## Tree

- *introduction*
- *binary tree*
- *complete binary tree*
  - *max heap, min heap*
  - *Chapter 7 – heap sorting*
  - *Chapter 9 - priority queues*
- ***binary search tree***

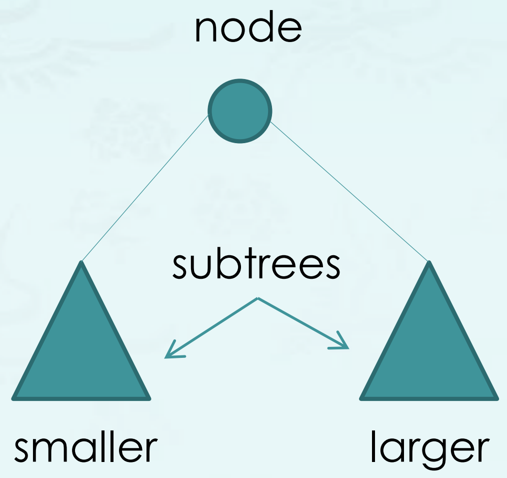
## Binary search trees

**Definition: A binary search tree is a binary tree in symmetric order.**

- A binary tree is either
  - empty
  - a key-value pair and two binary trees [neither of which contain that key]

equal keys ruled out

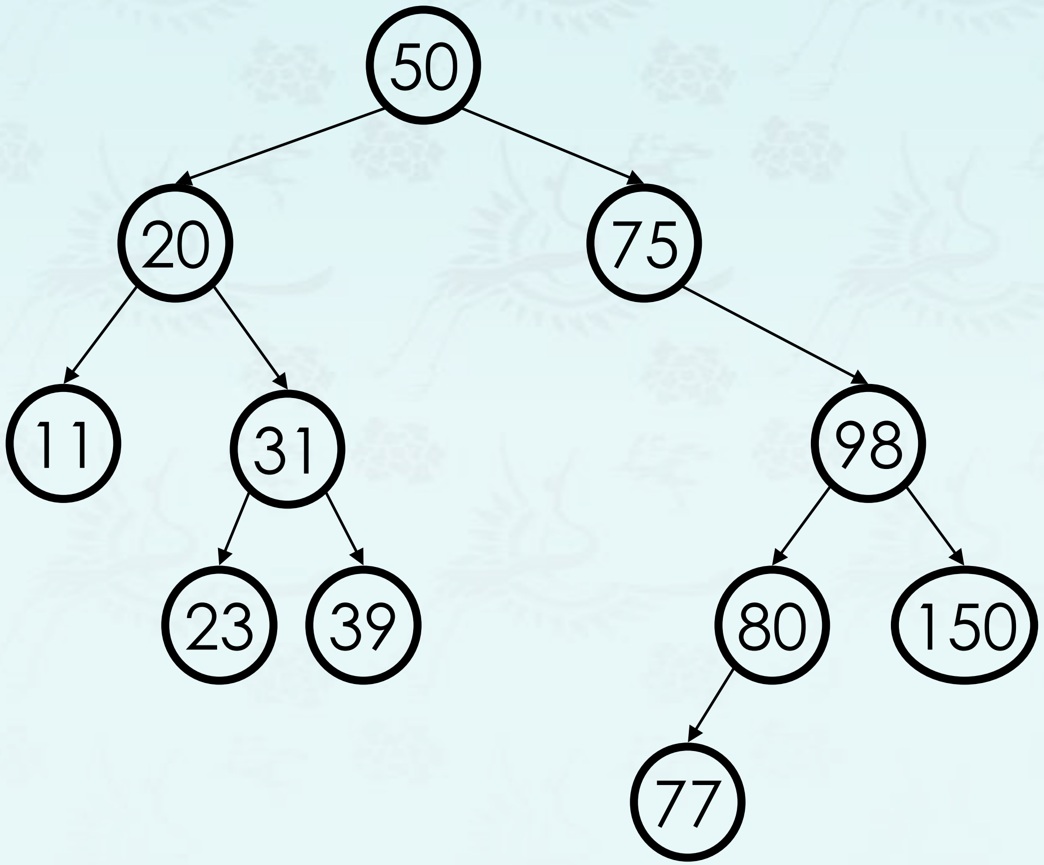
- Symmetric order means that
  - every node has a key
  - every node's key is
    larger than all keys in its left subtree
    smaller than all keys in its right subtree



## Operations: grow

- Q: Draw what a binary search tree would look like if the following values were added to an initially empty tree in this order:

50
20
75
98
80
31
150
39
23
11
77



## Operations: grow

- Q: Draw what a binary search tree would look like if the following values were added to an initially empty tree in this order:

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20
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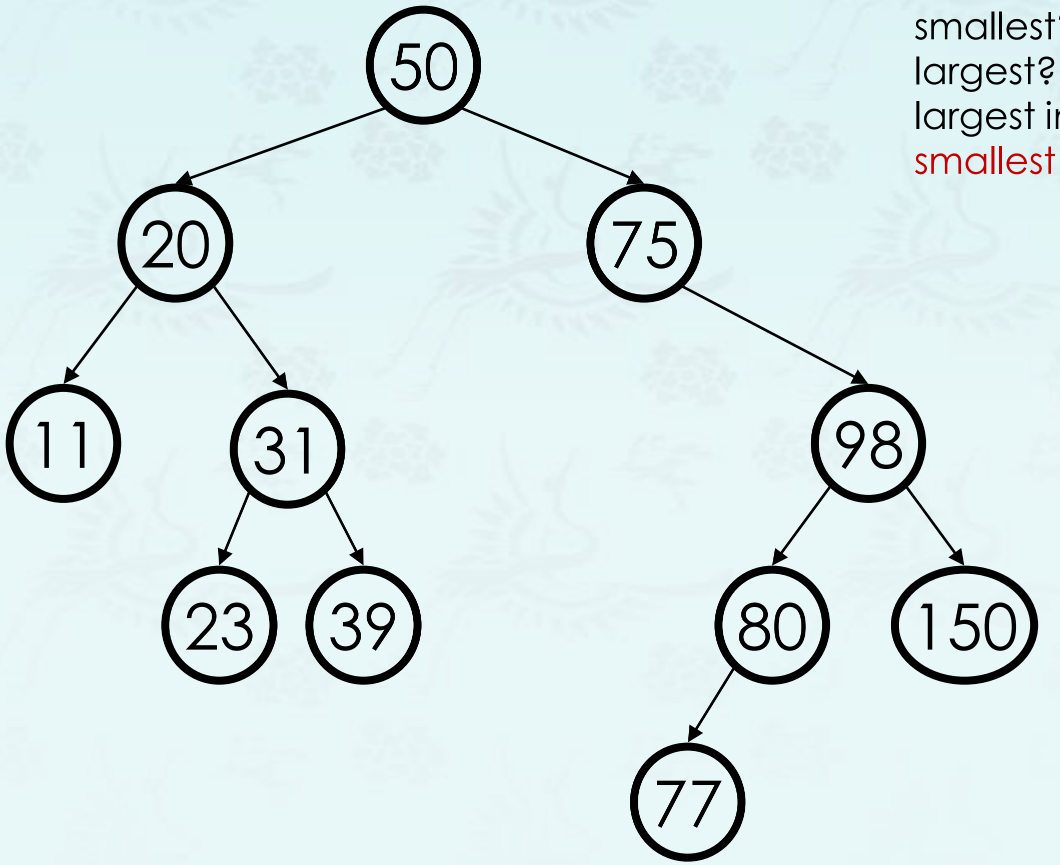
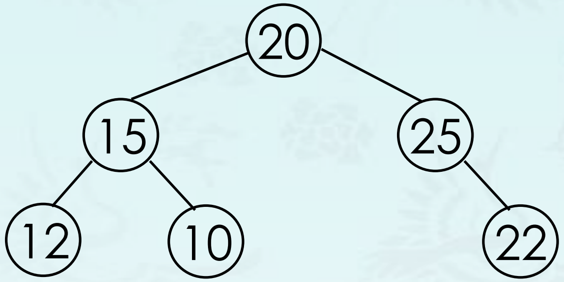


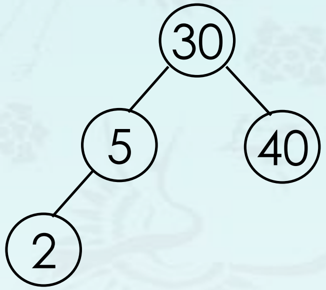

smallest?
largest?
largest in left?
smallest in right?

## Binary search trees

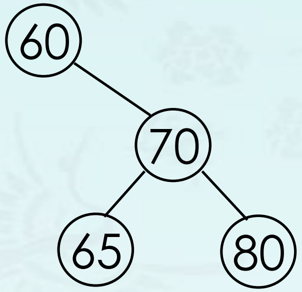
**Definition: A binary search tree is a binary tree in symmetric order.**
**Exercise: Identify non-BST(s) and correct them if not.**



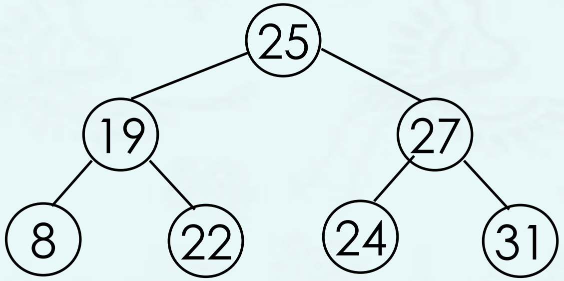

(a)




(b)



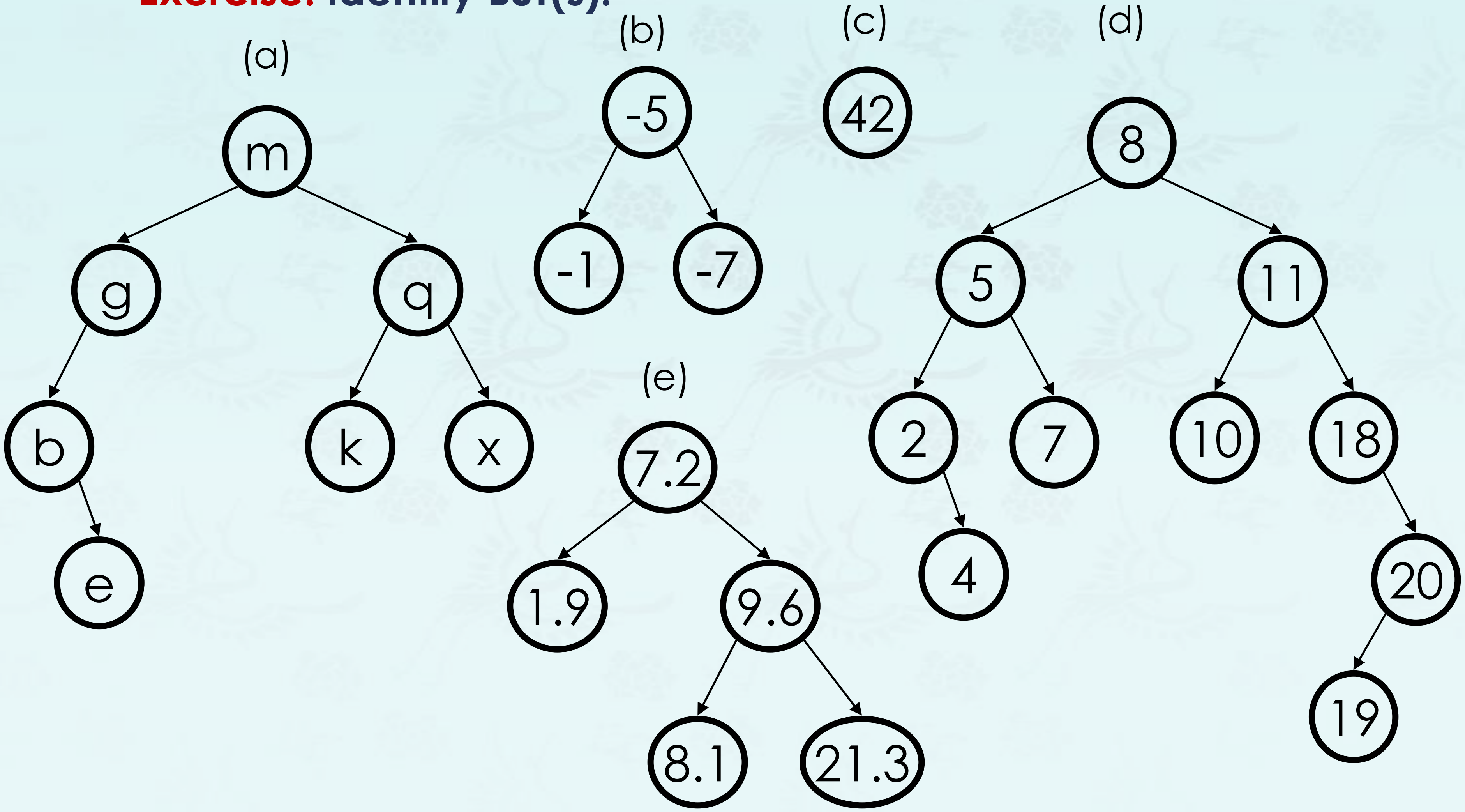

(c)

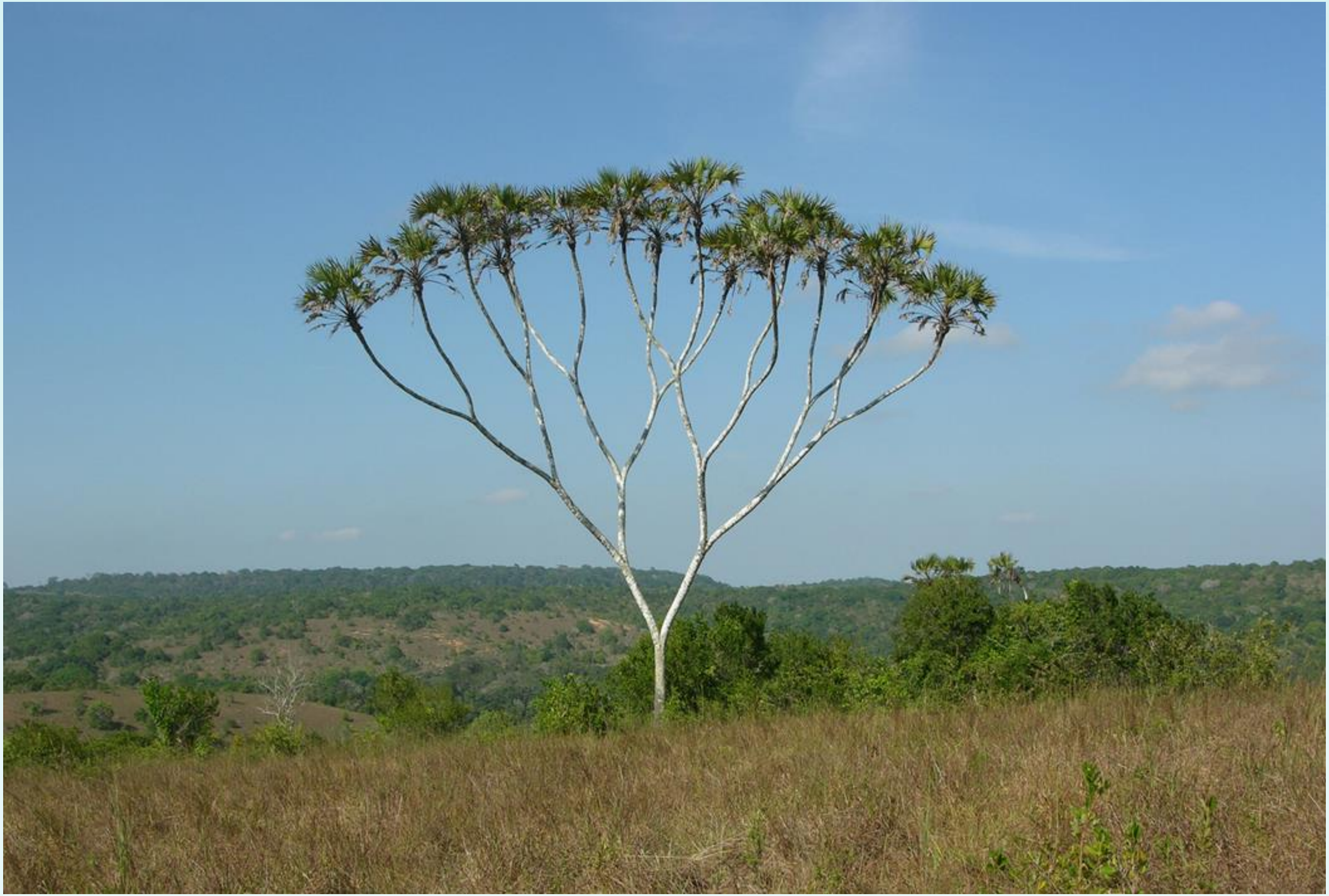



(d)

**Definition: A binary search tree is a binary tree in symmetric order.**

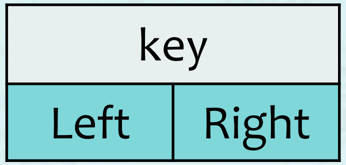
**Exercise: Identify BST(s).**





## Binary search trees

**Node structure:**




**Operations:**

- **Query – search, min/max, successor, predecessor**
- **grow – insert**
- **trim – delete**

**Binary search tree(BST) node structure:**

<table>
<tr><td colspan="2">key</td></tr>
<tr><td>tree left</td><td>tree right</td></tr>
</table>

```
struct TreeNode {
  int          key;    // key
  TreeNode*    left;   // left child
  TreeNode*    right;  // right child
};
using tree = TreeNode*;
```

# Binary search trees

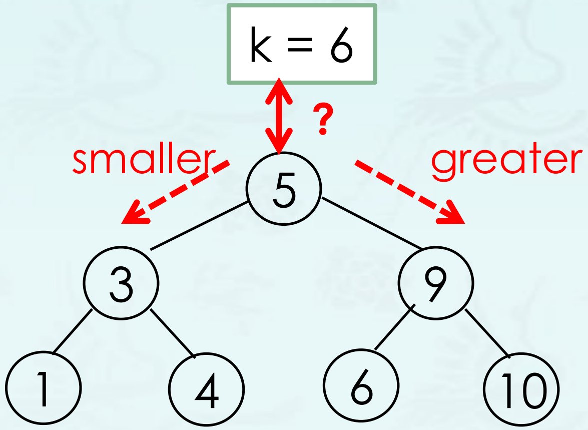
**Binary search tree(BST) node structure:**

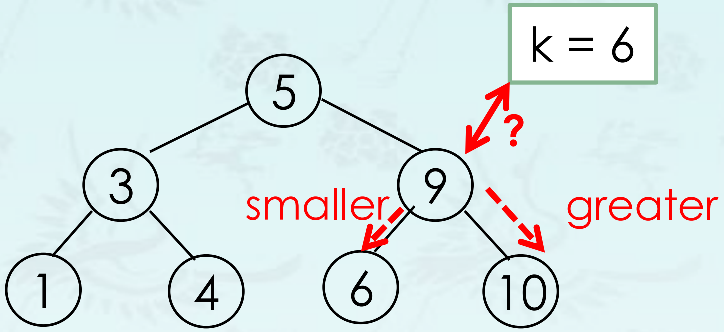
| key | |
|---|---|
| tree left | tree right |

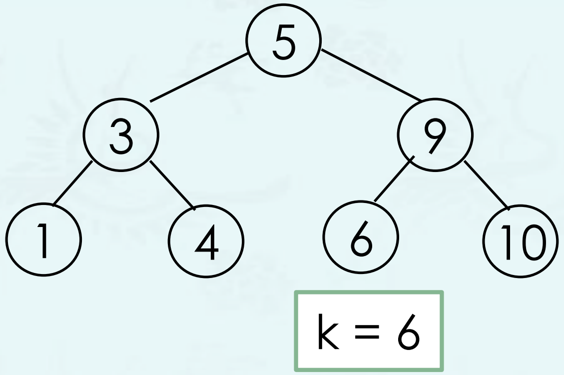
```
struct TreeNode{
  int       key;      // sorted by key
  TreeNode* left;     // left child
  TreeNode* right;    // right child
  TreeNode(int k, TreeNode* l, TreeNode* r) {
    key = k; left = l;right = r;
  }
  TreeNode(int k) : key(k), left(nullptr), right(nullptr) {}
  ~TreeNode(){}
};
using tree = TreeNode*;
```

## Operations: Search or "contains"

**Search(T, k) – search the BST, T for a key k**




❖ Search operation takes time O(h), where h is the height of a BST.

## Binary search trees

### Operations: Search or “contains”

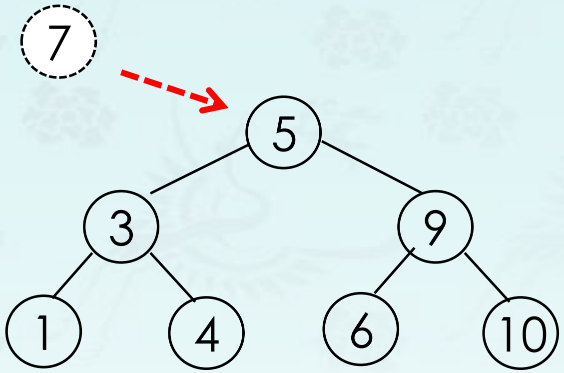
```
// does there exist a key-value pair with given key?
// search a key in binary search tree iteratively

bool containsIteration(tree node, int key)
{
    if (node == nullptr) return false;
    while (node) {
        if (key == node->key) return true;
        if (key  < node->key)
            node = node->left;
        else
            node = node->right;
    }
    return false;
}
```

## Operations: Search or "contains"

```
// does there exist a key-value pair with given key?
// search a key in binary search tree recursively

bool contains(tree node, int key)
{
    if (node == nullptr)   return false;

    if (key  == node->key) return true;

    if (key  <  node->key) return contains(node->left, key);

    return contains(node->right, key);
}
```

## Operations: grow

- grow(T, k)
  - Insert a node with Key = k into BST **T**
  - Time complexity? O(h)
- **Step 1**:
  if the tree is empty, then Root(**T**) = k
- **Step 2**:
  Pretending we are searching for k in BST, until we meet a nullptr node
- **Step 3**:
  Insert k




Q: Where is it inserted at?

## Operations: grow

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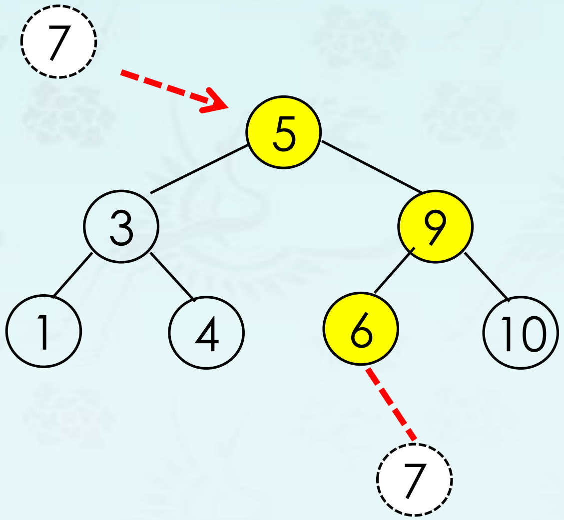


The light nodes are compared with key.

## Operations: grow

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  - Insert a node with Key = k into BST **T**
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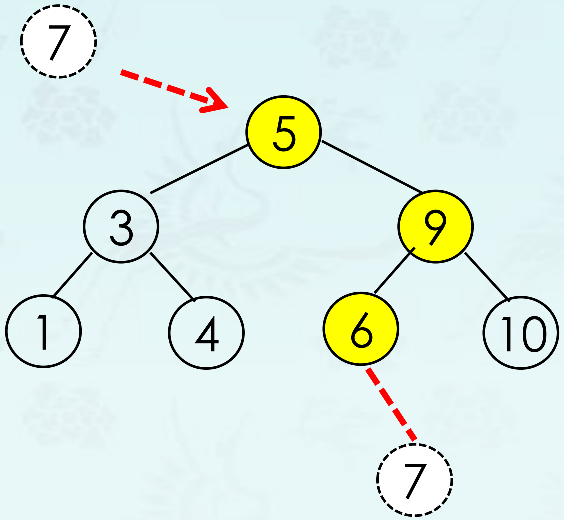


The light nodes are compared with key.

Q: Do you see the difference between the complete binary tree and binary search tree?

## grow: insert a new node with given key in BST

Something wrong?

```
tree grow(tree node, int key) {
  if (node == nullptr)
  if (key < node->key)  // recur down the tree
    grow(node->left, key);
  else
    grow(node->right, key);
  else
    cout << "grow: the same key " << key << " is ignored.\n";

  return node;
}
```

## grow: insert a new node with given key in BST

**Something wrong?**

```
tree grow(tree node, int key) {
  if (node == nullptr) return new Tree(key);
  if (key < node->key)  // recur down the tree
    grow(node->left, key);
  else
    grow(node->right, key);
  else
    cout << "grow: the same key " << key << " is ignored.\n";

  return node;
}
```

## grow: insert a new node with given key in BST

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```
tree grow(tree node, int key) {
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```

Something wrong?

## grow: insert a new node with given key in BST

Something wrong?

```
tree grow(tree node, int key) {
  if (node == nullptr) return new Tree(key);
  if (key < node->key)  // recur down the tree
    grow(node->left, key);
  else if (key > node->key)
    grow(node->right, key);
  else
    ;

  return node;
}
```

# grow: insert a new node with given key in BST

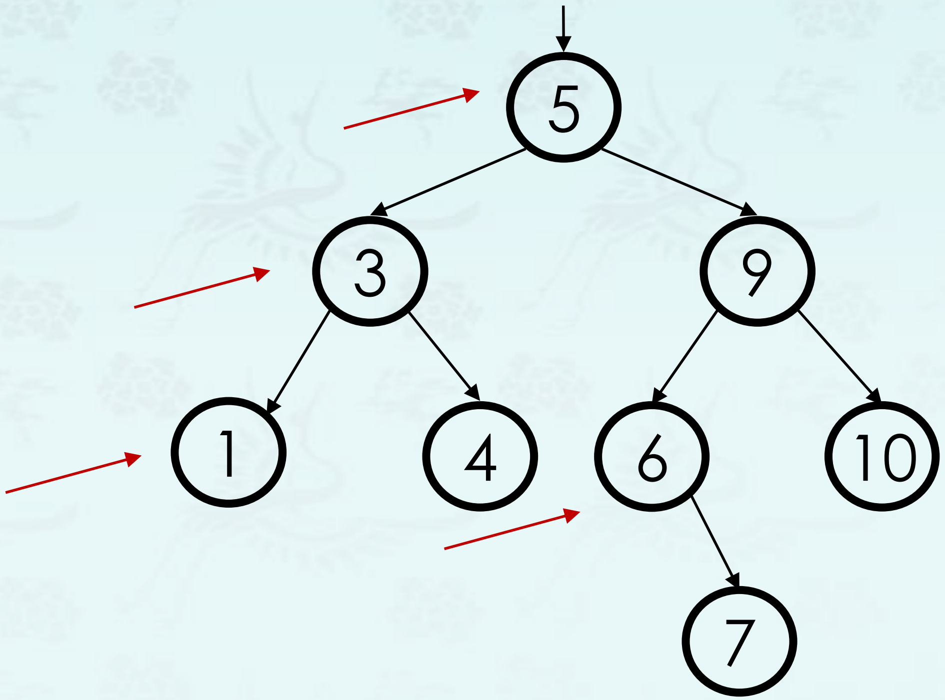
Something wrong?

```
tree grow(tree node, int key) {
  if (node == nullptr) return new Tree(key);
  if (key < node->key)  // recur down the tree
    node->left = grow(node->left, key);
  else if (key > node->key)
    node->right = grow(node->right, key);
  else
    ;

  return node;
}
```

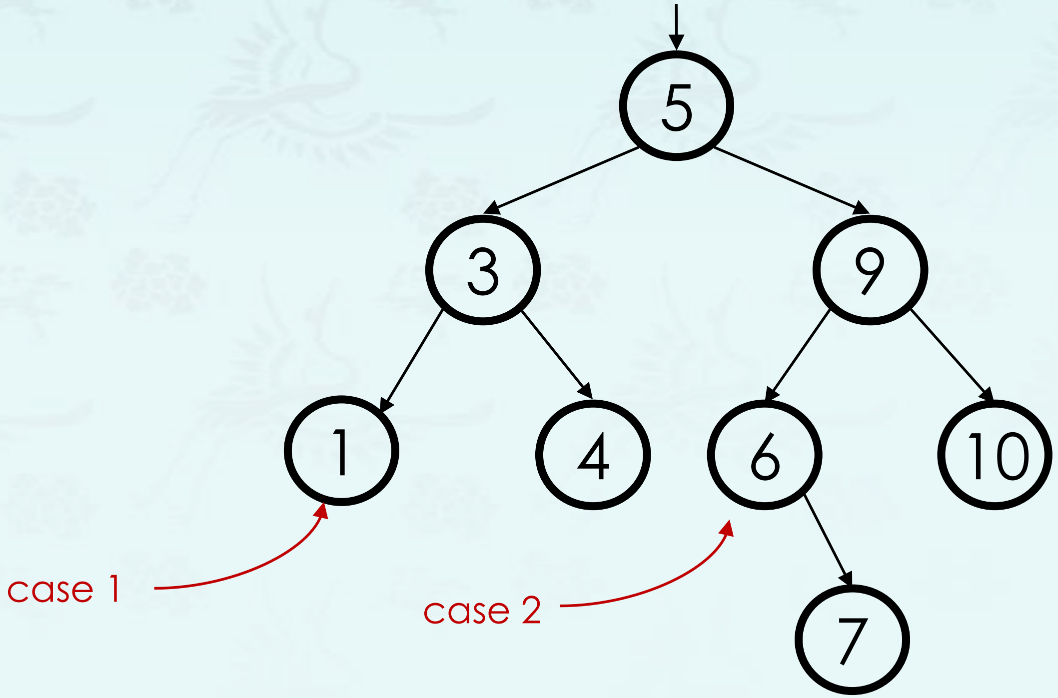
## Operations: trim

- How can we **trim** a node from a BST in such a way as to maintain proper BST ordering?
  - trim(1);
  - trim(3);
  - trim(6);
  - trim(5);



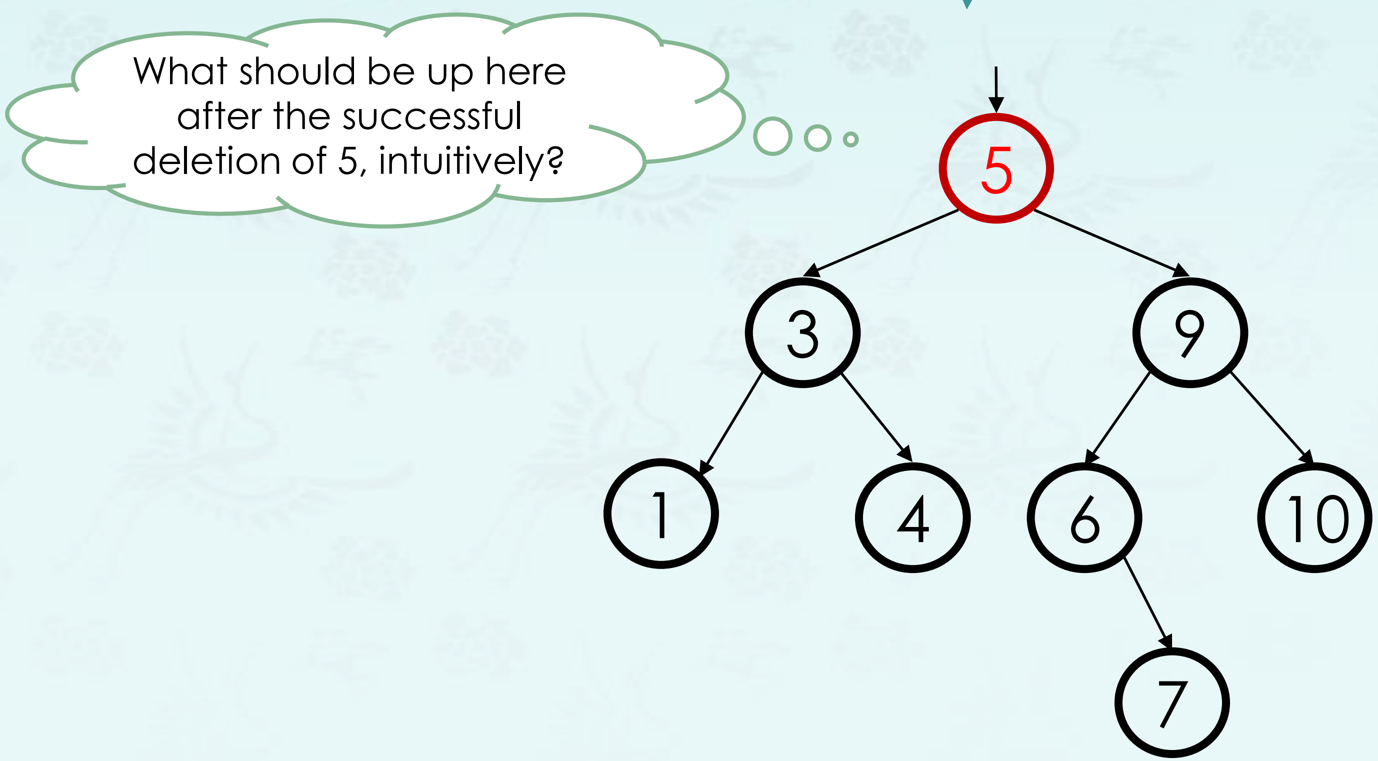
## Operations: trim

- **case 1**: leaf
  - a leaf - replace with nullptr
- **case 2**: one child case
  - a node with a left child only - replaced with left child
  - a node with a right child only - replaced with right child
- **case 3**: ?



## Operations: trim

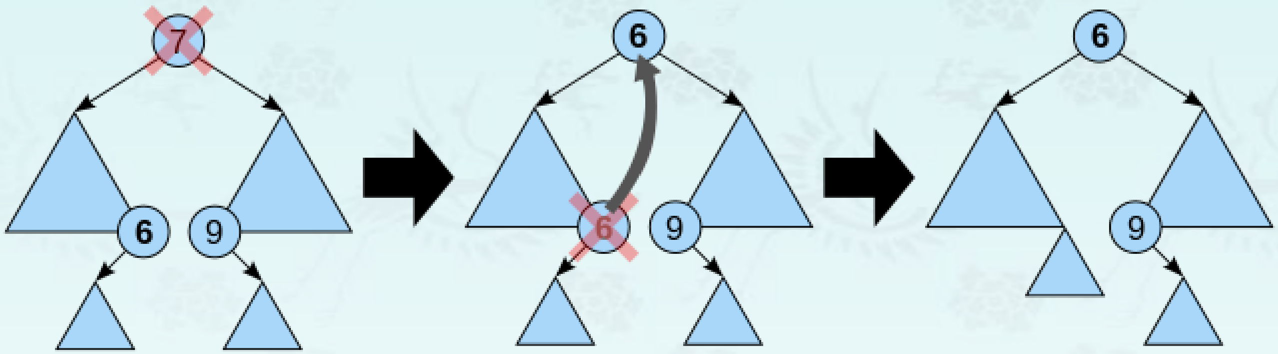
- **case 3**: two children case
  - What can we replace **5** with?



## Operations: trim

- **case 3**: two children case

  Where is predecessor or successor of root 7?




1. The rightmost node in the left subtree, the inorder **predecessor 6**, is identified.
2. Its value is copied into the node being trimmed.
3. The inorder **predecessor** can then be trimd because it has at most one child.

NOTE: The same method works symmetrically using the inorder **successor** labelled **9**.

## Operations: trim

- **case 3**: two children case

Idea: Replace the trimmed node with a value guaranteed to be between two child subtrees

Options:

- ***predecessor*** from left subtree: **maximum(** node -> left **)** 왼쪽으로 가서의 최댓값
- ***successor*** from right subtree: **minimum(** node -> right **)** 오늘쪽으로 가서의 최솟값
  - These are the easy cases of predecessor/successor

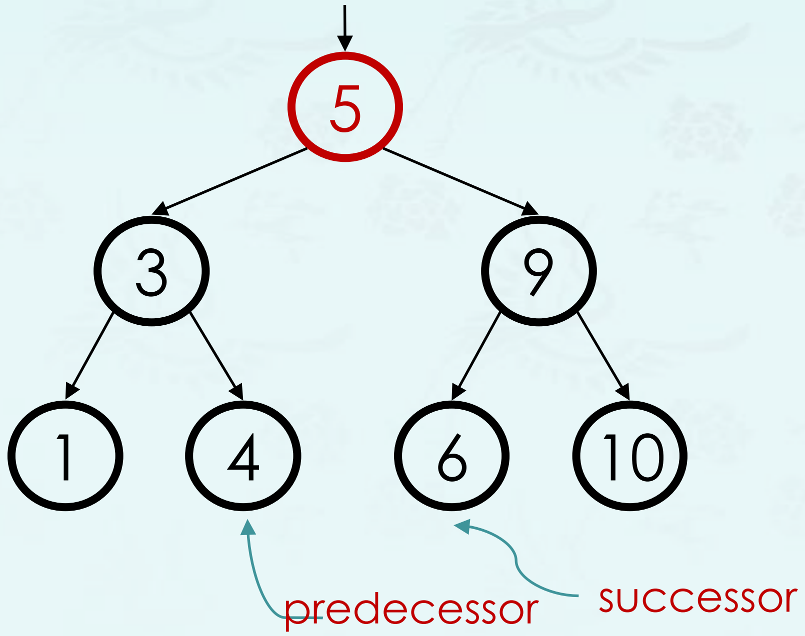
Now trim the original node containing *successor* or *predecessor*

- It becomes leaf or one child case – easy cases of trim!

## Operations: trim

- **case 3**: two children case
  - Replace with min from right or max from left
  - **Where is predecessor or successor of root 5?**

trim(5);

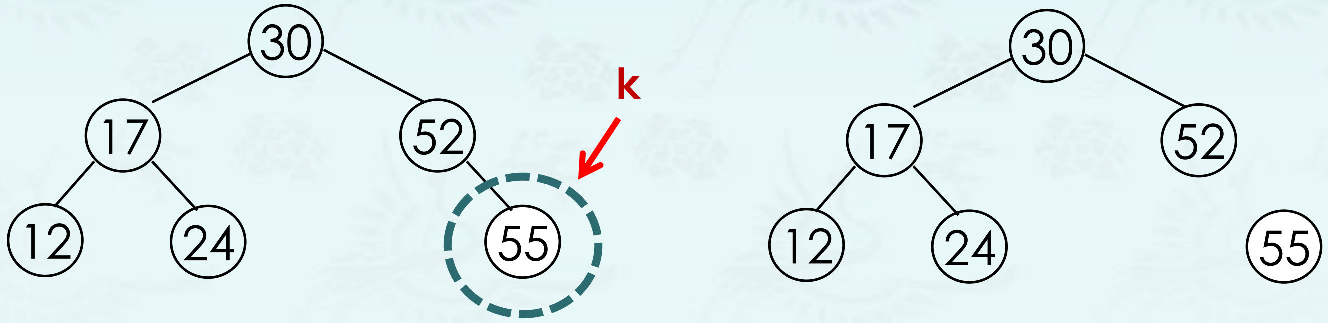


## Operations: trim

- trim(**T**, k)
  - trim a node with Key = k into BST **T**
  - Time complexity: O(h)

delete 해야할 것을 찾았을 때
자신을 삭제해주고 nullptr을 return 하면
nullptr이 저장이 될 것이다
그럼 삭제

**Case 1: k has no child**



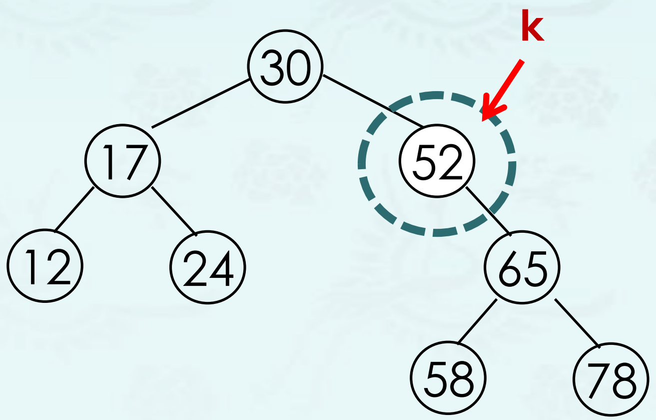

We can simply trim **55** from the tree.
1. delete **55**
2. **52→**right = nulltptr

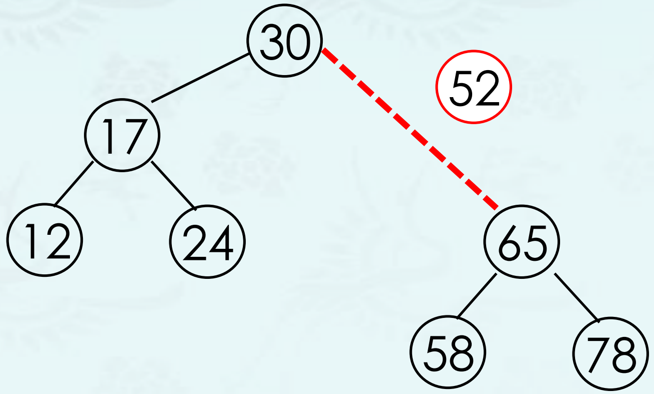
How?

## Operations: trim

- trim(**T**, k)
  - trim a node with Key = k into BST **T**
  - Time complexity: O(h)

**Case 2: k has one child**




After removing it, connect it's subtree to it's parent node.

How?

## Operations: trim

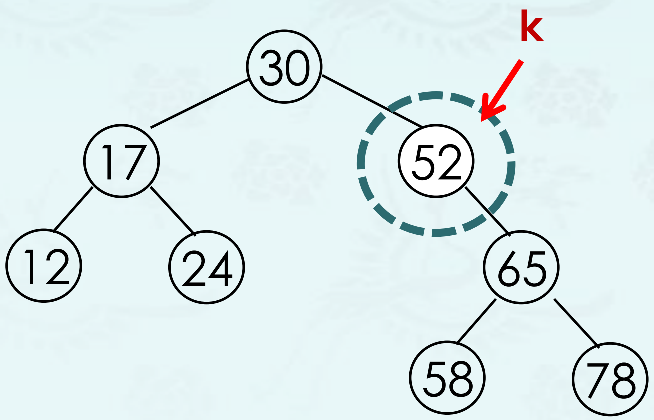
맨처음에 delete 할 주소 값을 먼저 저장한 뒤
그리고 delete 하는 방식

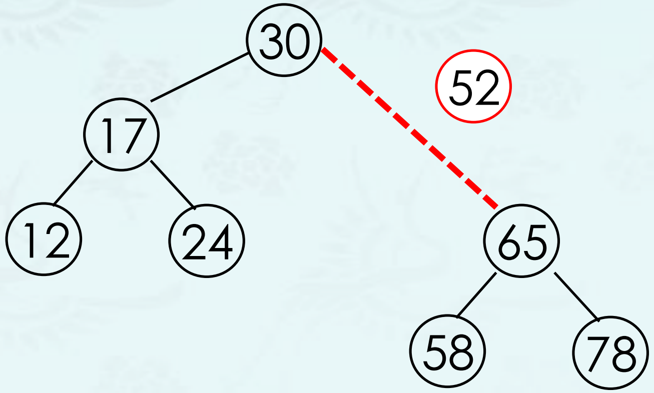
- trim(**T**, k)
  - trim a node with Key = k into BST **T**
  - Time complexity: O(h)

root = trim(root, 52)   // in main()

// in trim(root, key)
root→right = trim(root→right, 52)

**Case 2: k has one child**




1. delete 52
2. return 52→right

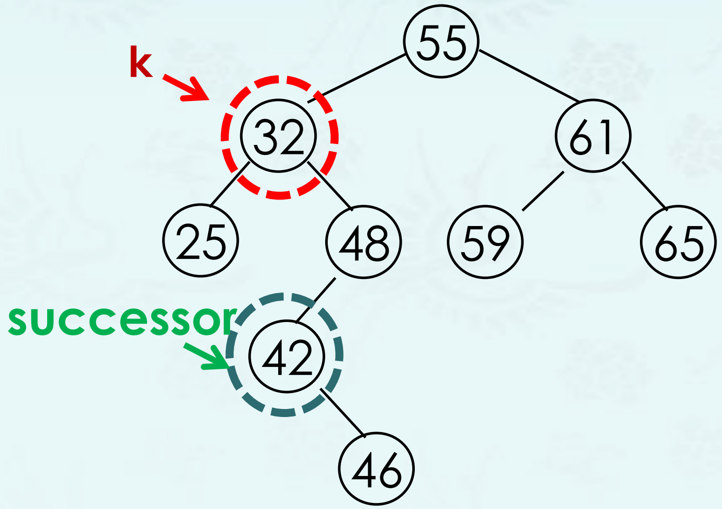
Don't forget to save 52→right before delete 52

## Operations: trim

- trim(**T**, k)
  - trim a node with Key = k into BST **T**
  - Time complexity: O(h)

맨처음에 successor을 찾는다
그리고 k의 key값과 successor의 key값
을 바꾼다
그리고 successor 같은 경우 반드시
child가 없거나 하나 밖에 없다
그렇기 때문에 trim하기 편함

**Case 3: k has two children**

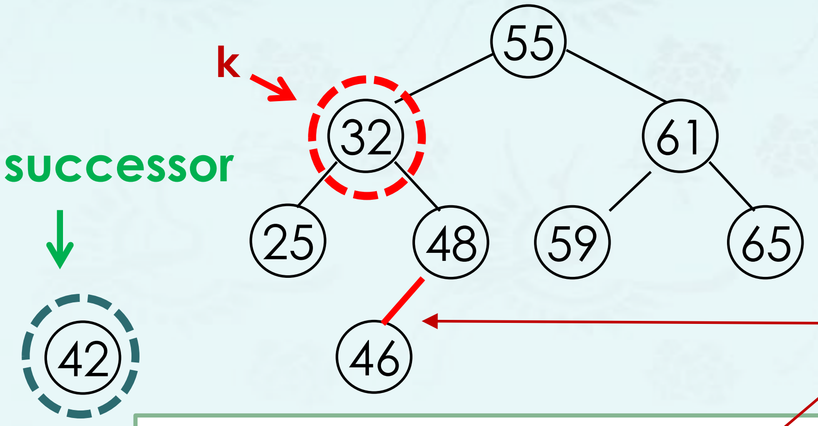



```
1. found the node 32
2. find the successor and its key 42.
3. replace node→32 with 42.
4. node->right = trim(node->right, 42);
```

## Operations: trim

- trim(**T**, k)
  - trim a node with Key = k into BST **T**
  - Time complexity: O(h)

**Case 3: k has two children**




```
int succ = value(minimum(root->right));
root->key = succ;
root->right = trim(root->right, succ);
```

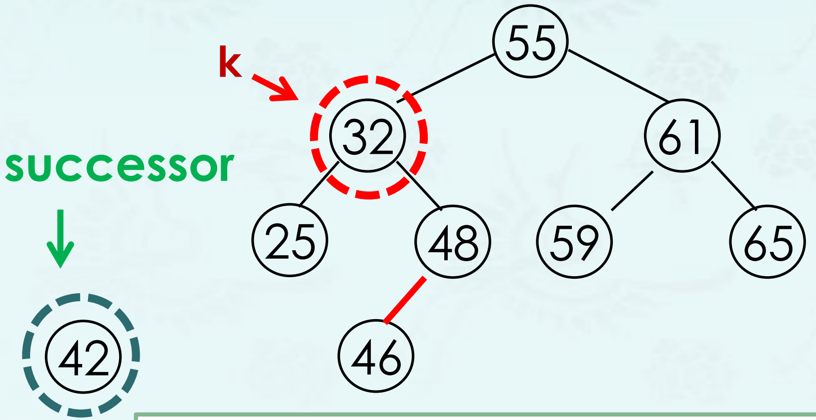
This is done by calling another trim() with succ key, recursively.

Pull out successor,
and connect the tree with it's child

## Operations: trim

- trim(**T**, k)
    - trim a node with Key = k into BST **T**
    - Time complexity: O(h)

**Case 3: k has two children**




Pull out successor,
and connect the tree with it's child

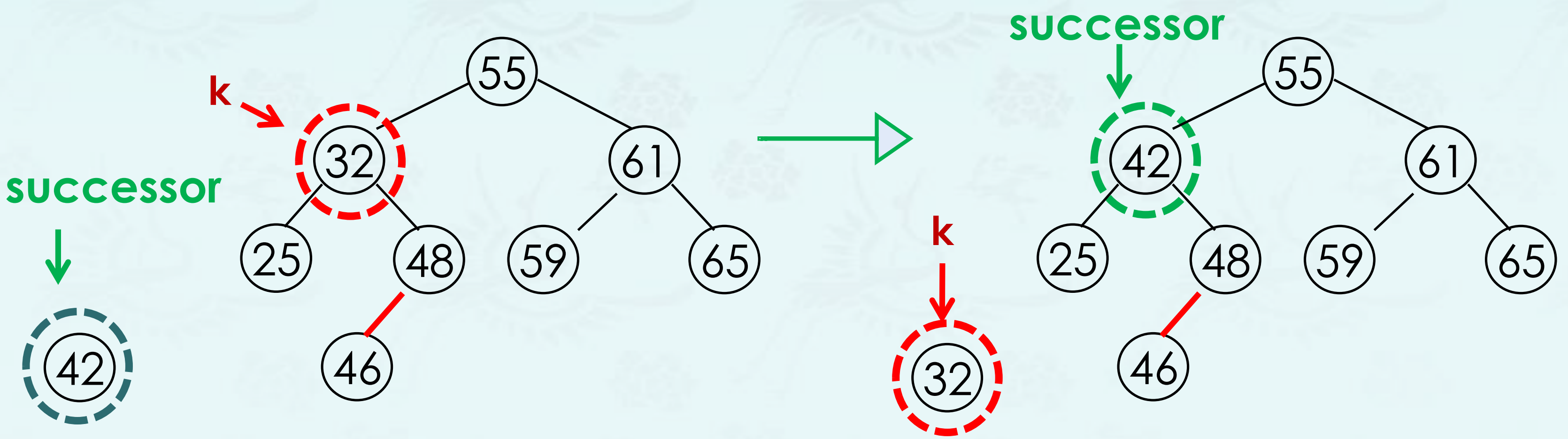
**A: Not possible !**
Because if it has two nodes, at least one of them is less than it, then in the process of finding successor, we won't pick it !

**Q:** What if successor has **two** children?

## Operations: trim

- trim(**T**, k)
  - trim a node with Key = k into BST **T**
  - Time complexity: O(h)

**Case 3: k has two children**




Replace the **key** with it's successor

**More Operations:**

- **Query – search, min/max, successor, predecessor**

**Min/max**

- For min, we simply follow the left pointer until we find a nullptr node. Why?
- Similar for Max
- Time complexity: O(h)

❖ Search operation takes time O(h), where h is the height of a BST.

# Binary Search Tree

- *Recursion Revisited*
- *binary search tree Implementation*
    - *size*
    - *height*
    - *traversal - inorder, preorder, postorder, levelorder*
    - *minimum, maximum,*
    - *predecessor, successor*

## bunnyEars(): counting bunny ears in recursion

```
// each bunny has two ears.
int bunnyEars(int bunnies) {

  return 2 + bunnyEars(bunnies-1);
}
```

## funnyEars(): counting funny ears in recursion

```
// even numbered funny has two ears, odd numbered funny three.
int funnyEars(int funnies) {
  if (bunnies == 0) return 0;

  if (funnies % 2 == 0)
    return
  else
    return
}
```

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  if (bunnies == 0) return 0;

  if (funnies % 2 == 0)
    return 2 + funnyEars(funnies - 1);
  else
    return 3 + funnyEars(funnies - 1);
}
```

## size(): in doubly linked list

```
int size(pList p) {
  int count = 0;
  for (pNode c = begin(p); c != end(p); c = c->next)
    count++;
  return count;
}
```

## size(): in singly linked list

```
int size(pNode node) {
  if (node->next == nullptr) return 0;
  return 1 + size(node->next);
}
```

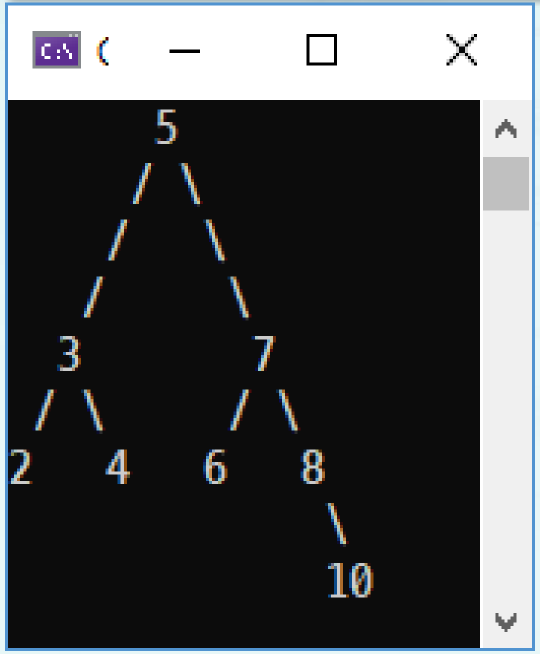
## Binary search trees

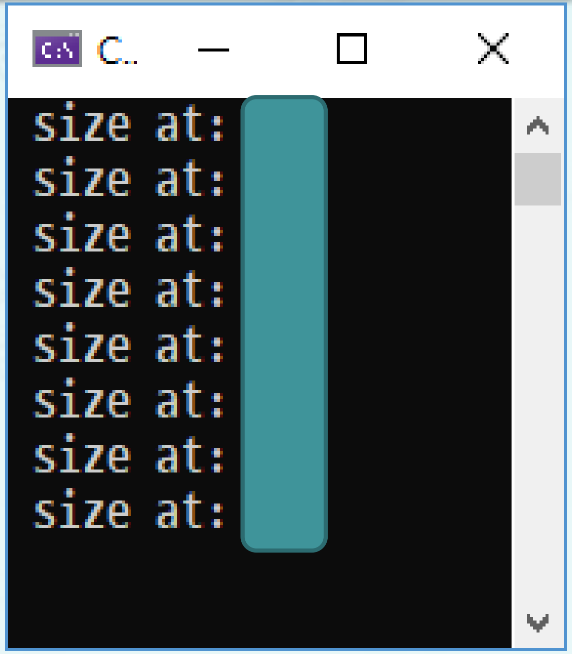
**size:** Count the number of nodes in the binary tree recursively.

```
int size(tree node) {
  if (node == nullptr) return 0;
  return 1 + size(node->left) + size(node->right);
}
```

**size:** Count the number of nodes in the binary tree recursively.

```
int size(tree node) {
  if (node == nullptr) return 0;
  cout << " size at: " << node->key << endl;
  return 1 + size(node->left) + size(node->right);
}
```



# Binary search trees

**size:** Count the number of nodes in the binary tree recursively.

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```
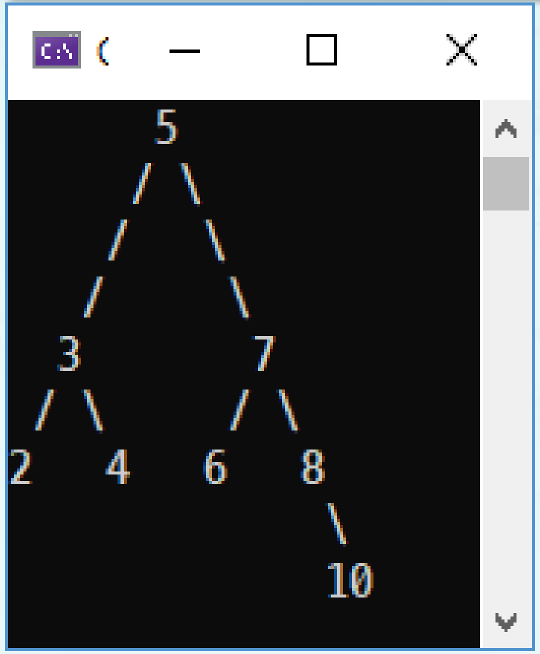
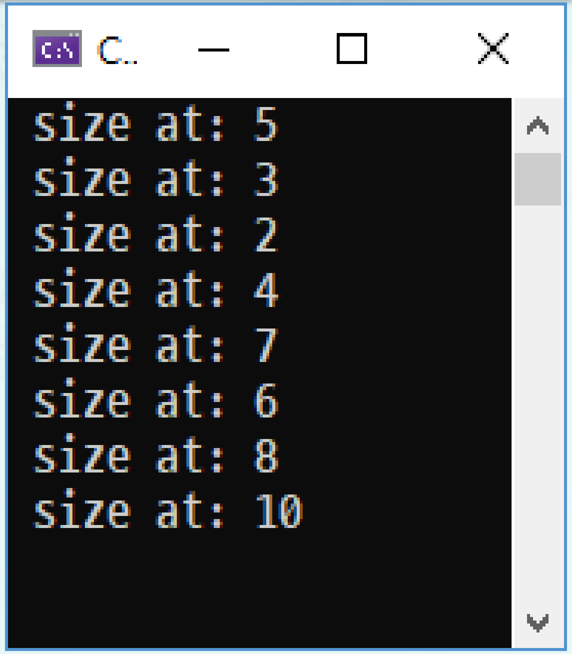



**size:** Count the number of nodes in the binary tree recursively.

```
int size(tree node) {
  if (node == nullptr) return 0;
  return 1 + size(node->left) + size(node->right);
}
```

**height:** compute the max height(or depth) of a tree.
// It is the number of nodes along the longest path from the root node
// down to the farthest leaf node.

```
int height(tree node) {


}
```

# inorder traversal: do inorder traversal of BST.

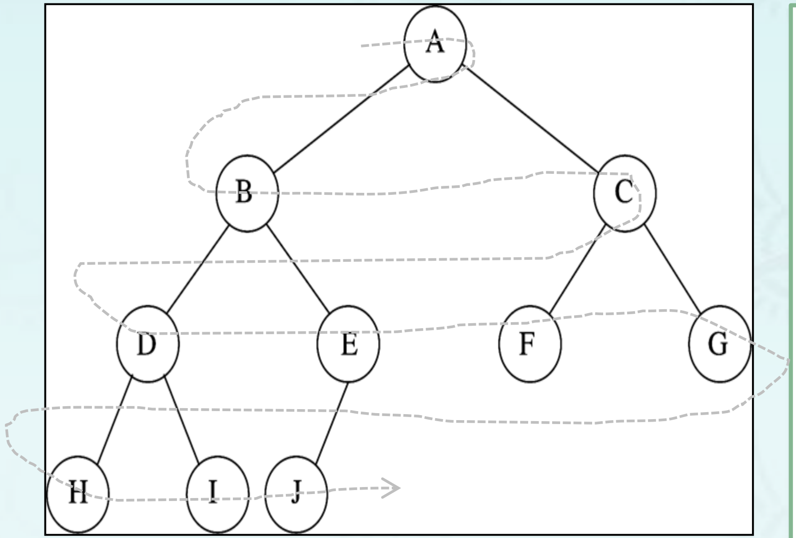
```
void inorder(tree node) {
    if (node == nullptr) return;

    inorder(node->left);
    cout << node->key;
    inorder(node->right);
}
```

```
void inorder(tree node, vector<int>& vec) {
  if (node == nullptr) return;

  inorder(node->left, vec);
  _
  inorder(node->right, vec);
}
```

```
case 'l':
  cout << "\n\tinorder:    ";
  vec.clear();
  inorder(root, vec);
  for (int i : vec)
    cout << i << " ";
```

## Level-order traversal

1. **Depth first** search(DFS) – preorder, inorder, postorder traversal
2. **Breadth first** search(BFS) - **level-order** traversal




```
#include <queue>
#include <vector>

void levelorder(tree root, vector<int>& vec)
•  Visit the root. if it is not null, enqueue it.
•  while queue is not empty
    1. que.front() – get the node in the queue
    2. save the key in vec.
    3. if its left child is not null, enqueue it.
    4. if its right child is not null, enqueue it.
    5. que.pop() – remove the node in the queue.
```

# Level-order traversal

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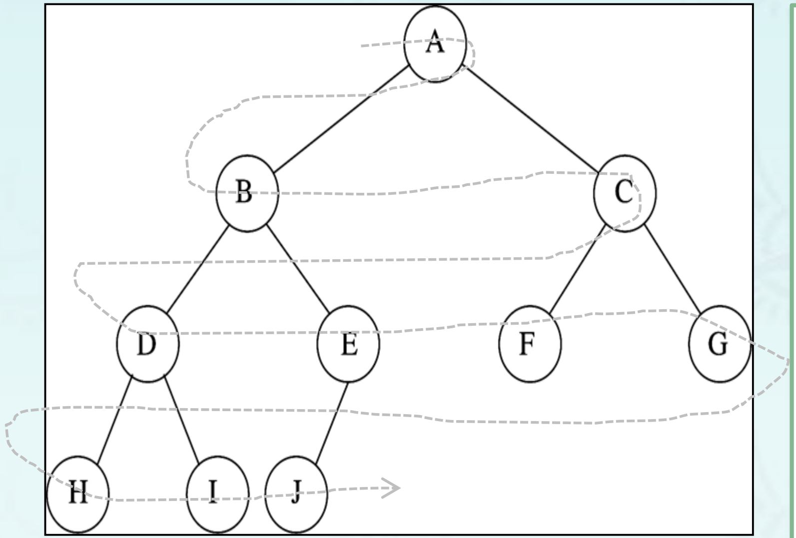



```
#include <queue>
#include <vector>

void levelorder(tree root, vector<int>& vec) {
  queue<tree> que;
  if (!root) return;
  que.push(root);
  while ...{


     cout << "your code here\n";


  }
}
```

## minimum, maximum:

returns the node with min or max key.
Note that the entire tree does not need to be searched.

```
tree minimum(tree node) {   // returns left-most node key


}
```

```
tree maximum(tree node) {   // returns right-most node key


}
```

## pred(), succ() – predecessor, successor:

```
Input: root node, key
output: predecessor node, successor node

1. If root is nullptr, then return

2. if key is found then
    a. If its left subtree is not nullptr
        Then predecessor will be the right most
        child of left subtree or left child itself.
    b. If its right subtree is not nullptr
        The successor will be the left most child
        of right subtree or right child itself.
    return
```

## trim**: trim node with the key and return the new root.

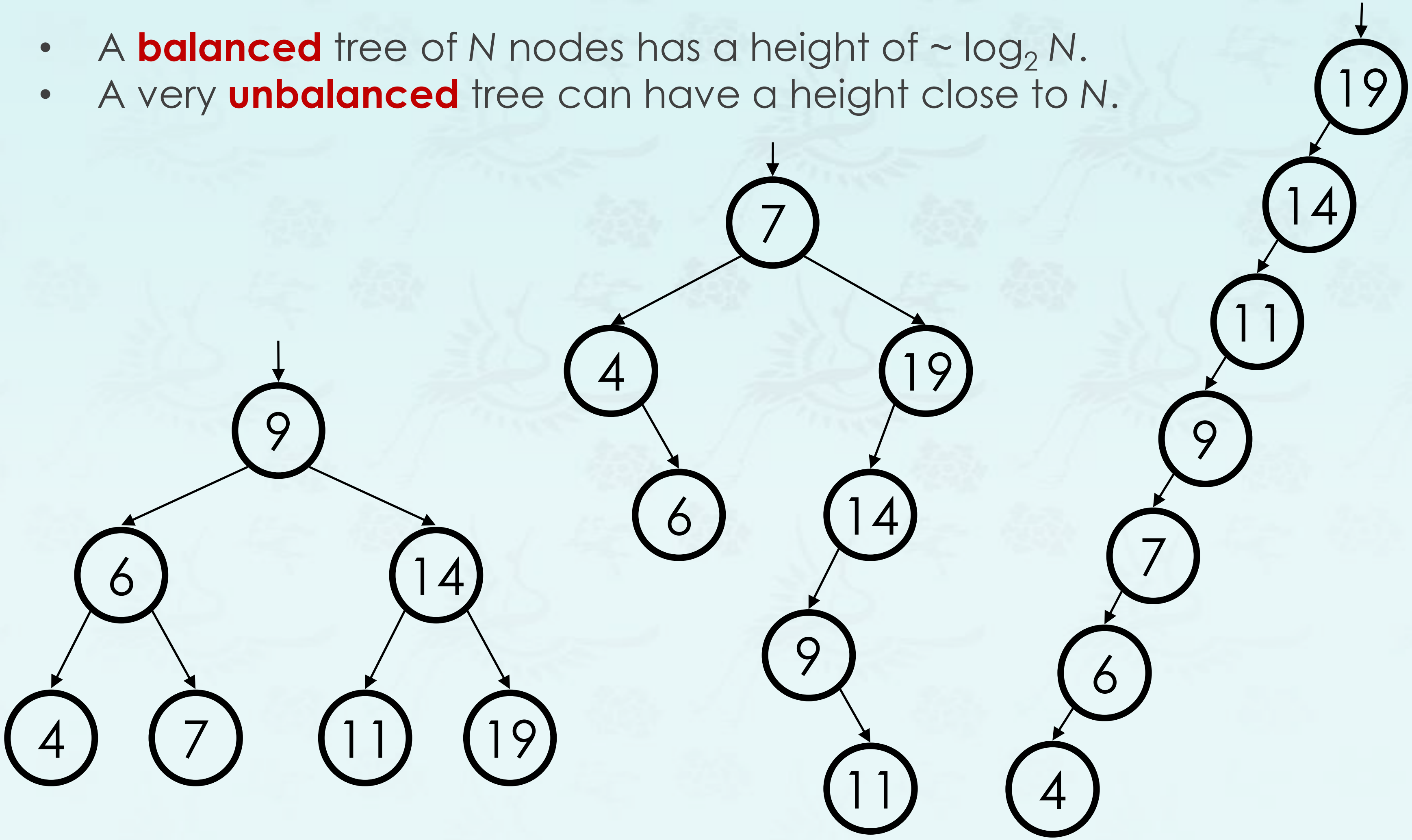
```
tree trim(tree node, int key) {
  if (node == nullptr) return node;// base case
  if (key < node->key)
    node->left = trim(node->left, key);
  else if (key > node->key) {
    node->right = trim(node->right, key);
  else {
    if (node->left == nullptr) {
       // your code here - trim the right one, return node
    }
    else if (node->right == nullptr) {
       // your code here - trim the left one, return node
    }
    else {// two children case
      // get the successor: smallest in right subtree
      // copy the successor's content to this "node" node
      // node->right = trim the successor recursively, which has one or no child case
    }
  }
  return node;
}
```

## Binary search trees

http://visualgo.net/bst

## Observations: What do you see in the following BSTs?

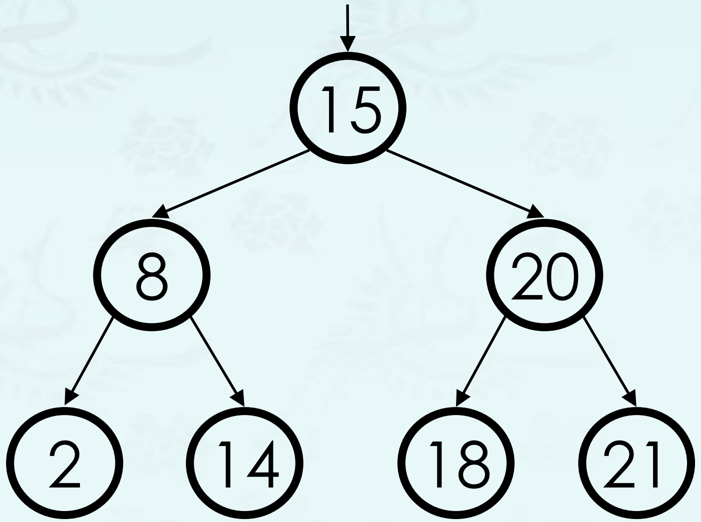
- A **balanced** tree of $N$ nodes has a height of ~ $\log_2 N$.
- A very **unbalanced** tree can have a height close to $N$.



## Observations: What do you see in the following BSTs?

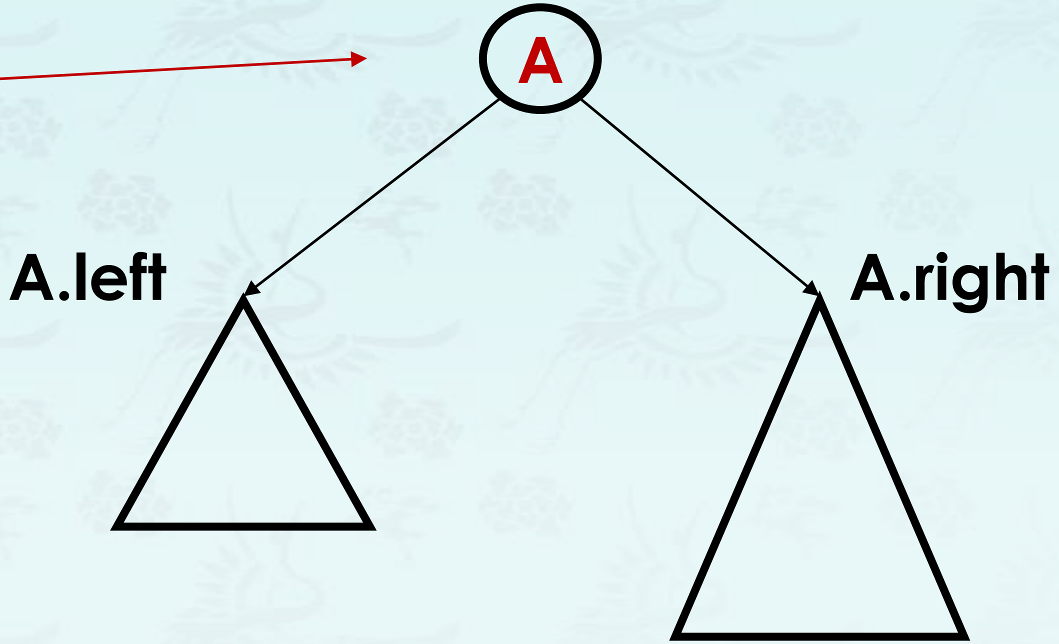
- *Observation:* The shallower the BST the better.
  - Average case height is O(log $N$)
  - Worst case height is O($N$)
  - Simple cases such as adding (1, 2, 3, ..., $N$), or the opposite order, lead to the worst case scenario: height O($N$).

- For binary tree of height $h$:
  - max # of leaves: $2^{h-1}$
  - max # of nodes: $2^h - 1$
  - min # of leaves: 1
  - min # of nodes: $h$



**Q: Calculate tree height.**

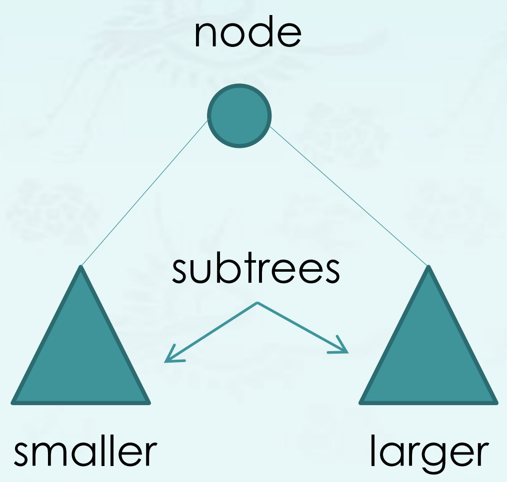
- ***Height*** *is* max number of nodes in path from root to any leaf.
  - height(nullptr) = 0
  - height(a leaf) = ?
  - height( **A** ) = ?
- ***Hint***:
  - use recursive.
  - use max(a, b).




- ***A:***
  - height(a leaf) = 1
  - height(A) = 1 + max(

## Conclusion:

- If you have a sorted sequence, and we want to design a data structure for it
- **Array or BST? and why?**



# Conclusion:

- If you have a sorted sequence, and we want to design a data structure for it
- **Array or BST? and why?**

| Time Complexity | |
|---|---|
| BST | $O(h)$ |
| Array | $O(\log n)$ |

## Conclusion:

**Q.** When searching, we're traversing a path (since we're always moving to one of the children); since the length of the longest path is the height h of the binary search tree, then finding an element takes $O(h)$.

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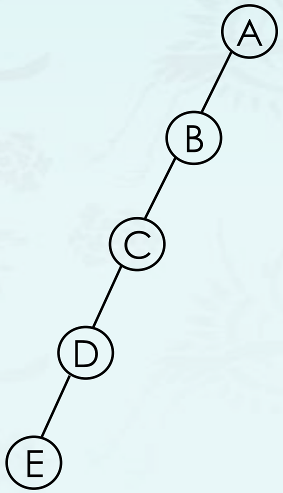
**A. The nodes could be arranged in linear sequence in BST, so the $\boldsymbol{height}$ $\boldsymbol{h}$ could be $\boldsymbol{n}$. In worst case, it is $\boldsymbol{O(n)}$ instead of $\boldsymbol{O(h)}$.**

# Conclusion:

- We already know that $n$ is fixed, but $h$ differs from how we insert those elements!
- So why we still need BST?
  - Easier insertion and deletion
  - And with some optimization, we can avoid the worst case!




$n = h$

a skew binary search tree

## Binary search trees

1. trim
   https://www.youtube.com/watch?v=gcULXE7ViZw

2. inorder
   https://www.youtube.com/watch?v=5cPbNCrdotA

3. binary search tree
   https://www.youtube.com/watch?v=pYT9F8_LFTM
   https://www.youtube.com/watch?v=COZK7NATh4k