10.007 Modelling the Systems World

Week 1 class 2: Systems of Linear Equations

Term 3, 2018

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Linear system

Definition: a *linear system* of m equations in n unknowns $(x_1, x_2, \dots, x_n)$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Each of the m equations above describes a *hyperplane* in n dimensions. Hyperplanes generalize lines and planes.

In the lecture example: $m = n = 2$, $x_1 = x$, $x_2 = y$, and the a_{ij}'s are constants:

$$\begin{aligned} \color{red}{a_{11}x_1 + a_{12}x_2 = b_1} &\qquad \longleftrightarrow \qquad \color{red}{2x + y = 8} \\ \color{blue}{a_{21}x_1 + a_{22}x_2 = b_2} &\qquad \longleftrightarrow \qquad \color{blue}{x - 3y = -3} \end{aligned}$$

Linear system as dot products

Each equation (hyperplane) in the linear system can be written as

$$\vec{a}_i \cdot \vec{x} = b_i,$$

where $\vec{a}_i = [a_{i1} \ a_{i2} \ \cdots \ a_{in}]$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Also, let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

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By stacking the row vectors $\vec{a}_i$ together, we have

$$\underbrace{\begin{bmatrix} \text{—} \ \vec{a}_1 \ \text{—} \\ \text{—} \ \vec{a}_2 \ \text{—} \\ \vdots \\ \text{—} \ \vec{a}_m \ \text{—} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}.$$

This allows us to write the m equations in the linear system as a single equation,

$$A\vec{x} = \vec{b}.$$

A is an $m \times n$ array of numbers, and is known as a **matrix**.
(The plural of matrix is 'matrices'.)

$A\vec{x} = \vec{b}$ is a matrix equation, and to solve this equation means to find $\vec{x}$, namely, to simultaneously solve for $x_1, x_2, \ldots, x_n$.

Activity 1 (10 minutes)

Consider this simplified scenario: taxi company A charges a flag-down fare of $\$3.99$ and subsequently $\$0.22$ for every 400m traveled, while taxi company B charges a flag-down fare of $\$3.00$ and subsequently $\$0.33$ for every 400m traveled.

Let us model this as a continuous problem, so if only 300m is traveled, then company B would charge $\$\big(3 + \frac{300}{400} \cdot 0.33\big) = \$3.25$.

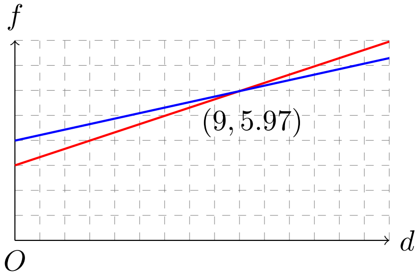
(a) On the same set of axes, sketch the fare versus distance traveled for both companies.

(b) For what distances is company A cheaper?

Activity 1 (solution)

(a) Let d be the distance traveled as a multiple of 400m, and let f be the fare in $\$$.

Company A's fare is given by $f = 3.99 + 0.22d$ (blue line); company B's fare is given by $f = 3 + 0.33d$ (red line).



Activity 1 (solution, continued)

(b) The point of intersection can be found by solving the two equations simultaneously: subtract the two equations to eliminate f, which gives

$$0 = 0.99 - 0.11d \quad \Rightarrow \quad d = 9.$$

So company A is cheaper when $d > 9$, that is, when the distance is greater than $9 \cdot 400 = 3600$m.

Activity 2 (10 minutes)

Consider the equation $2x + 4y - 3z = 5$.

(a) Find one solution to this equation.

(b) Subtract 2 from the x value of your solution and add 1 to the y value. Is the result still a solution?

(c) How many solutions are there to this equation? Can you find a way to write them all down?

Activity 2 (solution)

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(c) Infinitely many, since every point on the plane is a solution. Algebraically, you can start with any solution, and repeatedly apply the procedure in part (b) to generate infinitely many more.

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Let $x = s \in \mathbb{R}$ and $y = t \in \mathbb{R}$. Then $z = (2s + 4t - 5)/3$. Hence all solutions are given by $(s,\, t,\, (2s + 4t - 5)/3)$.

Here, s and t are called *parameters*, and they can take any real number as values.

Number of solutions

Consider the linear system for $m = n = 2$:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{aligned}
$$

Discussion: using a geometric interpretation, how many solutions can the system have?

Number of solutions

The system represents two lines in $\mathbb{R}^2$. Three possible cases can occur:

- When the lines are parallel and do not overlap, there are no solutions.
- When the lines are not parallel, there is 1 solution.
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Likewise, when $m = n = 3$, the linear system represents three planes in $\mathbb{R}^3$, and a solution to the system is a point common to all three planes.

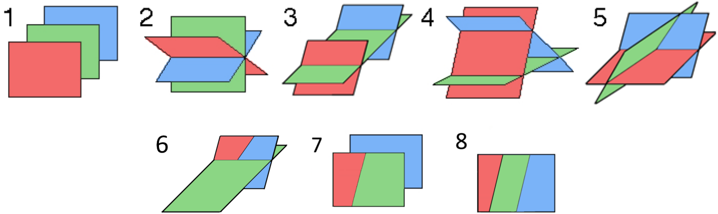
Informally speaking, if we pick three planes at random, then they 'usually' intersect at a unique point (think of two walls and the ceiling meeting at a point); but in 'rare' cases, other configurations can occur.

Activity 3(a) (15 minutes)

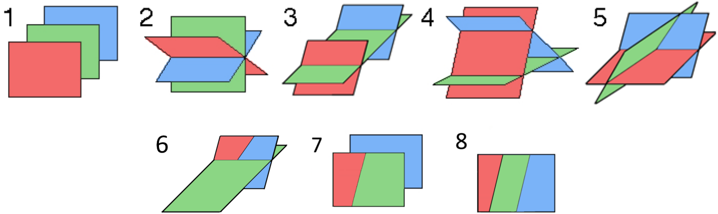
Describe or sketch all possible ways in which three planes can be configured.

Hint: there are 8 cases; one of them, in which three planes intersect at a point, has already been described to you.

Activity 3(a) (solution)



Activity 3(a) (solution)



Explanation:

- Case 6 is obtained by overlapping the red and blue planes in Case 3;
- Case 7 is obtained by overlapping the red and green planes in Case 1;
- Case 8 is obtained by overlapping all three planes in Case 1.

Types of linear systems

Definition: a linear system is called *consistent* if it has at least one solution, otherwise it has no solutions and is called *inconsistent*.

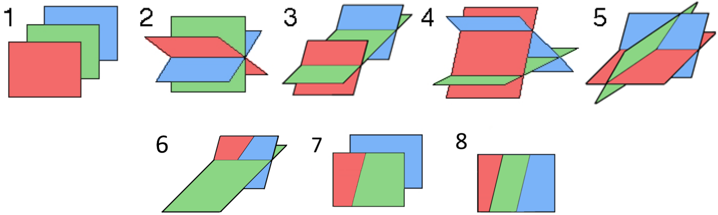
Definition: when a linear system has a unique solution, it is called nonsingular. Otherwise it has either 0 or infinitely many solutions (we will prove this next week), and is called singular.

Note: the word **singular** in this context means ‘exceptional’, and does *not* mean ‘single’.

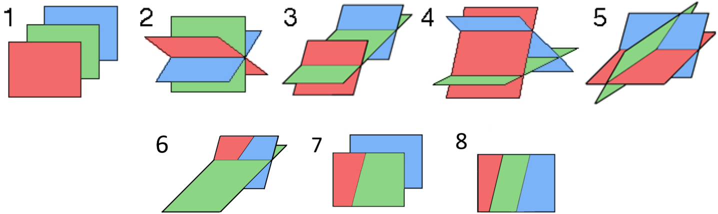
Activity 3(b) (5 minutes)

For each of the 8 ways that three planes can intersect in Activity 3(a), decide whether the underlying linear system is consistent, and whether it is singular.

Activity 3(b) (solution)



Activity 3(b) (solution)



1: Inconsistent, singular

2: Consistent, singular

3: Inconsistent, singular

4: Inconsistent, singular

5: Consistent, nonsingular

6: Consistent, singular

7: Inconsistent, singular

8: Consistent, singular

Activity 4 (15 minutes)

Consider the following linear system in $\mathbb{R}^3$:

$$u + v + w = 2 \tag{1}$$
$$u + 2v + 3w = 1 \tag{2}$$
$$v + kw = 0 \tag{3}$$

u, v, w are unknowns to be solved; k is a constant.

(a) Find all values of k such that the system is consistent. (Use elimination and substitution.)

(b) Solve the system.

(*Extension*) Investigate the system if the 0 on the right hand side of equation (3) is replaced by -1.

Activity 4 (solution)

Equation (2) $-$ equation (1) eliminates u:

$$v + 2w = -1. \tag{4}$$

Equation (3) $-$ equation (4) eliminates v:

$$(k-2)w = 1. \tag{5}$$

Activity 4 (solution)

Equation (2) − equation (1) eliminates u:

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Equation (3) − equation (4) eliminates v:

$$(k-2)w = 1. \tag{5}$$

So as long as $k \neq 2$, $w = 1/(k-2)$, which can be substituted into equation (4) to find v, which can then be used to find u.

Thus the system is consistent when $k \neq 2$, and the solution is

$$(u, v, w) = \left(\frac{3k-5}{k-2}, \frac{-k}{k-2}, \frac{1}{k-2} \right).$$

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(Check that $k = 2$ corresponds to Case 4 in the Activity 3 solutions.)

Activity 4 (extension solution)

Equation (2) $-$ equation (1) eliminates u:

$$v + 2w = -1. \tag{6}$$

Equation (3) $-$ equation (6) eliminates v:

$$(k - 2)w = 0. \tag{7}$$

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Equation (3) − equation (6) eliminates v:

$$(k-2)w = 0. \tag{7}$$

So as long as $k \neq 2$, $w = 0$, which can be substituted into equation (6) to find $v = -1$, which can then be used to find $u = 3$.

Thus for $k \neq 2$, the solution does not depend on k and is a single point:

$$(u, v, w) = (3, -1, 0).$$

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Thus for $k \neq 2$, the solution does not depend on k and is a single point:

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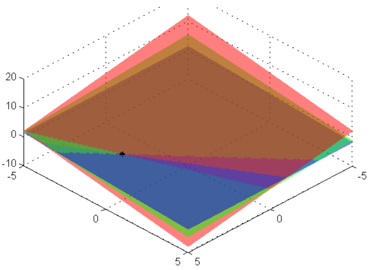
If $k = 2$, then w can be any number and equation (7) is still satisfied. Let $w = s$, for $s \in \mathbb{R}$. Then from (6), $v = -1 - 2s$, and from (1), $u = 3 + s$. So for $k = 2$, the solution is a *straight line*:

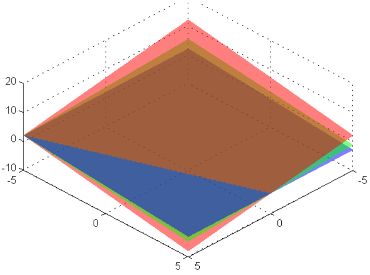
$$(u, v, w) = (3 + s, -1 - 2s, s) = (3, -1, 0) + s(1, -2, 1).$$

Activity 4 (extension solution)

Therefore the system is consistent for all k. For $k \neq 2$ the solution is unique (nonsingular, Case 5 in the Activity 3 solutions); for $k = 2$ there are infinitely many solutions on a straight line (singular, Case 2 in the Activity 3 solutions).



$k = 3$



$k = 2$

Activity 5 (5 minutes)

Write the following linear systems in the form $A\vec{x} = \vec{b}$.

(a)

$$\begin{aligned} 2x + y &= 8 \\ x - 3y &= -3 \end{aligned}$$

(b)

$$\begin{aligned} 2x_2 - 11x_4 &= 10 \\ x_3 + 4x_1 - 3x_2 &= \frac{3}{2} \\ -\frac{1}{3} + 5x_4 + 3x_2 &= 7x_1 - x_3 \end{aligned}$$

Activity 5 (solution)

(a)

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 8 \\ -3 \end{bmatrix}}_{\vec{b}}.$$

Activity 5 (solution)

(a)

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 8 \\ -3 \end{bmatrix}}_{\vec{b}}.$$

(b) Some care needs to be taken, since in the linear system, the unknowns are not written in the same order.

$$\begin{bmatrix} 0 & 2 & 0 & -11 \\ 4 & -3 & 1 & 0 \\ -7 & 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{3}{2} \\ \frac{1}{3} \end{bmatrix}.$$