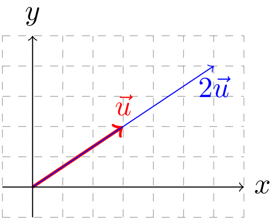
10.007 Modelling the Systems World

Week 1 class 1: Vectors

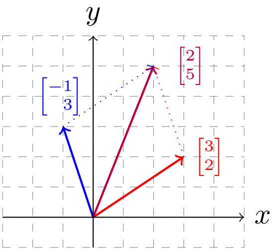
Term 3, 2018

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

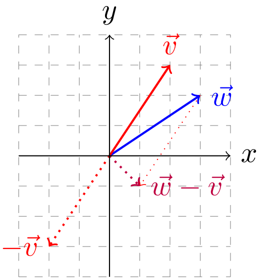
Vectors – operations



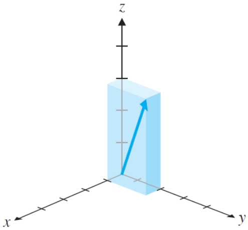
Scalar multiplication



Vector addition



Vector subtraction



A vector in $\mathbb{R}^3$

Dot product

If $\vec{u} = [u_1 \ u_2 \ \cdots \ u_n]$ and $\vec{v} = [v_1 \ v_2 \ \cdots \ v_n]$, then their **dot product** is given by

$$\vec{u} \cdot \vec{v} = \sum_{j=1}^{n} u_j v_j = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Reminder: $\displaystyle\sum_{j=a}^{b} f_j = f_a + f_{a+1} + \cdots + f_{b-1} + f_b.$

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The dot product is connected with lengths and angles:

The length, or **norm**, of a vector $\vec{v}$ is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}.$$

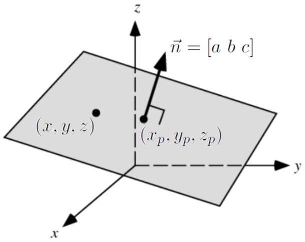
Let θ be the **angle** between non-zero vectors $\vec{u}$, $\vec{v}$; then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

Equation of a plane

Recall that the general equation of a plane in $\mathbb{R}^3$ is

$$ax + by + cz = d, \qquad \text{where}$$



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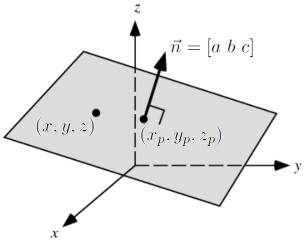

- The row vector $\vec{n} = [a\ b\ c]$ is perpendicular to the plane (it is called a *normal* vector);
- $d = ax_p + by_p + cz_p$, where (x_p, y_p, z_p) is any given point on the plane;
- with $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the equation can be written as $\vec{n} \cdot \vec{w} = d$.

Activity 1 (10 minutes)

(a) Find real numbers (*scalars*) a and b such that

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Give a geometric interpretation of your answer.

(b) Find the angle between $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Activity 1 (solution)

(a) Viewing the equation component-wise, we obtain a linear system:

$$2a - b = 4, \tag{1}$$

$$a + 3b = 6. \tag{2}$$

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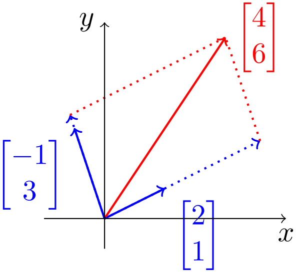
Equation (1) $-$ twice equation (2) gives $-7b = -8$, so $b = \frac{8}{7}$.

Substituting this back into either equation (1) or (2), we find that $a = \frac{18}{7}$.

Activity 1 (solution, continued)

Geometrically, the two blue vectors can be 'combined' (stretched by the right amounts then added) to give the red vector:



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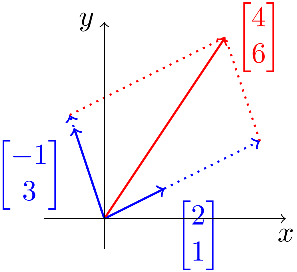

(b) Let $\vec{u} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}-1\\3\end{bmatrix}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-2+3}{\sqrt{5}\sqrt{10}} = \frac{1}{5\sqrt{2}},$$ so $\theta \approx 81.9^\circ$ or 1.43 radians.

Activity 2 (15 minutes)

Prove the following statement using vectors:

In $\mathbb{R}^2$, two lines with slopes m_1 and m_2 are perpendicular iff (if and only if) $m_1 m_2 = -1$.

Activity 2 (solution)

Recall that slope = 'rise over run'.

Therefore, a line with slope m_1 points along the direction of the vector $\begin{bmatrix} 1 \\ m_1 \end{bmatrix}$.

Similarly, a line with slope m_2 points along the direction of the vector $\begin{bmatrix} 1 \\ m_2 \end{bmatrix}$.

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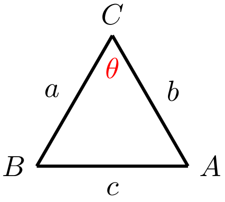
Two (non-zero) vectors are perpendicular iff the angle between them is 90°, iff their dot product is 0.

Therefore, the two lines are perpendicular iff $\begin{bmatrix} 1 \\ m_1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ m_2 \end{bmatrix} = 0$, that is, iff $1 \cdot 1 + m_1 \cdot m_2 = 0$, or $m_1 m_2 = -1$.

Activity 3 (15 minutes)

Recall the *cosine law*, which states that in a triangle,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$



Use the cosine law to *prove* the property $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$ in $\mathbb{R}^2$.

Hint: let $\vec{u} = \overrightarrow{CB}$ and $\vec{v} = \overrightarrow{CA}$.

Activity 3 (solution)

With $\vec{u} = \overrightarrow{CB}$ and $\vec{v} = \overrightarrow{CA}$, $\overrightarrow{BA} = \vec{v} - \vec{u}$. Then, applying the cosine law to the triangle ABC, we have:

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

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Expand out each of the squared norms explicitly, then simplify:

$$\begin{aligned} (v_1 - u_1)^2 + (v_2 - u_2)^2 &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta \\ -2v_1u_1 - 2v_2u_2 &= -2\|\vec{u}\|\|\vec{v}\| \cos\theta \\ \vec{u} \cdot \vec{v} &= \|\vec{u}\|\|\vec{v}\| \cos\theta. \end{aligned}$$

Dot product and inequalities

Since $|\cos\theta| \leq 1$, the formula $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$ leads to

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|, \quad \forall \text{ (for all) } \vec{u},\, \vec{v} \in \mathbb{R}^n.$$

This is known as the *Cauchy-Schwarz inequality* and is a surprisingly versatile result.

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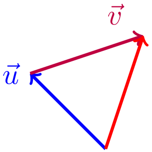
Among other things, the Cauchy-Schwarz inequality implies the *triangle inequality*,

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|, \quad \forall\ \vec{u},\, \vec{v} \in \mathbb{R}^n.$$

The triangle inequality is also a widely used mathematical result.

Discussion: why is it called the triangle inequality?

Triangle inequality



If $\vec{u}$ and $\vec{v}$ denote two sides of a triangle, then $\vec{u} + \vec{v}$ gives the third side.

The inequality states that in $\mathbb{R}^n$, the sum of the lengths of two sides of a triangle is at least as great as the length of the third side.

Triangle inequality – proof

Start with the square of the left hand side:

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2+2(\vec{u}\cdot\vec{v})+\|\vec{v}\|^2 \\
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&\le \|\vec{u}\|^2+2\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|^2,
\end{aligned}$$

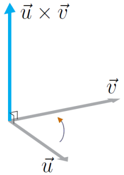
where we have applied the Cauchy-Schwarz inequality in the last line. The right hand side factorizes, so we obtain

$$\|\vec{u}+\vec{v}\|^2 \le \big(\|\vec{u}\|+\|\vec{v}\|\big)^2 \quad \Rightarrow \quad \|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|.$$

Cross product

Recall that the **cross product** of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}.$$



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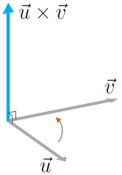

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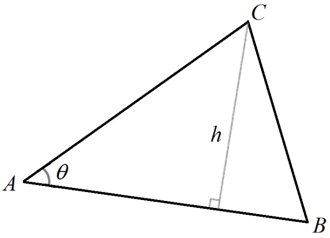
- $\vec{u} \times \vec{v}$ is *perpendicular* to both $\vec{u}$ and $\vec{v}$, and points in the direction given by the *right hand rule*.
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

Activity 4 (5 minutes)

Consider a triangle ABC in $\mathbb{R}^3$. Let $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AC}$.

Express the area of the triangle using a suitable cross product.

Activity 4 (solution)



Let $\overrightarrow{AB}$ be the base of the triangle. Then

$$\begin{aligned}
\text{area} &= \frac{1}{2} \cdot \text{base} \cdot \text{height} \\
&= \frac{1}{2}\|\overrightarrow{AB}\| \left(\|\overrightarrow{AC}\| \sin\theta\right) \\
&= \frac{1}{2}\|\vec{u}\|\|\vec{v}\| \sin\theta \\
&= \frac{1}{2}\|\vec{u} \times \vec{v}\|.
\end{aligned}$$

Activity 5 (20 minutes)

(a) Find a normal vector to the plane $4x + y + 2z = 15$.

(b) Find the equation of the plane that is perpendicular to $[1\ 1\ 1]$ and contains the point $(1, 2, 3)$.

(c) Check that the planes $4x - y + 5z = 2$ and $2x + 3y - z = 1$ are perpendicular.

(d) Find the equation of the plane that passes through $A = (1, 1, 1)$, $B = (4, 0, 2)$ and $C = (0, 1, -1)$.
Hint: use a suitable cross product.

Activity 5 (solution)

$$ax + by + cz = d = ax_p + by_p + cz_p$$

(a) A normal vector to $4x + y + 2z = 15$ is $[4\ 1\ 2]$.
(Can you find a different one?)

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(b) A normal vector is $[a\ b\ c] = [1\ 1\ 1]$; hence the equation of the plane is: $1 \cdot x + 1 \cdot y + 1 \cdot z = d = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3$, or $x + y + z = 6$.

You can *check* that the point $(1, 2, 3)$ indeed lies on the plane, since it satisfies the equation $x + y + z = 6$.

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The dot product of the normal vectors is $[4\ \ -1\ \ 5] \cdot [2\ \ 3\ \ -1]$ $= 8 - 3 - 5 = 0$, hence the planes are perpendicular.

Activity 5 (solution, continued)

(d) Consider the vectors $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$. They are both parallel to the plane, and hence their cross product is perpendicular to the plane.

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So a normal vector is given by $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

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The equation of the plane is given by

$$\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\Rightarrow\; 2x + 5y - z = 6.$$

Again, you can check your answer by verifying that A, B, C all satisfy this equation.