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MLE	Maximum Likelihood Estimate
cdf	cummulative density function
pdf	probability density function

1 Answer to Workbook Question

1.1 Question One

a) There are two variables X and Y. The realization of a sample of size 20 is given below (where X is the first variable and Y is the second):

X	Y
-8.504	-25.44
02.371	6.51
2.176	288.73
0.627	-421.32
-8.429	5.46
2.759	476.25
3.268	4.14
-9.362	-366.79
6.364	92.37
3.569	612.29
6.969	-511.51
-0.466	268.67
4.102	-350.29
-8.282	234.35
0.509	185.54
-0.442	-217.03
-3.841	-107.11
7.938	367.32
-1.062	-1019.37
-3.655	-338.54

Table 1: Table of X & Y values

You are to sketch an appropriate plot that displays the values of these points. Now calculate the sample covariance as well as the sample's expectations and variances of X and Y.
b) If a ball is thrown with a random angle $\theta \in (0, 360]$ (in degrees) and a random radius $r \in (0, 1]$ (in meters) both independent and uniform. Calculate the density of the variable X and Y (the cartesian coordinates of the point at angle θ and radius r as well as their expectation and variance.

1.1.1 Answer to Q1(a)

We will display the graphic in R using a scatter diagram because the random variables X and Y are suitable for a scatter plot.
The code for the generation of this graphic plot is (Using R):

```
x = c(-1.504, 0.371, 2.176, -3.627, -8.429, 2.759, 3.268,
-9.362, -2.364, 3.569, 6.969, -0.466, 4.102, -8.282, 0.509,
-0.422, -3.841, 7.938, -1.062, -3.655)

y = c(-235.44, 166.51, 288.73, -421.32, 95.46, 476.25,
```

```
4.14,-366.79, 92.37, 612.29, -511.51, 268.67, -350.29, 234.35,
185.54, -217.03, -107.11, 367.32, -1019.37, -338.54)

plot(y ~ x )
```

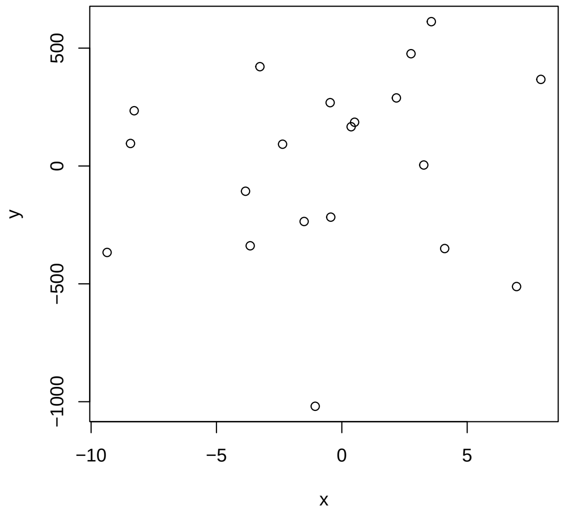

Figure 1: A graph of Probability density function for Question 2b.

```
abc = data.frame(x, y)
summary(abc)
```

X	Y
Min. :-9.3620	Min. :-1019.37
1st Qu.:-3.6340	1st Qu.: -341.48
Median :-0.4440	Median : 48.26
Mean :-0.5676	Mean : -38.79
3rd Qu.: 2.8862	3rd Qu.: 242.93
Max. : 7.9380	Max. : 612.29

Table 2: Table of summary of X & Y values

Thus,

$$\mu = \frac{\Sigma x_i}{n} \tag{1.1}$$

$$\mu_x = \frac{-11.37}{20} = -0.5686$$

$$\mu_y = \frac{-775.77}{20} = -38.79$$

And,

$$variance = \frac{\Sigma(x_i - \mu_x)^2}{n-1} \tag{1.2}$$

$$variance, \sigma_x = \frac{436.04}{19} = 22.95$$

$$variance, \sigma_y = \frac{2960947.5}{19} = 155839.3$$

We can calculate the covariance in R (Davies, 2016, p. 281)or by using equation 2.5.2 in (Hogg et al., 2019, p. 126) thus:

$$Cov(X, Y) = E(XY) - \mu_x \mu_y \tag{1.3}$$

Therefore,

$$Cov(X, Y) = 381.01 - (-38.7)(-0.5686)$$

$$Cov(X, Y) = 359.0$$

1.1.2 Answer to Q1(b)

From equation 2.1.5 in Hogg et al., 2019, the joint cummulative density function (cdf) of the random vector (θ, r) of space (0,360),(0,1):

$$F_{X,Y} = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x, y) dx dy \tag{1.4}$$

And the marginal PDF's are:

$$f(\theta) = \frac{1}{2\pi}, \theta \in [0, 2\pi)$$

$$f(r) = 1, r \in [0, 1)$$

And we can obtain the PDF for r and θ respectively,

$$f(\theta, r) = \frac{1}{2\pi}, r \in [0, 1), \theta \in [0, 2\pi)$$

We can invoke the polar to coordinates conversion for use here,

$$x = rcos\theta, y = rsin\theta, r^2 = \sqrt{x^2 + y^2}, \theta = \arctan \frac{y}{x} \tag{1.5}$$

We would obtain our Jacobian function and substitute into this expression:

$$f_(R, \Phi)(r, \phi) = f_(X, Y)(g^{-1}(x, y)).|J_g^{-1}(x, y)| \tag{1.6}$$

$$\begin{pmatrix} r \\ \\ \phi \end{pmatrix} = g \begin{pmatrix} x \\ \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2} \\ \\ \arctan \frac{y}{x} \end{pmatrix} \rightarrow Dg^{-1}(x, y) = \begin{pmatrix} \frac{\partial \sqrt{x^2+y^2}}{\partial x} & \frac{\partial \sqrt{x^2+y^2}}{\partial y} \\ \\ \frac{\partial \arctan \frac{y}{x}}{\partial x} & \frac{\partial \arctan \frac{y}{x}}{\partial y} \end{pmatrix} \rightarrow J = \sqrt{x^2 + y^2}$$

Which yields the joint pdf:

$$f_{x,y}(x, y) = \frac{1}{2\pi \cdot \sqrt{x^2 + y^2}}$$

and marginal PDF,

$$f_x(x) = \frac{1}{2\pi} \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2 + y^2}} dy = \frac{1}{4\pi} (log(\sqrt{1 - x^2} + 1) - log(1 - \sqrt{1 - x^2})) = 0, x \in [0, 1)$$

$$f_y(y) = \frac{1}{2\pi}\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{x^2+y^2}}dx = \frac{1}{4\pi}(log(\sqrt{1-y^2}+1) - log(1-\sqrt{1-y^2})) = \frac{3}{2\pi}, y \in [0,1)$$

And expectation of x yields,

$$E(X) = \int_{-\infty}^{\infty} x f_x(x,y)dy$$

But more precisely, we can utilize the formula for uniform distribution since the pdfs for x and y are both uniform distributions. Hence,

$$E(X) = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

$$E(X) = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

and the variance for x and y,

$$E(X) = \frac{(b-a)^2}{12} = \frac{(1-0)^2}{12} = 0.0833$$

$$E(Y) = \frac{(b-a)^2}{12} = \frac{(1-0)^2}{12} = 0.0833$$

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