10.007 Modelling the Systems World

Week 2 class 1: Gauss-Jordan Elimination

Term 3, 2018

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Augmented matrix

Definition: the *augmented matrix* of the linear system $A\vec{x} = \vec{b}$ is the matrix $[A \;\; \vec{b}]$, also denoted by $[A \,|\, \vec{b}]$.

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In Example 1: for the system

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 0 \\ -2 & 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix},$$

the augmented matrix is

$$[A \,|\, \vec{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & 2 & 0 & -2 \\ -2 & 7 & 4 & 1 \end{array}\right].$$

We denote the ith row of the augmented matrix by R_i.

When solving a linear system using matrices, the first goal is to use *elementary row operations* to bring the augmented matrix into a 'triangular', or more generally 'staircase' form:

$$\begin{bmatrix} * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reduction

When solving a linear system using matrices, the first goal is to use *elementary row operations* to bring the augmented matrix into a 'triangular', or more generally 'staircase' form:

$$\begin{bmatrix} * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- We call this a **row echelon form** (definition on next slide).
- The process of bringing a matrix into row echelon form is called *row reduction*.
- The first non-zero entry of each non-zero row is called the leading entry.

Row echelon form

Definition: a matrix is in *row echelon form* (ref) if

- Any rows of zeros are at the bottom.
- In each *non-zero row* (that is, a row with at least one non-zero entry), the leading entry is to the right of any leading entries above it.

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Gauss-Jordan elimination then further simplifies the matrix into **reduced row echelon form**.

Definition: a matrix is in *reduced row echelon form* (rref) if

- It is in row echelon form.
- The leading entry in each non-zero row is 1.
- Each column containing a leading 1 has zeros everywhere else.

Activity 1 (10 minutes)

Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & -2 & 3 & 1 \end{bmatrix}.$$

(a) Convert A first to row echelon form, then to reduced row echelon form.

(b) Swap the first and third rows of A, then repeat part (a).

(c) What do you notice? What is the rank of A?

Recall that the *rank* of a matrix is the number of non-zero rows in any of its row echelon forms.

Activity 1 (solution)

(a) $$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & -2 & 3 & 1 \end{bmatrix}$$

$$\Big\downarrow \begin{smallmatrix} R_3-3R_1 \\ R_3-R_2 \end{smallmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow R_1+R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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(b) $$\begin{bmatrix} 3 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

$$\Big\downarrow \begin{matrix} R_3 - \frac{1}{3}R_1 \\ R_3 + \frac{1}{3}R_2 \end{matrix}$$

$$\begin{bmatrix} 3 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow \begin{matrix} R_1 + 2R_2 \\ \frac{1}{3}R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Activity 1 (solution)

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$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) The ref's (red) are different, but the rref's (blue) are the same, as guaranteed by a theorem stated in the lecture. rank$(A) = 2$.

Activity 2 (15 minutes)

Imagine yourself in space, piloting a space pod. The pod is stationary, and with its current location as the origin, you would like to dock with the mother ship located at $(4, 10, 17)$.

You have 3 thrusters at your control. For each second you fire thruster A, the pod moves by $[1\ 2\ 3]$; for thruster B, the pod moves by $[1\ 3\ 6]$, while thruster C moves the pod by $[2\ 6\ 10]$.

(a) Set up a linear system for finding how many seconds each thruster needs to be fired to move the pod to the ship.

(b) Use Gauss-Jordan elimination to solve the system.

Activity 2 (solution)

(a) Let s_1, s_2, s_3 be the time in seconds for which each thruster is fired. With some care, the problem can be written in the form

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 6 & 10 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 17 \end{bmatrix}.$$

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(b) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & 3 & 6 & 10 \\ 3 & 6 & 10 & 17 \end{array}\right].$$

After the elimination steps $R_2 - 2R_1$, $R_3 - 3R_1$ and $R_3 - 3R_2$, we obtain a 'staircase' matrix (ref):

Activity 2 (solution, continued)

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -1 \end{array}\right].$$

Activity 2 (solution, continued)

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -1 \end{array}\right].$$

We continue the elimination with $R_2 + R_3$, $R_1 + R_3$, $R_1 - R_2$, and finally $-\frac{1}{2}\,R_3$.

Activity 2 (solution, continued)

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We continue the elimination with $R_2 + R_3$, $R_1 + R_3$, $R_1 - R_2$, and finally $-\frac{1}{2}\,R_3$.

The resulting matrix (rref) is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$$

Hence, thruster A needs to be fired for 2 seconds, thruster B for 1 second and thruster C for $\frac{1}{2}$ second (in any order).

Activity 2 (solution, continued)

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Hence, thruster A needs to be fired for 2 seconds, thruster B for 1 second and thruster C for $\frac{1}{2}$ second (in any order).

Extension: if the mother ship was at $(3, 10, 19)$, could you reach it?

Harder example

Gauss-Jordan elimination allows us to write down the general solution to any linear system, including singular systems.

Example 2

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \quad \Longrightarrow \\ -w + x - y &= -3 \end{aligned}$$

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Example 2

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y &= -3 \end{aligned} \implies$$

Augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \implies$$

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$$\boxed{R_2 - 2R_1,\ R_3 + R_1}$$

$$\left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right] \Longrightarrow$$

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$\boxed{R_2 - 2R_1,\ R_3 + R_1}$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right] \implies$$

$\boxed{R_3 + 2R_2}$

Row echelon form:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies$$

Leading and free variables

$\boxed{R_1 + R_2}$

Reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

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$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies$$

Simplified system:

$$\begin{aligned} w - x \quad + z &= 2 \\ y - z &= 1 \end{aligned} \quad \text{or}$$

$$\boxed{w = 2 + x - z, \quad y = 1 + z}$$

Leading and free variables

$\boxed{R_1 + R_2}$

Reduced row echelon form:

$$\left[\begin{array}{cccc|c} {\color{blue}1} & -1 & 0 & 1 & 2 \\ 0 & 0 & {\color{blue}1} & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies$$

Simplified system:

$$\begin{aligned} {\color{blue}w} - {\color{red}x} \quad + {\color{red}z} &= 2 \\ {\color{blue}y} - {\color{red}z} &= 1 \quad \text{or} \end{aligned}$$

$$\boxed{{\color{blue}w} = 2 + {\color{red}x} - {\color{red}z}, \quad {\color{blue}y} = 1 + {\color{red}z}}$$

w and y correspond to the leading entries of the rref; they are called leading variables.

Leading and free variables

$\boxed{R_1 + R_2}$

Reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

Simplified system:

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w and y correspond to the leading entries of the rref; they are called leading variables.

There are many ways to write the general solution. Since each leading variable appears in precisely *one* equation, we choose to *express the leading variables in terms of the other variables*.

The other variables, x and z, are free to take any value; they are called free variables.

General solution

We assign parameters to the free variables: $x = s$ and $z = t$, where $s,\ t \in \mathbb{R}$. Then $y = 1 + t$ and $w = 2 + s - t$.

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The general solution can be written as

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Activity 3 (15 minutes)

Use Gauss-Jordan elimination to solve the following linear systems.

(a)

$$\begin{aligned} x_1 + 2x_2 + 8x_3 - 7x_4 &= -2 \\ 3x_1 + 2x_2 + 12x_3 - 5x_4 &= 6 \\ -x_1 + x_2 + x_3 - 5x_4 &= -10 \end{aligned}$$

(b)

$$\begin{aligned} -w + x + 2y \qquad &= 3 \\ 2y + z &= 7 \\ 5w - 4x - 3y - 4z &= 9 \end{aligned}$$

Activity 3 (solution)

(a)

$$\left[\begin{array}{rrrr|r} 1 & 2 & 8 & -7 & -2 \\ 3 & 2 & 12 & -5 & 6 \\ -1 & 1 & 1 & -5 & -10 \end{array}\right]$$

$$\downarrow \quad \begin{array}{c} R_2-3R_1 \\ R_3+R_1 \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & 2 & 8 & -7 & -2 \\ 0 & -4 & -12 & 16 & 12 \\ 0 & 3 & 9 & -12 & -12 \end{array}\right]$$

$$\downarrow \quad R_3+\tfrac{3}{4}R_2$$

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$$\left[\begin{array}{rrrr|r} 1 & 2 & 8 & -7 & -2 \\ 0 & -4 & -12 & 16 & 12 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

Due to the the last row, which corresponds to the equation $0 = -3$, this system is inconsistent, therefore it has no solutions.

Activity 3 (solution, continued)

(b)

$$\left[\begin{array}{rrrr|r} -1 & 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 1 & 7 \\ 5 & -4 & -3 & -4 & 9 \end{array}\right]$$

$$\downarrow \quad R_3{+}5R_1$$

$$\left[\begin{array}{rrrr|r} -1 & 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 1 & 7 \\ 0 & 1 & 7 & -4 & 24 \end{array}\right]$$

$$\downarrow \quad R_2{\leftrightarrow}R_3$$

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$$\downarrow \quad \begin{array}{c} R_1{-}R_3 \\ R_2{-}\frac{7}{2}R_3 \end{array}$$

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$$\downarrow \quad \begin{array}{c} R_1-R_3 \\ R_2-\frac{7}{2}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & -1 & -4 \\ 0 & 1 & 0 & -\frac{15}{2} & -\frac{1}{2} \\ 0 & 0 & 2 & 1 & 7 \end{array}\right]$$

$$\downarrow \quad \begin{array}{c} R_1-R_2 \\ \frac{1}{2}\,R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & \frac{13}{2} & -\frac{7}{2} \\ 0 & 1 & 0 & -\frac{15}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{7}{2} \end{array}\right]$$

$$\downarrow \quad -1\,R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{13}{2} & \frac{7}{2} \\ 0 & 1 & 0 & -\frac{15}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{7}{2} \end{array}\right]$$

Activity 3 (solution, continued)

The simplified system is

$$\begin{aligned} w - \frac{13}{2}z &= \frac{7}{2} \\ x - \frac{15}{2}z &= -\frac{1}{2} \\ y + \frac{1}{2}z &= \frac{7}{2} \end{aligned}$$

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w, x, y are the leading variables and z is the free variable. We can parametrize $z = 2s$ (but there are many other parametrizations), so

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{2} + 13s \\ -\frac{1}{2} + 15s \\ \frac{7}{2} - s \\ 2s \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 7 \\ -1 \\ 7 \\ 0 \end{bmatrix} + s\begin{bmatrix} 13 \\ 15 \\ -1 \\ 2 \end{bmatrix}, \quad s \in \mathbb{R}.$$

This system is consistent and singular.

Free variables and rank

Based on our examples, we can make the following generalizations about a linear system in n variables, with coefficient matrix A:

- The number of leading variables $=$ rank(A).
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The last two statements imply that a linear system has either 0, 1 or infinitely many solutions.

Homogeneous and inhomogeneous systems

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- A homogeneous system has at least one solution, given by $\vec{x} = \vec{0}$; this solution is called the *trivial* solution. Any other solution is called *non-trivial*.

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- $\vec{0}$ denotes the *zero vector*; the zero vector in $\mathbb{R}^m$ is the vector with all m components equal to 0.
- A homogeneous system has at least one solution, given by $\vec{x} = \vec{0}$; this solution is called the *trivial* solution. Any other solution is called *non-trivial*.
- A homogeneous system is slightly easier to row reduce than an associated inhomogeneous system, since we can effectively ignore the last column of 0's in the augmented matrix $[A\ \vec{0}]$.

Solving an inhomogeneous system

If you are given (or have found) one *particular* solution $\vec{x}_p$ to the inhomogeneous system $A\vec{x} = \vec{b}$, then an efficient method to find the *general* solution can be done in two steps:

1. Find the general solution of the associated homogeneous system, $A\vec{x} = \vec{0}$.

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1. Find the general solution of the associated homogeneous system, $A\vec{x} = \vec{0}$.
2. The general solution of the inhomogeneous system $= \vec{x}_p +$ (the general solution from Step 1).

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If you are given (or have found) one *particular* solution $\vec{x}_p$ to the inhomogeneous system $A\vec{x} = \vec{b}$, then an efficient method to find the *general* solution can be done in two steps:

1. Find the general solution of the associated homogeneous system, $A\vec{x} = \vec{0}$.
2. The general solution of the inhomogeneous system $= \vec{x}_p +$ (the general solution from Step 1).

- In other words, any solution to the inhomogeneous system can be written as $\vec{x}_p +$ (a solution of the homogeneous system), and $\vec{x}_p +$ (any solution of the homogeneous system) is a solution to the inhomogeneous system.
- This is analogous to: the general solution to a non-homogeneous linear ODE $=$ (a particular solution) $+$ (the general solution of the associated homogeneous linear ODE); see Math2 Lecture 9.

Proof of why the method works

We want to prove:

$$\underbrace{\vec{x}_i \text{ is a solution to } A\vec{x} = \vec{b}}_{p} \quad \underbrace{\text{iff}}_{\Leftrightarrow} \quad \underbrace{\vec{x}_i = \vec{x}_p + (\text{a solution to } A\vec{x} = \vec{0})}_{q}$$

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Proof of $p \Rightarrow q$:

Let $\vec{x}_i$ be *any* solution to $A\vec{x} = \vec{b}$. Then

$$A(\vec{x}_i - \vec{x}_p) = A\vec{x}_i - A\vec{x}_p = \vec{b} - \vec{b} = \vec{0},$$

so $\vec{x}_i - \vec{x}_p$ is a solution to $A\vec{x} = \vec{0}$.

Therefore $\vec{x}_i = \vec{x}_p +$ (a solution to $A\vec{x} = \vec{0}$).

Proof of why the method works

We want to prove:

$$\underbrace{\vec{x}_i \text{ is a solution to } A\vec{x} = \vec{b}}_{p} \quad \underbrace{\text{iff}}_{\Leftrightarrow} \quad \underbrace{\vec{x}_i = \vec{x}_p + (\text{a solution to } A\vec{x} = \vec{0})}_{q}$$

Proof of $p \Rightarrow q$:

Let $\vec{x}_i$ be *any* solution to $A\vec{x} = \vec{b}$. Then

$$A(\vec{x}_i - \vec{x}_p) = A\vec{x}_i - A\vec{x}_p = \vec{b} - \vec{b} = \vec{0},$$

so $\vec{x}_i - \vec{x}_p$ is a solution to $A\vec{x} = \vec{0}$.

Therefore $\vec{x}_i = \vec{x}_p +$ (a solution to $A\vec{x} = \vec{0}$).

Proof of $q \Rightarrow p$:

Let $\vec{x}_h$ be *any* solution to $A\vec{x} = \vec{0}$. Then

$$A(\vec{x}_p + \vec{x}_h) = A\vec{x}_p + A\vec{x}_h = \vec{b} + \vec{0} = \vec{b},$$

so $\vec{x}_i = \vec{x}_p + \vec{x}_h$ is a solution to $A\vec{x} = \vec{b}$.

Activity 4 (10 minutes)

Consider the inhomogeneous system

$$
\begin{aligned}
w - x - y + 2z &= 2 \\
2w - 2x - y + 3z &= 3 \\
-w + x - y \phantom{{}+3z} &= 0
\end{aligned}
$$

(a) Guess a (simple) particular solution.

(b) Hence, write down the general solution.

Note: this system is very similar to Example 2 on Slide 10.

Activity 4 (solution)

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$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

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Parametrizing $x = s$, $z = t$, we obtain $w = s - t$, $y = t$.

The general solution is (a particular solution) + (the general solution of the homogeneous system), namely:

Activity 4 (solution, continued)

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} s - t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Activity 4 (solution, continued)

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A different particular solution would give a different looking general solution, but it would still represent the *same* set. E. g. with the particular solution $w = z = 0$, $x = y = -1$, the general solution would be

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad s_2, t_2 \in \mathbb{R},$$

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which equals the general solution at the top if we re-parametrize $s_2 = s + 1$, $t_2 = t + 1$.

Activity 5 (10 minutes)

Consider a homogeneous system $A\vec{x} = \vec{0}$ with m equations and n unknowns.

Prove that, if $n > m$, then the system has infinitely many solutions.

Hint: how large can the rank be?

Activity 5 (solution)

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Since a consistent system with at least one free variable has infinitely many solutions, the proof is complete.