surprised to see the inclusion of Elvis and Miles Davis – would never have guessed those two. And I'd assume various artists/va would include a number of hits from that era of rock.

So overall, the modeled rankings didn't match my expectations at the outset, but generally I found them coherent and plausible.

8 Problem 5: Reflections

8.1 Problem 5a

Discuss one advantage and one disadvantage of fitting a posterior using variational inference vs. sampling from the posterior using MCMC.

- Advantage: VI methods may be more computationally tractable here. For example, if we did HMC or Metropolis sampling, the chain could take days if not weeks to converge. Here, we get a quick and workable solution within minutes, and not having to supervise an MCMC chain may be a big plus.
- Disadvantage: We may not get the exact, or even a convergent, solution. Whereas MCMC is guaranteed to converge into infinity, there are no such guarantees with VI/ELBO methods. In fact, we may find ourselves stuck in a local minima, or in more general VI settings (e.g. VAEs) fail to achieve tight ELBO bounds. So the gain in computational viability may come at the expense of model accuracy.

Your answer here.

8.2 Problem 5b

First, explain why the assumption that $\mathbf{Z}, \mathbf{H}$ and $\boldsymbol{\Theta}$ are independent in the posterior will never hold.

Next, recall that maximizing the ELBO is equivalent to minimizing the KL divergence between the approximate posterior and the true posterior. In general, how will the approximate posterior differ from the true posterior, given that the variational family does not include the true posterior?

As set forth in our calculations for $\log p(\Theta|Z, H)$, we have that sub-count is explicitly dependent on Z, H, in violation of the assumption above. However, it may be a situation where we can get "good enough" results despite the violation of such an assumption.

In general, the approximate posterior will probably have a smaller variance than the true posterior. This is particularly true when approximating via a normal distribution, which: i.) is comparatively thinly-tailed (unlike a T, for instance) and ii.) especially true when the true posterior is outside the variational family. As but one example here, consider a multimodal setting. Recall here that our minimization problem w.r.t. the latent Z is of the form

$$
\begin{aligned}
ELBO(q; x, z, p) &= E_{z \sim q}[\log p(x, z; \theta) - \log q(z; \theta)] && (67)\\
&= p(x; \theta) - D_{KL}(q(z; \theta) || p(z|x; \theta)) && (68)\\
&= E_{z \sim q}[\log(p(x|z)p(z)) - \log q(z; \theta)] && (69)\\
&= E_{z \sim q}[\log(p(x|z; \theta)] + E_{z \sim q}[p(z; \theta) - \log q(z; \theta)] && (70)\\
&= E_{z \sim q}[\log(p(x|z; \theta)] - D_{KL}(q(z; \theta) || p(z; \theta)) && (71)
\end{aligned}
$$

In maximizing this, we have a two-way struggle: first, we aim to maximize $E_{z\sim q}[\log(p(x|z;\theta)]$, which would happen if we had q such that it put unit mass on the MAP of $p(x|z;\theta)$. Second, we also need q to stay reasonably close to p (governed by the $D_{KL}(q(z;\theta)||p(z;\theta))$), lest the KL divergence term blow things up.

Hence, in multimodal setting, the variational approximation will attach itself to one of the modes, since sitting in between will incur a $p(x|z;\theta)$ that is tiny (ruining maximization), and it will incur some least worst cost for the KL distance between q and p; that is, it will choose a lesser of the evils and just pick a mode. In turn, it's variance will form around that mode, meaning it will fail to capture the variance of the other modes, and hence underestimate variance in general.

8.3 Problem 5c

Suppose we are using this model to recommend new items to users. Describe one improvement that could be made to the model which you think would lead to better recommendations.

First, as noted above, some data wrangling may be necessary: we'll want to condense down the duplicate columns such as `v.a.` and `various artists` or `nick cave and the bad seeds` and `nick cave & the bad seeds`. This will allow us to fit on a better/more accurate dataset. Moreover, we may want to set up a tuning/CV procedure so we can better select priors and/or select an ideal value of `K`. Lastly, we may want to explore adding additional hierarchical components/features, to give us some way to correct for the fact that user preferences may change when/where they're listening to the music. For instance, someone's workout playlist may be different from their roadtrip playlist, and if we could embed workout/roadtrip proxy features, we may be able to better account/correct for seemingly disparate preferences.

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9 Submission Instructions

Formatting: check that your code does not exceed 80 characters in line width. If you're working in Colab, you can set *Tools* → *Settings* → *Editor* → *Vertical ruler column* to 80 to see when you've exceeded the limit.

Download your notebook in .ipynb format and use the following commands to convert it to PDF:

```
jupyter nbconvert --to pdf hw5_yourname.ipynb
```

Dependencies:

- `nbconvert`: If you're using Anaconda for package management,

```
conda install -c anaconda nbconvert
```

Upload your .pdf files to Gradescope.

[ ]:
```python
from torch.distributions import Uniform
data = X

alpha_theta=0.1
beta_theta=1.0
alpha_eta=0.1
beta_eta=1.0



data = data.float()
N, M = data.shape
K = 5       # short hand

# Initialize the variational posteriors.
q_eta = Gamma(Uniform(0.5 * alpha_eta, 1.5 * alpha_eta).sample((M, K)),
              Uniform(0.5 * beta_eta, 1.5 * beta_eta).sample((M, K)))
q_theta = Gamma(Uniform(0.5 * alpha_theta, 1.5 * alpha_theta).sample((N, K)),
                Uniform(0.5 * beta_theta, 1.5 * beta_theta).sample((N, K)))
q_z = BatchedMultinomial(data, logits=torch.zeros((N, M, K)))
```