1. Let's verify $(n-1)S_n^2 \sim \chi^2_{n-1}$ directly. Consider the standard normal random vector $X = (X_1, \ldots, X_n)$. Its covariance matrix is the identity matrix $\Sigma = I_n$. This means that X_i and X_j are independent and $\mathrm{Var}(X_i) = 1$.
   (a) Show that for a matrix $A \in \mathbb{R}^n$, if A is orthonormal (i.e., $A^T A = I_n$), then $Y = AX$ (it is a linear transformed random vector) is also a standard normal vector.
   (b) Let A be an orthonormal matrix and its first row be $(n^{-1/2}, \ldots, n^{-1/2})$.[1] So $Y_1 = \sqrt{n}\bar{X}, Y_2, \ldots, Y_n$ is a standard normal random vector. Then by the orthonormality of A, show that $\sum_{i=2}^n Y_i^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2$. Therefore, $(n-1)S^2$ is χ^2_{n-1}. (Hint: use the fact that $\sum_{i=1}^n Y_i^2 = (AX)^T AX = X^T A^T AX = \sum_{i=1}^n X_i^2$.)
2. (Hard) We have a rope of length 1m. We cut it n times, and each cutting point is independent Uniform$[0,1]$. Out of the $n+1$ pieces, what is the expected value of the lenght of the shortest piece?
   (a) Suppose $X_1 < \cdots < X_n$ are the positions of the cutting points. In addition, let $X_0 = 0$ and $X_{n+1} = 1$. Then $V_i = X_i - X_{i-1}$ are the length of the pieces. Explain that for $c_1, \ldots, c_{n+1}$ and $\sum_{i=1}^{n+1} c_i < 1$, $\mathsf{P}(V_1 \geq c_1, \ldots, V_{n+1} \geq c_{n+1}) = (1 - c_1 - \cdots - c_{n+1})^n$.
   (b) Find $\mathsf{E}[V_{(1)}]$.
3. Let $X \sim \mathit{Binomial}(100, 0.7)$. Approximate $30 \leq X \leq 40$ using normal distribution. (write in the form of $\Phi(x)$.)
4. Let $X_1, X_2, \cdots, X_n$ independently follows $\mathit{exponential}(1)$ and $X_{(n)} = max_{1 \leq i \leq n} X_i$. Find a sequence a_n so that $X_{(n)} - a_n$ converges in distribution.

[1] Why this is always possible? Think of each row of A as an orthornormal basis in $\mathbb{R}^n$. We fix one of them, we can always find the remaining $n-1$ bases in the subspace. Mathematically, we can use orthonormal decomposition to find rows of A sequentially.